Purity reigns supreme*

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... *Puro infino al primo giro* (Purgatorio, Canto 1)

The purpose of this note is to prove the Ramanujan conjecture for cuspidal representations π of $GL(n, \mathbb{A}_F)$ when F is either a totally real or a CM field, and π is a cohomological representation that is self–dual if F is totally real, or conjugate self–dual if F is CM. We will prove the conjecture only at primes v of F where all data are unramified. If p is the rational prime divided by v, this means that F is unramified at p, and that all factors $\pi_{v'}$ of π for the primes $v'|p$ are unramified.

That such a result is accessible has been known since the work of the author [7] relying on Kottwitz's description of particular Shimura varieties. Increasingly precise and general variants have been proved by Harris and Taylor [10] and then, recently, by S.–W. Shin [18] and as a result of the collective effort embodied in [S]. See in particular the final chapter [9] by Harris, Labesse and the author. S. Morel has proved [17], in particular cases, a result true for an unspecified set of primes.

A cohomological representation is associated to a finite – dimensional representation L of the reductive group G – here $GL(n, F)$ – being considered ; in the geometric cases L can also be seen as a coefficient system on the associated Shimura variety. At least for even n, the recent proofs [9, 18] of the Ramanujan conjecture require a regularity property of the highest weight of L : Shin calls L "mildly regular" in [18]. We will show that this assumption is unnecessary, at least at the primes of good reduction.

We refer to [6] for the notion of cohomological or algebraic representation, and for the attendant properties. If F is complex denote by c the complex conjugation ; it acts naturally on representations of $GL(n, \mathbb{A}_F)$.

*Provisional title

We now state the main result.

Theorem. *Assume F is either a totally real or a CM field. Assume π is a cuspidal representation of $GL(n, \mathbb{A}_F)$, $\pi = \bigotimes_v \pi_v$, and*

(i) $\pi \cong \widetilde{\pi}$ *(F real) or* $\pi \cong \widetilde{\pi}^c$ *(F CM).*

(ii) π *is cohomological.*

If p is a prime at which F is unramified, if the prime v of F divides p, and if $\pi_{v'}$ is unramified for all $v'|p$, π_v is a tempered representation.

Without the regularity assumption, Chenevier and Harris [4] have proved that there exists a compatible system (r_λ) of Galois representation of $\mathrm{Gal}(\overline{F}/F)$ associated to π, λ ranging over the set of finite primes of the field of coefficients E for π ; r_λ has image in $GL(n, \overline{\mathbb{Q}}_\ell)$ where ℓ is the rational prime divided by λ. Theorem 1 then implies (cf. § 3) :

Corollary. *r_λ is pure of weight $n-1$, i.e., for any prime v of F not dividing ℓ, and unramified in the previous sense, r_λ is unramified at v and the eigenvalues of a geometric Frobenius element* $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{F}/F)$ *are Weil numbers of weight* $(n-1)$.

Remark.– Even though the system of representations (r_λ) is pure, it is not "motivic". The representations of Chenevier and Harris are obtained by ℓ–adic interpolation, and one does not know, so far, how to exhibit a "motive" associated to π, even after a suitable base change.

Acknowledgement.– This theorem is for me the culmination of a story started in 1988. That this result could be proven became clear in the Spring 2009. Discussions with Michael Harris suggested however that technical problems might arise in the use of Labesse's trace formula results. By the end of 2009, it was evident that these fears were unwarranted. Indeed R. Taylor then informed me that he could also prove the theorem, using a slightly different method. The problem remains open at ramified primes.

I also recall that for n odd the full result, i.e., the fact that π_v is tempered at all primes, and the compatibility of the representation r_λ with the local Langlands correspondence, is proved by Shin [18]. (For even n, the same is true under the regularity assumption). I will therefore concentrate on the even case.

1 Some simple reductions

Assume first that F, π are as in the theorem and that F is totally real. As usual we reduce to the CM case. Our extra data are p and $v|p$. We can choose a quadratic imaginary field E unramified at p and replace F by $F' = EF$. By base change π defines a representation π' of $GL(n, \mathbb{A}_{F'})$ [1]. If $\pi \cong \widetilde{\pi}$, $\pi' \cong \widetilde{\pi}' \cong (\pi')^c$. The results of [1] imply that π' is cohomological. We are therefore reduced to the CM case.

We must however preserve the cuspidal character of π. If ε is the Artin character of F associated to EF/F, π' is cuspidal if $\pi \otimes \varepsilon \ncong \pi$. If $\pi \otimes \varepsilon \cong \pi$, we know from [1, Thm. 3.4.2] that, with the notations introduced there :

$$\pi' = \rho \boxplus \rho^c$$

where ρ is a cuspidal representation of $GL(m, \mathbb{A}_{F'})$ and c is complex conjugation on F'. Since $\widetilde{\pi'} \cong \pi'$ we have $\widetilde{\rho} \cong \rho^c$ or $\widetilde{\rho} \cong \rho$.

Consider a real prime v of F, and the associated complex prime w of F'. Since π_v is cohomological and self–dual, the Langlands parameter of π'_w is given by n characters of $\mathbb{C}^\times$ of the form

$$(z/\bar{z})^{p_1}, \ldots, (z/\bar{z})^{p_n}\ ,\ (z/\bar{z})^{-p_n}, \ldots, (z/\bar{z})^{-p_1}$$

with $p_i \in \frac{1}{2} + \mathbb{Z}$ and $(p_i, -p_i)$ all distinct. Up to a reordering, the Langlands parameter of ρ_w is then

$$(z/\bar{z})^{p_1}, \ldots, (z/\bar{z})^{p_n}$$

since the conjugate parameter appears in ρ^c.

The condition $\widetilde{\rho}_w \cong \rho_w$ would this imply that $p_i = -p_j$, which is impossible. Thus $\widetilde{\rho} \cong \rho^c$ and we are reduced to the Theorem for $m = n/2$. We are reduced to a smaller, even degree, or to the odd case which is known.

Note that there is certainly a choice of E such that π' is cuspidal. It is likely that this can be proved by using the associated L–functions. We can also use the existence of the Galois representation, proved by Chenevier and Harris. Indeed, if $\pi \otimes \varepsilon \cong \pi$, the Artin character ε occurs (for any λ) in the semi–simplification of $R_\lambda \otimes R_\lambda \otimes \omega^{-(n-1)}$. (Recall that there is a natural pairing $R_\lambda \times R_\lambda \to \overline{\mathbb{Q}}_\ell(n-1)$; $\omega : \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_\ell^\times$ is the cyclotomic character). This leaves a finite set of choices for ε, thus for FE and therefore for $E \subset FE$.

We now consider only the CM case. We write F^+ for the maximal real subfield of F. Our data are again p, v and π_v, all unramified. By replacing F

by FE where E is a quadratic imaginary field, split at p and linearly disjoint from F, we are reduced to a field $F = F^+E$ (change notation) with E split at p. We use the foregoing argument on the Galois representation to ensure that the new representation, obtained by base change, remains cuspidal. It is conjugate self–dual, cf. [1, Prop. 3.4.4]. We may further consider the primes v' of F where $\pi_{v'}$ is **ramified**, and the set S of rational primes below these primes. Clearly we can so choose E that each prime $q \in S$ splits in E. Finally, note that all our data (the new field F, the new representation π) remain unramified at p.

If we did not avail ourselves of the Galois representation, we would be bound to find a direct proof of the existence of E by using L–functions. (The argument, based on the Archimedean composants, that we used in the first reduction does not apply in the complex case). We leave this interesting exercise to the reader.

We are then reduced to prove

Proposition 1.1. *Assume F CM, $\widetilde{\pi} \cong \pi^c$, p unramified in F, and $F = EF^+$ with the previous conditions (cf. Hypothesis 2.1). Assume p unramified in F and $\pi_p = \bigotimes_{v|p} \pi_v$ unramified. Then, for all $v|p$, π_v is tempered.*

2 Preparation for the proof

2.1

For definiteness we assume n even. As we shew in § 1 we can make the following assumptions.

Hypothesis 2.1.

(i) $F = F^+E$ where E is quadratic imaginary.

(ii) p is a rational prime where F^+ is unramified and E split.

(iii) π_w is unramified for any place w of F dividing p.

(iv) Any place w of F where π is ramified divides a rational prime q split in E.

There exists a quasi–split unitary group G_0^*, relative to the extension F/F^+ and defined over F^+. At Archimedean primes v of F^+, $G_0^*(F_v^+)$ is isomorphic to $U(m, m)$ where $n = 2m$. We can twist G_0^*, by the results in [6, Ch. 2] so as to get a unitary group G_0/F^+, still quasi–split at finite primes, and of the following type at infinity :

(2.1) G_0 is compact, isomorphic to $U(n)$, at all Archimedean primes but one, denoted by v_0. Moreover $G_0(F^+_{v_0}) \cong (2, n-2)$.

If m and $d = [F^+ : \mathbb{Q}]$ are odd, it is possible to obtain G_0 verifying the same condition but with $G_0(F^+_{v_0}) \cong U(1, n-1)$. This leads to a stronger result, cf. [18, p. 6].

We denote by G the $\mathbb{Q}$–group obtained from G_0 by restriction of scalars. Denote, as in [9], by GU the group of unitary similitudes (with rational coefficient) deduced from G. If H is the vector space of dimension n over F, endowed with the Hermitian form $(.,.)$ defining G_0, we have for any commutative $\mathbb{Q}$–algebra R

$$GU(R) = \{G \in GL(H \bigotimes_{\mathbb{Q}} R) : (gv, gw) \equiv \lambda(g)(v, w),\ \lambda(g) \in R^+\}\,.$$

There is then a natural exact sequence of $\mathbb{Q}$–groups :

$$1 \to G \to GU \underset{\lambda}{\to} \mathbb{G}_m \to 1\,.$$

This sequence splits after extension of scalars to E. Indeed, denote by g^* the adjoint of $g \in GL(H)$ for the Hermitian form. The prime of $GU(\mathbb{Q})$ are given by

$$\{g \in GL(H) : gg^* = \lambda \in \mathbb{Q}^\times\}\,. \tag{2.2}$$

We have $G(E) = G_0(E \otimes F^+) = G_0(F) = GL(H)$ and the Galois automorphism on $G(E) \times E^\times$:

$$(g, \lambda) \longmapsto ((g^*)^{-1}\bar{\lambda}, \bar{\lambda}) \tag{2.3}$$

has for fixed points the set (2.2). This extends to any couple R, $R \otimes E$ where R is a $\mathbb{Q}$–algebra, whence our assertion by Galois descent :

$$GU_E \equiv G_E \times GL(1)_E$$

where $G_E = \mathrm{Res}_{F/E} GL(n)_F$.

A Shimura variety for GU is defined by the datum of $h : \mathbb{C}^\times = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GU(\mathbb{R})$. The group $GU(\mathbb{R})$ is the subgroup of

$$\prod_{v|\infty} GU(p_v, q_v) \qquad (v \text{ a prime of } F^+)$$

defined by the equality of the similitude ratios. We may assume $(p_{v_0}, q_{v_0}) = (2, n-2)$, $(p_v, q_v) = (0, n)$ for $v \neq v_0$, and $h = (h_v)$ is defined by

$$z \longmapsto (z, \bar{z}) \quad \text{(diagonal matrices of size } (2, \mathrm{n}-2))$$

at v_0,

$$z \longmapsto \bar{z} \quad \text{(diagonal) at } v \neq v_0\,.$$

Note that the image by ν is then

$$z \longmapsto z\bar{z}\,.$$

The reflex field for the family of Shimura varieties $S_K(G, h)$ associated to compact open subgroups $K \subset G(\mathbb{A}_f)$ is F.

Let π be our given representation of $G_0(\mathbb{A}_F) = G(\mathbb{A}_E) \cong GL(n, \mathbb{A}_F)$. Denote by θ the Galois automorphism of $GU(\mathbb{A}_E) \cong G(\mathbb{A}_E) \times \mathbb{A}_E^\times$ given by (2.3).

Lemma 2.2. *There exists an algebraic Grössencharakter χ of $\mathbb{A}_E^\times$ such that $\pi \otimes \chi$ (exterior tensor product) is stable by θ.*

Denote by $z \in \mathbb{A}_E^\times$ the similitude ratio λ of (2.3). The automorphism θ sends $\pi(g) \otimes \chi(z)$ to $\pi((g^*)^{-1}\bar{z}) \otimes \chi(\bar{z}) \cong \tilde{\pi}^c \otimes \omega_\pi(\bar{z})\chi(\bar{z})$. The condition is therefore $\omega_\pi(\bar{z}) = \chi(z)/\chi(\bar{z})$.(We have denoted complex conjugation on $\mathbb{A}_E$ by $z \to \bar{z}$).

At each real prime v, ($\equiv$ each complex prime w of F) the Langlands parameter of π_w is of the form

$$z \longmapsto ((z/\bar{z})^{p_1}, \ldots (z/\bar{z})^{p_n}) \qquad (z \in \mathbb{C}^\times)$$

with $p_i \in \frac{1}{2} + \mathbb{Z}$ [6]. At w, ω_π is therefore given by $z \longmapsto (z/\bar{z})^{P_v}$ with $P_v \in \mathbb{Z}$ since n is even.

Since $E^\times$ is embedded diagonally into $F^\times$, $\omega_\pi(z)$, for $z \in E_\infty^\times$, is of the form $(z/\bar{z})^P$. However, $\omega_\pi = {}^c\omega_\pi^{-1}$, so $\omega_\pi(z\bar{z}) = 1$ for $z \in \mathbb{A}_E^\times$. By Hilbert's theorem 90, we have the exact sequence

$$\longmapsto \mathbb{Q}^\times \setminus \mathbb{A}^\times \longrightarrow E^\times \setminus \mathbb{A}_E^\times \longrightarrow U(\mathbb{Q}) \setminus U(\mathbb{A}) \longrightarrow 1$$

where U is the torus of dimension 1 whose $\mathbb{Q}$–points are $\ker(E^\times \underset{N}{\longrightarrow} \mathbb{Q}^\times)$, and the map $E^\times \to U(\mathbb{Q})$ is $z \mapsto z/\bar{z}$. Since ω_π vanishes on $N_{E/\mathbb{Q}}\ \mathbb{A}_E^\times$ and also on $-1 \in \mathbb{R}^\times \subset \mathbb{Q}^\times \setminus \mathbb{A}^\times$, ω_π vanishes on $\mathbb{A}^\times$ and therefore comes from a character χ_U of $U(\mathbb{Q}) \setminus U(\mathbb{A})$, which we can extend to a Grössencharakter χ of $\mathbb{A}_E^\times$.

We have $\chi_\infty = z^p \bar{z}^q$ with $p - q \in \mathbb{Z}$. We can write $\chi_\infty = (z/\bar{z})^{\frac{p-q}{2}} (z\bar{z})^{\frac{p+q}{2}}$. We have $p - q = P$ since $\chi_\infty(z) = (z/\bar{z})^P$ if $z\bar{z} = 1$. If P is even, we can twist χ by a power of the idele norm to obtain $\chi_\infty = (z/\bar{z})^{P/2}$, an algebraic Grössencharakter that is unitary. If P is odd, $\chi_\infty = (z/\bar{z})^{P/2}(z\bar{z})^s$. We can twist so $s = \frac{1}{2}$ and $\chi_\infty = z^{\frac{P+1}{2}} \bar{z}^{\frac{1-P}{2}}$, an algebraic Grössencharakter which is, however, not unitary. The necessity of a weight translation in this case already appeared in [7, § 5.3].

The data (π, χ) determine an infinitesimal character for $GU(\mathbb{C})$. Since it is θ–invariant it gives by descent an infinitesimal character for $GU(\mathbb{R})$. The latter is regular since π is cohomological, and determines in turn an irreducible representation L of $GU(\mathbb{R})$; L is in fact defined over F. The parity constraint that occurred before is reflected in the nature of L. We have an exact sequence of $\mathbb{Q}$–groups

$$1 \longrightarrow (\pm 1) \longrightarrow G \times \mathbb{G}_m \longrightarrow GU \longrightarrow 1\,.$$

In the even case, the representation of G associated to Π is trivial on the kernel (± 1) and we can assume that the split center $\mathbb{G}_m$ acts trivially, giving on the Shimura variety a local system of weight 0. In the odd case, we can choose a local system of weight 1.

2.2

As in [9, §4], we now consider Kottwitz's expression for the alternating trace of Frobenius elements on $H^\bullet_{ét}(S_K \underset{F}{\times} \bar{\mathbb{Q}}\,,\ \mathcal{L}(\bar{\mathbb{Q}}_\ell))$, twisted by suitable Hecke correspondences. We have abridged $S_K(GU, h)$ as S_K. We consider a prime w of F dividing a rational prime p where our data, i.e. (F, π) are unramified (Proposition 1.1). Let Frob_w be a corresponding (geometric) Frobenius element in $\mathrm{Gal}(\bar{F}/F)$. Finally, $\mathcal{L}$ is the local system on S_K defined by L and ℓ is a prime distinct from p.

Kottwitz's formula is a sum over the set $\mathcal{E}$ of endoscopic groups for GU, modulo equivalence. $\mathcal{E}$ contains the quasi–split inner form GU^* of GU, and $\mathbb{Q}$–groups H associated to proper partitions

$$n = n_1 + n_2 \qquad (n_1 \geq n_2 > 0)\,.$$

The corresponding group H is the subgroup of $GU^*(n_1) \times GU^*(n_2)$ – product of quasi–split groups of unitary similitudes – given by the equality of the rational similitude ratios. There is a restriction on the pairs (n_1, n_2) : see [18, § 3.2].

The constructions that follow also necessitate, for all H, an "embedding" of L–groups over $\mathbb{Q}$,$^LH \to {}^LG$, i.e., an L–homomorphism. This is given by a Grössencharakter η of $\mathbb{A}_E^\times$ of a certain type. Cf. again [7, § 3.2] where our η is denoted by $\bar{\omega}$. We may assume that η is unramified at p. Recall that by assumption p is split in E, so this means that each component of η is unramified.

We change notation and denote by v a finite place of E ; u will denote a finite place of $\mathbb{Q}$. The group $GU \times E_v$ is isomorphic with $G_{E_v} \times GL(1)_{E_v}$, and $G_{E_v} = (\mathrm{Res}_{F/E} GL(n)_F) \underset{E}{\times} E_v$. Thus $G_{E_v} \cong \prod_{w|v} GL(n, F_w)$ where w runs over the places of F dividing v.

For $\alpha \geq 1$, Kottwitz defines functions $f^H_{p,\alpha}$ in the unramified Hecke algebra of $H(\mathbb{Q}_p)$ for each H, including of course GU^*. ($f^H_{p,\alpha}$ depends in fact on w, not only on p, but this would render our notation confusing). Similarly, the coefficient system L naturally defines functions f^H_∞ on (each) $H(\mathbb{R})$. Let f^p be an arbitrary function in the Hecke algebra of functions on $GU(\mathbb{A}^p_f)$ invariant by $K^p \subset GU(\mathbb{A}^p_f)$. We assume K decomposed, $K = K_p K^p$, where K_p is hyperspecial in $GU(\mathbb{Q}_p)$. (This was implicit in the earlier discussion : the variety S_K then has good reduction at the primes $w|p$ of F).

Endoscopy associates to f^p, and to the endoscopic data (embeddings), functions $f^{p,H}$ defined by their stable orbital integrals.

Let us write $H^\bullet(S_K, \mathcal{L})$ for the alternating sum of representations of $\mathrm{Gal}(\bar{F}/F)$:

$$\sum_{i=0}^{2D} (-1)^i \; H^i(S_K \underset{F}{\times} \bar{F} \,,\; \mathcal{L}(\bar{\mathbb{Q}}_\ell))\,.$$

where $D = \dim(S_K) = 2(n-2)$. Write q_w for the cardinality of the residue field of F at w.

Kottwitz's formula [13, Theorem 7.2] is then

$$q_w^{-\frac{D\alpha}{2}} \mathrm{trace}(\mathrm{Frob}_w^\alpha \times f^p \mid H^\bullet(S_K, \mathcal{L})) = \sum_{H \in \mathcal{E}} \iota(a, H) ST^*_e(f^H_\alpha) \tag{2.4}$$

where $f^H_\alpha = f^H_\infty \otimes f^H_{p,\alpha} \otimes f^{p,H}$ and ST^*_e is a sum of stable integral orbitals on the elliptic (GU, H)–regular elements in $H(\mathbb{Q})$ [12].

The definition of f_∞^H implies that the right-hand side has a simpler expression. Cf. [17, Thm. 6.2.1]. We give a simple argument, which is quite general. For $H = GU^*$ the regularity condition is void. For H a proper endoscopic subgroup, note that f_∞^H is defined, at (GU, H)–regular elements, by

$$SO_{\gamma_H}(f_\infty^H) =< \beta(\gamma), s > \Delta_\infty(\gamma_H, \gamma) e(I) \cdot \text{trace } \xi(\gamma) v^{-1} \tag{2.5}$$

where ξ is the finite–dimensional representation on L and the character $\gamma \mapsto < \beta(\gamma), s >$, the sign $e(I)$, or the volume v, need not concern us. Cf. [13, (7.4)]. The discriminant Δ_∞ can be taken equal to $\Delta_B(\gamma^{-1})\Delta_{B_H}(\gamma_H^{-1})^{-1}$ times, again, a sign and a character, cf. [13, p. 184]. Here B, B_H are complex Borel subgroups adapted to tori T, T_H containing γ, γ_H.

Now the pair (γ, γ_H) is (GU, H)–regular if $\gamma^\alpha \neq 1$ for any root α of (GU, T) not in H (in a natural sense). Cf. [12, p. 378]. Assume, on the contrary, γ is H–regular but not (GU, H)–regular. We have an identity

$$\Delta_{B_H}(\gamma_H^{-1}) SO_{\gamma_H}(f_\infty^H) = F(\gamma_H)\ \Delta_B(\gamma^{-1}) \,.$$

The discriminant $\Delta_{B_H}(\gamma_H^{-1})$ is non-zero, while $\Delta_B(\gamma^{-1})$ vanishes ; $SO_{\gamma_H}(h_\infty^H)$ is smooth in γ_H on the regular set, and so is $F(\gamma_H)$ up to a sign. We deduce that $SO_{\gamma_H}(f_\infty^H) = 0$. Thus $SO_{\gamma_H}(f_\infty^H)$ vanishes on all H–regular elements that are not (GU, H) regular.

Now f_∞^H is a linear combination of Euler–Poincaré functions associated to a family of rational representations of H (cf. [13, § 7], [8]). Its stable orbital integrals are linear combinations of rational characters affected by a sign (the same for each character). Take $\gamma_H \in T_H$. To say that γ_H is not (G, H)–regular is to say that $\gamma_H^\alpha = 1$ for a root α figuring in B_G but not in B_H. It suffices to show that γ_H is limit of elements γ'_H, H–regular and such that $(\gamma'_H)^\alpha = 1$. Clearly it suffices to solve the problem in the adjoint group, and then again in the unitary group. The following assertion is then obvious : let $t = (t_i)$ be an element of $T_G \equiv T_H \cong U(1)^n$. Assume $i_0 \neq j_0$, $t_{i_0} = t_{j_0}$. Then t is a limit of elements $t' \in U(1)^n$ such that $t'_i \neq t'_j$ for all $(i, j) \neq (i_0, j_0)$, $i \neq j$ and $t'_{i_0} = t'_{j_0} = t_{j_0}$.

We can therefore rewrite (2.4) as

$$q_w^{-\frac{D\alpha}{2}} \text{trace}(\text{Frob}_w^\alpha \times f \mid H^\bullet(S_K, \mathcal{L})) = \sum_H \iota(G, H)\ ST_e(f_\alpha^H) \,, \tag{2.6}$$

ST_e being the complete elliptic term of the stable trace formula.

We can now use the results of Labesse [15]. Recall that f^H_∞ is a sum of Euler–Poincaré functions associated to certain finite–dimensional representations of H. In particular it is associated to a sum of twisted Euler–Poincaré functions, φ^H_∞, on $H(E_\infty)$. Let us **assume** for the moment that $f^H_{p,\alpha}$ and $f^{p,H}$ are associated, by the identities of stable base change [14], to functions $\varphi^H_{p,\alpha}$ and $\varphi^{p,H}$. For each H define φ^H_α as the obvious tensor product. By [15, Théorème 5.8] we now have

$$ST_e(f^H_\alpha) = ST_{dis}(f^H_\alpha) = T^{\tilde{H}}_{dis}(\varphi^H_\alpha)\,. \tag{2.6}$$

The first equality is [15, Prop. 5.7], and is due to Arthur. The second is the quoted theorem of Labesse. The right–hand side is an ordinary twisted trace. Finally we obtain

$$q^{-\frac{D\alpha}{2}}\mathrm{trace}(\mathrm{Frob}^\alpha_w \times f^p \mid H^\bullet(S_K, \mathcal{L})) = \sum_H \iota(G,H) T^{\tilde{H}}_{dis}(\varphi^H_\alpha)\,. \tag{2.7}$$

2.3

We now return to the local functions defining φ^H_α. The function $f^H_{p,\alpha}$ is in the unramified Hecke algebra of H at p. We have assumed that p split in E, so base change between the unramified Hecke algebras of $H \times \mathbb{Q}_p$ and $(H \times E) \times \mathbb{Q}_p$ is split base change, i.e. convolution, which is surjective. At the other primes, however, we have to assume that the function f^H_u defined, via its stable orbital integrals, by f_u, is in the image of the base change map.

Since our datum will be, by Lemma 2.2, a representation of $GU(\mathbb{A}_E)$ we reverse the maps. So assume given a function

$$\varphi = \bigotimes \varphi_v \quad \text{(places of } E)$$

on $GU(\mathbb{A}_E)$. At the Archimedian place, φ_∞, an Euler–Poincaré (twisted) function, is associated to f_∞. At the prime p, $\varphi_{p,\alpha}$ is associated to $f_{p,\alpha}$ as before. ($\varphi_{p,\alpha} = \varphi_{v,\alpha} \otimes \varphi_{v',\alpha}$ where v, v' divide p. We can take $\varphi_{v,\alpha} = f_{p,\alpha}$ and $\varphi_{v',\alpha} = 1$). At any other finite prime we take f_u associated to $\varphi_u = \bigotimes_{v|u} \varphi_v$. This is possible by Labesse's results [14, Ch. III]. Thus we have defined $f^{\infty,p}$ from φ. At almost all primes, φ_p and f_p can be taken to be the units in the Hecke algebra by the (stable) fundamental lemma.

The function $f^{\infty,p}$, the power α of Frobenius, and the local system determine as above functions f^H for each (non principal) H. The function f^H_∞

determines a twisted Euler–Poincaré function φ^H_∞ ; we must show that at the finite primes f^H_u is in the image of the base change map. This is obvious at p since p is split. At other (inert) primes u we can avail ourselves of a result of Arthur converse to the one we used in the other direction :

Lemma 2.3 (Arthur)**.** *Assume that h, a C^∞_c function on $H(\mathbb{Q}_u)$, has vanishing stable orbital integrals on semi–simple regular elements not in the image of the norm between $H(E_u)$ and $H(\mathbb{Q}_u)$. Then h is associated to a function h_E on $H(E_u)$.*

Cf. Labesse [14, p. 80, Remarque]. Note that this depends on the general case of the fundamental lemma, now proven. At split primes, the existence of $\varphi^H_u = \bigotimes_{v|u} \varphi^H_v$ is, as usual, obvious. At inert primes we must so choose our functions that the condition in the Lemma is verified. We distinguish two cases.

First let u, and $v|u$, be arbitrary. Up to conjugacy, there are a finite number of maximal tori (over $\mathbb{Q}_u$) $T_H \subset H$. If f^H, a function on $H(\mathbb{Q}_u)$, is associated to f, $SO_{\gamma_H}(f^H)$, for γ_H regular in T_H, is a linear combination of orbital integrals $O_\gamma(f)$ at elements $\gamma \in T \subset G$ such that (γ, T) is stably conjugate to (γ_H, T_H).

We can choose an open, closed, invariant neighbourhood Ω of 1 in G such that the trace of Ω on any of the maximal tori $T \subset G$ is arbitrarily close to 1 (use the characteristic polynomial for a faithful representation of G). Assume f is supported on Ω. Suppose then that $SO_{\gamma_H}(f^H) \neq 0$. Thus there exists (γ, T) such that $< O_\gamma(f) \neq 0$. Suppose Ω is chosen so that $\gamma \in T$ is then a square. Since there is a $\mathbb{Q}_u$–isomorphism $T \to T_H$ sending γ to γ_H, γ_H is a square. Then γ_H is a norm in H. Therefore the condition of the Lemma is satisfied by f^H.

We also note that if f is associated by base change to φ on $G(E_u)$, the condition on f will be satisfied if the support of φ is sufficiently close to 1. (Again, if $\gamma = \mathcal{N}\delta$, $\gamma \in G(\mathbb{Q}_u)$, $\delta \in G(E_u)$, the characteristic polynomial of γ is a continuous function of that of $\delta \times \theta$).

We now consider a place u where all data are unramified. This means that F is unramified at u, and that G is therefore an unramified group over $\mathbb{Q}_u$; and that the endoscopic group H is also unramified, i.e., H is unramified and the map ${}^LH \to {}^LG$, given by

$$j_H : (h, w) \longmapsto (j(h, w), w) \tag{2.8}$$

($h \in \widehat{H}$, $w \in W_{\mathbb{Q}_u}$) factors through $\widehat{H} \times W(\mathbb{Q}_u^{nr}/\mathbb{Q}_u)$.

We simplify the notation by writing $K = \mathbb{Q}_u$, $L = E_v$. Our datum is φ_u, which we assume belongs to the unramified Hecke algebra of $G(L)$. By the fundamental lemma we can, and do, take f_u equal to the image of φ_u in the unramified Hecke algebra of $G(K)$.

Recall the formalities of base change[1]. We have

$$^L(G/K) = \widehat{G} \rtimes W_K \,,$$

the action of W_K factoring through $W_{nr} = W(K_{nr}/K)$. If $\widetilde{G} = \mathrm{Res}_{L/K}(G/L)$, we have

$$^L(\widetilde{G}/K) = (\widehat{G} \times \widehat{G}) \rtimes W_K := \widehat{\widetilde{G}} \rtimes W_K \,,$$

W_{nr} acting through $w(g_1, g_2) = (wg_2, wg_1)$, the componentwise action being defined by G. If $\mathcal{H}_H$, $\mathcal{H}_L$ are the unramified Hecke algebras, $\mathcal{H}_K$ is the ring of polynomial functions on $\widehat{G} \times$ Frob invariant by $\widehat{G}$; $\mathcal{H}_L$ is the ring of functions in $\widehat{\widetilde{G}} \times$ Frob invariant by $\widehat{\widetilde{G}}$. The diagonal map

$$\begin{array}{rccc} \beta_G : & ^L(G/K) & \longrightarrow & ^L(\widehat{G}/K) \\ & (g, w) & \longmapsto & (g, g, w) \end{array}$$

gives dually the stable base change map $b_G : \mathcal{H}_L \to \mathcal{H}_K$.

Of course this applies to H, yielding β_H and b_H. Furthermore – (cf. (2.8) – j_H defines naturally

$$\begin{array}{rcl} \widetilde{j}_H : \quad {}^L\widetilde{H} & \longrightarrow & {}^L\widetilde{G} \\ (h_1, h_2, w) & \longmapsto & (j(h, w), j(h, w), w) \,. \end{array}$$

The commutativity of the diagram

$$\begin{array}{ccc} {}^L H & \xrightarrow[\beta_H]{} & {}^L\widetilde{H} \\ {\scriptstyle j_H}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{j}_H} \\ {}^L G & \xrightarrow[\beta_G]{} & {}^L\widetilde{G} \end{array}$$

then implies the following : let $\lambda : \mathcal{H}_K \to \mathcal{H}_{H,K}$ be the natural homomorphism of unramified Hecke algebras, and $\widetilde{\lambda} : \mathcal{H}_L \to \mathcal{H}_{H,L}$ its analogue for

[1] See Langlands [16, §2] ; Borel [2].

L. Then $\lambda(b_G\varphi) = b_H(\widetilde{\lambda}\varphi)$. In particular f^H, which can be taken equal to $\lambda f = \lambda(b_G\varphi)$, is in the image of b_H.

We summarize this as :

Lemma 2.4. *Let S be a large set of finite primes such that $p \notin S$ and containing all places of ramification f or π, χ, GU and the endoscopic data. Suppose given a decomposed function φ on $GU(\mathbb{A}_E)$ such that*

(i) At the Archimedean prime v_∞, the function $\varphi^H_{v_\infty}$, for each H, is a twisted Euler–Poincaré functions as above.

(ii) As the split prime p, $\varphi^H_{p,\alpha} = \bigotimes_{v|p} \varphi^H_{v,\alpha}$ has $f^H_{p,\alpha}$ for image by base change.

(iii) At the places $u \in S$, φ_u is so chosen that f^H_u is in the image of the base change map (all H)

(iv) φ_u is unramified for $u \notin S$, $u \neq p$.

Then the identity (2.7) obtains, with, for all H :

$$\varphi^H_\alpha = \varphi^H_{v_\infty} \otimes \varphi^H_{p,\alpha} \otimes \bigotimes_{u \in S} \varphi^H_u \otimes \bigotimes_{u \notin S} \varphi^h_u\,,$$

with φ^H_u $(u \notin s)$ equal to the endoscopic image of φ_u, and $f^{p,\infty}$ equal to the base change image of $\varphi^{p,\infty}$.

Note that over E, all endoscopic group H are in fact isomorphic to $\mathrm{Res}_{F/E}(GL(n_1) \times GL(n_2)) \times \mathbb{G}_m$. Note also that the primes of ramification for GU are in fact the primes of ramification for F.

3 Proof of Proposition 1

3.1

We assume that Hypothesis 2.1 is verified, and we consider the given representation π of $GL(n, \mathbb{A}_F)$. In Lemma 2.1 we have constructed a representation $\Pi = \pi \otimes \chi$ of $GU(\mathbb{A}_E) = GL(n, \mathbb{A}_F) \times \mathbb{A}_E^\times$.

Let S be, as in §2, a set of primes of $\mathbb{Q}$ containing all ramified primes, including those for π. For brevity let us write for the moment H for GU^*, the principal endoscopic group. Let $\varphi = \varphi_\infty \otimes \varphi_{p,\alpha} \otimes \varphi^{\infty,p}$ where $\varphi^{\infty,p}$, a tensor product, is unramified at the places $u \notin S$. The main term in the contribution of H to (2.7) is

$$\sum_{(\rho,\eta)} \mathrm{trace}((\rho \otimes \eta)(\varphi) I_\theta) \tag{3.1}$$

where ρ is a cuspidal representation of $GL(n, \mathbb{A}_F)$ and η a Grössencharakter of $\mathbb{A}_E^\times$; I_θ is the intertwining operator given by Galois action. It is a finite sum if the ramification of φ_S is fixed. We take automorphic forms on the quotient of $H(\mathbb{A}_E) = GL(n, \mathbb{A}_F) \times \mathbb{A}_E^\times$ by $\mathbb{R}_+^\times \times \mathbb{R}_+^\times$ embedded diagonally, if P is even (§ 2.1) ; if P is odd we consider automorphic forms ψ such that $\psi((t_1, t_2)g) = t_2\psi(g)$ for$(t_1, t_2) \in \mathbb{R}_+^\times \times \mathbb{R}_+^\times$ and $g \in H(\mathbb{A}_E)$, as we must by § 2.1. Note that this space is invariant by θ, cf. (2.3). The trace formula, generally written for functions invariant by the neutral component $\mathbb{R}_+^\times \times \mathbb{R}_+^\times$ of the split centre, extends trivially to this case : the map $\psi(g) \mapsto |z|^{1/2}\psi(g)$, where $g = (g_1, z) \in GL(n, \mathbb{A}_F) \times \mathbb{A}_E^\times$ and $|z|$ is the idele norm, yields an isomorphism between our space and $L^2(\mathbb{R}_+^\times \times \mathbb{R}_+^\times \setminus H(\mathbb{A}_E))$.

The representation Π contributes one term to (3.1).

It is decomposed as a product of local, twisted traces. The twisted trace of φ_∞, i.e., the twisted Euler–Poincaré characteristic of $\pi \otimes \chi$, is non-zero : cf. [14, Lemme 4.7]. The twisted trace of $\varphi_{p,\alpha} = f_{p,\alpha} \otimes 1$ is given by the Langlands–Kottwitz construction of $f_{p,\alpha}$. Up to a sign – since the decomposition of I_θ, an involutive operator, into local factors is not uniquely defined[2] – we have

$$\mathrm{trace}(\Pi(\varphi_{p,\alpha}) I_\theta) = \mathrm{trace}\ R(t_{\Pi,p})$$

where $t_{\Pi,p}$ is the Hecke matrix of Π in $GL(n, \mathbb{C})^{(F:\mathbb{Q})} \times \mathbb{C}^\times$, and R the representation of the dual group defined by (GU, h). Given the expression of h (§ 2.1) we have, up to a sign :

$$\mathrm{trace}(\Pi(\varphi_{p,\alpha}) I_\theta) = \mathrm{trace}(\Lambda^2 t_{\Pi,W}^\alpha \chi^\alpha (N_{F_w/E_v} \bar{\omega}_w)$$

where w is the prime of F considered in § 2.2, $t_{\pi,w} \in GL(n, \mathbb{C})$ is the Hecke matrix of π at the prime w, and v is the prime of E below w ; $\bar{\omega}_w \in F_w$ is a uniformizer.

At primes $v \notin S$, the twisted trace is, again up to a sign – equal to 1 almost everywhere – equal to the trace and this yields the product

[2] We could obtain a decomposition into well -defined factors using Whittaker vectors : [3, 14, 8]. This is not necessary here.

$$\prod_{v\notin S}\Big(\prod_{w|v}\text{trace }\pi_w(\varphi'_w)\Big)\text{ trace }\chi_v(\varphi_v")$$

where we have written $\varphi_v = (\bigotimes_w \varphi'_w) \otimes \varphi''_v$, the first functions on the $GL(n)$–factors and the second on $E_v^\times$.

Consider now the places in S. Here we must enforce the restriction on the φ_v discussed in § 2.3, at least for v inert. That this is possible will be proved presently ; we state the result now. Write $\varphi_S = \bigotimes_{u\in S}\varphi_u$.

Lemma 3.1. *We can choose functions φ_u in an arbitrary neighbourhood of* 1 *(for all $u \in S$) such that* trace($\Pi_S(\varphi_S)I_\theta) \neq 0$.

We choose such a function φ_S – and therefore a level.

The contribution of Π to (3.1) is then, with $S_E = \{v|u, u\in S\}$:

$$C_0 \text{ trace}(\Lambda^2 t^\alpha_{\pi,w}\chi(N\bar\omega_w)^\alpha)\prod_{w\nmid S}\text{ trace }\Pi_w(\varphi'_w)\cdot\prod_{v\notin S_E}\chi_v(\varphi''_v) \tag{3.2}$$

with $C_0 \neq 0$, and $N = N_{F_w/E_v}$.

The other representations in the cuspidal spectrum (3.1) contribute a finite number of terms

$$C(\rho,\eta)\text{ trace}\Big(\Lambda^2 t^\alpha_{\rho,w}\eta(N\bar\omega_w)^\alpha\Big)\prod_{w\nmid S}\text{ trace }\rho_w(\varphi'_v)\cdot\prod_{v\notin S_E}\eta_v(\varphi''_v)\,. \tag{3.3}$$

Using the result of Jacquet–Shalika on the independence of Hecke eigenvalues for cuspidal representations [11], and the classical analogue for $\mathbb{A}_E^\times/E^\times$, we can choose the functions φ'_v, φ''_v for $v\notin S$ such that (3.3) vanishes and (3.2) is reduced to

$$C_0 \text{ trace}(\Lambda^2\ t^\alpha_{\pi,w}\ \chi(N\bar\omega_w)^\alpha)\,. \tag{3.4}$$

The same argument applies to the terms of $T^{\tilde H}_{dis}$ not described by (3.1) since they are associated to twisted traces in representations of $GL(n,\mathbb{A}_K)\times E^\times$ induced from parabolic subgroups, using this time the Jacquet–Shalita result for distinct values of n. These terms are explicitly described in [15, Prop. 3.3].

Finally, the endoscopic terms in (2.7), relative to groups $H \neq GU$, yield twisted traces associated to products $GL(n_1,F)\times GL(n_2,F)\times E^\times$, of smaller (semi–simple) rank. They can also be eliminated by the same argument based on the results of Jacquet–Shalika.

We still have to prove Lemma 3.1. Note that the problem is local. For a place $v \in S$, we must show that the twisted character of Π_v does not vanish identically in any neighbourhood of 1. The non–twisted analogue is obvious, but the result in the twisted case is surprisingly difficult. The assertion for Π easily reduces to the same for π. Note that $\pi_v = \bigotimes_{w|v} \pi_w$ (w a prime of F) is generic. Under this mere assumption Lemma 3.1 can be proven using the methods introduced in [3] by Chenevier and the author. In the context of this proof however, S can be taken as the set of primes of ramification of F, π, χ and the character η of $\mathbb{A}_E^\times$ used to define the endoscopic embeddings.

By Hypothesis 2.1, each prime of ramification for π is split in E. Thus we can assume π_w unramified. If w divides a prime w_+ of F^+ split in F, we are again in a split situation. Otherwise $G_0(F^+_{w_+})$ is a true unitary group split in F_w. The computation of the twisted character for the unramified, θ–stable principal series is standard, cf. e.g. [5] : π is an induced representation of the form $\pi(\chi_1, \chi_1^{-1}, \chi_2, \chi_2^{-1}, \dots \chi_m, \chi_m^{-1})$ $(n = 2m)$ which descends to a principal series π_+ of $G_0(F^+_{w_+})$ and the characters of π and π_+ are associated by the norm map.

3.2

We now return to our basic identity (2.7). We have Matsushima's decomposition of the cohomology with complex coefficients[3] :

$$H^i(S_K(\mathbb{C}), \mathcal{L}) = \bigoplus_{\Pi = \Pi_\infty \otimes \Pi_f} H^i(\mathfrak{g}, K_\infty, \Pi_\infty \otimes L) \otimes \pi_f^K \tag{3.5}$$

where the sum, in effect finite for a given level K, runs over a full decomposition of the space of automorphic forms on $L^2(AGU(\mathbb{Q}) \setminus GU(\mathbb{A}))$ when the weight (§ 2.1) is zero ; A denotes the connected component, at the Archimedean prime, of the split centre of GU. In the odd case we must take automorphic forms transforming under $A \cong \mathbb{R}_+^\times$ by the inverse of the weight character $t \mapsto t$.

The action of the Hecke algebra being naturally defined over $\bar{\mathbb{Q}}$ (as a field of **coefficients** for the cohomology) and commuting with the action of

[3]We free, for an instant, the letter Π in order to denote an arbitrary representation of $GU(\mathbb{A})$.

$\mathrm{Gal}(\bar{F}/F)$ on the étale cohomology with coefficients in $\bar{\mathbb{Q}}_\ell$, we can rewrite (3.5) as a decomposition of the representation of $\mathrm{Gal}(\bar{F}/F)$ if we group the terms giving the same factor Π_f :

$$H^i(\bar{S}_K, \mathcal{L}) := H^i(S_K \times \bar{F}, \mathcal{L}(\bar{\mathbb{Q}}_\ell)) = \bigoplus_{\Pi_f} H^i(\bar{S}_K, \mathcal{L})(\Pi_f) \tag{3.6}$$

where Π_f runs over the finite factors of all representations Π in (3.5), and, for an embedding $\bar{\mathbb{Q}}_\ell \subset \mathbb{C}$, each factor yields in Betti cohomology the sum

$$\bigoplus_{\Pi} H^i(\mathfrak{g}, K_\infty \,;\, \Pi_\infty)$$

where Π runs over the summands of (3.5) with given component Π_f.

We must now strengthen Lemma 3.1.

Lemma 3.2. *We can choose φ such that, at each place $u \in S$, f_u is the characteristic function of a small compact–open subgroup K_u of $G(\mathbb{Q}_n)$.*

This follows immediately from the proof : at places split in E, we can take $\varphi_u = 1_{K_{v'}} \otimes 1_{K_{v''}}$ where $K_{v'}$, $K_{v''}$ are permuted by the Galois action. At the other primes, π_v is unramified and we choose φ_v associated to the characteristic function of a small subgroup K_u. This is possible by Arthur's results on inverse base change : see Lemma 2.3. One could give a more direct proof of our specific assertion.

Choose φ, at the places $u \in S$, as in Lemma 3.2 ; φ_∞ and $\varphi_p = \varphi_p^\alpha$ have been described in § 2.2. Once φ_S is fixed, so is the level and the right–hand side (so also the left–hand side) of (2.7) involves a finite number of characters of the unramified Hecke algebra $H^S = \bigotimes_{\substack{v \nmid S \\ v \nmid \infty}} \mathcal{H}(GU(E_v), K_v)$ with $K_v \cong GL(n, \mathcal{O}_{(F\otimes E)_v}) \times \mathcal{O}_{E_v}^\times$. Note that each factor π_f^K of (3.5) defines by composition a character of $\mathcal{H}^S$. We choose $\varphi^S \in \mathcal{H}^S$ as described in § 3.1. The basic identity (2.7) now yields the equality of

$$q_w^{-\frac{D\alpha}{2}} \sum_{i=0}^{2D} (-1)^i \sum_{\Pi_f} \dim(\Pi_S^{K,S}) \ \mathrm{trace}(\mathrm{Frob}_w^\alpha \mid H^i(\bar{S}_K, \mathcal{L})(\Pi_f)) \tag{3.7}$$

and

$$C_0 \ \mathrm{trace}(\Lambda^2 \ t^\alpha_{\pi,\omega} \chi(N\bar{\omega}_w)^\alpha) \,, \ C_0 \neq 0 \,. \tag{3.8}$$

In (3.7), we have set $K_S = \prod_{u \in S} K_u$; Π_f runs over the representations of $GU(\mathbb{A}_f)$, unramified at p, such that Π_f^S is associated by base change to our given representation $\pi \otimes \chi$ (at rather $\Pi^S \otimes \chi^S$). This is an identity of the form

$$q_w^{-\frac{D\alpha}{2}} \sum_{i,j} (-1)^i \ \lambda_{ij}^{\alpha} = C \ \sum_k \mu_k^{\alpha} \qquad (\alpha \geq 1) \tag{3.9}$$

for some complex numbers λ_{ij}, μ_k. The λ_{ij} for different i, being Weyl numbers of different weights, are distinct. We deduce that C is a rational number and that the parity of i in the left–hand side is fixed.

Assume first, for simplicity, that the coefficient system $\mathcal{L}$ has weight 0, i.e. that P is even (§ 2.1). The λ_{ij} are Weil numbers of weight i. The character χ is unitary, and the μ_k are in fact parametrized by (a, b) $(a < b \leq n)$ and given by

$$\mu_{ab} = t_a t_b \ \chi(N\bar{\omega}_w) \, .$$

where (t_a) are the entries of the Hecke matrix $t_{\pi,w}$.

Thus we have for any $a \neq b$, χ being unitary :

$$|t_a t_b| = q_w^{i/2} \, , \quad i \in [-D, D] \, . \tag{3.10}$$

Suppose first that all weights i (translated by D) are even. Then $|t_a t_b| \in q_w^{\mathbb{Z}}$; but by the results of Jacquet–Shalika [11] and Tadič [19] :

$$q_{\omega}^{-1/2} < |t_a| < q_{\omega}^{1/2} \, . \tag{3.11}$$

Thus $q_w^{-1} < |t_a t_b| < q_w$ and we deduce that $|t_a t_b| = 1$; since $n \geq 4$ (3 would suffice) this implies that $|t_a|$ is independent of a and therefore equal to 1.

Suppose all weights are odd. Then $|t_a t_b| \in q_w^{1/2+\mathbb{Z}}$. Again, this implies $|t_a/t_b| \in q^{\mathbb{Z}}$. But (3.11) implies also $q_w^{-1} < |t_a/t_b| < q_w$, so $|t_a|$ is independent of a. Since the central character of π is unitary, this implies that $|t_a| = 1$. This however contradicts the fact that $|t_a t_b| \in q_w^{1/2+\mathbb{Z}}$, so this possibility does not occur. Since the dimension $D = 2(n-2)$ is even, we see that we have in fact proved that the cohomology associated to Π in S_K occurs only in even degrees.

If P is odd, the coefficient system has weight 1 and the weights on (3.10) are now in $[-D+1, D+1]$. However, the character χ also has weight 1, and $|\chi(N\bar{\omega}_w)| = q_w^{1/2}$. After taking care of this translation, the rest of the argument is the same. This proves Proposition 1.1, and the Theorem.

Finally, let us show that this implies the Corollary to the Theorem. Let E be the field of definition of Π_f ([6, § 3]). Then the normalized entries $\lambda_i = q_w^{\frac{n-1}{2}} t_i$ are the roots of a polynomial $P_w(X) \in E[X]$ of degree n. Moreover, for each embedding $\sigma : E \rightarrow \bar{\mathbb{Q}} \subset \mathbb{C}$, ${}^{\sigma}P_w(X)$ is associated to a cuspidal representation ${}^{\sigma}\Pi$ of $GL(n, \mathbb{A}_F)$, still cohomological [6, Thm. 3.13]. Since σ acts on the coefficients, ${}^{\sigma}\pi$ is still conjugate self–dual ; the ramification assumptions are obviously conserved. We see that $|\lambda_i| = q_w^{\frac{n-1}{2}}$ for each embedding $\mathbb{Q}(\lambda_i) \subset \bar{\mathbb{Q}} \subset \mathbb{C}$: the λ_i are Weyl numbers of the indicated weight, and the normalization of r_λ is such that they are the Frobenius eigenvalues.

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