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Specialisation sur le cône tangent et équisingularité à la Whitney

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It's a strange world, let's keep it that way.

Résumé

Cette thèse porte sur l'étude de la géométrie de l'espace de spécialisation $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ d'un germe de singularité analytique complexe $(X, 0)$ sur son cône tangent $(C_{X,0}, 0)$ du point de vue de l'équisingularité à la Whitney. L'application φ nous donne une famille plate des germes avec section tel que pour chaque $t \neq 0$ le germe $\varphi^{-1}(t)$ est isomorphe à $(X, 0)$ et la fibre spéciale est isomorphe au cône tangent. Le but est de établir des conditions sur les strates de la stratification de Whitney minimale de $(X, 0)$ qui assurent l'équisingularité du germe et son cône tangent, generalisant ainsi le résultat de Lê et Teissier pour les hypersurfaces de $\mathbb{C}^3$ qui prouve que l'absence des tangentes exceptionnelles est suffisant. Dans ce travail on montre que cette condition est nécessaire et suffisante dans le cas général pour la strate de codimension zero.

L'un des ingrédients clés dans la preuve est la théorie de la dépendance integrale sur des ideaux et des modules développé par Teissier, Lejeune, Gaffney, Kleiman, etc, qu'on rappelle au troisième chapitre et où l'on obtient des résultats spécifiques pour cette situation. Les deux premiers chapitres correspondent aux préliminaires, on commence par rappeller la modification de Nash et l'espace conormal d'un espace analytique plongé dans ses versions absolues et relatives à un morphisme et on donne une description explicite de la relation entre le conormal (Nash) relatif de $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ et le conormal (Nash) de $(X, 0)$. Dans le deuxième chapitre on définit le diagram normal/conormal, l'auréole du germe $(X, 0)$, les cônes exceptionnelles, et on énonce les résultats principaux correspondant à l'équisingularité à la Whitney en incluant la caractérisation des conditions de Whitney en termes du diagramme normal/conormal.

Mots-clefs

Spécialisation sur le cône tangent, équisingularité, conditions de Whitney.

Contents

Introduction

For a couple $(Y,0) \subset (X,0)$ formed by a smooth subgerm of a reduced equidimensional complex analytic singularity, the specialisation space of $(X,0)$ to its normal cone $(C_{X,Y},0)$ along Y has played an important role in the study of equisingularity conditions of X along Y.

The specialisation space $(\mathfrak{X},0) \to (\mathbb{C},0)$ has been used to study Whitney conditions in [Nav80], where an equivalent equisingularity condition was defined in terms of $\mathfrak{X}$ and Thom's a_f condition. It has also been used to study the structure of the set of limits of tangent spaces (see section 1.2) in [LT79] and [LT88] which plays an important role in the theory of equisingularity. It was these two articles that inspired the present work. In [LT88], the authors prove the existence of a finite family $\{V_\alpha\}$ of subcones of the reduced tangent cone $|C_{X,0}|$ that determines the set of limits of tangent spaces to X at 0.

To be more specific, we fix an embedding $(X,0) \subset (\mathbb{C}^{n+1},0)$ and consider the normal/conormal diagram,

$$\begin{tikzcd} E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[d, "\kappa'"'] \arrow[dr, "\xi"] & C(X) \arrow[d, "\kappa"] \\ E_0X \arrow[r, "e_0"'] & X \end{tikzcd}$$

where $E_0X \subset X \times \mathbb{P}^n$ is the blowup of X at the origin, $C(X) \subset X \times \check{\mathbb{P}}^n$ is the conormal space of X whose fiber over 0 determines the set of limits of tangent spaces, and $E_0C(X) \subset X \times \mathbb{P}^n \times \check{\mathbb{P}}^n$ is the blowup of $C(X)$ at $\kappa^{-1}(0)$. Then, consider the irreducible decomposition $|\xi^{-1}(0)| = \bigcup D_\alpha$ of the reduced fiber. The authors prove that the fiber $\xi^{-1}(0)$ is contained in the incidence variety $I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ and that each D_α establishes a projective duality of its images $V_\alpha \subset \mathbb{P}C_{X,0} = e_0^{-1}(0) \subset \mathbb{P}^n$ and $W_\alpha \subset \kappa^{-1}(0) \subset \check{\mathbb{P}}^n$.

In particular, the V_α's that are not irreducible components of the tangent cone are called exceptional cones and they appear in $\mathfrak{X}$ as an obstruction to the a_f stratification of the morphism $\mathfrak{X} \to \mathbb{C}$. They also prove that if the germ $(X,0)$ is a cone itself, then it does not have exceptional cones. So a natural question arises, if a germ of analytic singularity $(X,0)$ does not have exceptional cones, how close is it to being a cone?

A partial answer to this question was given in [LT79] in terms of Whitney equisingularity. The specialization space $(\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ has a canonical section where it picks the origin in each fiber. Let $Y \subset \mathfrak{X}$ be given by this section and let $\mathfrak{X}^0$ denote the non singular part of $\mathfrak{X}$. The authors prove that for a surface $(X, 0) \subset (\mathbb{C}^3, 0)$ with reduced tangent cone $C_{X,0}$, the absence of exceptional cones is a necessary and sufficient condition for it to be Whitney equisingular to its tangent cone. That is, they prove the existence of a Whitney stratification of $\mathfrak{X}$ which admits Y as a stratum. The main objective of this work is to take a step forward on the way of generalising this result.

In the first chapter, we begin by defining the specialisation space $\varphi : \mathfrak{X} \to \mathbb{C}$ of a germ $(X, 0)$ to its tangent cone $(C_{X,0}, 0)$ and develop its main properties. The special geometry of this space allows us to derive relations between its relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$ and the Nash modification of X, as well as between the relative conormal space $C_\varphi(\mathfrak{X})$ and the conormal space of X. Along the way we explicit in full generality how the conormal space of X and its Nash modification $\mathcal{N}X$ are related, while recalling how projective duality can be studied in terms of conormal spaces and symplectic geometry. Finally, we carefully define the concept of relative lagrangian, give a proof of the Lagrangian specialisation principle (Thm. 1.40) and prove that the relative conormal space $\kappa_{\varphi,\mathfrak{X}} : T^*_{\mathfrak{X}}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C}) \to \mathfrak{X}$ is always $\varphi-$lagrangian.

The aim of chapter 2 is basically to define the normal/conormal diagram of the pair $(X, Y, 0)$, where Y is a smooth subvariety of X, once a local embedding is chosen. We define the auréole of X along Y, the exceptional cones, and we give a proof in the special case when $Y = \{0\}$ of how the auréole of the germ $(X, 0)$ determines the set of limits of tangent hyperplanes. We state the definition and some of the main results concerning Whitney's conditions and Whitney equisingularity including their characterisation in terms of the normal/conormal diagram.

Another key ingredient in this work was the use of the theory of integral closure of ideals and modules in the study of equisingularity as developed by Lejeune and Teissier in [LJT08], and later by Gaffney in [Gaf92] and [Gaf97] and in conjunction with Kleiman in [GK99]. In chapter 3 we present the results concerning the use of the theory of integral closure of modules in the study of Whitney equisingularity in complex analytic geometry. We present complete proofs of the majority of them, mostly following and sometimes developing the proofs given by Gaffney and Kleiman in their papers. We finish by taking a closer look at what these results look like in the setting of the specialisation space $\varphi : \mathfrak{X} \to \mathbb{C}$ which allows us to derive the specific statements that will be used in the proofs of the main results.

The fourth chapter presents the main results. The first part of the chapter presents results valid in the general case. We prove in proposition 4.1 that in order to verify Whitney's conditions for the couple $(\mathfrak{X}^0, Y)$ at the origin it suffices to check for Whitney's condition a). That is, in this very specific setting Whitney's condition a) implies Whitney's condition b). Moreover, under the hypothesis of the tangent cone being reduced, we are able to prove in lemma 4.4 that the absence of exceptional cones for the germ $(X,0)$ is a necessary condition if we want the couple $(\mathfrak{X}^0, Y)$ to satisfy Whitney's conditions at the origin, and in proposition 4.5 we prove that the latter condition is equivalent to the absence of exceptional cones for the germ $(\mathfrak{X},0)$.

For the second part of the chapter we add the hypothesis of $(X,0)$ being not only reduced, but also irreducible. Nevertheless, the main result is proved in the general case:

Theorem 0.1. *(Thm 4.19) Let $(X,0)$ be a reduced and equidimensional germ of complex analytic singularity, and suppose that its tangent cone $C_{X,0}$ is reduced. Then the following statements are equivalent:*

1. *The germ $(X,0)$ does not have exceptional cones.*
2. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*
3. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions a) and b) at the origin.*
4. *The germ $(\mathfrak{X},0)$ does not have exceptional cones.*

The proof of this theorem shows us how everything fits together. On the one hand, the relation among the relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$, the Nash modification $\mathcal{N}X$ and the conormal space $C(X)$, allows us to prove (Prop. 4.14) $1 \Rightarrow 2$. We then use some sort of specialisation of integral dependence of modules, taking advantage of the absence of vertical components of the divisor D, together with the characterisation of Whitney's condition a) in terms of the integral closure of modules, and the fact that the pair $(\mathfrak{X}^0, Y)$ always satisfies Whitney's condition a) at every point with the only possible exception being the origin.

It must be said that this result is still far from constructing the Whitney stratification of $\mathfrak{X}$ we are looking for. However, it does gives us the first step it has to verify, and together with corollary 4.21 it also leaves us in a good position to continue building the stratification.

Chapter 1

Specialization to the Tangent Cone and Limits of Tangent Spaces

1.1 Specialization to the Tangent Cone

Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a representative of a reduced germ of analytic singularity of pure dimension d. That is, we are assuming that X is defined by an ideal $I = (f_1, \ldots, f_p)$ of $\mathbb{C}\{z_0, \ldots, z_n\}$ containing all analytic functions vanishing on X, and also that all its irreducible components have the same dimension d.

By definition, a singular point x of a reduced complex analytic space X is a point where the tangent space cannot be defined as usual. We are going to describe two possible substitutes for this; the tangent cone, and the set of limiting positions of tangent spaces at non-singular points tending to the given singular point x.

Let us start by the tangent cone. The canonical projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ restricted to X, induces the secant map

$$\begin{aligned} s_X \colon X \setminus \{0\} &\to \mathbb{P}^n \\ x &\mapsto [0x] \end{aligned}$$

which can be used to construct the blowup of 0 in X in the following way. Let E_0X be the closure in $X \times \mathbb{P}^n$ of the graph of s_X. It is an analytic subspace of dimension d, and together with the natural projection $e_0 \colon E_0X \to X$ induced by the first projection is isomorphic to the blowup of 0 in X. The fiber $e_0^{-1}(0)$ is a projective subvariety of $\mathbb{P}^n$ of dimension d-1, not necessarily reduced. (See [Whi65b, p.510-512])

Definition 1.1. *The cone with vertex* 0 *in* $\mathbb{C}^{n+1}$ *corresponding to the subset* $|e_0^{-1}(0)|$ *of projective space is the set-theoretic* ***tangent cone****.*

Again, set-theoretically, it is the set of limit directions of secant lines $0x$ for points $x \in X \setminus \{0\}$ tending to 0. This means more precisely that for each sequence $(x_i)_{i\in\mathbb{N}}$ of points of $X \setminus \{0\}$, tending to 0 as $i \to \infty$ we can, since $\mathbb{P}^{n-1}$ is compact, extract a subsequence such that the directions $[0x_i]$ of the secants $0x_i$ converge. The set of such limits is the underlying set of $e_0^{-1}(0)$. (See [Whi65a, thm 5.8])

Now, on to the algebraic definition. Let $O = O_{X,0} = \mathbb{C}\{z_1, \ldots, z_n\}/\langle f_1, \ldots, f_p\rangle$ be the local algebra of X at 0 and let $\mathfrak{m} = \mathfrak{m}_{X,0}$ be its maximal ideal. There is a natural filtration of $\mathcal{O}_{X,0}$ by the powers of $\mathfrak{m}$:

$$\mathcal{O}_{X,0} \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^i \supset \mathfrak{m}^{i+1} \supset \cdots,$$

which is *separated*(by Krull's intersection theorem [GP07, Cor. 2.1.35, pg 128]) in the sense that $\bigcap_{i=o}^{\infty} \mathfrak{m}^i = (0)$ because the ring $\mathcal{O}_{X,0}$ is noetherian.

Definition 1.2. *We define the* ***associated graded ring of*** O ***with respect to*** $\mathfrak{m}$*, to be the graded ring*

$$gr_{\mathfrak{m}}O\colon := \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

where $\mathfrak{m}^0 = O$*.*

Note that $gr_{\mathfrak{m}}O$ is generated as a $\mathbb{C}-$algebra by $\mathfrak{m}/\mathfrak{m}^2$, which is a finite dimensional vector space. So, $gr_{\mathfrak{m}}O$ is a finitely generated $\mathbb{C}-$algebra, to which we can associate a complex analytic space Specan $gr_{\mathfrak{m}}O$. Moreover, since $gr_{\mathfrak{m}}O$ is graded and finitely generated in degree one, the associated affine variety Specan $gr_{\mathfrak{m}}O$ is a cone. (For more on Specan, see [HIO88, Appendix I, 3.4.3 pg 480, and Appendix III 1.2 pgs 562-567].)

Definition 1.3. *We define the* ***tangent cone*** $C_{X,0}$ *as the complex analytic space Specan* $(gr_{\mathfrak{m}}O)$*.*

We have yet to establish the relation between the geometric and algebraic definitions of the tangent cone. This comes from the algebraic definition of the blowup and the **Projan** construction.(See [HIO88, Appendix III 1.4 pgs 570-572])

Let $i : \{0\} \hookrightarrow X$ be the inclusion of the point as a reduced closed complex subspace of a representative of $X \subset U$, where U is an open subset of $\mathbb{C}^{n+1}$, defined by the locally finite sheaf of ideals $\mathfrak{m} \subset O_X$. The blowup sheaf of O_X-algebras

$$B(\mathfrak{m}, O_X) := \bigoplus_{k\geq 0} \mathfrak{m}^k$$

is a graded, coherent sheaf of finitely presented O_X-algebras locally finitely generated in degree 1 such that:

$$i^*B(\mathfrak{m}, O_X) = gr_{\mathfrak{m}}O := \bigoplus_{i\geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

Moreover, according to theorem 1.4.4 [HIO88, pg 571], the projective analytic spectrum

$$e_0 : \mathrm{Projan}(B(\mathfrak{m}, O_X)) \longrightarrow X$$

is the blowup of X along $\{0\}$, and the fact that the Projan commutes with base change [HIO88, corollary 1.4.5, pg 572] gives us that $e_0^{-1}(0) = \mathrm{Projan}(gr_{\mathfrak{m}}O)$.

Now to round up our definition of the tangent cone, we are going to use the results of Appendix A to express the finitely generated $\mathbb{C}$-algebra $gr_{\mathfrak{m}}O$ in a more convenient way. Refering to the notation of the appendix, note that in our case the roles of R and J are played by the ring of convergent power series $\mathbb{C}\{z_0, \ldots, z_n\}$, and its maximal ideal $\mathfrak{M}$ respectively; while I corresponds to the ideal $< f_1, \ldots, f_p >$ defining the germ $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ and A corresponds to its analytic algebra O.

More importantly, the graded ring $gr_{\mathfrak{M}}R$, with this choice of R, is naturally isomorphic to the ring of polynomials $\mathbb{C}[z_0, \ldots, z_n]$ in such a way that definition A.1 coincides with the usual concept of initial form of a series and tells us that

$$gr_{\mathfrak{m}}O_{X,0} \cong \frac{\mathbb{C}[z_0, \ldots, z_n]}{\mathrm{In}_{\mathfrak{M}}I}$$

In other words, the germ $(C_{X,0}, 0)$ of the tangent cone at the origin is defined by the ideal generated by the initial forms of all elements of I with respect to the $\mathfrak{M}$-adic filtration.

Let us suppose that the generators $\langle f_1, \ldots, f_p \rangle$ for I, were chosen in such a way that their initial forms generate the ideal $\mathrm{In}_{\mathfrak{M}}I$ defining the tangent cone. Note that the f_i's are convergent power series in $\mathbb{C}^{n+1}$, so if m_i denotes the degree of the initial form of f_i, by defining

$$F_i(z_0, \ldots, z_n, t) := t^{-m_i} f_i(tz_0, \ldots, tz_n) \tag{1.1}$$

we obtain convergent power series, defining holomorphic functions on a suitable open subset U of $\mathbb{C}^{n+1} \times \mathbb{C}$. Moreover, we can define the analytic algebra

$$O_{\mathfrak{X},0} = \mathbb{C}\{z_0, \ldots, z_n, t\} / \langle F_1, \ldots, F_p \rangle$$

with a canonical morphism $\mathbb{C}\{t\} \longrightarrow O_{\mathfrak{X},0}$ coming from the inclusion $\mathbb{C}\{t\} \hookrightarrow \mathbb{C}\{z_0, \ldots, z_n, t\}$. Corresponding to this morphism of analytic algebras, we have the map germ $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ induced by the projection of $\mathbb{C}^{n+1} \times \mathbb{C}$ to the second factor.

Definition 1.4. *The germ of analytic space over* $\mathbb{C}$,

$$\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$$

is called the specialisation of $(X, 0)$ *to its tangent cone* $(C_{X,0}, 0)$.

There is another way of building this space that will allow us to derive some interesting properties. Let $E_{(0,0)}\mathbb{C}^{n+2}$ be the blowing up of the origin of $\mathbb{C}^{n+2}$, where we now have the coordinate system $(z_0, \ldots, z_n, t)$. Let $W \subset E_{(0,0)}\mathbb{C}^{n+2}$ be the chart where the invertible ideal defining the exceptional divisor is generated by t, that is, in this chart the blowing up map is given by $(z_0, \ldots, z_n, t) \mapsto (tz_0, \ldots, tz_n, t)$.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr] & E_{(0,0)}\mathbb{C}^{n+2} \arrow[d, "E_0"] \\ & \mathbb{C}^{n+2} \end{tikzcd}$$

Lemma 1.5. *Let* $X \times \mathbb{C} \subset \mathbb{C}^{n+2}$ *be a small enough representative of the germ* $(X \times \mathbb{C}, 0)$. *If* $(X \times \mathbb{C})'$ *denotes the strict transform of* $(X \times \mathbb{C})$ *in the blowing up* $E_{(0,0)}\mathbb{C}^{n+2}$, *then the space* $(X \times \mathbb{C})' \bigcap W$ *together with the map induced by the restriction of the map* $E_{(0,0)}\mathbb{C}^{n+2} \to \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ *is isomorphic to the specialisation space* $\varphi : \mathfrak{X} \to \mathbb{C}$.

Proof. We know that the strict transform $(X \times \mathbb{C})'$ is isomorphic to the blowing up of $X \times \mathbb{C}$ at the origin, and we are seeing it as a reduced analytic subvariety of $\mathbb{C}^{n+2} \times \mathbb{P}^{n+1}$. This means that the exceptional divisor $(X \times \mathbb{C})' \cap (\{0\} \times \mathbb{P}^{n+1})$ is equal to $\mathbb{P}(C_{X,0} \times \mathbb{C})$, and so the ideal defining it is generated by the ideal defining the tangent cone $C_{X,0}$ in $\mathbb{C}^{n+1}$, that is, the ideal of initial forms $\mathrm{In}_{\mathfrak{M}} I$. By hypothesis, $W \subset E_{(0,0)}\mathbb{C}^{n+2} \subset \mathbb{C}^{n+2} \times \mathbb{P}^{n+1}$ is set theoretically described by

$$W = \Big\{(tz_0, \dots, tz_n, t), [z_0 : \dots : z_n : 1] \,|\, (z_0, \dots, z_n, t) \in \mathbb{C}^{n+2}\Big\}$$

so in local coordinates the map E_0 restricted to W is given by $(z_0, \dots, z_n, t) \mapsto (tz_0, \dots, tz_n, t)$. Finally, since the ideal defining $X \times \mathbb{C}$ is generated in $\mathbb{C}\{z_0, \dots, z_n, t\}$ by the ideal $I = \langle f_1, \dots, f_p \rangle$ of $\mathbb{C}\{z_0, \dots, z_n\}$ defining X in $\mathbb{C}^{n+1}$, and since we have chosen the f_i's in such a way that their initial forms generate the ideal $\mathrm{In}_{\mathfrak{M}} I$, then the ideal defining the strict transform $(X \times \mathbb{C})'$ in W is given by

$$\mathfrak{I} O_W = \Big\langle t^{-m_1} f_1(tz_0, \dots, tz_n), \dots, t^{-m_p} f_p(tz_0, \dots, tz_n) \Big\rangle O_W$$

that is, we find the same functions $F_1, \dots, F_p$ which we used to define $\varphi : \mathfrak{X} \to \mathbb{C}$. □

Proposition 1.6. *Let $\varphi : \mathfrak{X} \to \mathbb{C}$ be a small enough representative of the germ, then:*

1. *The morphism φ is induced by the restriction of the projection $\mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$ to the closed subspace defined by $(F_1, \dots, F_p)$, and is faithfully flat.*
2. *The special fiber $\mathfrak{X}(0) \colon = \varphi^{-1}(0)$ is isomorphic to the tangent cone $C_{X,0}$.*
3. *The analytic space $\mathfrak{X} \setminus \varphi^{-1}(0)$ is isomorphic to $X \times \mathbb{C}^*$ as an analytic space over $\mathbb{C}^*$. In particular, for every $t \in \mathbb{C}^*$, the germ $(\varphi^{-1}(t), \{0\} \times t)$ is isomorphic to $(X, 0)$.*
4. *The germ $(\mathfrak{X}, 0)$ is reduced and of pure dimension $d+1$.*

that is, we have produced a 1-parameter flat family of germs of analytic spaces specializing $(X, 0)$ to $(C_{X,0}, 0)$.

Proof. First of all, note that the inclusion $\mathbb{C}\{t\} \hookrightarrow \mathbb{C}\{z_0, \dots, z_n, t\}$ can be seen as the stalk map at the origin of the holomorphic map defined by the linear projection onto the last coordinate $\mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$. This implies that φ is just the restriction to $\mathfrak{X}$ of this projection.

Now, to prove the (faithful) flatness of φ we must prove that $O_{\mathfrak{X},0}$ is faithfully flat as a $\mathbb{C}\{t\}$ module, but by [GLS07, Prop. B.3.3, pg 404] flat implies faithfully flat for local rings, and by [GP07, Corollary 7.3.5, pg 390] $O_{\mathfrak{X},0}$ is flat if and only if it is torsion free. In other words all we have to prove is that t is not a zero divisor in $O_{\mathfrak{X},0}$.

But by lemma 1.5, $\mathfrak{X}$ is isomorphic to an open subset of the blowing up of $X \times \mathbb{C}$ along the subspace $\{0\} \times \mathbb{C}$, where the ideal of the exceptional divisor is invertible, generated by t. Thus, by definition of blowing up, t is not a zero divisor, $\mathfrak{X}$ is of pure dimension $d+1$ (the dimension of $X \times \mathbb{C}$), and since the blowing up of a reduced space remains reduced then $\mathfrak{X}$ is reduced.

The biholomorphism $\phi : \mathbb{C}^{n+1} \times \mathbb{C}^* \to \mathbb{C}^{n+1} \times \mathbb{C}^*$ defined by $(\underline{z}, t) \mapsto (t\underline{z}, t)$, is also a direct consequence of lemma 1.5, it maps $\mathfrak{X} \setminus \mathfrak{X}(0)$ onto $\mathfrak{X} \times \mathbb{C}^*$, and for each $t \neq 0$ the fiber $\mathfrak{X}(t)$ is mapped bilohomorphically onto $X \times \{t\}$.

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr, "\varphi"'] & & X \times \mathbb{C} \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

Finally, the fact that the special fiber $\mathfrak{X}(0)$ is isomorphic to the tangent cone can be read directly from the analytic functions $F_1, \ldots, F_p$ defining $\mathfrak{X}$, since when setting $t = 0$ we have the initial forms $F_i(z, 0) = f_{m_i}$ which by hypothesis generate the ideal defining the tangent cone in $\mathbb{C}^{n+1}$.

□

A more detailed description of this space, relating it to a generalized Rees algebra and interpreting the space thus obtained as the open set of the blowup (1.5) of $X \times \mathbb{C}$ at the origin can be found in [LT79, pg 428-430] for surfaces, and [LT88, pg 556-557], or [Nav80, pg 200-202] in the general case.

Remark 1.7. *Note that:*

1. *The map $\phi : \mathfrak{X} \to X \times \mathbb{C}$ from proposition 1.6 is defined everywhere and maps the entire fiber $\mathfrak{X}(0)$ to the origin in $X \times \mathbb{C}$.*
2. *If we denote by $\mathfrak{X}(t)^0$ the non-singular part of the fiber, then the open dense subset $\bigcup_t \mathfrak{X}(t)^0 \subset \mathfrak{X}$ is called the* ***relative smooth locus of $\mathfrak{X}$ with respect to φ****.*

In a completely analogous way, if $i : Y \hookrightarrow X$ is an analytic subspace defined by a coherent ideal $J \subset O_X$ we can consider the blowup sheaf of J in O_X as the graded O_X-algebra:

$$B(J, O_X) = \bigoplus_{i \geq 0} J^i = O_X \oplus J \oplus J^2 \oplus \cdots$$

It is a graded, coherent sheaf of finitely presented O_X-algebras locally finitely generated in degree 1 and such that the pullback:

$$i^* B(J, O_X) = gr_J O_X = \bigoplus_{i \geq 0} J^i / J^{i+1}$$

is the associated sheaf of graded rings of O_X with respect to J. And so again by [HIO88, Thm 1.4.4 and Coro 1.4.5, pgs 571-572] the projective analytic spectrum

$$e_Y : \mathrm{Projan}(B(J, O_X)) \to X$$

is the blowup of X along Y, with $e_Y^{-1}(Y) = \mathrm{Projan}(gr_J O_X)$.

Definition 1.8. *We define the* ***normal cone of** X **along** Y*, *as the complex analytic space $C_{X,Y} := \mathrm{Specan}_Y(gr_J O_X)$.*

Note that we have a canonical inclusion $O_Y \hookrightarrow gr_J O_X$, which induces the structure of a locally finitely presented graded O_Y-algebra and consequently, by the Specan construction, a canonical analytic projection $C_{X,Y} \xrightarrow{\Pi} Y$, in which the fibers are cones.

We will be particularly interested in the case where $(Y,0) \subset (X,0) \subset (\mathbb{C}^{n+1},0)$ is smooth of dimension s. In this case, we can assume that Y is linear and that we have local coordinates $(z_0,\dots,z_{n-s},y_1,\dots,y_s)$ which gives us a canonical retraction $r:(X,0)\to(Y,0)$ that can be used to construct a representative of the blowup of X along Y in the following way. Define a map:

$$\begin{aligned} \sigma : X\setminus Y &\longrightarrow \mathbb{P}^{n-s} \\ x &\longmapsto \text{line } \overline{xr(x)} \end{aligned}$$

and then consider its graph in $(X\setminus Y)\times\mathbb{P}^{n-s}$. The map σ maps the point $x=(z_0,\dots,z_{n-s},y_1,\dots,y_s)\in X\setminus Y$ to the point $[z_0:\dots:z_{n-s}]\in\mathbb{P}^{n-s}$. The closure of the graph is the complex analytic space $E_YX\subset X\times\mathbb{P}^{n-s}$, and the restriction of the natural projection map gives us the map $e_Y := p\circ i$:

$$\begin{tikzcd} E_YX \arrow[r, hook, "i"] \arrow[dr, "e_Y"'] & X\times\mathbb{P}^{n-s} \arrow[d, "p"] \\ & X \end{tikzcd}$$

is proper. The map e_Y induces an isomorphism $E_YX\setminus e_Y^{-1}(Y)\to X\setminus Y$, and has the projectivized normal cone $\mathbb{P}C_{X,Y}$ as the fiber $e_Y^{-1}(Y)$. Using this, the underlying set of $(C_{X,Y},0)$ can be identified with the set of limiting positions of secant lines $x_ir(x_i)$ for $x_i\in X\setminus Y$ as x_i tends to $y\in Y$.

Moreover, by choosing adequate coordinates, we can now specifically determine the equations defining the normal cone, by interpreting definition A.1 and lemma A.2 of the appendix A in the following way. Using the notation of that section, let $R:=\mathbb{C}\{z_0,\dots,z_{n-s},y_1\dots,y_s\}$, $J=\langle z_0,\dots,z_{n-t}\rangle\subset R$ the ideal defining Y, $I=\langle f_1,\dots,f_p\rangle\subset R$ the ideal defining X and $A=R/I=O_{X,0}$. Then, the ring R/J is by definition $O_{Y,0}$ which is isomorphic to $\mathbb{C}\{y_1,\dots,y_s\}$, and its not hard to prove that

$$gr_JR\cong O_{Y,0}[z_0,\dots,z_{n-s}]$$

Now, given an element $f\in I\subset R$ we can write

$$f=\sum_{(\alpha,\beta)\in\mathbb{N}^s\times\mathbb{N}^{n-s}} c_{\alpha\beta}y^\alpha z^\beta$$

Define $\nu_Yf=\min\{|\beta|\mid c_{\alpha\beta}\neq 0\}$ and one can prove that

$$in_Jf=\sum_{|\beta|=\nu_Yf} c_{\alpha\beta}y^\alpha z^\beta$$

which after rearranging the terms with respect to z gives us a polynomial in the variables z_k with coefficients in $O_{Y,0}$, that is, an element of gr_JR. Note that these "polynomials" define analytic functions in $Y\times\mathbb{C}^{n-s+1}=\mathbb{C}^s\times\mathbb{C}^{n-s+1}$, and thus realize, by the Specan construction, the germ of the normal cone $(C_{X,Y},0)$ as a germ of analytic subspace of $(\mathbb{C}^{n+1},0)$ with a canonical analytic map to $(Y,0)$. Let us clarify all this with an example.

Example 1.9. *Take* $(X, 0) \subset (\mathbb{C}^3, 0)$ *defined by* $x^2 - y^2 z = 0$*, otherwise known as Whitney's Umbrella. Then from what we have discussed we obtain:*

i) The tangent cone at 0*,* $C_{X,0} \subset \mathbb{C}^3$*, is the analytic subspace defined by* $x^2 = 0$*.*

ii) For $Y =$ *z-axis, the normal cone along* Y*,* $C_{X,Y} \subset \mathbb{C}^3$*, is the analytic subspace defined by* $x^2 - y^2 z = 0$*, that is the entire space* X *viewed as a cone over* Y*.*

iii) For $Y =$ *y-axis, the normal cone along* Y*,* $C_{X,Y} \subset \mathbb{C}^3$*, is the analytic subspace defined by* $y^2 z = 0$*.*

The normal cone can give valuable geometrical information of X along Y as in the following result.

Proposition 1.10. *Given an s-dimensional closed nonsingular subspace* $Y \subset X$ *and a point* $0 \in Y$ *then for any local embedding* $(Y, 0) \subset (X, 0) \subset \mathbb{C}^n$*, the following conditions are equivalent:*

i) The multiplicity $m_y(X)$ *of* X *at the points* $y \in Y$ *is locally constant on* Y *near 0.*

ii) The dimension of the fibers of the map $C_{X,Y} \to Y$ *is locally constant on* Y *near 0.*

iii) For every point $y \in Y$ *there exists a dense open set of* $(n + 1 - d + s)$*-dimensional linear spaces containing* $T_y Y$ *(the tangent space to* Y *at* y*), and an open neighborhood* B *of* y *in* X*, such that if* W *is a non-singular* $(n+1-d+s)$*-dimensional space containing* Y *and whose tangent space is in that open set, then:*

$$|W \cap X \cap B| = Y \cap B$$

Proof.
See [HIO88, Appendix III, Thm 2.2.2, pg 584], or additionally [Tei82, Chapter I, 5.5 pg 347]. □

Example 1.11. *Let us look again at Whitney's Umbrella* $(X, 0) \subset (\mathbb{C}^3, 0)$ *defined by* $x^2 - y^2 z = 0$*, and let* $Y =$ *y-axis.* X *is not equimultiple along* Y *at 0, since the origin is a singular point of* X *and has multiplicity 2, while all the other points* y *in* Y *are smooth and so have multiplicity 1.*

Taking W *as the non-singular 2-dimensional space defined by* $z = ax$*, gives*

$$x(x - ay^2) = 0$$

so that whenever $a \neq 0$*, the intersection* $W \bigcap X$ *has two irreducible components: the y-axis, and the curve defined by the equations* $x = ay^2$*,* $z = ax$*.*

We can do the same for $Y =$ *z-axis, and* W *defined by* $y = ax$ *which gives*

$$x^2(1 - az^2)$$

that locally defines the z-axis. A simple calculation shows that in this case X *is equimultiple along* Y *at* 0*.*

Remark 1.12. *1. We can mimic the construction of lemma 1.5 to build the specialisation space* $\varphi : \mathfrak{X} \to \mathbb{C}$ *where we still have that the fiber* $(\mathfrak{X}(t), (0, t))_{t \neq 0}$ *is isomorphic to the germ* $(X, 0)$*, but this time the special fiber* $(\mathfrak{X}(0), (0, 0))$ *is isomorphic to the normal cone* $(C_{X,Y}, 0)$*. The map* φ *is again faithfully flat.*

2. *Let $Y \subset X$ be a linear subspace defined by the ideal $J = \langle z_0, \dots z_{n-s}\rangle \mathbb{C}\{z_0, \dots, z_{n-s}, y_1, \dots, y_s\}$ as before. We can choose analytic functions $f_1, \dots, f_p$ such that they generate the ideal I defining X in $\mathbb{C}^{n+1}$, and their initial forms $f_{m_i} = in_J f_i$ generate the ideal of initial forms $in_J I$. Then the ideal generated by the analytic functions $F_i(z, y, t) = t^{-m_i} f_i(tz_0, \dots, tz_{n-t}, y_1, \dots, y_s)$ will be the ideal defining the space $\mathfrak{X}$ in $\mathbb{C}^{n+1} \times \mathbb{C}$, where m_i is equal to $\nu_Y f_i$.*

We will end up this section by establishing the relation between the irreducible decomposition of the germ $(X, 0)$ and the irreducible decomposition of the specialisation space $(\mathfrak{X}, 0)$.

Lemma 1.13. *Let $X = \cup_{j=1}^r X_j$ be the irreducible decomposition of X. Then, the specialization $\mathfrak{X}_j$ of X_j is an analytic subspace of $\mathfrak{X}$, and the following diagram commutes.*

$$\begin{tikzcd} \mathfrak{X}_j \arrow[r, hook] \arrow[d, "\varphi_j"'] & \mathfrak{X} \arrow[r, hook] \arrow[d, "\varphi"'] & \mathbb{C}^n \times \mathbb{C} \arrow[d, "p_2"] \\ \mathbb{C} \arrow[r, "Id"'] & \mathbb{C} \arrow[r, "Id"'] & \mathbb{C} \end{tikzcd}$$

In particular, $\mathfrak{X} = \cup_{j=1}^r \mathfrak{X}_j$ is the irreducible decomposition of $\mathfrak{X}$.

Proof.

Note that X_j is a proper analytic subspace of X for $j = 1, \dots, r$, so we have a strict inclusion of their corresponding ideals in $O_{n+1} := \mathbb{C}\{z_0, \dots, z_n\}$, namely $I \subset J$, from which we immediately obtain that $In_{\mathfrak{M}} I \subset In_{\mathfrak{M}} J$ or equivalently $C_{X_j,0} \subset C_{X,0}$.

Now let us take as before, generators for I, $I = \langle f_1, \dots, f_p\rangle$, in such a way that their initial forms generate the ideal defining the tangent cone $In_{\mathfrak{M}} I$, and doing the same for J, we get $J = \langle g_1, \dots, g_s\rangle$ and $In_{\mathfrak{M}} J = \langle in_{\mathfrak{M}} g_1, \dots, in_{\mathfrak{M}} g_s\rangle$. But the previous inclusions tell us that we can choose as generators for $J = \langle f_1, \dots, f_p, g_1, \dots, g_s\rangle$, and still get that their initial forms generate the ideal $In_{\mathfrak{M}} J = \langle in_{\mathfrak{M}} f_i, in_{\mathfrak{M}} g_j\rangle$.

So finally, to build the specialization spaces $\mathfrak{X}$ and $\mathfrak{X}_j$ as we did before, we define the convergent series in O_{n+2}, $F_i(z, t) = t^{-m_i} f_i(tz_0, \dots, tz_n)$ and $G_j(z, t) = t^{-m_j} g_j(tz_0, \dots, tz_n)$, that give us the embedding $\mathfrak{X} := V(F_1, \dots, F_p) \subset \mathbb{C}^n \times \mathbb{C}$ and the embedding $\mathfrak{X}_j := V(F_1, \dots, F_p, G_1, \dots, G_s) \subset \mathbb{C}^n \times \mathbb{C}$. Moreover, since $\langle F_1, \dots, F_p\rangle \subset \langle F_i, G_j\rangle$ then we have a closed embedding $\mathfrak{X}_j \subset \mathfrak{X}$ compatible with the projection to the t axis.

And even more, since with respect to this embedding of $\mathfrak{X}$ in $\mathbb{C}^n \times \mathbb{C}$, the isomorphism ϕ is of the form:

$$\begin{aligned} \phi : \mathfrak{X} \setminus \varphi^{-1}(0) &\longrightarrow X \times \mathbb{C}^* \\ (z_0, \dots, z_n, t) &\longmapsto (tz_0, \dots, tz_n, t) \end{aligned}$$

Then, we also have compatibility with the isomorphism, that is $\phi_j = \phi|_{\mathfrak{X}_j}$.

$$\begin{tikzcd}
\mathfrak{X} \setminus \varphi^{-1}(0) \arrow[r, "\phi"] & X \times \mathbb{C}^* \\
\mathfrak{X}_j \setminus \varphi_j^{-1}(0) \arrow[r, "\phi_j"'] \arrow[u, hook] & X_j \times \mathbb{C}^* \arrow[u, hook]
\end{tikzcd}$$

□

1.2 Limits of Tangent Spaces and the Nash Modification.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^p, 0)$ be the germ of analytic map defined by the p series $f_1, \ldots, f_p \in \mathbb{C}\{z_0, \ldots, z_n\}$; note that $(X, 0) = (f^{-1}(0), 0)$. In the same way, let $F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^p, 0)$ denote the germ of analytic map defined by the p series $F_1, \ldots, F_p \in \mathbb{C}\{z_0, \ldots, z_n, t\}$, such that $(\mathfrak{X}, 0) = (F^{-1}(0), 0)$.

Definition 1.14. *A d-plane T of the Grassmannian $Gr(d, n+1)$ of directions of d-planes in $\mathbb{C}^{n+1}$, is a limit at $0 \in X$ of tangent spaces to the analytic space X if there exists a sequence $\{z_i\}$ of non-singular points of X and a sequence of d-planes $\{T_i\}$ of $Gr(d, n+1)$ such that for all i, the d-plane T_i is the direction of the tangent space to X at x_i, the sequence $\{x_i\}$ converges to 0 and the sequence T_i converges to T.*

Recall that if X is a reduced analytic space then the set $\operatorname{Sing} X$ of singular points of X, is also an analytic space and the non singular part $X^\circ :=X \setminus \operatorname{Sing} X$ is dense in X and has the structure of a complex manifold. How can we determine these limit positions? Consider the map:

$$\begin{aligned} \gamma : X^\circ &\longrightarrow Gr(d, n+1) \\ z &\longrightarrow T_z X^\circ \end{aligned}$$

where $T_z X^\circ$ denotes the tangent space to the manifold X° at the point z. The closure $\mathcal{N}X$ of the graph of γ in $X \times Gr(d, n+1)$ is an analytic subspace of dimension d ([Whi65b, Thm 16.4]), which is known as **the Nash modification of** X.

Proposition 1.15. *[Nob75, Thm 1, pg 299] The Nash modification $\nu : \mathcal{N}X \to X$ is locally a blowing-up with center a suitable ideal $J \subset O_X$. Moreover, if $(X, 0)$ is a complete intersection of dimension $n+1-p$ then we may take the ideal J to be the Jacobian ideal, formed by the $p \times p$ minors of the Jacobian matrix $[Df] = \left[\frac{\partial f_i}{\partial z_j}\right]_{j=0\ldots n}^{i=1\ldots p}$.*

Let us consider a representative $\varphi : \mathfrak{X} \to \mathbb{C}$ of the germ $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$. We will also be interested in considering the space of limiting tangent spaces to the fibers of this morphism. Recall that $(\mathfrak{X}, 0)$ is a germ of pure $(d+1)$ dimensional singularity, whose fibers $(\mathfrak{X}(t), (0, t)), t \neq 0$ are all isomorphic to $(X, 0)$, and the special fiber $(\mathfrak{X}(0), (0, 0))$ is isomorphic to the tangent cone $(C_{X,0}, 0)$, which is also of pure dimension d. Consider the map

$$\gamma_\varphi : \mathfrak{X}^\circ_\varphi \longrightarrow Gr(d, n+1)$$
$$(z,t) \longrightarrow T_{(z,t)}X^\circ_\varphi(t)$$

where $\mathfrak{X}^\circ_\varphi$ denotes the relative smooth locus of $\mathfrak{X}$ with respect to φ, $Gr(d, n+1)$ corresponds to the grassmannian of directions of $d-$planes of the hyperplane $\{t = 0\} \subset \mathbb{C}^{n+1} \times \mathbb{C}$, and $T_{(z,t)}\mathfrak{X}^\circ_\varphi(t)$ denotes the tangent space to the fiber $\mathfrak{X}^\circ_\varphi(t)$ at the point (z,t). The closure $\mathcal{N}_\varphi\mathfrak{X}$ of the graph of γ_φ in $\mathfrak{X} \times Gr(d, n+1)$ is an analytic space of dimension $d+1$, which is known as **the relative Nash modification of $\varphi : \mathfrak{X} \to \mathbb{C}$.**

The main ingredient of the proof of proposition 1.15 is the Plucker embedding of the grassmannian $G(d, n+1)$, in the projective space $\mathbb{P}^N$, where $N = \begin{pmatrix} n+1 \\ d \end{pmatrix}$. Thus, minor modifications of the proof immediately gives us an analogous result for the relative case. We will only state it in the case of $\varphi : \mathfrak{X} \to \mathbb{C}$.

Corollary 1.16. *The relative Nash modification $\nu_\varphi : \mathcal{N}_\varphi\mathfrak{X} \to \mathfrak{X}$ is locally a blowing-up with center a suitable ideal $J_\varphi \subset O_\mathfrak{X}$. Moreover, if $(\mathfrak{X}, 0)$ is a complete intersection of dimension $n+2-p$ then we may take the ideal $J_\varphi \subset O_\mathfrak{X}$ to be the relative Jacobian ideal, formed by the $p \times p$ minors of the relative Jacobian matrix $[D_\varphi F] = \left[\frac{\partial F_i}{\partial z_j}\right]^{i=1\ldots p}_{j=0\ldots n}$. (We are omitting the partial derivatives with respect to the parameters, which in this case correspond to the t-coordinate).*

Proof. Given integers $n+1 \geq r > 0$, $p \geq n+1-r$ and a $p \times (n+1)$ matrix A, let S(resp. S') denote the set of increasing sequences of $n+1-r$-positive integers less than $p+1$ (resp. $n+2$); if $\alpha = (\alpha_1, \ldots, \alpha_{n+1-r}) \in S$, $\beta = (\beta_1, \ldots, \beta_{n+1-r}) \in S'$, then $M_{\alpha\beta}$ will denote the minor of A obtained by considering the rows determined by α and the columns determined by β.

Following the proof of 1.15, let $\mathfrak{X} = \bigcup_{j=1}^k \mathfrak{X}_j$ be the irreducible decomposition of a small enough representative of $(\mathfrak{X}, 0)$. Let $[D_\varphi F] = \left[\frac{\partial F_i}{\partial z_j}\right]^{i=1\ldots p}_{j=0\ldots n}$ be the relative Jacobian matrix of the map $\varphi : \mathfrak{X} \to \mathbb{C}$. Then, by construction, there is an open dense set $U \subset \mathfrak{X}$, such that for every point (z_0, t_0) in U the matrix $[D_\varphi F(z_0, t_0)]$ has rank $n+1-d$. Since $\mathfrak{X}$ is reduced, each irreducible component $\mathfrak{X}_i$ is reduced and so for each $i = 1, \ldots, k$ there exists a pair $(\alpha^i, \beta^i) \in S \times S'$ such that the $(n+1-d) \times (n+1-d)$ minor $M_{\alpha^i\beta^i}$ of $[D_\varphi F]$ does not vanish identically on $\mathfrak{X}_i$. For each $i = 1, \ldots, k$, fix $H_i \in O_{\mathfrak{X},0}$ such that $H_i = 0$ on $\bigcup_{j\neq i} \mathfrak{X}_j$, and $H_i \neq 0$ on $\mathfrak{X}_i$. For each $\beta \in S'$ define the function $G_\beta = \sum_{i=1}^k H_i M_{\alpha^i\beta} \in O_{\mathfrak{X},0}$, and consider the ideal $J_\varphi \subset O_{\mathfrak{X},0}$ generated by the G_β's.

Note that the analytic subset $V(J_\varphi)$ of $\mathfrak{X}$ defined by the ideal J_φ contains the relative singular locus of $\varphi : \mathfrak{X} \to \mathbb{C}$. Moreover, the open set $W := \mathfrak{X} \backslash V(J_\varphi)$ is dense in $\mathfrak{X}$. Finally if we build a representative of this blowup using the functions G_β, then we will have it as an analytic subspace of $\mathfrak{X} \times \mathbb{P}^N$, with $N = \begin{pmatrix} n+1 \\ n+1-d \end{pmatrix} - 1 =$

$\binom{n+1}{d} - 1$, and for a point $(z,t) \in \mathfrak{X}_i \cap W$ we have that:

$$[G_\beta(z,t)] = [\sum_{j=1}^{k} H_j(z,t) M_{\alpha^j\beta}(z,t)] = [H_i(z,t) M_{\alpha^i\beta}(z,t)] = [M_{\alpha^i\beta}(z,t)] \in \mathbb{P}^N$$

which corresponds to the coordinates of the tangent space $T_{(z,t)}X^\circ_\varphi(t)$ for the Plucker embedding of the grassmannian $G(d, n+1)$, in the projective space $\mathbb{P}^N$. $\square$

Proposition 1.17. *There exists a natural surjective morphism* $\Gamma : \mathcal{N}_\varphi\mathfrak{X} \to \mathcal{N}X$*, making the following diagram commute:*

$$\begin{tikzcd} \mathcal{N}_\varphi\mathfrak{X} \arrow[r, "\Gamma"] \arrow[d, "\nu_\varphi"'] & \mathcal{N}X \arrow[d, "\nu"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

Proof. Algebraically, this result follows from the universal property of the blowup $\nu : \mathcal{N}X \to X$. We start with the diagram:

$$\begin{tikzcd} \mathcal{N}_\varphi\mathfrak{X} \arrow[d, "\nu_\varphi"'] & \mathcal{N}X \arrow[d, "\nu"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

where the map ϕ is defined by $(z_0, \dots, z_n, t) \to (tz_0, \dots, tz_n)$, and so it induces a morphism of analytic algebras $\phi^* : O_{X,0} \to O_{\mathfrak{X},0}$ defined by $z_i \to tz_i$.

Recall that the ideal of the germ $(\mathfrak{X}, 0)$ is generated by the series $F_i(z,t) = t^{-m_i} f_i(tz) \in \mathbb{C}\{z_0, \dots, z_n, t\}$, $i = 1, \dots, p$, where the series $f_j \in \mathbb{C}\{z_0, \dots, z_n\}$ are such that they generate the ideal of $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$ and their initial forms generate the ideal of $(C_{X,0}, 0)$.

By 1.15 there exists an ideal $JO_{X,0}$ whose blowup is isomorphic to the Nash modification of X. We have to prove that the ideal $\phi^*(J)O_{\mathfrak{X},0}$ is locally invertible when pulled back to $\mathcal{N}_\varphi\mathfrak{X}$.

Let $X = \bigcup_{j=1}^k X_j$ be the irreducible decomposition of a small enough representative of $(X, 0)$. Then the irreducible decomposition of a small enough representative of the germ $(\mathfrak{X}, 0)$ is of the form $\bigcup_{j=1}^k \mathfrak{X}_j$, where for each j the space $\widetilde{\mathfrak{X}_j}$, defined as the closure in $\mathfrak{X}$ of $\mathfrak{X}_j \setminus \mathfrak{X}(0)$, is isomorphic to the specialisation space of the X_j component to its tangent cone $C_{X_j,0}$. Now, by 1.15 the ideal $J \subset O_{X,0}$ can be constructed in the following way (see the proof of 1.16 for more details and notation): For each $i = 1, \dots, k$ there exists a pair $(\alpha^i, \beta^i) \in S \times S'$ such that the $(n+1-d) \times (n+1-d)$ minor $\mu_{\alpha^i\beta^i}$ of the jacobian matrix $[Df]$ does not vanish identically on X_i. Then for each $i = 1, \dots, k$, choose a function $h_i \in O_{X,0}$ such that $h_i = 0$ on $\bigcup_{j \neq i} X_j$, and $h_i \neq 0$ on X_i. By taking powers of the h_i's if necessary we can assume they are all of the same order γ. Finally for each $\beta \in S'$ define the function $g_\beta = \sum_{i=1}^{k} h_i \mu_{\alpha^i\beta} \in O_{X,0}$, and define J as the ideal generated by the g_β's.

Consider an $(n+1-d)\times(n+1-d)$ minor $\mu_{\alpha\beta}$ of the jacobian matrix $[Df]$

$$\mu_{\alpha,\beta} = \begin{vmatrix} \frac{\partial f_{\alpha_1}}{\partial z_{\beta_1}}(z) & \cdots & \frac{\partial f_{\alpha_1}}{\partial z_{\beta_{n+1-d}}}(z) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{\alpha_{n+1-d}}}{\partial z_{\beta_1}}(z) & \cdots & \frac{\partial f_{\alpha_{n+1-d}}}{\partial z_{\beta_{n+1-d}}}(z) \end{vmatrix}$$

Then, from the equalities $\phi^*(\frac{\partial f_i}{\partial z_j}(z)) = \frac{\partial f_i}{\partial z_j}(tz)$, and $\frac{\partial f_i}{\partial z_j}(tz) = t^{m_i-1}\frac{\partial F_i}{\partial z_j}(z,t)$, we have that the minor $\mu_{\alpha\beta}$ is mapped under ϕ^* to:

$$\begin{aligned}\phi^*(\mu_{\alpha\beta}) &= \begin{vmatrix} t^{m_{\alpha_1}-1}\frac{\partial F_{\alpha_1}}{\partial z_{\beta_1}}(z,t) & \cdots & t^{m_{\alpha_1}-1}\frac{\partial F_{\alpha_1}}{\partial z_{\beta_{n+1-d}}}(z,t) \\ \vdots & \vdots & \vdots \\ t^{m_{\alpha_{n+1-d}}-1}\frac{\partial F_{\alpha_{n+1-d}}}{\partial z_{\beta_1}}(z,t) & \cdots & t^{m_{\alpha_{n+1-d}}-1}\frac{\partial F_{\alpha_{n+1-d}}}{\partial z_{\beta n+1-d}}(z,t) \end{vmatrix} \\ &= t^{(\sum_1^{n+1-d} m_{\alpha_i})-(n+1-d)} \begin{vmatrix} \frac{\partial F_{\alpha_1}}{\partial z_{\beta_1}}(z,t) & \cdots & \frac{\partial F_{\alpha_1}}{\partial z_{\beta_{n+1-d}}}(z,t) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_{\alpha_{n+1-d}}}{\partial z_{\beta_1}}(z,t) & \cdots & \frac{\partial F_{\alpha_{n+1-d}}}{\partial z_{\beta n+1-d}}(z,t) \end{vmatrix} \\ &= t^{(\sum_1^{n+1-d} m_{\alpha_i})-(n+1-d)} M_{\alpha\beta}\end{aligned}$$

where $M_{\alpha\beta}$ is the $(n+1-d)\times(n+1-d)$ minor of the relative jacobian matrix $[D_\varphi F]$.

If we define $H_i \in O_{\mathfrak{X},0}$ by $H_i(z,t) = t^{-\gamma}h_i(tz)$, then each H_i satisfies that $H_i = 0$ on $\bigcup_{j\neq i}\mathfrak{X}_j$, and $H_i \neq 0$ on $\mathfrak{X}_i$ and so for each $\beta \in S'$ we have that

$$\phi^*(g_\beta) = \sum_{i=1}^{k}\phi^*(h_i)\phi^*(\mu_{\alpha^i\beta}) = t^{\left(\gamma+(\sum_1^{n+1-d} m_{\alpha_i})-(n+1-d)\right)}\sum_{i=1}^{k} H_i M_{\alpha^i\beta} = t^r G_\beta$$

and so

$$\phi^*(J)O_{\mathfrak{X},0} = \langle t^r\rangle\, J_\varphi O_{\mathfrak{X},0}$$

where by the proof of 1.16 $J_\varphi O_{\mathfrak{X},0}$ is an ideal whose blowup is isomorphic to the relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$. But by definition of the blowup, the ideal $J_\varphi O_{\mathfrak{X},0}$ is locally invertible when pulled back to $\mathcal{N}_\varphi\mathfrak{X}$. It follows that multiplying it by the invertible ideal $\langle t^r\rangle$ in $O_{\mathfrak{X},0}$ will remain locally invertible when pulled back to $\mathcal{N}_\varphi\mathfrak{X}$.

Finally, note that for the diagram to be commutative, the morphism Γ must map the point $(z,t,T_{(z,t)}\mathfrak{X}^\circ_\varphi(t)) \in \mathcal{N}_\varphi\mathfrak{X}$ to the point $(tz, T_{(tz)}X^\circ) \in \mathcal{N}X$. That is the tangent space $T_{(z,t)}\mathfrak{X}^\circ_\varphi(t)$ to the fiber $\mathfrak{X}(t)$ is canonically identified with the tangent space $T_{(tz)}X^\circ$ to X at the correspoding points. As it should be since we know that the restriction of the map ϕ to any fiber $(\mathfrak{X}(t),(0,t))$ for $t\neq 0$ is an isomorphism with $(X,0)$. □

1.3 The Conormal Space

The Nash modification is a little difficult to handle because of the fact that the rich geometry of the Grassmanian entails somewhat cumbersone computations.

There is a less intrinsic but more amenable way of encoding the limits of tangent spaces. The idea is to replace a tangent space to X° by the collection of all the hyperplanes of $\mathbb{C}^{n+1}$ which contain it. Tangent hyperplanes live in a projective space, namely the dual projective space $\check{\mathbb{P}}^n$, which is simpler to work with than the Grassmannian.

1.3.1 Some Symplectic Geometry

In order to describe this set of tangent hyperplanes, we are going to use the language of symplectic geometry and lagrangian submanifolds. So let us start by a couple of definitions.

Let M be any n-dimensional manifold, and let ω be a de Rham 2-form on M, that is, for each $p \in M$, the map

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is symplectic if it is closed and ω_p is non-degenerate for all $p \in M$. Naturally, a symplectic manifold is a pair (M, ω), where M is a manifold and ω is a symplectic form.

Now, for any manifold M, its cotangent bundle T^*M has a canonical symplectic structure as follows. Let

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ p = (x, \xi) &\longmapsto x \end{aligned}$$

where $\xi \in T^*_xM$, be the natural projection. The **Liouville 1-form** α on T^*M may be defined pointwise by:

$$\alpha_p(v) = \xi\left((d\pi_p)v\right), \quad for\ v \in T_p(T^*M)$$

Note that $d\pi_p : T_p(T^*M) \to T_xM$, so that α is well defined. Then, the **canonical symplectic 2-form** ω on T^*M is defined as

$$\omega = -d\alpha$$

And it is not hard to see, that if $(U, x_1, \ldots, x_n)$ is a coordinate chart for M with associated cotangent coordinates $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, then locally:

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i$$

and

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

Definition 1.18. *Let (M, ω) be a 2n-dimensional symplectic manifold. A submanifold Y of M is a **lagrangian submanifold** if, at each $p \in Y$, T_pY is a lagrangian subspace of T_pM, i.e., $\omega_p|_{T_pY} \equiv 0$ and $\dim T_pY = 1/2 \dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is **lagrangian** if and only if $i^*\omega = 0$ and $\dim Y = 1/2 \dim M$.*

Example 1.19. *The **zero section** of T^*M*

$$X := \{(x,\xi) \in T^*M | \xi = 0 \text{ in } T_x^*M\}$$

is an n-dimensional lagrangian submanifold of T^*M.

1.3.2 Conormal space.

Let now $X \subset M$ be a possibly singular complex subspace of pure dimension d, and let as before $X^\circ = X \setminus \operatorname{Sing} X$, be the non-singular part of X, so it is a submanifold of M.

Definition 1.20. *Set*

$$N_x^* X^\circ = \{\xi \in T_x^*M | \xi(v) = 0, \ \forall v \in T_x X^\circ\}$$

this means that the hyperplane $\{\xi = 0\}$ contains the tangent space to X° at x.
*The **conormal bundle** of X° is*

$$T_{X^\circ}^*M = \{(x,\xi) \in T^*M | x \in X^\circ, \ \xi \in N_x^* X^\circ\}$$

Proposition 1.21. *Let $i : T_{X^\circ}^*M \hookrightarrow T^*M$ be the inclusion, and let α be the Liouville 1-form in T^*M as before. Then $i^*\alpha = 0$. In particular the conormal bundle $T_{X^\circ}^*M$ is a lagrangian submanifold of T^*M, and of course has dimension n.*

Proof.
For a proof of this result look at [Can01, Proposition 3.6, Corollary 3.7, pg 16]. □

In the same context, we can define the **conormal space of X in M**, denoted as T_X^*M, as the closure of $T_{X^\circ}^*M$ in T^*M, with the **conormal map** $\kappa_X : T_X^*M \to X$, induced by the natural projection $\pi : T^*M \to M$. The conormal space may be singular, but it is of dimension n, and by proposition 1.21 α vanishes on every tangent vector at a nonsingular point, so it is by construction a lagrangian subspace of T^*M.

In words, the fiber of the conormal map $\kappa_X \colon T_X^*M \to X$ above a point $x \in X$ consists, if $x \in X^\circ$, of all the equations of hyperplanes tangent to X at x, in the sense that they contain the tangent space $T_x X^\circ$. If x is a singular point, the fiber consists of all equations of limits of hyperplane directions tangent at non singular points of X tending to x. In addition, if $x \in X^\circ$, the fiber $\kappa_X^{-1}(x)$ is isomorphic to $\mathbb{C}^{n-d}$, namely the space of linear forms vanishing on the tangent space $T_x X^\circ$.

Now the fibers of κ_X are invariant under multiplication by an element of $\mathbb{C}^*$, and we can divide by the equivalence relation this defines. The idea is to remember only the directions of tangent hyperplanes, and not a specific linear form defining it. That is, the conormal space is conical:= stable by vertical homotheties, so we can "projectivize" it. Since conormal varieties are conical, we may as well projectivize with respect to vertical homotheties of T^*M and work in $\mathbb{P}T^*M$, where it still makes sense to be lagrangian since α is homogeneous by definition. Moreover, we can characterize those subvarieties of the cotangent space which are the conormal spaces of their images in M.

Proposition 1.22. *([Pha79, Section 10.1, pg 91-92]) Let M be a non singular analytic variety of dimension n and let L be a closed conical irreducible analytic subvariety of T^*M of dimension n. The following conditions are equivalent:*
1) The variety L is the conormal space of its image in M
2) The Liouville 1-form α vanishes on all tangent vectors to L at every non singular point of L.
3) The symplectic 2-form $\omega = -\mathrm{d}\alpha$ vanishes on every pair of tangent vectors to L at every non singular point of L.

Proof.
First note that for an irreducible, pure dimensional analytic space X, the conormal space $T^*_X M$ satisfies all these hypothesis, i.e. it is a closed conical irreducible analytic subvariety of dimension n.

Secondly, if we have a subvariety $L \subset T^*M$ satisfying this hypothesis, then after projectivization of the cotangent space we still get a subvariety

$$\mathbb{P}L \subset \mathbb{P}T^*M \xrightarrow{\pi} M$$

where now π is a proper morphism, and by the Grauert-Remmert's mapping theorem [KK83, Thm. 45.17, pg 170] this implies that the image $\pi(L)$ is analytic subvariety of M.

Now, if L is the conormal space of its image in M, and $i : L \hookrightarrow T^*M$ is the inclusion map, then by proposition 1.21 the form $i^*\alpha \equiv 0$ on L^0, that is, it vanishes on all tangent vectors to L at every non singular point of L, this proves 1) $\Rightarrow$ 2).

To prove 2) $\Rightarrow$ 3), we must prove that $i^*\omega \equiv 0$ on L^0, but $\omega = -d\alpha$ and it is well known that the differential d commutes with pullback ([Bre93, Prop. 2.7, pg 264]) that is:

$$i^*\omega = i^*d\alpha = d(i^*\alpha) = d(0) = 0 \text{ on } L^0$$

To prove 3) implies 1) we must prove that for every $x \in \pi(L)^0$, and every $p = (x, \xi) \in \pi^{-1}(x)$, we have that the hyperplane $\{\xi = 0\}$ contains the tangent space $T_x\pi(L)$. Consider the restriction of the analytic map $\pi : \pi^{-1}(\pi(L)^0) \to \pi(L)^0$ between analytic manifolds. This map is generically a submersion, that is there is an open dense set $U \subset \pi^{-1}(\pi(L)^0)$ such that $\pi_{|U} : U \to \pi(U)$ is a submersion. It is important to note that U is dense in L as well.

Now, let $p = (x, \xi) \in U$, and let us choose a local chart $(V, x_1, \ldots, x_n)$ for M, with associated cotangent chart $(T^*V, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ such that $p = (0, \ldots, 0, \xi_1, \ldots, \xi_n)$ in T^*V. Since by hypothesis L is conical with respect to the ξ coordinates then for every $\lambda \in \mathbb{C}$ the point $(0, \lambda\xi_1, \ldots, \lambda\xi_n)$ is in L. This implies that the line $\lambda \cdot (0, \xi)$ is contained in the tangent space $T_pL \subset T_p(T^*M)$. In this set of coordinates, the map $\pi : T^*V \to V$ is just the projection on the x coordinates. Moreover, since we have chosen $p \in U$, the tangent map $D\pi : T_pL \to T_x\pi(L)$ is surjective, so for every vector $v_1 = (a_1, \ldots, a_n) \in T_x\pi(L)$ there exists a vector $v = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in T_pL$ with $D\pi(v) = v_1$. By hypothesis, the 2-form ω vanishes in every pair of tangent vectors to L at every non-singular point of L, in particular it vanishes for the vectors

v and $(0,\xi)$ and so from the expression of ω in local coordinates we get:

$$\omega_p(v,(0,\xi)) = \sum_{i=1}^{n} \xi_i a_i = \xi(D\pi_p(v)) = \xi(v_1) = 0$$

that is, the hyperplane $\{\xi = 0\}$ contains the tangent space $T_x\pi(L)$. This is true for every $p \in U$ which tells us that both U and its closure L are contained in the conormal space $C(\pi(L))$ of $\pi(L)$ in M, since the conormal space is closed. Finally, we have an inclusion $L \subset C(\pi(L))$ of irreducible analytic spaces of the same dimension, so they are equal.

□

Remark 1.23. *The irreducibility in the hypothesis comes from the fact that if we have two subvarieties $Y \subset X \subset M$, then the subvariety $T^*_Y M \bigcup T^*_X M$ satisfies all the hypothesis except irreducibility, however its image is clearly X, and generally we don't have the inclusion $T^*_Y M \subset T^*_X M$, which results in a contradiction of 1).*

Now, going back to our original problem we have $X \subset M = \mathbb{C}^{n+1}$, so $T^*M = \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$ and $\mathbb{P}T^*M = \mathbb{C}^{n+1} \times \check{\mathbb{P}}^n$. So we have the **(projective) conormal space** $\kappa_X : C(X) \to X$ with $C(X) \subset X \times \check{\mathbb{P}}^n$, where $C(X)$ denotes the projectivization of the conormal space $T^*_X M$. Note that we have not changed the name of the map κ_X after projectivizing since there is no ambiguity, and that the dimension of $C(X)$ is n, which shows immediately that it depends on the embedding of X in an affine space. We have the following result:

Proposition 1.24. *The (projective) conormal space $C(X)$ is a closed, reduced, analytic subspace of $X \times \check{\mathbb{P}}^n$, purely of dimension n.*

Proof.
For a proof of this result see [Tei82, Proposition 4.1, pg 379]. □

1.3.3 Conormal vs. Nash

Now we are going to describe the relation between the conormal space of $(X,0) \subset (\mathbb{C}^{n+1},0)$ and its Nash modification. Up to now we have:

$$\mathcal{N}X \subset X \times G(d,n+1) \subset X \times \mathbb{P}^N$$

But we know that the grassmannian $G(d,n+1)$ is isomorphic to the grassmannian $G(n+1-d,n+1)$ and the isomorphism is given by sending a $d-$plane T to the $n+1-d-$plane L of linear functionals in $\check{\mathbb{C}}^{n+1}$ that vanish on T. With this isomorphism, we have:

$$\mathcal{N}X \subset X \times G(n+1-d,n+1) \subset X \times \mathbb{P}^N$$

Let $\Xi \subset G(n+1-d,n+1) \times \check{\mathbb{P}}^n$ denote the tautological bundle, that is $\Xi = \{(L,[a]) \mid L \in G(n+1-d,n+1),\ [a] \in \mathbb{P}L \subset \check{\mathbb{P}}^n\}$, and consider the intersection

$$\begin{tikzcd}
E := \{X \times \Xi\} \bigcap \{\mathcal{N}X \times \check{\mathbb{P}}^n\} \arrow[r, hook] \arrow[d, "p_1"'] \arrow[drr, "p_2"] & X \times G(n+1-d,n+1) \times \check{\mathbb{P}}^n \arrow[dr] & \\
\mathcal{N}X & & X \times \check{\mathbb{P}}^n
\end{tikzcd}$$

with the vertical morphism p_2 being the morphism induced by the projection onto $X \times \check{\mathbb{P}}^n$. We then have the following result.

Proposition 1.25. *Let $p_2 : E \to X \times \check{\mathbb{P}}^n$ be as before. The set theoretical image $p_2(E)$ of the morphism p_2 coincides with the conormal space of X in $\mathbb{C}^{n+1}$*

$$C(X) \subset X \times \check{\mathbb{P}}^n$$

Moreover, the morphism $p_1 : E \to \mathcal{N}X$ is a locally trivial fiber bundle over $\nu^{-1}(X^0) \subset \mathcal{N}X$ with fiber $\mathbb{P}^{n-d}$.

Proof. By definition, the conormal space of X in $\mathbb{C}^{n+1}$ is an analytic space $C(X) \subset X \times \check{\mathbb{P}}^n$, together with a proper analytic map $\kappa_X : C(X) \to X$, where the fiber over a smooth point $x \in X^0$ is the set of tangent hyperplanes, that is the hyperplanes H containing the direction of the tangent space $T_x X$. That is, if we define $E^0 = \{(x, T, [a]) \in E \,|\, x \in X^0\}$, then by construction $E^0 = p_1^{-1}(\nu^{-1}(X^0))$, and $p_2(E^0) = C(X^0)$. Since the morphism p_2 is proper, in particular it is closed which finishes the proof. □

Corollary 1.26. *A hyperplane $H \in \check{\mathbb{P}}^n$ is a limit of tangent hyperplanes to X at 0, i.e. $H \in \kappa_X^{-1}(0)$ if and only if there exists a d-plane $(0, T) \in \nu^{-1}(0)$ such that $T \subset H$.*

Proof. Let $(0, T) \in \nu^{-1}(0)$ be a limit of tangent spaces to X at 0. The isomorphism between the grassmannians $G(d, n+1)$ and $G(n+1-d)$ identifies the d-plane T with the $n+1-d-$ plane $\check{T}$ of linear functionals in $\check{\mathbb{C}}^{n+1}$ that vanish on T. This implies by construction of E and proposition 1.25 that every hyperplane H containing T is in the fiber $\kappa_X^{-1}(0)$, that is it is a limit of tangent hyperplanes to X at 0.

On the other hand, by construction for any hyperplane $H \in \kappa_X^{-1}(0)$ there is a sequence of points $\{(x_i, H_i)\}_{i \in \mathbb{N}}$ in $\kappa_X^{-1}(X^0)$ converging to $p = (0, H)$. Now, to the sequence $\{(x_i)\}_{i \in \mathbb{N}} \subset X^0$ there corresponds the sequence $\left\{\left(x_i, \check{T}_i\right)\right\}_{i \in \mathbb{N}}$ living in $\nu^{-1}(X^0)$ where $\check{T}_i$ is the projective dual of the tangent space to X^0 at the point x_i. Since the grassmannian $G(n+1-d, n+1)$ is compact, there exists an $n+1-d$ plane $\check{T}$ in $G(n+1-d, n+1)$ and a subsequence $\left\{\left(x_j, \check{T}_j\right)\right\}_{j \in \Delta \subset \mathbb{N}}$, converging to the point $(0, \check{T})$.

The combination of these two sequences gives us a sequence $\left\{\left(x_j, \check{T}_j, H_j\right)\right\}_{j \in \Delta \subset \mathbb{N}}$ in E which converges to the point $(0, \check{T}, H)$. This means, that the hyperplane H contains the corresponding limit of tangent spaces $T \in \nu^{-1}(0) \subset \mathcal{N}X$.

□

If X is a hypersurface, the conormal map coincides with the Nash modification. In general, while it is true that the geometric structure of the inclusion $\kappa_X^{-1}(x) \subset \check{\mathbb{P}}^n$ determines the set of limit positions of tangent spaces, i.e., the fiber $\nu^{-1}(x)$ of the Nash modification, the correspondence is not so simple: by proposition 1.25 and its corollary, the points of $\nu^{-1}(x)$ correspond to projective subspaces $\check{\mathbb{P}}^{n-d}$ of $\check{\mathbb{P}}^n$ contained in $\kappa_X^{-1}(x)$.

1.3.4 Conormal spaces and projective duality

Let us assume for a moment that $X \subset \mathbb{C}^{n+1}$ is a cone over a projective algebraic variety. In the spirit of last section, let us take $M = \mathbb{C}^{n+1}$ with coordinates $(z_0, \dots, z_n)$, and consider the dual space $\check{\mathbb{C}}^{n+1}$ with coordinates $(\xi_0, \dots, \xi_n)$, i.e., its points are linear forms on $\mathbb{C}^{n+1}$.

Lemma 1.27. *Let $x \in X$ be a nonsingular point of X. Then the tangent space T_xX°, contains the line ℓ joining x to the origin. Moreover, the quotient T_xX°/ℓ is precisely the tangent space to the projective variety $\mathbb{P}X$ at the corresponding point.*

Proof.
This is due to Euler's identity for a homogeneous polynomial of degree m:

$$m.f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$$

and the fact that if $\{f_1, \dots, f_p\}$ is the set of homogeneous polynomials defining X, then T_xX° is the kernel of the matrix:

$$\begin{pmatrix} \nabla f_1(x) \\ \cdot \\ \cdot \\ \nabla f_p(x) \end{pmatrix}$$

□

It is also important to note that at all non-singular points of X in the same genera-ting line the tangent space to X° is constant since the partial derivatives are homogeneous again and the quotient by the generating line is the tangent space to $\mathbb{P}X$.

Let us consider the conormal space of X in $\mathbb{C}^{n+1}$, by construction $T^*_X\mathbb{C}^{n+1}$ is a subvariety of the cotangent space $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. Remark that globally the $1-$form α on $T^*\mathbb{C}^{n+1}$ has the form :

$$\alpha = \sum_{i=0}^{n} \xi_i dz_i$$

and since by proposition 1.21, $T^*_X\mathbb{C}^{n+1}$ is a Lagrangian subvariety of $\mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$, then by proposition 1.22 the 1-form α vanishes on all tangent vectors to $T^*_X\mathbb{C}^{n+1}$ at every non-singular point of $T^*_X\mathbb{C}^{n+1}$. Moreover, since X is a cone, the variety $T^*_X\mathbb{C}^{n+1}$ is bi-conical and by lemma 1.27 every point $(\underline{z}, \underline{\xi})$ in $T^*_X\mathbb{C}^{n+1}$ satisfies the equation $\sum_{i=0}^{n} z_i\xi_i = 0$.

Definition 1.28. *Define the* ***incidence variety*** *$I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ as the set of points satisfying:*

$$\sum_{i=0}^{n} z_i\xi_i = 0$$

where $[z_0 : \dots : z_n], [\xi_0 : \dots : \xi_n] \in \mathbb{P}^n \times \check{\mathbb{P}}^n$. It is a smooth subvariety of $\mathbb{P}^n \times \check{\mathbb{P}}^n$.

We have $T^*_X\mathbb{C}^{n+1} \subset I \subset \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$, where here we abusively denote again by I the cone over the incidence variety. Now since $\mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$ can also be identified with the cotangent space $T^*\check{\mathbb{C}}^{n+1}$ we have the canonical form

$$\check{\alpha} = \sum_{i=0}^{n} z_i d\xi_i$$

for which we have the following result.

Lemma 1.29. *([Kle84, Prop. 4.2, pg 188])*
Let $I \subset \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$ be the incidence variety as above, then $\alpha + \check{\alpha} = 0$ on I.

Proof.
The polynomial $P(\underline{z}, \underline{\xi}) := \sum_{i=0}^{n} z_i\xi_i$ defines an analytic function on $\mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. Using De Rham's differential, we get that:

$$dP = \sum_{i=0}^{n} \xi_i dz_i + \sum_{i=0}^{n} z_i d\xi_i = \alpha + \check{\alpha}$$

Note that $I \setminus \{0\} \hookrightarrow \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$ is a smooth subvariety, and the restriction of the function P is identically zero on I. Finally, since the pullback commutes with the differential we have that

$$i^*\alpha + i^*\check{\alpha} = i^*(\alpha + \check{\alpha}) = i^*(dP) = d(i^*P) = d(0) = 0$$

where i denotes the inclusion morphism of I. □

This lemma implies that if $L \subset I$ is an analytic variety and at some smooth point $p \in L$ the $1-$form α vanishes then $\check{\alpha}$ vanishes as well. This happens in particular for the conormal space $T^*_X\mathbb{C}^{n+1}$ whenever $X \subset \mathbb{C}^{n+1}$ is the cone over a projective subvariety of $\mathbb{P}^n$. This means, that if we have a a closed bi-conical irreducible analytic subvariety of $T^*\mathbb{C}^{n+1}$ contained in I, then by proposition 1.22 it is the conormal space of its image in $\mathbb{C}^{n+1}$ if and only if it is the conormal space of its image in $\check{\mathbb{C}}^{n+1}$. Note that in this case the conormal space $C(X) \subset X \times \check{\mathbb{P}}^n \subset \mathbb{C}^{n+1} \times \check{\mathbb{P}}^n$ is conical with respect to the $\mathbb{C}^{n+1}$ coordinates, so we can projectivize it to obtain $\mathbb{P}C(X) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$.

Definition 1.30. *Let us consider $\mathbb{P}C(X)$ as a projective subvariety contained in $I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$. The **dual variety** $\mathbb{P}\check{X}$ is the image of $\mathbb{P}C(X) \subset I$ in $\check{\mathbb{P}}^n$. So by construction $\mathbb{P}\check{X}$ is the closure in $\check{\mathbb{P}}^n$ of the set of hyperplanes tangent to $\mathbb{P}X^\circ$.*

Now, by symmetry, we immediately get that $\mathbb{P}\check{\check{X}} = \mathbb{P}X$. But more importantly, we see that duality is finding a lagrangian subvariety in I, its images in $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$ are necessarily dual.

So by considering $\mathbb{P}C(X)$ as a subvariety of $I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ we have the restriction of the two canonical projections:

$$\begin{tikzcd} & \mathbb{P}C(X) \subset I \arrow[dl, "p"'] \arrow[dr, "\check{p}"] & \\ \mathbb{P}X \subset \mathbb{P}^n & & \check{\mathbb{P}}^n \supset \mathbb{P}\check{X} \end{tikzcd}$$

But if we consider the projectivized conormal space $C(X) \subset \mathbb{C}^{n+1} \times \check{\mathbb{P}}^n$ of the cone $X \subset \mathbb{C}^{n+1}$, then we have that $C(X)$ has an image in $\check{\mathbb{P}}^n$ which is the projective dual of $\mathbb{P}X$.

$$\begin{tikzcd}[column sep=small] & \mathbb{P}C(X) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n \arrow[dl] \arrow[dr] & & C(X) \subset \mathbb{C}^{n+1} \times \check{\mathbb{P}}^n \arrow[dl] \arrow[dr, "\kappa_X"] & \\ \mathbb{P}X \subset \mathbb{P}^n & & \check{\mathbb{P}}^n \supset \mathbb{P}\check{X} & & X \subset \mathbb{C}^{n+1} \end{tikzcd}$$

The fiber over 0 of $\kappa_X : C(X) \to X$ as subvariety of $\check{\mathbb{P}}^n$, is equal to $\mathbb{P}\check{X}$: it is the set of limit positions at 0 of hyperplanes tangent to X°. There is one last thing that we would like to say about the incidence variety I.

Lemma 1.31. *The projectivized cotangent bundle of $\mathbb{P}^n$ is naturally isomorphic to I.*

Proof.
Let us first take a look at the cotangent bundle of $\mathbb{P}^n$:

$$\pi : T^*\mathbb{P}^n \longrightarrow \mathbb{P}^n$$

Remember that the fiber $\pi^{-1}(x)$ over a point x in $\mathbb{P}^n$ is by definition isomorphic to $\check{\mathbb{C}}^n$, that is, the vector space of linear forms over $\mathbb{C}^n$. Recall that projectivizing the cotangent bundle means projectivizing the fibers, and so we get a map:

$$\Pi : \mathbb{P}T^*\mathbb{P}^n \longrightarrow \mathbb{P}^n$$

where the fiber is isomorphic to $\check{\mathbb{P}}^{n-1}$. So we can see a point of $\mathbb{P}T^*\mathbb{P}^n$ as a pair $([x], [\chi])$ where $[x] \in \mathbb{P}^n$, and $[\chi] \in \check{\mathbb{P}}^{n-1}$.

On the other hand, if we fix a point $[x] \in \mathbb{P}^n$, then the equation defining the incidence variety I, tells us that for a fixed $[x]$, the set of points $([x], [\xi]) \in I$ is the set of hyperplanes of $\mathbb{P}^n$ that go through the point $[x]$, which we know is isomorphic to $\check{\mathbb{P}}^{n-1}$.

Now to explicitly define the map, take a chart $\mathbb{C}^n \times \left\{\check{\mathbb{C}}^n \setminus \{0\}\right\}$ of the manifold $T^*\mathbb{P}^n \setminus \{\text{zero section}\}$, where the $\mathbb{C}^n$ corresponds to a usual chart of $\mathbb{P}^n$ and $\check{\mathbb{C}}^n$ to it's associated cotangent chart. Define the map:

$$\begin{aligned} \phi_i : \mathbb{C}^n \times \left\{\check{\mathbb{C}}^n \setminus \{0\}\right\} &\longrightarrow \mathbb{P}^n \times \check{\mathbb{P}}^n \\ (z_1, \dots, z_n; \xi_1, \dots, \xi_n) &\longmapsto \left(\varphi_i(z), [\xi_1 : \cdots : \xi_{i-1} : -\sum_{j=1}^n z_j\xi_j : \xi_i : \cdots : \xi_n]\right) \end{aligned}$$

where $\varphi_i(z) = [z_1 : \cdots : z_{i-1} : 1 : z_i : \cdots : z_n]$.

An easy calculation shows that ϕ_i is injective, has its image in I and is well defined on the projectivization $\mathbb{C}^n \times \mathbb{P}^{n-1}$. It is also clear, that varying i from 1 to n we can reach any point in I. Thus, all we need to check now is that the ϕ_j's paste together correctly. For this, the important thing is to remember that if φ_i

and φ_j are charts of a manifold, and $h := \varphi_j^{-1}\varphi_i = (h_1, \ldots, h_n)$ then the change of coordinates in the associated cotangent charts $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ is given by:

$$\begin{tikzcd}
 & T^*M \arrow[dr, "\tilde{\varphi}_j{}^{-1}"] & \\
\mathbb{C}^n \times \check{\mathbb{C}}^n \arrow[ur, "\tilde{\varphi}_i"] \arrow[rr, "h"'] & & \mathbb{C}^n \times \check{\mathbb{C}}^n
\end{tikzcd}$$

$$(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n) \longrightarrow (h(x); (Dh^{-1}|_x)^T(\xi))$$

□

1.4 The Lagrangian Specialisation Principle

For this section our presentation will follow and adapt when necessary, the presentation of [LM87, Chapter II, pgs. 53-88]. Let $g : M \to S$ be a submersion between smooth complex analytic varieties, with M of dimension $n+1$. This means that each fiber $M(s) := g^{-1}(s)$ is a smooth subvariety of M, and as such we have the inclusion of tangent bundles $TM(s) \subset TM$. Moreover, the tangent space $T_mM(s)$ to the fiber $M(s)$ is the vector subspace of T_mM determined by the kernel of the tangent map $Dg(m) : T_mM \to T_{g(m)}S$ which is surjective for all $m \in M$.

Definition 1.32. *The subvector bundle $T(M/S)$ of the tangent bundle TM, defined by the kernel of the tangent map Dg, that is:*

$$T(M/S) = \{(m, v) \in TM \mid m \in M \text{ and } v \in TM(g(m))\}$$

is called the ***relative tangent bundle of* M *with respect to* g**.

$$\begin{tikzcd}
T(M/S) \arrow[r, hook] \arrow[dr] & TM \arrow[d] \\
 & M \arrow[d, "g"'] \\
 & S
\end{tikzcd}$$

By taking the duals of these vector bundles we obtain the ***relative cotangent bundle of* M *with respect to* g**, *together with a surjective map as shown in the following diagram:*

$$\begin{tikzcd}
T^*M \arrow[r, two heads] \arrow[d] & T^*(M/S) \arrow[dl, "\pi_g"] \\
M \arrow[d, "g"'] & \\
S &
\end{tikzcd}$$

where now, the fiber in $T^*(M/S)$ over a point $s \in S$ can be identified with the cotangent bundle $T^*M(s)$.

A section $M \to T^*(M/S)$ of the relative cotangent bundle is what we call a **relative 1-form on M with respect to g**, they are called vertical forms in [LM87]. Note that the restriction of α to a fiber $M(s)$ gives us a standard 1-form on the smooth variety $M(s)$.

Given two submersions $g_1 : M_1 \to S$ and $g_2 : M_2 \to S$, an holomorphic map $h : M_1 \to M_2$ such that $g_1 = g_2 \circ h$ will be called a morphism from M_1 to M_2 over S. In this setup, the tangent maps are all surjective, giving us the following commutative diagram:

$$\begin{tikzcd} TM_1 \arrow[rr, "Dh"] \arrow[dr, "Dg_1"'] & & TM_2 \arrow[dl, "Dg_2"] \\ & S & \end{tikzcd}$$

Remark 1.33. *The relative tangent bundle $T(M_1/S)$ is a subbundle of TM_1 and so we can define the* ***relative (or vertical) tangent map*** $\mathbf{D_v h}$ *as the restriction of Dh to this subbundle. If we additionally suppose that h is a submersion, then the previous diagram implies that the map $D_v h : T(M_1/S) \to T(M_2/S)$ is surjective onto the relative tangent bundle $T(M_2/S)$.*

Now, the map $q := (\pi_g \circ g) : T^*(M/S) \to S$ is again a submersion, so we can consider the relative cotangent bundle of $T^*(M/S)$ with respecto to q. Our objective now is to define the **relative Liouville 1-form on $\mathbf{T^*(M/S)}$** in order to give a "symplectic structure" to each fiber $q^{-1}(s) = T^*M(s)$ of $T^*(M/S)$ and so be able to generalize the concept of Lagrangian variety to the relative case.

Definition 1.34. *The* ***relative Liouville 1-form on*** *$T^*(M/S)$ can be defined pointwise by:*

$$\begin{aligned} \alpha_g : T^*(M/S) &\to T^*\left(T^*(M/S)/S\right) \\ (w,\xi) &\mapsto \alpha_g(w,\xi) \\ \alpha_g(w,\xi)(u) &= \xi\left((D_v\pi_g(w,\xi))(u)\right), \quad \text{for } (w,\xi,u) \in T(T^*(M/S)/S) \end{aligned}$$

Working locally, we can choose an appropiate system of coordinates $(x_0, \dots, x_{n-s}, y_1, \dots, y_s)$ such that the map $g : M \to S$ is the projection $g : \mathbb{C}^{n+1} \to \mathbb{C}^s$. In this case, the space $T^*(M/S)$ is just the space

$$\begin{tikzcd} \mathbb{C}^{n+1-s} \times \mathbb{C}^s \times \check{\mathbb{C}}^{n+1-s} \arrow[d, "\pi_g"'] \\ \mathbb{C}^{n+1} \end{tikzcd}$$

and a simple calculation then shows that locally:

$$\alpha_g = \sum_{i=0}^{n-s} \xi_i dx_i$$

where $(x_0, \dots, x_{n-s}, y_1, \dots, y_s, \xi_0, \dots, \xi_{n-s})$ is the associated coordinate system of $T^*(M/S)$.

From the definition of the relative cotangent bundle, we have a canonical map from the absolute cotangent bundle $T^*(T^*(M/S))$ to the relative cotangent bundle $T^*(T^*(M/S)/S)$ of $T^*(M/S)$. In local coordinates this map has the expression $(\underline{x}, \underline{y}, \underline{\xi}, \underline{\psi}) \mapsto (\underline{x}, \underline{y}, \underline{\xi})$, which shows, that α_g can then be equivalently defined as the image of the absolute Liouville 1-form α defined in section 1.3.1. It is actually this map, that allows us to define a **vertical differential** and consequently an exterior algebra of vertical differential forms as proven in [LM87, Chap. II, Sect. 6.9, pg 70]. For our purposes, it is sufficient to know that this vertical differential exists, giving us the relative (vertical) differential 2-form:

$$\omega_g = -d\alpha_g = \sum_{i=0}^{n-s} dx_i \wedge d\xi_i$$

which when restricted to the fiber $q^{-1}(s) = T^*\left(M(s)\right)$ gives us its canonical symplectic 2-form.

Let now $W \subset M$ be a possibly singular, pure dimensional complex analytic subvariety of M, such that the restriction $g|_W : W \to S$ has all its fibers $W(s)$ of pure dimension d, and such that there exists an analytic open dense set $W_\circ \subset W$ over which the restriction of g has smooth fibers.

Definition 1.35. *For $w \in W_\circ(s)$, let $T^*_{W_\circ(s)}M(s)$ denote the conormal bundle of $W_\circ(s)$ in $T^*M(s)$ as in definition 1.20. The* ***relative conormal bundle*** *of $W_\circ$ with respect to g is defined as*

$$T^*_{W_\circ}(M/S) = \{(w, \xi) \in T^*(M/S) \mid w \in W_\circ,\ \xi \in T^*_{W_\circ(s)}M(s)\}$$

This means that if the point $(w, \xi) \in T^*_{W_\circ}(M/S)$, then the hyperplane $\{\xi = 0\}$ contains the tangent space to the fiber $W_\circ(s)$ at the point w.

Definition 1.36. *A reduced analytic subspace $Y \subset T^*(M/S)$ is* ***g-Lagrangian*** *if for all $s \in S$ the reduced fiber $|Y(s)|$ of the morphism $q := g \circ \pi_g|_Y$ is a lagrangian subvariety of $T^*M(s)$. In other words, the fiber $|Y(s)|$ is of pure dimension equal to the dimension of $M(s)$ and the relative 2-form ω_g vanishes on every couple of tangent vectors at any non-singular point of the reduced fiber $|Y(s)|$.*

Proposition 1.37. *Let $i : T^*_{W_\circ}(M/S) \hookrightarrow T^*(M/S)$ be the inclusion, and let α_g be the relative Liouville $1-$form. Then $i^*\alpha_g = 0$. In particular the relative conormal bundle $T^*_{W_\circ}(M/S)$ is $g-$Lagrangian.*

Proof. By construction, the relative conormal bundle $T^*_{W_\circ}(M/S)$ is the union of the conormal bundles $T^*_{W_\circ(s)}M(s)$ of the fibers $W_\circ(s)$ in $M(s)$, and since the restriction of α_g to the cotangent space $T^*M(s)$ gives rise to its Liouville $1-$form α, the result follows from proposition 1.21. □

Definition 1.38. *For $W \subset M$ a possibly singular, reduced, pure dimensional complex analytic subvariety of M, such that the restriction $g|_W : W \to S$ has all its fibers $W(s)$ of pure dimension d, and such that there exists an analytic open dense* *set $W_\circ \subset W$ over which the restriction of g has smooth fibers, we define the* ***relative conormal space of*** W ***in*** M,

$$\kappa_{g,W} : T^*_W(M/S) \to W \xrightarrow{g} S$$

as the closure of $T^*_{W_\circ}(M/S)$ in the relative cotangent bundle $T^*(M/S)$.

Note that now the fiber of the relative conormal map $\kappa_{g,W} \colon T^*_W(M/S) \to W$ above a point $w \in W$ consists, if $w \in W_\circ$, of all the equations of hyperplanes tangent to the fiber $W(g(w))$ at w, in the sense that they contain the tangent space to the fiber $T_w W_\circ(g(w))$. If w is a singular point of the fiber $W(g(w))$, then the fiber $\kappa^{-1}_{g,W}(w)$ consists of all equations of limits of hyperplane directions tangent to the fibers $W(s)$ at points of $W_\circ$ tending to w.
Recall that in the setting we are working on we have the relative conormal space as a subset of $\mathbb{C}^{n+1-s} \times \mathbb{C}^s \times \check{\mathbb{C}}^{n+1-s}$, and just like in the case of the conormal space, the fibers of $\kappa_{g,W}$ are invariant under multiplication by an element of $\mathbb{C}^*$, so we can "projectivize" this space to define the **(projectivized) relative conormal space:**

$$\kappa_{g,W} : C_g(W) \to W \xrightarrow{g} S$$

as a subset of the projectivized relative cotangent bundle $\mathbb{P}T^*(M/S) = \mathbb{C}^{n+1-s} \times \mathbb{C}^s \times \check{\mathbb{P}}^{n-s}$.

Remark 1.39. *1. The relative conormal space $C_g(W)$ is of dimension $n + dim\ S = dim\ W + n - d$.*

2. The fiber $C_g(W)(s) = (g \circ \kappa_{g,W})^{-1}(s)$ contains the conormal space of $W(s)$ in $M(s)$, but the inclusion may be strict.

3. The relative conormal space $C_g(W)$, or strictly speaking $T^*_W(M/S)$, is not always g-Lagrangian and the reason is that the fibers $C_g(W)(s)$ may be too large.

For example, let $(X, 0)$ be a germ of reduced, isolated hypersurface singularity defined by the function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Next, consider the graph W of the function f in $\mathbb{C}^{n+1} \times \mathbb{C}$, and let $g : W \to \mathbb{C}$ be the restriction of the second projection to W. Then, the relative conormal space $C_g(W)$ is contained in $\mathbb{C}^{n+1} \times \mathbb{C} \times \check{\mathbb{P}}^n$, and the fiber $\kappa^{-1}_{g,W}(0)$ is the entire $\check{\mathbb{P}}^n$. The reason is that the space $C_g(W)$ can be identified with the blowing up of $\mathbb{C}^{n+1}$ along the jacobian ideal of f, whose zero set is only the origin by the isolated singularity hypothesis. Another proof of this fact can be found in [O'S89, Ex. 3, pg 231] and you can also find another example in [O'S89, Prop.6, pg 237].

Theorem 1.40 (The Lagrangian Specialisation Principle)**.** *Let $g : M \to S$ be a submersion between smooth complex analytic varieties, let Y be a closed analytic subspace of the relative cotangent bundle $\pi_g : T^*(M/S) \to M$, and let $q = (g \circ \pi_g)_{|Y}$ Then the following statements are equivalent:*

1. *The subspace $Y \subset T^*(M/S)$ is $g-$Lagrangian.*
2. *All the reduced fibers $|Y(s)| := |q^{-1}(s)|$ are of pure dimension equal to the dimension of $M(s)$, and there exists an open dense analytic subset $V \subset S$ such that $q^{-1}(V)$ is dense in Y, and for all $s \in V$, the fiber $|Y(s)|$ is a lagrangian subvariety of $T^*M(s) = \pi_g^{-1}(M(s))$.*

Proof. The implication 1) $\Rightarrow$ 2) is immediate by definition of $g-$Lagrangian.
2) $\Rightarrow$ **1)**
We must prove that the relative 2-form ω_g vanishes on every couple of tangent vectors at any non-singular point of the reduced fiber $|Y(s)|$ for all $s \in S \setminus V$. Since we are considering the reduced structure in Y and all the fibers $|Y(s)|$ are equidimensional then the relative smooth locus $W := \bigcup_{s\in S} Y(s)^0$ of the morphism $q : Y \to S$ is an analytic open dense subset of Y. Then the open set $U = W \cap q^{-1}(V)$ is dense in Y and both morphisms $q : W \to q(W)$, $q : U \to q(U)$ are submersions. This allows us to consider the relative tangent bundles $T(U/S)$ and $T(W/S)$ and identify them with subvarieties of $T\left(T^*(M/S)/S\right)$. Moreover, we have the inclusion $T(U/S) \subset T(W/S)$ as an open dense subset.

By definition the relative 2-form ω_g is a section of the bundle

$$T^*\left((T^*(M/S)/S\right) \bigwedge T^*\left((T^*(M/S)/S\right) \longrightarrow T^*(M/S)$$

therefore defining, for each point $p \in T^*(M/S)$, a bilinear form on the fiber over p of the relative tangent bundle $T\left(T^*(M/S)/S\right) \to T^*(M/S)$. As such, ω_g can be considered as an holomorphic function on the vector bundle

$$T\left(T^*(M/S)/S\right) \times_{T^*(M/S)} T\left(T^*(M/S)/S\right) \to T^*(M/S)$$

by hypothesis the restriction of ω_g to the subvariety $T(U/S) \times_U T(U/S)$ is identically zero, and by continuity this is also true for $T(W/S) \times_W T(W/S)$.

Let us take an $s \notin V$, if the fiber $Y(s)$ is reduced, then its smooth part $Y(s)^0$ is contained in the relative smooth locus W and we are finished. So we only have to consider the case when the fiber $Y(s)$ is not reduced, note that it is enough to prove that the relative 2-form ω_g vanishes on every couple of tangent vectors at any point of an open dense subset of the smooth locus of the reduced fiber $|Y(s)|^0$. What we are actually going to prove is that in an open dense subset of $|Y(s)|^0$ any couple of tangent vectors can be attained as a limit of couples of tangent vectors to the fibers in W, thus ending the proof by continuity of ω_g.

We will first prove it in the case where S is a smooth curve. By 2.26, the couple $(W \setminus |Y(s)|, |Y(s)|^0)$ satisfies Thom's condition a_q in an open dense analytic subset U_s of $|Y(s)|^0$. Let $y \in U_s$, then for any local embedding $(Y, y) \subset (\mathbb{C}^m, 0)$ and any sequence of points $\{x_j\} \subset W \setminus |Y(s)|$ tending to y we have

$$\lim_{n\to\infty} \delta\left(T_yY(s), T_{x_j}Y(f(x_j))\right) = 0$$

where δ is the distance between vector subspaces of $\mathbb{C}^m$ defined by

$$\delta(E, F) = \sup_{u\in E\setminus\{0\},\, v\in F^\perp\setminus\{0\}} \frac{|\langle u, v\rangle|}{\|u\|\|v\|}$$

$\langle u, v\rangle$ denotes the hermitian product in $\mathbb{C}^m$, and $F^\perp := \{v \in \mathbb{C}^m \mid \langle v, x\rangle = 0\ \forall x \in F\}$.

Let us fix a tangent vector $\overrightarrow{w} \in T_yY(s)$, and for each j let $\pi_j : \mathbb{C}^m \to T_{x_j}Y(f(x_j))$ be the projection with kernel $T_{x_j}Y(f(x_j))^\perp$. Then, for each j we can express $\mathbb{C}^m$ as a direct sum $T_{x_j}Y(f(x_j)) \bigoplus T_{x_j}Y(f(x_j))^\perp$, and so there is a unique expression $\overrightarrow{w} = \overrightarrow{w_j} + \overrightarrow{v_j}$, with $\overrightarrow{w_j} = \pi_j(\overrightarrow{w})$ and $\overrightarrow{v_j} = \overrightarrow{w} - \overrightarrow{w_j} \in T_{x_j}Y(f(x_j))^\perp$. Since $\langle \overrightarrow{w_j}, \overrightarrow{v_j} \rangle = 0$ we have that these two vectors are orthogonal as vectors of $\mathbb{R}^{2m}$, and so

$$\angle \overrightarrow{w}\,\overrightarrow{w_j} + \angle \overrightarrow{w}\,\overrightarrow{v_j} = \pi/2$$

where $\angle$ denotes the angle between the vectors. Moreover, since the real part of the hermitian product in $\mathbb{C}^m$ is equal to the dot product of the vectors in $\mathbb{R}^{2m}$ we have that for every j

$$\begin{aligned}
\delta\left(T_yY(s), T_{x_j}Y(f(x_j))\right) &\geq \sup_{v \in T_{x_j}Y(f(x_j))^\perp \setminus \{0\}} \frac{|\langle w, v\rangle|}{\|w\|\|v\|} \\
&\geq \frac{|\langle w, v_j\rangle|}{\|w\|\|v_j\|} \geq \frac{|\mathrm{Re}\,\langle w, v_j\rangle|}{\|w\|\|v_j\|} \\
&= \cos(\angle \overrightarrow{w}\,\overrightarrow{v_j})
\end{aligned}$$

This implies that the sequence $\{\cos(\angle \overrightarrow{w}\,\overrightarrow{v_j}\}$ tends to 0, so the angle $\angle \overrightarrow{w}\,\overrightarrow{v_j}$ tends to $\pi/2$ and the angle $\angle \overrightarrow{w}\,\overrightarrow{w_j}$ tends to 0. That is, the sequence of vectors $\{\overrightarrow{w_j}\}$ tends to the vector $\overrightarrow{w}$ as we wanted. This technique allows us to construct a sequence of points $(x_j, \overrightarrow{a_j}, \overrightarrow{b_j})$ in $T(W/S) \times_W T(W/S)$ tending to any given point $(y, \overrightarrow{a}, \overrightarrow{b})$ in $T\left(T^*(M/S)/S\right) \times_{T^*(M/S)} T\left(T^*(M/S)/S\right)$ where $\overrightarrow{a}, \overrightarrow{b}$ are tangent vectors to the fiber $|Y(s)|$ at the point y, and thus by continuity $\omega_g(y)(\overrightarrow{a}, \overrightarrow{b}) = 0$.

In the general case, since by hypothesis S is smooth, there locally exists an holomorphic embedding $i : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (S, S \setminus V, s)$. By considering the fibered product

$$\begin{tikzcd}
\mathbb{C} \times_S Y \arrow[r] \arrow[d, "\widetilde{q}"'] & Y \arrow[d, "q"] \\
\mathbb{C} \arrow[r, hook, "i"] & S
\end{tikzcd}$$

we obtain the morphism $\widetilde{q} : \mathbb{C} \times_S Y \to \mathbb{C}$, which has the same fibers as q and reduces our problem to the special case we just proved. $\square$

1.5 The Conormal Spaces $C(\mathfrak{X})$ and $C_\varphi(\mathfrak{X})$

We are now in the position to develop interesting properties of these two spaces that will be useful in the sequel.

Proposition 1.41. *Let $Y \subset X$ be a smooth analytic subvariety of dimension $0 \leq s < d$, let $\varphi : \mathfrak{X} \to \mathbb{C}$ denote the specialisation space of $X \subset \mathbb{C}^{n+1}$ to its normal cone along Y, and let $\phi : \mathfrak{X} \to X \times \mathbb{C}$ denote the canonical map obtained from the construction in lemma 1.5, then there exist isomorphisms $\psi : C(\mathfrak{X} \setminus \mathfrak{X}(0)) \to C(X) \times \mathbb{C}^*$; $P : C(\mathfrak{X} \setminus \mathfrak{X}(0)) \to C_\varphi(\mathfrak{X} \setminus \mathfrak{X}(0))$; and $\psi_\varphi : C_\varphi(\mathfrak{X} \setminus \mathfrak{X}(0)) \to C(X) \times \mathbb{C}^*$ making the following diagram commutative:*

$$\begin{tikzcd}
C(\mathfrak{X}\setminus\mathfrak{X}(0)) \arrow[r, "\psi"] \arrow[d, "P"] & C(X)\times\mathbb{C}^* \arrow[r, "\widetilde{pr_1}"] \arrow[d, "Id"] & C(X) \arrow[d, "Id"] \\
C_\varphi(\mathfrak{X}\setminus\mathfrak{X}(0)) \arrow[r, "\psi_\varphi"] \arrow[d, "\kappa_\varphi"] & C(X)\times\mathbb{C}^* \arrow[r, "\widetilde{pr_1}"] \arrow[d, "\kappa_X\times Id"] & C(X) \arrow[d, "\kappa_X"] \\
\mathfrak{X}\setminus\mathfrak{X}(0) \arrow[r, "\phi"'] \arrow[dr, "\varphi"'] & X\times\mathbb{C}^* \arrow[r, "pr_1"] \arrow[d] & X \\
 & \mathbb{C}^* &
\end{tikzcd}$$

Proof. We are working with a small enough representative of the germ $(X,0)\subset(\mathbb{C}^{n+1},0)$ embedded in such a way that $Y\subset X$ is linear, this implies that we will have:

1. $C(X)\subset\mathbb{C}^{n+1}\times\check{\mathbb{P}}^n$
2. $\mathfrak{X}\subset\mathbb{C}^{n+1}\times\mathbb{C}$.
3. $C(\mathfrak{X})\subset\mathbb{C}^{n+1}\times\mathbb{C}\times\check{\mathbb{P}}^{n+1}$
4. $C_\varphi(\mathfrak{X})\subset\mathbb{C}^{n+1}\times\mathbb{C}\times\check{\mathbb{P}}^n$

We will actually work with the non-projectivized versions of the conormal spaces, that is with the spaces $T^*_X(\mathbb{C}^{n+1})$, $T^*_{\mathfrak{X}}(\mathbb{C}^{n+1}\times\mathbb{C})$ and $T^*_{\mathfrak{X}}((\mathbb{C}^{n+1}\times\mathbb{C})/\mathbb{C})$ respectively. Moreover, we will fix a coordinate system $(z_0,\ldots,z_{n-s},y_1,\ldots,y_s,t,a_0,\ldots,a_{n-s},c_1,\ldots,c_s,b)$ of $\mathbb{C}^{n+1}\times\mathbb{C}\times\check{\mathbb{C}}^{n+1}\times\check{\mathbb{C}}$. By construction, the map $\phi:\mathfrak{X}\to X\times\mathbb{C}$ is an isomorphism when restricted to $\mathfrak{X}\setminus\mathfrak{X}(0)$ and has $X\times\mathbb{C}^*$ as its image. Actually, this alone implies that both the conormal space $C(\mathfrak{X}\setminus\mathfrak{X}(0))$ and the relative conormal space $C_\varphi(\mathfrak{X}\setminus\mathfrak{X}(0))$ are isomorphic to $C(X)\times\mathbb{C}^*$. However to verify that we have the commutative diagram we will specify these isomorphisms. Recall that the series

$$F_i=t^{-m_i}f_i(tz_0,\ldots,tz_{n-s},y_1,\ldots,y_s),\ i=1,\ldots,p$$

define the specialisation space $\mathfrak{X}$ in $\mathbb{C}^{n+1}\times\mathbb{C}$, where $m_i=\nu_Y f_i$.

Let $x=(z,y,t)$, $t\neq 0$, be a smooth point of $\mathfrak{X}$, then it is a smooth point of $\mathfrak{X}(t)$, and $\phi(x)=(tz,y,t)$ is a smooth point of $X\times\mathbb{C}^*$; consequently (tz,y) is a smooth point of X. Now, for any point (x,a,c,b) in $\kappa_{\mathfrak{X}}^{-1}(x)$ there exist constants $\lambda_1,\ldots,\lambda_p$ such that:

$$a_j=\sum_{i=1}^p\lambda_i\frac{\partial F_i}{\partial z_j}(x)=\sum_{i=1}^p\lambda_i t^{-m_i+1}\frac{\partial f_i}{\partial z_j}(tz,y) \tag{1.2}$$

$$c_j=\sum_{i=1}^p\lambda_i\frac{\partial F_i}{\partial y_j}(x)=\sum_{i=1}^p\lambda_i t^{-m_i}\frac{\partial f_i}{\partial y_j}(tz,y) \tag{1.3}$$

$$b=\sum_{i=1}^p\lambda_i\frac{\partial F_i}{\partial t}(x)=\sum_{i=1}^p\lambda_i\left((-m_i)t^{-m_i+1}f_i(tz,y)+t^{-m_i}(\sum_{k=0}^{n-s}z_k\frac{\partial f_i}{\partial z_k}(tz,y))\right) \tag{1.4}$$

$$=\sum_{i=1}^p\lambda_i\left(t^{-m_i}(\sum_{k=0}^{n-s}z_k\frac{\partial f_i}{\partial z_k}(tz,y))\right),\ \text{ because } f_i(tz,y)=0 \text{ on } X\times\mathbb{C}. \tag{1.5}$$

Analogously, for any point (x, a, c) in $\kappa_\varphi^{-1}(x)$, there exist constants $\lambda_1, \ldots, \lambda_p$ such that, the coordinates a_j and c_j are given by the corresponding equations 1.2 and 1.3. This implies that the natural projection $P : (z, y, t, a, c, b) \mapsto (z, y, t, a, c)$ induces a surjective morphism to $C_\varphi(\mathfrak{X} \setminus \mathfrak{X}(0))$ when restricted to $C(\mathfrak{X} \setminus \mathfrak{X}(0))$. But, from 1.5 we can see that $tb = \sum_{k=0}^{n-s} z_k a_k$, so as long as $t \neq 0$ the b coordinate is completely determined by the a and z coordinates which proves that the aforementioned map P is an isomorphism.

On the other hand, for the corresponding point $x' = (tz, y)$ of X, we have that for any point (x', a, c) in $\kappa_X^{-1}(x')$ there exists constants $\alpha_1, \ldots, \alpha_p$ such that:

$$
\begin{aligned}
a_j &= \sum_{i=1}^{p} \alpha_i \frac{\partial f_i}{\partial z_j}(tz, y) \\
c_j &= \sum_{i=1}^{p} \alpha_i \frac{\partial f_i}{\partial y_j}(tz, y)
\end{aligned}
$$

This implies that if $t \neq 0$, the automorphism of the ambient space $\Upsilon : \mathbb{C}^{n+1} \times \mathbb{C} \times \check{\mathbb{C}}^{n+1} \circlearrowleft$ defined by:

$$(z, y, t, a, c) \mapsto (tz_0, \ldots, tz_{n-s}, y_1, \ldots, y_s, t, a_0, \ldots, a_{n-s}, tc_1, \ldots, tc_s)$$

induces a surjective map $\psi_\varphi : C_\varphi(\mathfrak{X} \setminus \mathfrak{X}(0)) \to C(X) \times \mathbb{C}^*$ simply by setting $\lambda_i = t^{m_i - 1}\alpha_i$. Moreover, since the map Υ is biholomorphic in the open dense set $t \neq 0$, the map ψ_φ is an isomorphism. $\square$

Remark 1.42. *In regard to the previous diagrams, note that:*

1. *The map ϕ is defined on all of $\mathfrak{X}$, and the image of the special fiber $\mathfrak{X}(0)$ is just the origin in $X \times \mathbb{C}$. Note as well, that for a fixed $t \neq 0$, the morphism $pr_1 \circ \phi| : \mathfrak{X}(t) \to X$ is an isomorphism.*
2. *The obstruction to the extension of ψ to $C(\mathfrak{X})$ comes from the map $\check{\mathbb{P}}^{n+1} \to \check{\mathbb{P}}^n$, which is undefined at the point $[0 : \cdots : 0 : 1]$. This means that for any point $((\underline{z}, t), [\underline{a} : b])$ in $C(\mathfrak{X}) \cap \left(\mathfrak{X} \times \{\check{\mathbb{P}}^{n+1} \setminus [0 : 1]\}\right)$, the hyperplane $[\underline{a}] \in \check{\mathbb{P}}^n$ is tangent to X at the point $t\underline{z} = (tz_0, \ldots, tz_n)$. In particular, for $t = 0$ the hyperplane $[\underline{a}]$ is tangent to X at the origin.*

Proposition 1.43. *([Sab85, Lemma A.4.1, pg 190]) Let $Y \subset X$ be a smooth analytic subvariety of dimension $0 \leq s < d$, let $\varphi : \mathfrak{X} \to \mathbb{C}$ denote the specialisation space of X to its normal cone $C_{X,Y}$ along Y, and let $T^*_X(\mathbb{C}^{n+1}), T^*_Y(\mathbb{C}^{n+1})$ denote the conormal spaces of X and Y respectively, in $\mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. Then the map q from the relative conormal space to $\mathbb{C}$*

$$q : T^*_{\mathfrak{X}}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C}) \to \mathfrak{X} \to \mathbb{C}$$

is isomorphic to the specialisation space of $T^*_X(\mathbb{C}^{n+1})$ to its normal cone $C_{T^*_X(\mathbb{C}^{n+1}), T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})}$ along $T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})$. In particular, the fibre $q^{-1}(0)$ is isomorphic to the normal cone $C_{T^*_X(\mathbb{C}^{n+1}), T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})}$.

Proof.
Let $I \subset J$ be the coherent ideals of the structure sheaf of $\mathbb{C}^{n+1}$ that define the analytic subspaces X and Y respectively, and let $p : \mathfrak{D} \to \mathbb{C}$ be the specialization space of $T^*_X(\mathbb{C}^{n+1})$ to its normal cone along $T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})$. Note that, in this context, both spaces $\mathfrak{D}$ and $T^*_{\mathfrak{X}}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C})$ are analytic subspaces of $\mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. Let us consider a local chart, in such a way that $Y \subset X \subset \mathbb{C}^{n+1}$, locally becomes $\mathbb{C}^s \subset X \subset \mathbb{C}^{n+1}$ with local associated coordinates:

$$(t, z_0, \dots, z_{n-s}, y_1 \dots, y_s, a_0, \dots, a_{n-s}, c_1, \dots, c_s)$$

in $\mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$.

Note that, since locally $J = \langle z_0, \dots, z_{n-s} \rangle$ in $\mathbb{C}^{n+1}$, then the conormal space $T^*_Y(\mathbb{C}^{n+1})$ is given by the equations $(z_0, \dots, z_{n-s}, c_1, \dots, c_s)$ in $\mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. Thus, if we chose local equations $(g_1, \dots, g_r)$ for $T^*_X(\mathbb{C}^{n+1})$ like before, then the equations $G_i(t, z, y, a, c) = t^{-l_i} g_i(t, tz, y, a, tc)$ locally define the space $\mathfrak{D} \xrightarrow{p} \mathbb{C}$, where the fiber $p^{-1}(0)$ is the normal cone $C_{T^*_X(\mathbb{C}^{n+1}), T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})}$, and by lemma 1.5 the open set $\mathfrak{D} \setminus \mathfrak{D}(0)$ is isomorphic to $T^*_X(\mathbb{C}^{n+1}) \times \mathbb{C}^*$ via the morphism defined by $(t, z, y, a, c) \mapsto (t, tz, y, a, tc)$. Note that in this case the l_i's denote the order of the g_i's with respect to the variables $\{z_0, \dots, z_{n-s}, c_1, \dots, c_s\}$.

Note that this last isomorphism is defined by the restriction of the automorphism of the ambient space from proposition 1.41

$$\begin{aligned} \Upsilon : \mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1} &\longrightarrow \mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1} \\ (t, z, y, a, c) &\longmapsto (t, tz, y, a, tc) \end{aligned}$$

which we know is biholomorphic when restricted to the open dense set $\{t \neq 0\}$. So, if we take the analytic subspace $T^*_X(\mathbb{C}^{n+1}) \times \mathbb{C}^*$ in the image, then as a result of what we just said, we have the equality $\Upsilon^{-1}(T^*_X(\mathbb{C}^{n+1}) \times \mathbb{C}^*) = \mathfrak{D} \setminus \mathfrak{D}(0)$.

Finally, recall that both morphisms defining q are induced by the natural projections

$$\mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1} \to \mathbb{C} \times \mathbb{C}^{n+1} \to \mathbb{C}$$

and therefore, we have a commutative diagram

$$\begin{tikzcd}
T^*_{\mathfrak{X}}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C}) \arrow[r, hook] \arrow[d, "\kappa_\varphi"'] \arrow[ddrr, bend right=60, "q"'] & \mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1} \arrow[r, "\Upsilon"] \arrow[d, "\pi"] & \mathbb{C} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1} \arrow[d, "\pi"] \\
\mathfrak{X} \arrow[r, hook] \arrow[drr, "\varphi"'] & \mathbb{C} \times \mathbb{C}^{n+1} \arrow[r, "\phi"] \arrow[dr] & \mathbb{C} \times \mathbb{C}^{n+1} \arrow[d] \\
 & & \mathbb{C}
\end{tikzcd}$$

Finally, again by proposition 1.41, the restriction of the map Υ to the space $T^*_{\mathfrak{X} \setminus \mathfrak{X}(0)}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C})$ gives the isomorphism ψ_φ which has $T^*_X(\mathbb{C}^{n+1}) \times \mathbb{C}^*$ as image. Since we already know that $\Upsilon^{-1}(T^*_X(\mathbb{C}^{n+1}) \times \mathbb{C}^*) = \mathfrak{D} \setminus \mathfrak{D}(0)$ we have found an open dense set common to both spaces, and consequently the closures will be the same.

□

Corollary 1.44. *The relative conormal space* $\kappa_\varphi : T^*_{\mathfrak{X}}((\mathbb{C}^{n+1} \times \mathbb{C})/\mathbb{C}) \to \mathfrak{X}$ *is always* $\varphi-$*Lagrangian.*

Proof. We will use the notation of the proof of 1.43. From definition 1.35 we need to prove that every fiber $q^{-1}(s)$ is a lagrangian subvariety of $\{s\} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$. But, by proposition 1.41 we know that for $s \neq 0$, the fiber $q^{-1}(s)$ is isomorphic to $T^*_X(\mathbb{C}^{n+1})$ and so it is lagrangian. Thus, by theorem 1.40-2 all we need to prove is that the special fiber $q^{-1}(0)$ has the right dimension, which in this case is equal to $n + 1$.

Proposition 1.43 tells us that the fiber $q^{-1}(0)$ is isomorphic to the normal cone

$$C_{T^*_X(\mathbb{C}^{n+1}),T^*_Y(\mathbb{C}^{n+1})\cap T^*_X(\mathbb{C}^{n+1})}$$

Finally, since the projectivized normal cone $\mathbb{P}C_{T^*_X(\mathbb{C}^{n+1}),T^*_Y(\mathbb{C}^{n+1})\cap T^*_X(\mathbb{C}^{n+1})}$ is obtained as the divisor of the blowup of $T^*_X(\mathbb{C}^{n+1})$ along the $T^*_Y(\mathbb{C}^{n+1}) \cap T^*_X(\mathbb{C}^{n+1})$, then it has dimension n and so the cone over this projective variety has dimension $n + 1$ which finishes the proof. □

Chapter 2

Whitney Conditions and Exceptional Cones

The idea of stratification is to partition a singular space into a locally finite collection of locally closed *non singular* subspaces. The partition $X = \bigcup_{\alpha \in A} X_\alpha$ is useful if it helps to describe the geometry of the singular space and also if it allows us to do analysis on the singular space in spite of the absence of a tangent space at every point. Typically, we wish to define and integrate vector fields on a singular space, and classify the various local geometries that can be carried by a given singular space.

The first general ideas to create useful stratifications are due to Whitney [Whi65b, Sect. 19, pg 540], who defined the first example of "incidence conditions" between strata, and proved that every complex analytic space can be partitioned into non singular strata which satisfy these incidence conditions.

One of the important general facts about singularities in complex analytic geometry discovered by Whitney is that they are *locally conical*; a sufficiently small neighborhood of a singular point 0 of a space $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is "similar" to a cone with vertex 0 over the intersection of X with a small sphere around 0 in $\mathbb{C}^{n+1}$.

Lemma 2.1. ***Whitney's lemma.-*** *Given a representative $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ of a reduced complex analytic germ, purely of dimension d, for any sequence $(x_i)_{i\in\mathbb{N}}$ of non singular points of X, we may assume after extracting subsequences that the directions $[0x_i] \in \mathbb{P}^n$ of the secants and those $[T_{x_i}X] \in Gr(d, n+1)$ converge to limits ℓ, T. Then we have $\ell \subset T$.*

Proof. This lemma originally appeared in [Whi65b, Thm. 22.1, pg 547], but you can also find a proof due to Hironaka in [Lê81, Thm. 1.1.1]. We give a sketch of the proof. First of all consider the function δ defined on the set of non singular points X° by calculating the cosine of the angle between the line $[0x]$ and the tangent space T_xX. If $\langle u, v\rangle$ denotes the hermitian product of $\mathbb{C}^{n+1}$, then for any $x \in X^\circ$ the value δ can be calculated by the formula:

$$\delta(x) = \max_{u \in T_xX,\ v \in [0x]} \frac{|\mathrm{Re}\,\langle u, v\rangle\,|}{\|u\|\|v\|}$$

Note that the maximun is attained because to calculate $\delta(x)$ it suffices to fix the vector $\vec{v}$ and so the domain of the function can be seen as the projective space of directions of lines of T_xX which is compact.

The function δ can be pulled back to the analytic space built as the closure of the graph in $X \times \mathbb{P}^n \times Gr(d, n+1)$ of the map defined on X° by $x \mapsto (x, [0x], T_xX)$. By the curve selection lemma any point $(0, l, T)$ in this space can be attained as the endpoint $\gamma(0)$ of an analytic curve $\gamma(\tau)$ having its image over X°. Finally, the function

$$\delta_1(\gamma(\tau)) = \frac{|\mathrm{Re}\left\langle \gamma(\tau), \frac{d}{d\tau}\gamma(\tau)\right\rangle|}{\|\gamma(\tau)\|\|\frac{d}{d\tau}\gamma(\tau)\|}$$

satisfies $\delta_1(\gamma(\tau)) \leq \delta(\gamma(\tau))$ and it can be calculated that the limit of $\delta_1(\gamma(\tau))$ when τ tends to 0 is equal to 1, which implies that $\delta(0) = 1$ and so $\ell \subset T$. $\square$

This lemma, along with the characterisation of the Whitney conditions in the normal-conormal diagram, was the key to generalize the result describing the fiber $\nu^{-1}(x)$ over a singular point x of a surface S in $\mathbb{C}^3$ as the union of the projective dual variety of the tangent cone $C_{X,x}$ and a finite number of linear pencils having as axis special lines of the tangent cone called the exceptional tangents. It was first done for isolated singular points of surfaces in $\mathbb{C}^3$ [HL75], then for surfaces in $\mathbb{C}^3$ without the hypothesis of isolated singularity in [Lê81, Thm. 2.3.7]. The most general statement of this result was proven in [LT88, Thm 2.1.1 and its corollaries, pp. 559-561].

Definition 2.2. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of reduced, purely d-dimensional analytic singularity as before, and $(Y, 0) \subset (X, 0)$ a germ of nonsingular analytic subvariety of dimension $0 \leq s < d$. Then the normal conormal diagram of the pair $(X, Y, 0)$, is the following commutative diagram:*

$$\begin{tikzcd}
E_YC(X) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'_X"'] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa_X"] \\
E_YX \arrow[r, "e_Y"'] & X
\end{tikzcd}$$

where $e_Y : E_YX \to X$ denotes the blowup of X along Y, $\hat{e}_Y : E_YC(X) \to C(X)$ denotes the blowup of $C(X)$ along $\kappa^{-1}(Y)$, and the morphism $\kappa'_X : E_YC(X) \to E_YX$ comes from the universal property of the blowup of X along Y.

Remark 2.3. *By choosing an embedding $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ with an adequate coordinate system $(z_0, \ldots, z_{n-s}, y_1, \ldots, y_s)$, we have:*

1. *The projectivized conormal space $C(X) \subset X \times \check{\mathbb{P}}^n$, and if $[a_0 : \cdots : a_{n-s} : c_1, \ldots, c_s]$ are the coordinates on $\check{\mathbb{P}}^n$ corresponding to the dual coordinate system of $\check{\mathbb{C}}^{n+1}$ then $C(Y) = Y \times \check{\mathbb{P}}^{n-s}$ where $\check{\mathbb{P}}^{n-s}$ corresponds to the projective dual of $\mathbb{P}Y$, that is the algebraic set defined by $c_1 = \cdots = c_s = 0$.*
2. *The blowup space $E_YX \subset X \times \mathbb{P}^{n-s}$, where the morphism e_Y is induced by the restriction to E_YX of the first projection, just as in the discussion following definition 1.8.*

3. *The analytic space $E_Y C(X)$ can be built in the following way(See [EH00, Prop. IV-21, pg 167]). Consider the fibered product $E_Y X \times_X C(X) \subset X \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^n$, and let $\pi_2 : E_Y X \times_X C(X) \to C(X)$ denote the canonical projection. Then, the blowup $\hat{e}_Y : E_Y C(X) \to C(X)$ is isomorphic to the closure of $\pi_2^{-1}\left(C(X) \setminus \kappa_X^{-1}(Y)\right)$ in $E_Y X \times_X C(X)$, with the role of the morphism $\hat{e}_Y$ being played by the restriction of the projection π_2.*
4. *The construction of $E_Y C(X)$ as a subvariety of the fibered product $E_Y X \times_X C(X)$ implies that the map κ'_X is the restriction of the first projection:*
   $\pi_1 : E_Y X \times_X C(X) \to E_Y X$

In the case where Y is just a point, namely the origin, the normal conormal diagram will allow us to describe the set of limits of tangent hyperplanes $\kappa_X^{-1}(0)$ to the germ $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$.

Theorem 2.4. *([LT88, Thm. 2.1.1, pg 559])*
Let Y be the origin of $\mathbb{C}^{n+1}$, and consider the normal/conormal diagram:

$$\begin{tikzcd}
E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[d, "\kappa'_X"'] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa_X"] \\
E_0X \arrow[r, "e_0"'] & X
\end{tikzcd}$$

Let $D = |\zeta^{-1}(0)|$ denote the reduced divisor and consider the set of its irreducible components $\{D_\alpha\}$. Then:

a) *Each $D_\alpha \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ is in fact contained in the incidence variety $I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$.*

b) *Each D_α is lagrangian in I and therefore establishes a projective duality of its images:*

$$\begin{tikzcd}
D_\alpha \arrow[r] \arrow[d] & W_\alpha \subset \check{\mathbb{P}}^n \\
V_\alpha \subset \mathbb{P}^n &
\end{tikzcd}$$

Proof.

Referring back to remark 2.3, we have the space $E_0C(X)$ as a subvariety of $X \times \mathbb{P}^n \times \check{\mathbb{P}}^n$, and the morphisms κ'_X and $\hat{e}_0$ correspond to the projections to $X \times \mathbb{P}^n$ and $X \times \check{\mathbb{P}}^n$ respectively. By construction, for any point $p = (0, l, H) \in \zeta^{-1}(0)$ there is a sequence of points $\{(x_i, l_i, H_i)\}_{i \in \mathbb{N}}$ in $\pi_2^{-1}(C(X) \setminus \kappa_X^{-1}(0)) \subset E_0X \times_X C(X)$ converging to p, and since the non-singular part X^0 is dense in X, and also its inverse images $e_0^{-1}(X^0)$, and $\kappa_X^{-1}(X^0)$ are dense in E_0X and $C(X)$ respectively, we can assume that the points x_i are non-singular. So, to prove a), we must show that $\ell \subset H$

Now, using the sequence $\{x_i\}$ tending to 0, we can build a corresponding sequence $\left\{\left(x_i, \check{T}_i, H_i\right)\right\}_{i\in\mathbb{N}}$ living in the space E of section 1.3.3, where $\check{T}_i$ is the projective dual of the tangent space to X^0 at the point x_i. Since the space, $G(n+1-d, n+1) \times \check{\mathbb{P}}^n$ is compact, there exist an $n+1-d$ plane $\check{T}$ in $G(n+1-d, n+1)$ and a subsequence $\left\{\left(x_j, \check{T}_j, H_j\right)\right\}_{j\in\Delta\subset\mathbb{N}}$, such that it converges to the point $(0, \check{T}, H)$, where the hyperplane H is necessarily the hyperplane H corresponding to the point $p \in \zeta^{-1}(0)$. This means that the hyperplane H contains the corresponding limit of tangent spaces $T \in \nu^{-1}(0) \subset \mathcal{N}X$.

To finish the proof of **a)** note that the subsequence $\{(x_j, l_j, H_j)\}_{j\in\Delta}$ also converges to the point p, which means that the directions of secants $[0x_j]$ and those of tangent spaces T_jX converge to limits ℓ and T, so by Whitney's lemma 2.1, $\ell \subset T \subset H$.

To prove **b)**, note that for the reduced one point variety $Y = \{0\} \subset \mathbb{C}^{n+1}$ the projectivized conormal space $C(Y)$ is equal to $\{0\} \times \check{\mathbb{P}}^n$, and so we have the equality $\kappa_X^{-1}(0) = C(Y) \cap C(X)$ as complex spaces. This gives us the equality of normal cones $C_{C(X), C(Y)\cap C(X)} = C_{C(X), \kappa_X^{-1}(0)} = \zeta^{-1}(0)$ and so by proposition 1.43 and its corollary 1.44 the reduced fiber $D = |\zeta^{-1}(0)|$ is a lagrangian subvariety of $\{0\} \times \mathbb{C}^{n+1} \times \check{\mathbb{C}}^{n+1}$, which means by proposition 1.22 and section 1.3.4 that every irreducible component D_α is the conormal space of its image, thus realizing the projective duality of V_α and W_α.

□

Note that, from the commutativity of the diagram we obtain $\kappa_X^{-1}(0) = \bigcup W_\alpha$, and $e_0^{-1}(0) = \bigcup V_\alpha$. It is important to notice that these expressions are not the irreducible decompositions of $\kappa_X^{-1}(0)$ and $e_0^{-1}(0)$ respectively, since we can't assure that all the V_α (W_α) have the right dimension. However, it is true that they contain the respective irreducible decompositions.

In particular, note that if $\dim V_{\alpha_0} = d-1$, then the cone $O(V_{\alpha_0}) \subset \mathbb{C}^{n+1}$ is an irreducible component of the tangent cone $C_{X,0}$ and its projective dual $W_{\alpha_0} = \check{V}_{\alpha_0}$ is contained in $\kappa_X^{-1}(0)$. That is, any tangent hyperplane to the tangent cone is a limit of tangent hyperplanes to X at 0.

Definition 2.5. *The finite collection $\{V_\alpha\}$ of projective subvarieties of the projectivized tangent cone $\mathbb{P}C_{X,0}$ is called the **aureole of the germ** $(X, 0)$. We will abusively refer with the same name and notation to the corresponding collection of subcones of the tangent cone $C_{X,0}$.*

*The cones V_α that are not irreducible components of the tangent cone $C_{X,0}$ are called the **exceptional cones** of the germ $(X, 0)$.*

Corollary 2.6. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of reduced, purely d-dimensional analytic singularity as before, and let $(X, 0) = \bigcup_{i=1}^r (X_i, 0)$ be its irreducible decomposition. Then the germ $(X, 0)$ doesn't have exceptional cones if and only if for each $i \in \{1, \ldots, r\}$ the germ $(X_i, 0)$ does not have exceptional cones.*

Proof. First of all, for a small enough representative of $X \subset \mathbb{C}^{n+1}$, we have the equality $C(X) = \bigcup C(X_i)$ where $C(X_i)$ denotes the conormal space of the embedding $X_i \subset \mathbb{C}^{n+1}$, and so the conormal map κ_{X_i} is equal to the restriction of κ_X to $C(X_i)$. Moreover, we know that the strict transform $\overline{e_0^{-1}(X_i \setminus \{0\})}$ is equal to the blowing-up $E_0X_i \to X_i$, and since for every arc $\phi : (\mathbb{C}, 0) \to (X, 0)$ there exists a $j \in \{1, \ldots, r\}$ such that ϕ factorizes through X_j, we have the equality $\mathbb{P}C_{X,0} = \bigcup \mathbb{P}C_{X_i,0}$. All these imply that for each $i \in \{1, \ldots, r\}$ the normal/conormal diagram

$$\begin{tikzcd} E_0C(X_i) \arrow[r, "\hat{e}_0"] \arrow[d, "\kappa'_{X_i}"'] \arrow[dr, "\zeta"] & C(X_i) \arrow[d, "\kappa_{X_i}"] \\ E_0X_i \arrow[r, "e_0"'] & X_i \end{tikzcd}$$

is canonically embedded in the normal/conormal diagram of X:

$$\begin{tikzcd} E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[d, "\kappa'_{X}"'] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa_{X}"] \\ E_0X \arrow[r, "e_0"'] & X \end{tikzcd}$$

Now, the germ $(X, 0)$ doesn't have exceptional cones if and only if every irreducible component W_α of the fiber $|\kappa_X^{-1}(0)| = \bigcup |\kappa_{X_i}^{-1}(0)|$ is equal to the projective dual of an irreducible component V_α of the tangent cone $\mathbb{P}C_{X,0}$, that is an irreducible component of one of the tangent cones $\mathbb{P}C_{X_i,0}$. Finally, since for a reduced projective subvariety the double dual $\check{\check{Y}}$ is equal to Y, then two projective subvarieties Y_1 and Y_2 of $\mathbb{P}^n$ are different if and only if their duals are different $\check{Y}_1 \neq \check{Y}_2$. This prevents the appearance of a possible exceptional cone of X_j having the same dual as an irreducible component of $\mathbb{P}C_{X,0}$ which finishes the proof. $\square$

We should say that there is a method to "compute" the aureole of the germ $(X, 0)$ using the concept of **polar varieties** which we will describe in Appendix B.

Note that if $(X, 0) \subset (\mathbb{C}^{n+1})$ is the cone over a projective variety $\mathbb{P}X \subset \mathbb{P}^n$, then it coincides with its tangent cone, that is $(X, 0) = (C_{X,0}, 0)$, and so the germ has no exceptional tangents. Indeed, by proposition 2.23, the aureole of X can be recovered in the specialization space $\mathfrak{X}$ as the image by κ_φ of the irreducible components of the fiber $q^{-1}(0)$, where $q = \varphi \circ \kappa_\varphi : C_\varphi(\mathfrak{X}) \to \mathbb{C}$. But since X is a cone, then by construction the specialization space $\mathfrak{X}$ is equal to $X \times \mathbb{C}$, so the relative conormal space $C_\varphi(\mathfrak{X})$ is equal to $C(X) \times \mathbb{C}$, and the fiber $q^{-1}(0)$ is equal to the conormal space $C(X)$. Finally, by corollary 2.6, it is enough to consider the case when X is irreducible which implies that the conormal space $C(X) = q^{-1}(0)$ is irreducible.

One may then wonder if having no exceptional cones makes X look like a cone. This question was given an answer in the case of surfaces in [LT79], but it hasn't been studied in the general case.

Proposition 2.7. *[LT79, Thms. 2.1.1 and 2.2.1, pgs 438-445]*
Let $(X, 0)$ be a germ of a reduced, 2-dimensional analytic subspace of $\mathbb{C}^3$. If the tangent cone $C_{X,0}$ is reduced, then having no exceptional tangents makes X Whitney equisingular with its tangent cone.

We would like to point out that by saying X is Whitney equisingular with its tangent cone, we mean that if $\varphi : \mathfrak{X} \to \mathbb{C}$ denotes the specialisation space of $(X, 0)$ to its tangent cone $(C_{X,0}, 0)$, and denoting by $Y := 0 \times \mathbb{C} \subset \mathfrak{X}$ the parameter axis, then Y is a stratum in a Whitney stratification of $\mathfrak{X}$, in particular the couple $(\mathfrak{X}^0, Y)$ satisfies Whitney conditions (see section 2.1) at every point $y \in Y$.

2.1 Whitney Stratifications

Whitney had observed, as we can see from the statement of lemma 2.1, that "asymptotically" a germ $(X, 0)$ behaves like a cone with vertex 0, near 0. Suppose now, that we replace 0 by a non-singular subspace $Y \subset X$, and we want to force X to "look like a cone with vertex Y".

Definition 2.8. *A cone with vertex Y is a space C equipped with a map of complex spaces*

$$C \longrightarrow Y$$

and homotheties in the fibers. The space C is the Specan of a finitely presented graded sheaf of O_Y-algebras.

Let us take a look at the basic example we have thus far constructed.

Example 2.9.
The reduced normal cone $|C_{X,Y}| \longrightarrow Y$, with the canonical analytic projection mentioned after definition 1.8.

What does it mean that "asymptotically" X is cone-like over Y? Well, here is Whitney's answer:

As usual, let X be a reduced, pure dimensional analytic space of dimension d, let $Y \subset X$ be a nonsingular analytic subspace containing 0 of dimension s. Choose a local embedding $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ around 0, and a local retraction $\rho : (\mathbb{C}^{n+1}, 0) \longrightarrow (Y, 0)$. Note that, since Y is non-singular we can assume it is an open subset of $\mathbb{C}^s$, $(X, 0)$ is embedded in an open subset of $\mathbb{C}^{n-s+1} \times \mathbb{C}^s$ and the retraction ρ coincides with the second projection.

Definition 2.10. *Letting ρ be the aforementioned retraction, here we present the 2 conditions stated by Whitney in [Whi65b, Section 19, pg 540]:*

a) The pair $(X^0, Y)_0$ satisfies Whitney's condition a) at 0 if for any sequence of non singular points $\{x_i\}_{i\in\mathbb{N}} \subset X^0$ tending to 0, we have the inclusion

$$T_0Y \subset \lim_{i\to\infty} T_{x_i}X$$

as linear subspaces of $\mathbb{C}^{n+1}$.

b) The pair $(X^0, Y)_0$ satisfies Whitney's condition b) at 0 if for any sequence of non singular points $\{x_i\}_{i\in\mathbb{N}} \subset X^0$ tending to 0, we have the inclusion

$$\lim_{i\to\infty}[x_i\rho(x_i)] \subset \lim_{i\to\infty} T_{x_i}X$$

as linear subspaces of $\mathbb{C}^{n+1}$.

Remark 2.11. *1. The limiting of tangent spaces should be understood as in definition 1.14. We are implicitly using the fact that the grassmannian varieties are compact and thus every sequence has a convergent subsequence.*

2. The Whitney conditions are independent of the embedding. An algebraic proof consisting on characterizing them in terms of integral closure of modules was given first by Teissier in [Tei74] for hypersurfaces and then by Gaffney in [Gaf92, Thm. 2.5, p. 309] in the general case.

If we compare these conditions to Whitney's lemma 2.1, they are just spreading out along Y the fact observed when $Y = \{0\}$. Whitney conditions can also be characterized in terms of the normal conormal diagram of the pair $(X, Y, 0)$, as we will see later on.

Definition 2.12. *A locally finite partition $X = \bigcup_{\alpha\in A} X_\alpha$ is a Whitney stratification of X if:*

1. *The X_α are non singular analytic varieties, whose closure in X is a closed analytic subspace.*
2. *For each $\alpha \in A$, the boundary $\overline{X}_\alpha \setminus X_\alpha$ is a union of strata.*
3. *For each triple (X_α, X_β, x) such that $x \in X_\beta \subset \overline{X}_\alpha$, and every local embedding $(X, x) \subset (\mathbb{C}^{n+1}, 0)$ the pair $(X_\alpha, X_\beta)_0$ satisfies Whitney's conditions a) and b).*

The importance of these stratifying conditions lies in the following results.

Theorem 2.13. *[Whi65b, Thm. 19.2, pg 540] Let M be a reduced complex analytic space and let $X \subset M$ be a locally closed analytic subspace of M. Then, there exists a Whitney stratification of $M = \bigcup M_\alpha$ such that X is a union of strata.*

An important feature of this kind of stratifications is that they are preserved by generic hyperplane sections. To be more specific, consider a small enough representative X of a germ $(X, 0) \subset \mathbb{C}^{n+1}$ and a Whitney stratification of it $X = \bigcup_{\alpha\in A} X_\alpha$. Let H be a hyperplane of $\mathbb{C}^{n+1}$, such that $H \notin \kappa_X^{-1}(0)$, that is H, is not a limit of tangent hyperplanes to X at 0. Then H is transversal to every X_α at every point sufficiently close to 0, and the partition $\bigcup_{\alpha\in A}(X_\alpha \cap H)$ is a Whitney stratification of $X \cap H$. An exact reference of this result is given in theorem 2.20, or alternatively you can find it in [Gaf08, Thm. 2.2] where the author uses the theory of integral closure of modules. It can also be seen as a special case of the more general result stated in the following lemma.

Lemma 2.14. *[Che72],[Tei82, Lemma 4.2.2, pg 404]. Let Z be a non singular analytic space, and let $X = \bigcup_{\alpha\in A} X_\alpha$ and $Y = \bigcup_{\beta\in B} Y_\beta$ be two Whitney stratified analytic subspaces in Z. If for every $\alpha \in A$ and $\beta \in B$ the strata X_α and Y_β are transversal in Z, then the partition $\bigcup_{\alpha\in A,\beta\in B} (X_\alpha \cap Y_\beta)$ of $X \cap Y$ is a Whitney stratification of $X \cap Y$.*

The following fundamental result was proved by Thom in [Tho69] and by Mather in the widely circulated lecture notes [Mat70] in 1970.

Theorem 2.15. ***Thom-Mather isotopy theorem***. *Let X be a Whitney stratified space. Given a point $x \in X$, let X_β be the stratum containing x. For any local embedding $(X,x) \subset (\mathbb{C}^{n+1},0)$, and any choice of a local retraction $r\colon (\mathbb{C}^{n+1},0) \to (X_\beta,0)$ after possibly restricting to a smaller neighborhood of 0, there is a germ of homeomorphism of pairs $(\mathbb{C}^{n+1},X) \equiv (r^{-1}(0) \times X_\beta, X \cap r^{-1}(0) \times X_\beta)$ mapping the closures of strata $\overline{X_\alpha}$ which contain X_β to $(r^{-1}(0) \cap \overline{X_\alpha}) \times X_\beta$.*

A consequence of this result is that the local topological type of a stratified space X is locally constant along the strata of a Whitney stratification. Here, by local topological type, one understands the homeomorphy class of the pair $(\mathbb{C}^{n+1} \cap \mathbf{B}_\epsilon^{n+1}, X \cap \mathbf{B}_\epsilon^{n+1})$ where $\mathbf{B}_\epsilon$ is a ball of radius ϵ centered at the point x for a local embedding $(X,x) \subset (\mathbb{C}^{n+1},0)$. If X is compact, it can therefore display only finitely many topological types.

In short, each $\overline{X_\alpha}$, or if you prefer, the stratified set X, is locally topologically trivial along X_β in x. A natural question arises then, is the converse to the Thom-Mather theorem true? That is, does local topological triviality implies Whitney?

The answer is *NO*, in [BS75] Briancon and Speder showed that the family of surface germs
$$z^5 + ty^6z + y^7x + x^{15} = 0$$
(each member, for small t, having an isolated singularity at the origin) is locally topologically trivial, but not Whitney.

However there is a converse, proved by Lê and Teissier in [LT83, Thm. 5.3.1, pg 95](see also [Tei82, Thm. 4.4, pg 483]). Let us refer to the conclusion of the Thom-Mather theorem, as the condition (TT) (local topological triviality), so we can restate theorem 2.15 as Whitney implies (TT). Let us recall the notations of the theorem and let $d_\beta = dim\ X_\beta$. We say that a stratification satisfies the condition $(TT)^*$ (local topological triviality for the general sections) if in addition to the condition (TT), for every $x \in X_\beta$, there exists for every $k > d_\beta$ a Zariski open set Ω in $G(n+1-d_\beta, k-d_\beta)$ such that for any non-singular space E containing X_β and such that $T_xE \in \Omega$, the intersection $\overline{X_\alpha} \cap E$ satisfies (TT) for all X_α such that $\overline{X_\alpha} \supset X_\beta$.

Theorem 2.16. (Lê-Teissier)
For a stratification $X = \bigcup X_\alpha$ of a complex analytic space X, the following conditions are equivalent:

1) $X = \bigcup X_\alpha$ is a Whitney stratification.

2) $X = \bigcup X_\alpha$ satifies condition $(TT)^*$.

2.2 The Normal Conormal diagram.

Let $(Y,0) \subset (X,0)$ be a germ of nonsingular analytic subvariety of dimension $s < d$ as before. The Whitney conditions of the pair (X^0,Y) at 0 can be expressed in terms of the normal conormal diagram of the pair $(X,Y,0)$. We will choose an embedding $(X,0) \subset (\mathbb{C}^{n+1},0)$ such that the germ $(Y,0)$ is linear with coordinate system $(z_0,\ldots,z_{n-s},y_1,\ldots,y_s)$.

$$\begin{tikzcd} E_YC(X) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'_X"'] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa_X"] \\ E_YX \arrow[r, "e_Y"'] & X \end{tikzcd}$$

Proposition 2.17. *Let D denote the reduced divisor $|\zeta^{-1}(Y)| \subset E_YC(X)$, then:*

1. *The pair (X^0,Y) satisfies Whitney's condition a) at every point $y \in Y$ if and only if we have the set theoretical equality $|C(X) \cap C(Y)| = |\kappa_X^{-1}(Y)|$.*
2. *The pair (X^0,Y) satisfies Whitney's condition a) at every point $y \in Y$ if and only if D is contained in $Y \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$ where for every $y \in Y$, $\check{\mathbb{P}}^{n-s}$ denotes the space of hyperplanes containing T_yY. In particular, they satisfy Whitney's condition a) at 0 if and only if $\zeta^{-1}(0) \subset \{0\} \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$.*
3. *The pair (X^0,Y) satisfies Whitney's condition b) at $y \in Y$ if and only if $|\zeta^{-1}(y)|$ is contained in the incidence variety $I \subset \{y\} \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$.*

Proof. Whitney conditions are defined in terms of limit of tangent spaces. However, once we have fixed an embedding $(X,0) \subset (\mathbb{C}^{n+1},0)$, since a hyperplane H is a limit of tangent hyperplanes if and only if it contains a limit of tangent spaces (corollary 1.26) we can restate Whitney conditions(def. 2.10):

•) The pair $(X^0,Y)_0$ satisfies Whitney condition a) at 0 if for any sequence of non singular points $\{x_i\}_{i\in\mathbb{N}} \subset X^0$ tending to 0, and any sequence $\{H_i\}_{i\in\mathbb{N}}$ where H_i is a tangent hyperplane to X at the point x_i we have the inclusion

$$T_0Y \subset \lim_{i\to\infty} H_i$$

•) The pair (X^0,Y) satisfies Whitney condition b) at $y \in Y$ if for any sequence of non singular points $\{x_i\}_{i\in\mathbb{N}} \subset X^0$ tending to y, and any sequence $\{H_i\}_{\in\mathbb{N}}$ where H_i is a tangent hyperplane to X at the point x_i we have the inclusion

$$\lim_{i\to\infty}[x_i\rho(x_i)] \subset \lim_{i\to\infty} H_i$$

With this in mind **1)** is now only an observation. Note that we always have the inclusion $|C(X) \cap C(Y)| \subset |\kappa^{-1}(Y)|$. On the other hand, the inclusion $|\kappa^{-1}(Y)| \subset |C(Y)|$ means that for every $y \in Y$ every limit of tangent hyperplanes to X at y, $H \in \kappa_X^{-1}(y)$, is also a tangent hyperplane to Y at y, that is $T_yY \subset H$.

For **2)**, with the coordinate system we have fixed, we have from remark 2.3 the blowing up E_YX as a subspace of $X \times \mathbb{P}^{n-s}$, and the conormal space $C(Y)$ equal to $Y \times \check{\mathbb{P}}^{n-s}$ where $\check{\mathbb{P}}^{n-s}$ corresponds to the projective dual of $\mathbb{P}Y$, that is the algebraic set defined by $c_1 = \cdots = c_s = 0$. Then, from **1)** satisfying condition a) is then equivalent to the inclusion $|\kappa_X^{-1}(Y)| \subset Y \times \check{\mathbb{P}}^{n-s}$ which by construction of the normal conormal diagram is equivalent to the inclusion $|\zeta^{-1}(Y)| \subset Y \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$.

To prove **3)**, with the coordinate system we have fixed, we have the natural retraction $r : \mathbb{C}^{n+1} \to Y$ sending $(\underline{z}, \underline{y}) \to y$ which at the same time is used to build the underlying set of the blowup of X along Y, $E_Y X$. So, from the construction of $E_Y C(X)$ as a subspace of the fiber product, we have to take the closure of the set of points of this space of the form $(\underline{z}, \underline{y}, l, H)$ where $(\underline{z}, \underline{y})$ is a point in $X^0 \setminus Y$, $l \in \mathbb{P}^{n-s}$ is the line defined by $[(\underline{z}, \underline{y}) - r(\underline{z}, \underline{y})]$ and H is a tangent hyperplane to X at the point $(\underline{z}, \underline{y})$. Then, a point in the divisor $D = \zeta^{-1}(Y)$ is a point $(0, y, l, H)$, where $(0, y)$ is a point in Y, and l and H are a line and a hyperplane obtained in the way described in the definition of condition b) above. Finally the inclusion $l \subset H$ is just what it means that the pair (l, H) is in the incidence variety $I \subset \{y\} \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$ which finishes the proof. □

Let $JO_{C(X)}$ denote the ideal sheaf defining the subspace $\kappa_X^{-1}(Y)$ in $O_{C(X)}$, and $IO_{C(X)}$ the ideal sheaf defining the intersection $C(X) \cap C(Y)$ as a subspace of $C(X)$. Then the proposition tells us that the pair (X^0, Y) satisfies Whitney's condition a) if and only if both ideals have the same radical, i.e. $\sqrt{J}O_{C(X)} = \sqrt{I}O_{C(X)}$. There exists another stratifying condition, known as the **w)** condition or as the strict Whitney conditions ([Hir69, Def. 5.1, pg 135][Ver76, Def. 1.4, pg 296]), which has a similar characterization in the conormal space $C(X)$([LT88, Prop. 1.3.8, pg 550]). It is important to say that in the complex analytic case the $w)$ condition is equivalent to the Whitney conditions as proven in [Tei82, Chap. 5, Thm. 1.2, pg 455].

Definition 2.18. *The pair (X^0, Y) satisfies the **w)** condition at $y_0 \in Y$ if there exists an open neighborhood U of y_0 in X and a positive real constant K such that for every y in $Y \cap U$ and x in $X^0 \cap U$ we have:*

$$\delta(T_y Y, T_x X^0) \leq K \parallel x - y \parallel$$

where δ is the distance of vector subspaces of $\mathbb{C}^{n+1}$ defined by

$$\delta(T_y Y, T_x X^0) = \sup_{u \in T_x X^{\perp} \setminus \{0\},\, v \in T_y Y \setminus \{0\}} \frac{|\langle u, v \rangle|}{\|u\| \|v\|}$$

where $\langle u, v \rangle$ denotes the hermitian product of $\mathbb{C}^{n+1}$ and

$$T_x X^{\perp} = \{u \in \mathbb{C}^{n+1} \mid \langle u, v \rangle = 0,\ \forall v \in T_x X\}$$

Proposition 2.19. *[LT88, Prop. 1.3.8, pg 550]*
The following conditions are equivalent:

*i) The pair (X^0, Y) satisfies the **w)** condition at 0.*

ii) At every point of $\kappa_X^{-1}(0)$, the ideal $IO_{C(X)}$ defining the intersection $C(X) \cap C(Y)$ is integral over the ideal $JO_{C(X)}$ defining the subspace $\kappa_X^{-1}(Y)$ in $O_{C(X)}$.

This proposition is important because on the one hand is a good example of how the theory of integral closure of ideals (see Appendix C) and its generalization, the theory of integral closure of modules, can be used in analytic geometry to deal with limits of tangent spaces, and on the other hand it plays an important role in the generalization of theorem 2.4 which gives another characterization of Whitney conditions.

Theorem 2.20. *[LT88, Thm. 2.1.1 pg 559] Let $X \subset \mathbb{C}^{n+1}$ be a representative of a germ of reduced, equidimensional, analytic variety of dimension d. Let $Y \subset X$ be a non-singular subvariety of dimension $s < d$ passing through 0. Consider the normal conormal diagram of the pair $(X, Y, 0)$*

$$\begin{tikzcd} E_Y C(X) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'_X"] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa_X"] \\ E_Y X \arrow[r, "e_Y"'] & X \end{tikzcd}$$

and let $|D| = \bigcup D_\alpha$ be the irreducible decomposition of the reduced divisor $|D| = |\zeta^{-1}(Y)|$. The following conditions are equivalent:

i) The pair (X^0, Y) satisfies Whitney conditions a) and b) at 0.

ii) We have the equality dim $\zeta^{-1}(0) = n - 1 - s$.

iii) For each α, D_α is the relative conormal space of its image V_α in $E_Y X$ with respect to the canonical analytic projection $C_{X,Y} \to Y$ restricted to V_α, and all the fibers of the restriction $\zeta : D_\alpha \to Y$ have the same dimension near 0.

In particular, Whitney conditions are equivalent to the equidimensionality of the fibers and a relative duality:

$$\begin{tikzcd} D_\alpha \arrow[r] \arrow[d] & W_\alpha = Y - \text{dual of } V_\alpha \\ V_\alpha & \end{tikzcd}$$

That is, every irreducible component D_α is $Y-$Lagrangian in $Y \times \mathbb{C}^{n+1-s} \times \check{\mathbb{C}}^{n+1-s}$.

Just as in the case of the tangent cone (def.2.5), we can define the auréole of X along Y, as follows:

Definition 2.21. *[LT88, Def. 2.1.4, pg 562] The finite collection $\{V_\alpha\}$ of projective subvarieties of the projectivized normal cone $\mathbb{P}C_{X,Y}$ is called the **auréole of** X **along** Y. We will abusively refer with the same name and notation to the corresponding collection of subcones of the normal cone $C_{X,Y}$.*

*The cones V_α that are not irreducible components of the normal cone $C_{X,Y}$ are called the **exceptional cones** of X along Y.*

Remark 2.22.
1. *Every V_α is contained in the reduced, projectivized normal cone $|\mathbb{P}C_{X,Y}|$ and so inherits a projection $V_\alpha \to Y$.*
2. *The V_α's can be defined for every Y, however it is only in the case when the pair (X^0, Y) satisfies Whitney conditions that we are able to relate them with the set of limits of tangent hyperplanes.*

Proposition 2.23. *[LT88, Prop. 2.1.4.1, pg 562] In the setting of theorem 2.20 suppose that the pair (X^0, Y) satisfies Whitney conditions. Let $\varphi : \mathfrak{X} \to \mathbb{C}$ be the specialisation of X to the normal cone $C_{X,Y}$, and let $q : C_\varphi(\mathfrak{X}) \to \mathbb{C}$ be defined as the composition $q = \varphi \circ \kappa_\varphi$. Then, every V_α of the aureole of X along Y is the set theoretical image by κ_φ of an irreducible component of the fiber $q^{-1}(0)$.(Thus it is contained in the special fiber $\varphi^{-1}(0)$).*

In the setting of theorem 2.20, let $J \subset O_X$ be the ideal defining Y, and let H be a hyperplane of $\mathbb{C}^{n+1}$ defined by an equation $h = 0$. Note that if the initial form $in_J h$ is not a zero divisor in $gr_J O_X$, we have a canonical identification of the normal cone $C_{X\cap H, Y\cap H}$ with the subspace of $C_{X,Y}$ defined by this initial form $in_J h$. We will denote this subspace by $C_{X,Y} \cap H^Y$. In particular, this happens when H is not tangent to $C_{X,Y}$. We will denote $\mathfrak{i} : \mathbb{P}(C_{X,Y}) \cap \mathbb{P}(H^Y) \to \mathbb{P}(C_{|X\cap H|, Y\cap H})$ the corresponding identification after projectivisation.

Theorem 2.24. *[LT88, Thm. 2.3.2, pg 572] In the setting of theorem 2.20 suppose that the pair $(X^0, Y)_0$ satisfies Whitney conditions at 0. Let $H \in \check{\mathbb{P}}^n$ be a hyperplane such that H is not tangent to X at 0, i.e. $H \notin \kappa_X^{-1}(0)$, then:*

i) The hyperplane H is not tangent to any V_α in the aureole of X along Y.

ii) In a small enough neighborhood of 0 in X the intersection $X^0 \cap H$ is the non-singular part of $X \cap H$, and the pair $(X^0 \cap H, Y \cap H)_0$ satisfies Whitney conditions at 0.

iii) The identification $\mathfrak{i}$ identifies the family of non-empty $|V_\alpha \cap \mathbb{P}(H^Y)|$ with the aureole of $|X \cap H|$ along $Y \cap H$.

2.3 Thom's $\mathbf{a_f}$ condition

There exists a generalisation of Whitney's condition $a)$ to the relative case due to René Thom which we will now describe. Let $f : X \to S$ be a morphism between reduced complex analytic varieties, such that there exists an open dense set $U \subset X$, over which the morphism f has smooth fibers of constant dimension d. Let $Y \subset X$ be a smooth analytic subvariety of X over which the restriction of f has smooth fibers of constant dimension, and let y be a point of Y.

Definition 2.25. *If $x \in X$ is a smooth point of the fiber $X(f(x))$, let $T_x X(f(x))$ denote the tangent space to the fiber at this point. The pair (U, Y) satisfies Thom's condition a_f at the point $y \in Y$, if there exists a local embedding $(X, y) \subset (\mathbb{C}^n, 0)$ such that for every sequence $\{x_j\} \subset U$ tending to y, if the sequence of tangent spaces $\{T_{x_j} U(f(x_j))\}$ converges then:*

$$\lim_{j\to\infty} \delta\left(T_y Y(f(y)), T_{x_j} U(f(x_j))\right) = 0$$

where δ is the distance between vector subspaces of $\mathbb{C}^n$ defined by

$$\delta(E, F) = \sup_{u\in E\setminus\{0\},\, v\in F^\perp\setminus\{0\}} \frac{|\langle u, v\rangle|}{\|u\|\|v\|}$$

$\langle u, v\rangle$ denotes the hermitian product in $\mathbb{C}^n$, and $F^\perp := \{v \in \mathbb{C}^n \mid \langle v, x\rangle = 0 \; \forall x \in F\}$.

Note that $\delta(E, F) = 0$ is equivalent to $F^\perp \subset E^\perp$ which is equivalent to $E \subset F$. In other words, the pair (U, Y) satisfies Thom's condtion a_f at y if the the limit T of the tangent spaces $T_{x_j} U(f(x_j))$ contains the tangent space $T_y Y(f(y))$. If S is a point, this condition reduces to Whitney's condition $a)$.

Proposition 2.26. *In the current setup, if S is a non-singular curve and Y is the non-singular part of the reduced fiber $|f^{-1}(s)|$, then there exists an open dense subset $V \subset Y$, such that the pair (U, Y) satisfies Thom's condition a_f at every point $y \in V$.*

A proof due to Hironaka can be found in [Lê81, Thm. 1.2.4].

Chapter 3

Integral Closure of Modules

Our goal is to study the equisingularity of the specialisation space $\varphi : \mathfrak{X} \to \mathbb{C}$ along the parameter axis. If we want to generalise proposition 2.7 then the first step is to prove that if the tangent cone $C_{X,0}$ is reduced, then the absence of exceptional cones implies that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney conditions at the origin, where Y denotes the zero section defined by the map $\sigma : \mathbb{C} \to \mathfrak{X}$ picking the origin in each fiber $\mathfrak{X}(t)$.

One important tool is the theory of integral dependence of modules and its relation with equisingularity as developed by T. Gaffney in [Gaf92] and [Gaf97] and later in conjunction with S. Kleiman in [GK99]. Their work is a generalization of the theory of integral closure of ideals and its relation with equisingularity as developed by M. Lejeune-Jalabert and B. Teissier in [LJT08] and [Tei74] which deals with the hypersurface case.

There are several equivalent definitions of integral closure for modules. In our case, it is simpler to work with the following definition, as stated in [GK99, Section 3, pg 555]. **For this section,** $(\mathfrak{X}, 0)$ **will denote the germ of an arbitrary reduced complex analytic space.**

Definition 3.1. *Let $\mathcal{E} := O_{\mathfrak{X}}^p$ be a free module of rank $p \geq 1$, and let O_1 denote the structure sheaf of $\mathbb{C}$ as an analytic space. Let M be a coherent submodule of $\mathcal{E}$ and $h \in \mathcal{E}$. A map of germs $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0)$, induces a morphism of analytic algebras $O_{\mathfrak{X},0} \to O_1 = \mathbb{C}\{\tau\}$, which in turn defines a morphism $O_{\mathfrak{X}}^p \to O_1^p$. Denote by $h \circ \phi$ the induced section of O_1^p, and by $M \circ \phi$ the induced submodule. Call h integrally dependent (resp. strictly dependent) on M at 0 if, for every ϕ, the section $h \circ \phi \in O_1^p$ belongs to the submodule $M \circ \phi$ of O_1^p (resp. to the submodule $\mathfrak{m}_1(M \circ \phi)$), where $\mathfrak{m}_1$ is the maximal ideal of O_1. The submodule of $\mathcal{E}$ generated by all such h will be denoted by $\overline{M}$, respectively by $M^\dagger$.*

Moreover, we say that a submodule $N \subset M$ is a ***reduction*** *of M if $\overline{N} = \overline{M}$.*

It is worth saying that when $M \subset O_{\mathfrak{X}}$ is a coherent ideal, this definition of integral dependence is equivalent to the standard definition of integral dependence (Def. C.1), as proven in theorem C.3. However, even though the theory of integral dependence of ideals was used to study Whitney equisingularity of hypersurface isolated singularites in [Tei74], and a characterisation was given via the order function $\overline{\nu}$, the concept of strict dependence was not explicitly defined.

If the germ $(\mathfrak{X}, 0)$ is not irreducible, then for every irreducible component $\mathfrak{X}_i$ of $\mathfrak{X}$ the module M induces a submodule $M_{\mathfrak{X}_i}$ of $O^p_{\mathfrak{X}_i}$ via the morphism of analytic algebras $O_{\mathfrak{X},0} \to O_{\mathfrak{X}_i,0}$, and the same goes for a section h of $O^p_{\mathfrak{X}}$.

Lemma 3.2. *Let $(\mathfrak{X}, 0) = \bigcup_{i=1}^r (\mathfrak{X}_i, 0)$ be the irreducible decomposition of the germ. Then h is integrally dependent (respectively strictly dependent) on M at 0 if and only if for every irreducible component $\mathfrak{X}_i$ the induced section h_i is integrally dependent (respectively strictly dependent) on $M_{\mathfrak{X}_i}$ at 0.*

Proof. It is sufficient to prove is that there is a one to one correspondence between the set of all arcs $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0)$ and the union over $i \in \{1, \dots, r\}$ of the set of all arcs $\psi : (\mathbb{C}, 0) \to (\mathfrak{X}_i, 0)$. Now, any arc $\psi : (\mathbb{C}, 0) \to (\mathfrak{X}_i, 0)$ gives rise to an arc $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0)$ by composing with the canonical map $(\mathfrak{X}_i, 0) \hookrightarrow (\mathfrak{X}, 0)$. On the other hand, since the germ $(\mathbb{C}, 0)$ is irreducible, then its image by any map $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0)$ is irreducible and so it is contained in an irreducible component $\mathfrak{X}_j$. This implies that the arc ϕ factorizes through an arc $\psi : (\mathbb{C}, 0) \to (\mathfrak{X}_j, 0)$ which finishes the proof. □

We will first state a result linking the theories of the integral closure of modules and ideals, which will prove to be very useful later. Let M be a coherent submodule of $\mathcal{E}$ as before, and let $[M]$ be a matrix of generators of M for a small enough neighborhood of the origin in $(\mathfrak{X}, 0)$, that is the matrix describing the morphism μ of:

$$O^q_{\mathfrak{X}} \xrightarrow{\mu} O^r_{\mathfrak{X}} \longrightarrow O^p_{\mathfrak{X}}/M \longrightarrow 0$$

Let $J_k(M)$ denote the ideal of $O_{\mathfrak{X}}$ generated by the $k \times k$ minors of $[M]$. This is the same as the $(p-k)$-th Fitting ideal of $O^p_{\mathfrak{X}}/M$ and so is independent of the choice of generators of M. If $h \in \mathcal{E}$, let (h, M) denote the submodule of $\mathcal{E}$ generated by h and M.

Proposition 3.3. *[Gaf92, Prop 1.7, pg 304],and [Gaf97, Prop 1.5, pg 57]*
Suppose M is a submodule of $\mathcal{E}$, $h \in \mathcal{E}$ and the rank of (h, M) is k on each irreducible component of $(\mathfrak{X}, 0)$. Then h is integrally dependent (resp. strictly dependent) on M at 0 if and only if each minor in $J_k(h, M)$ which depends on h is integrally dependent (resp. strictly dependent) on $J_k(M)$.

Note that the module M is always contained in its integral closure $\overline{M}$, however in general the module M neither contains nor is contained in the module $M^\dagger$. Note as well that we always have the inclusion of ideals $J_k(M) \subset J_k(h, M)$. With this in mind, what proposition 3.3 is telling us is that $h \in \overline{M}$ if and only if $\overline{J_k(h, M)} = \overline{J_k(M)}$. And $h \in M^\dagger$ if and only if every minor m in $J_k(h, M)$ depending on h satisfies $m \in J_k(M)^\dagger$.

Proposition 3.4. *[GK99, Prop. 3.1, pg 556] If $N \subset M \subset \overline{N}$, then $\overline{M} = \overline{N}$ and $M^\dagger = N^\dagger$.*

An important property of the integral dependence is that to check for integral (resp. strict dependence), it suffices to use only those ϕ whose image meets any given dense Zariski open subset of $\mathfrak{X}$, for example the non-singular part $\mathfrak{X}^0$. This is proved in the following lemma.

Lemma 3.5. *Let $(W,0) \subset (\mathfrak{X},0)$ be a proper analytic subset of $\mathfrak{X}$. A section $h \in \mathcal{E}$ is in $\overline{M}$ (respectively $M^{\dagger}$) if and only if for every map of germs $\phi : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}, \mathfrak{X} \setminus W, 0)$, the section $h \circ \phi \in O_1^p$ belongs to the submodule $M \circ \phi$ of O_1^p (resp. to the submodule $\mathfrak{m}_1(M \circ \phi)$), where $\mathfrak{m}_1$ is the maximal ideal of 0 in $O_1 = \mathbb{C}\{\tau\}$.*

Proof. Since the necessity is clear, we prove the sufficiency. We will use the notation $\phi^*(h) := h \circ \phi$ and $\phi^*(M)O_1 := M \circ \phi$. Suppose there exists a map

$$\phi : (\mathbb{C}, 0) \to (W, 0) \hookrightarrow (\mathfrak{X}, 0)$$

such that $\phi^*(h) \notin \phi^*(M)O_1$, then we must prove the existence of a map

$$\widetilde{\phi} : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}, \mathfrak{X} \setminus W, 0)$$

such that $\widetilde{\phi}^*(h) \notin \widetilde{\phi}^*(M)O_1$. Note that having $\phi^*(h) \notin \phi^*(M)O_1$, is equivalent to having $\phi^*(M) \subsetneq \phi^*(M,h)$, where (M,h) denotes the submodule of $\mathcal{E}$ generated by M and h.

We will divide the proof in 3 steps:
Step 1]
We will say that the modules $\phi^*(M)$ and $\phi^*(M,h)$ are equivalent mod $\mathfrak{m}_1^k$, if we have the isomorphism:

$$\frac{\phi^*(M)}{\mathfrak{m}_1^k O_1^p \cap \phi^*(M)} \cong \frac{\phi^*(M,h)}{\mathfrak{m}_1^k O_1^p \cap \phi^*(M,h)}$$

and we will denote it by $\phi^*(M) \equiv \phi^*(M,h) \bmod \mathfrak{m}_1^k$. Note that the existence of a k such that $\phi^*(M) \not\equiv \phi^*(M,h) \bmod \mathfrak{m}_1^k$ implies that $\phi^*(M) \subsetneq \phi^*(M,h)$.

What we will first prove is that $\phi^*(M) \subsetneq \phi^*(M,h)$ implies the existence of a $\nu_0 \in \mathbb{N}$ such that for all $k > \nu_0$ we have $\phi^*(M) \not\equiv \phi^*(M,h) \bmod \mathfrak{m}_1^k$. Suppose it isn't so, then for every ν_0 there exists an $l > \nu_0$ such that $\phi^*(M) \equiv \phi^*(M,h) \bmod \mathfrak{m}_1^l$. This means that the canonical morphism

$$\phi^*(M) \longrightarrow \frac{\phi^*(M,h)}{\mathfrak{m}_1^l O_1^p \cap \phi^*(M,h)}$$

is surjective, and so for every $\varsigma \in \phi^*(M,h)$ there exists an $m \in \phi^*(M)$ such that $\varsigma_1 := \varsigma - m \in \mathfrak{m}_1^l O_1^p \cap \phi^*(M,h)$. This gives us the equality

$$\phi^*(M,h) = \phi^*(M) + \mathfrak{m}_1^l O_1^p \cap \phi^*(M,h) \tag{3.1}$$

Now, the Artin-Rees lemma ([GP07, Lemma 5.4.5, pg 332]), gives us the existence of a ν_0 such that for all $l > \nu_0$ we have the equality:

$$\mathfrak{m}_1^l O_1^p \cap \phi^*(M,h) = \mathfrak{m}_1^{l-\nu_0} \left(\mathfrak{m}_1^{\nu_0} O_1^p \cap \phi^*(M,h)\right)$$

which implies the inclusions $\mathfrak{m}_1^l O_1^p \cap \phi^*(M,h) \subset \mathfrak{m}_1^{l-\nu_0} \phi^*(M,h) \subset \mathfrak{m}_1 \phi^*(M,h)$. If we fix the ν_0 given by the Artin-Rees lemma then we can rewrite equation 3.1 in the form:

$$\phi^*(M,h) = \phi^*(M) + \mathfrak{m}_1 \phi^*(M,h)$$

which implies $\phi^*(M) = \phi^*(M,h)$ by Nakayama's lemma ([Sha90, Lemma 8.24, pg 158]), thus obtaining a contradiction with our initial hypothesis.

Step 2] Suppose $(\mathfrak{X},0)$ is smooth.
Choose a local coordinate system such that $(\mathfrak{X},0) = (\mathbb{C}^d,0)$, and fix a $k > \nu_0$ as given by step 1. We can alter ϕ by truncating it at level k and adding higher order terms from $\mathfrak{m}_1^{k+1}$ to obtain a $\widetilde{\phi} : (\mathbb{C},0) \to (\mathbb{C}^d,0)$ in such a way that the image of $\widetilde{\phi}$ does not intersect W away from the origin. Then by the way we chose k we have

$$\begin{aligned} \phi^*(M) &\not\equiv \phi^*(M,h) \bmod \mathfrak{m}_1^k \\ \phi^*(M) &\equiv \widetilde{\phi}^*(M) \bmod \mathfrak{m}_1^k \\ \phi^*(M,h) &\equiv \widetilde{\phi}^*(M,h) \bmod \mathfrak{m}_1^k \end{aligned}$$

which implies that $\widetilde{\phi}^*(M) \not\equiv \widetilde{\phi}^*(M,h) \bmod \mathfrak{m}_1^k$ and so $\widetilde{\phi}^*(M) \subsetneq \widetilde{\phi}^*(M,h)$ as we wanted.

Step 3] Suppose $(\mathfrak{X},0)$ is singular.
Let $\pi : \hat{\mathfrak{X}} \to \mathfrak{X}$ be a resolution of $\mathfrak{X}$. We will lift the arc ϕ with the help of the following diagram:

$$\begin{tikzcd} & \hat{\mathfrak{X}} \times_{\mathfrak{X}} \mathbb{C} \arrow[r, "\hat{\phi}"] \arrow[d, "\hat{\pi}"] & \hat{\mathfrak{X}} \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, bend left, "\theta"] \arrow[r, "\psi"] & \mathbb{C} \arrow[r, "\phi"] & \mathfrak{X} \end{tikzcd}$$

Take the fibered product $\hat{\mathfrak{X}} \times_{\mathfrak{X}} \mathbb{C}$ and choose a point $p \in \pi^{-1}(0)$. Since π is proper and surjective, then $\hat{\pi}$ is proper and surjective as well with the point $(p,0) \in \hat{\pi}^{-1}(0)$. By the curve selection lemma there exists an analytic arc

$$\theta : (\mathbb{C}, \mathbb{C} \setminus \{0\}) \to (\hat{\mathfrak{X}} \times_{\mathfrak{X}} \mathbb{C}, \hat{\mathfrak{X}} \times_{\mathfrak{X}} \mathbb{C} \setminus \{\hat{\pi}^{-1}(0)\}, (p,0))$$

which induces the arc $\psi : (\mathbb{C},0) \to (\mathbb{C},0)$ which can be seen as a reparametrisation of the arc ϕ and is defined by $\tau \mapsto \tau^s v$ where v is a unit in $\mathbb{C}\{\tau\}$. The composition $\hat{\phi_1} := \hat{\phi} \circ \theta : (\mathbb{C},0) \to (\hat{\mathfrak{X}}, (p,0))$ gives us an analytic map lifting $\phi_1 := \phi \circ \psi$. **Remark:** Since $\phi^*(h) \notin \phi^*(M)$ then $\phi_1^*(h) \notin \phi_1^*(M)$. We will prove this at the end.

We now have a commutative diagram

$$\begin{tikzcd} & (\hat{\mathfrak{X}}, p) \arrow[d, "\pi"] \\ (\mathbb{C},0) \arrow[ur, bend left, "\hat{\phi}_1"] \arrow[r, "\phi_1"] & (\mathfrak{X},0) \end{tikzcd}$$

where $\phi_1^*(h) \notin \phi_1^*(M)$ and $\hat{\phi}_1(\mathbb{C}) \subset \pi^{-1}(W)$. Moreover, since π is proper and surjective we have that $\pi^* : O_{\mathfrak{X},0} \to O_{\hat{\mathfrak{X}},p}$ is injective, and from the commutativity of the diagram we deduce that $\pi^*(h) \notin \pi^*(M)O_{\hat{\mathfrak{X}},p}$ and $\hat{\phi}_1^*(\pi^*(h)) \notin \hat{\phi}_1^*(\pi^*(M)O_{\hat{\mathfrak{X}},p})O_1$. But, since $(\hat{\mathfrak{X}},p)$ is smooth, by step 2 there exists an arc

$$\hat{\hat{\phi}} : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\hat{X}, \hat{X} \setminus \{\pi^{-1}(W)\}, p)$$

such that $\widetilde{\hat{\hat{\phi}}}^{*}(\pi^*(h)) \notin \widetilde{\hat{\hat{\phi}}}^{*}(\pi^*(M)O_{\hat{\mathfrak{X}},p})O_1$. Thus, by defining the map $\widetilde{\phi}$ as the composition $\pi \circ \widetilde{\hat{\hat{\phi}}}$ we obtain the map we were looking for.

To finish the proof we will prove the remark. By [Sha90, Thm. 10.5, pg 189] there exists a basis $\{\overrightarrow{f_i}\}_{i=1}^p$ for O_1^p, such that:

$$\phi^*(M) = \mathbb{C}\{\tau\}\tau^{n_1}\overrightarrow{f_1} + \cdots + \mathbb{C}\{\tau\}\tau^{n_p}\overrightarrow{f_p}$$

with $n_1 \leq n_2 \leq \cdots \leq n_p$ and some n_j's may be ∞, that is by convention $\tau^{n_j} = 0$. Moreover, if we consider the expresion of $\phi^*(h) := h \circ \phi$ in this basis, say $\phi^*(h) = a_1\overrightarrow{f_1} + \cdots + a_p\overrightarrow{f_p}$, then $\phi^*(h)$ belongs to $\phi^*(M)$ if and only if $a_j = b_j\tau^{n_j}$ for a suitable $b_j \in \mathbb{C}\{\tau\}$ for $1 \leq j \leq p$. Since $\phi^*(h) \notin \phi^*(M)$, there exists an $i \in \{1, \ldots, p\}$ such that $a_i \notin \langle\tau^{n_i}\rangle O_1$.

Now, the morphism of local algebras $\psi^* : \mathbb{C}\{\tau\} \to \mathbb{C}\{\tau\}$ defined by $\tau \to \tau^s\nu$ is injective and maps units to units. Thus, the induced morphism $\psi^* : O_1^p \to O_1^p$ is injective and maps the basis $\{\overrightarrow{f_i}\}_{i=1}^p$ of O_1^p to a basis $\{\overrightarrow{\psi^*(f_i)}\}_{i=1}^p$ of O_1^p such that:

$$\begin{aligned}
\phi_1^*(M) = \psi^*(\phi^*(M)) &= \mathbb{C}\{\tau\}\nu^{n_1}\tau^{sn_1}\overrightarrow{\psi^*(f_1)} + \cdots + \mathbb{C}\{\tau\}\nu^{n_p}\tau^{sn_p}\overrightarrow{\psi^*(f_p)} \\
\phi_1^*(h) = \psi^*(\phi^*(h)) &= \psi^*(a_1)\overrightarrow{\psi^*(f_1)} + \cdots + \psi^*(a_p)\overrightarrow{\psi^*(f_p)}
\end{aligned}$$

but, $\psi^*(a_i) \notin \langle\tau^{sn_i}\rangle O_1$, so $\phi_1^*(h) \notin \phi_1^*(M)$ which finishes the proof. □

Let h be a section of $O_{\mathfrak{X}}^p$ at 0 and consider maps $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0)$, and $\psi : (\mathbb{C}, 0) \to (\mathrm{Hom}(\mathbb{C}^p, \mathbb{C}), \lambda)$; then $h \circ \phi$ is a section of O_1^p, and as such can be considered as a germ $h \circ \phi : (\mathbb{C}, 0) \to (\mathbb{C}^p, q)$. We will denote by $\psi(h \circ \phi)$ the funcion $(\mathbb{C}, 0) \to (\mathbb{C}, y)$ defined by evaluating the linear functional $\psi(\tau)$ in the vector $(h \circ \phi)(\tau)$ of $\mathbb{C}^p$ for each sufficiently small $\tau \in \mathbb{C}$.

Lemma 3.6. *[GK99, Lemma 3.3, pg 557] For a section h of $\mathcal{E}$ to be integrally dependent, respectively strictly dependent, on M at 0, it is necessary that for all maps:*

$$\begin{aligned}
&\phi : (\mathbb{C}, 0) \to (\mathfrak{X}, 0) \\
&\psi : (\mathbb{C}, 0) \to (\mathrm{Hom}(\mathbb{C}^p, \mathbb{C}), \lambda), \quad \lambda \neq 0
\end{aligned}$$

the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belongs to the ideal $I_\psi(M \circ \phi)$ generated by applying $\psi(\tau)$ to the generators of $M \circ \phi$, respectively to the ideal $\mathfrak{m}_1 I_\psi(M \circ \phi)$.

Conversely it is sufficient that this condition is satisfied for every $\phi : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}, \mathfrak{X} \setminus W, 0)$, where $(W, 0) \subset (\mathfrak{X}, 0)$ is a proper analytic subset of $\mathfrak{X}$.

Proof.
$\Rightarrow]$
Suppose $h \in \overline{M}$ (respectively $h \in M^\dagger$), then by definition $h \circ \phi \in M \circ \phi$ (respectively $h \circ \phi \in \mathfrak{m}_1(M \circ \phi)$), and so $\psi(\tau)(h \circ \phi) \in I_\psi(M \circ \phi)$. Note that for a fixed τ, the form $\psi(\tau)$ is linear and so we have the equality $I_\psi(\mathfrak{m}_1(M \circ \phi)) = \mathfrak{m}_1 I_\psi(M \circ \phi)$ which proves the necessity for the strict dependence, that is $\psi(\tau)(h \circ \phi) \in \mathfrak{m}_1 I_\psi(M \circ \phi)$.

$\Leftarrow$]
Given any $\phi : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}, \mathfrak{X} \setminus W, 0)$, the module $\phi^*(M) := M \circ \phi$ is a submodule of the free module O_1^p over the principal ideal domain $\mathbb{C}\{\tau\}$ and so by [Sha90, Thm. 10.5, pg 189] there exists a basis $\{\overrightarrow{f_i}\}_{i=1}^p$ for O_1^p, such that:

$$\phi^*(M) = \mathbb{C}\{\tau\}\tau^{n_1}\overrightarrow{f_1} + \cdots + \mathbb{C}\{\tau\}\tau^{n_p}\overrightarrow{f_p}$$

with $n_1 \leq n_2 \leq \cdots \leq n_p$ and some n_j's may be ∞, that is by convention $\tau^{n_j} = 0$. Let us consider the expresion of $\phi^*(h) := h \circ \phi$ in this basis, say $\phi^*(h) = a_1\overrightarrow{f_1} + \cdots + a_p\overrightarrow{f_p}$. Then $\phi^*(h)$ belongs to $\phi^*(M)$ (respectively $\mathfrak{m}_1\phi^*(M)$) if and only if $a_j = b_j\tau^{n_j}$ (respectively $a_j = b_j\tau^{n_j+1}$) for a suitable $b_j \in \mathbb{C}\{\tau\}$ for $1 \leq j \leq p$.

Let $\{\overrightarrow{df_1}, \ldots, \overrightarrow{df_p}\}$ be the dual basis of the dual module $\mathrm{Hom}(O_1^p, \mathbb{C}\{\tau\})$, and define $\psi_{\overrightarrow{df_j}}$ as the morphism $\tau \mapsto \overrightarrow{df_j}$. Then, the ideal $I_{\psi_{\overrightarrow{df_j}}}(\phi^*(M))$ is equal to $\langle \tau^{n_j} \rangle O_1$ (respectively $\mathfrak{m}_1 I_{\psi_{\overrightarrow{df_j}}}(\phi^*(M)) = \langle \tau^{n_j+1} \rangle O_1$) and $\psi_{\overrightarrow{df_j}}(\phi^*(h)) = a_j$.

Hence $\phi^*(h) \in \phi^*(M)$ (respectively $\mathfrak{m}_1\phi^*(M)$) if and only if $\psi_{\overrightarrow{df_j}}(\phi^*(h))$ belongs to the ideal $I_{\psi_{\overrightarrow{df_j}}}(\phi^*(M))$ (respectively $\mathfrak{m}_1 I_{\psi_{\overrightarrow{df_j}}}(\phi^*(M))$) for the p maps $\psi_{\overrightarrow{df_j}}$, and for each of these we have $\lambda \neq 0$. It is enough to consider the maps missing the closed analytic subset W thanks to lemma 3.5. $\square$

The previous lemma directs us to work with the space $\mathfrak{X} \times \check{\mathbb{C}}^p$, or even with the space $\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$ since we ask that the image of ψ does not contain the point 0 in $\check{\mathbb{C}}^p$. These spaces can be seen respectively as the analytic spectrum (analytic proj) of the symmetric algebra of $O_{\mathfrak{X}}^p$, that is $O_{\mathfrak{X}}[u_1, \ldots, u_p]$. The section $h \in O_{\mathfrak{X}}^p$ and the submodule $M \subset O_{\mathfrak{X}}^p$ generate ideals in $O_{\mathfrak{X}}[u_1, \ldots, u_p]$ which we will denote by $\rho(h)$ and $\rho(M)$.

Remark 3.7. *Recall that the embedding of $O_{\mathfrak{X}}^p$ in $O_{\mathfrak{X}}[u_1, \ldots, u_p]$ is in degree 1, and is given by*

$$h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{pmatrix} \mapsto \rho(h) = u_1h_1 + \cdots + u_ph_p$$

The next result translates the notion of integral dependence of h on M at 0 on the germ $(\mathfrak{X}, 0)$, to the integral dependence of $\rho(h)$ on the ideal $\rho(M)$ on the more global space $\mathfrak{X} \times \check{\mathbb{C}}^p$ ($\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$).

Proposition 3.8. *[GK99, Prop. 3.4, pg 558] Let $Z \subset \mathfrak{X} \times \check{\mathbb{P}}^{p-1}$ be the analytic space defined by the ideal $\rho(M)$. Then, we have a canonical map $Z \to \mathfrak{X}$ induced by the projection $\mathfrak{X} \times \check{\mathbb{P}}^{p-1} \to \mathfrak{X}$. The section $h \in O_{\mathfrak{X}}^p$ is integrally dependent (resp. strictly dependent) on M at 0 if and only if for each point of Z lying over $0 \in \mathfrak{X}$ each generator of $\rho(h)$ is integrally dependent (resp. strictly dependent) on $\rho(M)$.*

Proof. Note that for every $h \in O_{\mathfrak{X}}^p$, its image $\rho(h) = u_1h_1 + \cdots u_ph_p$ in the symmetric algebra $O_{\mathfrak{X}}[u_1, \ldots, u_p]$ defines an analytic function, homogeneous in the u_i's, on $\mathfrak{X} \times \check{\mathbb{C}}^p$ provided we start with a small enough representative of $(\mathfrak{X}, 0)$. This implies, by the analytic proj construction (see [HIO88, Def. 1.2.8, pg 567]), that the u_i homogeneous ideal $\langle\rho(M)\rangle$ defines a closed complex subspace of $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$. Moreover, for any point $q=(0,[\lambda])$ in the analytic space $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$, its local ring is isomorphic to the ring $O_{\mathfrak{X},0}\{u_1/u_p,\ldots,u_{p-1}/u_p\}$ (after a possible reordering of the coordinates) and the ideal induced by $\langle\rho(M)\rangle$ is a proper ideal if and only if $q\in Z$.

The key point is that to give a map $\Xi:(\mathbb{C},0)\to(\mathfrak{X}\times\check{\mathbb{P}}^{p-1},q)$ is the same as to give a pair of maps $\phi:(\mathbb{C},0)\to(\mathfrak{X},0)$ and $\psi:(\mathbb{C},0)\to(\check{\mathbb{C}}^p,l)$, where l corresponds to $[\lambda]$ and ψ is determined only up to multiplication by a function that doesn't vanish at $0\in\mathbb{C}$. Moreover, we have that $\Xi^*(h)=\psi(\tau)(h\circ\phi)$ and $\Xi^*(\rho(M))O_1=I_\psi(M\circ\phi)$. Lemma 3.6 finishes the proof. □

As a corollary we can derive the analogue of the growth condition for integral dependence on ideals given in theorem C.3-5.

Corollary 3.9. *([Gaf92, Proposition 1.11, pg 306]) The section h is integrally dependent on M at 0 if and only if for each choice of generators $\{m_i\}$ of M there exists a neighborhood U of 0 in $\mathfrak{X}$, and a real constant C, such that for every section $\Psi:\mathfrak{X}\to\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ of the trivial bundle $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ and every point $z\in U$ we have:*

$$|\Psi(z)\cdot h(z)|\leq C\sup_i|\Psi(z)\cdot m_i(z)|$$

Proof. By proposition 3.8 and its proof we can see that the section h is integrally dependent on M at 0 if and only if $\rho(h)$ is integrally dependent on $\rho(M)$ at every point of $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ lying over $0\in\mathfrak{X}$. Now, if the set $\{m_i\}$ generates M, then the set $\{\rho(m_i)\}$ generates $\rho(M)$ and so by theorem C.3-5 we have that for every point $\{(0,[\lambda]\}\in\{0\}\times\check{\mathbb{P}}^{p-1}$ there exists a neighborhood U_λ, and a constant C_λ upon which

$$|\rho(h)|\leq C_\lambda\sup_i|\rho(m_i)|$$

that is:

$$|u_1h_1(z)+\ldots u_ph_p(z)|\leq C_\lambda\sup_i|u_1m_{i1}(z)+\ldots+u_pm_{ip}(z)|$$

for every point $(z,[u])\in U_\lambda$. The compactness of $\{0\}\times\check{\mathbb{P}}^{p-1}$ allows us to reduce this to a finite number of open neighborhoods and constants, $(U_{\lambda_1},C_{\lambda_1}),\ldots,(U_{\lambda_r},C_{\lambda_r})$. Since the map $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}\to\mathfrak{X}$ is open, defining U as the intersection of the images of the U_{λ_j}'s in $\mathfrak{X}$ and C as the maximum of the C_{λ_j}'s finishes the proof. □

This proposition allows us to generalize theorems C.3 and C.4 concerning integral dependence on ideals to the case of modules. For this, we will consider the normalized blowup of $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ along the subspace Z defined by the ideal $\rho(M)O_{\mathfrak{X}}[u_1,\ldots,u_p]$ which we will denote by

$$\pi:\overline{E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{p-1})}\to\mathfrak{X}\times\check{\mathbb{P}}^{p-1}\to\mathfrak{X}$$

Its exceptional divisor will be denoted by F

Proposition 3.10. *[GK99, Prop. 3.5, pg 558] Let $h\in\mathcal{E}$, and let Y be a closed analytic subset of the image of F in $\mathfrak{X}$. Then:*

1. *h is integrally dependent on M at 0 if and only if along each component of F, the ideal $\rho(h) \circ \pi$ vanishes to order at least the order of vanishing of $\rho(M) \circ \pi$.*
2. *h is strictly dependent on M at every $y \in Y$ if and only if along each component V of F, the ideal $\rho(h) \circ \pi$ lies in the product $I(Y,V)\rho(M) \circ \pi$, where $I(Y,V)$ denotes the ideal of the reduced preimage of Y in V.*

Proof. By taking a small enough neighborhood of 0 in X, we way assume that each component of F meets the fiber over 0.

By proposition 3.8 h is integrally dependent on M at 0 if and only if for each point of Z lying over $0 \in \mathfrak{X}$ each generator of $\rho(h)$ is integrally dependent on the ideal $\rho(M)$. But by theorem C.3, this is equivalent to $\overline{\nu}_{\rho(M)}(\rho(h)) \geq 1$, and by theorem C.4 this is equivalent to having the ideal $\rho(h) \circ \pi$ vanishing to order at least the order of vanishing of $\rho(M) \circ \pi$ along each component of F, which proves 1).

For each point $p \in F$, there is a neighborhood U_p of p in $\overline{E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})}$, such that the ideal $\langle \rho(M) \circ \pi \rangle \overline{E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})}$ is generated by a single section $\rho(g) \circ \pi$, where g is a suitable section of M. In this neighborhood we can define the meromorphic function $k := \frac{\rho(h)\circ\pi}{\rho(g)\circ\pi}$, and so $\rho(h) \circ \pi = k(\rho(g) \circ \pi)$. In these terms, the condition in 2) is equivalent to saying that the function k is holomorphic and vanishes at every point p lying over Y. Note that again by theorem C.3, the condition of the integral dependence of $\rho(h)$ on the ideal $\rho(M)$ is also equivalent to having $\rho(h) \circ \pi \in \langle \rho(M) \circ \pi \rangle \overline{E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})}$, or in this terms equivalent to k being a holomorphic function.

Suppose $\rho(h) \in \rho(M)^\dagger$ for every point in $\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$ lying over Y, in particular $\rho(h)$ is integrally dependent on $\rho(M)$ and the function k is holomorphic in a neighborhood of any point $b \in F$ lying over Y. If k does not vanish in b, then it doesn't vanish in neighborhood of b and so for every arc $\psi : (\mathbb{C}, 0) \to \left(\overline{E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})}, b\right)$, the germ $k \circ \psi$ is a unit in $\mathbb{C}\{\tau\}$, that is $(\rho(h) \circ \pi) \circ \psi \notin \mathfrak{m}_1 (\rho(M) \circ \pi \circ \psi) \mathbb{C}\{\tau\}$. The arc $\phi := \pi \circ \psi$ then gives a contradiction.

Consider an arc $\phi : (\mathbb{C}, 0) \to (\mathfrak{X} \times \check{\mathbb{P}}^{p-1}, (z, [\lambda]))$ such that the image of $\mathbb{C} \setminus \{0\}$ does not intersect the proper analytic subset $Z \cup \mathrm{Sing}\,(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$. By lemma 3.5, to check for strict dependence it is enough to check it for these arcs, and the advantage is that they have a unique lift ψ to $\overline{E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})}$. Now suppose k is holomorphic and vanishes at every point $b \in F$ over Y. Then $k \in I(Y,V)$ and so $k \circ \psi \in \mathfrak{m}_1 \mathbb{C}\{\tau\}$. But $\rho(h) \circ \pi = k(\rho(g) \circ \pi)$, and by definition $(\rho(g) \circ \pi) \circ \psi$ generates $(\rho(M) \circ \pi \circ \psi)\mathbb{C}\{\tau\}$. This implies $(\rho(h) \circ \pi) \circ \psi \in \mathfrak{m}_1 (\rho(M) \circ \pi \circ \psi) \mathbb{C}\{\tau\}$, and since by definition the arc ϕ is equal to $\pi \circ \psi$ we have finished the proof. □

3.1 Limits of tangent spaces and Whitney conditions

Going back to our subject of study let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a representative of a reduced germ of analytic singularity of pure dimension d, defined by the ideal $\langle f_1, \ldots, f_p \rangle \mathbb{C}\{z_0, \ldots z_n\}$, where the generators f_i are chosen in such a way that their initial forms $\{\mathrm{in} f_1, \ldots, \mathrm{in} f_p\}$ generate the ideal of initial forms defining the tangent cone $C_{X,0}$. Then, the germ $(\mathfrak{X}, 0) \subset (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ of the specialisation space of X to its tangent cone is defined by the ideal $\langle F_1, \ldots, F_p \rangle \, \mathbb{C}\{z_0, \ldots, z_n, t\}$. In other words, $(\mathfrak{X}, 0) = (F^{-1}(0), 0)$ where $F = (F_1, \ldots, F_p) : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^p, 0)$.

Definition 3.11. *Define the **Jacobian module of** F as the submodule $JM(F)$ of $O^p_{\mathfrak{X}}$ generated by all the partial derivatives of F, that is:*

$$JM(F) := O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_1}{\partial z_0} \\ \vdots \\ \frac{\partial F_p}{\partial z_0} \end{pmatrix} + \cdots + O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_1}{\partial z_n} \\ \vdots \\ \frac{\partial F_p}{\partial z_n} \end{pmatrix} + O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_1}{\partial t} \\ \vdots \\ \frac{\partial F_p}{\partial t} \end{pmatrix} \subset O^p_{\mathfrak{X}}$$

Let v be a vector in $\mathbb{C}^{n+1} \times \mathbb{C}$, then by $\frac{\partial F}{\partial v}$ we mean the directional derivative of F with respect to v. That is:

$$\frac{\partial F}{\partial v} := [DF]\,(v)$$

where $[DF]$ denotes the total derivative of F, or equivalently the jacobian matrix of F. In particular $\frac{\partial F}{\partial v}$ is a linear combination of the columns of F and so it belongs to the jacobian module $JM(F)$.

Definition 3.12. *Given an analytic map germ $g : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^l, 0)$, let $JM_g(F)$ denote the submodule of $JM(F)$ generated by the "partials" $\frac{\partial F}{\partial v}$ for all vector fields v on $\mathbb{C}^{n+1} \times \mathbb{C}$ tangent to the fibers of g, that is, for all v that map to the $0-$field on $\mathbb{C}^l$. Call $JM_g(F)$ the **Relative Jacobian Module** with respect to g.*

For example, if g is the projection onto the space of the last l variables of $\mathbb{C}^{n+1} \times \mathbb{C}$, then $JM_g(F)$ is simply the submodule generated by all the partial derivatives of F with respect to the first $n+2-l$ variables. That is:

$$JM_g(F) := O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_1}{\partial z_0} \\ \vdots \\ \frac{\partial F_p}{\partial z_0} \end{pmatrix} + \cdots + O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_1}{\partial z_{n+2-l}} \\ \vdots \\ \frac{\partial F_p}{\partial z_{n+2-l}} \end{pmatrix}$$

Note that if H is a hyperplane in $\mathbb{C}^{n+1} \times \mathbb{C}$ defined by the linear map $h : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$, then $JM_h(F)$ is the submodule of $JM(F)$ generated by the partials $\frac{\partial F}{\partial v}$ for all vectors $v \in H$.

Remember that our objective is to Whitney stratify the analytic space $(\mathfrak{X}, 0)$ in such a way that the zero section $(Y, 0)$ is a stratum. For that we need to control the set of limits of tangent spaces to $(\mathfrak{X}, 0)$ and for that purpose we will use the jacobian module $JM(F)$ together with the following proposition (3.13).

Proposition 3.13. *Let $E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1}) \subset \mathfrak{X} \times \check{\mathbb{P}}^{p-1} \times \check{\mathbb{P}}^{n+1}$ be the blowup of $\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$ along the subspace Z defined by the ideal $\rho(JM(F))O_{\mathfrak{X}}[u_1, \ldots, u_p]$. Then, there exists a surjective map $\eta : E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1}) \to C(\mathfrak{X})$, making the following diagram* *commutative:*

$$\begin{tikzcd}
E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{p-1}) \arrow[r, "\eta"] \arrow[d, "e_Z"'] & C(\mathfrak{X}) \arrow[d, "\kappa_{\mathfrak{X}}"'] \\
\mathfrak{X}\times\check{\mathbb{P}}^{p-1} \arrow[r] & \mathfrak{X} \arrow[d, "\varphi"'] \\
 & \mathbb{C}
\end{tikzcd}$$

Proof. Any "sufficiently general" point $(z,t,\vec{u})$ in $\mathfrak{X}\times\check{\mathbb{C}}^p$ defines a tangent hyperplane to $\mathfrak{X}$ at the point (z,t), that is a point in the conormal space $C(\mathfrak{X})$ in the following way. Let $\overrightarrow{dF_j}(z,t)$ denote the vector $(\frac{\partial F_j}{\partial z_0}(z,t),\ldots,\frac{\partial F_j}{\partial t}(z,t))\in\mathbb{C}^{n+1}\times\mathbb{C}$, and let (z,t) be a smooth point of $\mathfrak{X}$. If the vector $\overrightarrow{dF_j}(z,t)$ is not zero, then it defines a tangent hyperplane $[\overrightarrow{dF_j}(z,t)]\in\check{\mathbb{P}}^{n+1}$ to $\mathfrak{X}$ at (z,t). Moreover, we can find $n+1-d$ functions among the F_j's, $i=1,\ldots,p$, such that the tangent space $T_{(z,t)}\mathfrak{X}$ is obtained as the intersection of the $n+1-d$ hyperplanes $[\overrightarrow{dF_j}(z,t)]$. This implies that any tangent hyperplane $H=[a:b]$ to $\mathfrak{X}$ at (z,t) can be written as a linear combination of these $n+1-d$ hyperplanes $H=[\sum\beta_j\overrightarrow{dF_j}(z,t)]$, that is, they generate the fiber $\kappa_{\mathfrak{X}}^{-1}(z,t)$ over (z,t) in the conormal space $C(\mathfrak{X})$. So for any point $(z,t,u)\in\mathfrak{X}\times\check{\mathbb{C}}^p$ with $(z,t)\in\mathfrak{X}$ smooth, we can define the map

$$(z,t,u)\in\mathfrak{X}\times\check{\mathbb{C}}^p\mapsto(z,t),\left[\sum_{i=1}^p u_i\overrightarrow{dF_i}(z,t)\right]\in C(\mathfrak{X})\subset\mathfrak{X}\times\check{\mathbb{P}}^{n+1}$$

as long as the linear combination given by the point $u\in\check{\mathbb{C}}^p$ is not zero. Note that this map is invariant with respect to the homotheties of $\check{\mathbb{C}}^p$, so it defines a map in $\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$

On the other hand, from definition 3.11 and remark 3.7, we get that the ideal $\rho(JM(F))$ has the following system of homogeneous generators:

$$\rho(JM(F))=\left\langle u_1\frac{\partial F_1}{\partial z_0}+\cdots+u_p\frac{\partial F_p}{\partial z_0},\ldots,u_1\frac{\partial F_1}{\partial t}+\cdots+u_p\frac{\partial F_p}{\partial t}\right\rangle O_{\mathfrak{X}}[u_1,\ldots,u_p]$$

and so a point $(z,t,[u])\in\mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ is in Z if and only if

$$u_1\overrightarrow{dF_1}(z,t)+\cdots+u_p\overrightarrow{dF_p}(z,t)=\overrightarrow{0}$$

that is, Z is the set of points where the previously stated map

$$(z,t,[u])\in\mathfrak{X}\times\check{\mathbb{P}}^{p-1}\mapsto(z,t),\left[\sum_{i=1}^p u_i\overrightarrow{dF_i}(z,t)\right]\in C(\mathfrak{X})\subset\mathfrak{X}\times\check{\mathbb{P}}^{n+1}$$

is not defined. Thus, by blowing up the space Z in this set of coordinates, we obtain the space $E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{p-1})\subset\mathfrak{X}\times\check{\mathbb{P}}^{p-1}\times\check{\mathbb{P}}^{n+1}$ upon which the morphism

$$\eta:E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{p-1})\to C(\mathfrak{X})$$

is defined by the restriction to $E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$ of the projection $\mathfrak{X} \times \check{\mathbb{P}}^{p-1} \times \check{\mathbb{P}}^{n+1} \to \mathfrak{X} \times \check{\mathbb{P}}^{n+1}$. Moreover, since for any smooth point $(z,t) \in \mathfrak{X}$ and tangent hyperplane $H \in \kappa_{\mathfrak{X}}^{-1}(z,t)$ there exists a $[u] \in \check{\mathbb{P}}^{p-1}$ such that the point $(z,t,[u]) \notin Z$ and the point $(z,t,[u],H) \in E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$, then the morphism η is surjective. $\square$

Lemma 3.14. *([Gaf97, Lemma 2.1, pg 58]) Let $(\mathfrak{X}, 0) \subset (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ be defined by $F^{-1}(0)$ as before. Then a hyperplane $H = [a_0 : \cdots : a_n : b] \in \check{\mathbb{P}}^{n+1}$ is a limit of tangent hyperplanes to $(\mathfrak{X}, 0)$ if and only if there exists a pair of maps $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}^0, 0)$ and $\psi : (\mathbb{C}, 0) \to (\check{\mathbb{C}}^p, \lambda \neq 0)$ such that the point $(\phi(\tau), \psi(\tau)) \notin Z \subset \mathfrak{X} \times \check{\mathbb{C}}^p$ and for some k*

$$(a_0, \ldots, a_n, b) = \lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))}{\tau^k}$$

Proof. First of all, note that $DF(\phi(\tau))$ corresponds to the total derivative of F at the point $\phi(\tau) \in \mathfrak{X}^0$. Then $\psi(\tau)DF(\phi(\tau))$ gives us a linear combination of the rows of the jacobian matrix $[DF(\phi(\tau))]$ and so it gives us a linear functional defining a tangent hyperplane to $\mathfrak{X}^0$ at the point $\phi(\tau)$.

Take a sequence $\{(z_m, t_m, H_m)\}$ of points in the conormal space $C(\mathfrak{X})$, lying over the smooth part $\mathfrak{X}^0$ and tending to the point $(0,0,H) \in \kappa_{\mathfrak{X}}^{-1}(0)$. Then, by 3.13, for each $m \in \mathbb{N}$ there exists a point $[u] = [u_{m,1} : \cdots : u_{m,p}] \in \check{\mathbb{P}}^{p-1}$ such that $H_m = [\sum u_{m,i} \overrightarrow{dF_i}(z_m, t_m)]$, and from this we can obtain a convergent subsequence $(z_m, t_m, [u_m], H_m)$ in $E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$ tending to the point $(0,0,[u],H)$ for some $[u] \in \check{\mathbb{P}}^{p-1}$. By the curve selection lemma there exists a map

$$\Theta : (\mathbb{C}, 0) \to (E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1}), (0,0,[u],H))$$

lying over $\mathfrak{X}^0 \times \check{\mathbb{P}}^{p-1} \setminus Z$, such that for all $\tau \neq 0$, we have

$$\Theta(\tau) = (\phi(\tau), \psi(\tau), [\psi(\tau) DF(\phi(\tau))])$$

and

$$\Theta(0) = (\phi(0), \psi(0), \left[\lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))}{\tau^k}\right])$$

from which, upon composing with the blowing up map e_Z, we obtain the morphism

$$(\phi, \psi) : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}^0 \times \check{\mathbb{P}}^{p-1}, \mathfrak{X}^0 \times \check{\mathbb{P}}^{p-1} \setminus Z, (0,0,[u]))$$

which is equivalent to the maps we were looking for.

In the other direction, the existence of the maps ϕ, ψ gives us a map

$$(\phi, \psi) : (\mathbb{C}, 0) \to (\mathfrak{X} \times \check{\mathbb{C}}^p \setminus Z, (0, \lambda \neq 0))$$

which then, by 3.13, lifts to the blow up $\Theta : (\mathbb{C}, 0) \to E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$, in such a way that for each $\tau \neq 0$ the point $\Theta(\tau) = \left(\phi(\tau), \psi(\tau), [\sum \psi_i(\tau) \overrightarrow{dF_i}(\phi(\tau))]\right)$ gives us a tangent hyperplane to $\mathfrak{X}^0$ at the point $\phi(\tau)$. Then, the surjective map $\eta : E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1}) \to C(\mathfrak{X})$ of 3.13, sends the point

$$\Theta(0) = \left(0, \lambda, \left[\lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))}{\tau^k}\right]\right) = (0, \lambda, [a : b])$$

to the point $(0,[a:b]) \in \kappa_{\mathfrak{X}}^{-1}(0)$. That, is $H \in \kappa_X^{-1}(0)$.

$\square$

Corollary 3.15. *Let $\varphi : (\mathfrak{X},0) \to \mathbb{C}$ denote the specialisation of $(X,0)$ to its tangent cone $(C_{X,0},0)$. The hyperplane $\{t=0\}$ is not a limit of tangent hyperplanes to $\mathfrak{X}$ at (z,t) if and only if $\frac{\partial F}{\partial t} \in \overline{JM_\varphi(F)}$ in $O_{\mathfrak{X},(z,t)}$.*

Proof. From lemma 3.14, the hyperplane $\{t=0\}$ is a limit of tangent hyperplanes if and only if there exists a pair of maps $\phi : (\mathbb{C},0) \to (\mathfrak{X}^0,0)$ and $\psi : (\mathbb{C},0) \to (\check{\mathbb{C}}^p, \lambda \neq 0)$ such that the point $(\phi(\tau),\psi(\tau)) \notin Z \subset \mathfrak{X} \times \check{\mathbb{C}}^p$ and for some k

$$(0,\ldots,0,\alpha) = \lim_{\tau\to 0} \frac{\psi(\tau)DF(\phi(\tau))}{\tau^k}$$

But we can see that $\psi(\tau)DF(\phi(\tau))$ is equal to

$$\left(\rho\left(\frac{\partial F}{\partial z_0}\right)(\phi(\tau),\psi(\tau)),\ldots,\rho\left(\frac{\partial F}{\partial z_n}\right)(\phi(\tau),\psi(\tau)),\rho\left(\frac{\partial F}{\partial t}\right)(\phi(\tau),\psi(\tau))\right)$$

and so, if we denote by $\mathrm{ord}_0\gamma(\tau)$ the order of the series $\gamma(\tau)$ in $\mathbb{C}\{\tau\}$, the limit condition tells us that

$$\mathrm{ord}_0\rho\left(\frac{\partial F}{\partial t}\right)(\phi(\tau),\psi(\tau)) < \mathrm{ord}_0\rho\left(\frac{\partial F}{\partial z_j}\right)(\phi(\tau),\psi(\tau)), \text{ for } j=0,\ldots,n$$

This implies that for every $C \in \mathbb{R}$ there exists an $\epsilon \in \mathbb{R}$ such that for every $|\tau| < \epsilon$ we have that $|\rho\left(\frac{\partial F}{\partial t}\right)(\phi(\tau),\psi(\tau))| > C|\rho\left(\frac{\partial F}{\partial z_j}\right)(\phi(\tau),\psi(\tau))|$. Corollary 3.9 finishes the proof. $\square$

Remark 3.16. *From the proof of 3.14 we can see that any pair of maps $(\phi,\psi) : (\mathbb{C},0) \to \left(\mathfrak{X}\times\check{\mathbb{C}}^p,(0,\lambda\neq 0)\right)$ whose image does not intersect the space Z determines a limit of tangent hyperplanes to $(\mathfrak{X},0)$, just by lifting this map to the blown up space $E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{p-1})$.*

Proposition 3.17. *([Gaf97, Thm 2.2, pg 58] or [GK99, Lemma 4.1, pg 560]) Let $H \in \check{\mathbb{P}}^{n+1}$ be a hyperplane defined by the linear map $h : \mathbb{C}^{n+1}\times\mathbb{C} \to \mathbb{C}$. Then H is a limit of tangent hyperplanes to $(\mathfrak{X},0)$ if and only if the module $JM_h(F)$ is not a reduction of $JM(F)$.*

Proof. Let $H \in \check{\mathbb{P}}^{n+1}$, then by 3.14 H is a limit of tangent hyperplanes to $(\mathfrak{X},0)$ if and only if there exists a pair of maps $\phi : (\mathbb{C},0) \to (\mathfrak{X}^0,0)$ and $\psi : (\mathbb{C},0) \to (\check{\mathbb{C}}^p,\lambda\neq 0)$ such that the point $(\phi(\tau),[\psi(\tau)]) \notin Z \subset \mathfrak{X}\times\check{\mathbb{P}}^{p-1}$ and for some $k \geq 0$

$$(a_0,\ldots,a_n,b) = \lim_{\tau\to 0}\frac{\psi(\tau)DF(\phi(\tau))}{\tau^k}$$

But $\psi(\tau)DF(\phi(\tau) = (\tau^{r_0}w_0,\ldots,\tau^{r_{n+1}}w_{n+1}))$, and the limit condition tells us that $k = \min\{r_j \mid \tau_j^r w_j \neq 0\}$. Moreover, by definition the ideal $I_\psi(JM(F)\circ\phi)$ is the ideal in $\mathbb{C}\{\tau\}$ generated by applying $\psi(\tau)$ to the generators of $JMF\circ\phi$. That is :

$$I_\psi(JM(F)\circ\phi) = \langle\tau^{r_0}w_0,\ldots,\tau^{r_{n+1}}w_{n+1}\rangle = \left\langle\tau^k\right\rangle\mathbb{C}\{\tau\}$$

Recall, that the relative jacobian module $(JM_h(F) \circ \phi) \subset (JM(F) \circ \phi)$ is generated by the partials $\frac{\partial F}{\partial v} \circ \phi$ for all vectors $v \in H$. But $\frac{\partial F}{\partial v} \circ \phi = DF(\phi(\tau))(v)$, and since $v \in H$ we have that:

$$0 = \sum_0^n a_j v_j + b v_{n+1} = \lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))(v)}{\tau^k}$$

and since

$$\psi(\tau) DF(\phi(\tau))(v) = v_0 \tau^{r_0} w_0 \ + \cdots + v_{n+1} \tau^{r_{n+1}} w_{n+1} = \tau^r \tilde{w}$$

with $\tilde{w} \in \mathbb{C}\{\tau\}$ a unit, the limit being 0 implies that $r > k$. Translating this into ideals, we have that the ideal $I_\psi(JM_h(F) \circ \phi)$ is strictly contained in the ideal $I_\psi(JM(F) \circ \phi)$, so by 3.6 we have that $JM_h(F)$ is not a reduction of $JM(F)$.

In the other direction, if $JM_h(F)$ is not a reduction of $JM(F)$, then by 3.6 there exists a pair of maps ϕ, ψ as before, whose image in $\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$ misses Z, such that the ideal $I_\psi(JM_h(F) \circ \phi)$ is properly contained in, but not equal to the ideal $I_\psi(JM(F) \circ \phi) =< \tau^k >$. This means, that for any $v \in H$, if we apply ψ to the corresponding partial we obtain $\psi(\tau)(\frac{\partial F}{\partial v}) \circ \phi = \tau^r w$, with $w \in \mathbb{C}\{\tau\}$ a unit and $r > k$. But $\psi(\tau)(\frac{\partial F}{\partial v}) \circ \phi = \psi(\tau) DF(\phi(\tau)(v)$, so we get that

$$\lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))(v)}{\tau^k} = 0$$

Since this is true for all $v \in H$, then necessarily

$$\lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))}{\tau^k} = (a_0, \ldots, a_n, b)$$

and the result follows from 3.14. □

Theorem 3.18. *([Gaf97, Cor 2.4, pg 60] or [GK99, lemma 4.1, pg 560])*
Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an equidimensional, reduced, germ of analytic singularity as before, $X = f^{-1}(0)$, and let $(S, 0) \subset (X, 0)$ be a smooth subspace defined as the zero set of the analytic function $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^l, 0)$. Then the pair (X^0, S) satisfies Whitney's condition a) at the origin if and only if the module $JM_g(f)$ is contained in $JM(f)^\dagger$.

Proof. Let v be a vector of the tangent space $T_0 S$, and let H be a limiting tangent hyperplane to X at 0. Then Whitney condition $a)$ tells us that $v \in H$ and so by 3.14 we have that:

$$H = [a_0 : \cdots : a_n] = \lim_{\tau \to 0} \frac{\psi(\tau) DF(\phi(\tau))}{\tau^k}$$

Now, following the proof of proposition 3.17 we get that the ideal $I_\psi(JM(f) \circ \phi) = \left\langle \tau^k \right\rangle \mathbb{C}\{\tau\}$, and since $v \in H$ we have that

$$\lim_{\tau \to 0} \frac{\psi(\tau) Df(\phi(\tau))(v)}{\tau^k} = \sum_0^n a_j v_j = 0$$

But $Df(\phi(\tau))(v) = \frac{\partial f}{\partial v} \circ \phi$, so $\psi(\tau)(\frac{\partial f}{\partial v} \circ \phi) = \tau^r w$, with $w \in \mathbb{C}\{\tau\}$ a unit and $r > k$, that is

$$\psi(\tau)\left(\frac{\partial f}{\partial v} \circ \phi\right) \in \mathfrak{m}_1 I_\psi(JM(f) \circ \phi)$$

Since this happens for all $v \in T_0S$ and all limits of tangent hyperplanes H, which is equivalent by remark 3.16 to a statement for every pair of maps (ϕ, ψ) whose image in $X \times \check{\mathbb{P}}^{p-1}$ does not intersect Z, then by lemma 3.6 we have that $\frac{\partial f}{\partial v} \in JM(f)^\dagger$. Finally, since $JM_g(f) = \left\langle \frac{\partial f}{\partial v} \mid v \in T_0S \right\rangle$ then $JM_g(f) \subset JM(f)^\dagger$. In the other direction all we need to do is follow the proof backwards. □

Remark 3.19. *In our situation, if we look at the hypersurface case ($p = 1$) we have that the tangent plane at a point $(z,t) \in \mathfrak{X}^0$ to the specialisation space $\mathfrak{X}$, is represented in the dual projective space by the point $[\frac{\partial F}{\partial z_0}(z,t) : \cdots : \frac{\partial F}{\partial z_n}(z,t) : \frac{\partial F}{\partial t}(z,t)]$ and so the fact that $\frac{\partial F}{\partial t}$ is strictly dependent on the relative jacobian ideal $J_\varphi = \left\langle \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n} \right\rangle$ tells us that the function $\frac{\partial F}{\partial t}$ tends to 0 faster than the rest of the $\frac{\partial F}{\partial z_j}$ when approaching the origin. In particular this tells us that all the limits of tangent hyperplanes to $(\mathfrak{X}, 0)$ are of the form $[a_0 : \cdots : a_n : 0]$ and so to the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*

In the general case, we have that a tangent hyperplane at a point $(z,t) \in \mathfrak{X}^0$ is given by the point:

$$[\sum_{i=1}^{p} u_i \frac{\partial F_i}{\partial z_0}(z,t) : \cdots : \sum_{i=1}^{p} u_i \frac{\partial F_i}{\partial z_n}(z,t) : \sum_{i=1}^{p} u_i \frac{\partial F_i}{\partial t}(z,t)] \in \check{\mathbb{P}}^{n+1}$$

for a suitable $(u_1, \ldots, u_p) \in \check{\mathbb{C}}^p(\check{\mathbb{P}}^{p-1})$. Thus, in order to have the pair $(\mathfrak{X}^0, Y)$ satisfy Whitney's condition a) at the origin we need to prove that the function $\sum_{i=1}^{p} u_i \frac{\partial F_i}{\partial t}(z,t)$ tends faster to 0 than all of the $\sum_{i=1}^{p} u_i \frac{\partial F_i}{\partial z_j}(z,t)$.

Corollary 3.20. *In the same setup of 3.18, let the smooth subspace $(S, 0) \subset (X, 0)$ be linear and defined by the projection $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^l, 0)$ onto the first l coordinates. If $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1-l}, 0)$ denotes the retraction over $(S, 0)$, that is the projection onto the last $n + 1 - l$ coordinates, then the pair (X^0, S) satisfies Whitney's condition a) at the origin if and only if the module $JM_g(f)$ is contained in $JM_h(f)^\dagger$.*

Proof. Recall that

$$JM(f) = \left\langle \left(\frac{\partial f}{\partial z_0}\right), \cdots, \left(\frac{\partial f}{\partial z_n}\right) \right\rangle O_X^p$$

where $\left(\frac{\partial f}{\partial z_j}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial z_j} \\ \vdots \\ \frac{\partial f_p}{\partial z_j} \end{pmatrix}$. Then, according to definition 3.12 we have that:

$$JM_g(f) = \left\langle \left(\frac{\partial f}{\partial z_l}\right), \cdots, \left(\frac{\partial f}{\partial z_n}\right) \right\rangle O_X^p$$

and

$$JM_h(f) = \left\langle \left(\frac{\partial f}{\partial z_0}\right), \cdots, \left(\frac{\partial f}{\partial z_{l-1}}\right) \right\rangle O_X^p$$

Now, by definition, for a fixed map $(\phi, \psi) : (\mathbb{C}, 0) \to (X \times \check{\mathbb{C}}^p, 0)$, with $(\phi(\tau), [\psi(\tau)]) \notin Z$ for $\tau \neq 0$, we have that the ideal:

$$\begin{aligned} I_\psi(JM(f) \circ \phi) &= \left\langle \psi(\tau)\left(\frac{\partial f}{\partial z_0} \circ \phi\right), \cdots, \psi(\tau)\left(\frac{\partial f}{\partial z_n} \circ \phi\right) \right\rangle \mathbb{C}\{\tau\} \\ &= \langle \tau^{r_0} w_0, \ldots, \tau^{r_n} w_n \rangle \, \mathbb{C}\{\tau\}, \text{ with } w_j \in \mathbb{C}\{\tau\} \text{ unit} \\ &= \left\langle \tau^k \right\rangle \mathbb{C}\{\tau\} \end{aligned}$$

But, by theorem 3.18 we know that the pair (X^0, S) satisfies Whitney's condition a) at the origin if and only if $JM_g(f) \subset JM(f)^\dagger$. That is, for $j = l, \ldots, n$ we have that

$$\psi(\tau)\left(\frac{\partial f}{\partial z_j} \circ \phi\right) \in \mathfrak{m}_1 I_\psi(JM(f) \circ \phi) = \left\langle \tau^{k+1} \right\rangle \mathbb{C}\{\tau\}$$

so finally:

$$\begin{aligned} \langle \tau^{r_0} w_0, \ldots, \tau^{r_n} w_n \rangle \, \mathbb{C}\{\tau\} &= \langle \tau^{r_0} w_0, \ldots, \tau^{r_{l-1}} w_{l-1} \rangle \, \mathbb{C}\{\tau\} \\ &= I_\psi(JM_h(f) \circ \phi). \end{aligned}$$

and the result follows. □

Corollary 3.21. *Let $\varphi : (\mathfrak{X}, 0) \to \mathbb{C}$ denote the specialisation of $(X, 0)$ to its tangent cone $(C_{X,0}, 0)$. Then, the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin if and only if $\frac{\partial F}{\partial t} \in JM_\varphi(F)^\dagger$.*

Proof. For $(Y, 0) \subset (\mathfrak{X}, 0) \subset (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ we have that the projection

$$\varphi : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$$

onto the last coordinate can be seen as the retraction over $(Y, 0)$. Moreover, the subspace $(Y, 0)$ is defined by the projection

$$g : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$$

onto the first $n+1$ coordinates, so the module $JM_g(F) =< \frac{\partial F}{\partial t} > O_{\mathfrak{X}}^p$, and the result follows from 3.20. □

Remark 3.22. *Proposition 3.13 gives us a relation between the blowup space $E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1})$ and the limits of tangent hyperplanes for every point in a small enough neighborhood of the origin in $\mathfrak{X}$. Since it is this relation what gives the key to derive 3.14 to 3.20, this results are also valid for every point in a small enough neighborhood of the origin in $\mathfrak{X}$ and all we have to change is that the arcs $\phi : (\mathbb{C}, 0) \to (\mathfrak{X}^0, (z, t))$ arrive to the desired point. But more importantly, the characterization of Whitney's condition a) given in corollary 3.21 is valid as stated for any sufficiently close point $y \in Y$.*

So far we haven't said anything about Whitney's condition b), and the reason is that we begin chapter 4 by proving that for the pair $(\mathfrak{X}^0, Y)$ a) implies b) at the origin. The first result characterising Whitney's condition b) in terms of integral closure of ideals was given by Teissier in [Tei74, Prop. 3.6, pg 332] in the hypersurface case. It was subsequently generalised by Gaffney in [Gaf92, Thm. 2.5, pg 309] in the following setting.

Let $(X,0) \subset (\mathbb{C}^k \times \mathbb{C}^n, 0)$ be a germ of reduced and equidimensional analytic singularity, $X = f^{-1}(0)$, where $f : (\mathbb{C}^{k+n}, 0) \to (\mathbb{C}^p, 0)$. Let $Y = \mathbb{C}^k \times \{0\} \subset X$ a smooth subvariety, and let us fix a coordinate system $(y_1, \dots y_k, z_0, \dots, z_n)$ in $\mathbb{C}^k \times \mathbb{C}^n$. We will denote by $z : \mathbb{C}^{k+n} \to \mathbb{C}^k$ the analytic projection onto Y.

Theorem 3.23. *Let $g : \mathbb{C}^{n+k} \to \mathbb{C}^n$ be the analytic projection $(y,z) \mapsto (z)$, and let $I_Y = \langle z_0 \dots, z_n \rangle O_X$ denote the ideal sheaf generated by the coordinate functions of g. The pair (X^0, Y) satisfy Whitney's conditions a) and b) if and only if the module $JM_g(f)$ is contained in the module $\overline{I_Y \cdot JM_z(f)}$.*

3.1.1 Case of an Irreducible Germ

When the germ $(\mathfrak{X}, 0)$ is a complete intersection, the ideal $\rho(JM(F))$ defines the singular locus of $\mathfrak{X} \times \check{\mathbb{P}}^{p-1}$, and it can be proved that the map $\eta : E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{p-1}) \to C(\mathfrak{X})$ of proposition 3.13 is biholomorphic away from the exceptional divisor. Moreover, from corollary 1.16 the blowup of the ideal defined by the maximal minors of the relative Jacobian matrix $[D_\varphi F]$ gives the relative Nash modification $\mathcal{N}_\varphi \mathfrak{X} \to \mathfrak{X}$.

The purpose of this section is to derive similar results for a germ $(\mathfrak{X}, 0)$, when we replace the complete intersection hypothesis by an irreducibility hypothesis, without losing any of the results on limits of tangent spaces and Whitney conditions obtained in the previous section. Moreover, lemma 3.2 tells us that we don't lose anything by restricting ourselves to the irreducible case. We will need to introduce some notation.

Let c denote the codimension of $\mathfrak{X}$ in $\mathbb{C}^{n+1} \times \mathbb{C}$, and let S denote the set of increasing sequences of c positive integers less than $p+1$. For $\alpha \in S$ denote by $[DF]_\alpha$ the $c \times (n+2)$ submatrix of $[DF]$ formed by the $(\alpha_1, \dots, \alpha_c)$ lines of $[DF]$. That is the jacobian matrix, of the map $F_\alpha := (F_{\alpha_1}, \dots F_{\alpha_c}) : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}^c$.

Definition 3.24. *For $\alpha \in S$, define the* ***α-Jacobian module of*** *F as the submodule $JM(F)_\alpha$ of $O_{\mathfrak{X}}^c$ generated by the columns of the matrix $[DF]_\alpha$, that is:*

$$JM(F)_\alpha := O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_{\alpha_1}}{\partial z_0} \\ \vdots \\ \frac{\partial F_{\alpha_c}}{\partial z_0} \end{pmatrix} + \cdots + O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_{\alpha_1}}{\partial z_n} \\ \vdots \\ \frac{\partial F_{\alpha_c}}{\partial z_n} \end{pmatrix} + O_{\mathfrak{X}} \begin{pmatrix} \frac{\partial F_{\alpha_1}}{\partial t} \\ \vdots \\ \frac{\partial F_{\alpha_c}}{\partial t} \end{pmatrix} \subset O_{\mathfrak{X}}^c$$

It should now be clear what the notation $\frac{\partial F_\alpha}{\partial v}$ means for v a vector in $\mathbb{C}^{n+1} \times \mathbb{C}$, or what we mean by the **α-Relative Jacobian Module** with respect to an analytic map germ $g : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^l, 0)$ as in 3.12.

For the remainder of this section we will assume that the germ $(X,0)$ **is irreducible**, and as a consequence the associated specialisation space $(\mathfrak{X},0)$ **is irreducible** as well.

Remark 3.25. *1. For every non singular point $(z,t)\in\mathfrak{X}^0$, the matrix $[DF(z,t)]$ has rank $c:=n+1-d$, and so there exists an $\alpha\in S$ such that at least one of the maximal minors $(c\times c)$ of the matrix $[DF]_\alpha$ is not a zero divisor in $O_{\mathfrak{X},0}$.*

2. For every point (z,t) in the relative smooth locus $\bigcup\mathfrak{X}(t)^0$, the matrix $[D_\varphi F(z,t)]$ has rank $c:=n+1-d$, and so there exists a $\gamma\in S$ such that at least one of the maximal minors $(c\times c)$ of the matrix $[D_\varphi F]_\gamma$ is not identically zero in $O_{\mathfrak{X},0}$.

We are now in position to restate proposition 3.13 in these terms.

Proposition 3.26. *Let $E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{c-1})\subset\mathfrak{X}\times\check{\mathbb{P}}^{c-1}\times\check{\mathbb{P}}^{n+1}$ be the blowup of $\mathfrak{X}\times\check{\mathbb{P}}^{c-1}$ along the subspace Z defined by the ideal $\rho(JM(F)_\alpha)O_{\mathfrak{X}}[u_1,\dots,u_c]$. Then, there exists a surjective map $\eta: E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{c-1})\to C(\mathfrak{X})$, making the following diagram commutative:*

$$\begin{tikzcd}
E_Z(\mathfrak{X}\times\check{\mathbb{P}}^{c-1}) \arrow[r, "\eta"] \arrow[d, "e_Z"'] & C(\mathfrak{X}) \arrow[d, "\kappa_{\mathfrak{X}}"] \\
\mathfrak{X}\times\check{\mathbb{P}}^{c-1} \arrow[r] & \mathfrak{X} \arrow[d, "\varphi"] \\
 & \mathbb{C}
\end{tikzcd}$$

Proof. Let $\alpha\in S$ be as in remark 3.25. Since $\mathfrak{X}$ is irreducible, there exists an open dense set $\mathfrak{X}^0_\alpha\subset\mathfrak{X}^0$, where for any point $(z,t)\in\mathfrak{X}^0_\alpha$ the tangent space $T_{(z,t)}\mathfrak{X}$ is the kernel of the matrix $[DF]_\alpha$, that is, it is obtained as the intersection of the $c:=n+1-d$ hyperplanes $[\overrightarrow{dF_{\alpha_j}}(z,t)]$. Moreover, since c is the codimension of $\mathfrak{X}$, then any linear equation defining the tangent hyperplane $H=[a:b]$ to $\mathfrak{X}$ at (z,t) is expressed as a **unique** linear combination of these c hyperplanes $H=[\sum\beta_j\overrightarrow{dF_{\alpha_j}}(z,t)]$, that is, they form a base of the fiber $\kappa_{\mathfrak{X}}^{-1}(z,t)$ over (z,t) in the conormal space $C(\mathfrak{X})$. So for any point $(z,t,u)\in\mathfrak{X}\times\check{\mathbb{C}}^c$ with $(z,t)\in\mathfrak{X}^0_\alpha$ we have the map

$$(z,t,u)\in\mathfrak{X}\times\check{\mathbb{C}}^c\mapsto(z,t),\left[\sum_{i=1}^{c}u_i\overrightarrow{dF_{\alpha_i}}(z,t)\right]\in C(\mathfrak{X})\subset\mathfrak{X}\times\check{\mathbb{P}}^{n+1}$$

Note that this map is invariant with respect to the homotheties of $\check{\mathbb{C}}^c$, so it defines a map in $\mathfrak{X}\times\check{\mathbb{P}}^{c-1}$

The proof is now a word by word transcription of the proof of proposition 3.13, replacing p by c and putting the α where necessary. □

The proof of this proposition has the following result as an immediate corollary.

Corollary 3.27. *For each appropriately chosen $\alpha\in S$, the restriction of η to $e_Z^{-1}(\mathfrak{X}^0_\alpha)$ is an isomorphism. In other words, the analytic spaces $\mathfrak{X}^0_\alpha\times\check{\mathbb{P}}^{c-1}$ and $\kappa_{\mathfrak{X}}^{-1}(\mathfrak{X}^0_\alpha)$ are isomorphic.*

This corollary tells us that when $(X,0)$ is irreducible the space $E_Z(\mathfrak{X} \times \check{\mathbb{P}}^{c-1})$ built this way is a modification of the conormal space $C(X)$.

Remark 3.28. *In the same spirit of the proof of the previous proposition, we can see that by choosing a $\gamma \in S$ as in remark 3.25-2, then the irreducibility of $\mathfrak{X}$ together with the constructive proof of 1.16 implies that the blowup of the ideal $J_c(JM_\varphi(F)_\gamma)$ generated by the maximal minors of $[D_\varphi F]_\gamma$ gives the relative Nash modification $\mathcal{N}_\varphi \mathfrak{X}$.*

The key ingredient to prove lemma 3.14 is proposition 3.13, and so we can easily verify that the very same statement is valid if we replace DF by DF_α and consider the maps ψ as having their image in $\check{\mathbb{C}}^c$. In the same way all the rest of the results, from 3.15 to 3.21 build up on one another and so their analogous statements are valid by introducing the α and the c where necesary. We will just explicitly restate the analogous statement of corollary 3.21.

Corollary 3.29. *Let $\varphi : (\mathfrak{X},0) \to \mathbb{C}$ denote the specialisation of $(X,0)$ to its tangent cone $(C_{X,0},0)$. Then, the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin if and only if $\frac{\partial F_\alpha}{\partial t} \in JM_\varphi(F)^\dagger_\alpha$.*

Chapter 4

Whitney Equisingularity of $\mathfrak{X}$

Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a reduced germ of analytic singularity of pure dimension d, and let $\varphi : (\mathfrak{X},0) \to (\mathbb{C},0)$ denote the specialisation of X to its tangent cone $C_{X,0}$. Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the smooth subspace $0 \times \mathbb{C} \subset \mathfrak{X}$. Our aim is to generalize proposition 2.7, by studying the equisingularity of $\mathfrak{X}$ along Y, that is, we want to determine whether it is possible to find a Whitney stratification of $\mathfrak{X}$, in which the t-axis Y is a stratum.

The first step to find out if such a stratification is possible, is to verify that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions. Since $\mathfrak{X} \setminus \mathfrak{X}(0)$ is isomorphic to the product $X \times \mathbb{C}^*$, Whitney's conditions are automatically verified everywhere in $\{0\} \times \mathbb{C}$, with the possible exception of the origin. The following result tells us that in this particular case it is enough to check for Whitney's condition a).

Proposition 4.1. *If the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin, then it also satisfies Whitney's condition b) at the origin.*

Before proving proposition 4.1, we need the following lemma.

Lemma 4.2. *There exists a natural morphism $\omega : E_Y\mathfrak{X} \to E_0X$, making the following diagram commute:*

$$\begin{tikzcd} E_Y\mathfrak{X} \arrow[r, "\omega"] \arrow[d, "e_Y"'] & E_0X \arrow[d, "e_o"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

Moreover, when restricted to the exceptional divisor $e_Y^{-1}(Y) = \mathbb{P}C_{\mathfrak{X},Y}$ it induces the natural map $\mathbb{P}C_{\mathfrak{X},Y} = Y \times \mathbb{P}C_{X,0} \to \mathbb{P}C_{X,0}$.

Proof. Algebraically, this results from the universal property of the blowup E_0X. We start with the diagram:

$$\begin{tikzcd} E_Y\mathfrak{X} \arrow[d, "e_Y"'] & E_0X \arrow[d, "e_o"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

In this coordinate system, the maximal ideal $\mathfrak{m}$ of the analytic algebra $O_{X,0}$ is generated by $\langle z_0, \ldots, z_n \rangle$. The map ϕ, induces a morphism of analytic algebras

$O_{X,0} \to O_{\mathfrak{X},0}$ defined by $z_i \mapsto tz_i$. So we have to prove that the ideal $\langle tz_0, \ldots, tz_n \rangle \subset O_{\mathfrak{X},0}$ is locally invertible when pulled back to $E_Y\mathfrak{X}$. But as ideals we have the equality $\langle tz_0, \ldots, tz_n \rangle = \langle t \rangle \cdot \langle z_0, \ldots z_n \rangle$. And by definition of the blowup, the ideal $\langle z_0, \ldots, z_n \rangle \subset O_{\mathfrak{X},0}$ corresponding to Y is locally invertible when pulled back to $E_Y\mathfrak{X}$. After multiplication by a invertible ideal, it will remain locally invertible. Note that, for the diagram to be commutative the morphism ω must map the point $(z,t), [z] \in E_Y\mathfrak{X} \setminus \{Y \times \mathbb{P}^n\} \subset \mathfrak{X} \times \mathbb{P}^n$ to the point $(tz), [z] \in E_0X \subset X \times \mathbb{P}^n$ and the result follows. □

Remark 4.3. *Note that:*

1. *For any point $y \in Y$, the tangent cone $C_{\mathfrak{X},y}$ is isomorphic to $C_{X,0} \times Y$, and the isomorphism is uniquely determined once we have chosen a set of coordinates. The reason is that for any $f(z)$ vanishing on $(X,0)$, the function $F(z,t) = t^{-m}f(tz) = f_m + tf_{m+1} + t^2 f_{m+2} + \ldots$, vanishes in $(\mathfrak{X}, 0)$ and so for any point $y = (\underline{0}, t_0)$ the initial form of $F(z, t+t_0)$ in $\mathbb{C}\{z_0, \ldots, z_n, t\}$ is equal to the initial form of f at 0. That is $in_{(0,t_0)}F = in_0 f$.*
2. *The projectivized normal cone $\mathbb{P}C_{\mathfrak{X},Y}$ is isomorphic to $Y \times \mathbb{P}C_{X,0}$. This can be seen from the equations used to define $\mathfrak{X}$ (Chapter 1, eq. 1.1), where the initial form of F_i with respect to Y, is equal to the initial form of f_i at the origin. That is $in_Y F_i = in_0 f_i$.*

Now we can proceed to the proof of 4.1.

Proof. (*Proposition 4.1*)
We want to prove that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition b) at the origin. We are assuming that it already satisfies condition a), so in particular we have that $\zeta^{-1}(0)$ is contained in $\{0\} \times \mathbb{P}^n \times \check{\mathbb{P}}^n$. By proposition 2.17 it suffices to prove that any point $(0, l, H) \in \zeta^{-1}(0)$ is contained in the incidence variety $I \subset \{0\} \times \mathbb{P}^n \times \check{\mathbb{P}}^n$.

$$\begin{tikzcd}
E_Y C(\mathfrak{X}) \arrow[r, "\hat{e}_Y"] \arrow[d, "\kappa'_{\mathfrak{X}}"] \arrow[dr, "\zeta"] & C(\mathfrak{X}) \arrow[r, "\psi"] \arrow[d, "\kappa_{\mathfrak{X}}"] & C(X) \times \mathbb{C} \\
E_Y\mathfrak{X} \arrow[r, "e_Y"'] \arrow[d, "\omega"] & \mathfrak{X} & \\
E_0 X & &
\end{tikzcd}$$

By construction, there is a sequence (z_m, t_m, l_m, H_m) in $E_Y C(\mathfrak{X}) \hookrightarrow C(\mathfrak{X}) \times_{\mathfrak{X}} E_Y\mathfrak{X}$ tending to $(0, l, H)$ where (z_m, t_m) is not in Y. Through $\kappa'_{\mathfrak{X}}$, we obtain a sequence (z_m, t_m, l_m) in $E_Y\mathfrak{X}$ tending to $(0, l)$, and through $\hat{e}_Y$ a sequence (z_m, t_m, H_m) tending to $(0, H)$ in $C(\mathfrak{X})$.

Now, using the notation of proposition 1.41, through the map ψ we obtain the sequence $(t_m z_m, \widetilde{H}_m)$ and since by hypothesis we have $b = 0$, then by remark 1.42-2 both the sequence and its limit $(0, \widetilde{H})$ are in $C(X)$. Note that if H has coordinates $[a_0 : \cdots : a_n : 0]$, then $\widetilde{H} = [a_0 : \cdots : a_n] \in \check{\mathbb{P}}^n$. On the other hand, by lemma 4.2 we have that both the sequence $(t_m z_m, l_m)$ obtained through the map ω and its limit $(0, l)$ are in $E_0 X$. Finally, Whitney's lemma 2.1 tells us that in this situation we have that $l \subset \widetilde{H}$ and so the point $(0, l, H)$ is in the incidence variety.

If the sequence (z_m, t_m, l_m, H_m) in $E_Y C(\mathfrak{X})$ is contained in the special fiber, that is $t_m = 0$ for all m, then either the point $(z_m, 0)$ is a smooth point of $\mathfrak{X}$ and so the line $l_m = [z_m : 0]$ is contained in every tangent hyperplane H_m, or it is a singular point of $\mathfrak{X}$ and so by constructing a sequence of smooth points in $\mathfrak{X} \setminus \mathfrak{X}(0)$ tending to it and using the maps ψ and ω like before we prove that the line l_m is contained in H_m. In any case, what we have is that for any point in the sequence $(z_m, 0, l_m, H_m)$ we already have the inclusion $l_m \subset H_m$ and so the limit $(0, l, H)$ satisfies this condition as well. $\square$

The following result tells us that in order to generalize proposition 2.7, the condition of not having exceptional cones is necessary.

Lemma 4.4. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a reduced germ of analytic singularity of pure dimension d, and let $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ denote the specialisation of X to its tangent cone $C_{X,0}$. Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the smooth subspace $0 \times \mathbb{C} \subset \mathfrak{X}$. If the tangent cone $C_{X,0}$ is reduced and the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) then the germ $(X, 0)$ does not have exceptional cones.*

Proof. First of all, by hypothesis the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a), so by proposition 4.1 it also satisfies Whitney's condition b). Recall that the aureole of $(\mathfrak{X}, 0)$ along Y is a collection $\{V_\alpha\}$ of subcones of the normal cone $C_{\mathfrak{X},Y}$ whose projective duals determine the set of limits of tangent hyperplanes to $\mathfrak{X}$ at the points of Y in the case that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney conditions a) and b) at every point of Y (See Thm.2.20, Def. 2.21 and its remark). Among the V_α there are the irreducible components of $|C_{\mathfrak{X},Y}|$. Moreover:

1. By remark 4.3 we have that $C_{\mathfrak{X},Y} = Y \times C_{X,0}$ so its irreducible components are of the form $Y \times \widetilde{V}_\beta$ where $\widetilde{V}_\beta$ is an irreducible component of $|C_{X,0}|$.
2. For each α the projection $V_\alpha \to Y$ is surjective and all the fibers are of the same dimension. (See [LT88][Proposition 2.2.4.2, pg 570])
3. The hyperplane $H = [0 : 0 : \cdots : 1] \in \check{\mathbb{P}}^{n+1}$ is transversal to $(\mathfrak{X}, 0)$ by hypothesis, and so by theorem 2.24 the collection $\{V_\alpha \cap H\}$ is the aureole of $\mathfrak{X} \cap H$ along $Y \cap H$.

Notice that $(\mathfrak{X} \cap H, Y \cap H)$ is equal to $(\mathfrak{X}(0), 0)$, which is isomorphic to the tangent cone $(C_{X,0}, 0)$ and therefore does not have exceptional cones. This means that for each α either $V_\alpha \cap H$ is an irreducible component of $C_{X,0}$ or it is empty. But the intersection can't be empty because the projections $V_\alpha \to Y$ are surjective. Finally since all the fibers of the projection are of the same dimension then the V_α's are only the irreducible components of $C_{\mathfrak{X},Y}$. But this means, that if we define the affine hyperplane H_t as the hyperplane with the same direction as H and passing through the point $y = (0, t) \in Y$ for t small enough. Then H_t is transversal to $(\mathfrak{X}, y)$ and so we have again that the collection $\{V_\alpha \cap H_t\}$ is the aureole of $\mathfrak{X} \cap H_t$ along $Y \cap H_t$, that is the aureole of $(X, 0)$, so it does not have exceptional cones.

$\square$

The absence of exceptional cones allows us to deduce the following lemma which will turn out to be important for the proof of proposition 4.14.

We can now use lemma 4.4 to prove that the Whitney conditions of the pair $(\mathfrak{X}^0, Y)$ imply that the germ $(\mathfrak{X}, 0)$ does not have exceptional cones.

Proposition 4.5. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a reduced germ of analytic singularity of pure dimension d, and let $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ denote the specialisation of X to its tangent cone $C_{X,0}$. Let $\mathfrak{X}^0$ denote the open set of smooth points of $\mathfrak{X}$, and let Y denote the smooth subspace $0 \times \mathbb{C} \subset \mathfrak{X}$. Then*

1. *If the germ $(\mathfrak{X}, 0)$ does not have exceptional cones, then the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*
2. *Moreover, if the tangent cone $C_{X,0}$ is reduced and the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin then $(\mathfrak{X}, 0)$ does not have exceptional cones.*

Proof. Let us choose a representative of $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$, then $(\mathfrak{X}, 0) \subset (\mathbb{C}^{n+2}, 0)$. Let $C(\mathfrak{X}) \subset \mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$ denote the conormal space of $\mathfrak{X}$, and let us consider the following diagram:

$$\begin{tikzcd} C(\mathfrak{X}) \arrow[d, "\kappa_{\mathfrak{X}}"] & C(Y) \arrow[d, "h"] \\ \mathfrak{X} & Y \arrow[l, hook'] \end{tikzcd}$$

By proposition 2.17, Whitney's condition a) at the origin is equivalent to the set theoretic inclusion

$$|\kappa_{\mathfrak{X}}^{-1}(0)| \subset |h^{-1}(0)|$$

Let $((z_0, \ldots, z_n, t), [a_0 : a_1 : \ldots : a_n : b])$ be the coordinates of $\mathbb{C}^{n+2} \times \check{\mathbb{P}}^{n+1}$ as before. Now, since Y is the t axis, the conormal space $C(Y)$ is defined by the equations $z_0 = \cdots = z_n = b = 0$, and for $h^{-1}(0)$ we just add the equation $t = 0$.

1) By hypothesis $(\mathfrak{X}, 0)$ does not have exceptional cones, which means that $|\kappa_{\mathfrak{X}}^{-1}(0)|$ is just the dual of the tangent cone $C_{\mathfrak{X},0} = C_{X,0} \times \mathbb{C}$. In particular, every tangent hyperplane to $C_{\mathfrak{X},0}$ contains the t axis, that is $b = 0$, so is contained in $h^{-1}(0)$, and we have Whitney's condition a).

2) By lemma 4.4 we know that $(X, 0)$ does not have exceptional cones. Since every point in $\kappa_{\mathfrak{X}}^{-1}(0)$, that is every tangent hyperplane to $\mathfrak{X}$ at the origin satisfies $b = 0$, the remark 1.42-2 tells us that the morphism $(\widetilde{pr_1} \circ \psi) : C(\mathfrak{X} \setminus \mathfrak{X}(0)) \to C(X)$ of proposition 1.41, sending $(z, t), [a : b] \to (tz), [a]$ can be extended to $C(\mathfrak{X})$. In particular the point, $(0), [a]$ is in $\kappa_X^{-1}(0) \subset C(X)$, and since $(X, 0)$ does not have exceptional cones, then $[a]$ is in the dual of the tangent cone $C_{X,0}$, which implies that $\kappa_{\mathfrak{X}}^{-1}(0)$ is just the dual of the tangent cone $C_{\mathfrak{X},0}$, and $(\mathfrak{X}, 0)$ does not have exceptional cones.

□

There is a useful generalization of proposition 4.1. Let Σ be the singular locus of X, and suppose that it is smooth at 0 and that the pair (X^0, Σ) satisfies Whitney's conditions a) and b) at the origin. Now, consider the subspace $\widetilde{\Sigma}$ of $\mathfrak{X}$ corresponding to the specialisation of $(\Sigma, 0)$ to its tangent cone. It corresponds to the topological closure in $\mathfrak{X}$ of the inverse image of $\Sigma \times \mathbb{C}^* \subset X \times \mathbb{C}^*$ with respect to the isomorphism $\phi : \mathfrak{X} \setminus \mathfrak{X}(0) \to X \times \mathbb{C}^*$. That is, $\widetilde{\Sigma} := \overline{\phi^{-1}(\Sigma \times \mathbb{C}^*)}$. It is easy to see that $(\widetilde{\Sigma}, 0)$ is smooth and is contained in the singular locus of $\mathfrak{X}$.

Proposition 4.6. *If the pair $(\mathfrak{X}^0, \widetilde{\Sigma})$ satisfies Whitney's condition a) at the origin, then it also satisfies Whitney's condition b) at the origin.*

Proof. The proof is analogous to that of 4.1, with some minor modifications, and it goes as follows. By making a change of coordinates if necessary, we can assume that $(\Sigma, 0) \subset (X, 0)$ is linear, say

$$\Sigma = \{(z_0 \dots, z_{n-s}, y_1, \dots, y_s) \in \mathbb{C}^{n+1} | z_0 = z_1 = \dots = z_{n-s} = 0\}$$

then $\widetilde{\Sigma} = \Sigma \times \mathbb{C} \subset \mathfrak{X}$. We will use the characterisation of the Whitney conditions given by proposition 2.17 in terms of the normal conormal diagram of the pair $(\mathfrak{X}, \widetilde{\Sigma}, 0)$.

$$\begin{tikzcd}
E_{\widetilde{\Sigma}}C(\mathfrak{X}) \arrow[r, "\hat{e}_{\widetilde{\Sigma}}"] \arrow[d, "\kappa'_{\mathfrak{X}}"] \arrow[dr, "\zeta"] & C(\mathfrak{X}) \arrow[r, "\psi"] \arrow[d, "\kappa_{\mathfrak{X}}"] & C(X) \times \mathbb{C} \\
E_{\widetilde{\Sigma}}\mathfrak{X} \arrow[r, "e_{\widetilde{\Sigma}}"'] \arrow[d, "\omega"] & \mathfrak{X} & \\
E_{\Sigma}X & &
\end{tikzcd}$$

We want to prove that the pair $(\mathfrak{X}^0, \widetilde{\Sigma})$ satisfies Whitney's condition b) at the origin. We are assuming that it already satisfies condition a), so in particular we have that $\zeta^{-1}(0)$ is contained in $\{0\} \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$, so to prove condition b) it suffices to prove that any point $(0, l, H) \in \zeta^{-1}(0)$ is contained in the incidence variety $I \subset \{0\} \times \mathbb{P}^{n-s} \times \check{\mathbb{P}}^{n-s}$.

By construction, there is a sequence (z_j, t_j, l_j, H_j) in $E_{\widetilde{\Sigma}}\mathfrak{X} \hookrightarrow C(\mathfrak{X}) \times_{\mathfrak{X}} E_{\widetilde{\Sigma}}\mathfrak{X}$ tending to $(0, l, H)$ where (z_j, t_j) is not in $\widetilde{\Sigma}$. Through $\kappa'_{\mathfrak{X}}$, we obtain a sequence (z_j, t_j, l_j) in $E_{\widetilde{\Sigma}}\mathfrak{X}$ tending to $(0, l)$, and through $\hat{e}_{\widetilde{\Sigma}}$ a sequence (z_j, t_j, H_j) tending to $(0, H)$ in $C(\mathfrak{X})$.

Now, using the notation of proposition 1.41, through the map ψ we obtain the sequence $(t_j z_j, \widetilde{H}_j)$ and since by hypothesis we have $c_1 = \dots = c_s = b = 0$, then by remark 1.42-2 both the sequence and its limit $(0, \widetilde{H})$ are in $C(X)$. Note that if H has coordinates $[a_0 : \dots : a_{n-s} : 0 : \dots : 0] \in \check{\mathbb{P}}^{n+1}$, then $\widetilde{H} = [a_0 : \dots : a_{n-s} : 0 : \dots : 0] \in \check{\mathbb{P}}^n$. On the other hand, by lemma 4.7 we have that both the sequence $(t_j z_j, l_j)$ obtained through the map ω and its limit $(0, l)$ are in $E_\Sigma X$. Finally, since the pair (X^0, Σ) satisfies Whitney's condition b) at the origin we have that $l \subset \widetilde{H}$ and so the point $(0, l, H)$ is in the incidence variety.

If the sequence (z_j, t_j, l_j, H_j) in $E_Y C(\mathfrak{X})$ is contained in the special fiber, that is $t_j = 0$ for all j, then note that for every point $(z_j, 0)$ in the special fiber we can construct sequence of smooth points in $\mathfrak{X} \setminus \mathfrak{X}(0)$ tending to it and using the maps ψ and ω like before we can prove that the line l_j is contained in H_j. That is, for any point in the sequence $(z_j, 0, l_j, H_j)$ we already have the inclusion $l_j \subset H_j$ and so the limit $(0, l, H)$ satisfies this condition as well.

□

To complete the proof we state the following lemma, which generalizes 4.2.

Lemma 4.7. *There exists a natural morphism $\omega : E_{\widetilde{\Sigma}}\mathfrak{X} \to E_\Sigma X$, making the following diagram commute:*

$$\begin{tikzcd} E_{\widetilde{\Sigma}} \arrow[r, "\omega"] \arrow[d, "e_{\widetilde{\Sigma}}"'] & E_\Sigma X \arrow[d, "e_\Sigma"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

Proof. Algebraically, this results from the universal property of the blowup $E_\Sigma X$. We start with the diagram:

$$\begin{tikzcd} E_{\widetilde{\Sigma}}\mathfrak{X} \arrow[d, "e_{\widetilde{\Sigma}}"'] & E_\Sigma X \arrow[d, "e_\Sigma"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

In this coordinate system, the maximal ideal $\mathfrak{m}$ of the analytic algebra $O_{X,0}$ is generated by $\langle z_0, \ldots, z_{n-s}, y_1, \ldots, y_s \rangle$. The map ϕ, induces a morphism of analytic algebras $O_{X,0} \to O_{\mathfrak{X},0}$ defined by $z_i \mapsto tz_i$; $y_j \mapsto ty_j$. So we have to prove that the ideal $\langle tz_0, \ldots, tz_{n-s} \rangle \subset O_{\mathfrak{X},0}$ is locally invertible when pulled back to $E_{\widetilde{\Sigma}}\mathfrak{X}$. But as ideals we have the equality $\langle tz_0, \ldots, tz_{n-s} \rangle = \langle t \rangle \cdot \langle z_0, \ldots, z_{n-s} \rangle$. And by definition of the blowup, the ideal $\langle z_0, \ldots, z_{n-s} \rangle \subset O_{\mathfrak{X},0}$ corresponding to $\widetilde{\Sigma}$ is locally invertible when pulled back to $E_{\widetilde{\Sigma}}\mathfrak{X}$. It follows immediately that multiplying it by an invertible ideal will remain locally invertible. Note that, for the diagram to be commutative the morphism ω must map the point $(z, y, t), [z] \in E_{\widetilde{\Sigma}}\mathfrak{X} \setminus \{\widetilde{\Sigma} \times \mathbb{P}^{n-s}\} \subset \mathfrak{X} \times \mathbb{P}^{n-s}$ to the point $(tz, ty), [z] \in E_\Sigma X \subset X \times \mathbb{P}^{n-s}$ and the result follows. □

We would like to point out the type of difficulties one encounters if we remove the hypothesis that the tangent cone $C_{X,0}$ is reduced. Consider the specialisation space $(\mathfrak{X}, 0)$ and recall that $\mathrm{Sing}\mathfrak{X} \cap \mathfrak{X}(0) \subset \mathrm{Sing}C_{X,0}$. Suppose that the tangent cone $C_{X,0}$ is not reduced, and that it has at least one (non-embedded) irreducible component that is not reduced. Then we have the following two cases:

1. **The special fiber $\mathfrak{X}(0)$ has an irreducible component $\widetilde{V}_\beta$ completely contained in the singular locus of $\mathfrak{X}$.** Then, it is possible that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions at the origin. However, since $\mathfrak{X}^0 \cap \widetilde{V}_\beta = \emptyset$, then there is an irreducible component V_α of $\mathrm{Sing}\mathfrak{X}$ that contains $\widetilde{V}_\beta$. But $\mathfrak{X}(0)$ is transversal to Y, and so no stratification of $|\mathfrak{X}(0)|$ can be Whitney compatible with Y. This implies, that Y cannot be a stratum of a Whitney stratification of $(\mathfrak{X}, 0)$.

2. **The singular locus of $\mathfrak{X}$ does not contain any irreducible component of $\mathfrak{X}(0)$.** In this case, let $\widetilde{V}_\beta$ be a non-reduced irreducible component of $\mathfrak{X}(0)$. Since no irreducible component of $\mathfrak{X}(0)$ is contained in the singular locus of $\mathfrak{X}$ this means that the intersection $\mathrm{Sing}\mathfrak{X} \cap \widetilde{V}_\beta$ is a closed analytic subspace of $\widetilde{V}_\beta$. Thus there is an open dense set U_β of $\widetilde{V}_\beta$ such that $U_\beta \subset \mathfrak{X}^0$. Now, for every point $p = (\underline{z}, 0) \in U_\beta$, p is smooth in $\mathfrak{X}$, but it is singular in $\mathfrak{X}(0)$ which implies that the hyperplane $H := \{t = 0\}$ is tangent to $\mathfrak{X}$ at p. Finally, since U_β is dense in $\widetilde{V}_\beta$, and $0 \in \widetilde{V}_\beta$ we get that H is a limit of tangent hyperplanes to $\mathfrak{X}$ at 0, and so the pair $(\mathfrak{X}^0, Y)$ does not satisfy Whitney's condition a).

As we have said before, the first step in our objective of constructing a Whitney stratification of $(\mathfrak{X}, 0)$ having the parameter axis $(Y, 0)$ as a stratum, is proving that the pair $(\mathfrak{X}^\circ, Y)$ satisfies Whitney conditions $a)$ and $b)$ at the origin. But by corollary 3.21 this is equivalent to proving that $\frac{\partial F}{\partial t} \in JM_\varphi(F)^\dagger$.

Now let $(X, 0) = \bigcup_{j=1}^r (X_j, 0)$ be its irreducible decomposition, then by lemma 1.13 $(\mathfrak{X}, 0) = \bigcup_{j=1}^r (\mathfrak{X}_j, 0)$ is the irreducible decomposition of the specialisation space $\mathfrak{X}$, where $(\mathfrak{X}_j, 0)$ is the specialisation space of the irreducible component $(X_j, 0)$ to its tangent cone $(C_{X_j,0}, 0)$. Moreover, if the germ $(X, 0)$ doesn't have exceptional cones, then by corollary 2.6, the germs $(X_j, 0)$ don't have exceptional cones either. These two results allow us to restrict ourselves to the case where **the germ** $(X, 0)$ **is irreducible**, which we will assume from this point on.

We should say that the following results are also valid if we replace the irreducible hypothesis by asking the germ $(X, 0)$ to have a reduced, complete intersection tangent cone $(C_{X,0}, 0)$. This implies ([dP00, Def. 7.2.3 & Coro. 7.2.7, pg 284]) that $(X, 0)$ is a complete intersection, and so by construction $(\mathfrak{X}, 0)$ is a complete intersection singularity as well.

Let $c = n + 1 - d$ be the codimension of $\mathfrak{X}$ in $\mathbb{C}^{n+2}$, and let us fix an $\alpha = (\alpha_1, \ldots, \alpha_c) \in S$ having the properties of remark 3.25-2.

Lemma 4.8. *The α-Jacobian module $JM(F)_\alpha$ has rank c on $(\mathfrak{X}, 0)$.*

Proof. By definition, the rank of a module over the integral domain $O_{\mathfrak{X},0}$ is the dimension as a vector space over the quotient field $Q(O_{\mathfrak{X},0})$ of the vector space $Q(O_{\mathfrak{X},0}) \bigotimes JM(F)_\alpha O_{\mathfrak{X},0}$.

Consider the presentation

$$O_{\mathfrak{X},0}^r \longrightarrow O_{\mathfrak{X},0}^{n+2} \overset{[DF]_\alpha}{\longrightarrow} JM(F)_\alpha O_{\mathfrak{X},0} \to 0$$

where $[DF]_\alpha$ denotes the jacobian matrix of the map $F_\alpha : \mathbb{C}^{n+2} \to \mathbb{C}^c$, which defines this map. By tensorizing this sequence by the field $Q(O_{\mathfrak{X},0})$, we obtain the sequence

$$Q(O_{\mathfrak{X},0})^r \longrightarrow Q(O_{\mathfrak{X},0})^{n+2} \overset{[DF]_\alpha}{\longrightarrow} Q(O_{\mathfrak{X},0}) \bigotimes JM_\alpha(F) O_{\mathfrak{X},0} \to 0$$

where the map defined by the jacobian matrix remains surjective. Remark that we now have that the rank of the module $JM_\alpha(F)O_{\mathfrak{X},0}$ is equal to the rank of the matrix $[DF]_\alpha$ when considering its entries as members of the quotient field $Q(O_{\mathfrak{X},0})$.

Our choice of α garanties the existance of a non zero $c \times c$ minor in $O_{\mathfrak{X},0}$. This implies that the ideal $J_c(JM(F)_\alpha O_{\mathfrak{X},0})$ of $O_{\mathfrak{X},0}$ generated by all the $c \times c$ minors of the matrix $[DF]_\alpha$ is different from zero. Moreover since the matrix $[DF]_\alpha$ is of size $c \times (n+2)$, then the ideal $J_{c+1}(JM(F)_\alpha O_{\mathfrak{X},0})$ is equal to the zero ideal. This remains true when considering the minors as elements of the quotient field $Q(O_{\mathfrak{X},0})$, and so the rank of the matrix $[DF]_\alpha$ is equal to c which finishes the proof. $\square$

Since we are assuming $(\mathfrak{X}, 0)$ irreducible, what we have to prove, according to corollary 3.29, is that $\frac{\partial F_\alpha}{\partial t} \in JM_\varphi(F)^\dagger_\alpha$. So in terms of 3.3, what we must prove is that every minor M in $J_c(JM_\alpha(F))$ depending on $\frac{\partial F_\alpha}{\partial t}$ satisfies $M \in J_c(JM_\varphi(F)_\alpha)^\dagger$. We will prove this using 3.10, and since we are working with ideals, it leads us to consider the normalized blowup of $\mathfrak{X}$ along the ideal $J_c(JM_\varphi(F)_\alpha)$. Moreover, by remark 3.28, the blowup of $\mathfrak{X}$ along the ideal $J_c(JM_\varphi(F)_\alpha)$ gives the relative Nash modification $\nu_\varphi : \mathcal{N}_\varphi\mathfrak{X} \to \mathfrak{X}$.

When $(\mathfrak{X}, 0)$ is a complete intersection, there is no choice of α to make, the ideal $J_c(JM_\varphi(F))$ is the relative jacobian ideal, and by corollary 1.16, the blowup of $\mathfrak{X}$ along the relative jacobian ideal J_φ also gives us the relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$.

We know that the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions at every point y of Y with the possible exception of the origin, so we have by 3.22 that every minor M in $J_c(JM(F)_\alpha)$ depending on $\frac{\partial F_\alpha}{\partial t}$ satisfies $M \in J_c(JM_\varphi(F)_\alpha)^\dagger$ in $O_{\mathfrak{X},y}$ for all these points. What we are going to prove in proposition 4.14 is that this condition carries over to the origin under the assumption that $(X, 0)$ does not have exceptional cones.

Remark 4.9.

1. *The fact proven in proposition 1.41, that the isomorphism between the conormal space $C(\mathfrak{X} \setminus \mathfrak{X}(0))$ and $C(X) \times \mathbb{C}^*$ is given by a natural projection implies that the vertical hyperplane $\{t = 0\} := [0 : \cdots : 0 : 1] \in \check{\mathbb{P}}^{n+1}$ is not tangent to $\mathfrak{X} \setminus \mathfrak{X}(0)$ at any point $(z, t) \in \mathfrak{X} \setminus \mathfrak{X}(0)$. This is equivalent, by corollary 3.15, to $\frac{\partial F_\alpha}{\partial t} \in \overline{JM_\varphi(F)_\alpha}$ in $O_{\mathfrak{X},(z,t)}$ for every point $(z, t) \in \mathfrak{X} \setminus \mathfrak{X}(0)$.*
2. *When $(\mathfrak{X}, 0)$ is a complete intersection, the center of the blowup defined by the ideal $J_c(JM_\varphi(F))$ is set theoretically the relative singular locus of $\mathfrak{X}$. Moreover, since in this case, the tangent cone $(C_{X,0}, 0)$ is a complete intersection, the equality $\frac{\partial F_i}{\partial z_j}(z, 0) = \frac{\partial f_{m_i}}{\partial z_j}(z)$ implies that the restriction of the ideal $J_c(JM_\varphi(F))$ to the special fiber is equal to the jacobian ideal $J_{C_{X,0}}$ of the tangent cone $C_{X,0}$ in $O_{C_{X,0}}$. This implies that the strict transform of $\mathfrak{X}(0)$ with respect to this blowup is equal to the Nash modification $\mathcal{N}C_{X,0}$ of the fiber.*
3. *Even though we are considering that $(X, 0)$ and as a result $(\mathfrak{X}, 0)$ are irreducible germs, this doesn't mean that the tangent cone $(C_{X,0})$ is irreducible. The problem with this is that the restriction of the ideal $J_c(JM_\alpha(F))$ to the special fiber $\mathfrak{X}(0)$ may vanish in an irreducible component of the tangent cone $(C_{X,0}, 0)$ and so its strict transform will no longer be equal to the Nash modification of $C_{X,0}$.*

Lemma 4.10. *For a reduced and irreducible germ $(X, 0)$ of analytic singularity with reduced tangent cone $(C_{X,0}, 0)$, there exists an ideal $I \subset O_{\mathfrak{X},0}$ such that:*

1. *The analytic subset $V(I) \subset \mathfrak{X}$ defined by I contains the relative singular locus $\mathrm{Sing}_\varphi\mathfrak{X} := \bigcup_t \mathrm{Sing}\mathfrak{X}(t)$.*

2. *The blowup of $\mathfrak{X}$ along I is equal to the relative Nash modification of $\mathfrak{X}$, that is $E_I\mathfrak{X} \cong \mathcal{N}_\varphi\mathfrak{X}$.*
3. *The blowup of the special fiber $\mathfrak{X}(0)$ along the ideal $IO_{\mathfrak{X}(0),0}$ defined by the restriction of I to $\mathfrak{X}(0)$ is isomorphic to the Nash modification $\mathcal{N}C_{X,0}$.*

Proof. Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^p, 0)$ denote the germ of analytic map defined by the p series $F_1, \ldots, F_p \in \mathbb{C}\{z_0, \ldots, z_n, t\}$, such that $(\mathfrak{X}, 0) = (F^{-1}(0), 0)$. Let $[D_\varphi F]$ denote the relative jacobian matrix, and define the $p \times (n+1)$ matrix A by setting the t coordinate to 0, that is $A = [D_\varphi F(z, 0)]$. By definition, A is the jacobian matrix of the map $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^p, 0)$ defined by the homogeneous polynomials $g_i = F_i(z, 0)$ such that $(C_{X,0}, 0) = (g^{-1}(0), 0)$. Let c be the codimension of $\mathfrak{X}$ in $\mathbb{C}^{n+1} \times \mathbb{C}$, then c is also the codimension of $C_{X,0}$ in $\mathbb{C}^{n+1}$, and let S(resp. S') denote the set of increasing sequences of c-positive integers less than $p+1$ (resp. $n+2$). For $\alpha = (\alpha_1, \ldots, \alpha_{n+1-r}) \in S$, and $\beta = (\beta_1, \ldots, \beta_{n+1-r}) \in S'$, $g^{\alpha\beta}$ will denote the minor of A obtained by considering the rows determined by α and the columns determined by β.

Let $C_{X,0} = \bigcup_{j=1}^{l} V_j$ be the irreducible decomposition of the tangent cone. By the proof of proposition 1.15 (or its corollary 1.16) there exist $\alpha^1, \ldots, \alpha^l$ in S and functions $h_1, \ldots, h_l \in O_{C_{X,0},0}$, with $h_i = 0$ on $\bigcup_{j \neq i} V_j$ and $h_i \neq 0$ on V_i, such that the blowup of $C_{X,0}$ along the ideal $J = \left\langle \sigma_\beta := \sum_{i=1}^{l} h_i g^{\alpha^i,\beta},\ \beta \in S' \right\rangle$ gives the Nash modification $\mathcal{N}C_{X,0}$.

Now, since for each α^i there is a non-zero minor of the matrix $[Dg]_{\alpha^i}$, then the corresponding minor of the matrix $[D_\varphi F]_{\alpha^i}$ is not identically zero. Since by hypothesis $\mathfrak{X}$ is irreducible then the proof of 1.16 tells us that this condition is enough to prove that the blowup of $\mathfrak{X}$ along the ideal $J_c(JM_\varphi(F)_{\alpha^i})$ gives the relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$.

Let $F^{\alpha\beta}$ denote the minor of $[D_\varphi F]$ obtained by considering the rows determined by α and the columns determined by β, and define the ideal $I = \left\langle \rho_\beta := \sum_{i=1}^{l} h_i F^{\alpha^i,\beta},\ \beta \in S' \right\rangle$ where the h_i's are the same we used for the tangent cone. Now, by construction, the blowup of the special fiber $\mathfrak{X}(0)$ along the ideal $IO_{\mathfrak{X}(0),0}$ is isomorphic to the Nash modification $\mathcal{N}C_{X,0}$, and since for any point (z,t) in the relative singular locus all the $c \times c$ minors of $[D_\varphi F]$ vanish, then we have the inclusion $\mathrm{Sing}_\varphi\mathfrak{X} \subset V(I)$. All that is left to prove, is that the blowup of I gives $\mathcal{N}_\varphi\mathfrak{X}$.

Let $x = (z,t)$ be a point in the relative smooth locus of $\mathfrak{X}$ and $T_x\mathfrak{X}(t)^0 = [a_0 : \ldots : a_N]$ denote the coordinates of the point of $\mathbb{P}^N$ corresponding to the direction of the tangent space to the fiber $\mathfrak{X}(t)$ at x by the Plucker embedding of the grassmannian $G(d, n+1)$ in the projective space $\mathbb{P}^N$. If (z,t) is sufficiently general then for each of the α^i's we have:

$$[F^{\alpha^i,\beta^0} : \cdots : F^{\alpha^i,\beta^N}] = [a_0 : \ldots : a_N]$$

where we have ordered the β's lexicographically. This means that there exist $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ such that for every $\alpha^1, \ldots, \alpha^l$ and $\beta^k \in S'$ we have:

$$F^{\alpha^i,\beta^k} = \lambda_i a_k$$

which implies that for each $\beta^k \in S'$:

$$\rho_{\beta^k}(x) = \sum_{i=1}^{l} h_i F^{\alpha^i,\beta^k}(x) = \sum_{i=1}^{l} h_i \lambda_i a_k = a_k \sum_{i=1}^{l} \lambda_i h_i$$

and so $[\rho_\beta(x)] = [a]$ in $\mathbb{P}^N$. Finally, since the $\lambda' s$ are non-zero constants, the function $\sum_{i=1}^{l} \lambda_i h_i$ cannot be identically zero. This implies that the equation $[\rho_\beta(x)] = [a]$ in $\mathbb{P}^N$ is true for every point x in an open dense set $U \subset \mathfrak{X}$ which finishes the proof. $\square$

Proposition 4.11. *Let $\nu_\varphi : \mathcal{N}_\varphi\mathfrak{X} \to \mathfrak{X}$ be the relative Nash modification of $\varphi : \mathfrak{X} \to \mathbb{C}$. Let $Z \subset \mathfrak{X}$ be the subspace defined by the ideal I of 4.10, and let D be the divisor defined by I in $\mathcal{N}_\varphi\mathfrak{X}$, that is $D = \nu_\varphi^{-1}(Z)$. If the germ $(X, 0)$ does not have exceptional cones, then $\nu_\varphi^{-1}(Z \setminus \mathfrak{X}(0))$ is dense in D. That is, the exceptional divisor D of $\mathcal{N}_\varphi\mathfrak{X}$ does not have* **vertical components** *over $\mathfrak{X}(0)$.*

Proof. We know that $\mathfrak{X}(0)$ is isomorphic to the tangent cone $C_{X,0}$. Now, by 4.10 the strict transform of $\mathfrak{X}(0)$ in $\mathcal{N}_\varphi(\mathfrak{X})$ is isomorphic to the Nash modification $\upsilon : \mathcal{N}C_{X,0} \to C_{X,0}$. Moreover, by the definition of blowup, $\upsilon^{-1}(Z \cap \mathfrak{X}(0))$ is a divisor (of dimension $d - 1$).

Now, by 1.17 if $(z, 0, T) \in \nu_\varphi^{-1}(z, 0) \subset \mathcal{N}_\varphi\mathfrak{X}$ then the $d-$plane T is via Γ a limit of tangent spaces to X at 0, that is the point $(0, T) \in \nu^{-1}(0) \subset \mathcal{N}X$. But, since by hypothesis the germ $(X, 0)$ does not have exceptional cones , then T is tangent to the tangent cone $C_{X,0}$.

We want to prove that the total transform $\nu_\varphi^{-1}(\mathfrak{X}(0))$ coincides with the strict transform $\mathcal{N}C_{X,0}$, that is, we need to prove that the point $(z, 0, T)$ is in $\mathcal{N}C_{X,0}$. For this purpose all that is now left to prove is that T is tangent to $C_{X,0}$ at the point $p = (z)$.

Let $\alpha : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (X, X^0, 0)$ be an arc such that its lift $\widetilde{\alpha}$ to $\mathcal{N}X$ has the point $(0, T) \in \mathcal{N}X$ as endpoint.

$$\begin{tikzcd}
 & & E \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \mathcal{N}X \arrow[dr, "\nu"'] & & C(X) \arrow[dl, "\kappa_X"] \\
\mathbb{C} \arrow[ur, "\widetilde{\alpha}"] \arrow[rr, "\alpha"] \arrow[uurr, dashed, bend left, "\alpha_H"] & & X &
\end{tikzcd}$$

By construction $\alpha(\mathbb{C} \setminus \{0\})$ is contained in the smooth locus X°, and if we denote by E° the inverse image $p_1^{-1}(\nu^{-1}(X^0))$, then by 1.25 the open subset E° is dense in E, and it defines a locally trivial fiber bundle over $\nu^{-1}(X^\circ)$. This implies that for any point $(0, T, H) \in p_1^{-1}(0, T)$ the arc $\widetilde{\alpha}$ can be lifted to an arc α_H having the point $(0, T, H)$ as endpoint. So now we have transformed the problem into proving that any hyperplane $H \in \check{\mathbb{P}}^n$, such that $T \subset H$, is a tangent hyperplane to $C_{X,0}$ at the point $p = z$.

Going back again to the diagram of 1.17:

$$\begin{tikzcd} \mathcal{N}_\varphi\mathfrak{X} \arrow[r, "\Gamma"] \arrow[d, "\nu_\varphi"'] & \mathcal{N}X \arrow[d, "\nu"] \\ \mathfrak{X} \arrow[r, "\phi"'] & X \end{tikzcd}$$

we have that for any sequence $\{(z_m, t_m)\}$ in the smooth part of $\mathfrak{X} \setminus \mathfrak{X}(0)$ tending to the point $(z, 0)$ in the special fiber $\mathfrak{X}(0)$, we have a corresponding sequence $\{(t_m z_m)\}$ tending to the origin in X. The final step of the proof is now a consequence of the projective duality obtained from the normal-conormal diagram:

$$\begin{tikzcd} E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[d, "\kappa'"'] \arrow[dr, "\zeta"] & C(X) \arrow[d, "\kappa"] \\ E_0X \arrow[r, "e_0"'] & X \end{tikzcd}$$

since the sequence $\{t_m z_m\} \subset X \setminus \{0\}$ tending to the origin gives us the sequence $\{(t_m z_m), [z_m]\}$ in E_0X, the blowup of X at 0, which tends to the point $(0, [z])$ in the exceptional divisor $\mathbb{P}C_{X,0}$. In the same way, we obtain the sequence $\{(t_m z_m, [z_m], H_m)\}$ in $E_0C(X) \subset X \times \mathbb{P}^n \times \check{\mathbb{P}}^n$ tending to the point $(0, [z], H)$ in $F = \zeta^{-1}(0)$. Recall that if $|F| = \bigcup_\alpha F_\alpha$ is the irreducible decomposition of the reduced space $|F|$, then each F_α is the conormal space of an irreducible component of $\mathbb{P}C_{X,0}$. To finish the proof, note that so far we have proved that $\nu_\varphi^{-1}(\mathfrak{X}(0))$ is just $\mathcal{N}C_{X,0}$ and so $\nu_\varphi^{-1}(Z(0))$ is of dimension $d-1$, whereas an irreducible component of D is of dimension d. $\square$

Corollary 4.12. *Let* $\mathrm{Sing}\mathfrak{X}(0)$ *denote the singular locus of the special fiber, then the dimension of* $\nu_\varphi^{-1}(\mathrm{Sing}\mathfrak{X}(0))$ *is less or equal than* $d-1$.

Proof. By definition of the ideal I, the analytic subset $\mathrm{Sing}\mathfrak{X}(0)$ is contained in the subspace Z defined by I. Then we have the inclusion $\nu_\varphi^{-1}(\mathrm{Sing}\mathfrak{X}(0)) \subset \nu_\varphi^{-1}(Z(0))$ and by proposition 4.11 the dimension of $\nu_\varphi^{-1}(Z(0))$ is equal to $d-1$ which finishes the proof. $\square$

Note that the following result does not uses the irreducible hypothesis, and so is valid in a more general setting.

Lemma 4.13. *Let* Y *denote the smooth subspace* $0 \times \mathbb{C} \subset \mathfrak{X}$ *as before, let* $\nu : \mathcal{N}X \to X$ *be the Nash modification of* X*, and let* $\widetilde{\nu_\varphi} : \widetilde{\mathcal{N}_\varphi\mathfrak{X}} \to \mathfrak{X}$ *be the normalized relative Nash modification of* $\mathfrak{X}$*. Then:*

1. *If the germ* $(X, 0)$ *does not have exceptional cones we have the set-theoretical equality:*
$$\left|\nu_\varphi^{-1}(Y)\right| = \left|Y \times \nu^{-1}(0)\right|$$
2. *The set theoretical inverse image* $|\widetilde{\nu_\varphi}^{-1}(Y \setminus \{0\})|$ *is dense in* $|\widetilde{\nu_\varphi}^{-1}(Y)|$.

Proof. From proposition 1.17 we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{N}_\varphi\mathfrak{X} & \xrightarrow{\Gamma} & \mathcal{N}X \\ \downarrow{\scriptstyle \nu_\varphi} & & \downarrow{\scriptstyle \nu} \\ \mathfrak{X} & \xrightarrow[\phi]{} & X \end{array}$$

where ϕ and Γ are surjective. The morphism ϕ is the restriction to $\mathfrak{X}$ of the map $\mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}^{n+1}$ defined by $(z_0, \ldots, z_n, t) \mapsto (tz_0, \ldots, tz_n)$ which is an isomorphism on $\mathbb{C}^{n+1} \times \mathbb{C}^*$. This implies in particular that the restriction of the differential $D\phi$ to the tangent space $T_{(z,t)}\mathfrak{X}(t)$ maps it isomorphically to $T_{(tz)}X$, where (z,t) is a smooth point of the fiber $\mathfrak{X}(t)$ with $t \neq 0$. But the restriction of $D\phi$ to $T_{(z,t)}\mathfrak{X}(t)$ is t times the identity Id, which implies that $\nu_\varphi^{-1}(Y \setminus \{(0,0)\} = Y \setminus \{(0,0)\} \times \nu^{-1}(0)$ and as a consequence $\nu_\varphi^{-1}(0,0)$ contains $\nu^{-1}(0)$. Finally, from the proof of proposition 4.11 we know that the fiber $\nu_\varphi^{-1}(\mathfrak{X}(0))$ is equal to the Nash modification of the tangent cone $C_{X,0}$, so the fiber $\nu_\varphi^{-1}(0,0)$ is equal to the set of limits of tangent spaces to $C_{X,0}$ which coincides with $\nu^{-1}(0)$ since the germ $(X,0)$ doesn't have exceptional cones.

To prove 2), note that since $\nu_\varphi^{-1}(Y)$ has a product structure we already have that $\nu_\varphi^{-1}(Y \setminus \{0\})$ is dense in Y, and so we need to study how the normalisation $n : \widetilde{\mathcal{N}_\varphi\mathfrak{X}} \to \mathcal{N}_\varphi\mathfrak{X}$ affects this subspace. Let $(0,0,T) \in \mathcal{N}_\varphi\mathfrak{X}$ be a point over the origin in $\mathfrak{X}$. Since by hypothesis $\mathfrak{X}$ is irreducible, the space $\mathcal{N}_\varphi\mathfrak{X}$ is also irreducible, however it may not be locally irreducible so the germ $(\mathcal{N}_\varphi\mathfrak{X}, (0,0,T))$ may have an irreducible decomposition of the form $(\mathcal{N}_\varphi\mathfrak{X}, (0,0,T)) = \bigcup_j (W_j, (0,0,T))$. Now, by [dP00, Section 4.4], we have that the normalisation map is finite, and over $(\mathcal{N}_\varphi\mathfrak{X}, (0,0,T))$ in the normalised space $\widetilde{\mathcal{N}_\varphi\mathfrak{X}}$ we have a multigerm $\bigsqcup_j (\widetilde{W_j}, p_j)$ such that:

1. The germ $(\widetilde{W_j}, p_j)$ is irreducible, and corresponds to the normalisation of $(W_j, (0,0,T))$.
2. For every j we have that $n^{-1}(0,0,T) \cap \widetilde{W_j} = \{p_j\}$.

This implies that if $\nu_\varphi(\nu_\varphi^{-1}(Y) \cap W_j) = Y$, then set-theoretically $\widetilde{\nu_\varphi}^{-1}(Y \setminus \{0\}) \cap \widetilde{W_j}$ is dense in $\widetilde{\nu_\varphi}^{-1}(Y) \cap \widetilde{W_j}$, and so all we have to prove is that every W_j satisfies this condition.

Since the open set of relative smooth points $\mathfrak{X}_\varphi^0 \setminus \mathfrak{X}(0)$ is dense in $\mathfrak{X}$, then its preimage $\nu_\varphi^{-1}(\mathfrak{X}_\varphi^0 \setminus \mathfrak{X}(0))$ is dense in $\mathcal{N}_\varphi\mathfrak{X}$ and so it intersects every irreducible component W_j in an open dense set U_j. This means that there exists an arc contained in U_j

$$\begin{aligned} \mu : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) &\to (W_j, U_j, (0,0,T)) \\ \tau &\mapsto (z(\tau), t(\tau), T(\tau)) \end{aligned}$$

having $(0,0,T)$ as endpoint, moreover by composing it with ν_φ we get an arc

$$\tilde{\mu} : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}, \mathfrak{X}_\varphi^0 \setminus \mathfrak{X}(0), (0,0))$$

contained in $\mathfrak{X}^0_\varphi \setminus \mathfrak{X}(0)$ having the origin as endpoint.

Let $\widetilde{\mu} = (z(\tau), t(\tau))$ and let $\alpha \in \mathbb{C}^*$, then by propositions 1.41 and 1.17, this arc can be "verticalized" to an arc $\widetilde{\mu_\alpha} : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathfrak{X}(\alpha), \mathfrak{X}^0(\alpha), (0, \alpha))$ as follows:

$$\begin{gathered}(\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to \Big(\mathfrak{X}, \mathfrak{X}^0_\varphi \setminus \mathfrak{X}(0), (0,0)\Big) \longrightarrow (X, X^0, 0) \longrightarrow \Big(\mathfrak{X}(\alpha), \mathfrak{X}(\alpha)^0, (0, \alpha)\Big) \\ \tau \mapsto (z(\tau), t(\tau)) \longmapsto (t(\tau)z(\tau)) \longmapsto \left(\frac{t(\tau)z(\tau)}{\alpha}, \alpha\right)\end{gathered}$$

Since the canonical isomorphism between two fibers $\mathfrak{X}(\alpha_1)$ and $\mathfrak{X}(\alpha_2)$ used here is given by $(z, \alpha_1) \mapsto (\frac{\alpha_1}{\alpha_2} z, \alpha_2)$, for every smooth point the tangent map acts as $\frac{\alpha_1}{\alpha_2}$ times the identity on the embedded tangent space leaving it invariant. Now, since the arc is contained in the smooth locus $\mathfrak{X}^0(\alpha)$ it has a unique lift to an arc

$$\mu_\alpha : (\mathbb{C}, \mathbb{C} \setminus \{0\}, 0) \to (\mathcal{N}_\varphi \mathfrak{X}, \nu_\varphi^{-1}(\mathfrak{X}^0_\varphi), (0, \alpha, T))$$

having as endpoint the point $(0, \alpha, T)$. Moreover for every τ_0 close enough to the origin in $\mathbb{C}$ the point $(z(\tau_0), t(\tau_0), T(\tau_0))$ is in W_j and since the arc $\mu_{t(\tau_0)}$ passes through this point, then it is completely contained in W_j, in particular the endpoint $(0, t(\tau_0), T)$ is in W_j which finishes the proof. $\square$

We are now in position to prove that $\frac{\partial F_\alpha}{\partial t}$ is strictly dependent on $JM_\varphi(F)_\alpha$ at 0.

Proposition 4.14. *If the germ $(X, 0)$ does not have exceptional cones then every minor M in $J_c(JM(F)_\alpha)$ depending on $\frac{\partial F_\alpha}{\partial t}$ satisfies $M \in J_c(JM_\varphi(F)_\alpha)^\dagger$ in $O_{X,0}$.*

Proof. Let M be a minor in $J_c(JM(F))$ that depends on $\frac{\partial F_\alpha}{\partial t}$, and let $W \subset \mathfrak{X}$ be the subspace defined by the ideal $J_c(JM_\varphi(F)_\alpha)$. Note that by definition, not only the t-axis Y, but the entire relative singular locus $\mathrm{Sing}_\varphi \mathfrak{X}$ is contained in W. Let $\widetilde{\nu_\varphi} : \widetilde{\mathcal{N}_\varphi \mathfrak{X}} \to \mathfrak{X}$ be the normalized blowup of $\mathfrak{X}$ along $J_c(JM_\varphi(F)_\alpha)$, and let $\overline{D}$ be its exceptional divisor. By considering a small enough neighborhood of the origin in $\mathfrak{X}$, or in other words a small enough representative of the germ $(\mathfrak{X}, 0)$ we can assume that the divisor $\overline{D}$ has a finite number of irreducible components, and every irreducible component of $\overline{D}$ intersects $\widetilde{\nu_\varphi}^{-1}(0)$. Thanks to the fact that each irreducible component $\overline{D}_k$ is mapped by the normalisation map $n : \widetilde{\mathcal{N}_\varphi \mathfrak{X}} \to \mathcal{N}_\varphi(\mathfrak{X})$ to an irreducible component D_j of $D = |\nu_\varphi^{-1}(W)|$ we can check these conditions in $\mathcal{N}_\varphi(\mathfrak{X})$.

Let $b \in \overline{D}$ be a point in the exceptional divisor lying over $W(0)$. Now, since $\overline{D}$ is a divisor, the ideal $J_c(JM_\varphi(F)_\alpha) \circ \widetilde{\nu_\varphi}$ is locally invertible, so at each $b \in \overline{D}(0)$ it is generated by a single element $g \circ \widetilde{\nu_\varphi}$, where $g \in J_c(JM_\varphi(F)_\alpha)$. By proposition 3.10, we need to prove that for every such b the function $M \circ \widetilde{\nu_\varphi}$ lies in the product $I(Y, \overline{D}_k) J_c(JM_\varphi(F)_\alpha) \circ \widetilde{\nu_\varphi}$, or equivalently (from the proof of the proposition) that the meromorphic function k locally defined by $\frac{M \circ \widetilde{\nu_\varphi}}{g \circ \widetilde{\nu_\varphi}}$ is holomorphic and vanishes at b if b lies over $(0,0) \in Y$.

Note that if $\widetilde{\nu_\varphi}(b)$ is not in Y then the ideal $I(Y, \overline{D}_k) O_{\widetilde{\mathcal{N}_\varphi \mathfrak{X}}, b}$ is not a proper ideal and so all we need to prove is that $M \circ \widetilde{\nu_\varphi}$ belongs to the ideal $J_c(JM_\varphi(F)_\alpha) \circ \widetilde{\nu_\varphi}$, which by proposition 3.10 is equivalent to k being holomorphic and also to $M \in \overline{J_c(JM_\varphi(F)_\alpha)}$. Now, by remark 4.9-1, for any point $(z,t) \in \mathfrak{X} \setminus \mathfrak{X}(0)$ we already have $M \in \overline{J_c(JM_\varphi(F)_\alpha)}$ which implies that the function k is holomorphic on $\overline{D} \setminus \overline{D}(0)$, and so its polar locus is contained in $\overline{D}(0)$.

Let $(z,0) \in W$ such that $(z,0)$ is not in $\mathrm{Sing}_\varphi \mathfrak{X}$, that is $(z,0)$ is a smooth point of both the space $\mathfrak{X}$ and the special fiber $\mathfrak{X}(0)$. Then, the vertical hyperplane $H = [0 : \cdots : 0 : 1] \in \check{\mathbb{P}}^{n+1}$ cannot be tangent to $\mathfrak{X}$ at $(z,0)$ and so by remark 4.9-1 we have $M \in \overline{J_c(JM_\varphi(F)_\alpha)}$ and k holomorphic. Indeed, if H is tangent to $\mathfrak{X}$ at the point $(z,0)$, then the point $(z,0)$ is a singular point of $\mathfrak{X} \cap H = \mathfrak{X}(0)$. This implies that the polar locus of k is contained in $\widetilde{\nu_\varphi}^{-1}(\mathrm{Sing}\mathfrak{X}(0))$, but by corollary 4.12 the dimension of $\nu_\varphi^{-1}(\mathrm{Sing}\mathfrak{X}(0))$ is less than or equal to $d-1$, and since the normalisation map is finite we also have dim $\widetilde{\nu_\varphi}^{-1}(\mathrm{Sing}\mathfrak{X}(0)) < d$, that is, it has codimension at least 2. However, in a normal space the polar locus of a meromorphic function is of codimension 1 or empty ([KK83, Thm. 71.12, pg 307]), which implies that k is holomorphic at every point $b \in \overline{D}$.

All that is left to prove is that the holomorphic function k vanishes at every point $b \in \overline{D}$ lying over Y. Since for any point $y \neq 0 \in Y$ the pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at y we have that k vanishes on $\widetilde{\nu_\varphi}^{-1}(Y \setminus \{(0,0)\})$, and by continuity it vanishes on its closure in $\widetilde{\mathcal{N}_\varphi \mathfrak{X}}$. But by lemma 4.13-2 the closure of the reduced inverse image $|\widetilde{\nu_\varphi}^{-1}(Y \setminus \{(0,0)\})|$ is equal to $|\widetilde{\nu_\varphi}^{-1}(Y)|$, and so we have that the function k vanishes at any point b lying over $(0,0) \in Y$. □

Let $Z \subset \mathfrak{X}$ be the subspace defined by the ideal I of 4.10 as before. Note that the key point in proving the previous proposition is the inequality dim $\nu_\varphi^{-1}(\mathrm{Sing}\mathfrak{X}(0)) < d$ which was a consequence of 4.11 and this gives us the following result.

Proposition 4.15. *Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a reduced and irreducible d dimensional germ of analytic singularity such that the tangent cone is reduced. Then $(X,0)$ does not have exceptional cones if and only if $\nu_\varphi^{-1}(Z)$ does not have vertical components over $\mathfrak{X}(0)$.*

Proof. If $(X,0)$ does not have exceptional cones, then it is proposition 4.11. On the other hand, if $\nu_\varphi^{-1}(Z)$ does not have vertical components over $\mathfrak{X}(0)$ then corollary 4.12 and the proof of proposition 4.14 gives us that the pair $(\mathfrak{X}^0, Y)_0$ satisfies Whitney's condition a) at the origin, and by 4.5 this is equivalent to $(\mathfrak{X},0)$ having no exceptional cones. Finally, this implies that $(X,0)$ does not have exceptional cones either. □

This proposition is also valid, with the same proof, if $(X,0)$ is reduced with a reduced complete intersection tangent cone.

Remark 4.16. *Note that if $(X,0)$ has exceptional cones then, $(\mathfrak{X},0)$ also has exceptional cones.*

Indeed, if $\kappa_\mathfrak{X} : C(\mathfrak{X}) \to \mathfrak{X}$ is the conormal space of $\mathfrak{X}$ and $\kappa_X : C(X) \to X$ the conormal space of X, then $\kappa_\mathfrak{X}^{-1}(Y \setminus \{0\}) = Y \setminus \{0\} \times \kappa_X^{-1}(0)$ and so $\kappa_\mathfrak{X}^{-1}(Y)$ contains $Y \times \kappa_X^{-1}(0)$. In particular, if $H = [a_0 : \cdots a_n] \in \kappa_X^{-1}(0) \subset \check{\mathbb{P}}^n$, but H is not tangent to the tangent cone $C_{X,0}$, then $\widetilde{H} = [a_0 : \cdots : a_n : 0] \in \kappa_{\mathfrak{X}}^{-1}(0) \subset \check{\mathbb{P}}^{n+1}$ and it can not be tangent to the tangent cone $C_{\mathfrak{X},0} = C_{X,0} \times \mathbb{C}$.

Proposition 4.11 will allow us to derive further properties of the space Z. Consider the map $h : Z \to \mathbb{C}$ defined by the inclusion $i : Z \hookrightarrow \mathfrak{X}$ followed by $\varphi : \mathfrak{X} \to \mathbb{C}$.

Lemma 4.17. *If the map $h : (Z, 0) \to (\mathbb{C}, 0)$ is not flat, then the analytic germ $(Z, 0)$ has a possibly embedded irreducible component contained in the special fiber $Z(0)$.*

Proof. By definition, the map $h : (Z, 0) \to (\mathbb{C}, 0)$ is flat if the corresponding map of analytic algebras $h^* : \mathbb{C}\{t\} \to O_{Z,0}$ gives $O_{Z,0}$ the structure of a flat $\mathbb{C}\{t\}$-module. But a $\mathbb{C}\{t\}$-module is flat if and only if $h^*(t)$ is not a zero divisor Note that in our case $h^*(t) = t \in O_{Z,0} = O_{\mathfrak{X},0}/J_\varphi$.

Let $I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ be a minimal primary decomposition of I in $O_{\mathfrak{X},0}$. Now, t is a zero divisor in $O_{Z,0}$ if there exists an element $a \in O_{\mathfrak{X},0}$ such that $a \notin I$ but $ta \in I$. So there exists $i \in \{1, \ldots, r\}$ such that $a \notin Q_i$ but $ta \in Q_i$, since Q_i is a primary ideal this implies that $t^s \in Q_i$ for some $s \in \mathbb{N}$. Finally this implies that t belongs to the prime ideal $\sqrt{Q_i} = P_i$, and so the irreducible component $V(P_i)$ of $(Z, 0)$ is contained in $t = 0$. $\square$

Suppose h is a flat map, then the open set $Z \setminus Z(0)$ is dense in Z. The next result tells us that, even if we may not have flatness, the absence of vertical components of the exceptional divisor D implies that set-theoretically $Z \setminus Z(0)$ is dense in Z.

Corollary 4.18. *Let $\nu_\varphi : \mathcal{N}_\varphi(\mathfrak{X}) \to \mathfrak{X}$ and $(Z, 0) \subset (\mathfrak{X}, 0)$ be as before. Let $D = \nu_\varphi^{-1}(Z)$ be the exceptional divisor. If D does not have vertical components over $\mathfrak{X}(0)$, then set-theoretically, the closure of $Z \setminus Z(0)$ in $\mathfrak{X}$ is equal to Z.*

Proof. Lets consider the map $h : (Z, 0) \to (\mathbb{C}, 0)$ as before. If h is flat, we have nothing to prove, so suppose h is not flat. Then, by lemma 4.17, we can find a minimal primary decomposition of I in $O_{\mathfrak{X},0}$:

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_s$$

such that $t^{n_i} \in Q_i$ for $1 < r \leq i \leq s$ with $n_i > 0$, so each of these Q_i correspond to a possibly embedded irreducible component of the germ $(Z, 0)$ contained in the special fiber $Z(0)$.

Let $I = Q \cap B$, where $B = Q_r \cap \cdots \cap Q_s$. There exists a small neighbourhood of the origin $U \subset \mathfrak{X}$, such that $I(U) = Q(U) \cap B(U)$, and for every $x \in U$ we have the equality $I_x = Q_x \cap B_x$ in $O_{\mathfrak{X},x}$. But, for any open set $V \subset U$ such that $0 \notin V$, since $t^m \in B(V)$ and t^m is a unit in $O_{\mathfrak{X}}(V)$ we have that $I_x = Q_x$ in $O_{\mathfrak{X},x}$ for any point $x \in Z \setminus \{0\}$, so their integral closures are equal $\overline{I_x} = \overline{Q_x}$ for every point $x \in V$

Let $\widetilde{\nu_\varphi} : \widetilde{\mathcal{N}_\varphi(\mathfrak{X})} \xrightarrow{n} \mathcal{N}_\varphi(\mathfrak{X}) \xrightarrow{\nu_\varphi} \mathfrak{X}$ be the composition of ν_φ and the normalisation of $\mathcal{N}_\varphi(\mathfrak{X})$. By hypothesis, D does not have vertical components over the origin, and since the normalisation is a finite map, we have that $\overline{D} = \widetilde{\nu_\varphi}^{-1}(Z) = n^{-1}(D)$ does not have vertical components over the origin either. Let $w \in Q$, then for U

sufficiently small $w \in Q(U)$. Now, we know that the coherent ideal $\widetilde{I} := IO_{\widetilde{\mathcal{N}_\varphi(\mathfrak{X})}}$ is locally invertible, so in particular for any point $p \in \overline{D}$ there exists an open neighborhood V_p of p in $\widetilde{\mathcal{N}_\varphi(\mathfrak{X})}$ such that $\widetilde{I}(V_p) = \langle g_p \rangle \, O_{\widetilde{\mathcal{N}_\varphi(\mathfrak{X})}}(V_p)$.

For any such neighborhood, we can consider the meromorphic function $q := (w \circ \widetilde{\nu_\varphi})/g_p$. The polar locus of q is contained in $\overline{D}$, more precisely, since the ideal $\widetilde{I}$ and $\widetilde{Q}$ coincide outside $\widetilde{\nu_\varphi}^{-1}(0)$, we have that the polar locus of q is contained in $\widetilde{\nu_\varphi}^{-1}(Z(0))$. But $\overline{D}$ does not have vertical components over $\mathfrak{X}(0)$ so $\widetilde{\nu_\varphi}^{-1}(Z(0))$ is of codimension at least 2. Since in a normal space the polar locus of a meromorphic function is of codimension one or empty ([KK83, Thm. 71.12,pg 307]), q is actually holomorphic and $\widetilde{I} = \widetilde{Q}$ in $\widetilde{\mathcal{N}_\varphi(\mathfrak{X})}$, which implies by theorem C.3 that the integral closures $\overline{I} = \overline{Q}$ are equal in $O_{\mathfrak{X},0}$.

Finally, since the integral closure of an ideal is contained in its radical, then set theoretically Z is the zero locus of $\overline{I}$, that its $|Z| = V(\overline{I}) = V(\overline{Q}) = V(Q)$ and it does not have vertical components over the origin.

□

We can summarize all we have done so far with the following theorem:

Theorem 4.19. *Let $(X,0)$ be a reduced and equidimensional germ of complex analytic singularity, and suppose that its tangent cone $C_{X,0}$ is reduced. Then the following statements are equivalent:*

1. *The germ $(X,0)$ does not have exceptional cones.*
2. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's condition a) at the origin.*
3. *The pair $(\mathfrak{X}^0, Y)$ satisfies Whitney's conditions a) and b) at the origin.*
4. *The germ $(\mathfrak{X},0)$ does not have exceptional cones.*

Proof. Let $(X,0) = \bigcup_{i=1}^r (X_i,0)$ be the irreducible decomposition of $(X,0)$. Then by corollary 2.6, and lemma 1.13 it is enough to verify these equivalences for each irreducible component $(X_j,0)$ and its specialisation space $(\mathfrak{X}_j,0)$. Now for an irreducible germ we have:
1) $\Rightarrow$ 2) by proposition 4.14.
2) $\Rightarrow$ 3) by proposition 4.1.
3) $\Rightarrow$ 4) by 4.5.
4) $\Rightarrow$ 1) by remark 4.16. □

In the case when $(X,0)$ is reduced with a reduced complete intersection tangent cone, the ideal I can be taken to be the ideal $J_c(JM_\varphi(F))$ as established in remark 4.9-2. The fact that the analytic set defined by this ideal is the relative singular locus $\mathrm{Sing}_\varphi \mathfrak{X}$ allows us to get some information regarding the exceptional cones.

Corollary 4.20. *If both $(X,0)$ and its tangent cone are complete interserction singularities, then the equivalent conditions of theorem 4.19 are also equivalent to:*
The divisor $D = \nu_\varphi^{-1}(Z)$ of the relative Nash modification $\mathcal{N}_\varphi \mathfrak{X} \to \mathfrak{X}$, does not have vertical components over $Z(0)$, where Z is set-theoretically the relative singular locus $\mathrm{Sing}_\varphi \mathfrak{X}$.

Proof. In this case proposition 4.15 gives us the equivalence of this condition with the absence of exceptional cones. ☐

Corollary 4.21. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a reduced germ of singularity such that the tangent cone $C_{X,0}$ is a reduced complete intersection. Let $|\mathrm{Sing}\, C_{X,0}| = \bigcup E_\alpha$ be the irreducible decomposition of the singular locus of the tangent cone. If there exists an α, such that E_α is not completely contained in the reduced tangent cone $|C_{\mathrm{Sing\,X},0}|$, then it is contained in an exceptional cone. In particular we have the inclusion*

$$|\mathrm{Sing}\, C_{X,0}| \subset \left|C_{|\mathrm{Sing\,X}|,0}\right| \bigcup \{\text{Exceptional cones}\}$$

Proof. Let $\varphi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ be the specialization space of X to its tangent cone $(C_{X,0}, 0)$, and let $\nu_\varphi : \mathcal{N}_\varphi(\mathfrak{X}) \to \mathfrak{X}$ be considered as the blowup of $\mathfrak{X}$ with center $Z \subset \mathfrak{X}$ defined by the ideal $J_c(JM_\varphi(F))$, and exceptional divisor $D \subset \mathcal{N}_\varphi(\mathfrak{X})$. Since set-theoretically Z is the relative singular locus of $\mathfrak{X}$, then if we set W as the closure of $Z \setminus Z(0)$ in $\mathfrak{X}$, then set theoretically $W(0)$ is $|C_{|\mathrm{Sing\,X}|,0}|$, so the existence of the E_α in the hypothesis amounts to Z having a vertical (irreducible) component Z_β over the origin.

The existence of such a Z_β implies by 4.18 the existence of a vertical component D_β of $|D|$, which then implies by 4.11 that the germ $(X, 0)$ has exceptional cones. Now for any point $z \in Z_\beta \setminus W$ there exists an open neighborhood $z \in U_z \subset \mathfrak{X}$ such that $U_z \cap W = \emptyset$ and $Z_\beta \setminus W$ is dense in Z_β. That is, there exists an open neighborhood U of $Z_\beta \setminus W$ in $\mathfrak{X}$, such that $U \cap W = \emptyset$, and so $\nu_\varphi^{-1}(U \cap W) = \nu_\varphi^{-1}(U) \cap \nu_\varphi^{-1}(W) = \emptyset$. But $\nu_\varphi^{-1}(W)$ contains $\overline{D \setminus D(0)}$, and $\nu_\varphi^{-1}(U) \cap D$ is not empty, so there is necessarily an irreducible component D_β of D, such that $D_\beta \supset \nu_\varphi^{-1}(Z_\beta)$ and D_β is completely contained in $D(0)$. All that is left to prove is that the component D_β is mapped by ν_φ into an exceptional cone.

By remark 4.9, the strict transform $\overline{\nu_\varphi^{-1}(\mathfrak{X}(0) \setminus Z)}$ is equal to the Nash modification of the fiber $\mathfrak{X}(0)$ which has dimension d, on the other hand since D_β is an irreducible component of the divisor D it is also of dimension d and so cannot be contained in $\mathcal{N}\mathfrak{X}(0)$, i.e. $D_\beta \nsubseteq \mathcal{N}\mathfrak{X}(0)$.

Now, by [LT88][Proposition 2.1.4.1, pg 562], the cones of the aureole are set theoretically the images by κ_φ of the irreducible components of $|\kappa_\varphi^{-1}(\mathfrak{X}(0))|$. So let us consider the relative version of the diagram given in proposition 1.25, relating the relative Nash modification $\mathcal{N}_\varphi\mathfrak{X}$ with the relative conormal space $C_\varphi(\mathfrak{X})$.

$$\begin{tikzcd}
 & E_\varphi \arrow[r, hook] \arrow[dl, "p_1"'] \arrow[drr, "p_2"] & \mathfrak{X} \times G(n+1-d, n+1) \times \check{\mathbb{P}}^n & \\
\mathcal{N}_\varphi\mathfrak{X} \arrow[dr, "\nu_\varphi"'] & & & C_\varphi(\mathfrak{X}) \arrow[dll, "\kappa_\varphi"] \\
 & \mathfrak{X} & &
\end{tikzcd}$$

By commutativity of the diagram, we have the equality $p_2(p^{-1}(\mathcal{N}\mathfrak{X}(0))) = C(\mathfrak{X}(0))$, where $C(\mathfrak{X}(0))$ denotes the conormal space of the fiber $\mathfrak{X}(0)$ and it is equal to

$\overline{\kappa_\varphi^{-1}(\mathfrak{X}(0) \setminus Z)}$. This implies that the space $\widetilde{D_\beta} := p_2(p_1^{-1}(D_\beta))$ cannot be contained in $C(\mathfrak{X}(0))$. Now, the conormal space $C(\mathfrak{X}(0))$ is of dimension n, and since $C_\varphi(\mathfrak{X}) \to \mathfrak{X} \to \mathbb{C}$ is isomorphic to the specialization space of $C(X)$ to its normal cone along $\kappa_X^{-1}(0)$ (see Prop. 1.43), then the dimension of $\kappa_\varphi^{-1}(\mathfrak{X}(0))$ is also n. This means that $\widetilde{D_\beta}$ is contained in an irreducible component of $|\kappa_\varphi^{-1}(\mathfrak{X}(0))|$ outside of $C(\mathfrak{X}(0))$ and so is mapped by κ_φ into an exceptional cone. $\square$

Note that we always have the inclusion $\left|C_{|\mathrm{Sing\,X}|,0}\right| \subset |\mathrm{Sing}\, C_{X,0}|$, so the absence of exceptional cones together with 4.21 tells us that in this setting not only do the relative singular locus, and the singular locus of $\mathfrak{X}$ coincide, but also that $\overline{\mathrm{Sing}\,\mathfrak{X} \setminus \mathfrak{X}(0)} = \mathrm{Sing}\,\mathfrak{X}$. In particular we have $\left|C_{|\mathrm{Sing\,X}|,0}\right| = |\mathrm{Sing}\, C_{X,0}|$ and this leaves us in a good position to continue building a Whitney stratification of $\mathfrak{X}$ having Y as a stratum.

Corollary 4.22. *Let $(X,0)$ satisfy the hypothesis of theorem 4.19. If $(X,0)$ has an isolated singularity and its tangent cone is a complete intersection singularity, then the absence of exceptional cones implies that $C_{X,0}$ has an isolated singularity and $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.*

Proof. Proposition 4.21 tells us that $\left|C_{|\mathrm{Sing\,X}|,0}\right| = |\mathrm{Sing}\, C_{X,0}|$, and since $(X,0)$ has an isolated singularity then $\left|C_{|\mathrm{Sing\,X}|,0}\right| = \{0\}$ and so the tangent cone $(C_{X,0},0)$ also has an isolated singularity. This implies, that $\mathrm{Sing}\mathfrak{X} = Y$, and theorem 4.19 finishes the proof. $\square$

There is a partial converse to 4.22, in which we can construct a Whitney stratification of $\mathfrak{X}$ under the assumption that the tangent cone has an isolated singularity at the origin.

Corollary 4.23. *Let $(X,0)$ satisfy the hypothesis of theorem 4.19. If the tangent cone $(C_{X,0},0)$ has an isolated singularity at the origin, then $(X,0)$ has an isolated singularity and $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.*

Proof. The first step is to prove that $(X,0)$ doesn't have exceptional cones, however by [LT88, Prop. 2.1.4.2, p. 563] this is always the case when the tangent cone has an isolated singularity at the origin.

Now, by theorem 4.19, it is enough to prove that the singular locus of $\mathfrak{X}$ is Y. It is a general fact that the relative singular locus $\mathrm{Sing}_\varphi\mathfrak{X}$ of $\mathfrak{X}$, contains the singular locus $\mathrm{Sing}\mathfrak{X}$, and they coincide away from the special fiber. In other words, the space $W := \mathrm{Sing}_\varphi\mathfrak{X} \setminus \{\mathfrak{X}(0)\}$ is isomorphic via $\phi : \mathfrak{X} \setminus \mathfrak{X}(0) \to X \times \mathbb{C}^*$ to $\mathrm{Sing}X \times \mathbb{C}^*$, and so the map induced by φ to its closure $\overline{W} \to \mathbb{C}$ can be identified with the specialization space of $|\mathrm{Sing}X|$ to its tangent cone. In view of this, the hypothesis tells us that the only singular point of $\mathfrak{X}$ in the special fiber is the origin $(0,0)$; this implies $\overline{W}(0) = \{0\}$ and since it is isomorphic to the tangent cone $C_{|\mathrm{Sing}X|,0}$, then $(X,0)$ has an isolated singularity and $\mathrm{Sing}\mathfrak{X} = Y$ which finishes the proof. $\square$

Example 4.24. Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a reduced and irreducible germ of isolated hypersurface singularity ($n \geq 2$), then the projectivized tangent cone $\mathbb{P}C_{X,0}$ is smooth if and only if $(X,0)$ is a superisolated singularity and it is equisingular with its tangent cone $C_{X,0}$.

By definition a germ of isolated hypersurface singularity $(X, 0)$ is superisolated if the blowup of the point $E_0X \to X$ is a resolution of singularities. Let us fix a representative $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$, with defining equation $f \in \mathbb{C}\{z_0, \ldots, z_n\}$ of order m, that is $f = f_m + f_{m+1} + \cdots$, where f_i is an homogeneous polynomial of degree i. Then being superisolated is also equivalent to the projectivized tangent cone $\mathbb{P}C_{X,0}$ being reduced and $\operatorname{Sing} \mathbb{P}C_{X,0} \cap \{f_{m+1} = 0\} = \emptyset$ in $\mathbb{P}^n$.

Now, suppose the projectivized tangent cone $\mathbb{P}C_{X,0}$ is smooth, then we obviously have that $\mathbb{P}C_{X,0} \cap \{f_{m+1} = 0\} = \emptyset$ in $\mathbb{P}^n$ and so $(X, 0)$ is superisolated. Moreover, since the tangent cone to the singular locus of X is contained in the singular locus of $C_{X,0}$ we have that both $(X, 0)$ and its tangent cone have isolated singularities and corollary 4.23 tells us that $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.

On the other hand, suppose that $(X, 0)$ is superisolated, and Whitney equisingular with its tangent cone. If $\mathbb{P}C_{X,0} \subset \mathbb{P}^n$ is not smooth, then it has a singular point $[l]$. The corresponding line l in $\mathbb{C}^{n+1}$ is then singular in $C_{X,0}$ and is not contained in $C_{|\operatorname{Sing} X|,0} = \{0\}$, so by 4.21 it is in an exceptional cone of $(X, 0)$, which contradicts the equisingularity of $(X, 0)$ with its tangent cone by theorem 4.19.

Example 4.25. Let $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a reduced and irreducible isolated complete intersection variety defined by an homogeneous ideal $I_0 = \langle h_{m_1}, \ldots, h_{m_k} \rangle$, where m_i is the degree of the polynomial. That is, V is the cone over a smooth, complete intersection, projective variety.

Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ defined by the ideal $I = \langle h_1, \ldots, h_k \rangle$, where $h_i = h_{m_i} + P_i$ and $P_i \in \mathbb{C}\{z_0, \ldots, z_n\}$ is such that $\operatorname{ord}_0 P_i(z) > m_i$. Then:

– The germ $(X, 0)$ is a reduced complete intersection.

– The tangent cone $C_{X,0}$ is defined by the ideal I_0 and so it is isomorphic to V.

That X is a complete intersection can be seen by considering the analytic family $\{X_t\}$ defined by the $h_i^t := h_{m_i} + tP_i$ and the upper semicontinuity of fiber dimension. For the other assertion consider the radical idea $\widetilde{I} := \sqrt{I}$ defining $|X|$. This gives us the following inclusion of initial ideals

$$\mathrm{In}_{\mathfrak{M}} I_0 = I_0 \subset \mathrm{In}_{\mathfrak{M}} I \subset \mathrm{In}_{\mathfrak{M}} \widetilde{I}$$

and as a result the surjective morphism of analytic algebras:

$$\frac{\mathbb{C}\{z_0, \ldots, z_n\}}{I_0} \longrightarrow \frac{\mathbb{C}\{z_0, \ldots, z_n\}}{\mathrm{In}_{\mathfrak{M}} \widetilde{I}}$$
$$O_{V,0} \longrightarrow O_{C_{|X|,0}}$$

But V is irreducible, so $O_{V,0}$ is an integral domain and since both algebras have krull dimension $n + 1 - k$ they are isomorphic and $I_0 = \mathrm{In}_{\mathfrak{M}} \widetilde{I}$. Finally, this tells us that $\mathrm{In}_{\mathfrak{M}} \widetilde{I} = \langle \mathrm{In}_{\mathfrak{M}} h_1, \ldots, \mathrm{In}_{\mathfrak{M}} h_k \rangle$, which implies that $\widetilde{I} = \langle h_1, \ldots, h_k \rangle = I$ and so X is reduced and $C_{X,0} = V$.

Now, by construction, the specialization space $\varphi : \mathfrak{X} \to \mathbb{C}$ is defined by the equations $H_i(z, t) = t^{-m_i} h_i(tz)$ in $\mathbb{C}^{n+1} \times \mathbb{C}$ and since the tangent cone $C_{X,0}$ is reduced and has an isolated singularity at the origin, corollary 4.23 tells us that $\{\mathfrak{X} \setminus Y, Y\}$ is a Whitney stratification of $\mathfrak{X}$.

Appendix A

Graded Rings and Ideal of Initial Forms

Let R be a noetherian ring, and $I \subset J \subset R$ ideals such that

$$R \supset J \supset \cdots \supset J^i \supset J^{i+1} \supset \cdots$$

is a separated filtration in the sense that $\bigcap_{i=o}^{\infty} J^i = (0)$.

Take the quotient ring $A = R/I$, define the ideal $\tilde{J}_i := (J^i + I)/I \subset A$ and consider the induced filtration

$$A \supset \tilde{J} \supset \cdots \supset \tilde{J}_i \supset \tilde{J}_{i+1} \supset \cdots$$

note that in fact $\tilde{J}_i = \tilde{J}^i$.

Consider now the associated graded rings

$$gr_J R = \bigoplus_{i=0}^{\infty} J^i/J^{i+1}$$

$$gr_{\tilde{J}} A = \bigoplus_{i=0}^{\infty} \tilde{J}_i/\tilde{J}_{i+1}$$

Definition A.1. *Let $f \in I$, since $\bigcap_{i=o}^{\infty} J^i = (0)$, there exists a largest natural number k such that $f \in J^k$. Define the initial form of f with respect to J as*

$$in_J f := f \pmod{J^{k+1}} \in gr_J R$$

Using this define, the ***ideal of initial forms of* I** *as the ideal of $gr_J R$ generated by the initial forms of all the element of I.*

$$In_J I := < in_J f >_{f \in I} \subset gr_J R$$

Lemma A.2. *Using the notations defined above, the following sequence is exact:*

$$0 \longrightarrow In_J I \hookrightarrow gr_J R \xrightarrow{\phi} gr_{\tilde{J}} A \longrightarrow 0$$

that is, $gr_{\tilde{J}} A \cong gr_J R / In_J I$.

Proof.
First of all, note that

$$\mathfrak{I}_i/\mathfrak{I}_{i+1} \cong \frac{\frac{J^i+I}{I}}{\frac{J^{i+1}+I}{I}} \cong \frac{J^i+I}{J^{i+1}+I} \cong \frac{J^i}{I \cap J^i + J^{i+1}}$$

where the first isomorphism is just the definition, the second one is one of the classical isomorphism theorems and the last one comes from the onto map $J^i \to \frac{J^i+I}{J^{i+1}+I}$ defined by sending $x \mapsto x + J^{i+1} + I$. This last map, tells us that there are natural onto morphisms:

$$\begin{aligned} \varphi_i : \frac{J^i}{J^{i+1}} &\longrightarrow \frac{\mathfrak{I}_i}{\mathfrak{I}_{i+1}} \cong \frac{J^i}{I \cap J^i + J^{i+1}} \\ x + J^{i+1} &\longmapsto x + I \cap J^i + J^{i+1} \end{aligned}$$

which we use to define the onto graded morphism of graded rings $\phi : gr_J R \to gr_{\mathfrak{J}} A$. Now, all that is left to prove is that the kernel of ϕ is exactly $In_J I$.

Let $f \in I$ be such that $in_J f = f + J^{k+1} \in J^k/J^{k+1}$, then

$$\phi(in_J f) = \varphi_k(f + J^{k+1}) = f + I \cap J^k + J^{k+1} = 0$$

because $f \in I \cap J^k$, and since varying $f \in I$ we get a set of generators of the ideal in question then $In_J I \subset Ker\ \phi$.

For the other inclusion, let $g = \bigoplus \overline{g_k} \in Ker\ \phi$, where we use the notation $\overline{g_k} := g_k + J^{k+1} \in J^k/J^{k+1}$. Then, $\phi(g) = 0$ implies by homogeneity $\phi(\overline{g_k}) = \varphi_k(g_k + J^{k+1}) = 0$ for all k. Now, suppose $\overline{g_k} \neq 0$ then

$$\varphi(g_k + J^{k+1}) = g_k + I \cap J^k + J^{k+1} = 0$$

implies $g_k = f + h$, where $0 \neq f \in (I \cap J^k) \setminus J^{k+1}$ and h belongs to J^{k+1}. But, this means that $g_k \equiv f \pmod{J^{k+1}}$, which implies $g_k + J^{k+1} = in_J f$ and concludes the proof. $\square$

Appendix B

Polar Varieties

Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be a representative of a reduced germ of analytic singularity of pure dimension d. Let $D_{d-k+1} \subset \mathbb{C}^{n+1}$ be a linear subspace of codimension $d-k+1$, for $0 \leq k \leq d-1$, and let $L^{d-k} \subset \check{\mathbb{P}}^n$ be the dual space of D_{d-k+1}, that is the linear space of hyperplanes of $\mathbb{C}^{n+1}$ that contain D_{d-k+1}. Let $\kappa_X : C(X) \to X$ be the conormal space of X in $\mathbb{C}^{n+1}$, and consider the following diagram:

$$\begin{tikzcd} C(X) \arrow[r, hook] \arrow[d, "\kappa_X"] \arrow[dr, "\lambda"] & X \times \check{\mathbb{P}}^n \arrow[d, "pr_2"] \\ X & \check{\mathbb{P}}^n \end{tikzcd}$$

Proposition B.1. *For a sufficiently general D_{d-k+1}, the image $\kappa_X(\lambda^{-1}(L^{d-k}))$ is the closure in X of the set of points of X^0 which are critical for the projection*

$$\pi|_{X^0} : X^0 \to \mathbb{C}^{d-k+1}$$

induced by the projection $\mathbb{C}^{n+1} \to \mathbb{C}^{d-k+1}$ with kernel D_{d-k+1}.

Proof.
Note that $x \in X^0$ is critical for π, iff the tangent map $d_x\pi : T_xX^0 \to \mathbb{C}^{d-k+1}$ is not onto, that is iff dim $ker\ d_x\pi \geq k$ since dim $T_xX^0 = d$, and $ker\ d_x\pi = D_{d-k+1} \cap T_xX^0$. Now, note that the conormal space $C(X^0)$ of the nonsingular part of X is equal to $\kappa_X^{-1}(X^0)$ so by definition:

$$\lambda^{-1}(L^{d-k})\bigcap C(X^0) = \{(x,H) \in C(X) | x \in X^0,\ H \in L^{d-k},\ T_xX^0 \subset H\}$$

equivalently:

$$\lambda^{-1}(L^{d-k})\bigcap C(X^0) = \{(x,H), \in C(X) | x \in X^0,\ H \in \check{D},\ H \in \check{T}_xX^0\}$$

thus $H \in \check{D} \cap \check{T}_xX^0$, and from the equation $\check{D} \cap \check{T}_xX^0 = (D+T)\check{}$ we deduce that the intersection is not empty iff $D + T \neq \mathbb{C}^{n+1}$, which implies that dim $D \cap T \geq k$, and consequently $\kappa_X((x,H)) = x$ is a critical point.

Now, according to [Tei82, Chapter IV,1.3 pg 419], there exists an open dense set U_k in the grasmmannian of $n-d+k$-planes of $\mathbb{C}^{n+1}$ such that if $D \in U_k$, the intersection $\lambda^{-1}(L^{d-k})\bigcap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$. So, for any $D \in U$, since κ_X is a proper map, and as such closed, we have that $\kappa_X(\lambda^{-1}(L^{d-k})) = \kappa_X\left(\overline{\lambda^{-1}(L^{d-k}) \cap C(X^0)}\right) = \overline{\kappa_X(\lambda^{-1}(L^{d-k}) \cap C(X^0))}$, which finishes the proof.

(See [Tei82, Chap. 4, 4.1.1 pg 432] for a complete proof of a more general statement.) □

Definition B.2. *Under the notation and hypothesis of proposition B.1, define the* ***local polar variety****.*

$$P_k(X; L^{d-k}) = \kappa_X(\lambda^{-1}(L^{d-k}))$$

A priori, we have just defined $P_k(X; L^{d-k})$ set-theoretically, however we have the following result, for which a proof can be found in [Tei82, Chapter IV, 1.3.2 pg 421].

Proposition B.3. *The local polar variety $P_k(X; L^{d-k}) \subseteq X$ is a closed analytic subspace of X either of pure codimension k in X, or empty.*

Remark B.4. *It is important to have in mind the following easily verifiable facts:*

a) The fiber $\kappa_X^{-1}(x)$ over a regular point $x \in X^0$ in the (projectivized) conormal space $C(X)$ is of dimension $n-d$, so by semicontinuity of fiber dimension we have that dim $\kappa_X^{-1}(0) \geq n-d$.

b) The analytic set $\lambda^{-1}(L^{d-k})$ can be obtained by the intersection of $C(X)$ and $\mathbb{C}^{n+1} \times L^{d-k}$ in $\mathbb{C}^{n+1} \times \check{\mathbb{P}}^n$. However, the space $\mathbb{C}^{n+1} \times L^{d-k}$ is "linear", defined by $n-d+k$ linear equations, namely it is the intersection of this same number of "hyperplanes". Thus for a general L^{d-k}, this intersection is of pure dimension $n-n+d-k = d-k$.

c) Note that for a fixed L^{d-k}, the germ $(P_k(X; L^{d-k}), 0)$ is empty if and only if the intersection $\kappa_X^{-1}(0) \cap \lambda^{-1}(L^{d-k})$ is empty. Now, from a) we know that dim $\kappa_X^{-1}(0) = n-d+r$ with $r \geq 0$. Thus, by the exact same argument as in b), this implies that the polar variety is not empty, that is dim $\kappa_X^{-1}(0) \cap \lambda^{-1}(L^{d-k}) \geq 0$, if and only if $r \geq k$.

We have thus far defined a local polar variety that depends on both, the choice of the embedding $(X,0) \subset (\mathbb{C}^{n+1}, 0)$ and the choice of the linear space D_{d-k+1}. However, an important information that can be extracted from these polar varieties is its multiplicity at 0, and this number is an invariant provided an appropiate choice of the linear spaces used to define them.

Proposition B.5. *Let $(X,0) \subset (\mathbb{C}^{n+1}, 0)$ be as before, then for every $0 \leq k \leq d-1$ and a sufficiently general linear space $D_{d-k+1} \subset \mathbb{C}^{n+1}$ the multiplicity of the polar variety $P_k(X; L^{d-k})$ at 0 depends only on the analytic type of $(X,0)$.*

Proof.
See [Tei82, Chapter IV, Thm 3.1 pg 425]. □

This last result allows us to associate to any reduced, pure d-dimensional, analytic local algebra $O_{X,0}$ a sequence of d integers $(m_0, \ldots, m_{d-1})$, where m_k is the multiplicity of the polar variety $P_k(X; L^{d-k})$ at 0 calculated from any given embedding $(X,0) \subset (\mathbb{C}^{n+1}, 0)$, and a wise choice of D_{d-k+1}.

Remark B.6. *Seeing as how we have taken for a linear space L^{d-k} to be "sufficiently general" mean that it belongs to an open dense subset specified by certain conditions, we can just as well take a sufficiently general flag*

$$L^1 \subset L^2 \subset \cdots \subset L^{d-2} \subset L^{d-1} \subset \check{\mathbb{P}}^n$$

which by definition of a polar variety and proposition B.5, gives us a chain

$$P_{d-1}(L^1; X) \subset P_{d-2}(L^2; X) \subset \cdots \subset P_1(L^{d-1}; X) \subset X$$

each of which with generic multiplicity at the origin. This implies, that if the germ of a general polar variety $(P_k(X; L^{d-k}), 0)$ is empty for a fixed k, then it will be empty for all $d-1 \geq l \geq k$. This fact can also be deduced, from B.4 c) by counting dimensions.

Example B.7.
Let $X := y^2 - x^3 - t^2x^2 = 0 \subset \mathbb{C}^3$, so dim $X = 2$, and thus $k = 0, 1$. An easy calculation shows that the singular loci of X is the $t-$axis, and $m_0(X) = 2$.

Now, note that for $k = 0$, D_3 is just the origin in $\mathbb{C}^3$, so the projection

$$\pi : X^0 \to \mathbb{C}^3$$

with kernel D_3 is the restriction to X^0 of the identity map, and as a result of the dimension we get that the whole X^0 is the critical set of such map. Thus,

$$P_0(X, L^2) = X.$$

For $k = 1$, D_2 is of dimension 1. So let us take for instance $D_2 = y-$axis, so we get the projection

$$\pi : X^0 \to \mathbb{C}^2 \quad (x, y, t) \mapsto (x, t)$$

and we obtain that the set of critical points of the projection is given by

$$P_1(X, L^1) = \begin{cases} x = -t^2 \\ y = 0 \end{cases}$$

The following result gives us a method to compute the aureole of the germ $(X, 0)$ using polar varieties.

Proposition B.8. *Let $\{V_\alpha\}$ be the aureole of the reduced and purely d-dimensional germ $(X, 0)$. Let $O(V_\alpha) \subset \mathbb{C}^{n+1}$ denote the cone over the projective variety $V_\alpha \subset \mathbb{P}^n$. Then, for any integer k, $0 \leq k \leq d-1$ and sufficiently general $L^{d-k} \subset \check{\mathbb{P}}^n$ the tangent cone $C_{P_k(X,L),0}$ of the polar variety $P_k(X, L)$ at the origin consists of:*

- *The union of the cones $O(V_\alpha)$ which are of dimension d-k (= dim $P_k(X, L)$).*
- *The polar varieties of dimension d-k for the projection p associated to L of the cones $O(V_\beta)$ such that dim $O(V_\beta) = d - k + j$, that is $P_j(O(V_\beta), L)$.*

Proof. [LT88, Propo. 2.2.1, pg 565]
Using the normal/conormal diagram

$$\begin{tikzcd}
E_0C(X) \arrow[r, "\hat{e}_0"] \arrow[dd, "\kappa'_X"'] \arrow[ddr, "\xi"] & C(X) \arrow[r, hook] \arrow[dr, "\lambda"] \arrow[dd, "\kappa_X"] & X \times \check{\mathbb{P}}^n \arrow[d, "pr_2"] \\
 & & \check{\mathbb{P}}^n \\
E_0X \arrow[r, "e_0"'] & X &
\end{tikzcd}$$

recall that we can obtain the blowup $E_0(P_k(X,L))$ of the polar variety $P_k(X,L)$ by taking its strict transform under the morphism e_0, and as such we will get the projectivized tangent cone $\mathbb{P}C_{P_k(X,L),0}$ as the fiber over the origin.

The first step is to prove that set-theoretically the projectivized tangent cone can also be expressed as

$$|\mathbb{P}C_{P_k(X,L),0}| = \bigcup_\alpha \kappa'_X(\hat{e}_0^{-1}(\lambda^{-1}(L) \cap W_\alpha)) = \bigcup_\alpha \kappa'_X(D_\alpha \cap \mathbb{P}^n \times L)$$

Now recall that the intersection $P_k(X,L) \cap X^0$ is dense in $P_k(X,L)$, so for any point $(0,[l]) \in \mathbb{P}C_{P_k(X,L),0}$ there exists a sequence of points $\{x_n\} \subset X^0$ converging to it. So, by definition of a polar variety, if $D_{d-k+1} = \check{L}$ and $T_n = T_{xn}X^0$ then by B.1 we know that dim $T_n \cap D_{d-k+1} \geq k$ which is a closed condition. In particular if T is a limit of tangent spaces obtained from the sequence $\{T_n\}$, then $T \cap D_{d-k+1} \geq k$ also. But if this is the case then, since the dimension of T is d, there exists a limit of tangent hyperplanes $H \in \kappa_X^{-1}(0)$ such that $T + D_{d-k+1} \subset H$ which is equivalent to $H \in \kappa_X^{-1}(0) \cap \lambda^{-1}(L) \neq \emptyset$. Consequently, the point $(0,[l],H) \in \bigcup_\alpha \hat{e}_0^{-1}(\lambda^{-1}(L) \cap W_\alpha)$, and so we have the inclusion:

$$|\mathbb{P}C_{P_k(X,L),0}| \subset \bigcup_\alpha \kappa'_X(\hat{e}_0^{-1}(\lambda^{-1}(L) \cap W_\alpha))$$

For the other inclusion, the proof relies on the fact that since L is general, not only is it transversal to all W_α in $\check{\mathbb{P}}^n$, but $\mathbb{C}^{n+1} \times L$ is also transversal in $\mathbb{C}^{n+1} \times \check{\mathbb{P}}^n$ to the strata of any Whitney fixed stratification of $C(X)$, for a small enough representative of X. From this and Proposition 5.2 of [Tei82, Chap. 3, pg 411] applied with $B_k = \kappa^{-1}(0)$, it follows that the inverse image of $\lambda^{-1}(L) = C(X) \cap (\mathbb{C}^{n+1} \times L)$ by $\hat{e}_0$ is equal to its strict transform $\overline{\hat{e}_0^{-1}(\lambda^{-1}(L) \setminus \kappa_X^{-1}(0))}$.

Now, we have that the normal/conormal diagram is commutative, the spaces $X \setminus 0$ and $E_0X \setminus e_0^{-1}(0)$ are isomorphic via the map e_0, and the same goes for the spaces $C(X) \setminus \kappa^{-1}(0)$ and $E_0C(X) \setminus \hat{e}_0^{-1}(\kappa^{-1}(0))$ via $\hat{e}_0$. This implies that the image by κ' of $\hat{e}_0^{-1}(\lambda^{-1}(L) \setminus \kappa_X^{-1}(0))$ is equal to $e_0^{-1}(P_k(X,L) \setminus \{0\})$. On the other hand, we have that the image by κ of $\lambda^{-1}(L)$ is equal to the polar variety $P_k(X,L)$, and the strict transform $\overline{e_0^{-1}(P_k(X,L) \setminus \{0\})}$ has the projectivized tangent cone $\mathbb{P}C_{P_k(X,L),0}$ as fiber over the origin. But since κ' is a proper morphism, and so particularly a closed morphism we have that the image by κ' of the strict transform $\overline{\hat{e}_0^{-1}(\lambda^{-1}(L) \setminus \kappa_X^{-1}(0))}$ is the strict transform of $P_k(X,L)$ by e_0. Taking fibers over the origin gives the result.

The second and final step of the proof is to use that from theorem 2.4 a) and b) we have that each $D_\alpha \subset I \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ is the conormal space of V_α in $\mathbb{P}^n$, with the restriction of κ'_X to D_α being its conormal morphism.

Note that D_α is of dimension $n-1$, and since all the maps involved are just projections, we can take the cones over the V_α's and proceed like in section 1.3.4. In this setting, we get that since L is sufficiently general, by proposition B.1 and definition B.2:

- For the D_α's corresponding to cones $O(V_\alpha)$ of dimension d-k (= dim $P_k(X, L)$), the intersection $D_\alpha \cap \{\mathbb{C}^{n+1} \times L\}$ is not empty and as such its image is a polar variety $P_0(O(V_\alpha), L) = O(V_\alpha)$.
- For the D_α's corresponding to cones $O(V_\alpha)$ of dimension d-k+j, the intersection $D_\alpha \cap \{\mathbb{C}^{n+1} \times L\}$ is either empty or of dimension $d-k$ and as such its image is a polar variety of dimension d-k,that is $P_j(O(V_\alpha), L)$.

$\square$

Note that the polar variety $P_k(X, L)$ is not unique, since it varies with L, but we are saying that their tangent cones have things in common. The V_α's are fixed, so the first part is the fixed part of $C_{P_k(X,L),0}$ because it is independent of L, the second part is the mobile part, since we are again talking of polar varieties of certain spaces which of course depend on L.

So for any complex germ $(X, 0)$ reduced and purely $d-$dimensional, we have a method to “compute” or rather describe, the set of limiting positions of tangent hyperplanes:

1) For all integers k, $0 \leq k \leq d-1$, compute the “general” polar varieties $P_k(X, L)$, leaving in the computation the coefficients of the equations of L as indeterminates. (Partial derivatives)
2) Compute the tangent cones $C_{P_k(X,L),0}$. (Grobner basis)
3) Sort out those components of the tangent cone of each $P_k(X, L)$ which are independent of L. (Irreducible components)
4) Take the projective duals of the corresponding projective varieties. (Elimination)

Remark B.9. *Be careful, even if the dimension of $\kappa_X^{-1}(0)$ is greater than $n-d$, that is the polar hypersurface $P_1(X)$ is not empty, this does not mean that the germ $(X, 0)$ has exceptional cones. This can very well happen just by having an irreducible component V_α of the tangent cone $C_{X,0}$ whose dual variety W_α is sufficiently big, that is of dimension $n-d+1$ or bigger.*

Appendix C

Integral Closure of Ideals

We will state the basic definitions as well as the results we will need of the theory of integral closure of ideals and its relation with equisingularity as developed by M. Lejeune-Jalabert and B. Teissier in [LJT08] and [Tei74].

Definition C.1. *Let O be a commutative ring with unity and let $I \subsetneq O$ an ideal. We say that an element $h \in O$ is integral over I if it satisfies an equation:*

$$h^k + a_1 h^{k-1} + \cdots + a_k = 0$$

where $a_j \in I^j$. We denote by $\overline{I}$ the set of all elements of O which are integral over O and call it the integral closure of I in O. We say that I is integrally closed if $I = \overline{I}$.

Definition C.2. *Let O be a commutative ring with unity, and let I, J be two ideals of O. Set*

$$\nu_I(J) = \sup \{n \mid n \in \mathbb{N}, J \subset I^n\}$$

and

$$\overline{\nu}_I(J) = \lim_{k\to\infty} \frac{\nu_I(J^k)}{k}$$

See [LJT08, pg 787] for a proof that $\overline{\nu}_I$ is well defined.

The link between these two concepts, in the setting of complex analytic geometry is given by the following results.

Theorem C.3. *([LJT08, Thm 2.1, pg 799]) Let X be a reduced complex analytic space. Let $Y \subset X$ be a closed, nowhere dense, analytic subspace of X, and x a point in Y. Let $\mathfrak{I} \subset O_X$ be the coherent ideal defining Y, and let $\mathfrak{J} \subset O_X$ be another coherent ideal. Let I (resp J) be the stalk of $\mathfrak{I}$(resp $\mathfrak{J}$) at x. Then the following statements are equivalent:*

1. *$J \subset \overline{I}$*
2. *$\overline{\nu}_I(J) \geq 1$*
3. *For every germ of morphism $\phi : (\mathbb{D}, 0) \to (X, x)$*

$$\phi^* J \cdot O_{\mathbb{D},0} \subset \phi^* I \cdot O_{\mathbb{D},0}$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$.

4. *For every morphism $\pi : X' \to X$ such that X' is a normal analytic space, π is proper and surjective, and $\mathfrak{I} \cdot O_{X'}$ is locally invertible, there exists an open subset $U \subset X$ containing x, such that:*

$$\mathfrak{J} \cdot O_{X'}|_{\pi^{-1}(U)} \subset \mathfrak{I} \cdot O_{X'}|_{\pi^{-1}(U)}$$

4*. *If $\Pi : \widetilde{E_{\mathfrak{I}}X} \to X$ denotes the normalized blowup of X along $\mathfrak{I}$, then there exists an open subset $U \subset X$ containing x, such that:*

$$\mathfrak{J} \cdot O_{\widetilde{E_{\mathfrak{I}}X}}|_{\Pi^{-1}(U)} \subset \mathfrak{I} \cdot O_{\widetilde{E_{\mathfrak{I}}X}}|_{\Pi^{-1}(U)}$$

5. *Let $V \subset X$ be a neighborhood of x, where both $\mathfrak{J}$ and $\mathfrak{I}$ are generated by their global sections. Then for every system of generators $g_1, \ldots, g_m$ of $\Gamma(V, \mathfrak{I})$ and every $f \in \Gamma(V, \mathfrak{J})$, there is a neighborhood V' of x and a constant C such that:*

$$|f(y)| \leq C \sup_{i=1,\ldots,m} |g_i(y)|$$

for every $y \in V'$.

Let X be a reduced complex analytic space and J a coherent O_X ideal such that the support of O_X/J is nowhere dense on X. Let $\pi_0 : X'_0 \to X$ be the blowup of X along J, and let $\pi : X' \to X$ be the normalized blowup of X along J, defined as the composition of π_0 with the normalization $n : X' \to X'_0$ of X'_0.

The ideal $J \cdot X'$ is locally invertible in the normal space X', and defines the exceptional divisor D of π. Now for every open set $U \subset X$ we can define order functions on the ring $\Gamma(U, X)$ as follows:

Consider the irreducible components of $|D \cap \pi^{-1}(U)| = \bigcup_\alpha |D_\alpha|$ and for every $f \in \Gamma(U, X)$ define $\nu_\alpha(f)$ as the vanishing order of $f \circ \pi$ on D_α. For an ideal $I \subset \Gamma(U, X)$ define $\nu_\alpha(I) = \inf_{f \in I} \nu_\alpha(f)$.

Theorem C.4. *Let $x \in X$, then for every $f \in O_{X,x}$ there exists a neighborhood W of x in X and a representative $\tilde{f}$ of f on W such that:*

$$\overline{\nu}_J(f) = \inf_\alpha \frac{\nu_\alpha(\tilde{f})}{\nu_\alpha(J|W)}$$

Proof. See [LJT08, Thm 4.6, pg 812] for a proof of a general result. □

Let $\mathfrak{A}_{X,x}$ be the set of morphisms $\phi : (\mathbb{D}, 0) \to (X, x)$ such that the associated morphism of local algebras $\phi^* : O_{X,x} \to \mathbb{C}\{\tau\}$ is not zero. If $\mathfrak{m}_1$ denotes the maximal ideal of $\mathbb{C}\{\tau\}$, then the order function $\nu_{\mathfrak{m}_1}$ coincides with the standard valuation of $\mathbb{C}\{\tau\}$ associating to a series $h \in \mathbb{C}\{\tau\}$ its order.

Proposition C.5. *[LJT08, Thm. 5.2] Let X be a complex analytic space, $x \in X$ and $\mathfrak{I} \subset O_X$ a coherent ideal with stalk I. If $h \in O_{X,x}$ then:*

$$\overline{\nu}_I(h) = \inf_{\phi \in \mathfrak{A}_{X,x}} \left\{ \frac{\nu_{\mathfrak{m}_1}(\phi^*(h))}{\nu_{\mathfrak{m}_1}(\phi^*(I)\mathbb{C}\{\tau\})} \right\}$$

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