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Contributions à la théorie des jeux d'évolution et de congestion

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dirigée par Sylvain Sorin

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À mes parents et notre chat Maugham.

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Résumé

Résumé

Cette thèse porte sur les jeux d'évolution et de congestion.

Après une revue des études sur les jeux de congestion dans les réseaux dans le chapitre 1, nous étudions la relation entre la composition des joueurs (non-atomiques, atomiques, composites) et les coûts d'équilibre dans les chapitres 2 et 3. En particulier, l'impact de la formation des coalitions est examiné.

Les chapitres 4 et 5 introduisent le comportement de délégation dans les jeux composites et les jeux divisibles en entiers. Plusieurs jeux et processus de délégation dans des contextes différents sont définis et étudiés.

Enfin, nous nous penchons sur l'aspect dynamique des jeux. Le chapitre 6 est consacré à une dynamique à deux échelles qui modélise le phénomène de sélection à niveaux multiples. La thèse est conclue par une revue des études sur les dynamiques de type réplicateur dans le chapitre 7.

Mots-clefs

jeu de congestion, réseau, joueur non-atomique, joueur atomique, jeu composite, équilibre composite, coalition, délégation, jeu de délégation à un coup, processus de délégation à meilleures réponses alternatives, joueur divisible en entiers, jeu de délégation, chaine compatible de paiements d'équilibre, sélection à niveaux multiples, dynamique réplicateur

Contributions to Evolutionary and Congestion Game Theory

Abstract

This thesis is contributed to evolutionary games and congestion games.

After a survey of the studies on network congestion games in Chapter 1, Chapters 2 and 3 consider the relation between the composition of the players (nonatomic, atomic, composite) and the equilibrium cost. In particular, the impact of the formation of coalitions is examined.

Chapters 4 and 5 introduce the behavior of delegation in composite games and integer-splittable games. Several delegation games and a delegation process are defined and studied in different contexts.

Finally, dynamic aspects in games are considered. Chapter 6 focuses on a two-level dynamics which models the phenomenon of multilevel selection. The thesis is concluded by a survey of the studies on the dynamics of replicator type in Chapter 7.

Keywords

congestion game, network, nonatomic player, atomic player, composite game, composite equilibrium, coalition, delegation, one-shot delegation game, alternating best reply delegation process, integer-splittable player, delegation game, consistent chain of delegation equilibrium payoff, multilevel selection, replicator dynamics

Table des matières

Introduction

Cette thèse porte sur les jeux de congestion et les jeux de population.

Afin de fixer l'idée, nous considérons les jeux de congestion dans les réseaux finis. Un joueur contrôle un stock à envoyer de son origine à sa destination. Le joueur est non-atomique (*resp.* atomique) si son stock est infinitésimal (*resp.* de poids strictement positif). Un joueur atomique est divisible (*resp.* non-divisible) s'il peut (*resp.* ne peut pas) diviser son stock en plusieur parties et les envoyer par des chemins différents. Dans un jeu de congestion composite, les joueurs sont atomiques divisibles ou non-atomiques. Le coût d'un chemin dépend du poids de stock sur chacun de ses arcs. L'espace de stratégies d'un joueur non-atomique ou atomique non-divisible est fini, tandis que celui d'un joueur atomique divisible est un continuum, en fait, un simplexe. Plus généralement, on appelle jeux de population une situation interactive où le paiement de chaque joueur dépend de son choix/type et de la distribution des choix/types des autres joueurs.

C'est dans ce contexte que cette thèse est effectuée. Elle est composée de sept chapitres. Chaque chapitre est issu d'un article (accepté, soumis ou en cours) de l'auteur (seule pour les chapitres 1 à 4, en collaboration pour les chapitres 5 à 7). Afin de conserver la structure et la logique de ces travaux, ils sont reproduits ici sans modification. Les notations et la mise en page ont été homogénéisées. Par ailleurs, trois paragraphes initialement retirés sont rajoutés dans le chapitre 2 comme appendice.

Le chapitre 1 est issu de l'article soumis « *Jeux de congestion : modèles et propriétés* ». Il présente les jeux de congestion dans un cadre général. Différents modèles et leurs propriétés sont traités. Après avoir introduit les jeux de congestion à la Rosenthal, un modèle avec stocks discrets qui est une des origines des jeux de congestion, nous nous focalisons sur les modèles avec stocks continus : non-atomiques, atomiques divisibles et composites. Nous détaillons la définition, la caractérisation, l'existence et l'unicité des équilibres dans différentes circonstances avant de mentionner deux passages d'approximation d'un nombre fini de joueurs à un continuum de joueurs. Ensuite, nous nous penchons sur l'efficacité des équilibres au niveau social comme au niveau individuel, leurs différentes mesures et les problématiques associées. Enfin, nous résumons des études sur des processus dynamiques dans les jeux de congestion dans différents modèles.

Le chapitre 2 est issu de l'article « *Coalitions in network congestion games* » à paraître dans le journal *Mathematics of Operations Research.* Il est consacré aux propriétés statiques des jeux de congestion composites. En nous restreignant aux réseaux composés de deux sommets reliés par deux arcs parallèles, nous nous intéressons à l'impact de la formation des coalitions entre les joueurs sur les coûts en équilibre. Sous une hypothèse standard sur la convexité des fonctions de coût, nous montrons que la formation des coalitions entre les joueurs non-atomiques réduit le coût social comme celui de chaque joueur. Rappelons que le coût des membres d'une coalition est défini comme la moyenne du coût de la coalition. Dans le cas d'une seule coalition, le coût des joueurs non-atomiques, le coût moyen de la coalition et le coût social diminuent tous par rapport à la taille de la coalition. Dans le cas de multiples coalitions, si un groupe de joueurs non-atomiques forme ou rejoignt une coalition, le coût des joueurs non-atomiques diminue.

Le chapitre 3 est issu de l'article « *Composition of the players in two-singleton-choice congestion games* ». Nous poursuivons l'analyse du chapitre 2 en étudiant comment les coûts d'équilibre évoluent en fonction de la composition des joueurs. En nous restreignant aux réseaux composés de deux sommets reliés par deux arcs parallèles, nous montrons que, si un joueur atomique divisible est remplaçé par un ensemble composite de joueurs, à savoir des joueurs atomiques divisibles et/ou des joueurs non-atomiques, dont le poids total est le même que celui qui est remplaçé, alors le coût social et le coût des autres joueurs augmentent à l'équilibre.

Le chapitre 4 est issu de l'article « *One-shot delegation games and delegation processes* ». Dans ce chapitre, nous introduisons la notion de délégation aux jeux de congestion composites. On dit qu'un joueur atomique divisible délègue s'il divise son stock en plusieurs parties puis les confie à des joueurs indépendants, et son coût est la somme des coûts de ses délégués. Remarquons qu'un délégué atomique divisible peut lui aussi déléguer. Nous nous plaçons toujours dans un réseau de deux sommets reliés par deux arcs parallèles. D'abord, nous montrons que, face aux autres joueurs atomiques divisibles et/ou non-atomiques, un joueur atomique divisible dispose toujours de meilleure réponse en termes de délégation (ce qui n'est pas évident car l'espace de ses choix est le produit d'un nombre dénombrable de simplexes). En particulier, toutes ses stratégies de délégation sont faiblement dominées par les stratégies dites uni-atomiques, à savoir celles qui désignent un seul délégué atomique en dehors des délégués nonatomiques. Puis, nous établissons l'existence des équilibres de Nash purs dans les jeux de délégation à un coup, dans le cas où il y a deux joueurs atomiques divisibles et les fonctions de coût des deux arcs sont affines. Enfin, nous considérons un processus de délégation dans lequel les joueurs atomiques divisibles délèguent alternativement. Nous étudions sa vitesse de convergence et l'ensemble de ses issues, afin de comparer ces issues avec les équilibres du jeu de congestion initial et ceux du jeu de délégation en un coup.

Le chapitre 5 est issu de l'article soumis « *Delegation equilibrium payoffs in integer-splitting games* » en collaboration avec Sylvain Sorin. Nous considérons le comportement de délégation dans les jeux de congestion dits divisibles en entiers, à savoir que le poids du stock de chaque joueur est un nombre entier et il ne peux le diviser qu'en plusieurs parties de poids en nombre entier. Contrairement au chapitre 4, un joueur ne dispose que d'un nombre fini de stratégies de délégation, et le processus de délégation ne peut pas durer infiniment. Ce jeu est aussi différent des jeux sous forme normale et des jeux extensifs bien connus, car les joueurs sont crées au fur et à mesure. Nous étudions d'abord ses propriétés statiques. Nous construisons le jeu par une structure d'arbre et nous définissons les coûts d'équilibre et les coûts d'équilibre compatibles. Puis, nous cherchons son lien avec la sélection d'équilibre, l'induction en amont et l'induction en aval.

Le chapitre 6 est issu de l'article « *A dynamical model of a two-scale interaction* » en collaboration avec Mario Bravo. Dans ce chapitre, nous modélisons un phénomène nommé « sélection à multiples niveaux » en biologie mais aussi courant dans la société humaine. Supposons qu'il y a un ensemble de groupes dont chacun est composé d'individus de deux types : C (coopérateur) et D (défecteur). Au sein d'un groupe, le type D a toujours une fitness supérieure à celle de type C, quelle que soit la composition du groupe. En revanche, la fitness d'un groupe croît avec la proportion de type C qu'elle contient. Ces deux effets pourraient se compenser. Dans un état stationnaire, tous les groupes dans la société pourraient avoir des propositions de type C identiques ou différentes. Ce phénomène peut être modélisé de manières variées. Nous commençons par un processus dynamique avec un nombre fini de proportions possibles de type C et en temps discret. Nous montrons qu'un état stationaire existe et la dynamique converge vers un état où les groupes ont différentes proportions de type C. Une variante du modèle dans laquelle les individus de type D pourraient muter en devenant de type C est également étudiée.

Le chapitre 7 est issu de l'article « *Replicator dynamics in game theory* » en collaboration avec Mario Bravo et Sylvain Sorin. Il consiste en une revue de la dynamique réplicateur dans les jeux. Nous prenons les jeux de congestion comme un cas particulier et comparons les mêmes équations de la dynamique réplicateur dans plusieurs contextes distincts. Plus précisément, nous étudions ses différentes formes d'application dans les jeux de (une ou multiples) population(s) non-atomique(s) avec auto-interaction (au sein d'une population) ou externe-interaction (entre les populations), les jeux matriciels, et les jeux à un nombre fini de joueurs si les espaces de stratégies sont des espaces compacts.

This thesis investigates the congestion games and the population games.

To fix the idea, let us consider network congestion games. Each player holds a stock to be sent from its origin to its destination. The player is nonatomic (*resp.* atomic) if her stock is infinitesimal (*resp.* of strictly positive weight). An atomic player is splittable (*resp.* unsplittable) if she can (*resp.* cannot) divide her stock into several parts and send them by different paths. In a composite congestion game, the players are atomic splittable or nonatomic. The cost of a path depends on the total weight of the stocks on each of the arc component of the path. The strategy space of a nonatomic player or an atomic unsplittable player is finite, while that of an atomic splittable player is a continuum or, in fact, a simplex. More generally, a population game describes an interactive situation where each player's payoff depends on her choice/type and the distribution of the choices/types among the other players.

It is in this framework that the thesis is written. It is composed of seven chapters. Each chapter is based on a paper (which is accepted or submitted or finished but not yet submitted) of the author (by herself for the first four chapters, and in collaboration with others for the rest). In order to keep the structure and the logic of these works, they are reproduced here without modification. The notations and the formatting are homogenized. Besides, three paragraphs initially removed from Chapter 2 are restored as appendices.

Chapter 1 is based on the submitted paper "*Jeux de congestion : modèles et propriétés*". It presents congestion games in a general framework. Different models and their properties are treated. After the introduction of Rosenthal congestion games (a model with discrete stocks which is one of the origins of congestion games), it focuses on the models with continuous stocks: nonatomic, atomic splittable and composite. The definition, the characterization, the existence and the uniqueness of the equilibria in different settings are provided in detail, and two approximations of a continuum of players by a large number of players are mentioned. Then, it discusses the efficiency of the equilibria at the social level as well as at the individual level, their different measures and the associated topics. Finally, it summarizes the studies on the dynamical processes in congestion games in different models.

Chapter 2 is based on the paper "*Coalitions in network congestion games*" which is to appear in the journal *Mathematics of Operations Research*. It is contributed to the static properties of composite congestion games. In a network composed of two vertices connected by parallel arcs, the impact of the formation of coalitions between the players on the equilibrium costs is considered. Under a standard assumption on the convexity of the cost functions, it is shown that the formation of coalitions between nonatomic players reduces the social cost as well as each player's cost. Recall that the cost to the members of coalition is defined as the average cost to the coalition. In the case of a unique coalition, the nonatomic players' common cost, the average cost to the coalition and the social cost are strictly decreasing with respect to the size of the coalition. In the case of multi-coalition, if a group of nonatomic players form or join a coalition, the nonatomic players' common cost is reduced.

Chapter 3 is based on the paper "*Composition of the players in two-singleton-choice congestion games*". It extends the analysis in Chapter 2 by studying how the equilibrium costs evolve with respect to the composition of the players. In a network composed of two vertices connected by two parallel arcs, it is shown that, if an atomic splittable player is replaced by a composite set of players, namely, a set of atomic splittable players and/or nonatomic players whose total weight is the same as that of the replaced player, then the social cost and the cost to the other players are increased at the equilibrium.

Chapter 4 is based on the paper "*One-shot delegation games and delegation processes*". It introduces the notion of delegation in composite congestion games. An atomic splittable player is said to delegate if she divides her stock to several parts and commits them respectively to independent players, so that her cost is the sum of the costs to her delegates. Notice that an atomic splittable player can also delegate. This chapter is always restrained to two-terminal two-parallel-arc networks. First, it is shown that, facing the other atomic splittable and/or nonatomic players, an atomic splittable player always has a best reply in terms of delegation (This result is not evident because the set of such choices is the product of a countable number of simplices.). In particular, all her delegation strategies are weakly dominated by the so-called single-atomic strategies, i.e. those that appoint a single atomic delegate in addition to nonatomic delegates. Then, the existence of pure Nash equilibria is established in one-shot delegation games, in the case where there are two atomic splittable players and the cost functions of the two arcs are affine. Finally, it considers a delegation process where the atomic splittable players delegate in turn. The convergence speed and the set of the outcomes of the process are studied. And these outcomes are compared with the equilibria of the initial congestion game and the equilibria of the one-shot delegation game.

Chapter 5 is based on the submitted paper "*Delegation equilibrium payoffs in integer-splitting games*" in collaboration with Sylvain Sorin. It considers the behavior of delegation in integer-splittable congestion games, namely, the congestion games where each player's weight is an integer and she can only divide her stock into several parts of integer weights. Unlike in Chapter 4, a player has only a finite number of delegation strategies, and the delegation process cannot continue infinitely. This game is also different from the well-know normal form games or extensive form games, because the players are created gradually. The static properties of such an integer-splitting delegation game are first studied. The game is constructed by a tree structure. The definition of equilibrium costs and that of consistent equilibrium costs are given. Then, the links of this model with the selection of equilibria, forward induction and backward induction are discussed.

Chapter 6 is based on the paper "*A dynamical model of a two-scale interaction*" in collaboration with Mario Bravo. It models the phenomenon called "multi-level selection" in biology which is also quite usual in human society. Suppose that there are a family of groups. Each of them is composed of individuals of one of the two types: C (cooperator) and D (defector). Within a group, type D always has a higher fitness than type C, whatever the composition of the group is. On the contrary, the fitness of a group increases with the proportion of type C in it. These two effects might compensate each other. At a stationary state, all the groups in the society might have the same proportion or different proportions of type C. This phenomenon can be modeled in different ways. Here, one begins by a dynamical process in discrete time with a finite number of possible proportions of type C. It is shown that a stationary state exists and the dynamics converges to a state where the groups have different proportions of type C. A variant of the model where the individuals of type D can mutate by becoming type C is also studied.

Chapter 7 is based on the paper "*Replicator dynamics in game theory*" in collaboration with Mario Bravo and Sylvain Sorin. It is a survey on replicator dynamics studied in game theory. It takes congestion games as examples and compares the same equations of replicator dynamics in different contexts. Explicitely, it studies different forms of the application of replicator dynamics in (one or multiple) nonatomic population games with self-interaction (within each population) or external-interaction (between the populations), in matrix games, and in N-player games where the action spaces are compact sets.

Notations

$\mathbb{R}^d_+ = \{\mathbf{x} \in \mathbb{R}^d \,|\, x^i \geq 0,\, 1 \leq i \leq d\}$.

$\mathbb{R}_{++} = \{x \in \mathbb{R} \,|\, x > 0\}$.

$\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

$\mathbb{N} = \{0,\, 1,\, 2,\, \cdots\}$.

$\mathbb{N}^* = \{1,\, 2,\, \cdots\}$.

For $m, n \in \mathbb{N}$, $[\![m, n]\!] = \{m, m+1, \ldots, n-1, n\}$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\langle\, \mathbf{x}\,, \mathbf{y}\,\rangle = \sum_{i=1}^d x_i y_i$, i.e. the inner product of $\mathbf{x}$ and $\mathbf{y}$.

For $\mathbf{x} = (x_i)_{i=1}^d$ and $\mathbf{y} = (y_i)_{i=1}^d \in \mathbb{R}^d$, $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for $i = 1, \ldots, d$.

For $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$, $x^- = \min\{-x, 0\}$, and $|x| = \max\{x^+, x^-\}$, i.e. the absolute value of x.

For $\mathbf{x} \in \mathbb{R}^d$, $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq d}\{|x^i|\}$, i.e. its uniform norm in $\mathbb{R}^d$.

For any set A, $|A|$ is the cardinality of A.

$\Delta^{d-1} = \{\mathbf{x} = (x_i)_{i=1}^d \in \mathbb{R}^d \,|\, x_i \geq 0,\, i = 1, \ldots, d,\, \sum_{i=1}^d x_i = 1\}$, i.e. the simplex of dimension $d - 1$.

For any finite set A, $\Delta(A) = \Delta^{|A|-1}$.

For any matrix A, A^T is the transpose of A.

Chapter 1

Network congestion games: models and properties

This chapter is translated from the paper *Jeux de congestion : modèles et propriétés* particularly for the English version of this PhD thesis.

Abstract. *This paper provides a survey of some important results on network congestion games. Rosenthal congestion games, the associated potential games and a nonatomic version of them, as well as the classical counter-examples or paradoxes are introduced. Network congestion games are discussed for different versions of actors. The definitions of equilibrium and their static properties such as characterization, existence and uniqueness are provided. The inefficiency of the equilibria is discussed at both the social level and the individual level. Some results on dynamic processes are presented.*

1.1 Introduction

This work presents a survey of the important results on network congestion games. It begins by Rosenthal congestion games, the associated potential game, a nonatomic version and the classical counter-examples or paradoxes. Next, the congestion model with different versions of actors are developed: a finite number of atomic players with unsplittable stocks, atomic players with splittable stocks and/or nonatomic players who control each an infinitesimal quantity of stock. The definitions of equilibria in these games will be formulated, then their static properties, such as their characterization, their existence and their uniqueness, will be discussed. Two approximation models from a finite number of players to a continuum of players are mentioned. Then, we study the efficiency of the equilibria in terms of the social cost as well as the individuals' costs, their different measures and the associated problems. Finally, the studies on the dynamical processes in congestion games in the different models mentioned above are reviewed.

1.2 Congestion games: definition, proprieties, examples

1.2.1 Rosenthal congestion games

Rosenthal [74] introduced a class of strategic games with a finite number of players and strategies. Their structure is very rich. In particular, they possess Nash equilibria in pure strategies.

Definition 1.1. A *Rosenthal congestion game*, $\Gamma(A, (l_a)_{a\in A}, \mathcal{N}, (P^i)_{i\in\mathcal{N}}, (u^i)_{i\in\mathcal{N}})$, is specified by the following elements.

- $\mathcal{N} = \{1, 2, \ldots, N\}$, a finite set of players.
- A, a finite set of resources.
- l_a, the per-unit cost function of resource $a \in A$, defined from $\mathbb{R}_+ \cap \mathbb{N}$ to $\mathbb{R}_+$. When there are x players who use a, the cost to each of them for using it is $l_a(x)$.
- $P^i \subset 2^A$, the pure strategy set of player $i \in \mathcal{N}$. A pure strategy $p^i \in P^i$ is a subset of A.
- u^i, the cost function of player $i \in \mathcal{N}$, defined on the set of pure strategy profiles $\mathbf{P} = \prod_{i\in\mathcal{N}} P^i$ as follows:
$$u^i(\mathbf{p}) = \sum_{a\in p^i} l_a(\phi_a(\mathbf{p})),$$
where $\phi_a(\mathbf{p})$ is the total number of the players using resource a, when the strategy profile is $\mathbf{p}$.

Let us first cite a simple example, then make some remarks on the definition of Rosenthal potential games.

Example 1.2. In a public goods game, each of the N players chooses one and only one piece of public goods from a set A. The cost to a player is determined by the number of the players who make the same choice as her.

Remark 1.3.

Separability of the cost. By definition, given a strategy profile, the cost associated to a strategy played by someone is the sum of the costs of the resources used by it. This property is called the *separability* of the cost.

Non externality of the cost. The cost of a resource depends only on the number of its users, but not on the number of the users of the others resources. In other words, there is no *externality*.

Anonymous game. The cost to player i depends only on her own strategy p^i and the set of the strategies played, but not the identity of the carrier of a certain strategy p^j. In other words, Rosenthal congestion games belong to the class of anonymous games, where each player's payoff depends on her own strategy and the vector $(x_p)_{p\in\cup_{i\in\mathcal{N}} P^i}$, where x_p is the number of the players choosing strategy p. If the players have specific weights, then "number" should be replaced by "weight".

Notice that, in a Rosenthal congestion game, the players have the same weight and the same cost functions for the resources, which need not be the case in either anonymous games or congestion games in the general case.

Network congestion game. An important family of congestion games take place *in a network*, namely, a finite directed graph. The resources there are arcs. A pair of vertices are associated to each player, which are her origin and her destination, respectively. Each strategy of her is a directed elementary path (i.e., it does not pass twice by the same vertex), which connects her origin to her destination. For example, in Figure 1.1, the players having o_1/d_1 (*resp.* o_2/d_2) as origin/destination have 7 (*resp.* 2) pure strategies. The cost associated to an arc can be interpreted as the travel time, which is usually nondecreasing with the traffic intensity on the arc because of the congestion.

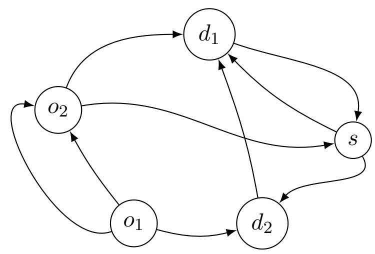


Figure 1.1: A congestion game in a network.

1.2.2 Potential games

The proof by Rosenthal [74] of the existence of pure Nash equilibria in Rosenthal games is based on a potential function. Nevertheless, it is only 23 years later that Monderer and Shapley [62] formally defined *potential games* (with a finite number of players).

Definition 1.4. Suppose that $\Gamma(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (u^i)_{i\in\mathcal{N}})$ is a strategic game, where $\mathcal{N} = \{1, \dots, N\}$ is the finite set of players, S^i the (probably infinite) set of pure strategies of player i, and u^i the payoff function of player i, defined on the set of pure strategy profiles $\mathcal{S} = \prod_{i\in\mathcal{N}} S^i$. Game Γ is a *potential game* (finite, *cf.* Chapter 7) if there exists a real-valued function P defined on $\mathcal{S}$ such that, for all player i, for all strategies s^i and t^i in S^i, for all profile s^{-i} in $S^{-i} = \prod_{j\in\mathcal{N}\setminus\{i\}} S^j$, one has:

$$u^i(s^i, s^{-i}) - u^i(t^i, s^{-i}) \;=\; P(s^i, s^{-i}) - P(t^i, s^{-i}). \tag{1.1}$$

And P is called a *potential function* of the game Γ.

The connection between potential games and Rosenthal congestion games is clarified by the following theorem (Monderer and Shapley [62]).

Theorem 1.5. *1. Each Rosenthal congestion game* $\Gamma(A, (l_a)_{a\in A}, \mathcal{N}, (P^i)_{i\in\mathcal{N}}, (u^i)_{i\in\mathcal{N}})$ *is a potential game.*

A potential function of Γ *is*

$$P(\mathbf{p}) = \sum_{a\in A} \sum_{k=0}^{\phi_a(\mathbf{p})} l_a(k). \tag{1.2}$$

2. Conversely, a potential game with a finite number of strategies is isomorphic to a Rosenthal congestion game.

Proof. 1. Suppose that $p^i, q^i \in P^i$ are two strategies of player i, and $p^{-i} \in P^{-i} = \prod_{j\in\mathcal{N}\setminus\{i\}} P^j$ is a strategy profile of the other players. Suppose that $B = (p^i \cap q^i) \cup (A \setminus (p^i \cup q^i))$ is the subset of A which contains all the common components of p^i and q^i as well as the resources used neither by p^i nor by q^i. Then, for a resource $b \in B$, the number of its users ϕ_b does not change when the strategy profile changes from (p^i, p^{-i}) to (q^i, p^{-i}).

On the one hand,

$$\begin{aligned}
&P(p^i, p^{-i}) - P(q^i, p^{-i}) \\
&= \sum_{a \in p^i \setminus q^i} \left(\sum_{k=0}^{\phi_a(p^i, p^{-i})} l_a(k) - \sum_{k=0}^{\phi_a(q^i, p^{-i})} l_a(k) \right) - \sum_{b \in q^i \setminus p^i} \left(\sum_{k=0}^{\phi_b(q^i, p^{-i})} l_b(k) - \sum_{k=0}^{\phi_b(p^i, p^{-i})} l_b(k) \right) \\
&= \sum_{a \in p^i \setminus q^i} l_a(\phi_a(p^i, p^{-i})) - \sum_{b \in q^i \setminus p^i} l_b(\phi_b(q^i, p^{-i})),
\end{aligned}$$

where $p^i \setminus q^i = \{a \in A \,|\, a \in p^i, a \notin q^i\}$.

On the other hand,

$$\begin{aligned}
u^i(p^i, p^{-i}) - u^i(q^i, p^{-i}) &= \sum_{a \in p^i} l_a(\phi_a(p^i, p^{-i})) - \sum_{b \in q^i} l_b(\phi_b(q^i, p^{-i})) \\
&= \sum_{a \in p^i \setminus q^i} l_a(\phi_a(p^i, p^{-i})) - \sum_{b \in q^i \setminus p^i} l_b(\phi_b(q^i, p^{-i})).
\end{aligned}$$

Therefore,

$$u^i(p^i, p^{-i}) - u^i(q^i, p^{-i}) \;=\; P(p^i, p^{-i}) - P(q^i, p^{-i}). \tag{1.3}$$

2. See Monderer and Shapley [62]. □

Theorem 1.6. *All Rosenthal congestion game admits a pure Nash equilibrium.*

Proof. In a Rosenthal congestion game $\Gamma(A, (l_a)_{a\in A}, \mathcal{N}, (P^i)_{i\in\mathcal{N}}, (u^i)_{i\in\mathcal{N}})$, according to Theorem 1.5, there is a potential function P such that, for all player i, for all strategies p^i and q^i in P^i, for all profile $p^{-i} \in P^{-i}$, equality (1.3) holds.

Since $\mathbf{P}$ is finite set and cost functions $(l_a)_{a\in A}$ are positives, there exists a global minimum point $\mathbf{p}$ of function P in $\mathbf{P}$. If $\mathbf{p}$ was not a Nash equilibrium of game G, then a player i could have reduced her cost by choosing another strategy q^i, i.e. $u^i(p^i, p^{-i}) > u^i(q^i, p^{-i})$. However, according to (1.3), this implies that $P(p^i, p^{-i}) > P(q^i, p^{-i})$, which contradicts the global minimality of $\mathbf{p}$. □

Remark 1.7. The proof of Theorem 1.6 implies that, in a Rosenthal congestion game, a global minimum of the potential function is a Nash equilibrium. The converse is not true. In particular, there is no uniqueness. In the following example (Rosenthal [74]), the game admits two pure Nash equilibria, and one of them does not minimize the potential function.

Example 1.8. In Figure 1.2, the cost functions are written on the arcs.

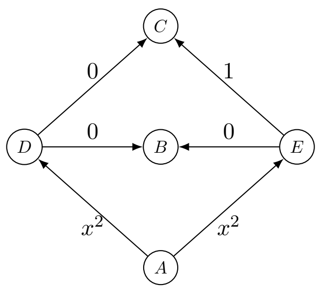


Figure 1.2

Player 1 goes from A to B, while player 2 goes from A to C. At one of the two pure Nash equilibria, player 1 takes path $A \to E \to B$, while player 2 takes path $A \to D \to C$. The potential function attains its minimum 3. The other pure Nash equilibrium is attained when player 1 takes path $A \to D \to B$, while player 2 takes path $A \to E \to C$. The value of the potential function is 4, hence it is not minimized.

Remark 1.9. By definition, the players have the same weight in a Rosenthal congestion game. If the players have different weights, it is called a *weighted* congestion game. The existence of a potential function and thus that of a pure Nash equilibrium are not guaranteed in weighted congestion games. Here is an example from Libman and Orda [51].

Example 1.10. The arc cost functions in Figure 1.3 are given in Table 1.1.

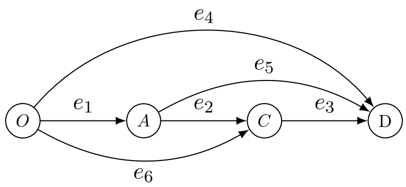


Figure 1.3

arc a	$l_a(1)$	$l_a(2)$	$l_a(3)$
e_1	5	10	40
e_2	5	200	1000
e_3	10	50	60
e_4	60	600	2000
e_5	80	90	100
e_6	50	60	70

Table 1.1

The weight of player 1 is 2, while that of player 2 is 1. Both of them travel from O to D. One can verify that the game has no pure Nash equilibrium.

However, there are particular cases where a weighted game admits a potential function, for example, if the arc cost functions are affine (*cf.* Fotakis et al. [30], Mavroniclas et al. [53]).

Remark 1.11. If the players have the same weight but not the same cost functions for the resources, they are said to have *specific* costs. In this case, a potential function does not necessarily exists either. There is an exception when the cost functions of the same arc for different players are identical up to an additive constant, i.e. if, for all arc a, there exists a function $x \mapsto l_a(x)$ such that, for all player i, her cost function for arc a is $l_a^i(\cdot) = l_a(\cdot) + \theta_a^i$, where $\theta_a^i > 0$ (Mavroniclas et al. [53]).

1.2.3 Nonatomic congestion games

When the number of the players is so large that the influence of a player on the others becomes negligible, one can represent the set of players by a continuum, for example, a real interval endowed with Lebesgue measure. Each player is reduced to a point. That is the origin of the name "nonatomic". Here, we will only give the definition of a nonatomic public goods game, which is a congestion game, and some important properties of the game. A more general analysis on the nonatomic congestion games will be made in §1.3.4.

In a *nonatomic public goods game* Γ, a population of nonatomic players is represented by the real interval [0, 1] endowed with Lebesgue measure λ. The players have a common set of strategies A, which is a finite set of public goods. A per-unit cost function l_a is associated to each piece of goods $a \in A$, which depends only on the total weight of the players choosing it. A strategy profile $\boldsymbol{\theta}$ induces a *flow vector* $\mathbf{x} = (x_a)_{a \in A}$ in the $(|A|-1)$-simplex $\Delta^{|A|-1}$, where x_a stands for the measure of the set of the players choosing a. Hence, by abuse of notation, the cost to the users of a is $l_a(\mathbf{x}) = l_a(x_a)$.

An equilibrium of this game is called a *Wardrop equilibrium* because Wardrop [94] was the first one to formulate it in the following way.

Definition 1.12. In a nonatomic public goods game with a set of goods A, a strategy profile $\boldsymbol{\theta}$ is a *Wardrop equilibrium* if, for all $a, b \in A$ such that a is used by a nonnegligible set of players, i.e. $x_a > 0$, one has

$$l_a(\mathbf{x}) \leq l_b(\mathbf{x}),$$

where $\mathbf{x}$ is the flow vector induced by $\boldsymbol{\theta}$.

Theorem 1.13. *Suppose that, in a nonatomic public goods game Γ with a set of goods A, for all $a \in A$, the cost function l_a is continuous and finite on a neighborhood of* $[0, 1]$. *Then, a real-valued function P called the potential* (*in the sense of population games,* cf. *Chapter 7*) *of game Γ is defined on $\Delta^{|A|-1}$ by*

$$P(\mathbf{x}) = \sum_{a \in \mathcal{A}} \int_0^{x_a} l_a(s)\, \mathrm{d}s. \tag{1.4}$$

All optimal solution of the nonlinear program

$$\min_{\mathbf{x} \in \Delta^{|A|-1}} P(\mathbf{x}) \tag{1.5}$$

is induced by a Wardrop equilibrium.

Conversely, if, in addition to the above assumptions on the arc cost functions, for all $a \in A$, l_a is weakly monotone, then all Wardrop equilibrium induces an optimal solution of program (1.5)*.*

Remark 1.14. One notices immediately the similarity between (1.2) and (1.4), the potential function in the discreet model and that in the continuous model. As in a (discreet) Rosenthal congestion game, one can find Wardrop equilibria in a (continuous) nonatomic congestion game by minimizing its potential function. The continuity of the goods cost functions and the compactness of $\Delta^{|A|-1}$ imply immediately the existence of a Wardrop equilibrium. On the contrary, the goods cost functions have to be monotone in a nonatomic congestion game, which is not the case in a Rosenthal congestion game.

The *social cost* C_s is defined as the sum of the costs to all the players:

$$C_s(\mathbf{x}) = \sum_{a \in \mathcal{A}} x_a\, l_a(x_a). \tag{1.6}$$

If the cost functions are furthermore assumed to be differentiable on their definition domain, then, by defining the *modified cost* of the piece of goods a by

$$\tilde{l}_a(s) = l_a(s) + s\, l'_a(s), \tag{1.7}$$

the social cost can be written as

$$C_s(\mathbf{x}) = \sum_{a \in \mathcal{A}} \int_0^{x_a} \tilde{l}_a(s)\,\mathrm{d}s. \tag{1.8}$$

By comparing (1.4) and (1.8), one sees that the social cost function in a nonatomic congestion game Γ is equivalent to the potential function in another game $\tilde{\Gamma}$ with the modified costs. In consequence, the social cost minimum in Γ would be attained at a Wardrop equilibrium in $\tilde{\Gamma}$.

In particular, if the cost functions are all of forme $l_a = \beta_a x^\alpha$, where α and $\{\beta_a\}_{a\in A}$ are positive constants, then, according to (1.7), $\tilde{l}_a = (1+\alpha)l_a$. Thus, $C_s(\mathbf{x}) = (1+\alpha)P(\mathbf{x})$. This implies that the social optimum and the Wardrop equilibrium coincide in this case. This example was first cited by Dafermos and Sparrow [28].

1.2.4 Braess's paradox and Pigou's example

Braess's paradox It is the most famous paradox in transport analysis [16]. This paradox shows that adding an arc in a network may be disadvantageous to all the passengers.

Suppose that a continuum of nonatomic players of total weight 1 travel from A to C in the left network in Figure 1.4. Two paths are available: $A \to B \to C$ and $A \to D \to C$. The cost of arc AD and that of BC are always 1, while the arc cost functions of AB and DC are x, where x is the total weight on the arc in question. Then, at the unique Wardrop equilibrium, the players equally share the two paths. The cost is common for all, which is $1 + \frac{1}{2} = \frac{3}{2}$.

Suppose that a new arc of cost zero is added in the network to connect B to D as illustrated on the right in Figure 1.4. At the unique Wardrop equilibrium, all the players take path $A \to B \to D \to C$. Their common cost is then $1 + 1 = 2$.

Conclusion: Adding an arc has increased all the players' costs!

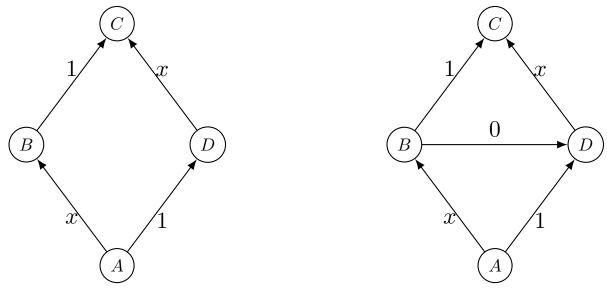


Figure 1.4: The paradox of Braess.

This paradox also exists in Rosenthal's discreet model. Suppose that there are $2n$ players, all of weight 1, traveling from A to C. One has just to replace the per-unit cost of arc AB and that of DC by $\frac{x}{2n+1}$, where x is the total number of the players on the arc. Then, in the left network, at the unique Nash equilibrium, n players take path $A \to B \to C$ and the others take $A \to D \to C$, each having a cost $\frac{3n+1}{2n+1}$. In the right network, at the unique Nash equilibrium, all the $2n$ players take path $A \to B \to D \to C$, and each has a cost $\frac{4n}{2n+1}$.

In a minimization problem, relaxing constraints on the variable or, in other words, extending the definition domain of the objective function, can only improve the optimal solution. The Braess's paradox shows that this is not the case in a game: it is possible that the cost to each player and the social cost are increased when more choices are available.

Pigou's example A well-know example of the sub-optimality of Nash equilibria was established by Pigou. Suppose that a continuum of players of total weight 1 play in a network composed of their origin O, their destination D, and two arcs connecting the two vertices as illustrated in Figure 1.5. The cost of the upper arc is always 1, while that of the lower arc is equal to the total weight of the passengers on the arc.

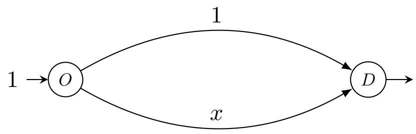


Figure 1.5: Pigou's example.

At the unique Wardrop equilibrium, all the players take the lower arc and their common cost is 1. The social cost is 1. On the contrary, if half of the players take the lower arc and the other half take the upper one, the social cost is only $1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$.

As Braess's paradox, Pigou's example also has a discreet Rosenthal version. Consider $2n$ players, all of weight 1. Now, let the cost of the lower arc by $\frac{x}{2n+1}$. Then, at the unique Nash equilibrium, all the players take the lower arc and have a common cost $\frac{2n}{2n+1}$, thus the average social cost is also $\frac{2n}{2n+1}$. On the contrary, the average social cost attains its minimum $\frac{(3n+1)/2}{2n+1}$ if n players go by the upper arc and the other half go by the lower one.

Pigou's example points out the probable inefficiency of Wardrop or Nash equilibria at the social level in congestion games, caused by the lack of coordination among the players.

1.3 Network congestion

1.3.1 General model

The underlying network of a congestion game is a finite directed graph $G = (V, A, \mathbf{l})$, where V is the vertex set, A the set of directed arcs, $\mathbf{l}$ the vector of arc costs $\mathbf{l} = (l_a)_{a \in A}$, and l_a is the per-unit cost function of arc a defined from $]-\eta, +\infty[$ to $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where $\eta > 0$. The per-unit cost of arc a depends only on the total traffic weight on a. The player set $\mathcal{N}$ can be finite or infinite. A player i is characterized by her weight m^i, i.e. the stock that she controls, and a pair of vertices $(o^i, d^i) \in V \times V$ which are, respectively, the origin and the destination of her stock. In other words, she has to send a quantity m^i of stocks o^i to d^i.

From now on, we consider only network congestion games so that they will be simply called congestion games. Furthermore, we consider only the case where players have a common cost function for each arc. The case where the players have specific costs will be discussed at the end.

Four versions will be treated according to the composition of the players.

Atomic games with unsplittable stocks (*cf.* §1.3.2) Each player controls a finite quantity which cannot be split, hence she has to send it by a single path.

Atomic games with splittable stocks (*cf.* §1.3.3) Each player controls a finite quantity which she can split arbitrarily and send each part by a different path.

Nonatomic games (*cf.* §1.3.4) Each player controls an infinitesimal quantity (Thus the weight of a nonatomic player is zero).

Composites games (*cf.* §1.3.5) Atomic players with splittable stocks and nonatomic players coexist.

1.3.2 Atomic congestion games with unsplittable stocks

A congestion game with a finite number of players who have each a finite quantity to send by a single path is called *atomic with unsplittable stocks*. Such a game is specified by an underlying graph $G = (V, A, \mathbf{l})$, a finite set of players $\mathcal{N} = \{1, \dots, N\}$, a vector of weight $\mathbf{m} = (m^i)_{i\in\mathcal{N}}$, and a vector of origin/destination pairs $(o^i, d^i)_{i\in\mathcal{N}}$. A strategy of player i is a directed elementary path connecting o^i to d^i. Let the set of such paths available to player i be denoted by P^i, and the set of pure strategy profiles be denoted by $\mathbf{P} = \prod_{i\in\mathcal{N}} P^i$. The game is denoted by $\Gamma_{nd}(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, where the subscript nd stands for "unsplittable".

The players' cost functions are slightly different from those in Rosenthal congestion games because the players are weighted. Still let $\phi_a(\mathbf{p})$ denote the total weight of players using arc a, while the strategy profile is $\mathbf{p} = (p^i)_{i\in\mathcal{N}}$, where p^i is the path taken by player i. Then, player i's cost function is defined by

$$u^i(\mathbf{p}) = m^i \sum_{a\in p^i} l_a(\phi_a(\mathbf{p})).$$

Game Γ is a matrix game, thus it admits mixed Nash equilibria which correspond to random choices of paths. On the contrary, except that the players have the same weight (and thus the game is simply a Rosenthal one), the existence of a pure Nash equilibrium is not guaranteed (*cf.* Example 1.10).

1.3.3 Atomic congestion games with splittable stocks

Definitions and notations

If the (finitely many) atomic players can arbitrarily split their stock and send different parts by different paths, the game is called *atomic with splittable stocks*. In such a game $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ (where the subscript d stands for "splittable"), for player i in $\mathcal{N}$ of weight m^i, of origin o^i and of destination d^i, P^i still denotes the set of elementary paths connecting o^i to d^i which are available to her.

A strategy θ^i of player i is defined by a distribution of her stock onto the paths. Explicitly, for path p in P^i, let f^i_p denote the quantity that player i sends by p. Vector $\mathbf{f}^i = (f^i_p)_{p\in P^i}$ is also called the *configuration on the paths* of player i's stock. A configuration on the paths induced a configuration on the arcs. For arc a, denote

$$x^i_a = \sum_{p\colon p\in P^i,\, a\in p} f^i_p,$$

the total weight of the stock that player i sends on arc a. To avoid confusion, let us call vector $\mathbf{x}^i = (x^i_a)_{a\in A}$ player i's *flow*, and reserve the term *configuration* for *paths* only. The *incidence matrix* $D^i \in \mathbb{R}^{|A|\times|P^i|}$ is defined by

$$D^i_{a,p} = (\delta_{a,p})_{a\in A,\, p\in P^i},$$

where $\delta_{a,p} = 1$ if arc a belongs to path p, and 0 otherwise. Then, the connection between a configuration and the flow induced by it is given by the following equation:

$$\mathbf{x}^i \;=\; D^i\,\mathbf{f}^i. \tag{1.9}$$

Unlike $\mathbf{f}^i$, $\mathbf{x}^i$ does not characterize player i's pure strategy. Different configurations can induce the same flow, and matrix D^i is not invertible in the general case.

The *set of feasible configurations* of player i is her (pure) strategy set. This is a convex compact set in $\mathbb{R}^{|P^i|}$, defined by

$$F^i = \{\, \mathbf{f}^i = (f^i_p)_{p\in P^i} \in \mathbb{R}^{|P^i|} \mid \mathbf{f}^i \geq \mathbf{0}\,;\; \sum_{p\in P^i} f^i_p = m^i \,\}. \tag{1.10}$$

The *set of feasible flows* of player i is a convex compact set in $\mathbb{R}^{|A|}$, defined by

$$F^i_l = \{\, \mathbf{x}^i \in \mathbb{R}^{|\mathcal{A}|} \mid \exists\, \mathbf{f}^i \in F^i \text{ such that } \mathbf{x}^i = D^i\, \mathbf{f}^i \,\}. \tag{1.11}$$

The strategy profile of the N players is specified by the vector of players' configurations or, the *system configuration*, $\mathbf{f} = (\mathbf{f}^i)_{i\in\mathcal{N}}$. It induces the vector of players' flows or, the *system flow*, $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}}$. Let us define the system incidence matrix by

$$\mathbf{D} = diag\,\{\, D^1, \,\ldots,\, D^N \,\}.$$

Explicitly, $\mathbf{D} = (d_{st})_{st}$ is a matrix of dimension $N|A| \times \sum_{i\in\mathcal{N}} |P^i|$. The bloc composed of the elements d_{st} such that $(i|A| - |A| + 1) \leq s \leq i|A|$ and $(\sum_{j=1}^{i-1} |P|^j) \leq t \leq (\sum_{j=1}^{i} |P|^j)$ corresponds to player i. In this bloc, the element d_{st} is 1 if $s = (i-1)|A| + a$ and $t = \sum_{j=1}^{i-1} |P|^j + p$ for certain $a \in A$ and certain $p \in P^i$ (i.e., arc a is a component of path p), otherwise d_{st} is 0. The elements not belonging to these N blocs are all zero.

Then,

$$\mathbf{x} \,=\, \mathbf{D}\,\mathbf{f}. \tag{1.12}$$

The set of strategy profiles or, the *set of feasible system configurations*, is a convex compact set in $\mathbb{R}^{\sum_{i\in\mathcal{N}} |P^i|}$, defined by $F = F^1 \times \cdots \times F^N$. The *set of feasible system flow* is a convex compact set in $\mathbb{R}^{|A|\times N}$ defined by $F_l = F^1_l \times \cdots \times F^N_l$.

Let us define the vector of aggregate arc flows or, simply, the *aggregate flow* by

$$\boldsymbol{\xi} = (\xi_a)_{a\in A}, \text{ where } \xi_a = \sum_{i\in\mathcal{N}} x^i_a. \tag{1.13}$$

The aggregate flow $\boldsymbol{\xi}$ describes the total weight of traffic on each arc induced by the system flow $\mathbf{x}$.

Given a system configuration $\mathbf{f}$, as well as the system flow $\mathbf{x}$ and the aggregate flow $\boldsymbol{\xi}$ induced by it, the vector of arc costs is $\mathbf{l}\,(\mathbf{x}) = \big(l_a(\xi_a)\big)_{a\in A}$. This determines the cost of path $p \in \bigcup_{i\in\mathcal{N}} P^i$:

$$c_p(\mathbf{f}) = \sum_{a\in p} l_a(\xi_a).$$

Player i's path costs vector is defined by

$$\mathbf{c}^i(\mathbf{f}) = \big(c^i_p(\mathbf{f})\big)_{p\in P^i} = {D^i}^T\, \mathbf{l}\,(\mathbf{x}), \tag{1.14}$$

where the notation M^T stands for the transpose of matrix M.

The *vector of system path costs*, which is a function of the system configuration $\mathbf{f}$, is defined by $\underline{\mathbf{c}}(\mathbf{f}) = \big(\mathbf{c}^i(\mathbf{f})\big)_{i\in\mathcal{N}}$, and the *vector of system arc costs* is defined by $\underline{\mathbf{l}}\,(\mathbf{x}) = \big(\underbrace{\mathbf{l}\,(\mathbf{x}), \ldots, \mathbf{l}\,(\mathbf{x})}_{N \text{ times}}\big)$, then

$$\underline{\mathbf{c}}(\mathbf{f}) = \mathbf{D}^T\, \underline{\mathbf{l}}\,(\mathbf{x}). \tag{1.15}$$

Now, one can determine the players' cost functions. Given a strategy profile $\mathbf{f}$, player i's cost function is defined by

$$u^i(\mathbf{f}) = \langle\, \mathbf{f}^i \,,\, \mathbf{c}^i(\mathbf{f}) \,\rangle \;=\; \sum_{p\in P^i} f_p^i\, c_p^i(\mathbf{f}), \tag{1.16}$$

where $\langle , \rangle$ stands for the inner product.

One can also compute player i's cost by the system flow:

$$v^i(\mathbf{x}) \;=\; \langle\, \mathbf{x}^i \,,\, \mathbf{l}\,(\mathbf{x}) \,\rangle \;=\; \sum_{a\in\mathcal{A}} x_a^i\, l_a(\xi_a). \tag{1.17}$$

Remark 1.15. At first glance, the $(|P^i|-1)$ dimensional simplex $\Delta(P^i)$, which is the set of *mixed* strategies of player i in a game with *unsplittable* stocks, is identical to F^i, the set of *pure* strategies of player i in a game with *splittable* stocks. Rigourously, $F^i = m^i\Delta(P^i)$, in the sense that $\mathbf{x}^i \mapsto m^i\mathbf{x}^i$ is a bijection from $\Delta(P^i)$ to F^i. However, the interpretation is quite different: in the first case, x_p^i is the probability with which player i sends a quantity m_i on path p, thus it describes a random choice; in the second case, $m^i x_p^i$ is the quantity that she puts on path p, hence it describes a determinist choice. In particular, player i's cost functions are different in the two cases and the two games are of fairly different natures. Let us cite a simple example to understand the difference.

Example 1.16. Suppose that an atomic player of weight 1 has to send her stock from vertex o to vertex d. Two parallel arcs connect the two vertices. If the stock is unsplittable, a mixed strategy $(\frac{1}{2},\frac{1}{2})$ means that the player sends all her stock by arc 1 with probability $\frac{1}{2}$, and by arc 2 with the same probability. If the stock is splittable, a pure strategy $(\frac{1}{2},\frac{1}{2})$ means that the player sends half of the stock by arc 1 and the rest by arc 2. If per-unit cost functions of the two arcs are both $x \mapsto x$, then the expected cost of the player is 1 in the unsplittable case, but $\frac{1}{2}$ in the splittable case.

Nash equilibrium: definition and characterizations

Let us give the definition of a Nash equilibrium (in pure strategies) in two formulations: via paths and via arcs.

Definition 1.17. In an atomic congestion game with splittable stocks $\Gamma_d(G,\mathcal{N},\mathbf{m},\mathbf{P})$, a system configuration $\mathbf{f}^* \in F$ is a *Nash equilibrium* if, for all player i, her configuration $\mathbf{f}^{*i} \in F^i$ is an optimal solution of the program

$$\min_{\mathbf{f}^i\in F^i} u^i(\,\mathbf{f}^i\,,\,\mathbf{f}^{*-i}\,), \tag{1.18}$$

where $\mathbf{f}^{*-i} = (\mathbf{f}^{*j})_{j\in\mathcal{N}\setminus\{i\}} \in F^{-i} = \prod_{j\in\mathcal{N}\setminus\{i\}} F^j$.

In the same game, a system flow $\mathbf{x}^* \in F_l$ is *induced by a Nash equilibrium* if, for all player i, $\mathbf{x}^{*i} \in F_l^i$ is an optimal solution of the program

$$\min_{\mathbf{x}^i\in F_l^i} v^i(\,\mathbf{x}^i\,,\,\mathbf{x}^{*-i}\,), \tag{1.19}$$

where $\mathbf{x}^{*-i} = (\mathbf{x}^{*j})_{j\in\mathcal{N}\setminus\{i\}} \in F_l^{-i} = \prod_{j\in\mathcal{N}\setminus\{i\}} F_l^j$.

In this framework, some assumptions on the arc cost functions are necessary. Let $M = \sum_{i\in\mathcal{N}} m^i$ be the total weight of all the players.

A 1.18. *For all arc $a \in A$, the cost function l_a is continuous and finite on a neighborhood U of interval $[0,\, M]$, and positive on $U \cap \mathbb{R}_+$.*

A1.18 is assumed to be true throughout this work.

A 1.19. *For all arc $a \in A$, the cost function l_a is of class $\mathcal{C}^1$ on U.*

Remark 1.20. Under A1.19, according to equation (1.12) and the fact that $c_p(\mathbf{f}) = \sum_{a \in p} l_a(\xi_a)$, for all player i and all path $p \in P^i$, $c_p(\mathbf{f})$ is a function of class $\mathcal{C}^1$ on a neighborhood of F.

Now, one can define the marginal costs of the paths and those of the arcs.

Definition 1.21. In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, under A1.19, the *marginal cost of path $p \in P^i$* for player i is a function of the system configuration defined by

$$\hat{c}^i_p(\mathbf{f}) = c_p(\mathbf{f}) + \sum_{l \in P^i} f^i_l \frac{\partial\, c_l(\mathbf{f})}{\partial\, f^i_p}.$$

The vector of marginal path costs for player i is $\hat{\mathbf{c}}^i(\mathbf{f}) = \left(\hat{c}^i_p(\mathbf{f})\right)_{p \in P^i}$. The system vector of marginal path costs is $\hat{\mathbf{c}}\,(\mathbf{f}) = \left(\hat{\mathbf{c}}^i(\mathbf{f})\right)_{i \in \mathcal{N}}$.

The *marginal cost of arc $a \in A$* for player i is a function of the system flow defined by

$$\hat{l}^i_a(\mathbf{x}) = l_a(\xi_a) + x^i_a the_a(\xi_a).$$

The vector of marginal arc costs for player i is $\hat{\mathbf{l}}^i(\mathbf{x}) = \left(\hat{l}^i_a(\mathbf{x})\right)_{a \in A}$. The system vector of marginal arc costs is $\hat{\mathbf{l}}\,(\mathbf{x}) = \left(\hat{\mathbf{l}}^i(\mathbf{x})\right)_{i \in \mathcal{N}}$.

Remark 1.22. It is easy to verify that

$$\hat{\mathbf{c}}^i\,(\mathbf{f}) = \nabla_i\, u^i(\mathbf{f}), \quad \hat{\mathbf{l}}^i(\mathbf{x}) = \nabla_i\, v^i(\mathbf{x}), \tag{1.20}$$

where $\nabla_i\, u^i(\mathbf{f})$ stands for the gradient of function $u^i(\mathbf{f}^i,\, \mathbf{f}^{-i})$ with respect to $\mathbf{f}^i$, while $\nabla_i v^i(\mathbf{x})$ stands for the gradient of function $v^i(\mathbf{x}^i,\, \mathbf{x}^{-i})$ with respect to $\mathbf{x}^i$.

Next, let us formulate the conditions of Nash equilibrium via variational inequalities under A1.19. To this end, first recall a classical result which characterizes a solution of a (convex) optimization problem by a variational inequality.

Proposition 1.23. *Suppose that X is a closed convex set in $\mathbb{R}^n$, and f is a real-valued function of class $\mathcal{C}^1$ defined on X.*

1. All optimal solution $\mathbf{x}^*$ of the program

$$\min_{\mathbf{x} \in X} f(\mathbf{x}) \tag{1.21}$$

satisfies the following variational inequality:

$$\langle\, \nabla f(\mathbf{x}^*)\,,\, \mathbf{x} - \mathbf{x}^*\, \rangle \geq 0, \qquad \forall\, \mathbf{x} \in X. \tag{1.22}$$

2. If f is convex on X, then all point $\mathbf{x}^*$ in X satisfying (1.22) is an optimal solution of the program (1.21).

Proof. See Kinderlehrer and Stampacchia [49, Chapter 1, Proposition 5.1] for statement 1, and [49, Chapter 1, Proposition 5.2] for statement 2. □

The following theorem characterizes a Nash equilibrium in an atomic congestion game with splittable stocks by a family of variational inequalities.

Theorem 1.24. *Suppose that in an atomic congestion game with splittable stocks $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, A1.19 holds.*

1. If a system configuration $\mathbf{f}^* \in F$ is a Nash equilibrium, then $\mathbf{f}^*$ satisfies the following variational inequality:

$$\langle\, \hat{\mathbf{c}}\,(\mathbf{f}^*)\,,\, \mathbf{f} - \mathbf{f}^*\,\rangle \geq 0, \qquad \forall\, \mathbf{f} \in F. \tag{1.23}$$

Conversely, if, for all player i and for all fixed $\mathbf{f}^{-i} \in F^{-i}$, $u^i\,(\mathbf{f}^i\,,\, \mathbf{f}^{-i})$ is convex with respect to $\mathbf{f}^i$, then all $\mathbf{f}^* \in F$ satisfying (1.23) is a Nash equilibrium.

2. If a system flow $\mathbf{x}^* \in F_l$ is induced by a Nash equilibrium, then $\mathbf{x}^*$ satisfies the following variational inequality:

$$\langle\, \hat{\mathbf{l}}\,(\mathbf{x}^*)\,,\, \mathbf{x} - \mathbf{x}^*\,\rangle \geq 0, \qquad \forall\, \mathbf{x} \in F_l. \tag{1.24}$$

Conversely, if, for all player i and for all fixed $\mathbf{x}^{-i} \in F_l^{-i}$, $v^i\,(\mathbf{x}^i\,,\, \mathbf{x}^{-i})$ is convex with respect to $\mathbf{x}^i$, then all $\mathbf{x}^*$ satisfying (1.24) is induced by a Nash equilibrium.

Proof. Only statement 1 will be proved. Statement 2 can be proved in a similar way.

According to Proposition 1.23, on the one hand, if $\mathbf{f}^{*i}$ is an optimal solution of (1.18), then

$$\langle\, \hat{\mathbf{c}}^i\,(\mathbf{f}^{*i}, \mathbf{f}^{*-i})\,,\, \mathbf{f}^i - \mathbf{f}^{*i}\,\rangle \geq 0, \qquad \forall\, \mathbf{f}^i \in F^i; \tag{1.25}$$

on the other hand, if $u^i\,(\mathbf{f}^i\,,\, \mathbf{f}^{*-i})$ is convex with respect to $\mathbf{f}^i$, then all $\mathbf{f}^{*i} \in F^i$ satisfying (1.23) is an optimal solution of (1.18).

It remains to show that (1.25) is equivalent to (1.23).

On the one hand, if (1.25) is true for all i, then (1.23) follows immediately; on the other hand, if (1.23) is true, one can take a specific configuration $\mathbf{f} \in F$ such that $\mathbf{f}^j = \mathbf{f}^{*j}$ for all $j \neq i$ to obtain (1.25) for player i. □

Remark 1.25. Variational inequalities were used by Haurie and Marcotte [36] to characterize the Nash equilibria in atomic congestion games with splittable stocks.

Nash equilibrium: existence and uniqueness

The formulation of Nash equilibria via variational inequalities is essential to study their existence and uniqueness.

Let us first recall the definition of a monotone map, then give some classical results on the existence and the uniqueness of solutions of variational inequalities.

Definition 1.26. Suppose that X is a subset of $\mathbb{R}^n$ and $\mathbf{F}$ is a map defined on X to $\mathbb{R}^n$.

Map $\mathbf{F}$ is *monotone* on X if

$$\langle\, \mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\,,\, \mathbf{x} - \mathbf{y}\,\rangle \geq 0, \qquad \forall\, \mathbf{x} \in X, \forall\, \mathbf{y} \in X;$$

Map $\mathbf{F}$ is *strictly monotone* on X if

$$\langle\, \mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\,,\, \mathbf{x} - \mathbf{y}\,\rangle > 0, \qquad \forall\, \mathbf{x} \in X, \forall\, \mathbf{y} \in X,\ \mathbf{x} \neq \mathbf{y}.$$

Theorem 1.27. *Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. Suppose that $\mathbf{F}$ is a continuous map defined on X to $\mathbb{R}^n$. Then, there exists $\mathbf{x}^* \in X$ which satisfies the following variational inequality:*

$$\langle\, \mathbf{F}(\mathbf{x}^*)\,,\, \mathbf{x} - \mathbf{x}^*\,\rangle \geq 0, \qquad \forall\, x \in X. \tag{1.26}$$

If $\mathbf{F}$ is strictly monotone on X, then the solution of variational inequality (1.26) is unique.

Proof. For the existence, see Kinderlehrer and Stampacchia [49, Chapter 1, Theorem 3.1].

For the uniqueness under the assumption that $\mathbf{F}$ is strictly monotone on X, suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ in X are two solutions of variational inequality (1.26), then

$$\langle\, \mathbf{F}(\mathbf{x}_1)\,,\, \mathbf{x}_2 - \mathbf{x}_1 \,\rangle \geq 0,\ \langle\, \mathbf{F}(\mathbf{x}_2)\,,\, \mathbf{x}_1 - \mathbf{x}_2 \,\rangle \geq 0 \Rightarrow \langle\, \mathbf{F}(\mathbf{x}_1) - \mathbf{F}(\mathbf{x}_2)\,,\, \mathbf{x}_1 - \mathbf{x}_2 \,\rangle \leq 0,$$

so that $\mathbf{x}_1 = \mathbf{x}_2$. □

The following theorem is an immediate consequence of Theorem 1.27.

Theorem 1.28. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, under A1.19, if, for all player i and for all profile* $\mathbf{f}^{-i} \in F^{-i}$ (resp. $\mathbf{x}^{-i} \in F_l^{-i}$), $u^i(\mathbf{f}^i, \mathbf{f}^{-i})$ (resp. $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$) *is convex with respect to* $\mathbf{f}^i$ (resp. $\mathbf{x}^i$), *then the game admits a Nash equilibrium.*

If $\hat{\mathbf{l}}\,(\mathbf{x})$ is strictly monotone, then the flow induced by the Nash equilibria is unique.

Proof. Under A1.19, variational inequality (1.23) admits a solution $\mathbf{f}^*$ in F. According to Theorem 1.24, if, for all i and all $\mathbf{f}^{-i} \in F^{-i}$, $u^i(\mathbf{f}^i, \mathbf{f}^{-i})$ is convex with respect to $\mathbf{f}^i$, then $\mathbf{f}^*$ is a Nash equilibrium of the game.

If $\hat{\mathbf{l}}\,(\mathbf{x})$ is strictly monotone, then the variational inequality (1.23) admits only one solution. According to Theorem 1.24, all flows induced by a Nash equilibrium of the game is a solution of (1.23). Consequently, the flow induced by the Nash equilibria is unique. □

Naturally, one would like to know when $u^i(\mathbf{f}^i, \mathbf{f}^{-i})$ (*resp.* $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$) is convex with respect to $\mathbf{f}^i$ (*resp.* $\mathbf{x}^i$). Here are two examples.

Lemma 1.29. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, under A1.19, if, for all arc $a \in A$, the cost function l_a is nondecreasing and convex on U, then $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex with respect to $\mathbf{x}^i$ for all fixed $\mathbf{x}^{-i} \in F^{-i}$.*

Proof. Recall that $\nabla_i v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ stands for the gradient of $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ with respect to $\mathbf{x}^i$, i.e.

$$\nabla_i v^i(\mathbf{x}) = \Big(l_a(x_a) + x_a^i l_a'(x_a) \Big)_{a \in \mathcal{A}}.$$

In order to show that $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex with respect to $\mathbf{x}^i$, it is enough to show that

$$v^i(\mathbf{y}^i, \mathbf{x}^{-i}) \geq v^i(\mathbf{x}^i, \mathbf{x}^{-i}) + \langle\, \nabla_i v^i(\mathbf{x}^i, \mathbf{x}^{-i}), \mathbf{y}^i - \mathbf{x}^i \,\rangle, \quad \forall\, \mathbf{x}^i, \mathbf{y}^i \in F^i. \tag{1.27}$$

For all arc a, l_a is convex and nondecreasing. This implies

$$\begin{aligned}
& l_a(y_a^i + x_a^{-i}) &\geq\ & l_a(x_a^i + x_a^{-i}) + (y_a^i - x_a^i)\, l_a'(x_a^i + x_a^{-i}) \\
\Rightarrow\quad & y_a^i\, l_a(y_a^i + x_a^{-i}) &\geq\ & y_a^i\, l_a(x_a) + y_a^i\, (y_a^i - x_a^i)\, l_a'(x_a) \\
& &\geq\ & y_a^i\, l_a(x_a) + x_a^i\, (y_a^i - x_a^i)\, l_a'(x_a) \\
& &=\ & x_a^i\, l_a(x_a) + (y_a^i - x_a^i)\, [\, l_a(x_a) + x_a^i\, l_a'(x_a)\,]. \\
\Rightarrow\ & \sum_{a \in \mathcal{A}} y_a^i\, l_a(y_a^i + x_a^{-i}) &\geq\ & \sum_{a \in \mathcal{A}} x_a^i\, l_a(x_a) + \sum_{a \in \mathcal{A}} (y_a^i - x_a^i)\, [\, l_a(x_a) + x_a^i\, l_a'(x_a)\,],
\end{aligned}$$

which is simply a reformulation of (1.27). □

Lemma 1.30. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, if, for all arc $a \in A$, the cost function l_a is of class $\mathcal{C}^2$ on U, and*

$$2\, l_a'(x) + y\, l_a''(x) \geq 0, \qquad \forall\, x \in U \cap \mathbb{R}_+,\ \forall\, y \in [0, x],$$

then $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex with respect to $\mathbf{x}^i$ for all fixed $\mathbf{x}^{-i} \in F^{-i}$.

Proof. If the conditions in the theorem hold, then the Hessien matrix of $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$, as a function of $\mathbf{x}^i$, is a diagonal matrix, and the element corresponding to arc a is $2\,l'^i_a(x_a) + x^i_a\, l''^i(x_a) \geq 0$. Consequently, $v^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex with respect to $\mathbf{x}^i$. □

Theorem 1.28 and Lemmas 1.29 and 1.30 entail immediately the following theorems.

Theorem 1.31. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, under A1.19, if, for all arc $a \in A$, the cost function l_a is nondecreasing and convex on U, then the game admits a Nash equilibrium.*

Theorem 1.32. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, if the conditions in Lemma 1.30 hold, then the game admits a Nash equilibrium.*

Next, one would like to find the sufficient conditions for $\hat{\mathbf{l}}$ to be strictly monotone. Here is a particular case.

Theorem 1.33. *In $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, if, for all $a \in \mathcal{A}$, $l_a(s) = b_a\, s + c_a$ for $s \in U$ with $b_a \geq 0$ and $c_a \geq 0$, and there exists $\hat{a} \in A$ such that $b_{\hat{a}} > 0$, then the system flow induced by the Nash equilibria is unique.*

Proof. For all $a \in A$, $\hat{l}^i_a(\mathbf{x}) = b_a(\xi_a + x^i_a) + c_a$.

Let $\mathbf{x}$ and $\mathbf{x}'$ be two distinct points in $\mathcal{F}_l$. Then,

$$\begin{aligned}\langle\, \hat{\mathbf{l}}\,(\mathbf{x}) - \hat{\mathbf{l}}\,(\mathbf{x}')\,,\, \mathbf{x} - \mathbf{x}'\,\rangle &= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{N}} (\, b_a(\xi_a + x^i_a) - b_a(\xi'_a + x'^i_a)\,)\,(x^i_a - x'^i_a) \\ &= \sum_{a\in\mathcal{A}} b_a \Big[(\xi_a - \xi'_a)\sum_{i\in\mathcal{N}}(x^i_a - x'^i_a) + \sum_{i\in\mathcal{N}}(x^i_a - x'^i_a)^2\Big] \\ &= \sum_{a\in\mathcal{A}} b_a \Big[(\xi_a - \xi'_a)^2 + \sum_{i\in\mathcal{N}}(x^i_a - x'^i_a)^2\Big] \\ &> 0.\end{aligned}$$

The last inequality is due to the assumption that there exists $\hat{a} \in A$ such that $b_{\hat{a}} > 0$.

The conclusion follows from Theorem 1.28. □

Remark 1.34. If, for all $a \in \mathcal{A}$, $l_a(s) = b_a\, s + c_a$ for $s \in U$ with $b_a \geq 0$ and $c_a \geq 0$, game $\Gamma_d(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ is a potential game in the sense of Monder and Shapley. A potential function is

$$P(\mathbf{x}) \;=\; \frac{1}{2}\sum_{a\in\mathcal{A}} b_a \left(\xi^2_a + \sum_{i\in\mathcal{N}} x^{i2}_a\right) + \sum_{a\in\mathcal{A}} c_a\xi_a.$$

Indeed, player i's cost function is

$$v^i(\,\mathbf{x}^i\,,\, \mathbf{x}^{-i}\,) = \sum_{a\in\mathcal{A}} x^i_a\,(b_a\xi_a + c_a) = \sum_{a\in\mathcal{A}} b_a {x^i_a}^2 + (\sum_{a\in\mathcal{A}} b_a x^{-i}_a + c_a)x^i_a, \quad \mathbf{x}^i \in \mathcal{F}^i_l,\ \mathbf{x}^{-i} \in \mathcal{F}^{-i}_l,$$

where $x^{-i}_a = \sum_{j\in\mathcal{N}\setminus\{i\}} x^i_a$.

One can verify that, for all $\mathbf{x}^i$ and $\mathbf{y}^i$ in $\mathcal{F}^i_l$ and all $\mathbf{x}^{-i}$ in $\mathcal{F}^{-i}_l$,

$$\begin{aligned} v^i(\mathbf{x}^i, \mathbf{x}^{-i}) - v^i\left(\mathbf{y}^i, \mathbf{x}^{-i}\right) &= P(\mathbf{x}^i, \mathbf{x}^{-i}) - P(\mathbf{y}^i, \mathbf{x}^{-i}) \\ &= \sum_{a\in\mathcal{A}} b_a({x^i_a}^2 - {y^i_a}^2) + \sum_{a\in\mathcal{A}}(b_a x^{-i}_a + c_a)(x^i_a - y^i_a).\end{aligned}$$

Remark 1.35. In the general case where the arc cost functions are not affine, or the players have specific costs, the uniqueness of the Nash equilibrium is not guaranteed. Nevertheless, Orda et al. [66] showed that, if the network is composed of two vertices connected by parallel arcs or if the atomic players are *symmetric*, i.e. if they have interchangeable stocks, the equilibrium is unique under much less stringent conditions on the convexity of the arc cost functions. Richman and Shimkin [71] and Bhaskar et al. [10] extended, respectively, the result (on parallel networks) of [66] to *nearly parallel* networks (a notion introduced by Milchtaich [61]), and to *generalized parallel* networks. Altman et al. [3] proved the uniqueness of the Nash equilibrium for a specific class of arc cost functions.

1.3.4 Nonatomic congestion games

Model

A *nonatomic* congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ (where the subscript n stands for "nonatomic") is specified by the following elements.

- $G = (V, A, \mathbf{l})$, a directed finite graph equipped with a per-unit cost function l_a for each arc $a \in A$.
- $\mathcal{N} = \{1, 2, \ldots, N\}$, the finite set of populations of nonatomic players. Population i is described by a real interval I^i endowed with Lebesgue measure λ. Each player in population i is specified by a point in I^i. The players in the same population are identical and anonymous.
- $\mathbf{m} = (m^i)_{i \in \mathcal{N}}$, where $m^i = \lambda(I^i)$ is the total weight of population i.
- $\mathbf{P} = \prod_{i \in \mathcal{N}} P^i$, where $P^i \in 2^A$ is the set of elementary paths from vertex o^i to vertex d^i, which are, respectively, the origin and the destination of population i.

A (pure) strategy of a player in population i is a path in P^i. A (pure) strategy profile of population i is described by a measurable map θ^i defined from interval I^i to P^i such that player $t \in I^i$ takes path $\theta^i(t) \in P^i$. Map θ^i induces a configuration $\mathbf{f}^i = (f^i_p)_{p \in P^i}$, where $f^i_p = \lambda(\theta^{-i}(p))$ is the total weight of the players in population i choosing path p. Since the players in population i are identical and anonymous, the configuration $\mathbf{f}^i$ characterizes the strategy profile θ^i in the sense that all the profiles inducing the same configuration are equivalent. From now on, *strategy profile* of population i means its configuration.

One notices immediately the similarity between the configuration of a population in a nonatomic game and that of a player in an atomic game with splittable stocks. Even though a population and an atomic player with splittable stocks have strategically different objectives, both have the same kind of behavior, namely, distributing the stock on available paths.

The flow $\mathbf{x}^i = D^i \mathbf{f}^i$ of population i is defined in the same way. Its set of feasible configurations F^i, its set of feasible flows F^i_l, the system configuration $\mathbf{f}$, the system flow $\mathbf{x}$, the set of feasible system configurations F, the set of feasible system flows F_l and the aggregate flow $\boldsymbol{\xi}$ are all defined as their counterparts in §1.3.3. Population i's flow still does not characterize its strategy profile. The cost for a player to take path p is $c_p(\mathbf{f}) = \sum_{a \in p} l_a(\xi_a)$. The vector of the costs of the paths available to population i is defined by $\mathbf{c}^i(\mathbf{f})$, the vector of system path costs $\underline{\mathbf{c}}(\mathbf{f})$, the vector of arc costs $\mathbf{l}(\mathbf{x})$ and the vector of system arc costs $\underline{\mathbf{l}}(\mathbf{x})$ are defined in the same way as their counterparts in §1.3.3.

Wardrop equilibrium: definition and characterizations

At an equilibrium of a nonatomic congestion game, no player has incentive to change her path unilaterally. In other words, the cost of a path chosen by a player is lower than or equal to the cost of any other path that she could have chosen, where the cost of the first path is in the actual configuration, while the cost of the second path is in the configuration after the deviation. However, the fact that a player of measure zero change her path does not change the system configuration. One deduces that a path used by a nonatomic player at an equilibrium has a cost lower than or equal to any other path between the same pair of origin and destination. This result was first formulated by Wardrop [94] (*cf.* §1.2.3).

Definition 1.36. In $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, a system configuration $\mathbf{f}$ is a *Wardrop equilibrium* if, for all population i and all path $p \in P^i$,

$$\text{if} \quad f_p^i > 0, \quad \text{then} \quad c_p(\mathbf{f}) \leq c_q(\mathbf{f}) \ \text{ for all } \ q \in P^i. \tag{1.28}$$

Similar to a Nash equilibrium in an atomic game with splittable stocks, a Wardrop equilibrium in a nonatomic game is also characterized by a variational inequality, but for a different reason. Recall that variational inequalities (1.23) and (1.24) are obtained as first order conditions of some nonlinear programs. On the contrary, the variational inequality characterizing a Wardrop equilibrium is deduced from its definition (1.28). This is shown by the proof of the following theorem which extends Theorem 1.5.

Theorem 1.37. *In* $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$*,*
1. a system configuration $\mathbf{f}^* \in F$ *is a Wardrop equilibrium if, and only if, it satisfies the following variational inequality:*

$$\langle\, \underline{\mathbf{c}}(\mathbf{f}^*)\,,\ \mathbf{f} - \mathbf{f}^* \,\rangle \geq 0, \qquad \forall\, \mathbf{f} \in \mathcal{F}; \tag{1.29}$$

2. a system flow $\mathbf{x}^* \in F_l$ *is induced by a Wardrop equilibrium if, and only if, it satisfies the following variational inequality:*

$$\langle\, \underline{\mathbf{l}}\,(\mathbf{x}^*)\,,\ \mathbf{x} - \mathbf{x}^* \,\rangle \geq 0, \qquad \forall\, \mathbf{x} \in \mathcal{F}_l. \tag{1.30}$$

Proof. 1. Define $\mu^i = \min_{p \in \mathcal{P}^i} c_p^i(\mathbf{f}^*)$. Then,

$$\begin{aligned}
\sum_{p \in \mathcal{P}^i} \left(c_p(\mathbf{f}^*) - \mu^i\right)\left(f_p^i - f_p^{*i}\right) &= \sum_{p \in \mathcal{P}^i} c_p(\mathbf{f}^*)\left(f_p^i - f_p^{*i}\right) - \mu^i \sum_{p \in \mathcal{P}^i} \left(f_p^i - f_p^{*i}\right) \\
&= \sum_{p \in \mathcal{P}^i} c_p(\mathbf{f}^*)\left(f_p^i - f_p^{*i}\right) - \mu^i(m^i - m^i) \\
&= \langle\, c_p(\mathbf{f}^*),\ \mathbf{f}^i - \mathbf{f}^{*i} \,\rangle,
\end{aligned}$$

which implies that (1.29) is equivalent to

$$\sum_{p \in \mathcal{P}^i} \left(c_p(\mathbf{f}^*) - \mu^i\right)\left(f_p^i - f_p^{*i}\right) \geq 0. \tag{1.31}$$

It remains to show the equivalence between (1.31) and (1.28).
(1.28) $\Rightarrow$ (1.31): According to (1.28),

$$\left(c_p(\mathbf{f}^*) - \mu^i\right)\left(f_p^i - f_p^{*i}\right) = \begin{cases} \left(c_p(\mathbf{f}^*) - \mu^i\right) f_p^i \geq 0, & \text{if } f_p^{*i} = 0, \\ 0, & \text{if } f_p^{*i} > 0, \end{cases}$$

and (1.31) is satisfied.

(1.31) $\Rightarrow$ (1.28): Define an auxiliary configuration $\mathbf{f}^i$ of player i as follows:

$$f_p^i = \begin{cases} 0, & \text{if } c_p(\mathbf{f}^*) > \mu^i, \\ m^i/\kappa^i, & \text{if } c_p(\mathbf{f}^*) = \mu^i, \end{cases}$$

where $\kappa^i = \left|\{\, p \in \mathcal{P}^i \,|\, c_p^i(\mathbf{f}^*) = \mu^i \,\}\right|$ is the number of the paths in P^i which have minimal cost at $\mathbf{f}^*$. By taking this $\mathbf{f}^i$ in (1.31), one has

$$\sum_{p\in\mathcal{P}^i:\, c_p^i(\mathbf{f}^*)>\mu^i} \left(c_p^i(\mathbf{f}^*\,) - \mu^i\right)\left(-f_p^{*i}\right) \geq 0,$$

which implies that f_p^{*i} can only be zero if $c_p^i(\mathbf{f}^*) > \mu^i$, then (1.28) follows.

2. Suppose that a system flow $\mathbf{x}^*$ is induced by a Wardrop equilibrium $\mathbf{f}^*$, i.e. $\mathbf{x}^* = \mathbf{D}\,\mathbf{f}^*$. According to statement 1, $\mathbf{f}^*$ satisfies (1.29).

For $\mathbf{x} \in F_l$, there exists $\mathbf{f}$ in F such that $\mathbf{x} = \mathbf{D}\,\mathbf{f}$. Then, according to (1.29) and (1.15),

$$\begin{aligned} &\left\langle\, \underline{\mathbf{c}}(\mathbf{f}^*)\,,\, \mathbf{f} - \mathbf{f}^* \,\right\rangle \geq 0 \;\Rightarrow\; \left\langle\, \mathbf{D}^T\, \underline{\mathbf{l}}\,(\mathbf{x}^*)\,,\, \mathbf{f} - \mathbf{f}^* \,\right\rangle \geq 0 \\ \Rightarrow\; &\left\langle \underline{\mathbf{l}}\,(\mathbf{x}^*)\,,\, \mathbf{D}\,\mathbf{f} - \mathbf{D}\,\mathbf{f}^* \,\right\rangle \geq 0 \;\Rightarrow\; \left\langle \underline{\mathbf{l}}\,(\mathbf{x}^*)\,,\, \mathbf{x} - \mathbf{x}^* \,\right\rangle \geq 0. \end{aligned}$$

Conversely, suppose that $\mathbf{x}^* \in F_l$ satisfies (1.30). First, there exists $\mathbf{f}^*$ (not necessarily unique) in $\mathcal{F}$ such that $\mathbf{x}^* = \mathbf{D}\,\mathbf{f}^*$. For $\mathbf{f} \in F$ and $\mathbf{x} = \mathbf{D}\,\mathbf{f}$, according to (1.30),

$$\begin{aligned} &\left\langle\, \mathbf{l}\,(\mathbf{x}^*)\,,\, \mathbf{x} - \mathbf{x}^* \,\right\rangle \geq 0 \;\Rightarrow\; \left\langle\, \mathbf{l}\,(\mathbf{x}^*)\,,\, \mathbf{D}\,\mathbf{f} - \mathbf{D}\,\mathbf{f}^* \,\right\rangle \geq 0 \\ \Rightarrow\; &\left\langle\, \mathbf{D}^T\,\mathbf{l}\,(\mathbf{x}^*)\,,\, \mathbf{f} - \mathbf{f}^* \,\right\rangle \geq 0 \;\Rightarrow\; \left\langle\, \mathbf{c}\,(\mathbf{f}^*)\,,\, \mathbf{f} - \mathbf{f}^* \,\right\rangle \geq 0. \end{aligned}$$

Thus, according to statement 1, $\mathbf{f}^*$ is a Wardrop equilibrium. □

Remark 1.38. Smith [85] and Dafermos [27] characterized a Wardrop equilibrium via a variational inequality. Aashtiani and Magnanti [1] identified a Wardrop equilibrium with a solution of a nonlinear complementarity problem.

In addition to the characterization via a variational inequality, a Wardrop equilibrium can be identified with an optimal solution of a convex program under A1.19, where a *potential function* is to be minimized. This formulation first appeared in Beckmann et al. [9].

Theorem 1.39. *In $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ under A1.19, a potential function P of game Γ_n is defined on F_l by*

$$P(\mathbf{x}) \;=\; \sum_{a\in\mathcal{A}} \int_0^{\xi_a} l_a(s)\,\mathrm{d}s. \tag{1.32}$$

1. Each optimal solution of the nonlinear program

$$\min_{\mathbf{x}\in\mathcal{F}_l} P(\mathbf{x}) \tag{1.33}$$

is a system flow induced by a Wardrop equilibrium.

2. If, for all arc a, the cost function l_a is weakly monotone on U, then all Wardrop equilibrium induces a system flow in F_l which is an optimal solution of nonlinear program (1.33).

Proof. According to the definition of potential function P, its gradient with respect to $\mathbf{x}$ is the vector of system arc costs $\underline{\mathbf{l}}(\mathbf{x})$.

Besides, if, for all arc $a \in A$, the cost function l_a is monotone on U, then $\underline{\mathbf{l}}(\mathbf{x})$ is a monotone map on F_l. Indeed, given $\mathbf{x}$ and $\mathbf{y}$ in F_l,

$$\begin{aligned}\langle \underline{\mathbf{l}}(\mathbf{x}) - \underline{\mathbf{l}}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle &= \sum_{a\in\mathcal{A}} \sum_{i\in\mathcal{N}} (l(x_a) - l(y_a))(x_a^i - y_a^i) \\ &= \sum_{a\in\mathcal{A}} (l(x_a) - l(y_a))(x_a - y_a) \geq 0.\end{aligned}$$

Consequently, $P(\mathbf{x})$ is convex on F_l.

The proof can be achieved by applying Theorems 1.23 and 1.37. □

Remark 1.40. If the characterization of a Wardrop equilibrium via variational inequalities extends, without difficulty, to the case where the populations have specific costs, this is not the case for its characterization by the minimum of a potential function. In particular, such a function whose gradient is $\underline{\mathbf{l}}(\mathbf{x})$ does not always exist.

Recall that the potential function in a Rosenthal game has form (1.2), which is simply a discreet version of (1.32). Indeed, in order that a potential function exists in an atomic game with unsplittable stocks, the players must have the same weight and the same arc cost functions. The counterpart of the first condition is automatically satisfied in a nonatomic game because the stocks are continuous. The counterpart of the second condition is simply that the populations have the same per-unit cost functions.

As in Rosenthal games, if the cost functions of different populations for a same arc are identical up to an additive constant, then a potential function always exists (*cf.* Remark 1.11).

Finally, it should be stressed that the potential function in this nonatomic congestion game (thus with a continuum of players) does not have the property (1.1) as in a potential game with N players in the sense of Monderer and Shapley.

Wardrop equilibrium: existence and uniqueness

In the first place, the existence and the uniqueness of Wardrop equilibria can be studied with the help of its formulation via a variational inequality.

Theorem 1.41. *All nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ admits a Wardrop equilibrium.*

If the system vector of arc costs $\underline{\mathbf{l}}$ is strictly monotone on F_l, then the flow induced by the Wardrop equilibria is unique.

Proof. Since $\mathcal{F}_l$ is nonempty, compact and convex, and $\underline{\mathbf{l}}(\mathbf{x})$ is continuous on $\mathcal{F}_l$, Theorem 1.27 implies that variational inequality (1.30) admits a solution which is a flow induced by a Wardrop equilibrium according to Theorem 1.37. The uniqueness of the flow induced by the Wardrop equilibria under the assumption that $\underline{\mathbf{l}}$ is strictly monotone follows immediately from Theorem 1.27. □

If a potential function exists in a nonatomic congestion game, one can show the existence and the uniqueness of the Wardrop equilibrium by its characterization as a minimum of the potential function. Let us begin by the following general result (Mangasarian [52]).

Lemma 1.42. *Suppose that X is a convex set in $\mathbb{R}^n$, and Ω is a neighborhood of X. Let f be a real-valued function defined on Ω, which is convex and of class $\mathcal{C}^2$. If $\bar{X}$ is the set of optimal solutions of the program $\min_{x\in X} f(x)$, then the gradient $\nabla f(x)$ is constant on $\bar{X}$.*

Theorem 1.43. *The game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ admits a Wardrop equilibrium.*

Under A1.19, if, for all $a \in A$, the cost function l_a is weakly monotone on U, then the Wardrop equilibrium is unique in $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, in the sense that the cost for a population i is the same at all equilibria.

If, in addition, for all $a \in A$, the monotonicity of l_a is strict, then the system flow induced by the Wardrop equilibria is unique.

Proof. Theorem 1.39 states that all minimum of the potential function P on F_l is induced by a Wardrop equilibrium. But, P is continuous on the convex compact set F_l, thus a minimum exists.

In the proof of Theorem 1.39, one has seen that P is convex in $\mathbf{x}$ on F_l, if l_a is monotone on U for all $a \in A$. A1.19 assures that P is of class $\mathcal{C}^2$ on a neighborhood V of F_l. Then, Lemma 1.42 implies that $\underline{\mathbf{l}}$, the gradient of P, is constant on the set of its minima. In other words, if $\mathbf{x}$ and $\mathbf{x}'$ are two system flows induced by two Wardrop equilibria, the cost for a nonatomic player of population i is the same at $\mathbf{x}$ and at $\mathbf{x}'$.

If the monotonicity of l_a is strict for all a, then, for two different system flows $\mathbf{x}$ and $\mathbf{y}$ in F_l,

$$\langle\, \underline{\mathbf{l}}\,(\mathbf{x}) - \underline{\mathbf{l}}\,(\mathbf{y})\,,\, \mathbf{x} - \mathbf{y}\,\rangle = \sum_{a\in\mathcal{A}} (\, l(x_a) - l(y_a)\,)\,(x_a - y_a) > 0,$$

hence $\mathbf{l}$ is strictly monotone. Therefore, P is strictly convex on convex compact set F_l, hence its minimum is unique there. □

Remark 1.44. The result on the uniqueness of the Wardrop equilibrium in Theorem 1.43 does not hold when the populations have specific costs. Nevertheless, Milchtaich [61] showed that the uniqueness of populations' costs at Wardrop equilibria is guaranteed under some weak convexity and monotonicity conditions if the network has a nearly parallel structure (*cf.* Remark 1.35).

1.3.5 Composite congestion games

After the introduction of atomic games with splittable stocks and nonatomic games, their similarity can be observed in several aspects. In particular, the equilibria in the two classes of games are characterized by variational inequalities. Only, marginal path or arc costs are used for atomic players with splittable stocks, while paths or arc costs are used for nonatomic players. Based on this idea, one can define a more general class of games, called composite congestion games, which contains the two previous classes. By *composite*, one means that atomic players with splittable stocks and nonatomic players coexist while their strategic behaviors are different (Nash and Wardrop, respectively). Composite games were first studied by Harker [32] then by Boulogne et al. [15]. They called such games *mixed games* (This terminology is avoided here because it can be confused with the mixed extension of a game).

Formally, a *composite* congestion game $\Gamma_c(G, \mathcal{N} \cup \mathcal{M}, \{m^i\}_{i\in\mathcal{N}} \cup \{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}} \cup \{P^j\}_{j\in\mathcal{M}})$ (where the subscript c stands for "composite") is specified by the following elements.

– $G = (V, A, \mathbf{l})$, a directed finite graph equipped with a cost function l_a for all arc $a \in A$.

– $\mathcal{N}$, a set of N atomic players with splittable stocks; $\mathcal{M}$, a set of M populations of nonatomic players.
– m^i, the weight of atomic player i; μ^i, the total weight of population j.
– $P^i \in 2^A$, the set of elementary paths connecting vertex o^i to vertex d^i, where o^i and d^i are, respectively, the origin and the destination of atomic player i; $P^j \in 2^A$, the set of elementary paths connecting vertex o^j to vertex d^j, where o^j and d^j are, respectively, the origin and the destination of population j.

Composite equilibrium

Subscripts d and n will be added to distinguish the atomic players with splittable stocks from the populations when it is necessary. Denote the configuration of the atomic players by $\mathbf{f}_d = (\mathbf{f}^i)_{i\in\mathcal{N}}$, the configuration of the populations by $\mathbf{f}_n = (\mathbf{f}^j)_{j\in\mathcal{M}}$, the set of feasible configurations for the atomic players by F_d, and the set of feasible configurations for the populations by F_n. Notice that, the system configuration is $\mathbf{f} = (\mathbf{f}_d, \mathbf{f}_n)$, and the set of feasible system configurations is $F = F_d \times F_n$. The system flow and the corresponding set of feasible flows are defined in a similar way. The marginal path or arc cost functions for the atomic players, their cost functions in the game, and the path and arc cost functions for the populations are also defined as in §1.3.3 and §1.3.4.

The definition of an equilibrium, its characterization via variational inequalities, and the existence and the uniqueness of the equilibria can be obtained as a simple generalization of the results in two previous sections. Only the results are collected here without proof.

Definition 1.45. In $\Gamma_c(G, \mathcal{N}\cup\mathcal{M}, \{m^i\}_{i\in\mathcal{N}} \cup \{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}} \cup \{P^j\}_{j\in\mathcal{M}})$, a system configuration $\mathbf{f}^*$ in F is a *composite equilibrium* if the following two conditions are satisfied.

(1.i) For all player i, $\mathbf{f}^{*i} \in F^i$ is an optimal solution of the program $\min_{\mathbf{f}^i\in F^i} u^i(\mathbf{f}^i, \mathbf{f}^{*-i})$.

(1.ii) For all population j and all path $p \in P^j$,

$$\text{if} \quad f_p^{*j} > 0, \quad \text{then} \quad c_p(\mathbf{f}^*) \le c_q(\mathbf{f}^*) \;\text{ for all }\; q \in P^j.$$

A system flow $\mathbf{x}^* \in F_l$ is *induced by a composite equilibrium* $\mathbf{f}^*$ if the two conditions below are satisfied:

(2.i) For all player i, $\mathbf{x}^{*i} \in F_l^i$ is an optimal solution of the program $\min_{\mathbf{x}^i\in F_l^i} v^i(\mathbf{x}^i, \mathbf{x}^{*-i})$.

(2.ii) For all population j and all path $p \in P^j$,

$$\text{if} \quad f_p^{*j} > 0, \quad \text{then} \quad c_p(\mathbf{f}^*) \le c_q(\mathbf{f}^*) \;\text{ for all }\; q \in P^j.$$

Theorem 1.46. *Suppose that, in a composite congestion game* $\Gamma_c(G, \mathcal{N}\cup\mathcal{M}, \{m^i\}_{i\in\mathcal{N}} \cup \{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}} \cup \{P^j\}_{j\in\mathcal{M}})$, *A1.19 holds.*

1. If a system configuration $\mathbf{f}^* = (\mathbf{f}_d^*, \mathbf{f}_n^*) \in F$ *is a composite equilibrium, then* $\mathbf{f}^*$ *satisfies the following variational inequality:*

$$\langle\, \hat{\mathbf{c}}_d(\mathbf{f}^*), \mathbf{f}_d - \mathbf{f}_d^* \,\rangle + \langle\, \underline{\mathbf{c}}_n(\mathbf{f}^*), \mathbf{f}_n - \mathbf{f}_n^* \,\rangle \ge 0, \qquad \forall\, \mathbf{f} = (\mathbf{f}_d, \mathbf{f}_n) \in F. \tag{1.34}$$

If, for all atomic player i, *for all profile* $\mathbf{f}^{-i} \in F^{-i} = \prod_{k\in\mathcal{N}\cup\mathcal{M}\setminus\{i\}} F^k$, $u^i(\mathbf{f}^i, \mathbf{f}^{-i})$ *is convex with respect to* $\mathbf{f}^i$, *then all system configuration* $\mathbf{f}^*$ *satisfying variational inequality* (1.34) *is a composite equilibrium.*

2. If a system flow $\mathbf{x}^* = (\mathbf{x}^*_d, \mathbf{x}^*_n) \in F_l$ *is induced by a composite equilibrium, then* $\mathbf{x}^*$ *satisfies the following variational inequality:*

$$\langle\, \hat{\mathbf{l}}_d(\mathbf{x}^*)\,,\, \mathbf{x}_d - \mathbf{x}^*_d \,\rangle \,+\, \langle\, \underline{\mathbf{l}}_n(\mathbf{x}^*)\,,\, \mathbf{x}_n - \mathbf{x}^*_n \,\rangle \,\geq\, 0, \qquad \forall\, \mathbf{x} = (\mathbf{x}_d, \mathbf{x}_n) \in F_l, \tag{1.35}$$

If, for all atomic player i *and for all profile* $\mathbf{x}^{-i} \in F_l^{-i} = \prod_{k\in\mathcal{N}\cup\mathcal{M}\setminus\{i\}} F_l^k$, $v^i\left(\mathbf{x}^i\,,\,\mathbf{x}^{-i}\right)$ *is convex with respect to* $\mathbf{x}^i$*, then all system flow* $\mathbf{x}^*$ *satisfying variational inequality (1.35) is induced by a composite equilibrium.*

Theorem 1.47. *In* $\Gamma_c(G, \mathcal{N}\cup\mathcal{M}, \{m^i\}_{i\in\mathcal{N}}\cup\{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}}\cup\{P^j\}_{j\in\mathcal{M}})$ *under A1.18, if, for all atomic player* i *and for all* $\mathbf{f}^{-i} \in F^{-i}$ *(*resp. $\mathbf{x}^{-i} \in F_l^{-i}$*),* $u^i(\mathbf{f}^i,\, \mathbf{f}^{-i})$ (resp. $v^i(\mathbf{x}^i,\, \mathbf{x}^{-i})$) *is convex with respect to* $\mathbf{f}^i$ (resp. $\mathbf{x}^i$)*, then the game admits a composite equilibrium.*

If the map $\left(\hat{\mathbf{l}}_d(\mathbf{x}), \underline{\mathbf{l}}_n(\mathbf{x})\right)$ *is strictly monotone in* $\mathbf{x}$*, then the flow induced by the composite equilibria is unique.*

Existence of a potential function in composite games

The analogy between the variational inequalities characterizing, respectively, a Wardrop equilibrium, a Nash equilibrium and a composite equilibrium (*cf.* Theorems 1.37, 1.24 and 1.46) implies that a Nash equilibrium (*resp.* a composite equilibrium) in an atomic game with splittable stocks (*resp.* a composite game) could be identified with a Wardrop equilibrium in a nonatomic game, only by replacing each atomic player i by a population of nonatomic players holding the same stocks and, in addition, by imposing on them a per-unit cost $\hat{c}^i_a$ for each arc a, which was the marginal cost of arc a for the replaced player i. However, once such a virtual nonatomic game is defined, the populations have, in general, specific costs (because of the term $x^i_a l'_a$ in $\hat{c}^i_a$). In consequence, a potential function does not necessarily exist except in some particular cases, for example, when the paths are all composed of a single arc with an affine cost function.

In the particular case where a potential function exists, its minima are Wardrop equilibria of the virtual nonatomic game. Nevertheless, in order that they are Nash or composite equilibria in the initial atomic or composite game, the cost function $v^i(\mathbf{x})$ for each atomic player i should also be convex with respect to her flow $\mathbf{x}^i$ (*cf.* (1.24) and (1.46)). This is true in the case where the paths are all composed of a single arc with an affine cost function.

1.3.6 Social optimum

In the four previous sections, the congestion problem is studied from the angle of the players' strategic behaviors. Now, the social cost, which is the sum of the costs to all the players, will be considered, and its minimal level will be studied.

Definition 1.48. The *social cost* in $\Gamma_c(G, \mathcal{N}\cup\mathcal{M}, \{m^i\}_{i\in\mathcal{N}}\cup\{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}}\cup\{P^j\}_{j\in\mathcal{M}})$ is defined by

$$\beta\,(\mathbf{f}) \,=\, \langle\, \underline{\mathbf{c}}(\mathbf{f})\,,\, \mathbf{f} \,\rangle \,=\, \sum_{i\in\mathcal{N}\cup\mathcal{M}} \sum_{p\in\mathcal{P}^i} f^i_p\, c_p(\mathbf{f}) \tag{1.36}$$

as a function of the system configuration $\mathbf{f} \in F$ or, equivalently, by

$$C_s\,(\mathbf{x}) \,=\, \langle\, \mathbf{l}(\boldsymbol{\xi})\,,\, \boldsymbol{\xi} \,\rangle \,=\, \sum_{a\in A} \xi_a\, l_a(\xi) \tag{1.37}$$

as a function of the system flow $\mathbf{x} \in F_l$.

Definition 1.49. In $\Gamma_c(G, \mathcal{N} \cup \mathcal{M}, \{m^i\}_{i\in\mathcal{N}} \cup \{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}} \cup \{P^j\}_{j\in\mathcal{M}})$, a system configuration $\mathbf{f} \in F$ (*resp.* a system flow $\mathbf{x} \in F_l$) is a *social optimum* if it is an optimal solution of the nonlinear program

$$\min_{\mathbf{f}\in F} \beta(\mathbf{f}), \tag{1.38}$$

$$(\textit{resp.} \qquad \min_{\mathbf{x}\in F_l} C_s(\mathbf{x}).) \tag{1.39}$$

Theorem 1.50. *All composite congestion game* $\Gamma_c(G, \mathcal{N}\cup\mathcal{M}, \{m^i\}_{i\in\mathcal{N}}\cup\{\mu^j\}_{j\in\mathcal{M}}, \{P^i\}_{i\in\mathcal{N}}\cup \{P^j\}_{j\in\mathcal{M}})$ *admits a social optimum. In addition, the optimal social cost is unique.*

Proof. The continuous function β (*resp.* C_s) attains its global minimum in the nonempty and compact set F (*resp.* F_l). $\square$

Unlike the existence of the social optimum which requires only the continuity of the cost functions, its uniqueness is not always guaranteed, except in some particular cases.

1.4 Comparisons

1.4.1 Approximation and convergence

It is shown that, when there are a great number of atomic players such that each individual has only a negligible influence on the others, they can be approximated by a continuum of nonatomic players, each of whom has measure zero. Parallel results are also obtained in atomic and nonatomic models. Let us return to two themes: the formulation of an equilibrium in terms of variational inequalities, and the potential function.

Formulation of equilibria in terms of variational inequalities

The Nash equilibria in atomic games with splittable stocks are formulated by (1.23) and (1.24), while the Wardrop equilibria in nonatomic games by (1.29) and (1.30). The two formulations are nearly identical except that, in the atomic case, the marginal costs replace the initial costs. Haurie and Marcotte [36] showed that the Nash equilibria in a sequence of atomic games with splittable stocks converge to the Wardrop equilibrium in a nonatomic game, if the atomic players split themselves into smaller and smaller players. To this end, they showed that the sequence of variational inequalities corresponding to the Nash equilibria of the atomic games converges to the variational inequality corresponding to the Wardrop equilibrium of the nonatomic game. Wan [91] (*cf.* Chapter 2) generalized this result to composite games, by allowing, in addition, the players to split themselves into small players of different sizes.

Potential functions in Rosenthal games and nonatomic games

The potential function exists in (discreet) Rosenthal games and in (continuous) nonatomic games, under the condition that the players have the same arc cost functions and, furthermore, they have the same weight in the first case. The potential functions' forms are similar in the two cases: it is a partial sum (1.2) in the discreet case and an integral (1.4) in the continuous case. This is not a coincidence. Sandholm [79] showed that such a nonatomic game or, more generally, a *population game*, is indeed the limit of a sequence of potential games with a finite number of players in the sense of Monderer and Shapley.

1.4.2 Inefficiency of the equilibria and price of anarchy

Pigou's example (*cf.* §1.2.4) illustrates the probable gap between the social cost at (Nash, Wardrop, composite) equilibria and the optimal social cost. Koutsoupias and Papadimitriou [50] first suggested to evaluate the inefficiency of equilibria by the *price of anarchy* [67], defined as the ratio between the social cost of the worst equilibrium and the optimal social cost.

For atomic games with unsplittable stocks, Christocoulou and Koutsoupias [19] showed that, in a Rosenthal congestion game with N players, the upper bound of the price of anarchy is 2.5 if the arc cost functions are affine, and that this bound is improved to $\frac{5N-2}{2N+1}$ if the players have the same strategy set. Awerbuch et al. [8] found that the upper bound is 2.618 in the case where the cost functions are affine but the players are weighted. Both [19] and [8] also treated the case where the cost functions are polynomial. Since then, a large number of studies have followed and improvements on the bounds have been made.

Let us focus on the price of anarchy in nonatomic games and atomic games with splittable stocks.

Price of anarchy in nonatomic congestion games

Roughgarden and Tardos [78] were the first to study the price of anarchy in nonatomic congestion games, followed by a series of articles, for example, Roughgarden [75] and Correa et al. [22, 23]. Some important results and proof sketches are presented here.

Definition 1.51. The *price of anarchy* of a nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ is defined by

$$\rho(\Gamma_n) = \frac{\beta(\mathbf{f}^*)}{\beta(\tilde{\mathbf{f}})} = \frac{C_s(\mathbf{x}^*)}{C_s(\tilde{\mathbf{x}})},$$

where $\mathbf{f}^*$ is the Wardrop equilibrium at which the social cost is the highest (among all the Wardrop equilibria), $\tilde{\mathbf{f}}$ the optimal configuration in terms of the social cost, $\mathbf{x}^*$ and $\tilde{\mathbf{x}}$ the flows induced by, respectively, $\mathbf{f}^*$ and $\tilde{\mathbf{f}}$.

Recall that $\beta(\mathbf{f})$ and $C_s(\mathbf{x})$ are the social cost functions defined by (1.36) and (1.37).

Lemma 1.52. *[78] Suppose that, in a nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ under A1.19, for each arc $a \in A$, the cost function l_a is nondecreasing on U, and*

$$x\, l_a(x) \le M \int_0^x l_a(s)\, ds, \quad \forall x \in \mathbb{R}^+ \cap U.$$

Then, $\rho(\Gamma_n)$, the price of anarchy of the game, satisfies that $\rho(\Gamma_n) \le M$.

Proof. Suppose that $\mathbf{x}^*$ is the flow induced by a Wardrop equilibrium, $\mathbf{x}$ is a feasible flow, and $\boldsymbol{\xi}^*$ and $\boldsymbol{\xi}$ are the aggregate flow induced by, respectively, $\mathbf{x}^*$ and $\mathbf{x}$. Then,

$$C_s(\mathbf{x}^*) = \sum_{a \in A} \xi_a^* l_a(\xi_a^*) \le \sum_{a \in A} M \int_0^{\xi_a^*} l_a(s) \mathrm{d}s = M\, P(\mathbf{x}^*) \le M\, P(\mathbf{x}) \le M C_s(\mathbf{x}).$$

Here, P is the potential function introduced in (1.32). The last inequality follows from the fact that $\int_0^x l_a(s)\mathrm{d}s \le x\, l_a(x)$, because l_a is nondecreasing. Finally, let $\mathbf{x}$ be the social optimum $\tilde{\mathbf{x}}$. □

The following parameter is needed.

Definition 1.53. [22, 75] Given a family of functions $\mathcal{L}$, the *Pigou's bound* of $\mathcal{L}$ is defined by

$$\alpha(\mathcal{L}) = \sup_{l\in\mathcal{L}} \sup_{x,s\geq 0} \frac{s\, l(s)}{x\, l(x) + (s-x)\, c(s)}.$$

First, one has the following result which bounds the price of anarchy [22, 75].

Theorem 1.54. *Given a family of functions $\mathcal{L}$, suppose that, in a nonatomic congestion game $\Gamma_n(G,\mathcal{N},\mathbf{m},\mathbf{P})$, the arc cost functions are nondecreasing and in $\mathcal{L}$. Then, $\rho(\Gamma_n)$, the price of anarchy of the game, satisfies that $\rho(\Gamma_n) \leq \alpha(\mathcal{L})$.*

Proof. Suppose that $\boldsymbol{\xi}^*$ is the aggregated flow induced by the Wardrop equilibrium $\mathbf{x}^*$, and $\boldsymbol{\xi}$ the aggregate flow induced by a feasible $\mathbf{x}$. Then,

$$C_s(\mathbf{x}^*) = \sum_{a\in A} \xi_a^*\, l_a(\xi_a^*) \geq \Big[\frac{1}{\alpha(\mathcal{L})} \sum_{a\in A} \xi_a\, l_a(\xi_a)\Big] + \sum_{a\in A} (\xi_a^* - \xi_a)\, l_a(\xi_a) \geq \frac{C_s(\mathbf{x})}{\alpha(\mathcal{L})},$$

where the first inequality is due to the definition of $\alpha(\mathcal{L})$, and the second is due to variational inequality (1.30). Finally, let $\mathbf{x}$ be the social optimum $\tilde{\mathbf{x}}$. $\square$

A corollary of Theorem 1.54 (*cf.* Roughgarden and Tardos [78, Theorem 4.5]) provides the least upper bound of the price of anarchy in a nonatomic congestion game with affine arc cost functions.

Theorem 1.55. *Suppose that, in a nonatomic congestion game $\Gamma_n = \langle G,\mathcal{N},\mathbf{m},\mathbf{P}\rangle$, the arc cost functions are affine: for all arc a in $\mathcal{A}$, $l_a(s) = \zeta_a s + \theta_a$, where $\zeta_a \geq 0, \theta_a \geq 0$. Then, $\rho(\Gamma_n)$, the price of anarchy of the game, satisfies that $\rho(\Gamma_n) \leq 4/3$.*

Proof. Let us cite a proof by Correa et al. [23, 24] which is extremely concise and elegant.

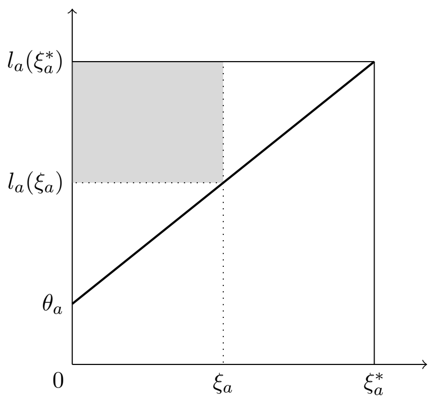


Figure 1.6

Let $\mathbf{x}^*$ be the Wardrop equilibrium, and $\boldsymbol{\xi}^*$ the aggregate flow induced by $\mathbf{x}^*$. Let $\boldsymbol{\xi}$ be the aggregate flow induced by a feasible flow $\mathbf{x}$. Then, according to (1.30),

$$\begin{aligned} C_s(\mathbf{x}^*) &= \sum_{a\in A} \xi_a^*\, l_a(\xi_a^*) \leq \sum_{a\in A} \xi_a^*\, l_a(\xi_a) \\ &= \sum_{a\in A} \xi_a\, l_a(\xi_a) + \sum_{a\in A} (\xi_a^* - \xi_a)\, l_a(\xi_a) \leq \sum_{a\in A} \xi_a\, l_a(\xi_a) + \sum_{a\in A:\xi_a^*>\xi_a} (\xi_a^* - \xi_a)\, l_a(\xi_a) \\ &\leq \sum_{a\in A} \xi_a\, l_a(\xi_a) + \frac{1}{4} \sum_{a\in A} \xi_a^*\, l_a^*(\xi_a^*), \end{aligned}$$

where the last inequality is due to fact that the gray part in Figure 1.6 occupies at most 1/4 of the area of the large rectangle. Finally, let $\mathbf{x}$ be the social optimum $\tilde{\mathbf{x}}$. □

On the contrary, if the arc cost functions in a nonatomic game are nonlinear but still nondecreasing and of class $\mathcal{C}^\infty$, Roughgarden and Tardos [78, p.247 §3] showed that the price of anarchy can be probably infinite. Explicitly, they used an example similar to that of Pigou, by replacing the cost function of the lower arc by x^p and letting p tend to $+\infty$.

Nevertheless, even though the price of anarchy can no longer be bounded by a numerical value in this case, the following theorem [78, Theorem 3.1] shows that the social cost at an equilibrium of a nonatomic game does not exceed the social cost at any configuration in another game where the total weight of each population is doubled.

Theorem 1.56. *Suppose that, in a nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$, the arc cost functions are nondecreasing on a neighborhood U' of interval $[0, 2M]$. Let $\mathbf{x}^*$ be the flow induced by the Wardrop equilibrium of Γ_n. Let $\mathbf{x}$ be a feasible flow in the nonatomic congestion game $\Gamma'_n(G, \mathcal{N}, \mathbf{2m}, \mathbf{P})$ where the size of each population is doubled with respect to Γ_n. Then,*

$$C_s(\boldsymbol{x}^*) \leq C_s(\boldsymbol{x}).$$

Proof. Let us still cite the proof of Correa et al. [23, 24] which is in the same esprit as that of Theorem 1.55.

For all $a \in A$, because l_a is nondecreasing on U', one has, for all z_a, $y_a \in U' \cap \mathbb{R}^+$,

$$y_a\, l_a(z_a) \leq \max\{z_a\, l_a(z_a),\, y_a\, l_a(y_a)\} \leq z_a\, l_a(z_a) + y_a\, l_a(y_a).$$

In this inequality, let us take $\mathbf{z} = \boldsymbol{\xi}^*$, the aggregate flow induced by $\mathbf{x}^*$, and $\mathbf{y} = \boldsymbol{\xi}$, the aggregate flow induced by $\mathbf{x}$. Notice that $\mathbf{x}/2$ is a feasible system aggregate flow in game Γ_n. Thus,

$$C_s(\mathbf{x}^*) = 2C_s(\mathbf{x}^*) - C_s(\mathbf{x}^*) \leq 2\sum_{a\in A} \frac{\xi_a}{2}\, l_a(\xi_a^*) - C_s(\mathbf{x}^*) = \sum_{a\in A} \xi_a\, l_a(\xi_a^*) - C_s(\mathbf{x}^*) \leq C_s(\mathbf{x}).$$

□

Remark 1.57. Only those congestion games where the cost functions of strategies are separable, i.e., the cost of a strategy is the sum of the costs of the resources used by it are discussed so far. The price of anarchy is also studied in the nonseparable case, for example, in Chau and Sim [18] for symmetric players, and in Perakis [69] for more general cases.

Price of anarchy in atomic games with splittable stocks and in composite games

Cominetti et al. [20] showed that the bounds of the price of anarchy in nonatomic games cannot be extended directly to atomic games with splittable stocks. For example, in such a game with N atomic players, if the arc cost functions are affine, the price of anarchy is bounded by $\frac{3N+1}{2N+2} \leq \frac{3}{2}$, and it tends to $\frac{3}{2}$ when N tends to infinity. In particular, as long as $N > 5$, $\frac{3N+1}{2N+2}$ is greater than $\frac{4}{3}$, which is the bound in nonatomic games with affine costs [20, Proposition 3.5]. We will return to this point in §1.4.3.

For atomic games with splittable stocks with more general cost functions, and for composite games, the results are less general than in the case of nonatomic games. Harks [33] showed that, if the network is composed of two vertices connected by parallel arcs whose cost functions are nondecreasing, convex and of class $\mathcal{C}^1$, the price of anarchy does not exceed the number of atomic players. Bhaskar et al. [11] extended this result to *series parallel* networks (see Duffin[29] for a description). Roughgarden and Tardos [78], Roughgarden [76], Cominetti et al. [20], Harks [33, 34], and Roughgarden and Schoppmann [77] obtained the bounds of the price of anarchy for more general network topology and more general cost functions.

Besides, the price of anarchy in an atomic game with splittable stocks or in a composite game can be studied by comparing the social cost at Nash or composite equilibria with that at Wardrop equilibria in a nonatomic game with the same stocks. This approach will be developed in §1.4.3.

1.4.3 Comparison between nonatomic games and composite games

Single-commodity games

For atomic games with splittable stocks and composite games, a simple case called *single-commodity*, where all the players have the same origin o, the same destination d and the same set P of available paths, was widely studied. Some results are collected here. For the sake of simplicity, given Γ, a single-commodity atomic game with splittable stocks or a single-commodity composite game, its *corresponding nonatomic game* is the nonatomic game where the players have the same set of stocks as in Γ.

Altman et al. [3] showed that, in a single-commodity atomic game, if, for each arc $a \in A$, the cost function is $l_a(x) = \alpha_a x^\gamma$, where $\alpha_a > 0$ and $\gamma > 0$ (γ is common for all a), then the Nash equilibrium in this atomic game, the Wardrop equilibrium in its corresponding nonatomic game, and the social optimum coincide.

For more general cost functions, one has the following results concerning the social cost.

Theorem 1.58. *Suppose that, in a single-commodity atomic game Γ_d with splittable stocks, under A1.19, the total weight of the N players is M. Then, in each of the three cases below, the social cost at the Nash equilibrium of game Γ_d does not exceed the social cost at the Wardrop equilibrium of its corresponding nonatomic game.*

1. For all arc $a \in A$, the cost function l_a is nondecreasing and convex on U. All the players have the same weight, namely, M/N.

2. Path P contains a finite number of parallel arcs, i.e, $P = A$. For all arc a, the cost function of l_a is nondecreasing on U, and function $s \mapsto s\, l_a(s)$ is convex on U.

3. The network is series parallel. For all arc $a \in A$, the cost function l_a is nondecreasing and convex on U.

Proof. 1. See Cominetti et al. [20, Corollaire 4.1].
2. See Hayrapetyan et al. [37, Theorem 2.3].
3. See Bhaskar et al. [11, Theorem 1]. □

The three results in Theorem 1.58 compare the Nash equilibrium and the Wardrop equilibrium in terms of the social cost. The following result, obtained by Wan [91] (*cf.* Chapter 2) in the framework of composite games, shows that not only the social cost, but also the per-unit cost of each atomic player and the nonatomic players' cost at the composite equilibrium are reduced with respect to the Wardrop equilibrium cost.

Theorem 1.59. *Suppose that, in a composite game Γ_c under A1.19, N atomic players with splittable stocks of total weight m and a population of nonatomic players of total weight μ have* *all origin o, destination d and a set A of parallel arcs connecting o to d as available paths. If, for all arc a, the cost function l_a is strictly increasing and convex on a neighborhood of $[0, m+\mu]$, the social cost at the composite equilibrium of Γ_c, the per-unit cost of each atomic player and the common cost to the nonatomic players do not exceed the Wardrop equilibrium cost of the corresponding nonatomic game.*

To every composite game, one can associate a nonatomic game and a one-player atomic game with splittable stocks such that the three games have the same set of stocks. In the one-player game, the unique outcome is the social optimum. More generally, for a set of splittable stocks $\mathcal{S}$ which have origins, destinations and sets of available paths in a finite directed graph, one can define a structure of poset (i.e. a partially ordered set) $\Xi(\mathcal{S})$ on the composite games where the set of stocks controlled by the (atomic and nonatomic) players is $\mathcal{S}$: a game Γ_1 is higher than a game Γ_2 if an atomic player in Γ_1 represents a family of atomic players and a mass of nonatomic players in Γ_2. Therefore, the game with an atomic player is the highest and the nonatomic game is the lowest in $\Xi(\mathcal{S})$.

Theorems 1.58 and 1.59 imply that, in the single-commodity case and, in particular, where the paths are (series-)parallel arcs, all composite game in $\Xi(\mathcal{S})$ is more efficient than the nonatomic game as far as the social cost is concerned, and it is even more efficient in terms of individual's cost in Theorem 1.59 (under some conditions on the arc cost functions). These results compare the costs at the equilibrium of a composite game with the equilibrium cost of its corresponding nonatomic game. However, one can also make the comparison between different composite games with the same set of stocks. Wan [92] (*cf.* Chapter 3) showed that, if one considers only single-commodity games in a graph composed of two vertices connected by two parallel arcs, there is a monotonicity in terms of the social cost in the poset $\Xi(\mathcal{S})$. More precisely, the social cost at the composite equilibrium of a game is less than that of a game lower than it in the poset.

Theorem 1.60. *Suppose that G is a network composed of two vertices connected by a finite number of parallel arcs, and all the arc cost functions are strictly increasing, convex, and of class $\mathcal{C}^1$ on a neighborhood of $[0, M]$, where $M > 0$. Let Γ_1 be a composite congestion game played in G such that the total weight of the players is M. Suppose that a player i of weight m^i is replaced by a composite set of players, namely, a finite number L (probably 0) of atomic players with splittable stocks, in addition to a (probably empty) set of nonatomic players whose total weight is m^i. In this way, game Γ_1 becomes game Γ_2. Then, the social cost and the costs to all the players except i (and her replacers) at the unique equilibrium of Γ_2 are not lower than those in Γ_1.*

Nevertheless, the result that the social cost is reduced after each formation of coalition(s) is no longer valid when there are more than two parallel arcs. Huang [46] provided a counter-example where a single-commodity congestion game takes place in a network composed of two vertices connected by three parallel arcs whose cost functions are nondecreasing, convex and of class $\mathcal{C}^1$. The social cost at the equilibrium is increased when two of the three players form a coalition.

Huang [46] gave the necessary and sufficient conditions for the social cost to be reduced after each formation of coalition(s). His result [46, Theorem 1] is reformulated here.

Theorem 1.61. *In a single-commodity atomic congestion game with splittable stocks, suppose that the arc cost functions are nondecreasing, convex and of class $\mathcal{C}^1$. The following two conditions are necessary and sufficient for that, after each formation of coalition(s) between some players, the social cost at the equilibrium does not exceed the social cost before.*

(i) The network is well designed*.*

(ii) The are cost functions are affine.

Huang's definition of a well designed network is omitted here. For example, he showed that a series parallel network is well designed [46, Proposition 2].

Nevertheless, for the costs of the players not involved in coalition, whether the results in Theorem 1.60 hold in more general networks remains a open problem.

Multi-commodity games

As soon as we leave the framework of single-commodity stocks, the impact of the composition of the players on the social cost and the individuals' costs is even more complicated. For example, in §1.4.2, it is shown that the price of anarchy of an atomic game with splittable stocks in a network where the arc costs are affine can well exceed the upper bound of the price of anarchy in nonatomic games in the same network. Here is another interesting example cited by Cominetti et al. [20].

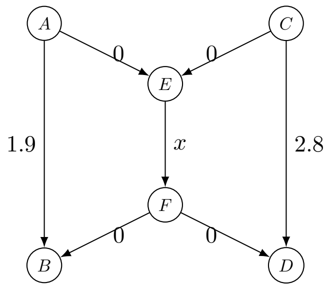


Figure 1.7

Example 1.62. In Figure 1.7, the cost functions are written on the corresponding arcs. There are two splittable stocks, each of weight one. The stock on the left has origin A, destination B, and paths set $\{A \to B, A \to E \to F \to B\}$, while the stock on the right has origin C, destination D, and path set $\{C \to D, C \to E \to F \to D\}$.

At the social optimum, all the stock on the left goes by arc AB, while all the stock on the right goes by path $C \to E \to F \to D$. The minimal social cost is 2.9.

If the two stocks are controlled by two populations of nonatomic players 1 and 2, respectively, then, at the Wardrop equilibrium, 0.9 of the players of population 1 take path $A \to B$ and 0.1 of them take path $A \to E \to F \to B$, while the whole population 2 takes path $C \to E \to F \to D$. The social cost is 3.8.

Now, suppose that the stock on the right is controlled by an atomic player, while the stock on the left is still controlled by population 1. At the composite equilibrium, the whole population 1 takes path $A \to E \to F \to B$, while the atomic player sends 0.9 of her stock by path $C \to E \to F \to D$, and the rest by arc $C \to D$. The social cost is then 3.89, larger than 3.8. In particular, the price of anarchy of this composite game is $\frac{3.89}{2.9} \approx 1.341 > \frac{4}{3}$.

Besides, the total cost of the stock on the right is 1.9 at the Wardrop equilibrium when it is controlled by a population, but 1.99 at the composite equilibrium when it is controlled by an atomic player. This means that the formation of a coalition by population 2 has increased the average cost of its members.

Cominetti et al. [20] also showed that, if the stock on the right is controlled by 94 or more identical atomic players, then the price of anarchy in this atomic game will also exceed

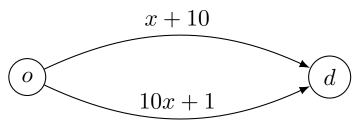


Figure 1.8

$\frac{4}{3}$. In other words, the formation of a coalition by atomic players with splittable stocks may produce similar effects. Altman et al. [5] also cited an example where the formation of the coalitions by atomic players with splittable stocks reduces not only the social cost but also the average cost of the members of each coalition.

Variation of one part of the stocks

Let $\mathcal{S}$ be a set of single-commodity stocks. The social optimum is attained in the one-player game which is the highest one in the poset $\Xi(\mathcal{S})$ (*cf.* §1.4.3). Moreover, in the particular case where the network is composed of two vertices connected by two parallel arcs whose cost functions are nondecreasing, convex and of class $\mathcal{C}^1$, the social cost increases when one goes down in $\Xi(\mathcal{S})$.

Now, suppose that the stocks in S are divided into two parts, I and II, and the composition of part II is fixed (By "the *composition* of the stocks", one means that the distribution of these stocks among the atomic and nonatomic players). We are only interested in the total cost of part I. Let Γ_1 denote the composite game played by an atomic player who controls part I and the players who share (in a disjointed way) part II. Then, one has to go down from Γ_1 in the poset $\Xi(\mathcal{S})$ so as to find the best composition of part I. Is it still true that the total cost of part I attains its minimum when I is controlled by a single player or, furthermore, her total cost keeps increasing while one goes down in $\Xi(\mathcal{S})$? The reply depends on the composition of part II. Here is an example in Wan [91] (*cf.* Chapter 2).

Example 1.63. As illustrated in Figure 1.8, a set of stocks of total weight 1 should be sent from vertex o to vertex d by two parallel arcs. The cost function of the upper arc is $x \mapsto x + 10$ and that of the lower arc is $x \mapsto 10x + 1$. The stocks are divided into two parts I and II, each of weight $\frac{1}{2}$.

Let us consider two scenarios according to the composition of part II of the stocks. In each scenario, different compositions of part I of the stocks facing the composition of part II will be compared.

SCENARIO 1. Part II is controlled by a population of nonatomic players. Theorem 1.59 shows that the total cost to part I is lower when it is controlled by a single player than when it is controlled by a population of nonatomic players.

SCENARIO 2. Part II is controlled by a single atomic player. If part I is controlled by a single atomic player, her total cost at the Nash equilibrium is 4.227. If it is controlled by a population of nonatomic players, her total cost at the composite equilibrium is 4.136, lower than 4.227. This result is in contrast to the results for the social cost. Besides, if part I is controlled by two atomic players, both of weight $\frac{1}{4}$, their total cost at the Nash equilibrium is 4.125, even better than the two previous compositions. This shows that the total cost to part I is not monotone with respect to the order in the poset $\Gamma(\mathcal{S})$, which is still different from the result for the social cost.

1.5 Dynamics

So far, only static properties concerning the equilibria in congestion games have been discussed. Now, let us turn to the dynamic issues.

1.5.1 Dynamics in atomic games with unsplittable stocks

In a Rosenthal congestion game which is also a potential game, Monderer and Shapley [62] showed that an alternating best reply dynamics, where the players adapt their pure strategy in turn, converges to a minimum point of the potential function, which is an equilibrium of the game. Milchtaich [60] developed this idea.

A weighted atomic game with unsplittable stocks is reduced to a finite game. A model of learning process is proposed by Cominetti et al. [21] for such a game, where each player's only information is her own payoff at each stage. Throughout the time, a player adapts her mixed strategy according to her information. An application of this model to a Rosenthal congestion game is discussed in detail in the same paper. In this N-player game, every day, everyone chooses one of the M parallel paths, and her only information is her own cost, namely, her travel time, which depends only on the number of the players choosing the same path as her. She updates an M dimensional vector of scores, each component of which corresponds to a different path. Then, she updates her mixed strategy in the $(M-1)$-simplex according to this vector. The almost sure convergence of the dynamics to the unique rest point is obtained under certain conditions on the path cost functions and the adaptation mechanism.

1.5.2 Dynamics in nonatomic games

Potential game approach

Recall that, in §1.3.4, a Wardrop equilibrium is formulated as a minimum point of a potential function (1.32). Sandholm [79] started to study a class of dynamics in nonatomic games where a potential function exists. The existence of such a function is essential, because it is used as a Lyapunov function for all the dynamics satisfying certain conditions. As an example, let us take the BNN (Brown-von Neumann-Nash) dynamics as an example to cite some principle results in [79]. They are collected in Theorem 1.64 below.

First, recall that the BNN dynamics in a nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ is

$$\dot{f}^i_p = m^i k^i_p - f^i_p \sum_{q \in P^i} k^i_q, \quad \forall\, p \in P^i, \forall\, i \in \mathcal{N}, \tag{1.40}$$

where, with the notation $x^+ = \max\{x, 0\}$,

$$k^i_p = \Big[\frac{1}{m^i} \sum_{q \in P^i} f^i_q c_q(\mathbf{f}) - c_p(\mathbf{f})\Big]^+,$$

which compares the cost of path p to the average cost of population i.

Theorem 1.64. *In $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P})$ under A1.19, one has*

1. *$-P$, the negative of the potential function P defined by (1.32), is a global Lyapunov function for the BNN dynamics;*
2. *the Wardrop equilibria of the game coincide with the rest points of the BNN dynamics;*

3. *the set of local minimum points is stable in the sense of Lyapunov;*
4. *all limit set $\omega(\mathbf{f})$ ($\mathbf{f} \in F$) of the BNN dynamics is a closed and connected set of Wardrop equilibria of the game.*

Proof. See the proofs of Lemma 4.1, Proposition 4.2, Proposition 4.3, Theorem 4.4 and Theorem 4.5 in [79]. □

Stables game approach

Remark 1.40 states that a potential function does not always exist if the populations have specific costs. Therefore, the dynamics for the potential games introduced earlier cannot be applied directly in this case. Nevertheless, if the vector of system path costs (whose definition will be given below) is a continuous and monotone map, then the game belongs to the class of stable games introduced by Hofbauer and Sandholm [40], where they studied dynamics for this class. Before, Smith [86] had already applied this idea in a dynamics, which is called *Smith dynamics* later. Smith applied it to nonatomic congestion games.

First, let us define a nonatomic game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P}, \mathbf{l})$ by specifying the *specific* arc cost functions: $\mathbf{l} = (\mathbf{l}^i)_{i\in\mathcal{N}}$, where $\mathbf{l}^i = (l_a^i)_{a\in A}$ is the vector of arc costs for population i, and $x \mapsto l_a^i(x)$ is continuous and finite on U, positive on $U \cap \mathbb{R}_+$ for all a and all i. The cost of path p for population i is denoted by $c_p^i = \sum_{a\in p} l_a^i$. The vector of path costs for population i is thus $\mathbf{c}^i = (c_p^i)_{p\in P^i}$, and the vector of system path costs is $\mathbf{c} = (\mathbf{c}^i)_{i\in\mathcal{N}}$. Let the system flow still be denoted by $\mathbf{x}$, and the system configuration by $\mathbf{f}$. Then, $\mathbf{l}$ is a map defined from F_l, the set of feasible system flows, to $\mathbb{R}^{N|A|}$, and $\mathbf{c}$ is a map defined on F, the set of feasible system configurations, to $\mathbb{R}^{\sum_{i\in\mathcal{N}}|P^i|}$.

If $-\mathbf{c}$ is a monotone map on F, Γ_n is a *stable game* defined by Hofbauer and Sandholm, in the sense that

$$\langle\, \mathbf{g} - \mathbf{f},\, \mathbf{c}\,(\mathbf{g}) - \mathbf{c}\,(\mathbf{f})\,\rangle \le 0, \qquad \forall\, \mathbf{f}, \mathbf{g} \in F.$$

Smith [86] proposed the following dynamics for a nonatomic congestion game $\Gamma_n(G, \mathcal{N}, \mathbf{m}, \mathbf{P}, \mathbf{l})$: $\dot{\mathbf{f}} = \Phi(\mathbf{f})$ for $\mathbf{f} \in F$, where the equation for the component f_p^i is

$$\dot{f}_p^i = \sum_{l\in P^i} f_l^i \big(c_l(\mathbf{f}) - c_p(\mathbf{f})\big)^+ - f_p^i \sum_{l\in P^i} \big(c_p(\mathbf{f}) - c_l(\mathbf{f})\big)^+. \tag{1.41}$$

Remark 1.65. The Smith dynamics belongs to the *pairwise comparison* dynamics (*cf.* [40, 80, 81]).

With the help of the following Lyapunov function V, Smith [86] obtained the convergence and the stability of dynamics (1.41) as shown by Theorem 1.66:

$$V(\mathbf{f}) = \sum_{i\in\mathcal{N}} \sum_{p,l\in P^i} f_p^i \big(c_p(\mathbf{f}) - c_l(\mathbf{f})\big)^{+2}.$$

Theorem 1.66. *If the vector of system path costs* $\mathbf{c}$ *is of class* $\mathcal{C}^1$*, and if* $-\mathbf{c}$ *is monotone on* F*, then, for all initial system configuration* $\mathbf{f}_0 \in F$*, dynamics* (1.41) *admits a unique solution which converges to a nonempty set of equilibria.*

A particular case where map $-\mathbf{c}$ is monotone is when each path is composed of a single arc, and the cost functions of an arc a for different populations are equal to a monotone function l_a up to an additive constant.

1.5.3 Atomic games with splittable stocks and composite games

In §1.3.5, it is shown that a potential function exists in a composite game if the paths are all composed of a single arc with affine cost function. In this case, an atomic player with splittable stocks can be replaced by a population of nonatomic players, who control the same stocks and whose per-unit cost of an arc is the marginal arc cost of the replaced atomic player. In this way, the game is transformed into a virtual nonatomic game, and its Wardrop equilibrium coincides with the composite equilibrium of the initial composite game. The dynamics in the potential games approach can then be applied to this virtual nonatomic game so as to obtain the convergence to the equilibrium. Altman et al. [4] applied the replicator dynamics to this kind of atomic games with the help of the corresponding potential function.

The approach of stable games can also be applied to the case where the paths are all composed of a single arc, and the cost functions of each arc for different players and different populations are equal to an affine function up to an additive constant. It can be shown that the vector of marginal costs is monotone on F_l. The Smith's dynamics can be applied to the nonatomic game induced by the initial composite game by replacing each atomic player by a population, whose costs in this virtual nonatomic game are defined to be the marginal costs of the replaced player in the initial game.

As for the dynamics linked to the atomic players' rationality, one first thinks about the best-reply dynamics. Orda et al. [66] considered this dynamics in a two-atomic-player game with splittable stocks in a network composed of two parallel arcs with cost functions which are nondecreasing, convex and of class $\mathcal{C}^1$. The two players play in turn a best reply to their opponent. A result on the convergence of the dynamics to the unique Nash equilibrium was obtained. Mertzios [59] extended this result to series parallel networks. Altman et al. [2] applied the best-reply dynamics to N-player atomic games in the same structure of network, but with linear arc costs. They obtained the results of convergence for several scenarios, for example, where the players play one by one or two by two a best reply.

However, except such cases where the networks have specific structures and cost functions, the dynamics in atomic games with splittable stocks and in composite games remains an open problem.

1.5.4 Two-level dynamics

In the two previous sections, the composition of the players or, more precisely, the distribution of the stocks among the atomic players with splittable stocks and the populations of nonatomic players is fixed during the dynamical process. However, one can consider situations where this distribution varies as well.

A first example is inspired by a model in Wan [93] (*cf.* §1.4.3 and Chapter 4). An atomic player i might have interest in distributing her stocks among some small atomic players and/or a mass of nonatomic players, facing the structure of the stocks not controlled by her. This structure will be called player i's *environment*. The following two-level scenario can then be considered. At the first level, the initial atomic players change the composition of their stocks and, at the second level, the initial nonatomic players and the new atomic and nonatomic players created by the initial atomic players play a composite game. This scenario then entails a two-level dynamical process of two levels, probably with different speeds at the two levels. For example, the dynamics of the composite game at the second level is faster than the dynamics of the the recomposition of the stocks at the first level.

Another example comes from the congestion game model with coalitions [20, 37, 91] (*cf.*

Chapter 2). In a composite game, atomic and/or nonatomic players may have incentive to form a coalition which behaves like an atomic player with splittable stocks, then these coalitions and the independent players belonging to no coalition play a composite game. Here, a two-level scenario is such that, at the first level, the initial players form coalitions and, at the second level, the coalitions and the independent players play a composite game. In a dynamical process of two levels with different speeds induced by this scenario, the dynamics for the composite game at the second level might be faster than the dynamics for the formation of coalitions at the first level. Bravo and Wan [17] (*cf.* Chapter 6) modeled this scenario by a discreet dynamics.

1.6 Remarks and possible extensions

1.6.1 Splittability of the stocks

An atomic player with an unsplittable stock can only send all her stock by a single path. An atomic player with a splittable stock can arbitrarily split her stock into infinitesimally small parts. In addition to these two extreme cases (unsplittable and infinitesimally splittable), other intermediate levels of divisibility of the stocks can also be considered.

An atomic player i with *generalized* splittable stocks of weight m^i is specified by the unsplittable components and a splittable component of her stocks: $\mathbf{m^i} = (m^{i,1}, \ldots, m^{i,n^i}, \mu^i)$, where $n^i \in \mathbb{N}^*$, $m^{i,l} \in \mathbb{R}_{++}$ for all l and $\sum_{l=1}^{n^i} m^{i,l} + \mu^i = m^i$. Her stocks has total weight m^i, where n^i components are unsplittable (which are of weight, respectively, $m^{i,1}, \ldots, m^{i,n^i}$), and a splittable component of weight μ^i. She can send different unsplittable components by different paths, but an unsplittable component can no longer be divided. On the contrary, she can arbitrarily split the splittable component.

For example, in a traffic network, a train can be split into wagons but no smaller components; in a telecommunication network, a flow can only be decomposed into packets.

1.6.2 Type of the stocks

It is mentioned that the players may have specific costs. More rigorously, it is the stocks which have specific cost functions. This idea is to be specified and extended here.

In a congestion game, there can be a finite number of types of stocks. A *type* τ is characterized by two elements:

1. P^τ, the set of available paths for the stock, and
2. $(l_a^\tau)_{a \in A}$, the vector of arc costs for the stock.

Remark 1.67. Set P^τ determines not only the origin and the destination of the stock, but also the set of the paths connecting these vertices which are *available* to it. Two stocks which have the same pair of origin/destination do not necessarily have access to the same paths.

Remark 1.68. It is not rare that, on the same arc, stocks of different types have different per-unit costs. For example, a truck, a car, and a bike do not experience the congestion on a path in the same way.

An atomic player can control several types of stocks. Let us define a composite congestion game with T types of stocks by specifying, on the one hand, for each atomic player, unsplittable components and a splittable component for each type of her stocks and, on the other hand, for each population of nonatomic players, the type of their stocks:

Nonatomic players: there are T populations of nonatomic players of distinct types: population τ have stocks of type τ and their total weight is θ_τ;

Atomic players: for each atomic player i, her stocks are composed of T parts corresponding to the T types. The part corresponding to type τ is specified by a vector $\mathbf{m}_\tau^i = (m_\tau^{i,1}, \ldots, m_\tau^{i,n^i}, \mu_\tau^i)$, where $n_\tau^i \in \mathbb{N}$ is the number of unsplittable components of stocks of type τ (they are of weight, respectively, $m_\tau^{i,1}, \ldots, m^{i,n_\tau^i})$, and μ_τ^i is the weight of the splittable component of stock of type i.

As the definition of a composite equilibrium in §1.3.5, a composite equilibrium in such a composite congestion game with T types of stocks and N atomic players is defined as a system configuration which

(i) minimizes the cost of each of the N atomic players given her opponents' configurations, and which

(ii) equalizes, for each of the T populations of nonatomic players, the per-unit costs of all the paths used by it.

Chapter 2

Coalitions in network congestion games

This chapter is based on the paper *Coalitions in network congestion games.*

Abstract. *This work shows that, in a two-terminal parallel-arc network, the formation of a finite number of coalitions in a nonatomic network congestion game benefits everyone. At the equilibrium of the composite game played by coalitions and individuals, the average cost to each coalition and the individuals' common cost are all lower than in the corresponding nonatomic game (without coalitions). The individuals' cost is lower than the average cost to any coalition. Similarly, the average cost to a coalition is lower than that to any larger coalition. Whenever some members of a coalition become individuals, the individuals' payoff is increased. In the case of a unique coalition, both the average cost to the coalition and the individuals' cost are decreasing with respect to the size of the coalition. In a sequence of composite games, if a finite number of coalitions are fixed, while the size of the remaining coalitions goes to zero, the equilibria of these games converge to the equilibrium of a composite game played by the same fixed coalitions and the remaining individuals.*

2.1 Introduction

This paper considers the impact of introducing coalitions in network congestion games played by nonatomic individuals, namely, nonatomic routing games. These games belong to a more general class of noncooperative games played by a continuum of anonymous identical players, each of whom has a negligible effect on the others.

First, let us cite some historic references on routing games, in particular, on coalitions in such games.

Beckman, McGuire and Winston [9] first formulated Wardrop equilibrium (Wardrop [94]) in nonatomic congestion games as an optimal solution of a convex programming problem, and thus proved its existence under weak conditions on the cost functions.

A coalition of nonatomic individuals of total weight T behaves the same way as an atomic player who holds a flow of weight T that can be split and sent by different paths. Routing games with finitely many atomic players holding splittable flow (called atomic splittable games) were first examined by Haurie and Marcotte [36]. They focused on the asymptotic behavior of Nash equilibria in such games. By characterizing a Nash equilibrium in an atomic splittable game and a Wardrop equilibrium in the corresponding nonatomic game by two variational inequalities, they proved that the former converges to the latter, when the number of atomic players tends to infinity. This result will be extended in this paper.

Harker [32] first studied *composite games* (that he called mixed games), where atomic players holding splittable flow (or coalitions) and nonatomic individuals play together. He characterized a composite-type equilibrium by a variational inequality, and thus proved the existence of a solution under some weak conditions on the cost functions as well as its uniqueness under more stringent conditions.

Orda, Rom and Shimkin [66] made a detailed study on the uniqueness and other properties of Nash equilibria in atomic splittable games on *two-terminal parallel-link* networks. This specific setting will be adopted in this paper, where their results will be extended. Richman and Shimkin [71] extended their results to composite congestion games in nearly parallel-link networks.

For the impact of coalitions on the equilibrium costs, Cominetti, Correa and Stier-Moses [20] showed that, in the atomic splittable case where the atomic players are identical, the social cost at the equilibrium of the game is bounded by that of the corresponding nonatomic game, under weak conditions on the cost functions.

Hayrapetyan, Tardos and Wexler [37] proved that the formation of coalitions (that they called collusion) reduces the social cost in a two-terminal parallel-link network. Although stronger conditions on the cost functions are needed in this paper, our results prove that the formation of coalitions benefits everyone.

Apart from the consequence of the formation of coalitions on the equilibrium costs, this paper also studies how this impact varies with the structure of coalitions.

2.1.1 A Sketch of the model

A continuum of nonatomic individuals are commuters in a two-terminal parallel-arc (directed) network. Their common origin and common destination are the only two vertices, which are connected by a finite set of parallel arcs. The per-unit traffic cost of an arc depends only on the total weight of the flow on it. A pure strategy of an individual is an arc by which she goes from the origin to the destination. Nash equilibria in such nonatomic games are usually called *Wardrop equilibria* (WE for short) [94]. At a WE, the arc chosen by an individual costs no more than any other available arc, hence it has the lowest cost in the network. The individuals have the same cost at a WE.

A composite routing game is played by a finite number of disjoint coalitions formed by some of the individuals and the remaining individuals. A coalition is specified by its size. Within a coalition, a coordinator assigns an arc to each member, with the objective of minimizing their total cost. An equilibrium in this game is called *composite equilibrium* (CE for short), because it is Nash type for the coalitions and Wardrop type for the individuals. All the individuals have the same cost at a CE, while the average costs to the coalitions may differ.

2.1.2 Main results

After recalling the *existence* and the *uniqueness* of the CE of a composite game under certain conditions on the cost functions, five main results are obtained:

1. At the CE, the average social cost, the individuals' cost and the average cost to each coalition are lower than the equilibrium cost at the WE of the corresponding nonatomic game.

2. At the CE, the average cost to a coalition is lower than that to any other larger coalition. If a coalition sends flow on a certain arc, then any other larger coalition sends more on it.
3. If some members quit a coalition to become individuals, the individuals' cost is increased at the corresponding CE.
4. If there is only one coalition, the social cost, the average cost to the coalition, and the individuals' cost at the CE are all decreasing with respect to the size of the unique coalition.
5. If, in a sequence of composite games, a finite number of coalitions are fixed, and the maximum size of the remaining coalitions tends to zero, the sequence of equilibrium of these games converges to the equilibrium of a game played by the same fixed coalitions and the remaining individuals.

2.1.3 Organization of the work

The paper is organized in the following way. Section 2.2 provides a detailed description of the model as well as characterizations of the CE in different formulations. The existence and the uniqueness of the CE will be recalled. Section 2.3 analyzes some important properties of the CE. Section 2.4 deals with the impact of the formation of coalitions by comparing the players' costs at the WE of the corresponding nonatomic game and those at the CE. Section 2.5 considers the impact of the composition of the players on the CE costs: first, how the equilibrium costs vary with the size of a unique coalition; second, how the individuals' cost varies when some members of a coalition become individuals. Section 2.6 focuses on the asymptotic behavior of CE, by fixing some coalitions while letting the remaining coalitions vanish. Section 2.7 discusses some problems for future research.

2.2 The model and characterization of an equilibrium

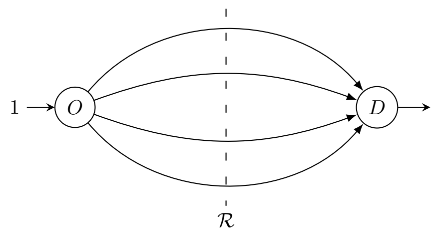
2.2.1 Model and notations

Network and arc costs $(\mathcal{R},\, \mathbf{c})$. Let the set of identical anonymous nonatomic individuals be described by the unit real interval $I = [\,0,\, 1]$, endowed with the Lebesgue measure μ. The players' common origin is vertex O, and their common destination is vertex D. The finite set of parallel arcs between O and D is denoted by $\mathcal{R}$, with $R = |\mathcal{R}|$ its cardinality. Let $\mathbf{c} = (c_r)_{r\in\mathcal{R}}$ be the vector of the per-unit arc cost functions: for every arc r, $x \mapsto c_r(x)$ is a real function defined on a neighborhood U of $[\,0,\, 1]$. The per-unit cost of an arc only depends on the total weight of the flow on it. The network is characterized by the pair $(\mathcal{R},\, \mathbf{c})$.

The following assumption is made *throughout* this paper.

A 2.1. *For every arc r in $\mathcal{R}$, the cost function c_r is strictly increasing, convex and continuously differentiable on U, and nonnegative on $[\,0,\, 1]$.*

Composite routing game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$. Suppose that K coalitions are formed in the set of individuals I, with $K \in \mathbb{N} = \{0, 1, 2, \cdots\}$. The family of coalitions is denoted by $\mathcal{K} = \{1,\, \ldots,\, K\}$. Every coalition behaves like an atomic player holding a splittable flow. The remaining individuals are independent nonatomic players. For a coalition $k \in \mathcal{K}$, the measurable set of its members is denoted by I^k, a subset of I, and its total weight is denoted by $T^k = \mu(I^k)$. Let I^0 denote the set of individuals so that $I^0 = [\,0,\, 1]\setminus \cup_{k\in\mathcal{K}} I^k$,



and its weight is $T^0 = \mu(I^0) = 1 - \sum_{k\in\mathcal{K}} T^k$. Without loss of generality, it is assumed that $T^1 \geq T^2 \geq \cdots \geq T^K$. Let us define $\mathbf{T} = (\,T^0;\, T^1, \ldots, T^K\,)$. Let $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$ be the composite routing game played by these K coalitions and the remaining individuals in the network $(\mathcal{R},\, \mathbf{c})$.

Two particular cases should be mentioned. First, if $I^0 = [\,0,\,1]$ and $K = 0$, there is no coalition so that the game is a nonatomic one, denoted simply by $\Gamma(\mathcal{R}, \mathbf{c})$, and the equilibria there are WE. Second, if I^0 is empty, i.e. $T^0 = 0$, the game is an atomic splittable one with K atomic players, and the equilibria there are *Nash equilibria* (NE for short) in its usual sense; in particular, if $K = 1$, i.e, there is a global coalition, the equilibrium is obtained by solving the optimization problem of searching for the social optimum.

Strategies and flow configurations In the game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$, as the individuals are identical and anonymous, only the total weight sent by each coalition on each arc counts. The strategy profile of the individuals (*resp.* the strategy of coalition k) is specified by the *flow configuration* (flow for short) $\mathbf{x}^0$ (*resp.* $\mathbf{x}^k$) defined by

$$\mathbf{x}^0 = (x_r^0)_{r\in\mathcal{R}} \quad (resp. \quad \mathbf{x}^k = (x_r^k)_{r\in\mathcal{R}}),$$

where x_r^0 (*resp.* x_r^k) is the total weight of the individuals (*resp.* of coalition k) on arc r.

A strategy profile is specified by $\mathbf{x} = (\mathbf{x}^0,\, \mathbf{x}^1,\, \ldots,\, \mathbf{x}^K)$, a point in $\mathbb{R}^{(1+K)\times R}$.

The *feasible flow set* of the individuals (*resp.* of coalition k) is a convex compact subset of $\mathbb{R}^R$, defined by

$$F^0 = \{\mathbf{x}^0 \in \mathbb{R}^R \mid \forall\, r \in \mathcal{R},\, x_r^0 \geq 0\,;\, \sum_{r\in\mathcal{R}} x_r^0 = T^0\},$$

$$(resp. \quad F^k = \{\mathbf{x}^k \in \mathbb{R}^R \mid \forall\, r \in \mathcal{R},\, x_r^k \geq 0\,;\, \sum_{r\in\mathcal{R}} x_r^k = T^k\}).$$

The *feasible flow set* F of the game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$ is a convex compact subset of $\mathbb{R}^{(1+K)\times R}$, defined by $F = F^0 \times F^1 \times \cdots \times F^K$.

The *aggregate flow* $\mathbf{x}'$ induced by $\mathbf{x}$ is a vector in $\mathbb{R}^R$, defined by $\mathbf{x}' = (x_r)_{r\in\mathcal{R}}$, where $x_r = x_r^0 + \sum_{k\in\mathcal{K}} x_r^k$ is the aggregate weight on arc r.

For coalition k, the vector $\mathbf{x}^{-k}$ is a point in $F^{-k} = \prod_{l\in\{0\}\cup\mathcal{K}\setminus\{k\}} F^l$, defined by $\mathbf{x}^{-k} = (\mathbf{x}^l)_{l\in\{0\}\cup\mathcal{K}\setminus\{k\}}$. For all arc r, define $x_r^{-k} = x_r^0 + \sum_{l\in\mathcal{K}\setminus\{k\}} x_r^l$.

Average costs and marginal costs The *average cost* to the individuals, the *average cost* to coalition k and the *average social cost* are respectively defined by

$$Y^0(\mathbf{x}) = \frac{1}{T^0}\sum_{r\in\mathcal{R}} x_r^0 c_r(x_r), \quad Y^k(\mathbf{x}) = \frac{1}{T^k}\sum_{r\in\mathcal{R}} x_r^k c_r(x_r), \quad Y(\mathbf{x}) = \sum_{r\in\mathcal{R}} x_r c_r(x_r).$$

Recall that the total weight of the players is normalized to one, hence the average social cost is just the social cost.

The total cost to coalition k is denoted by $u^k(\mathbf{x}) = T^k \cdot Y^k(\mathbf{x}) = \sum_{r\in\mathcal{R}} x_r^k c_r(x_r)$.

Following Harker [32], the *marginal cost* function of coalition k is defined by

$$\hat{c}^k(\mathbf{x}) = (\hat{c}_r^k(\mathbf{x}))_{r\in\mathcal{R}}, \text{ where } \hat{c}_r^k(\mathbf{x}) = c_r(x_r) + x_r^k c_r'(x_r).$$

Notice that $\hat{c}^k(\mathbf{x})$ is the gradient of $u^k(\mathbf{x})$ with respect to $\mathbf{x}^k$. More precisely,

$$\hat{c}^k(\mathbf{x}) = \nabla_{\mathbf{x}^k}\, u^k(\mathbf{x}^k, \mathbf{x}^{-k}) = \Big(\frac{\partial u^k}{\partial x_r^k}(\mathbf{x})\Big)_{r\in\mathcal{R}}.$$

2.2.2 Characterizing equilibria: existence and uniqueness

The following definition of a CE (Harker [32]) consists of two parts: the first for the individuals and the second for the coalitions.

Definition 2.2 (Composite equilibrium)**.** A point $\mathbf{x}^* = (\mathbf{x}^{*0}, \mathbf{x}^{*1}, \ldots, \mathbf{x}^{*K})$ in F is a *CE* of the game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$ if

$$\forall\, r \in \mathcal{R}, \quad \text{if } x_r^{*0} > 0, \text{ then } r \in \arg\min_{s\in\mathcal{R}} c_s(x_s^*); \tag{2.1}$$

$$\forall\, k \in \mathcal{K}, \quad \mathbf{x}^{*k} \text{ minimizes } u^k(\mathbf{x}^k, \mathbf{x}^{*-k}) \text{ on } F^k. \tag{2.2}$$

Proposition 2.3 (Characterization of a CE)**.** *The following are equivalent:*

(i) $\mathbf{x}^* = (\mathbf{x}^{*0}, \mathbf{x}^{*1}, \ldots, \mathbf{x}^{*K})$ *in* F *is a CE;*

(ii) (marginal cost formulation) $\mathbf{x}^* = (\mathbf{x}^{*0}, \mathbf{x}^{*1}, \ldots, \mathbf{x}^{*K})$ *in* F *satisfies*

$$\forall\, r \in \mathcal{R}, \quad \textit{if } x_r^{*0} > 0, \quad \textit{then } \forall s \in \mathcal{R},\ c_r(x_r^*) \le c_s(x_s^*); \tag{2.3}$$

$$\begin{aligned} \forall\, k \in \mathcal{K}, \quad \textit{if } x_r^{*k} > 0, \quad &\textit{then } \forall s \in \mathcal{R},\ \hat{c}_r^k(x_r^*) \le \hat{c}_s^k(x_s^*), \\ \textit{i.e.} \quad & c_r(x_r^*) + x_r^{*k} c_r'(x_r^*) \le c_s(x_s^*) + x_s^{*k} c_s'(x_s^*); \end{aligned} \tag{2.4}$$

(iii) (variational inequality formulation) $\mathbf{x}^* = (\mathbf{x}^{*0}, \mathbf{x}^{*1}, \ldots, \mathbf{x}^{*K})$ *in* F *satisfies*

$$\langle\, \mathbf{c}(\mathbf{x}^*), \mathbf{x}^0 - \mathbf{x}^{*0} \,\rangle + \sum_{k\in\mathcal{K}} \langle\, \hat{\mathbf{c}}^k(\mathbf{x}^*), \mathbf{x}^k - \mathbf{x}^{*k} \,\rangle \ge 0, \quad \forall\, \mathbf{x} = (\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^K) \in F, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ *stands for the standard inner product operator on the Euclidean spaces.*

Proof. (i) $\Leftrightarrow$ (ii): For the individuals, (2.3) is simply a reformulation of (2.1). For the coalitions, in order to show that (2.2) is equivalent to (2.4), let us first prove that for coalition k, $u^k(\mathbf{x}^k, \mathbf{x}^{-k})$ is convex in $\mathbf{x}^k$ for any given $\mathbf{x}^{-k}$ in F^{-k}.

Indeed, for any r in $\mathcal{R}$, the fact that c_r is convex and strictly increasing implie that

$$\begin{aligned} & c_r(y_r^k + x_r^{-k}) \ge c_r(x_r^k + x_r^{-k}) + (y_r^k - x_r^k)\, c_r'(x_r^k + x_r^{-k}) \\ \Rightarrow\quad & y_r^k\, c_r(y_r^k + x_r^{-k}) \ge y_r^k\, c_r(x_r) + y_r^k\, (y_r^k - x_r^k)\, c_r'(x_r) \ge y_r^k\, c_r(x_r) + x_r^k\, (y_r^k - x_r^k)\, c_r'(x_r) \\ & \qquad = x_r^k\, c_r(x_r) + (y_r^k - x_r^k)\, \Big[c_r(x_r) + x_r^k\, c_r'(x_r)\Big]. \\ \Rightarrow\quad & \sum_{r\in\mathcal{R}} y_r^k\, c_r(y_r^k + x_r^{-k}) \ge \sum_{r\in\mathcal{R}} x_r^k\, c_r(x_r) + \sum_{r\in\mathcal{R}} (y_r^k - x_r^k)\, \Big[c_r(x_r) + x_r^k\, c_r'(x_r)\Big], \end{aligned}$$

which further implies that

$$u^k(\mathbf{y}^k,\, \mathbf{x}^{-k}) \geq u^k(\mathbf{x}^k,\, \mathbf{x}^{-k}) + \langle\, \nabla_{\mathbf{x}^k}\, u^k(\mathbf{x}^k,\, \mathbf{x}^{-k}),\, \mathbf{y}^k - \mathbf{x}^k \,\rangle, \quad \forall\, \mathbf{x}^k,\, \mathbf{y}^k \,\in F^k.$$

Thus, $\mathbf{x}^{*k}$ minimizes the convex function $u^k(\mathbf{x}^k,\, \mathbf{x}^{*-k})$ on the convex compact set F^k if, and only if, $\langle\, \nabla_{\mathbf{x}^k}\, u^k(\mathbf{x}^*),\, \mathbf{x}^k - \mathbf{x}^{*k} \,\rangle \geq 0$ for all $\mathbf{x}^k \,\in F^k$ or, equivalently,

$$\langle\, \hat{\mathbf{c}}^k(\mathbf{x}^*),\, \mathbf{x}^k - \mathbf{x}^{*k} \,\rangle \geq 0, \quad \forall\, \mathbf{x}^k \,\in F^k. \tag{2.6}$$

Let us set $\hat{c}^k = \min_{r\in\mathcal{R}} \hat{c}^k_r(\mathbf{x}^*)$. Then, $\sum_{r\in\mathcal{R}} (\, \hat{c}^k_r(\mathbf{x}^*) - \hat{c}^k\,)\, (x^k_r - x^{*k}_r) = \sum_{r\in\mathcal{R}}\, \hat{c}^k_r(\mathbf{x}^*)\, (x^k_r - x^{*k}_r) - \hat{c}^k \sum_{r\in\mathcal{R}} (x^k_r - x^{*k}_r) = \sum_{r\in\mathcal{R}}\, \hat{c}^k_r(\mathbf{x}^*)\, (x^k_r - x^{*k}_r) - \hat{c}^k (T^k - T^k) = \langle\, \hat{\mathbf{c}}^k(\mathbf{x}^*),\, \mathbf{x}^k - \mathbf{x}^{*k} \,\rangle$. Consequently, (2.6) is equivalent to

$$\sum_{r\in\mathcal{R}} (\, \hat{c}^k_r(\mathbf{x}^*) - \hat{c}^k\,)\, (x^k_r - x^{*k}_r) \geq 0. \tag{2.7}$$

It remains to show that (2.7) is equivalent to (2.4).

(2.4) $\Rightarrow$ (2.7): According to (2.4),

$$(\, \hat{c}^k_r(\mathbf{x}^*) - \hat{c}^k\,)\, (x^k_r - x^{*k}_r) = \begin{cases} (\, \hat{c}^k_r(\mathbf{x}^*) - \hat{c}^k\,)\, x^k_r \geq 0, & \text{if } x^{*k}_r = 0, \\ 0, & \text{if } x^{*k}_r > 0. \end{cases}$$

Thus, $\sum_{r\in\mathcal{R}} (\, \hat{c}^k_r(\mathbf{x}^*) - \hat{c}^k\,)\, (x^k_r - x^{*k}_r) \geq 0$.

(2.7) $\Rightarrow$ (2.4): Let us define an auxiliary flow $\mathbf{x}^k$ in F^k as follows: $x^k_r = 0$ if $\hat{c}^k_r(\mathbf{x}^*) > \hat{c}^k$, and $x^k_r = \frac{T^k}{m}$ if $\hat{c}^k_r(\mathbf{x}^*) = \hat{c}^k$. Here $m = |\{r \in \mathcal{R} \,|\, \hat{c}^k_r(\mathbf{x}^*) = \hat{c}^k\}|$, the number of arcs whose marginal cost to coalition k at $\mathbf{x}^*$ are the smallest in the network. Then, for this specific $\mathbf{x}^k$, (2.7) implies that $\sum_{r\in\mathcal{R}, \hat{c}^k_r(\mathbf{x}^*)>\hat{c}^k} (\, \hat{c}^k_r(\mathbf{x}^*\,) - \hat{c}^k\,)\, (-x^{*k}_r) \geq 0$. Consequently, $x^{*k}_r = 0$ if $\hat{c}^k_r(\mathbf{x}^*) > \hat{c}^k$, which leads to (2.4).

(ii) $\Leftrightarrow$ (iii): By the same argument used above for the equivalence between (2.4) and (2.6), one can show that (2.3) is equivalent to

$$\Big\langle\, \mathbf{c}(\mathbf{x}^*),\, \mathbf{x}^0 - \mathbf{x}^{*0} \,\Big\rangle \geq 0, \quad \forall\, \mathbf{x}^0 \in F^0. \tag{2.8}$$

The variational inequalities (2.6) and (2.8) imply immediately (2.5). For the converse, it is enough to take an $\mathbf{x} = (\, \mathbf{x}^0,\, \mathbf{x}^1,\, \ldots,\, \mathbf{x}^K\,)$ in F such that $\mathbf{x}^l = \mathbf{x}^{*l}$ for all l in $\mathcal{K}$ (*resp.* $\mathbf{x}^l = \mathbf{x}^{*l}$ for all l in $\{0\} \cup \mathcal{K} \setminus \{k\}$) to get (2.8) (*resp.* (2.6)).

Thus, one has shown that (2.3) and (2.4) are equivalent to (2.5). □

Remark 2.4. (iii) has been proven for the specific cases of NE and WE as well as for CE: a WE was characterized as the solution of a variational inequality problem by Smith [85] and Dafermos [27], and as the solution of a nonlinear complementarity problem by Aashtiani and Magnanti [1]. Variational inequalities were used to characterize a NE in atomic splittable games by Haurie and Marcotte [36], and a CE in composite games by Harker [32].

Condition (2.4) shows that the marginal costs $(\hat{c}^k_r)_{r\in\mathcal{R}}$ play the same role for coalition k as $(c_r)_{r\in\mathcal{R}}$ for the individuals: at the CE, all the arcs used by coalition k have the lowest marginal cost and, *a fortiori*, the same one. For flow $\mathbf{x} \in F$, $\hat{c}^k_r(\mathbf{x}) = c_r(x_r) + x^k_r c'_r(x_r)$ is a function of only two variables x^k_r and x_r. Besides, according to Assumption 2.1, it is strictly increasing in both of them.

Theorem 2.5 (Existence and uniqueness of CE)**.** *In a composite game, a CE exists, and it is unique.*

Proof. The variational inequality formulation for CE (2.5) is used to prove its existence. Theorem 3.1 in Kinderlehrer and Stampacchia [49, p.12] states that the variational inequality problem (2.5) admits a solution if F is a convex compact set, and if $\hat{\mathbf{c}}^k$ and $\mathbf{c}$ are continuous. According to Assumption 2.1, these conditions are satisfied.

For the uniqueness of CE, see Richman and Shimkin [71, Theorem 4.1]. $\square$

Remark 2.6. For the nonatomic routing game $\Gamma(\mathcal{R}, \mathbf{c})$, a WE exists if the cost functions c_r's are continuous. If they are furthermore strictly increasing on U, then the WE is unique. See Patriksson [68, Theorems 2.4, 2.5] for a proof.

2.3 A detailed study on CE

Let us consider a composite game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$. This section focuses on the properties of its unique CE, denoted by $\mathbf{x}$ here and in Section 2.4.

First, some notations are recalled or given.

Notation. $\mathcal{R}^0(\mathbf{x}) = \{r \in \mathcal{R} \,|\, x_r^0 > 0\} \subset \mathcal{R}$ is the support of $\mathbf{x}^0$;
$\mathcal{R}^k(\mathbf{x}) = \{r \in \mathcal{R} \,|\, x_r^k > 0\} \subset \mathcal{R}$ is the support of $\mathbf{x}^k$, for coalition k;
$c^0(\mathbf{x})$ is the lowest arc cost in the network;
$\hat{c}^k(\mathbf{x})$ is the marginal cost to coalition k of every arc used by it;
$Y^0(\mathbf{x})$ is the common cost to all the individuals; $Y^0(\mathbf{x}) = c^0(\mathbf{x})$;
$Y^k(\mathbf{x})$ is the average cost to coalition k;
$\underline{Y}^k(\mathbf{x}) = \min_{r \in \mathcal{R}^k} c_r(x_r)$ is the lowest arc cost of the arcs used by coalition k;
$Y(\mathbf{x})$ is the social cost.

All the statements made in this section and Section 2.4 are to be understood at the CE $\mathbf{x}$. And $\mathbf{x}$ will often be omitted if it does not cause confusion.

The following facts follow immediately from (2.3) and (2.4). They will be repeatedly referred to in this work without further explanation:

$$\begin{aligned} & c_r(x_r) = c^0, \ \text{ if } r \in \mathcal{R}^0; \quad c_r(x_r) \geq c^0, \ \text{ if } r \in \mathcal{R} \setminus \mathcal{R}^0; \\ \forall\, k \in \mathcal{K}, \quad & \hat{c}_r^k(\mathbf{x}) = \hat{c}^k, \ \text{ if } r \in \mathcal{R}^k; \quad \hat{c}_r^k(\mathbf{x}) \geq \hat{c}^k, \ \text{ if } r \in \mathcal{R} \setminus \mathcal{R}^k. \end{aligned}$$

The following lemma states that an arc used by a coalition costs less than any arc not used by it.

Lemma 2.7. *For any coalition k, for any arc r in $\mathcal{R}^k$ and any arc s in $\mathcal{R} \setminus \mathcal{R}^k$, $c_r(x_r) < c_s(x_s)$.*

Proof. Indeed, $c_r(x_r) < c_r(x_r) + x_r^k c_r'(x_r) = \hat{c}^k \leq c_s(x_s)$ because $x_r^k > 0$ and $x_s^k = 0$,. $\square$

The next lemma shows that an arc used by individuals is also used by all the coalitions. Besides, the average cost to any coalition is not lower than the individuals' cost.

Lemma 2.8. *For any coalition k,*

(i) $\mathcal{R}^0 \subset \mathcal{R}^k$, *i.e. for all* $r \in \mathcal{R}$, *if* $x_r^0 > 0$, *then* $x_r^k > 0$*;*

(ii) $c^0 < \hat{c}^k$*;*

(iii) $Y^0 = \underline{Y}^k \leq Y^k$*.*

Proof. (i) Suppose that $x_r^0 > 0$. If $x_r^k = 0$, there is another arc s such that $x_s^k > 0$. Then, $c_r(x_r) \geq \hat{c}^k(\mathbf{x}) = c_s(x_s) + x_s^k c_s'(x_s) > c_s(x_s)$. However, $x_r^0 > 0$, hence $c_r(x_r) \leq c_s(x_s)$, a contradiction.

(ii) Take r in $\mathcal{R}^0$. By (i), $x_r^k > 0$. Thus, $\hat{c}^k = c_r(x_r) + x_r^k c'(x_r) > c_r(x_r) = c^0$.

(iii) The individuals take the arcs with the lowest cost, hence $Y^0 \leq \underline{Y}^k \leq Y^k$. And (i) implies that $Y^0 = \underline{Y}^k$. □

The next lemma states that an arc used by a coalition is also used by any larger coalition, and the larger one sends more flow on it.

Lemma 2.9. *Let two coalitions k and l be such that $T^k < T^l$. Then, the following are true:*

(i) $\mathcal{R}^k \subset \mathcal{R}^l$, i.e. for all $r \in \mathcal{R}$, if $x_r^k > 0$, then $x_r^l > 0$;

(ii) $\hat{c}^k < \hat{c}^l$;

(iii) For any arc r, $x_r^k \leq x_r^l$, and the inequality is strict if $x_r^k > 0$;

(iv) $Y^k \leq Y^l$, and the equality holds if, and only if, $Y^l = Y^k = Y^0$.

If $T^k = T^l$, all these inequalities or inclusions become equalities.

Proof. (i) Suppose that $T^k < T^l$. If $\mathcal{R}^k \not\subset \mathcal{R}^l$, there is some r such that $x_r^k > 0$ but $x_r^l = 0$. Hence, $\hat{c}^k = c_r(x_r) + x_r^k c_r'(x_r) > c_r(x_r) \geq \hat{c}^l$. In particular, $\hat{c}^k > \hat{c}^l$.

For all s in $\mathcal{R}\backslash\mathcal{R}^k$, $c_s(x_s) \geq \hat{c}^k > \hat{c}^l$, which implies that $x_s^l = 0$. As a result, $\mathcal{R}\backslash\mathcal{R}^k \subset \mathcal{R}\backslash\mathcal{R}^l$ or, equivalently, $\mathcal{R}^l \subset \mathcal{R}^k$.

For all r in $\mathcal{R}^l$ and, *a fortiori*, in $\mathcal{R}^k$, $\hat{c}^k = c_r(x_r) + x_r^k c_r'(x_r)$ and $\hat{c}^l = c_r(x_r) + x_r^l c_r'(x_r)$. Hence, $x_r^k - x_r^l = (\hat{c}^k - \hat{c}^l)/c_r'(x_r) > 0$, so that $x_r^k > x_r^l$. As a result, $T^l = \sum_{r\in\mathcal{R}^l} x_r^l < \sum_{r\in\mathcal{R}^l} x_r^k \leq T^k$, a contradiction.

Therefore, $\mathcal{R}^k \subset \mathcal{R}^l$.

Suppose that $T^k = T^l$. The above proof is still valid. Thus, $\mathcal{R}^k \subset \mathcal{R}^l$ and, by symmetry, $\mathcal{R}^l \subset \mathcal{R}^k$. This leads to $\mathcal{R}^k = \mathcal{R}^l$.

(ii) and (iii) Suppose that $T^k < T^l$. By (i), $\mathcal{R}^k \subset \mathcal{R}^l$. There are two cases.

CASE 1. $\mathcal{R}^k = \mathcal{R}^l$. Given r in $\mathcal{R}^k = \mathcal{R}^l$, $\hat{c}^k = c_r(x_r) + x_r^k c_r'(x_r)$ and $\hat{c}^l = c_r(x_r) + x_r^l c_r'(x_r)$. It follows that $x_r^l - x_r^k = (\hat{c}^l - \hat{c}^k)/c_r'(x_r)$. Thus, $0 < T^l - T^k = \sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k) = (\hat{c}^l - \hat{c}^k)\sum_{r\in\mathcal{R}^k} 1/c_r'(x_r)$ and, consequently, $\hat{c}^l > \hat{c}^k$.

CASE 2. $\mathcal{R}^k \subset \mathcal{R}^l$ but $\mathcal{R}^k \neq \mathcal{R}^l$. Take s in $\mathcal{R}^l \setminus \mathcal{R}^k$. Then, $\hat{c}^l = c_s(x_s) + x_s^l c_s'(x_s) > c_s(x_s) \geq \hat{c}^k$.

In both cases, $\hat{c}^l > \hat{c}^k$. For all r in $\mathcal{R}^k$, $x_r^l - x_r^k = (\hat{c}^l - \hat{c}^k)/c_r'(x_r) > 0$; in particular, $x_r^k < x_r^l$. And for all r in $\mathcal{R} \setminus \mathcal{R}^k$, $0 = x_r^k \leq x_r^l$.

Suppose that $T^k = T^l$. By (i), $\mathcal{R}^k = \mathcal{R}^l$. On the one hand, the same argument as for Case 1 leads to $\hat{c}^l = \hat{c}^k$ and $x_r^l = x_r^k$ for all r in $\mathcal{R}^k = \mathcal{R}^l$. On the other hand, $x_r^l = x_r^k = 0$ for all r in $\mathcal{R} \setminus \mathcal{R}^k$.

(iv) Suppose that $T^k < T^l$. According to (i), $\mathcal{R}^k \subset \mathcal{R}^l$.

Set $\tilde{Y}^l = \sum_{r\in\mathcal{R}^k} x_r^l c_r(x_r)/\sum_{r\in\mathcal{R}^k} x_r^l$, the average cost to coalition l on $\mathcal{R}^k$. By Lemma 2.7, the arcs in $\mathcal{R}^k$ cost strictly less than those in $\mathcal{R} \setminus \mathcal{R}^k$. One deduces that $\tilde{Y}^l = Y^l$ if $\mathcal{R}^k$ is equal to $\mathcal{R}^l$, and $\tilde{Y}^l < Y^l$ if $\mathcal{R}^k$ is a proper subset of $\mathcal{R}^l$.

Now, let us show that $Y^k \leq \tilde{Y}^l$:

$$\begin{aligned}
\tilde{Y}^l &= \frac{\sum_{r\in\mathcal{R}^k} x_r^l c_r(x_r)}{\sum_{r\in\mathcal{R}^k} x_r^l} \\
&= \frac{\sum_{r\in\mathcal{R}^k} x_r^k c_r(x_r) + \sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)c_r(x_r)}{\sum_{r\in\mathcal{R}^k} x_r^k + \sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)} \\
&= \frac{Y^k T^k + \sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)c_r(x_r)}{T^k + \sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)}.
\end{aligned}$$

It follows from (iii) that, for all r in $\mathcal{R}^k$, $x_r^l - x_r^k > 0$. The relation $Y^k \leq \tilde{Y}^l$ is thus equivalent to the inequality

$$Y^k \leq \frac{\sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)c_r(x_r)}{\sum_{r\in\mathcal{R}^k}(x_r^l - x_r^k)}. \tag{2.9}$$

For r in $\mathcal{R}^k$, $x_r^l - x_r^k = (\hat{c}^l - \hat{c}^k)/c_r'(x_r)$ and $c_r(x_r) = \hat{c}^k - x_r^k c_r'(x_r)$. Inequality (2.9) can thus be written as

$$\begin{aligned}
&\frac{\sum_{r\in\mathcal{R}^k} x_r^k c_r(x_r)}{\sum_{r\in\mathcal{R}^k} x_r^k} \leq \frac{\sum_{r\in\mathcal{R}^k} c_r(x_r)(\hat{c}^l - \hat{c}^k)/c_r'(x_r)}{\sum_{r\in\mathcal{R}^k}(\hat{c}^l - \hat{c}^k)/c_r'(x_r)} = \frac{\sum_{r\in\mathcal{R}^k} c_r(x_r)/c_r'(x_r)}{\sum_{r\in\mathcal{R}^k} 1/c_r'(x_r)} \\
\Leftrightarrow\ & \sum_{r\in\mathcal{R}^k} x_r^k c_r(x_r) \sum_{r\in\mathcal{R}^k} \frac{1}{c_r'(x_r)} \leq \sum_{r\in\mathcal{R}^k} x_r^k \sum_{r\in\mathcal{R}^k} \frac{c_r(x_r)}{c_r'(x_r)} \\
\Leftrightarrow\ & \sum_{r\in\mathcal{R}^k} x_r^k \left(\hat{c}^k - x_r^k c_r'(x_r)\right) \sum_{r\in\mathcal{R}^k} \frac{1}{c_r'(x_r)} \leq \sum_{r\in\mathcal{R}^k} x_r^k \sum_{r\in\mathcal{R}^k} \frac{\hat{c}^k - x_r^k c_r'(x_r)}{c_r'(x_r)} \\
\Leftrightarrow\ & \sum_{r\in\mathcal{R}^k} \left(x_r^k\right)^2 c_r'(x_r) \sum_{r\in\mathcal{R}^k} \frac{1}{c_r'(x_r)} \geq \left(\sum_{r\in\mathcal{R}^k} x_r^k\right)^2.
\end{aligned} \tag{2.10}$$

Inequality (2.10) follows from Cauchy-Schwarz inequality. Furthermore, the equality holds (or, equivalently, $Y^k = \tilde{Y}^l$) if, and only if, $x_r^k c_r'(x_r)$ is constant for all r in $\mathcal{R}^k$. When this is the case, $c_r(x_r) = \hat{c}^k - x_r^k c_r'(x_r)$ is also a constant for all r in $\mathcal{R}^k$. According to Lemma 2.8 (iii), this constant must be equal to c^0.

The relations $Y^k(\mathbf{x}) \leq \tilde{Y}^l(\mathbf{x}) \leq Y^l(\mathbf{x})$ is now established. Suppose, moreover, that $Y^k(\mathbf{x}) = Y^l(\mathbf{x})$. On the one hand, $\tilde{Y}^l(\mathbf{x}) = Y^l(\mathbf{x})$, implying that $\mathcal{R}^k = \mathcal{R}^l$. On the other hand, $Y^k(\mathbf{x}) = \tilde{Y}^l(\mathbf{x})$, implying that every arc in $\mathcal{R}^k$ costs c^0.

Suppose that $T^k = T^l$. The result follows directly from (iii). □

Remark 2.10. (i) and (iii) of Lemma 2.9 were also proven by Orda, Rom and Shimkin [66] with another formulation for atomic splittable games. Lemma 1 in [66] claims that, at the NE, if $x_r^k < x_r^l$ for some arc r, then $x_s^k \leq x_s^l$ for all arc s, and the inequality is strict if $x_s^k > 0$.

The following corollary of Lemma 2.9 shows that the behavior of a coalition at the CE is specified by its weight.

Corollary 2.11. *Two coalitions send the same weight on every arc if, and only if, they have the same weight. In this case, they have the same average cost.*

2.4 Comparison between CE and WE

The previous section was contributed to the basic properties of the CE $\mathbf{x}$ of the game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$. This section will compare it with the WE $\mathbf{w} = (w_r)_{r\in\mathcal{R}}$ of the corresponding nonatomic game $\Gamma(\mathcal{R}, \mathbf{c})$. The equilibrium cost at $\mathbf{w}$ is denoted by $W \in \mathbb{R}$. One says that $\mathbf{x}$ *induces* $\mathbf{w}$ if $\mathbf{x}' = \mathbf{w}$, i.e. $x_r = w_r$ for all $r \in \mathcal{R}$.

Following Hayrapetyan, Tardos and Wexler [37], let $\mathcal{R}_- = \{r \in \mathcal{R} \,|\, x_r < w_r\}$, $\mathcal{R}_+ = \{r \in \mathcal{R} \,|\, x_r > w_r\}$ and $\mathcal{R}_J = \{r \in \mathcal{R} \,|\, x_r = w_r\}$ be, respectively, the set of *underloaded arcs*, the set of *overloaded arcs* and the set of *justly-loaded arcs*.

Lemma 2.12. *If* $\mathbf{x}$ *does not induce* $\mathbf{w}$*, then the following are true:*

(i) for all $s \in \mathcal{R}_-$ *and for all* $r \in \mathcal{R}_+$, $c_s(x_s) < W < c_r(x_r)$*;*

(ii) $\mathcal{R}^0 \subset \mathcal{R}_-$*, i.e. for all* $r \in \mathcal{R}$*, if* $x_r^0 > 0$*, then* $x_r < w_r$*;*

(iii) $\mathcal{R}_+ \subset \mathcal{R}^1$*, i.e. for all* $r \in \mathcal{R}$*, if* $x_r > w_r$*, then* $x_r^1 > 0$*.*

Proof. (i) As $\mathbf{x}' \neq \mathbf{w}$, both $\mathcal{R}_-$ and $\mathcal{R}_+$ are nonempty. Take s in $\mathcal{R}_-$ and r in $\mathcal{R}_+$, then $w_s > x_s \geq 0$ and $w_r < x_r$. In particular, $w_s > 0$, which implies that s is used at the WE. Then, $c_s(x_s) < c_s(w_s) = W \leq c_r(w_r) < c_r(x_r)$.

(ii) The individuals take the arcs of the lowest cost at $\mathbf{x}$. According to (i), these arcs must be in $\mathcal{R}_-$, hence $\mathcal{R}^0 \subset \mathcal{R}_-$.

(iii) For all r in $\mathcal{R}_+$, r is used at $\mathbf{x}$ because $x_r > w_r \geq 0$,. According to Lemma 2.8 and Lemma 2.9, it is used by the largest coalition, coalition 1. Thus, $\mathcal{R}_+ \subset \mathcal{R}^1$. □

The following theorem compares the equilibrium costs at $\mathbf{x}$ with the equilibrium cost at $\mathbf{w}$.

Theorem 2.13. *If* $\mathbf{x}$ *does not induce* $\mathbf{w}$*, then* $Y^0(\mathbf{x}) < W$ *and* $Y^k(\mathbf{x}) < W$ *for each coalition* k*. Consequently,* $Y(\mathbf{x}) < W$*.*

Proof. For the individuals, for all $r \in \mathcal{R}^0$, $c_r(x_r) = Y^0(\mathbf{x})$. Lemma 2.12(ii) implies that r is in $\mathcal{R}_-$, and Lemma 2.12(i) shows that $c_r(x_r) < W$.

For the coalitions, it is enough to show that $Y^1(\mathbf{x}) < W$ for the largest coalition, coalition 1. Once this is proven, the remaining results follow from Lemma 2.9.

Let us define an auxiliary flow $\mathbf{z}$ in F, such that it induces $\mathbf{w}$ and satisfies the following conditions:

$$\begin{cases} z_r^1 > x_r^1, \quad z_r^k \geq x_r^k, \quad z_r^0 = x_r^0, & k \in \mathcal{K} \setminus \{1\}, \quad r \in \mathcal{R}_-, \\ z_r^1 < x_r^1, \quad z_r^k \leq x_r^k, \quad z_r^0 = x_r^0, & k \in \mathcal{K} \setminus \{1\}, \quad r \in \mathcal{R}_+, \\ z_r^k = z_r^k, & k = 0 \text{ or } k \in \mathcal{K}, \quad r \in \mathcal{R}_J. \end{cases}$$

For example, one can define, for all $k \in \mathcal{K}$ and $r \in \mathcal{R}_+$, $z_r^k = x_r^k - d_r^k$, where $d_r^k = (x_r - w_r)x_r^k / \sum_{l\in\mathcal{K}} x_r^l$, while for all $r \in \mathcal{R}_-$, $z_r^k = x_r^k + d_r^k$, where $d_r^k = (w_r - x_r) \sum_{t\in\mathcal{R}_+} d_t^k / \sum_{s\in\mathcal{R}_-}(w_s - x_s)$. The above conditions are satisfied according to Lemma 2.12(iii).

Let us define another auxiliary flow $\mathbf{y}$ in F as follows. For all $r \in \mathcal{R}$,

$$y_r^k = \begin{cases} z_r^k, & \text{if } k \neq 1, \\ x_r^1, & \text{if } k = 1. \end{cases}$$

In other words, at $\mathbf{y}$, the individuals and all coalitions, except coalition 1, behave like at $\mathbf{w}$, while coalition 1 behaves like at $\mathbf{x}$. Let us show that $u^1(\mathbf{x}) \leq u^1(\mathbf{y}) < u^1(\mathbf{z})$.

Some preliminary results are needed.

For all $s \in \mathcal{R}_-$ and for all $r \in \mathcal{R}_+$,
i) $y_s \geq x_s$, because $y_s - x_s = \sum_{\mathcal{K}\setminus\{1\}} z_s^k - x_s^k \geq 0$:
ii) $x_r \geq y_r$, because $x_r - y_r = \sum_{\mathcal{K}\setminus\{1\}} x_r^k - z_r^k \geq 0$:
iii) by Lemma 2.12(iii), coalition 1 takes arc r. Thus,

$$c_r(x_r) + x_r^1 c_r'(x_r) = \hat{c}^1(\mathbf{x}) \leq c_s(x_s) + x_s^1 c_s'(x_s). \tag{2.11}$$

Moreover, according to Assumption 2.1, for all $0 < x < x_r$ and $y > 0$,

$$c_r(x_r - x) + (x_r^1 - x)\, c_r'(x_r - x) < \hat{c}^1(\mathbf{x}) < c_s(x_s + y) + (x_s^1 + y)\, c_s'(x_s + y). \tag{2.12}$$

Finally, $c_s(x_s) < c_r(x_r)$ by Lemma 2.12(i). Then, it follows from (2.11) that $x_r^1 c_r'(x_r) < x_s^1 c_s'(x_s)$. Let B be a constant such that $\max_{t\in\mathcal{R}_+}\{x_t^1\, c_t'(x_t)\} \leq B \leq \min_{t\in\mathcal{R}_-}\{x_t^1\, c_t'(x_t)\}$. Then, by Assumption 2.1, for all x and y such that $0 \leq x < x_r$ and $y > x_s$,

$$x_r^1\, c_r'(x) \leq B \leq x_s^1\, c_s'(y). \tag{2.13}$$

Now, let us show that $u^1(\mathbf{x}) \leq u^1(\mathbf{y}) < u^1(\mathbf{z})$:

$$\begin{aligned}
u^1(\mathbf{y}) - u^1(\mathbf{x}) &= \sum_{r\in\mathcal{R}} y_r^1 c_r(y_r) - \sum_{r\in\mathcal{R}} x_r^1 c_r(x_r) = \sum_{r\in\mathcal{R}} x_r^1 c_r(y_r) - \sum_{r\in\mathcal{R}} x_r^1 c_r(x_r) \\
&= \sum_{s\in\mathcal{R}_-} \left[\, x_s^1 c_s(y_s) - x_s^1 c_s(x_s)\,\right] - \sum_{r\in\mathcal{R}_+} \left[\, x_r^1 c_r(x_r) - x_r^1 c_r(y_r)\,\right] \\
&= \sum_{s\in\mathcal{R}_-} \int_{x_s}^{y_s} x_s^1\, c_s'(x)\, \mathrm{d}x - \sum_{r\in\mathcal{R}_+} \int_{y_r}^{x_r} x_r^1\, c_r'(x)\, \mathrm{d}x \\
&\geq \sum_{s\in\mathcal{R}_-} (y_s - x_s)B - \sum_{r\in\mathcal{R}_+} (x_r - y_r)B = \sum_{r\in\mathcal{R}} (y_r - x_r)B = 0,
\end{aligned}$$

where the inequality is due to (2.13) and the fact that $y_s \geq x_s$ for all s in $\mathcal{R}_-$ and $x_r \geq y_r$ for all r in $\mathcal{R}_+$;

$$\begin{aligned}
u^1(\mathbf{z}) - u^1(\mathbf{y}) &= \sum_{r\in\mathcal{R}} z_r^1 c_r(w_r) - \sum_{r\in\mathcal{R}} y_r^1 c_r(y_r) = \sum_{r\in\mathcal{R}} z_r^1 c_r(w_r) - \sum_{r\in\mathcal{R}} x_r^1 c_r(w_r - z_r^1 + x_r^1) \\
&= \sum_{s\in\mathcal{R}_-} \left[\, z_s^1 c_s(w_s) - x_s^1 c_s(w_s - z_s^1 + x_s^1)\,\right] - \sum_{r\in\mathcal{R}_+} \left[\, x_r^1 c_r(w_r - z_r^1 + x_r^1) - z_r^1 c_r(w_r)\,\right] \\
&= \sum_{s\in\mathcal{R}_-} \int_{x_s^1}^{z_s^1} \frac{\partial}{\partial x}\left[\, x c_s(w_s - z_s^1 + x)\,\right] \mathrm{d}x - \sum_{r\in\mathcal{R}_+} \int_{z_r^1}^{x_r^1} \frac{\partial}{\partial x}\left[\, x c_r(w_r - z_r^1 + x)\,\right] \mathrm{d}x \\
&= \sum_{s\in\mathcal{R}_-} \int_{x_s^1}^{z_s^1} \left[\, c_s(w_s - z_s^1 + x) + x c_s'(w_s - z_s^1 + x)\,\right] \mathrm{d}x \\
&\quad - \sum_{r\in\mathcal{R}_+} \int_{z_r^1}^{x_r^1} \left[\, c_r(w_r - z_r^1 + x) + x c_r'(w_r - z_r^1 + x)\,\right] \mathrm{d}x \\
&\geq \sum_{s\in\mathcal{R}_-} \int_{x_s^1}^{z_s^1} \left[\, c_s(x_s - x_s^1 + x) + x c_s'(x_s - x_s^1 + x)\,\right] \mathrm{d}x \qquad (2.14) \\
&\quad - \sum_{r\in\mathcal{R}_+} \int_{z_r^1}^{x_r^1} \left[\, c_r(x_r - x_r^1 + x) + x c_r'(x_r - x_r^1 + x)\,\right] \mathrm{d}x \\
&> \sum_{s\in\mathcal{R}_-} (z_s^1 - x_s^1)\, \hat{c}^1(\mathbf{x}) - \sum_{r\in\mathcal{R}_+} (x_r^1 - z_r^1)\, \hat{c}^1(\mathbf{x}) \qquad (2.15) \\
&= \sum_{r\in\mathcal{R}} (z_r^1 - x_r^1)\, \hat{c}^1(\mathbf{x}) = (T^1 - T^1)\, \hat{c}^1(\mathbf{x}) = 0.
\end{aligned}$$

Inequality (2.14) is due to the following facts which follow immediately from the definition of $\mathbf{z}$. For s in $\mathcal{R}_-$, $z_s^1 > x_s^1$ and $w_s - z_s^1 \geq x_s - x_s^1$, while for r in $\mathcal{R}_+$, $x_r^1 > z_r^1$ and $x_r - x_r^1 \geq w_r - z_r^1$. Inequality (2.15) is due to (2.12).

Thus, one has proved that $u^1(\mathbf{x}) < u^1(\mathbf{z})$ or, equivalently, $Y^1(\mathbf{x}) < Y^1(\mathbf{z})$. Besides, every arc used at $\mathbf{z}$ costs W because $\mathbf{z}$ induces $\mathbf{w}$. Therefore, $Y^1(\mathbf{z}) = W$, which completes the proof. □

Remark 2.14. Cominetti, Correa and Stier-Moses [20, Corollary 4.1] proved that, if the cost functions are non-decreasing, convex and differentiable, and the atomic splittable players are identical, then the social cost at any NE in an atomic splittable game is lower than that at the corresponding WE. Hayrapetyan, Tardos and Wexler [37, Theorem 2.3] proved that, in a two-terminal parallel-arc network, if the cost functions are non-decreasing, convex and differentiable, then the social cost at any NE in an atomic splittable game is lower than that at the corresponding WE. In this work, stronger convexity conditions on the cost functions allow to prove that not only the social average cost, but also the average cost to any coalition and the individuals' cost are lower at the CE than at the WE.

Remark 2.15. Cominetti, Correa and Stier-Moses [20, §2.1] provided an example where two groups of individuals have *different* origin/destination pairs. They showed that, when one of the two groups forms a coalition, both the social cost and the average cost to this coalition are increased. Altman, Kameda and Hayerl [5] showed that a similar phenomenon can exist when atomic players with splittable flow form a coalition. These imply that further studies are needed for more general cases where the network is not two-terminal parallel-arc type. In particular, the study on *price of collusion*, a notion introduced by Hayrapetyan, Tardos and Wexler [37] and developped by Altman, Kameda and Hayerl [5], should be carried further.

2.5 Impact of the composition of the population on the CE costs

This section focuses on the relation between the costs at the CE and the composition of the set of the players, i.e. its partition into coalitions and individuals. In the first part, one considers a unique coalition of weight $T \in [0, 1]$, and studies the variation of the coalition's cost and the remaining individuals' cost with respect to T. In the second part, for a general composition of the set of the players, one shows that, whenever a coalition decreases, i.e. some of its members become individuals, the individuals' cost is increased.

2.5.1 CE costs as functions of the size of the unique coalition

Suppose that a unique coalition of weight $T \in [0, 1]$ is formed.

Every horizontal line in Figure 2.1 represents a composition of the set of the players: the unique coalition is presented by the plain part on the left, and the individuals by the dashed part on the right. From bottom to top, the unique coalition decreases. The top (dashed) line stands for the WE $\mathbf{w}$, the bottom (plain) line stands for the social optimum, and any horizontal line between them stands for the CE of a one-coalition composite game.

Lemma 2.16. *There exists a number $\tilde{T}$ in $[0, 1]$ such that the CE in $\Gamma(\mathcal{R}, \mathbf{c}, (1-T; T))$ induces $\mathbf{w}$ if, and only if, $T \leq \tilde{T}$.*

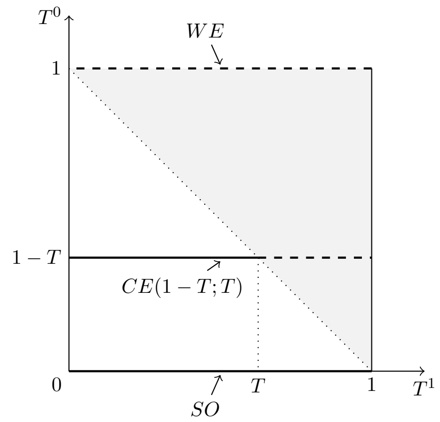


Figure 2.1: Composition of the players

Proof. Let $\mathcal{R}_a = \{r \in \mathcal{R} \,|\, w_r > 0\}$ be the set of used arcs at $\mathbf{w}$, and $\mathcal{R}_i = \mathcal{R} \backslash \mathcal{R}_a = \{r \in \mathcal{R} \,|\, w_r = 0\}$ the set of unused arcs, which may be empty. Set $A = \sum_{r \in \mathcal{R}_a} \frac{1}{c_r'(w_r)}$. Then, the following constant

$$\tilde{T} = \min\Big\{\min_{r \in \mathcal{R}_a} w_r c_r'(w_r) A,\ \min_{r \in \mathcal{R}_i} (c_r(0) - W)\, A\Big\}, \tag{2.16}$$

is the threshold. This can be proven in two cases.

CASE 1. For all $r \in \mathcal{R}_i$, $c_r(0) > W$.

In this case, $\tilde{T} > 0$. Let us show that the CE induces $\mathbf{w}$ if, and only if, $T \leq \tilde{T}$.

On the one hand, if $T \leq \tilde{T}$, the following flow $\mathbf{x}$ is the CE:

$$\begin{cases} x_r^1 = \frac{T}{Ac_r'(w_r)}, \quad x_r^0 = w_r - x_r^1, & r \in \mathcal{R}_a; \\ x_r^1 = x_r^0 = 0, & r \in \mathcal{R}_i. \end{cases}$$

Indeed, $\mathbf{x}$ is well-defined because of the definition of $\tilde{T}$, and $\sum_{r \in \mathcal{R}} x_r^1 = T$. Next, as $x_r = w_r$ for all r, the individuals do take the arcs of the lowest cost. Finally, it follows from the definition of $\tilde{T}$ that, for all $r \in \mathcal{R}_i$, $(c_r(0) - W)A \geq \tilde{T} \geq T$. It is not difficult to see that, for all $r \in \mathcal{R}_a$, $c_r(x_r) + x_r^1 c_r'(w_r) = W + \frac{T}{A}$ and, for all $r \in \mathcal{R}_i$, $c_r(0) \geq W + \frac{T}{A}$. The equilibrium condition (2.2) is thus satisfied for the coalition. One deduces that $\mathbf{x}$ is the CE. Besides, it induces a WE, because $x_r = w_r$ for all r.

On the other hand, if the CE $\mathbf{x}$ induces $\mathbf{w}$, i.e. $x_r = w_r$ for all r, then for all $r \in \mathcal{R}_i$, $x_r = w_r = 0$, which implies that there exists an arc $s \in \mathcal{R}_a$ such that $x_s^1 > 0$. However, for all $r \in \mathcal{R}_a$ such that $x_r^1 = 0$, one has $W = c_r(w_r) = c_r(x_r) \geq \hat{c}^1(\mathbf{x}) = c_s(x_s) + x_s^1 c_s'(x_s) > c_s(x_s) = c_s(w_s) = W$, a contradiction. Therefore, for all $r \in \mathcal{R}_a$, $x_r^1 > 0$ and $c_r(x_r) + x_r^1 c_r'(x_r) = \hat{c}^1(\mathbf{x})$. As a result, $x_r^1 = \frac{\hat{c}^1(\mathbf{x}) - c_r(x_r)}{c_r'(x_r)} = \frac{\hat{c}^1(\mathbf{x}) - W}{c_r'(w_r)}$. The constraint $x_r^1 \leq w_r$ implies that $\hat{c}^1 - W < w_r c_r'(w_r)$. Consequently, $T = \sum_{r \in \mathcal{R}_a} x_r^1 = \sum_{r \in \mathcal{R}_a} \frac{\hat{c}^1 - W}{c_r'(w_r)} = (\hat{c}^1 - W)A < w_r c_r'(w_r)$, for all $r \in \mathcal{R}_a$.

Besides, for all $r \in \mathcal{R}_i$, $\hat{c}^1 \leq c_r(0)$, which implies that $T = (\hat{c}^1 - W)A \leq (c_r(0) - W)A$. Thus, $T \leq \tilde{T}$ is proven.

CASE 2. There exists some $t \in \mathcal{R}_i$ such that $c_t(0) = W$.

In this case, $\tilde{T} = 0$. Let us show that the CE does not induce a WE as long as $T > 0$. Otherwise, suppose that for some $T > 0$, the CE $\mathbf{x}$ induces a WE. By the same reasoning as in Case 1, there exists an arc $s \in \mathcal{R}_a$ such that $x_s^1 > 0$. Then, $c_t(0) = W = c_s(w_s) < c_s(w_s) + x_s^1 c_s'(w_s) = \hat{c}^1$. However, $x_t = w_t = 0$, which implies that $c_t(0) \geq \hat{c}^1$, a contradiction. □

Example 2.17. There are two parallel arcs r_1 and r_2, whose cost functions are, respectively, $c_1(x) = x + 10$ and $c_2(x) = 10x + 1$. A computation shows that the threshold is $\tilde{T} = \frac{1}{10}$. If less than one tenth of the players join the coalition, the coalition changes actually nothing in the game equilibrium.

Theorem 2.18. *Let $\tilde{T}$ be defined by (2.16). The individuals' cost $Y^0(T)$, the average cost to the unique coalition $Y^1(T)$, and the social cost $Y(T)$ in $\Gamma(\mathcal{R}, \mathbf{c}, (1-T;\, T))$ have the following properties:*

(i) for $T \in [0, \tilde{T}]$, $Y^0(T) = Y^1(T) = Y(T) = W$;

(ii) for $T \in (\tilde{T}, 1]$, $Y^0(T) < Y^1(T) < W$, $Y(T) < W$. In particular, $Y^1(1) < Y^0(0) = W$;

(iii) $Y^0(T)$, $Y^1(T)$ and $Y(T)$ are all strictly decreasing with respect to T on $[\tilde{T}, 1]$.

Proof. First, notice that $Y^1(T)$ is not defined for $T = 0$, and that $Y^0(T)$ is not defined for $T = 1$. However, as the cost functions satisfy Assumptions 2.1, one can extend $Y^1(T)$ to $T = 0$ and $Y^0(T)$ to $T = 1$ without difficulty.

(i) See Lemma 2.16.

(ii) According to Lemma 2.16, when $\tilde{T} < T \leq 1$, the CE does not induce $\mathbf{w}$. Therefore, according to Lemma 2.8(iii) and Theorem 2.13, $Y^0(T) \leq Y^1(T) < W$. It remains to show that $Y^0(T) < Y^1(T)$.

Suppose that $Y^0(T) = Y^1(T)$. Then, $Y^0(T) = \underline{Y}^1(T) = Y^1(T)$ by Lemma 2.8, where $\underline{Y}^1(T)$ is the lowest cost of the arcs used by the coalition. This means that every arc used by the coalition has the lowest cost $Y^0(T)$. Therefore, the CE does induce $\mathbf{w}$, a contradiction.

(iii) Suppose that $\tilde{T} < S < T \leq 1$. Let $\mathbf{x} = (\mathbf{x}^0, \mathbf{x}^1)$ and $\mathbf{y} = (\mathbf{y}^0, \mathbf{y}^1)$ be, respectively, the CE of the game $\Gamma(\mathcal{R}, \mathbf{c}, (1-T;\, T))$ and that of the game $\Gamma(\mathcal{R}, \mathbf{c}, (1-S;\, S))$. The other notations are as listed at the beginning of Section 2.3.

Let $\mathcal{R}_-$ be the set of underloaded or justly-loaded arcs, and $\mathcal{R}_+ = \mathcal{R} \setminus \mathcal{R}_-$ be the set of overloaded arcs. In other words,

$$\mathcal{R}_- = \{r \in \mathcal{R} \,|\, y_r \leq x_r\}, \quad \mathcal{R}_+ = \{r \in \mathcal{R} \,|\, y_r > x_r\}.$$

If $\mathcal{R}_+ = \emptyset$, then $\mathbf{x}' = \mathbf{y}'$, i.e. $x_r = y_r$ for all $r \in \mathcal{R}$. Let us prove that this is impossible.

Define $\mathcal{R}^\flat = \{r \in \mathcal{R} \,|\, c_r(x_r) = c^0(\mathbf{x})\}$ and $\mathcal{R}^\sharp = \{r \in \mathcal{R} \,|\, x_r > 0,\, c_r(x_r) > c^0(\mathbf{x})\}$. Then, for all $r \in \mathcal{R}^\sharp, x_r^0 = 0$. As $T > S > \tilde{T}$, according to (ii), $\mathbf{x}$ and $\mathbf{y}$ do not induce $\mathbf{w}$. Therefore, $\mathcal{R}^\sharp$ is nonempty.

For all $r \in \mathcal{R}^\flat$, $c_r(y_r) = c_r(x_r) = c^0(\mathbf{x})$ while, for all $r \in \mathcal{R}^\sharp$, $c_r(y_r) = c_r(x_r) > c^0(\mathbf{x})$. Hence, $c^0(\mathbf{x})$ is the minimal arc cost at $\mathbf{y}$. One deduces that $c^0(\mathbf{y}) = c^0(\mathbf{x})$ and, for all $r \in \mathcal{R}^\sharp$, $y_r^0 = 0, y_r^1 = y_r = x_r$.

On the one hand, $\hat{c}^1(\mathbf{x})$ and $\hat{c}^1(\mathbf{y})$ are both equal to $c_r(x_r) + x_r c_r'(x_r)$ for all $r \in \mathcal{R}^\sharp$. On the other hand, for all $r \in \mathcal{R}^\flat$, as $\hat{c}^1(\mathbf{x}) = c_r(x_r) + x_r^1 c_r'(x_r)$ and $\hat{c}^1(\mathbf{y}) = c_r(x_r) + y_r^1 c_r'(x_r)$, it follows from $\hat{c}^1(\mathbf{x}) = \hat{c}^1(\mathbf{y})$ that $x_r^1 = y_r^1$. Therefore, $T = \sum_{r\in\mathcal{R}^\flat} x_r^1 + \sum_{r\in\mathcal{R}^\sharp} x_r = \sum_{r\in\mathcal{R}^\flat} y_r^1 + \sum_{r\in\mathcal{R}^\sharp} y_r = S$, a contradiction. Hence, $\mathcal{R}_+ \neq \emptyset$, and there exists some $r \in \mathcal{R}_-$ such that $y_r < x_r$.

Now, we will show that $Y^0(T) < Y^0(S)$, $Y^1(T) < Y^1(S)$ and $Y(T) < Y(S)$ in eight steps.

(a) Let us prove that there exists some $s \in \mathcal{R}_+$ such that $y_s^0 > 0$.

If for all $s \in \mathcal{R}_+$, $y_s^0 = 0$, then $y_s^1 = y_s > x_s \geq x_s^1$ and, consequently, $\hat{c}^1(\mathbf{y}) = c_s(y_s) + y_s^1 c'(y_s) > c_r(x_s) + x_s^1 c'(x_s) = \hat{c}^1(\mathbf{x})$. Moreover, $\sum_{s\in\mathcal{R}_+} y_s^1 > \sum_{s\in\mathcal{R}_+} x_s^1$. But $\sum_{r\in\mathcal{R}} y_r^1 = S <$

$T = \sum_{r\in\mathcal{R}} x_r^1$. Therefore, $\sum_{t\in\mathcal{R}_-} y_t^1 < \sum_{t\in\mathcal{R}_-} x_t^1$. In particular, there exists some $r \in \mathcal{R}_-$ such that $y_r^1 < x_r^1$. Because $y_r \le x_r$, $\hat{c}^1(\mathbf{y}) \le c_r(y_r) + y_r^1 c'(y_r) < c_r(x_r) + x_r^1 c'(x_r) = \hat{c}^1(\mathbf{x})$, a contradiction.

(b) Let us show that $c^0(\mathbf{y}) > c^0(\mathbf{x})$.

Choose the previous $s \in \mathcal{R}_+$ with $y_s^0 > 0$, and recall that $y_s > x_s$. Then, $Y^0(S) = c^0(\mathbf{y}) = c_s(y_s) > c_s(x_s) \ge c^0(\mathbf{x}) = Y^0(T)$. One deduces that $Y^0(T)$ is strictly decreasing in T on $[\tilde{T}, 1]$.

(c) For all $r \in \mathcal{R}_-$, $x_r^0 = 0$, because $y_r \le x_r$ and, consequently, $c_r(x_r) \ge c_r(y_r) \ge c^0(\mathbf{y}) > c^0(\mathbf{x})$.

(d) Let us show that $\hat{c}^1(\mathbf{y}) < \hat{c}^1(\mathbf{x})$.

Recall that there exists $r \in \mathcal{R}_-$ such that $y_r < x_r$. Then, $y_r^1 \le y_r < x_r$, and $x_r^1 = x_r$ according to (c). Therefore, $\hat{c}^1(\mathbf{y}) \le c_r(y_r) + y_r^1 c'(y_r) < c_r(x_r) + x_r^1 c'(x_r) = \hat{c}^1(\mathbf{x})$.

(e) One can show that, for all $s \in \mathcal{R}_+$, $y_s^1 < x_s^1$ and, consequently, $y_s^0 > 0$, $x_s^1 > 0$.

Indeed, for all $s \in \mathcal{R}_+$, $y_s > x_s \ge 0$, hence $y_s^1 > 0$. If there exists some $s \in \mathcal{R}_+$ such that $y_s^1 \ge x_s^1$, then $\hat{c}^1(\mathbf{y}) = c_s(y_s) + y_s^1 c'(y_s) > c_s(x_s) + x_s^1 c'(x_s) \ge \hat{c}^1(\mathbf{x})$, i.e. $\hat{c}^1(\mathbf{y}) > \hat{c}^1(\mathbf{x})$. This contradicts (d).

It follows from the fact that $y_s > x_s$ that $y_s^0 > x_s^0 \ge 0$. Besides, $x_s^1 > y_s^1 \ge 0$.

(f) For all $r \in \mathcal{R}_-$ and $s \in \mathcal{R}_+$, $c_r(x_r) > c_s(x_s)$, because $c_r(x_r) \ge c_r(y_r) \ge c^0(\mathbf{y}) = c_s(y_s) > c_s(x_s)$.

(g) Let us define an auxiliary flow $\mathbf{z}$ in the game $\Gamma(\mathcal{R}, \mathbf{c}, (1-T;\, T))$ by

$$\begin{cases} z_s^1 = y_s - x_s^0,\ z_s^0 = x_s^0, & s \in \mathcal{R}_+; \\ z_r^1 = y_r,\ z_r^0 = 0, & r \in \mathcal{R}_-. \end{cases}$$

Clearly, $\mathbf{z}' = \mathbf{y}'$, i.e. for all $r \in \mathcal{R}$, $z_r = y_r$, and

$$\begin{aligned} &x_s^1 < z_s^1 \le y_s, \quad z_s^0 = x_s^0, && s \in \mathcal{R}_+, \\ &z_r^1 = z_r \le x_r = x_r^1, \quad z_r^0 = x_r^0, && r \in \mathcal{R}_-. \end{aligned}$$

Now, we are ready to prove that the total cost to the coalition of weight T at $\mathbf{z}$ is higher than that at $\mathbf{x}$, i.e. $u_T^1(\tilde{\mathbf{y}}) > u_T^1(\mathbf{x})$ (the subscript T is added to stress the weight of the coalition in question). Indeed, for all $s \in \mathcal{R}_+$ and for all $r \in \mathcal{R}_-$ such that $x_r > 0$,

$$\hat{c}^1(\mathbf{x}) = c_s(x_s) + x_s^1 c'(x_s) = c_r(x_r^1) + x_r^1 c'(x_r^1). \tag{2.17}$$

One deduces that, for all $s \in \mathcal{R}_+$, $r \in \mathcal{R}_-$ such that $x_r > 0$, and for all $x > x_s^1$ and y such that $0 \le y \le x_r^1$,

$$c_s(x + x_s^0) + x\, c_s'(x + x_s^0) > \hat{c}^1(\mathbf{x}) \ge c_r(y) + y\, c_r'(y). \tag{2.18}$$

Then,

$$\begin{aligned} u_T^1(\mathbf{z}) - u_T^1(\mathbf{x}) &= \sum_{s\in\mathcal{R}_+} \left[\, z_s^1 c_s(z_s) - x_s^1 c_s(x_s) \,\right] - \sum_{r\in\mathcal{R}_-} \left[\, x_r^1 c_r(x_r^1) - z_r^1 c_r(z_r^1) \,\right] \\ &= \sum_{s\in\mathcal{R}_+} \int_{x_s^1}^{z_s^1} \frac{\partial}{\partial x} [x\, c_s(x + x_s^0)]\, \mathrm{d}x - \sum_{r\in\mathcal{R}_-} \int_{z_r^1}^{x_r^1} \frac{\partial}{\partial x} [x\, c_r(x)]\, \mathrm{d}x \\ &= \sum_{s\in\mathcal{R}_+} \int_{x_s^1}^{z_s^1} [c_s(x + x_s^0) + x\, c_s'(x + x_s^0)]\, \mathrm{d}x - \sum_{r\in\mathcal{R}_-} \int_{z_r^1}^{x_r^1} [c_r(x) + x\, c_r'(x)]\, \mathrm{d}x \\ &> \sum_{s\in\mathcal{R}_+} (z_s^1 - x_s^1)\, \hat{c}^1(\mathbf{x}) - \sum_{r\in\mathcal{R}_-} (x_r^1 - z_r^1)\, \hat{c}^1(\mathbf{x}) = \sum_{r\in\mathcal{R}} (z_r^1 - x_r^1)\, \hat{c}^1(\mathbf{x}) = (T - T)\, \hat{c}^1(\mathbf{x}) = 0. \end{aligned}$$

The inequality above is due to (2.18).

Next, notice the following three facts.

1) For all $r \in \mathcal{R}$, $z_r = y_r$ by the definition of $\mathbf{z}$,

2) For all $s \in \mathcal{R}_+$, $c_s(y_s) = c^0(\mathbf{y})$ by (e), and

3) For all $r \in \mathcal{R}_-$, either $z_r^1 = y_r = y_r^1$ or $z_r^1 = y_r > y_r^1$. In the second case, $y_r^0 > 0$, which implies that $c_r(y_r) = c^0(\mathbf{y})$.

These facts induce the relation between the total cost to the coalition of weight T at $\mathbf{z}$ and the total cost to the coalition of weight S at $\mathbf{y}$:

$$\begin{aligned}
u_T^1(\mathbf{z}) &= \sum_{s\in\mathcal{R}_+} z_s^1\, c_s(z_s) + \sum_{r\in\mathcal{R}_-} z_r^1\, c_r(z_r) \\
&= \sum_{s\in\mathcal{R}_+} [\, y_s^1 + (z_s^1 - y_s^1)\,] c_s(z_s) + \sum_{r\in\mathcal{R}_-} [\, y_r^1 + (z_r^1 - y_r^1)\,] c_r(z_r) \\
&= \sum_{s\in\mathcal{R}_+} y_s^1\, c_s(y_s) + \sum_{r\in\mathcal{R}_-} y_r^1\, c_r(y_r) + \sum_{s\in\mathcal{R}_+} (z_s^1 - y_s^1)\, c_s(y_s) + \sum_{r\in\mathcal{R}_-} (z_r^1 - y_r^1)\, c_r(y_r) \\
&= u_S^1(\mathbf{y}) + \sum_{r\in\mathcal{R}} (z_r^1 - y_r^1)\, c^0(\mathbf{y}) \\
&= u_S^1(\mathbf{y}) + (T-S)\, c^0(\mathbf{y}) = S\cdot Y^1(S) + (T-S)\, c^0(\mathbf{y}).
\end{aligned}$$

Recall that $u_T^1(\mathbf{z}) > u_T^1(\mathbf{x})$. Then,

$$T\cdot Y^1(T) = u_T^1(\mathbf{x}) \;<\; u_T^1(\mathbf{z}) = S\cdot Y^1(S) + (T-S)\, c^0(\mathbf{y}). \tag{2.19}$$

This implies that $Y^1(S) > Y^1(T)$. Because, otherwise, according to (ii), $Y^0(S) = c^0(\mathbf{y}) < Y^1(S)$. Then,

$$S\cdot Y^1(S) + (T-S)\, c^0(\mathbf{y}) < T\cdot Y^1(S) \le T\cdot Y^1(T),$$

which contradicts (2.19).

Therefore, $Y^1(S) > Y^1(T)$. One deduces that $Y^1(T)$ is strictly decreasing in T on $[\,\tilde{T}, 1\,]$.

(h) Finally, let us prove that $Y(S) > Y(T)$, i.e. the social cost at $\mathbf{y}$ is higher than at $\mathbf{x}$. In other words, $Y(T)$ is strictly decreasing in T on $[\,\tilde{T}, 1\,]$.

Indeed, (2.17) implies that, for all $s \in \mathcal{R}_+$ and $r \in \mathcal{R}_-$ such that $x_r > 0$,

$$c_s(x_s) + x_s c'(x_s) \ge \hat{c}^1(\mathbf{x}) \ge c_r(x_r) + x_r c'(x_r).$$

Then, for all $s \in \mathcal{R}_+$ and $r \in \mathcal{R}_-$ such that $x_r > 0$, for all $u > x_s$ and v such that $0 \le v \le x_r$,

$$c_s(u) + u\, c_s'(u) > \hat{c}^1(\mathbf{x}) \ge c_r(v) + v\, c_r'(v). \tag{2.20}$$

Thus,

$$\begin{aligned}
Y(S) - Y(T) &= \sum_{s\in\mathcal{R}_+} [\, y_s c_s(y_s) - x_s c_s(x_s)\,] - \sum_{r\in\mathcal{R}_-} [\, x_r c_r(x_r) - y_r c_r(y_r)\,] \\
&= \sum_{s\in\mathcal{R}_+} \int_{x_s}^{y_s} \frac{\partial}{\partial u} u\, c_s(u)\, \mathrm{d}u - \sum_{r\in\mathcal{R}_-} \int_{y_r}^{x_r} \frac{\partial}{\partial v} v\, c_r(v)\, \mathrm{d}v \\
&= \int_{x_s}^{y_s} [\, c_s(u) + u\, c_s'(u)\,]\, \mathrm{d}u - \sum_{r\in\mathcal{R}_-} \int_{y_r}^{x_r} [\, c_r(v) + v\, c_r'(v)\,]\, \mathrm{d}v \\
&> \sum_{s\in\mathcal{R}_+} (y_s - x_s)\, \hat{c}^1(\mathbf{x}) - \sum_{r\in\mathcal{R}_-} (x_r - y_r)\, \hat{c}^1(\mathbf{x}) = 0,
\end{aligned}$$

where the inequality is due to (2.20). $\square$

2.5.2 Individuals' cost and the composition of the players

The previous results can be partially extended to the multiple coalitions case. Consider the following two composite games:

$$\Gamma_0 = \Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T}), \quad \mathbf{T} = (T^0; T^1, \ldots, T^K),$$
$$\Gamma_1 = \Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T}'), \quad \mathbf{T}' = (T^0 + \delta T; T^1, \ldots, T^{l-1}, T^l - \delta T, T^{l+1}, \ldots, T^K),$$

with $K \geq 1$, $1 \leq l \leq K$ and $0 < \delta T < T^l$. Profile $\mathbf{T}'$ can be seen as obtained by $\mathbf{T}$ after the withdrawal from coalition l of a group of members of total weight δT who become individuals. Let $\mathbf{x}$ and $\mathbf{y}$ be, respectively, the CE of the game Γ_0 and that of the game Γ_1. The other notations are as before.

The following theorem shows that the individuals' cost is (weakly) higher at the CE in the game Γ_1 than in the game Γ_0.

Theorem 2.19. $c^0(\mathbf{x}) \leq c^0(\mathbf{y})$.

Proof. The case $K = 1$ is proven in Theorem 2.18, only the case $K \geq 2$ is treated here.

First, define two sets of arcs $\mathcal{R}_- = \{r \in \mathcal{R} \,|\, y_r \leq x_r\}$ and $\mathcal{R}_+ = \mathcal{R} \setminus \mathcal{R}_- = \{r \in \mathcal{R} \,|\, y_r > x_r\}$.

If $\mathcal{R}_+ = \emptyset$, then $\mathbf{x}' = \mathbf{y}'$ and $c^0(\mathbf{x}) = c^0(\mathbf{y})$.

If $\mathcal{R}_+ \neq \emptyset$, let us first prove that, for all $k \in \mathcal{K} \setminus \{l\}$, $\sum_{r\in\mathcal{R}_-} y_r^k \geq \sum_{r\in\mathcal{R}_-} x_r^k$.

Suppose that $\sum_{r\in\mathcal{R}_-} y_r^k \leq \sum_{r\in\mathcal{R}_-} x_r^k$ and, consequently, $\sum_{s\in\mathcal{R}_+} y_s^k \geq \sum_{s\in\mathcal{R}_+} x_s^k$. Therefore, there is an arc $r \in \mathcal{R}_-$ and an arc $s \in \mathcal{R}_+$ such that $y_r^k \leq x_r^k$ and $y_s^k \geq x_s^k$.

For all such r and s, if $x_r^k > 0$ and $y_s^k > 0$, then $\hat{c}^k(\mathbf{y}) \leq c_r(y_r) + y_r^k c_r'(y_r) \leq c_r(x_r) + x_r^k c_r'(x_r) = \hat{c}^k(\mathbf{x}) \leq c_s(x_s) + x_s^k c_s'(x_s) < c_s(y_s) + y_s^k c_s'(y_s) = \hat{c}^k(\mathbf{y})$, a contradiction. Consequently, either $y_r^k = x_r^k = 0$ or $y_s^k = x_s^k = 0$. For this to be true, there can be two cases.

Case 1. For all $s \in \mathcal{R}_+$ such that $y_s^k \geq x_s^k$, $y_s^k = x_s^k = 0$. Then, there is no $s \in \mathcal{R}_+$ such that $y_s^k < x_s^k$ because, otherwise, $\sum_{s\in\mathcal{R}_+} y_s^k < \sum_{s\in\mathcal{R}_+} x_s^k$, which contradicts the hypothesis that $\sum_{r\in\mathcal{R}_-} y_r^k \leq \sum_{r\in\mathcal{R}_-} x_r^k$. Thus, for all $s \in \mathcal{R}_+$, $y_s^k = x_s^k = 0$ and, consequently, $\sum_{r\in\mathcal{R}_-} y_r^k = \sum_{r\in\mathcal{R}_-} x_r^k = T^k$.

Case 2. There exists some $s \in \mathcal{R}_+$ such that $y_s^k \geq x_s^k$ and $y_s^k > 0$. Then, for all $r \in \mathcal{R}_-$ such that $y_r^k \leq x_r^k$, $y_r^k = x_r^k = 0$. Therefore, there is no $r \in \mathcal{R}_-$ such that $y_r^k > x_r^k$ because, otherwise, $\sum_{r\in\mathcal{R}_-} y_r^k > \sum_{r\in\mathcal{R}_-} x_r^k$, which again contradicts the hypothesis. Thus, for all $r \in \mathcal{R}_-$, $y_r^k = x_r^k = 0$ and, in consequence, $\sum_{r\in\mathcal{R}_-} y_r^k = \sum_{r\in\mathcal{R}_-} x_r^k = 0$.

Hence, $\sum_{r\in\mathcal{R}_-} y_r^k \geq \sum_{r\in\mathcal{R}_-} x_r^k$ and, consequently, $\sum_{s\in\mathcal{R}_+} y_s^k \leq \sum_{s\in\mathcal{R}_+} x_s^k$. Besides, the equalities hold if, and only if, $\sum_{r\in\mathcal{R}_-} y_r^k = \sum_{r\in\mathcal{R}_-} x_r^k = T^k$ or 0.

From the fact that $\sum_{s\in\mathcal{R}_+} y_s > \sum_{s\in\mathcal{R}_+} x_s$ and $\sum_{s\in\mathcal{R}_+} y_s^k \leq \sum_{s\in\mathcal{R}_+} x_s^k$ for all $k \in \mathcal{K} \setminus \{l\}$, one can deduce that $\sum_{s\in\mathcal{R}_+} (y_s^0 + y_s^l) > \sum_{s\in\mathcal{R}_+} (x_s^0 + x_s^l) \geq 0$.

Let us show that there exists some $t \in \mathcal{R}_+$ such that $y_t^0 > 0$. Indeed, if for all $s \in \mathcal{R}_+$, $y_s^0 = 0$, then $\sum_{s\in\mathcal{R}_+} y_s^l > \sum_{s\in\mathcal{R}_+} (x_s^l + x_s^0) \geq \sum_{s\in\mathcal{R}_+} x_s^l$. Besides, $\sum_{r\in\mathcal{R}} y_r^l = T - \delta T < T = \sum_{r\in\mathcal{R}} x_r^l$, hence $\sum_{t\in\mathcal{R}_-} y_t^l < \sum_{t\in\mathcal{R}_-} x_t^l$. In particular, there exists $r \in \mathcal{R}_-$ such that $y_r^l < x_r^l$ and $s \in \mathcal{R}_+$ such that $y_s^l > x_s^l$. Then, $\hat{c}^l(\mathbf{y}) \leq c_r(y_r) + y_r^l c'(y_r) < c_r(x_r) + x_r^l c'(x_r) = \hat{c}^l(\mathbf{x})$, and $\hat{c}^l(\mathbf{y}) = c_s(y_s) + y_s^l c'(y_s) > c_s(x_s) + x_s^l c'(x_s) \geq \hat{c}^l(\mathbf{x})$, a contradiction. Thus, there exists $t \in \mathcal{R}_+$ such that $y_t^0 > 0$ and, consequently, $c^0(\mathbf{y}) = c_t(y_t) > c_t(x_t) \geq c^0(\mathbf{x})$. □

However, the average cost to the coalition of weight T^l (called coalition l) in Γ_0 is not necessarily lower than that to the coalition of weight $T^l - \delta T$ (called coalition l') in Γ_1, and the former is not necessarily lower than the average cost to the group composed of coalition l' and a set of individuals of total weight δT in Γ_1 (the group corresponding to coalition l in Γ_0). In other words, the other two results in Theorem 2.18 (iii) cannot be extended to the multiple coalitions case. Here is an example.

Example 2.20. Composite game Γ takes place in the same network as in Example 2.17, where the cost functions of arcs r_1 and r_2 are, respectively, $c_1(x) = x+10$ and $c_2(x) = 10x+1$. Coalition 1 has weight T with $T \in (0, \frac{1}{2}]$; coalition 2 has weight $\frac{1}{2}$; the total weight of the individuals is $\frac{1}{2} - T$. Then, the average cost to coalition 1 at the CE is $\frac{91}{11}$ for $T \in (0, \frac{1}{5}]$ and $\frac{1}{99}[919 - 4(25T + \frac{4}{T})]$ for $T \in [\frac{1}{5}, \frac{1}{2}]$, which is constant in T on $(0, \frac{1}{5}]$, strictly increasing on $[\frac{1}{5}, \frac{2}{5}]$ and strictly decreasing on $[\frac{2}{5}, \frac{1}{2}]$. Therefore, it is not always decreasing in the size of the coalition. The average cost to the group of total weight $\frac{1}{2}$ composed of coalition 1 and all the individuals is $\frac{91}{11}$ for $T \in (0, \frac{1}{5}]$ and $\frac{1}{99}[400(T - \frac{11}{40})^2 + 816.75]$ for $T \in [\frac{1}{5}, \frac{1}{2}]$, which is constant in T on $(0, \frac{1}{5}]$, strictly decreasing in $(\frac{1}{5}, \frac{11}{40}]$ and strictly increasing on $[\frac{11}{40}, \frac{1}{2}]$. As a result, it is not always decreasing in T.

2.6 Asymptotic behavior of composite games

This subsection studies the asymptotic behavior of composite games, when some coalitions are fixed and the size of the others vanish.

Definition 2.21 (Admissible sequence of composite games and its limit game)**.** A sequence of composite games $\{\Gamma_n\}_{n\in\mathbb{N}^*}$, with $\Gamma_n = \Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T}_n)$ and $\mathbf{T}_n = (T_n^0; T_n^1, T_n^2, \dots, T_n^{K_n})$, is called *admissible* if $\{\mathbf{T}_n\}_{n\in\mathbb{N}^*}$ satisfies the following conditions:

(i) there is a constant $L \in \mathbb{N}$, and L strictly positive constants $\{T^1, T^2, \dots, T^L\}$ such that $\sum_{k=1}^L T^k < 1$. For all n, $K_n > L$, and $T_n^i = T^i$ for $i = 1, \dots, L$;

(ii) $\delta_n = \max_{L<k\le K_n} T_n^k$. And $\delta_n \to 0$ as $n \to \infty$.

The L-coalition composite game $\Gamma_0 = \Gamma(\mathcal{R}, \mathbf{c}, (\tilde{T}^0; T^1, T^2, \dots, T^L))$ is called the *limit game* of the sequence $\{\Gamma_n\}_{n\in\mathbb{N}^*}$, where $\tilde{T}^0 = 1 - \sum_{k=1}^L T^k$.

Remark 2.22. Condition (i) means that there are L coalitions fixed all along the sequence $\{\Gamma_n\}_{n\in\mathbb{N}^*}$, and the total weight of the remaining coalitions and the individuals is fixed to $\tilde{T}^0$. Condition (ii) means that the other coalitions are vanishing along the sequence and, necessarily, K_n tends to infinity.

Notation. As before, in the game Γ_n, $\mathbf{x}_n^* = (\mathbf{x}_n^{*k})_{k=0}^{K_n}$ is the CE, where $\mathbf{x}_n^{*0}$ is the flow of the individuals, and $\mathbf{x}_n^{*k}$ the flow of coalition k. Besides, $Y^0(\mathbf{x}_n^*)$ is the individuals' cost and $Y^k(\mathbf{x}_n^*)$ the average cost to coalition k at CE.

The aggregate flows are defined as $\mathbf{x}_n^{*\prime} = (\mathbf{y}_n^*, \mathbf{x}_n^{*1}, \mathbf{x}_n^{*2}, \dots, \mathbf{x}_n^{*L})$, where $\mathbf{y}_n^* = (y_{n,r}^*)_{r\in\mathcal{R}}$, $y_{n,r}^* = x_{n,r}^{*0} + \sum_{k=L+1}^{K_n} x_{n,r}^{*k}$. Thus, $\mathbf{y}_n^*$ is the aggregate flow of the individuals in addition to all the coalitions different from the L fixed ones. Notice that this is different from the definition of aggregate flow in the previous sections.

The feasible flow set is $F_n = \{\mathbf{x} \in \mathbb{R}^{R\times(1+K_n)} \,|\, \mathbf{x} \ge \mathbf{0};\ \forall\, k = 0 \text{ or } k \in \mathcal{K},\ \sum_{r\in\mathcal{R}} x_r^k = T_n^k\}$ and $F_0 = \{\mathbf{x} \in \mathbb{R}^{R\times(1+L)} \,|\, \mathbf{x} \ge \mathbf{0};\ \forall\, k \in \mathcal{K},\ \sum_{r\in\mathcal{R}} x_r^k = T^k;\ \sum_{r\in\mathcal{R}} x_r^0 = \tilde{T}^0\}$ is the feasible aggregate flow set. Notice that it is common to all the games in $\{\Gamma_n\}_{n\in\mathbb{N}^*}$.

In Γ_0, $\mathbf{x}^* = (\mathbf{y}^*, \mathbf{x}^{*1}, \dots, \mathbf{x}^{*L})$ is the CE, where $\mathbf{y}^* = \mathbf{x}^{*0}$ is the flow of the individuals, and $\mathbf{x}^{*k}$ the flow of coalition k. $Y^0(\mathbf{x}^*)$ is the individuals' cost and $Y^k(\mathbf{x}^*)$ the average cost to coalition k at CE. The feasible flow set is F_0.

The following theorem states that the CE of Γ_n converges to the CE of Γ_0. Hence, it justifies the name 'limit game'.

Theorem 2.23 (Convergence of admissible composite games). *Suppose that $\{\Gamma_n\}_{n\in\mathbb{N}^*}$ is a sequence of admissible games satisfying Assumption 2.1. Let Γ_0 be its limit game. Then, $\mathbf{x}_n^{*\prime} \to \mathbf{x}^*$ as $n \to \infty$. In particular, $Y^k(\mathbf{x}_n^*) \to Y^k(\mathbf{x}^*)$ for $k = 1, \ldots, L$, and $Y^k(\mathbf{x}_n^*) \to Y^0(\mathbf{x}^*)$ for $k = 0$ and $k > L$.*

Proof. Let us begin by writing the variational inequality condition for the CE's $\mathbf{x}_n^*$ and $\mathbf{x}^*$.

By Proposition 2.3, $\mathbf{x}_n^*$ is the CE of Γ_n if, and only if,

$$\Big\langle\, \mathbf{c}(\mathbf{x}_n^*),\, \mathbf{x}_n^0 - \mathbf{x}_n^{*0} \,\Big\rangle + \sum_{k=1}^{K_n} \Big\langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\Big\rangle \geq 0, \quad \forall\, \mathbf{x}_n \in F_n, \tag{2.21}$$

and $\mathbf{x}^*$ is the CE of Γ_0 if, and only if,

$$\langle\, \mathbf{c}(\mathbf{x}^*),\, \mathbf{y} - \mathbf{y}^* \,\rangle + \sum_{k=1}^{L} \Big\langle\, \hat{\mathbf{c}}^k(\mathbf{x}^*),\, \mathbf{x}^k - \mathbf{x}^{*k} \,\Big\rangle \geq 0, \quad \forall\, \mathbf{x} = (\mathbf{y},\, \mathbf{x}^1,\, \ldots,\, \mathbf{x}^L) \in F_0. \tag{2.22}$$

According to Assumption 2.1, one can find a constant M such that $M > \sup_{r\in\mathcal{R},\, x\in[0,\,1]}\{|c_r'(x)|\}$. Set $\epsilon_n = 2\delta_n MR$ so that ϵ_n tends to 0. Let us show that, for all n, the aggregate flow $\mathbf{x}_n^{*\prime}$ in F_0 satisfies

$$\Big\langle\, \mathbf{c}(\mathbf{x}_n^{*\prime}),\, \mathbf{y} - \mathbf{y}_n^* \,\Big\rangle + \sum_{k=1}^{L} \Big\langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^{*\prime}),\, \mathbf{x}^k - \mathbf{x}_n^{*k} \,\Big\rangle \geq -\epsilon_n, \quad \forall\, \mathbf{x} = (\mathbf{y},\, \mathbf{x}^1,\, \ldots,\, \mathbf{x}^L) \in F_0. \tag{2.23}$$

Indeed, for any $\mathbf{x} = (\mathbf{y},\, \mathbf{x}^1,\, \ldots,\, \mathbf{x}^L) \in F_0$, one can find $\mathbf{x}_n = (\mathbf{x}_n^k)_{k=0}^{K_n} \in F_n$ such that

$$\begin{cases} \mathbf{x}_n^k = \mathbf{x}^k, & k = 1, \ldots, L; \\ x_{n,r}^0 + \sum_{k=L+1}^{K_n} x_{n,r}^k = y_{n,r}, & \forall\, r \in \mathcal{R}. \end{cases} \tag{2.24}$$

For example, take $x_{n,r}^0 = y_{n,r} T^0/\tilde{T}^0$, $x_{n,r}^k = y_{n,r} T^k/\tilde{T}^0$ for $k = L+1,\, \ldots,\, K_n$. Then, by (2.21),

$$\begin{aligned}
&\langle\, \mathbf{c}(\mathbf{x}_n^*),\, \mathbf{x}_n^0 - \mathbf{x}_n^{*0} \,\rangle + \sum_{k=1}^{K_n} \langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle \geq 0 \\
\Rightarrow\, &\langle\, \mathbf{c}(\mathbf{x}_n^*),\, \mathbf{x}_n^0 - \mathbf{x}_n^{*0} \,\rangle + \sum_{k=1}^{L} \langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle + \sum_{k=L+1}^{K_n} \langle\, \mathbf{c}(\mathbf{x}_n^*) + \mathbf{x}_n^{*k} \dot{\mathbf{c}}(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle \geq 0 \\
\Rightarrow\, &\Big\langle\, \mathbf{c}(\mathbf{x}_n^*),\, \mathbf{x}_n^0 + \sum_{k=L+1}^{K_n} \mathbf{x}_n^k - \mathbf{x}_n^{*0} - \sum_{k=L+1}^{K_n} \mathbf{x}_n^{*k} \,\Big\rangle + \sum_{k=1}^{L} \langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle \\
&\geq -\sum_{k=L+1}^{K_n} \langle\, \mathbf{x}_n^{*k} \dot{\mathbf{c}}(\mathbf{x}_n^*),\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle \geq -\sum_{k=L+1}^{K_n} \langle\, \delta_n M,\, \mathbf{x}_n^k - \mathbf{x}_n^{*k} \,\rangle \\
&= -\Big\langle\, \delta_n M,\, \sum_{k=L+1}^{K_n} \mathbf{x}_n^k - \sum_{k=L+1}^{K_n} \mathbf{x}_n^{*k} \,\Big\rangle \geq -2\delta_n MR.
\end{aligned}$$

By (2.24), this is just $\langle\, \mathbf{c}(\mathbf{x}_n^{*\prime}),\, \mathbf{y} - \mathbf{y}_n^* \,\rangle + \sum_{k=1}^{L} \langle\, \hat{\mathbf{c}}^k(\mathbf{x}_n^{*\prime}),\, \mathbf{x}^k - \mathbf{x}_n^{*k} \,\rangle \geq -\epsilon_n$.

The fact that F_0 is a compact subset of $\mathbb{R}^{R\times(1+L)}$ implies that $\{\mathbf{x}_n^{*\prime}\}_{n\in\mathbb{N}^*}$ admits accumulation points in F_0. For any convergent subsequence of $\{\mathbf{x}_n^{*\prime}\}_{n\in\mathbb{N}^*}$ (which is still denoted by $\{\mathbf{x}_n^{*\prime}\}_{n\in\mathbb{N}^*}$ for simplicity), let $\tilde{\mathbf{x}} = (\tilde{\mathbf{y}}, \tilde{\mathbf{x}}^1, \ldots, \tilde{\mathbf{x}}^L)$ be its accumulation point. Let n tend to infinity in (2.23). Then, by the continuity of the marginal cost functions and the fact that ϵ_n tends to 0,

$$\Big\langle \mathbf{c}(\tilde{\mathbf{x}}^L), \mathbf{y} - \tilde{\mathbf{y}} \Big\rangle + \sum_{k=1}^{L} \Big\langle \hat{\mathbf{c}}^k(\tilde{\mathbf{x}}), \mathbf{x}^k - \tilde{\mathbf{x}}^k \Big\rangle \geq 0, \quad \forall\, \mathbf{x} = (\,\mathbf{y}, \mathbf{x}^1, \ldots, \mathbf{x}^L\,) \in F_0.$$

According to (2.22), this implies that $\tilde{\mathbf{x}} = \mathbf{x}^*$.

Therefore, $\mathbf{x}_n^{*\prime}$ converges to $\mathbf{x}^*$ as n tends to infinity. This induces immediately that $Y^k(\mathbf{x}_n^*)$ tends to $Y^k(\mathbf{x}^*)$ for $k = 1, \ldots, L$, and $Y^k(\mathbf{x}_n^*)$ tends to $Y^0(\mathbf{x}^*)$ for $k = 0$ and $k > L$. □

Remark 2.24. When $T^0 = 0$ and $L = 0$, Theorem 2.23 shows that the NE of an atomic splittable game with only coalitions and no individuals converges to the WE of the corresponding nonatomic game, when the coalitions split into smaller and smaller ones. This result is obtained by Haurie and Marcotte [36], but only for the case where the coalitions split into equal-size ones. Theorem 2.23 is an extension of their result in three aspects. First, the coalitions do not have equal size. Second, the games are composite. Finally, some coalitions are fixed at a nonnegligible weight.

2.7 Some problems for future research

This section presents some directions for further studies.

2.7.1 Backward induction

Consider a two-stage extensive form game with an underlying network $(\mathcal{R}, \mathbf{c})$ and a set of nonatomic individuals $[0, 1]$. At the first stage, the individuals are given $K+1$ choices $\{s_0, s_1, \ldots, s_K\}$, where K is a fixed number in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The players who choose s_0 are called individuals, and those who choose s_k $(1 \leq k \leq K)$ are considered as members of coalition k. If there are L coalitions having nonnegligible weights $(0 \leq L \leq K)$ then, at the second stage, the individuals and L coalitions play the composite routing game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$. By Lemma 2.8, the players who choose s_0 at the first stage have the lowest cost at the end. Therefore, by a backward induction, the only subgame perfect equilibrium of this two-stage game consists in having all the players choosing s_0 at the first stage.

2.7.2 Composition-decision games

In a two-player composition-decision game, each player is atomic with a splittable flow. The weight of player I is T, while that of player II is $1 - T$. Each player chooses a pair of representatives consisting of a coalition and a group of individuals, whose total weight is her own weight. The cost to each player is defined as the average equilibrium cost to her representatives in a composite routing game played by all the representatives. Consider two simple models where the game reduces to a one-player game.

Model 1: One atomic player faces individuals. Player I has two strategies. Strategy 1 consists in choosing a coalition of weight T, while strategy 2 consists in choosing a group of individuals of weight T. Player II always chooses a group of individuals of weight $1-T$.

If player I chooses strategy 1, the costs to the two players are, respectively, the equilibrium cost to the unique coalition and that to the individuals in the composite game $\Gamma(\mathcal{R}, \mathbf{c}, (1-T; T))$, i.e. $Y^1(T)$ for player I, $Y^0(T)$ for player II. If player I chooses strategy 2, both the costs to the two players are W, the equilibrium cost in the nonatomic game $\Gamma(\mathcal{R}, \mathbf{c})$.

Theorem 2.18(ii) shows that, if $0 < T \leq \tilde{T}$, then the two strategies make no difference to player I. If $\tilde{T} < T < 1$, player I's only best reply is strategy 1, and her cost is $Y^1(T)$, which is lower than W. Strategy 1 dominates strategy 2.

Model 2: One group faces a coalition. Player I has the same two strategies as in Model 1. Player II always chooses a coalition of weight $1-T$.

If player I chooses strategy 1, the costs to the two players are the equilibrium costs in the two-coalition game $\Gamma(\mathcal{R}, \mathbf{c}, (0; T, 1-T))$. If player I chooses strategy 2, the costs to the two players are, respectively, the equilibrium cost to the individuals and that to the unique coalition in the composite game $\Gamma(\mathcal{R}, \mathbf{c}, (T; 1-T))$, i.e. $Y^0(1-T)$ for player I, $Y^1(1-T)$ for player II.

Does strategy 1 still dominate strategy 2? The answer is negative. Here is a counter-example.

Example 2.25. Still take the network where two parallel arcs r_1 and r_2 have cost functions $c_1(x) = x + 10$ and $c_2(x) = 10x + 1$. $T = 1/2$, i.e. player I and player II both have weight $1/2$. Player II always chooses a coalition of weight $1/2$.

If player I chooses strategy 1, at the NE of the atomic splittable game $\Gamma(\mathcal{R}, \mathbf{c}, (0;\ 1/2,\ 1/2))$, the average cost to the coalition of player I is $93/11 = 8.455$. (The average cost to the coalition of player II is $93/11 = 8.455$. Both two coalitions send weight $2/11$ on r_1, and the rest $7/22$ on r_2.)

If player I chooses strategy 2, at the CE of the composite game $\Gamma(\mathcal{R}, \mathbf{c}, (1/2; 1/2))$, the average cost to the individuals of player I is $91/11 = 8.273$. (The average cost to the coalition of player II is $103/11 = 9.364$. The coalition of player II sends weight $3/11$ on r_1, and the rest $5/22$ on r_2, while the individuals of player I all take r_2.)

$8.273 < 8.455$, so that the only best reply of player I is strategy 2.

Furthermore, Example 2.20 shows that, if player I can choose an arbitrary size $S \in [0, 1/2]$ for her coalition, her best choice is $S = 11/40$.

2.8 Appendix

2.8.1 Group deviation

Let us return to the one-coalition games studied in Section 2.5. Suppose that there is one coalition of size T formed. Theorem 2.18 (ii) states that, if $T > \tilde{T}$, then the individuals' cost $Y^0(T)$ is strictly lower than the average cost to the coalition. Because of the individuals' weightlessness, being the only individual who quits or enters in the coalition will only affect her own cost. If everybody believes herself to be the only smart one, and no contact is allowed between them, and if one is free to quit or to stay in the coalition before the game is repeated, the coalition will no longer exist. Theorem 2.18 confirms that it is the worst result for everyone.

However, if the individuals are allowed to talk to each other before the game, and if a group of individuals of total weight $dT > 0$ agree to enter in the coalition simultaneously, their group deviation might decrease their own costs as well as those of the others. This is actually the case when $Y^1(T + dT) < Y^0(T)$.

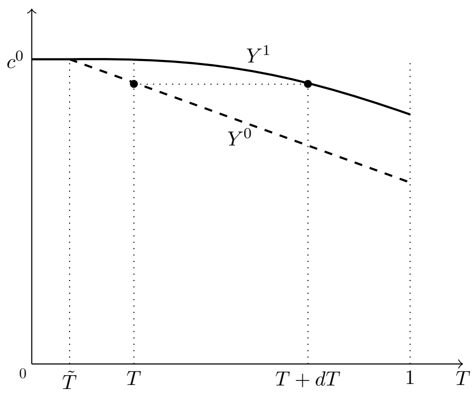


Figure 2.2

Example 2.26. As in Example 2.17, two parallel arcs r_1 and r_2 have, respectively, cost functions $c_1(x) = x + 10$ and $c_2(x) = 10x + 1$. Then, $\tilde{T} = 1/10$, and (*cf.* Figure 2.2)

$$\begin{cases} Y^0(T) = Y^1(T) = W = \frac{111}{11}, & \text{if } 0 \le T \le \frac{1}{10}; \\ Y^0(T) = \frac{111}{11} - \frac{50}{11}(T - \frac{1}{10}),\ Y^1(T) = \frac{111}{11} - \frac{25}{11T}(T - \frac{1}{10})^2, & \text{if } \frac{1}{10} < T < 1. \end{cases}$$

For $T \in (\tilde{T}, 1)$, let

$$d\,(T) = \inf\left\{d \ge 0 \,\middle|\, Y^1(T + d) \le Y^0(T)\right\}$$

be the infimum of group size such that their group deviation reduces their cost. Then

$$d\,(T) = \begin{cases} \sqrt{T^2 - \frac{1}{100}}, & \text{if } \frac{1}{10} < T \le \frac{101}{200}; \\ +\infty, & \text{if } \frac{101}{200} < T < 1. \end{cases}$$

In other words, when a coalition of weight between 1/10 and 101/200 is already formed, the group deviation of a group of noncoalitional individuals of total weight larger than $\sqrt{T^2 - 1/100}$ will reduce the cost of each of them. This infimum of deviation group size increases with the size of the actual coalition.

However, it is well known that, lack of trust in each other, such kind of cheap talk before the game will actually change nothing in the choices of the players.

2.8.2 Cost levels and social marginal costs

This section studies the relation between the cost of an arc and its contribution to the social cost at the CE in a composite game $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$.

Definition 2.27. The *social marginal cost* of arc r at $\mathbf{y} \in F$ is $\tilde{c}_r(\mathbf{y}) = c_r(y_r) + y_r c'_r(y_r)$.

From now on, all the definitions and results are to be understood at $\mathbf{x}$, the unique CE of $\Gamma(\mathcal{R}, \mathbf{c}, \mathbf{T})$.

Definition 2.28. The *cost level* of an arc r is denoted by i_r, where i is a function from $\mathcal{R}$ to $\{1, \dots, K, +\infty\}$ defined by

$$i_r = \begin{cases} 0, & \text{if } \ r \in \mathcal{R} \setminus \mathcal{R}^1(\mathbf{x}); \\ k, & \text{if } \ r \in \mathcal{R}^k(\mathbf{x}) \ \text{ and } \ r \notin \mathcal{R}^l(\mathbf{x}), \ \forall\, l > k; \\ +\infty, & \text{if } \ r \in \mathcal{R}^0(\mathbf{x}), \end{cases}$$

where $k \in \mathcal{K}, l \in \mathcal{K} \cup \{0\}$.

The following corollary follows immediately from Lemmas 2.8 and 2.9.

Corollary 2.29. *For any arcs $r, s, t \in \mathcal{R}$,*

(i) if $i_r = i_s = +\infty$ and $i_t < +\infty$, then $c_r(\mathbf{x}) = c_s(\mathbf{x}) \leq c_t(\mathbf{x})$;

(ii) if $0 \leq i_r < i_s < +\infty$, then $c_s(\mathbf{x}) < c_r(\mathbf{x})$.

The following theorem shows that, at the CE, the lower the cost of an arc, the higher its social marginal cost.

Proposition 2.30. *For any arcs $r, s \in \mathcal{R}$,*

(i) if $i_s < i_r \leq +\infty$, or if $1 < i_s = i_r < +\infty$ and $c_r(\mathbf{x}) < c_s(\mathbf{x})$, then $\tilde{c}_r(\mathbf{x}) > \tilde{c}_s(\mathbf{x})$ and consequently $x_r\, c'(x_r) > x_s\, c'(x_s)$;

(ii) if $i_s = i_r = 1$, or if $1 < i_r = i_s < +\infty$ and $c_r(\mathbf{x}) = c_s(\mathbf{x})$, then $\tilde{c}_r(\mathbf{x}) = \tilde{c}_s(\mathbf{x})$.

Proof. First recall that if $x_r^k > 0$ and $x_s^k > 0$ for some $k \in \mathcal{K}$,

$$c_r(\mathbf{x}) + x_r^k c_r'(\mathbf{x}) = c_s(\mathbf{x}) + x_s^k c_s'(\mathbf{x}). \tag{2.25}$$

(i) If $i_s < i_r \leq +\infty$, then $c_s(\mathbf{x}) \geq c_r(\mathbf{x})$. By taking the sum of (2.25) for all $k \leq i_s$, one has

$$\begin{aligned} i_s\, c_r(\mathbf{x}) + \sum_{k=1}^{i_s} x_r^k\, c_r'(\mathbf{x}) \ &= \ i_s\, c_s(\mathbf{x}) + x_s\, c_s'(\mathbf{x}) \\ \Rightarrow (i_s - 1 + 1)\, c_r(\mathbf{x}) + x_r\, c_r'(\mathbf{x}) \ &= \ (i_s - 1 + 1)\, c_s(\mathbf{x}) + x_s\, c_s'(\mathbf{x}) + \sum_{k \in \mathcal{K} \cup \{+\infty\}, k > i_s} x_r^k c_r'(\mathbf{x}) \\ \Rightarrow \tilde{c}_r(\mathbf{x}) - \tilde{c}_s(\mathbf{x}) \ &= \ (i_s - 1)\, [\, c_s(\mathbf{x}) - c_r(\mathbf{x})\,] + \sum_{k \in \mathcal{K} \cup \{+\infty\}, k > i_s} x_r^k\, c_r'(\mathbf{x}) > 0 \qquad (2.26) \\ \Rightarrow \tilde{c}_r(\mathbf{x}) \ &> \ \tilde{c}_s(\mathbf{x}). \end{aligned}$$

Here (2.26) is because $\sum_{k \in \mathcal{K} \cup \{+\infty\}, k > i_s} x_r^k \geq x_r^{i_r} > 0$.

If $1 < i_s = i_r < +\infty$ and $c_r(\mathbf{x}) < c_s(\mathbf{x})$, summing (2.25) over all $k \leq i_s$ leads to

$$\begin{aligned} i_s\, c_r(\mathbf{x}) + x_r\, c_r'(\mathbf{x}) \ &= \ i_s\, c_s(\mathbf{x}) + x_s\, c_s'(\mathbf{x}) \qquad (2.27) \\ \Rightarrow \quad \tilde{c}_r(\mathbf{x}) - \tilde{c}_s(\mathbf{x}) \ &= \ (i_s - 1)\, [\, c_s(\mathbf{x}) - c_r(\mathbf{x})\,] > 0. \end{aligned}$$

(2) If $i_s = i_r = 1$, or if $1 < i_r = i_s < +\infty$ and $c_r(\mathbf{x}) = c_s(\mathbf{x})$, then, still by (2.27),

$$\tilde{c}_r(\mathbf{x}) - \tilde{c}_s(\mathbf{x}) \ = \ (i_s - 1)\, [\, c_s(\mathbf{x}) - c_r(\mathbf{x})\,] = 0.$$

□

2.8.3 The name of "composite games"

Composite games were first studied by Harker [32] then by Boulogne et al. [15]. They called such games *mixed games*. This terminology is avoided here because it can be confused with the mixed extension of a game. However, since the term "mixed equilibrium" of Harker is already very much in use, it may be wise to keep this name for games played by atomic players with splittable stocks and nonatomic players, while reserve "composite games" for a more general class of games where the atomic players can hold several types of stocks, some arbitrarily splittable, some unsplittable, and some splittable but not arbitrarily etc. For a detailed discussion on the types of the stocks, the reader is referred to §1.6.

Chapter 3

Composition of the players in two-singleton-choice congestion games

This chapter is based on the paper *Composition of the players in two-singleton-choice congestion games.*

Abstract. *In a two-terminal two-parallel-arc network, a finite number of atomic players with splittable stocks and a group of nonatomic players have a common origin/destination pair and common arc cost functions. This paper shows that, under a standard condition on the cost functions, whenever an atomic player l is replaced by a composite set $\mathcal{T}$ of players who, together, hold the same stock as l, the social cost as well as the cost to each of the other players at the unique equilibrium of the second game (played by the players in $\mathcal{T}$ and the others) are increased or do not change with respect to the first game (played by player l and the others). However, the per-unit cost to an atomic player in $\mathcal{T}$ in the second game can be higher or lower than the per-unit cost to player l in the first game.*

3.1 Introduction

3.1.1 Composite congestion games and the composition of the players

In a congestion game [74], each player has a certain quantity of stock and a (specific) finite set of choices. A choice is a subset of common facilities. A player holding a stock of infinitesimal weight is called a nonatomic player or an individual. She has to affect her stock to one choice. A player holding a stock of strictly positive weight is called an atomic player. Furthermore, she (more rigorously, her stock) is splittable if she can divide it into several parts and affect each part to a different choice. This work considers only splittable stocks so that the word *splittable* is often omitted. Each facility entails a cost to the stocks affected to it, and the cost depends on the total weight of these stocks. The cost to a player is the total cost to her stock, and she wishes to minimize it. The social cost is the total cost to all the players or, equivalently, to all the stocks.

A game with nonatomic (*resp.* atomic, *resp.* both nonatomic and atomic) players is called a nonatomic (*resp.* atomic, *resp.* composite) congestion game. Given an atomic or a composite congestion game Γ, the nonatomic congestion game with the same stocks as in Γ is called the corresponding nonatomic game of Γ. With continuously differentiable cost functions of the facilities, an equilibrium exists in a composite congestion game, and it is called a composite equilibrium (CE for short) [15, 32, 91]. An equilibrium in a nonatomic (*resp.* atomic) game is often called Wardrop equilibrium [94] (*resp.* Nash equilibrium). Notice that an equilibrium is not necessarily a social optimum, namely, a strategy profile that minimizes the social cost.

This work aims to examine the relation between the composition of the players and their costs at equilibria. With a fixed set of stocks, do the costs at equilibria depend on how these stocks are distributed between the players? And if so, how? If some atomic or nonatomic players form a coalition which behaves like an atomic player, will the total cost of the cooperators be reduced? Will the costs to the non-cooperators be increased? Will the social cost be reduced or increased? Or, on the contrary, if an atomic player is replaced by a composite set of players, i.e. a group of atomic and/or nonatomic players, but the total weight remains the same, how will the equilibrium costs change?

Network congestion games, also called routing games, constitute an important class of congestion games. They take place in a directed graph, where a facility is an arc and a choice is a directed path. A player has to send her stock from its origin to its destination, and the set of her choices is the set of directed paths between these two vertices. Routing games will be taken as an example in this work to study general congestion games.

In a congestion game with two singleton-choices, all the players have the same two choices, and each choice contains a single facility. In a routing game, it corresponds to the case where all the players have to send their stock from a common origin to a common destination, and the two vertices are connected by two parallel arcs.

3.1.2 Related works

For the sake of simplicity, the following results are presented under the assumption that the cost functions of the arcs are *standard*, which means that they are strictly increasing, convex and continuously differentiable on a domain large enough so that there is no capacity constraints. Besides, a game has *single commodity* if all the stocks are to be sent from the same origin to the same destination by the same available paths, on which they experience the same per-unit cost. Otherwise, it has *multi-commodity*.

Roughgarden and Tardos [78], Roughgarden [76], Cominetti et al. [20], Harks [33, 34] and Roughgarden and Schoppmann [77] studied the *price of anarchy* in atomic congestion games. Recall that the *price of anarchy* [50, 67] is the ratio between the social cost at the worst Nash equilibrium and the minimal social cost.

Hayrapetyan et al. [37] defined the *price of collusion* of a nonatomic game to be the ratio between the social cost at the worst Nash equilibrium of all the atomic games induced by a formation of coalitions (which they called *collusion*) and the social cost at the worst Wardrop equilibrium. They showed that any cooperation reduces the social cost at the unique equilibrium in a two-terminal parallel-arc network. Therefore, the price of collusion in nonatomic games in such networks is bounded by 1. Bhaskar et al. [11] extended this result to single-origin single-destination series-parallel networks. Recall that a series-parallel network is obtained by merging, in series or in parallel, some two-terminal parallel-arc networks. Wan [91] (*cf.* Chapter 2) proved the same result for composite games in two-terminal parallel-arc networks, where individuals can remain independent without joining any coalition. For a general network, Cominetti et al. [20] obtained that, in an atomic congestion game, if all the atomic players have the same weight, the social cost at the unique equilibrium is bounded by that in the corresponding nonatomic game. For the multi-commodity case, Cominetti et al. [20] gave two examples where the social cost is increased after the formation of coalition(s) of the individuals.

As for the cooperation between the atomic players and its impact on the social cost, Cominetti et al. [20] showed that, in single commodity case, the social cost at the unique equilibrium in an atomic game where K atomic players equally share the stocks is bounded by the social cost in a game where $\tilde{K}$ atomic players equally share the same stocks, and $\tilde{K} > K$. Huang [46] proved that, in single commodity case, the social cost after the formation of coalition(s) (not necessarily of the same size) is bounded by that before the cooperation if, and only if, the network is well-designed (*cf.* §3.5) and the cost functions are affine (in which case the equilibrium is always unique). Altman et al. [5] gave examples of multi-commodity atomic congestion games where the formation of coalitions by atomic players increases the social cost.

The above works compare the social cost at equilibria in an atomic or composite congestion game with that in the corresponding nonatomic game. The impact of the cooperation on the equilibrium cost to each player, either cooperator or non-cooperator, is more complex. Wan [91] (*cf.* Chapter 2) showed that each atomic player's per-unit cost and the individuals' cost at the unique equilibrium of a composite game in a two-terminal parallel-arc network is lower than the unique equilibrium cost in the corresponding nonatomic game. The author also showed that, in the same context, if there is a unique coalition, the individuals' common cost and the average cost to the coalition are both decreasing with respect to the coalition's weight. This result is partially extended to multi-coalition cases in the same paper. It is proved that, whenever a coalition l is replaced by a composite set of players of the same total weight as l, which contains a smaller coalition and a set of individuals, the individuals' common cost is increased. On the contrary, an example in [91] shows that the impact of the cooperation on the cooperators themselves can be either positive or negative, even in a two-parallel network with affine costs. For multi-commodity case, Cominetti et al. [20] and Altman et al. [5] gave examples where the formation of coalition(s) by, respectively, nonatomic and atomic players increases their own members' costs.

3.1.3 Main results

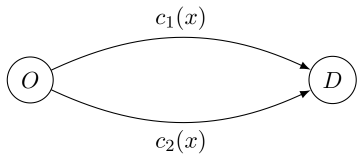
This work shows that, in a composite congestion game with two-singleton choices or, equivalently, in a two-terminal two-parallel-arc composite routing game, with standard cost functions for each choice, whenever an atomic player of weight m is replaced by a composite set of players, i.e. L (L can be 0) atomic players of weight $m^1, \ldots, m^L$ and a set of nonatomic players of total weight m^0, such that $m^0 \geq 0$, $m^i > 0$ for $i = 1, \ldots, L$, and $\sum_{i=0}^{L} m^i = m$, the social cost at the unique equilibrium is increased or does not change, and the cost to each of the other players not involved in this replacement is increased or does not change. In an equivalent way, after each formation of coalition between certain atomic and nonatomic players, the social cost and the cost to each of the players not involved in the coalition are decreased or do not change at the unique equilibrium.

The paper is organized in the following way. In Section 3.2, the model of the two-terminal two-parallel-arc composite routing game is presented, and some preliminary results are recalled. Section 3.3 is contributed to the properties of the composite equilibria in the specific setting of two-terminal two-parallel-arc networks. In Section 3.4, the main results are obtained. Section 3.5 concludes by some remarks and some topics for future research.

3.2 Model and known results

In this work, $[\![m, n]\!]$ stands for the set of successive positive integers $[m, n] \cap \mathbb{N} = \{m, m+1, \dots, n-1, n\}$. Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

3.2.1 Model of a two-terminal two-parallel-arc composite routing game



Network. Vertex O and vertex D are linked by two arcs. The per-unit cost function of arc r is c_r, for $r = 1, 2$, and $\mathbf{c} = (c_1, c_2)$. When the total weight of stocks on arc r is x, the cost to each unit of them is $c_r(x)$. Real-valued cost functions c_1 and c_2 are defined on the real interval $(-\eta, M+\eta)$, where $M > 0$ and $\eta > 0$, and they satisfy the following (standard) assumption *throughout this work.*

A 3.1. *Cost functions c_1 and c_2 are strictly increasing, convex and continuously differentiable on $(-\eta,\ M+\eta)$, and non-negative on $[\,0, M\,]$.*

Players and strategies. There is a group of individuals of total weight T^0, and N atomic splittable players, respectively, of weight $T^1, T^2, \dots, T^N$, where $N \in \mathbb{N}$, $T^0 \geq 0$, and $T^i > 0$ for all $i \in [\![1, N]\!]$ if $N > 0$. The profile of the players is described by the vector $\mathbf{T} = (T^0, T^1, T^2, \dots, T^N)$. The total weight of the players, i.e. the total weight of their stocks, is $M = \sum_{i=0}^N T^i$. Without loss of generality, suppose that $T^1 \geq \cdots \geq T^N$. An atomic player is said to be *larger* than another if she has a larger weight. Denote $T^{[p]} = \sum_{i=1}^p T^i$, the total weight of the p largest atomic players, for $p \in [\![1, N]\!]$.

The profile of the individuals' strategies is specified by a vector $\mathbf{x}^0 = (x_1^0, x_2^0)$, called their flow, where x_r^0 is the total weight of the individuals on arc r. The strategy of the atomic player i is specified by a vector $\mathbf{x}^i = (x_1^i, x_2^i)$, called her flow, where x_r^i is the weight that she sends by arc r. Denote $\xi_r = \sum_{i=0}^N x_r^i$, the total weight on arc r. The vector $\mathbf{x} = (\mathbf{x}^i)_{i=0}^N$ is called the system *flow* (induced by the players' strategies) or, flow for short.

For $i \in [\![1, N]\!]$, denote $\mathbf{x}^{-i} = \prod_{0 \leq j \leq N, j \neq i} \mathbf{x}^j$ and $x_r^{-i} = \xi_r - x_r^i$. Besides, for $i \in \{0\} \cup [\![1, N]\!]$, denote $X^i = \{\mathbf{x}^i \in \mathbb{R}^2 \,|\, x_1^i \geq 0,\ x_2^i \geq 0,\ x_1^i + x_2^i = T^i\}$ and $X = \prod_{i=0}^N X^i$. These are the spaces of feasible flows for the individuals, the atomic players and the whole system.

Costs. When the flow is $\mathbf{x}$, the cost to an individual taking arc r is $c_r(\xi_r)$. The cost to the atomic player i is $u^i(\mathbf{x}) = x_1^i c_1(\xi_1) + x_2^i c_2(\xi_2)$. Her per-unit cost is denoted by $Y^i(\mathbf{x}) = u^i(\mathbf{x})/T^i$. The social cost is defined as $v(\mathbf{x}) = \xi_1 c_1(\xi_1) + \xi_2 c_2(\xi_2)$.

From now on, the network and its cost functions $\mathbf{c}$ are fixed. A composite game taking place there is specified by the profile of its players $\mathbf{T}$ and thus denoted by $\Gamma(\mathbf{T})$.

Composite equilibrium (CE).

Definition 3.2. In the game $\Gamma(\mathbf{T})$, a flow $\mathbf{x} \in X$ is a *composite equilibrium* if it meets the two conditions below:

(i) For all $r \in \{1,2\}$, if $x_r^0 > 0$, then $c_r(\xi_r) \leq c_t(\xi_t)$ for all $t \in \{1,2\}$.
(ii) For all $i \in [\![1,N]\!]$, $\mathbf{x}^i$ minimizes $Y^i(\,\cdot\,,\mathbf{x}^{-i})$ on X^i.

By definition, the individuals take the arcs of the lowest cost at the CE $\mathbf{x}$. When $T^0 = 0$, i.e. there are no individuals, $Y^0(\mathbf{x})$ will still be used to denote the lowest arc cost in the network.

3.2.2 Auxiliary functions and notations

Let ϵ be a strictly positive constant. Two functions h, a, and a family of functions $\{F_n\}_{n\in\mathbb{N}}$ are defined on the real line $\mathbb{R}$ as below:

$$h(x) = \begin{cases} \dfrac{c_2(M-x)-c_1(x)}{c_1'(x)}, & \text{if } 0 \leq x \leq M; \\ h(0) - \epsilon x, & \text{if } x < 0; \\ h(M) - \epsilon(x-M), & \text{if } x > M, \end{cases} \qquad a(x) = \begin{cases} \dfrac{c_2'(M-x)}{c_1'(x)}, & \text{if } 0 \leq x \leq M; \\ a(0), & \text{if } x < 0; \\ a(M), & \text{if } x > M, \end{cases}$$

$$F_n(x) = (M-x)\left(1+a(x)\right) + n\,h(x), \qquad n \in \mathbb{N}.$$

Remark 3.3. It follows immediately from A3.1 that h and F_n's are strictly decreasing and continuous on $\mathbb{R}$, and their ranges are all $\mathbb{R}$. Their inverse functions h^{-1} and F_n^{-1}'s are also strictly decreasing and continuous on $\mathbb{R}$.

Besides, a is non-increasing, strictly positive and continuous on $\mathbb{R}$.

Proposition 3.4. *The following two statements are equivalent.*
(i) There exists $\hat{\xi} \in [\,0,M\,]$ such that $c_1(\hat{\xi}) = c_2(M-\hat{\xi})$.
(ii) $c_1(M) \geq c_2(0)$ and $c_2(M) \geq c_1(0)$ or, equivalently, $h(M) \leq 0$ and $h(0) \geq 0$.
Furthermore, there is at most one $\hat{\xi} \in [\,0,M\,]$ such that $c_1(\hat{\xi}) = c_2(M-\hat{\xi})$.

Proof. Clear by A3.1. □

Notation. $H = h(M)$.
If there exists $\hat{\xi} \in [\,0,M\,]$ such that $c_1(\hat{\xi}) = c_2(M-\hat{\xi})$, then denote $A = a(\hat{\xi})$.

Remark 3.5. When $\hat{\xi}$ exists, $h(\hat{\xi}) = 0$, $F_k(\hat{\xi}) = (M-\hat{\xi})\,(1+A) = F_0(\hat{\xi})$ for all $k \in \mathbb{N}$.

3.2.3 Properties of CE

Let us recall some important results on composite routing games. The following three theorems are cited from Wan [91] (*cf.* Chapter 2). They are reformulated in our context.

Theorem 3.6. *A vector $\mathbf{x} \in X$ is a CE of the game $\Gamma(\mathbf{T})$ if, and only if, for all $r \in \{1,2\}$,*

$$x_r^0 > 0 \;\Rightarrow\; c_r(\xi_r) = \min_{s\in\{1,2\}} c_s(\xi_s) \tag{3.1}$$

and, for all $r \in \{1,2\}$ and $i \in [\![1,N]\!]$,

$$x_r^i > 0 \;\Rightarrow\; c_r(\xi_r) + x_r^i\, c'(\xi_r) = \min_{s\in\{1,2\}} c_s(\xi_s) + x_s^i\, c'(\xi_s). \tag{3.2}$$

Theorem 3.7. *The game $\Gamma(\mathbf{T})$ admits one and only one CE.*

Theorem 3.8. *At the CE $\mathbf{x}$ of the game $\Gamma(\mathbf{T})$, for all $r \in \{1,2\}$,*
1. *if $x_r^0 > 0$, then $x_r^i > 0$ for all $i \in [\![1,N]\!]$;*
2. *for all $i,j \in [\![1,N]\!]$, if $T^i \geq T^j$, then $x_r^i \geq x_r^j$, and the equality holds if, and only if, $T^i = T^j$ or $x_r^i = x_r^j = 0$;*
3. *for all $i \in [\![1,N]\!]$ and $s \in \{1,2\} \setminus \{r\}$, if $x_r^i > 0$ and $x_s^i = 0$, then $c_r(\xi_r) < c_s(\xi_s)$.*

3.3 Three modes of the CE

3.3.1 Preliminary results

The following assumption will also be made *throughout this work*.

A 3.9. *One and only one of the following two conditions holds:*

(i) $c_1(M) < c_2(0)$ *or, equivalently,* $H > 0$.

(ii) $c_1(M) \geq c_2(0), c_2(M) \geq c_1(0)$ *or, equivalently,* $H \leq 0, h(0) \geq 0$; *and*

$$\hat{\xi}\, c_1'(\hat{\xi}) \geq (M - \hat{\xi})\, c_2'(M - \hat{\xi}). \tag{3.3}$$

Remark 3.10. Clearly, the case where $c_2(M) < c_1(0)$ and the case where $c_1(M) \geq c_2(0)$, $c_2(M) \geq c_1(0)$ and $\hat{\xi}\, c_1'(\hat{\xi}) \leq (M - \hat{\xi})\, c_2'(M - \hat{\xi})$ are, respectively, symmetric to the two cases in A3.9. Thus, A3.9 does not lose generality.

Remark 3.11. Without difficulty, one can deduce from (3.3) that

$$(M - \hat{\xi})(1 + A) \ \leq \ M \ \leq \ \hat{\xi}\, \frac{1 + A}{A}. \tag{3.4}$$

On account of A3.9, the flow of the CE $\mathbf{x}$ takes some special form as the following proposition shows.

Proposition 3.12. *At the CE* $\mathbf{x}$ *of* $\Gamma(\mathbf{T})$,

1. $c_1(\xi_1) \leq c_2(\xi_2)$;
2. *if* $c_2(0) \leq c_1(M)$ *or, equivalently,* $H \leq 0$, *then* $\xi_1 \leq \hat{\xi}$;
3. $h(\xi_1) \geq 0$, *and the equality holds if, and only if,* $c_2(0) \leq c_1(M)$ *and* $\xi_1 = \hat{\xi}$.

Proof. 1. In the case where $c_1(M) < c_2(0)$, $c_1(\xi_1) \leq c_2(\xi_2)$ is always true.

In the case where $c_1(M) \geq c_2(0)$ and $c_2(M) \geq c_1(0)$, suppose that $c_1(\xi_1) > c_2(\xi_2)$. If $\xi_1 \leq \hat{\xi}$ and thus $M - \xi_1 \geq M - \hat{\xi}$, according to the monotonicity of c_1 and c_2 and the definition of $\hat{\xi}$, $c_1(\xi_1) \leq c_1(\hat{\xi}) = c_2(M - \hat{\xi}) \leq c_2(\xi_2)$, which contradicts the hypothesis that $c_1(\xi_1) > c_2(\xi_2)$. Therefore, $\xi_1 > \hat{\xi}$. In particular, $\xi_1 > 0$.

Because $\xi_1 > \hat{\xi}$ and, consequently, $\xi_2 < M - \hat{\xi}$, according to A3.1 and A3.9, $\xi_1\, c_1'(\xi_1) > \hat{\xi}\, c_1'(\hat{\xi}) \geq (M - \hat{\xi})\, c_2'(M - \hat{\xi}) > \xi_2\, c_2'(\xi_2)$. Thus, $\xi_1\, c_1'(\xi_1) > \xi_2\, c_2'(\xi_2)$.

In this case, notice that $N \geq 1$ because, otherwise, $T^0 = M$, i.e. all the players are individuals, and the hypothesis that $c_1(\xi_1) > c_2(\xi_2)$ implies that all the individuals are on arc 2 because of (3.1), hence $c_1(0) > c_2(M)$, which contradicts A3.9.

Because $c_1(\xi_1) > c_2(\xi_2)$, it follows from Theorem 3.8 that there exists some $l \leq N$ such that $x_1^i > 0, x_2^i > 0$ for $i \in [\![1, l]\!]$ and, if $l < N$, $x_1^i = 0, x_2^i = T^i$ for $i \in [\![l+1, N]\!]$. According to (3.2),

$$c_1(\xi_1) + x_1^i\, c_1'(\xi_1) = c_2(\xi_2) + x_2^i\, c_2'(\xi_2), \qquad i \in [\![1, l]\!]. \tag{3.5}$$

Besides, if $T^0 > 0$, then $x_1^0 = 0$ and $x_2^0 = T^0$ because of (3.1). Summing (3.5) over all $i \in [\![1, l]\!]$ leads to

$$l\, c_1(\xi_1) + \xi_1\, c_1'(\xi_1) \ = \ l\, c_2(\xi_2) + (\xi_2 - \sum_{i=l+1}^{N} T^i - T^0)\, c_1'(\xi_1) \ \leq \ l\, c_2(\xi_2) + \xi_2\, c_1'(\xi_1),$$

and the equality holds if, and only if, $T^0 = 0$ and $l = N$. But this is impossible because, otherwise, $c_1(\xi_1) = c_2(\xi_2)$, which contradicts the hypothesis that $c_1(\xi_1) > c_2(\xi_2)$.

Thus, $c_1(\xi_1) \leq c_2(\xi_2)$.

2. If $c_2(0) \leq c_1(M)$ and $\xi_1 > \hat{\xi}$, then by A3.1, $c_1(\xi_1) > c_1(\hat{\xi}) = c_2(M - \hat{\xi}) > c_2(\xi_2)$, which contradicts the fact that $c_1(\xi_1) \leq c_2(\xi_2)$.

3. Recall that h is strictly decreasing. If $c_1(M) < c_2(0)$ or, equivalently, $H > 0$, the fact that $\xi_1 \leq M$ implies that $h(\xi_1) \geq h(M) = H > 0$. If $c_2(0) \leq c_1(M)$, the fact that $\xi_1 \leq \hat{\xi}$ and the monotonicity of h imply that $h(\xi_1) \geq h(\hat{\xi}) = 0$, and the equality holds if and only if $\xi_1 = \hat{\xi}$. □

Combining the previous results, only three modes of $\mathbf{x}$ of the game $\Gamma(\mathbf{T})$ are possible. They will be analyzed below.

3.3.2 Mode 1

The CE $\mathbf{x}$ is of *mode 1* if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); \\ x_1^i = T^i, \qquad i \in 0 \cup [\![1, N]\!]. \end{cases} \tag{3.6}$$

Mode 1	T^1	$\cdots$	T^N	T^0
arc 1	T^1	$\cdots$	T^N	T^0
arc 2	0	$\cdots$	0	0

Because $\xi_1 = M$ and $c_1(\xi_1) < c_2(\xi_2)$, $c_1(M) < c_2(0)$ or, equivalently, $H > 0$.

If $N \geq 1$, it follows from (3.2) that, for $i \in [\![1, N]\!]$, $c_1(M) + T^i\, c_1'(M) \leq c_2(0)$ or, equivalently, $T^i \leq H$.

These results yield the following proposition.

Proposition 3.13. *The sufficient and necessary conditions for the CE* $\mathbf{x}$ *of* $\Gamma(\mathbf{T})$ *to be of mode 1* (cf. (3.6)) *are the following:*

$$H > 0; \tag{3.7}$$

$$\textit{if } N \geq 1, \textit{ then } T^i \leq H \textit{ for all } i \in [\![1, N]\!]. \tag{3.8}$$

3.3.3 Mode 2

The CE $\mathbf{x}$ is of *mode 2* if

$$c_1(\xi_1) = c_2(\xi_2). \tag{3.9}$$

Mode 2	T^1	$\cdots$	T^N	T^0
arc 1	x_1^1	$\cdots$	x_1^N	x_1^0
arc 2	x_2^1	$\cdots$	x_2^N	x_2^0

According to Proposition 3.4 and A3.9, the fact that $c_1(\xi_1) = c_2(\xi_2)$ implies that $H \leq 0$ and $\xi_1 = \hat{\xi}$.

If $N \geq 1$, then it follows from (3.2) that $c_1(\xi_1) + x_1^i\, c_1'(\xi_1) = c_2(\xi_2) + x_2^i\, c_2'(\xi_2)$ for $i \in [\![1, N]\!]$. This is equivalent to $x_1^i = \frac{AT^i}{1+A}$ because $c_1(\xi_1) = c_2(\xi_2)$. The flow is

$$x_1^i = \frac{A\,T^i}{1 + A}, \qquad i \in [\![1, N]\!]; \tag{3.10a}$$

$$x_1^0 = \hat{\xi} - \frac{A\,T^{[N]}}{1 + A}. \tag{3.10b}$$

Because $0 \leq x_1^0 \leq T^0$, one deduces from (3.10) and (3.4) that $T^{[N]} < (M - \hat{\xi}\,)(1 + A)$. These results give rise to the following proposition.

Proposition 3.14. *The sufficient and necessary conditions for the CE* $\mathbf{x}$ *of* $\Gamma(\mathbf{T})$ *to be of mode 2* (cf. (3.9)) *are the following:*

$$H \leq 0; \tag{3.11}$$

$$\textit{if } N \geq 1, \textit{ then } T^{[N]} \leq (M - \hat{\xi}\,)(1 + A). \tag{3.12}$$

3.3.4 Mode 3

The CE $\mathbf{x}$ is of *mode 3 and specified by* k if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); \\ N \geq 1,\ 1 \leq k \leq N; \\ x_2^i > 0, \qquad i \in [\![1, k]\!]; \\ x_2^i = 0, \qquad i \in \{0\} \text{ and, if } k < N,\ i \in [\![k+1, N]\!]. \end{cases} \tag{3.13}$$

Mode 3	T^1	$\cdots$	T^k	T^{k+1}	$\cdots$	T^N	T^0
arc 1	x_1^1	$\cdots$	x_1^k	T^{k+1}	$\cdots$	T^N	T^0
arc 2	x_2^1	$\cdots$	x_2^k	0	$\cdots$	0	0

The following two results follow from (3.2):
(i) if $N > k$, then, for $i \in [\![k{+}1, N]\!]$, $c_1(\xi_1) {+} T^i\, c_1'(\xi_1) \leq c_2(\xi_2)$ or, equivalently, $T^i \leq h(\xi_1)$;
(ii) for $i \in [\![1, k]\!]$, $c_1(\xi_1) {+} x_1^i\, c_1'(\xi_1) = c_2(\xi_2) {+} x_2^i\, c_2'(\xi_2)$ or, equivalently, $x_1^i = \frac{T^i\, a(\xi_1) + h(\xi_1)}{1 + a(\xi_1)}$.

It is not difficult to deduce the following two results from the constraint that $0 < x_1^i < T^i$:
(i) $T^i > -h(\xi_1)/a(\xi_1)$, which is always true, because $h(\xi_1) \geq 0$ and $a(\xi_1) > 0$;
(ii) $T^i > h(\xi_1)$.
Besides, $\xi_1 = \sum_{i=1}^k x_1^i + M - T^{[k]} = M - \frac{T^{[k]} - k\, h(\xi_1)}{1 + a(\xi_1)}$ or, equivalently,

$$F_k(\xi_1) = T^{[k]}.$$

On account of the strict monotonicity of F_k, ξ_1 is also the unique solution to the equation $F_k(\cdot) = T^{[k]}$.

Therefore, the total weight on arc 1 is the unique solution to equation (3.14) below, and the weights of the atomic (*resp.* nonatomic) players on arc 1 is given by (3.15) and (3.16) (*resp.* (3.17)):

$$\xi_1 = M - \frac{T^{[k]} - k\, h(\xi_1)}{1 + a(\xi_1)}; \tag{3.14}$$

$$x_1^i = \frac{T^i\, a(\xi_1) + h(\xi_1)}{1 + a(\xi_1)}, \qquad i \in [\![1, k]\!]; \tag{3.15}$$

$$\text{If } k < N, \quad x_1^i = T^i, \qquad i \in [\![k+1, N]\!]; \tag{3.16}$$

$$x_1^0 = T^0. \tag{3.17}$$

If $H \leq 0$, then $\xi_1 < \hat{\xi}$ according to Proposition 3.12. This implies that $T^{[k]} = F_k(\xi_1) > F_k(\hat{\xi}) = (M - \hat{\xi})(1 + A)$.
These results entail the following proposition.

Proposition 3.15. *The sufficient and necessary conditions for the CE* $\mathbf{x}$ *of* $\Gamma(\mathbf{T})$ *to be of mode 3 and specified by* k (cf. (3.13)) *are the following:*

$$\text{if } H \le 0, \quad T^{[k]} > (M - \hat{\xi})(1 + A) \;\; (\Leftrightarrow \xi_1 < \hat{\xi}\,); \tag{3.18}$$

$$T^i > h(\xi_1), \qquad i \in [\![1, k]\!]; \tag{3.19}$$

$$\text{if } k < N, \quad T^i \le h(\xi_1), \qquad i \in [\![k+1, N]\!], \tag{3.20}$$

where ξ_1 *is the unique solution to the equation* $F_k(\cdot) = T^{[k]}$*.*

Besides, the flow at $\mathbf{x}$ *is given by* (3.14)-(3.17)*.*

The following lemma will be needed later.

Lemma 3.16. *Suppose that the CE* $\mathbf{x}$ *of the* $\Gamma(\mathbf{T})$ *is of mode 3 and specified by* k*, and* $\mathbf{x}$*,* ξ_1 *are given by* (3.14)-(3.17)*. Let a function* ν *be defined on the real interval* $(-\eta,\, M + \eta\,)$ *by* $\nu(x) = x\, c_1(x) + (M - x)\, c_2(M - x)$*. Then, if* $x > \xi_1$*,* $\nu(x) > \nu(\xi_1)$*.*

Proof. It follows from (3.2) that $c_1(\xi_1) + x_1^i\, c_1'(\xi_1) = c_2(\xi_2) + x_2^i\, c_2'(\xi_2)$ for $i \in [\![1, k]\!]$. Summing the equation over all $i \in [\![1, k]\!]$ leads to $k\, c_1(\xi_1) + [\, \xi_1 - (M - T^{[k]})\,]\, c_1'(\xi_1) = k\, c_2(\xi_2) + \xi_2\, c_2'(\xi_2)$. Consequently, $c_1(\xi_1) + \xi_1\, c_1'(\xi_1) = c_2(\xi) + \xi_2\, c_2'(\xi_2) + (k-1)[\, c_2(\xi_2) - c_1(\xi_1)\,] + (M - T^{[k]})\, c_1'(\xi_1)$.

According to Proposition 3.12, $c_1(\xi_1) \le c_2(\xi_2)$. Besides, $k \ge 1$. Thus, there exists a constant $B > 0$ such that $c_1(\xi_1) + \xi_1\, c_1'(\xi_1) \ge B \ge c_2(\xi_2) + \xi_2\, c_2'(\xi_2)$. According to A3.1, c_1 and c_2 are both strictly increasing while c_1' and c_2' are non-decreasing. Hence, for any s such that $\xi_1 < s \le M$ and any t such that $0 \le t < \xi_2$,

$$c_1(s) + s\, c_1'(s) > B > c_2(t) + t\, c_2'(t). \tag{3.21}$$

For any $x \in (\,\xi_1, M\,]$,

$$\begin{aligned}
&\nu(x) - \nu(\xi_1) \\
&= [\, x\, c_1(x) + (M - x)\, c_2(M - x)\,] - [\, \xi_1\, c_1(\xi_1) + (M - \xi_1)\, c_2(M - \xi_1)\,] \\
&= [\, x\, c_1(x) - \xi_1\, c_1(\xi_1)\,] - [\, (M - \xi_1)\, c_2(M - \xi_1) - (M - x)\, c_2(M - x)\,] \\
&= \int_{\xi_1}^{x} [u\, c_1(u)]'\, \mathrm{d}u - \int_{M-x}^{M-\xi_1} [u\, c_2(u)]'\, \mathrm{d}u \\
&= \int_{\xi_1}^{x} [c_1(u) + u\, c_1'(u)]\, \mathrm{d}u - \int_{M-x}^{M-\xi_1} [c_2(u) + u\, c_2'(u)]\, \mathrm{d}u \\
&> (x - \xi_1)\, B - (M - \xi_1 - M + x)\, B \\
&= 0,
\end{aligned}$$

where the inequality is due to (3.21). □

3.4 The composition of the players and the CE costs

3.4.1 Two scenarios

The arc cost functions $\mathbf{c}$ and the total weight M of the players are fixed. We will focus on the relation between the composition of the players and the social cost, the individuals' cost as well as the (per-unit) costs to the atomic players at the CE. In particular, we will be interested in how these values change when an atomic player l is replaced by a composite set of players, i.e. a group of individual and several atomic players, who together hold the same stocks T^l as her.

Let us consider two scenarios.

Scenario 1 The atomic player l is replaced by a group of individuals of total weight T^l. The new profile of $N-1$ atomic players (with non-increasing sizes) and a group of individuals is denoted by $\boldsymbol{\gamma} = (\gamma^0, \gamma^1, \ldots, \gamma^{N-1})$ such that there is a bijection σ from $[\![1, N]\!] \setminus \{l\}$ to $[\![1, N-1]\!]$ defined by

$$\sigma(i) = \begin{cases} i, & \text{if } i \in [\![1, l-1]\!]; \\ i-1, & \text{if } l \leq N-1 \text{ and } i \in [\![l+1, N]\!]. \end{cases} \tag{3.22}$$

Then, $T^i = \gamma^{\sigma(i)}$ for all $i \in [\![1, N]\!] \setminus \{l\}$. The inverse function of σ is denoted by σ^{-1}. In other words, for any atomic player i in the profile $\mathbf{T}$ other than l, $\sigma(i)$ is her name in the profile $\boldsymbol{\gamma}$. Conversely, for any atomic player i in $\boldsymbol{\gamma}$, $\sigma^{-1}(i)$ is her name in $\mathbf{T}$. As a result,

$$\begin{cases} \gamma^i = T^i, & \gamma^{[i]} = T^{[i]}, & \text{if } i \in [\![1, l-1]\!]; \\ \gamma^i = T^{i+1}, & \gamma^{[i]} = T^{[i+1]} - T^l, & \text{if } l \leq N-1 \text{ and } i \in [\![l, N-1]\!]; \\ \gamma^0 = T^0 + T^l, & & \end{cases} \tag{3.23}$$

where $\gamma^{[p]} = \sum_{i=1}^p \gamma^i$.

In the game $\Gamma(\boldsymbol{\gamma})$, let us denote the unique CE by $\mathbf{y}$, and $\eta_r = \sum_{i=0}^N y_r^i$. The social cost $v(\mathbf{y})$ and the players' costs $Y^0(\mathbf{y})$, $Y^i(\mathbf{y})$, $u^i(\mathbf{y})$ are defined in the same way for $\mathbf{y}$ as for $\mathbf{x}$.

Scenario 2 The atomic player l is replaced by two atomic players of total weight T^l. The new profile of $N+1$ atomic players (with non-increasing sizes) and a (possibly empty) set of individuals is denoted by $\boldsymbol{\tau} = (\tau^0, \tau^1, \ldots, \tau^{N+1})$ such that there is an injection π from $[\![1, N]\!] \setminus \{l\}$ to $[\![1, N+1]\!]$ which satisfies:

$$\begin{cases} T^i = \tau^{\pi(i)}, & \text{for } i \in [\![1, N]\!] \setminus \{l\}; \\ T^l = \tau^p + \tau^q, & \text{where } \{p, q\} = [\![1, N+1]\!] \setminus \pi[\,[\![1, N]\!] \setminus \{l\}\,], \end{cases} \tag{3.24}$$

where $\pi[\,[\![1, N]\!] \setminus \{l\}\,]$ stands for the image of π.

Indeed, π is a bijection from $[\![1, N]\!] \setminus \{l\}$ to $[\![1, N+1]\!] \setminus \pi[\,[\![1, N]\!] \setminus \{l\}\,]$, and the inverse of π from the latter to the former is denoted by π^{-1}. The interpretation of π is similar to that of σ in Scenario 1.

Notice that π can be defined in such a way that

$$\begin{cases} i \leq \pi(i) \leq i+2, & \text{if } i \in [\![1, l-1]\!]; \\ \pi(i) = i+1, & \text{if } l \leq N-1 \text{ and } i \in [\![l+1, N]\!]. \end{cases}$$

As a result,

$$\begin{cases} \tau^i = T^i, & \tau^{[i]} = T^{[i]}, & \text{if } i \in [\![\,1, l-1\,]\!]; \\ T^{i+1} \leq \tau^i \leq T^{i-1}, & T^{[i-1]} \leq \tau^{[i]} < T^{[i]}, & \text{if } l \leq N-1 \text{ and } i \in [\![\,l, N\,]\!]; \\ \tau^0 = T^0,\ \tau^{N+1} \leq T^N, & \tau^{[N+1]} = T^{[N]}, & \end{cases} \tag{3.25}$$

where $\tau^{[p]} = \sum_{i=1}^p \tau^i$.

In the game $\Gamma(\boldsymbol{\tau})$, let us denote the CE by $\mathbf{z}$, and $\zeta_r = \sum_{i=0}^N z_r^i$. The social cost $v(\mathbf{z})$ and the players' costs $Y^0(\mathbf{z})$, $Y^i(\mathbf{z})$, $u^i(\mathbf{z})$ are defined in the same way for $\mathbf{z}$ as for $\mathbf{x}$.

The following two sections study how the equilibrium costs in $\Gamma(\boldsymbol{\gamma})$ and $\Gamma(\boldsymbol{\tau})$ change with respect to those in $\Gamma(\mathbf{T})$. In §3.4.3, the cases where the equilibrium costs do not change are discussed, while §3.4.2 is focused on the cases where they change.

Before this, let us recall that, according to Proposition 3.12, $c_1(\xi_1) \leq c_2(\xi_2)$, $c_1(\eta_1) \leq c_2(\eta_2)$ and $c_1(\zeta_1) \leq c_2(\zeta_2)$.

3.4.2 Cases where the equilibrium costs do not change

In three cases, the equilibrium costs do not change after the replacement of the atomic player l in both scenarios: if $\mathbf{x}$ is of mode 1, or mode 2, or mode 3 and specified by k but $k < l$.

Lemma 3.17. *Suppose that the CE $\mathbf{x}$ of $\Gamma(\mathbf{T})$ is of mode 1* (cf. (3.6))*. For any $l \in [\![1, N]\!]$, let two profiles $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ be defined by* (3.23) *and* (3.24) *respectively. Then,*

$$v(\mathbf{x}) = v(\mathbf{y}) = v(\mathbf{z});\ Y^0(\mathbf{x}) = Y^0(\mathbf{y}) = Y^0(\mathbf{z});\ Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y}) = Y^{\pi(i)}(\mathbf{z}),\ \forall i \in [\![1, N]\!]\backslash\{l\},$$

where $\mathbf{y}$ is the CE of $\Gamma(\boldsymbol{\gamma})$ and $\mathbf{z}$ is the CE of $\Gamma(\boldsymbol{\tau})$.

Proof. According to (3.22),$\gamma^i = T^{\sigma^{-1}(i)} \leq H$ for $i \in [\![1, N-1]\!]$, where the inequality is due to fact that $\mathbf{x}$ is of mode 1. Similarly, $\tau^i = T^{\pi^{-1}(i)} \leq H$ for $i \in \pi[\,[\![1, N]\!] \setminus \{l\}\,]$, and $\tau^i < T^l \leq H$ for $i \in [\![1, N+1]\!] \setminus \pi[\,[\![1, N]\!] \setminus \{l\}\,]$.

Therefore, the conditions (3.7) and (3.8) are satisfied in $\Gamma(\boldsymbol{\gamma})$ and $\Gamma(\boldsymbol{\tau})$, with T^i being replaced by, respectively, γ^i $(1 \leq i \leq N-1)$ and τ^i $(1 \leq i \leq N+1)$. It follows from Proposition 3.13 that both $\mathbf{y}$ and $\mathbf{z}$ are of mode 1. Consequently, $v(\mathbf{x}) = v(\mathbf{y}) = v(\mathbf{z}) = M\,c_1(M)$, $Y^0(\mathbf{x}) = Y^0(\mathbf{y}) = Y^0(\mathbf{z}) = c_1(M)$ and $Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y}) = Y^{\pi(i)}(\mathbf{z}) = c_1(M)$ for $i \in [\![1, N]\!] \setminus \{l\}$. □

Lemma 3.18. *Suppose that the CE $\mathbf{x}$ of $\Gamma(\mathbf{T})$ is of mode 2* (cf. (3.9))*. For any $l \in [\![1, N]\!]$, let two profiles $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ be defined by* (3.23) *and* (3.24) *respectively. Then,*

$$v(\mathbf{x}) = v(\mathbf{y}) = v(\mathbf{z});\ \ Y^0(\mathbf{x}) = Y^0(\mathbf{y}) = Y^0(\mathbf{z});\ \ Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y}) = Y^{\pi(i)}(\mathbf{z}),\ \forall i \in [\![1, N]\!]\backslash\{l\},$$

where $\mathbf{y}$ is the CE of $\Gamma(\boldsymbol{\gamma})$ and $\mathbf{z}$ is the CE of $\Gamma(\boldsymbol{\tau})$.

Proof. According to (3.23) and (3.12), $0 \leq \gamma^{[N-1]} = T^{[N]} - T^l < T^{[N]} < (M - \hat{\xi}\,)(1 + A)$. Similarly, $0 \leq \tau^{[N+1]} = T^{[N]} < (M - \hat{\xi}\,)(1 + A)$.

Therefore, the conditions (3.11) and (3.12) are satisfied in $\Gamma(\boldsymbol{\gamma})$ and $\Gamma(\boldsymbol{\tau})$, with $T^{[N]}$ being replaced by, respectively, $\gamma^{[N-1]}$ and $\tau^{[N+1]}$. Then, it follows from Proposition 3.14 that $\mathbf{y}$ and $\mathbf{z}$ are of mode 2. Consequently, $v(\mathbf{x}) = v(\mathbf{y}) = v(\mathbf{z}) = M\,c_1(\hat{\xi})$, $Y^0(\mathbf{x}) = Y^0(\mathbf{y}) = Y^0(\mathbf{y}) = c_1(\hat{\xi})$, and $Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y}) = Y^{\pi(i)}(\mathbf{y}) = c_1(\hat{\xi})$ for $i \in [\![1, N]\!] \setminus \{l\}$. □

Lemma 3.19. *Suppose that the CE $\mathbf{x}$ of $\Gamma(\mathbf{T})$ is of mode 3 and specified by k, where $k < N$* (cf. (3.13))*, and $\mathbf{x}$ is given by* (3.14)*-*(3.17)*. For any $l \in [\![k+1, N]\!]$, let two profiles $\boldsymbol{\gamma}$ and $\boldsymbol{\tau}$ be defined by* (3.23) *and* (3.24) *respectively. Then,*

$$v(\mathbf{x}) = v(\mathbf{y}) = v(\mathbf{z});\ \ Y^0(\mathbf{x}) = Y^0(\mathbf{y}) = Y^0(\mathbf{z});\ \ Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y}) = Y^{\pi(i)}(\mathbf{z}),\ \forall i \in [\![1, N]\!]\backslash\{l\},$$

where $\mathbf{y}$ is the CE of $\Gamma(\boldsymbol{\gamma})$ and $\mathbf{z}$ is the CE of $\Gamma(\boldsymbol{\tau})$.

Proof. Let us prove that the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$ is of mode 3 and specified by k. For $\Gamma(\boldsymbol{\tau})$, the proof is similar.

For $i \in [\![1, k]\!]$, $\gamma^i = T^i$ by the definition (3.22); in particular, $\gamma^{[k]} = T^{[k]}$. Besides, it follows from Proposition 3.15 that $T^i > h(\xi_1)$, thus, $\gamma^i > h(\xi_1)$.

For $i \in [\![k+1, N-1]\!]$, $\gamma^i = T^i$ or $\gamma^i = T^{i+1}$ by the definition (3.22). According to Proposition 3.15, $T^{i+1} \leq T^i < h(\xi_1)$ and thus $\gamma^i < h(\xi_1)$

If $H \leq 0$, $\gamma^{[k]} = T^{[k]} \geq (M - \hat{\xi})(1 + A)$, where the inequality is due to Proposition 3.15.

By combining these three results and the fact that ξ_1 is the unique solution to the equation $F_{k+1}(\cdot) = T^{[k]} = \gamma^{[k]}$, one deduces from Proposition 3.15 that the CE $\mathbf{y}$ of the game $\Gamma(\boldsymbol{\gamma})$ is of mode 3 and specified by k. Furthermore, for $y_1^i = \frac{\gamma^i a(\xi_1)+h(\xi_1)}{1+a(\xi_1)} = x_1^i$ for $i \in [\![1,k]\!]$, $y_1^i = \gamma^i$ for $i \in [\![k+1, N-1]\!]$, $y_1^0 = T^0 + T^l$ and $\eta_1 = \xi_1$.

Because $\eta_1 = \xi_1$ and thus $\eta_2 = \xi_2$, $v(\mathbf{x}) = \xi_1\, c_1(\xi_1) + \xi_2\, c_2(\xi_2) = \eta_1\, c_1(\eta_1) + \eta_2\, c_2(\eta_2) = v(\mathbf{y})$. Therefore, the social costs $v(\mathbf{x})$ and $v(\mathbf{y})$ are equal.

For the individuals, $Y^0(\mathbf{x}) = c_1(\xi_1) = c_1(\eta_1) = Y^0(\mathbf{y})$.

For the atomic player $i \in [\![1,k]\!]$, according to (3.22), $\sigma(i) = i$. Because $y_1^i = x_1^i$ and, consequently, $y_2^i = x_2^i$, $u^i(\mathbf{x}) = x_1^i c_1(\xi_1) + x_2^i c_2(\xi_2) = y_1^i c_1(\eta_1) + y_2^i c_2(\eta_2) = u^{\sigma(i)}(\mathbf{y})$ or, equivalently, $Y^i(\mathbf{x}) = Y^{\sigma(i)}(\mathbf{y})$.

For the atomic player $i \in [\![k+1, N]\!] \setminus \{l\}$, according to (3.22), $\sigma(i) > k$. Thus, $Y^i(\mathbf{x}) = c_1(\xi_1) = c_1(\eta_1) = Y^{\sigma(i)}(\mathbf{y})$. □

3.4.3 Cases where the equilibrium costs change

If $\mathbf{x}$ is of mode 3 and specified by k but $l \le k$, then the equilibrium costs change after the replacement of player l, both in $\Gamma(\boldsymbol{\gamma})$ and $\Gamma(\boldsymbol{\tau})$.

First, let us consider Scenario 1.

Lemma 3.20. *Suppose that the CE $\mathbf{x}$ of $\Gamma(\mathbf{T})$ is of mode 3 and specified by k* (cf. (3.13)), *and $\mathbf{x}$ is given by* (3.14)-(3.17). *For any l such that $1 \le l \le k$, let profile $\boldsymbol{\gamma}$ be defined by* (3.23). *Then,*

$$v(\mathbf{x}) < v(\mathbf{y});\;\; Y^0(\mathbf{x}) < Y^0(\mathbf{y});\;\; Y^i(\mathbf{x}) < Y^{\sigma(i)}(\mathbf{y}),\ \forall i \in [\![1,N]\!] \setminus \{l\},$$

where $\mathbf{y}$ is the CE of $\Gamma(\boldsymbol{\gamma})$.

Proof. The proof is made in 3 steps.
1) Let us show that the equation

$$F_{k-1}(x) = \gamma^{[k-1]} \tag{3.26}$$

admits a unique solution w_1. Besides, $\xi_1 < w_1 < M$ if $k > 1$, and $w_1 = M$ if $k = 1$.

Notice that $\gamma^{[k-1]} = T^{[k]} - T^l$ according (3.24), and $F_k(\xi_1) = T^{[k]}$ according to (3.14).

On the one hand,

$$F_{k-1}(\xi_1) = F_k(\xi_1) - h(\xi_1) = T^{[k]} - h(\xi_1) > T^{[k]} - T^l \begin{cases} = 0, & \text{if } k = 1; \\ > 0, & \text{if } k > 1, \end{cases}$$

where the first inequality is due to (3.19).

On the other hand,

$$F_{k-1}(M) = (k-1)\, h(M) \begin{cases} = 0 = T^{[k]} - T^l, & \text{if } k = 1; \\ < (k-1)\, h(\xi_1) < T^{[k]} - T^l, & \text{if } k > 1, \end{cases}$$

where the first inequality follows from the fact that $\xi_1 < M$ and the strict monotonicity of h, and the second inequality is due to (3.19).

By combining these two results while noticing that F_{k-1} is strictly decreasing, one deduces that the equation (3.26) admits a unique solution w_1, and $\xi_1 < w_1 < M$ if $k > 1$ while $w_1 = M$ if $k = 1$.

2) Suppose that the following assumption A3.21 holds.

A 3.21. *For some $p \in [\![1, N-k+1]\!]$, a sequence of numbers $w_1, w_2, \dots, w_p$ is obtained such that*

(i) $\xi_1 < w_p < \cdots < w_1 \le M$*;*

(ii) for $q = 1, \dots, p$, $F_{k+q-2}(w_q) = \gamma^{[k+q-2]}$*;*

(iii) if $H > 0$ *and* $p \ge 2$ *for* $q = 1, \dots, p-1$, $\gamma^{k+q-1} > h(w_q)$*;*

(iv) if $H \le 0$ *and* $p \ge 2$*, for* $q = 1, \dots, p-1$*, either* $\gamma^{k+q-1} > h(w_q)$ *or* $w_q \ge \hat{\xi}$*.*

Let us prove the two statements below by analyzing the five possible cases.

1. If $p = N-k+1$, one can find the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$.

2. If $p \le N-k$, either one can find the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$, or one can go on and find w_{p+1} such that A3.21 holds for $p+1$.

CASE 1. $p \le N-k$ and $\gamma^{k+p-1} > h(w_p)$.

One can show that the equation

$$F_{k+p-1}(\cdot) = \gamma^{[k+p-1]} \tag{3.27}$$

admits a unique solution w_{p+1}, and $\xi_1 < w_{p+1} < w_p$.

Indeed, on the one hand, $F_{k+p-1}(\xi_1) = F_k(\xi_1) + (p-1)\,h(\xi_1) = T^{[k]} + (p-1)\,h(\xi_1) \ge T^{[k]} + T^{k+1} + \cdots + T^{k+p-1} = T^{[k+p-1]} = \gamma^{[k+p-2]} + T^l = \gamma^{[k+p-1]} - T^{k+p} + T^l > \gamma^{[k+p-1]}$, where the first inequality is due to (3.20), and the last two equalities are due to (3.23).

On the other hand, $F_{k+p-1}(w_p) = F_{k+p-2}(w_p) + h(w_p) = \gamma^{[k+p-2]} + h(w_p) < \gamma^{[k+p-2]} + \gamma^{k+p-1} = \gamma^{[k+p-1]}$, where the second equality is due A3.21(iii) and the inequality is due to the hypothesis that $\gamma^{k+p-1} > h(w_p)$.

By combining these two results while noticing that F_{k+p-1} is strictly decreasing, one deduces that the equation (3.27) admits a unique solution w_{p+1}, and $\xi_1 < w_{p+1} < w_p$.

CASE 2. $H \le 0$; $p \le N-k$ and $w_p \ge \hat{\xi}$.

One can show that the equation $F_{k+p-1}(\cdot) = \gamma^{[k+p-1]}$ admits a unique solution w_{p+1}, and $\xi_1 < w_{p+1} < w_p$.

Indeed, on the one hand, $F_{k+p-1}(\xi_1) > \gamma^{[k+p-1]}$ as in Case 1. On the other hand, $F_{k+p-1}(w_p) = F_{k+p-2}(w_p) + h(w_p) = \gamma^{[k+p-2]} + h(w_p) \le \gamma^{[k+p-2]} + h(\hat{\xi}) = \gamma^{[k+p-2]} < \gamma^{[k+p-1]}$, where the inequality is due to the hypothesis that $w_p \ge \hat{\xi}$.

CASE 3. $H > 0$; either $p \le N-k$ and $\gamma^{k+p-1} \le h(w_p)$, or $p = N-k+1$.

Suppose that $w_p = M$. Then, according to A3.21(i), $p = 1$. Step 1 of the proof shows that $k = 1$. In other words, there is no atomic player in the profile $\boldsymbol{\gamma}$, and the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$ is thus of mode 1. Consequently, $\eta_1 = M > \xi_1$. If $N \ge 2$, $y_1^{\sigma(i)} = x_1^i = T^i$ for $i \in [\![2, N]\!]$.

Suppose that $w^p < M$. First, let us show that $\gamma^{k+p-2} > h(w_p)$. There are two cases: $p = 1$ and $p \ge 2$.

In the case where $p = 1$, $\gamma^{k-1} = T^k > h(\xi_1) > h(w_1)$ according to (3.23), (3.19), A3.21(i) and the strict monotonicity of h.

In the case where $p \ge 2$. By A3.21(ii), $F_{k+p-2}(w_p) = \gamma^{[k+p-2]}$, hence $F_{k+p-3}(w_p) + h(w_p) = \gamma^{[k+p-3]} + \gamma^{k+p-2}$. If $\gamma^{k+p-2} \le h(w_p)$, then $F_{k+p-3}(w_p) \le \gamma^{[k+p-3]} = F_{k+p-3}(w_{p-1})$, where the equation is due to A3.21(ii). As a result, $w_p \ge w_{p-1}$ because F_{k+p-3} is decreasing. But this contradicts A3.21(i). Thus, $\gamma^{k+p-2} < h(w_p)$.

Therefore, for all $i \in [\![1, k+p-2]\!]$, $\gamma^i \ge \gamma^{k+p-2} > h(w_p)$.

If $p \le N-k$, for all $i \in [\![k+p-1, N-1]\!]$, $\gamma^i \le \gamma^{k+p-1} \le h(w_p)$ according to the hypothesis that $\gamma^{k+p-1} \le h(w_p)$.

By combining these two results and the fact that $F_{k+p-2}(w_p) = \gamma^{[k+p-2]}$, it follows from Proposition 3.15 that the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$ is of mode 3 and specified by $k+p-2$. The same proposition further implies that

$$y_1^i = \frac{\gamma^i\, a(w_p) + h(w_p)}{1 + a(w_p)}, \qquad \text{if } i \in [\![1, k+p-2]\!]; \tag{3.28}$$

$$y_1^i = \gamma^i, \qquad \text{if } p \le N-k, \text{ and } i \in [\![k+p-1, N-1]\!]; \tag{3.29}$$

$$y_1^0 = \gamma^0, \tag{3.30}$$

$$\eta_1 = w_p. \tag{3.31}$$

Next, let us prove the two statements below.

1. For $i \in [\![1, k+p-1]\!] \setminus \{l\}$, $T^i \ge x_1^i > y_1^{\sigma(i)}$.
2. If $N \ge k+p$, then $x_1^i = y_1^{\sigma(i)} = T^i$ for $i \in [\![k+p, N]\!]$.

If $N \ge k+1$, then, according to (3.22), $\sigma(i) = i-1$ for $i \in [\![k+1, N]\!]$.

On the one hand, if $p \le N-k$, for $i \in [\![k+p, N]\!]$, it follows from (3.29) that $x_1^i = T^i = \gamma^{i-1} = y_1^{\sigma(i)}$.

On the other hand, if $p \ge 2$, for $i \in [\![k+1, k+p-1]\!]$, $x_1^i = T^i = \gamma^{i-1} > \frac{\gamma^{i-1}\, a(\eta_1) + h(\eta_1)}{1+a(\eta_1)} = y_1^{\sigma(i)}$, where the inequality is due to the fact that $\gamma^{i-1} > h(w_p) = h(\eta_1)$, and the equality is due to (3.28).

For $i \in [\![1, k]\!] \setminus \{l\}$, $x_1^i = \frac{T^i\, a(\xi_1) + h(\xi_1)}{1+a(\xi_1)}$ by (3.15). Besides, as $\sigma(i) \le k+p-2$ by (3.22), $y_1^{\sigma(i)} = \frac{T^i\, a(\eta_1) + h(\eta_1)}{1+a(\eta_1)}$ by (3.28). One can show that $x_1^i > y_1^{\sigma(i)}$. Indeed, this is equivalent to $T^i - x_1^i < T^i - y_1^{\sigma(i)}$ or, still, $\frac{T^i - h(\xi_1)}{1+a(\xi_1)} < \frac{T^i - h(\eta_1)}{1+a(\eta_1)}$. The last inequality is true because, since $\xi_1 < \eta_1$, one has $a(\xi_1) \ge a(\eta_1)$ and $h(\xi_1) > h(\eta_1)$.

Case 4. $H \le 0$; either $p \le N-k$, $\gamma^{k+p-1} \le h(w_p)$ and $w_p < \hat{\xi}$, or $p = N-k+1$ and $w_p < \hat{\xi}$.

The same analysis as in Case 3 and the fact that $\gamma^{[k+p-2]} = F_{k+p-2}(w_p) > F_{k+p-2}(\hat{\xi}) = (M - \hat{\xi})(1+A)$ show that the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$ is given by (3.28)-(3.30).

Case 5. $H \le 0$; $p = N-k+1$ and $w_p \ge \hat{\xi}$.

It follows from A3.21(ii), the monotonicity of F_{N-1} and the hypothesis that $w_p \ge \hat{\xi}$ that,

$$\gamma^{[N-1]} = F_{N-1}(w_p) \le F_{N-1}(\tilde{\xi}) = (M - \tilde{\xi}\,)(1 + A\,).$$

Proposition 3.14 implies that the CE $\mathbf{y}$ of $\Gamma(\boldsymbol{\gamma})$ is of mode 2. Besides, for $i \in [\![1, N-1]\!]$, $y_1^i = \frac{A\gamma^i}{1+A}$; $y_1^0 = \hat{\xi} - \frac{A\gamma^{[N-1]}}{1+A}$; $\eta_1 = \sum_{i=0}^N y_1^i = \hat{\xi} > \xi_1$.

Next, let us show that $x_1^i > y_1^{\sigma(i)}$ for all $i \in [\![1, N]\!] \setminus \{l\}$.

If $k < N$, for $i \in [\![k+1, N]\!]$, $\sigma(i) = i-1$, $\gamma^{i-1} = T^i$. Thus, $x_1^i = T^i > \frac{A\,T^i}{1+A} = \frac{A\,\gamma^{i-1}}{1+A} = y_1^{\sigma(i)}$.

For $i \in [\![1, k]\!] \setminus \{l\}$, $x_1^i = \frac{T^i\, a(\xi_1) + h(\xi_1)}{1+a(\xi_1)}$ and $y_1^{\sigma(i)} = \frac{A\gamma^{\sigma(i)}}{1+A} = \frac{T^i\, a(\hat{\xi}) + h(\hat{\xi})}{1+a(\hat{\xi})}$. One can show that $x_1^i > y_1^{\sigma(i)}$. Indeed, this is equivalent to $T^i - x_1^i < T^i - y_1^{\sigma(i)}$ or, equivalently, $\frac{T^i - h(\xi_1)}{1+a(\xi_1)} < \frac{T^i - h(\hat{\xi})}{1+a(\hat{\xi})}$. The last inequality is true because, since $\xi_1 < \hat{\xi}$, one has $a(\xi_1) \ge a(\hat{\xi})$ and $h(\xi_1) > h(\hat{\xi})$.

3) Till now, one has shown that $\eta_1 > \xi_1$, $y_1^{\sigma(i)} < x_1^i$ for all $i \in [\![1, k]\!] \setminus \{l\}$, and, if $N > k$, $y_1^{\sigma(i)} \le x_1^i$ for all $i \in [\![k+1, N]\!]$.

The fact that $\eta_1 > \xi_1$ immediately implies that $Y^0(\mathbf{y}) = c_1(\eta_1) > c_1(\xi_1) = Y^0(\mathbf{x})$ and, according to Lemma 3.16, $v(\mathbf{y}) = \nu(\eta_1) > \nu(\xi_1) = v(\mathbf{x})$.

Let us show that $Y^{\sigma(i)}(\mathbf{y}) > Y^i(\mathbf{x})$ for all $i \in [\![1, N]\!] \setminus \{l\}$.

For $i \in [\![1, k]\!] \setminus \{l\}$, (3.2) implies that $c_1(\xi_1) + x_1^i c_1'(\xi_1) = c_2(\xi_2) + x_2^i c_2'(\xi_2)$ and, consequently, $x_1^i c_1'(\xi_1) > x_2^i c_2'(\xi_2)$ because $c_1(\xi_1) < c_2(\xi_2)$. Let B be a constant such that $x_1^i c_1'(\xi_1) > B > x_2^i c_2'(\xi_2)$. Thus, by the monotonicity of c_1 and c_2, for all $s \in (x_1^{-i}, M - x_1^i]$ and all $t \in [-x_2^i, x_2^{-i})$,

$$x_1^i c_1'(x_1^i + s) > B > x_2^i c_2'(x_2^i + t). \tag{3.32}$$

It follows from the relation $\eta_1 > \xi_1$ that $\eta_1 - x_1^i > x_1^{-i}$ and $\eta_2 - x_2^i < x_2^{-i}$. Therefore,

$$\begin{aligned}
&\left[\, x_1^i\, c_1(\eta_1) + x_2^i\, c_2(\eta_2) \,\right] - \left[\, x_1^i\, c_1(\xi_1) + x_2^i\, c_2(\xi_2) \,\right] \\
&= \left[\, x_1^i\, c_1(x_1^i + \eta_1 - x_1^i) + x_2^i\, c_2(x_2^i + \eta_2 - x_2^i) \,\right] - \left[\, x_1^i\, c_1(x_1^i + x_1^{-i}) + x_2^i\, c_2(x_2^i + x_2^{-i}) \,\right] \\
&= x_1^i \left[\, c_1(x_1^i + \eta_1 - x_1^i) - c_1(x_1^i + x_1^{-i}) \,\right] - x_2^i \left[\, c_1(x_2^i + x_2^{-i}) - c_2(x_2^i + \eta_2 - x_2^i) \,\right] \\
&= \int_{x_1^{-i}}^{\eta_1 - x_1^i} x_1^i\, c_1'(x_1^i + s)\, \mathrm{d}s - \int_{\eta_2 - x_2^i}^{x_2^{-i}} x_2^i\, c_2'(x_2^i + t)\, \mathrm{d}t \\
&> [\eta_1 - x_1^i - x_1^{-i}]\, B - [x_2^{-i} - \eta_2 + x_2^i]\, B = 0,
\end{aligned}$$

where the inequality is due to (3.32), and

$$\begin{aligned}
&\left[\, y_1^{\sigma(i)}\, c_1(\eta_1) + y_2^{\sigma(i)}\, c_2(\eta_2) \,\right] - \left[\, x_1^i\, c_1(\eta_1) + x_2^i\, c_2(\eta_2) \,\right] \\
&= \left[\, y_1^{\sigma(i)} - x_1^i \,\right] c_1(\eta_1) + \left[\, y_2^{\sigma(i)} - x_2^i \,\right] c_2(\eta_2) = \left[\, y_1^{\sigma(i)} - x_1^i \,\right] \left[\, c_1(\eta_1) - c_2(\eta_2) \,\right] \geq 0
\end{aligned}$$

because $y_1^{\sigma(i)} < x_1^i$ and $c_1(\eta_1) \leq c_2(\eta_2)$. As a result,

$$\begin{aligned}
u^{\sigma(i)}(\mathbf{y}) - u^i(\mathbf{x}) =& \left[\, y_1^{\sigma(i)}\, c_1(\eta_1) + y_2^{\sigma(i)}\, c_2(\eta_2) \,\right] - \left[\, x_1^i\, c_1(\xi_1) + x_2^i\, c_2(\xi_2) \,\right] \\
=& \left[\, y_1^{\sigma(i)}\, c_1(\eta_1) + y_2^{\sigma(i)}\, c_2(\eta_2) \,\right] - \left[\, x_1^i\, c_1(\eta_1) + x_2^i\, c_2(\eta_2) \,\right] \\
&+ \left[\, x_1^i\, c_1(\eta_1) + x_2^i\, c_2(\eta_2) \,\right] - \left[\, x_1^i\, c_1(\xi_1) + x_2^i\, c_2(\xi_2) \,\right] > 0.
\end{aligned}$$

If $N > k$, $x_1^i = T^i \geq y_1^{\sigma(i)}$ for $i \in [\![k+1, N]\!]$. Recall that $c_2(\eta_2) \geq c_1(\eta_1)$ and $\eta_1 > \xi_1$. Therefore,

$$u^{\sigma(i)}(\mathbf{y}) - u^i(\mathbf{x}) = \left[\, y_1^{\sigma(i)}\, c_1(\eta_1) + y_2^{\sigma(i)}\, c_2(\eta_2) \,\right] - T^i\, c_1(\xi_1) \geq T^i\, c_1(\eta_1) - T^i\, c_1(\xi_1) > 0.$$

□

Next, let us consider Scenario 2.

Lemma 3.22. *Suppose that the CE* $\mathbf{x}$ *of* $\Gamma(\mathbf{T})$ *is of mode 3 and specified by* k (cf. (3.13)), *and* $\mathbf{x}$ *is given by* (3.15)-(3.17). *For any* l *such that* $1 \leq l \leq k$, *let profile* $\boldsymbol{\tau}$ *be defined by* (3.24). *Then,*

$$v(\mathbf{x}) < v(\mathbf{z});\;\; Y^0(\mathbf{x}) < Y^0(\mathbf{z});\;\; Y^i(\mathbf{x}) < Y^{\pi(i)}(\mathbf{z}),\ \forall i \in [\![1, N]\!] \setminus \{l\},$$

where $\mathbf{z}$ *is the CE of* $\Gamma(\boldsymbol{\tau})$.

Proof. There are two cases.

Case 1. $\tau^{[k]} > kH$.

First, let us show that the equation $F_k(\cdot) = \tau^{[k]}$ admits a unique solution w_0, and $\xi_1 < w_0 < M$.

Indeed, $F_k(\xi_1) = T^{[k]} > \tau^{[k]}$ according to (3.25), and $F_k(M) = kH < \tau^{[k]}$ by the hypothesis.

Next, suppose that for some $p \in \{0\} \cup [\![1, N-k+1]\!]$, a sequence of numbers $w_0, w_1, \ldots, w_p$ is obtained such that (i) $\xi_1 < w_p < \cdots < w_0 < M$, (ii) for $q = 0, 1, \ldots, p$, $F_{k+q}(w_q) = \tau^{[k+q]}$, (iii) if $H > 0$ and $p > 0$, for $q = 0, 1, \ldots, p-1$, $\gamma^{k+q+1} > h(w_q)$, and (iv) if $H \leq 0$ and $p > 0$, for $q = 0, 1, \ldots, p-1$, either $\gamma^{k+q+1} > h(w_q)$ or $w_p > \hat{\xi}$.

The rest of the proof for this case is nearly the same as the proof for Lemma 3.20. The only difference is that $\mathbf{z}$ cannot have mode 1, because $w^0 < M$.

Case 2. $\tau^{[k]} \leq kH$. Notice that this is only possible if $H > 0$.

According to (3.25) and condition (3.19), for $i \in [\![1, k-1]\!]$, $\tau^i \geq T^{i+1} \geq T^k > h(\xi_1)$, which implies that $\tau^{[k-1]} > (k-1)\, h(\xi_1) > (k-1)H$, where the second inequality is owing to the fact that $\xi_1 < M$ (otherwise $\mathbf{x}$ is of mode 1, contradiction); in particular,

$$\tau^{[k-1]} > (k-1)H. \tag{3.33}$$

The relation (3.33) and the hypothesis $\tau^{[k]} \leq kH$ imply that

$$\tau^k < H. \tag{3.34}$$

Let us show that the equation $F_{k-1}(\cdot) = \tau^{[k-1]}$ admits a unique solution w_1. Moreover, $\xi_1 < w_1 < M$ if $k > 1$, and $w_1 = M$ if $k = 1$.

Indeed, on the one hand,

$$F_{k-1}(\xi_1) = F_k(\xi_1) - h(\xi_1) = T^{[k]} - h(\xi_1) > T^{[k]} - T^k = \begin{cases} T^{[k-1]} \geq \tau^{[k-1]}, & \text{if } k > 1, \\ 0 = \tau^{[k-1]}, & \text{if } k = 1, \end{cases}$$

where the first inequality is due to condition (3.19) and the second due to (3.25). On the other hand,

$$F_{k-1}(M) = (k-1)H \begin{cases} < (k-1)\, h(\xi_1) < \tau^{[k-1]}, & \text{if } k > 1, \\ = 0 = \tau^{[k-1]}, & \text{if } k = 1. \end{cases}$$

If $k = 1$ and, consequently, $w_1 = M$ because $H > 0$ and $\tau^1 = \tau^{[1]} \leq H$ by hypothesis, it follows from Proposition 3.13 that the CE $\mathbf{z}$ of $\Gamma(\boldsymbol{\tau})$ is of mode 1.

If $k > 1$ and, consequently, $w_1 < M$, let us show that the CE $\mathbf{z}$ of $\Gamma(\boldsymbol{\tau})$ is of mode 3 and specified by $k-1$.

For $i \in [\![1, k-1]\!]$, $\tau^i \geq \tau^{k-1} \geq T^k > h(\xi_1) > h(w_1)$ according to (3.25), (3.19) and the fact that $w_1 > \xi_1$. For $i \in [\![k, N+1]\!]$, $\tau^i \leq \tau^k < H \leq h(w_1)$ because of (3.25), (3.34) and the fact that $w_1 \leq M$.

Combining these two results and the fact that w_1 is the unique solution to the equation $F_{k-1}(\cdot) = \tau^{[k-1]}$, one deduces from Proposition 3.15 that $\mathbf{z}$ is of mode 3 and specified by $k-1$. Moreover, for $i \in [\![1, k-1]\!]$, $z_1^i = \frac{\tau^i\, a(w_1) + h(w_1)}{1 + a(w_1)}$; for $i \in [\![k, N+1]\!]$ or $i = 0$, $z_1^i = \tau^i$; $\zeta_1 = \sum_{i=0}^{N+1} z_1^i = w_1 > \xi_1$.

The rest of the proof is similar to that for Lemma 3.20. $\square$

3.4.4 Main result

The following theorem combines Lemma 3.17, Lemma 3.18, Lemma 3.19, Lemma 3.20 and Lemma 3.22.

Theorem 3.23. *Let* $\mathbf{x}$ *be the CE of* $\Gamma(\mathbf{T})$*. For any* l *such that* $1 \leq l \leq N$*, let two profiles* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\tau}$ *be defined by* (3.23) *and* (3.24) *respectively. Then,*

$$v(\mathbf{x}) \leq v(\mathbf{y}),\ v(\mathbf{x}) \leq v(\mathbf{z});\quad Y^0(\mathbf{x}) \leq Y^0(\mathbf{y}),\ Y^0(\mathbf{x}) \leq Y^0(\mathbf{z});$$
$$Y^i(\mathbf{x}) \leq Y^{\sigma(i)}(\mathbf{y}),\ Y^i(\mathbf{x}) \leq Y^{\pi(i)}(\mathbf{z}),\quad \forall\, i \in [\![1, N]\!] \setminus \{l\},$$

where $\mathbf{y}$ *is the CE of* $\Gamma(\boldsymbol{\gamma})$ *and* $\mathbf{z}$ *is the CE of* $\Gamma(\boldsymbol{\tau})$*.*

Besides, the equalities hold if, and only if, $Y^l(\mathbf{x}) = Y^0(\mathbf{x})$*.*

The family of the composite routing games played, in the fixed two-terminal two-parallel-arc network specified by $\mathbf{c}$, by a finite number of atomic players and a set of individuals such that the total weight of all the players is M is denoted by $\Xi = \{\, \Gamma(\mathbf{T}) \,|\, \mathbf{T} = (T^0, T^1, \ldots, T^N),\ N \in \mathbb{N},\ T^0 + \sum_{i=0}^{N} T^i = M,\ T^i > 0\, \forall\, i \in [\![1, N]\!],\ T^0 \geq 0 \,\}$.

Definition 3.24. Let $\Gamma(\mathbf{T})$ and $\Gamma(\mathbf{R})$ be two games in Ξ, with $\mathbf{T} = (T^0, T^1, \ldots, T^N)$ and $\mathbf{R} = (R^0, R^1, \ldots, R^M)$. The game $\Gamma(\mathbf{R})$ is called a *direct successor* of the game $\Gamma(\mathbf{T})$, if there exists some $l \in [\![1, N]\!]$, a subset $\mathcal{R}$ of $[\![1, M]\!]$ and a non-negative number $\tilde{R}$ such that $T^l = \sum_{j \in \mathcal{R}} R^j + \tilde{R}$, $R^0 = T^0 + \tilde{R}$, and there exists a bijection θ from $[\![1, N]\!] \setminus \{l\}$ to $[\![1, M]\!] \setminus \mathcal{R}$ such that $R^{\theta(i)} = T^i$ for all $i \in [\![1, N]\!] \setminus \{l\}$.

In other words, the atomic player l is replaced by a composite set of players consisting of the atomic players in $\mathcal{R}$ and a group of individuals of total weight $\tilde{R}$.

Definition 3.25. Let $\Gamma(\mathbf{T})$ and $\Gamma(\mathbf{R})$ be two games in Ξ. The game $\Gamma(\mathbf{R})$ is called a *successor* of the game $\Gamma(\mathbf{T})$ if there exists a finite sequence of games $\Gamma(\mathbf{T}_0), \Gamma(\mathbf{T}_1), \ldots, \Gamma(\mathbf{T}_k)$ in Ξ such that $\Gamma(\mathbf{T}_0) = \Gamma(\mathbf{T})$, $\Gamma(\mathbf{T}_k) = \Gamma(\mathbf{R})$, and $\Gamma(\mathbf{T}_l)$ is a direct successor of $\Gamma(\mathbf{T}_{l-1})$ for all $l \in [\![1, k]\!]$.

Remark 3.26. If $\Gamma(\mathbf{R})$ (with M atomic players) is a successor of $\Gamma(\mathbf{T})$ (with N atomic players), then there exists a subset $\mathcal{T}$ of $[\![1, N]\!]$ such that, for all $l \in \mathcal{T}$, there is a distinct subset $\mathcal{R}^l$ of $[\![1, M]\!]$ as well as a non-negative number $\tilde{R}^l$ satisfying that, for all $l \in \mathcal{T}$, $T^l = \sum_{j \in \mathcal{R}^l} R^j + \tilde{R}^l$, and $T^0 = R^0 + \sum_{l \in \mathcal{T}} \tilde{R}^l$. Besides, there exists a bijection from θ from $[\![1, N]\!] \setminus \mathcal{T}$ to $[\![1, M]\!] \setminus \cup_{l \in \mathcal{T}} \mathcal{R}^l$ such that $R^{\theta(i)} = T^i$ for all $i \in [\![1, N]\!] \setminus \mathcal{T}$.

Theorem 3.27. *Suppose that* $\Gamma(\mathbf{T})$ *and* $\Gamma(\mathbf{R})$ *are two games in* Ξ*, and* $\Gamma(\mathbf{R})$ *is a successor of* $\Gamma(\mathbf{T})$*. Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be, respectively, the CE of* $\Gamma(\mathbf{T})$ *and that of* $\Gamma(\mathbf{R})$*. The sets* N*,* M*,* $\mathcal{T}$*,* $\mathcal{R}^l$*'s and the bijection* θ *are those described in Remark 3.26. Then,*

$$v(\mathbf{x}) \leq v(\mathbf{y});\ \ Y^0(\mathbf{x}) \leq Y^0(\mathbf{y});\ \ Y^i(\mathbf{x}) \leq Y^{\theta(i)}(\mathbf{y}),\ \ \forall\, i \in [\![1, N]\!] \setminus \mathcal{T}.$$

Besides, the equalities hold if, and only if, for all $l \in \mathcal{T}$*,* $Y^l(\mathbf{x}) = Y^0(\mathbf{x})$*.*

3.4.5 Coalition members

Theorem 3.27 can also be interpreted in this way: the formation of coalitions benefits the social cost and the players not involved in these coalitions. However, whether the formation of a coalition benefits or does not benefit its own members remains an open problem. Explicitly, for $l \in \mathcal{T}$ and $j \in \mathcal{R}^l$ in Theorem 3.27, the relation between $Y^l(\mathbf{x})$ and $Y^j(\mathbf{y})$ is not clear. On the one hand, one can cite an example where the coalition benefits its members: all the nonatomic players form a coalition, then their average cost is the minimal social cost which cannot be higher than the Wardrop equilibrium cost. On the other hand, the following two examples, which are based on Example 5.2 in [91] (*cf. Example 2.20*), show that, sometimes, it may be better not to form a coalition.

Example 3.28. The cost functions of the two arcs are, respectively, $c_1(x) = x + 10$ and $c_2(x) = 10x + 1$.

Consider two games: $\Gamma_I = \Gamma(\frac{1}{10}, \frac{1}{2}, \frac{2}{5})$ and $\Gamma_{II} = \Gamma(\frac{3}{10}, \frac{1}{2}, \frac{1}{5})$. The atomic player of weight $\frac{2}{5}$ (called $I.2$) in Γ_I can be seen as the coalition of the atomic player of weight $\frac{1}{5}$ (called $II.2$) and a group of individuals of weight $\frac{1}{5}$ in Γ_{II}. The per-unit cost to player $I.2$ is $\frac{839}{99} \approx 8.47$ at the CE of Γ_I, and the per-unit cost to player $II.2$ is $\frac{91}{11} \approx 8.27$ at the CE of Γ_{II}. Thus, the coalition with nonatomic players has increased the per-unit cost to $II.2$.

Consider two other games: $\Gamma_{III} = \Gamma(0, \frac{1}{2}, \frac{1}{2})$ and $\Gamma_{IV} = \Gamma(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The second atomic player of weight $\frac{1}{2}$ (called $III.2$) in Γ_{III} can be seen as a coalition of the two atomic players of weight $\frac{1}{4}$ (called $IV.2$ and $IV.3$) in Γ_{III}. The per-unit cost to player $III.2$ is $\frac{91}{11} \approx 8.273$ at the CE of Γ_{III}, and the per-unit costs to players $IV.2$ and $IV.3$ are both $\frac{33}{4} = 8.250$ at the CE of Γ_{IV}. Thus, both members of the coalition have seen their per-unit cost increased after the alliance.

3.5 Remarks and discussion

3.5.1 Social cost in the case of more than two singleton-choices

When there are more than two parallel arcs in a routing game or, equivalently, more than two singleton-choices in a single commodity congestion game, Theorem 3.27 is no longer valid. Huang [46, Theorem 1] provides the necessary and sufficient conditions for the social cost to be reduced after every formation of coalitions in a single commodity atomic congestion game. His result is reformulated here.

Theorem 3.29. *In a single commodity atomic routing game where arc cost functions satisfy A3.1, the following two conditions are necessary and sufficient for that the equilibrium social cost is reduced or do not change after each formation of disjoint coalition(s):*

(i) The network is well-designed.

(ii) All the cost functions of the arcs are affine.

For the definition of a well-designed network, the reader is referred to [46]. Here, let us just mention that a single-origin single-destination series-parallel network with cost functions satisfying A3.1 is always well-designed [46, Proposition 1].

For this, Huang defines a partial order on the profiles of atomic players with a fixed set of single commodity stocks. He shows that, if the network is well-designed and the cost functions are affine, the social cost with profile A is bounded by that with profile B whenever A majorizes B according to the partial order. This partial order is close to the partial order defined on the composite profiles of players developed in this paper.

Huang [47] gave two counter-examples. In each of them, one of the two conditions in Theorem 3.29 is violated, and the social cost is increased after the formation of a coalition between some players. In particular, the counter-example A.2 in [47] shows that, in a two-terminal three-parallel-arc network where the arc cost functions are, respectively, $20x+5000$,

$x^2 + 500$ and x^{11}, the social cost at the unique equilibrium with three atomic players of weight, respectively, 200, 20.9 and 0.1 is lower than that with two atomic players of weight, respectively, 200 and 21. This example also shows that our results cannot be extended to the case where there are more than two parallel arcs.

3.5.2 Price of anarchy and price of collusion

The results in §3.4.5 and §3.5.1 imply that one might define a more general notion of price of collusion or price of anarchy in congestion games, which takes the following two questions into consideration. First, for the impact of the coalitions on the social cost, the game before the formation of coalitions need not be nonatomic. Second, the study of the impact of the coalitions need not be limited to the social cost, but how each player's cost is affected should also be measured. In the same vein as our analysis, Altman et al. [5] suggested several alternative definitions of price of collusion.

3.5.3 Delegation

In the two-parallel-arc routing game discussed in this work, although the other players' costs at the equilibrium are increased when one atomic player l is replaced by a composite set of players, the *sum* of the costs to these players at the equilibrium after the replacement can be lower than player l's cost at the equilibrium before the replacement. This is the case in the second part in Example 3.28, where the cost to player $III.2$ is $\frac{91}{22} \approx 4.136$, while the sum of the costs to player $IV.2$ and player $IV.3$ is $\frac{33}{8} = 4.125$. Notice that this is different from the discussion in §3.4.5. There, one compares the average cost to a coalition (after its formation) and the cost to each of its members (before the formation of their coalition), so as to deduce whether it is beneficial for the players to cooperate. Here, one compares the total cost to a coalition (after its formation) and the sum of the costs to its members (before their cooperation).

The problem may be easier to understand in the converse direction. Let us see the replacement of player l by a composite set of players as a choice of player l to delegate her stock to these ones, so that they play independently in the congestion game instead of her. If player l's cost is counted as the sum of the cost to her delegates, it may be advantageous for her to delegate. For instance, in Example 3.28, by delegating her stock to players $IV.2$ and $IV.3$, player $III.2$ gets a lower cost (4.125) than what she would have got if she had played herself (4.136).

One can define a delegation game associated to a composite game, where each atomic player's choice is to delegate her stock to a group of atomic and/or nonatomic delegates, and her cost is the sum of the cost to these ones in the composite routing game played by all the delegates and the initial individuals (those who do not delegate because their stock is already infinitesimal). The existence of an equilibrium in this delegation game is not sure. As a matter of fact, even if the number of atomic delegates of an atomic player is restricted to finite integers, the space of her choices of delegation is not only infinite but even not a compact set in a Euclidean space. As a result, even whether an atomic player has a best choice of delegation, in response to the other players, remains a question. Furthermore, an atomic delegate may well have incentive to delegate as well. In this way, the delegation should not be restricted to only one step. Sorin and Wan [88] (*cf.* Chapter 5) defined a delegation game associated to an integer-splittable congestion game, where each player has integer weight, and she can only split her stock into parts of integer weight. Each player has thus only a finite ways to delegate, and after at most M turns of delegation, where M

is the total weight of all the players, all the players have weight 1 and the delegation cannot continue. Therefore, a delegation game can be well defined in this case. However, in our model of composite congestion game with infinitesimally splittable stocks, even the definition of delegation remains an open problem.

3.5.4 Assumptions on the cost functions and the topology of the network

Finally let us give some explanation for the assumptions made in this work on the arc cost functions and those on the topology of the network.

In a network with general topology, it is sufficient to assume that the arc cost functions are of class $\mathcal{C}^1$ for a composite congestion game to admit an equilibrium. For this, one can use the variational inequality formulation for composite equilibrium (*cf* Chapter 2, equation (2.5)). Theorem 3.1 in Kinderlehrer and Stampacchia [49, p.12] states that the variational inequality problem (2.5) admits a solution if the space of feasible flows is a convex compact set, and if the arc costs functions as well as the marginal arc cost functions are continuous.

However, our objective is to compare the equilibrium costs for different composition of the players. To this end, multiple equilibria for each composition may cause the problem of equilibrium selection. Thus, one would like to guarantee the uniqueness of the composite equilibrium by adding more stringent conditions on the arc cost functions or the network topology. That is why the model of two-terminal parallel-arc network with strictly increasing and convex arc costs is used here. Orda et al. [66] proved the uniqueness of the Nash equilibrium in atomic splittable congestion games taking place in such network. The same property for composite congestion games can be obtained by an analysis in the same vein (*cf.* Richman and Shimkin [71]). Furthermore, as stated at the end of §3.5.1, our result does not hold for the case with more than two parallel links.

As a matter of fact, Richman and Shimkin [71] and Bhaskar et al. [10] extended the result of [66] to some more general network topologies under the same assumptions on arc cost functions, while Altman et al. [3] proved the uniqueness for a specific class of arc cost functions but with no constraints on the network topology. These more general settings are not discussed in this work (and neither in Chapter 2) because the computation of the equilibrium costs would become much more complicated. However, it may well be an interesting issue for further research work.

Chapter 4

One-shot delegation games and delegation processes

This chapter is based on the paper *One-shot delegation games and delegation processes.*

Abstract. *This work studies the behavior of delegation and its impact on the equilibrium costs in composite congestion games. Any atomic player can split her stock into several parts in order to delegate them to independent atomic splittable or nonatomic players. Her cost is the sum of the costs to her delegates. It is shown that, if all the players have the same two choices which are singletons, and the choice cost functions meet a standard convexity condition, then, facing the composition of the other players, an atomic player's delegation strategies are dominated by single-atomic ones, which consist in delegating to at most one atomic player in addition to nonatomic ones. A fortiori, she possesses at least one best reply in terms of delegation strategies. In a one-shot delegation game where the atomic players play each a delegation strategy once and simultaneously, if the cost functions of the two choices are affine and there are only two atomic players, the game admits pure equilibria. In the same context, if the atomic players play a single-atomic best reply in an alternating way so that there are always at most two players in the network, the process either stops after a finite number of steps when there are no more atomic players willing to delegate, or it converges to a limit at an exponential rate. The outcomes of the alternating best reply process, called myopic ones, are different from the equilibria of the one-shot delegation game. At each myopic outcome, the initial atomic players may have a lower or higher cost than if they had not delegated, while the nonatomic players' cost and the social cost are all higher than if there was no delegation.*

4.1 Introduction

In a network congestion game, namely, a routing game, if a player holds a stock of strictly positive weight which can be arbitrarily divided, she is called atomic arbitrarily splittable (or atomic splittable for short). A player holding a stock of infinitesimal weight is called nonatomic, and a player who cannot split her stock of strictly positive weight is called atomic unsplittable. Each player has to send her stock from its origin to its destination and, in the meanwhile, minimize her cost. The cost of using a certain path depends upon the quantity of stocks on it. While a nonatomic player or an atomic unsplittable player has to choose only one path, an atomic splittable player can split her stock so as to send different parts by different paths.

In Sorin and Wan [88] (*cf.* Chapter 5), the players are integer-splittable, in the sense that they each have an integer weight and they can split their stock into several parts of integer weight. A player can commit each part to an independent delegate who ensures its transportation, and her cost will be the sum of the costs to her delegates. Sorin and Wan showed that, when there are more than one player, a player may be interested in delegating her stock. They further introduced a delegation game associated to the original congestion game, where a delegate can also delegate her stock and this continues until that all the players have weight 1. They proved the existence of equilibria and so-called consistent equilibria in a delegation game.

This paper studies the behavior of delegation in a more general setting where the atomic players are arbitrarily splittable and there can also be nonatomic players. A congestion game with such players is called composite [15, 32, 91]. Two difficulties arise. The first one is that, because of the fact that the stock is arbitrarily splittable, the strategies of delegation of an atomic player are no longer finitely many as in the integer-splittable case. As one will see later, the space of such strategies is not even a convex compact subset of a Euclidean space. The second difficulty is that, unlike the integer-splittable case where no one continues to delegate when all the players have weight 1, in our setting, the delegation can always go on as long as the players are not all nonatomic.

In view of these two difficulties, the delegation game associated to a composite congestion game cannot be defined and studied in exactly the same way as for integer-splittable games. This paper is a first tentative to deal with the problem. All the players are assumed to have the same two choices which are singletons. In the setting of network congestion games, it is equivalent to saying that the network contains two vertices which are the common origin and the common destination of the players, and they are linked by two parallel arcs.

The main results of the paper are:

1. Facing the composition of the other players, an atomic player's delegation strategies are dominated by single-atomic ones, which consist in delegating to at most one atomic player in addition to nonatomic ones. A fortiori, she possesses at least one best reply in terms of delegation strategies.

2. In a one-shot delegation game where each atomic player plays a delegation strategy, if the cost functions of the two arcs are affine and there are only two atomic players, the game admits pure equilibria.

3. In the same context as for the second result, suppose that the atomic players play a single-atomic best reply in an alternating way, so that there are always at most two players. The process stops when there are no more atomic players who would like to delegate. Such a process either stops after a finite number of steps, or converges to a limit at an exponential rate.

4. The outcomes of this alternating best reply process are called myopic ones. They are different from the equilibria in the one-shot delegation game.

5. At a myopic outcome, the initial atomic players may have a lower or higher cost than at the equilibrium of the original game (without delegation), while the nonatomic players' cost and the social cost are all higher than at the equilibrium of the original game.

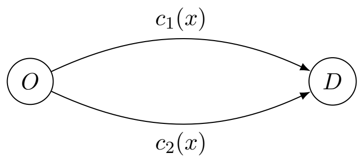
The paper is organized as follows. The next section describes the network setting of our model, and recalls the properties of composite routing games in such a setting. In Section 4.3, the definition of a one-shot delegation game and that of the delegation strategies are provided. The special role of the single-atomic strategies is pointed out. In Section 4.4, it is shown that an atomic player possesses a best reply to the composition of the other players in the network, in terms of delegation strategies, under a standard condition on the convexity of the cost functions. Section 4.5 establishes the existence of (pure) Nash equilibria in a specific class of one-shot delegation games, where there are only two atomic players in addition to the nonatomic players, and the arc cost functions are affine. A dynamical process in the same class of games as in Section 4.5 is analyzed in Section 4.6, where the atomic players play, in an alternating way, a best delegation strategy in reply to the composition of her opponents. Section 4.7 is contributed to remarks and general discussions.

4.2 Model and known results

In this work, $[\![m, n]\!]$ stands for the set of successive positive integers $[m, n] \cap \mathbb{N} = \{m, m+1, \ldots, n-1, n\}$. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

First, let us recall the model of a composite routing game in a two-terminal two-parallel-arc and some preliminary results.

4.2.1 Model of a two-terminal two-parallel-arc composite routing game



Vertices O and D are linked by two arcs 1 and 2. Their per-unit cost functions are, respectively, c_1 and c_2, which are real-valued functions defined on the real interval $(-\eta, M + \eta)$, where $M > 0$ and $\eta > 0$. When the total weight of stocks on arc r is x, the cost to each unit of them is $c_r(x)$. Denote $\mathbf{c} = (c_1, c_2)$. The following assumption on the cost functions is made *throughout this work.*

A 4.1. *Cost functions c_1 and c_2 are strictly increasing, convex and continuously differentiable on $(-\eta,\ M + \eta)$, and non-negative on $[0, M]$.*

The profile of the players is described by a vector $\mathbf{T} = (T^0, T^1, T^2, \ldots, T^N)$ with $N \in \mathbb{N}$, where T^0 is the total weight of the nonatomic players and, for $p \geq 1$, T^p is the weight of the atomic player p. The total weight of the players is $M = \sum_{i=0}^N T^i$.

Each player aims to send her stock from O to D at a minimal cost. A nonatomic player takes one of the two arcs. The profile of the nonatomic players' strategies is specified by a vector $\mathbf{x}^0 = (x_1^0, x_2^0)$, where x_r^0 is the total weight of the nonatomic players on arc r. An atomic player i splits her stock into two parts and send them by two arcs respectively. Her strategy is specified by a vector $\mathbf{x}^i = (x_1^i, x_2^i)$, where x_r^i is the weight that she sends by arc r. Denote $\xi_r = \sum_{i=0}^N x_r^i$, the total weight on arc r. The vector $\mathbf{x} = (\mathbf{x}^i)_{i=0}^N$ is called the *flow* (induced by the players' strategies).

For $i \in [\![1, N]\!]$, denote $\mathbf{x}^{-i} = \prod_{0 \leq j \leq N,\, j \neq i} \mathbf{x}^j$ and $x_r^{-i} = \xi_r - x_r^i$. For $i \in \{0\} \cup [\![1, N]\!]$, denote $X^i = \{\mathbf{x}^i \in \mathbb{R}^2 \,|\, x_1^i \geq 0,\, x_2^i \geq 0,\, x_1^i + x_2^i = T^i\}$. Finally, denote $X = \prod_{i=0}^N X^i$.

When the flow is $\mathbf{x}$, the cost to a nonatomic player taking arc r is $c_r(\xi_r)$. The cost to the atomic player i is $u^i(\mathbf{x}) = x_1^i c_1(\xi_1) + x_2^i c_2(\xi_2)$. The social cost is defined as $v(\mathbf{x}) = \xi_1 c_1(\xi_1) + \xi_2 c_2(\xi_2)$.

A composite game taking place in a two-terminal two-parallel-arc network with cost functions $\mathbf{c}$ with the profile of the players being $\mathbf{T}$ is denoted by $\Gamma(\mathbf{c}, \mathbf{T})$. For the sake of simplicity, if the network is fixed and only the profile of the players changes, one can simply write $\Gamma(\mathbf{T})$.

Definition 4.2 (Harker [32]). In the game $\Gamma(\mathbf{T})$, a flow $\mathbf{x} \in X$ is a *composite equilibrium* if it meets the two conditions below:

(i) For all $r \in \{1, 2\}$, if $x_r^0 > 0$, then $c_r(\xi_r) \leq c_t(\xi_t)$ for all $t \in \{1, 2\}$.

(ii) For all $i \in [\![1, N]\!]$, $\mathbf{x}^i$ minimizes $Y^i(\,\cdot, \mathbf{x}^{-i})$ on X^i.

4.2.2 Notations and known results

The following auxiliary functions and notations are introduced in Wan [92] (*cf.* Chapter 3).

Two functions h, a, and a family of functions $\{F_n\}_{n\in\mathbb{N}}$ are defined on the real line $\mathbb{R}$. Let ϵ be a strictly positive constant.

$$h(x) = \begin{cases} \dfrac{c_2(M-x) - c_1(x)}{c_1'(x)}, & \text{if } 0 \leq x \leq M; \\ h(0) - \epsilon x, & \text{if } x < 0; \\ h(M) - \epsilon(x - M), & \text{if } x > M, \end{cases} \qquad a(x) = \begin{cases} \dfrac{c_2'(M-x)}{c_1'(x)}, & \text{if } 0 \leq x \leq M; \\ a(0), & \text{if } x < 0; \\ a(M), & \text{if } x > M, \end{cases}$$

$$F_n(x) = (M - x)\,(1 + a(x)) + n\,h(x), \qquad n \in \mathbb{N}.$$

Functions h and F_n's are strictly decreasing and continuous on $\mathbb{R}$, and their ranges are all $\mathbb{R}$. Hence, their inverse functions h^{-1} and F_n^{-1}'s are all strictly decreasing and continuous on $\mathbb{R}$. Function a is non-increasing, strictly positive and continuous on $\mathbb{R}$.

Proposition 4.3. *The following two statements are equivalent.*

(i) There exists $\hat{\xi} \in [0, M]$ such that $c_1(\hat{\xi}) = c_2(M - \hat{\xi})$.

(ii) $c_1(M) \geq c_2(0)$ and $c_2(M) \geq c_1(0)$ or, equivalently, $h(M) \leq 0$ and $h(0) \geq 0$.

Furthermore, there is at most one $\hat{\xi} \in [0, M]$ such that $c_1(\hat{\xi}) = c_2(M - \hat{\xi})$.

Proof. Clear by A4.1. □

Notation. $H = h(M)$. $A = a(\hat{\xi})$ if there exists $\hat{\xi} \in [0, M]$ such that $c_1(\hat{\xi}) = c_2(M - \hat{\xi})$.

The following results, cited from Wan [91] (*cf.* Chapter 2), have been reformulated for our context.

Theorem 4.4. *A vector $\mathbf{x} \in X$ is a CE of the game $\Gamma(\mathbf{T})$ if, and only if, for all $r \in \{1, 2\}$,*

$$x_r^0 > 0 \;\Rightarrow\; c_r(\xi_r) = \min_{s\in\{1,2\}} c_s(\xi_s) \tag{4.1}$$

and, for all $r \in \{1, 2\}$ and $i \in [\![1, N]\!]$,

$$x_r^i > 0 \;\Rightarrow\; c_r(\xi_r) + x_r^i\, c'(\xi_r) = \min_{s\in\{1,2\}} c_s(\xi_s) + x_s^i\, c'(\xi_s). \tag{4.2}$$

Theorem 4.5. *The game $\Gamma(\mathbf{T})$ admits one and only one CE.*

By definition, the nonatomic players take the arcs of the lowest cost at the CE $\mathbf{x}$.

Theorem 4.6. *At the CE* $\mathbf{x}$ *of the game* $\Gamma(\mathbf{T})$*, for* $r \in \{1,2\}$*,*

1. *if* $x_r^0 > 0$*, then* $x_r^i > 0$ *for all* $i \in [\![1,N]\!]$*;*
2. *for* $i,j \in [\![1,N]\!]$*, if* $T^i \geq T^j$*, then* $x_r^i \geq x_r^j$*, and the equality holds if, and only if,* $T^i = T^j$ *or* $x_r^i = x_r^j = 0$*;*
3. *for* $i \in [\![1,N]\!]$ *and* $s \in \{1,2\} \setminus \{r\}$*, if* $x_r^i > 0$ *and* $x_s^i = 0$*, then* $c_r(\xi_r) < c_s(\xi_s)$*.*

As in Wan [92] (*cf.* A3.9), the following assumption will also be made *throughout this work* and it does not lose generality.

A 4.7. *One and only one of the following two conditions holds:*

(i) $c_1(M) < c_2(0)$ *or, equivalently,* $H > 0$*;*

(ii) $c_1(M) \geq c_2(0), c_2(M) \geq c_1(0)$ *or, equivalently,* $H \leq 0, h(0) \geq 0$*; and*

$$\hat{\xi}\, c_1'(\hat{\xi}) \geq (M - \hat{\xi})\, c_2'(M - \hat{\xi}); \tag{4.3}$$

consequently,

$$(M - \hat{\xi})(1 + A) \ \leq \ M \ \leq \ \hat{\xi}\, \frac{1+A}{A}. \tag{4.4}$$

Because of to A4.7, the flow of the CE $\mathbf{x}$ takes some special form as the following proposition shows.

Proposition 4.8 (Wan [92] (*cf.* Proposition 3.12))**.** *At the CE* $\mathbf{x}$ *of the game* $\Gamma(\mathbf{T})$*,*

1. $c_1(\xi_1) \leq c_2(\xi_2)$*;*
2. *if* $c_2(0) \leq c_1(M)$ *or, equivalently,* $H \leq 0$*, then* $\xi_1 \leq \hat{\xi}$*;*
3. $h(\xi_1) \geq 0$*, and the equality holds if, and only if,* $c_2(0) \leq c_1(M)$ *and* $\xi_1 = \hat{\xi}$*.*

4.3 Delegation

4.3.1 Incentive of delegation

The following example shows that an atomic player may have incentive to delegate.

Example 4.9. The cost functions of the two arcs are, respectively, $c_1(x) = x + 10$ and $c_2(x) = 10x + 1$. Suppose that there are two atomic players both of weight $\frac{1}{2}$ and there are no nonatomic players. Each atomic player's cost at the CE is $\frac{91}{22} \approx 4.136$. Now, the atomic player 2 commits her stock to two atomic delegates so that they both have weight $\frac{1}{4}$. At the CE of the routing game played by the atomic player 2's two delegates and the atomic player 1 of weight $\frac{1}{2}$, the sum of the costs to the two delegates is $\frac{33}{4} = 4.125$. Therefore, the atomic player 2 benefits from her delegation.

4.3.2 Delegation strategies

Let us give a rigorous definition of a *one-shot delegation game* $D_1(\mathbf{T})$ associated to the composite routing game $\Gamma(\mathbf{T})$.

The players of the game are the same as in $\Gamma(\mathbf{T})$.

Definition 4.10. The delegation strategy space of any nonatomic player is $\emptyset$, i.e. they have no choice.

Definition 4.11. An atomic player p of weight T^p *delegates* her stock if she is replaced by a finite number $n \in \mathbb{N}$ of atomic players of weight, respectively, $\alpha^1, \alpha^2, \dots, \alpha^n$ (in a non-increasing order) and a group of nonatomic players of total weight α^0, where $\alpha^i > 0$ for all $i \in [\![1, n]\!]$, $\alpha^0 \geq 0$, and $\sum_{i=1}^n \alpha^i + \alpha^0 = T^p$.

These nonatomic players of total weight α^0 and the n atomic players are called *delegates* of the atomic player p.

A *delegation strategy* of the atomic player p is a profile $\boldsymbol{\alpha} = (\alpha^0, \alpha^1, \dots, \alpha^n)$ according to which she designates her delegates.

The space of delegation strategies of the atomic player p is denoted by $\mathcal{S}^p$. Then,

$$\mathcal{S}^p = \bigcup_{n=0}^{+\infty} \mathcal{S}^{p,n}, \text{ where}$$

$$\mathcal{S}^{p,n} = \{\boldsymbol{\alpha} = (\alpha^0, \alpha^1, \dots, \alpha^n) \in \mathbb{R}^{n+1} \,|\, \alpha^0 \geq 0,\ \alpha^1 \geq \alpha^2 \geq \dots \geq \alpha^n > 0;\ \sum_{i=0}^n \alpha^i = T^p\}, \quad \forall n \in \mathbb{N}.$$

Remark 4.12. The set $\mathcal{S}^{p,n}$ contains all the delegation strategies that designate n atomic delegates (in addition to a set of nonatomic delegates). For all $n \in \mathbb{N}^*$, $\mathcal{S}^{p,n}$ is a proper subset of the simplex of n dimension Δ^n. The set $\mathcal{S}^{p,0}$ is a singleton set.

From now on, a delegation strategy will simply be called a strategy.

Definition 4.13. Let $\boldsymbol{\alpha} = (\alpha^0,\ \alpha^1,\ \dots,\ \alpha^n)$ be in $\mathcal{S}^p$.

Strategy $\boldsymbol{\alpha}$ is the *nonatomic strategy*, denoted by $\boldsymbol{\alpha}_0$, if $\alpha^0 = T^p$ or, equivalently, $n = 0$. It is the unique element in $\mathcal{S}^{p,0}$.

Strategy $\boldsymbol{\alpha}$ is the *trivial strategy*, denoted by $\boldsymbol{\alpha}_1$, if $n = 1$ and $\alpha^1 = T^p$. (Rigorously, by choosing the trivial strategy, the atomic player p does not delegate.)

Strategy $\boldsymbol{\alpha}$ is a *single-atomic strategy*, if $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$ or $n = 1$ and $\alpha^1 \in (0,\ T^p\,]$. The set of single-atomic strategies is $\mathcal{S}^{p,0} \cup \mathcal{S}^{p,1}$.

A single-atomic strategy is determined and denoted by $\alpha^1 \in [\,0,\ T^p\,]$ so that the space of single-atomic strategies is isometric to the closed interval $S^p = [0,\ T^p]$. The value of the strategy α^1 corresponds to the weight of the unique atomic delegate. (Rigorously, there is no atomic delegate when $\alpha^1 = 0$.)

Suppose that the profile of the delegation strategies of the N atomic players in the profile $\mathbf{T}$ is $(\boldsymbol{\alpha}^p)_{p=1}^N$, where $\boldsymbol{\alpha}^p \in \mathcal{S}^p$. This induces a profile of atomic and nonatomic delegates. In order to define the cost to a specific atomic player p in $\mathbf{T}$, let her strategy be denoted by $\boldsymbol{\alpha} = (\alpha^0, \alpha^1, \dots, \alpha^n)$, and the profile of all the atomic and nonatomic delegates appointed by the other $N-1$ atomic players in addition to the nonatomic players in $\mathbf{T}$ be denoted by $\boldsymbol{\beta} = (\beta^0, \beta^1, \dots, \beta^m)$. Explicitly, β^0 is the total weight of the nonatomic players in $\mathbf{T}$ and all the nonatomic delegates appointed by the atomic players in $\mathbf{T}$ *except* p, while $\beta^1, \dots, \beta^m$ are the weights, in a non-increasing order, of all the atomic delegates appointed by the atomic players in $\mathbf{T}$ *except* p. Let $T^{-p} = M - T^p$ be the total weight of all the players other than p.

Notation. For $\boldsymbol{\alpha} = (\alpha^0, \alpha^1, \dots, \alpha^n) \in \mathcal{S}^p$, denote $\alpha^{[k]} = \sum_{i=1}^k \alpha^i$, for $k \in [\![1, n]\!]$. Similarly, $\beta^{[l]} = \sum_{j=1}^l \beta^j$, for $l \in [\![1, m]\!]$.

Let $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the composite routing game whose profile of players is $(\alpha^0 + \beta^0, \alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^m)$. According to Theorem 4.5, it admits a unique CE. Let it be denoted by $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The following notations are adopted.

At $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta}) = ((\mathbf{x}^i)_{i=0}^n, (\mathbf{y}^j)_{j=0}^m)$, with $\mathbf{x}^i = (x_1^i, x_2^i)$ and $\mathbf{y}^j = (y_1^j, y_2^j)$, the atomic player α^i sends a stock of weight x_r^i on arc r, and nonatomic players of total weight x_r^0 among α^0 choose arc r, $r = 1, 2$. Let $x_r = \sum_{i=0}^n x_r^i$ be the total weight put on arc r by the delegates of the atomic player p.

Notations y_r^j (for $j \in [\![1,m]\!]$), y_r^0 and y_r have similarly meanings for $\beta^1, \ldots, \beta^m$ and β^0. Let $\xi_r = x_r + y_r$ be the total weight on arc r. The per-unit cost of arc r is thus $c_r(\xi_r)$.

The cost function of the atomic player p in $D_1(\mathbf{T})$ is defined by

$$u^p(\boldsymbol{\alpha},\boldsymbol{\beta}) = x_1\, c_1(\xi_1) + x_2\, c_2(\xi_2). \tag{4.5}$$

In other words, the atomic player p's cost is the total cost to her delegates at the CE of the game $\Gamma(\boldsymbol{\alpha},\boldsymbol{\beta})$, which is induced by the delegation strategies of the atomic players in $\mathbf{T}$.

The cost function of the nonatomic players or, more rigorously, the lowest arc cost in the network in $D_1(\mathbf{T})$ is denoted by $c^0(\boldsymbol{\alpha},\boldsymbol{\beta})$. According to Proposition 4.8, arc 1 always costs less than arc 2, hence $c^0(\boldsymbol{\alpha},\boldsymbol{\beta}) = c_1(\xi_1)$.

The one-shot delegation game $D_1(\mathbf{T})$ is thus defined.

Notice that, by Definition 4.5, the atomic player p's cost in $D_1(\mathbf{T})$ depends only on ξ_1 and x_1. This property inspires the following definition of equivalence between the delegation strategies.

Definition 4.14. Two strategies $\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\alpha}}$ in $\mathcal{S}^p$ are *equivalent with respect to* a profile $\boldsymbol{\beta}$ if the flows of $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\mathbf{x}(\tilde{\boldsymbol{\alpha}},\boldsymbol{\beta})$ satisfy

$$\xi_1 = \tilde{\xi}_1, \qquad x_1 = \tilde{x}_1,$$

and, consequently, $u^p(\boldsymbol{\alpha},\boldsymbol{\beta}) = u^p(\tilde{\boldsymbol{\alpha}},\boldsymbol{\beta})$.

4.3.3 Five modes of the CE of the induced routing game

This section studies the flow of the CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ of the composite routing game $\Gamma(\boldsymbol{\alpha},\boldsymbol{\beta})$ induced by the delegation strategies of the atomic players in $\mathbf{T}$. According to Theorem 4.6, A4.7 and Proposition 4.8, $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ can have five modes, which will be analyzed below.

Mode 1

The CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is of *mode 1* if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); & \\ x_1^i = \alpha^i, & i \in \{0\} \cup [\![1,n]\!]; \\ y_1^j = \beta^j, & j \in \{0\} \cup [\![1,m]\!]. \end{cases} \tag{4.6}$$

Mode 1	α^1	$\cdots$	α^n	α^0	β^1	$\cdots$	β^m	β^0
arc 1	α^1	$\cdots$	α^n	α^0	β^1	$\cdots$	β^m	β^0
arc 2	0	$\cdots$	0	0	0	$\cdots$	0	0

In this mode,

$$x^1 = T^p, \quad y^1 = T^{-p}, \quad \xi^1 = M. \tag{4.7}$$

Because $\xi_1 = M$ and $c_1(\xi_1) < c_2(\xi_2)$, one has $c_1(M) < c_2(0)$ or, equivalently, $H > 0$.

If $n \geq 1$, then it follows from (4.2) that, for $i \in [\![1,n]\!]$, $c_1(M) + \alpha^i\, c_1'(M) \leq c_2(0)$ or, equivalently, $\alpha^i \leq H$. Similarly, if $m \geq 1$, then $\beta^j \leq H$ for $j \in [\![1,m]\!]$.

These results entail the following proposition.

Proposition 4.15. *Assume that $H > 0$.*

(1) A necessary and sufficient condition on the profile $\boldsymbol{\beta}$ for the atomic player p to have a strategy $\boldsymbol{\alpha}$ such that the CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is of mode 1 (cf. (4.6)) *is that*

$$m = 0, \quad \text{or } m \geq 1 \text{ but } \beta^1 \leq H. \tag{4.8}$$

(2) Suppose that condition (4.8) is satisfied. Then, a strategy $\boldsymbol{\alpha}$ of the atomic player p is in the following subset of $\mathcal{S}^1$:

$$\mathcal{S}_1^p = \{\, \boldsymbol{\alpha} \in \mathcal{S}^p \,|\, n = 0, \quad \text{or } n \geq 1 \text{ but } \alpha^1 \leq H \,\}$$

if, and only if, it induces a CE $\mathbf{x}$ of mode 1.
Furthermore, the strategies in $\mathcal{S}_1^p$ are equivalent to each other with respect to $\boldsymbol{\beta}$. They induce the same total weight on arc 1, which is M, the same total weight put by the delegates of the atomic player p on arc 1, which is T^p, and thus the same cost to the atomic player p.

(3) In particular, the nonatomic strategy $\boldsymbol{\alpha}_0$ is in $\mathcal{S}_1^p$.

Mode 2

The CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is of *mode 2* and *specified by* k and l, if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); \\ n \geq 1,\ 1 \leq k \leq n; \quad x_2^i > 0,\ i \in [\![1,k]\!]; \quad x_2^i = 0,\ i \in [\![k+1,n]\!] \cup \{0\}; \\ m \geq 1,\ 1 \leq l \leq m; \quad y_2^j > 0,\ j \in [\![1,l]\!]; \quad y_2^j = 0,\ j \in [\![l+1,m]\!] \cup \{0\}. \end{cases} \tag{4.9}$$

Mode 2	α^1	$\cdots$	α^k	α^{k+1}	$\cdots$	α^n	α^0	β^1	$\cdots$	β^l	β^{l+1}	$\cdots$	β^m	β^0
arc 1	x_1^1	$\cdots$	x_1^k	α^{k+1}	$\cdots$	α^n	α^0	y_1^1	$\cdots$	y_1^l	β^{l+1}	$\cdots$	β^m	β^0
arc 2	x_2^1	$\cdots$	x_2^k	0	$\cdots$	0	0	y_2^1	$\cdots$	y_2^l	0	$\cdots$	0	0

The following two results follow from (4.2).
(i) If $n > k$, then, for $i \in [\![k+1, N]\!]$, $c_1(\xi_1) + T^i c_1'(\xi_1) \leq c_2(\xi_2)$ or, equivalently, $\alpha^i \leq h(\xi_1)$.
(ii) For $i \in [\![1,k]\!]$, $c_1(\xi_1) + x_1^i c_1'(\xi_1) = c_2(\xi_2) + x_2^i c_2'(\xi_2)$ or, equivalently,

$$x_1^i = \frac{\alpha^i a(\xi_1) + h(\xi_1)}{1 + a(\xi_1)}. \tag{4.10}$$

For $i \in [\![1,k]\!]$, it is not difficult to deduce the following two results from (4.10) and the constraint that $0 < x_1^i < \alpha^i$:
(i) $\alpha^i > -h(\xi_1)/a(\xi_1)$, which is always true, because $h(\xi_1) \geq 0$ and $a(\xi_1) > 0$;
(ii)

$$\alpha^i > h(\xi_1). \tag{4.11}$$

Similarly, if $l < m$, then, for $j \in [\![l+1, m]\!]$,

$$\beta^j \leq h(\xi_1) \tag{4.12}$$

and, for $j \in [\![1, l]\!]$,

$$y_1^j = \frac{\beta^j a(\xi_1) + h(\xi_1)}{1 + a(\xi_1)}, \tag{4.13}$$

$$\beta^j > h(\xi_1). \tag{4.14}$$

According to (4.10) and (4.13),

$$x_1 = \sum_{i=1}^{k} x_1^i + T^p - \alpha^{[k]} = T^p - \frac{\alpha^{[k]} - k\,h(\xi_1)}{1 + a(\xi_1)}, \tag{4.15}$$

$$y_1 = \sum_{j=1}^{l} y_1^j + T^{-p} - \beta^{[l]} = T^{-p} - \frac{\beta^{[l]} - l\,h(\xi_1)}{1 + a(\xi_1)}, \tag{4.16}$$

$$\Rightarrow \quad \xi_1 = x_1 + y_1 = M - \frac{\alpha^{[k]} + \beta^{[l]} - (k+l)\,h(\xi_1)}{1 + a(\xi_1)} \tag{4.17}$$

$$\Leftrightarrow \quad F_{k+l}(\xi_1) = \alpha^{[k]} + \beta^{[l]}. \tag{4.18}$$

Because F_{k+l} is strictly decreasing, ξ_1 is the unique solution to the equation $F_{k+l}(\cdot) = \alpha^{[k]} + \beta^{[l]}$.

Four constraints on ξ_1 can be deduced.

(i) Relation (4.11) implies that

$$\alpha^{[k]} > k\,h(\xi_1) \tag{4.19}$$

or, equivalently, according to (4.18),

$$\begin{aligned} & F_{k+l}(\xi_1) = (M - \xi_1)(1 + a(\xi_1)) + (k+l)\,h(\xi_1) > k\,h(\xi_1) + \beta^{[l]} \\ \Rightarrow \quad & F_l(\xi_1) = (M - \xi_1)(1 + a(\xi_1)) + l\,h(\xi_1) > \beta^{[l]} \\ \Rightarrow \quad & \xi_1 < F_l^{-1}(\beta^{[l]}). \end{aligned} \tag{4.20}$$

(ii) Relation (4.14) implies

$$\xi_1 > h^{-1}(\beta^l). \tag{4.21}$$

(iii) It follows from (4.12) that, if $l < m$, then

$$\xi_1 \leq h^{-1}(\beta^{l+1}). \tag{4.22}$$

(iv) If $H \leq 0$, then, according to Proposition 4.8,

$$\xi_1 < \hat{\xi}. \tag{4.23}$$

These four constraints on ξ_1 (*cf.* (4.20)-(4.23)), together with (4.11), (4.18) and (4.19), imply that

$$T^p \geq \alpha^{[k]} > kh(\xi_1) > k\,h(F_l^{-1}(\beta^{[l]})); \tag{4.24}$$

$$\alpha^{[k]} = F_{k+l}(\xi_1) - \beta^{[l]} < F_{k+l}(h^{-1}(\beta^l)) - \beta^{[l]}; \tag{4.25}$$

$$\text{if } \ l < m, \text{ then } \ T^p \geq \alpha^{[k]} = F_{k+l}(\xi_1) - \beta^{[l]} \geq F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]};$$

$$\text{if } \ H \leq 0, \text{ then } \ T^p \geq \alpha^{[k]} = F_{k+l}(\xi_1) - \beta^{[l]} \geq F_0(\hat{\xi}) - \beta^{[l]}.$$

Among these relations, (4.24) and (4.25) imply that

$$\begin{aligned} k\,h(F_l^{-1}(\beta^{[l]})) < F_{k+l}(h^{-1}(\beta^l)) - \beta^{[l]} & \Rightarrow \beta^{[l]} + k\,h(F_l^{-1}(\beta^{[l]})) < F_l(h^{-1}(\beta^l)) + k\,\beta^l \\ & \Rightarrow F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l). \end{aligned}$$

These results entail the following proposition.

Proposition 4.16. *(1) A set of necessary and sufficient conditions on the profile $\boldsymbol{\beta}$ for the atomic player p to have a strategy $\boldsymbol{\alpha}$ such that the CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is of mode 2 and specified by some k and l in $\mathbb{N}^*$* (cf. (4.9)) *is that*

$$\begin{cases} m \geq l; \\ F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l); \\ T^p > k\,h(F_l^{-1}(\beta^{[l]})); \\ \textit{if} \ \ m \geq l+1, \ \textit{then} \ \ F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} \leq T^p; \\ \textit{if} \ \ H \leq 0, \ \textit{then} \ \ F_0(\hat{\xi}) - \beta^{[l]} < T^p. \end{cases} \tag{4.26}$$

(2) Suppose that the conditions in (4.26) *are satisfied. Given $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq T^p$ and*

$$\begin{cases} \alpha > k\,h(F_l^{-1}(\beta^{[l]})); \\ \alpha < F_{k+l}(h^{-1}(\beta^l)) - \beta^{[l]}; \\ \textit{if} \ \ m \geq l+1, \ \textit{then} \ \ \alpha \geq F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]}; \\ \textit{if} \ \ H \leq 0, \ \textit{then} \ \ \alpha > F_0(\hat{\xi}) - \beta^{[l]}, \end{cases} \tag{4.27}$$

denote $\xi_1 = F_{k+l}^{-1}(\alpha + \beta^{[l]})$. Then, a strategy $\boldsymbol{\alpha}$ of the atomic player p is in the following subset of $\mathcal{S}^p$:

$$\mathcal{S}_2^p(\alpha,\boldsymbol{\beta};k,l) = \{\, \boldsymbol{\alpha} \in \mathcal{S}^p \,|\, n \geq k;\ \alpha^{[k]} = \alpha;\ \forall\, i \in [\![1,k]\!], \alpha^i > h(\xi_1);\ \forall\, i \in [\![k{+}1,n]\!],\ \alpha^i \leq h(\xi_1)\}$$

if, and only if, it induces a CE $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$ of mode 2 and specified by k and l and, at $\mathbf{x}(\boldsymbol{\alpha},\boldsymbol{\beta})$, the total weight on arc 1 is ξ_1 while the total weight put by the delegates of the atomic player p on arc 1 is $T^p - \frac{\alpha - kh(\xi_1)}{1+a(\xi_1)}$.

As a result, the strategies in $\mathcal{S}_2^p(\alpha,\boldsymbol{\beta};k,l)$ are equivalent to each other with respect to $\boldsymbol{\beta}$, and they induce the same cost to the atomic player p.

(3) The single-atomic strategy $\alpha - (k-1)h(\xi_1)$ is equivalent to the strategies in $\mathcal{S}_2^p(\alpha,\boldsymbol{\beta};k,l)$ with respect to $\boldsymbol{\beta}$.

Proof. Only (3) needs to be proved.

In five steps, let us show that, if the conditions in (4.26) and those in (4.27) are satisfied for k and α, then they are also satisfied when k is replaced by 1 and α is replaced by $\alpha - (k-1)h(\xi_1)$.

3.1) If $h(F_l^{-1}(\beta^{[l]})) < 0$, then $T^p \geq \alpha > h(F_l^{-1}(\beta^{[l]}))$. If $h(F_l^{-1}(\beta^{[l]})) > 0$, $T^p \geq \alpha > k\,h(F_l^{-1}(\beta^{[l]})) > h(F_l^{-1}(\beta^{[l]}))$.

3.2) If $l < m$, then

$$\begin{aligned} T^p &\geq F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} = F_{1+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} + (k-1)\beta^{l+1} \\ &\geq F_{1+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]}, \end{aligned}$$

where the second inequality is due to the fact that $k \geq 1$.

3.3) According to (4.21), $\xi_1 > h^{-1}(\beta^l)$, and it implies that $F_{1+l}(\xi_1) < F_{1+l}(h^{-1}(\beta^l))$. Besides, by the definition of ξ_1, $\alpha-(k-1)h(\xi_1) = F_{k+l}(\xi_1)-\beta^{[l]}-(k-1)h(\xi_1) = F_{1+l}(\xi_1)-\beta^{[l]}$. Therefore, $\alpha - (k-1)h(\xi_1) < F_{1+l}(h^{-1}(\beta^l)) - \beta^{[l]}$.

3.4) If $l < m$, the relation $\xi_1 \leq h^{-1}(\beta^{l+1})$ (*cf.* (4.22)) implies that $F_{1+l}(\xi_1) \geq F_{1+l}(h^{-1}(\beta^{l+1}))$. As a result, $\alpha - (k-1)h(\xi_1) = F_{1+l}(\xi_1) - \beta^{[l]} \geq F_{1+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]}$.

3.5) If $H \leq 0$, the relation $\xi_1 \leq \hat{\xi}$ (*cf.* (4.23)) implies that $\alpha - (k-1)h(\xi_1) = F_{1+l}(\xi_1) - \beta^{[l]} \geq F_0(\hat{\xi}) - \beta^{[l]}$.

Therefore, the CE $\mathbf{x}(\alpha - (k-1)h(\xi_1), \boldsymbol{\beta})$ of the composite routing game induced by the single-atomic strategy $\alpha - (k-1)h(\xi_1)$ and the profile $\boldsymbol{\beta}$, denoted by $\tilde{\mathbf{x}}$, is of mode 3 and specified by 1 and l. Statement (2) can thus be applied to compute the total weight on arc 1 and the total weight put by the delegates of the atomic player p on arc 1 at $\tilde{\mathbf{x}}$.

Firstly, the definition of ξ_1 implies that $\alpha + \beta^{[l]} = F_{k+l}(\xi_1) = F_{1+l}(\xi_1) + (k-1)h(\xi_1)$. One deduces that $\alpha - (k-1)h(\xi_1) + \beta^{[l]} = F_{1+l}(\xi_1)$ or, equivalently, $\xi_1 = F_{1+l}^{-1}(\alpha - (k-1)h(\xi_1))$. Then, according to statement (2), the total weight on arc 1 at $\tilde{\mathbf{x}}$ is also ξ_1.

Secondly, in statement (2), by replacing α by $\alpha - (k-1)h(\xi_1)$ and k by 1 in $T^p - \frac{\alpha - kh(\xi_1)}{1+a(\xi_1)}$, one obtains that, at $\tilde{\mathbf{x}}$, the total weight put by the delegates of the atomic player p on arc 1 is also $T^p - \frac{\alpha - kh(\xi_1)}{1+a(\xi_1)}$.

These two results imply that the single-atomic strategy $\alpha - (k-1)h(\xi_1)$ is equivalent to the strategies in $\mathcal{S}_2^p(\alpha, \boldsymbol{\beta}; k, l)$ with respect to $\boldsymbol{\beta}$. □

Mode 3

The CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of *mode 3* and *specified by* l, if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); \\ n \geq 0; \quad x_2^i = 0,\ i \in \{0\} \cup [\![1, n]\!]; \\ m \geq 1,\ 1 \leq l \leq m; \quad y_2^j > 0,\ j \in [\![1, l]\!]; \quad y_2^j = 0,\ j \in [\![l+1, m]\!] \cup \{0\}. \end{cases} \tag{4.28}$$

Mode 3	α^1	$\cdots$	α^n	α^0	β^1	$\cdots$	β^l	β^{l+1}	$\cdots$	β^m	β^0
arc 1	α^1	$\cdots$	α^n	α^0	y_1^1	$\cdots$	y_1^l	β^{l+1}	$\cdots$	β^m	β^0
arc 2	0	$\cdots$	0	0	y_2^1	$\cdots$	y_2^l	0	$\cdots$	0	0

An analysis similar to that for mode 2 yields the following results.

For $j \in [\![1, l]\!]$, $y_1^j = \frac{\beta^j\, a(\xi_1) + h(\xi_1)}{1+a(\xi_1)}$ and

$$\beta^j > h(\xi_1). \tag{4.29}$$

If $l < m$, then $\beta^j \leq h(\xi_1)$ for $j \in [\![l+1, m]\!]$. If $n > 0$, then $\alpha^i \leq h(\xi_1)$ for $i \in [\![1, n]\!]$.

Besides, $x_1 = T^p$, $y_1 = T^{-p} - \frac{\beta^{[l]} - l\, h(\xi_1)}{1+a(\xi_1)}$, $\xi_1 = x_1 + y_1 = M - \frac{\beta^{[l]} - l\, h(\xi_1)}{1+a(\xi_1)}$ or, equivalently, $F_l(\xi_1) = \beta^{[l]}$. Moreover, ξ_1 is the unique solution to the equation $F_l(\cdot) = \beta^{[l]}$.

If $H \leq 0$, it follows from Proposition 4.8 that $\xi_1 < \hat{\xi}_1$ or, equivalently, $\beta^{[l]} = F_l(\xi_1) > F_l(\hat{\xi}) = (M - \hat{\xi})(1 + A)$.

One deduces the following proposition.

Proposition 4.17. *(1) A set of necessary and sufficient conditions on the profile $\boldsymbol{\beta}$ for the atomic player p to have a strategy $\boldsymbol{\alpha}$ such that the CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 3, specified by $l \in [\![1, m]\!]$* (cf. (4.28))*, is that*

$$\begin{cases} m \geq l; \\ F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l); \\ \text{if } m \geq l+1, \text{ then } F_l^{-1}(\beta^{[l]}) \leq h^{-1}(\beta^{l+1}); \\ \text{if } H \leq 0, \text{ then } \beta^{[l]} > F_0(\hat{\xi}). \end{cases} \tag{4.30}$$

(2) Suppose that the conditions in (4.30) are satisfied. Then, a strategy $\boldsymbol{\alpha}$ of the atomic player p is in the following subset of $\mathcal{S}^p$:

$$\mathcal{S}_3^p(\boldsymbol{\beta}; l) = \{\, \boldsymbol{\alpha} \in \mathcal{S}^1 \,|\, n = 0, \text{ or } n \geq 1 \text{ but } \alpha^1 \leq h(F_l^{-1}(\beta^{[l]}))\,\}$$

if, and only if, it induces a CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of mode 3 and specified by l and, at $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, the total weight on arc 1 is $F_l^{-1}(\beta^{[l]})$ and the total weight put by the delegates of the atomic player p on arc 1 is T^p.

As a result, the strategies in $\mathcal{S}_3^p(\boldsymbol{\beta}; l)$ are equivalent to each other with respect to $\boldsymbol{\beta}$ and they induce the same cost to the atomic player p.

(3) In particular, the nonatomic strategy $\boldsymbol{\alpha}_0$ is in $\mathcal{S}_3^p(\boldsymbol{\beta}; l)$.

Mode 4

The CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of *mode 4* and *specified by k*, if

$$\begin{cases} c_1(\xi_1) < c_2(\xi_2); \\ n \geq 1,\, 1 \leq k \leq n; \quad x_2^i > 0,\ i \in [\![1, k]\!]; \quad x_2^i = 0,\ i \in [\![k+1, n]\!] \cup \{0\}. \\ m \geq 0; \quad y_2^i = 0,\ y \in \{0\} \cup [\![1, m]\!]. \end{cases} \tag{4.31}$$

Mode 4	α^1	$\cdots$	α^k	α^{k+1}	$\cdots$	α^n	α^0	β^1	$\cdots$	β^m	α^0
arc 1	x_1^1	$\cdots$	x_1^k	α^{k+1}	$\cdots$	α^n	α^0	β^1	$\cdots$	β^m	β^0
arc 2	x_2^1	$\cdots$	x_2^k	0	$\cdots$	0	0	0	$\cdots$	0	0

An analysis similar to that for mode 2 yields the following results.
For $i \in [\![1, k]\!]$, $x_1^i = \frac{\alpha^j\, a(\xi_1) + h(\xi_1)}{1 + a(\xi_1)}$, and

$$\alpha^i > h(\xi_1). \tag{4.32}$$

This further implies that $\alpha^{[k]} > kh(\xi_1)$.
If $k < n$, then $\alpha^i \leq h(\xi_1)$ for $i \in [\![k+1, n]\!]$.
If $m \geq 1$, then $\beta^j \leq h(\xi_1)$ for $j \in [\![1, m]\!]$. In particular, $\xi_1 \leq h^{-1}(\beta^1)$.

$$x_1 = \sum_{i=1}^{l} x_1^i + T^p - \alpha^{[k]} = T^p - \frac{\alpha^{[k]} - k\,h(\xi_1)}{1 + a(\xi_1)} \tag{4.33}$$

$$y_1 = T^{-p} \tag{4.34}$$

$$\xi_1 = x_1 + y_1 = M - \frac{\alpha^{[k]} - k\,h(\xi_1)}{1 + a(\xi_1)} \quad \text{or, equivalently,} \quad F_k(\xi_1) = \alpha^{[k]}.$$

Moreover, ξ_1 is the unique solution to the equation $F_k(\cdot) = \alpha^{[k]}$.
If $H \leq 0$, $\xi_1 < \hat{\xi}_1$.

One deduces the following proposition.

Proposition 4.18. *(1) A set of necessary and sufficient conditions on the profile $\boldsymbol{\beta}$ for the atomic player p to have a strategy $\boldsymbol{\alpha}$ such that the CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 4 and specified by k (*cf. *(4.31)), is that*

$$\begin{cases} T^p > kH; \\ \textit{if } m \geq 1, \textit{ then } T^p \geq F_k(h^{-1}(\beta^1)); \\ \textit{if } H \leq 0, \textit{ then } T^p > F_0(\hat{\xi}). \end{cases} \tag{4.35}$$

(2) *Suppose that the conditions in (4.35) are satisfied. Given* $\alpha \in \mathbb{R}$ *such that* $0 < \alpha \leq T^p$ *and*

$$\begin{cases} kH < \alpha \leq T^p; \\ \text{if } m \geq 1,\ \alpha \geq F_k(h^{-1}(\beta^1)); \\ \text{if } H \leq 0,\ \alpha > F_0(\hat{\xi}), \end{cases} \tag{4.36}$$

denote $\xi_1 = F_k^{-1}(\alpha)$. *Then, a strategy* $\boldsymbol{\alpha}$ *of the atomic player* p *is in the following subset of* $\mathcal{S}^p$

$$\mathcal{S}_4^p(\alpha, \boldsymbol{\beta}; k) = \{\, \boldsymbol{\alpha} \in \mathcal{S}^1 \,|\, n \geq k;\ \alpha^{[k]} = \alpha;\ \forall\, i \in [\![1, k]\!],\ \alpha^i > h(\xi_1);\ \forall\, i \in [\![k+1, n]\!],\ \alpha^i \leq h(\xi_1)\}$$

if, and only if, it induces a CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *of mode 4 and specified by* k *and, at* $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, *the total weight on arc 1 is* ξ_1 *and the total weight put by the delegates of the atomic player* p *on arc 1 is* $T^p - \frac{\alpha - k\,h(\xi_1)}{1+a(\xi_1)}$.

As a result, the strategies in $\mathcal{S}_4^p(\alpha, \boldsymbol{\beta}; k)$ *are equivalent to each other with respect to* $\boldsymbol{\beta}$ *and they induce the same cost to the atomic player* p.

(3) *The single-atomic strategy* $\alpha - (k-1)h(\xi_1)$ *is equivalent to the strategies in* $\mathcal{S}_4^p(\alpha, \boldsymbol{\beta}; k)$.

Mode 5

The CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of *mode 5* if

$$c_1(\xi_1) = c_2(\xi_2). \tag{4.37}$$

Mode 5	α^1	$\cdots$	α^n	α^0	β^1	$\cdots$	β^m	β^0
arc 1	x_1^1	$\cdots$	x_1^n	x_1^0	y_1^1	$\cdots$	y_1^m	y_1^0
arc 2	x_2^1	$\cdots$	x_2^n	x_2^0	y_2^1	$\cdots$	y_2^m	y_2^0

Clearly, in this case, $\xi_1 = \hat{\xi}$ and, according to Proposition 4.3, $H \leq 0$.

If $n \geq 1$, then it follows from (4.2) that $c_1(\xi_1) + x_1^i\, c_1'(\xi_1) = c_2(\xi_2) + (x_2^i)\, c_2'(\xi_2)$ for $i \in [\![1, n]\!]$. This implies that $x_1^i = \frac{A\alpha^i}{1+A}$ because $c_1(\xi_1) = c_2(\xi_2)$. Similarly, if $m \geq 1$, then $y_1^j = \frac{A\beta^j}{1+A}$ for $j \in [\![1, m]\!]$.

These two results imply that

$$x_1^0 + y_1^0 = \xi_1 - \sum_i^n x_1^i - \sum_j^m y_1^j = \hat{\xi} - \frac{A(\alpha^{[n]} + \beta^{[m]})}{1+A}. \tag{4.38}$$

It follows from $0 \leq x_1^0 \leq \alpha^0$ and $0 \leq y_1^0 \leq \beta^0$ that

$$0 \leq x_1^0 + y_1^0 \leq T^p - \alpha^{[n]} + T^{-p} - \beta^{[m]} = M - \alpha^{[n]} - \beta^{[m]}.$$

One deduces, by considering (4.38), that $\alpha^{[n]} + \beta^{[m]} \leq \hat{\xi} \cdot \frac{1+A}{A}$ and $\alpha^{[n]} + \beta^{[m]} \leq (M - \hat{\xi})(1+A)$. But, $\hat{\xi} \cdot \frac{1+A}{A} \geq (M - \hat{\xi})(1+A)$ according to (4.4). Therefore,

$$\alpha^{[n]} + \beta^{[m]} \leq (M - \hat{\xi})(1+A) = F_0(\hat{\xi}), \tag{4.39}$$

These results lead to the following proposition.

Proposition 4.19. *Assume that* $H \leq 0$.

(1) A necessary and sufficient condition on the profile $\boldsymbol{\beta}$ for the atomic player p to have a strategy $\boldsymbol{\alpha}$ such that the CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 5 (cf. (4.37)) *is that*

$$m = 0, \ \text{or } m \geq 1 \text{ but } \beta^{[m]} \leq F_0(\hat{\xi}). \tag{4.40}$$

(2) Suppose that the condition (4.40) *is satisfied. Then, a strategy $\boldsymbol{\alpha}$ of the atomic player p is in the following subset of $\mathcal{S}^p$*

$$\mathcal{S}_5^p = \{\, \boldsymbol{\alpha} \in \mathcal{S}^1 \,|\, n = 0, \ \text{or } n \geq 1 \ \text{ but } \alpha^{[n]} \leq F_0(\hat{\xi}) - \beta^{[m]} \}$$

if, and only if, it induces a CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of mode 5.

Furthermore, facing $\boldsymbol{\beta}$, all the strategies in $\mathcal{S}_5^p$ induce the same cost to the atomic player p, which is $T^p\, c_1(\hat{\xi})$.

(3) In particular, the nonatomic strategy $\boldsymbol{\alpha}_0$ is in $\mathcal{S}_5^p$.

4.3.4 Special role of single-atomic strategies

By combining Propositions 4.15-4.19, one deduces the following theorem on the special role of single-atomic strategies in one-shot delegation games.

Theorem 4.20. *Given a profile $\boldsymbol{\beta}$, for any strategy $\boldsymbol{\alpha} \in \mathcal{S}^p$, the atomic player p can obtain the cost $u^p(\boldsymbol{\alpha}, \boldsymbol{\beta})$ by playing a single-atomic strategy. In other words, the delegation strategies in $\mathcal{S}^p$ are weakly dominated by the single-atomic strategies in S^p.*

Therefore, facing a profile $\boldsymbol{\beta}$, if the atomic player p has best replies to $\boldsymbol{\beta}$, then at least one of them is a single-atomic one. In the following section, one will show that, given any $\boldsymbol{\beta}$, a best reply, which is a single-atomic strategy, always exists.

4.4 Best replies

Notation. For a profile $\boldsymbol{\beta} = (\beta^0, \beta^1, \ldots, \beta^m)$ induced by the delegation strategies of all the atomic players except p in addition to the nonatomic players in $\mathbf{T}$, denote $\beta^0 = \beta^{[0]} = 0$, and

$$B_l = F_{l+1}(h^{-1}(\beta^{l+1})) - \beta^{[l]}, \quad \text{for } \ l = 0 \ \text{ and, if } \ m > 1, \ \text{ for all } \ l \in [\![1, m-1]\!]. \tag{4.41}$$

4.4.1 Preliminary results

Lemma 4.21. *Suppose that $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are, respectively, the CE of the composite routing games $\Gamma(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and that of $\Gamma(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$. If $\mathbf{x}$ is of mode 2* (cf. (4.9)) *or mode 4* (cf. (4.31))*, and $x_1 < \tilde{x}_1$, $y_1 \leq \tilde{y}_1$, then*

$$u^p(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) > u^p(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

Proof. Suppose that $\mathbf{x}$ is specified by $k \in \mathbb{N}^*$. According to (4.2), for the k largest atomic delegates of the atomic player p, i.e. those who have weight $\alpha^1, \ldots, \alpha^k$,

$$c_1(\xi_1) + x_1^i\, c_1'(\xi_1) = c_2(\xi_2) + x_2^i\, c_2'(\xi_2), \quad i \in [\![1, k]\!].$$

Summing over all $i \in [\![1, k]\!]$ leads to

$$k\, c_1(\xi_1) + [\, x_1 - (T^p - \alpha^{[k]})\,]\, c_1'(\xi_1) = k\, c_2(\xi_2) + x_2\, c_2'(\xi_2).$$

Consequently,

$$c_1(\xi_1) + x_1\, c_1'(\xi_1) \;=\; c_2(\xi_2) + x_2\, c_2'(\xi_2) + (k-1)\big[\, c_2(\xi_2) - c_1(\xi_1)\,\big] + (T^p - \alpha^{[k]})\, c_1'(\xi_1).$$

Recall that, according to Proposition 4.8, $c_1(\xi_1) \le c_2(\xi_2)$. Then, one deduces from the previous equation that $c_1(\xi_1) + x_1\, c_1'(\xi_1) \ge c_2(\xi_2) + x_2\, c_2'(\xi_2)$. Moreover, there exists $B > 0$ such that

$$c_1(\xi_1) + x_1\, c_1'(\xi_1) \ge B \ge\; c_2(\xi_2) + x_2\, c_2'(\xi_2). \tag{4.42}$$

Because of the strict monotonicity and the convexity of c_1 and c_2, (4.42) and the fact that $c_1(\xi_1) \le c_2(\xi_2)$ imply that, for all $s \in (x_1, T^p\,]$ and $t \in [\,0, x_2)$,

$$\begin{aligned} &c_1(s+y_1) + s\, c_1'(s+y_1) > B > c_2(t+y_2) + t\, c_2'(t+y_2), \\ &c_1(s+y_1) < c_1(\xi_1) \le c_2(\xi_2) < c_2(t+y_2). \end{aligned} \tag{4.43}$$

They further imply that $s\, c_1'(s+y_1) > t\, c_2'(t+y_2)$ and, moreover, there exists $C > 0$ such that for any $w \in (\,y_1, T^{-p}\,]$ and $z \in [\,0, y_2\,)$,

$$s\, c_1'(s+w) > C > t\, c_2'(t+z). \tag{4.44}$$

Let us compare $u^p(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ and $u^p(\boldsymbol{\alpha}, \boldsymbol{\beta})$:

$$\begin{aligned} u^p(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) - u^p(\boldsymbol{\alpha}, \boldsymbol{\beta}) &= \big[\, \tilde{x}_1\, c_1(\tilde{\xi}_1) + \tilde{x}_2\, c_2(\tilde{\xi}_2)\,\big] - \big[\, x_1\, c_1(\xi_1) + x_2\, c_2(\xi_2)\,\big] \\ &= \big[\, \tilde{x}_1\, c_1(\tilde{\xi}_1) - \tilde{x}_1\, c_1(\tilde{x}_1 + y_1)\,\big] + \big[\, \tilde{x}_1\, c_1(\tilde{x}_1 + y_1) - x_1\, c_1(\xi_1)\,\big] \\ &\quad + \big[\, \tilde{x}_2\, c_2(\tilde{\xi}_2) - \tilde{x}_2\, c_2(\tilde{x}_2 + y_2)\,\big] + \big[\, \tilde{x}_2\, c_2(\tilde{x}_2 + y_2) - x_2\, c_2(\xi_2)\,\big] \\ &= \int_{y_1}^{\tilde{y}_1} [\, \tilde{x}_1\, c_1(\tilde{x}_1 + t)\,]'\, \mathrm{d}t + \int_{x_1}^{\tilde{x}_1} [\, t\, c_1(t + y_1)\,]'\, \mathrm{d}t \\ &\quad - \int_{\tilde{y}_2}^{y_2} [\, \tilde{x}_2\, c_2(\tilde{x}_2 + t)\,]'\, \mathrm{d}t - \int_{\tilde{x}_2}^{x_2} [\, t\, c_2(t + y_2)\,]'\, \mathrm{d}t \\ &= \int_{y_1}^{\tilde{y}_1} \tilde{x}_1\, c_1'(\tilde{x}_1 + t)\, \mathrm{d}t + \int_{x_1}^{\tilde{x}_1} [\, c_1(t + y_1) + t\, c_1'(t + y_1)\,]\, \mathrm{d}t \\ &\quad - \int_{\tilde{y}_2}^{y_2} \tilde{x}_2\, c_2'(\tilde{x}_2 + t)\, \mathrm{d}t - \int_{\tilde{x}_2}^{x_2} [\, c_2(t + y_2) + t\, c_2'(t + y_2)\,]\, \mathrm{d}t \\ &> (\tilde{y}_1 - y_1)\, C + (\tilde{x}_1 - x_1)\, B - (y_2 - \tilde{y}_2)\, C - (x_2 - \tilde{x}_2)\, B \\ &= 0, \end{aligned}$$

where the inequality is due to (4.43) and (4.44). □

Lemma 4.22. *If H and $\boldsymbol{\beta} = (\beta^0,\, \beta^1,\, \ldots,\, \beta^m)$ satisfy one of the two conditions below:*

(i) $H > 0,\ m \ge 1,\ \beta^1 > H$*;*

(ii) $H \le 0,\ m \ge 1,\ \beta^{[m]} \ge F_0(\hat{\xi})$*,*

then one has:

(1) The CE $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ induced by the atomic player p's nonatomic strategy and the profile $\boldsymbol{\beta}$ is of mode 3 and specified by l_0, where l_0 is the unique number in $[\![1, m]\!]$ that meets the conditions in (4.30)*.*

(2)

$$F_1^{-1}(\beta^{[1]}) > F_2^{-1}(\beta^{[2]}) > \cdots > F_{l_0}^{-1}(\beta^{[l_0]}); \tag{4.45}$$
$$F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l), \qquad \forall\, l \in [\![1, l_0]\!]. \tag{4.46}$$

If $l_0 < m$, then

$$F_{l_0}^{-1}(\beta^{[l_0]}) \le F_{l_0+1}^{-1}(\beta^{[l_0+1]}) \le \cdots \le F_m^{-1}(\beta^{[m]}); \tag{4.47}$$
$$F_l^{-1}(\beta^{[l]}) \le h^{-1}(\beta^l), \qquad \forall\, l \in [\![l_0+1, m]\!]. \tag{4.48}$$

(3) For all $k \in \mathbb{N}$,

$$F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} \begin{cases} > k\, h(F_l^{-1}(\beta^{[l]})), & \text{if } l_0 > 1 \text{ and } l \in [\![1, l_0-1]\!], \\ \le k\, h(F_l^{-1}(\beta^{[l]})), & \text{if } l_0 < m \text{ and } l = l_0. \end{cases} \tag{4.49}$$

In particular, if $l_0 > 1$, then $B_l > h\big(F_l^{-1}(\beta^{[l]})\big)$ for all $l \in [\![1, l_0-1]\!]$.

(4) If $l_0 > 1$, then, for all $k \in \mathbb{N}$ and $l \in [\![1, l_0-1]\!]$,

$$F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} > k\, h\big(F_{l_0}^{-1}(\beta^{[l_0]})\big). \tag{4.50}$$

(5) If $l_0 > 1$, then $B_0 \ge B_1 \ge \cdots \ge B_{l_0-1}$.
For $l \in [\![1, l_0-1]\!]$, $B_l = B_{l-1}$ if, and only if, $\beta^{l+1} = \beta^l$.

(6) $B_{l_0-1} > h\big(F_{l_0}^{-1}(\beta^{[l_0]})\big)$.

(7) For all strategy $\boldsymbol{\alpha} \in \mathcal{S}^p$, one and only one of the three statements below is true:
(i) $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 3 and specified by l_0. Besides, $\boldsymbol{\alpha}$ is equivalent to $\boldsymbol{\alpha}_0$ with respect to $\boldsymbol{\beta}$;
(ii) $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 4 and specified by some $k \in \mathbb{N}^*$;
(iii) $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 2 and specified by some $k \in \mathbb{N}^*$ and some $l \in [\![1, l_0]\!]$;

Proof. (1) The CE $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ is not of mode 1 when $H > 0$, because (4.8) is not satisfied. And it is not mode 5 when $H \le 0$, because (4.40) is not satisfied. It is neither of mode 2 nor of mode 4, because $n = 0$. Thus, it is of mode 3. Proposition 4.17 shows that $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ is specified by l if, and only if, l meets the conditions in (4.30). Such an $l_0 \in [\![1, m]\!]$ exists and it is unique because, otherwise, the composite routing game $\Gamma(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ has no CE or more than one CE, which contradicts Theorem 4.5.

(2) One has only to prove for the case where $m > 1$.

First, suppose that $l_0 < m$. Let us prove (4.47) and (4.48) by induction.

The fact that $m \ne l_0$ implies that m does not meet all the four conditions in (4.30). However, in this case, only the second condition can be violated. Thus, $F_m^{-1}(\beta^{[m]}) \le h^{-1}(\beta^m)$.

Now, suppose that for some $l \in [\![l_0+1, m]\!]$,

$$F_l^{-1}(\beta^{[l]}) \le F_{l+1}^{-1}(\beta^{[l+1]}) \le \cdots \le F_m^{-1}(\beta^{[m]});$$
$$F_p^{-1}(\beta^{[p]}) \le h^{-1}(\beta^p), \qquad \forall\, p \in [\![l, m]\!].$$

The relation $F_l^{-1}(\beta^{[l]}) \le h^{-1}(\beta^l)$ implies that $h(F_l^{-1}(\beta^{[l]})) \ge \beta^l$ and, consequently,

$$\begin{aligned} &\beta^{[l]} = F_l(F_l^{-1}(\beta^{[l]})) = F_{l-1}(F_l^{-1}(\beta^{[l]})) + h(F_l^{-1}(\beta^{[l]})) \ge F_{l-1}(F_l^{-1}(\beta^{[l]})) + \beta^l \\ \Rightarrow\ &\beta^{[l-1]} \ge F_{l-1}(F_l^{-1}(\beta^{[l]})) \\ \Rightarrow\ &F_{l-1}^{-1}(\beta^{[l-1]}) \le F_l^{-1}(\beta^{[l]}) \le h^{-1}(\beta^l). \end{aligned}$$

In particular, $F_{l-1}^{-1}(\beta^{[l-1]}) \leq F_l^{-1}(\beta^{[l]})$ and $F_{l-1}^{-1}(\beta^{[l-1]}) \leq h^{-1}(\beta^l)$.

If $F_{l-1}^{-1}(\beta^{[l-1]}) > h^{-1}(\beta^{l-1})$, then all the conditions in (4.30) are satisfied so that $l_0 = l-1$. Otherwise, one continues the induction by considering $l-1$. In this way, (4.47) and (4.48) are proved.

Next, let us prove (4.45) and (4.46) by induction.

According to (4.30), $F_{l_0}^{-1}(\beta^{[l_0]}) > h^{-1}(\beta^{l_0})$. Suppose that $l_0 > 1$ and, for some $l \in [\![2, l_0]\!]$,

$$\begin{aligned}
&F_l^{-1}(\beta^{[l]}) > F_{l+1}^{-1}(\beta^{[l+1]}) > \cdots > F_{l_0}^{-1}(\beta^{[l_0]});\\
&F_p^{-1}(\beta^{[p]}) > h^{-1}(\beta^p), \qquad \forall\, p \in [\![l, l_0]\!];
\end{aligned}$$

If $F_{l-1}^{-1}(\beta^{[l-1]}) \leq F_l^{-1}(\beta^{[l]})$, then

$$\begin{aligned}
& F_{l-1}(F_{l-1}^{-1}(\beta^{[l-1]})) \geq F_{l-1}(F_l^{-1}(\beta^{[l]}))\\
\Rightarrow\quad & \beta^{[l-1]} \geq F_l(F_l^{-1}(\beta^{[l]})) - h(F_l^{-1}(\beta^{[l]})) = \beta^{[l]} - h(F_l^{-1}(\beta^{[l]}))\\
\Rightarrow\quad & h(F_l^{-1}(\beta^{[l]})) \geq \beta^l\\
\Rightarrow\quad & F_l^{-1}(\beta^{[l]}) \leq h^{-1}(\beta^l).
\end{aligned}$$

It contradicts the hypothesis that $F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l)$. Therefore, $F_{l-1}^{-1}(\beta^{[l-1]}) > F_l^{-1}(\beta^{[l]})$.

Furthermore, $F_{l-1}^{-1}(\beta^{[l-1]}) > F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l) \geq h^{-1}(\beta^{l-1})$. In particular, $F_{l-1}^{-1}(\beta^{[l-1]}) > h^{-1}(\beta^{l-1})$. In this way, (4.45) and (4.46) are proven.

(3) For all $k \in \mathbb{N}$,

$$\begin{aligned}
&F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} - k\,h(F_l^{-1}(\beta^{[l]}))\\
&= F_l(h^{-1}(\beta^{l+1})) - \beta^{[l]} + k\big[\,\beta^{l+1} - h(F_l^{-1}(\beta^{[l]}))\,\big] \begin{cases} > 0, & \text{if } h^{-1}(\beta^{l+1}) < F_l^{-1}(\beta^{[l]}),\\ \leq 0, & \text{if } h^{-1}(\beta^{l+1}) \geq F_l^{-1}(\beta^{[l]}).\end{cases}
\end{aligned} \tag{4.51}$$

If $l_0 > 1$, then, for all $l \in [\![1, l_0-1]\!]$, (4.45) and (4.46) show that $F_l^{-1}(\beta^{[l]}) > F_{l+1}^{-1}(\beta^{[l+1]}) > h^{-1}(\beta^{l+1})$; in particular, $F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^{l+1})$.

If $l_0 < m$, then, according to (4.30), $F_{l_0}^{-1}(\beta^{[l_0]}) \leq h^{-1}(\beta^{l_0+1})$.

These two inequalities and (4.51) lead to the conclusion.

(4) For $l \in [\![1, l_0-1]\!]$, it is proven in (3) that $F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^{l+1})$ and, consequently, $\beta^{[l]} < F_l(h^{-1}(\beta^{l+1}))$.

Besides, (4.30) implies that $h^{-1}(\beta^{l_0}) < F_{l_0}^{-1}(\beta^{[l_0]})$ and, in consequence, $\beta^{l_0} > h(F_{l_0}^{-1}(\beta^{[l_0]}))$. Therefore, $\beta^{l+1} \geq \beta^{l_0}$ because $\beta^{l+1} > h(F_{l_0}^{-1}(\beta^{[l_0]}))$.

These two results imply that

$$F_{k+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} - k\,h(F_{l_0}^{-1}(\beta^{[l_0]})) = F_l(h^{-1}(\beta^{l+1})) - \beta^{[l]} + k\,\big[\,\beta^{l+1} - h(F_{l_0}^{-1}(\beta^{[l_0]}))\,\big] > 0,$$

which gives rise to the conclusion.

(5) For $l \in \mathbb{N}$ such that $0 \leq l \leq l_0 - 2$,

$$\begin{aligned}
B_l - B_{l+1} &= \big[\,F_{l+1}(h^{-1}(\beta^{l+1})) - \beta^{[l]}\,\big] - \big[\,F_{l+2}(h^{-1}(\beta^{l+2})) - \beta^{[l+1]}\,\big]\\
&= \big[\,F_{l+2}(h^{-1}(\beta^{l+1})) - \beta^{l+1} - \beta^{[l]}\,\big] - \big[\,F_{l+2}(h^{-1}(\beta^{l+2})) - \beta^{[l]} - \beta^{l+1}\,\big]\\
&= F_{l+2}(h^{-1}(\beta^{l+1})) - F_{l+2}(h^{-1}(\beta^{l+2})) \geq 0
\end{aligned}$$

because $\beta^{l+1} \geq \beta^{l+2}$. Clearly, the equality holds if, and only if, $\beta^{l+1} = \beta^{l+2}$.

(6) According to (4.30), $F_{l_0}^{-1}(\beta^{[l_0]}) > h^{-1}(\beta^{l_0})$. As a result, $\beta^{[l_0]} < F_{l_0}(h^{-1}(\beta^{l_0}))$ and $h(F_{l_0}^{-1}(\beta^{[l_0]})) < \beta^{l_0}$. This implies that

$$\begin{aligned} & F_{l_0}(h^{-1}(\beta^{l_0})) > \beta^{[l_0]} = \beta^{[l_0-1]} + \beta^{l_0} > \beta^{[l_0-1]} + h(F_{l_0}^{-1}(\beta^{[l_0]})) \\ \Rightarrow \quad & B_{l_0-1} = F_{l_0}(h^{-1}(\beta^{l_0})) - \beta^{[l_0-1]} > h(F_{l_0}^{-1}(\beta^{[l_0]})). \end{aligned}$$

(7) Given an arbitrary strategy $\boldsymbol{\alpha} \in \mathcal{S}^p$. Because $\beta^1 > H$ in the case where $H > 0$, and $\beta^{[m]} \geq F_0(\hat{\xi})$ in the case where $H \leq 0$, $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ cannot be of mode 1 or mode 5 according to Propositions 4.15 and 4.19.

If $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 3 and specified by l, then l meets the conditions in (4.30). However, l_0 is the unique number in $[\![1, m]\!]$ that meets the conditions in (4.30), hence $l = l_0$.

If $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 2 and specified by k and l, then $F_l^{-1}(\beta^{[l]}) > h^{-1}(\beta^l)$ by (4.26). According to (4.46) and (4.48), $l \in [\![1, l_0]\!]$. □

Lemma 4.23. *Suppose that $H \leq 0$. If $\boldsymbol{\beta} = (\beta^0, \beta^1, \ldots, \beta^m)$ and T^p are such that $m \geq 1$, $\beta^{[m]} \leq F_0(\hat{\xi})$ and $F_0(\hat{\xi}) - \beta^{[m]} < T^p < F_1(h^{-1}(\beta^1))$, then one has:*

(1) The CE $\mathbf{x}(\boldsymbol{\alpha}_1, \boldsymbol{\beta})$ induced by the atomic player p's trivial strategy and profile $\boldsymbol{\beta}$ is of mode 2 and specified by 1 and some $l_1 \in [\![1, m]\!]$.

(2) $F_m^{-1}(\beta^{[m]}) > h^{-1}(\beta^m)$.
If $m > 1$, then, for all $l \in [\![1, m]\!]$, $h^{-1}(\beta^l) < F_l^{-1}(\beta^{[l]}) < F_{l-1}^{-1}(\beta^{[l-1]})$.

(3) $B_0 \geq B_1 \geq \cdots \geq B_{m-1}$.
If $m > 1$, then, for all $l \in [\![1, m-1]\!]$, $B_l = B_{l-1}$ if, and only if, $\beta^{l+1} = \beta^l$.

(4) $F_0(\hat{\xi}) - \beta^{[m]} < B_{m-1}$.
If $m > 1$, then, for all $l \in [\![1, m-1]\!]$, $F_0(\hat{\xi}) - \beta^{[l]} < B_l$.

(5) If $m = 1$, or if $m > 1$ but $T^p < B_{m-1}$, then $l_1 = m$. If $m > 1$ and $T^p \geq B_{m-1}$, then $l_1 < m$ and $B_{l_1} \leq T^p < B_{l_1-1}$.

Proof. (1) Clear by Propositions 4.15–4.19.

(2) On the one hand, $\beta^m > 0 = h(\hat{\xi})$, hence $h^{-1}(\beta^m) < \hat{\xi}$. On the other hand, by hypothesis, $\beta^{[m]} \leq F_0(\hat{\xi}) = F_m(\hat{\xi})$ and thus $F_m^{-1}(\beta^{[m]}) \geq \hat{\xi}$. Therefore, $F_m^{-1}(\beta^{[m]}) > h^{-1}(\beta^m)$. The rest of the proof is similar to that of Lemma 4.22 (2).

(3) Similar to the proof of Lemma 4.22 (5).

(4) In order to prove that $B_{m-1} = F_m(h^{-1}(\beta^m)) - \beta^{[m-1]} > F_0(\hat{\xi}) - \beta^{[m]}$, it is enough to show that $F_m(h^{-1}(\beta^m)) + \beta^m > F_0(\hat{\xi})$. In (2), it is shown that $h^{-1}(\beta^m) < \hat{\xi}$, hence $F_m(h^{-1}(\beta^m)) > F_m(\hat{\xi}) = F_0(\hat{\xi})$. As a result, $F_m(h^{-1}(\beta^m)) + \beta^m > F_0(\hat{\xi})$. For $l \in [\![1, m-1]\!]$, the proof is similar.

(5) The conclusion follows from Proposition 4.16, in particular, condition (4.27), and the previous statements. □

In the following two sections, the atomic player p's best replies in the case $H > 0$ and in the case $H \leq 0$ will be discussed respectively.

4.4.2 Case $H > 0$

One will see that, face to different $\boldsymbol{\beta}$'s, the atomic player p can have three types of best replies. In the first case, which will be called *nonatomic*, all her delegation strategies, in particular, the nonatomic one $\boldsymbol{\alpha}_0$, are best replies. In the second case, which will be called *trivial*, the trivial strategy $\boldsymbol{\alpha}_1$ is the *unique* best reply. In the third case, which will be called *nontrivial*, there exists a best reply which is a single-atomic strategy, but it is not necessarily the trivial one or the nonatomic one.

Nonatomic case

Lemma 4.24. *Suppose that $H > 0$. If $\boldsymbol{\beta} = (\beta^0, \beta^1, \dots, \beta^m)$ and T^p satisfy one of the two conditions below:*

(i) $m = 0$, or $m \geq 1$ but $\beta^1 \leq H$; $T^p \leq H$;

(ii) $m \geq 1$ and $\beta^1 > H$; $H < T^p \leq h(F_{l_0}^{-1}(\beta^{[l_0]}))$, where l_0 is the one in Lemma 4.22,

then every strategy in $\mathcal{S}^p$ is equivalent to the nonatomic strategy $\boldsymbol{\alpha}_0$ with respect to $\boldsymbol{\beta}$. As a result, every strategy of the atomic player p is a best reply to $\boldsymbol{\beta}$.

Besides, in case (i), $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ is of mode 1, while in case (ii), it is of mode 3 and specified by l_0.

Proof. The results follow immediately from Propositions 4.15 and 4.17. □

Trivial case

Lemma 4.25. *Suppose that $H > 0$. If $\boldsymbol{\beta} = (\beta^0, \beta^1, \dots, \beta^m)$ and T^p are such that $T^p > H$, $m = 0$ or $m \geq 1$ but $\beta^1 \leq H$, then the atomic player p's unique best reply is the trivial strategy $\boldsymbol{\alpha}_1$.*

Besides, $\mathbf{x}(\boldsymbol{\alpha}_1, \boldsymbol{\beta})$ is of mode 4 and specified by 1.

Proof. For any strategy $\boldsymbol{\alpha} \in \mathcal{S}^p$, the CE $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ cannot be of mode 5, because $H > 0$. It cannot be of mode 2 (*resp.* of mode 3) because, otherwise, (4.14) (*resp.* (4.29)) implies that $\beta^1 > h(\xi_1) > h(M) = H$, which contradicts the hypothesis that $\beta^1 \leq H$. Therefore, $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 1 or mode 4.

According to Proposition 4.15, $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 1 if, and only if, $\boldsymbol{\alpha} = (\alpha^0, \alpha^1, \dots, \alpha^n)$ is in $\mathcal{S}_1^p$, i.e. $n = 0$, or $n \geq 1$ but $\alpha^1 \leq H$. In particular, a single-atomic strategy α is in $\mathcal{S}_1^p$ if, and only if, $\alpha \leq H$.

For any $\boldsymbol{\alpha} \notin \mathcal{S}_1^p$, $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is of mode 4. It follows from Proposition 4.18 that there exists some $\alpha \in (0, T^p]$ and some $k \in \mathbb{N}^*$ which satisfy the conditions in (4.36) so that $\boldsymbol{\alpha} \in \mathcal{S}_4^p(\alpha, \boldsymbol{\beta}; k)$ and $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is specified by k.

The rest of the proof is made in four steps.

1) It is easy to verify that $\boldsymbol{\alpha}_1$ is in $\mathcal{S}_4^p(T^p, \boldsymbol{\beta}; 1)$ so that $\mathbf{x}(\boldsymbol{\alpha}_1, \boldsymbol{\beta})$, denoted by $\mathbf{x}^*$ from now on, is of mode 4 and specified by 1. According to Proposition 4.18 and equation (4.34), $x_1^* = T^p - \frac{T^p - h(\xi_1^*)}{1 + a(\xi_1^*)} < T^p$, $y_1^* = T^{-p}$ and $\xi_1^* = F_1^{-1}(T^p)$.

2) Consider a strategy $\boldsymbol{\alpha} \in \mathcal{S}_1^p$, i.e. $n = 0$, or $n \geq 1$ but $\alpha^1 \leq H$. Let $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be denoted by $\mathbf{x}$, which is of mode 1. Then, $x_1 = T^p$ and $y_1 = T^{-p}$ by (4.7). Because $x_1 > x_1^*$ and $y_1 = y_1^*$, it follows from Lemma 4.21 that $u^1(\mathbf{x}) > u^1(\mathbf{x}^*)$. In other words, no strategy in $\mathcal{S}_1^p$ is a best reply.

3) Consider a single-atomic strategy $\alpha \in (H, T^p]$. It is not in $\mathcal{S}_1^p$, hence $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 4. Besides, $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is specified by 1 because α is single-atomic. In order to verify that $\alpha \in (H, T^p]$ and $k = 1$ meet the conditions in (4.36), one has only to show that $\alpha > H \geq F_1(h^{-1}(\beta^1))$ if $m > 0$. Indeed, the hypothesis $\beta^1 \leq H = h(M)$ implies that $h^{-1}(\beta^1) \geq M$ and, consequently, $F_1(h^{-1}(\beta^1)) \leq F_1(M) = H$.

For any $\alpha \in (H, T^p]$, the total weight on arc 1 at $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is $\xi_1 = F_1^{-1}(\alpha)$ by Proposition 4.18. Because F_1 is strictly decreasing, a bijection θ can be defined from interval $(H, T^p]$, the domain of α, to the interval $[\,F_1^{-1}(T^p), M)$, the domain of ξ_1, such that $\theta = F_1^{-1}$. Then, for this fixed profile $\boldsymbol{\beta}$, the atomic player p's cost $u^p(\alpha, \boldsymbol{\beta})$ induced by $(\alpha, \boldsymbol{\beta})$ can be written as a function v of ξ_1 on $[F_1^{-1}(T^p), M)$: $v(\xi_1) = u^p(\theta^{-1}(\xi_1), \boldsymbol{\beta}) = u^p(F_1(\xi_1), \boldsymbol{\beta})$, where θ^{-1} denotes the inverse function of θ.

According to (4.33), $x_1 = T^p - \frac{F_1(\xi_1) - h(\xi_1)}{1 + a(\xi_1)} = T^p - M + \xi_1$. Then, for all $\xi_1 \in [F_1^{-1}(T^p), M)$,

$$v(\xi_1) = x_1 c_1(\xi_1) + (T^p - x_1)\, c_2(M - \xi_1) = (T^p - M + \xi_1) c_1(\xi_1) + (M - \xi_1) c_2(M - \xi_1).$$

Its derivative function is

$$\begin{aligned} v'(\xi_1) &= c_1(\xi_1) + (T^p - M + \xi_1)\, c_1'(\xi_1) - c_2(M - \xi_1) - (M - \xi_1) c_2'(M - \xi_1) \\ &= c_1'(\xi_1)\,(T^p - F_1(\xi_1)) = c_1'(\xi_1)\,(T^p - \alpha) \geq 0 \end{aligned}$$

and the equality holds if, and only if, $\alpha = T^p$ or, equivalently, $\xi_1 = F_1^{-1}(T^p)$.

Therefore, $v(\xi_1)$ attains its unique minimum on the interval $[F_1^{-1}(T^p), M)$ at $F_1^{-1}(T^p)$, and it is strictly increasing on $[F_1^{-1}(T^p), M)$. Equivalently, as a function of α, $u^p(\alpha, \boldsymbol{\beta})$ attains its unique minimum on the interval $(H, T^p]$ at T^p, and it is strictly decreasing on $(H, T^p]$.

Therefore, the trivial strategy $\boldsymbol{\alpha}_1$ is a best reply, and this is the unique single-atomic strategy that is a best reply.

4) Finally, let us show that $\boldsymbol{\alpha}_1$ is the unique best reply. It is proved that no strategy in $\mathcal{S}_1^p$ is a best reply, hence it is enough to show that no strategy in $\mathcal{S}^p \setminus \mathcal{S}_1^p$ other than the trivial one is a best reply.

Given an arbitrary strategy $\boldsymbol{\alpha} \in \mathcal{S}^p \setminus \mathcal{S}_1^p$, suppose that it is in $\mathcal{S}_4^p(\alpha, \boldsymbol{\beta}; k)$ for some $\alpha \in (0, T^p]$ and some $k \in \mathbb{N}^*$. According to Proposition 4.18, $\boldsymbol{\alpha}$ is equivalent to the single-atomic strategy $\alpha - (k-1)\, h(\xi_1)$, where $\xi_1 = F_k^{-1}(\alpha)$. As a result, $\boldsymbol{\alpha}$ induces the same cost to the atomic player p as the single-atomic strategy $\alpha - (k-1)\, h(\xi_1)$.

If the single-atomic strategy $\alpha - (k-1)\, h(\xi_1)$ is not the trivial strategy, then, according to (3), it is not a best reply to $\boldsymbol{\beta}$, hence neither is $\boldsymbol{\alpha}$.

If the single-atomic strategy $\alpha - (k-1)\, h(\xi_1)$ is the trivial strategy, i.e. $\alpha - (k-1)\, h(\xi_1) = T^p$, then $\boldsymbol{\alpha}$ induces the same total weight on arc 1, i.e. $\xi_1 = \xi_1^* = F_1^{-1}(T^p)$. As a result, $\alpha = T^p + (k-1) h(F_1^{-1}(T^p))$.

On the one hand, $\xi_1^* = F_1^{-1}(T^p) < M$ and, consequently, $h(F_1^{-1}(T^p)) > h(M) = H > 0$. It follows that $\alpha = T^p + (k-1)\, h(F_1^{-1}(T^p)) \geq T^p$, and the equality holds if, and only if, $k = 1$. On the other hand, $\alpha \leq T^p$. Therefore, $k = 1$ and $\alpha = T^p$. In other words, $\boldsymbol{\alpha}$ is nothing else but $\boldsymbol{\alpha}_1$.

One concludes that the unique best reply of the atomic player p is the trivial strategy $\boldsymbol{\alpha}_1$. $\square$

Nontrivial case

Lemma 4.26. *Suppose that $H > 0$. If $\boldsymbol{\beta} = (\beta^0, \beta^1, \ldots, \beta^m)$ is such that $m \geq 1$, $\beta^1 > H$, and $T^p > h(F_{l_0}^{-1}(\beta^{[l_0]}))$, where l_0 is the one in Lemma 4.22, then the atomic player p has at least one best reply which is a single-atomic strategy.*

Besides, if $\boldsymbol{\alpha} \in \mathcal{S}^p$ is a best reply, then $\mathbf{x}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be of mode 3 and specified by l_0, or of mode 2 and specified by some $k \in \mathbb{N}^*$ and some $l \in [\![1, l_0]\!]$, or of mode 4 and specified by some $k \in \mathbb{N}^*$.

Proof. Theorem 4.20 implies that all the strategies in $\boldsymbol{\beta}$ are dominated by the single-atomic strategies in $S^p = [0, T^p]$. One can thus consider only the single-atomic strategies α ($0 \leq \alpha \leq T^p$) to find the minimum cost that the atomic player p can get. The rest of the proof is made up of six parts.

1) For an arbitrary $\alpha \in [0, T^p]$, $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is not of mode 5, because $H > 0$. It is not of mode 1, because the hypothesis $\beta^1 > H$ violates the condition (4.8). It is not of mode 4 because, by hypothesis, $T^p < B_0$, which violates the second condition in (4.35). Therefore, $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 2 or 3.

2) Suppose that a single-atomic strategy $\alpha \in [0, T^p]$ induces a CE $\mathbf{x}(\alpha, \boldsymbol{\beta})$ of mode 3.

According to Lemma 4.22 (7), $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is specified by l_0, and it is equivalent to the nonatomic strategy $\boldsymbol{\alpha}_0$ with respect to $\boldsymbol{\beta}$.

It follows from Proposition 4.17 that a necessary and sufficient condition on α for $\mathbf{x}(\alpha, \boldsymbol{\beta})$ to be of mode 3 is $\alpha \leq h(F_{l_0}^{-1}(\beta^{[l_0]}))$, which is always possible since, by hypothesis, $T^p > h(F_{l_0}^{-1}(\beta^{[l_0]}))$. Therefore, this incurs no more conditions on T^p.

Still according to Proposition 4.17, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, the weight on arc 1 is $\xi_1 = F_{l_0}^{-1}(\beta^{[l_0]})$, and the total weight put by the delegates of the atomic player p on arc 1 is T^p. The cost to the atomic player p is thus $u^p(\alpha; \boldsymbol{\beta}) = T^p\, c_1(F_{l_0}^{-1}(\beta^{[l_0]}))$.

3) Suppose that a single-atomic strategy $\alpha \in [0, T^p]$ induces a CE $\mathbf{x}(\alpha, \boldsymbol{\beta})$ of mode 4.

The necessary and sufficient conditions on T^p for the atomic player p to possess a strategy which induces a CE of mode 4 and specified by 1 are given by (4.35) in Proposition 4.18: $T^p > H$ and $T^p \geq F_1(h^{-1}(\beta^1)) = B_0$. The condition $T^p > H$ always holds. Indeed, recall that $F_{l_0}^{-1}(\beta^{[l_0]})$ is the total weight on arc 1 at $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$, hence $F_{l_0}^{-1}(\beta^{[l_0]}) < M$. One deduces from this result and the hypothesis $T^p > h(F_{l_0}^{-1}(\beta^{[l_0]}))$ that $T^p > h(M) = H$. Thus, the only condition on T^p is $T^p \geq B_0$.

Notice that $F_1(h^{-1}(\beta^1)) > F_1(h^{-1}(H)) = H$ because $\beta^1 > H$.

When $T^p \geq B_0$, according to (4.36) and the fact that $F_1(h^{-1}(\beta^1)) > H$, the sufficient and necessary condition on the single-atomic strategy α for $\mathbf{x}(\alpha, \boldsymbol{\beta})$ to be of mode 4 is $F_1(h^{-1}(\beta^1)) \leq \alpha \leq T^p$, i.e. $B_0 \leq \alpha \leq T^p$.

Proposition 4.18 implies that, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, the total weight on arc 1 is $\xi_1 = F_1^{-1}(\alpha)$. As in the trivial case discussed in Lemma 4.25, one can define a continuous and strictly decreasing bijection θ from the interval $[B_0, T^p]$, the domain of α, to the interval $[\,F_1^{-1}(T^p), h^{-1}(\beta^1)\,]$, the domain of ξ_1, such that $\theta = F_1^{-1}$ and $\xi_1 = \theta(\alpha)$. The atomic player p's cost $u^p(\alpha, \boldsymbol{\beta})$ at $\mathbf{x}(\alpha, \boldsymbol{\beta})$ can then be written as a function v of ξ_1 on $[\,F_1^{-1}(T^p), h^{-1}(\beta^1)\,]$: $v(\xi_1) = u^p(\theta^{-1}(\xi_1), \boldsymbol{\beta})$. Explicitly, because the total weight put by the delegates of the atomic player p on arc 1 is $x_1 = T^p - \frac{F_1(\xi_1) - h(\xi_1)}{1 + a(\xi_1)}$ according to Proposition 4.18, $v(\xi_1) = x_1\, c_1(\xi_1) + x_2\, c_2(\xi_2) = T^p\, c_1(\xi_1) + (M - \xi_1)\,\big(c_2(M - \xi_1) - c_1(\xi_1)\big)$.

4) Suppose that a single-atomic strategy $\alpha \in [0, T^p]$ induces a CE $\mathbf{x}(\alpha, \boldsymbol{\beta})$ of mode 2.

According to Lemma 4.22 (7), $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is specified by 1 and some $l \in [\![1, l_0]\!]$.

First, consider the case where $m > 1$, $l_0 > 1$ and $l \in [\![1, l_0 - 1]\!]$. Two necessary conditions on T^p for the atomic player p to possess a strategy which induces a CE of mode 2 and specified by 1 and l are given by (4.26): $T^p > h(F_l^{-1}(\beta^{[l]}))$ and $T^p \geq F_{1+l}(h^{-1}(\beta^{l+1})) - \beta^{[l]} = B_l$. According to Lemma 4.22 (3), $B_l > h(F_l^{-1}(\beta^{[l]}))$. Thus, the only necessary condition on T^p is $T^p \geq B_l$.

When $T^p \geq B_l$, according to (4.27), the sufficient and necessary conditions on the single-atomic strategy α for $\mathbf{x}(\alpha, \boldsymbol{\beta})$ to be specified by 1 and l are $F_{l+1}(h^{-1}(\beta^{l+1})) - \beta^{[l]} \leq \alpha < F_{l+1}(h^{-1}(\beta^l)) - \beta^{[l]}$, $\alpha > h(F_l^{-1}(\beta^{[l]}))$ and $\alpha \leq T^p$. Notice that $F_{l+1}(h^{-1}(\beta^l)) - \beta^{[l]} = F_l(h^{-1}(\beta^l)) + \beta^l - \beta^{[l]} = F_l(h^{-1}(\beta^l)) - \beta^{[l-1]} = B_{l-1}$, and $B_l > h(F_l^{-1}(\beta^{[l]}))$ as shown in the previous paragraph. Thus, the sufficient and necessary conditions on α are $B_l \leq \alpha < B_{l-1}$ and $0 \leq \alpha \leq T^p$.

Proposition 4.16 implies that, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, the total weight on arc 1 is $\xi_1 = F_{1+l}^{-1}(\alpha + \beta^{[l]})$. Thus, one can define a continuous and strictly decreasing bijection from the interval $[B_l, B_{l-1}) \cap [0, T^p]$, the domain of α, to the interval $\big(h^{-1}(\beta^l), h^{-1}(\beta^{l+1})\big] \cap \big[F_{l+1}^{-1}(T^p + \beta^{[l]}), +\infty)$, the domain of ξ_1, such that $\theta = F_{1+l}^{-1}(\cdot + \beta^{[l]})$ and $\xi_1 = \theta(\alpha)$. The atomic player p's cost $u^p(\alpha, \boldsymbol{\beta})$ at $\mathbf{x}(\alpha, \boldsymbol{\beta})$ can then be written as a function v of ξ_1 on $\big(h^{-1}(\beta^l), h^{-1}(\beta^{l+1})\big] \cap \big[F_{l+1}^{-1}(T^p + \beta^{[l]}), +\infty)$: $v(\xi_1) = u^p(\theta^{-1}(\xi_1), \boldsymbol{\beta})$. Explicitly, as $x_1 = T^p - \frac{\alpha - h(\xi_1)}{1 + a(\xi_1)}$ according to Proposition 4.16, $v(\xi_1) = x_1\, c_1(\xi_1) + x_2\, c_2(\xi_2) = T^p\, c_1(\xi_1) + \Big[\,(M - \xi_1) - \frac{\beta^{[l]} - l h(\xi_1)}{1 + a(\xi_1)}\Big]\big(c_2(M - \xi_1) - c_1(\xi_1)\big)$.

In the case where $l = l_0$, by similar arguments, the necessary and sufficient condition on T^p for $\mathbf{x}(\alpha, \boldsymbol{\beta})$ to be of mode 2 and specified by 1 and l_0 is $T^p > h(F_{l_0}^{-1}(\beta^{[l_0]}))$, which is always satisfied by hypothesis. The sufficient and necessary conditions on α for $\mathbf{x}(\alpha, \boldsymbol{\beta})$ to be of mode 2 and specified by 1 and l_0 are $h(F_{l_0}^{-1}(\beta^{[l_0]})) < \alpha < B_{l_0 - 1}$ and $0 \leq \alpha \leq T^p$.

Besides, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, the total weight on arc 1 is $\xi_1 = F_{l_0+1}^{-1}(\alpha + \beta^{[l_0]})$. One can define a continuous and strictly decreasing bijection θ from the interval $(h(F_{l_0}^{-1}(\beta^{[l_0]})), B_{l_0-1}) \cap [0, T^p]$, the domain of α, to the interval $\big(h^{-1}(\beta^{l_0}), F_{l_0}^{-1}(\beta^{[l_0]})\big) \cap \big[F_{l_0+1}^{-1}(T^p + \beta^{[l_0]}), +\infty)$, the domain of ξ_1, such that $\theta = F_{l_0+1}^{-1}(\cdot + \beta^{[l_0]})$ and $\xi_1 = \theta(\alpha)$. The atomic player p's cost $u^p(\alpha, \boldsymbol{\beta})$ at $\mathbf{x}(\alpha, \boldsymbol{\beta})$ can then be written as a function v of ξ_1 on $\big(h^{-1}(\beta^{l_0}), F_{l_0}^{-1}(\beta^{[l_0]})\big) \cap \big[F_{l_0+1}^{-1_0}(T^p + \beta^{[l_0]}), +\infty)$. Its explicit form is the same as in the case where $l \in [\![1, l_0 - 1]\!]$, except that l is replaced by l_0.

Table 4.1 summarizes the necessary and sufficient conditions on T^p and α for that, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, there are k delegates of the atomic player p and l other delegates who use arc 2. The third line (for $l \in [\![1, l_0 - 1]\!]$) is possible only if $m \geq l_0 > 1$.

mode	k	l	T^p $(> h(F_{l_0}^{-1}(\beta^{[l_0]})))$	α $(0 \leq \alpha \leq T^p)$
3	0	l_0		$\alpha \leq h(F_{l_0}^{-1}(\beta^{[l_0]}))$
2	1	l_0		$h(F_{l_0}^{-1}(\beta^{[l_0]})) < \alpha < B_{l_0-1}$
2	1	$l_0 - 1, l_0 - 2, \ldots, 1$	$T^p \geq B_l$	$B_l \leq \alpha < B_{l-1}$
4	1	0	$T^p \geq B_0$	$\alpha \geq B_0$

Table 4.1: necessary and sufficient conditions on T^p and α

5) In parts 2–4, a map θ is defined from $[0, T^p]$, the domain of α, to J, the domain of

ξ_1, where $J = \big[\, F_{\hat{l}+1}^{-1}(T^p + \beta^{[\hat{l}]}),\, F_{l_0}^{-1}(\beta^{[l_0]}) \,\big]$, $\hat{l} = l_0$ if $T^p < B_{l_0-1}$, and $\hat{l} = \arg\min\{\, l \in \{0, 1, \ldots, l_0 - 1\} \,|\, T^p \geq B_l \,\}$ if $T^p \geq B_{l_0-1}$. In particular, θ has the following properties: (i) for $\alpha \in \big[\, 0,\, h(F_{l_0}^{-1}(\beta^{[l_0]})) \,\big]$, $\theta(\alpha) = F_{l_0}^{-1}(\beta^{[l_0]})$; (ii) θ is strictly decreasing in α on $\big(\, h(F_{l_0}^{-1}(\beta^{[l_0]})),\, T^p \,\big]$ and, in consequence, θ is a bijection between $\alpha \in \big(\, h(F_{l_0}^{-1}(\beta^{[l_0]})),\, T^p \,\big]$ and $\xi_1 \in J \backslash \{F_{l_0}^{-1}(\beta^{[l_0]})\}$; (iii) θ is piecewise continuous and, indeed, one can easily verify that it is continuous on $[0, T^p]$. Besides, recall that $\theta^{-1}(\xi_1) = \{\, \alpha \in \big[0,\, h(F_{l_0}^{-1}(\beta^{[l_0]}))\,\big] \,|\, \theta(\alpha) = \xi_1 \,\}$ for all $\xi_1 \in J$. In particular, $\theta^{-1}(\beta^{[l_0]})$ is the interval $\big[\, 0,\, h(F_{l_0}^{-1}(\beta^{[l_0]})) \,\big]$ and, for any $\xi_1 \in J \setminus \{F_{l_0}^{-1}(\beta^{[l_0]})\}$, $\theta^{-1}(\xi_1)$ is a singleton.

6) In parts 2–4, a one-variable cost function $v(\xi_1)$ is defined for the atomic player p for $\xi_1 \in J$. Here is its explicit expression:

- $\xi_1 = F_{l_0}^{-1}(\beta^{[l_0]})$ and, correspondingly, $\alpha \in \big[0,\, h(F_{l_0}^{-1}(\beta^{[l_0]}))\,\big]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 3 and specified by l_0):

$$v(\xi_1) = T^p\, c_1(\xi_1)$$

- $\xi_1 \in \big(\, h^{-1}(\beta^{l_0}),\, F_{l_0}^{-1}(\beta^{[l_0]}) \,\big) \cap \big[\, F_{l_0+1}^{-1}(T^p + \beta^{[l_0]}),\, +\infty \big)$ and, correspondingly, $\alpha = F_{l_0+1}(\xi_1) - \beta^{[l_0]} \in \big(\, h(F_{l_0}^{-1}(\beta^{[l_0]})),\, B_{l_0-1} \big) \cap [0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 2 and specified by 1 and l_0):

$$v(\xi_1) = T^p\, c_1(\xi_1) + \Big[\, (M - \xi_1) - \frac{\beta^{[l_0]} - l_0 h(\xi_1)}{1 + a(\xi_1)} \,\Big] \big(\, c_2(M - \xi_1) - c_1(\xi_1) \,\big)$$

- $l = l_0 - 1,\, l_0 - 2,\, \ldots,\, 1,\quad \xi_1 \in \big(\, h^{-1}(\beta^l),\, h^{-1}(\beta^{l+1}) \,\big] \cap \big[\, F_{l+1}^{-1}(T^p + \beta^{[l]}),\, +\infty \big)$ and, correspondingly, $\alpha = F_{l+1}(\xi_1) - \beta^{[l]} \in [B_l, B_{l-1}) \cap [0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 2 and specified by 1 and l):

$$v(\xi_1) = T^p\, c_1(\xi_1) + \Big[\, (M - \xi_1) - \frac{\beta^{[l]} - l\, h(\xi_1)}{1 + a(\xi_1)} \,\Big] \big(\, c_2(M - \xi_1) - c_1(\xi_1) \,\big)$$

- $\xi_1 \in \big[\, F_1^{-1}(T^p),\, h^{-1}(\beta^1) \,\big]$ and, correspondingly, $\alpha = F_1(\xi_1) \in [B_0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 4 and specified by 1):

$$v(\xi_1) = T^p\, c_1(\xi_1) + (M - \xi_1)\, \big(\, c_2(M - \xi_1) - c_1(\xi_1) \,\big)$$

Therefore, one has shown that $v(\xi_1)$ is a piecewise continuous function on the closed interval J. It is not difficult to verify that it is in fact continuous. As a result, it attains its minimum on J. Suppose that ξ_1^* is a minimum. If $\xi_1^* = F_{l_0}^{-1}(\beta^{[l_0]})$, then any $\alpha \in \big[\, 0,\, h(F_{l_0}^{-1}(\beta^{[l_0]})) \,\big]$ is a best reply of the atomic player p. If $\xi_1^* \in J \setminus \{F_{l_0}^{-1}(\beta^{[l_0]})\}$, then $\alpha^* = \theta^{-1}(\xi_1^*)$ is a best reply. □

4.4.3 Case $H \leq 0$

Lemma 4.27. *Suppose that $H \leq 0$. If $\boldsymbol{\beta} = (\beta^0,\, \beta^1,\, \ldots,\, \beta^m)$ and T^p satisfy one of the three conditions below:*

(i) $m = 0$ and $T^p \leq F_0(\hat{\xi})$;

(ii) $m \geq 1$, $\beta^{[m]} \leq F_0(\hat{\xi})$, and $T^p \leq F_0(\hat{\xi}) - \beta^{[m]}$;

(iii) $m \geq 1$, $\beta^{[m]} > F_0(\hat{\xi})$, and $T^p \leq h(F_{l_0}^{-1}(\beta^{[l_0]}))$, where l_0 the one in Lemma 4.22,

then every strategy in $\mathcal{S}^p$ is equivalent to the nonatomic strategy $\boldsymbol{\alpha}_0$. As a result, every strategy of the atomic player p is a best reply to $\boldsymbol{\beta}$.

Beside, in case (i) and case (ii), $\mathbf{x}(\boldsymbol{\alpha}_0, \boldsymbol{\beta})$ is of mode 5 while, in case (iii), it is of mode 3 and specified by l_0.

Proof. The results follow immediately from Propositions 4.17 and 4.19. □

Lemma 4.28. *Suppose that $H \leq 0$, and $\boldsymbol{\beta} = (\beta^0, \beta^1, \ldots, \beta^m)$ and T^p are such that $m = 0$ and $T^p > F_0(\hat{\xi})$, then the atomic player p's unique best reply is the trivial strategy $\boldsymbol{\alpha}_1$.*

Besides, $\mathbf{x}(\boldsymbol{\alpha}_1, \boldsymbol{\beta})$ is of mode 4 and specified by 1.

Proof. Similar to the proof of Lemma 4.25. □

Lemma 4.29. *Suppose that $H \leq 0$, and $\boldsymbol{\beta} = (\beta^0, \beta^1, \ldots, \beta^m)$ and T^p satisfy one of the two conditions below:*

(i) $m \geq 1$, $\beta^{[m]} \leq F_0(\hat{\xi})$, $T^p > F_0(\hat{\xi}) - \beta^{[m]}$;

(ii) $m \geq 1$, $\beta^{[m]} > F_0(\hat{\xi})$, $T^p > h(F_{l_0}^{-1}(\beta^{[l_0]}))$,

then the atomic player p has at least one best reply which is a single-atomic strategy.

Proof. (i) Similar analysis to that in the proof of Lemma 4.26 yields Table 4.2, which summarizes the necessary and sufficient conditions on T^p and α for that, at $\mathbf{x}(\alpha, \boldsymbol{\beta})$, there are k delegates of the atomic player p and l other delegates who use arc 2. The third line (for $l \in [\![1, m-1]\!]$) is possible only if $m > 1$.

mode	k	l	T^p $(> F_0(\hat{\xi}) - \beta^{[m]})$	α $(0 \leq \alpha \leq T^p)$
5	0	0		$\alpha \leq F_0(\hat{\xi}) - \beta^{[m]}$
2	1	m		$F_0(\hat{\xi}) - \beta^{[m]} < \alpha < B_{m-1}$
2	1	$m-1, m-2, \ldots, 1$	$T^p \geq B_l$	$B_l \leq \alpha < B_{l-1}$
4	1	0	$T^p \geq B_0$	$\alpha \geq B_0$

Table 4.2: necessary and sufficient conditions on T^p and α

Besides, a continuous non-increasing function θ can be defined from $[0, T^p]$, the domain of α, to J, the domain of ξ_1, where $J = \big[\, F_{\hat{l}+1}^{-1}(T^p + \beta^{[\hat{l}]}),\, \hat{\xi} \,\big]$, $\hat{l}$ is equal to l_1 which is defined in Lemma 4.23 if $T^p < B_0$, and $\hat{l} = 0$ if $T^p \geq B_0$. For $\alpha \in \big[\, 0,\, F_0(\hat{\xi}) - \beta^{[m]} \,\big]$, $\theta(\alpha) = \hat{\xi}$. Function θ is strictly decreasing in α on $\big(\, F_0(\hat{\xi}) - \beta^{[m]},\, T^p \,\big]$, thus it is a bijection between $\alpha \in \big(\, F_0(\hat{\xi}) - \beta^{[m]},\, T^p \,\big]$ and $\xi_1 \in J \setminus \{\hat{\xi}\}$. For all $\xi_1 \in J$, $\theta^{-1}(\xi_1) = \{\, \alpha \in [0, T^p] \,|\, \theta(\alpha) = \xi_1 \,\}$. In particular, $\theta^{-1}(\hat{\xi})$ is the interval $\big[\, 0,\, F_0(\hat{\xi}) - \beta^{[m]} \,\big]$, while $\theta^{-1}(\xi_1)$ is a singleton for any $\xi_1 \in J \setminus \{\hat{\xi}\}$.

Finally, a one-variable cost function $v(\xi_1)$ is defined for the atomic player p for $\xi_1 \in J$, whose explicit expression is as follows:

- $\xi_1 = \hat{\xi}$ and, correspondingly, $\alpha \in \big[\, 0,\, F_0(\hat{\xi}) - \beta^{[m]} \,\big]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 5):

$$v(\xi_1) = T^p\, c_1(\hat{\xi})$$

- $\xi_1 \in \big(\, h^{-1}(\beta^m),\, \hat{\xi} \,\big) \cap \big[\, F_{m+1}^{-1}(T^p + \beta^{[m]}), +\infty \big)$ and, correspondingly, $\alpha = F_{m+1}(\xi_1) - \beta^{[m]} \in (\, F_0(\hat{\xi}) - \beta^{[m]},\, B_{m-1}) \cap [0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 2 and specified by

1 and m):

$$v(\xi_1) = T^p\, c_1(\xi_1) + \Big[\,(M-\xi_1) - \frac{\beta^{[m]} - mh(\xi_1)}{1+a(\xi_1)}\,\Big]\,(\,c_2(M-\xi_1) - c_1(\xi_1)\,)$$

• $l = m-1, m-2, \ldots, 1, \quad \xi_1 \in \big(\,h^{-1}(\beta^l),\, h^{-1}(\beta^{l+1})\,\big] \cap \big[\,F_{l+1}^{-1}(T^p + \beta^{[l]}),\, +\infty\big)$ and, correspondingly, $\alpha = F_{l+1}(\xi_1) - \beta^{[l]} \in [B_l, B_{l-1}) \cap [0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 2 and specified by 1 and l):

$$v(\xi_1) = T^p\, c_1(\xi_1) + \Big[\,(M-\xi_1) - \frac{\beta^{[l]} - l\,h(\xi_1)}{1+a(\xi_1)}\,\Big]\,(\,c_2(M-\xi_1) - c_1(\xi_1)\,)$$

• $\xi_1 \in \big[\,F_1^{-1}(T^p),\, h^{-1}(\beta^1)\,\big]$ and, correspondingly, $\alpha = F_1(\xi_1) \in [B_0, T^p]$ (so that the flow $\mathbf{x}(\alpha, \boldsymbol{\beta})$ is of mode 4 and specified by 1):

$$v(\xi_1) = T^p\, c_1(\xi_1) + (M-\xi_1)\,(\,c_2(M-\xi_1) - c_1(\xi_1)\,)$$

Similar to Lemma 4.26, $v(\xi_1)$ is a continuous function on the closed interval J. As a result, it attains its minimum on J.

(ii) Similar to the proof of Lemma 4.26. □

This section is concluded by the following theorem which combines Lemmas 4.24-4.29.

Theorem 4.30. *In the one-shot delegation game $D_1(\mathbf{T})$ associated to the composite routing game $\Gamma(\mathbf{T})$, every player has a best reply, in particular, a single-atomic one, facing the strategies of the other players.*

4.5 One-shot delegation game with two atomic players and affine costs

Based on Theorem 4.30, this section will show that, in a one-shot delegation game taking place in a two-parallel-arc network, if the cost functions of the two arcs are affine and there are two atomic players, a Nash equilibrium (in pure strategies) always exists. This specific case is adopted because the equilibria can be computed explicitly.

4.5.1 A model with affine costs and two atomic players

The cost functions of the two arcs are assumed to be affine. The assumptions A4.1 and A4.7 must hold. Only the case where $H > 0$ is considered as an example. For all these considerations, the following assumption is made in §4.5 and §4.6:

A 4.31. *The arc cost functions are, respectively, $c_1(x) = a_1 x + b_1$ and $c_2(x) = a_2 x + b_2$, where $a_1 > 0$, $a_2 > 0$ and $b_2 \geq b_1 > 0$.*
The parameter $H > 0$, where $H = \frac{b_2 - b_1}{a_1} - M$.
In the definition of the function h (cf. §4.2.2), $\epsilon = 1 + A$.

Consider an atomic player 1 of weight $\alpha \in (0, M]$, who faces either another atomic player 2 of weight $\beta \in (0, M-\alpha]$ and a set of nonatomic players of total weight $M - \alpha - \beta$, or a set of nonatomic players of total weight $M - \alpha$ (which corresponds to the case where $\beta = 0$). Let the set-valued function of α's single-atomic best replies be denoted by α^*, which is a function in β for $\beta \in [0, M-\alpha]$.

After some computation, the results in §4.4.2 applied to this affine model yield the explicit expression of α^* stated in the following proposition.

Proposition 4.32. *Suppose that A4.31 holds. The explicit expression of the set-valued function α^* is as follows:*

(1) If $\beta \leq H$ and $\alpha \leq H$, then $\alpha^*(\beta)$ is the interval $[0, \alpha]$.

(2) If $\beta \leq H$ and $\alpha > H$, then $\alpha^*(\beta) = \{\alpha\}$.

(3) If $\beta > H$ and $\alpha \leq \frac{\beta+H}{2}$, then $\alpha^*(\beta)$ is the interval $[0, \alpha]$.

(4) If $\beta > H$, $\frac{\beta+H}{2} < \alpha \leq \beta + H$ and $\alpha < 2\beta - H$, then $\alpha^*(\beta)$ is the interval $[0, \frac{\beta+H}{2}]$.

(5) If $\beta > 2H$ and $\beta + H < \alpha < 2\beta - H$, then $\alpha^*(\beta) = \{\frac{3}{4}(\alpha - \frac{\beta+H}{3})\}$.

(6) If $H < \beta \leq 2H$ and $2\beta - H \leq \alpha \leq \beta + H$, then

$$\alpha^*(\beta) = \begin{cases} [0, \frac{\beta+H}{2}], & \text{if } \alpha < \beta + \sqrt{\beta^2 - H^2}; \\ [0, \frac{\beta+H}{2}] \cup \{\alpha\}, & \text{if } \alpha = \beta + \sqrt{\beta^2 - H^2}; \\ \{\alpha\}, & \text{if } \alpha > \beta + \sqrt{\beta^2 - H^2}. \end{cases}$$

(7) If $\beta > H$, $2\beta - H \leq \alpha < 3\beta - H$ and $\alpha > \beta + H$, then

$$\alpha^*(\beta) = \begin{cases} \{\frac{3}{4}(\alpha - \frac{\beta+H}{3})\}, & \text{if } \alpha < (\sqrt{2}+1)\beta - H; \\ \{\frac{3}{4}(\alpha - \frac{\beta+H}{3}), \alpha\}, & \text{if } \alpha = (\sqrt{2}+1)\beta - H; \\ \{\alpha\}, & \text{if } \alpha > (\sqrt{2}+1)\beta - H. \end{cases}$$

(8) If $\beta > H$ and $\alpha \geq 3\beta - H$, then $\alpha^*(\beta) = \{\alpha\}$.

The simplex $\Delta = \{\, (\alpha, \beta) \,|\, \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta = M \,\}$ can thus be divided into eight disjoint regions:

$R_1^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta \leq H,\ \alpha \leq H \,\}$
$R_2^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta \leq H,\ \alpha > H \,\}$
$R_3^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta > H,\ \alpha \leq \frac{\beta+H}{2} \,\}$
$R_4^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta > H,\ \frac{\beta+H}{2} < \alpha \leq \beta + H,\ \alpha < 2\beta - H \,\}$
$R_5^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta > 2H,\ \beta + H < \alpha < 2\beta - H \,\}$
$R_6^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ H < \beta \leq 2H,\ 2\beta - H \leq \alpha \leq \beta + H \,\}$
$R_7^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta > H,\ 2\beta - H \leq \alpha < 3\beta - H,\ \alpha > \beta + H \,\}$
$R_8^\alpha = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \beta > H,\ \alpha \geq 3\beta - H \,\}$

Each of the statements in Proposition 4.32 corresponds to one region. In the same way, if $\beta > 0$, one can divide Δ into eight regions corresponding to β^*, the set-valued functions of the best replies of the atomic player 2, which are denoted by $R_1^\beta, \ldots, R_8^\beta$. The definition of R_i^β is symmetric to that of R_i^α, in the sense that one has only to swap the roles of α and β in the conditions. For example, $R_3^\beta = \{\, (\alpha, \beta) \,|\, (\alpha, \beta) \in \Delta,\ \alpha > H,\ \beta \leq \frac{\alpha+H}{2} \,\}$.

Remark 4.33. It is not difficult to see that the two partitions of the simplex Δ, $\{R_i^\alpha\}_{i=1}^8$ and $\{R_j^\beta\}_{j=1}^8$, have the following properties.

1. $R_1^\alpha = R_1^\beta,\ \ R_3^\alpha = R_2^\beta \cup R_6^\beta \cup R_7^\beta \cup R_8^\beta,\ \ R_5^\beta \subset R_4^\alpha,\ \ R_4^\alpha \cup R_5^\alpha = R_4^\beta \cup R_5^\beta$.
2. If $M \leq H$, then $R_1^\alpha \neq \emptyset$ and, for $i = 2, 3, 4, 5, 6, 7$, $R_i^\alpha = \emptyset$.
3. If $H < M \leq 2H$, then, for $i = 1, 2, 3$, $R_i^\alpha \neq \emptyset$ and, for $i = 4, 5, 6, 7, 8$, $R_i^\alpha = \emptyset$.
4. If $2H < M \leq 3H$, then, for $i = 1, 2, 3, 4, 6$, $R_i^\alpha \neq \emptyset$ and, for $i = 5, 7, 8$, $R_i^\alpha = \emptyset$.
5. If $3H < M \leq 5H$, then, for $i = 1, 2, 3, 4, 6, 7, 8$, $R_i^\alpha \neq \emptyset$ and $R_5^\alpha = \emptyset$.
6. If $M > 5H$, then $R_i^\alpha \neq \emptyset$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$.

7. All these statements are also true if one swaps the roles of α and β.

Recall that the interior of the simplex Δ is $\operatorname{int}\Delta = \{ (\alpha, \beta) \,|\, \alpha > 0,\ \beta > 0,\ \alpha + \beta = M \}$. The following particular case will be used as an example in the rest of the paper.

A 4.34. *In addition to A4.31, $M > 3H$. Besides, the atomic player 1's weight α_0 and the atomic player 2's weight β_0 are such that $(\alpha_0, \beta_0) \in \operatorname{int}\Delta$, $H < \beta_0 \leq 2H$, $2\beta_0 - H \leq \alpha_0 \leq \beta_0 + H$, $\alpha_0 < \beta_0 + \sqrt{(\beta_0)^2 - H^2}$ and $2H < \alpha_0 < (\sqrt{2}+1)H$.*

4.5.2 Equilibria

Let us consider a one-shot delegation game $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$, where there are two atomic players of weight α_0 and β_0, where $(\alpha_0, \beta_0) \in \operatorname{int}\Delta$, and the total weight of the nonatomic players is $M - \alpha_0 - \beta_0$.

Theorem 4.35. *Suppose that A4.31 holds. For all $(\alpha_0, \beta_0) \in \operatorname{int}\Delta$, the one-shot delegation game $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$ admits a Nash equilibrium.*

Proof. It is sufficient to show that a Nash equilibrium with two atomic players using single-atomic strategies exists. As an example, let us prove the result for the case where A4.34 holds.

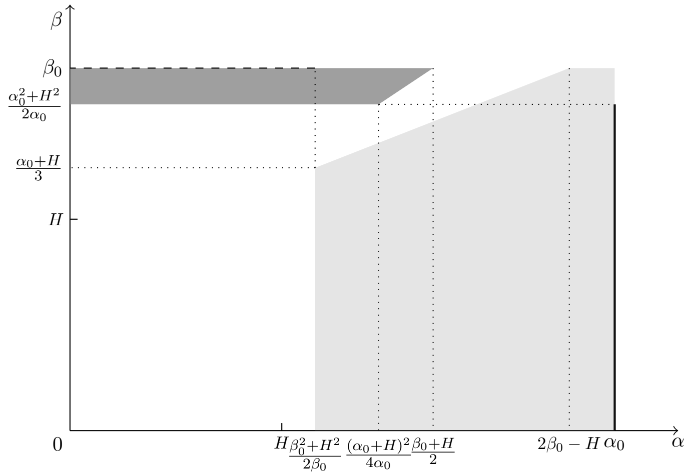


Figure 4.1: best replies of the two atomic players

For the atomic player 1 of weight α_0, facing an atomic player of weight β and a set of nonatomic players of total weight $M - \alpha_0 - \beta$, the set-valued function of her best single-atomic replies is

$$\alpha^*(\beta) = \begin{cases} \{\alpha_0\}, & \text{if } 0 \leq \beta < \frac{\alpha_0^2+H^2}{2\alpha_0}; \\ \{\alpha_0\} \cup [0, \frac{\beta+H}{2}] & \text{if } \beta = \frac{\alpha_0^2+H^2}{2\alpha_0}; \\ [0, \frac{\beta+H}{2}], & \text{if } \frac{\alpha_0^2+H^2}{2\alpha_0} < \beta \leq \beta_0. \end{cases}$$

The graph of $\alpha^*(\beta)$ contains thus the black segment on the right and the darkly shadowed area on the top.

For the atomic player 2 of weight β_0, facing an atomic player of weight α and a set of nonatomic players of total weight $M - \alpha - \beta_0$, the set-valued function of her best single-atomic replies is

$$\beta^*(\alpha) = \begin{cases} \{\beta_0\}, & \text{if } 0 \le \alpha < \frac{\beta_0^2+H^2}{2\beta_0}; \\ \{\beta_0\} \cup [0, \frac{\alpha+H}{2}], & \text{if } \alpha = \frac{\beta_0^2+H^2}{2\beta_0}; \\ [0, \frac{\alpha+H}{2}], & \text{if } \frac{\beta_0^2+H^2}{2\beta_0} \le \alpha \le 2\beta_0 - H; \\ [0, \beta_0], & \text{if } 2\beta_0 - H < \alpha \le \alpha_0. \end{cases}$$

The graph of $\beta^*(\alpha)$ contains thus the dashed segment on the top and the lightly shadowed area on the right.

Clearly, the intersection of the two graphs is the black segment on the right, denoted by I_α, and the dashed segment on the top, denoted by I_β. In other words, the set of the Nash equilibria (in pure strategies) of the game $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$ is $I_\alpha \cup I_\beta$, where

$$I_\alpha = \{\, (\alpha, \beta) \in \Delta \,|\, \alpha = \alpha_0,\ 0 \le \beta \le \tfrac{\alpha_0^2+H^2}{2\alpha_0} \,\}, \quad I_\beta = \{\, (\alpha, \beta) \in \Delta \,|\, 0 \le \alpha \le \tfrac{\beta_0^2+H^2}{2\beta_0},\ \beta = \beta_0 \,\}.$$

□

Notation. In a one-shot delegation game $D_1(M-\alpha_0-\beta_0, \alpha_0, \beta_0)$, let $E_1(M-\alpha_0-\beta_0, \alpha_0, \beta_0)$ be the set of equilibrium cost profiles of the players. Explicitly,

$$E_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0) = \bigcup_{(\alpha,\beta)} \{\, (c^0(\alpha, \beta), u^1(\alpha, \beta), u^2(\alpha, \beta)) \,\},$$

where the union is taken over all the equilibria of $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$. Recall that $c^0(\alpha, \beta)$, $u^1(\alpha, \beta)$ and $u^2(\alpha, \beta)$ are, respectively, the costs to the nonatomic players and the two atomic players 1 and 2, when these two play delegation strategies α and β.

For the particular case where A4.34 holds, a computation shows that, for all $(\alpha, \beta) \in I_\alpha$,

$$\begin{cases} c^0(\alpha, \beta) \equiv a_1 M + b_1 - a_1 \frac{\alpha_0 - H}{2(1+A)}, \\ u^1(\alpha, \beta) \equiv (a_1 + a_2) \big[\frac{\alpha_0 - H}{2(1+A)} \big]^2 - (a_1 M + b_1 - b_2 + a_1 \alpha_0) \frac{\alpha_0 - H}{2(1+A)} + \alpha_0 (a_1 M + b_1), \\ u^2(\alpha, \beta) \equiv \beta_0 \big[a_1 M + b_1 - a_1 \frac{\alpha_0 - H}{2(1+A)} \big]. \end{cases} \tag{4.52}$$

Let this profile of costs be denoted by $(c^0_{I_\alpha}, u^1_{I_\alpha}, u^2_{I_\alpha})$.

For all $(\alpha, \beta) \in I_\beta$,

$$\begin{cases} c^0(\alpha, \beta) \equiv a_1 M + b_1 - a_1 \frac{\beta_0 - H}{2(1+A)}, \\ u^1(\alpha, \beta) \equiv \alpha_0 \big[a_1 M + b_1 - a_1 \frac{\beta_0 - H}{2(1+A)} \big], \\ u^2(\alpha, \beta) \equiv (a_1 + a_2) \big[\frac{\beta_0 - H}{2(1+A)} \big]^2 - (a_1 M + b_1 - b_2 + a_1 \beta_0) \frac{\beta_0 - H}{2(1+A)} + \beta_0 (a_1 M + b_1). \end{cases} \tag{4.53}$$

Let this profile of costs be denoted by $(c^0_{I_\beta}, u^1_{I_\beta}, u^2_{I_\beta})$.

4.5.3 Robustness of the equilibrium

However, in a one-shot delegation game, if, at an equilibrium, at least one of the players plays neither a trivial strategy nor a nonatomic strategy, then her unique atomic delegate may, in her turn, have her own interest to delegate her stock. Thus, such an equilibrium need not to be *robust*.

Example 4.36. Suppose that $H = 0.28$, $M = 2$, and the two atomic players both have weight one, i.e. $\alpha_0 = \beta_0 = 1$. There are no nonatomic players. Then, an equilibrium is $\alpha = \beta = \frac{3-H}{5} = 0.544$, where α and β are, respectively, the single-atomic strategy of player 1 and that of player 2.

However, player 1's delegate, who has weight α, is not glad with the actual situation. Some computation allows to see that, facing an atomic player of weight β and a set of nonatomic players of total weight $M - \alpha - \beta$, her best reply is not the trivial strategy. Indeed, the pair (α, β) is in region R_4^α. Consequently, the set of her best replies is the interval $[0, \frac{\beta+H}{2}]$. The delegate of player 2 is faced with the same situation.

This example shows that it is not enough to consider one-shot delegation games in our study of the delegation behavior in congestion games. An outcome of the game is *robust* only if no atomic player has incentive to delegate any more, or if there are no more atomic players. For example, all equilibrium in $I_\alpha \cup I_\beta$ in the proof of Theorem 4.35 is robust because there is at least one atomic player playing a trivial strategy.

If all atomic delegates are allowed to delegate in their turn, what will happen? As long as there is an atomic player, no matter how small her weight is, she can always delegate. Will the game come to an end?

In §4.6, a dynamic approach will be employed to study the possible outcomes of a delegation process.

4.6 Alternating best reply delegation processes

From now on, an atomic player is called by her weight.

4.6.1 Rules of the delegation process

Initially, two atomic players and a set of nonatomic players coexist in the network. At every stage, an atomic player plays a best single-atomic delegation strategy in reply to the composition of the other players around her. By playing this strategy, she is replaced by an atomic player and/or a set of nonatomic players. At the next stage, the atomic player different from her atomic delegate or herself (if she has played a trivial strategy) plays.

Suppose that, the initial two atomic players have weight, respectively, α_0 and β_0, where $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$. The total weight of the nonatomic players is $M - \alpha_0 - \beta_0$.

In the first process, the atomic player α_0 begins. Here are the rules.

1. At the first stage, the atomic player α_0 plays a single-atomic best delegation strategy $\alpha_1 \in [0, \alpha_0]$ in reply to the composition of the rest of the players. By playing this strategy, she is replaced by an atomic delegate α_1 and a set of nonatomic players of total weight $\alpha_1 - \alpha_0$ if $\alpha_1 > 0$, or a set of nonatomic players of total weight α_0 if $\alpha_1 = 0$. If $\alpha_1 = \alpha_0$, i.e. it is the trivial strategy, player α_2 is just player α_1 herself.
2. At the second stage, the atomic player β_0 plays a single-atomic best reply $\beta_1 \in [0, \beta_0]$, so that she is replaced by her delegates if $\beta_1 < \beta_0$, or she goes on to play herself if $\beta_1 = \beta_0$.
3. Suppose that, for $m \geq 3$, the process is not yet stopped after stage $m - 2$. In each of the two cases below, the game stops after stage $m - 1$, otherwise it continues.
   (i) If the strategies played at stage $m - 2$ and stage $m - 1$ are both trivial ones.
   (ii) If the strategy played at stage $m - 2$ is a nonatomic one, and the strategy played at stage $m - 1$ is a trivial one.

If the game stops after stage $m-1$, and $m = 2n+1$ or $2n+2$ for some $n \in \mathbb{N}^*$, then, for all $k \geq n$, $\alpha_k \equiv \alpha_n$, $\beta_k \equiv \beta_n$.

4. Suppose that, at stage $m \geq 3$, the process is not yet stopped. If $m = 2n+1$ (*resp.* $m = 2n+2$) for some $n \in \mathbb{N}^*$, and if $\alpha_n = 0$ (*resp.* $\beta_n = 0$), then nobody plays and the process goes on to stage $m+1$; otherwise, if $\alpha_n > 0$ (*resp.* $\beta_n > 0$), then the atomic player α_n (*resp.* the atomic player β_n) plays a single-atomic best reply $\alpha_{n+1} \in [0, \alpha_n]$ (*resp.* $\beta_{n+1} \in [0, \beta_n]$).

The second process where the atomic player β_0 begins is defined in the same way.

Remark 4.37. Case (i) in rule 3 means that, since the player who plays at stage m is the one who plays at stage $m-2$ (because she has chosen the trivial strategy at stage $m-2$), and she faces the same opponents as at stage $m-2$ (because the other atomic player has chosen the trivial strategy at stage $m-1$), she will do as at stage $m-2$, i.e. to choose the trivial strategy. The same argument holds for the player who plays at stage $m+1$, and so on. Therefore, there is no need to continue the process. Similar interpretation can be made for case (ii).

Definition 4.38. Suppose that $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$. The two processes described above are called the *alternating best reply delegation processes* associated to the composite routing game $\Gamma(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$. The sequence $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ induced by an alternating best reply delegation process is called a *trajectory* with initial point (α_0, β_0).

Proposition 4.39. *Suppose that $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$. All trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ of an alternating best reply delegation process associated to $\Gamma(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$ converges.*

Proof. The sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are both bounded and decreasing, hence they converge. □

Notation. Suppose that $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ is a trajectory of one of the two alternating best reply delegation processes. Let its limit be denoted by $(\alpha_\infty, \beta_\infty)$.

For all $n \geq 0$, let $c^0(\alpha_n, \beta_n)$ (*resp.* $u^1(\alpha_n, \beta_n)$, *resp.* $u^2(\alpha_n, \beta_n)$, *resp.* $v(\alpha_n, \beta_n)$) be the nonatomic players' cost (*resp.* the atomic player α_0's cost, *resp.* the atomic player β_0's cost, *resp.* the social cost) in the one-shot delegation game $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$, if α_0 plays the single-atomic strategy α_n and β_0 plays the single-atomic strategy β_n. The cost profile $(c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))$ and the social cost $v(\alpha_\infty, \beta_\infty)$ are defined in the same way.

By the continuity of the cost functions, the sequence of the cost profiles $\big\{(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n))\big\}_{n\in\mathbb{N}}$ converges to $(c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))$. The latter is called a *myopic outcome* of the process.

Let $\mathcal{E}(\alpha_0, \beta_0)$ be the set of all the myopic outcomes of the two alternating best reply delegation processes associated to the composite routing game $\Gamma(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$. In other words,

$$\begin{aligned}\mathcal{E}(\alpha_0, \beta_0) = \{\ (c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty)) \,|\, (\alpha_\infty, \beta_\infty) &\text{ is the limit of} \\ \text{some trajectory } \{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}} \text{ with initial point } &(\alpha_0, \beta_0)\ \}\end{aligned}$$

4.6.2 Trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$

To illustrate the idea, let us take the case where A4.34 holds to obtain all the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ induced by the two alternating best reply processes. All the best replies are deduced according to Proposition 4.32, hence the argument will not be repeated every time it is used.

First, consider the process where the atomic player α_0 begins.

At stage 1, any number in the interval $[0, \frac{\beta_0+H}{2}]$ is a best reply of player α_0 because $H < \beta_0 \leq 2H$, $2\beta_0 - H \leq \alpha_0 \leq \beta_0 + H$, $\alpha_0 < \beta_0 + \sqrt{(\beta_0)^2 - H^2}$. Three cases are possible.

CASE 6.1: $0 \leq \alpha_1 \leq H$.

At stage 2, player β_0 plays. The pair (α_1, β_0) is in R_2^β because $\alpha_1 \leq H$ and $\beta_0 > H$. The unique best reply is the trivial one, i.e. $\beta_1 = \beta_0$.

If $\alpha_1 = 0$, the process stops after stage 2. For all $n \geq 1$, $\alpha_n \equiv 0$, $\beta_n \equiv \beta_0$.

If $0 < \alpha_1 \leq H$, then player α_1 plays at stage 3.

Let us show that, for all $n \geq 1$,

$$H \geq \alpha_1 \geq \alpha_2 \geq \cdots \alpha_n \geq 0, \quad \beta_0 = \beta_1 = \cdots = \beta_n. \tag{4.54}$$

This is true for $n = 1$. Suppose that the process has not yet stopped after stage $2n$, and that (4.54) holds for some $n \geq 1$.

At stage $2n+1$, it is player α_n's turn to play. The pair (α_n, β_n) is in R_3^α because $\alpha_n \leq H$ and $\beta_n = \beta_0 > 2H$. The set of the best replies of player α_n is the interval $[0, \alpha_n]$.

If player α_n chooses the trivial strategy, i.e. $\alpha_{n+1} = \alpha_n$, the process stops. Clearly, (4.54) holds for all $m > n$.

If $0 \leq \alpha_{n+1} < \alpha_n$, it is not difficult to verify that the pair (α_{n+1}, β_n) is in region R_2^β. The unique best reply of player β_n is the trivial one, i.e. $\beta_{n+1} = \beta_n$. Therefore, (4.54) holds for $n + 1$. Besides, if $\alpha_{n+1} = 0$, the process stops, and (4.54) holds for all $m > n$. Otherwise, one can continue the induction for $n + 2$.

In summary, once player α_0 has chosen $0 \leq \alpha_1 \leq H$, the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ can only take one of the two forms below:

(i) there exists some $m \in \mathbb{N}^*$ such that $\alpha_0 > H \geq \alpha_1 > \alpha_2 > \cdots > \alpha_m \geq 0$ and, for all $n \geq m$, $\alpha_n = \alpha_m \in [0, H]$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$; hence, $\alpha_\infty = \alpha_m$, $\beta_\infty = \beta_0$;

(ii) $\alpha_0 > H \geq \alpha_1 > \alpha_2 > \cdots$, hence, $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to a limit α_∞ in $[0, H)$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$, hence $\beta_\infty = \beta_0$.

CASE 6.2: $H < \alpha_1 \leq \frac{(\beta_0)^2+H^2}{2\beta_0}$.

At stage 2, player β_0 plays. Firstly, $H < \alpha_1 \leq \frac{3H}{2} < 2H$ because $H < \beta_0 \leq 2H$. Secondly, $\beta_0 \geq 2\alpha_1 - H$, because $\alpha_1 \leq \frac{(\beta_0)^2+H^2}{2\beta_0} < \frac{\beta_0+H}{2}$. Thirdly, $\beta_0 \leq \alpha_1 + H$. Indeed, otherwise, $\beta_0 > \alpha_1 + H > 2H$, which contradicts the fact that $\beta_0 \leq 2H$. Fourthly, $\beta_0 \geq \alpha_1 + \sqrt{(\alpha_1)^2 - H^2}$ because of the relation $\frac{(\beta_0)^2+H^2}{2\beta_0} \geq \alpha_1$. These four relations imply that the pair (α_1, β_0) is in region R_6^β, and the unique best reply of player β_0 is the trivial one if $\alpha_1 < \frac{(\beta_0)^2+H^2}{2\beta_0}$. If $\alpha_1 = \frac{(\beta_0)^2+H^2}{2\beta_0}$, the trivial strategy is one of the best replies, and it is assumed to be adopted. Hence, $\beta_1 = \beta_0$.

At stage 3, player α_1 plays. Because of the fact that $\beta_0 > H$ and $\alpha_1 < \frac{\beta_0+H}{2} = \frac{\beta_1+H}{2}$, the pair (α_1, β_1) is in region R_3^α and, consequently, any number in $[0, \alpha_1]$ is a best reply. Two subcases are to be discussed.

CASE 6.2.1: $0 \leq \alpha_2 \leq H$

By a similar analysis to that in Case 6.1, the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ takes one of the following forms:

(i) there exists some $m \geq 2$ such that $\alpha_2 > \alpha_3 > \cdots > \alpha_m \geq 0$ and, for all $n \geq m$, $\alpha_n \equiv \alpha_m$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$; hence, $\alpha_\infty = \alpha_m$, $\beta_\infty = \beta_0$;

(ii) $\alpha_2 > \alpha_3 > \cdots$, and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to a limit α_∞ in $[0, H)$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$, hence $\beta_\infty = \beta_0$.

CASE 6.2.2: $H < \alpha_2 \leq \alpha_1$

If $\alpha_2 = \alpha_1$, the process stops after stage 5.

If $H < \alpha_2 < \alpha_1$, it is not difficult to see that, as at the beginning of case 6.2, the pair (α_2, β_1) is in region R_6^β, and the set of the best reply of the atomic player β_1 is the trivial one, i.e. $\beta_2 = \beta_1$. The actual situation for (α_2, β_2) is the same as for (α_1, β_1) in the case 6.2, except that $H < \alpha_2 < \alpha_1$ and $\beta_2 = \beta_1$. There is no need to repeat the analysis.

In summary, if player α_0 has chosen $H < \alpha_1 \leq \frac{(\beta_0)^2+H^2}{2\beta_0}$, and in the case where $\alpha_1 = \frac{(\beta_0)^2+H^2}{2\beta_0}$, player β_0 chooses the trivial strategy, i.e. $\beta_1 = \beta_0$, then the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ takes one of the four forms below:

(i) there exists some $k \geq 2$ and $m \geq k$ such that $H \geq \alpha_k > \alpha_{k+1} > \cdots > \alpha_m \geq 0$ and, for all $n \geq m$, $\alpha_n \equiv \alpha_m$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$; hence, $\alpha_\infty = \alpha_m$, $\beta_\infty = \beta_0$;

(ii) there exists some $m \geq 2$ such that $H \geq \alpha_m > \alpha_{m+1} > \cdots > 0$, and α_n converges to a limit α_∞ in $[0, H)$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$, hence $\beta_\infty = \beta_0$;

(iii) there exists some $m \geq 1$ such that $\frac{(\beta_0)^2+H^2}{2\beta_0} \geq \alpha_1 > \alpha_2 > \cdots > \alpha_m > H$ and, for all $n \geq m$, $\alpha_n \equiv \alpha_m \in \big(H, \frac{(\beta_0)^2+H^2}{2\beta_0}]$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$; hence, $\alpha_\infty = \alpha_m$, $\beta_\infty = \beta_0$;

(iv) $\frac{(\beta_0)^2+H^2}{2\beta_0} \geq \alpha_1 > \alpha_2 > \cdots > H$, and α_n converges to a limit α_∞ in $\big[H, \frac{(\beta_0)^2+H^2}{2\beta_0}\big)$; for all $n \in \mathbb{N}$, $\beta_n \equiv \beta_0$, hence $\beta_\infty = \beta_0$.

CASE 6.3: $\frac{(\beta_0)^2+H^2}{2\beta_0} \leq \alpha_1 \leq \frac{\beta_0+H}{2}$.

A similar argument to the one at the beginning of case 6.2 implies that the pair (α_1, β_0) is in region R_6^β, and the set of the best replies of player β_0 is the interval $[0, \frac{\alpha_1+H}{2}]$ if $\alpha_1 > \frac{(\beta_0)^2+H^2}{2\beta_0}$. If $\alpha_1 = \frac{(\beta_0)^2+H^2}{2\beta_0}$, the set of the best replies of β_0 is the union of the trivial strategy and the interval $[0, \frac{\alpha_1+H}{2}]$. The trivial strategy is studied in Case 6.2. Here, let us assume that $\beta_1 \in [0, \frac{\alpha_1+H}{2}]$. Three subcases are to be considered.

CASE 6.3.1: $0 \leq \beta_1 \leq H$.

The situation is the same as in the beginning of case 6.1, except that α_1 there is replaced by β_1 here, and β_0 there is replaced by α_1 here. Therefore, the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ takes one of the two forms below:

(i) there exists some $m \in \mathbb{N}^*$ such that $\beta_0 > H \geq \beta_1 > \beta_2 > \cdots > \beta_m \geq 0$ and, for all $n \geq m$, $\beta_n \equiv \beta_m \in [0, H]$; for all $n \geq 1$, $\alpha_n \equiv \alpha_1 \in [\frac{(\beta_0)^2+H^2}{2\beta_0}, \frac{\beta_0+H}{2}]$; hence, $\alpha_\infty = \alpha_1$, $\beta_\infty = \beta_m$;

(ii) $\beta_0 > H \geq \beta_1 > \beta_2 > \cdots \geq 0$, and there exists $\beta_\infty \in [0, H)$ such that $\beta_n \to \beta_\infty$; for all $n \geq 1$, $\alpha_n \equiv \alpha_1 \in [\frac{(\beta_0)^2+H^2}{2\beta_0}, \frac{\beta_0+H}{2}]$, hence $\alpha_\infty = \alpha_1$.

CASE 6.3.2: $H < \beta_1 \leq \frac{(\alpha_1)^2+H^2}{2\alpha_1}$ and, in the case where $\beta_1 = \frac{(\alpha_1)^2+H^2}{2\alpha_1}$, $\alpha_2 = \alpha_1$.

The situation is the same as at the beginning of case 6.2, except that α_1 there is replaced by β_1 here, and β_0 there is replaced by α_1 here. Therefore, the sequence $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ takes one of the four forms below:

(i) there exists some $k \geq 2$ and $m \geq k$ such that $H \geq \beta_k > \beta_{k+1} > \cdots > \beta_m \geq 0$ and, for all $n \geq m$, $\beta_n \equiv \beta_m$; for all $n \in \mathbb{N}$, $\alpha_n \equiv \alpha_1$; hence, $\alpha_\infty = \alpha_1$, $\beta_\infty = \beta_m$;

(ii) there exists some $m \geq 2$ such that $H \geq \beta_m > \beta_{m+1} > \cdots > 0$, and β_n converges to a limit β_∞ in $[0, H)$; for all $n \in \mathbb{N}$, $\alpha_n \equiv \alpha_1$, hence $\alpha_\infty = \alpha_1$;

(iii) there exists some $m \geq 1$ such that $\frac{(\alpha_1)^2+H^2}{2\alpha_0} \geq \beta_1 > \beta_2 > \cdots > \beta_m > H$ and, for all $n \geq m$, $\beta_n \equiv \beta_m \in \big(H, \frac{(\alpha_1)^2+H^2}{2\alpha_1}]$; for all $n \in \mathbb{N}$, $\alpha_n \equiv \alpha_1$; hence, $\alpha_\infty = \alpha_1$, $\beta_\infty = \beta_m$;

(iv) $\frac{(\alpha_1)^2+H^2}{2\alpha_1} \geq \beta_1 > \beta_2 > \cdots > H$, and β_n converges to a limit β_∞ in $[H, \frac{(\alpha_0)^2+H^2}{2\alpha_0})$; for all $n \in \mathbb{N}$, $\alpha_n \equiv \alpha_1$, hence $\alpha_\infty = \alpha_1$.

CASE 6.3.3: $\frac{(\alpha_1)^2+H^2}{2\alpha_1} \leq \beta_1 \leq \frac{\alpha_1+H}{2}$ and, in the case where $\beta_1 = \frac{(\alpha_1)^2+H^2}{2\alpha_1}$, $\alpha_2 \in [0, \frac{\beta_1+H}{2}]$.

A similar argument to the one at the beginning of case 6.3 implies that $\alpha_2 \in [0, \frac{\beta_1+H}{2}]$.

The situation is the same as that at stage 1, except that (α_0, β_0) is replaced by (α_1, β_1), where $\frac{(\beta_0)^2+H^2}{2\beta_0} \leq \alpha_1 \leq \frac{\beta_0+H}{2}$ and $\frac{(\alpha_1)^2+H^2}{2\alpha_1} \leq \beta_1 \leq \frac{\alpha_1+H}{2}$. The rest of the analysis is the same, hence one can conclude by recurrence.

To this end, let us first define three real sequences $\{B_n\}_{n\in\mathbb{N}}$, $\{A_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}^*}$ such that $A_0 = B_0 = \beta_0$ and, for $n \geq 1$,

$$\begin{cases} B_n = \frac{B_{n-1}+3H}{4} = \frac{\beta_0-H}{4^n} + H, \\ A_n = [(\frac{(A_{n-1})^2+H^2}{2A_{n-1}})^2 + H^2]/[\frac{(A_{n-1})^2+H^2}{A_{n-1}}], \\ C_n = \frac{(A_{n-1})^2+H^2}{2A_{n-1}}. \end{cases} \tag{4.55}$$

Remark 4.40. It is not difficult to see that, for all $n \in \mathbb{N}^*$, $B_n > H$, $A_n > H$, $C_n > H$, and all the three sequences are strictly decreasing and converge to H.

Proposition 4.41. *Suppose that A4.34 holds. Then, any trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}^*}$ of the alternating best reply delegation process associated to the composite routing game $\Gamma(M-\alpha_0-\beta_0, \alpha_0, \beta_0)$, where player α_0 begins, takes one of the following five forms (The parameters A_n, B_n and C_n are defined by (4.55)):*

Form 1 *There exists $l \in \mathbb{N}$ which satisfies the following three conditions:*

1.1) *if $l \geq 1$, then, for all $n \in [\![1, l]\!]$, $\frac{(\beta_{n-1})^2+H^2}{2\beta_{n-1}} \leq \alpha_n \leq \frac{\beta_{n-1}+H}{2}$, $\frac{(\alpha_n)^2+H^2}{2\alpha_n} \leq \beta_n \leq \frac{\alpha_n+H}{2}$ and, consequently, $C_n \leq \alpha_n \leq \frac{B_{n-1}+H}{2}$, $A_n \leq \beta_n \leq B_n$;*

1.2) *for all $n \geq l$, $\beta_n \equiv \beta_l$, hence $\beta_\infty = \beta_l$;*

1.3) *there exists $m \geq l+1$ such that $\frac{(\beta_l)^2+H^2}{2\beta_l} \geq \alpha_{l+1} > \alpha_{l+2} > \cdots > a_m > H \geq \alpha_{m+1} \geq 0$, and $\alpha_n \to \alpha_\infty$ as $n \to \infty$, where $\alpha_\infty \in [0, H]$.*

Form 2 *There exists $l \in \mathbb{N}$ which satisfies the following three conditions:*

2.1) *if $l \geq 1$, then, for all $n \in [\![1, l]\!]$, $\frac{(\beta_{n-1})^2+H^2}{2\beta_{n-1}} \leq \alpha_n \leq \frac{\beta_{n-1}+H}{2}$, $\frac{(\alpha_n)^2+H^2}{2\alpha_n} \leq \beta_n \leq \frac{\alpha_n+H}{2}$ and, consequently, $C_n \leq \alpha_n \leq \frac{B_{n-1}+H}{2}$, $A_n \leq \beta_n \leq B_n$;*

2.2) *for all $n \geq l$, $\beta_n \equiv \beta_l$, hence $\beta_\infty = \beta_l$;*

2.3) *$\frac{(\beta_l)^2+H^2}{2\beta_l} \geq \alpha_{l+1} > H$, and $\alpha_n \to \alpha_\infty$ as $n \to \infty$, where $\alpha_\infty \in [H, \frac{(\beta_l)^2+H^2}{2\beta_l}] \subset [H, \frac{(B_l)^2+H^2}{2B_l}]$.*

Form 3 *There exists $l \in \mathbb{N}^*$ which satisfies the following three conditions;*

3.1) *if $l \geq 2$, then, for all $n \in [\![1, l-1]\!]$, $\frac{(\beta_{n-1})^2+H^2}{2\beta_{n-1}} \leq \alpha_n \leq \frac{\beta_{n-1}+H}{2}$, $\frac{(\alpha_n)^2+H^2}{2\alpha_n} \leq \beta_n \leq \frac{\alpha_n+H}{2}$ and, consequently, $C_n \leq \alpha_n \leq \frac{B_{n-1}+H}{2}$, $A_n \leq \beta_n \leq B_n$;*

3.2) *for all $n \geq l$, $\alpha_n \equiv \alpha_l$, hence $\alpha_\infty = \alpha_l$, where $\frac{(\beta_{l-1})^2+H^2}{2\beta_{l-1}} \leq \alpha_l \leq \frac{\beta_{l-1}+H}{2}$ and, in consequence, $C_l \leq \alpha_l \leq \frac{B_{l-1}+H}{2}$;*

3.3) *$\beta_{l-1} > H \geq \beta_l \geq 0$, and $\beta_n \to \beta_\infty$ as $n \to \infty$, where $\beta_\infty \in [0, H)$.*

Form 4 *There exists $l \in \mathbb{N}^*$ which satisfies the following three conditions:*

4.1) *if $l \geq 2$, then, for all $n \in [\![1, l-1]\!]$, $\frac{(\beta_{n-1})^2+H^2}{2\beta_{n-1}} \leq \alpha_n \leq \frac{\beta_{n-1}+H}{2}$, $\frac{(\alpha_n)^2+H^2}{2\alpha_n} \leq \beta_n \leq \frac{\alpha_n+H}{2}$ and, consequently, $C_n \leq \alpha_n \leq \frac{B_{n-1}+H}{2}$, $A_n \leq \beta_n \leq B_n$;*

4.2) *for all* $n \geq l$, $\alpha_n \equiv \alpha_l$, *hence* $\alpha_\infty = \alpha_l$, *where* $\frac{(\beta_{l-1})^2+H^2}{2\beta_{l-1}} \leq \alpha_l \leq \frac{\beta_{l-1}+H}{2}$ *and, in consequence,* $C_l \leq \alpha_l \leq \frac{B_{l-1}+H}{2}$;

4.3) $\frac{(\alpha_l)^2+H^2}{2\alpha_l} \geq \beta_l > H$, *and* $\beta_n \to \beta_\infty$ *as* $n \to \infty$, *where* $\beta_\infty \in [H, \frac{(\alpha_l)^2+H^2}{2\alpha_l}] \subset \left[H, \frac{(\frac{B_l+H}{2})^2+H^2}{2(\frac{B_l+H}{2})}\right]$.

Form 5 *For all* $n \geq 1$, $\frac{(\beta_{n-1})^2+H^2}{2\beta_{n-1}} \leq \alpha_n \leq \frac{\beta_{n-1}+H}{2}$, $\frac{(\alpha_n)^2+H^2}{2\alpha_n} \leq \beta_n \leq \frac{\alpha_n+H}{2}$ *and, consequently,* $C_n \leq \alpha_n \leq \frac{B_{n-1}+H}{2}$, $A_n \leq \beta_n \leq B_n$.

By a similar analysis, one can obtain that, any trajectory $\{(\alpha_n, \beta_n)\}_{n \in \mathbb{N}^*}$ of the alternating best reply process where player β_0 begins converges, and it takes one of the following forms.

1. 1.1) For all $n \geq 0$, $\alpha_n \equiv \alpha_0$;
   1.2) there exists $m \geq 1$ such that $\beta_{m-1} > H \geq \beta_m \geq 0$, and $\beta_n \to \beta_\infty$ as $n \to \infty$, with $\beta_\infty \in [0, H]$.
2. 2.1) For all $n \geq 0$, $\alpha_n \equiv \alpha_0$;
   2.2) there exists $m \geq 1$ such that $\beta_{m-1} > \frac{\alpha_0+H}{3} \geq \beta_m \geq H$, and $\beta_n \to \beta_\infty$ as $n \to \infty$, with $\beta_\infty \in [H, \frac{\alpha_0+H}{3}]$.
3. 3.1) For all $n \geq 0$, $\alpha_n \equiv \alpha_0$;
   3.2) $\beta_1 > \alpha - H > \beta_2 > \beta_3 \cdots > \frac{\alpha_0+H}{3}$, and $\beta_n \to \beta_\infty$ as $n \to \infty$, with $\beta_\infty \in [\frac{\alpha_0+H}{3}, \alpha - H)$.
4. For all $n \in \mathbb{N}$, denote $\tilde{\beta}_n = \beta_{n+1}$. Then, $\tilde{\beta}_0 \in [\frac{(\alpha_0)^2+H^2}{2\alpha_0}, \beta_0]$, and $\{(\alpha_n, \tilde{\beta}_n)\}_{n \in \mathbb{N}^*}$ may take any form in Proposition 4.41.

4.6.3 Sequence of cost profiles

In order to compute the set of all the myopic outcomes of the two alternating best reply delegation processes in the case where A4.34 holds, let us discuss the myopic outcomes of trajectories of each of the five forms in Proposition 4.41.

Form 1 Suppose that the sequence $\{(\alpha_n, \beta_n)\}_{n \in \mathbb{N}^*}$ has form 1 and l is the natural number that satisfied the three conditions. Then, at the CE of $\Gamma(M - \alpha_\infty - \beta_\infty, \alpha_\infty, \beta_\infty)$, which is of mode 3, the total weight on arc 1 is $F^{-1}(\beta_\infty) = M - \frac{\beta_\infty - H}{2(1+A)}$, the total weight of the stocks on arc 1 initially belonging to α_0 is α_0, and the total weight of the stocks on arc 1 initially belonging to β_0 is $\beta_0 - \frac{\beta_\infty - H}{2(1+A)}$.

The costs to the nonatomic players, atomic players α_0 and β_0 are, respectively,

$$\begin{cases} c^0(\alpha_\infty, \beta_\infty) = a_1 M + b_1 - a_1 \frac{\beta_\infty - H}{2(1+A)}, \\ u^1(\alpha_\infty, \beta_\infty) = \alpha_0 \big[\, a_1 M + b_1 - a_1 \frac{\beta_\infty - H}{2(1+A)} \,\big], \\ u^2(\alpha_\infty, \beta_\infty) = (a_1 + a_2) \big[\, \frac{\beta_\infty - H}{2(1+A)} \,\big]^2 - (a_1 M + b_1 - b_2 + a_1 \beta_0) \frac{\beta_\infty - H}{2(1+A)} + \beta_0 (a_1 M + b_1). \end{cases}$$

Besides, it is clear that, for all $n \geq l$, $(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n)) = (c^0_\alpha(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))$. In other words, the sequence of cost profiles $\{(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n))\}_{n \in \mathbb{N}^*}$ is a constant one except the first finite number of terms.

Finally, a simple computation shows that $c^0(\alpha_\infty, \beta_\infty)$, $u^1(\alpha_\infty, \beta_\infty)$ and $u^2(\alpha_\infty, \beta_\infty)$ are all strictly decreasing in β_∞ for $\beta_\infty \in (H, \beta_0]$. Therefore, for all the trajectories $\{(\alpha_n, \beta_n)\}_{n \in \mathbb{N}^*}$

of form 1, the range of the myopic outcomes are as follows:

$$\begin{cases} a_1M + b_1 - a_1\frac{\beta_0-H}{2(1+A)} \le c^0(\alpha_\infty,\beta_\infty) < a_1M + b_1, \\ \alpha_0\big[\, a_1M + b_1 - a_1\frac{\beta_0-H}{2(1+A)}\,\big] \le u^1(\alpha_\infty,\beta_\infty) < \alpha_0(a_1M + b_1), \\ (a_1+a_2)\big[\,\frac{\beta_0-H}{2(1+A)}\,\big]^2 - (a_1M + b_1 - b_2 + a_1\beta_0)\frac{\beta_0-H}{2(1+A)} + \beta_0(a_1M + b_1) \\ \qquad\qquad\qquad\qquad \le u^2(\alpha_\infty,\beta_\infty) < \beta_0(a_1M + b_1). \end{cases} \tag{4.56}$$

Form 2 Same as form 1.

Form 3 Suppose that the trajectory $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ has form 3 and l is the natural number that satisfied the three conditions. Then, at the CE of $\Gamma(M-\alpha_\infty-\beta_\infty,\alpha_\infty,\beta_\infty)$, which is of mode 4, the total weight on arc 1 is $F^{-1}(\alpha_\infty) = M - \frac{\alpha_\infty-H}{2(1+A)}$, the total weight of the stocks on arc 1 initially belonging to α_0 is $\alpha_0 - \frac{\alpha_\infty-H}{2(1+A)}$, and the total weight of the stocks on arc 1 initially belonging to β_0 is β_0.

The costs to the nonatomic players, atomic players α_0 and β_0 are, respectively,

$$\begin{cases} c^0(\alpha_\infty,\beta_\infty) = a_1M + b_1 - a_1\frac{\alpha_\infty-H}{2(1+A)}, \\ u^1(\alpha_\infty,\beta_\infty) = (a_1+a_2)\big[\,\frac{\alpha_\infty-H}{2(1+A)}\,\big]^2 - (a_1M + b_1 - b_2 + a_1\alpha_0)\frac{\alpha_\infty-H}{2(1+A)} + \alpha_0(a_1M + b_1), \\ u^2(\alpha_\infty,\beta_\infty) = \beta_0\big[\, a_1M + b_1 - a_1\frac{\alpha_\infty-H}{2(1+A)}\,\big]. \end{cases}$$

Besides, for all $n \ge l$, $\big(c^0(\alpha_n,\beta_n),u^1(\alpha_n,\beta_n),u^2(\alpha_n,\beta_n)\big) = \big(c^0(\alpha_\infty,\beta_\infty),u^1(\alpha_\infty,\beta_\infty), u^2(\alpha_\infty,\beta_\infty)\big)$, the cost sequence $\big\{\big(c^0(\alpha_n,\beta_n),u^1(\alpha_n,\beta_n),u^2(\alpha_n,\beta_n)\big)\big\}_{n\in\mathbb{N}^*}$ is a constant one except the first finite number of terms.

Similar to the analysis for form 1, one can show that $c^0(\alpha_\infty,\beta_\infty)$, $u^1(\alpha_\infty,\beta_\infty)$ and $u^2(\alpha_\infty,\beta_\infty)$ are all strictly decreasing in α_∞ for $\alpha_\infty \in (H,\frac{\beta_0+H}{2}]$. Therefore, for all the trajectories $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ of form 3, the range of the myopic outcomes are as follows:

$$\begin{cases} a_1M + b_1 - a_1\frac{\beta_0-H}{4(1+A)} \le c^0(\alpha_\infty,\beta_\infty) < a_1M + b_1, \\ (a_1+a_2)\big[\,\frac{\beta_0-H}{4(1+A)}\,\big]^2 - (a_1M + b_1 - b_2 + a_1\alpha_0)\frac{\beta_0-H}{4(1+A)} + \alpha_0(a_1M + b_1) \\ \qquad\qquad\qquad\qquad \le u^1(\alpha_\infty,\beta_\infty) < \alpha_0(a_1M + b_1), \\ \beta_0\big[\, a_1M + b_1 - a_1\frac{\beta_0-H}{4(1+A)}\,\big] \le u^2(\alpha_\infty,\beta_\infty) < \beta_0(a_1M + b_1). \end{cases} \tag{4.57}$$

Form 4 Same as form 3.

Form 5 Suppose that the trajectory $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ has form 5. Then, at the CE of $\Gamma(M-\alpha_\infty-\beta_\infty,\alpha_\infty,\beta_\infty)$, which is of mode 1, the total weight on arc 1 is M, the total weight of the stocks on arc 1 initially belonging to α_0 is α_0, and the total weight of the stocks on arc 1 initially belonging to β_0 is β_0.

The costs to the nonatomic players, atomic players α_0 and β_0 are, respectively,

$$\begin{cases} c^0(\alpha_\infty,\beta_\infty) = a_1M + b_1 \\ u^1(\alpha_\infty,\beta_\infty) = \alpha_0(a_1M + b_1) \\ u^2(\alpha_\infty,\beta_\infty) = \beta_0(a_1M + b_1) \end{cases} \tag{4.58}$$

Besides, recall that, for all $n \in \mathbb{N}$, $\beta_n \le B_n = \frac{\beta_0-H}{4^n} + H$ and $\alpha_n \le \frac{\beta_{n-1}+H}{2} = \frac{\beta_0-H}{2\cdot 4^{n-1}} + H$, α_n and β_n converge to H at an exponential rate. By the linearity of the cost functions, $\big(c^0(\alpha_n,\beta_n),u^1(\alpha_n,\beta_n),u^2(\alpha_n,\beta_n)\big)$ also converges to $\big(c^0(\alpha_\infty,\beta_\infty),u^1(\alpha_\infty,\beta_\infty),u^2(\alpha_\infty,\beta_\infty)\big)$ at an exponential rate. This idea is formulated in the following lemma.

Lemma 4.42. *Suppose that (α_0, β_0) satisfies A4.34. If the trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ of an alternating best reply delegation process with initial point (α_0, β_0) is of form 5* (cf. *Proposition 4.41*)*, then there exist a constant $C > 0$ such that, for all $n \in \mathbb{N}^*$,*

$$\|(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n)) - (c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))\|_\infty < C/4^n,$$

where $\|\cdot\|_\infty$ stands for the uniform norm in $\mathbb{R}^3$.

Proof. On the one hand, for all $n \in \mathbb{N}$, the CE of $\Gamma(M - \alpha_n - \beta_n; \alpha_n, \beta_n)$ is of mode 4, and the costs to the nonatomic players is $c^0(\alpha_n, \beta_n) = a_1 M + b_1 - a_1 \frac{\alpha_n - H}{2(1+A)}$. On the other hand, according to (4.58), $c^0(H, H) = a_1 M + b_1$.

According to (4.55), $\alpha_n \le \frac{\beta_{n-1}+H}{2}$ and $\beta_{n-1} \le B_{n-1} = \frac{\beta_0 - H}{4^{n-1}} + H$.Thus,

$$|\, c^0(\alpha_n, \beta_n) - c^0(H, H)\,| = |\, a_1 \tfrac{\alpha_n - H}{2(1+A)}\,| < |\, a_1 \tfrac{\beta_0 - H}{2(1+A)4^{n-1}}\,| = |\, \tfrac{a_1(\beta_0 - H)}{1+A}\,| \, \tfrac{1}{4^n} < \tfrac{C_0}{4^n},$$

where C_0 is a positive constant greater than $\frac{a_1(\beta_0 - H)}{1+A}$.

In the same way, one can show that $|\, u^1(\alpha_n, \beta_n) - u^1(H, H)\,| < C_1/4^n$, $|\, u^2(\alpha_n, \beta_n) - u^2(H, H)\,| < C_2/4^n$, where C_1 and C_2 are two positive constants.

Finally, it is enough to take $C = \max\{\, C_0, C_1, C_2\,\}$. □

More generally, by a similar analysis for all the initial points (α_0, β_0), one can obtain the following theorem on the trajectories of the alternating best reply delegation processes.

Theorem 4.43. *Suppose that A4.31 holds and $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$. Then, for any trajectory $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ of an alternating best reply process associated to $\Gamma(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$, one has:*

(1) The sequence of cost profiles $\{(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n))\}_{n\in\mathbb{N}}$ converges to $(c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))$, where $(\alpha_\infty, \beta_\infty)$ is the limit of $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$.

(2) One of the following holds:

(i) There exists some $l \in \mathbb{N}^*$ such that, for all $n \ge l$, $(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n)) = (c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))$.

(ii) There exist a constant $C > 0$ such that, for all $n \in \mathbb{N}^*$, $\|\,(c^0(\alpha_n, \beta_n), u^1(\alpha_n, \beta_n), u^2(\alpha_n, \beta_n)) - (c^0(\alpha_\infty, \beta_\infty), u^1(\alpha_\infty, \beta_\infty), u^2(\alpha_\infty, \beta_\infty))\,\|_\infty < C/4^n$.

Besides, in the particular case where A4.34 holds, by gathering together the previous results on the range of the myopic outcomes for the five forms, one can compute the range of all the myopic outcomes of the delegation process where player α_0 begins. Similar computation shows that the range of the myopic outcomes of the delegation process where β_0 begins is the same.

Lemma 4.44. *Suppose that $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$ satisfies A4.34, then the set of myopic outcomes $\mathcal{E}(\alpha_0, \beta_0)$ is a convex compact set in $\mathbb{R}^3$. Explicitly,*

$$\mathcal{E}(\alpha_0, \beta_0) = \Big\{\, (c^0, u^1, u^2) \in \mathbb{R}^3 \quad \textit{such that}$$
$$a_1 M + b_1 - a_1 \tfrac{\beta_0 - H}{2(1+A)} \le c^0 \le a_1 M + b_1,\ \alpha_0\big[\, a_1 M + b_1 - a_1 \tfrac{\beta_0 - H}{2(1+A)}\,\big] \le u^1 \le \alpha_0(a_1 M + b_1),$$
$$(a_1 + a_2)\big[\, \tfrac{\beta_0 - H}{2(1+A)}\,\big]^2 - (a_1 M + b_1 - b_2 + a_1\beta_0)\tfrac{\beta_0 - H}{2(1+A)} + \beta_0(a_1 M + b_1) \le u^2 \le \beta_0(a_1 M + b_1)\,\Big\}$$

Proof. According to (4.56), (4.57) and (4.58), it is enough to prove that $a_1 M + b_1 - a_1 \frac{\beta_0 - H}{2(1+A)} < a_1 M + b_1 - a_1 \frac{\beta_0 - H}{4(1+A)}$, $\alpha_0[a_1 M + b_1 - a_1 \frac{\beta_0 - H}{2(1+A)}] < (a_1 + a_2)[\frac{\beta_0 - H}{4(1+A)}]^2 - (a_1 M + b_1 - b_2 + a_1\alpha_0)\frac{\beta_0 - H}{4(1+A)} + \alpha_0(a_1 M + b_1)$, and $(a_1 + a_2)[\frac{\beta_0 - H}{2(1+A)}]^2 - (a_1 M + b_1 - b_2 + a_1\beta_0)\frac{\beta_0 - H}{2(1+A)} + \beta_0(a_1 M + b_1) < \beta_0[a_1 M + b_1 - a_1 \frac{\beta_0 - H}{4(1+A)}]$. The computation is omitted. □

4.6.4 Comparison of the costs

Comparison with the equilibrium costs without delegation

In order to know whether the behavior of delegation benefits or, on the contrary, harm the players, one can compare the costs to the nonatomic players and the two atomic players α_0, β_0 in all the myopic outcomes with the costs in the original composite game without delegation, i.e. $\Gamma(M-\alpha_0-\beta_0,\alpha_0,\beta_0)$. The case where A4.34 holds will still be taken as an example.

Let us first compute the players' costs at the CE of $\Gamma(M-\alpha_0-\beta_0,\alpha_0,\beta_0)$. A simple computation shows that it is of mode 4. At the CE, the total weight on arc 1 is $F^{-1}(\alpha_0)=M-\frac{\alpha_0-H}{2(1+A)}$, the total weight of the stocks on arc 1 sent by player α_0 is $\alpha_0-\frac{\alpha_0-H}{2(1+A)}$, and the total weight of the stocks on arc 1 sent by player β_0 is β_0. Then,

$$\begin{cases} c^0(\alpha_0,\beta_0)=a_1M+b_1-a_1\frac{\alpha_0-H}{2(1+A)}, \\ u^1(\alpha_0,\beta_0)=(a_1+a_2)\big[\frac{\alpha_0-H}{2(1+A)}\big]^2-(a_1M+b_1-b_2+a_1\alpha_0)\frac{\alpha_0-H}{2(1+A)}+\alpha_0(a_1M+b_1), \\ u^2(\alpha_0,\beta_0)=\beta_0\big[a_1M+b_1-a_1\frac{\alpha_0-H}{2(1+A)}\big]. \end{cases} \tag{4.59}$$

Now, let us compare the costs in (4.59) to the outcomes of the trajectories of form 1 and form 2. Recall that, for trajectories $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ of form 1 and form 2, the costs in the myopic outcome $c^0_\alpha(\alpha_\infty,\beta_\infty)$, $u^1_\alpha(\alpha_\infty,\beta_\infty)$ and $u^2_\alpha(\alpha_\infty,\beta_\infty)$ are all independent of α_∞ and strictly decreasing in β_∞.

For the atomic player α_0, one can obtain that

$$u^1(\alpha_\infty,\beta_\infty)\begin{cases} >u^1(\alpha_0,\beta_0), & \text{if } H<\beta_\infty<\frac{(\alpha_0)^2+H^2}{2\alpha^0}; \\ =u^1(\alpha_0,\beta_0), & \text{if } \beta_\infty=\frac{(\alpha_0)^2+H^2}{2\alpha^0}; \\ <u^1(\alpha_0,\beta_0), & \text{if } \frac{(\alpha_0)^2+H^2}{2\alpha^0}<\beta_\infty\le\beta_0. \end{cases}$$

For all $\beta_\infty\in(H,\beta_0]$, $u^2(\alpha_\infty,\beta_\infty)>u^2(\alpha_0,\beta_0)$, $c^0(\alpha_\infty,\beta_\infty)>c^0(\alpha_0,\beta_0)$.

By similar computations, one finds that, for any trajectory $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ of form 3 or form 4, for all $\alpha_\infty\in(H,\frac{\beta_0+H}{2}]$, $c^0(\alpha_\infty,\beta_\infty)>c^0(\alpha_0,\beta_0)$, $u^1(\alpha_\infty,\beta_\infty)>u^1(\alpha_0,\beta_0)$, and $u^2(\alpha_\infty,\beta_\infty)>u^2(\alpha_0,\beta_0)$.

Finally, for any trajectory $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}^*}$ of form 5, $c^0(H,H)>c^0(\alpha_0,\beta_0)$, $u^1(H,H)>u^1(\alpha_0,\beta_0)$, and $u^2(H,H)>u^2(\alpha_0,\beta_0)$.

Lemma 4.45. *Suppose that $(\alpha_0,\beta_0)\in\operatorname{int}\Delta$ satisfies A4.34 and $(c^0(\alpha_\infty,\beta_\infty),u^1(\alpha_\infty,\beta_\infty),u^2(\alpha_\infty,\beta_\infty))$ is a myopic outcome of the alternating best reply delegation processes associated to $\Gamma(M-\alpha_0-\beta_0,\alpha_0,\beta_0)$. Then, $c^0(\alpha_\infty,\beta_\infty)>c^0(\alpha_0,\beta_0)$, $u^2(\alpha_\infty,\beta_\infty)>u^2(\alpha_0,\beta_0)$, while $u^1(\alpha_\infty,\beta_\infty)$ may be greater than, or less than, or equal to $u^1(\alpha_0,\beta_0)$.*

In other words, the behavior of delegation does not always benefit or harm the atomic players.

For the impact of delegation on the social costs and the nonatomic players' costs, a more general result in Wan [92] (*cf.* Theorem 3.23) gives rise to the following theorem.

Theorem 4.46. *Suppose that A4.31 holds and $(\alpha_0,\beta_0)\in\operatorname{int}\Delta$. Then, for any trajectory $\{(\alpha_n,\beta_n)\}_{n\in\mathbb{N}}$ of an alternating best reply delegation process associated to $\Gamma(M-\alpha_0-$*

$\beta_0, \alpha_0, \beta_0)$, *for all* $n \in \mathbb{N}^*$,

$$v(\alpha_{n+1}, \beta_{n+1}) \geq v(\alpha_n, \beta_n),\ c^0(\alpha_{n+1}, \beta_{n+1}) \geq c^0(\alpha_n, \beta_n),$$
$$v(\alpha_n, \beta_n) \geq v(\alpha_0, \beta_0),\ c^0(\alpha_n, \beta_n) \geq c^0(\alpha_0, \beta_0),$$
$$v(\alpha_\infty, \beta_\infty) \geq v(\alpha_n, \beta_n),\ c^0(\alpha_\infty, \beta_\infty) \geq c^0(\alpha_n, \beta_n).$$

Proof. See Theorem 4.7 in Wan [92]. □

Comparison with the equilibrium costs in the one-shot delegation game

Recall that, in the proof of Theorem 4.35, the set of equilibrium costs of the one-shot delegation game $D_1(M - \alpha_0 - \beta_0, \alpha_0, \beta_0)$, denoted by $E_1(\alpha_0, \beta_0)$, contains two elements: $(c^0_{I_\alpha}, u^1_{I_\alpha}, u^2_{I_\alpha})$ and $(c^0_{I_\beta}, u^1_{I_\beta}, u^2_{I_\beta})$ whose explicit expressions are given by (4.52) and (4.53).

By a simple computation, one can see that $(c^0_{I_\alpha}, u^1_{I_\alpha}, u^2_{I_\alpha}) \notin \mathcal{E}(\alpha_0, \beta_0)$, while $(c^0_{I_\beta}, u^1_{I_\beta}, u^2_{I_\beta}) \in \mathcal{E}(\alpha_0, \beta_0)$.

This implies that $\mathcal{E}(\alpha_0, \beta_0)$ neither contains nor is contained in $E_1(\alpha_0, \beta_0)$.

Theorem 4.47. *Suppose that A4.31 holds. There exists* $(\alpha_0, \beta_0) \in \operatorname{int} \Delta$ *such that none of the two relations below is true:*

(i) $\mathcal{E}(\alpha_0, \beta_0) \subset E_1(\alpha_0, \beta_0)$;

(ii) $E_1(\alpha_0, \beta_0) \subset \mathcal{E}(\alpha_0, \beta_0)$.

4.7 Remarks and discussion

4.7.1 More general delegation games and dynamic programming

In this paper, the behavior of delegation in the congestion games are studied in some specific cases. Firstly, the network is restrained to a two-terminal two-parallel-arc one. Secondly, a one-shot delegation game is defined for a general composition of players and general cost functions, but the existence of equilibria are proved only for the case where there are two atomic players in addition to nonatomic players, and the cost functions are affine. Thirdly, the atomic players use single-atomic delegation strategies only. The extension of the results of this paper to more general contexts remains an open problem.

Besides, Example 4.9 shows that, in a one-shot delegation game, an equilibrium might be not robust, in the sense that the delegates created by the delegation strategies may have incentive to delegate in their turn. Therefore, other definitions of delegation games should be considered. A dynamical approach is adopted in this paper, where the atomic players play a best reply in an alternating way. However, one need not consider only this kind of myopic behavior. A discrete dynamic programming model could be considered, where the players optimize in the long run. Nevertheless, there are two differences with a usual dynamic programming problem. First, there is not only one decision maker, but at least two players who interact with each other. Moreover, if the atomic players use not only simple-atomic delegation strategies, the number of players may even increase throughout the process. Second, in a usual dynamic programming process, the decision maker anticipates what she will do at later stages, and she makes her decisions based upon her anticipation. In our context, when an atomic player plays a delegation strategy, she is replaced by her delegates so that she does not play any longer herself. As a result, she has to do her decisions based upon her anticipation of the behavior of her delegates, and the delegates of her delegates, and so on.

4.7.2 Discretization

Sorin and Wan [88] (*cf.* Chapter 5) defined a delegation game associated to integer-splittable congestion games. In such games, the players have integer weights, and they can only split their stock into integer weight parts so as to delegate. The process of delegation will come to an end after a finite number of stages when every player has weight 1. The game and its equilibria are constructed by induction. More precisely, in a delegation game $\mathcal{D}(G)$, when all the players have chosen their delegation strategy, a delegation game $\mathcal{D}(G')$ played by their delegates, together with the set of its equilibrium costs $\mathcal{E}(G')$, are supposed to have already been defined. Then, each player in $\mathcal{D}(G)$ uses the costs in $\mathcal{E}(G')$ to compute the sum of the costs to her delegates, which is her own cost in $\mathcal{D}(G)$. In this way, a set of equilibrium costs of $\mathcal{D}(G)$ can be computed. And this is done for all the possible profiles of delegation strategies in $\mathcal{D}(G)$, so that all its equilibrium costs are found. The set of equilibrium costs of $\mathcal{D}(G)$ is denoted by $\mathcal{E}(G)$. The induction begins by defining a delegation game where all the players have weight 1.

More generally, a delegation game can be defined in this way when the stock of each player is made up of several parts which can be separated from one another, but each part itself cannot be divided anymore.

This definition cannot be applied immediately in the framework of (arbitrarily) splittable games, because an atomic player can always delegate, no matter how small her weight is. To construct an analogue model, one might begin by discretizing the continuous weight. Explicitly, given a composite game Γ, by fixing a small constant ϵ as the unit weight, an atomic player's weight T^p is reset to be $n\epsilon$ with $n \in \mathbb{N}$, such that $|n\epsilon - T^p| \leq |m\epsilon - T^p|$ for all $m \in \mathbb{N}$. Then, one defines an approximate game of Γ, denoted by Γ_ϵ, which is an ϵ-splittable game. In Γ_ϵ, an atomic player is allowed to split her weight into several parts such that the weight of each part is $l\epsilon$ for different l's in $\mathbb{N}^*$. The delegation game $\mathcal{D}(\Gamma_\epsilon)$ associated to Γ_ϵ and its set of equilibrium costs $\mathcal{E}(\Gamma_\epsilon)$ can thus be defined. By letting $\epsilon \to 0$, one can study the behavior of $\mathcal{E}(\Gamma_\epsilon)$.

4.7.3 Efficiency of the delegation

Because of the specific property of the two-terminal two-parallel-arc networks, every time an atomic player delegates, the social cost is increased or stays the same at the equilibrium. However, this is not necessarily the case for other network topologies. Huang [47] provided an example where there are three parallel arcs with cost functions satisfying A4.1, but when an atomic player commits her stock to two atomic delegates, the social cost is decreased at the equilibrium.

Therefore, it will be interesting to study the impact of the behavior of delegation on the social costs in more general settings. Recall that the *price of anarchy* [50] and the *price of collusion* [37] are the indices which measure, respectively, the impact of the lack of coordination and the impact of the formation of coalitions on the social cost in congestion games. To measure the impact of the behavior of delegation, one can define the *price of delegation* as the ratio between the worst social cost at a myopic outcome and the equilibrium social cost in the original game.

Chapter 5

Delegation equilibrium payoffs in integer-splitting games

This chapter is based on the paper *Delegation equilibrium payoffs in integer-splitting games* in collaboration with Sylvain Sorin.

Abstract. *This work studies a new strategic game called delegation game. A delegation game is associated to a basic game with a finite number of players where each player has a finite integer weight and her strategy consists in dividing it into several integer parts and assigning each part to one of finitely many choices. In the associated delegation game, a player divides her weight into several integer parts, commits each part to an independent delegate and collects the sum of their payoffs in the basic game played by these delegates. Delegation equilibrium payoffs, consistent delegation equilibrium payoffs and consistent chains inducing these ones in a delegation game are defined. Several examples are provided.*

5.1 Motivation

In an N-player routing game, each player holds a certain quantity of stock that she has to send by one or several directed paths from its origin vertex to its destination vertex. The cost of using a certain path depends upon the quantity of stock passing through it. Every player wishes to minimize her cost. If the stocks are unsplittable, a player must send all her stock through a single path, otherwise she can split it into several parts so as to send them by different paths. In this work an integer-splittable case will be considered: every player has a stock of integer weight which can be divided into several parts of integer weight. For example, a player holding a stock of weight 2 can only split it into two parts, each of weight 1, but not one part of weight 0.5 and another of weight 1.5. If a player has R available paths, then her pure strategy set is the finite set of the divisions of her stock into R integer weight parts.

Suppose that, instead of deciding by herself how to send her stock, a player n divides it into integer weight parts and commits each part to a different delegate. The delegates are then independent players who ensure the transportation of the stocks committed to them. The cost to player n is the sum of the costs to her delegates. This procedure is called (*integer*) *delegation*.

Why should a player be interested in delegating her stock? If there is only one player, the minimum cost that she obtains at the equilibrium of this one-player game is just the social optimum cost. If she commits her stock to some delegates, then whatever the outcome is, it will not be better than the social optimum. Hence she has no incentive to delegate.

However, when there are several players, the situation might be different. Here is an example which shows the potential advantage of producing delegates.

Example 5.1. Consider the following two-player routing game with integer-splittable stock where player I has a stock of weight 2 and player II has a stock of weight 1. Both players have to send their stock from vertex O to vertex D in Figure 5.1. Two paths are available and each path is just a single arc. The per-unit cost function of the upper path is $c_1(x) = x$, which means that the cost to each unit on the path is x if the total weight is x. The per-unit cost of the lower path is $c_2(x) = 0.1x + 2.3$.

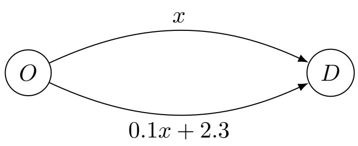


Figure 5.1

First let us consider a routing game without delegation. Player I has three pure strategies: (u, u), i.e. to send both units by the upper path, (u, l), i.e. to send one unit by the upper path and the other the lower path, and (l, l), i.e. to send both units by the lower path. Player II has two pure strategies: u, i.e. to send her unit by the upper path, and l, i.e. to send it by the lower path. In the cost matrix of the game in Table 5.1, player I is the row player and player II the column player. The only pair of pure Nash equilibrium costs (4.4, 2) is starred. Besides, as the strategy (u, l) is a dominant one for player I, the pure equilibrium is also the unique equilibrium.

	u	l
u,u	6, 3	4, 2.4
u,l	4.4, 2 *	3.5, 2.5
l,l	5, 1	5.2, 2.6

Table 5.1

Now suppose that player I splits her stock into two parts, each of weight 1, and commits them to two delegates called respectively i_I and ii_I. In the routing game played by the three players i_I, ii_I and II, each of them has a stock of weight 1 and has two pure strategies u and l. The left (*resp.* right) matrix in Table 5.2 corresponds to the choice of player II.

The three pure Nash equilibria are starred. At two of them, (u, l, u) and (l, u, u), the pair of the total costs to player I and player II is the same as in the previous model without delegation: (4.4, 2). At the third one (u, u, l), the pair of total costs to player I and player II is (4, 2.4). The cost to player I is *lower* than in the case without delegation. Conclusion: delegation can be advantageous!

Remark 5.2. Notice that in the three-player game, the equilibrium (u, u, l) exists because player II is facing the strategy profile (u, u) of the two delegates of player I. But if player I knew that player II would choose l, she would not delegate but rather assign by herself one unit weight of stock to u and the other to l so as to get a cost 3.5 instead of 4. Then the best

	u	l		u	l
u	3, 3, 3	2, 2.4, 2 *	u	2, 2, 2.4 *	1, 2.5, 2.5
l	2.4, 2, 2 *	2.5, 2.5, 1	l	2.5, 1, 2.5	2.6, 2.6, 2.6
	u			l	

Table 5.2

reply of player II would be u. This underlines the fact that the stability of the equilibria relies on the independence of the delegates.

Since a delegate is an independent player, she may well delegate her stock in her turn. By induction, the procedure can continue until all the players/delegates have a stock of weight 1.

This work will establish a rigorous model of *integer-splitting delegation game* and the corresponding delegation equilibrium payoffs.

5.2 Basic (integer-splitting) game

5.2.1 Model

Before introducing the notion of delegation, let us first define formally the *basic (integer-splitting) games*. For example, the basic game on a network introduced in Example 5.1 is a routing game with integer-splittable stocks.

Definition 5.3. An *basic (integer-splitting) game* $G(\mathcal{N}, P)$ is defined by the following elements.

- P is a non-empty finite set of *choices*.
- $\mathcal{N}$ is a finite set of players.
  A player $n \in \mathcal{N}$ is characterized by three data:
  1) her *integer weight* $m^n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$,
  2) the non-empty set of her available *choices* $P^n \subset P$,
  3) a vector function $\psi^n = (\psi_p^n)_{p \in P^n}$, where ψ_p^n, her *per-unit payoff function* for choice p, is a real-valued function defined on $[0, M] \cap \mathbb{N}$, where $M = \sum_{n \in \mathcal{N}} m^n$.
  In particular a player's *type* is specified by the pair (P^n, ψ^n).
- A pure strategy $\mathbf{f}^n$ of player n is an *integer partition* of her weight onto P^n. Explicitly, it is a $|P^n|$-dimensional vector ($|A|$ denote the cardinality of a finite set A) with integer components: $\mathbf{f}^n = (f_p^n)_{p \in P^n}$, $f_p^n \in \mathbb{N}$ being the part of her weight that she assigns to choice $p \in P^n$. The finite set of player n's pure strategies is thus $F^n = \{ \mathbf{f}^n = (f_p^n)_{p \in P^n} \in \mathbb{N}^{|P^n|} \mid \sum_{p \in P^n} f_p^n = m^n \}$.
- $F = \prod_{n \in \mathcal{N}} F^n$ is the space of pure-strategy profiles. A profile of pure strategies $\mathbf{f} = (\mathbf{f}^n)_{n \in \mathcal{N}}$ induces a vector $\xi(\mathbf{f}) = \big(\xi_p(\mathbf{f})\big)_{p \in P}$ called *aggregated configuration*, where $\xi_p(\mathbf{f}) = \sum_{n \in \mathcal{N}: p \in P^n} f_p^n$ is the total weight assigned to p. Denote X the set of feasible aggregated configurations. It is a subset of $\mathbb{N}^{|P|}$.
- The *payoff function* of player n is defined as follows:

$$u^n(\mathbf{f}) = u^n(\mathbf{f}^n, \xi(\mathbf{f})) = \sum_{p \in P^n} f_p^n \, \psi_p^n(\xi_p(\mathbf{f})). \tag{5.1}$$

Remark 5.4. One can let the set of available choices for each player be P and then define $\psi_p^n \equiv -\infty$ for all $p \notin P^n$.

Remark 5.5. Every player's payoff is *additive*: the payoff associated to the weight that player n assigns to choice p is counted separately for each choice $p \in P$, then her total payoff u^n is the sum of these payoffs.

Remark 5.6. A basic game is an *anonymous aggregate game* according to the definition of ψ^n's. This means that only the total weight assigned to a certain choice is taken into account, but not the identity of the players who send them nor the specific decomposition.

$G(\mathcal{N}, P)$ is a finite game: the number of players and the number of strategies for each player is finite. Hence there exists an equilibrium (in mixed strategies).

Notation. The set of equilibrium payoff vectors is denoted by $E\big(G(\mathcal{N}, P)\big)$. It is a non-empty subset of $\mathbb{R}^{|\mathcal{N}|}$.

In the current analysis we fix a choice set P and consider different profiles of players with specific available choices in P. The set of players of a basic game G is denoted by $\mathcal{N}_G$ and by abuse of notation, $|G|$ will denote its cardinality.

5.2.2 Delegation

Consider a basic game G.

Definition 5.7. A player n in G, with weight m^n and type (P^n, ψ^n), *delegates* if she is replaced by two players n' and n'' of the same type as her and of strictly positive integer weights respectively $m^{n'}$ and $m^{n''}$ with $m^{n'} + m^{n''} = m^n$.

Definition 5.8. A basic game G' is a *direct successor* of another basic game G if G' is obtained by the delegation of one of the players in G.

Remark 5.9. $|G^{'}| = |G| + 1$.

Definition 5.10. A basic game G' is a *successor* of another basic game G if there exists a finite sequence of basic games $G_0, G_1, \dots, G_k$ such that $G = G_0$, $G' = G_k$, and G_l is a direct successor of G_{l-1} for $l = 1, 2, \dots, k$.

Remark 5.11. $|G^{'}| = |G| + k$.

Definition 5.12. Denote the family consisting of a basic game G and all the successors of G by Ξ_G. This set has a natural tree structure with root G. The *derived game* $\bar{G}$ of G is the element in Ξ_G which is the last on each branch. It thus has the largest cardinality, i.e. each player in $\bar{G}$ has weight 1.

Remark 5.13. $|\bar{G}| = \sum_{n \in \mathcal{N}_G} m^n$.

5.3 Equilibrium payoffs in delegation games

In this section we will define and construct the set of delegation equilibrium payoffs.

5.3.1 Game form

The game form of the delegation game $\mathcal{D}(G)$ associated to the basic game G, i.e. its set of players and their strategy sets, is defined as follows.

The player set is $\mathcal{N}_G$, the same as that of G.

For each player $n \in \mathcal{N}_G$ of weight m^n, her *strategy set* S^n is the finite set of integer-partitions of m^n. Explicitly, for each $\mathbf{s}^n \in S^n$, there is an integer $l_n \in \mathbb{N}^*$ such that

$$\mathbf{s}^n = \{m_i^n\}_{i=1}^{l_n} \in (\mathbb{N}^*)^{l_n}, \quad \sum_{i=1}^{l_n} m_i^n = m^n.$$

The interpretation is that player n creates l_n delegates with delegate i having weight $m_i^n, i := 1, \cdots, l_n$. In particular, the strategy set S^n is finite.

Remark 5.14. A profile of strategies $\{\mathbf{s}^n\}_{n\in\mathcal{N}}$ induces a successor G' of G if there exists $n \in \mathcal{N}$ such that $l_n > 1$, i.e. at least one player delegates: it is the basic game with player profile defined by $\{\{m_i^n\}_{i=1}^{l_n}\}_{n\in\mathcal{N}}$. Otherwise, if $l_n = 1, \mathbf{s}^n = \{m^n\}$ for all $n \in \mathcal{N}$, the profile $\{\mathbf{s}^n\}_{n\in\mathcal{N}}$ induces the basic game G.

5.3.2 Set of delegation equilibrium payoffs

The payoff functions and the set of *delegation equilibrium payoffs* associated to the delegation game $\mathcal{D}(G)$, denoted by $\mathcal{E}(G)$, are defined by induction on the tree Ξ_G.

For the derived basic game $\bar{G}$, let $\mathcal{E}(\bar{G})$ be $E(\bar{G})$, the set of equilibrium payoffs.

Let $\hat{G} \in \Xi_G$ and assume that the set of delegation equilibrium payoffs $\mathcal{E}(G')$ in $\mathcal{D}(G')$ is defined for all successors G' of $\hat{G}$. We define $\mathcal{E}(\hat{G})$ in three steps.

1) Let e be a selection from $E(\hat{G})$, i.e. $e(\hat{G}) \in E(\hat{G}) \subset \mathbb{R}^{|\hat{G}|}$.

Let y be a selection from the sets of delegation equilibrium payoffs $\{\mathcal{E}(G')\}_{G'\in\Xi_{\hat{G}}\setminus\{\hat{G}\}}$. In other words, for any successor G' of $\hat{G}$, $y(G') \in \mathcal{E}(G') \subset \mathbb{R}^{|G'|}$. Let $Y(\hat{G})$ denote the collection of all such selections of $\{\mathcal{E}(G')\}_{G'\in\Xi_{\hat{G}}\setminus\{\hat{G}\}}$.

2) For each pair $(e, y) \in E(\hat{G}) \times Y(\hat{G})$, consider an auxiliary game $\mathcal{G}(\hat{G}; e, y)$ associated to the game form of $\mathcal{D}(G)$. The payoff F^k to player $k \in \mathcal{N}_{\hat{G}}$ as a function of the strategy profile $\mathbf{s} = \{\mathbf{s}^k\}_{k\in\mathcal{N}_{\hat{G}}}$ is defined as follows.

If $\mathbf{s}$ induces the basic game $\hat{G}$, $F^k(\mathbf{s}) = e^k(\hat{G})$.

If $\mathbf{s}$ induces a successor G' of $\hat{G}$ so that the player profile of G' is $\{\{m_i^k\}_{i=1}^{l_k}\}_{k\in\mathcal{N}_{\hat{G}}}$, then

$$F^k(\mathbf{s}) = \sum_{i=1}^{l_k} y_i^k(G').$$

Namely the payoff to player k is the sum of the delegation equilibrium payoffs to her delegates according to y.

The auxiliary game $\mathcal{G}(\hat{G}; e, y)$ is a finite game. We denote by $E(\hat{G}; e, y) \subset \mathbb{R}^{|\hat{G}|}$ its non empty subset of equilibrium payoff profiles.

3) The set of delegation equilibrium payoff profiles in the delegation game $\mathcal{D}(\hat{G})$ is finally defined by

$$\mathcal{E}(\hat{G}) = \bigcup_{e\in E(\hat{G}),\, y\in Y(\hat{G})} E(\hat{G}; e, y).$$

Proposition 5.15. *The set of delegation equilibrium payoff profiles $\mathcal{E}(G)$ is a non-empty subset of $\mathbb{R}^{|G|}$.*

5.3.3 Consistent delegation equilibrium payoffs

Now we define, for a delegation game $\mathcal{D}(G)$, the set of *consistent chains of delegation equilibrium payoffs*, denoted by $\mathcal{H}(G) \subset \prod_{G' \in \Xi_G} \mathbb{R}^{|G'|}$, and the associated set of *consistent delegation equilibrium payoffs*, denoted by $\tilde{\mathcal{E}}(G) \subset \mathcal{E}(G)$.

The definition of a consistent chain of equilibrium payoffs $h = \big(h(G')\big)_{G' \in \Xi_G}$ is obtained by induction as in the previous section.

For $\bar{G}$, let $h(\bar{G})$ be an arbitrary element in $\mathcal{E}(\bar{G})$.

Let $\hat{G} \in \Xi_G \setminus \{G\}$ and assume that $h(G')$ is defined for all successors G' of $\hat{G}$ with $h(G') \in \mathbb{R}^{|G'|}$. As above, for any selection e of $E(\hat{G}) \subset \mathbb{R}^{|\hat{G}|}$, the auxiliary game $\mathcal{G}(\hat{G}; e, h)$ has a non-empty equilibrium payoff profile set $E(\hat{G}; e, h) \subset \mathbb{R}^{|\hat{G}|}$. Choose $h(\hat{G})$ as an element in $\bigcup_{e \in E(\hat{G})} E(\hat{G}; e, h) \subset \mathcal{E}(\hat{G})$. In this way a consistent chain of delegation equilibrium h is defined for the delegation game $\mathcal{D}(G)$. $\mathcal{H}(G)$ is the collection of all such chains.

For any $h \in \mathcal{H}(G)$, $h(G) \in \mathcal{E}(G)$ is called a consistent delegation equilibrium payoff profile *induced by the consistent chain h*. The set of consistent delegation equilibrium payoff profiles in $\mathcal{D}(G)$, denoted by $\tilde{\mathcal{E}}(G)$, is defined as a subset of $\mathcal{E}(G)$, by

$$\tilde{\mathcal{E}}(G) = \{\, h(G) \,\}_{h \in \mathcal{H}(G)}.$$

Proposition 5.16. *The collection of consistent chains of delegation equilibrium payoffs $\mathcal{H}(G)$ is a non-empty set in $\prod_{G' \in \Xi_G} \mathbb{R}^{|G'|}$.*

The set of consistent delegation equilibrium payoffs $\tilde{\mathcal{E}}(G)$ is a non-empty subset of $\mathcal{E}(G) \subset \mathbb{R}^{|G|}$.

5.4 Example

In this section a delegation game on a network is studied in detail. A delegation equilibrium, a chain of consistent delegation equilibrium profiles and the consistent delegation equilibrium profile that it induces will be obtained. A construction of a non consistent delegation equilibrium profile will also be provided.

The basic game G takes place in a network composed of two vertices O and D, with two parallel arcs r_1 and r_2 connecting O to D. Their per-unit cost functions are respectively $l_1(x) = x + 1$, $l_2(x) = 0.1x + 3.6$. Two players both hold a stock to send from O to D. The weight of player I's stock is 3 and that of player II's is 2. The profile of the players is $\mathcal{N}_G = \{I|3,\, II|2\}$, where the name of a player is her type and the number after the name of a player is her weight. G has five successors A, B, C, D and $\bar{G}$. The set of players of the six basic games and some of their equilibrium cost profiles are given below. The type of a player depends on the initial possessor of her stock. In each profile, players of the same type are put in the same pair of brackets.

$\mathcal{N}_G = \{\, \{I|3\},\, \{II|2\} \,\}$ $\qquad$ $\mathcal{N}_A = \{\, \{a^1|2,\, a^2|1\},\, \{a^3|2\} \,\}$

$\mathcal{N}_B = \{\, \{b^1|1,\, b^2|1,\, b^3|1\},\, \{b^4|2\} \,\}$ $\qquad$ $\mathcal{N}_C = \{\, \{c^1|3\},\, \{c^2|1,\, c^3|1\} \,\}$

$\mathcal{N}_D = \{\, \{d^1|2,\, d^2|1\},\, \{d^3|1,\, d^4|1\} \,\}$ $\qquad$ $\mathcal{N}_{\bar{G}} = \{\, \{q^1|1,\, q^2|1,\, q^3|1\},\, \{q^4|1,\, q^5|1\} \,\}$

$E(G) \supset \{(10.8,\ 6.9),\ (9.9,\ 7.8)\}$

$E(A) \supset \{(7.8,\ 3,\ 6.9),\ (6.9,\ 3,\ 7.8),\ (6,\ 3.9,\ 7.8),\ (7.8,\ 3.9,\ 6),\ (6.9,\ 3.9,\ 6.9)\}$

$E(B) \supset \{(3,\ 3,\ 3.9,\ 7.8),\ (3,\ 3.9,\ 3,\ 7.8),\ (3.9,\ 3.9,\ 3.9,\ 6),\ (3.9,\ 3.9,\ 3,\ 6.9)\}$

$E(C) \supset \{(10.8,\ 3.9,\ 3),\ (10.8,\ 3,\ 3.9),\ (9.9,\ 3.9,\ 3.9)\}$

$E(D) \supset \{(7.8,\ 3,\ 3,\ 3.9),\ (7.8,\ 3.9,\ 3,\ 3),\ (6,\ 3.9,\ 3.9,\ 3.9),\ (6.9,\ 3,\ 3.9,\ 3.9)\}$

$E(\bar{G}) \supset \{(3.9, 3.9, 3.9, 3, 3), (3.9, 3.9, 3, 3.9, 3), (3.9, 3, 3.9, 3.9, 3), (3, 3, 3.9, 3.9, 3.9)\}$

Figure 5.2 illustrates Ξ_G by its tree structure.

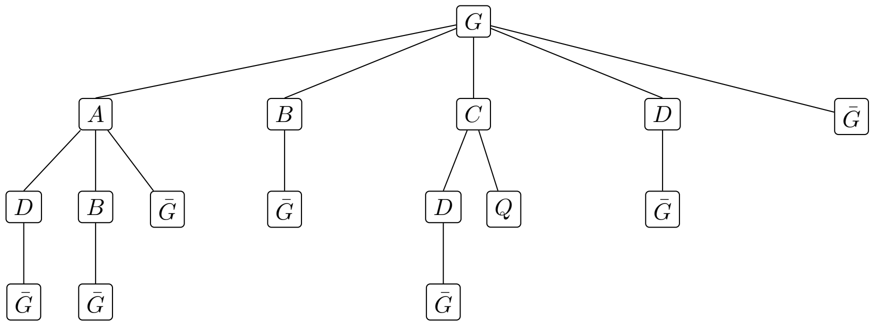


Figure 5.2

1) In the delegation game $\mathcal{D}(\bar{G})$, select $h(\bar{G}) = (3.9,\ 3.9,\ 3.9,\ 3,\ 3) \in \mathcal{E}(\bar{G}) = E(\bar{G})$. Note that another possible selection $y(\bar{G}) = (3,\ 3,\ 3.9,\ 3.9,\ 3.9)$.

2) In the basic game D, select $e(D) = (7.8,\ 3.9,\ 3,\ 3) \in E(D)$.

3) In the delegation game $\mathcal{D}(D)$, the payoff matrix of the auxiliary game $\mathcal{G}(D; e, h)$ determined by $(e(D), h(\bar{G}))$ is shown on the left in Table 5.3, while the one determined by $(e(D), y(\bar{G}))$ is shown on the right in Table 5.3. Player d^1 is the row player. Players d^2, d^3 and d^4 have no actions.

{2}	$D:\ (7.8,\ 3.9,\ 3,\ 3)$ *
{1,1}	$\bar{G}:\ (3.9 + 3.9,\ 3.9,\ 3,\ 3)$ *

{2}	$D:\ (7.8,\ 3.9,\ 3,\ 3)$
{1,1}	$\bar{G}:\ (3 + 3,\ 3.9,\ 3.9,\ 3.9)$ *

Table 5.3: $\mathcal{G}(D; e, h)$ and $\mathcal{G}(D; e, y)$.

The pure equilibria outcomes are starred. In the auxiliary game on the left, select $h(D) = (7.8,\ 3.9,\ 3,\ 3) \in \mathcal{E}(D)$. In the auxiliary game on the right, select $y(D) = (6,\ 3.9,\ 3.9,\ 3.9) \in \mathcal{E}(D)$.

4) In the basic game C, select $e(C) = (10.8,\ 3.9,\ 3) \in E(C)$.

5) In the delegation game $\mathcal{D}(C)$, the payoff matrix of the auxiliary game determined by $(e(C), h(D), h(\bar{G}))$ is shown in Table 5.4. Player c^1 is the row player. Players c^2 and c^3 have no choice.

Select $h(C) = (10.8,\ 3.9,\ 3) \in \mathcal{E}(C)$.

6) In the basic game B, select $e(B) = (3,\ 3.9,\ 3,\ 7.8) \in E(B)$.

7) In the delegation game $\mathcal{D}(B)$, the payoff matrix of the auxiliary game determined by $(e(B), h(\bar{G}))$ is shown in Table 5.5. Player b^4 is the row player. Players b^1, b^2 and b^3 have no action.

Select $h(B) = (3.9,\ 3.9,\ 3.9,\ 6) \in \mathcal{E}(B)$.

{3}	$C:$ (10.8, 3.9, 3) *
{2,1}	$D:$ $(7.8+3.9,$ 3, 3) *
{1,1,1}	$\bar{G}:$ $(3.9+3.9+3.9,$ 3, 3) *

Table 5.4: $\mathcal{G}(C;e,h)$.

{2}	$B:$ (3, 3.9, 3, 7.8)
{1, 1}	$\bar{G}:$ (3.9, 3.9, 3.9, $3+3$) *

Table 5.5: $\mathcal{G}(B;e,h)$.

8) In the basic game A, select $e(A) = (6.9, 3, 7.8) \in E(A)$.

9) In the delegation game $\mathcal{D}(A)$, the payoff matrix of the auxiliary game determined by $(e(A), h(D), h(B), h(\bar{G}))$ is shown in Table 5.6. Player a^1 is the row player and player a^3 is the column player. Player a^2 has no action.

	{2}	{1,1}
{2}	$A:$ (6.9, 3, 7.8)	$D:$ (7.8, 3.9, $3+3$) *
{1,1}	$B:$ $(3.9+3.9$, 3.9, 6)	$\bar{G}:$ $(3.9+3.9$, 3.9, $3+3)$*

Table 5.6: $\mathcal{G}(A;e,h)$.

Select $h(A) = (7.8, 3.9, 6) \in \mathcal{E}(A)$.

10) In the basic game G, select $e(G) = (9.9, 7.8) \in E(G)$.

11) In the delegation game $\mathcal{D}(G)$, the payoff matrix of the auxiliary game determined by $(e(G), h(A), h(B), h(C), h(D), h(\bar{G}))$ is shown in Table 5.7. Player I is the row player and player II is the column player.

	{2}	{1,1}
{3}	$G:$ (9.9, 7.8)	$C:$ (10.8, $3.9+3$) *
{2, 1}	$A:$ $(7.8+3.9$, 6)	$D:$ $(7.8+3.9$, $3+3)$
{1, 1, 1}	$B:$ $(3.9+3.9+3.9$, 6)	$\bar{G}:$ $(3.9+3.9+3.9$, $3+3)$

Table 5.7: $\mathcal{G}(G;e,h)$.

Select $h(G) = (10.8, 6.9)$. Then the construction of a consistent chain of delegation equilibrium profiles h is completed, and (10.8, 6.9) is a consistent delegation equilibrium profile in $\mathcal{D}(G)$.

Alternatively, the auxiliary game $\mathcal{G}(G;e,y)$ which uses $y(D)$ instead of $h(D)$ has the payoff matrix in Table 5.8.

The only equilibrium is attained if player I plays {3} with probability 2/3 and {2, 1} with probability $\frac{1}{3}$, and player II plays {2} with probability 1/3 and {1, 1} with probability $\frac{2}{3}$. The equilibrium cost profile is (10.5, 7.2). This is an equilibrium cost profile in $\mathcal{D}(G)$ but is not constructed by a consistent chain.

	$\{2\}$	$\{1,1\}$
$\{3\}$	$G:\ (9.9,\ 7.8)$	$C:(10.8,\ 3.9+3)$
$\{2,1\}$	$A:\ (7.8+3.9,\ 6)$	$D:\ (6+3.9,\ 3.9+3.9)$
$\{1,1,1\}$	$B:\ (3.9+3.9+3.9,\ 6)$	$\bar{G}:\ (3.9+3.9+3.9,\ 3+3)$

Table 5.8: $\mathcal{G}(G;e,y)$.

5.5 Comments and extensions

5.5.1 Nash equilibria

One could define equilibrium payoffs $F(G)$ in the delegation game $\mathcal{D}(G)$ by induction using first an auxiliary game $\mathcal{G}(G;e,z)$, where e is a selection of $E(G)$ and, for each successor G', $z(G')$ is a selection of feasible payoff in G'. Then one would require that an equilibrium σ in the auxiliary game satisfies the coherency condition: for each G' in the support of σ, $z(G')$ belongs to $F(G')$. However this would assume that off the equilibrium path the delegate players would play a specific “threat" which is not an equilibrium and is thus difficult to justify.

5.5.2 Subgame perfection

The induction procedure used to define delegation equilibria is reminiscent of subgame perfection [84]. However there are important differences. First the tree structure involves new players and the consistency argument used in subgame perfection (the (same) players should play an equilibrium at each node of the tree) is replaced by an argument involving the rationality of the delegates. Moreover, for the notion of consistent delegation equilibrium, one goes even one step further since one asks for a same perception of the play in the successor games. One could relate the construction of delegation equilibrium to the usual selection of subgame perfect equilibria through forward induction: if G' is reached, play an equilibrium in G'. Then the notion of consistent delegation equilibria would add a backward induction property: the behavior in a successor game G' of G is independent of the way G' is reached during the play of G and depends only on G' (and its successors).

5.5.3 Composite equilibria

The same concepts can be defined in games where each player has a finitely divisible stock, even in the presence of a set of nonatomic players, using the notion of composite equilibria [32, 91]. However the general case where each player n can divide arbitrarily her stock m^n composed of finitely (or countably) many atoms $m_i^n, i \geq 1$ and a nonatomic part m_0^n with $\sum_{j=0}^{+\infty} m_j^n = m^n$ deserves further study, since the definition of delegation equilibrium by backward induction is no longer available.

5.5.4 A special case

A special case where there are only two choices in the basic game is studied in [92] (*cf.* Chapter 3). In such a setting, all players have the same payoff function. In a basic game, a player holding a stock of strictly positive weight can arbitrarily divide her stock into two parts and affect them to two choices respectively, so she is called atomic splittable.

A player holding an infinitesimal stock is called nonatomic. Under a standard concavity assumption on the payoff functions, a unique equilibrium exists in each basic game. In a delegation game, each atomic splittable player can arbitrarily delegate her stock to finitely many atomic delegates and a set of nonatomic delegates. Her payoff is the sum of the payoffs to her delegates at the equilibrium of the basic game played by all the delegates. It is shown that, in a basic game Γ_1, if one or several atomic players delegate so that the basic game after the delegation is Γ_2, then the payoff to any player not delegating is reduced or does not change in Γ_2 with respect to Γ_1, and the social payoff, i.e. the sum of the payoffs to all the players, is reduced or does not change in Γ_2 with respect to Γ_1.

Chapter 6

A dynamical model of a two-scale interaction

This chapter is based on the paper *A dynamical model of a two-scale interaction* in collaboration with Mario Bravo.

Abstract. *Assume that a society is composed of a family of populations, where each population consists of two types of individuals: workers and free-riders. A population is characterized by its type, interpreted as the proportion of the workers in it. This paper aims to model a twofold dynamical phenomenon. On the one hand, populations possessing a larger proportion of workers are more efficient in terms of reproduction rate. On the other hand, workers have incentives to become free-riders within each population. A discrete deterministic dynamic model on the distribution of types over the society is proposed. The main result is that, under some natural assumptions, the discrete dynamics converges to a stationary distribution which need not be a Dirac mass, and the most efficient populations can disappear at this state. Some numerical simulations are also presented to underline the scope of this result. A variant of the original model is studied. It considers mutations so that a very small proportion of free-riders may become workers. Finally, some interesting lines of future research are discussed.*

6.1 Introduction

This work presents a dynamic model of a two-scale interaction in a family of populations characterized by their types. The type of a population is the proportion of the *workers* (or *cooperators*) within it. The nonworkers are called *free-riders*. The workers contribute to the growth of a population, while the free-riders do nothing but exploit these ones' production. Therefore, the fitness of a population, namely, the reproduction rate of its members, depends on its type.

The first interaction is a local one, which takes place within a population. As the free-riders profit from the workers' production all by doing nothing, their net gain is greater. Therefore, either by a strategic interpretation that workers will imitate the free-riders' behavior, or by an evolutionary biological interpretation that free-riders have a higher fitness and thus a higher reproduction rate, the proportion of the workers in the population will reduce and, consequently the fitness of the whole population will drop.

The second interaction is a global one, which takes place among the different populations. Populations with higher types or, equivalently, with a larger proportion of workers, have greater fitness, and thus reproduce more quickly.

Two interactions of different scales tie up each other. If there was only one population, the free-riders would invade the whole society. If the populations' types, i.e. their compositions, are fixed, the population with the highest type would win the competition between the populations. However, when local and global interactions are combined together, it is not clear which mechanism will get the upper hand.

In biology, a similar phenomenon is studied under the name of *group selection* or, more generally, *multilevel selection* [54, 56, 96–98]. For example, in an ant colony, worker ants work for the survival of the whole population, while the queen ant does not work and just reproduces. However, a population made up of only queen ants cannot survive. Similarly, in a group of meerkats, those who are on guard while the others sleep have a net loss. But the whole group's survival will be in danger if all the meerkats sleep.

Similar phenomenon also exists in economics and sociology. In public good games [31], free-riders always gain more than cooperators in a society, but if the whole society are filled with free-riders, the public good will be exhausted. The formation of coalition is a largely studied topic in economics [12, 13, 35], in cooperative game theory and in the network formation [7, 48, 63], etc.

In this work, this two-scale interaction is studied via a simple deterministic model where type and time are both discrete. Locally, within each population, the fact that free-riders have a higher fitness so that the composition of the population changes accordingly can be interpreted in different ways. For instance, workers imitate the free-riders' behavior or the free-riders have a higher reproduction rate, hence the proportion of the free-riders grows after a renormalization of the population's size. In our model, this is interpreted by a *migration* phenomenon among the populations. Explicitly, within each population, some members leave to join other populations of lower types. The global comparison among the different types is modeled via their fitness by assuming that higher types are better fitted to reproduce, i.e. they have higher reproduction rates.

In nonatomic congestion games [20, 37, 91], this phenomenon appears in the following way. Assume that a fixed proportion of individuals form a coalition. Each of the remaining individuals minimizes her own cost independently. In Wan [91] (*cf.* Chapter 2), two results are established. On the one hand, the average cost to the coalition, the common cost to the individual players, and the total cost to the population, all decrease with the size of the coalition. Thus, the population's efficiency or its competence, characterized by its total cost, increases with the size of the coalition. On the other hand, the individuals' cost is always lower than the average cost to the coalition, whatever the coalition size is. This provides the cooperators an incentive to leave the coalition and become free-riders.

The paper is organized as follows: In Section 6.2.1, we present a dynamic model where the state variable is the distribution of the types over the society. In Section 6.2.2, the existence of rest points in this dynamics, which correspond to *stationary* distributions, is established. Section 6.2.3 shows that the process converges to a stationary distribution under some natural assumptions. To give an idea of this result, suppose that the initial distribution has full support. Then, the dynamics converges to a particular stationary distribution whose support has form $\{1, 2, \ldots, L\}$, where L is determined by the maximal product of a population's growth rate and its proportion of non-migrants. Populations of high type but low product of the growth rate and the proportion of non-migrants may disappear. Numerical simulations are also provided to conform the above result. Section 6.3 is contributed to a variant of the original model with mutations. An individual mutates if she leaves her population to join another one of higher type. It is shown that, if the mutation rates are very small but not zero, the stationary state is unique, and it has full support. In Section 6.4, some qualitative analysis and extensions of the model to the case of a continuum of types are mentioned.

6.2 Basic model

6.2.1 Notation and dynamics

Assume that there are K *types* of individuals, where $K \in \mathbb{N}^*$. Let the set of types be denoted by $\mathcal{K} = \{1, \ldots, K\}$. The individuals of type $k \in \mathcal{K}$ form a population k. The K populations form a *society*.

At instant n, the size of population k is denoted by $\hat{\mu}_n^k$, and the size of the society by $M_n = \sum_{k\in\mathcal{K}} \hat{\mu}_n^k$. The distribution of different types in the society is denoted by a vector $\mu_n = (\mu_n^k)_{k\in\mathcal{K}}$, where $\mu_n^k = \hat{\mu}_n^k / M_n$. Thus, $\mu_n \in \Delta^{K-1}$, the $(K-1)$-dimensional simplex.

The type of a population is characterized by the proportion of the cooperators (or workers) in it, while the remaining members are free-riders. The fitness of the population is strictly increasing in its type, so that the vector of reproduction rates of the populations $\beta = (\beta^k)_{k\in\mathcal{K}}$ is such that $0 < \beta^1 < \cdots < \beta^K$.

The reproduction rate of the society is $\langle \beta, \mu_n \rangle = \sum_{k=1}^K \mu_n^k \beta^k$, where $\langle \cdot\,, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^K$.

For $k \in \mathcal{K}$, denote $\gamma^k = 1 + \beta^k$ and call it the *growth rate* of population k. Denote $\gamma = (\gamma^k)_{k\in\mathcal{K}}$. Then, $1 < \gamma^1 < \cdots < \gamma^K$, and $\langle \gamma, \mu_n \rangle = 1 + \langle \beta, \mu_n \rangle$. One has:

$$\frac{M_{n+1}}{M_n} = \frac{\sum_{k=1}^K \hat{\mu}_n^k (1 + \beta^k)}{M_n} = 1 + \sum_{k=1}^K \frac{\hat{\mu}_n^k}{M_n} \beta^k = 1 + \sum_{k=1}^K \mu_n^k \beta^k = \langle \gamma, \mu_n \rangle.$$

Therefore, the dynamics on the size of the society is

$$M_{n+1} = M_n \langle \gamma, \mu_n \rangle. \tag{6.1}$$

At every instant, for any type $k \geq 2$, there is a proportion $1 - \alpha^k$ of the individuals in population k who change their type to $k-1$ by joining population $k-1$. The remaining α^k of them stay as type k. Let $\alpha^k \in [0, 1]$ be called the *staying rate* of population k. Assume that, the higher the type, the greater the difference between the net gain of a free-rider and a worker within the population. As a result, the staying rate is non-increasing in $k \in \mathcal{K}$, i.e. $\alpha^1 = 1 \geq \alpha^2 \geq \cdots \geq \alpha^K \geq 0$. Denote $\alpha = (\alpha^k)_{k\in\mathcal{K}}$.

One has the following discrete dynamics on the populations' sizes:

$$\begin{cases} \hat{\mu}_{n+1}^k = \hat{\mu}_n^k \, \alpha^k \, (1 + \beta^k) + \hat{\mu}_n^{k+1} (1 - \alpha^{k+1}) \, (1 + \beta^{k+1}), & 1 \leq k \leq K-1, \\ \hat{\mu}_{n+1}^K = \hat{\mu}_n^K \, \alpha^K \, (1 + \beta^K). \end{cases}$$

Together with (6.1), they imply that

$$\begin{cases} \dfrac{\hat{\mu}_{n+1}^k}{M_{n+1}} = \dfrac{\hat{\mu}_n^k \, \alpha^k \, \gamma^k}{M_n \, \langle \gamma, \mu_n \rangle} + \dfrac{\hat{\mu}_n^{k+1} \, (1 - \alpha^{k+1}) \, \gamma^{k+1}}{M_n \, \langle \gamma, \mu_n \rangle}, & 1 \leq k \leq K-1, \\ \dfrac{\hat{\mu}_{n+1}^K}{M_{n+1}} = \dfrac{\hat{\mu}_n^K \, \alpha^K \, \gamma^K}{M_n \, \langle \gamma, \mu_n \rangle}, \end{cases}$$

which yields the following discrete dynamics on the distribution of the types in the society:

$$\begin{cases} \mu_{n+1}^k = \mu_n^k \, \dfrac{\alpha^k \, \gamma^k}{\langle \gamma, \mu_n \rangle} + \mu_n^{k+1} \, \dfrac{(1 - \alpha^{k+1}) \, \gamma^{k+1}}{\langle \gamma, \mu_n \rangle}, & 1 \leq k \leq K-1, \\ \mu_{n+1}^K = \mu_n^K \, \dfrac{\alpha^K \, \gamma^K}{\langle \gamma, \mu_n \rangle}. \end{cases} \tag{6.2}$$

Let us define a mapping $F = (F^k)_{k\in\mathcal{K}}$ from Δ^{K-1} to Δ^{K-1} by

$$F^k(\mu) = \begin{cases} \mu^k \dfrac{\alpha^k \gamma^k}{\langle \gamma,\, \mu\rangle} + \mu^{k+1} \dfrac{(1-\alpha^{k+1})\gamma^{k+1}}{\langle \gamma,\, \mu\rangle}, & \text{if } 1 \le k \le K-1, \\ \mu^K \dfrac{\alpha^K \gamma^K}{\langle \gamma,\, \mu\rangle}, & \text{if } k = K. \end{cases} \tag{6.3}$$

Then, (6.2) can be written as

$$\mu_{n+1} = F(\mu_n). \tag{6.4}$$

Proposition 6.1. *In the dynamical system (6.4), consider a trajectory $\{(\mu_n^k)_{k\in\mathcal{K}}\}_{n\in\mathbb{N}}$. The following hold.*

(1) For all $k \in \mathcal{K}$, if $\mu_t^k > 0$ at some instant $t \in \mathbb{N}^*$, then $\mu_n^k > 0$ for all $n \ge t$.

(2) If there is an $L \le K$ and some $t \in \mathbb{N}^*$ such that $\mu_t^k = 0$ for all type $k \ge L+1$, then $\mu_n^k = 0$ for all $k \ge L+1$ and all $n \ge t$.

(3) If there is an $L \le K$ and some $t \in \mathbb{N}^*$ such that $\mu_t^L > 0$, then $\mu_{t+L-l}^l > 0$ for all $l \le L$.

Remark 6.2. Given a trajectory $\{(\mu_n^k)_{k\in\mathcal{K}}\}_{n\in\mathbb{N}}$ with an initial point $(\mu_0^k)_{k\in\mathcal{K}}$ such that $\mu_0^L > 0$ and $\mu_0^k = 0$ for all type $k \le L$, Proposition 6.1 implies that it is equivalent to study a trajectory $\{(\nu_n^k)_{k\in\mathcal{L}}\}_{n\in\mathbb{N}}$ of the dynamical system

$$\begin{cases} \nu_{n+1}^k = \nu_n^k \dfrac{\alpha^k \gamma^k}{\langle \gamma,\, \nu_n\rangle} + \nu_n^{k+1} \dfrac{(1-\alpha^{k+1})\,\gamma^{k+1}}{\langle \gamma,\, \nu_n\rangle}, & 1 \le k \le L-1, \\ \nu_{n+1}^L = \nu_n^L \dfrac{\alpha^L \gamma^L}{\langle \gamma,\, \nu_n\rangle}, \end{cases}$$

with an initial point $(\nu_0^k)_{k\in\mathcal{L}} = (\mu_L^k)_{k\in\mathcal{L}}$ with full support $\mathcal{L} = \{1, \dots, L\}$.

6.2.2 Stationary states

Definition 6.3. A vector $\mu \in \Delta^{K-1}$ is a *stationary distribution* of K types in the society if μ is a rest point of the dynamical system (6.4) or, equivalently, if μ is a fixed point of mapping F.

A society with a stationary distribution of types is said to be at a *stationary* state.

Suppose that $\mu = (\mu^k)_{k\in\mathcal{K}}$ is a fixed point of F. Then, by (6.2),

$$\begin{cases} \mu^k = \mu^k \dfrac{\alpha^k \gamma^k}{\langle \gamma,\, \mu\rangle} + \mu^{k+1} \dfrac{(1-\alpha^{k+1})\,\gamma^{k+1}}{\langle \gamma,\, \mu\rangle}, & 1 \le k \le K-1, \\ \mu^K = \mu^K \dfrac{\alpha^K \gamma^K}{\langle \gamma,\, \mu\rangle}, \end{cases}$$

which yields

$$\mu^k\,(\,\langle \gamma,\, \mu\rangle - \alpha^k\,\gamma^k\,) \;=\; \mu^{k+1}\,(1-\alpha^{k+1})\,\gamma^{k+1}, \qquad 1 \le k \le K-1 \tag{6.5a}$$

$$\mu^K\,(\,\langle \gamma,\, \mu\rangle - \alpha^K\,\gamma^K\,) \;=\; 0. \tag{6.5b}$$

Remark 6.4. There always exists a (trivial) fixed point $\bar{\mu}$ of F, where $\bar{\mu}^1 = 1$ and $\bar{\mu}^k = 0$ for $k \ge 2$.

The relation (6.5a) implies that if μ is a stationary distribution and $\mu^l = 0$ for some type l, then $\mu^k = 0$ for all type $k \geq l$. Therefore, $\mu^1 \neq 0$, and one deduces the following.

Proposition 6.5. *A fixed point μ of F must have a support of the form $\{1, 2, \ldots, L\}$ for some $L \in \mathcal{K}$.*

Proposition 6.6. *The distribution $\mu \in \Delta^{K-1}$ is a fixed point of F with support $\{1, 2, \ldots, L\}$ for some $L \in \mathcal{K}$ if, and only if, μ is a solution to the following equations:*

$$\mu^k = \mu^{k+1} q^k, \qquad 1 \leq k \leq L-1 \tag{6.6a}$$
$$\mu^k > 0, \qquad 1 \leq k \leq L \tag{6.6b}$$
$$\mu^k = 0, \qquad \text{if } L < K,\ L+1 \leq k \leq K \tag{6.6c}$$
$$\langle \gamma, \mu \rangle = \alpha^L \gamma^L, \tag{6.6d}$$

where

$$q^k = \frac{(1 - \alpha^{k+1})\, \gamma^{k+1}}{\langle \gamma, \mu \rangle - \alpha^k \gamma^k}, \qquad 1 \leq k \leq L-1. \tag{6.7}$$

Proof. It is clear that a solution to the equations (6.6) is a fixed point of F with support $\{1, 2, \ldots, L\}$.

For the converse result, apply the equation (6.5a) to L, and one gets $\mu^L \cdot \frac{\langle \gamma, \mu \rangle - \alpha^L \gamma^L}{(1-\alpha^{L+1})\, \gamma^{L+1}} = 0$, which implies that $\langle \gamma, \mu \rangle = \alpha^L \gamma^L$. One concludes by combining the above results with (6.5a). $\square$

The following corollary is a direct consequence of Proposition 6.6. It provides a natural necessary condition for the existence of stationary distributions and a uniqueness property.

Corollary 6.7.

(1) If F has a non-trivial fixed point μ with support $\{1, 2, \ldots, L\}$, where $L \geq 2$, then,

$$\alpha^k \gamma^k < \langle \gamma, \mu \rangle = \alpha^L \gamma^L, \qquad 1 \leq k \leq L-1. \tag{6.8}$$

(2) For each $L \in \mathcal{K}$, if F has a fixed point with support $\{1, 2, \ldots, L\}$, then it is the unique one with this support. In particular, F has at most K fixed points.

Proof. (1) Equations (6.6a) and (6.6b) imply that $q^k > 0$ for $k \leq L-1$. Then, (6.8) follows from the definition of q^k's in (6.7).

(2) If two vectors μ and η are both fixed points of F with support $\{1, 2, \ldots, L\}$, then it follows from Proposition 6.6 that $\mu^k/\mu^{k+1} = \eta^k/\eta^{k+1} = q^k$ for $k \leq L-1$, where q^k's are defined by (6.7). Then, one must have $\mu = \eta$ because $\sum_{k=1}^L \mu^k = \sum_{k=1}^L \eta^k = 1$. $\square$

Remark 6.8. According to Corollary 6.7 and the fact that $\alpha^1 = 1$, if $\alpha^k \gamma^k \leq \gamma^1$ for all $k \geq 2$,then F has a unique fixed point which is the trivial one.

When condition (6.8) is satisfied, the explicit form of the fixed point of F with support $\{1, 2, \ldots, L\}$ is provided by the following proposition.

Proposition 6.9. *Suppose that $2 \leq L \leq K$ and*

$$\alpha^k \gamma^k < \alpha^L \gamma^L, \qquad 1 \leq k \leq L-1. \tag{6.9}$$

Let μ be a vector in Δ^{K-1} such that

$$\mu^k = \begin{cases} \mu^L \rho^k, & 1 \le k \le L-1, \\ 0, & L+1 \le k \le K, \end{cases}$$

where

$$\rho^L = 1, \tag{6.10a}$$
$$\rho^k = \rho^{k+1} q^k, \qquad 1 \le k \le L-1, \tag{6.10b}$$
$$q^k = \frac{(1-\alpha^{k+1})\,\gamma^{k+1}}{\alpha^L\,\gamma^L - \alpha^k\,\gamma^k}, \qquad 1 \le k \le L-1. \tag{6.10c}$$

Then, μ is a fixed point of F with support $\{1, 2, \dots, L\}$.

Proof. It is enough to show that μ is well-defined and, for $k \le L$, $\mu_k > 0$. The fact that it is a rest point follows immediately from Proposition 6.6.

For $k \le L$, condition (6.9) ensures that $q^k > 0$, thus $\rho^k > 0$.

By definition, $\mu^k = \mu^L \rho^k$ for all $k \le L$, thus $1 = \sum_{k=1}^K \mu^k = \mu^L\,(\rho^1 + \cdots + \rho^{L-1} + 1)$ because μ is in Δ^{K-1}. Consequently, $\mu^L = \left(\rho^1 + \cdots + \rho^{L-1} + 1\right)^{-1}$. But $\mu^k = \mu^L \rho^k$ for $k \le L$ and $\mu^k = 0$ for $k \ge L+1$, this determines the unique rest point of F with support $\{1, 2, \dots, L\}$. □

For any non-trivial fixed point μ, the fact that $\gamma^1 < \cdots < \gamma^L$ implies that there must be a type $l \ge 2$ such that the weighted average value $\sum_{k \in \mathcal{K}} \mu^k \gamma^k$ is situated between γ^{l-1} and γ^l, i.e. $\gamma^k \ge \langle \gamma, \mu \rangle$ for all $k \ge l$ and $\gamma^k < \langle \gamma, \mu \rangle$ for all $k < l$. This provides a clue on the shape of a fixed point.

Proposition 6.10. *Suppose that μ is a non-trivial fixed point of F with support $\{1, 2, \dots, L\}$, where $L \ge 2$. Let l be such that $\gamma^k \ge \langle \gamma, \mu \rangle$ for all $k \ge l$ and $\gamma^k < \langle \gamma, \mu \rangle$ for all $k < l$. Then, for $k \ge l$, $\mu^{k+1} < \mu^k$, while, for $k \le l-1$,*

$$\begin{cases} \mu^{k+1} > \mu^k, & \textit{if } \alpha^k\,\gamma^k + (1-\alpha^{k+1})\,\gamma^{k+1} < \langle \gamma, \mu \rangle, \\ \mu^{k+1} < \mu^k, & \textit{if } \alpha^k\,\gamma^k + (1-\alpha^{k+1})\,\gamma^{k+1} > \langle \gamma, \mu \rangle, \\ \mu^{k+1} = \mu^k, & \textit{if } \alpha^k\,\gamma^k + (1-\alpha^{k+1})\,\gamma^{k+1} = \langle \gamma, \mu \rangle. \end{cases}$$

Proof. For $k \ge l$, according to (6.7),

$$\frac{\mu^{k+1}}{\mu^k} = \frac{1}{q^k} = \frac{\langle \gamma, \mu \rangle - \alpha^k\,\gamma^k}{(1-\alpha^{k+1})\,\gamma^{k+1}} \le \frac{(1-\alpha^k)\,\gamma^k}{(1-\alpha^{k+1})\,\gamma^{k+1}} < 1.$$

This means that, for all the types greater than l, the proportion of a type in the society decreases with its value. The proof for $k < l$ is similar. □

6.2.3 Convergence to stationary distributions

This section is dedicated to the convergence of the dynamical system (6.4). Proposition 6.11 below shows that not all the stationary distributions can be the limit of such a process.

Proposition 6.11. *In the dynamical system* (6.4), *if a trajectory* $\{\mu_n\}_{n\in\mathbb{N}}$ *with initial state* $(\mu_0^k)_{k\in\mathcal{K}}$ *such that* $\mu_0^{L_0} > 0$ *for some* $L_0 \geq L$ *converges to a fixed point* μ *of* F *with support* $\{1, 2, \dots, L\}$, *then*

$$\alpha^k \gamma^k < \alpha^L \gamma^L, \qquad 1 \leq k \leq L \tag{6.11a}$$
$$\alpha^k \gamma^k \leq \alpha^L \gamma^L, \qquad L+1 \leq k \leq L_0. \tag{6.11b}$$

Proof. The trajectory $\{\mu_n\}_{n\in\mathbb{N}}$ converges to μ, thus, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}^*$ such that, for all $n > N$, $|\mu - \mu_n| < \epsilon$ and, consequently, $|\langle\gamma, \mu\rangle - \langle\gamma, \mu_n\rangle| < \epsilon$.

For all type $k \in \{L+1, \dots, L_0\}$, according to (6.2), for all $n > N$,

$$\mu_{n+1}^k \geq \mu_n^k \cdot \frac{\alpha^k \gamma^k}{\langle\gamma, \mu_n\rangle} > \mu_n^k \cdot \frac{\alpha^k \gamma^k}{\langle\gamma, \mu\rangle + \epsilon}.$$

If $\alpha^k \gamma^k > \langle\gamma, \mu\rangle$, one can take an ϵ such that $\frac{\alpha^k \gamma^k}{\langle\gamma, \mu\rangle+\epsilon} > 1$ so that $\mu_n^k \to +\infty$ as $n \to +\infty$. This contradicts the fact that μ_n^k tends to 0. Therefore, $\alpha^k \gamma^k \leq \langle\gamma, \mu\rangle$. One concludes by combining this result with Proposition 6.1 and condition (6.8). □

The following theorem is the main result of this section. It shows that dynamics (6.4) converges, and its limit distribution combines, in a non trivial way, the two interactions at different scales.

Theorem 6.12. *In the dynamical system* (6.4), *suppose that* $1 \leq L_0 \leq K$. *If there is a type* L *such that* $L \leq L_0$ *and*

$$\alpha^L \gamma^L > \alpha^k \gamma^k, \qquad 1 \leq k \leq L_0,\ k \neq L, \tag{6.12}$$

then a trajectory $\{\mu_n\}_{n\in\mathbb{N}}$ *with initial state* $(\mu_0^k)_{k\in K}$ *such that* $\mu_0^{L_0} > 0$ *and* $\mu_0^k = 0$ *for all* $k \geq L_0$ *converges to a fixed point* μ *of* F *with support* $\{1, 2, \dots, L\}$.

Proof. See the appendix. □

6.2.4 Numerical example

Let us see a simple numerical example that underlines the scope of the above result. Assume that there are $K = 50$ types. The reproduction rates β and the staying rates α are given by the following formulae:

$$\begin{cases} \beta^k &= \beta^0 + \dfrac{(k-1)^{0.8}}{K}\beta, \\ \alpha^k &= 1 - \dfrac{(k-1)^2}{K^2}\alpha, \end{cases} \qquad 1 \leq k \leq 50\,,$$

where $\alpha = 0.8, \beta = 2$ and $\beta^0 = 0.3$. Figure 6.1 presents the shape of the vector $u = (u^k)_{k\in\mathcal{K}}$, where $u^k = \alpha^k(1+\beta^k) = \alpha^k\gamma^k$. The maximum of u is attained by $k = 27$.

Given that the limit value of the process strongly depends on the initial state, two simple cases are considered here. In Figure 6.2, the initial state μ_0 is taken to be the uniform distribution, i.e. $\mu_0 = (1/K, \dots, 1/K)$. In Figure 6.3, the case where $\tilde{\mu}_0 = (0, 0, 0, 0, 0, 1/5, 1/5, 1, 5, 1/5, 1/5, 0, \dots, 0)$ is presented.

Observe that, when the initial condition is μ_0, the limit distribution μ has support $\{1, \dots, 27\}$ even if the last coordinates of μ are very small with respect to the others. On the other hand, when the initial condition $\tilde{\mu}_0$ is considered, the support of the limit distribution is $\{1, \dots, 10\}$ because $u^1 < \dots < u^{10}$ and $\tilde{\mu}_0^k = 0$ for $k > 10$.

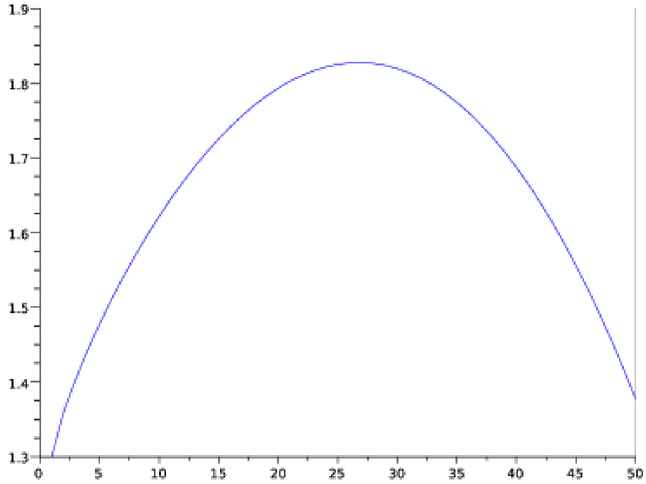


Figure 6.1: The vector of total growth rates $\alpha_k\gamma_k$.

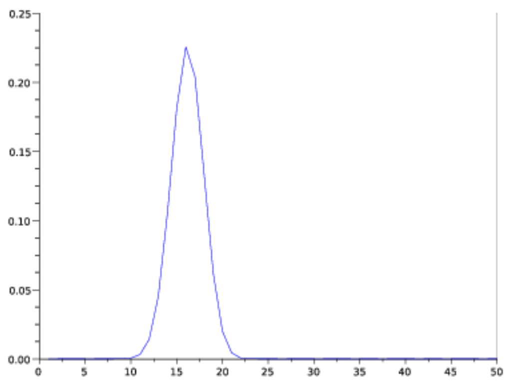


Figure 6.2: Uniform initial condition μ_0

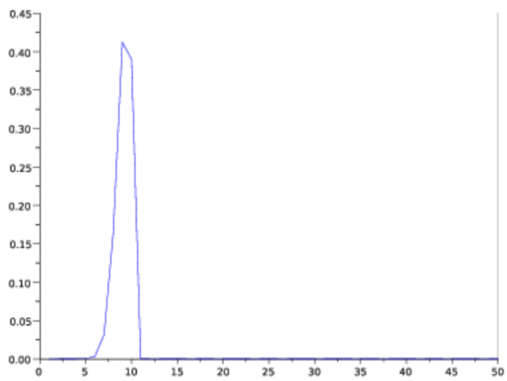


Figure 6.3: Initial condition $\tilde{\mu}_0$.

6.3 Mutations

6.3.1 Model with mutation

In this section, the rare event that a free-rider becomes a cooperator by mutation is taken into account.

Assume that, at every instant n, in all population $k \leq K-1$, a proportion ϵ^k of the individuals mutate by adopting type $k+1$, where $0 < \epsilon^k < 1$. Let ϵ^K be equal to 0. Among the remaining $1-\epsilon^k$ nonmutants, as before, a proportion α^k of them stay as type k, while the others leave to join population $k-1$. Still, one assumes that $1 = \alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^K$. The other notations remain unchanged.

Then, the discrete dynamics on the populations' sizes is

$$\begin{cases} \hat{\mu}^1_{n+1} = \hat{\mu}^1_n \,(1-\epsilon^1)\,\alpha^1\,\gamma^1 + \hat{\mu}^2_n\,(1-\epsilon^2)\,(1-\alpha^2)\,\gamma^2, \\ \hat{\mu}^k_{n+1} = \hat{\mu}^k_n\,(1-\epsilon^k)\,\alpha^k\,\gamma^k + \hat{\mu}^{k+1}_n\,(1-\epsilon^{k+1})\,(1-\alpha^{k+1})\,\gamma^{k+1} + \hat{\mu}^{k-1}_n\,\epsilon^{k-1}\,\gamma^{k-1}, \\ \hfill 2 \leq k \leq K-1, \\ \hat{\mu}^K_{n+1} = \hat{\mu}^K_n\,(1-\epsilon^K)\,\alpha^K\,\gamma^K + \hat{\mu}^{K-1}_n\,\epsilon^{K-1}\,\gamma^{K-1}. \end{cases} \tag{6.13}$$

They entail the following discrete dynamics on the populations' proportions in the society.

$$\mu_{n+1} = \tilde{F}(\mu_n), \tag{6.14}$$

where the mapping $\tilde{F}$ is defined from Δ^{K-1} to Δ^{K-1} by

$$\tilde{F}^1(\mu) = \begin{cases} \mu^1\,\dfrac{(1-\epsilon^1)\,\alpha^1\,\gamma^1}{\langle\gamma,\,\mu\rangle} + \mu^2\,\dfrac{(1-\epsilon^2)\,(1-\alpha^2)\,\gamma^2}{\langle\gamma,\,\mu\rangle}, \\ \mu^k\,\dfrac{(1-\epsilon^k)\,\alpha^k\,\gamma^k}{\langle\gamma,\,\mu\rangle} + \mu^{k+1}\,\dfrac{(1-\epsilon^{k+1})\,(1-\alpha^{k+1})\,\gamma^{k+1}}{\langle\gamma,\,\mu\rangle} + \mu^{k-1}\,\dfrac{\epsilon^{k-1}\,\gamma^{k-1}}{\langle\gamma,\,\mu\rangle}, \\ \hfill 2 \leq k \leq K-1, \\ \mu^K\,\dfrac{(1-\epsilon^K)\,\alpha^K\,\gamma^K}{\langle\gamma,\,\mu\rangle} + \mu^{K-1}\,\dfrac{\epsilon^{K-1}\,\gamma^{K-1}}{\langle\gamma,\,\mu\rangle}. \end{cases} \tag{6.15}$$

A fixed point $\mu = (\mu^k)_{k\in\mathcal{K}}$ of $\tilde{F}$ or, equivalently, a stationary state of the model with mutation meets the following conditions:

$$\begin{cases} \mu^1\,[\langle\gamma,\,\mu\rangle - (1-\epsilon^1)\,\alpha^1\,\gamma^1] - \mu^2\,(1-\epsilon^2)\,(1-\alpha^2)\,\gamma^2 &= 0, \\ -\mu^{k-1}\,\epsilon^{k-1}\,\gamma^{k-1} + \mu^k\,[\langle\gamma,\,\mu\rangle - (1-\epsilon^k)\,\alpha^k\,\gamma^k] & \\ \qquad - \mu^{k+1}\,(1-\epsilon^{k+1})\,(1-\alpha^{k+1})\,\gamma^{k+1} &= 0, \quad 2 \leq k \leq K-1, \\ -\mu^{K-1}\,\epsilon^{K-1}\,\gamma^{K-1} + \mu^K\,[\,\langle\gamma,\,\mu\rangle - (1-\epsilon^K)\,\alpha^K\,\gamma^K\,] &= 0. \end{cases} \tag{6.16}$$

6.3.2 Some properties of stationary states

This part provides some properties of a fixed point of $\tilde{F}$.

Proposition 6.13. *$\tilde{F}$ has a fixed point $\mu \in \Delta^{K-1}$.*
In addition, all the fixed point of $\tilde{F}$ have full support.

Proof. Function $\tilde{F} : \Delta^{K-1} \to \Delta^{K-1}$ is continuous, hence the first statement follows immediately from Brower's fixed point theorem. The second statement is clear because of (6.16). □

Proposition 6.14. *Suppose that $\mu = (\mu^1, \dots, \mu^K)$ is a fixed point of $\tilde{F}$ in Δ^{K-1}, and*

$$\begin{aligned}
&a_{k,k} = \langle \gamma, \mu \rangle - (1-\epsilon^k)\,\alpha^k\,\gamma^k, && 1 \le k \le K;\\
&a_{k-1,k} = -(1-\epsilon^k)\,(1-\alpha^k)\,\gamma^k, && 2 \le k \le K;\\
&a_{k,k-1} = -\epsilon^{k-1}\,\gamma^{k-1}, && 2 \le k \le K;\\
&\tilde{a}_{1,1} = a_{1,1};\\
&\tilde{a}_{k,k} = \begin{cases} a_{k,k} - \dfrac{a_{k-1,k}\,a_{k,k-1}}{\tilde{a}_{k-1,k-1}}, & \text{if } \tilde{a}_{k-1,k-1} \neq 0,\\ 0, & \text{if } \tilde{a}_{k-1,k-1} = 0, \end{cases} && 2 \le k \le K.
\end{aligned}$$

Then, one has:

(1) $\langle \gamma, \mu \rangle > \max_{k\in\mathcal{K}}(1-\epsilon^k)\,\alpha^k\,\gamma^k$;

(2) for all $k \le K-1$, $\tilde{a}_{k,k} > 0$, and $\mu^k = -\mu^{k+1}\frac{a_{k,k+1}}{\tilde{a}_{k,k}}$;

(3) $\tilde{a}_{K,K} = 0$.

Proof. (1) The conditions in (6.16) can be rewritten as

$$a_{1,1}\mu^1 + a_{1,2}\mu^2 = 0, \tag{6.17}$$
$$a_{k,k-1}\mu^{k-1} + a_{k,k}\mu^k + a_{k,k+1}\mu^{k+1} = 0, \quad k = 2, \dots, K-1, \tag{6.18}$$
$$a_{K,K-1}\mu^{K-1} + a_{K,K}\mu^K = 0. \tag{6.19}$$

By definition, for all $k \ge 2$, $a_{k-1,k} < 0$ and $a_{k,k-1} < 0$. Proposition 6.13 states that $\mu^k > 0$ for all k. Then, (6.17)-(6.19) imply that $a_{1,1} = -a_{1,2}\frac{\mu^2}{\mu^1} > 0$, $a_{k,k} = -a_{k,k-1}\frac{\mu^{k-1}}{\mu^k} - a_{k,k+1}\frac{\mu^{k+1}}{\mu^k} > 0$ for all $k \in \{2, \dots, K-1\}$, and $a_{K,K} = -a_{K,K-1}\frac{\mu^{K-1}}{\mu^K} > 0$.

Hence, $\langle \gamma, \mu \rangle - (1-\epsilon^k)\,\alpha^k\,\gamma^k = a_{k,k} > 0$ for all k and, consequently, $\langle \gamma, \mu \rangle > \max_{k\in\mathcal{K}}(1-\epsilon^k)\,\alpha^k\,\gamma^k$.

(2) The proof is made by induction.

Equation (6.17) implies that $\mu^1 = -\mu^2\frac{a_{1,2}}{\tilde{a}_{1,1}}$.

Suppose that, for some $k \le K-1$, one has shown that $\mu^{k-1} = -\mu^k\frac{a_{k-1,k}}{\tilde{a}_{k-1,k-1}}$ and $\tilde{a}_{k-1,k-1} > 0$. Then, according to (6.18), $(a_{k,k} - \frac{a_{k-1,k}\,a_{k,k-1}}{\tilde{a}_{k-1,k-1}})\mu^k + a_{k,k+1}\mu^{k+1} = 0$ or, equivalently,

$$\tilde{a}_{k,k}\,\mu^k + a_{k,k+1}\mu^{k+1} = 0. \tag{6.20}$$

In (6.20), $\mu^k > 0, \mu^{k+1} > 0$ and $a_{k,k+1} < 0$. As a result, $\tilde{a}_{k,k} > 0$ and $\mu^k = -\mu^{k+1}\frac{a_{k,k+1}}{\tilde{a}_{k,k}}$.

In this way, one has proved that $\mu^k = -\mu^{k+1}\frac{a_{k,k+1}}{\tilde{a}_{k,k}}$ and $\tilde{a}_{k,k} > 0$ for $k \le K-1$.

(3) By replacing μ^{K-1} by $-\mu^K\frac{a_{K-1,K}}{\tilde{a}_{K-1,K-1}}$ in (6.19), one has $(a_{K,K} - \frac{a_{K-1,K}\,a_{K,K-1}}{\tilde{a}_{K-1,K-1}})\mu^K = 0$ or, equivalently, $\tilde{a}_{K,K}\mu^K = 0$. But $\mu^K > 0$, hence $\tilde{a}_{K,K} = 0$. □

6.3.3 Uniqueness of the stationary states

In order to study the uniqueness of the fixed points of $\tilde{F}$, one needs some auxiliary functions.

Suppose that μ is a fixed point of $\tilde{F}$. Notice that it exists according to Proposition 6.13.

First, let us define a $K \times K$ tridiagonal matrix function $A(x) = \big(a_{ij}(x)\big)_{1\le i,j\le K}$ on $\mathbb{R}$ by

$$\begin{cases} a_{k,k}(x) = x - (1-\epsilon^k)\,\alpha^k\,\gamma^k, & 1 \le k \le K, \\ a_{k-1,k}(x) = a_{k-1,k}, & 2 \le k \le K, \\ a_{k,k-1}(x) = a_{k,k-1}, & 2 \le k \le K, \\ a_{i,j}(x) = 0, & (i,j) \ne (k,k) \text{ or } (k-1,k) \text{ or } (k,k-1), \end{cases} \tag{6.21}$$

where $a_{k-1,k}$'s and $a_{k,k-1}$'s are those defined in Proposition 6.14.

For $x \in \mathbb{R}$, define $b_{1,1}(x) = a_{1,1}(x)$ and, for $k = 2, \ldots, K$,

$$b_{k,k}(x) = \begin{cases} a_{k,k}(x) - \dfrac{a_{k-1,k}\,a_{k,k-1}}{b_{k-1,k-1}(x)}, & \text{if } b_{k-1,k-1}(x) \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\mathcal{R} = \{x \in \mathbb{R} \,|\, b_{k,k}(x) \ne 0, \text{ for } k = 1, \ldots, K-1\}$. It is not empty because, according to Proposition 6.14, $\langle\gamma,\mu\rangle$ belongs to $\mathcal{R}$.

Next, let us define a $K \times K$-dimensional upper bidiagonal matrix function $\tilde{A}(x) = \big(\tilde{a}_{ij}(x)\big)_{1\le i,j\le K}$ on $\mathcal{R}$ in the following way.

$$\begin{cases} \tilde{a}_{k,k}(x) = b_{k,k}(x), & 1 \le k \le K, \\ \tilde{a}_{k-1,k}(x) = a_{k-1,k}, & 2 \le k \le K, \\ \tilde{a}_{i,j}(x) = 0, & (i,j) \ne (k,k) \text{ or } (k-1,k). \end{cases} \tag{6.22}$$

Remark 6.15. For all $x \in \mathcal{R}$, $\tilde{A}(x)$ is obtained from $A(x)$ by a series of element transformation. Hence, there exists a $K \times K$ invertible matrix $P(x)$ such that $\tilde{A}(x) = P(x)A(x)$.

According to (6.17)-(6.19), $A(\langle\gamma,\,\mu\rangle)\,\mu = 0$. Because of the previous statement, $\tilde{A}(\langle\gamma,\,\mu\rangle)\,\mu = 0$.

Finally, let us define K real-valued functions $h_1, \ldots, h_K$ on the closed interval $[0, \gamma^K]$ as follows:

$$h_1(x) = \big[\,x - (1-\epsilon^1)\,\alpha^1\,\gamma^1\,\big]^+, \tag{6.23}$$

where the notation x^+ stands for $\max\{x, 0\}$; for $k = 2, \ldots, K$,

$$h_k(x) = \begin{cases} \left[\,x - (1-\epsilon^k)\,\alpha^k\,\gamma^k - \dfrac{a_{k-1,k}\,a_{k,k-1}}{h_{k-1}(x)}\,\right]^+, & \text{if } h_{k-1}(x) > 0, \\ 0, & \text{if } h_{k-1}(x) = 0. \end{cases} \tag{6.24}$$

Define a constant

$$\bar{\pi} = \max_{1\le k\le K}(1-\epsilon^k)\,\alpha^k\,\gamma^k + 2\max_{2\le k\le K}\big[\,(1-\epsilon^k)(1-\alpha^k)\,\gamma^k\,\epsilon^{k-1}\,\gamma^{k-1}\,\big]^{1/2}. \tag{6.25}$$

Let us make the following assumption which holds as long as the mutation rates $(\epsilon^k)_{k\in\mathcal{K}}$ are sufficiently small.

A 6.16. $\bar{\pi} < \gamma^K$.

Lemma 6.17. *Suppose that A6.16 holds. Then, there exists K numbers $\pi_1, \ldots, \pi_K$ such that $0 \le \pi_1 < \pi_2 < \cdots < \pi_K < \bar{\pi}$ and, for all $k \in K$, the following three conditions of h_k hold:*

(1) $h_k = 0$ *on* $[0, \pi_k]$*;*

(2) h_k *is strictly positive and strictly increasing on* $(\pi_k, \gamma^K]$*;*

(3) h_k *is continuous on* $[0, \gamma^K]$*.*

Proof. The proof is made by induction.

Let π_1 be equal to $(1-\epsilon^1)\,\alpha^1\,\gamma^1$. Clearly, $\pi_1 < \bar\pi$. It is easy to see that, firstly, $h_1(x) = 0$ on $[0, \pi_1]$; secondly, $h_1(x) = x - (1-\epsilon^1)\,\alpha^1\,\gamma^1 > 0$ and it is strictly increasing on $(\pi_1, \gamma^K]$; thirdly, h_1 is continuous on $[0, \gamma^K]$.

For later use, define a real-valued function $\tilde h_1(x) = x - (1-\epsilon^1)\,\alpha^1\,\gamma^1$ on $[0, \gamma^K]$.

Now, suppose that, for some $k \ge 2$, $k-1$ numbers $\pi_1, \dots, \pi_{k-1}$ such that $0 \le \pi_1 < \pi_2 < \cdots < \pi_{k-1} < \bar\pi$ are found, and the three conditions of h_l hold for all $l \le k-1$. Let us find π_k such that $\pi_{k-1} < \pi_k < \bar\pi$ and the three conditions hold for h_k.

By hypothesis, h_{k-1} is continuous in a neighborhood of π_{k-1}, hence $\lim_{x\to\pi_{k-1}+} h_{k-1}(x) = h_{k-1}(\pi_{k-1}) = 0$ and, consequently, $\lim_{x\to\pi_{k-1}+} -\frac{1}{h_{k-1}(x)} = -\infty$. Still by hypothesis, h_{k-1} is continuous, strictly positive and strictly increasing on the interval $(\pi_{k-1}, \gamma^K]$, hence $-\frac{1}{h_{k-1}(x)}$ is continuous, strictly negative and strictly increasing on $(\pi_{k-1}, \gamma^K]$.

Let us define a real-valued function $\tilde h_k$ on $(\pi_{k-1}, \gamma^K]$ by

$$\tilde h_k(x) = x - (1-\epsilon^k)\,\alpha^k\,\gamma^k - \frac{a_{k-1,k}\,a_{k,k-1}}{h_{k-1}(x)}. \tag{6.26}$$

On account of the results in the previous paragraph, $\tilde h_k$ is continuous and strictly increasing on $(\pi_{k-1}, \gamma^K]$, and

$$\lim_{x\to\pi_{k-1}+} \tilde h_k(x) = -\infty. \tag{6.27}$$

By definition, $h_k \equiv 0$ on the interval $[0, \pi_{k-1}]$ and $h_k = [\tilde h_k(x)]^+$ on $(\pi_{k-1}, \gamma^K]$. Therefore, according to the properties of the function $\tilde h_k$, one has only to show that there exists π_k such that $\pi_{k-1} < \pi_k < \bar\pi$ and $\tilde h_k(\pi_k) = 0$. Indeed, if this is true, then, for all x in the interval (π_{k-1}, π_k), $\tilde h_k(x) < 0$, thus $h_k(x) = [\tilde h_k(x)]^+ = 0$, while $h_k = \tilde h_k \ge 0$ on $[\pi_k, \gamma^K]$, and the equality holds only in π_k.

In order to prove the existence of π_k, it is enough to show that $\tilde h_k(\bar\pi) > 0$. Indeed, when this is proved, by considering (6.27) and the fact that $\tilde h_k$ is strictly increasing on $(\pi_{k-1}, \gamma^K]$, one obtains immediately the existence of a unique $\pi_k \in (\pi_{k-1}, \bar\pi)$ such that $\tilde h_k(\pi_k) = 0$.

Denote $\xi = 2\max_{2\le k\le K}\left[(1-\epsilon^k)(1-\alpha^k)\,\gamma^k\,\epsilon^{k-1}\,\gamma^{k-1}\right]^{1/2}$. Let us show, by induction, that $\tilde h_l(\bar\pi) > \xi/2$ for $l \le k$.

First, $\tilde h_1(\bar\pi) = \bar\pi - (1-\epsilon^1)\,\alpha^1\,\gamma^1 > \xi$ by the definition of $\bar\pi$.

Suppose that $\tilde h_{l-1}(\bar\pi) > \xi/2$ for some $l \in \{2, \dots, k\}$. Then,

$$h_{l-1}(\bar\pi) = [\tilde h_{l-1}(\bar\pi)]^+ = \tilde h_{l-1}(\bar\pi) > \xi/2, \tag{6.28}$$

and

$$\tilde h_l(\bar\pi) = \bar\pi - (1-\epsilon^l)\,\alpha^l\,\gamma^l - \frac{a_{l-1,l}\,a_{l,l-1}}{h_{l-1}(\bar\pi)} > \xi - \frac{(1-\epsilon^l)(1-\alpha^l)\,\gamma^l\,\epsilon^{l-1}\,\gamma^{l-1}}{h_{l-1}(\bar\pi)} > \xi - \frac{(\xi/2)^2}{\xi/2} = \xi/2,$$

where the first inequality is due to the definitions of $\bar\pi$ and ξ, and the second one is due to the definition of ξ and (6.28).

In this way, one has proved that $\tilde h_k(\bar\pi) > \xi/2 > 0$. $\square$

Corollary 6.18. *Suppose that A6.16 holds. Then,* *(1)* $h_k(\pi_K) > 0$ *for* $k \le K-1$*, and* $h_K(\pi_K) = \tilde{h}_K(\pi_K) = 0$*;*

(2) if, for some $x \in [0, \bar{\pi}) \cup (\bar{\pi}, \gamma^K]$*,* $h_k(x) > 0$ *for* $k \le K-1$ *and* $h_K(x) = 0$*, then* $\tilde{h}_K(x) < 0$*.*

Here π_K *is the real constant given in Lemma 6.17, and* $\tilde{h}_K$ *is the function defined on* $(\pi_{K-1}, \gamma^K]$ *by (6.26).*

Proof. According to Lemma 6.17, $h_K(\pi_K) = \tilde{h}_K(\pi_K) = 0$ and, for $k \le K-1$, $\pi_K > \pi_k$, hence $h_k(\pi_K) > 0$. If $x \in [0, \bar{\pi}) \cup (\bar{\pi}, \gamma^K]$ and $h_k(x) > 0$ for $k \le K-1$, then $x > \pi_{K-1}$ by Lemma 6.17 (2). Thus, $\tilde{h}_K$ is well defined at x. If $h_k(x) = 0$, it follows from the proof of Lemma 6.17 that $x < \pi_K$ and $\tilde{h}_K(x) < 0$. □

One is ready to write, explicitly, a fixed point of $\tilde{F}$ defined by (6.15) and show that this is the unique one.

Theorem 6.19. *Suppose that A6.16 holds. Then,* $\tilde{F}$ *has a unique fixed point in* Δ^{K-1}*.*

Proof. Suppose that $\mu \in \Delta^{K-1}$ is a fixed point of (6.15). Two matrix functions $A(x)$ and $\tilde{A}(x)$ are defined by (6.21) and (6.22), and K real-valued functions $h_1, \dots, h_K$ are defined by (6.23) and (6.24).

According to Proposition 6.14,

$$A(\langle\gamma,\, \mu\rangle)\mu = 0 \ \text{ or, equivalently, } \ \tilde{A}(\langle\gamma,\, \mu\rangle)\mu = 0, \tag{6.29}$$

and $\tilde{a}_{k,k}(\langle\gamma,\, \mu\rangle) > 0$ for $k \le K-1$, $\tilde{a}_{K,K}(\langle\gamma,\, \mu\rangle) = 0$. By the definition of the functions h_k's and that of $\tilde{h}_k$'s in Lemma 6.17, this is equivalent to

$$h_k(\langle\gamma,\, \mu\rangle) = \tilde{h}_k(\langle\gamma,\, \mu\rangle) \begin{cases} > 0, & \text{if } 1 \le k \le K-1, \\ = 0, & \text{if } k = K. \end{cases}$$

Corollary 6.18 implies that the only number on $[0, \gamma^K]$ which meets the above conditions is π_K, the one given in Lemma 6.17. Therefore, $\langle\gamma,\, \mu\rangle = \pi_K$. According to (6.29), μ is a solution in Δ^{K-1} to the following system of equations:

$$\begin{cases} \tilde{A}(\pi^K)\, \mathbf{x} = 0, \\ \langle\gamma,\, \mathbf{x}\rangle = \pi_K. \end{cases} \tag{6.30}$$

Let us define a vector $\nu \in \Delta^{K-1}$ and show that it is the unique solution in Δ^{K-1} to (6.30).

Denote $\rho^K = 1$. For $k \le K-1$, denote $q^k = -\frac{a_{k,k+1}}{\tilde{a}_{k,k}(\pi_K)}$, $\rho^k = q^k \rho^{k+1}$. The vector $\nu = (\nu^1, \dots, \nu^K) \in \Delta^{K-1}$ is defined by

$$\mu^k = \begin{cases} \left(\sum_{l=1}^K \rho^l\right)^{-1}, & \text{if } k = K, \\ \mu^K \rho^k, & \text{if } 1 \le k \le K-1. \end{cases}$$

Indeed, it is not difficult to verify that ν is a solution to $\tilde{A}(\pi^K)\, \mathbf{x} = 0$ or, equivalently, to $A(\pi^K)\mathbf{x} = 0$. In other words,

$$\begin{aligned} &a_{1,1}(\pi_K)\mu^1 + a_{1,2}\mu^2 = 0, \\ &a_{k,k-1}\mu^{k-1} + a_{k,k}(\pi_K)\mu^k + a_{k,k+1}\mu^{k+1} = 0, \quad 2 \le k \le K-1, \\ &a_{K,K-1}\mu^{K-1} + a_{K,K}(\pi_K)\mu^K = 0. \end{aligned}$$

By taking the sum of the above K equations and replacing $a_{k,k}(\pi_K)$ by their definitions in (6.21), one obtains $\pi_K = \langle\gamma, \mu\rangle$.

Besides, the fact that $\tilde{a}_{k,k}(\pi^K) > 0$ for $k \leq K-1$ and $\tilde{a}_{K,K}(\pi^K) = 0$ implies that the rank of the upper bidiagonal matrix $\tilde{A}(\pi^K)$ is $K-1$. Therefore, the dimension of its solution space is 1. As a result, ν is the only solution to (6.30) in Δ^{K-1}. □

Finally, one has the following estimation of the average growth rate at the unique stationary state of the society.

Corollary 6.20. *Suppose that A6.16 holds, and $\mu = (\mu^1, \ldots, \mu^K) \in \Delta^{K-1}$ is the unique fixed point of $\tilde{F}$. Then, $\max_{1\leq k\leq K}(1-\epsilon^k)\,\alpha^k\,\gamma^k < \langle\gamma, \mu\rangle < \bar{\pi}$.*

Proof. It follows immediately from Proposition 6.14, Lemma 6.17 and Theorem 6.19. □

6.4 Qualitative analysis and extensions

6.4.1 Critical points in the parameters

In the initial model without mutation, it is shown that the limit point of a trajectory is determined by the parameters α and γ as well as the initial state. The highest type L in the support of the limit state is such that $\alpha^L\gamma^L$ is the largest among all the $\alpha^k\gamma^k$'s for k in the support $\{1, \ldots, L_0\}$ of initial state. In other words, when the initial state is fixed, the support of the limit state is uniquely determined by α and γ. If α and γ evolve in a continuous way (while keeping the increasing or decreasing order of their elements), there are critical points of α and γ where the support of the limit state changes abruptly.

To cite a simple example, let us consider the case where $K = 2$. Theorems 6.11 and 6.12 entail the following result.

Corollary 6.21. *Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a trajectory in the dynamical system (6.2) with $K = 2$. If the initial state μ_0 is such that $\mu_0^2 > 0$, then*

(i) if $\alpha^2 < \gamma^1/\gamma^2$, $\{\mu_n\}_{n\in\mathbb{N}}$ converges to $(1,0)$;

(ii) if $\alpha^2 > \gamma^1/\gamma^2$, $\{\mu_n\}_{n\in\mathbb{N}}$ converges to $\left(\frac{(1-\alpha^2)\,\gamma^2}{\gamma^2-\gamma^1}, \frac{\gamma^2\,\alpha^2-\gamma^1}{\gamma^2-\gamma^1}\right)$.

This corollary shows that, when γ^1 and γ^2 are fixed, γ^1/γ^2 is a critical point for α^2. At this point, the support of the limit changes.

6.4.2 Towards a model with continuous types

Extended model

One can consider a more general model where an individual in population k may leave it to join any population with a type smaller than k, instead of only population $k-1$.

Formally, α^k is the *staying rate* of type k for $k \in \mathcal{K}$, and $1 = \alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^K$. At every instant n, in population k, a proportion α^k of the individuals stay as type k at the next instant $n+1$ and the rest $1-\alpha^k$ of them change their types. Among those who leave, a strictly positive proportion ϕ_l^k join population l, for all type $l \leq k-1$, and $\sum_{l=1}^{k-1}\phi_l^k = 1$. The other notations remain unchanged.

By the same arguments as before, one can show that the discrete dynamical system on the distribution of the types in the society is

$$\mu_{n+1} = G(\mu_n), \tag{6.31}$$

where G is a mapping from Δ^{K-1} to Δ^{K-1} defined by

$$G^k(\mu) = \begin{cases} \mu^k \dfrac{\alpha^k \gamma^k}{\langle \gamma, \mu \rangle} + \displaystyle\sum_{l=k+1}^{K} \mu^l \dfrac{(1-\alpha^l)\phi_k^l \gamma^l}{\langle \gamma, \mu \rangle} & \text{if } 1 \le k \le K-1, \\ \mu^K \dfrac{\alpha^K \gamma^K}{\langle \gamma, \mu \rangle} & \text{if } k = K. \end{cases} \tag{6.32}$$

A fixed point $\mu = (\mu^k)_{k \in \mathcal{K}}$ of G or, equivalently, a stationary state of the generalized model satisfies

$$\mu^k \left(\langle \gamma, \mu \rangle - \alpha^k \gamma^k \right) = \sum_{l=k+1}^{K} \mu^l (1-\alpha^l)\phi_k^l \gamma^l, \qquad 1 \le k \le K-1 \tag{6.33a}$$

$$\mu^K \left(\langle \gamma, \mu \rangle - \alpha^K \gamma^K \right) = 0. \tag{6.33b}$$

The existence of fixed points and the convergence of the dynamical system to a fixed point in this generalized model are analog to those of the original one. They are given here without proof.

Theorem 6.22. *Suppose that $1 \le L \le K$ and $\alpha^k \gamma^k < \alpha^L \gamma^L$ for all $k \le L-1$. Let μ be a vector in Δ^{K-1} such that $\mu^k = \mu^L \rho^k$ for all $k \le L-1$ and $\mu^k = 0$ for all $k \ge L+1$, where $\rho^L = 1$ and, for all $k \le L-1$, $\rho^k = \sum_{l=k+1}^{L} \rho^l q_k^l$, $q_k^l = \frac{\alpha_k^l \gamma^l}{\alpha^L \gamma^L - \alpha^k \gamma^k}$.*

Then, μ is a fixed point of G with support $\{1, 2, \ldots, L\}$, and it is the only rest point with this support.

Theorem 6.23. *In the dynamical system (6.31), suppose that $1 \le L_0 \le K$. If there is a type L such that $L \le L_0$ and*

$$\alpha^k \gamma^k < \alpha^L \gamma^L, \qquad 1 \le k \le L_0,\ k \ne L, \tag{6.34}$$

then a trajectory $\{\mu_n\}_{n \in \mathbb{N}}$ with initial state $(\mu_0^k)_{k \in \mathcal{K}}$ such that $\mu_0^{L_0} > 0$ and $\mu_0^k = 0$ for $k \ge L_0 + 1$ converges to a fixed point μ of G with support $\{1, 2, \ldots, L\}$.

The model with mutations can be equally extended to the case where a mutate individual in population k may join any population instead of the population $k+1$ only.

Continuous types

The above extensions of the discrete model might be the first step towards an extension to a model with continuous types. Explicitly, by discretizing a closed interval of types $[0, 1]$ as well as the reproduction rate function β, the staying rate function α, transition rate functions ϕ and the mutation rate function ϵ on it, one might get a continuous-type dynamical system as the limit of a sequence of discrete-type dynamical systems.

6.5 Appendix

Proof of Theorem 6.12. Without loss of generality, let $\{1, 2, \ldots, L_0\}$ be the support of the initial state $\mu_0 \in \Delta^{K-1}$ (see Remark 6.2). Let $\mu \in \Delta^{K-1}$ be the fixed point of F with support $\{1, 2, \ldots, L\}$, where L is given by (6.12) and $L \le L_0$. Then, $\mu_n^L > 0$ for all $n \in \mathbb{N}^*$.

For all type $k \leq L_0$ and $k \neq L$, denote $\theta^k = \frac{\alpha^k \gamma^k}{\alpha^L \gamma^L}$ and $\rho_n^k = \mu_n^k / \mu_n^L$ for $n \in \mathbb{N}^*$. Notice that $\theta^k < 1$.

The rest of the proof is made up of two parts.

1) This part proves that, if $L < L_0$, then $\rho_n^k \to 0$ for all $k \in \{L+1, \ldots, L_0\}$.

1.1) For L_0,

$$\rho_n^{L_0} = \frac{\mu_n^{L_0}}{\mu_n^L} = \frac{\mu_{n-1}^{L_0}\, \alpha^{L_0}\, \gamma^{L_0}}{\mu_{n-1}^L\, \alpha^L\, \gamma^L + \mu_{n-1}^{L+1}\,(1-\alpha^{L+1})\,\gamma^{L+1}} \leq \frac{\rho_{n-1}^{L_0}\, \alpha^{L_0}\, \gamma^{L_0}}{\alpha^L\, \gamma^L} = \rho_{n-1}^{L_0}\, \theta^{L_0} \leq \rho_0^{L_0}\, (\theta^{L_0})^n.$$

Therefore, $\rho_n^{L_0} \to 0$ as $n \to +\infty$ because $\theta^{L_0} < 1$.

1.2) Suppose that, for type $k \in \{L+1, \ldots, L^0\}$, $\rho_n^k \to 0$ as $n \to +\infty$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\rho_n^k < \epsilon \cdot \frac{\alpha^L \gamma^L}{(1-\alpha^k)\gamma^k} \cdot \frac{1-\theta^{k-1}}{2}$. Consequently, for all $n > N$,

$$\begin{aligned}
\rho_n^{k-1} = \frac{\mu_n^{k-1}}{\mu_n^L} &= \frac{\mu_{n-1}^{k-1}\, \alpha^{k-1}\, \gamma^{k-1} + \mu_{n-1}^k\,(1-\alpha^k)\,\gamma^k}{\mu_{n-1}^L\, \alpha^L\, \gamma^L + \mu_{n-1}^{L+1}\,(1-\alpha^{L+1})\,\gamma^{L+1}} \leq \frac{\rho_{n-1}^{k-1}\, \alpha^{k-1}\, \gamma^{k-1} + \rho_{n-1}^k\,(1-\alpha^k)\,\gamma^k}{\alpha^L\, \gamma^L} \\
&= \rho_{n-1}^{k-1}\, \theta^{k-1} + \rho_{n-1}^k \frac{(1-\alpha^k)\,\gamma^k}{\alpha^L\, \gamma^L} < \rho_{n-1}^{k-1}\, \theta^{k-1} + \frac{\epsilon}{2}\,(1-\theta^{k-1}) \\
&< \rho_N^{k-1}\, (\theta^{k-1})^{n-N} + \frac{\epsilon}{2}\,(1-\theta^{k-1})\,(1+\theta^{k-1}+\cdots+(\theta^{k-1})^{n-N-1}) \\
&< \rho_N^{k-1}\, (\theta^{k-1})^{n-N} + \frac{\epsilon}{2}.
\end{aligned}$$

There exists $N' > N$ such that $\rho_N^{k-1}\,(\theta^{k-1})^{n-N} < \frac{\epsilon}{2}$ for all $n \geq N'$ because $\theta^{k-1} < 1$. Thus, $\rho_n^{k-1} < \epsilon$. This implies that $\rho_n^{k-1} \to 0$ as $n \to +\infty$.

It follows from the fact that $0 < \mu_n^L \leq 1$ and $\rho_n^k = \mu_n^k/\mu_n^L \to 0$ that $\mu_n^k \to 0$ as $n \to +\infty$. By induction, this is the case for all $k \in \{L+1, \ldots, L_0\}$.

2) This part proves that $\rho_n^k \to \rho^k$ for all type $k < L$, where ρ^k's are defined by (6.10a).

2.1) Take a strictly positive constant ϵ such that $0 < \frac{1}{1-\theta^{L-1}} - \epsilon\frac{\alpha^L \gamma^L}{(1-\alpha^L)\gamma^L} < \frac{1}{1-\theta^{L-1}}$. Consider a function $f(x) = \frac{(1-x)^2}{1-\theta^{L-1}(1-x)}$ defined for $x \geq 0$. Clearly, f is decreasing close to $x = 0$ and $f(0) = \frac{1}{1-\theta^{L-1}}$. Therefore, there exists $\epsilon_0 > 0$ such that

$$f(\epsilon_0) > \frac{1}{1-\theta^{L-1}} - \epsilon \frac{\alpha^L\, \gamma^L}{(1-\alpha^L)\,\gamma^L}. \tag{6.35}$$

Because $\rho_n^{L+1} \to 0$, there exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$,

$$\left[1 + \rho_{n-1}^{L+1}\, \frac{\alpha^{L+1}\, \gamma^{L+1}}{(1-\alpha^L)\,\gamma^L}\right]^{-1} > 1 - \epsilon_0. \tag{6.36}$$

Furthermore, because $\theta^{L-1} < 1$, there exists $N_1 > N_0$ such that, for all $n \geq N_1$,

$$\rho_{N_0}^{L-1}\, (\theta^{L-1})^{n-N_0} < \epsilon, \tag{6.37}$$

$$\left[\theta^{L-1}(1-\epsilon_0)\right]^{n-N_0} < \epsilon_0. \tag{6.38}$$

Thus, for all $n \geq N_1$,

$$\rho_n^{L-1} = \frac{\mu_n^{L-1}}{\mu_n^L} = \frac{\mu_{n-1}^{L-1}\,\alpha^{L-1}\,\gamma^{L-1} + \mu_{n-1}^{L}\,(1-\alpha^L)\,\gamma^L}{\mu_{n-1}^{L}\,\alpha^L\,\gamma^L + \mu_{n-1}^{L+1}\,(1-\alpha^{L+1})\,\gamma^{L+1}} = \frac{\rho_{n-1}^{L-1}\,\alpha^{L-1}\,\gamma^{L-1} + (1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L + \rho_{n-1}^{L+1}\,(1-\alpha^{L+1})\,\gamma^{L+1}}$$
$$= \left[\rho_{n-1}^{L-1}\,\theta^{L-1} + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\right] \cdot \left[1 + \rho_{n-1}^{L+1}\,\frac{(1-\alpha^{L+1})\,\gamma^{L+1}}{\alpha^L\,\gamma^L}\right]^{-1} \tag{6.39}$$

On the one hand, (6.39) implies that

$$\begin{aligned}
\rho_n^{L-1} &< \rho_{n-1}^{L-1}\,\theta^{L-1} + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L} \;<\; \rho_{n-2}^{L-1}\,(\theta^{L-1})^2 + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,(1+\theta^{L-1}) \\
&< \rho_{N_0}^{L-1}\,(\theta^{L-1})^{n-N_0} + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,(1+\theta^{L-1}+\cdots+(\theta^{L-1})^{n-N_0-1}) \\
&= \rho_{N_0}^{L-1}\,(\theta^{L-1})^{n-N_0} + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,\frac{1-(\theta^{L-1})^{n-N_0}}{1-\theta^{L-1}} \\
&< \epsilon + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,\frac{1}{1-\theta^{L-1}} \;=\; \epsilon + \rho^{L-1},
\end{aligned}$$

where the last inequality is due to (6.37). Hence,

$$\rho_n^{L-1} < \epsilon + \rho^{L-1}, \qquad n \geq N_1. \tag{6.40}$$

On the other hand, it follows from (6.39) and (6.36) that

$$\rho_n^{L-1} > \left[\rho_{n-1}^{L-1}\,\theta^{L-1} + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\right](1-\epsilon_0), \qquad n \geq N_1,$$

which implies that, for $n \geq N_1$,

$$\begin{aligned}
\frac{\rho_n^{L-1}}{1-\epsilon_0} &> \frac{\rho_{n-1}^{L-1}}{1-\epsilon_0}\,\theta^{L-1}(1-\epsilon_0) + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L} \\
&> \frac{\rho_{N_0}^{L-1}}{1-\epsilon_0}\,\left[\theta^{L-1}(1-\epsilon_0)\right]^{n-N_0} \\
&\quad + \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\Big(1+\theta^{L-1}(1-\epsilon_0)+\cdots+\left[\theta^{L-1}(1-\epsilon_0)\right]^{n-N_0-1}\Big) \\
&> \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,\frac{1-\left[\theta^{L-1}(1-\epsilon_0)\right]^{n-N_0}}{1-\theta^{L-1}(1-\epsilon_0)} \;>\; \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,\frac{1-\epsilon_0}{1-\theta^{L-1}(1-\epsilon_0)},
\end{aligned}$$

where the last inequality is due to (6.38).

Consequently, for $n \geq N_1$,

$$\begin{aligned}
\rho_n^{L-1} &> \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,\frac{(1-\epsilon_0)^2}{1-\theta^{L-1}(1-\epsilon_0)} \;=\; \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\,f(\epsilon_0) \\
&> \frac{(1-\alpha^L)\,\gamma^L}{\alpha^L\,\gamma^L}\left[\frac{1}{1-\theta^{L-1}} - \epsilon\,\frac{\alpha^L\,\gamma^L}{(1-\alpha^L)\,\gamma^L}\right] \;=\; \rho^{L-1} - \epsilon,
\end{aligned}$$

where the second inequality is due to (6.35).

Combining this result with (6.40), one deduces that $\rho_n^{L-1} \to \rho^{L-1}$ as $n \to +\infty$.

2.2) Assume that, for some $k < L$, $\rho_n^k \to \rho^k$. As before, take $\epsilon > 0$ small enough so that $0 < \frac{1}{1-\theta^{k-1}} - \epsilon \frac{\alpha^L \gamma^L}{\rho^k (1-\alpha^k) \gamma^k} < \frac{1}{1-\theta^{k-1}}$. Consider a function $g(x) = \frac{(1-x)^2}{1-\theta^{k-1}(1-x)}$. As before, there exists $\epsilon_1 > 0$ such that

$$g(\epsilon_1) > \frac{1}{1-\theta^{k-1}} - \frac{\epsilon}{2} \frac{\alpha^L \gamma^L}{\rho^k (1-\alpha^k) \gamma^k}.$$

It is not difficult to see that, for all y such that

$$0 \leq y < \frac{\epsilon \, \alpha^L \gamma^L}{\frac{2\rho^k (1-\alpha^k) \gamma^k}{1-\theta^{k-1}} - \epsilon \, \alpha^L \gamma^L},$$

one has

$$\left[\frac{1}{1-\theta^{k-1}} - \frac{\epsilon}{2} \frac{\alpha^L \gamma^L}{\rho^k (1-\alpha^k) \gamma^k} \right] (1-y) > \frac{1}{1-\theta^{k-1}} - \epsilon \frac{\alpha^L \gamma^L}{\rho^k (1-\alpha^k) \gamma^k}.$$

Let ϵ_2 be a constant such that

$$0 < \epsilon_2 < \min \left\{ \frac{\epsilon}{2} \frac{\alpha^L \gamma^L}{(1-\alpha^k) \gamma^k} (1-\theta^{k-1}), \; \rho^k \, \frac{\epsilon \, \alpha^L \gamma^L}{\frac{2(1-\alpha^L) \gamma^L}{1-\theta^{k-1}} - \epsilon \, \alpha^L \gamma^L} \right\}.$$

On account of the fact that $\rho_n^{L+1} \to 0$ and $\rho_n^k \to \rho^k$, there exists $N_2 \in \mathbb{N}$ such that $|\rho_n^k - \rho^k| < \epsilon_2$ and $\rho_n^{L+1} < \frac{\epsilon_1}{1-\epsilon_1} \frac{\alpha^L \gamma^L}{(1-\alpha^{L+1}) \gamma^{L+1}}$ for all $n \geq N_2$, and thus

$$\left[1 + \rho_n^{L+1} \frac{(1-\alpha^{L+1}) \gamma^{L+1}}{\alpha^L \gamma^L} \right]^{-1} > 1 - \epsilon_1.$$

The rest of the proof follows exactly the same lines as above to prove that $\rho_n^{k-1} \to \rho^{k-1}$.

2.3) On has already shown that, for $k \in \{L+1, \ldots, L_0\}$, $\rho_n^k \to 0$, and, for $k \in \{1, \ldots L-1\}$, $\rho_n^k \to \rho^k$. Therefore, $\mu_n^L \to 1/S$ and, consequently, $\mu_n^k \to \rho^k/S$ for all $k \leq L-1$, where $S \triangleq 1 + \sum_{k=1}^{L-1} \rho^k$. By the definition of ρ^k's and Proposition 6.9, this limit of the trajectory is precisely μ, the rest point of the dynamical system with support $\{1, \ldots, L\}$. $\square$

Chapter 7

Replicator dynamics in game theory

This chapter is based on the paper *Replicator dynamics in game theory* in collaboration with Mario Bravo and Sylvain Sorin.

Abstract. *This work focuses on the replicator dynamics: its origin, deduction, form, properties, interpretation and applications in different contexts in game theory. Three main frameworks are treated: nonatomic population games with/without self-interaction within each population, N-player finite games and N-player splittable games with finitely many choices.*

7.1 Introduction

Let S be a finite set of alternatives. The replicator dynamics describes the evolution of a vector $(x^s)_{s\in S}$ in the simplex $\Delta(S) = \{\mathbf{x} = (x^s)_{s\in S} \in \mathbb{R}^{|S|} \mid x^s \geq 0$ for all $s \in S$, $\sum_{s\in S} x^s = 1\}$. In the following different contexts in game theory, $(x^s)_{s\in S}$ has different interpretations and the replicator dynamics has different properties.

1) A population of nonatomic players (or, equivalently, individuals) is represented by the unit interval $[0, 1]$, endowed with the Lebesgue measure (Schmeidler [82]). Each individual corresponds to a point in the interval, thus has weight zero. She chooses an action (or has a type) from a finite set S. The proportion of the individuals associated to $s \in S$ is x^s. The payoff (or fitness) of an action (or type) depends on the composition of the entire population $\mathbf{x} = (x^s)_{s\in S}$ (Maynard Smith [55]). This corresponds to the *self-interaction* case. This model can be extended to N-population games. Each individual in population $i \in \mathcal{N} = \{1, \ldots, N\}$ selects an action (or has a type) from a finite set S^i. Her payoff (or fitness) depends, in addition, on the composition of each population, i.e. on the vector $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}}$, where $\mathbf{x}^i = (x^{is})_{s\in S^i} \in \Delta(S^i)$.

2) In a model of two populations of nonatomic players, $\mathbf{x}$ and $\mathbf{y}$ stand for the compositions of the two populations. The payoff (or fitness) to an action (or a type) within a population is a function of the composition of the other population. This is the case *with external-interaction*. A typical example corresponds to the random matching between the two populations (Taylor [89]). This model can also be extended to multi-population case.

From now on, population always means "non atomic" population.

3) In an N-player game ($N \geq 2$) with finitely many actions, S is the action set of one of the players, and $\mathbf{x} = (x^s)_{s\in S}$ describes her mixed action. Explicitly, x^s is the probability with which she plays the action $s \in S$. Her payoff for playing s depends on the strategies of the others but not on the probabilities with which she plays other actions, i.e. not on $\mathbf{x}$. This model is thus *with external-interaction*. It will also be studied in a more general setting, leading to the so-called unilateral replicator dynamics.

4) In an N-player splittable game ($N \geq 1$), each player splits a stock in several parts and distributes each part on some facility s from the finite set S. A player's action is specified by a vector $\mathbf{x} = (x^s)_{s\in S}$, where x^s is the proportion of her stock allocated to the facility s. The payoff to her stock on the facility s depends on the stocks sent there by all the players and, in particular, on x^s. Hence, this framework allows for *self-interaction*.

In the sequel, these four contexts will be presented one after the other.

7.2 Nonatomic population games with self-interaction

7.2.1 One-population case

First, let us recall the basic one-population game where the players are *playing the field* (Maynard Smith [55, 57]).

Each individual from the interval $[0,1]$ chooses an action s from a finite set S. The proportion of the individuals choosing strategy s is denoted by x^s. The vector $\mathbf{x} = (x^s)_{s\in S}$ is called the *state* of the population. The payoff function is $F = (F^s)_{s\in S} : \Delta(S) \to \mathbb{R}^{|S|}$, so that the *payoff* to s at state $\mathbf{x}$ is $F^s(\mathbf{x})$. Let this one-population game be denoted by $\mathcal{G}(S,F)$.

Definition 7.1. A *Wardrop equilibrium* of the game $\mathcal{G}(S,F)$ is a state $\mathbf{x} \in \Delta(S)$ such that

$$\forall s \in S,\ x^s > 0 \ \Rightarrow\ F^s(\mathbf{x}) \geq F^t(\mathbf{x}),\ \forall\, t \in S. \tag{7.1}$$

The notion of Wardrop equilibrium is widely used in traffic analysis and congestion games since the seminal work of Wardrop [94].

The following proposition shows that a Wardrop equilibrium can be characterized as a solution of a variational inequality problem.

Proposition 7.2. *In a one-population game $\mathcal{G}(S,F)$, $\mathbf{x} \in \Delta(S)$ is a Wardrop equilibrium if and only if*

$$\langle\, \mathbf{y},\, F(\mathbf{x})\,\rangle \leq \langle\, \mathbf{x},\, F(\mathbf{x})\,\rangle, \qquad \forall\, \mathbf{y} \in \Delta(S). \tag{7.2}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the standard inner product in a Euclidean space.

When F is linear, i.e. $F(\mathbf{x}) = A\mathbf{x}$ with $A \in \mathbb{R}^{|S|\times|S|}$, the following *pairwise random matching* game model is used to interpret the payoff function in this one-population *linear* game denoted by $\mathcal{G}(S,A)$.

Consider an auxiliary two-player symmetric matrix game where S is the finite set of actions for both players, and the payoff matrix is $A = (A_{sr})_{s,r\in S}$. In other words, when the line player plays strategy s and the column player plays strategy r, the former's payoff is A_{sr} and the latter's payoff is A_{rs}. Suppose that the individuals in the population are simultaneously matched, two by two, to play game A. As x^r is the proportion of the individuals choosing strategy r, the average payoff to strategy s is $\sum_{r\in S} A_{sr}x^r = e_s A\mathbf{x}$ since the population is large and the matching is random. Here, e_s denotes the s-th unit vector.

Definition 7.3. A strategy $\mathbf{x} \in \Delta(S)$ is a symmetric Nash equilibrium in a symmetric matrix game $A = (A_{sr})_{s,r\in S}$ if

$$\mathbf{y}A\mathbf{x} \leq \mathbf{x}A\mathbf{x}, \qquad \forall\, \mathbf{y} \in \Delta(S).$$

The following fact links Definition 7.1 and Definition 7.3.

Proposition 7.4. *In a one-population linear game $\mathcal{G}(S, A)$, $\mathbf{x} \in \Delta(S)$ is a Wardrop equilibrium if, and only if, it is a symmetric Nash equilibrium in the symmetric matrix game A.*

Now, let us consider the evolution of the state of the population at time t, i.e. $\mathbf{x}_t$.

The *one-population replicator dynamics* is defined by the following differential equation:

$$\dot{x}^s_t = x^s_t\,[\,F^s(\mathbf{x}_t) - \langle\, \mathbf{x}_t,\, F(\mathbf{x}_t)\,\rangle\,], \qquad \forall\, s \in S. \tag{7.3}$$

When F is linear, the replicator dynamics (7.3) takes its classical form (Taylor and Jonker [90]):

$$\dot{x}^s_t = x^s_t\,[\,e_s A\mathbf{x}_t - \mathbf{x}_t A\mathbf{x}_t\,], \qquad \forall\, s \in S. \tag{7.4}$$

The first interpretation of the replicator dynamics is a pure evolutionary biological one as in Taylor and Jonker's seminal paper [90]. This paper first introduced the replicator dynamics into game theoretical models of population evolution. The total size of the subset of individuals of type s is ξ^s, thus the frequency of type s in the population is $x^s = \xi^s / \sum_{r\in S} \xi^r$. Assume that the average growth rate of ξ^s_t is $F^s(\mathbf{x}_t)$, the *average fitness* of type s, i.e. $\dot{\xi}^s_t = \xi^s_t F^s(\mathbf{x}_t)$. A simple computation shows that $\mathbf{x}_t$ is governed by (7.3). In other words, the logarithmic time derivative of x^s, i.e. $\dot{x}^s/x^s$, is the difference between the average fitness $F^s(\mathbf{x}_t)$ of type s and the population's average fitness $\langle\, \mathbf{x}_t,\, F(\mathbf{x}_t)\,\rangle$.

7.2.2 Stationary configurations and ESS

A rest point of (7.3) is an *equalizer* of the fitness of the types *present* in the population. Let it be called a *stationary state*. It satisfies:

$$F^s(\mathbf{x}) = \text{constant}, \quad \text{for all } \; s \in S \text{ such that } x^s > 0. \tag{7.5}$$

Proposition 7.5. *In a one-population game, a Wardrop equilibrium is a rest point of the replicator dynamics (7.3), while an interior rest point is a Wardrop equilibrium.*

For the sake of completeness, let us recall some notions concerning the stability of dynamical systems.

Definition 7.6. In the dynamical system $\dot{z}_t = g(z_t)$,

(i) z is *Lyapunov stable* if for every neighborhood $\mathcal{U}$ of z, there exists a neighborhood $\mathcal{V}$ of z such that, if $z_0 \in \mathcal{V}$, then $z_t \in \mathcal{U}$ for all $t > 0$;

(ii) z is *attracting* if there exists a neighborhood $\mathcal{V}$ of z such that, if $z_0 \in \mathcal{V}$, then $z_t \to z$ as $t \to +\infty$;

(iii) z is *asymptotically stable* if it is Lyapunov stable and attracting. If, furthermore, V is the entire space, then z is *globally asymptotically stable.*

Besides, a function V is a Lyapunov function for the dynamics if it is decreasing along trajectories. More precisely, denote $W_t = V(z_t)$. If $\dot{W}_t = \langle \nabla V(z_t), g(z_t)\rangle \leq 0$, then V is a Lyapunov function.

The following results are often referred to as *folk theorems in evolutionary game theory.* For proofs and more details, see [14, 43, 44, 64, 95].

Theorem 7.7. *Suppose that $\mathcal{G}(S, A)$ is a one-population linear game, where A is its auxiliary two-player symmetric matrix game. Then, the following hold.*

1. *A strict symmetric Nash equilibrium of A is an asymptotically stable rest point of the replicator dynamics (7.4) in $\mathcal{G}(S, A)$.*
2. *If a stationary state $\mathbf{x}$ in $\mathcal{G}(S, A)$ is the limit of an interior orbit, i.e. an orbit $\mathbf{x}_t$ in $\operatorname{int}\Delta$, for the replicator dynamics (7.4), then $\mathbf{x}$ is a symmetric Nash equilibrium of A.*
3. *If a stationary state $\mathbf{x}$ is a stable rest point in the replicator dynamics (7.4) in $\mathcal{G}(S, A)$, then it is a symmetric Nash equilibrium of A.*

Theorem 7.7 can be extended to the case where payoff functions F are not linear. The reader is referred to Hofbauer and Sigmund [44, Section 3.1] for more references.

ESS Another important notion in one-population linear games is that of evolutionarily stable strategies (Maynard Smith and Price [58], Maynard Smith [57]).

Definition 7.8. In a one-population linear game $\mathcal{G}(S, A)$, $\mathbf{x} \in \Delta$ is an *evolutionarily stable strategy* (ESS for short) if, for all $\mathbf{y} \in \Delta$, either $\mathbf{y}A\mathbf{x} \leq \mathbf{x}A\mathbf{x}$, or $\mathbf{y}A\mathbf{x} = \mathbf{x}A\mathbf{x}$ and $\mathbf{x}A\mathbf{y} < \mathbf{y}A\mathbf{y}$.

The fundamental relationship between ESS and the replicator dynamics is given by the following theorem.

Theorem 7.9. *If $\mathbf{x} \in \Delta$ is an ESS in one-population linear game $\mathcal{G}(S, A)$, then it is an asymptotically stable rest point of the replicator dynamics (7.4).*

If $\mathbf{x} \in \operatorname{int}\Delta$ is an ESS, then it is a globally asymptotically stable rest point for (7.4).

As a matter of fact, Theorem 7.9 is a consequence of the following proposition (*cf.* Hofbauer et al. [41]).

Proposition 7.10. *In a one-population linear game $\mathcal{G}(S, A)$, $\mathbf{x} \in \Delta(S)$ is an* ESS *if and only if $V(z) = \prod_{s \in S} (z^s)^{\mathbf{x}^s}$ is locally a Lyapunov function for the replicator dynamics (7.3).*

The notion of ESS extends to the case where F is nonlinear. The main idea is to linearize payoff functions at a neighborhood of an ESS. The reader is again referred to Hofbauer and Sigmund [44, Section 2.6 and Section 3.1] for a concise review.

7.2.3 Extension to the N-population case $(N \geq 2)$

For a population $i \in \mathcal{N} = \{1, \ldots, N\}$, S^i is its finite set of actions and $\Delta(S^i)$ is its state space. Let us denote $S = \prod_i S^i$ and $X = \prod_j \Delta(S^i)$. In an N-population game with self-interaction [26, 83, 89], the payoff for a population $i \in \mathcal{N}$ is defined through a function $F^i : X \to \mathbb{R}^{|S^i|}$. In other words, $F^{is}(\mathbf{x})$ is the payoff to an individual in population i who chooses $s \in S^i$, when the state of all the populations is $\mathbf{x} = (\mathbf{x}^j)_{j \in \mathcal{N}}$, where $\mathbf{x}^j \in \Delta(S^j)$.

The following is a straightforward extension of the one-population model.

A population' state $\mathbf{x}$ is a *Wardrop equilibrium* if

$$\forall i \in \mathcal{N}, \forall s \in S^i, \; x^{is} > 0 \;\Rightarrow\; F^{is}(\mathbf{x}) \geq F^{it}(\mathbf{x}), \; \forall t \in S^i.$$

The *N-population replicator dynamics* on X is given by

$$\dot{x}_t^{is} = x_t^{is} \left[F^{is}(\mathbf{x}_t) - \langle x_t^i, F^i(\mathbf{x}_t) \rangle \right], \quad \forall s \in S^i, \forall i \in \mathcal{N}. \tag{7.6}$$

7.2.4 Application: nonatomic network congestion games

One-population case

An important class of population games with nonlinear payoffs is that of *congestion games* and, in particular, those taking place in a network, called *network congestion games* or *routing games*. As an example, let us consider a one-population routing game $\mathcal{G}(S, \mathbf{c})$ where the common origin O and the common destination D of all the players are linked by a finite set S of parallel arcs.

A population (of nonatomic individuals of total weight 1) travels from O to D by the arcs in S. A choice of a player is an arc $s \in S$. The proportion of players choosing arc s, x^s, is also the total weight on it. The common cost to all individuals choosing s, when the population state is $\mathbf{x} = (x^s)_{s\in S}$, is given by $c^s(x^s)$. This means that the cost of any arc s depends only on the total weight on s, but not on the congestion on the other arcs. Assume that c^s, the per-unit cost function of arc s, is defined on a neighborhood of $[0, 1]$, and it is continuous, nonnegative and non-decreasing. In general, the function $\mathbf{c}(\mathbf{x}) = (c^s(x^s))_{s\in S}$ is not linear in $\mathbf{x}$. This game is a playing-the-field population game.

The game described above turns out to be a *population potential game* (*cf.* §7.8.2), since the the gradient of the function

$$f(\mathbf{x}) = \sum_{s\in S} \int_0^{x^s} c^s(y)\,\mathrm{d}y \tag{7.7}$$

is equal to $\mathbf{c}(\mathbf{x})$. This function was first proposed by Beckman et al. [9] to prove the existence of a Wardrop equilibrium, which is a minimum of f on $\Delta(S)$.

The function f is also a Lyapunov function for the replicator dynamics (7.3), by replacing F by $-\mathbf{c}$.

For a proof of the following result, see, for instance, Sandholm [81].

Theorem 7.11. *For the replicator dynamics in $\mathcal{G}(S, \mathbf{c})$, the limit set of $\mathbf{x_t} \in \Delta(S)$ is a closed, connected set of stationary states for any initial condition. If $\mathbf{z} \in \operatorname{int}\Delta(S)$ is a limit point of $\mathbf{x}_t$, then it is a Wardrop equilibrium.*

In fact, this result holds for a general population potential game (see §7.8.2 for references).

N-population case ($N \geq 2$)

In an N-population network congestion game, the individuals of different populations may have different origin/destination pairs, and different cost functions for the same arc.

Consider an N-population congestion game $\mathcal{G}_n(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\mathbf{c}^i)_{i\in\mathcal{N}})$ (the subscript n for *nonatomic*) taking place in the same network as in §7.2.4. Population $i \in \mathcal{N}$ has weight m^i. Suppose that a proportion x^{is} of the individuals in population i choose arc $s \in S$, and $\mathbf{x}^i = (x^{is})_{s\in S} \in \Delta(S)$ is the population i's state. Denote $X = \Delta(S)^N$ and $M = \sum_{i\in\mathcal{N}} m^i$.

At a state $\mathbf{x} \in X$, the per-unit cost of arc s to the individuals in population i is $c^{is}(\sum_{j\in\mathcal{N}} m^j x^{js})$, where c^{is} is of class $\mathcal{C}^1$, nonnegative and defined on a neighborhood of $[0, M]$. Thus, $\mathbf{c}^i(\mathbf{x}) = (c^{is}(\sum_{j\in\mathcal{N}} m^j x^{is}))_{s\in\mathcal{S}}$.

In this case, $\mathbf{x} \in X$ is a Wardrop equilibrium if

$$\forall\, i \in \mathcal{N},\ \forall\, s \in S,\ x^{is} > 0 \Rightarrow c^{is}(\mathbf{x}) \leq c^{ir}(\mathbf{x}), \quad \forall\, r \in S, \tag{7.8}$$

and the equation of the replicator dynamics on the populations' state $\mathbf{x}_t$ is

$$\dot{x}_t^{is} = x_t^{is}\,[\,\langle\, \mathbf{x}_t^i,\, \mathbf{c}^i(\mathbf{x}_t)\,\rangle - c^{is}(\mathbf{x}_t)\,], \quad \forall\, s \in S,\ \forall\, i \in \mathcal{N}. \tag{7.9}$$

7.3 N-population games with external-interaction ($N \geq 2$)

In an N-population game with the same notations as in §7.2.3, let the state of all the populations except i be denoted by $\mathbf{x}^{-i} \in \prod_{j \neq i} \Delta(S^j)$. If the payoff to an action s used in population i depends only on $\mathbf{x}^{-i}$ but not on the population i's state $\mathbf{x}^i$, the game is *with external-interaction.*

7.3.1 Two-population case

From the biological evolutionary point of view, an individual of type s from population 1 has fitness A_{sr} facing an individual of type r from population 2, while the latter has fitness B_{sr} facing the former. The fitnesses A_{sr} and B_{sr} are determined by two matrices $A, B \in \mathbb{R}^{|S^1| \times |S^2|}$. The average fitness of type s in population 1 is a linear function of the state $\mathbf{x}^2$ of population 2, i.e. $F^{1s}(\mathbf{x}^2) = e_s A \mathbf{x}^2$. Similarly, $F^{2r}(\mathbf{x}^1) = \mathbf{x}^1 B e_r$. Let this two-population linear game be denoted by $\mathcal{G}(S^1, S^2, A, B)$.

Notice that the process of random matching between two populations here differs from the process of random matching within one population mentioned in §7.2.1.

The relation between Wardrop equilibria of the two-population game $\mathcal{G}(S^1, S^2, A, B)$ and its auxiliary two-player bimatrix game denoted by (A, B) is similar to the one-population case.

Proposition 7.12. *In a two-population linear game $\mathcal{G}(S^1, S^2, A, B)$, $(\mathbf{x}, \mathbf{y}) \in \Delta(S^1) \times \Delta(S^2)$ is a Wardrop equilibrium if, and only if, it is a mixed Nash equilibrium in the auxiliary two-player bimatrix game (A, B).*

The replicator dynamics in the two-population game $\mathcal{G}(S^1, S^2, A, B)$ is:

$$\begin{cases} \dot{x}_t^{1s} = x_t^{1s} \left[e_s A \mathbf{x}_t^2 - \mathbf{x}_t^1 A \mathbf{x}_t^2 \right], & \forall\, s \in S^1, \\ \dot{x}_t^{2r} = x_t^{2r} \left[\mathbf{x}_t^1 B e_r - \mathbf{x}_t^1 B \mathbf{x}_t^2 \right], & \forall\, r \in S^2. \end{cases} \tag{7.10}$$

The link between the rest points of the dynamics (7.10) and the Wardrop equilibria of $\mathcal{G}(S^1, S^2, A, B)$ or, equivalently, the Nash equilibria of its auxiliary bimatrix game (A, B) goes exactly as in the one-population case (*cf.* Hofbauer [38, 39]). Furthermore, an analog of Theorem 7.7 exists in this setting (see Chapter 3 in Cressman[25] for details).

In the case where $S^1 = S^2 = S$ and $A = B$, by comparing (7.4) and (7.10), it is clear that, if $\mathbf{x}$ is a Wardrop equilibrium of the one-population game $\mathcal{G}(S, A)$, then $(\mathbf{x}, \mathbf{x})$ is a Wardrop equilibrium of the two-population game $\mathcal{G}(S, S, A, A)$. However, $\mathcal{G}(S, S, A, A)$ can also have equilibria that do not lie on the diagonal set $D = \{(z, z) \,|\, z \in \Delta(S)\} \subseteq \Delta(S) \times \Delta(S)$ or, equivalently, the bimatrix game (A, A) may have nonsymmetric mixed Nash equilibria. An example is the 2×2 complementarity-type game in Table 7.1.

	s	t
s	$(0,0)$	$(1,1)$
t	$(1,1)$	$(0,0)$

Table 7.1: Complementarity-type game

Observe that the strategy profile $\mathbf{x}^1 = (0,1), \mathbf{x}^2 = (1,0)$ and the strategy profile $\mathbf{x}^1 = (1,0), \mathbf{x}^2 = (0,1)$ are nonsymmetric equilibria, while the strategy profile $\mathbf{x}^1 = (1/2, 1/2), \mathbf{x}^2 = (1/2, 1/2)$ is a symmetric equilibrium.

In general, the two replicator dynamics (7.4) and (7.10) can have very different asymptotic and stability properties. For instance, equation (7.10) is volume-preserving (Hofbauer [38]) and, consequently, it cannot have an interior asymptotically stable point, while interior ESS are globally stable for (7.4) (Hofbauer and Sigmund [44]).

7.3.2 N-population case ($N \geq 2$)

For the general case where $N \geq 2$, the replicator dynamics is

$$\dot{x}_t^{is} = x_t^{is} \left[F^{is}(\mathbf{x}_t^{-i}) - \langle \mathbf{x}_t^i , F^i(\mathbf{x}_t^{-i}) \rangle \right], \quad \forall\, i \in \mathcal{N},\ \forall\, s \in S^i. \tag{7.11}$$

This extension was first studied by Ritzberger [72] and Ritzberger and Weibull [73].

Besides, from a mathematical point of view, the case $N = 2$ and the case $N \geq 3$ can show quite different dynamical behavior. The reader is referred to Plank [70] for explicit examples.

7.4 Unilateral version and N-player finite games

7.4.1 Unilateral replicator dynamics

The *unilateral version* of the replicator dynamics can be stated in the following way. Let S be a finite set, and $(U_t)_{t\geq 0}$ a bounded measurable process from $\mathbb{R}_+$ to $\mathbb{R}^{|S|}$. The unilateral (or U-based) replicator dynamics (Hofbauer et al. [45]) is given by:

$$\dot{x}_t^s = x_t^s \left[U_t^s - \langle \mathbf{x}_t , U_t \rangle \right], \quad \forall\, s \in S. \tag{7.12}$$

Proposition 7.13 ([45])**.** *Let $V_t = \int_0^t U_s ds$. Then, the process $\mathbf{x}_t = (x_t^s)_{s\in S} \in \Delta(S)$, defined by*

$$x_t^s = \frac{\exp(V_t^s)}{\sum\limits_{r\in S} \exp(V_t^r)}, \quad \forall\, s \in S, \tag{7.13}$$

follows the unilateral replicator dynamics (7.12).

A biological interpretation. From a biological point of view, (7.12) can be interpreted in the same vein as before for the evolution of a type in its environment. At time $t \geq 0$, type $s \in S$ has a fitness U_t^s coming from an interaction with a changing environment.

Learning and game theoretical interpretation. Equation (7.12) also appears in the framework of a *learning* procedure (Sorin [87], Hofbauer et al. [45]). Assume that an agent is facing an unknown process $(U_t)_{t\geq 0}$. Her finite action set is S. At time $t \geq 0$, she selects an action $s_t \in S$ based on the past history $\{U_\tau , \tau \leq t\}$. Then, she is informed of the whole vector of payoffs $U_t = (U_t^s)_{s\in S} \in \mathbb{R}^{|S|}$, where U_t^s is the payoff to the action s if it had been played. The outcome is $U_t^{s_t}$.

7.4.2 N-player finite game

Let us consider an N-player finite game $\Gamma(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (G^i)_{i\in\mathcal{N}})$ where the set of players is $\mathcal{N} = \{1, \ldots, N\}$, the finite pure-strategy set of player i is S^i, and her payoff function is $G^i : \mathbf{S} \to \mathbb{R}$, where $\mathbf{S} = \Pi_{j\in\mathcal{N}} S^j$. By a usual abuse of notation, G^i is also used for her expected payoff when the players use mixed strategies, i.e. $G^i(\mathbf{x}^i, \mathbf{x}^{-i}) = \mathbb{E}_{\mathbf{x}}(G^i)$ with $\mathbf{x}^i \in \Delta(S^i)$ and $\mathbf{x}^{-i} \in \prod_{j\neq i} \Delta(S^j)$. Explicitly

$$G^i(\mathbf{x}^i, \mathbf{x}^{-i}) = \sum_{\mathbf{s}=(s^l)_{l\in\mathcal{N}}\in\mathbf{S}} \big(\prod_j x^{js^j}\big) G^i(\mathbf{s}).$$

Then, by setting $U^i_t = G^i(\cdot, \mathbf{x}^{-i})$ for all $i \in \mathcal{N}$, Proposition 7.13 leads to the following proposition.

Proposition 7.14. *In the N-player finite game $\Gamma(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (G^i)_{i\in\mathcal{N}})$ repeated in continuous time, if every player adopts the strategy (7.13), then the evolution of their mixed strategies is described by the replicator dynamics*

$$\dot{x}^{is}_t = x^{is}_t \left[G^i(s, \mathbf{x}^{-i}_t) - G^i(\mathbf{x}^i, \mathbf{x}^{-i}) \right], \quad \forall\, s \in S^i, \forall\, i \in \mathcal{N}. \tag{7.14}$$

Remark 7.15. This dynamics is exactly the same as the one in the N-population linear games with external-interaction (7.11), with $F^{is}(\mathbf{x}^{-i}) = G^i(s, \mathbf{x}^{-i})$.

7.4.3 Application to congestion games

Let us take a finite congestion game as an example. The game takes place in the same network as in §7.2.4. There are N players, each of whom has weight 1. Each player has to go from O to D by one of the arcs in S, i.e. S is their common set of pure strategies. This game belongs to the class of *Rosenthal congestion games* [74], which is a subclass of *atomic unsplittable congestion games*, where the players cannot split their weight on several paths. This is by opposition to the atomic splittable congestion games which will be discussed in §7.5. The cost function l^s of arc s is an increasing function defined on $\mathbb{N} \cap [0, N]$ so that $l^s(u)$ is the common cost of arc s to all the u players on it.

When the action profile is $\mathbf{s} = (s^i)_{i\in\mathcal{N}}$, the cost to player i is defined by $c^i(\mathbf{s}) = l^k(u^k(\mathbf{s}))$, where $k = s^i$, and $u^k(\mathbf{s}) = \#\{j \in \mathcal{N} \,|\, s^j = k\}$ is the number of players on k when the pure strategy profile is $\mathbf{s}$. As above, when the players use mixed strategies $(\mathbf{x}^i)_{i\in\mathcal{N}}$, the expected cost to player i is denoted by $c^i(\mathbf{x}^i, \mathbf{x}^{-i})$. Denote $F^{is}(\mathbf{x}^{-i}) = -c^i(s, \mathbf{x}^{-i})$.

Proposition 7.14 shows that, if the above congestion game is repeated in continuous time, and if all the N players use (7.13) to determine their mixed strategies at every moment, then the profile of strategies $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}}$ follows the replicator dynamics (7.14).

Besides, a Lyapunov function is given by

$$\Lambda(\mathbf{x}) = \mathbb{E}\Big[\sum_{s\in S} \sum_{u=1}^{Z^s} l^s(u)\Big], \tag{7.15}$$

where the expectation is taken with respect to the random variables $Z^s = \sum_{i\in\mathcal{N}} X^{is}$ with X^{is} being independent Bernoulli variables such that $\mathbb{P}(X^{is} = 1) = x^{is}$, for all $s \in S$. This function was found by Cominetti et al. [21] in a different context.

7.5 N-player splittable games

In §7.2.3, the replicator dynamics (7.6) is presented in the framework of a N-population game with self-interaction. This section will show that the same equation appears in a class of N-player games where the strategy space of each player is a simplex in a Euclidean space.

Here, let us take a specific class called *atomic splittable congestion games* as an example. Consider the same network as in §7.2.4, i.e. vertex O and vertex D being connected by a finite set S of parallel arcs. Suppose that player $i \in \mathcal{N}$ holds a stock of strictly positive weight m^i that she has to send from O to D. To this end, she can divide the stock into $|S|$ parts of arbitrary positive weight provided that their sum is m^i, and then send each part through one of the arcs. Hence, each player has a continuum of pure strategies. Indeed, for each player $i \in \mathcal{N}$, the set $\Delta(S)$ is her action set: $\mathbf{x}^i = (x^{is})_{s\in S} \in \Delta(S)$ means sending a fraction x^{is} of her stock by arc s. The space of pure strategy profiles is thus $X = \Delta(S)^N$. Denote $M = \sum_{i\in\mathcal{N}} m^i$. Let this game be denoted by $\mathcal{G}_a(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\mathbf{c}^i)_{i\in\mathcal{N}})$ (the subscript a for atomic).

Similar to the N-population nonatomic routing game, given a profile of strategies $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}} \in X$, the per-unit cost of arc s to player i is $c^{is}(\sum_{i\in\mathcal{N}} m^i x^{is})$ with the same hypotheses on the functions c^{is}'s as in §7.2.4. Furthermore, for all $i \in \mathcal{N}$ and $s \in S^i$, c^{is} is assumed to be convex. The cost to player i is

$$v^i(\mathbf{x}) = \sum_{s\in S} m^i x^{is}\, c^{is}\Big(\sum_{j\in\mathcal{N}} m^j x^{js}\Big).$$

Definition 7.16. In an atomic splittable congestion game $\mathcal{G}_a(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\mathbf{c}^i)_{i\in\mathcal{N}})$, the *marginal cost* function of arc $s \in S$ to player $i \in \mathcal{N}$ is defined on a neighborhood of Δ by

$$\tilde{c}^{is}(\mathbf{x}) = c^{is}\Big(\sum_{j\in\mathcal{N}} m^j x^{js}\Big) + m^i x^{is}\,(c^{is})'\Big(\sum_{j\in\mathcal{N}} m^j x^{js}\Big)$$

The following result can be found in Haurie and Marcotte [36] or Harker [32]. It states that Nash equilibria are characterized by the solutions of a variational inequality problem.

Proposition 7.17. *In $\mathcal{G}_a(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\mathbf{c}^i)_{i\in\mathcal{N}})$, a profile of strategies $\mathbf{x}$ is a (pure) Nash equilibrium if, and only if,*

$$\forall\, i \in \mathcal{N},\, \forall\, s \in \mathcal{S},\, x^{is} > 0, \Rightarrow\, \tilde{c}^i_s(\mathbf{x}) \le \tilde{c}^i_r(\mathbf{x}), \quad \forall\, r \in \mathcal{S}. \tag{7.16}$$

By comparing the above condition and the characterization of Wardrop equilibria (7.8) in N-population routing games, one observes that the two conditions are the same, except that the arc costs in (7.8) are replaced by the marginal costs in (7.16). The similarity between (7.8) and (7.16) gives rise to the following proposition.

Proposition 7.18. *A repartition $\mathbf{x} \in X$ is a pure Nash equilibrium of the game $\mathcal{G}_a(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\mathbf{c}^i)_{i\in\mathcal{N}})$ if, and only if, it is a Wardrop equilibrium of the game $\mathcal{G}_n(\mathcal{N}, (m^i)_{i\in\mathcal{N}}, (\tilde{\mathbf{c}}^i)_{i\in\mathcal{N}})$.*

Remark 7.19. Although the two equilibria are described by the same vector $\mathbf{x} \in X$, the interpretations of $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}}$ are different:

In the population game, each individual holds independently a stock of infinitesimal weight. Thus, $\mathbf{x}^i$ is the repartition of these independent individuals (or, equivalently, the total stocks that they hold) in population i on different arcs.

In the atomic splittable game, an atomic player holds a stock of finite weight. Hence, $\mathbf{x}^i$ is the repartition of player i's stock on different arcs, i.e. it is her pure strategy.

The interpretation is also different from the N-player routing game in §7.4.3, where $\mathbf{x}^i = (x^{is})_{s\in\mathcal{S}}$ is a mixed strategy of player i, with x^{is} being the probability with which she selects s.

According to Proposition 7.18, one can write the following replicator dynamics for the atomic congestion game with splittable stocks:

$$\dot{x}_t^{is} = x_t^{is}\,[\,\langle\, x_t^i,\, \tilde{c}^i(\mathbf{x}_t)\,\rangle - \tilde{c}^{is}(\mathbf{x}_t)\,], \quad \forall\, s \in S,\ \forall\, i \in \mathcal{N}. \tag{7.17}$$

One cannot expect to find a Lyapunov function for (7.17). Indeed, a potential function like (7.7) exists if, and only if, the Jacobian matrix of $\tilde{\mathbf{c}}$ is symmetric. This condition is hard to verify in general, except in some specific cases.

For example, if the per-unit cost functions of the arcs are affine and common to all the players, i.e. $c^{is} = a^s x + b^s$ for all $i \in \mathcal{N}$ and $s \in S^i$, then a Lyapunov function exists (*cf.* Altman et al. [4]).

Remark 7.20. One can introduce a more general class of N-player games, where the pure-strategy space of player i is $\Delta(S^i)$ with S^i a finite set. The payoff function of player i is defined as $T^i(\mathbf{x}) = \sum_{s\in S^i} x^{is} F^{is}(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}^j)_{j=1}^N$ is the strategy profile, $\mathbf{x}^j = (x^{js})_{s\in S^j}$ is player j's strategy, and F^{is} is a real-valued continuously differentiable function defined a neighborhood of $\prod_{j=1}^N \Delta(S^j)$. Then, Propositions 7.17 and 7.18 still hold if one replaces $\tilde{c}^{is}$ by $\frac{\partial T^i(\mathbf{x})}{\partial x^{is}}$.

7.6 Extension: composite games

Consider a game played by N atomic players in $\mathcal{N} = \{1, \ldots, N\}$ and M nonatomic populations in $\mathcal{M} = \{1, \ldots, M\}$. For all $i \in \mathcal{N} \cup \mathcal{M}$, S^i is a finite set. The pure strategy set of the atomic player $i \in \mathcal{N}$ is $\Delta(S^i)$, while the action set of a nonatomic player in population $j \in \mathcal{M}$ is S^j. Denote the vector $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{N}\cup\mathcal{M}}$, where $\mathbf{x}^i$ is the pure strategy of the atomic player i for $i \in \mathcal{N}$, and $\mathbf{x}^j$ is the state of the population j for $j \in \mathcal{M}$. Denote $X = \prod_{i\in\mathcal{N}\cup\mathcal{M}} \Delta(S^i)$. The payoff function of the atomic player $i \in \mathcal{N}$ is defined as $T^i(\mathbf{x}) = \sum_{s\in S^i} x^{is} F^{is}(\mathbf{x})$, where F^{is} is a function of class $\mathcal{C}^1$ defined a neighborhood of X. The payoff function for a nonatomic player in population $j \in \mathcal{M}$ who plays action $s \in S^j$ is F^{js}, a function of class $\mathcal{C}^1$ defined a neighborhood of X. This game is called a *composite game*. Let it be denoted by $\mathcal{G}_c(\mathcal{N}, \mathcal{M}, (F^i)_{i\in\mathcal{N}}, (F^j)_{j\in\mathcal{M}})$. For all $i \in \mathcal{M} \cup \mathcal{N}$, denote $F^i = (F^{is})_{s\in S^i}$.

In the congestion game framework, a composite game is played by atomic splittable players (*cf.* §7.5) and nonatomic players (*cf.* §7.2.4). This model was first studied by Harker [32].

Definition 7.21. In the composite game $\mathcal{G}_c(\mathcal{N}, \mathcal{M}, (F^i)_{i\in\mathcal{N}}, (F^j)_{j\in\mathcal{M}})$, a point $\mathbf{x} \in X$ is a *composite equilibrium* if the following two conditions hold:

1. for all $i \in \mathcal{N}$, $\mathbf{x}^i$ minimizes $T^i(\mathbf{x}^i, \mathbf{x}^{-i})$ on $\Delta(S^i)$, where $\mathbf{x}^{-i} = (\mathbf{x}^j)_{j\in\mathcal{N}\cup\mathcal{M}\setminus\{i\}}$;
2. for all $j \in \mathcal{M}$ and all $s \in S^j$, if $x^{js} > 0$, then $F^{is}(\mathbf{x}) \geq F^{it}(\mathbf{x})$ for all $t \in S^j$,.

Similar to Wardrop equilibria in §7.2 and Nash equilibria in §7.5, a composite equilibrium is characterized as a solution of a variational inequality problem.

Proposition 7.22. *In a composite game $\mathcal{G}_c(\mathcal{N}, \mathcal{M}, (F^i)_{i\in\mathcal{N}}, (F^j)_{j\in\mathcal{M}})$, $\mathbf{x} \in X$ is a composite equilibrium if and only if*

$$\begin{cases} \langle\, \mathbf{y}^i,\, \nabla_i T^i(\mathbf{x})\,\rangle \leq \langle\, \mathbf{x}^i,\, \nabla_i T^i(\mathbf{x})\,\rangle, & \forall\, \mathbf{y}^i \in \Delta(S^i), \quad \forall\, i \in \mathcal{N}; \\ \langle\, \mathbf{y}^j,\, F^j(\mathbf{x})\,\rangle \leq \langle\, \mathbf{x}^j,\, F^j(\mathbf{x})\,\rangle, & \forall\, \mathbf{y}^j \in \Delta(S^j),\ \forall\, j \in \mathcal{M}, \end{cases}$$

where $\nabla_i T^i(\mathbf{x}) = \left(\frac{\partial T^i(\mathbf{x})}{\partial x^{is}}\right)_{s\in S^i}$.

The replicator dynamics in this model is

$$\begin{cases} \dot{x}_t^{is} = x_t^{is}\,[\,\frac{\partial T^i(\mathbf{x}_t)}{\partial x^{is}} - \langle\, x_t^i,\, \nabla_i T^i(\mathbf{x}_t)\,\rangle\,], & \forall\, s \in S^i,\ \forall\, i \in \mathcal{N}. \\ \dot{x}_t^{js} = x_t^{js}\,[\,F^{js}(\mathbf{x}_t) - \langle\, x_t^j,\, F^j(\mathbf{x}_t)\,\rangle\,], & \forall\, s \in S^j,\ \forall\, j \in \mathcal{M}. \end{cases} \tag{7.18}$$

As in the atomic unsplittable case, a Lyapunov function does not exist in general, except in some particular cases. For example, if there is a finite set S such that $S^i = S$ for all $i \in \mathcal{N} \cup \mathcal{M}$, and there are positive constants a^s, b^s for all $s \in S$, and strictly positive constants m^i for all $i \in \mathcal{N} \cup \mathcal{M}$, such that, for all $s \in S$, $F^{is}(\mathbf{x}) = m^i(a^s\xi^s + b^s)$ for all $i \in \mathcal{N}$, $F^{js}(\mathbf{x}) = a^s\xi^s + b^s$ for all $j \in \mathcal{M}$, where $\xi^s = \sum_{i\in\mathcal{N}\cup\mathcal{M}} m^i\, x^{is}$, then a Lyapunov function for the replicator dynamics (7.18) exists. Consequently, a convergence result similar to Theorem 7.11 holds.

7.7 Summary

So far, two forms of the replicator dynamics, (7.19) and (7.20), have been analyzed.

$$\dot{x}_t^{is} = x_t^{is}\,[\,F^{is}(\mathbf{x}_t) - \langle\, \mathbf{x}_t^i,\, F^i(\mathbf{x}_t)\,\rangle\,], \qquad \forall\, s \in S^i,\ \forall\, i \in \mathcal{N}. \tag{7.19}$$

$$\dot{x}_t^{is} = x_t^{is}\,[\,F^{is}(\mathbf{x}_t^{-i}) - \langle\, \mathbf{x}_t^i\,,\, F^i(\mathbf{x}_t^{-i})\,\rangle\,], \quad \forall\, s \in S^i,\ \forall\, i \in \mathcal{N}. \tag{7.20}$$

They mainly appear in the following three contexts.

(i) N-population games with or without self-interaction.

(ii) N-player games where the pure strategy spaces are simplices in a Euclidean space (for example, N-player splittable congestion games).

(iii) N-player finite games (for example, N-player unsplittable congestion games).

These three categories are related with each other in the following way.

(1) In (i) and (ii), the variable x^{is} stands either for the proportion of the individuals of type s in the population i, or for the proportion of the stock sent by player i on the facility s. In both cases, self-interaction is possible, since the fitness or the payoff to $s \in S^i$ can depend upon the whole vector $\mathbf{x}$ and, in particular, upon $\mathbf{x}^i$. The corresponding replicator dynamics is (7.19).

(2) In (i), the fitness of a given type s in population i can be independent of $\mathbf{x}^i \in \Delta(S^i)$ (*cf.* §7.3). This property of external-interaction is also present in (iii), since x^{is} refers to the probability of player i to play action $s \in S^i$ so that the payoff to s depends only on $\mathbf{x}^{-i}$. The corresponding replicator dynamics is (7.20).

(3) In both (ii) and (iii), the model refers to an N-player game, hence there is no population involved.

Population games with self-interaction is more general than population games with external-interaction. In the case of multilinear F, the second model reduces to the N-player case with finite strategies, hence all general results in the case of population games with self-interaction extend to N-player finite games.

7.8 Appendix: Potential games

7.8.1 Definitions

Two types of potential games are to be defined.

First, consider the N-player game $\Gamma(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (G^i)_{i\in\mathcal{N}})$ defined in §7.4.2. Monderer and Shapley [62] introduced the following notion of N-player potential games.

Definition 7.23. An N-player game $\Gamma(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (G^i)_{i\in\mathcal{N}})$ is a *finite potential game* (FPG for short) if there exists a real function Φ defined on $\mathbf{S} = \prod_{i\in\mathcal{N}} S^i$ such that, for all player $i \in \mathcal{N}$, for all $s^i, t^i \in S^i$ and, for all $s^{-i} \in S^{-i} = \prod_{j\neq i} S^j$, one has

$$G^i(s^i, s^{-i}) - G^i(t^i, s^{-i}) = \Phi(s^i, s^{-i}) - \Phi(t^i, s^{-i}).$$

Such a function Φ is called a *potential function* of Γ.

Note that, in this definition, set S^i is not necessarily finite.

Remark 7.24. A particular class of finite N-player potential games contains the *partnership games*, see, for example, Hofbauer and Sigmund [42], where all players have the same payoff function.

Next, consider the N-population game $\mathcal{G}_n(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (F^i)_{i\in\mathcal{N}})$ defined in §7.2. The following definition is introduced by Sandholm [79].

Definition 7.25. An N-population game $\mathcal{G}_n(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (F^i)_{i\in\mathcal{N}}))$ is a *population potential game* (PPG for short) if there exists a real-valued function f of class $\mathcal{C}^1$, defined on a neighborhood of X, such that

$$\frac{\partial f}{\partial x^{is}} = F^{is}(\mathbf{x}), \quad \forall\, s \in S^i, \forall\, i \in \mathcal{N}. \tag{7.21}$$

Such a function f is called a *potential function* of $\mathcal{G}_n(\mathcal{N}, (S^i)_{i\in\mathcal{N}}, (F^i)_{i\in\mathcal{N}}))$.

Remark 7.26. Consider an N-player finite game Γ where all the players have the same payoff function $G : X \to \mathbb{R}$. For all $i \in \mathcal{N}$ and $s \in S^i$, let function $F^{is} : X^{-i} \to \mathbb{R}$ be defined as $F^{is}(\mathbf{x}^{-i}) = G(s, \mathbf{x}^{-i})$. Then, the real-valued function f defined on X by $f(\mathbf{x}) = G(\mathbf{x}) = \sum_s x^{is} G(s, \mathbf{x}^{-i})$ (for all $i \in \mathcal{N}$) is a potential function for the population game defined by the payoff functions F^{is}.

7.8.2 Properties

In a game where a potential function exists, either a FPG or a PPG, the potential function can play two roles in the analysis of the game: first, as a tool to show the existence of equilibria in both cases; second, as a Lyapunov function for the replicator dynamics.

Equilibrium aspects

Proposition 7.27. *Consider an N-player FPG Γ with potential function Φ. If an action profile $\mathbf{s} \in \prod_i S^i$ is a maximizer of Φ, then $\mathbf{s}$ is a Nash equilibrium of Γ. In particular, every FPG possesses a pure Nash equilibrium.*

Note that the inverse statement is not true, i.e. a pure Nash equilibrium is not necessarily a maximizer of Φ. The reader is referred to Rosenthal [74] for a counter-example.

Remark 7.28. Neyman [65] obtains a characterization of correlated equilibria (Aumann [6]) in the case where the action spaces are convex and the potential Φ is of class $\mathcal{C}^1$ and concave. Precisely, any correlated equilibrium is a combination of pure action profiles that maximize Φ.

A Rosenthal congestion game (§7.4.3) is an N-player FPG. Indeed, a potential function is:

$$\Phi(\mathbf{s}) = \sum_{s \in S} \sum_{k=0}^{u^s(\mathbf{s})} l^s(k). \tag{7.22}$$

This congestion game is first introduced by Rosenthal in [74], where he used the function (7.22) to prove the existence of pure Nash equilibria.

Remark 7.29. In general, the multilinear extension of the potential function Φ in an N-player finite potential game is not a Lyapunov function for the replicator dynamics (7.20) which describes the evolution of the mixed strategies of the players. In the particular congestion game described in §7.4, a Lyapunov function exists (equation (7.15)).

In N-player atomic splittable congestion games, if Φ is a potential function in the sense of Definition 7.23, then

$$\begin{aligned}\Phi\left(\mathbf{x}^i, \mathbf{x}^{-i}\right) - \Phi\left(\mathbf{y}^i, \mathbf{x}^{-i}\right) &= v^i\left(\mathbf{x}^i, \mathbf{x}^{-i}\right) - v^i\left(\mathbf{y}^i, \mathbf{x}^{-i}\right) \\ &= \sum_{s \in S} m^i x^{is} c^{is}(m^i x^{is} + \sum_{j \neq i} m^j x^{js}) - \sum_{s \in S} m^i y^{is} c^{is}(m^i y^{is} + \sum_{j \neq i} m^j x^{js})\end{aligned}$$

for all $\mathbf{x}^i, \mathbf{y}^i \in \Delta^i$, and all $\mathbf{x}^{-i} \in \prod_{j \neq i} \Delta^j$.

A potential function does not exist except for some particular cases. For example, if, for all arc $s \in S$, all the players' cost functions for s are affine and are equivalent up to a constant, i.e. $c^{is}(x) = a^s x + b^{is}$, for all $i \in \mathcal{N}$, or, if all the atomic players have the same amount of stock to send from the same origin to the same destination and they have the same cost functions, i.e. the players are identical (Cominetti et al. [20]), then a potential function exists.

The following result, established in Sandholm [79], is an analog of Proposition 7.27 for PPG.

Proposition 7.30. *Consider an $N - PPG$ $\mathcal{G}_n(\mathcal{N}, (S^i)_{i \in \mathcal{N}}, (F^i)_{i \in \mathcal{N}})$ with potential function f. Then, $\mathbf{x}$ is a local maximizer of f on X if, and only if, it is a Wardrop equilibrium of the game.*

Dynamical aspects

In a FPG and a PPG, the potential is a Lyapunov function for the replicator dynamics. The following result applies to the finite N-player case or the N-population case with external and multi-linear interaction.

Proposition 7.31. *In a* FPG*, the potential Φ is Lyapunov function for the replicator dynamics.*

Proof. Let $f_t = \Phi(\mathbf{x}_t)$. Then, $\dot{f}_t = \Sigma_i \Phi(\dot{\mathbf{x}}_t^i, \mathbf{x}_t^{-i})$ by linearity. On the one hand, for all $i \in \mathcal{N}$, $\Phi(\dot{\mathbf{x}}_t^i, \mathbf{x}_t^{-i}) = \Sigma_s \dot{x}_t^{is} \Phi(s, \mathbf{x}_t^{-i}) = \Sigma_s x_t^{is} [\Phi(s, \mathbf{x}_t^{-i}) - \Phi(\mathbf{x}_t)] \Phi(s, \mathbf{x}_t^{-i})$. On the other hand, $0 = \Sigma_s x_t^{is} [\Phi(s, \mathbf{x}_t^{-i}) - \Phi(\mathbf{x}_t)] \Phi(\mathbf{x}_t)$. By adding the previous two equations, one has $\Phi(\dot{\mathbf{x}}_t^i, \mathbf{x}_t^{-i}) = \Sigma_s x_t^{is} [\Phi(s, \mathbf{x}_t^{-i}) - \Phi(\mathbf{x}_t)]^2$. Therefore, $\dot{f}_t = \Sigma_{is} x_t^{is} [\Phi(s, \mathbf{x}_t^{-i}) - \Phi(\mathbf{x}_t)]^2$, and the minimum 0 is reached, and only reached, on the stationary states of the replicator dynamics. $\square$

The following theorem is a particular case of the results in Sandholm [79], where a more general class of dynamics are considered.

Theorem 7.32. *Consider an* $N-PPG$, $\mathcal{G}_n(\mathcal{N},(S^i)_{i\in\mathcal{N}},(F^i)_{i\in\mathcal{N}})$, *with potential function* f. *Then, the following hold for the replicator dynamics* (7.6)*:*

1. f *is a global Lyapunov function;*
2. *the limit set of a trajectory* $(\mathbf{x}_t)_{t\geq 0}$ *is a closed, connected set of rest points.*

Proof. One has

$$\dot{f}(\mathbf{x}_t)=\sum_{i,s}x_t^{is}\,[\,F^{is}(\mathbf{x}_t)-\langle\,\mathbf{x}_t^i,\,F^i(\mathbf{x}_t)\,\rangle\,]^2+\sum_{i,s}x_t^{is}\,[\,F^{is}(\mathbf{x}_t)-\langle\,\mathbf{x}_t^i,\,F^i(\mathbf{x}_t)\,\rangle\,]\langle\,\mathbf{x}_t^i,\,F^i(\mathbf{x}_t)\,\rangle.$$

But, for each $i\in\mathcal{N}$, $\sum_s x_t^{is}\,[\,F^{is}(\mathbf{x}_t)-\langle\,\mathbf{x}_t^i,\,F^i(\mathbf{x}_t)\,\rangle\,]\langle\,\mathbf{x}_t^i,\,F^i(\mathbf{x}_t)\,\rangle=0$. The conclusion follows. $\square$

Remark 7.33. Naturally, Theorem 7.11 is a corollary of Theorem 7.32.

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