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Surfaces in 3D contact sub-Riemannian manifolds and controllability of nonlinear ODEs

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Abstract

The four chapters of this thesis present original results about smooth surfaces in a 3-dimensional contact sub-Riemannian manifold, and properties involving controllability in geometric control theory. They are preceded by an introduction, which gives an overview of the results and the previous literature.

The topic of smooth surfaces is studied from two viewpoints. First, given a surface in a 3-dimensional contact sub-Riemannian manifold, we investigate the metric structure induced on the surface, in the sense of length spaces. We define a new metric coefficient at any characteristic point, which determines locally the characteristic foliation of the surface, and we identify some global conditions for the induced distance to be finite. In particular, we prove that the induced distance is finite for surfaces with the topology of a sphere embedded in a tight coorientable distribution, with isolated characteristic points.

Second, we study a new canonical stochastic process on such surfaces. Precisely, employing the Riemannian approximations with respect to the Reeb vector field of the sub-Riemannian manifold, we obtain a second order partial differential operator on the surface arising as a limit of Laplace–Beltrami operators. The stochastic process associated with this limiting operator moves along the characteristic foliation induced on the surface by the contact distribution. For this stochastic process we show that elliptic characteristic points are inaccessible, while hyperbolic characteristic points are accessible from the separatrices. We illustrate this process with examples, and we recognise some well-known stochastic processes appearing on certain surfaces embedded in the canonical model spaces for sub-Riemannian structures on 3-dimensional Lie groups.

Concerning controllability, we show that a control system on a connected manifold satisfying the local reachability property is controllable, as it was somehow implicitly expected. Herein we say that a control systems satisfies the local reachability property if the attainable sets from any initial state are a neighbourhood of the respective initial states, while a system is controllable if the attainable set from every state is the entire state manifold. Quite surprisingly, the question of whether local reachability implies controllability seems not to have been considered in the literature, despite the apparent absence of elementary arguments justifying the implication.

Finally, we show that a bilinear control system is approximately controllable if and only if it is controllable in $\mathbb{R}^n \setminus \{0\}$. We approach this property by looking at the foliation made by the orbits of the system, and by showing that there does not exist a codimension-one foliation in $\mathbb{R}^n \setminus \{0\}$ with dense leaves that are everywhere transversal to the radial direction. The proposed geometric approach allows to extend the result to homogeneous systems that are angularly controllable.

Keywords: contact geometry, sub-Riemannian manifold, Stochastic process, controllability, local reachability, approximate controllability, bilinear control systems, foliations, length space, Riemannian approximation, Gaussian curvature, Heisenberg group

Resumé

Les quatre chapitres de cette thèse contiennent des résultats originaux relatifs aux surfaces dans une variété sous-riemanienne de contact de dimension trois, et à certaines propriétés concernant la contrôlabilité en théorie géométrique du contrôle. Ils sont précédés par une introduction, qui donne un aperçu de ces résultats et de la littérature antérieure.

Nous avons étudié le sujet des surfaces de deux points de vue. En premier lieu, étant donnée une surface dans une variété sous-riemannienne de contact de dimension trois, nous examinons la structure métrique induite sur la surface, au sens des espaces de longueur. Nous définissons un nouveau coefficient métrique en tout point caractéristique de la surface, et nous identifions des conditions globales pour que la distance induite soit finie. En particulier, nous montrons que la distance induite est finie pour des surfaces avec la topologie d'une sphère, plongées dans une distribution coorientable tendue, et avec des points caractéristiques isolés.

En second lieu, nous étudions un nouveau processus stochastique sur des telles surfaces. Précisément, en utilisant l'approximation riemanienne par rapport au champ de Reeb de la structure sous-riemanienne, nous obtenons un opérateur différentiel d'ordre deux sur la surface résultant de la limite d'opérateurs de Laplace–Beltrami. Le processus stochastique associé avec cet opérateur se déplace le long du feuilletage caractéristique induit sur la surface par la distribution de contact. Pour ce processus stochastique nous montrons que les points caractéristiques elliptiques sont inaccessibles, tandis que les points caractéristiques hyperboliques sont accessibles à travers les séparatrices. Nous illustrons ce processus avec des exemples, et nous reconnaissons des processus stochastiques classiques qui apparaissent sur certaines surfaces plongées dans les espaces modèles de structure sous-riemanienne sur les groupes de Lie de dimension trois.

Quant à la contrôlabilité, nous montrons qu'un système qui satisfait la propriété d'atteignabilité locale dans une variété connexe est contrôlable. Ci-dessus nous disons qu'un système de contrôle satisfait la propriété d'atteignabilité locale si les ensembles atteignables à partir de tout état sont un voisinage de l'état de départ, tandis que le système est contrôlable si les ensembles atteignables à partir de tout état coïncident avec la variété entière. Étrangement, le fait que l'atteignabilité locale implique la contrôlabilité globale semble ne pas avoir été considéré dans la littérature, en dépit de l'apparente absence d'arguments élémentaires justifiants cette implication.

Pour conclure, nous montrons qu'un système de contrôle bilinéaire est contrôlable de façon approchée si et seulement s'il est contrôlable en $\mathbb{R}^n \setminus \{0\}$. Nous étudions ce problème en analysant le feuilletage défini par les orbites du système, et en montrant qu'il n'existe pas de feuilletage de codimension un en $\mathbb{R}^n \setminus \{0\}$ dont les feuilles sont denses et partout transversales à la direction radiale. L'approche géométrique ainsi proposée permet d'étendre ce résultat aux systèmes homogènes qui sont contrôlables angulairement.

Mots-clés : géométrie de contact, variété sous-riemanienne, processus stochastique, contrôlabilité, atteignabilité locale, contrôlabilité approchée, système de contrôle bilinéaire, feuilletage, espace de longueur, approximation riemanienne, courbure de Gauss, groupe d'Heisenberg

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Contents

Chapter 1

Introduction

Contents

In this chapter we present some original results about controllability in geometric control theory, and about properties of smooth surfaces in a three-dimensional contact sub-Riemannian manifold. The results are accompanied by the relevant literature and are proven in the forthcoming chapters.

Precisely, in Section 1.1 we introduce control systems, and we present some new relations:

- between controllability and local reachability (in Subsection 1.1.1),
- between controllability and approximate controllability (in Subsection 1.1.3).

The former result is discussed in Chapter 5, following my paper [BCFS21] (joint work with U. Boscain, V. Franceschi and M. Sigalotti), soon to be submitted for review. The latter is developed in Chapter 4 and published in [CS21] (joint work with M. Sigalotti).

Next, in Section 1.2 we present some new metric properties of surfaces embedded in 3D contact sub-Riemannian manifolds. Indeed, we present:

- a new metric invariant $\widehat{K}$ at the characteristic points (in Subsection 1.2.2),
- an analysis of the length distance induced on surfaces (in Subsection 1.2.3).

These results are discussed more extensively in Chapter 2 and published in my paper [BBC21] (joint work with D. Barilari and U. Boscain).

Finally, in Section 1.3 we discuss the properties of a new canonical stochastic process defined on such surfaces. These results are discussed in Chapter 3 and published in my paper [BBCH21] (joint work with D. Barilari U. Boscain and K. Habermann).

Notations. In what follows M is a smooth n-dimensional manifold. We denote the vector fields of M by capital letters such as X, Y and Z. The notations $X(f) = Xf$ and $X(q)$ indicate, respectively, the derivative of a smooth function f in $C^\infty(M)$ with respect to X, and the derivation based at a point q in M defined by X. The $C^\infty(M)$-module of vector fields of M is denoted $\Gamma(TM)$, which with the Lie bracket $[\cdot,\cdot]$ is a Lie algebra. The flow at time t of a vector field X is denoted by e^{tX}. For notational simplicity we assume that the vector fields are complete, i.e., their flows are defined for all t in $\mathbb{R}$.

1.1 Control systems associated with a family of vector fields

Let $\mathcal{F} \subset \Gamma(TM)$ be any set of smooth vector fields on a manifold M. Let Ω be a set of indices for the family $\mathcal{F}$, i.e., $\mathcal{F} = \{X_u \mid u \in \Omega\}$. The *control system* associated to the family $\mathcal{F}$ is the system

$$\dot{p}(t) = X_{u(t)}(p(t)), \qquad p \in M,\ X_{u(t)} \in \mathcal{F}, \tag{C}$$

where as *control* we use maps $u \in \mathcal{U}_{pc}$, where $\mathcal{U}_{pc} = \bigcup_{T\geq 0}\{u : [0,T] \to \Omega \mid u \text{ piecewise constant}\}$. Following the vocabulary of control systems, an element $u \in \Omega$ is called a *control parameter*, Ω is the *space of control parameters*, points in M are *states*, and M is the *state space*.

Since (for the moment) we use piecewise constant controls, the differential equation (C) is defined in the classical sense, up to a discrete set of times in which the control is not continuous. Therefore, once a control u is fixed, the solution of (C) is determined by the initial conditions. Let us note $\phi(t,p,u)$ the value at time t of the solution of (C) starting from p. Precisely, consider a control $u\colon [0,T] \to \Omega$ such that there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$ and control parameters $u_1, \ldots, u_k \in \Omega$ satisfying

$$u(t) = u_i, \qquad \forall t \in (t_{i-1}, t_i),\ i = 1, \ldots, k.$$

If $t \in [t_{i-1}, t_i]$ for a certain $i = 1, \ldots, k$, then

$$\phi(t,p,u) = e^{(t-t_{i-1})X_{u_i}} \circ e^{(t_{i-1}-t_{i-2})X_{u_{i-1}}} \circ \cdots \circ e^{t_1 X_{u_1}}(p),$$

namely $\phi(t,p,u)$ is constructed by concatenating the flows of the vector fields in $\mathcal{F}$ indexed by the control u. The *attainable set* $\mathcal{A}_p$ from a state p in M for system (C) is the set of points reached by solutions of (C) starting from p using positive times; precisely,

$$\mathcal{A}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1, \ldots, t_k \geq 0,\ X_1, \ldots, X_k \in \mathcal{F}\}.$$

Similarly, the set of points reached using positive and negative times is called the *orbit* $\mathcal{O}_p$ of a state x; precisely,

$$\mathcal{O}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1, \ldots, t_k \in \mathbb{R},\ X_1, \ldots, X_k \in \mathcal{F}\}.$$

Attainable sets are of greater interest than orbits from the point of view of control theory, since they are obtained following strictly the vector fields in $\mathcal{F}$. Indeed, to follow the flow of a vector field $X \in \mathcal{F}$ for negative time is equivalent to follow the flow of $-X$, but $-X$ need not to be in the family $\mathcal{F}$. Note that orbits and attainable sets coincide if the family $\mathcal{F}$ is symmetric, i.e., if $-\mathcal{F} = \mathcal{F}$.

Sometimes we want to use measurable, essentially bounded functions as controls, instead of piecewise constant maps. When this is the case, we implicitly assume that Ω is a subset of $\mathbb{R}^m$, for some $m \in \mathbb{N}$, and that a smooth function $X : M \times \Omega \to TM$ parametrises $\mathcal{F}$, i.e., $\mathcal{F} = \{X(\cdot, u) \mid u \in \Omega\}$. To adhere to the notation in system (C), we continue to write $X_u = X(\cdot, u)$ for all u in Ω. We denote the set of essentially bounded controls as

$$\mathcal{U}_\infty = \bigcup_{T\geq 0} L^\infty([0,T], \Omega).$$

Fixed a control u in $\mathcal{U}_\infty$, the non-autonomous differential equation (C) is well-posed in the space of absolutely continuous functions. More precisely, for any given initial condition p in M, there exist

$T > 0$ and a neighbourhood V of p such that $\phi(t, q, u)$ is defined for $(t, q) \in [0, T] \times V$ and absolutely continuous with respect to time. Moreover, for $t \in [0, T]$, the flow $\phi(t, \cdot, u)$ restricted to V is a local diffeomorphism (see, e.g., [Jea17, Thm. 6.2] or [Son90, Thm. 1]).

In what follows $\mathcal{U}$ denotes one of the control set presented above. (Observe that $U_{pc} \subset \mathcal{U}_\infty$.) The attainable set from a state p in M is written in full generality as

$$\mathcal{A}_p = \{\phi(T, p, u) \mid T \geq 0,\ u \in \mathcal{U},\ \phi(\cdot, p, u) \text{ is defined on } [0, T]\}.$$

System (C) is said to be *controllable* if the attainable set from any state in M coincides with the entire state space, i.e.,

$$\mathcal{A}_p = M, \qquad \forall\, p \in M.$$

1.1.1 Local reachability and controllability

System (C) satisfies the *local reachability* property if, for each $p \in M$ the attainable set $\mathcal{A}_p$ contains a neighbourhood of p, i.e.,

$$p \in \operatorname{Int} \mathcal{A}_p, \qquad \forall\, p \in M. \tag{LR}$$

The property of local reachability has been studied extensively in the literature, especially in the stronger forms of *small-time* local reachability (for which the attainable set $\mathcal{A}_p$ is replaced by the set of points reachable from p within an arbitrarily small positive time) and *localised* local reachability (for which one considers the set of points reachable from p by admissible trajectories that stay in an arbitrarily small neighbourhood of p). If both properties hold at the same time, we stay that system (C) satisfies *small-time localised local reachability.* Interest in small time local reachability is motivated, for example, by its relation with the continuity of the minimum time function, as explained in [Sus87]. (Observe that in this reference the term local controllability is used instead of local reachability: here we preferred the latter not to confuse it with the one used, e.g., in [Cor07, Def. 3.2].)

Although it is somehow expected that controllability can be deduced by its local counterparts, we found that the issue has not been really discussed in the literature. An exception is [CLH+07, Sec. 12.3], where it is stated (without proof) that small-time localised local reachability implies controllability. However, in some situation it is more natural to consider local reachability instead of localised local reachability: for instance, a linear control system $\dot{x} = Ax + Bu$ satisfies localised reachability only if the range of B is the entire state space. (See [BS83] for more results on controllability and local reachability of control-affine systems with unbounded controls.) Thus, in Chapter 5 we prove the following more general statement.

Theorem 1.1 [BCFS21, Thm. 1]

Assume that M is connected and that system (C) is equipped with controls in $\mathcal{U}_{pc}$ or $\mathcal{U}_\infty$. If system (C) satisfies the local reachability property, then it is controllable.

Our proof of Theorem 1.1 is based on the following property: if system (C) satisfies the local reachability property, then for all p, q in M one has that $p \in \mathcal{A}_q$ if and only if $q \in \mathcal{A}_p$ (see Lemma 5.2). This property is shown by proving that the trajectories of (C) can be retraced back by finding a control driving their endpoints to their starting points. More precisely, assume $q \in \mathcal{A}_p$ and consider a control u such that $q = \phi(T, p, u)$. For t in a left neighbourhood of T, the states $\phi(t, p, u)$ can be reached from q due to local reachability. By repeating this argument and concatenating controls, one can find smaller and smaller times $t \geq 0$ such that $\phi(t, p, u)$ can be reached from q. In order to reach $p = \phi(0, p, u)$ one has to show that the sequence of times t found following such a procedure eventually attains zero, unlike the situation depicted in Figure 1.1.

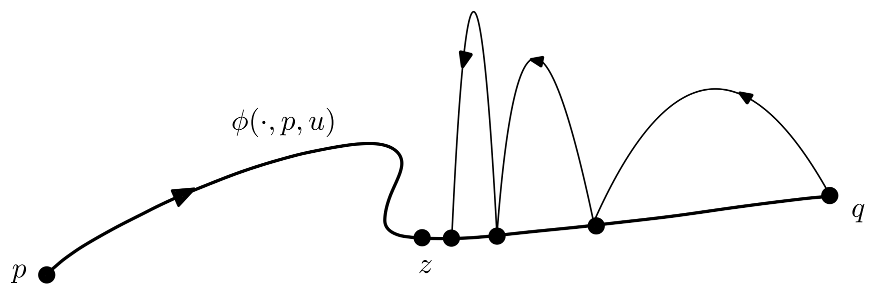

Figure 1.1: When retracing back the trajectory $\phi(\cdot,p,u)$ the reachable sets might get smaller and smaller and collapse to a point z before attaining x, since a priori their size is not lower semi-continuous. Lemma 5.2 shows that this situation cannot happen, proving a key step for the proof of Theorem 1.1.

Remark 1.2. The argument of the proof of Theorem 1.1 generalises to more general classes of controls, provided that the control system remains well-posed in the space of absolutely continuous functions, in the sense expressed in Section 1.1. Here we decided to use essentially bounded controls rather than to stick with piecewise constant controls in order to show that the differences which arise between controls in $\mathcal{U}_\infty$ and $\mathcal{U}_{pc}$ do not affect Theorem 1.1.

We mention that if localised local reachability holds, then the proof that (C) is controllable follows by a simpler argument than the one in Theorem 1.1 (see Proposition 5.4 and Remark 5.3). Still the fact that (C) is controllable does not follow immediately from the definitions, since reachability is not a symmetric property. Indeed, the fact that one can reach an open neighbourhood of any initial state does not imply directly that one can control the neighbourhood back to the initial state.

Finally, Theorem 1.1 implies that any sufficient condition for local reachability also yields controllability. Actually, in the literature it is more common to find conditions for (small-time) localised local reachability, since those can be deduced from Lie algebraic arguments (see, e.g., [KN21] and references therein).

1.1.2 Orbits, distributions and Lie-brackets

In this section we recall some important classical properties of orbits and attainable sets. First, orbits are immersed submanifolds of M. Precisely, an *immersed k-dimension submanifold* is a subset $S \subset M$ with a structure of smooth k-dimensional manifold (not necessarily with the topology inherited by M) such that the pushforward of the inclusion $i : S \hookrightarrow M$ satisfies $\dim i_*(T_pS) = k$ for all $p \in S$, i.e., the inclusion is an *immersion.*

Theorem 1.3 (Orbit theorem, [Sus73])**.** *For every $q \in M$, the orbit $\mathcal{O}_q$ is a connected, immersed submanifold of M. Moreover, for all $p \in \mathcal{O}_q$,*

$$T_p\mathcal{O}_q = \operatorname{span}_{\mathbb{R}}\big\{(e_*^{t_kX_k} \circ \cdots \circ e_*^{t_1X_1}Y)(p) \mid k \in \mathbb{N},\ t_1,\dots,t_k \in \mathbb{R},\ X_1,\dots,X_k,Y \in \mathcal{F}\big\}. \qquad (1.1)$$

It follows from the orbit theorem that the orbits define a *foliation*, i.e., a partition of X in connected, immersed submanifolds (called *leaves*), possibly of varying dimension. Furthermore, the foliation described by the orbits is a *smooth* foliation, i.e., for any p in M and vector $v \in T_pM$ tangent to the leaf through p can be extended to a smooth vector field everywhere tangent to the leafs of the foliation. Indeed, the fact that foliations arising as orbits of control systems as in (C) are smooth is a consequence of the orbit theorem and formula (1.1). Conversely, any smooth foliation is the orbit partition of a control system. For this, it suffices to take as family of admissible vector fields the collection of all vector fields that are everywhere tangent to the leaves.

Remark 1.4. In some texts the term foliation describes what we shall call here a *regular foliation*, i.e., a foliation which admits locally around each point a chart (U,x) such that $x(U)$ can be written as

$x(U) = V' \times V'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, for a fixed $k \in \mathbb{N}$, and such that the intersection of a leaf with U is either empty or the countable union of sets of the form $x^{-1}(V' \times \{c\})$ for $c \in V''$. Such a chart is called a *foliated* chart, and k is the dimension of the foliation.

We will call a *distribution* D the arbitrary assignment, for each point $p \in M$, of a linear subspace $D_p \subset T_pM$. Given a vector field $X\colon M \to TM$ we say that X is a section of D if $X_p \in D_p$ for all $p \in M$. We denote by $\Gamma(D)$ the set of sections of D. A distribution D is *smooth* if, for all points $p \in M$, there exist smooth sections $X_1, \ldots, X_k$ of D such that $(X_1)_p, \ldots, (X_k)_p$ is a basis of D_p. In such a case, we call k the *rank* of D in p.

Now, define $\operatorname{Lie}\mathcal{F}$ to be the smallest Lie subalgebra of $\Gamma(TM)$ containing $\mathcal{F}$. Precisely,

$$\operatorname{Lie}\mathcal{F} = \operatorname{span}_{C^\infty(U)}\big\{[X_1, \ldots [X_{k-1}, X_k] \ldots] \mid k \in \mathbb{N},\ X_1, \ldots, X_k \in \mathcal{F}\big\}.$$

For any family $\mathcal{G}$ of vector fields, denote by $D^{\mathcal{G}}$ the smooth distribution defined by assigning, for all $p \in M$, the subspace $D^{\mathcal{G}}_p = \operatorname{span}\{X_p \mid X \in \mathcal{G}\}$. Since the Lie bracket of two vector fields X and Y can be expressed as

$$[X, Y]_p = \frac{d}{dt}\Big|_{t=0} e^{-\sqrt{t}Y} \circ e^{-\sqrt{t}X} \circ e^{\sqrt{t}Y} \circ e^{\sqrt{t}X}(p),$$

formula (1.1) for the tangent space of an orbit implies that

$$D^{\mathcal{F}}_p \subset D^{\operatorname{Lie}\mathcal{F}}_p \subset T_p\mathcal{O}_q, \qquad \forall\, q \in M,\ p \in \mathcal{O}_q. \tag{1.2}$$

We say that the control system (C) is Lie-determined if the last inclusion in (1.2) is an equality, i.e.,

$$D^{\operatorname{Lie}\mathcal{F}} = T\mathcal{O}. \tag{1.3}$$

Under some rather general hypotheses, system (C) is Lie-determined. In this regard, let us recall some additional definitions. As before, let $\mathcal{G}$ be a family of vector fields of M.

(i) $\mathcal{G}$ is *analytic* if M is an analytic manifold and, for all $X \in \mathcal{G}$, X is analytic.

(ii) $\mathcal{G}$ is *locally finitely generated* if for every $q \in M$ there exist an open neighbourhood U of q and finite vector fields $X_1, \ldots, X_k$ in $\mathcal{G}$ such that $\mathcal{G}|_U \subset \operatorname{span}_{C^\infty(U)}\{X_1|_U, \ldots, X_k|_U\}$.

(iii) $D^{\mathcal{G}}$ is *$\mathcal{G}$-invariant* if for all X in $\mathcal{G}$ one has $e^{tX}_*(D^{\mathcal{G}}_p) = D^{\mathcal{G}}_{e^{tX}p}$ for all $p \in M$ and t.

The theorem of Nagano [Nag66], the theorem of Hermann [Her62], and the theorem of Stefan-Sussmann [Sus73, Ste74] state that if, respectively, $\mathcal{F}$ is analytic, $\operatorname{Lie}\mathcal{F}$ is locally finitely generated, $D^{\operatorname{Lie}\mathcal{F}}$ is $\operatorname{Lie}\mathcal{F}$-invariant, then system (C) is Lie-determined. We recall here also the renewed Frobenius theorem [Fro77] stating that if $D^{\operatorname{Lie}\mathcal{F}}$ is a constant rank distribution, then system (C) is Lie-determined and the foliation described by the orbits is regular. For additional details to the subject we refer to [Lav18].

Under the assumption that system (C) is Lie-determined, it can be shown that orbits cannot be of a greater dimension than attainable sets. This is the content of Krener's theorem.

Theorem 1.5 (Krener's theorem [Kre74]). *Assume that* (C) *is Lie-determined. Then, for every p in M, the attainable set $\mathcal{A}_p$ has a nonempty interior in the orbit $\mathcal{O}_p$.*

The family $\mathcal{F}$ is said to satisfy the *Lie algebra rank condition at* $p \in M$ if the evaluation at p of the Lie algebra generated by $\mathcal{F}$ has maximal dimension, i.e.,

$$D^{\operatorname{Lie}\mathcal{F}}_p = T_pM. \tag{1.4}$$

Remark 1.6. When M is a real-analytic manifolds and the vector fields in $\mathcal{F}$ are real-analytic, the following holds true: if system (C) satisfies the local reachability property, then $\mathcal{F}$ satisfies the Lie-algebra rank condition at any point. See for instance [SJ72, Thm. 3.1].

1.1.3 Approximate controllability for bilinear control systems

A first result obtained studying approximate controllability is that, if system (C) is approximately controllable, then the foliation described by its orbits is a regular foliation of M.

Lemma 1.7 ([CS21, Lem. 4]). *Assume that system* (C) *is approximately controllable. Then, the orbits of* (C) *form a regular foliation of M with dense leaves.*

The proof of this lemma, which is given in Chapter 4, follows from the lower semi-continuity of the dimension of the orbits. Lemma 1.7 provides no direct information about controllability. Indeed, there might be only one orbit and still the attainable sets might not coincide with M. However, with the additional hypothesis that (C) is Lie-determined, one deduces the following.

Corollary 1.8 ([CS21, Cor. 5]). *Assume that system* (C) *is Lie-determined and approximately controllable. Then, exactly one of the following alternatives holds:*

(a) *$\mathcal{F}$ satisfies the Lie algebra rank condition at all points in M; hence, system* (C) *is controllable.*

(b) *There exists an integer k with $0 < k < n$ such that the orbits of* (C) *form a regular k-dimensional foliation of N with dense leaves.*

Corollary 1.8 turns out to be useful if one can exclude the existence of a regular foliation of M with the properties described in *(b)*; this might be possible thanks to the particular form of system (C) or some topological properties of M. This is what we managed to do for the control systems of the form

$$\dot{p} = A(t)p, \qquad p \in \mathbb{R}^n \setminus \{0\},\ A(t) \in \mathcal{M} \tag{BL}$$

for piecewise constant controls $A : [0, +\infty) \to \mathcal{M}$ taking values in a subset $\mathcal{M}$ of the space $M_n(\mathbb{R})$ of $n \times n$ matrices with real coefficients, $n \geq 1$. By a slight abuse of notation, we refer to control systems such as (BL) as *bilinear control systems*, although the latter term usually denotes systems for which $M(t) = A + u^1(t)B_1 + \cdots + u^m(t)B_m$ for some fixed $A, B_1, \ldots, B_m \in M_n(\mathbb{R})$, with control $t \mapsto (u^1(t), \ldots, u^m(t))$ taking values in some subset Ω of $\mathbb{R}^m$. For an introduction to bilinear control system we refer to [CK00, Ell09]. Precisely, we proved the following result.

Theorem 1.9 [CS21, Thm. 1]

Consider the bilinear control system (BL) of $\mathbb{R}^n \setminus \{0\}$. System (BL) is approximately controllable if and only if it is controllable.

The proof, which is given in Chapter 4, is as follows. First, we deduce from [BS20, BV13] that, if the projection of (BL) onto $\mathbb{RP}^{n-1}$ is approximately controllable, then the orbits of (BL) are transversal to the radial direction. Corollary 1.8 applies since (BL) is Lie-determined due to Nagano theorem. If (BL) is not controllable, then the orbit foliation has codimension one with leaves transversal to the radial direction. Next, we show in Lemma 4.5 that such a foliation cannot have dense leaves, giving the desired result. As a byproduct of the above method, we extend Theorem 1.9 to angularly controllable homogenous control systems; see Corollary 4.6.

This result shows that the a priori weaker notion of approximate controllability implies controllability with no additional assumption, other than that the systems being a finite-dimensional bilinear control system. A possible way of applying Theorem 1.9 is the following: if for a bilinear system one is able to identify vector fields compatible with (BL), in the sense of [AS02, Def. 8.4], which lead to approximate controllability when added to the admissible ones, then controllability of (BL) follows without the need of checking the Lie algebra rank condition. Such an extension argument by compatible vector fields is at the core, for instance, of the results in [Che05], which can therefore be improved by our result. In particular, Theorem 1.9 implies that the hypotheses ii) and iii) on the existence of stable and antistable equilibria can be dropped from [Che05, Theorem 4.3]. Similarly, [Che05, Prop. 3.3 and 3.6] can be strengthened by replacing in their conclusions approximate and practical controllability by controllability.

The result in Theorem 1.9 is in sharp contrast with the case of bilinear systems in infinite-dimension: when the controlled operators B_i appearing in the representation $M(t) = A + u^1(t)B_1 + \cdots + u^m(t)B_m$ are bounded, these systems cannot be controllable (see [BMS82, Thm 3.6] and also [BCC20] for recent extensions), while there exist some criteria for approximate controllability (see, e.g., [Kha10, Chap. 4 and 9] and [CFK17, BCS14]).

Theorem 1.9 does not hold for general finite-dimensional systems (to which the notions of controllability and approximate controllability straightforwardly extend). Indeed, while controllability clearly implies approximate controllability, the converse may fail to hold. A standard example can be provided using the irrational winding of a line in the torus $\mathbb{T}^n$, $n \geq 2$. On the other hand, the equivalence stated in Theorem 1.9 is known to hold for some other classes of control systems. This is the case for *linear control systems*, that is, systems of the form

$$\dot{x} = Ax + Bu(t), \qquad u : [0, +\infty) \to \mathbb{R}^m, \quad x \in \mathbb{R}^n, \tag{1.5}$$

with $A \in M_n(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$. Indeed, the approximate controllability of (1.5) implies in particular that the attainable set from the origin $\mathcal{A}_0$ is dense in $\mathbb{R}^n$. Since $\mathcal{A}_0$ is a linear space (and in particular it is closed), it follows that $\mathcal{A}_0 = \mathbb{R}^n$, which is well known to be equivalent to the controllability of (1.5) due to the linear structure of the system. Few other classes of control systems for which approximate controllability implies controllability are known: closed quantum systems on S^{n-1} [BGRS15, Thm. 17]; right-invariant control systems on simple Lie groups (as it follows from [JS72, Lem. 6.3] and [Smi42, Note at p. 312]); control systems obtained by projecting onto $\mathbb{RP}^{n-1}$ systems of the form of (BL) [BS20, Prop. 44].

1.2 Sub-Riemannian manifolds

Let $\mathcal{F}$ be a family of vector fields on a smooth manifold M. Assume that the distribution defined by $\mathcal{F}$ has constant rank, i.e., there exists $k \in N$ such that

$$\dim D_p^{\mathcal{F}} = k \qquad \forall p \in M. \tag{1.6}$$

We say that a locally Lipschitz curve $\gamma\colon I \to M$ is *horizontal* with respect to $\mathcal{F}$ if $\dot\gamma(t) \in D^{\mathcal{F}}$ for almost every $t \in I$. Recall that a Lipschitz curve admits a derivative almost everywhere; e.g., see [Hei04, p. 18]. We call a curve *admissible* if it is Lipschitz and horizontal. In other words, a curve is admissible if and only if for any sufficiently small intervals $I' \subset I$ there exist $X_1, \ldots, X_k$ in $\mathcal{F}$ and measurable, essentially bounded functions $u^1, \ldots, u^k\colon I' \to \mathbb{R}$ such that

$$\dot\gamma(t) = u^1(t)X_1(\gamma(t)) + \cdots + u^k(t)X_k(\gamma(t)), \qquad \text{for almost every } t \in I'. \tag{1.7}$$

This is a version of the control equation (C), with essentially bounded controls and symmetric control parameters. Now, assume that a smooth scalar product g has been chosen on the distribution $D^{\mathcal{F}}$. In this case, the *sub-Riemannian length* of an admissible curve γ is defined as

$$L_{sR}(\gamma) = \int_I \sqrt{g(\dot\gamma, \dot\gamma)}. \tag{1.8}$$

Ultimately, one would like to define the *sub-Riemannian distance* between two points p and q in M as

$$d_{sR}(p, q) = \inf\{L_{sR}(\gamma) \mid \gamma : [a, b] \to M \text{ admissible}, \ \gamma(a) = p, \ \gamma(b) = q\}. \tag{1.9}$$

A property which is required is that the topology defined by d_{sR} should coincide with the topology of M. This is the case when the family $\mathcal{F}$ to satisfy the Lie algebra rank condition at every point. The latter property depends uniquely on the distribution $D^{\mathcal{F}}$, motivating the following definition. A smooth constant rank distribution D is said to be *bracket-generating* if, for any family $\mathcal{F}$ of vector fields with $D = D^{\mathcal{F}}$, the family $\mathcal{F}$ satisfies the Lie algebra rank condition at every point. Finally, a *sub-Riemannian structure* on a manifold M consists of a smooth, bracket-generating distribution D, and a smooth scalar product g defined on D. The triple (M, D, g) is a *sub-Riemannian manifold.*

The above definition of sub-Riemannian manifold is not the most general one can give, but it is sufficient for the purposes of this thesis. For a general introduction to sub-Riemannian geometry we refer the reader to the monographs [ABB20], [LD17], [Jea14], [Mon02] and [Rif14]. In what follows it is assumed that M is a three-dimensional manifold and D is a distribution of rank two.

1.2.1 Surfaces in 3D contact manifolds

From now on, the manifold M is supposed to be 3-dimensional. The distribution D is said to be *coorientable* if there exists a one-form ω on M such that

$$\ker \omega_p = D_p, \qquad \forall p \in M. \tag{1.10}$$

Under the assumption that D is coorientable, the pair (M, D) is called a *contact manifold* if any one-form ω satisfying locally (1.10) satisfies also $\mathrm{d}\omega|_D \neq 0$, or equivalently

$$\omega \wedge \mathrm{d}\omega \neq 0. \tag{1.11}$$

Such one-form ω is called a *contact form.* One can verify that if (M, D) is a contact manifold then, for every p in M and $X_1, X_2 \in \Gamma(D)$, one has

$$\operatorname{span}\{X_1(p), X_2(p)\} = D_p \implies [X_2, X_1]_p \notin D_p. \tag{1.12}$$

In this section we recall some relevant facts about contact manifolds, and we refer to [Etn03] and [Gei08] for an introduction to the subject.

Les S be an embedded surface in M. A point p in S is a *characteristic point* if the tangent space T_pS coincides with the distribution D_p. The set of characteristic points of S is the *characteristic set*, noted $\Sigma(S)$. The characteristic set is closed due to the lower semi-continuity of the rank, and it cannot contain open sets due to the contact condition. Moreover, since the distribution D is contact, the set $\Sigma(S)$ is contained in a 1-dimensional submanifold of S (see Lemma 2.5) and, generically, it is composed of isolated points (see [Gei08, Par. 4.6]).

Outside the characteristic set, the intersection $TS \cap D$ is a one-dimensional distribution and defines (due to Frobenius theorem) a regular one-dimensional foliation on $S \setminus \Sigma(S)$. This foliation extends to a smooth foliation (cf. Subsection 1.1.2) of S by adding a singleton at every characteristic point. The resulting foliation is the *characteristic foliation* of S. Characteristic foliations of surfaces in 3D contact manifolds are studied in numerous references; in this regard we refer to [Gir91, Gir00, Ben83]. See Figure 1.3 (at the end of the current chapter) for a graphical representation of the characteristic foliation on a sphere in the Heisenberg group.

We call characteristic vector fields the vector fields of S whose orbit partition coincides with the characteristic foliation of S. Precisely, given an open set U in S, a vector field X is a *characteristic vector field* of S in U if, for all q in U,

$$\operatorname{span}_{\mathbb{R}} X(q) = \begin{cases} \{0\}, & \text{if } q \in \Sigma(S), \\ T_qS \cap D_q, & \text{otherwise,} \end{cases} \tag{1.13}$$

and satisfies the condition

$$\operatorname{div} X(p) \neq 0, \qquad \forall\, p \in \Sigma(S) \cap U. \tag{1.14}$$

Notice that $\operatorname{div} X(p)$ is well-defined since $X(p) = 0$, i.e., p is a characteristic point, and it is independent on the volume form; in particular $\operatorname{div} X(p) = \operatorname{tr} DX(p)$. Under some classical hypothesis, one can assure the existence of a global characteristic vector field as recalled in the following lemma. Moreover, these hypotheses always hold locally, in an open neighbourhood of any point.

Lemma 1.10 ([Gei08, Par. 4.6]). *Assume that S is orientable and that D is coorientable. Then, S admits a global characteristic vector field; moreover, the characteristic vector fields of S are the vector fields X for which there exists a volume form Ω of S such that*

$$\Omega(X, Y) = \omega(Y) \qquad \textit{for all } \; Y \in TS. \tag{1.15}$$

Indeed, in [Gei08] it is shown that, if a vector field satisfies (1.15), then it satisfies (1.13) and (1.14). Reciprocally, a vector field $\bar{X}$ satisfying (1.13) is a multiple of any other vector field X satisfying (1.15) for some function ϕ with $\phi|_{S \setminus \Sigma(S)} \neq 0$; additionally, if (1.14) holds, then $\phi|_{\Sigma(S)} \neq 0$; thus, $\bar{X}$ satisfies (1.15) with $\frac{1}{\phi}\Omega$ as volume form of S.

Remark 1.11. Formula (1.15) means that the characteristic vector fields are dual to the restrictions of the contact forms $\omega|_S$ with respect to the volume forms Ω of S. Since the volume forms of S are proportional by nowhere-zero functions, the same holds for the characteristic vector fields. In particular, if X is a characteristic vector field, then also $-X$ is a characteristic vector field.

Let us provide another way to find, locally, an explicit expression for a local characteristic vector field. Any point in S admits a neighbourhood U in M in which there exists an orthonormal frame (X_1, X_2) for $D|_U$, and a submersion u of class C^2 for which S is a level set, i.e.,

$$S \cap U = \{q \in U : u(q) = 0\}, \quad \text{and } \mathrm{d}u \neq 0 \text{ on } S \cap U\,. \tag{1.16}$$

Observe that for any point $p \in U \cap S$, one has

$$p \in \Sigma(S) \quad \text{if and only if} \quad X_1 u(p) = X_2 u(p) = 0. \tag{1.17}$$

Here we used that a vector V is in TS if and only if $Vu = 0$. Now, the vector field X_u defined by

$$X_u = (X_1 u) X_2 - (X_2 u) X_1, \tag{1.18}$$

is a characteristic vector field of S. Indeed, X_u satisfies (1.13) since it follows from the definition that, for all q in S, the vector $X_u(q) \in T_q S \cap D_q$, and $X_u(p) = 0$ if and only if $p \in \Sigma(S)$, due to (1.17). Moreover, X_u satisfies (1.14) since the divergence of X_u at the characteristic points is nonzero due to the contact condition (1.12) and the following expression

$$\operatorname{div} X_u(p) = X_2 X_1 u(p) - X_1 X_2 u(p) = [X_2, X_1] u(p), \qquad \forall\, p \in \Sigma(S).$$

In the literature, the study of surfaces in three-dimensional contact manifolds has found a lot of interest since, amongst others, the characteristic foliations of surfaces provide an important invariant used to classify contact structures. Moreover, the following theorem holds.

Theorem 1.12 (Giroux, [Gei08, Thm. 2.5.22 and 2.5.23]). *Let S_i be closed surfaces in contact three-dimensional manifolds (M_i, D_i), $i = 0, 1$, with D_i cooriented. Assume that and $\phi : S_0 \to S_1$ is a diffeomorphism between the respective characteristic foliations. Then there is a contactomorphism $\psi : N(S_0) \to N(S_1)$ of suitable neighbourhoods $N(S_i)$ of S_i, i.e., $\psi_* D_1|_{N(S_1)} = D_0|_{N(S_0)}$, with $\psi|_{S_0} = \phi$.*

Tight contact structures

Roughly, a distribution is *tight* if it does not admit an *overtwisted disk*, i.e., an embedding of a disk with horizontal boundary such that the distribution does not twists along the boundary. The notion of tight distribution will be necessary to state Theorem 1.16. However, a reader who is satisfied with the above definition might skip ahead.

To define an overtwisted disk, let us first consider an embedding of $\Delta = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ in M, and denote $\Gamma = \partial\Delta$. Let Γ be horizontal with respect to the contact distribution D, i.e., $T\Gamma \subset D$. Then, the normal bundle $N\Gamma = TM|_\Gamma/T\Gamma$ can be decomposed in two ways: the first with respect to the tangent space of Δ, i.e.,

$$N\Gamma \cong {}^{TM}\!/_{T\Delta} \oplus {}^{T\Delta}\!/_{T\Gamma}, \tag{1.19}$$

and the second with respect to the contact distribution D, i.e.,

$$N\Gamma \cong {}^{TM}\!/_{D} \oplus {}^{D}\!/_{T\Gamma}. \tag{1.20}$$

A frame (Y_1, Y_2) of $N\Gamma$ is called a *surface frame* if it respects the splitting (1.19), i.e., $Y_1 \in TM \setminus T\Delta$ and $Y_2 \in T\Delta \setminus T\Gamma$; similarly, it is called a *contact frame* if it respects the splitting (1.20). Since the contact distribution is cooriented near Δ, both bundles (1.19) and (1.20) are trivial, thus one can always find contact and surface frames.

The Thurston–Bennequin invariant of Γ, denoted by $\mathrm{tb}(\Gamma)$, is the number of twists of a contact frame of Γ with respect to a surface frame: the right-handed twists are counted positively, and the left-handed twists negatively (cf. for instance [Gei08, Def. 3.5.4]). Note that $\mathrm{tb}(\Gamma)$ is independent of the orientation of Γ. The requirement that the distribution D does not twist along the boundary of Δ is equivalent to $\mathrm{tb}(\partial\Delta) = 0$, i.e., the Thurston–Bennequin invariant of $\partial\Delta$ being zero.

An embedded disk Δ in a cooriented contact manifold (M, D) with smooth boundary $\partial\Delta$ is an *overtwisted disk* if $\partial\Delta$ is a horizontal curve of D, $\mathrm{tb}(\partial\Delta) = 0$, and there is exactly one characteristic point in the interior of the disk. Note that the elimination lemma of Giroux allows to remove the condition that there is only one characteristic point in the interior of the overtwisted disk, as discussed for instance in [Gei08, Def. 4.5.2].

1.2.2 The Riemannian approximation

The construction of Riemannian approximation of a sub-Riemannian structure is a key tool used in this thesis, and their use to define sub-Riemannian geometric invariants is a well-known technique. For a general description of the properties of the Riemannian approximation in the Heisenberg group we refer to [CDPT07]. In this thesis, we use the Riemannian approximation for two purposes. First to associate with each characteristic point a real number $\widehat{K}$, as explained below in Theorem 1.13. Second, as explained in Section 1.3, to construct a canonical stochastic process on S.

Assume that (M, D, g) is a three-dimensional manifold equipped with a cooriented distribution D of rank two. Under this assumption, one can fix a vector field X_0 transverse to the distribution, i.e.,

$$\mathrm{span}\{D, X_0\} = TM.$$

Once this choice has been made, one can extend the sub-Riemannian metric g to a Riemannian metric g^{X_0} by defining X_0 to be unitary and orthogonal to D. The Riemannian metrics $g^{\varepsilon X_0}$, for $\varepsilon > 0$, are the *Riemannian approximations* of (D, g) with respect to X_0. Precisely, for every $\varepsilon > 0$, one has

$$\langle D, X_0 \rangle_{g^{\varepsilon X_0}} = 0, \quad |X_0|_{g^{\varepsilon X_0}} = \frac{1}{\varepsilon}, \qquad \forall \varepsilon > 0.$$

To simplify the notation, once it is clear which transversal vector field has been chosen we might drop the dependance on X_0 in the superscript, writing $g^\varepsilon = g^{\varepsilon X_0}$.

Let K^{X_0} be the Gaussian curvature of S with respect to g^{X_0}. Fixed a characteristic point $p \in \Sigma(S)$, the coefficient $\widehat{K}$ will be defined as the limit for $\varepsilon \to 0$ of $K^{\varepsilon X_0}$, using a suitably normalisation of the

Lie bracket structure on the distribution. Precisely, let B^{X_0} be the bilinear form $B^{X_0}: D \times D \to \mathbb{R}$ defined by

$$B^{X_0}(X,Y) = \alpha \qquad \text{if} \qquad [X,Y] = \alpha X_0 \mod D.$$

Since D is endowed with the metric g, the bilinear form B^{X_0} admits a well-defined determinant.

Theorem 1.13 [BBC21, Thm. 1.1]

Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold. Let p be a characteristic point of S, and let X_0 be a vector field transverse to the distribution D in a neighbourhood of p. Then, in the notations defined above, the limit

$$\widehat{K}_p = \lim_{\varepsilon \to 0} \frac{K_p^{\varepsilon X_0}}{\det B_p^{\varepsilon X_0}} \tag{1.21}$$

is finite and independent on the vector field X_0.

Notice that in the previous literature the Riemannian approximation is employed to define sub-Riemannian geometric invariants *outside* of the characteristic set. For instance, in [BTV17] the authors defined the sub-Riemannian Gaussian curvature at a point $x \in S \setminus \Sigma(S)$ as

$$\mathcal{K}_S(x) = \lim_{\varepsilon \to 0} K_x^{\varepsilon X_0}, \tag{1.22}$$

and they proved that a Gauss-Bonnet type theorem holds; here the authors worked in the setting of the Heisenberg group, and with X_0 equals to the Reeb vector field of the Heisenberg group. This construction is extended in [WW20] to the affine group and to the group of rigid motions of the Minkowski plane, and in [Vel20] to a general sub-Riemannian manifold. In the latter, the author linked $\mathcal{K}_S$ with the curvature introduced in [DV16], and, when $\Sigma(S) = \emptyset$, they proved a Gauss-Bonnet theorem by Stokes formula. A Gauss-Bonnet theorem (in a different setting) was also proven in [ABS08]. We finally notice that the invariant $\mathcal{K}_S$ also appears in [Lee13], where it is called curvature of transversality. An expression for $\mathcal{K}_S$ is provided also in Proposition 1.18.

As we shall see, the coefficient $\widehat{K}_p$ determines the qualitative behaviour of the characteristic foliation near a characteristic point p. Following the terminology of contact geometry (cf. for instance [Gei08, Par. 4.6]), given a characteristic point $p \in \Sigma(S)$ and a characteristic vector field X, the point p is called *non-degenerate* if $\det DX(p) \neq 0$. Furthermore, p is called *elliptic* if $\det DX(p) > 0$, and *hyperbolic* if $\det DX(p) < 0$. In the theory of dynamical systems, saddles and hyperbolic points are, respectively, what we call hyperbolic characteristic points and non-degenerate characteristic points.

Proposition 1.14 ([BBC21, Prop. 1.2]). *Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold. Given a characteristic point p in $\Sigma(S)$, let X be a characteristic vector field X near p. Then,* $\operatorname{tr} DX(p) \neq 0$ *and*

$$\widehat{K}_p = -1 + \frac{\det DX(p)}{(\operatorname{tr} DX(p))^2}. \tag{1.23}$$

Thus, p is hyperbolic if and only if $\widehat{K}_p < -1$, and p it is elliptic if and only if $\widehat{K}_p > -1$.

This equality links $\widehat{K}_p$ to the eigenvalues of $DX(p)$, which determine the qualitative behaviour of the characteristic foliation around the characteristic point p. This relation is made explicit in Corollary 2.10 for a non-degenerate characteristic point, and in Corollary 2.12 for a degenerate characteristic point. Moreover, equation (1.23) shows that $\widehat{K}_p$ is independent on the sub-Riemannian metric, and depends only on the line field defined by D on S.

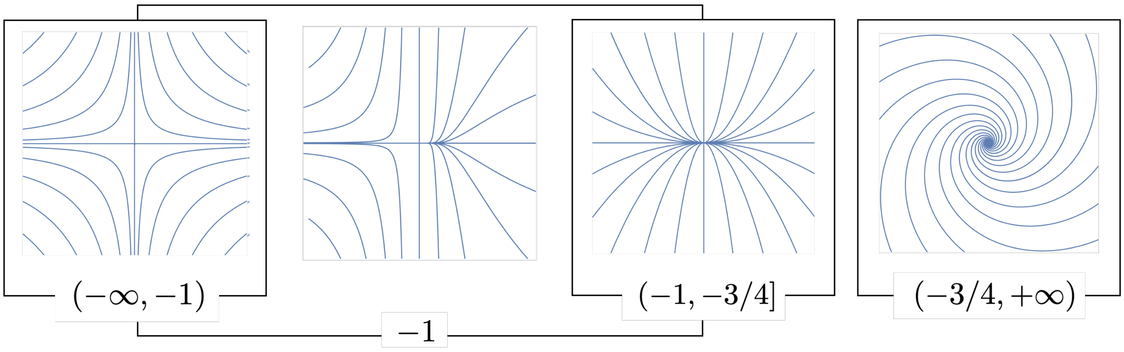

Figure 1.2: The qualitative picture for the characteristic foliation at an *isolated* characteristic point, for the corresponding values of $\widehat{K}$. Left to right, we recognise a saddle, a saddle-node, a node, and a focus.

1.2.3 Induced distance on surfaces

The study of the geometry of submanifolds S of an ambient manifold M with a given geometric structure is a classical subject. A familiar example, whose study goes back to Gauss, is that of a surface S embedded in the Euclidean space $\mathbb{R}^3$. In such a case, S inherits its natural Riemannian structure by restricting the metric tensor to the tangent space of S. The distance induced on S by this metric tensor is not the restriction of the distance of $\mathbb{R}^3$ to points on S, but rather the length space structure induced on S by the ambient space.

Things are less straightforward for a smooth 3-manifold M endowed with a contact sub-Riemannian structure (D, g). Indeed, for a two-dimensional submanifold S, the intersection $T_xS \cap D_x$ is one-dimensional for most points x in S; thus, $TS \cap D$ is not a bracket-generating distribution and there is no well-defined sub-Riemannian distance induced by (M, D, g) on S. This fact is indeed more general, as already observed in [Gro96, Sec. 0.6.B]. Nevertheless, one can still define a distance on S following the length space viewpoint: the sub-Riemannian distance d_{sR} defines the *length* of any continuous curve $\gamma : [0,1] \to M$ as

$$L_{sR}(\gamma) = \sup\left\{\sum\nolimits_{i=1}^{N} d_{sR}(\gamma(t_i), \gamma(t_{i+1})) \mid 0 = t_0 \leq \ldots \leq t_N = 1\right\},$$

and one can define $d_S : S \times S \to [0, +\infty]$ by

$$d_S(x, y) = \inf\{L_{sR}(\gamma) \mid \gamma : [0,1] \to S,\ \gamma(0) = x,\ \gamma(1) = y\}.$$

The space (S, d_S) is called a *length space*, and d_S the *induced distance* defined by (M, d_{sR}). (In the theory of length metric spaces, the induced distance d_S is called intrinsic distance, emphasising that it depends uniquely on lengths of curves in S, see [BBI01].) We stress that the induced distance d_S is not the restriction $d_{sR}|_{S\times S}$ of the sub-Riemannian distance to S.

We started the investigation by looking for necessary and sufficient conditions on the surface S so that the induced distance d_S is finite. i.e., $d_S(x, y) < +\infty$ for all points x, y in S; this is equivalent to (S, d_S) being a metric space. This can be formulated through the use of the characteristic foliation of S. Precisely, consider a continuous curve $\gamma : [0,1] \to S$. Its length is finite, i.e., $L_{sR}(\gamma) < +\infty$, if and only if γ is a reparametrisation of a curve $\bar\gamma$ horizontal with respect to D; in such case, the length of γ coincides with the sub-Riemannian length of $\bar\gamma$. We refer to [BBI01, Ch. 2] and [ABB20, Sec. 3.3] for more details. In conclusion, the distance $d_S(x, y)$ between two points x and y in S is finite if and only if there exists a finite-length continuous concatenation of leaves of the characteristic foliation of S connecting them.

From a local point of view, for the induced distance to be finite one needs characteristic points to be accessible from their complement. In this regard, we were able to prove that the one-dimensional leaves of the characteristic foliation of S which converge to a characteristic point have finite length. Precisely, let ℓ be a leaf of the characteristic foliation of S; we say that a point p in S is a *limit point* of ℓ if there exists a point q in ℓ and a characteristic vector field X of S such that

$$e^{tX}(q) \to p \qquad \text{for } t \to +\infty, \tag{1.24}$$

where e^{tX} is the flow of X. In such case, we denote the semi-leaf $\ell_X^+(q) = \{e^{tX}(q) \mid t \geq 0\}$. With the above definition, a leaf can have at most two limit points: one for each extremity. Finally, notice that a limit point of a leaf must be a zero of the corresponding characteristic vector field X, i.e., a characteristic point of S.

Proposition 1.15 ([BBC21, Thm. 1.3]). *Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold, and let p be a limit point of a one-dimensional leaf ℓ. Let $x \in \ell$, and X be a characteristic vector field such that $e^{tX}(x) \to p$ for $t \to +\infty$. Then, the length of $\ell_X^+(x)$ is finite.*

This result is a consequence of the sub-Riemannian structure being contact. Indeed, for a non-contact distribution this conclusion is false; for instance, in [ZZ95, Lem. 2.1] the authors prove that the length of the semi-leaves of the characteristic foliation of a Martinet surface converging to an elliptic point is infinite.

On the global side, we determine some conditions for the induced metric d_S to be finite under the assumption that there exists a global characteristic vector field of S. In such a case, for a compact, connected surface S with isolated characteristic points, in Proposition 2.16 we show that d_S is finite in the absence of the following classes of leaves in the characteristic foliation of S: nontrivial recurrent trajectories, periodic trajectories, and sided contours. Those conditions are satisfied by spheres in coorientable, tight contact spaces,

Theorem 1.16 [BBC21, Thm. 1.4]

Let (M, D, g) be a tight coorientable sub-Riemannian contact structure, and let S be a C^2 embedded surface with isolated characteristic points, homeomorphic to a sphere. Then the induced distance d_S is finite.

We stress that having isolated characteristic points is a generic property for a surface in a contact manifold. Example 2.5.4 and Example 2.5.5 in the Heisenberg distribution show that, if S is not a topological sphere, then S presents possibly nontrivial recurrent trajectories or periodic trajectories, cases in which d_S is not finite. Moreover, if one removes the hypothesis of the contact structure being tight, then a sphere S might present a periodic trajectory, hence the induced distance d_S would not be finite. The compactness hypothesis is also important, as one can see in Example 2.5.1.

1.3 Stochastic processes on sub-Riemannian surfaces

Let (M, D, g) be a contact, cooriented sub-Riemannian space. We shall assume that the distribution D is *free*, i.e., globally generated by a pair of vector fields (X_1, X_2) which constitute an orthonormal frame for D oriented with respect to the volume form vol_g on D defined by g. Moreover, we choose the contact form ω to be normalised so that

$$\mathrm{d}\omega|_D = \mathrm{vol}_g\,. \tag{1.25}$$

Associated with such a contact form ω, we have a canonical choice of a vector field everywhere transversal to D: this is the *Reeb* vector field X_0, which is the unique vector field on M satisfying

$$\mathrm{d}\omega(X_0, \cdot) \equiv 0, \qquad \text{and} \qquad \omega(X_0) \equiv 1. \tag{1.26}$$

Let g^ε, for $\varepsilon > 0$, be the Riemannian approximations of (M, D, g) with respect to the Reeb vector field X_0, as described in Section 1.2.2. Recall that g^ε is defined by requiring $(X_1, X_2, \varepsilon X_0)$ to be a global orthonormal frame for g^ε. Let S be a surface embedded in M, and assume that S is globally defined by a submersion u in the sense of equation (1.16). Namely, one has

$$S = \{x \in M : u(x) = 0\}, \qquad \text{and } \mathrm{d}u \neq 0 \text{ on } S. \tag{1.27}$$

This implies that the characteristic vector field X_u introduced in (1.18) is defined globally on S. Let $\widehat{X}_S$ be the vector field on $S \setminus \Sigma(S)$ defined by $\widehat{X}_S = X_u / |X_u|_g$. Explicitly,

$$\widehat{X}_S = \frac{(X_1 u) X_2 - (X_2 u) X_1}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{1.28}$$

Even though $\widehat{X}_S$ is expressed in terms of X_1, X_2 and u, it only depends on the sub-Riemannian manifold (M, D, g), the embedded surface S and a choice of sign. Let $b \colon S \setminus \Sigma(S) \to \mathbb{R}$ be the function given by

$$b = \frac{X_0 u}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{1.29}$$

Similarly to the vector field $\widehat{X}_S$, the function b can be understood intrinsically. Indeed, let $\widehat{X}_S^\perp$ be such that $(\widehat{X}_S, \widehat{X}_S^\perp)$ is an oriented orthonormal frame for $D|_{S\setminus\Sigma(S)}$. Then, the function b is uniquely defined by requiring $b\widehat{X}_S^\perp - X_0$ to be a vector field on $S \setminus \Sigma(S)$. Finally, define

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ , \tag{1.30}$$

which is a second order partial differential operator on $S \setminus \Sigma(S)$. The operator Δ_0 is invariant under multiplications of u by nonzero functions. As stated in the theorem below, it arises as the limiting operator of the Laplace–Beltrami operators Δ_ε of the Riemannian approximations g^ε, for $\varepsilon \to 0$.

Theorem 1.17 ([BBCH21, Thm. 1.1]). *For any twice differentiable function $f \in C_c^2(S \setminus \Sigma(S))$ compactly supported in $S \setminus \Sigma(S)$, the functions $\Delta_\varepsilon f$ converge uniformly on $S \setminus \Sigma(S)$ to $\Delta_0 f$ as $\varepsilon \to 0$.*

Following the definition in Balogh, Tyson and Vecchi [BTV17] for surfaces in the Heisenberg group, the intrinsic Gaussian curvature $\mathcal{K}_S$ of a surface in a general three-dimensional contact sub-Riemannian manifold is defined as the limit of the Gaussian curvatures with respect to the Riemannian metrics g_ε, as described in (1.22). In the following proposition we derive an expression for $\mathcal{K}_S$, employing the same orthogonal frame exhibited to prove Theorem 1.17.

Proposition 1.18 ([BBCH21, Prop. 1.2]). *Uniformly on compact subsets of $S \setminus \Sigma(S)$, we have*

$$\mathcal{K}_S := \lim_{\varepsilon \to 0} K^\varepsilon = -\widehat{X}_S(b) - b^2 \ .$$

We now consider the *canonical stochastic process* on $S \setminus \Sigma(S)$ whose generator is $\frac{1}{2}\Delta_0$. Assuming that it starts at a certain point then, up to explosion, the process moves along the unique leaf of the characteristic foliation picked out by the starting point. This follows from the fact that the vector field $\widehat{X}_S$ is tangent to the characteristic foliation of S. As shown by the next theorem and the following proposition, for this stochastic process, elliptic characteristic points are inaccessible, while hyperbolic characteristic points are accessible from the separatrices.

Theorem 1.19 [BBCH21, Thm. 1.3]

The set of elliptic characteristic points in a surface S embedded in M is inaccessible for the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$.

In Section 3.3.3, we discuss an example of a surface in the Heisenberg group whose induced stochastic process is killed in finite time if started along the separatrices of the characteristic point. Indeed, this phenomena always occurs in the presence of a hyperbolic characteristic point.

Theorem 1.20 [BBCH21, Prop. 1.4]

Suppose that the surface S embedded in M has a hyperbolic characteristic point. Then the stochastic process having generator $\frac{1}{2}\Delta_0$ and started on the separatrices of the hyperbolic characteristic point reaches that characteristic point with positive probability.

Sections 3.3 and 3.4 are devoted to illustrating the various behaviours shown by the canonical stochastic process induced on the surface S. Besides illustrating Proposition 1.20, we show in Theorem 1.21 below that three classes of familiar stochastic processes arise when considering a natural choice for the surface S in the three classes of model spaces for three-dimensional sub-Riemannian structures, which are the Heisenberg group in $\mathbb{R}^3$, and the special unitary group $\mathrm{SU}(2)$ and the special linear group $\mathrm{SL}(2,\mathbb{R})$ equipped with sub-Riemannian contact structures with scalar products differing by a constant multiple. In all these cases, the orthonormal frame (X_1, X_2) for the distribution D is formed by two left-invariant vector fields which together with the Reeb vector field X_0 satisfy, for some $\kappa \in \mathbb{R}$, the commutation relations $[X_2, X_1] = X_0$, $[X_1, X_0] = \kappa X_2$, and $[X_2, X_0] = -\kappa X_1$, with $\kappa = 0$ in the Heisenberg group, $\kappa > 0$ in $\mathrm{SU}(2)$ and $\kappa < 0$ in $\mathrm{SL}(2,\mathbb{R})$. Associated with each of these Lie groups and their Lie algebras, we have the group exponential map exp for which we identify a left-invariant vector field with its value at the origin.

Theorem 1.21 ([BBCH21, Thm. 1.5]). *Fix $\kappa \in \mathbb{R}$. For $\kappa \neq 0$, let $k \in \mathbb{R}$ with $k > 0$ be such that $|\kappa| = 4k^2$. Set $I = (0, \frac{\pi}{k})$ if $\kappa > 0$ and $I = (0, \infty)$ otherwise. In the model space for three-dimensional sub-Riemannian structures corresponding to κ, we consider the embedded surface S parameterised as*

$$S = \{\exp(r\cos\theta X_1 + r\sin\theta X_2) : r \in I \text{ and } \theta \in [0, 2\pi)\} .$$

Then, the limiting operator Δ_0 on S is given by $\Delta_0 = \frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r}$, where

$$b(r) = \begin{cases} 2k\cot(kr) & \text{if } \kappa = 4k^2 \\ \frac{2}{r} & \text{if } \kappa = 0 \\ 2k\coth(kr) & \text{if } \kappa = -4k^2 \end{cases} .$$

The stochastic process induced by the operator $\frac{1}{2}\Delta_0$ moving along the leaves of the characteristic foliation of S is a Bessel process of order 3 if $\kappa = 0$, a Legendre process of order 3 if $\kappa > 0$ and a hyperbolic Bessel process of order 3 if $\kappa < 0$.

The stochastic processes we recover here are all related to one-dimensional Brownian motion by the same type of Girsanov transformation, with only the sign of a parameter distinguishing between them. For the details, see Revuz and Yor [RY99, p. 357]. Let us recall here that a Bessel process of order 3 arises by conditioning a one-dimensional Brownian motion started on the positive real line to never hit the origin, whereas a Legendre process of order 3 is obtained by conditioning a Brownian motion started inside an interval to never hit either endpoint of the interval. The examples making up Theorem 1.21 can be considered as model cases for our setting, and all of them illustrate Theorem 1.19.

Finally, notice that the limiting operator we obtain on the leaves is not the Laplacian associated with the metric structure restricted to the leaves as the latter has no drift term. However, the operator Δ_0 restricted to a leaf can be considered as a weighted Laplacian. For a smooth measure $\mu = h^2\,\mathrm{d}x$ on an interval I of the Euclidean line $\mathbb{R}$, the weighted Laplacian applied to a scalar function u yields

$$\mathrm{div}_\mu\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{2h'(x)}{h(x)}\frac{\partial f}{\partial x} .$$

In the model cases above, we have

$$h(r) = \begin{cases} \sin(kr) & \text{if } \kappa = 4k^2 \\ r & \text{if } \kappa = 0 \\ \sinh(kr) & \text{if } \kappa = -4k^2 \end{cases} .$$

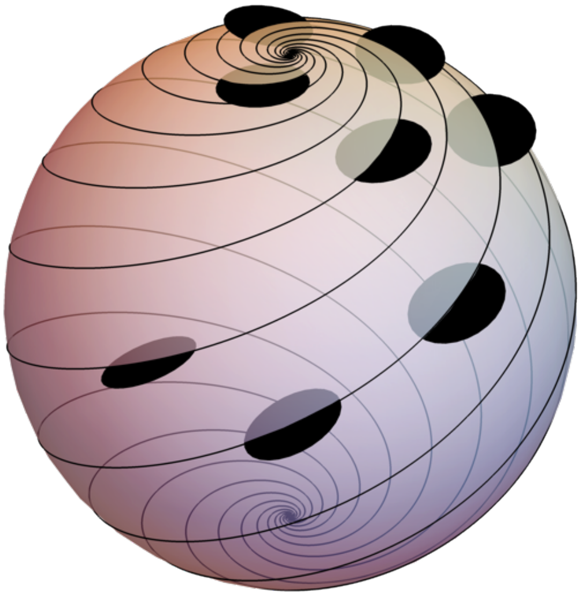

Figure 1.3: The characteristic foliation defined by the Heisenberg distribution $(\mathbb{R}^3, \ker(dz+\frac{1}{2}(ydx-xdy))$ on an Euclidean sphere centred at the origin: any horizontal curve connecting points on different spirals goes though one of the characteristic points, at the North or the South pole. The sub-Riemannian length of the leaves spiralling around the characteristic points is finite because of Proposition 1.15. Thus, the induced distance d_S is finite: this is a particular case of Theorem 1.16. The canonical stochastic process started outside the characteristic points never hits neither the north pole nor the south pole, and it induces a one-dimensional process on the unique leaf of the characteristic foliation picked out by the starting point, due to Theorem 1.19.

Chapter 2

On the induced geometry on surfaces in 3D contact sub-Riemannian manifolds

Contents

This chapter describes the results in the paper [BBC21], joint work with Davide Barilari and Ugo Boscain, and it is currently submitted for publication.

First, in Section 2.1 we discuss the limit for $\varepsilon \to 0$ of the Gaussian curvature K^ε of a surface S in the Riemannian approximations $g^{\varepsilon X_0}$; this will give the asymptotic presented in Theorem 1.13 at the characteristic points. Next, we focus on the characteristic foliation of a surface S and its local properties around a characteristic point, and we prove Proposition 1.14 (pictured in Figure 1.2 for isolated characteristic points), and Proposition 1.15. Next, in Section 2.3 we discuss the global features of the characteristic foliations which might prevent the intrinsic distance d_S to be finite, and we prove Proposition 2.16. In Section 2.4 we show that such conditions are satisfied by spheres with the hypothesis that the distribution D is contact, proving Theorem 1.16. Finally, in Section 2.5 we discuss some examples.

2.1 Riemannian approximations and Gaussian curvature

Following the notations introduced in the introduction, let M be a smooth 3-dimensional manifold, and (D, g) be a smooth contact sub-Riemannian structure on M. As in Subsection 1.2.2, assume having fixed a vector field X_0 transverse to the distribution, and extend the sub-Riemannian metric g to a family of Riemannian metrics $g^\varepsilon = g^{\varepsilon X_0}$, $\varepsilon > 0$, for which X_0 is transversal to D and with norm $1/\varepsilon$. Let $\overline{\nabla}^\varepsilon$ be the Levi-Civita connection of (M, g^ε). Let us express this connection locally, in a domain equipped with an orthonormal frame (X_1, X_2) of D oriented with respect to vol_g; thus, $(\varepsilon X_0, X_1, X_2)$ is an orthonormal basis of g^ε. Due to the Koszul formula, for all $i, j, k = 0, 1, 2$, one has

$$\left\langle \overline{\nabla}^\varepsilon_{X_i} X_j, X_k \right\rangle_{g^\varepsilon} = \frac{1}{2}\Big(- \langle X_i, [X_j, X_k]\rangle_{g^\varepsilon} + \langle X_k, [X_i, X_j]\rangle_{g^\varepsilon} + \langle X_j, [X_k, X_i]\rangle_{g^\varepsilon} \Big).$$

This identity enables us to describe $\overline{\nabla}^\varepsilon$ using the frame (X_0, X_1, X_2), which is independent from ε. This is done using the Lie bracket structure of the frame, i.e., the C^∞ functions c_{ij}^k called the *structure constants* of the frame (X_0, X_1, X_2) defined by

$$[X_j, X_i] = c_{ij}^1 X_1 + c_{ij}^2 X_2 + c_{ij}^0 X_0 \qquad \text{for } i, j = 0, 1, 2. \tag{2.1}$$

It follows immediately from Koszul formula that

$$\begin{aligned}
\overline{\nabla}^\varepsilon_{X_0} X_0 &= \frac{c_{01}^0}{\varepsilon^2} X_1 + \frac{c_{02}^0}{\varepsilon^2} X_2 \qquad\qquad \forall \varepsilon > 0 \qquad (2.2)\\
\overline{\nabla}^\varepsilon_{X_0} X_1 &= -c_{01}^0 X_0 + \frac{1}{2}\left(c_{02}^1 - c_{01}^2 + \frac{c_{12}^0}{\varepsilon^2} \right) X_2\\
\overline{\nabla}^\varepsilon_{X_0} X_2 &= -c_{02}^0 X_0 + \frac{1}{2}\left(c_{01}^2 - c_{02}^1 - \frac{c_{12}^0}{\varepsilon^2} \right) X_1\\
\overline{\nabla}^\varepsilon_{X_1} X_0 &= c_{01}^1 X_1 + \frac{1}{2}\left(c_{02}^1 + c_{01}^2 + \frac{c_{12}^0}{\varepsilon^2} \right) X_2\\
\overline{\nabla}^\varepsilon_{X_1} X_1 &= -c_{01}^1 \varepsilon^2 X_0 + c_{12}^1 X_2\\
\overline{\nabla}^\varepsilon_{X_1} X_2 &= \frac{1}{2}\left(-c_{02}^1 \varepsilon^2 - c_{01}^2 \varepsilon^2 - c_{12}^0 \right) X_0 - c_{12}^1 X_1\\
\overline{\nabla}^\varepsilon_{X_2} X_0 &= \frac{1}{2}\left(c_{01}^2 + c_{02}^1 - \frac{c_{12}^0}{\varepsilon^2} \right) X_1 + c_{02}^2 X_2\\
\overline{\nabla}^\varepsilon_{X_2} X_1 &= \frac{1}{2}\left(-c_{01}^2 \varepsilon^2 - c_{02}^1 \varepsilon^2 + c_{12}^0 \right) X_0 + c_{12}^2 X_2\\
\overline{\nabla}^\varepsilon_{X_2} X_2 &= -c_{02}^2 \varepsilon^2 X_0 - c_{12}^2 X_1.
\end{aligned}$$

Now, let S be an embedded surface in M. The *Gaussian curvature* $K^\varepsilon = K^{\varepsilon X_0}$ of S in (M, g^ε) is defined by the Gauss formula

$$K^\varepsilon = K^\varepsilon_{\mathrm{ext}} + \det(\mathrm{II}^\varepsilon), \tag{2.3}$$

where, given a frame (X, Y) of TS, the *extrinsic curvature* $K^\varepsilon_{\mathrm{ext}}$ is

$$K^\varepsilon_{\mathrm{ext}} = \frac{\left\langle \overline{\nabla}^\varepsilon_X \overline{\nabla}^\varepsilon_Y Y - \overline{\nabla}^\varepsilon_Y \overline{\nabla}^\varepsilon_X Y - \overline{\nabla}^\varepsilon_{[X,Y]} Y, X \right\rangle_{g^\varepsilon}}{|X|^2_{g^\varepsilon} |Y|^2_{g^\varepsilon} - \langle X, Y\rangle^2_{g^\varepsilon}}, \tag{2.4}$$

and the determinant $\det \mathrm{II}^\varepsilon$ of the second fundamental form is

$$\det \mathrm{II}^\varepsilon = \frac{\left\langle \mathrm{II}^\varepsilon(X, X), \mathrm{II}^\varepsilon(Y, Y) \right\rangle_{g^\varepsilon} - \left\langle \mathrm{II}^\varepsilon(X, Y), \mathrm{II}^\varepsilon(X, Y) \right\rangle_{g^\varepsilon}}{|X|^2_{g^\varepsilon} |Y|^2_{g^\varepsilon} - \langle X, Y\rangle^2_{g^\varepsilon}}. \tag{2.5}$$

In this last formula, the *second fundamental form* II^ε of S is defined as the projection of the Levi-Civita connection $\overline{\nabla}^\varepsilon$ on the orthogonal to the tangent space of the surface. Both quantities (2.4) and (2.5) are independent on the frame (X, Y) of TS chosen to compute them.

2.1.1 Proof of Theorem 1.13

To prove the theorem, we explicitly compute the asymptotic of the quantities in limit (1.21). Let us fix a characteristic point p, and, in a neighbourhood of p, let us fix an oriented orthonormal frame (X_1, X_2) of D and a submersion u defining S in the sense of (1.16).

The determinant of the bilinear form $B_p^{\varepsilon X_0}$ is homogeneous in ε, and satisfies

$$\det B_p^{\varepsilon X_0} = \frac{\det B_p^{X_0}}{\varepsilon^2} = \frac{B_p^{X_0}(X_1, X_2)^2}{\varepsilon^2} = \frac{(c_{12}^0(p))^2}{\varepsilon^2}, \tag{2.6}$$

where c_{12}^0 is defined in (2.1). Therefore, in order to prove the convergence of the limit in (1.21), it suffices to show that the Gaussian curvature $K_p^{\varepsilon X_0}$ at p diverges at most as $1/\varepsilon^2$.

Let us start with the computation of the determinant (2.5) of the second fundamental form at a characteristic point. It is convenient to write the second fundamental form as

$$\mathrm{II}^\varepsilon(X, Y) = \langle \overline{\nabla}^\varepsilon_X Y, N^\varepsilon \rangle N^\varepsilon.$$

where N^ε is the Riemannian unitary gradient of u, i.e.,

$$N^\varepsilon = \frac{(X_1u)X_1 + (X_2u)X_2 + \varepsilon(X_0u)X_0}{\sqrt{(X_1u)^2 + (X_2u)^2 + \varepsilon(X_0u)^2}}.$$

At the characteristic point p, the gradient $N^\varepsilon(p)$ simplifies to

$$N^\varepsilon(p) = \varepsilon \ \mathrm{sign}(X_0u) \ X_0(p). \tag{2.7}$$

To compute (2.5) one needs to choose a frame of TS; we will use the frame (F_1, F_2) with

$$F_i = (X_0u)X_i - (X_iu)X_0 \qquad \text{for } i = 1, 2. \tag{2.8}$$

This frame is well-defined for $X_0u \neq 0$; in particular, it is suited to calculate the Gaussian curvature at the characteristic points. Recall that the horizontal Hessian of u is

$$\mathrm{Hess}_H(u) = \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u \end{pmatrix}. \tag{2.9}$$

Lemma 2.1. *Let $p \in S$ be a characteristic point. Then, in the previous notations, for every $\varepsilon > 0$, the determinant (2.5) of the second fundamental form in p is*

$$\det \mathrm{II}^\varepsilon(p) = \frac{1}{\varepsilon^2}\left(\frac{\det \mathrm{Hess}_H u(p)}{(X_0u(p))^2} - \frac{(c_{12}^0(p))^2}{4}\right) + O(1).$$

Proof. Let p be a characteristic point. Because $X_1u(p) = X_2u(p) = 0$, one can show that,

$$\overline{\nabla}^\varepsilon_{F_i} F_j(p) = \Big((X_0u)^2 \overline{\nabla}^\varepsilon_{X_i} X_j + (X_0u)(X_iX_0u)X_j - (X_0u)(X_iX_ju)X_0\Big)\Big|_p,$$

for $i, j = 1, 2$. Using formula (2.7) for N^ε, one finds that only the component along X_0 plays a role in the second fundamental form in p. Thus, using the covariant derivatives in (2.2),

$$\langle \overline{\nabla}^\varepsilon_{F_i} F_j, N^\varepsilon \rangle\big|_p = -\frac{|X_0u(p)|}{\varepsilon}\left(X_iX_ju + (X_0u)\frac{c_{ij}^0}{2} + (X_0u)\varepsilon^2 \frac{c_{0i}^j + c_{0j}^i}{2}\right)\Bigg|_p,$$

for $i, j = 1, 2$. This, together with $\big(|F_1|^2|F_2|^2 - \langle F_1, F_2\rangle^2\big)\big|_p = (X_0u(p))^4$, gives the result. $\square$

Next, the extrinsic curvature (2.4) is the sectional curvature of the plane T_pS in M, which is known when X_0 is the Reeb vector field and $\varepsilon = 1$; this can be found for instance in [BBL20, Prop. 14]. In our setting, the resulting expression for $\varepsilon \to 0$ is the following.

Lemma 2.2. *Let $p \in S$ be a characteristic point. Then, for every $\varepsilon > 0$,*

$$K^{\varepsilon}_{\mathrm{ext}}(p) = -\frac{3}{4\varepsilon^2}(c^0_{12}(p))^2 + O(1).$$

Proof. To compute the extrinsic curvature we use the frame (X_1, X_2) of TM, which coincides with $T_pS = D_p$ at the characteristic point p. Then, to compute

$$K^{\varepsilon}_{\mathrm{ext}}(p) = \langle \overline{\nabla}^{\varepsilon}_{X_1}\overline{\nabla}^{\varepsilon}_{X_2}X_2 - \overline{\nabla}^{\varepsilon}_{X_2}\overline{\nabla}^{\varepsilon}_{X_1}X_2 - \overline{\nabla}^{\varepsilon}_{[X_1,X_2]}X_2, X_1\rangle\Big|_p$$

it suffices to use the expressions (2.2). ☐

Remark 2.3. Following the proof of Lemma 2.1 and Lemma 2.2, the exact expressions for $\det \mathrm{II}^{\varepsilon}(p)$ and $K^{\varepsilon}_{\mathrm{ext}}(p)$ at a characteristic point p are, for all $\varepsilon > 0$,

$$\begin{aligned}
\det \mathrm{II}^{\varepsilon}(p) = &+ \frac{1}{\varepsilon^2}\left(\frac{\det \mathrm{Hess}_H f}{(X_0u)^2} - \frac{(c^0_{12})^2}{4}\right)\Big|_p + \varepsilon^2\left(c^1_{01}c^2_{02} - \frac{\left(c^1_{02} + c^2_{01}\right)^2}{4}\right)\Big|_p \\
&+ \frac{1}{X_0u(p)}\left(c^2_{02}X_1X_1u + c^1_{01}X_2X_2u - \frac{c^2_{01} + c^1_{02}}{2}(X_2X_1u + X_1X_2u)\right)\Big|_p, \\
K^{\varepsilon}_{\mathrm{ext}}(p) = &- \frac{3}{4}\frac{(c^0_{12}(p))^2}{\varepsilon^2} - \varepsilon^2\left(c^1_{01}c^2_{02} - \frac{(c^2_{01} + c^1_{02})^2}{4}\right)\Big|_p \\
&+ \left(X_2(c^1_{12}) - X_1(c^2_{12}) - (c^1_{12})^2 - (c^2_{12})^2 + c^0_{12}\frac{c^2_{01} - c^1_{02}}{2}\right)\Big|_p.
\end{aligned}$$

If one chooses as transversal vector field the Reeb vector field of the contact sub-Riemannian manifold, then one recognises the first and the second functional invariants of the sub-Riemannian structure, defined in [ABB20, Ch. 17]. Finally, notice that these expressions are still valid for non-contact distributions.

Proof of Theorem 1.13. In the previous notations, due to the Gauss formula (2.3), Lemma 2.1 and Lemma 2.2, the Gaussian curvature at a characteristic point p satisfies

$$K^{\varepsilon}_p = K^{\varepsilon X_0}_p = \frac{(c^0_{12}(p))^2}{\varepsilon^2}\left(-1 + \frac{\det \mathrm{Hess}_H u(p)}{[X_2, X_1]u(p)^2}\right) + O(1).$$

Here we have used that $c^0_{12}(p)X_0u(p) = [X_2, X_1]u(p)$ at p, which holds due to definition (2.1) and $X_1u(p) = X_2u(p) = 0$. Using formula (2.6) for the determinant of $B^{\varepsilon X_0}_p$, one finds that

$$\frac{K^{\varepsilon X_0}_p}{\det B^{\varepsilon X_0}_p} = -1 + \frac{\det \mathrm{Hess}_H u(p)}{[X_2, X_1]u(p)^2} + O(\varepsilon^2), \tag{2.10}$$

which shows that the limit (1.21) is finite. Moreover, $\widehat{K}_p$ is independent of X_0 because the transversal vector field X_0 is absent in the constant term of equation (2.10). ☐

Formula (2.10) is useful to compute $\widehat{K}_p$ explicitly, as it contains only derivatives of the submersion u; thus, let us enclose it with the following corollary.

Corollary 2.4. *Let p be a characteristic point of S. Let u be a local submersion of class C^2 describing S, and let (X_1, X_2) be a local oriented orthonormal frame of D. Then,*

$$\widehat{K}_p = -1 + \frac{\det \mathrm{Hess}_H u(p)}{[X_2, X_1]u(p)^2}. \tag{2.11}$$

Note that both $\det \mathrm{Hess}_H u(p)$ and $[X_2, X_1]u(p)$ calculated at the characteristic point p are invariant with respect to the frame (X_1, X_2). Moreover, we emphasise that their ratio, which appears in (2.11), is independent on the choice of u.

2.2 Local study near a characteristic point

In this section, we prove Proposition 1.14, and we discuss the local qualitative behaviour of the characteristic foliation near $\Sigma(S)$ in relation to the metric coefficient $\widehat{K}$; next, we estimate the length of a semi-leaf converging to a point, proving Proposition 1.15.

However, let us begin by mentioning a fact which was already known in the literature, but whose proof is straightforward with setting explained in Subsection 1.2.1. For a more general discussion on the size of the characteristic set, we refer to [Bal03] and references therein.

Lemma 2.5 ([BBC21, Lem. 2.1])**.** *The characteristic set $\Sigma(S)$ of a surface S of class C^2 is contained in a 1-dimensional submanifold of S of class C^1.*

Proof. It suffices to show that for every point p in $\Sigma(S)$ there exists a neighbourhood V of p such that $V \cap \Sigma(S)$ is contained in an embedded C^1 curve. Let us fix a point p in $\Sigma(S)$, and a neighbourhood U of p in M equipped with a frame (X_1, X_2) and a function u with the properties described above. Because of (1.17), the characteristic points in $V = U \cap S$ are the solutions of the system $X_1 u = X_2 u = 0$.

Due to the implicit function theorem, it suffices to show that $d_p(X_1 u) \neq 0$ or $d_p(X_2 u) \neq 0$. Thanks to the contact condition, we have that $[X_2, X_1]u(p) \neq 0$. As a consequence, since $X_2 X_1 u(p) = [X_2, X_1]u(p) + X_1 X_2 u(p)$, at least one of the following is true: $X_2 X_1 u(p) \neq 0$, or $X_1 X_2 u(p) \neq 0$. Assume that the first is true; then $d_p(X_1 u)(X_2) = X_2 X_1 u(p) \neq 0$. The other case being similar, the lemma is proved. □

Let us fix a characteristic point p in $\Sigma(S)$, and a characteristic vector field X. Since $X(p) = 0$, there exists a well-defined linear map $DX(p) : T_pS \to T_pS$. Indeed, let e^{tX} be the flow of X. The pushforward of the flow gives, for every x in S, a family of linear maps $e^{tX}_* : T_xS \to T_{e^{tX}(x)}S$. Since $e^{tX}(p) = p$ for all t, then the preceding gives the linear flow $e^{tX}_* : T_pS \to T_pS$, whose infinitesimal generator is the differential $DX(p)$.

Definition 2.6. A characteristic point $p \in \Sigma(S)$ is *non-degenerate* if, given a characteristic vector field X of S, the differential $DX(p)$ is invertible. Otherwise, p is called *degenerate*.

Remark 2.7. Condition (1.14) in the definition of characteristic vector field ensures that the degeneracy of a characteristic point is independent on the choice of characteristic vector field.

Since T_pS coincides with D_p at the characteristic point p, we can endow T_pS with a metric; thus, $DX(p)$ admits a well-defined determinant and trace. Now, let X be the vector field $X = a_1 X_1 + a_2 X_2$, where (X_1, X_2) is an orthonormal oriented frame of D and $a_i \in C^1(S)$, for $i = 1, 2$. Then, in the frame defined by (X_1, X_2) one has

$$DX = \begin{pmatrix} X_1 a_1 & X_2 a_1 \\ X_1 a_2 & X_2 a_2 \end{pmatrix}, \tag{2.12}$$

and the formulas for the determinant and the trace are

$$\det DX = (X_1 a_1)(X_2 a_2) - (X_1 a_2)(X_2 a_1), \tag{2.13}$$

$$\operatorname{tr} DX = \operatorname{div} X = (X_1 a_1) + (X_2 a_2). \tag{2.14}$$

2.2.1 Proof of Proposition 1.14

Let us fix a characteristic point p in $\Sigma(S)$. We claim that the right-hand side of (1.23) is independent on the choice of the characteristic vector field X. Indeed, due to Remark 1.11 any two characteristic vector fields are multiples by nonzero functions, thus, at characteristic point p, their differentials are multiples by nonzero scalars; precisely, if $Y = \phi X$, for ϕ in $C^1(S)$, then one has $DY(p) = \phi(p) DX(p)$. Thus, the claim follows because both determinant and trace-squared are homogenous of the degree two.

Thus, we fix a local submersion u defining S near p, and the characteristic vector field $X_f = (X_1u)X_2 - (X_2u)X_1$ defined in (1.18). Using expression (2.12) for the differential of a vector field, we get

$$DX_f(p) = \begin{pmatrix} -X_1X_2u(p) & -X_2X_2u(p) \\ X_1X_1u(p) & X_2X_1u(p) \end{pmatrix}.$$

Thus, using expressions (2.13) and (2.14) for the determinant and the trace, we find that $\det DX_f(p) = \det \operatorname{Hess}_H u(p)$, and $\operatorname{tr} DX_f(p) = [X_2, X_1]u(p)$. In conclusion,

$$\frac{\det DX_f(p)}{\operatorname{tr} DX_f(p)^2} = \frac{\det \operatorname{Hess}_H u(p)}{[X_2, X_1]u(p)^2},$$

which, together with Corollary 2.4, gives the desired result.

□

The eigenvalues of the linearisation $DX(p)$ of a characteristic vector field X can be written as a function of $\widehat{K}_p$ by rearranging equation (1.23), as in the following corollary.

Corollary 2.8. *In the hypothesis of Proposition 1.14, let $\lambda_+(X,p)$ and $\lambda_-(X,p)$ be the two eigenvalues of $DX(p)$. Then*

$$\lambda_\pm(X,p) = \operatorname{tr} DX(p)\left(\frac{1}{2} \pm \sqrt{-\frac{3}{4} - \widehat{K}_p}\right). \tag{2.15}$$

Proof. Let us note $\lambda_\pm = \lambda_\pm(X,p)$, and $\alpha = \operatorname{tr} DX(p)$. Equation (1.23) reads

$$\widehat{K}_p = -1 + \frac{\lambda_+\lambda_-}{\alpha^2}.$$

Using that $\lambda_+ + \lambda_- = \alpha$, equation (1.23) implies that the eigenvalues satisfy the equation $z^2 - \alpha z + \alpha^2(\widehat{K}_p + 1) = 0$, which implies (2.15). □

Remark 2.9. It is possible to choose canonically a characteristic vector field with trace 1. Indeed, in the notations used to define X_u in (1.18), let us define the characteristic vector field

$$X_S = \frac{(X_1u)X_2 - (X_2u)X_1}{Zf}, \tag{2.16}$$

where Z is the Reeb vector field of the contact form ω of D normalised as in (1.25). Recall that the Reeb vector field was defined in (1.26) as the unique vector field satisfying $\omega(Z) = 1$ and $d\omega(Z, \cdot) = 0$. The vector field X_S is a characteristic vector field in a neighbourhood of p because it is a nonzero multiple of X_f near $\Sigma(S)$, since $Zu(p) = [X_2, X_1]u(p) \neq 0$. Using the latter, one can verify that $\operatorname{div} X_S(p) = \operatorname{tr} DX_S(p) = 1$.

It is worth mentioning that the vector field X_S is independent on u and on the frame (X_1, X_2), i.e., it depends uniquely on S and (M, D, g). Moreover, the norm of X_S satisfies $|X_S|_g^{-1} = |p_S|$, where p_S is the *degree of transversality* defined in [Lee13]; in the case of the Heisenberg group, p_S coincides with the *imaginary curvature* introduced in [AF07, AF08].

Expression (2.15) for the eigenvalues of the linearisation $DX(p)$ implies the following relations between the eigenvalues and the metric coefficient $\widehat{K}_p$:

(i) $\widehat{K}_p < -1$ if and only if $\lambda_\pm \in \mathbb{R}^*$ with different signs;

(ii) $\widehat{K}_p = -1$ if and only if $\lambda_- = 0$ and $\lambda_+ \in \mathbb{R}^*$;

(iii) $-1 < \widehat{K}_p \leq -3/4$ if and only if $\lambda_\pm \in \mathbb{R}^*$ with same sign;

(iv) $-3/4 < \widehat{K}_p$ if and only if $\Re(\lambda_\pm) \neq 0 \neq \Im(\lambda_\pm)$ and $\lambda_- = \overline{\lambda}_+$.

Notice that the characteristic point p is degenerate if and only if $\widehat{K}_p = -1$, which is case *(ii)*.

Assume that p is a non-degenerate characteristic point. Then, the linear dynamical system defined by $DX(p)$ is a saddle, a node, and a focus respectively in case *(i)*, *(iii)* and *(iv)*. In these cases there exists a local C^1-diffeomorphism near p which sends the flow of X to the flow of $DX(p)$ in $\mathbb{R}^2$, i.e., the flows are C^1-*conjugate*, as proven by Hartman in [Har60]. For this theorem to hold, one needs the characteristic vector field X to be of class C^2. For this reason, in the following corollary we assume the surface S to be of class C^3.

Corollary 2.10. *Assume that the surface S is of class C^3, and let p be a non-degenerate characteristic point in $\Sigma(S)$. Then, $\widehat{K}_p \neq -1$, and the characteristic foliation of S in a neighbourhood of p is C^1-conjugate to*

- *a saddle if and only if $\widehat{K}_p < -1$;*
- *a node if and only if $-1 < \widehat{K}_p \leq -3/4$;*
- *a focus if and only if $-3/4 < \widehat{K}_p$.*

The cases are depicted, respectively, in the first, third and fourth image in Figure 1.2 in Subsection 1.2.2.

Remark 2.11. For surfaces of class C^2, i.e., with characteristic vector fields of class C^1, one can use the Hartman-Grobman theorem, by which one recovers a C^0-conjugation to the corresponding linearisation. However, under this hypothesis, a node and a focus become indistinguishable. For the Hartman-Grobman theorem we refer to [Per12, Par. 2.8]. Finally, for a C^∞ surface some informations can be found in [GHR03].

Next, if p is a degenerate characteristic point, then we are in case *(ii)*. Thus, $\widehat{K}_p = -1$, and the differential $DX(p)$ has a zero eigenvalue with multiplicity one. In this situation, the qualitative behaviour of the characteristic foliation does not depend uniquely on the linearisation, but also on the nonlinear dynamic along a *center manifold*, i.e., an embedded curve $\mathcal{C} \subset S$ with the same regularity as X, invariant with respect to the flow of X, and tangent to the zero eigenvector of $DX(p)$. The analogue of Corollary 2.10 is the following.

Corollary 2.12. *Assume that the surface S is of class C^2, and let p be a degenerate characteristic point in $\Sigma(S)$. Then, $\widehat{K}_p = -1$, and the characteristic foliation in a neighbourhood centred at p is C^0-conjugate at the origin to the orbits of a system of the form*

$$\begin{cases} \dot{u} = \phi(u) \\ \dot{v} = v \end{cases}, \tag{2.17}$$

for a function ϕ with $\phi(0) = \phi'(0) = 0$. If p is isolated, then the characteristic foliation described in (2.17) *at the origin is either a saddle, a saddle-node, or a node; those cases are depicted, respectively, in the first, second, and third image in Figure 1.2.*

The proof of Corollary 2.12 follows from considerations on the center manifold of the dynamical system defined by X, which we recall in Section 2.6.

Remark 2.13. A node and a focus are not distinguishable by a conjugation C^0. However, the center manifold of the characteristic point p is an embedded curve of class C^1, thus it does not spiral around p. Therefore, the existence of a center manifold gives further properties then what is expressed in Corollary 2.12.

To justify the last sentence of Corollary 2.12 let us get a sense of the qualitative properties of a system as (2.17). The line $\{v = 0\}$, parametrised by u, is a center manifold of (2.17), and the function ϕ determines the dynamic of (2.17); this illustrates the fact that the nonlinear terms on a center manifold determine the dynamic near a degenerate characteristic point.

The equilibria of (2.17) occur only in $\{v = 0\}$, i.e., on a center manifold, and a point $(u, 0)$ is an equilibrium if and only if $\phi(u) = 0$. Thus, if the characteristic point p is isolated, then $u_0 = 0$ is an isolated zero of ϕ. In such case, let us note $\phi^+ = \phi|_{u>0}$ and $\phi^- = \phi|_{u<0}$, and without loss of generality let us suppose that the signs of ϕ^+ and ϕ^- are constant.

- If $\phi^+ > 0$ and $\phi^- < 0$, then the origin is a topological node.
- If $\phi^+ < 0$ and $\phi^- > 0$, then the origin is a a topological saddle.
- If ϕ^+ and ϕ^- have the same sign, then the two half spaces $\{u > 0\}$ and $\{u < 0\}$ have two different behaviours: one is a node, and the other one is a saddle. This gives the characteristic foliation called saddle-node.

Remark 2.14. For an isolated characteristic point, combining Corollary 2.10 and Corollary 2.12, we obtain the four characteristic foliations depicted in Figure 1.2.

2.2.2 Proof of Proposition 1.15

In this section we prove the finiteness of the sub-Riemannian length of a semi-leaf converging to a point. Since we are interested in a local property, it is not restrictive to assume the existence of a global characteristic vector field X of S.

Let ℓ be a one-dimensional leaf of the characteristic foliation of S, and $x \in \ell$ such that $e^{tX}(x) \to p$ as $t \to +\infty$. The limit point p has to be an equilibrium of X, i.e., $X(p) = 0$, hence p is a characteristic point of S. Let U be a small open neighbourhood of p in S for which we have a coordinate chart $\Phi : U \to B \subset \mathbb{R}^2$ with $\Phi(p) = 0$, where B is the open unit ball. Let y be the point of last intersection between $\ell^+_X(x)$ and the boundary ∂U. Since $L_{sR}(\ell^+_X(x)) = L_{sR}(\ell|_{[x,y]}) + L_{sR}(\ell^+_X(y))$ and $L_{sR}(\ell|_{[x,y]})$ is finite, it suffices to show that $L_{sR}(\ell^+_X(y))$ is finite. We claim that there exists a constant $C > 0$ such that

$$\frac{1}{C}|V|_{\mathbb{R}^2} \leq |V|_g \leq C|V|_{\mathbb{R}^2} \qquad \forall\, V \in D \cap TS|_U, \tag{2.18}$$

where we have dropped Φ_* in the notation. Indeed, let $\tilde{g}$ be any Riemannian extension of g on the surface S (for example $\tilde{g} = g^{X_0}|_S$). Since $\tilde{g}$ is an extension, one has $|v|_g = |v|_{\tilde{g}}$ for all v in $D \cap TS$. Equivalence (2.18) follows from the local equivalence of $\tilde{g}$ with the pullback by Φ of the Euclidean metric of $\mathbb{R}^2$. Now, inequality (2.18) implies that

$$L_{sR}(\ell^+_X(y)) = \int_0^{+\infty} |X(e^{tX}(y))|_g \, dt \leq C \int_0^{+\infty} |X(e^{tX}(y))|_{\mathbb{R}^2} \, dt. \tag{2.19}$$

At this point the proof of the finiteness of the sub-Riemannian length of $\ell^+_X(y)$ differs depending on whether p is a non-degenerate or a degenerate characteristic point.

First, assume that p is a non-degenerate characteristic point. Since p is non-degenerate, then the set of point w with $e^{tX}(w) \to p$ for $t \to +\infty$ form a manifold, called the *stable manifold* at p for the dynamical system defined by X. In our case, since $e^{tX}(y) \to p$ for $t \to +\infty$, the semi-leaf $\ell^+_X(y)$ is contained in the stable manifold at p. Moreover, the stable manifold convergence property, precisely stated in [Per12, Par. 2.8], shows that each trajectory inside the stable manifold converges to p sub-exponentially in t. Precisely, if α satisfies $|\Re(\lambda_\pm(p, X))| > \alpha$, then there exists constants $C, t_0 > 0$ such that

$$|e^{tX}(y) - p|_{\mathbb{R}^2} \leq Ce^{-\alpha t} \qquad \forall\, t > t_0. \tag{2.20}$$

Since $X(p) = 0$, for all $t > 0$ one has

$$\left|X(e^{tX}(y))\right|_{\mathbb{R}^2} = \left|X(e^{tX}(y)) - X(p)\right|_{\mathbb{R}^2} \leq \sup_B ||DX(x)|| \; |e^{tX}(y) - p|_{\mathbb{R}^2}.$$

Due to the inequality (2.19) and (2.20), this shows that $L_{sR}(\ell_X^+(y))$ is finite.

Next, assume that p is a degenerate characteristic point. As we said in the introduction of Corollary 2.12, there exists a center manifold $\mathcal{C}$ at p for the dynamical system defined by X. The asymptotic approximation property of the center manifold, recalled in Proposition 2.26, shows that if a trajectory converges to p, then it approximates any center manifold exponentially fast. Precisely, since $e^{tX}(y) \to p$, then there exist constants $C, \alpha, t_0 > 0$ and a trajectory $e^{tX}(z)$ contained in $\mathcal{C}$, such that

$$|e^{tX}(y) - e^{tX}(z)|_{\mathbb{R}^2} \leq Ce^{-\alpha t} \qquad \forall\, t \geq t_0. \tag{2.21}$$

The triangle inequality implies that

$$|X(e^{tX}(y))|_{\mathbb{R}^2} \leq |X(e^{tX}(y)) - X(e^{tX}(z))|_{\mathbb{R}^2} + |X(e^{tX}(z))|_{\mathbb{R}^2}. \tag{2.22}$$

Due to inequality (2.19), to prove that $L_{sR}(\ell_X^+(y))$ is finite, it suffices to show that the two terms on the right-hand side of (2.22) are integrable for $t \geq 0$. Thanks to (2.21) and

$$|X(e^{tX}(y)) - X(e^{tX}(z))|_{\mathbb{R}^2} \leq \sup_B ||DX||\ |e^{tX}(y) - e^{tX}(z)|_{\mathbb{R}^2},$$

the first term in (2.22) is integrable. Next, because $e^{tX}(z)$ is a regular parametrisation of a bounded interval inside a C^1 embedded curve (the center manifold $\mathcal{C}$), then its derivative $|X(e^{tX}(z))|_{\mathbb{R}^2}$ is integrable. □

Remark 2.15. Let X be a characteristic vector field of a compact surface S. If the ω-limit set with respect to X of a non-periodic leaf ℓ contains more then one point, then $L_{sR}(\ell_X^+) = +\infty$. Therefore, if a leaf ℓ does not converge to a point in any of its extremities, then the points in ℓ have infinite distance from the points in $S \setminus \ell$.

In particular, if the characteristic set of a surface S is empty, then the induced distance d_S is not finite. For a discussion on non-characteristic domains we refer to [DGN06, Ch. 3].

2.3 Global study of the characteristic foliation

The main goal of this section is to identify a sufficient condition for the induced distance d_S to be finite. As explained in the introduction, this is done by excluding the existence of certain leaves in the characteristic foliation of S, as in Proposition 2.16. In this section we assume the existence of a global characteristic vector field X of S.

The leaves the characteristic foliation of S are precisely the orbits of the dynamical system defined by X, therefore we are going to call them *trajectories*, stressing that they are parametrised by the flow of X. Moreover, the vector field X enables us to use the notions of ω-limit set and α-limit set of a point y in S, which are, respectively,

$$\omega(y, X) = \left\{ q \in S \;\middle|\; \exists\, t_n \to +\infty \text{ such that } e^{t_n X}(y) \to q \right\}, \qquad \alpha(y, X) = \omega(y, -X).$$

The points y in a leaf ℓ have the same limit sets, thus one can define $\omega(\ell, X)$ and $\alpha(\ell, X)$.

Proposition 2.16. *Let S be a compact, connected surface C^2 embedded in a contact sub-Riemannian structure. Assume that S has isolated characteristic points, and that the characteristic foliation of S is described by a global characteristic vector field of S which does not contain any of the following trajectories:*

- *nontrivial recurrent trajectories,*

- *periodic trajectories,*

- *sided contours.*

Then, d_S is finite.

Let us give a formal definition of these objects. A *periodic* trajectory is a leaf of the characteristic foliation homeomorphic to a circle. A periodic trajectory has infinite distance from its complementary, hence it is necessary to exclude its presence for d_S to be finite.

Next, a leaf ℓ is *recurrent* if $\ell \subset \omega(\ell, X)$ and $\ell \subset \alpha(\ell, X)$. A *nontrivial* recurrent trajectory is a recurrent trajectory which is not an equilibrium nor a periodic trajectory. Because the ω-limit and the α-limit set of a nontrivial recurrent trajectory contains more then one point, then, due to Remark 2.15, those trajectories have infinite distance from their complementary.

Lastly, a *sided contour* is either a left-sided or right-sided contour. A right-sided contour (resp. left-sided) is a family of points $p_1, \dots, p_s$ in $\Sigma(S)$ and trajectories $\ell_1, \dots, \ell_s$ such that:

- for all $j = 1, \dots, s$, we have $\omega(\ell_j, X) = p_j = \alpha(\ell_{j+1}, X)$ (where $\ell_{s+1} = \ell_1$);
- for every $j = 1, \dots, s$, there exists a neighbourhood U_j of p_j such that U_j is a right-sided hyperbolic sector (resp. left-sided) for p_j with respect to ℓ_j and ℓ_{j+1}.

Let us give a precise definition of a hyperbolic sector. Note that, given a non-characteristic point $x \in S$, and a curve T going through x and transversal to the flow of X, the orientation defined by X defines the right-hand and the left-hand connected component of $T \setminus \{x\}$, denoted T^r and T^l respectively.

Definition 2.17. Let p be a characteristic point, and ℓ_1 and ℓ_2 be two trajectories such that $\omega(\ell_1, X) = p = \alpha(\ell_2, X)$. A neighbourhood U of p homeomorphic to a disk is a *right-sided hyperbolic sector* (resp. left-sided) with respect to ℓ_1 and ℓ_2 iu, for every point $x_i \in \ell_i \cap U$, for $i = 1, 2$, there exists a curve T_i going through x_i and transversal to the flow of X such that:

- for every point $y \in T_1^r$ (resp. T_1^l) the positive semi-trajectory $\ell_X^+(y)$ starting from y intersects T_2^r (resp. T_2^l) before leaving U;
- the point of first intersection of $\ell_X^+(y)$ and T_2^r (resp. T_2^l) converges to x_2, for $y \to x_1$.

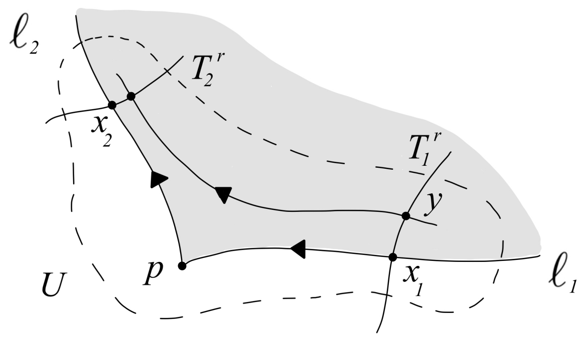

Figure 2.1: The illustration of a right-sided hyperbolic sector

Note that a right-sided hyperbolic sector for X is a left-sided hyperbolic sector for $-X$. An illustration of hyperbolic sector can be found in Figure 2.1, an example of sided contours can be found in Figure 2.4, and for the general theory we refer to [ABZ96, Par. 2.3.5].

2.3.1 Topological structure of the characteristic foliation

Now, assume that S does not contain any nontrivial recurrent trajectories. To prove Proposition 2.16 we are going to use the topological structure of a flow. We resume here the relevant theory, following the exposition in [ABZ96, Par. 3.4].

The *singular trajectories* of the characteristic foliation of S are precisely the following:

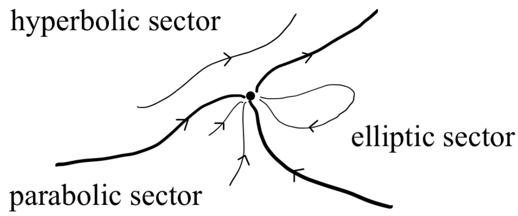

Figure 2.2: The sectors of an isolated equilibrium of a dynamical system.

- characteristic points;
- separatrices of characteristic points (see [ABZ96, Par. 2.3.3]);
- isolated periodic trajectories;
- periodic trajectories which contain in every neighbourhood both periodic and non-periodic trajectories.

The union of the singular trajectories is noted $ST(S)$, and it is closed. The open connected components of $S \setminus ST(S)$ are called *cells*. The leaves of the characteristic foliation of S contained in the same cell have the same behaviour, as shown in the following proposition.

Proposition 2.18 ([ABZ96, Par. 3.4.3])**.** *Assume that the flow of X has a finite number of singular trajectories. Let R be a cell filled by non-periodic trajectories; then:*

(i) R is homeomorphic to a disk, or to an annulus;

(ii) the trajectories contained in R have all the same ω-limit and α-limit sets;

(iii) the limit sets of any trajectory in R belongs to ∂R;

(iv) each connected component of ∂R contains points of the ω-limit or α-limit sets.

Using this proposition, we show the following lemma.

Lemma 2.19. *Let S be surface satisfying the hypothesis of Proposition 2.16 . Then, for every cell R of the characteristic foliation of S, we have that*

$$d_S(x, y) < +\infty \qquad \forall x, y \in R \cup \partial R.$$

Proof. Since the surface S is compact and the characteristic points in $\Sigma(S)$ are isolated, there is a finite number of characteristic points. Moreover, there are no periodic trajectories. This implies that there is a finite number of singular trajectories, hence we can apply Proposition 2.18.

Let R be a cell of the characteristic foliation of S, and let Γ be one of the connected components of the boundary ∂R (of which there are either one or two, due to Proposition 2.18). The curve Γ is the union of characteristic points and separatrices. If all characteristic points have a hyperbolic sector towards R (right-sided or left-sided), then Γ would be a sided contour, which is excluded. Therefore, there exists a characteristic point $p \in \Gamma$ without a hyperbolic sector towards R. As shown in [ALGM73, Par. 8.18], around an isolated equilibrium there are only the three kinds of sectors depicted in Figure 2.2. Since there is no elliptic sector due to Remark 2.14, the point p has a parabolic sector towards R.

Due to Proposition 2.18, the point p is the ω-limit or the α-limit of every trajectory in R. Then, for every point $x \in R$, there exists a semi-leaf $\ell^+_X(x)$ or $\ell^+_{-X}(x)$ starting from x and converging to p. Due to Proposition 1.15, this semi-leaf has finite sub-Riemannian length, hence $d_S(x, p)$ is finite.

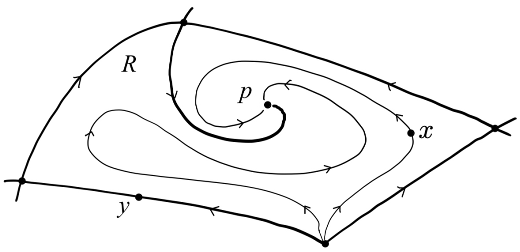

Figure 2.3: How to connect the points of a cell with the points in the boundary.

Next, for every point $y \in \Gamma$, note that

$$d_S(x, y) \leq d_S(x, p) + d_S(p, y).$$

We have already proven that $d_S(x, p)$ is finite, and the same holds for $d_S(p, y)$. Indeed, one can find a horizontal curve of finite length connecting p and y using a concatenation of the separatrices contained in Γ.

If the boundary of R has a second connected component, then the above argument holds also for the other connected component because it suffices to repeat the above argument for it. Thus, we have shown that

$$d_S(x, y) < +\infty \qquad \forall\, x \in R,\ y \in \partial R,$$

which implies the statement of the lemma. $\square$

Lemma 2.20. *Let S be surface satisfying the hypothesis of Proposition 2.16. Then, for every x in S, there exists an open neighbourhood U of x such that, for all y in U,*

$$d_S(x, y) < +\infty.$$

Proof. Let x be a point of S. If x does not belong to the union of the singular trajectories, then it is in the interior of a cell R. Thus, due to Lemma 2.19, one can choose $U = R$. Otherwise, the point x belongs to a separatrix, or it is a characteristic point of S.

Assume that x belongs to a separatrix Γ. Then, there exists a neighbourhood U of x which is divided by Γ in two connected components. Those two connected components are contained in some cell R_1 and R_2, which contain Γ in their boundary. For every $y \in U$, then either $y \in R_i$, for $i = 1, 2$, or $y \in \Gamma$. If $y \in R_i$, then it suffices to apply Lemma 2.19. Otherwise, if $y \in \Gamma$, the separatrix Γ itself connects x and y.

Finally, assume that x is a characteristic point. Due to Corollary 2.10, Remark 2.11, and Corollary 2.12, there exists a neighbourhood U of x in which the characteristic foliation of S is topologically conjugate to a saddle, a node or a saddle-node. Thus, one can repeat the same argument as before: for every $y \in U$, if y belongs to a cell then one applies Lemma 2.19; otherwise, if y belongs to a separatrix one can connect x and y directly. $\square$

The proof of Proposition 2.16 is a corollary of Lemma 2.20.

Proof of Proposition 2.16 . The property of having finite distance is an equivalence relation on the points of S. Because of Lemma 2.20, the equivalence classes are open. Thus, because S is connected, there is only one class. $\square$

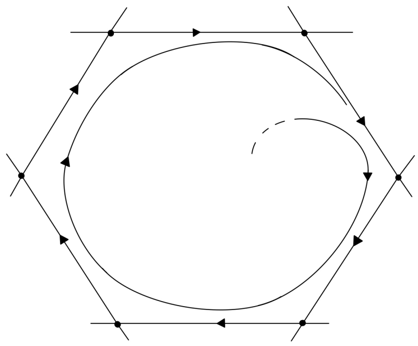

Figure 2.4: An embedded polygon which bounds a right-sided contour

2.4 Spheres in a tight contact distribution

In this section we prove Theorem 1.16, i.e., in a tight coorientable contact distribution the topological spheres have finite induced distance. This is done by showing that the hypothesis of Proposition 2.16 are satisfied in this setting.

An overtwisted disk, precisely defined at the end of Section 1.2.1, is an embedding of a disk with horizontal boundary such that the distribution does not twist along the boundary. A contact distribution is called *overtwisted* if it admits a overtwisted disk, and it is called *tight* if it is non-overtwisted.

Remark 2.21. Note that if the boundary of a disk is a periodic trajectory of its characteristic foliation, then the disk is overtwisted. Indeed, since a periodic trajectory does not contain characteristic points, then the plane distribution never coincides with the tangent space of the disk, thus the distribution can't perform any twists.

Lemma 2.22. *Let (M, D) be a tight contact 3-manifold, and S an embedded surface with the topology of a sphere. Then, the characteristic foliation of S does not contain periodic trajectories.*

Proof. Assume that the characteristic foliation of S has a periodic trajectory ℓ. Then, because ℓ does not have self-intersections, the leaf ℓ divides S in two topological half-spheres Δ_1 and Δ_2. The disks Δ_i, for $i = 1, 2$, are overtwisted, which contradicts the hypothesis that the distribution is tight because Remark 2.21. □

Now, let us discuss the sided contours.

Lemma 2.23. *Let (M, D) be a tight contact 3-manifold, and $S \subset M$ an embedded surface with the topology of a sphere. Then the characteristic foliation of S does not contain sided contours.*

Proof. Assume that the characteristic foliation presents a sided contour Γ. Its complementary $S \setminus \Gamma$ has two connected components, which are topologically half-spheres. Let us call Δ the component on the same side of Γ, i.e., if Γ is right-sided (resp. left-sided) then Δ is on the right (resp. left). For instance, if Γ is right-sided, then the characteristic foliation of Δ looks like that of the polygon in Figure 2.4.

Let p be one of the vertices of Δ, let ℓ_1 and ℓ_2 be the separatrices adjacent to p, and let U be a neighbourhood of p such that we are in the condition of Definition 2.17. Let us fix two points $x_i \in \ell_i \cap U$, for $i = 1, 2$. Due to the definition of hyperbolic sector, in a neighbourhood of x_1 the leaves pass arbitrarily close to x_2.

We are going to give the idea of how to perturb the surface near x_1 and x_2 so that the separatrices ℓ_1 and ℓ_2 are diverted to the same nearby leau, therefore bypassing p. In other words, via a C^∞-small perturbation of S supported in a neighbourhood of x_1 and x_2, we obtain a sphere which contains a sided contour with one less vertex, see Figure 2.5. By repeating such perturbation for every vertex, one

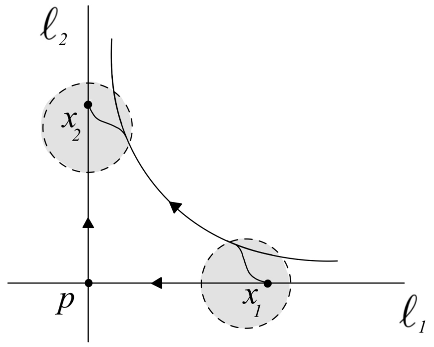

Figure 2.5: The characteristic foliation of the perturbed surface.

obtains a new surface with a periodic trajectory in its characteristic foliation, which is excluded due to Lemma 2.22.

Consider the Heisenberg distribution $(\mathbb{R}^3, \ker(dz + \frac{1}{2}(ydx - xdy))$. Let $\mathcal{P}$ be the vertical plane $\mathcal{P} = \{x = 0\}$, and q a point in $\mathcal{P}$ contained in the y-axis. As one can see in Example 2.5.1, the characteristic foliation of $\mathcal{P}$ is made up of parallel horizontal lines.

Locally, it is possible to rectify the surface S into the plane $\mathcal{P}$ using a contactomorphism of the respective ambient spaces, as explained in the following lines. Due to the rectification theorem of dynamical systems, the characteristic foliation of S in a neighbourhood of x_1 is diffeomorphic to that of a neighbourhood of q in $\mathcal{P}$. A generalisation of a theorem of Giroux [Gei08, Thm. 2.5.23] implies that the C^1-conjugation between the characteristic foliations of the two surfaces can be extended, in a smaller neighbourhood, to a contactomorphism. Precisely, there exists a contactomorphism ψ from a neighbourhood $V \subset M$ of x_1 to a neighbourhood of q in $\mathbb{R}^3$, with $\psi(S) \subset \mathcal{P}$.

For what it has been said above, the image of ℓ_1 by ψ is contained in the y-axis. By creating a small bump in $\mathcal{P}$ after the point q, we will be able to divert the leaf going through q to any other parallel line. Precisely, for any curve $\gamma(t) = (x(t), y(t))$, defining

$$z(t) = \frac{1}{2}\int_{t_1}^{t} x(s)y'(s) - y(s)x'(s)ds \qquad \forall\, t \in [t_1, t_2],$$

we obtain a horizontal curve $(x(t), y(t), z(t))$. Now, let γ be a smooth curve which joins smoothly to the y-axis at its end points $\gamma(t_1) = q$ and $\gamma(t_2)$, and let Ω be the set between γ and the y-axis. One can verify that $z(t_2) = \text{Area}\big(\Omega\big)$, where the area is a signed area. By choosing an appropriate curve γ, we can connect the y-axis from q to any other parallel line in $\mathcal{P}$ via a horizontal curve (Figure 2.6). Next, by creating a small bump in $\mathcal{P}$ in order to include this horizontal curve one has successfully diverted the leaf. This procedure can be done C^∞-small, provided one wants to connect to parallel lines sufficiently close to the y-axis. Thus, one can make sure that no new characteristic points are created. Finally, this perturbation has to be transposed to a perturbation of S using ψ.

The same argument has to be repeated *mutatis mutandis* in a neighbourhood of x_2, ensuring that one connects x_2 exactly to the leaf coming from x_1. This is possible due to the continuity property of a hyperbolic sector, which ensures that the leaf coming from x_1 intersects the domain of the rectifying contactomorphism of x_2. $\square$

We can finally prove Theorem 1.16.

Proof of Theorem 1.16. The surface S admits a global characteristic vector field, due to Lemma 1.10. Next, a surface with the topology of a sphere doesn't allow flows with nontrivial recurrent trajectories, see [ABZ96, Lem. 2.4]. Indeed, from a nontrivial recurrent trajectory one can construct a closed curve transversal to the flow which does not separate the surface, which contradicts the Jordan curve theorem.

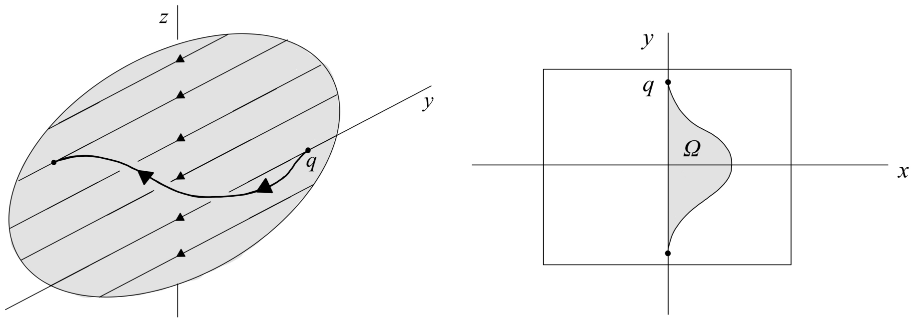

Figure 2.6: The lift to an horizontal curve connecting different leaves.

Then, Lemma 2.22 and Lemma 2.23 imply that the flow of a characteristic vector field of S does not contain periodic trajectories and sided contours, thus the hypothesis of Proposition 2.16 are satisfied. Consequently, d_S is finite. $\square$

2.5 Examples of surfaces in the Heisenberg structure

In this section we present some examples of surfaces in the Heisenberg sub-Riemannian structure, that is the contact, tight, sub-Riemannian structure of $\mathbb{R}^3$ for which (X_1, X_2) is a global orthonormal frame, where

$$X_1 = \partial_x - y/2\ \partial_z, \qquad X_2 = \partial_y + x/2\ \partial_z.$$

If $(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$ is a parametrisation of a surface S, then the characteristic vector field X in coordinates u, v becomes

$$X = -\left(z_v + x_v \frac{y}{2} - y_v \frac{x}{2}\right) \frac{\partial}{\partial u} + \left(z_u + x_u \frac{y}{2} - y_u \frac{x}{2}\right) \frac{\partial}{\partial v}, \tag{2.23}$$

where have used the subscripts to denote a partial derivative. When the surface is the graph of a function $S = \{z = h(x, y)\}$, then in the graph coordinates

$$X = \left(\frac{x}{2} - \partial_y h\right) \frac{\partial}{\partial x} + \left(\partial_x h + \frac{y}{2}\right) \frac{\partial}{\partial y},$$

and, at a characteristic point $p = (x, y, z)$, the metric coefficient $\widehat{K}_p$ is computed by

$$\widehat{K}_p = -3/4 + \partial^2_{xx} h(x, y)\ \partial^2_{yy} h(x, y) - \partial^2_{xy} h(x, y)\ \partial^2_{yx} h(x, y).$$

2.5.1 Planes.

Let us consider affine planes in Heisenberg. Thanks to the left-invariance, it is not restrictive to consider a plane $\mathcal{P}$ going throughout the origin. Thus,

$$\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\} \qquad \text{with } (a, b, c) \neq (0, 0, 0).$$

If $c = 0$, i.e., the plane is vertical, then $\mathcal{P}$ does not contain characteristic points. Every characteristic vector field is parallel to the vector $(b, -a,\ 0)$, therefore the characteristic foliation of $\mathcal{P}$ consists of lines that are parallel to the xy-plane. This implies that points with different z-coordinate are not at finite distance from each other, see Figure 2.7 (left).

Otherwise, if $c \neq 0$, then $\mathcal{P}$ has exactly one characteristic point $p = (-2b/c, 2a/c, 0)$. One has that

$$\widehat{K}_p = -\frac{3}{4}.$$

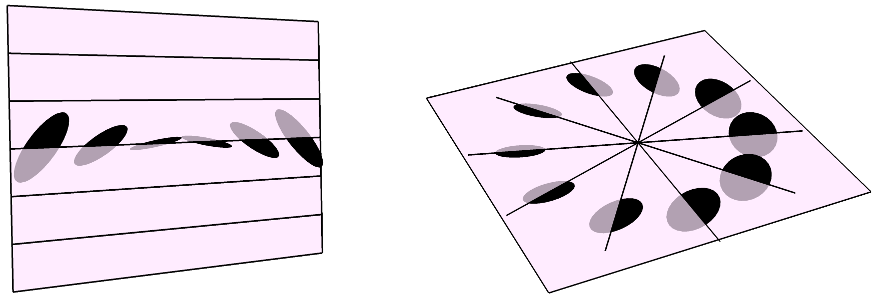

Figure 2.7: The qualitative picture of the characteristic foliation of a vertical plane (left), and of a non-vertical plane (right).

Thus, because of formula (2.15), there is one eigenvalue of multiplicity two. Due to Corollary 2.10, the characteristic foliation of $\mathcal{P}$ has a node at p. An explicit computation of X_S shows that

$$X_S(q) = \frac{q-p}{2} \qquad \forall\, q \in \mathcal{P},$$

which shows that the characteristic foliation of $\mathcal{P}$ is composed of Euclidean half-lines radiating out of p. The metric $d_{\mathcal{P}}$ induced by the Heisenberg group on $\mathcal{P}$ satisfies the following relation: for all $q, q' \in \mathcal{P}$, one has

$$d_{\mathcal{P}}\big(q,q'\big) = \begin{cases} |(x,y)-(x',y')|_{\mathbb{R}^2}, & \text{if } (q-p) \mathbin{/\!/} (q'-p) \\ d_{\mathcal{P}}(q,p) + d_{\mathcal{P}}(q',p), & \text{otherwise,} \end{cases}$$

where we have written $q = (x,y,z)$ and $q' = (x',y',z')$. This distance is sometimes called British Rail metric. See Figure 2.7 (right).

2.5.2 Ellipsoids

Fix $a, b, c > 0$, and consider the surface $\mathcal{E} = \mathcal{E}_{a,b,c}$ defined by

$$\mathcal{E}_{a,b,c} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \right\}.$$

This surface has exactly two characteristic points $p_1 = (0,0,c)$ and $p_2 = (0,0,-c)$, respectively at the North and the South pole. For both points, one has

$$\widehat{K}_{p_i} = -\frac{3}{4} + \frac{c^2}{a^2b^2}, \qquad i = 1,2.$$

Because of Corollary 2.10, the characteristic foliation of $\mathcal{E}$ spirals around the two poles, as in Figure 1.3. Due to Proposition 1.15, the spirals converging to the poles have finite sub-Riemannian length, thus the length distance d_S is finite. Indeed, d_S is realised by the length of the curves joining the points with either the North, or the South pole. Here, the finiteness of d_S is also a particular case of Theorem 1.16.

2.5.3 Symmetric paraboloids

Let $a \in \mathbb{R}$, and consider the paraboloid $\mathcal{P}_a$ with

$$\mathcal{P}_a = \big\{ (x,y,z) \in \mathbb{R}^3 \mid z = a\big(x^2+y^2\big) \big\}.$$

The origin p is the unique characteristic point of $\mathcal{P}_a$. Note that

$$\widehat{K}_p = -\frac{3}{4} + 4a^4,$$

therefore the characteristic foliation is a focus.

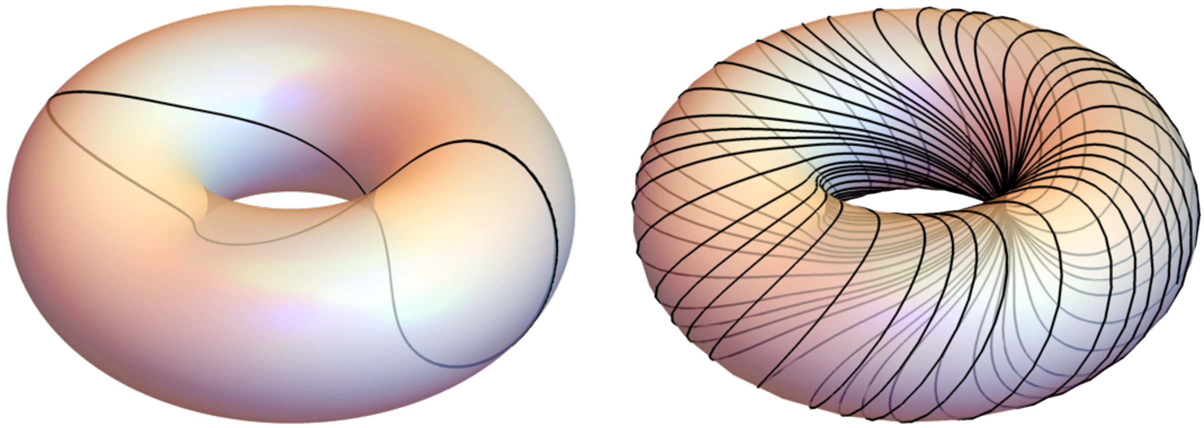

Figure 2.8: A leaf of the characteristic foliation of two Horizontal tori. On the left-hand side the leaf is periodic, and on the right-hand side there is a portion of an everywhere dense leaf .

2.5.4 Horizontal torus

Fix $R > r > 0$, and consider the torus parametrised by

$$\Phi(u, v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big).$$

This is the torus obtained by revolving a circle of radius $r > 0$ in the xz-plane around a circle of radius $R > r$ surrounding the z-axis. Using formula (2.23), a characteristic vector field X in the coordinates (u, v) is

$$X = \frac{(R + r\cos(u))^2}{2}\frac{\partial}{\partial u} - \frac{r\cos(u)}{2}\frac{\partial}{\partial v}. \tag{2.24}$$

It is immediate to see that the characteristic set is empty. Thus, no point can be a limit point of any leaves of the characteristic foliation; due to Remark 2.15, this implies that the length distance is infinite.

Lemma 2.24. *The characteristic foliation of a horizontal torus is filled either with periodic trajectories, or with everywhere dense trajectories.*

Proof. Using expression (2.24), in the coordinates u, v the trajectories of X satisfy

$$\begin{cases} \dot{u} = (R + r\cos(u))^2/2 \\ \dot{v} = -r\cos(u)/2. \end{cases} \tag{2.25}$$

Because the Heisenberg distribution and the horizontal torus are invariant under rotations around the z-axis, the same applies to the characteristic foliation. Thus, the solutions of (2.25) are v-translations of the solution $\gamma_0(t) = (u(t), v(t))$ with initial condition $\gamma_0(0) = (0, 0)$.

Note that $(r+R)^2/2 \geq \dot{u}(t) \geq (R-r)^2/2$. Thus, there exists a time t_0 in which the trajectory $\gamma_0(t)$, satisfies $u(t_0) = 2\pi$. Define $\alpha_{r,R} = v(t_0)$. If $\alpha_{r,R}/(2\pi) = m/n$ is rational, then $\gamma_0(nt_0) = 0 \pmod{2\pi}$. This shows that $\gamma_0(t)$ is periodic, as every other trajectory. On the other hand, if $\alpha_{r,R}/(2\pi)$ is irrational, then a classical argument shows that $\gamma(t)$ is dense in the torus, see for instance [ABZ96, E.g .2.3.1]. $\square$

See Figure 2.8 for a picture of a leaf in these two cases.

2.5.5 Vertical torus

Fix $R > r > 0$, and consider the torus $\mathcal{T} = \mathcal{T}_{r,R}$ parametrised by

$$\Phi(u, v) = \big(r\sin u, (R + r\cos u)\cos v, (R + r\cos u)\sin v\big).$$

This is the torus obtained by turning a circle of radius r in the xy-plane around a circle of radius R surrounding the x-axis. Due to formula (2.23), a characteristic vector field X in coordinates u, v is

$$X = (R + r\cos u)\Big(\cos v + \frac{r}{2}\sin v \sin u\Big)\frac{\partial}{\partial u} + \frac{r}{2}\Big(2\sin u \sin v - R\cos u \cos v - r\cos v\Big)\frac{\partial}{\partial v}.$$

The characteristic points are critical points of the vector field X. If $\cos v = \sin u = 0$, then (u, v) corresponds to a solution; this gives 4 characteristic points

$$F_\pm = \big(0, 0, \pm(R+r)\big), \qquad V_\pm = \big(0, 0, \pm(R-r)\big).$$

The other critical points of X occur if and only if

$$\tan v = -\frac{2}{r\sin u}, \qquad \cos u = -\frac{4+r^2}{rR}. \tag{2.26}$$

System (2.26) has solutions if and only if $R > 4$ and $|2r - R| \leq \sqrt{R^2 - 16}$, in which case we have 4 additional characteristic points $S_i(r, R)$, for $i = 1, 2, 3, 4$. Now, the metric coefficient at the characteristic points $F_\pm$ and $V_\pm$ is

$$\widehat{K}_{F_\pm} = -\frac{3}{4} + \frac{1}{r(R+r)},$$
$$\widehat{K}_{V_\pm} = -\frac{3}{4} - \frac{1}{r(R-r)}.$$

Note that $\widehat{K}_{F_\pm} > -3/4$, thus, due to Corollary 2.10, $F_\pm$ is a focus for all value of r and R. On the other hand, $\widehat{K}_{V_\pm}$ can attain any value between $-\infty$ and $-3/4$; precisely:

- if $R < 4$ or $|2r - R| > \sqrt{R^2 - 16}$, then $\widehat{K}_{V_\pm} < -1$ and $V_\pm$ are saddles.
- if $|2r - R| = \sqrt{R^2 - 16}$, then $\widehat{K}_{V_\pm} = -1$ and $V_\pm$ is a degenerate characteristic point; due to the Poincaré Index theorem, the points $V_\pm$ are saddles.
- if $|2r - R| < \sqrt{R^2 - 16}$, then $-1 < \widehat{K}_{V_\pm} < -3/4$ and $V_\pm$ are nodes.

The values for which $|2r - R| = \sqrt{R^2 - 16}$ are a bifurcation of the dynamical system X, because the number of characteristic point changes from 4 to 8. The characteristic points S_i which appears at this bifurcation are saddles, due to the Poincaré Index theorem. The bifurcation which takes place is the one presented in [Per12, E.g. 4.2.6].

2.6 Appendix on the center manifold theorem

In the language of dynamical systems, a non-degenerate characteristic point p is a *hyperbolic equilibrium* for any characteristic vector field X, i.e., an equilibrium for which the real parts of the eigenvalues of $DX(p)$ are non-zero. For a hyperbolic equilibrium p, the Hartman-Grobman theorem and the Hartman theorem give a conjugation between the flow of X and the flow of $DX(p)$, see [Per12, Par. 2.8] and [Har60].

Let us discuss here the case of a non-hyperbolic equilibrium, i.e., of a degenerate characteristic point. Let E be an open set of $\mathbb{R}^n$ containing the origin, and let X be a vector field in $C^1(E, \mathbb{R}^n)$ with $X(0) = 0$. Due to the Jordan decomposition theorem, we can assume that the linearisation of X at the origin is

$$DX(0) = \begin{pmatrix} C & & \\ & P & \\ & & Q \end{pmatrix},$$

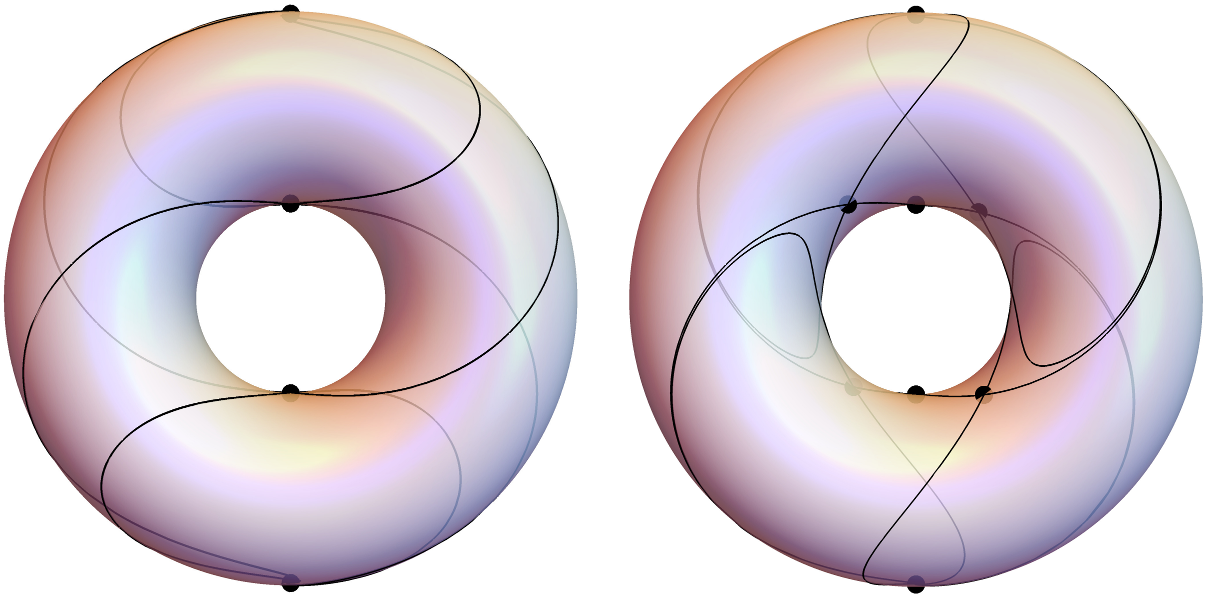

Figure 2.9: The topological skeleton, i.e., the singular trajectories, of the characteristic foliations of two vertical tori: the torus on the left-hand side has four characteristic points, and the torus on the right-hand side has eight characteristic points.

where C is a square $c \times c$ matrix with c complex (generalised) eigenvalues with zero real part, P with p complex (generalised) eigenvalues with positive real part, and Q with q complex (generalised) eigenvalues with negative real part. Thus, the dynamical system $\dot{\gamma} = X(\gamma)$ can be rewritten as

$$\begin{cases} \dot{x} = Cx + F(x, y, z) \\ \dot{y} = Py + G(x, y, z) \\ \dot{z} = Qz + H(x, y, z) \end{cases}$$

for $(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$, and for suitable functions F, G and H with $F(0) = G(0) = H(0) = 0$ and $DF(0) = DG(0) = DH(0) = 0$.

The origin is a non-hyperbolic characteristic point if and only if $c \geq 1$. Under these hypotheses, the following theorem shows that there exists an embedded submanifold $\mathcal{C}$ of dimension c, tangent to $\mathbb{R}^c$, and invariant for the flow of X. Such manifold is called a *central manifold* of X at the origin.

Proposition 2.25 ([Per12, Par. 2.12])**.** *Under the previous notations, there exists an open set $U \subset \mathbb{R}^c$ containing the origin, and two functions $h_1 : U \to \mathbb{R}^p$ and $h_2 : U \to \mathbb{R}^q$ of class C^1 with $h_1(0) = h_2(0) = 0$ and $Dh_1(0) = Dh_2(0) = 0$, and such that the map $x \mapsto (x, h_1(x), h_2(x))$ parametrises a submanifold invariant for the flow of X. Moreover, the flow of X is C^0-conjugate to the flow of*

$$\begin{cases} \dot{x} = Cx + F(x, h_1(x), h_2(x)) \\ \dot{y} = Py \\ \dot{z} = Qz. \end{cases} \tag{2.27}$$

In general, the central manifold $\mathcal{C}$ is non-unique. Note that the dynamic of the x-variable in equation (2.27) is simply the restriction of X to the center manifold $\mathcal{C}$. One can show that the trajectory converging to the origin approaches $\mathcal{C}$ exponentially fast: this is the asymptotic approximation property we used in (2.21).

Proposition 2.26 ([Bre07, p. 330])**.** *Under the previous assumptions, let us denote $\mathcal{C}$ a center manifold of the flow of X at the origin. Then, for every trajectory $l(t)$ such that $l(t) \to 0$ as $t \to +\infty$, there exists $\eta > 0$ and a trajectory $\zeta(t)$ in the center manifold $\mathcal{C}$, such that*

$$e^{\eta t} |l(t) - \zeta(t)|_{\mathbb{R}^n} \to 0, \qquad as \quad t \to +\infty.$$

Chapter 3

Stochastic processes on embedded surfaces in sub-Riemannian manifolds

Contents

This chapter presents the results in [BBCH21] as a joint work with Davide Barilari, Ugo Boscain and Karen Habermann in the journal Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.

First, in Section 3.1 we prove Theorem 1.17 and Proposition 1.18. Their proofs rely on the expression of $\Delta_\varepsilon|_{S\setminus\Sigma(S)}$ given in Lemma 3.2 in terms of an orthogonal frame for $T(S\setminus\Sigma(S))$. In Section 3.2, we prove Theorem 1.19 and Proposition 1.20 using Lemma 3.3 and Lemma 3.4, which expand the function $b\colon S\setminus\Sigma(S)\to\mathbb{R}$ from (1.29) in terms of the arc length along the integral curves of $\widehat{X}_S$. These results are illustrated in the last two sections with some examples. In Section 3.3, we study quadric surfaces in the Heisenberg group, whereas in Section 3.4, we consider canonical surfaces in $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ equipped with the standard sub-Riemannian contact structures. The examples establishing Theorem 1.21 are discussed in Section 3.3.1, Section 3.4.1 and Section 3.4.2, with a unified viewpoint presented in Section 3.4.3.

Given a coorientable sub-Riemannian contact manifold (M,D,g), in this section we make the choice to normalise the contact form ω using $\mathrm{d}\omega|_D = -\operatorname{vol}_g$; this in order to change the sign of the Reeb vector field. This will ease the references to [BBCH21], where this convention is used.

3.1 Family of Laplace–Beltrami operators on the embedded surface

Let (M, D, g) be a sub-Riemannian contact manifold, and assume that distribution D is free, that is, globally generated by a pair of vector fields X_1 and X_2, which we can choose so that (X_1, X_2) is an oriented orthonormal frame for D. Let (M, g_ε) be the Riemannian approximations constructed (canonically) using the Reeb vector field X_0 associated to the contact sub-Riemannian structure. By the Cartan formula and due to $\mathrm{d}\omega|_D = -\operatorname{vol}_g$, we have

$$\omega([X_1, X_2]) = -\mathrm{d}\omega(X_1, X_2) = 1 \ .$$

Since X_0 is the Reeb vector field, we have that

$$\omega([X_0, X_i]) = -\mathrm{d}\omega(X_0, X_i) = 0 \quad \text{for } i \in \{1, 2\} \ .$$

It follows that the structure constant defined in (2.1) are given by

$$[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 + X_0 \ , \tag{3.1}$$

$$[X_0, X_1] = c_{01}^1 X_1 + c_{01}^2 X_2 \ , \tag{3.2}$$

$$[X_0, X_2] = c_{02}^1 X_1 + c_{02}^2 X_2 \ . \tag{3.3}$$

In particular, it becomes clear that vector fields X_1, X_2 and $[X_1, X_2]$ are linearly independent everywhere.

Following the introduction, let Δ_ε be the Laplace–Beltrami operator of the Riemannian manifold $(S, i^* g_\varepsilon)$, where $i\colon S \hookrightarrow M$ is the natural immersion. We now express Δ_ε in terms of two vector fields on the surface S (independent on ε) which are orthogonal for each of the Riemannian approximations. Using these expressions of the Laplace–Beltrami operators Δ_ε where only the coefficients and not the vector fields depend on $\varepsilon > 0$, we prove Theorem 1.17. The orthogonal frame exhibited further allows us to establish Proposition 1.18.

For a vector field X on the manifold M, the property $Xu|_S \equiv 0$ ensures that $X(x) \in T_x S$ for all $x \in S$. Therefore, we see that F_1 and F_2 given by

$$F_1 = \frac{(X_2 u) X_1 - (X_1 u) X_2}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \qquad \text{and} \tag{3.4}$$

$$F_2 = \frac{(X_0 u)(X_1 u) X_1 + (X_0 u)(X_2 u) X_2}{(X_1 u)^2 + (X_2 u)^2} - X_0 \ , \tag{3.5}$$

are indeed well-defined vector fields on $S \setminus \Sigma(S)$ due to (1.17) and because we have $F_1 u|_{S \setminus \Sigma(S)} \equiv 0$ as well as $F_2 u|_{S \setminus \Sigma(S)} \equiv 0$. Here, $S \setminus \Sigma(S)$ is a manifold itself because the characteristic set $\Sigma(S)$ is a closed subset of S. We observe that both F_1 and F_2 remain unchanged if the function u defining the surface S is multiplied by a positive function, whereas F_1 changes sign and F_2 remains unchanged if u is multiplied by a negative function. Since the zero set of the twice differentiable submersion defining S needs to remain unchanged, these are the only two options which can occur. Observe that the vector field F_1 on $S \setminus \Sigma(S)$ is opposite to the vector field $\widehat{X}_S$ defined in (1.28).

Recalling that g_ε is obtained by requiring $(X_1, X_2, \varepsilon X_0)$ to be a global orthonormal frame, we further obtain

$$g_\varepsilon(F_1, F_2) = 0$$

as well as

$$g_\varepsilon(F_1, F_1) = 1 \quad \text{and} \quad g_\varepsilon(F_2, F_2) = \frac{(X_0 u)^2}{(X_1 u)^2 + (X_2 u)^2} + \frac{1}{\varepsilon^2} \ . \tag{3.6}$$

Thus, (F_1, F_2) is an orthogonal frame for $T(S \setminus \Sigma(S))$ for each Riemannian manifold $(S, i^* g_\varepsilon)$. While in general, the frame (F_1, F_2) is not orthonormal it has the nice property that it does not depend on $\varepsilon > 0$, which aids the analysis of the convergence of the operators Δ_ε in the limit $\varepsilon \to 0$. Since F_1 and F_2 are vector fields on $S \setminus \Sigma(S)$, there exist functions $b_1, b_2 \colon S \setminus \Sigma(S) \to \mathbb{R}$, not depending on $\varepsilon > 0$, such that

$$[F_1, F_2] = b_1 F_1 + b_2 F_2 \ . \tag{3.7}$$

Whereas determining the functions b_1 and b_2 explicitly from (3.4) and (3.5) is a painful task, we can express them nicely in terms of, following the notations in [BK19], the characteristic deviation h and a tensor η related to the torsion. Let $J \colon D \to D$ be the linear transformation induced by the contact form ω by requiring that, for vector fields X and Y in the distribution D,

$$g\left(X, J(Y)\right) = \mathrm{d}\omega(X, Y) \ . \tag{3.8}$$

Under the assumption of the existence of the global orthonormal frame (X_1, X_2) this amounts to saying that

$$J(X_1) = X_2 \quad \text{and} \quad J(X_2) = -X_1 \ . \tag{3.9}$$

For a unit-length vector field X in the distribution D, we use $[X, J(X)]|_D$ to denote the restriction of the vector field $[X, J(X)]$ on M to the distribution D and we set

$$\begin{aligned} h(X) &= -g\left([X, J(X)]|_D, X\right) \ , \\ \eta(X) &= -g\left([X_0, X], X\right) \ , \end{aligned}$$

where the expression for η is indeed well-defined because according to (3.2) and (3.3), the vector field $[X_0, X]$ lies in the distribution D.

Lemma 3.1. *For $b \colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29), we have*

$$[F_1, F_2] = -\left(bh(F_1) + \eta(F_1)\right) F_1 - bF_2 \ ,$$

that is, $b_1 = -bh(F_1) - \eta(F_1)$ and $b_2 = -b$.

Proof. We first observe that due to (3.9), we can write

$$F_2 = bJ(F_1) - X_0 \ .$$

Using (3.2) and (3.3) as well as (3.8), it follows that

$$\omega\left([F_1, F_2]\right) = \omega\left([F_1, bJ(F_1) - X_0]\right) = -\mathrm{d}\omega(F_1, bJ(F_1)) = -g(F_1, bJ^2(F_1)) = b \ .$$

On the other hand, from (3.4), (3.5) and (3.7), we deduce

$$\omega\left([F_1, F_2]\right) = \omega\left(b_2 F_2\right) = -b_2 \ ,$$

which implies that $b_2 = -b$, as claimed. It remains to determine b_1. From (3.8), we see that

$$g(F_1, J(F_1)) = -\omega([F_1, F_1]) = 0 \ .$$

Together with (3.7) this yields

$$b_1 = g\left([F_1, F_2], F_1\right) = g\left([F_1, bJ(F_1) - X_0], F_1\right) = bg\left([F_1, J(F_1)]|_D, F_1\right) + g\left([X_0, F_1], F_1\right) \ ,$$

and therefore, we have $b_1 = -bh(F_1) - \eta(F_1)$, as required. $\square$

To derive an expression for the Laplace–Beltrami operators Δ_ε of $(S, i^* g_\varepsilon)$ restricted to $S \setminus \Sigma(S)$ in terms of the vector fields F_1 and F_2, it is helpful to consider the normalised frame associated with the orthogonal frame (F_1, F_2). For $\varepsilon > 0$ fixed, we define $a_\varepsilon \colon S \setminus \Sigma(S) \to \mathbb{R}$ by

$$a_\varepsilon = \left(\frac{(X_0 u)^2}{(X_1 u)^2 + (X_2 u)^2} + \frac{1}{\varepsilon^2}\right)^{-\frac{1}{2}} \tag{3.10}$$

and we introduce the vector fields E_1 and $E_{2,\varepsilon}$ on $S \setminus \Sigma(S)$ given by

$$E_1 = F_1 \quad \text{and} \quad E_{2,\varepsilon} = a_\varepsilon F_2 \ . \tag{3.11}$$

In the Riemannian manifold (S, g_ε), this yields the orthonormal frame $(E_1, E_{2,\varepsilon})$ for $T(S \setminus \Sigma(S))$.

Lemma 3.2. *For $\varepsilon > 0$, the operator Δ_ε restricted to $S \setminus \Sigma(S)$ can be expressed as*

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = F_1^2 + a_\varepsilon^2 F_2^2 + \left(b - \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) F_1 - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 \ .$$

Proof. Fix $\varepsilon > 0$ and let $\operatorname{div}_\varepsilon$ denote the divergence operator on the Riemannian manifold (S, g_ε) with respect to the corresponding Riemannian volume form. Since $(E_1, E_{2,\varepsilon})$ is an orthonormal frame for $T(S \setminus \Sigma(S))$, we have

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = E_1^2 + E_{2,\varepsilon}^2 + (\operatorname{div}_\varepsilon E_1)\, E_1 + (\operatorname{div}_\varepsilon E_{2,\varepsilon})\, E_{2,\varepsilon} \ . \tag{3.12}$$

Let $(\nu_1, \nu_{2,\varepsilon})$ denote the dual to the orthonormal frame $(E_1, E_{2,\varepsilon})$. Proceeding, for instance, in the same way as in [Bar13, Proof of Proposition 11], we show that, for any vector field X on $S \setminus \Sigma(S)$,

$$\operatorname{div}_\varepsilon X = \nu_1\left([E_1, X]\right) + \nu_{2,\varepsilon}\left([E_{2,\varepsilon}, X]\right) \ .$$

This together with (3.11) and Lemma 3.1 implies that

$$\operatorname{div}_\varepsilon E_1 = \nu_{2,\varepsilon}\left([a_\varepsilon F_2, F_1]\right) = -\nu_{2,\varepsilon}\left(a_\varepsilon [F_1, F_2] + F_1(a_\varepsilon) F_2\right) = b - \frac{F_1(a_\varepsilon)}{a_\varepsilon}$$

as well as

$$\operatorname{div}_\varepsilon E_{2,\varepsilon} = \nu_1\left([F_1, a_\varepsilon F_2]\right) = \nu_1\left(a_\varepsilon [F_1, F_2] + F_1(a_\varepsilon) F_2\right) = -a_\varepsilon \left(bh(F_1) + \eta(F_1)\right) \ .$$

The desired result follows from (3.11) and (3.12). ☐

Note that $\Delta_\varepsilon|_{S\setminus\Sigma(S)}$ in Lemma 3.2 can equivalently be written as

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = F_1^2 + a_\varepsilon^2 F_2^2 + \left(b - \frac{F_1\left(a_\varepsilon^2\right)}{2a_\varepsilon^2}\right) F_1 - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 \ .$$

Using Lemma 3.2 we can prove Theorem 1.17.

Proof of Theorem 1.17. From (1.29) and (3.10), we obtain that

$$a_\varepsilon^2 = \left(b^2 + \frac{1}{\varepsilon^2}\right)^{-1} = \frac{\varepsilon^2}{\varepsilon^2 b^2 + 1} \ , \tag{3.13}$$

which we use to compute

$$\frac{F_1(a_\varepsilon)}{a_\varepsilon} = \frac{F_1(a_\varepsilon^2)}{2a_\varepsilon^2} = -\frac{\varepsilon^2 b F_1(b)}{\varepsilon^2 b^2 + 1} \ .$$

It follows that

$$a_\varepsilon^2 \leq \varepsilon^2 \quad \text{as well as} \quad \left|\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right| \leq \varepsilon^2 \left|bF_1(b)\right| \ . \tag{3.14}$$

Since $u \in C^2(M)$ by assumption, both $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ and $F_1(b)\colon S \setminus \Sigma(S) \to \mathbb{R}$ are continuous and therefore bounded on compact subsets of $S \setminus \Sigma(S)$. In a similar way, we argue that the function $b_1 = -bh(F_1) - \eta(F_1)$ is bounded on compact subsets of $S \setminus \Sigma(S)$. Due to (3.14), this implies that, uniformly on compact subsets of $S \setminus \Sigma(S)$,

$$\lim_{\varepsilon\to 0} a_\varepsilon^2 = 0 \ , \quad \lim_{\varepsilon\to 0} \frac{F_1(a_\varepsilon)}{a_\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) = 0 \ . \tag{3.15}$$

Let $f \in C_c^2(S \setminus \Sigma(S))$. We then have $F_1 f, F_2 f \in C_c^1(S \setminus \Sigma(S))$ and $F_1^2 f, F_2^2 f \in C_c^0(S \setminus \Sigma(S))$. Since the expression (1.30) for Δ_0 can be rewritten as

$$\Delta_0 = F_1^2 + bF_1$$

and since the convergence in (3.15) is uniformly on compact subsets of $S \setminus \Sigma(S)$, we deduce from Lemma 3.2 that

$$\lim_{\varepsilon\to 0} \|\Delta_\varepsilon f - \Delta_0 f\|_{\infty, S\setminus\Sigma(S)} = \lim_{\varepsilon\to 0} \left\| a_\varepsilon^2 F_2^2 f - \frac{F_1(a_\varepsilon)}{a_\varepsilon} F_1 f - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 f \right\|_{\infty, S\setminus\Sigma(S)} = 0 ,$$

that is, the functions $\Delta_\varepsilon f$ indeed converge uniformly on $S \setminus \Sigma(S)$ to $\Delta_0 f$. □

Using the orthonormal frames $(E_1, E_{2,\varepsilon})$, we easily derive the expression given in Proposition 1.18 for the intrinsic Gaussian curvature K_0 of the surface S in terms of the vector field $\widehat{X}_S$ and the function b. Unlike the reasoning presented in [BTV17], which further exploits intrinsic symmetries of the Heisenberg group $\mathbb{H}$, our derivation does not rely on the cancellation of divergent quantities and holds for surfaces in any three-dimensional contact sub-Riemannian manifold, cf. [BTV17, Remark 5.3].

Proof of Proposition 1.18. From Lemma 3.1 and due to (3.7) as well as (3.11), we have

$$[E_1, E_{2,\varepsilon}] = [F_1, a_\varepsilon F_2] = a_\varepsilon [F_1, F_2] + F_1(a_\varepsilon) F_2 = a_\varepsilon b_1 E_1 + \left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) E_{2,\varepsilon} .$$

According to the classical formula for the Gaussian curvature of a surface in terms of an orthonormal frame, see e.g. [ABB20, Proposition 4.40], the Gaussian curvature K_ε of the Riemannian manifold (S, g_ε) is given by

$$K_\varepsilon = F_1 \left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) - a_\varepsilon F_2 \left(a_\varepsilon b_1\right) - \left(a_\varepsilon b_1\right)^2 - \left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right)^2 . \tag{3.16}$$

We deduce from (3.13) that

$$a_\varepsilon F_2 \left(a_\varepsilon\right) = \frac{1}{2} F_2 \left(a_\varepsilon^2\right) = -\frac{\varepsilon^4 b F_2(b)}{\left(\varepsilon^2 b^2 + 1\right)^2}$$

as well as

$$F_1 \left(\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) = -F_1 \left(\frac{\varepsilon^2 b F_1(b)}{\varepsilon^2 b^2 + 1}\right) = -\frac{\varepsilon^2 F_1 \left(b F_1(b)\right)}{\varepsilon^2 b^2 + 1} + \frac{2\varepsilon^4 b^2 \left(F_1(b)\right)^2}{\left(\varepsilon^2 b^2 + 1\right)^2} ,$$

which, in addition to (3.14), implies

$$\left|a_\varepsilon F_2 \left(a_\varepsilon\right)\right| \le \varepsilon^4 \left|b F_2(b)\right| \quad \text{and} \quad \left|F_1 \left(\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right)\right| \le \varepsilon^2 \left|F_1 \left(b F_1(b)\right)\right| + 2\varepsilon^4 b^2 \left(F_1(b)\right)^2 .$$

By passing to the limit $\varepsilon \to 0$ in (3.16), the desired expression follows. □

Notice that, by construction, the function b and the intrinsic Gaussian curvature K_0 are related by the Riccati-like equation

$$\dot{b} + b^2 + K_0 = 0 ,$$

with the notation $\dot{b} = \widehat{X}_S(b)$, which is independent on the convection used to determine the sign of the Reeb vector field.

3.2 Canonical stochastic process on the embedded surface

We study the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$. After analysing the behaviour of the drift of the process around non-degenerate characteristic points, we prove Theorem 1.19 and Proposition 1.20.

By construction, the process with generator $\frac{1}{2}\Delta_0$ moves along the characteristic foliation of S, that is, along the integral curves of the vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ defined in (1.28). Around a fixed non-degenerate characteristic point $x \in \Sigma(S)$, the behaviour of the canonical stochastic process is determined by how $b\colon S\setminus\Sigma(S) \to \mathbb{R}$ given in (1.29) depends on the arc length along integral curves emanating from x. Since the vector fields X_1, X_2 and the Reeb vector field X_0 are linearly independent everywhere, the function $X_0u\colon S \to \mathbb{R}$ does not vanish near characteristic points. In particular, we may and do choose the function $u \in C^2(M)$ defining the surface S such that $X_0u \equiv 1$ in a neighbourhood of x.

Understanding the expression for the horizontal Hessian $\operatorname{Hess} u$ in (2.9) as a matrix representation in the dual frame of (X_1, X_2), and noting that the linear transformation $J\colon D \to D$ defined in (3.8) has the matrix representation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

we see that

$$(\operatorname{Hess} u)\, J = \begin{pmatrix} X_1X_2u & -X_1X_1u \\ X_2X_2u & -X_2X_1u \end{pmatrix} .$$

The dynamics around the characteristic point $x \in \Sigma(S)$ is uniquely determined by the eigenvalues λ_1 and λ_2 of $((\operatorname{Hess} u)(x))J$. Since $x \in \Sigma(S)$ is non-degenerate by assumption both eigenvalues are non-zero, and due to $X_0u \equiv 1$ in a neighbourhood of x, we further have

$$\lambda_1 + \lambda_2 = \operatorname{Tr}\left(((\operatorname{Hess} u)(x))J\right) = (X_1X_2u)\,(x) - (X_2X_1u)\,(x) = (X_0u)\,(x) = 1 \,. \tag{3.17}$$

Thus, one of the following three cases occurs, where we use the terminology from [Rob95, Section 4.4] to distinguish between them. In the first case, where the eigenvalues λ_1 and λ_2 are complex conjugate, the characteristic point x is of focus type and the integral curves of $\widehat{X}_S$ spiral towards the point x. In the second case, where both eigenvalues are real and of positive sign, we call $x \in \Sigma(S)$ of node type, and all integral curves of $\widehat{X}_S$ approaching x do so tangentially to the eigendirection corresponding to the smaller eigenvalue, with the exception of the separatrices of the larger eigenvalue. In the third case with the characteristic point x being of saddle type, the two eigenvalues are real but of opposite sign, and the only integral curves of $\widehat{X}_S$ approaching x are the separatrices.

Note that an elliptic characteristic point is of focus type or of node type, whereas a hyperbolic characteristic point is of saddle type. Depending on which of theses cases arises, we can determine how the function b depends on the arc length along integral curves of $\widehat{X}_S$ emanating from x. The choice of the function $u \in C^2(M)$ such that $X_0u \equiv 1$ in a neighbourhood of x fixes the sign of the vector field $\widehat{X}_S$. In particular, an integral curve γ of $\widehat{X}_S$ which extends continuously to $\gamma(0) = x$ might be defined either on the interval $[0, \delta)$ or on $(-\delta, 0]$ for some $\delta > 0$. As the derivation presented below works irrespective of the sign of the parameter of γ, we combine the two cases by writing $\gamma\colon I_\delta \to S$ for integral curves of $\widehat{X}_S$ extended continuously to $\gamma(0) = x$.

The expansion around a characteristic point of focus type is a result of the fact that the real parts of complex conjugate eigenvalues satisfying (3.17) equal $\frac{1}{2}$.

Lemma 3.3. *Let $x \in \Sigma(S)$ be a non-degenerate characteristic point and suppose that $u \in C^2(M)$ is chosen such that $X_0u \equiv 1$ in a neighbourhood of x. For $\delta > 0$, let $\gamma\colon I_\delta \to S$ be an integral curve of the vector field $\widehat{X}_S$ extended continuously to $\gamma(0) = x$. If the eigenvalues of $((\operatorname{Hess} u)(x))J$ are complex conjugate then, as $s \to 0$,*

$$b(\gamma(s)) = \frac{2}{s} + O(1) \,.$$

Proof. Since $X_0u \equiv 1$ in a neighbourhood of x, we may suppose that $\delta > 0$ is chosen small enough such that, for $s \in I_\delta \setminus \{0\}$,

$$b(\gamma(s)) = \frac{1}{\sqrt{\left((X_1u)\,(\gamma(s))\right)^2 + \left((X_2u)\,(\gamma(s))\right)^2}} \,.$$

A direct computation shows

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right)=\widehat{X}_S\left(b(\gamma(s))^{-1}\right)=\left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right),\widehat{X}_S\left(\gamma(s)\right)\right)\,.$$

By the Hartman–Grobman theorem, it follows that, for $s\to 0$,

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right)=\left((\operatorname{Hess} u)(x)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right),\widehat{X}_S\left(\gamma(s)\right)\right)+O(s)\,.$$

As complex conjugate eigenvalues of $((\operatorname{Hess} u)(x))J$ have real part equal to $\frac{1}{2}$ and due to $\widehat{X}_S$ being a unit-length vector field, the previous expression simplifies to

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right)=\frac{1}{2}+O(s)\,. \tag{3.18}$$

Since $(X_1u)(x)=(X_2u)(x)=0$ at the characteristic point x, we further have

$$\lim_{s\to 0}\frac{1}{b(\gamma(s))}=0\,. \tag{3.19}$$

A Taylor expansion together with (3.18) and (3.19) then implies that, as $s\to 0$,

$$\frac{1}{b(\gamma(s))}=\frac{s}{2}+O\left(s^2\right)\,,$$

which yields, for $s\to 0$,

$$b(\gamma(s))=\frac{2}{s}\left(1+O(s)\right)^{-1}=\frac{2}{s}+O(1)\,,$$

as claimed. □

The expansion of the function b around characteristic points of node type or of saddle type depends on along which integral curve of $\widehat{X}_S$ we are expanding. By the discussions preceding Lemma 3.3, all possible behaviours are covered by the next result.

Lemma 3.4. *Fix a non-degenerate characteristic point $x\in\Sigma(S)$. For $\delta>0$, let $\gamma\colon I_\delta\to S$ be an integral curve of the vector field $\widehat{X}_S$ which extends continuously to $\gamma(0)=x$. Assume $u\in C^2(M)$ is chosen such that $X_0u\equiv 1$ in a neighbourhood of x and suppose $((\operatorname{Hess} u)(x))J$ has real eigenvalues. If the curve γ approaches x tangentially to the eigendirection corresponding to the eigenvalue λ_i, for $i\in\{1,2\}$, then, as $s\to 0$,*

$$b(\gamma(s))=\frac{1}{\lambda_i s}+O(1)\,.$$

Proof. As in the proof of Lemma 3.3, we obtain, for $\delta>0$ small enough and $s\in I_\delta\setminus\{0\}$,

$$\widehat{X}_S\left(b(\gamma(s))^{-1}\right)=\left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right),\widehat{X}_S\left(\gamma(s)\right)\right)\,.$$

Since γ is an integral curve of the vector field $\widehat{X}_S$, we deduce that

$$\frac{\partial}{\partial s}\left(\frac{1}{b(\gamma(s))}\right)=\left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\gamma'(s)\right),\gamma'(s)\right)\,.$$

By Taylor expansion, this together with (3.19) yields, for $s\to 0$,

$$\frac{1}{b(\gamma(s))}=\left((\operatorname{Hess} u)\left(x\right)\right)\left(J\left(\gamma'(0)\right),\gamma'(0)\right)s+O\left(s^2\right)\,.$$

By assumption, the vector $\gamma'(0) \in T_xS$ is a unit-length eigenvector of $((\operatorname{Hess} u)(x))J$ corresponding to the eigenvalue λ_i, which has to be non-zero because x is a non-degenerate characteristic point. It follows that

$$((\operatorname{Hess} u)(x)) \left(J\left(\gamma'(0)\right), \gamma'(0)\right) = \lambda_i \neq 0 \ ,$$

which implies, for $s \to 0$,

$$b(\gamma(s)) = \frac{1}{\lambda_i s}\left(1 + O(s)\right)^{-1} = \frac{1}{\lambda_i s} + O(1) \ ,$$

as required. $\square$

Remark 3.5. We stress Lemma 3.4 does not contradict the positivity of the function b near the point x ensured by the choice of $u \in C^2(M)$ such that $X_0 u \equiv 1$ in neighbourhood of x. The derived expansion for b simply implies that on the separatrices corresponding to the negative eigenvalue of a hyperbolic characteristic point, the vector field $\widehat{X}_S$ points towards the characteristic point for that choice of u, that is, we have $s \in (-\delta, 0)$. At the same time, we notice that

$$\frac{\partial^2}{\partial s^2} + b\left(\gamma(s)\right)\frac{\partial}{\partial s}$$

remains invariant under a change from s to $-s$. Therefore, in our analysis of the one-dimensional diffusion processes induced on integral curves of $\widehat{X}_S$, we may again assume that the integral curves are parameterised by a positive parameter.

With the classification of singular points for stochastic differential equations given by Cherny and Engelbert in [CE05, Section 2.3], the previous two lemmas provide what is needed to prove Theorem 1.19 and Proposition 1.20. One additional crucial observation is that for a characteristic point of node type both eigenvalues of $((\operatorname{Hess} u)(x))\,J$ are positive and less than one, whereas for a characteristic point of saddle type, the positive eigenvalue is greater than one.

Proof of Theorem 1.19. Fix an elliptic characteristic point $x \in \Sigma(S)$. For $\delta > 0$, let $\gamma\colon [0, \delta] \to S$ be an integral curve of the vector field $\widehat{X}_S$ extended continuously to $x = \lim_{s \downarrow 0} \gamma(s)$. Following Cherny and Engelbert [CE05, Section 2.3], since the one-dimensional diffusion process on γ induced by $\frac{1}{2}\Delta_0$ has unit diffusivity and drift equal to $\frac{1}{2}b$, we set

$$\rho(t) = \exp\left(\int_t^\delta b(\gamma(s))\,\mathrm{d}s\right) \quad \text{for } t \in (0, \delta] \ . \tag{3.20}$$

If the characteristic point x is of node type the real positive eigenvalues λ_1 and λ_2 of $((\operatorname{Hess} u)(x))J$ satisfy $0 < \lambda_1, \lambda_2 < 1$ by (3.17). As x is of focus type or of node type by assumption, Lemma 3.3 and Lemma 3.4 establish the existence of some $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ such that, as $s \downarrow 0$,

$$b(\gamma(s)) = \frac{1}{\lambda s} + O(1) \ .$$

We deduce, for $\delta > 0$ sufficiently small,

$$\rho(t) = \exp\left(\int_t^\delta \left(\frac{1}{\lambda s} + O\left(1\right)\right)\mathrm{d}s\right) = \exp\left(\frac{1}{\lambda}\ln\left(\frac{\delta}{t}\right) + O(\delta - t)\right) = \left(\frac{\delta}{t}\right)^{\frac{1}{\lambda}}\left(1 + O(\delta - t)\right).$$

Due to $\frac{1}{\lambda} > 1$, this implies that

$$\int_0^\delta \rho(t)\,\mathrm{d}t = \infty \ .$$

According to [CE05, Theorem 2.16 and Theorem 2.17], it follows that the elliptic characteristic point x is an inaccessible boundary point for the one-dimensional diffusion processes induced on the integral curves of $\widehat{X}_S$ emanating from x. Since $x \in \Sigma(S)$ was an arbitrary elliptic characteristic point, the claimed result follows. $\square$

Proof of Proposition 1.20. We consider the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$ near a hyperbolic point $x \in \Sigma(S)$. Let γ be one of the four separatrices of x parameterised by arc length $s \geq 0$ and such that $\gamma(0) = x$. Let λ_1 be the positive eigenvalue and λ_2 be the negative eigenvalue of $((\operatorname{Hess} u)(x))J$. From the trace property (3.17), we see that $\lambda_1 > 1$. By Lemma 3.4 and Remark 3.5, we have, for $i \in \{1, 2\}$ and as $s \downarrow 0$,

$$b(\gamma(s)) = \frac{1}{\lambda_i s} + O(1) \ .$$

As in the previous proof, for $\delta > 0$ sufficiently small and $\rho\colon (0, \delta] \to \mathbb{R}$ defined by (3.20), we have

$$\rho(t) = \left(\frac{\delta}{t}\right)^{\frac{1}{\lambda_i}} (1 + O(\delta - t)) \ .$$

However, this time, due to $\frac{1}{\lambda_i} < 1$ for $i \in \{1, 2\}$, we obtain

$$\int_0^\delta \rho(t) \,\mathrm{d}t < \infty \ .$$

Using $\frac{1}{\lambda_1} > 0$, we further compute that, on the separatrices corresponding to the positive eigenvalue,

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} \,\mathrm{d}t = \int_0^\delta \frac{t^{\frac{1}{\lambda_1} - 1}}{2\lambda_1 \delta^{\frac{1}{\lambda_1}}} (1 + O(t)) \,\mathrm{d}t < \infty$$

and

$$\int_0^\delta \frac{|b(\gamma(t))|}{2} \,\mathrm{d}t = \infty \ .$$

On the separatrices corresponding to the negative eigenvalue, we have, due to $\frac{1}{\lambda_2} < 0$,

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} \,\mathrm{d}t = \int_0^\delta \frac{t^{\frac{1}{\lambda_2} - 1}}{2\lambda_2 \delta^{\frac{1}{\lambda_2}}} (1 + O(t)) \,\mathrm{d}t = \infty$$

as well as

$$s(t) = \int_0^t \rho(s) \,\mathrm{d}s = \frac{\lambda_2 \delta^{\frac{1}{\lambda_2}}}{\lambda_2 - 1} t^{1 - \frac{1}{\lambda_2}} (1 + O(t))$$

and

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} s(t) \,\mathrm{d}t = \int_0^\delta \frac{1}{2\,(\lambda_2 - 1)} (1 + O(t)) \,\mathrm{d}t < \infty \ .$$

Hence, as a consequence of the criterions [CE05, Theorem 2.12 and Theorem 2.13], the hyperbolic characteristic point x is reached with positive probability by the one-dimensional diffusion processes induced on the separatrices. Thus, the canonical stochastic process started on the separatrices is killed in finite time with positive probability. $\square$

3.3 Stochastic processes on quadric surfaces in the Heisenberg group

Let $\mathbb{H}$ be the first Heisenberg group, that is, the Lie group obtained by endowing $\mathbb{R}^3$ with the group law, expressed in Cartesian coordinates,

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right) \ .$$

On $\mathbb{H}$, we consider the two left-invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \ ,$$

and the contact form

$$\omega = \mathrm{d}z - \frac{1}{2}\left(x\,\mathrm{d}y - y\,\mathrm{d}x\right) \ .$$

We note that the vector fields X and Y span the contact distribution D corresponding to ω, that they are orthonormal with respect to the smooth fibre inner product g on D given by

$$g_{(x,y,z)} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y \ ,$$

and that

$$\mathrm{d}\omega|_D = -\mathrm{d}x \wedge \mathrm{d}y = -\operatorname{vol}_g \ .$$

Therefore, the Heisenberg group $\mathbb{H}$ understood as the three-dimensional contact sub-Riemannian manifold $(\mathbb{R}^3, D, g)$ falls into our setting, with $X_1 = X$, $X_2 = Y$ and the Reeb vector field

$$X_0 = \frac{\partial}{\partial z} = [X_1, X_2] \ .$$

In Section 3.3.1 and in Section 3.3.2, we discuss paraboloids and ellipsoids of revolution admitting one or two characteristic points, respectively, which are elliptic and of focus type. For these examples, the characteristic foliations can be described by logarithmic spirals in $\mathbb{R}^2$ lifted to the paraboloids and spirals between the poles on the ellipsoids, which are loxodromes, also called rhumb lines, on spheres. The induced stochastic processes are the Bessel process of order 3 for the paraboloids and Legendre-like processes for the ellipsoids moving along the leaves of the characteristic foliation. In Section 3.3.3, we consider hyperbolic paraboloids where, depending on a parameter, the unique characteristic point is either of saddle type or of node type, and we analyse the induced stochastic processes on the separatrices.

3.3.1 Paraboloid of revolution

For $a \in \mathbb{R}$, let S be the Euclidean paraboloid of revolution given by the equation $z = a(x^2 + y^2)$ for Cartesian coordinates (x, y, z) in the Heisenberg group $\mathbb{H}$. This corresponds to the surface given by (1.27) with $u\colon \mathbb{R}^3 \to \mathbb{R}$ defined as

$$u(x,y,z) = z - a\left(x^2 + y^2\right) \ .$$

We compute

$$X_0 u \equiv 1 \ , \quad \left(X_1 u\right)(x,y,z) = -2ax - \frac{y}{2} \quad \text{and} \quad \left(X_2 u\right)(x,y,z) = -2ay + \frac{x}{2} \ ,$$

which yields

$$\left((X_1 u)(x,y,z)\right)^2 + \left((X_2 u)(x,y,z)\right)^2 = \frac{1}{4}\left(1 + 16a^2\right)\left(x^2 + y^2\right) \ . \tag{3.21}$$

Thus, the origin of $\mathbb{R}^3$ is the only characteristic point on the paraboloid S. It is elliptic and of focus type because $X_0 u \equiv 1$ and

$$(\operatorname{Hess} u)\, J \equiv \begin{pmatrix} \frac{1}{2} & 2a \\ -2a & \frac{1}{2} \end{pmatrix}$$

has eigenvalues $\frac{1}{2} \pm 2a\,\mathrm{i}$. On $S \setminus \Sigma(S)$, the vector field $\widehat{X}_S$ defined by (1.28) can be expressed as

$$\widehat{X}_S = \frac{1}{\sqrt{\left(1 + 16a^2\right)\left(x^2 + y^2\right)}} \left((x - 4ay)\frac{\partial}{\partial x} + (y + 4ax)\frac{\partial}{\partial y} + 2a\left(x^2 + y^2\right)\frac{\partial}{\partial z} \right) \ . \tag{3.22}$$

Changing to cylindrical coordinates (r, θ, z) for $\mathbb{R}^3 \setminus \{0\}$ with $r > 0$, $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$ and using

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{as well as} \quad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ ,$$

the expression (3.22) for the vector field $\widehat{X}_S$ simplifies to

$$\widehat{X}_S = \frac{1}{\sqrt{1+16a^2}}\left(\frac{\partial}{\partial r} + \frac{4a}{r}\frac{\partial}{\partial \theta} + 2ar\frac{\partial}{\partial z}\right) \ .$$

From (3.21), we further obtain that the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29) can be written as

$$b(r, \theta, z) = \frac{1}{\sqrt{1+16a^2}}\frac{2}{r} \ .$$

Characteristic foliation The characteristic foliation induced on the paraboloid S of revolution by the contact structure D of the Heisenberg group $\mathbb{H}$ is described through the integral curves of the vector field $\widehat{X}_S$, cf. Figure 3.1. Its integral curves are spirals emanating from the origin which can be indexed by $\psi \in [0, 2\pi)$ and parameterised by $s \in (0, \infty)$ as follows

$$s \mapsto \left(\frac{s}{\sqrt{1+16a^2}}, 4a\ln\left(\frac{s}{\sqrt{1+16a^2}}\right) + \psi, \frac{as^2}{1+16a^2}\right) \ . \tag{3.23}$$

By construction, the vector field $\widehat{X}_S$ is a unit vector field with respect to each metric induced on the surface S from Riemannian approximations of the Heisenberg group. In particular, it follows that the parameter $s \in (0, \infty)$ describes the arc length along the spirals (3.23).

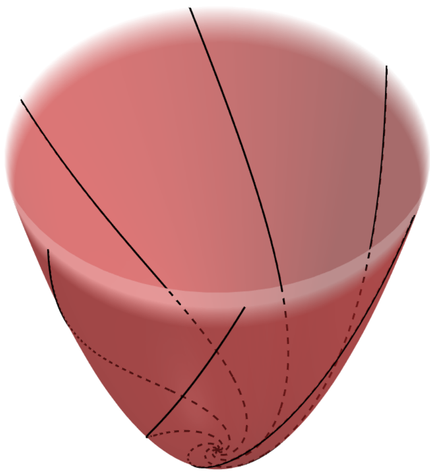

Figure 3.1: Characteristic foliation described by logarithmic spirals

Remark 3.6. The spirals on S defined by (3.23) are logarithmic spirals in $\mathbb{R}^2$ lifted to the paraboloid of revolution. In polar coordinates (r, θ) for $\mathbb{R}^2$, a logarithmic spiral can be written as

$$r = \mathrm{e}^{k(\theta+\theta_0)} \quad \text{for } k \in \mathbb{R} \setminus \{0\} \text{ and } \theta_0 \in [0, 2\pi) \ . \tag{3.24}$$

Therefore, the spirals in (3.23) correspond to lifts of logarithmic spirals (3.24) with $k = \frac{1}{4a}$. The arc length $s \in (0, \infty)$ of a logarithmic spiral (3.24) measured from the origin satisfies

$$s = \sqrt{1 + \frac{1}{k^2}}\, r \ ,$$

which for $k = \frac{1}{4a}$ yields $s = \sqrt{1+16a^2}\, r$. Note that this is the same relation between arc length and radial distance as obtained for integral curves (3.23) of the vector field $\widehat{X}_S$. For further information on logarithmic spirals, see e.g. Zwikker [Zwi63, Chapter 16].

Using the spirals (3.23) which describe the characteristic foliation on the paraboloid of revolution, we introduce coordinates (s, ψ) with $s > 0$ and $\psi \in [0, 2\pi)$ on the surface $S \setminus \Sigma(S)$. The vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ and the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ are then given by

$$\widehat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s} .$$

Thus, the canonical stochastic process induced on $S \setminus \Sigma(S)$ has generator

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\left(\widehat{X}_S^2 + b\widehat{X}_S\right) = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} .$$

This gives rise to a Bessel process of order 3 which out of all the spirals (3.23) describing the characteristic foliation on S stays on the unique spiral passing through the chosen starting point of the induced stochastic process. In agreement with Theorem 1.19, the origin is indeed inaccessible for this stochastic process because a Bessel process of order 3 with positive starting point remains positive almost surely. It arises as the radial component of a three-dimensional Brownian motion, and it is equal in law to a one-dimensional Brownian motion started on the positive real line and conditioned to never hit the origin. We further observe that the operator Δ_0 coincides with the radial part of the Laplace–Beltrami operator for a quadratic cone, cf. [BN20, BP16] for $\alpha = -2$, where the self-adjointness of Δ_0 is also studied.

As the limiting operator Δ_0 does not depend on the parameter $a \in \mathbb{R}$, the behaviour described above is also what we encounter on the plane $\{z = 0\}$ in the Heisenberg group $\mathbb{H}$, where the spirals (3.23) degenerate into rays emanating from the origin. We note that the stochastic process induced by $\frac{1}{2}\Delta_0$ on the rays differs from the singular diffusion introduced by Walsh [Wal78] on the same type of structure, but that it falls into the setting of Chen and Fukushima [CF15].

3.3.2 Ellipsoid of revolution

For $a, c \in \mathbb{R}$ positive, we study the Euclidean spheroid, also called ellipsoid of revolution, in the Heisenberg group $\mathbb{H}$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2c^2} = 1$$

in Cartesian coordinates (x, y, z). To shorten the subsequent expressions, we choose $u\colon \mathbb{R}^3 \to \mathbb{R}$ defining the Euclidean spheroid S through (1.27) to be given by

$$u(x, y, z) = x^2 + y^2 + \frac{z^2}{c^2} - a^2 .$$

Proceeding as in the previous example, we first obtain

$$(X_0 u)(x, y, z) = \frac{2z}{c^2}$$

as well as

$$(X_1 u)(x, y, z) = 2x - \frac{yz}{c^2} \quad \text{and} \quad (X_2 u)(x, y, z) = 2y + \frac{xz}{c^2} ,$$

which yields

$$((X_1 u)(x, y, z))^2 + ((X_2 u)(x, y, z))^2 = \left(x^2 + y^2\right)\left(4 + \frac{z^2}{c^4}\right) . \tag{3.25}$$

This implies the north pole $(0, 0, ac)$ and the south pole $(0, 0, -ac)$ are the only two characteristic points on the spheroid S. We further compute that

$$(X_2 u)X_1 - (X_1 u)X_2 = \left(2y + \frac{xz}{c^2}\right)\frac{\partial}{\partial x} - \left(2x - \frac{yz}{c^2}\right)\frac{\partial}{\partial y} - \left(x^2 + y^2\right)\frac{\partial}{\partial z} . \tag{3.26}$$

Using adapted spheroidal coordinates (θ,φ) for $S\setminus\Sigma(S)$ with $\theta\in(0,\pi)$ and $\varphi\in[0,2\pi)$, which are related to the coordinates (x,y,z) by

$$x=a\sin(\theta)\cos(\varphi)\ ,\quad y=a\sin(\theta)\sin(\varphi)\ ,\quad z=ac\cos(\theta)\ ,$$

we have

$$\begin{aligned}\frac{a\sin(\theta)}{c}\frac{\partial}{\partial\theta}&=\frac{xz}{c^2}\frac{\partial}{\partial x}+\frac{yz}{c^2}\frac{\partial}{\partial y}-\left(x^2+y^2\right)\frac{\partial}{\partial z}\quad\text{and}\\ \frac{\partial}{\partial\varphi}&=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\ .\end{aligned}$$

It follows that (3.26) on the surface $S\setminus\Sigma(S)$ simplifies to

$$(X_2u)X_1-(X_1u)X_2=\frac{a\sin(\theta)}{c}\frac{\partial}{\partial\theta}-2\frac{\partial}{\partial\varphi}\ ,$$

whereas (3.25) on $S\setminus\Sigma(S)$ rewrites as

$$((X_1u)(\theta,\varphi))^2+((X_2u)(\theta,\varphi))^2=a^2\left(\sin(\theta)\right)^2\left(4+\frac{a^2\left(\cos(\theta)\right)^2}{c^2}\right)\ .$$

This shows that the vector field $\widehat{X}_S$ on $S\setminus\Sigma(S)$ defined by (1.28) is given as

$$\widehat{X}_S=\frac{1}{\sqrt{4c^2+a^2\left(\cos(\theta)\right)^2}}\left(\frac{\partial}{\partial\theta}-\frac{2c}{a\sin(\theta)}\frac{\partial}{\partial\varphi}\right)\ . \tag{3.27}$$

For the function $b\colon S\setminus\Sigma(S)\to\mathbb{R}$ defined by (1.29), we further obtain that

$$b(\theta,\varphi)=\frac{2\cot(\theta)}{\sqrt{4c^2+a^2\left(\cos(\theta)\right)^2}}\ . \tag{3.28}$$

As in the preceding example, in order to understand the canonical stochastic process induced by the operator $\frac{1}{2}\Delta_0$ defined through (1.30), we need to express the vector field $\widehat{X}_S$ and the function b in terms of the arc length along the integral curves of $\widehat{X}_S$. Since both $\widehat{X}_S$ and b are invariant under rotations along the azimuthal angle φ, this amounts to changing coordinates on the spheroid S from (θ,φ) to (s,φ) where $s=s(\theta)$ is uniquely defined by requiring that

$$\frac{\partial}{\partial s}=\frac{1}{\sqrt{4c^2+a^2\left(\cos(\theta)\right)^2}}\left(\frac{\partial}{\partial\theta}-\frac{2c}{a\sin(\theta)}\frac{\partial}{\partial\varphi}\right)\qquad\text{and}\qquad s(0)=0\ .$$

This corresponds to

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}=\frac{1}{\sqrt{4c^2+a^2\left(\cos(\theta)\right)^2}}\ , \tag{3.29}$$

which together with $s(0)=0$ yields

$$s(\theta)=\int_0^\theta\sqrt{4c^2+a^2\left(\cos(\tau)\right)^2}\,\mathrm{d}\tau=\int_0^\theta\sqrt{(4c^2+a^2)-a^2\left(\sin(\tau)\right)^2}\,\mathrm{d}\tau\quad\text{for }\theta\in(0,\pi)\ .$$

Hence, the arc length s along the integral curves of $\widehat{X}_S$ is given in terms of the polar angle θ as a multiple of an elliptic integral of the second kind. Consequently, the question if θ can be expressed explicitly in terms of s is open. However, for our analysis, it is sufficient that the map $\theta \mapsto s(\theta)$ is invertible and that (3.28) as well as (3.29) then imply

$$b(s,\varphi) = 2\cot\left(\theta(s)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s} .$$

Therefore, using the coordinates (s,φ), the operator $\frac{1}{2}\Delta_0$ on $S\setminus\Sigma(S)$ can be expressed as

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \left(\cot\left(\theta(s)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\frac{\partial}{\partial s} ,$$

which depends on the constants $a, c \in \mathbb{R}$ through (3.29). Without the Jacobian factor $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ appearing in the drift term, the canonical stochastic process induced by the operator $\frac{1}{2}\Delta_0$ and moving along the leaves of the characteristic foliation would be a Legendre process, that is, a Brownian motion started inside an interval and conditioned not to hit either endpoint of the interval. The reason for the appearance of the additional factor $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ is that the integral curves of $\widehat{X}_S$ connecting the two characteristic points are spirals and not just great circles. For some further discussions on the characteristic foliation of the spheroid, see the subsequent Remark 3.8.

The emergence of an operator which is almost the generator of a Legendre process moving along the leaves of the characteristic foliation motivates the search for a surface in a three-dimensional contact sub-Riemannian manifold where we do exhibit a Legendre process moving along the leaves of the characteristic foliation induced by the contact structure. This is achieved in Section 3.4.1.

Remark 3.7. The northern hemisphere of the spheroid could equally be defined by the function

$$u(x,y,z) = z - c\sqrt{a^2 - x^2 - y^2} .$$

With this choice we have $X_0 u \equiv 1$. We further obtain

$$\left(\left(\operatorname{Hess} u\right)(0,0,ac)\right) J = \begin{pmatrix} \frac{1}{2} & -\frac{c}{a} \\ \frac{c}{a} & \frac{1}{2} \end{pmatrix} ,$$

whose eigenvalues are $\frac{1}{2} \pm \frac{c}{a}\mathrm{i}$. A similar computation on the southern hemisphere implies that both characteristic points are elliptic and of focus type. Thus, by Theorem 1.19, the stochastic process with generator $\frac{1}{2}\Delta_0$ hits neither the north pole nor the south pole, and it induces a one-dimensional process on the unique leaf of the characteristic foliation picked out by the starting point.

Remark 3.8. With respect to the Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^3$, we have for the adapted spheroidal coordinates (θ,φ) of $S\setminus\Sigma(S)$ as above that

$$\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = a^2\left(\cos(\theta)\right)^2 + a^2c^2\left(\sin(\theta)\right)^2 \quad \text{and} \quad \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \right\rangle = a^2\left(\sin(\theta)\right)^2 .$$

It follows that the angle α formed by the vector field $\widehat{X}_S$ given in (3.27) and the azimuthal direction satisfies

$$\cos\left(\alpha(\theta,\varphi)\right) = -\frac{2c}{\sqrt{a^2\left(\cos(\theta)\right)^2 + a^2c^2\left(\sin(\theta)\right)^2 + 4c^2}} .$$

Notably, on spheres, that is, if $c = 1$, the angle α is constant everywhere. Hence, the integral curves of $\widehat{X}_S$ considered as Euclidean curves on an Euclidean sphere are loxodromes, cf. Figure 1.3, which are also called rhumb lines. They are related to logarithmic spirals through stereographic projection. Loxodromes arise in navigation by following a path with constant bearing measured with respect to the north pole or the south pole, see Carlton-Wippern [CW92].

3.3.3 Hyperbolic paraboloid

For $a \in \mathbb{R}$ positive and such that $a \neq \frac{1}{2}$, we consider the Euclidean hyperbolic paraboloid S in the Heisenberg group $\mathbb{H}$ given by (1.27) with $u\colon \mathbb{R}^3 \to \mathbb{R}$ defined as

$$u(x,y,z) = z - axy \; ,$$

for Cartesian coordinates (x,y,z). We compute

$$X_0 u \equiv 1 \; , \quad (X_1 u)(x,y,z) = -ay - \frac{y}{2} \quad \text{as well as} \quad (X_2 u)(x,y,z) = -ax + \frac{x}{2} \; , \tag{3.30}$$

and further that

$$(\operatorname{Hess} u)\, J \equiv \begin{pmatrix} \frac{1}{2} - a & 0 \\ 0 & \frac{1}{2} + a \end{pmatrix} . \tag{3.31}$$

Due to

$$((X_1 u)(x,y,z))^2 + ((X_2 u)(x,y,z))^2 = \left(\frac{1}{2} - a\right)^2 x^2 + \left(\frac{1}{2} + a\right)^2 y^2 \; ,$$

the hyperbolic paraboloid S has the origin of $\mathbb{R}^3$ as its unique characteristic point. By (3.31), this characteristic point is elliptic and of node type if $0 < a < \frac{1}{2}$, and hyperbolic and therefore of saddle type if $a > \frac{1}{2}$. The reason for having excluded the case $a = \frac{1}{2}$ right from the beginning is that it gives rise to a line of degenerate characteristic points.

We note that the x-axis and the y-axis lie in the hyperbolic paraboloid S. From (3.30), we see that the positive and negative x-axis as well as the positive and negative y-axis are integral curves of the vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$. In the following, we restrict our attention to studying the behaviour of the canonical stochastic process on these integral curves, which nevertheless nicely illustrates Theorem 1.19 and Proposition 1.20.

We start by analysing the one-dimensional diffusion process induced on the positive y-axis γ_y^+, which by symmetry is equal in law to the process induced on the negative y-axis. For all positive $a \in \mathbb{R}$ with $a \neq \frac{1}{2}$, we have

$$\widehat{X}_S|_{\gamma_y^+} = \frac{\partial}{\partial y} \; ,$$

implying that the arc length $s > 0$ along γ_y^+ is given by $s = y$. This yields, for all $s > 0$,

$$b\left(\gamma_y^+(s)\right) = \frac{1}{\left(\frac{1}{2} + a\right) s} \; .$$

Thus, the one-dimensional diffusion process on γ_y^+ induced by $\frac{1}{2}\Delta_0$ has generator

$$\frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{1}{(1+2a)\, s}\frac{\partial}{\partial s} \; ,$$

which gives rise to a Bessel process of order $1 + \frac{2}{1+2a}$. If started at a point with positive value this diffusion process stays positive for all times almost surely if $1 + \frac{2}{1+2a} > 2$ whereas it hits the origin with positive probability if $1 + \frac{2}{1+2a} < 2$. This is consistent with Theorem 1.19 and Proposition 1.20 because for $a > \frac{1}{2}$ the positive y-axis is a separatrix for the hyperbolic characteristic point at the origin and

$$2 < 1 + \frac{2}{1+2a} \quad \text{if } 0 < a < \frac{1}{2} \qquad \text{as well as} \qquad 2 > 1 + \frac{2}{1+2a} \quad \text{if } a > \frac{1}{2} \; .$$

Some more care is needed when studying the diffusion process induced on the positive x-axis γ_x^+. As before, this process is equal in law to the process induced on the negative x-axis. We obtain

$$\widehat{X}_S|_{\gamma_x^+} = \begin{cases} \frac{\partial}{\partial x} & \text{if } 0 < a < \frac{1}{2} \\ -\frac{\partial}{\partial x} & \text{if } a > \frac{1}{2} \end{cases}$$

as well as, for $x>0$,

$$b(x,0,0)=\begin{cases}\dfrac{1}{\left(\frac{1}{2}-a\right)x} & \text{if } 0<a<\frac{1}{2}\\ -\dfrac{1}{\left(\frac{1}{2}-a\right)x} & \text{if } a>\frac{1}{2}\end{cases}.$$

It follows that the one-dimensional diffusion process on γ_x^+ induced by $\frac{1}{2}\Delta_0$ has generator

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{(1-2a)\,x}\frac{\partial}{\partial x}.$$

This yields a Bessel process of order $1+\frac{2}{1-2a}$. In agreement with Theorem 1.19 and Proposition 1.20, if started at a point with positive value this process never reaches the origin if $0<a<\frac{1}{2}$ which ensures $1+\frac{2}{1-2a}>3$, whereas the process reaches the origin with positive probability if $a>\frac{1}{2}$ as this corresponds to $1+\frac{2}{1-2a}<1$.

3.4 Stochastic processes on canonical surfaces in $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$

In Section 3.3.1, we establish that for a paraboloid of revolution embedded in the Heisenberg group $\mathbb{H}$, the operator $\frac{1}{2}\Delta_0$ induces a Bessel process of order 3 moving along the leaves of the characteristic foliation, which is described by lifts of logarithmic spirals emanating from the origin. As discussed in Revuz and Yor [RY99, Chapter VIII.3], the Legendre processes and the hyperbolic Bessel processes arise from the same type of Girsanov transformation as the Bessel process, where these three cases only differ by the sign of a parameter. We further recall that in Section 3.3.2 we encounter a canonical stochastic process which is almost a Legendre process moving along the leaves of the characteristic foliation induced on a spheroid in the Heisenberg group $\mathbb{H}$. This motivates the search for surfaces in three-dimensional contact sub-Riemannian manifolds where the canonical stochastic process is a Legendre process of order 3 or a hyperbolic Bessel process of order 3 moving along the leaves of the characteristic foliation.

We consider surfaces in the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ endowed with standard sub-Riemannian structures. Together with the Heisenberg group, these sub-Riemannian geometries play the role of model spaces for three-dimensional contact sub-Riemannian manifolds. In the first two subsections, we find, by explicit computations, the canonical stochastic processes induced on certain surfaces in these groups, when expressed in convenient coordinates. The last subsection proposes a unified geometric description, justifying the choice of our surfaces.

3.4.1 Special unitary group $\mathrm{SU}(2)$

One obstruction to recovering Legendre processes moving along the characteristic foliation in Section 3.3.2 is that the characteristic foliation of a spheroid in the Heisenberg group is described by spirals connecting the north pole and the south pole instead of great circles. This is the reason for considering S^2 as a surface embedded in $\mathrm{SU}(2)\simeq S^3$ understood as a contact sub-Riemannian manifold because this gives rise to a characteristic foliation on S^2 described by great circles.

The special unitary group $\mathrm{SU}(2)$ is the Lie group of 2×2 unitary matrices of determinant 1, that is,

$$\mathrm{SU}(2)=\left\{\begin{pmatrix} z+w\,\mathrm{i} & y+x\,\mathrm{i}\\ -y+x\,\mathrm{i} & z-w\,\mathrm{i}\end{pmatrix} : x,y,z,w\in\mathbb{R}\text{ with } x^2+y^2+z^2+w^2=1\right\},$$

with the group operation being given by matrix multiplication. Using the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\quad\text{and}\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

we identify SU(2) with the unit quaternions, and hence also with S^3, via the map

$$\begin{pmatrix} z + w\,\mathrm{i} & y + x\,\mathrm{i} \\ -y + x\,\mathrm{i} & z - w\,\mathrm{i} \end{pmatrix} \mapsto zI_2 + x\,\mathrm{i}\,\sigma_1 + y\,\mathrm{i}\,\sigma_2 + w\,\mathrm{i}\,\sigma_3 \,.$$

The Lie algebra $\mathfrak{su}(2)$ of SU(2) is the algebra formed by the 2×2 skew-Hermitian matrices with trace zero. A basis for $\mathfrak{su}(2)$ is $\{\frac{\mathrm{i}\sigma_1}{2}, \frac{\mathrm{i}\sigma_2}{2}, \frac{\mathrm{i}\sigma_3}{2}\}$ and the corresponding left-invariant vector fields on the Lie group SU(2) are

$$\begin{aligned} U_1 &= \frac{1}{2}\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} - w\frac{\partial}{\partial y} + y\frac{\partial}{\partial w}\right) \,, \\ U_2 &= \frac{1}{2}\left(-y\frac{\partial}{\partial z} + w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - x\frac{\partial}{\partial w}\right) \,, \\ U_3 &= \frac{1}{2}\left(-w\frac{\partial}{\partial z} - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}\right) \,, \end{aligned}$$

which satisfy the commutation relations $[U_1, U_2] = -U_3$, $[U_2, U_3] = -U_1$ and $[U_3, U_1] = -U_2$. Thus, any two of these three left-invariant vector fields give rise to a sub-Riemannian structure on SU(2). To streamline the subsequent computations, we choose $k \in \mathbb{R}$ with $k > 0$ and equip SU(2) with the sub-Riemannian structure obtained by setting $X_1 = 2kU_1$, $X_2 = 2kU_2$ and by requiring (X_1, X_2) to be an orthonormal frame for the distribution D spanned by the vector fields X_1 and X_2. The appropriately normalised contact form ω for the contact distribution D is

$$\omega = \frac{1}{2k^2}\left(w\,\mathrm{d}z + y\,\mathrm{d}x - x\,\mathrm{d}y - z\,\mathrm{d}w\right)$$

and the associated Reeb vector field X_0 satisfies

$$X_0 = [X_1, X_2] = -4k^2U_3 = 2k^2\left(w\frac{\partial}{\partial z} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - z\frac{\partial}{\partial w}\right) \,.$$

In SU(2), we consider the surface S given by the function $u\colon \mathrm{SU}(2) \to \mathbb{R}$ defined by

$$u(x, y, z, w) = w \,.$$

The surface S is isomorphic to S^2 because

$$S = \left\{\begin{pmatrix} z & y + x\,\mathrm{i} \\ -y + x\,\mathrm{i} & z \end{pmatrix} : x, y, z \in \mathbb{R} \text{ with } x^2 + y^2 + z^2 = 1\right\} \,.$$

We compute

$$(X_0u)(x, y, z, w) = -2k^2z \,, \quad (X_1u)(x, y, z, w) = ky \quad \text{and} \quad (X_2u)(x, y, z, w) = -kx \,,$$

which yields

$$((X_1u)(x, y, z, w))^2 + ((X_2u)(x, y, z, w))^2 = k^2\left(x^2 + y^2\right) \,.$$

Due to $x^2 + y^2 + z^2 = 1$, it follows that a point on S is characteristic if and only if $z = \pm 1$. Thus, the characteristic points on S are the north pole $(0, 0, 1)$ and the south pole $(0, 0, -1)$. The vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ defined by (1.28) is given as

$$\widehat{X}_S = \frac{k}{\sqrt{x^2 + y^2}}\left(\left(x^2 + y^2\right)\frac{\partial}{\partial z} - xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y}\right) \,, \tag{3.32}$$

and for the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29), we obtain

$$b(x, y, z) = -\frac{2kz}{\sqrt{x^2 + y^2}} \,. \tag{3.33}$$

We now change coordinates for $S \setminus \Sigma(S)$ from (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $z \neq \pm 1$ to (θ, φ) with $\theta \in (0, \frac{\pi}{k})$ and $\varphi \in [0, 2\pi)$ by

$$x = \sin(k\theta)\cos(\varphi) \ , \quad y = \sin(k\theta)\sin(\varphi) \quad \text{and} \quad z = \cos(k\theta) \ .$$

We note that

$$\frac{\partial}{\partial \theta} = k\cos(k\theta)\cos(\varphi)\frac{\partial}{\partial x} + k\cos(k\theta)\sin(\varphi)\frac{\partial}{\partial y} - k\sin(k\theta)\frac{\partial}{\partial z}$$

as well as

$$xz = \sin(k\theta)\cos(k\theta)\cos(\varphi) \ , \quad yz = \sin(k\theta)\cos(k\theta)\sin(\varphi) \quad \text{and} \quad \sqrt{x^2 + y^2} = \sin(k\theta) \ .$$

This together with (3.32) and (3.33) implies that

$$\widehat{X}_S = -\frac{\partial}{\partial \theta} \quad \text{and} \quad b(\theta, \varphi) = -2k\cot(k\theta) \ .$$

We deduce that the integral curves of $\widehat{X}_S$ are great circles on S and that

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial \theta^2} + k\cot(k\theta)\frac{\partial}{\partial \theta} \ ,$$

which indeed, on each great circle, induces a Legendre process of order 3 on the interval $(0, \frac{\pi}{k})$. These processes first appeared in Knight [Kni69] as so-called taboo processes and are obtained by conditioning Brownian motion started inside the interval $(0, \frac{\pi}{k})$ to never hit either of the two boundary points, see Bougerol and Defosseux [BD19, Section 5.1]. As discussed in Itô and McKean [IM74, Section 7.15], they also arise as the latitude of a Brownian motion on the three-dimensional sphere of radius $\frac{1}{k}$.

3.4.2 Special linear group $\mathrm{SL}(2, \mathbb{R})$

The appearance of the Bessel process on the plane $\{z = 0\}$ in the Heisenberg group $\mathbb{H}$ and of the Legendre processes on a compactified plane in $\mathrm{SU}(2)$ understood as a contact sub-Riemannian manifold suggests that the hyperbolic Bessel processes arise on planes in the special linear group $\mathrm{SL}(2, \mathbb{R})$ equipped with a sub-Riemannian structure. This is indeed the case if we consider the standard sub-Riemannian structures on $\mathrm{SL}(2, \mathbb{R})$ where the flow of the Reeb vector field preserves the distribution and the fibre inner product.

The special linear group $\mathrm{SL}(2, \mathbb{R})$ of degree two over the field $\mathbb{R}$ is the Lie group of 2×2 matrices with determinant 1, that is,

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{R} \text{ with } xw - yz = 1 \right\} \ ,$$

where the group operation is taken to be matrix multiplication. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$ is the algebra of traceless 2×2 real matrices. A basis of $\mathfrak{sl}(2, \mathbb{R})$ is formed by the three matrices

$$p = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad q = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad j = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

whose corresponding left-invariant vector fields on $\mathrm{SL}(2, \mathbb{R})$ are

$$\begin{aligned} X &= \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w} \right) \ , \\ Y &= \frac{1}{2}\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \right) \ , \\ K &= \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \right) \ . \end{aligned}$$

These vector fields satisfy the commutation relations $[X,Y]=K$, $[X,K]=Y$ and $[Y,K]=-X$. For $k\in\mathbb{R}$ with $k>0$, we equip $\mathrm{SL}(2,\mathbb{R})$ with the sub-Riemannian structure obtain by considering the distribution D spanned by $X_1=2kX$ and $X_2=2kY$ as well as the fibre inner product uniquely given by requiring (X_1,X_2) to be a global orthonormal frame. The appropriately normalised contact form corresponding to this choice is

$$\omega=\frac{1}{4k^2}\left(z\,\mathrm{d}x+w\,\mathrm{d}y-x\,\mathrm{d}z-y\,\mathrm{d}w\right)\,,$$

and the Reeb vector field X_0 associated with the contact form ω satisfies

$$X_0=[X_1,X_2]=4k^2K=2k^2\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-w\frac{\partial}{\partial z}+z\frac{\partial}{\partial w}\right)\,.$$

The plane in $\mathrm{SL}(2,\mathbb{R})$ passing tangentially to the contact distribution through the identity element is the surface S given as (1.27) by the function $u\colon \mathrm{SL}(2,\mathbb{R})\to\mathbb{R}$ defined by

$$u(x,y,z,w)=y-z\,.$$

Observe that, on S, we have the relation $xw=1+y^2\geq 1$. Therefore, if a point (x,y,z,w) lies on the surface S then so does the point $(-x,y,z,-w)$, and neither x nor w can vanish on S. Thus, the function $u\colon \mathrm{SL}(2,\mathbb{R})\to\mathbb{R}$ induces a surface consisting of two sheets. By symmetry, we restrict our attention to the sheet containing the 2×2 identity matrix, henceforth referred to as the upper sheet. We compute

$$(X_1u)(x,y,z,w)=-k\,(y+z)\quad\text{and}\quad(X_2u)(x,y,z,w)=k\,(x-w)\,,$$

as well as

$$(X_0u)(x,y,z,w)=2k^2\,(x+w)\,.$$

We note that

$$((X_1u)(x,y,z,w))^2+((X_2u)(x,y,z,w))^2=k^2\,(y+z)^2+k^2\,(x-w)^2$$

vanishes on S if and only if $y=z=0$ and $x=w$. From $xw=1+y^2$, it follows that the surface S admits the two characteristic points $(1,0,0,1)$ and $(-1,0,0,-1)$, that is, one unique characteristic point on each sheet. Following Rogers and Williams [RW00, Section V.36], we choose coordinates (r,θ) with $r>0$ and $\theta\in[0,2\pi)$ on the upper sheet of $S\setminus\Sigma(S)$ such that

$$\begin{aligned}x&=\cosh(kr)+\sinh(kr)\cos(\theta)\,,\\ w&=\cosh(kr)-\sinh(kr)\cos(\theta)\,,\quad\text{and}\\ y&=\sinh(kr)\sin(\theta)\,.\end{aligned}$$

On the upper sheet of $S\setminus\Sigma(S)$, we obtain

$$(X_1u)(r,\theta)=-2k\sinh(kr)\sin(\theta)\quad\text{and}\quad(X_2u)(r,\theta)=2k\sinh(kr)\cos(\theta)\,,$$

which yields

$$\sqrt{((X_1u)(r,\theta))^2+((X_2u)(r,\theta))^2}=2k\sinh(kr)\,,$$

as well as

$$(X_0u)(r,\theta)=4k^2\cosh(kr)\,.$$

A direct computation shows that on the upper sheet of $S\setminus\Sigma(S)$, we have

$$\widehat{X}_S=\frac{\partial}{\partial r}\quad\text{and}\quad b(r,\theta)=2k\coth(kr)\,,$$

which implies that

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial r^2} + k\coth(kr)\frac{\partial}{\partial r}\ .$$

Hence, we recover all hyperbolic Bessel processes of order 3 as the canonical stochastic processes moving along the leaves of the characteristic foliation of the upper sheet of $S \setminus \Sigma(S)$, and similarly on its lower sheet. For further discussions on hyperbolic Bessel processes, see Borodin [Bor08], Gruet [Gru00], Jakubowski and Wiśniewolski [JW13], and Revuz and Yor [RY99, Exercise 3.19]. As for the Bessel process of order 3 and the Legendre processes of order 3, the hyperbolic Bessel processes of order 3 can be defined as the radial component of Brownian motion on three-dimensional hyperbolic spaces.

3.4.3 A unified viewpoint

The surfaces considered in the last two examples together with the plane $\{z = 0\}$ in the Heisenberg group are particular cases of the following construction.

Let G be a three-dimensional Lie group endowed with a contact sub-Riemannian structure whose distribution D is spanned by two left-invariant vector fields X_1 and X_2 which are orthonormal for the fibre inner product g defined on D. Assume that the commutation relations between X_1, X_2 and the Reeb vector field X_0 are given by, for some $\kappa \in \mathbb{R}$,

$$[X_1, X_2] = X_0\ , \quad [X_0, X_1] = \kappa X_2\ , \quad [X_0, X_2] = -\kappa X_1\ .$$

Under these assumptions the flow of the Reeb vector field X_0 preserves not only the distribution, namely $\mathrm{e}_*^{tX_0} D = D$, but also the fibre inner product g. The examples presented in Section 3.3.1 and in Sections 3.4.1 and 3.4.2 satisfy the above commutation relations with $\kappa = 0$ in the Heisenberg group, and for a parameter $k > 0$, with $\kappa = 4k^2$ in $\mathrm{SU}(2)$ and $\kappa = -4k^2$ in $\mathrm{SL}(2,\mathbb{R})$. These are the three classes of model spaces for three-dimensional sub-Riemannian structures on Lie groups with respect to local sub-Riemannian isometries, see for instance [ABB20, Chapter 17] and [AB12] for more details.

In each of the examples concerned, the surface S that we consider can be parameterised as

$$\begin{aligned} S &= \{\exp(x_1X_1 + x_2X_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \{\exp(r\cos\theta X_1 + r\sin\theta X_2) : r \geq 0, \theta \in [0, 2\pi)\}\ . \end{aligned}$$

Observe that S is automatically smooth, connected, and contains the origin of the group. Under these assumptions, the sub-Riemannian structure is of type $\mathbf{d} \oplus \mathbf{s}$ in the sense of [ABB20, Section 7.7.1], and for θ fixed, the curve $r \mapsto \exp(r\cos\theta X_1 + r\sin\theta X_2)$ is a geodesic parameterised by length. Hence, $r \geq 0$ is the arc length parameter along the corresponding trajectory. It follows that the surface S is ruled by geodesics, each of them having vertical component of the initial covector equal to zero. We refer to [ABB20, Chapter 7] for more details on explicit expressions for sub-Riemannian geodesics in these cases, see also [BR08].

Chapter 4

Approximately controllable finite-dimensional bilinear systems are controllable

Contents

This chapter presents the results in the paper [CS21], joint work with Mario Sigalotti and published in the journal Systems & Control Letters.

We begin in Section 4.1 with the proofs for Lemma 1.7 and Corollary 1.8. Next, in Section 4.2 we specialise in bilinear control systems, proving Theorem 1.9.

4.1 Properties of approximately controllable systems

Consider a control system defined as in (C) by a fixed family $\mathcal{F}$ on vector fields on M. We now prove Lemma 1.7, which shows that if (C) is approximately controllable, then the orbits of (C) form a regular foliation of M.

Proof of Lemma 1.7. As noticed above, the orbits of system (C) form a partition of M in immersed submanifolds. Since the attainable sets are contained in the orbits, the approximate controllability implies that the orbits are dense. Finally, due to the expression of the tangent space of the orbits in Theorem 1.3, the dimension of the orbits is lower semi-continuous, i.e., for all x in M, there exists a neighbourhood $V(x)$ of x such that

$$\dim \mathcal{O}_x \leq \dim \mathcal{O}_y, \qquad \forall\, y \in V(x).$$

Now let $\mathcal{O}_x$ be an orbit of maximal dimension; since $\mathcal{O}_x$ is dense, all other orbits have the same dimension as $\mathcal{O}_x$. Finally, an integral foliation of constant rank is a regular foliation. $\square$

Recall that the family $\mathcal{F}$ is said to satisfy the *Lie algebra rank condition at* $x \in M$ if the evaluation at x of the Lie algebra generated by $\mathcal{F}$ has maximal dimension, i.e., $D_x^{\mathrm{Lie}\,\mathcal{F}} = T_xM$.

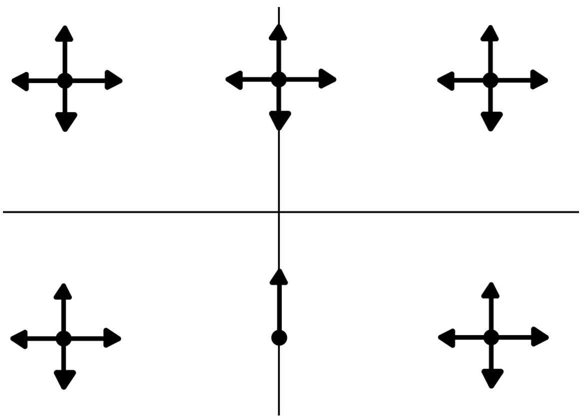

Figure 4.1: The admissible vector fields for the control system introduced in Example 4.2.

Proof of Corollary 1.8. If there exists a point x in M at which $\mathcal{F}$ satisfies the Lie algebra rank condition, then (C) has a single orbit. Indeed, Krener's theorem implies that the interior of $\mathcal{A}_x$ is nonempty. Hence, due to the approximate controllability assumption, the attainable set from any other point intersects $\mathcal{A}_x$, and therefore is contained in $\mathcal{O}_x$. Thus, (C) has a single orbit and, due to the Lie-determinedness property (1.3), every point in M satisfies the Lie algebra rank condition. The controllability follows as a corollary of Krener's theorem (see, e.g., [AS02, Cor. 8.3]).

Otherwise, assume that $\mathcal{F}$ does not satisfy the Lie algebra rank condition at any point. Then, Lemma 1.7 shows that the orbits of (C) form a regular foliation, whose leaves are dense since they contain the attainable sets. Finally, the dimension of the orbits is less than the dimension of M, otherwise $\mathcal{F}$ would satisfy the Lie algebra rank condition due to (1.3). □

Remark 4.1. Corollary 1.8 is useful if one can exclude the existence of a foliation with the properties described in *(b)*; this might be possible thanks to the particular form of system (C) or some topological properties of M. Most of the results in this direction are for codimension-one regular foliations: for example, it is known that even dimensional spheres do not admit codimension-one regular foliations [Dur72]. We recall also that a compact manifold with finite fundamental group has no analytic regular foliations of codimension one [Hae57]; some additional results can be found in [Law74].

The hypothesis of Lie-determinedness in Corollary 1.8 is necessary. Indeed, if we drop the hypothesis that the system is Lie-determined, it is possible to construct an approximately controllable and not controllable system having only one orbit. This is shown in the following example.

Example 4.2. Let $M = \mathbb{R}^2$ and consider the family $\mathcal{F} = \{f_1,\ f_2,\ f_3^+,\ f_3^-\}$ with

$$f_1 = \frac{\partial}{\partial y}, \quad f_2 = -\phi(x,y) f_1, \quad f_3^{\pm} = \pm\phi(x,y)\frac{\partial}{\partial x},$$

where $\phi : \mathbb{R}^2 \to [0,+\infty)$ is a smooth function such that, for all $(x,y) \in \mathbb{R}^2$, $\phi(x,y) = 0$ if and only if $x = 0$ and $y \leq 0$. The four vector fields are illustrated in Figure 4.1. It is not hard to check that this system has a single orbit, is approximately controllable, and still it is not controllable.

Finally, there are examples of controllable systems which are nowhere Lie-determined. For instance, one can consider the following system.

Example 4.3. Let $M = \mathbb{R}^3$, and consider the family $\mathcal{F} = \{f_1^+,\ f_1^-,\ f_2^+\ f_2^-,\ f_3^+,\ f_3^-\}$ with

$$f_1^{\pm} = \pm\frac{\partial}{\partial x}, \quad f_2^{\pm} = \pm\psi(x)\frac{\partial}{\partial y}, \quad f_3^{\pm} = \pm\psi(-x)\frac{\partial}{\partial z},$$

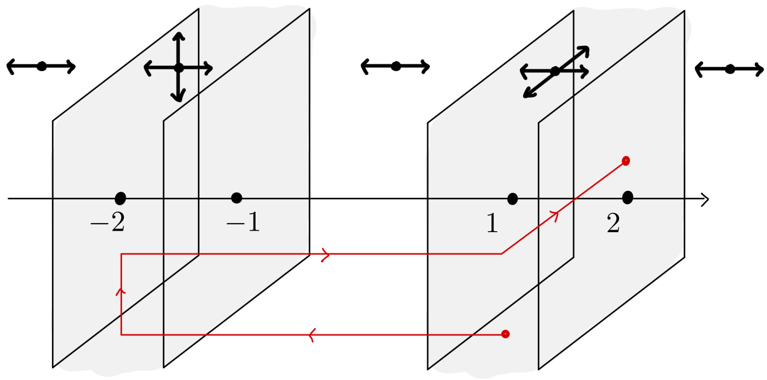

Figure 4.2: The vector fields of the control system of $\mathbb{R}^3$ in Example 4.3.

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function with support in $\mathbb{R} \setminus [1, 2]$ and such that $\psi|_{(1,2)} > 0$. It can be easily verified that this system is controllable (see Figure 4.2), but the dimension of $\operatorname{Lie}\mathcal{F}$ is everywhere less or equal to 2.

4.2 Bilinear control systems

Consider a bilinear control system, as defined in (BL). Given a matrix A in $\mathcal{M}$, let us denote by f_A the associated vector field

$$f_A : x \mapsto Ax, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

Since the vector fields f_A are $\mathbb{R}$-homogenous for each A in $\mathcal{M}$, the set $\mathcal{F}^{\mathcal{M}} = \{f_A \mid A \in \mathcal{M}\}$ is a family of analytical, homogenous vector fields in $\mathbb{R}^n \setminus \{0\}$.

Let us introduce two systems which can be naturally associated with (BL): the projections of system (BL) onto S^{n-1} and $\mathbb{RP}^{n-1}$. First, consider the projection $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $\pi(x) = x/|x|$. For every x in S^{n-1} and every $v \in T_x\mathbb{R}^n$, consider the pushforward $\pi_*(v) = v - \langle x, v\rangle x$, and let us introduce the control system

$$\dot{x} = \pi_*(A(t)x), \qquad A(t) \in \mathcal{M},\ x \in S^{n-1}. \tag{SΣ}$$

Due to homogeneity, the trajectories of (SΣ) are the image of the trajectories of (BL) via π; thus, the orbits $\mathcal{O}^S$ of (SΣ) are projections of the orbits of (BL), that is

$$\mathcal{O}^S_{\pi(y)} = \pi\big(\mathcal{O}_y\big), \qquad \forall y \in \mathbb{R}^n \setminus \{0\}. \tag{4.1}$$

We say that (BL) is *angularly controllable* if (SΣ) is controllable. Similarly, consider the canonical projection $\omega : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ and the system

$$\dot{q} = \omega_*(A(t)x), \qquad A(t) \in \mathcal{M},\ q = \omega(x) \in \mathbb{RP}^{n-1}. \tag{$\mathbb{P}\Sigma$}$$

This system is well-defined because $\omega_*(A(t)x)$ depends only on q and not on the choice of the specific $x \in \mathbb{R}^n \setminus \{0\}$ such that $q = \omega(x)$.

4.2.1 Proof of Theorem 1.9

Assume that system (BL) is approximately controllable and that $n \geq 2$, the case $n = 1$ being trivial. System (BL) is Lie-determined due to the analyticity of each linear vector field and the already cited Nagaro theorem [Nag66]. Thus, Corollary 1.8 applies: either system (BL) is controllable, or the partition of $\mathbb{R}^n \setminus \{0\}$ into the orbits of (BL) forms a regular foliation of positive codimension as described in *(b)*. Let us assume the latter, and show that this leads to a contradiction.

As mentioned in the introduction, Theorem 1.9 holds if one replaces (BL) by ($\mathbb{P}\Sigma$), as shown in [BS20, Prop. 44]. Using this result, let us deduce the following property.

Lemma 4.4. *If* (BL) *is approximately controllable then* (BL) *is angularly controllable.*

Proof. Since the projection of the trajectories of (BL) are trajectories of ($\mathbb{P}\Sigma$), if the former is approximately controllable then the same holds for the latter. Due to [BS20, Prop. 44], if ($\mathbb{P}\Sigma$) is approximately controllable then it is controllable. In [BV13, Thm. 1] the authors show that system ($\mathbb{P}\Sigma$) is controllable if and only if system (SΣ) is controllable. Therefore, system (SΣ) is controllable, meaning that system (BL) is angularly controllable. $\square$

Let us denote by $\mathcal{O} = \{\mathcal{O}_x \mid x \in \mathbb{R}^n \setminus \{0\}\}$ the orbit partition of system (BL). Due to (4.1) and Lemma 4.4, one has

$$\pi_*\big(T_y\mathcal{O}_x\big) = T_{\pi(y)}\mathcal{O}^S_{\pi(x)} = T_{\pi(y)}S^{n-1}, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ \forall\, y \in \mathcal{O}_x.$$

Therefore, we have that

$$T_y\mathcal{O}_x + \mathbb{R}y = \mathbb{R}^n, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ \forall\, y \in \mathcal{O}_x, \tag{4.2}$$

which is to say that the orbits are transversal to the radial direction. Additionally, since we assumed to be in case *(b)* of Corollary 1.8, this implies that $\dim T_y\mathcal{O}_x = n-1$. It follows that $\mathcal{O}$ is a codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction and with dense leaves. In the following lemma we show that such a foliation cannot exist.

Lemma 4.5. *Assume that $n \geq 2$. Then, there does not exist a homogenous, codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction and with dense leaves.*

The hypothesis of transversality between the foliation and the radial direction is necessary, as counterexamples can be constructed otherwise. For instance, in [Hec76] the author presents an example of codimension-one regular foliation of $\mathbb{R}^3$ with dense leaves; this construction is presented with additional details in [CN13, Chap. 4], and in this thesis in Example 4.8 for completeness.

Proof of Lemma 4.5. By contradiction, suppose there exists a codimension-one regular foliation $\mathcal{L} = \{L_\alpha \mid \alpha \in A\}$ of $\mathbb{R}^n \setminus \{0\}$ with dense leaves transversal to the radial direction.

Let us first consider the case $n = 2$. Orienting the foliation using the clockwise direction and applying Whitney's theorem (see [ABZ96, Thm. 2.3 at p. 23]), the foliation can be identified with the set of trajectories of a vector field. Using the stereographic projection, the flow of such a vector field can be pushed to the sphere S^2 minus two points. However, a flow with dense trajectories on S^2 minus finitely many points does not exist: see, for instance, [ABZ96, Lem. 2.4 at p. 56].

Assume that $n \geq 3$. Let us fix the point $p = (0, \dots, 0, 1) \in \mathbb{R}^n$, and denote by S^{n-2} the embedded sphere $S^{n-2} \times \{0\} \subset \mathbb{R}^n$. For every θ in S^{n-2}, let $\mathcal{P}_\theta$ be the plane

$$\mathcal{P}_\theta = \operatorname{span}\{p, \theta\},$$

as depicted in Figure 4.3. Because of the transversality between the leaves of $\mathcal{L}$ and the radial direction, the linear subbundle $I_\theta = \mathcal{P}_\theta \cap T\mathcal{L}|_{\mathcal{P}_\theta \setminus \{0\}}$ is a one-dimensional distribution on $\mathcal{P}_\theta \setminus \{0\}$ satisfying

$$I_\theta(x) \oplus \mathbb{R}x = T_x\mathcal{P}_\theta, \qquad \forall\, x \in \mathcal{P}_\theta \setminus \{0\}. \tag{4.3}$$

By definition, the intersection $L_p \cap \mathcal{P}_\theta$ contains the integral curve to I_θ starting at p. The integral curves of I_θ are nothing but the leaves of the one-dimensional foliation defined by the distribution I_θ in $\mathcal{P}_\theta \setminus \{0\}$. We claim that such a foliation is orientable. Indeed, for each x in $\mathcal{P}_\theta \setminus \{0\}$, we can say that a nonzero vector v in $I_\theta(x)$ has a positive orientation if (v, x) is an oriented frame of $\mathcal{P}_\theta$. It follows from

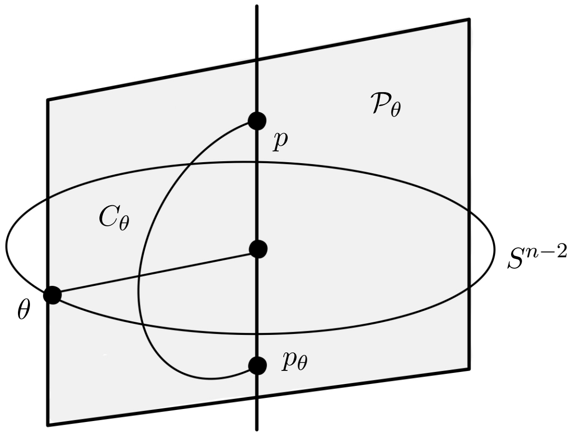

Figure 4.3: A graphic representation of the construction in the proof of Lemma 4.5

the already cited Whitney's theorem that the foliation defined by I_θ is the orbit partition of $\mathcal{P}_\theta \setminus \{0\}$ by the flow of a vector field g_θ everywhere transversal to the radial direction.

Because of the transversality condition and the homogeneity, the flow of g_θ spirals around the origin. Let us choose the vector field g_θ such that, starting from p, one intersects $\mathbb{R}_{>0}\theta$ before $-\mathbb{R}_{>0}\theta$. Define p_θ to be the point of first intersection between the integral curve of g_θ starting at p and the ray $-\mathbb{R}_{>0}p$, and C_θ to be the arc between p and p_θ (see Figure 4.3).

Now, let us define the map $\Phi : S^{n-2} \to -\mathbb{R}_{>0}p$ by $\Phi(\theta) = p_\theta$. The map Φ is well-defined, in the sense that p_θ does not depend on the vector field g_θ (once the latter is chosen with the appropriate orientation). Moreover, the map Φ is continuous, as it follows from the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2}. In addition, the image of Φ is contained in the intersection $L_p \cap -\mathbb{R}_{>0}p$, which has empty interior because of the transversality between $T\mathcal{L}$ and the radial direction. Since S^{n-2} is connected ($n > 2$), it follows that Φ is constant. Let us define

$$S = \bigcup_{\theta \in S^{n-2}} C_\theta.$$

By the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2} it follows that we can parameterize C_θ as a continuous arc on $[0,1]$ continuously with respect to θ. Hence, the topology of S as a subset of $\mathbb{R}^n$ is that of $(S^{n-2} \times [0,1])/\sim$, where $\sim$ is the equivalence relation which identifies the points in $S^{n-2} \times \{0\}$ to a single equivalence class, and analogously for the points in $S^{n-2} \times \{1\}$. That is, S is a topological manifold homeomorphic to the sphere S^{n-1}. In particular, $S \subset L_p$ is closed in $\mathbb{R}^n$, and therefore it is closed in L_p. Since S has the same topological dimension as L_p, we have that S is open in the topology of L_p. Since L_p is connected, we can conclude that $S = L_p$. This is contradicts the assumptions that the leaves are dense. □

Lemma 4.5 shows that the supposition that we are in case *(b)* of Corollary 1.8 leads to a contradiction. Therefore, we are in case *(a)*, and (BL) is controllable. This concludes the proof of Theorem 1.9. □

4.3 Complementary remarks

The result in Lemma 4.5 implies that Theorem 1.9 generalises for control systems which are Lie-determined, homogeneous, and angularly controllable. We say that (C) is *homogeneous* if $X = \mathbb{R}^n \setminus \{0\}$ and for every $x \in X$, $u \in U$, and $\lambda > 0$, $\pi_*(f_u(\lambda x)) = \pi_*(f_u(x))$, where $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ still denotes the canonical projection. Just as in the bilinear case, the projection of such systems on the sphere S^{n-1} is

$$\dot{x} = \pi_*(f_{u(t)}(x)), \qquad f_{u(t)} \in \mathcal{F}, \ x \in S^{n-1}, \tag{SC}$$

and we say that (C) is angularly controllable if (SC) is controllable.

Corollary 4.6. *Let $n \geq 2$. Assume that the control system* (C) *is Lie-determined, homogenous, and angularly controllable. Then,* (C) *is approximately controllable if and only if it is controllable.*

Since for $n = 2$ system (C) projects to S^1, in this case the hypothesis of angular controllability can be easily removed.

Remark 4.7. For all n odd, $n = 2k+1$ with $k \geq 1$, the hypothesis of angular controllability in Corollary 4.6 is superfluous. Indeed, if system (C) is Lie-determined, homogenous, and approximately controllable, then Corollary 1.8 implies that either (C) is angularly controllable, or the projection of its orbits forms a nontrivial regular foliation of the even-dimensional sphere S^{2k}. Since even dimensional spheres do not admit nontrivial regular foliations (indeed their tangent spaces do not admit any nontrivial subbundles; see, e.g., [MS74, Problem 9C]), the angular controllability follows.

However, it has not been possible to fully remove the hypothesis of angular controllability in Corollary 4.6. In this regard, let us discuss the case $n = 4$. Due to Corollary 1.8, if (C) is Lie-dermined, homogeneous, approximately controllable, and (SC) is not controllable, then the orbits of (SC) form a regular foliation of S^3 of either dimension one or codimension one. The latter gives a contradiction, since the Novikov compact leaf theorem implies that any codimension-one regular foliation of the sphere S^3 has a compact leaf [Nov65]. The former implies that the orbits of (SC) are given by the flow of a minimal vector field, i.e., a vector field whose orbits are dense. The existence of such flows has been raised as an open question in [Got58] for compact metric spaces, and for the sphere S^3 has been mentioned by Smale in [Sma98] under the name Gottschalk conjecture; further details can be found in [FPW07]. If the Gottschalk conjecture were to be true, it would imply the existence of Lie-determined, homogenous, approximate controllable, yet not controllable systems, showing that Corollary 1.8 fails to hold if we remove the angular controllability hypothesis.

Finally, we mention that the hypotheses of transversality to the radial direction in Lemma 4.5 is necessary. Indeed, the following example gives a sketch of the construction of a smooth foliation of $\mathbb{R}^3$ with dense leaves, from which one can obtain a regular foliation of $\mathbb{R}^3 \setminus \{0\}$ by subtracting the origin.

Example 4.8 (Hector's example, [Hec76]). This example is due to Hector. The first part of the construction is to associate to certain diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ a foliation ξ_f of the three-dimensional cylinder $D^2 \times \mathbb{R}$, where D^2 is a closed two-dimensional disk; the hypothesis demanded on f is the existence of an interval $[a, b]$ such that $f(x) = x$ for all $x \in [a, b]$. Given such a diffeomorphism f, one first constructs a foliation in the solid cylinder $D^2 \times \mathbb{R} \setminus \{0\} \times [(-\infty, a) \cup (b, +\infty)]$ with the property that (r, θ, z) and $(r, \theta, f(z))$ are in the same leaf, for all $r \in (0, 1]$, $\theta \in S^1$ and $z \in \mathbb{R}$ (see the image on the left in Figure 4.4). Next, one can obtain a foliation on the whole cylinder by performing a C^∞ deformation supported on $\frac{2}{3}D^2 \times \mathbb{R}$ (i.e., not changing a neighbourhood of $S^1 \times \mathbb{R}$) sending a and b to $-\infty$ and $+\infty$ respectively (see the image on the right in Figure 4.4). Denote this foliation by ξ_f.

Now, suppose having two solid cylinders $D_1^2 \times \mathbb{R}$ and $D_2^2 \times \mathbb{R}$ with two respective foliations ξ_{f_1} and ξ_{f_2}. Fix two compact arcs $V_i \subset S^1$, for $i = 1, 2$, and take a diffeomorphism $\phi : V_1 \times \mathbb{R} \to V_2 \times \mathbb{R}$ for which $\phi_*(\xi_{f_2}|_{V_2\times\mathbb{R}}) = \xi_{f_1}|_{V_1\times\mathbb{R}}$. The cylinders $D_1^2 \times \mathbb{R}$ and $D_2^2 \times \mathbb{R}$ in can be glued together along ϕ to obtain the foliation ξ_{f_1,f_2} on $(D_1^2 \cup_\phi D_2^2) \times \mathbb{R}$. Observe that the intersection of the leaf of ξ_{f_1,f_2} going thought a point z in S_1 (denoted by $\xi_{f_1,f_2}(z)$) and the vertical line $\{z\} \times \mathbb{R}$ is

$$\xi_{f_1,f_2}(z) \cap \big(\{z\} \times \mathbb{R}\big) = \big\{f_1^{i_1} \circ f_2^{j_1} \circ \cdots \circ f_1^{i_k} \circ f_2^{j_k}(z) \;\big|\; i_1, j_1, \ldots, i_k, j_k \in \mathbb{Z}\big\}.$$

This can be verified by looking at the fundamental group of the bouquet of two circles $S^1 \wedge S^1$; see Figure 4.5. Notice that $D_1^2 \cup_\phi D_2^2$ is diffeomorphic to a disk, and that the gluing described above can

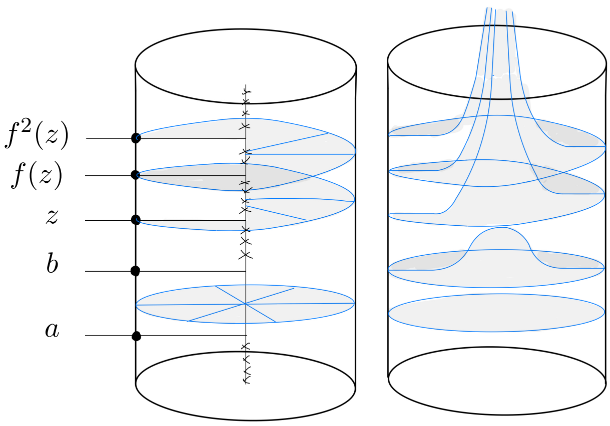

Figure 4.4: On the left-hand side the foliation constructed from the *suspension* by the diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, and on the right-hand side the *whirlwind* defined by f on the whole cylinder.

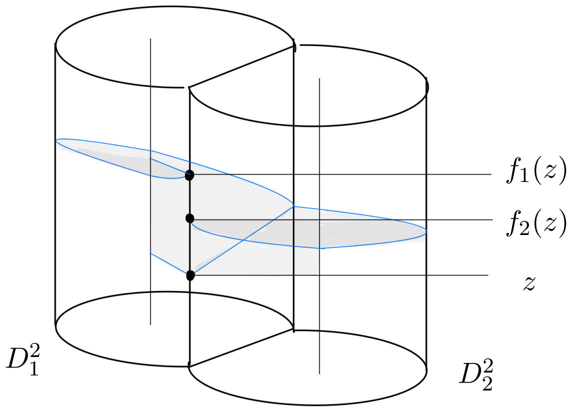

Figure 4.5: A portion of a leaf of the foliation ξ_{f_1,f_2}.

be repeated multiple times to obtain a foliation $\xi_{f_1,\dots,f_k}$ on $D^2 \times \mathbb{R}$. Moreover, for all z in S^1, the intersection of the leaf going through z and the vertical line $\{z\} \times \mathbb{R}$ is

$$\xi_{f_1,\dots,f_k}(z) \cap \big(\{z\} \times \mathbb{R}\big) = \big\{f_{i_1}^{j_1} \circ \dots \circ f_{i_n}^{j_n}(z) \mid i_1, \dots, i_n \in \{1, \dots, k\},\ j_1, \dots, j_n \in \mathbb{Z}\big\}.$$

Observe that if the intersection $\xi_{f_1,\dots,f_k}(z) \cap \big(\{z\} \times \mathbb{R}\big)$ is dense, then the leaf $\xi_{f_1,\dots,f_k}(z)$ is dense in the cylinder $D^2 \times \mathbb{R}$.

Lemma 4.9. *There exists $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ diffeomorphism for which the above construction can be applied, and such that, for all z, the set*

$$\big\{f_{i_1}^{j_1} \circ \dots \circ f_{i_n}^{j_n}(z) \mid i_1, \dots, i_n \in \{1, \dots, 4\},\ j_1, \dots, j_n \in \mathbb{Z}\big\}$$

is dense in $\mathbb{R}$.

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ smooth and increasing, with $\phi|_{(-\infty,0]} \equiv 0$ and $\phi|_{[1,+\infty)} \equiv 1$. Define $f_1(x) = x + \alpha\phi(x)$ and $f_2(x) = f_1^{-1}(\alpha + x)$, and $f_3(x) = x + \phi(x)$ and $f_4(x) = f_1^{-1}(x + 1)$. By definition, $f_1 \circ f_2(x) = x + \alpha$ and $f_3 \circ f_4 = x + 1$, therefore

$$(f_1 \circ f_2)^n \circ (f_3 \circ f_4)^m(x) = x + \alpha n + m, \qquad \forall n, m \in \mathbb{Z},\ \forall x \in \mathbb{R}.$$

If α is irrational, then the set of numbers $\{\alpha n + m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. □

Thus, using the functions constructed in Lemma 4.9, one obtains a foliation of $D^2 \times \mathbb{R}$ for which every leaf is dense. Finally, by sending the boundary S^1 to the infinity of $\mathbb{R}^2$ one has a foliation of $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with dense leaves.

Chapter 5

Control systems satisfying local reachability are controllable

Contents

This chapter presents the results collected in the paper [BCFS21], joint work with Ugo Boscain, Valentina Franceschi and Mario Sigalotti and currently submitted for publication.

5.1 Introduction

As in the introduction, let M be a connected smooth manifold. Consider the control system C with piecewise constant controls or essentially bounded controls. The proof of Theorem 1.1 relies on the following lemmas.

Lemma 5.1. *If* (C) *satisfies the local reachability property, then it is approximately controllable.*

The proof of this lemma, presented in the next section, relies on the regularity of the flow of (C) for a fixed control, and on the connectedness of M. Lemma 5.1 is a key step in the proof of the following key property.

Lemma 5.2. *Assume* (C) *satisfies the local reachability property. Then, for any state x and y in M,*

$$y \in \mathcal{A}_x \implies x \in \mathcal{A}_y.$$

Given a state x in M, the *controllable set to x* is the set of states from which x is reached, i.e., $\mathcal{A}_x^- = \{y \in M \mid x \in \mathcal{A}_y\}$. Observe that $\mathcal{A}_x^-$ is the attainable set from x for the control system defined by $-F$, whose solutions are the solutions of (C) followed in the opposite time direction.

Remark 5.3. Assume that (C) is approximately controllable (recalling that, according to Lemma 5.1, this is the case if the system satisfies local reachability). If a state x in M satisfies $\operatorname{Int} \mathcal{A}_x^- \neq \emptyset$, then the state x can be reached from any other state. Indeed, for any y in M, since $\mathcal{A}_y$ is dense in M it intersects the interior of A_x^-. Thus, there exists $z \in A_y \cap A_x^-$. Since $z \in \mathcal{A}_x$ and $z \in \mathcal{A}_y$, one can steer the system from y to x (see Figure 5.1).

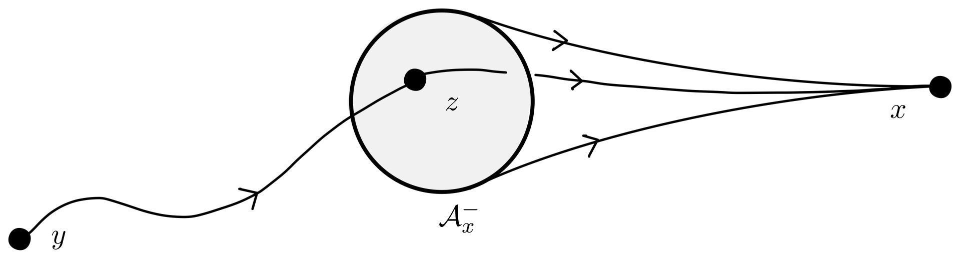

Figure 5.1: Situation in Remark 5.3. The gray ball is contained in the interior of $\mathcal{A}_x^-$ and it contains reachable points from any $y \in M$ by approximate controllability. Thus $x \in \mathcal{A}_y$.

Remark 5.3 together with the following proposition shows that a system satisfying the localised local reachability property is controllable.

Proposition 5.4. *If system* (C) *is localized locally reachable, then for any state $x \in M$ the set $\mathcal{A}_x^-$ has nonempty interior.*

Proof. The argument mimics the proof of Krener's theorem [Kre74]. Let x in M. We claim that there exists $X_1 \in \mathcal{F}$ such that $X_1(x) \neq 0$. Indeed, if that was the case, any solution $\phi(\cdot, x, u)$ for $u \in \mathcal{U}$ would be constant. Let

$$N_1 = \{e^{-tX_1}(x) \mid t \in (0, \delta)\}$$

for $\delta > 0$. If M is one-dimensional, then we have concluded. Otherwise, we claim there exist $y_1 \in N_1$ and $X_2 \in \mathcal{F}$ a such that $X_1(y_1)$ and $X_2(y_1)$ are transverse. Indeed, let $\mathcal{V}_1$ be a neighbourhood of $e^{-\delta/2X_1}(x)$ not containing x nor $e^{-\delta X_1}(x)$ and assume that every $f \in \mathcal{F}$ is tangent to $V_1 \cap N$. Then the trajectories of (C) starting from $N_1 \cap V_1$ and staying in V_1 cannot quit $N_1 \cap V_1$. This contradicts the localized local reachability property.

Thus, define the embedded two-dimensional submanifold

$$N_2 = \{e^{-t_2X_2} \circ e^{-tX_1}(x) \mid (t_1, t_2) \in I_2 \times (0, \delta_2)\},$$

for a suitable nonempty open subinterval I_2 of $(0, \delta)$ and a suitable $\delta_2 > 0$. If the dimension of M is equal to 2 the proof is concluded, otherwise there exist $y_2 \in N_2$ and $X_3 \in \mathcal{F}$ such that $X_3(y_2)$ is transversal to N_2 and we iterate the construction up to reaching the dimension of M. $\square$

Remark 5.5. As an alternative of the self-contained proof proposed above, Proposition 5.4 could have been directly deduced from [Gra92, Theorem 5.3], since the property of localized local reachability implies, in the terminology of [Gra92], that (C) has the nontangency property.

5.2 Local reachability implies controllability

Once Lemma 5.2 is proven, Theorem 1.1 follows from the following classical argument.

Proof of Theorem 1.1. Assume that (C) satisfies the local reachability property. Define the equivalence relation $\sim$ on M by requiring $x \sim y$ if and only if $x \in \mathcal{A}_y$. Thanks to Lemma 5.2, this is indeed an equivalence relation. Due to local reachability, the equivalence classes are open. The classes are also closed, since one class is the complementary of the union of the other classes, and this union is open. Due to the connectedness of M, there is only one class and system (C) is controllable.

$\square$

5.2.1 Proof of Lemma 5.1

Assume that (C) satisfies the local reachability property. Let $x \in M$. We want to show that $\mathrm{cl}(\mathcal{A}_x)$ is open. By connectedness of M, this implies that $\mathrm{cl}(\mathcal{A}_x) = M$, thus proving the lemma.

Let $y \in \mathrm{cl}(\mathcal{A}_x)$. We claim that for every control $u \in \mathcal{U}$ and $t > 0$ such that $\phi(t, y, u)$ is defined, we have that

$$\phi(t, y, u) \in \mathrm{cl}(\mathcal{A}_x). \tag{5.1}$$

This concludes the proof of the lemma. Indeed, from (5.1) it follows that $\mathcal{A}_y \subset \mathrm{cl}(\mathcal{A}_x)$; since $\mathcal{A}_y$ contains y in its interior due to (LR), this proves that $\mathrm{cl}(\mathcal{A}_x)$ is open.

In order to prove (5.1), take y, u, t as in the assumptions and fix any neighbourhood V of $\phi(t, y, u)$: we show that V has nonempty intersection with $\mathcal{A}_x$. Consider a neighbourhood W of y such that the map $\varphi\colon W \ni z \mapsto \phi(t, z, u) \in \varphi(W)$ is a diffeomorphism. In particular, $\varphi(W)$ is a neighbourhood of $\phi(t, y, u)$, and the set $W' = \varphi^{-1}(V \cap \varphi(W))$ is a neighborhood of y. Since y is in the closure of $\mathcal{A}_x$, there exists $y_1 \in W' \cap \mathcal{A}_x$. Consider an admissible control from x to y_1: by concatenating this control with u one finds that $\phi(t, y_1, u)$ is in $\mathcal{A}_x$. This implies that $\phi(t, y_1, u)$ belongs to $V \cap \mathcal{A}_x$, proving that $V \cap \mathcal{A}_x$ is nonempty, as required. $\square$

Observe that if system (C) satisfies local controllability, then the attainable sets are open. Let us now proceed with the proof of Lemma 5.2.

5.2.2 Proof of Lemma 5.2

Let x and y in M be such that $y \in \mathcal{A}_x$. We argue by contradiction supposing that $x \notin \mathcal{A}_y$. We claim that this implies the existence of a state z in M (actually $z \in \mathcal{A}_x$) such that

$$z \notin \mathcal{A}_y \quad \text{and} \quad \mathrm{Int}\,\mathcal{A}_z^- \neq \emptyset. \tag{5.2}$$

This is a contradiction, since the assertions in (5.2) cannot hold both at the same time due to Remark 5.3. The rest of the proof is dedicated to proving the existence of a point z satisfying (5.2).

Consider a control $u \in \mathcal{U}$ and $T > 0$ such that $\phi(T, x, u) = y$. Define the absolutely continuous curve $\gamma : [0, T] \to M$ by $\gamma(t) = \phi(t, x, u)$. Let

$$\tau = \inf\{t \in [0, T] \mid \gamma(t) \in \mathcal{A}_y\}.$$

We claim that $\gamma([0, T]) \cap \mathcal{A}_y = \gamma((\tau, T])$. Indeed, $\gamma^{-1}(\gamma([0, T]) \cap \mathcal{A}_y)$ is open since $\mathcal{A}_y$ is open, and its complementary is nonempty since it contains zero (we supposed that $x \notin \mathcal{A}_y$). Moreover, if a certain $s \in [0, T]$ satisfies $\gamma(s) \in \mathcal{A}_y$, then, for all t in $[s, T]$, one has $\gamma(t) \in \mathcal{A}_y$ since it suffices to concatenate the control from y to $\gamma(s)$ with $u|_{[s,t]}$ in order to to attain $\gamma(t)$. Up to renaming $\gamma(\tau)$ as x, we can assume that $\tau = 0$. Namely, without loss of generality, one can assume (see Figure 1.1)

$$x \notin \mathcal{A}_y \quad \text{and} \quad \phi(t, x, u) \in \mathcal{A}_y \text{ for all } t \in (0, T].$$

Let V be a neighbourhood of x contained in $\mathcal{A}_x$. We now construct a parametrisation $C_n : I_n \to M$ ($I_n \subset \mathbb{R}^n$) of a n-dimensional embedded sub-manifold of M satisfying $C_n(I_n) \subset \mathcal{A}_x^-$, and therefore x (or more exactly $\gamma(\tau)$) satisfies (5.2). This will be done by a recursive argument, by constructing parametrisations $C_k : I_k \to M$ ($I_k \subset \mathbb{R}^k$), for $k = 1, \dots, n$, with

$$C_k(s) \in V, \quad \text{and} \quad x \in \mathcal{A}_{C_k(s)} \quad \text{for all } s \in I_k. \tag{5.3}$$

Let us begin with $k = 1$. Let $v \in \Omega$ (constant) such that $X_v(x) \neq 0$, and let I_1 an open interval of the form $(0, \delta_1)$ such that the map $C_1\colon I_1 \to M$ defined by $C_1(t) = e^{-tX_v}(x)$ parametrises an embedded curve. Since the constant control defined by v belongs to $\mathcal{U}$, then one can reach x from any point in C_1. Moreover, one has that C_1 is contained in V, up to choosing δ_1 sufficiently small. Thus, C_1 satisfies (5.3) for $k = 1$.

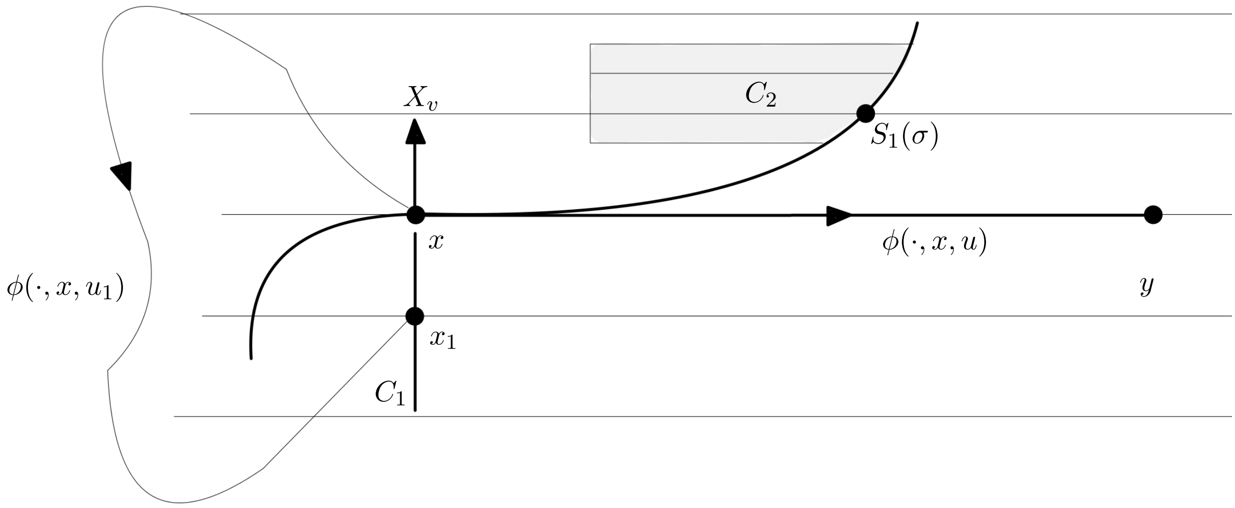

Figure 5.2: A graphic representation of the iterations in the proof of Lemma 5.2 in dimension two, and for piecewise constants controls. On can control any point in C_2 to x by first attaining S_1, then attaining C_1 via the control u_1 from which one can reach x.

Now, suppose having constructed a k-dimensional parametrisation C_k satisfying (5.3), with $1 \leq k \leq n-1$. Fix a point $x_k \in C_k(I_k)$, and consider a control u_k in $\mathcal{U}$ and a time $T_k \geq 0$ such that $x_k = \phi(T_k, x, u_k)$. Let W_k be a neighbourhood of x such that $\varphi_k : W_k \ni y \mapsto x(T_k, y, u_k) = \varphi_k(W_k)$ is a diffeomorphism. Define $S_k = \varphi_k^{-1} \circ C_k$, this parametrises an embedded submanifold of dimension k containing x. Moreover,

$$x \in \mathcal{A}_{S_k(s)}, \qquad \forall s \in I_k, \tag{5.4}$$

since $x \in \mathcal{A}_{C_k(s)}$, and $C_k(s) \in \mathcal{A}_{S_k(s)}$ using u_k as control. In particular, we have that $S_k(s) \notin \mathcal{A}_y$ for all $s \in I_k$. As a consequence, since $\phi(t, x, u) \in \mathcal{A}_y$ for all $t \in (0, T]$, we have that

$$S_k(I_k) \cap \{\phi(t, x, u) \mid t \in (0, T]\} = \emptyset. \tag{5.5}$$

This implies the existence of $t_k > 0$ and of $\sigma \in I_k$ with $S_k(\sigma)$ arbitrarily close to x and such that $X(S_k(\sigma), u(t_k))$ and $T_{S_k(\sigma)} S_k$ are transverse, i.e.,

$$\dim \left(\operatorname{span}\{X(S_k(\sigma), u(t_k))\} \oplus T_{S_k(\sigma)} S_k\right) = k + 1. \tag{5.6}$$

Indeed, if one had $X(S_k(\sigma), u(t)) \subset T_{S_k(s)} S_k$ for all $s \in I_k$ and $t \in [0, T]$, then, by uniqueness of solutions, the flow starting from x would belong to $S_k(I_k)$, at least for $t > 0$ sufficiently small. However, this contradicts (5.5). Moreover, σ can be chosen so that $S_k(\sigma)$ belongs to V. Let X_k be the vector field $X(\cdot, u(t_k))$

Therefore, there exists an open neighbourhood $I'_k \subset I_k$ containing σ and a $\delta_{k+1} > 0$ such that the map $C_{k+1} : I_{k+1} := I'_k \times (-\delta_{k+1}, \delta_{k+1}) \to M$ defined by

$$C_{k+1}(s, t) = e^{tX_k} \circ S_k(s), \qquad \forall (s, t) \in I'_k \times (-\delta_{k+1}, \delta_{k+1}),$$

is a parametrisation of an embedded submanifold of dimension $k+1$. This is due to the fact that the differential of C_{k+1} at $(\sigma, 0)$ has full rank, as it follows from (5.6). Moreover, since $C_{k+1}(\sigma, 0) = S_k(\sigma) \in V$, the set I_{k+1} can be chosen so that $C_{k+1}(I_{k+1}) \subset V$. We are now left to observe that, for all $(s, t) \in I'_k \times (-\delta_{k+1}, 0)$, we have that $x \in \mathcal{A}_{C_{k+1}(s,t)}$. In fact, starting from $C_{k+1}(s, t)$ one can reach $S_k(s)$ using the (constant) control corresponding to X_k, and x can be reached from $S_k(s)$ due to (5.4). This concludes the iteration, since $C_{k+1}|_{I'_k \times (-\delta_{k+1}, 0)}$ satisfies (5.3). $\square$

5.3 Complementary remarks

As mentioned in Remark 1.2, Lemma 5.2 (and consequently Theorem 1.1) generalises to other classes of controls provided that for any fixed control one still has existence and uniqueness in the class of absolutely continuous functions, and regularity on the initial conditions. In particular, uniqueness in necessary to prove the base case of the induction, and to find the transverse vector field in (5.6); on the other hand, regularity is used to define the submanifolds S_k starting from the submanifolds C_k.

In Theorem 1.1 we have shown that if system C satisfies local reachability then it is controllable. The property of local reachability means that one can reach an open neighbourhood of any initial condition. A slight modification of Example 4.2 gives a system which satisfies local reachability at every point *except* for one. The resulting system is not controllable, showing that the lack of local reachability even only at a point might impair global cotrollability.

Example 5.6. Let $M = \mathbb{R}^2$, and denote by V the half-line $V = \{0\} \times \mathbb{R}_{\leq 0}$. Consider the control system defined by the family $\mathcal{F} = \{f_1,\ f_2,\ f_3^+,\ f_3^-\}$ with

$$f_1 = \frac{\partial}{\partial y}, \quad f_2 = -\psi(x,y)\frac{\partial}{\partial y}, \quad f_3^{\pm} = \pm\phi(x,y)\frac{\partial}{\partial x},$$

for two smooth functions $\phi, \psi : \mathbb{R}^2 \to [0, +\infty)$ satisfying, for all $(x,y) \in \mathbb{R}^2$, $\phi(x,y) = 0$ if and only if $(x,y) \in V$, and $\psi(x,y) = 0$ if and only if $x = y = 0$, respectively. For any $p \in \mathbb{R}^2 \setminus V$ one has that $\mathcal{A}_p = \mathbb{R}^2 \setminus V$, while for any $p \in V \setminus \{0\}$ one has that $\mathcal{A}_p = \mathbb{R}^2$. Thus, at these points the system satisfies the local controllability property. However, for $p_0 = 0$ one has $\mathcal{A}_{p_0} = \{p_0\} \cup \mathbb{R}^2 \setminus V$ which does not contain p_0 in its interior.

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