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Catégorifications d'algèbres amassées et représentations d'algèbres de Hecke carquois

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Contents

Résumé

Cette thèse porte sur l'étude de diverses conséquences des résultats de catégorifications monoïdales d'algèbres amassées par les algèbres de Hecke carquois, établis dans les travaux de Kang-Kashiwara-Kim-Oh [69]. Nous nous intéresserons en particulier à trois aspects de cette théorie: en premier lieu celui de la combinatoire, puis de la géométrie polytopale, et enfin celui de la théorie des représentations géométrique.

Nous étudierons tout d'abord certaines relations combinatoires entre objets de nature a priori différentes: d'une part, les g-vecteurs au sens de Fomin-Zelevinsky, et d'autre part les partitions de racines qui paramétrisent les représentations simples de dimension finie des algèbres de Hecke carquois de type fini. Ces relations proviennent directement de certaines compatibilités remarquables entre différents ordres partiels naturels issus respectivement de la théorie des algèbres amassées et de la théorie des représentations. Nous montrons l'existence de telles relations dans le cas d'algèbres de Hecke carquois de type A_n. Nous établissons également une expression explicite pour les partitions de racines associées aux modules déterminantaux qui catégorifient une graine standard particulière de $\mathbb{C}[N]$.

La deuxième partie de cette thèse est consacrée à la construction de polytopes de Newton-Okounkov en utilisant de manière naturelle la théorie des représentations des algèbres de Hecke carquois. Nous commencerons par étendre les résultats de la partie précédente au cas d'algèbres de Hecke carquois de tout type (fini) simplement lacé, et ce grâce aux récents résultats de Kashiwara-Kim [72]. Ceci joue un rôle important dans la preuve de plusieurs propriétés combinatoires et géométriques de ces polytopes. Nous montrons ainsi que les volumes de certains de ces polytopes sont reliés à des formules des équerres (colorée) issues de la théorie combinatoire des éléments complètement commutatifs des groupes de Weyl.

Enfin, nous étudierons les modules déterminantaux catégorifiant les graines standard de $\mathbb{C}[N]$ à l'aide d'une notion géométrique a priori non reliée à la théorie des algèbres de Hecke carquois ni aux algèbres amassées et appelée multiplicité équivariante, introduite par Joseph [63], Rossmann [108] et Brion [17]. Baumann-Kamnitzer-Knutson [6] ont récemment défini un morphisme d'algèbre $\overline{D}$ sur $\mathbb{C}[N]$ relié aux multiplicités équivariantes des cycles de Mirković-Vilonen via la correspondance de Satake géométrique. Nous montrons qu'en types A_n et D_4, l'évaluation de $\overline{D}$ sur les mineurs drapeaux de $\mathbb{C}[N]$ prend une forme distinguée, semblable aux valeurs prises par $\overline{D}$ sur les éléments de la base canonique duale correspondant aux modules fortement homogènes des algèbres de Hecke carquois selon la construction de Kleshchev-Ram [78]. Ceci soulève également la question de certaines propriétés de lissité des cycles MV correspondant aux mineurs drapeaux de $\mathbb{C}[N]$. Nous mettons également en évidence certaines relations entre les images par $\overline{D}$ des mineurs drapeaux d'une même graine standard et nous montrons qu'en tous types ADE ces relations sont préservées par mutation d'une graine standard à une autre.

Abstract

The purpose of this thesis is to investigate various consequences of Kang-Kashiwara-Kim-Oh's monoidal categorifications of cluster algebras via quiver Hecke algebras [69]. We are interested in three different aspects of this theory: combinatorics, polytopal geometry, and geometric representation theory.

We begin by studying some combinatorial relationships between objects of different natures: the g-vectors in the sense of Fomin-Zelevinsky on the one hand, and the root partitions parametrizing irreducible finite-dimensional representations of finite type quiver Hecke algebras on the other hand. These relationships arise from certain compatibilities between various natural partial orderings respectively coming from cluster theory and representation theory. We prove the existence of such relationships in the case of quiver Hecke algebras of type A_n. We also provide an explicit description of the root partitions associated to the determinantial modules categorifying a particular standard seed in $\mathbb{C}[N]$.

The second part of this thesis is devoted to constructing Newton-Okounkov polytopes in a natural way using the representation theory of quiver Hecke algebras. We begin by extending the results of the previous part to any (finite) simply-laced type using recent results of Kashiwara-Kim [72]. This plays a key role for proving several combinatorial and geometric properties of these polytopes. In particular, we show that the volumes of certain of these polytopes are related to (colored) hook formulae coming from the combinatorics of fully-commutative elements of Weyl groups.

Finally, we study the determinantial modules categorifying the standard seeds of $\mathbb{C}[N]$ using certain a priori unrelated geometric tools, called equivariant multiplicities, introduced by Joseph [63], Rossmann [108] and Brion [17]. Baumann-Kamnitzer-Knutson [6] recently defined an algebra morphism on $\mathbb{C}[N]$ related to the equivariant multiplicities of Mirković-Vilonen cycles via the geometric Satake correspondence. We show that in types A_n and D_4, the evaluation of $\overline{D}$ on the flag minors of $\mathbb{C}[N]$ takes a distinguished form, similar to the values of $\overline{D}$ on the elements of the dual canonical basis corresponding to Kleshchev-Ram's [78] strongly homogeneous modules over quiver Hecke algebras. This also raises the question of certain smoothness properties of the MV cycles corresponding to the flag minors of $\mathbb{C}[N]$. We also exhibit certain identities relating the images under $\overline{D}$ of the flag minors belonging to the same standard seed and we show that in any ADE type these relations are preserved under cluster mutation from one standard seed to another.

Chapter 1

Introduction

1.1 Contexte historique et motivations

1.1.1 Groupes quantiques et anneaux de coordonnées unipotents

Introduits dans les années 1985 par Drinfeld [33] et Jimbo [62], les groupes quantiques peuvent être vus comme des déformations des algèbres enveloppantes d'algèbres de Lie simples complexes de dimension finie. Etant donnée une telle algèbre de Lie $\mathfrak{g}$ simple de type fini, son algèbre enveloppante $U(\mathfrak{g})$ est en un sens la plus petite algèbre associative contenant $\mathfrak{g}$. Bien que considérée comme un objet de nature classique d'un point de vue algébrique, elle peut elle-même être construite comme quantification de l'algèbre des fonctions sur le groupe de Poisson-Lie correspondant à $\mathfrak{g}$ (on renvoie aux livres de Chari-Pressley et d'Etingof-Schiffmann [30, 34] pour plus de détails sur ce point). Le groupe quantique $U_q(\mathfrak{g})$ est une déformation de $U(\mathfrak{g})$ selon un paramètre q qui en général est supposé appartenir au corps des complexes. Le groupe quantique $U_q(\mathfrak{g})$ peut être muni d'une structure d'algèbre de Hopf ce qui confère naturellement une structure monoïdale à la catégorie des $U_q(\mathfrak{g})$-modules. La théorie des représentations des groupes quantiques est très riche: en premier lieu, les représentations irréductibles de dimension finie de $U_q(\mathfrak{g})$ admettent une paramétrisation remarquable en termes de plus haut poids. Par ailleurs, l'étude de ces représentations a permis d'exhiber certaines *R-matrices* (ou opérateurs d'entrelacement) dans la catégorie des $U_q(\mathfrak{g})$-modules de dimension finie, c'est-à-dire des isomorphismes de $U_q(\mathfrak{g})$-modules

$$r_{U,V} : U \otimes V \longrightarrow V \otimes U$$

satisfaisant l'équation de Yang-Baxter i.e. rendant commutatif le diagramme suivant:

$$\begin{tikzcd}
 & U \otimes V \otimes W \arrow[dl, "r_{U,V}"'] \arrow[dr, "r_{V,W}"] & \\
V \otimes U \otimes W \arrow[d, "r_{U,W}"'] & & U \otimes W \otimes V \arrow[d, "r_{U,W}"] \\
V \otimes W \otimes U \arrow[dr, "r_{V,W}"'] & & W \otimes U \otimes V \arrow[dl, "r_{U,V}"] \\
 & W \otimes V \otimes U &
\end{tikzcd}$$

Prouver l'existence de morphismes non triviaux solutions de cette équation est en général un problème difficile. Dans le cas des groupes quantiques, Drinfeld a en fait montré l'existence (et l'unicité) d'une *R-matrice universelle*, i.e. un élément $\mathcal{R} \in U_q(\mathfrak{g})\hat{\otimes}U_q(\mathfrak{g})$ où $\hat{\otimes}$ désigne un produit tensoriel

complété. Cette R-matrice universelle se spécialise sur les représentations de dimension finie de $U_q(\mathfrak{g})$ en des morphismes comme ci-dessus qui sont alors toujours inversibles.

Il existe une généralisation de $U_q(\mathfrak{g})$ obtenue par un procédé d'affinisation et appelée *algèbre affine quantique*. Drinfeld a montré que cette algèbre pouvait en fait être réalisée comme déformation quantique de l'algèbre enveloppante d'une algèbre de Lie complexe $\hat{\mathfrak{g}}$ appelée algèbre de Kac-Moody affine. Cette algèbre de Lie est de dimension infinie dont un sous-quotient est l'algèbre des lacets $L\mathfrak{g}$ de $\mathfrak{g}$ définie par

$$L\mathfrak{g} := \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$$

où t est le paramètre d'affinisation. L'algèbre affine quantique correspondante est alors notée $U_q(\hat{\mathfrak{g}})$. Elle peut être regardée autant comme un affinisation du groupe quantique $U_q(\mathfrak{g})$ que comme une quantification de $U(\hat{\mathfrak{g}})$ selon le diagramme suivant.

$$\begin{tikzcd}
 & U(\hat{\mathfrak{g}}) \arrow[dr, squiggly, "quantification"] & \\
U(\mathfrak{g}) \arrow[ur, squiggly, "affinisation"] \arrow[dr, squiggly, "quantification"'] & & U_q(\hat{\mathfrak{g}}) \\
 & U_q(\mathfrak{g}) \arrow[ur, squiggly, "affinisation"'] &
\end{tikzcd}$$

On peut également construire des opérateurs d'entrelacement dans certaines catégories de représentations de dimension finie de l'algèbre affine quantique $U_q(\hat{\mathfrak{g}})$. Cependant, contrairement au cas des groupes quantiques issus d'algèbres de Lie de type fini, ces R-matrices ne sont pas des isomorphismes en général. Chari-Pressley ont donné une classification des représentations irréductibles de dimension finie de l'algèbre affine quantique $U_q(\hat{\mathfrak{g}})$ en termes de *polynômes de Drinfeld*, d'abord dans le cas $\mathfrak{g} = \mathfrak{sl}_2$ [29], puis dans le cas général [30]. Par les travaux de Frenkel-Reshetikhin [44], on dispose également d'une notion de *q-caractère*, i.e. on a un morphisme d'anneau injectif

$$\chi_q : K_0\left(U_q(\hat{\mathfrak{g}}) - mod\right) \longrightarrow \mathbb{C}[Y_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times]$$

où $Y_{i,a}$ est une indéterminée pour chaque $i \in I$ et $a \in \mathbb{C}^*$. La notation K_0 désigne l'anneau de Grothendieck de la catégorie considérée. Pour certains modules simples, il est possible de calculer explicitement ces q-caractères en utilisant un algorithme appelé algorithme de Frenkel-Mukhin.

Pour une algèbre de Lie ou de Kac-Moody affine $\mathfrak{g}$, on note $\mathfrak{n}$ la sous-algèbre nilpotente de $\mathfrak{g}$, et N le groupe pro-unipotent associé à $\mathfrak{n}$. Le groupe quantique (ou l'algèbre affine quantique) $U_q(\mathfrak{g})$ admet une décomposition triangulaire, c'est-à-dire qu'on a un isomorphisme

$$U_q(\mathfrak{g}) \simeq U_q^+(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})$$

La partie positive $U_q^+(\mathfrak{g})$ dans la décomposition ci-dessus est isomorphe (en tant que $\mathbb{Q}(q)$-algèbre) à une q-déformation de l'anneau $\mathbb{C}[N]$ des fonctions sur N. Cette déformation est notée $\mathcal{A}_q(\mathfrak{n})$ et appelée anneau de coordonnée unipotent quantique.

L'étude et la caractérisation de bonnes bases dans l'algèbre $\mathcal{A}_q(\mathfrak{n})$ sont devenues des questions importantes dès le début des années 1990 avec les travaux de Kashiwara et Lusztig. Kashiwara [70] introduisit la notion de *cristal* comme modèle combinatoire pour décrire la structure des représentations irréducibles de dimension finie de $U_q(\mathfrak{g})$. Il définit la base globale inférieure (resp. base globale supérieure) comme certaines bases cristallines du groupe quantique $U_q^-(\mathfrak{g})$ (resp. de l'anneau de coordonnée unipotent quantique $\mathcal{A}_q(\mathfrak{n})$). Lusztig [88] utilisa des méthodes géométriques en étudiant des catégories de faisceaux pervers sur certaines variétés carquois. Il définit la base canonique (resp. base canonique duale) de $U_q^-(\mathfrak{g})$ (resp. $\mathcal{A}_q(\mathfrak{n})$). Grojnowski-Lusztig [54] et Kashiwara-Saito [73] ont prouvé que la base canonique duale coincide en fait avec la base globale supérieure. Une des

propriétés essentielles de la base canonique (ou globale inférieure) est qu'elle permet de construire naturellement des bases de toutes les représentations de dimension finie de $U_q(\mathfrak{g})$.

A la suite de l'introduction de ces bases remarquables, Berenstein et Zelevinsky remarquèrent que certains éléments de la base canonique duale admettaient des propriétés combinatoires intéressantes, et notamment que dans certains cas leurs produits étaient encore des éléments de la base canonique duale. Dès lors se posa la question d'une description algorithmique des éléments de la base canonique duale. Ceci fut l'une des motivations principales pour l'introduction des algèbres amassées par Fomin et Zelevinsky [41]. Leurs observations les avaient amenés à formuler la conjecture que les éléments de la base canonique duale pouvaient être décrits comme les produits de certains éléments irréductibles distingués regroupés en ensembles finis de même cardinalité appelés amas. Ces produits, appelés monômes d'amas, peuvent être décrits de manière combinatoire grâce aux outils de la théorie des algèbres amassées développés dans [43]. Berenstein et Zelevinsky [10] ont ensuite considéré certaines déformations non commutatives des algèbres amassées, appelées algèbres amassées quantiques.

Geiss-Leclerc-Schröer [52] ont prouvé que l'anneau de coordonnée unipotent quantique $\mathcal{A}_q(\mathfrak{n})$ admet une structure d'algèbre amassée quantique. Plus généralement, ils définissent une famille $\{\mathcal{A}_q(\mathfrak{n}(w)), w \in W\}$ de sous -algèbres de $\mathcal{A}_q(\mathfrak{n})$ paramétrée par le groupe de Weyl W correspondant à $\mathfrak{g}$, et ce pour n'importe quelle algèbre de Kac-Moody $\mathfrak{g}$. Ces sous-algèbres $\mathcal{A}_q(\mathfrak{n}(w))$ sont appelées sous-groupes unipotents quantiques de $\mathcal{A}_q(\mathfrak{n})$. Lorsque le paramètre q tend vers 1, on obtient une sous-algèbre de $\mathbb{C}[N]$ notée $\mathbb{C}[N(w)]$. Lorsque $\mathfrak{g}$ est simple de type fini, la cellule $\mathcal{A}_q(\mathfrak{n}(w_0))$ associée au plus long élément w_0 de W coincide avec $\mathcal{A}_q(\mathfrak{n})$. Geiss-Leclerc-Schröer prouvent que chaque sous-algèbre $\mathcal{A}_q(\mathfrak{n}(w))$ de $\mathcal{A}_q(\mathfrak{n})$ admet une structure d'algèbre amassée quantique. En s'inspirant de constructions dues à Berenstein-Fomin-Zelevinsky [7], ils construisent également de manière explicite de nombreux amas dans $\mathcal{A}_q(\mathfrak{n}(w))$, paramétrés par l'ensemble des expressions réduites de w. Leur construction repose sur des techniques de catégorification additive par des catégories de représentations de dimension finie de l'algèbre préprojective issues de [51] ainsi que des travaux de Buan-Marsh-Reiten-Reineke-Todorov [21].

1.1.2 Algèbres amassées

Les algèbres amassées furent introduites par Fomin et Zelevinsky [41] dans les années 2000 dans le but d'étudier des questions de positivité totale et de comprendre les propriétés combinatoires des bases canoniques et canoniques duales des groupes quantiques. Ce sont des sous-algèbres commutatives du corps $\mathbb{Q}(x_1, \ldots, x_N)$ des fonctions rationnelles à N variables indépendantes. Elles sont engendrées (en tant que $\mathbb{Q}$-algèbres) par certaines fonctions rationnelles appelées *variables d'amas* définies de manière inductive comme suit. On part d'une donnée initiale constituée des éléments suivants:

1. le N-uplet de variables $(x_1, \ldots, x_N)$,
2. un carquois Q à N sommets sans boucle et sans 2-cycle.

Le N-uplet de variables $(x_1, \ldots, x_N)$ est appelé un *amas* et la donnée d'un amas et d'un carquois Q est appelée une *graine*. On note $J = \{1, \ldots, N\}$ l'ensemble des sommets de Q. On décompose J en

$$J = J_{ex} \sqcup J_{fr}.$$

Les sommets des ensembles J_{fr} et J_{ex} sont respectivement appelés sommets *gelés* et sommets *échangeables*.

Pour chaque sommet échangeable k, on définit une nouvelle graine à partir de la graine initiale $\mathcal{S} = ((x_1, \ldots, x_N), Q)$ en remplaçant Q par un nouveau carquois Q' et x_k par une nouvelle variable x'_k (les variables $x_j, j \neq k$ sont laissées inchangées). Le carquois Q' a le même ensemble de sommets

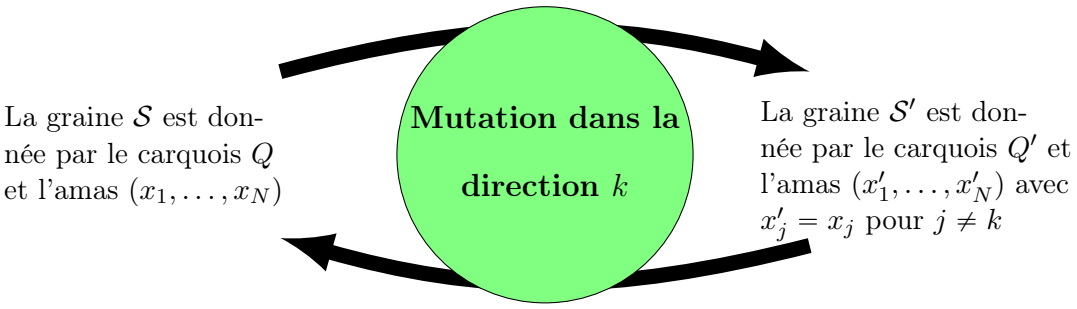


Figure 1: Une mutation dans la direction k.

que Q mais a des flèches différentes, qui sont obtenues à partir de Q par un certain algorithme. La nouvelle variable x'_k est entièrement déterminée par les variables d'amas $x_j, j \neq k$ et le carquois Q. Elle est donnée par la *relation d'échange*

$$x'_k := \frac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{k \to j} x_j\right). \tag{1.1}$$

Le procédé qui consiste à produire la graine $\mathcal{S}' := ((x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_N), Q')$ à partir de la graine $\mathcal{S} = ((x_1, \ldots, x_N), Q)$ est appelé *mutation dans la direction* k de la graine $\mathcal{S}$.

La première propriété essentielle de ce procédé est son involutivité: si l'on mute deux fois consécutivement dans une même direction k, on retrouve la graine de départ. On peut ensuite itérer ce processus en réalisant des suites (finies) de mutations dans des directions échangeables arbitraires à partir d'une graine initiale $\mathcal{S}$ fixée. On obtient ainsi pour toute suite quelconque $\mathbf{t} = (t_1, \ldots, t_r)$ d'entiers appartenant à J_{ex} une graine $\mathcal{S}^{\mathbf{t}}$ constituée d'un amas $(x_1^{\mathbf{t}}, \ldots, x_N^{\mathbf{t}})$ et d'un carquois $Q^{\mathbf{t}}$. Les nouvelles variables obtenues ainsi (i.e. les variables $x_i^{\mathbf{t}}$ pour tous les $i \in J$ et toutes les suites $\mathbf{t}$ arbitrairement longues) sont appelées *variables d'amas*. L'algèbre amassée engendrée par la graine $\mathcal{S}$ est par définition la sous-$\mathbb{Q}$-algèbre de $\mathbb{Q}(x_1, \ldots, x_N)$ engendrée par les variables d'amas:

$$\mathcal{A}(\mathcal{S}) := \mathbb{Q}[x_i^{\mathbf{t}}, i \in J, r \in \mathbb{Z}_{\geqslant 0}, \mathbf{t} = (t_1, \ldots, t_r) \in J_{ex}^r].$$

Les monômes ne faisant intervenir que des variables d'amas appartenant à un même amas sont appelés *monômes d'amas*.

Une algèbre amassée peut contenir un nombre fini ou infini de variables d'amas. Autrement dit le processus de mutations successives à partir d'une graine initiale fixée peut ne produire qu'un nombre fini de graines distinctes, ou au contraire peut générer une quantité infinie de nouvelles graines. Fomin-Zelevinsky [42] ont classifié les algèbres amassées *de type fini*, i.e. ne contenant qu'un nombre fini de variables d'amas. Cette classification remarquable se fait en termes de diagrammes de Dynkin. Les variables d'amas sont alors en bijection avec les racines presque positives correspondant (c'est-à-dire la réunion des racines positives et des opposées des racines simples).

Le premier résultat fondamental de la théorie des algèbres amassés est le *phénomène de Laurent*, dû à Fomin et Zelevinsky [41]: pour une graine $\mathcal{S}$ fixée quelconque, toute variable d'amas x de l'algèbre amassée $\mathcal{A}(\mathcal{S})$ s'écrit comme polynôme de Laurent à coefficients entiers en les variables d'amas de $\mathcal{S}$. On appelle cela la *décomposition amassée* de x par rapport à $\mathcal{S}$. Fomin et Zelevinsky conjecturèrent que les coefficients de ces polynômes de Laurent étaient en fait des entiers naturels. Cette conjecture de positivité a d'abord été prouvée par Musiker-Schiffler-Williams [93] pour une classe importante d'algèbres amassées intensément étudiées [48, 39, 81]: celles provenant de surfaces de Riemann triangulées. Lee-Schiffler [85] et Gross-Hacking-Keel-Kontsevitch [55] prouvèrent la conjecture de positivité dans le cas général. Une autre conjecture importante de la théorie des algèbres amassées est l'indépendance linéaire des monômes d'amas, prouvée en toute généralité

par Cerulli-Irelli-Keller-Labardini-Fragoso-Plamondon [28]. Comme on le verra dans la section suivante, cette conjecture peut être prouvée lorsque l'on dispose d'une catégorification monoïdale au sens de Hernandez-Leclerc [56] de l'algèbre amassée considérée.

Il est en fait possible de préciser la forme de la décomposition amassée d'une variable d'amas x par rapport à une graine $\mathcal{S} = ((x_1, \ldots, x_N), Q)$. En effet, l'ensemble des monômes de Laurent en $(x_1, \ldots, x_N)$ peut être muni d'un ordre partiel $\preccurlyeq_{\mathcal{S}}$ appelé *ordre de dominance pour* $\mathcal{S}$ selon la terminologie de F. Qin [106] par analogie avec l'ordre de Nakajima dans la théorie des algèbres affines quantiques (voir Section 1.1.3). On peut alors montrer que la décomposition amassée de x par rapport à $\mathcal{S}$ s'écrit

$$x = x_1^{g_1} \cdots x_N^{g_N} + \sum_{\mathfrak{m} \prec_{\mathcal{S}} \mathbf{x}^{\mathbf{g}}} a_{\mathfrak{m}} \mathfrak{m}$$

où les $a_{\mathfrak{m}}$ sont des entiers et $\mathbf{x}^{\mathbf{g}} := x_1^{g_1} \cdots x_N^{g_N}$. Le N-uplet $(g_1, \ldots, g_N) \in \mathbb{Z}^N$ est appelé *g-vecteur de x par rapport à $\mathcal{S}$* et est noté $\mathbf{g}_{\mathcal{S}}(x)$.

Autrement dit l'ensemble des monômes de Laurent apparaissant dans la décomposition amassée de x par rapport à $\mathcal{S}$ admet un unique monôme minimal et (de manière duale un unique monôme maximal) pour $\preccurlyeq_{\mathcal{S}}$. Ce résultat avait été conjecturé par Fomin-Zelevinsky [43] et a été prouvé par Derksen-Weyman-Zelevinsky [32] dans le cas anti-symétrique en utilisant la théorie des carquois à potentiels, et par Gross-Hacking-Keel-Kontsevitch [55] en toute généralité. La décomposition amassée de x par rapport à $\mathcal{S}$ peut alors s'écrire

$$x = F\left(\hat{y_1}, \ldots, \hat{y_n}\right) x_1^{g_1} \cdots x_N^{g_N}$$

où $\hat{y_1}, \ldots, \hat{y_n}$ sont des monômes de Laurent en $x_1, \ldots, x_N$ qui ne dépendent que de la graine de référence $\mathcal{S}$ (et pas de x) et F est un polynôme de terme constant 1 appelé F-polynôme de x par rapport à $\mathcal{S}$. Si x est une variable d'amas quelconque et $\mathcal{S}$ et $\mathcal{S}'$ sont deux graines reliées par une mutation, les g-vecteurs $\mathbf{g}_{\mathcal{S}}(x)$ et $\mathbf{g}_{\mathcal{S}'}(x)$ sont reliés par une *relation d'échange tropicale*. Ces dernières relations apparaissent dans les travaux de Fock-Goncharov [38] comme des formules de changement de base: à chaque graine $\mathcal{S}$ de $\mathcal{A}$ est associée une base d'un réseau, de telle sorte que la base associée à une graine $\mathcal{S}'$ mutée de $\mathcal{S}$ soit reliée à la base initiale par ces relations d'échange tropicales. Celles-ci jouent un rôle crucial dans la construction des variétés amassées dans [38] comme recollement de tores. Dans le cas des algèbres amassées avec coefficients, les mutations des coefficients sont également encodées par des vecteurs à coefficients entiers appelés c-vecteurs. La conjecture prouvée par [32, 55] ci-dessus est équivalent à celle dite de cohérence de signes, i.e. chaque c-vecteur a toutes ses composantes non nulles de même signe (voir par exemple [97]). Nakanishi-Zelevinsky [98] ont également mis en évidence une dualité entre c-vecteurs et g-vecteurs, qui peut s'interpréter en termes de géométrie polytopale (voir [61]): les c-vecteurs peuvent être vus comme les vecteurs normaux aux facettes d'un polytope dont les sommets correspondent aux g-vecteurs.

Berenstein-Zelevinsky [10] ont introduit certaines déformations non commutatives des algèbres amassées, appelées algèbres amassées quantiques. Dans ce cadre, les variables d'amas d'un même amas ne commutent pas, mais q-commutent, i.e. satisfont des relations de la forme

$$x_i x_j = q^{\lambda_{ij}} x_j x_i.$$

Les résultats principaux de la théorie classique des algèbres amssées, tels que le phénomène de Laurent ou la positivité des coefficients des décompositions amassées, admettent des analogues quantiques (voir [10, 31]). Plus de détails sur les algèbres amassées quantiques sont donnés dans la Section 1.1.5.

1.1.3 Catégorifications des algèbres amassées

Très vite après l'introduction des algèbres amassées par Fomin et Zelevinsky, l'idée d'utiliser des techniques de catégorifications s'est révélée particulièrement fructueuse. On peut regrouper ces différentes catégorifications en deux grandes classes: celles dites *additives* et celles dites *monoïdales*.

Les catégorifications additives sont apparues peu de temps après l'introduction des algèbres amassées par Fomin-Zelevinsky. Elles font intervenir des catégories de représentations carquois telles que la somme directe de représentations correspond au produit dans une algèbre amassée. Les catégories amassées furent introduites par Buan-Marsh-Reineke-Reiten-Todorov [21] et furent ensuite intensément étudiées par Caldero-Keller [22, 23], Amiot [1], Plamondon [101, 102]. Geiss-Leclerc-Schröer [50] ont également mis en évidence de telles catégorifications, bien que de nature un peu différente. Ils montrent que certaines catégories de modules de dimension finie sur les algèbres préprojectives de types de Dynkin finis simplement lacés catégorifient une structure amassée naturelle sur les cellules unipotentes $\mathbb{C}[N(w)]$ de $\mathbb{C}[N]$.

Les catégories amassées admettent un nombre fini d'objets simples ainsi qu'un nombre fini d'objets projectifs indécomposables. Les premiers sont en bijection avec les racines simples et les seconds avec les racines positives du système de racines correspondant. En ce qui concerne les représentations d'algèbres préprojectives de type de Dynkin fini, les résultats de Geiss-Leclerc-Schröer [50] montrent que les modules qui catégorifient les monômes d'amas des anneaux de coordonée unipotents appartiennent à une certaine classe de modules appelés *rigides*.

Ici nous nous intéressons à un autre point de vue sur les catégorifications d'algèbres amassées dû à Hernandez-Leclerc [56], et sensiblement différent du précédent.

Catégorifications monoïdales

Partant d'une algèbre amassèe $\mathcal{A}$, l'idée est de réaliser $\mathcal{A}$ comme l'anneau de Grothendieck $K_0(\mathcal{C})$ d'une catégorie monoïdale $\mathcal{C}$ via un isomorphisme compatible avec la structure amassée sur $\mathcal{A}$. Plus précisément, on requiert les deux conditions suivantes

1. Il y a un isomorphisme d'anneaux $\mathcal{A} \xrightarrow{\simeq} K_0(\mathcal{C})$.
2. L'image d'un monôme d'amas de $\mathcal{A}$ par cet isomorphisme est la classe d'un objet simple dans $\mathcal{C}$.

On peut ainsi utiliser la théorie des algèbres amassées pour comprendre la catégorie $\mathcal{C}$, ou au contraire utiliser la structure de $\mathcal{C}$ pour en déduire des résultats non triviaux au niveau de $\mathcal{A}$. On peut déjà constater que cette définition impose implicitement certaines restrictions sur les objets simples de $\mathcal{C}$ correspondant aux monômes d'amas dans $\mathcal{A}$. En effet, par définition, le carré (et plus généralement toute puissance) d'un monôme d'amas est encore un monôme d'amas. En revanche, le produit tensoriel dans $\mathcal{C}$ d'un objet simple par lui-même n'a aucune raison d'être simple. Une condition nécesssaire pour que la classe d'un objet simple M de $\mathcal{C}$ soit un monôme d'amas dans $\mathcal{A}$ est donc que M soit *réel* dans la terminologie de Hernandez-Leclerc [56], i.e. que $M \otimes_{\mathcal{C}} M$ soit simple dans $\mathcal{C}$. Notons que la définition originale de [56] demande la condition plus forte d'avoir une bijection

$$\{\text{objets réels simples de } \mathcal{C}\}/\sim \quad \longleftrightarrow \quad \{\text{monômes d'amas de } \mathcal{A}\}.$$

Il résulte également de la définition ci-dessus que les catégorifications monoïdales d'algèbres amassées admettent une infinité d'objets simples à isomorphisme près, ce qui les distingue des catégorifications additives qui n'admettent qu'un nombre fini d'objets simples.

Algèbres affines quantiques

Le premier exemple de catégorification monoïdale d'algèbre amassée a été construit par Hernandez-Leclerc [56] en utilisant des catégories de représentations de dimension finie d'algèbres affines quantiques. Etant donnée une algèbre de Lie $\mathfrak{g}$ de type fini simplement lacé, on note $\hat{\mathfrak{g}}$ l'algèbre de Kac-Moody affine correspondante et on considère la catégorie $\mathcal{C}$ des représentations de dimension finie de $U_q(\hat{\mathfrak{g}})$. Comme dans le cas des groupes quantiques de type fini, on peut construire des opérateurs d'entrelacement dans la catégorie $\mathcal{C}$, i.e. des morphismes de $U_q(\hat{\mathfrak{g}})$-modules

$$r_{U,V} : U \otimes V \longrightarrow V \otimes U$$

satisfaisant l'équation de Yang-Baxter (cf. Section 1.1.1). Cependant, contrairement au cas des groupes quantiques, ces morphismes ne sont plus inversibles en général. Ils donnent donc lieu à des suites exactes courtes dans $\mathcal{C}$ de la forme

$$0 \longrightarrow \text{Ker } r_{U,V} \longrightarrow U \otimes V \longrightarrow \text{Im } r_{U,V} \longrightarrow 0.$$

Dans l'anneau de Grothendieck $K_0(\mathcal{C})$, ceci équivaut à

$$[U] \cdot [V] = [\text{Ker } r_{U,V}] + [\text{Im } r_{U,V}].$$

L'idée de Hernandez-Leclerc est que certaines de ces relations peuvent interpréter comme les relations d'échange associées aux mutations d'une algèbre amassée isomorphe (en tant qu'anneau) à $K_0(\mathcal{C})$.

L'ensemble des polynômes de Drinfeld qui paramétrisent l'ensemble des objets simples de $\mathcal{C}$ peut être muni d'une structure de monoïde. Ce monoïde est partiellement ordonné selon un ordre appelé *ordre de Nakajima*. Hernandez-Leclerc [56] étudient en particulier une sous-catégorie $\mathcal{C}_1$ de $\mathcal{C}$ dont les objets simples sont donnés par certains polynômes de Drinfeld remarquables (cf. [56, Sections 3.7-3.8]). Ils construisent un isomorphisme d'anneaux de $K_0(\mathcal{C}_1)$ vers une algèbre amassée de type fini (i.e. contenant un nombre fini de variables d'amas) et formulent la conjecture que cet isomorphisme induit une bijection entre les monômes d'amas et les classes d'objets réels simples de $\mathcal{C}_1$. Ils prouvent cette conjecture pour les types A_n et D_4. Cette conjecture fut prouvée en tout type ADE par Nakajima [96] en utilisant la géométrie des faisceaux pervers sur certaines variétés carquois (appelées variétés de Nakajima).

Hernandez-Leclerc [56] utilisent aussi ces résultats de catégorification monoïdale pour établir des correspondances entre objets issus de la théorie des représentations d'une part et de celle des algèbres amassées d'autre part:

Théorie des représentations	**Théorie des algèbres amassées**
Ordre de Nakajima	ordre de dominance
Polynômes de Drinfeld	g-vecteurs
q-caractères	F-polynômes.

D'autres résultats de catégorifications monoïdales ont été obtenus pour d'autres catégories de représentations de $U_q(\hat{\mathfrak{g}})$ [57, 20, 11]. Enfin Cautis-Williams [27] ont montré que la catégorie de Satake cohérente pour la Grassmanienne affine était une catégorification monoïdale de certains anneaux de coordonnées affines.

Ici nous nous intéresserons tout particulièrement à d'autres exemples importants de catégorifications monoïdales d'algèbres amassées apparus dans les travaux de Kang-Kashiwara-Kim-Oh [66, 67, 69, 72]. Les catégories qui interviennent dans ce cas sont des catégories de modules sur certaines algèbres appelées *algèbres de Hecke carquois* ou algèbres KLR.

1.1.4 Algèbres de Hecke carquois

On considère une algèbre de Kac-Moody symétrique $\mathfrak{g}$ et on note $\Pi = \{\alpha_i, i \in I\}$ l'ensemble des racines simples de $\mathfrak{g}$ où I est un ensemble fini. On note également

$$Q_+ := \bigoplus_i \mathbb{Z}_{\geqslant 0}\alpha_i$$

Les algèbres de Hecke carquois (ou algèbres KLR) ont été introduites par Khovanov-Lauda [76] et Rouquier [110]. Il s'agit d'une famille $\{R(\beta), \beta \in Q_+\}$ indexée par Q_+ d'algèbres associatives $\mathbb{Z}$-graduées. Pour chaque $\beta \in Q_+$, l'algèbre $R(\beta)$ est engendrée par trois familles de générateurs: des générateurs polynomiaux $x_1, \ldots, x_n$, des générateurs de tresse $\tau_1, \ldots, \tau_{n-1}$, et des idempotents $e(w)$ indexés par un ensemble fini (entièrement déterminé par β) de mots sur l'alphabet I. Ces idempotents commutent avec les générateurs polynomiaux et sont orthogonaux entre eux dans le sens suivant:

$$e(w)e(w') = \delta_{w,w'}e(w)$$

pour tous w, w'. Les autres relations généralisent celles qui définissent les algèbres de Hecke affines, d'où la terminologie d'algèbres de Hecke carquois. La principale propriété de ces algèbres est de catégorifier la partie négative du groupe quantique $U_q(\mathfrak{g})$ associé à $\mathfrak{g}$ au sens suivant. Pour chaque $\beta \in Q_+$, on désigne par $R(\beta) - gmod$ la catégorie des $R(\beta)$-modules gradués de dimension finie. On pose alors

$$R - gmod := \bigoplus_{\beta \in Q_+} R(\beta) - gmod.$$

Cette catégorie peut être munie d'une structure de catégorie monoïdale, notée $\circ$, définie par induction parabolique. Le groupe de Grothendieck de $R - gmod$ hérite par conséquent d'une structure d'anneau. On a également un endofoncteur de décalage: pour tout objet $M = \bigoplus_{n\in\mathbb{Z}} M_n$ dans $R - gmod$ on définit $q \cdot M = \bigoplus_{n\in\mathbb{Z}}(q \cdot M)_n$ par $(q \cdot M)_n := M_{n+1}$. Ceci confère à l'anneau de Grothendieck $K_0(R - gmod)$ une structure de $\mathbb{Z}[q^{\pm 1}]$-module. Khovanov-Lauda [76] et Rouquier [110] prouvent qu'il existe un isomorphisme de $\mathbb{Z}[q^{\pm 1}]$-modules

$$K_0(R - gmod) \xrightarrow{\simeq} \mathcal{A}_q(\mathfrak{n}). \tag{1.2}$$

La seconde propriété cruciale des algèbres de Hecke carquois est que cet isomorphisme induit une bijection entre les classes d'isomorphisme d'objets simples de $R - gmod$ et la base canonique duale (ou base globale supérieure) de $\mathcal{A}_q(\mathfrak{n})$. Ce résultat est dû à Rouquier [110] et Varagnolo-Vasserot [113].

Kleshchev et Ram [79] ont donné une classification des objets simples de $R - gmod$ pour les algèbres de Hecke carquois de type fini, c'est-à-dire lorsque $\mathfrak{g}$ est une algèbre de Lie simple de dimension finie. Cette paramétrisation d'objets simples utilise la combinatoire des *mots Lyndon* et peut être vue comme une catégorification des bases de Lyndon de $\mathcal{A}_q(\mathfrak{n})$ étudiées par Rosso [109] et Leclerc [84]. Soit I l'ensemble indexant les racines simples de $\mathfrak{g}$ et soit $<$ un ordre total arbitraire sur I. On note encore $<$ l'ordre lexicographique induit sur l'ensemble $\mathcal{M}$ des mots (finis) sur l'alphabet I. Pour chaque mot $\mu \in \mathcal{M}$, le *poids* de μ est l'élément de Q_+ défini par

$$\mathrm{wt}(\nu) := \sum_{i\in I} \sharp\{k, h_k = i\}\alpha_i.$$

Kleshchev-Ram construisent une classe remarquable d'objets simples dans $R - gmod$ appelés modules *cuspidaux*, paramétrés de manière combinatoire par un sous-ensemble de $\mathcal{M}$ noté $\mathcal{GL}$ dont les éléments sont appelés *bons mots Lyndon*. On a une bijection

$$\begin{array}{ccc} \mathcal{GL} & \longrightarrow & \Phi_+ \\ \mathbf{j} & \longmapsto & \mathrm{wt}\,(\mathbf{j}) \end{array}$$

où Φ_+ désigne l'ensemble des racines positives du système de racine de $\mathfrak{g}$. Les modules simples dans $R-gmod$ sont alors réalisés comme quotients de produits (pour la structure monoïdale $\circ$ de $R-gmod$) de modules cuspidaux; en termes combinatoires, ils sont paramétrés par l'ensemble

$$\mathbf{M} := \{\mathbf{j}_1 \cdots \mathbf{j}_k \mid \mathbf{j}_1, \ldots, \mathbf{j}_k \in \mathcal{GL}, \mathbf{j}_1 \geqslant \cdots \geqslant \mathbf{j}_k\} \subset \mathcal{M}. \tag{1.3}$$

Pour $\mu = \mathbf{j}_1 \cdots \mathbf{j}_k \in \mathbf{M}$, l'unique (à isomorphisme près) module simple de dimension finie $L(\mu)$ correspondant à μ est donné par

$$L(\mu) = \mathrm{hd}\left(L(\mathbf{j}_1) \circ \cdots \circ L(\mathbf{j}_k)\right).$$

De plus, si μ est de la forme $\mathbf{j}^n$ pour un certain $\mathbf{j} \in \mathcal{GL}$, alors on a $L(\mu) = L(\mathbf{j})^{\circ n}$. Les éléments de $\mathbf{M}$ sont appelés *mots dominants* ou *partitions de racines*. Une classification similaire a été étudiée par McNamara [90]. Lorsque $\mathfrak{g}$ est une algèbre de Kac-Moody affine (non tordue), Kleshchev [77] a établi une classification analogue des objets simples de $R-gmod$. Dans ce cas, il y a une infinité de modules cuspidaux. Par ailleurs, on dispose alors seulement d'un étiquetage (c'est-à-dire une bijection ensembliste) de ces modules cuspidaux par des partitions d'entiers, et non d'une description explicite de chaque module cuspidal comme en type fini.

L'étude des modules cuspidaux fut l'une des motivations principales à la définition par Kleshchev-Ram [78] (en types simplement lacés) de certaines représentations irréductbles remarquables, appelées *représentations homogènes*. Ces modules vérifient la propriété d'être concentrés en un seul degré pour la graduation naturelle des algèbres de Hecke carquois. Ces représentations sont classifiées par la combinatoire des éléments *complètement commutatifs* des groupes de Weyl. Les propriétés combinatoires de ces éléments est très riche et a été étudiée par Proctor [104, 105], Stembridge [111], et Nakada [95]. On note $\mathcal{FC}$ l'ensemble des éléments complètement commutatifs de W. La construction de Kleshchev-Ram [78] donne une bijection

$$\begin{array}{ccc} \mathcal{FC} & \longrightarrow & \mathcal{H}om \\ w & \longmapsto & S(w) \end{array}$$

entre l'ensemble $\mathcal{H}om$ des classes d'isomorphisme de ces modules homogènes et le sous-ensemble (fini) $\mathcal{FC}$ des éléments complètement commutatifs de W. Ces modules sont en un sens les seuls objets simples de $R-gmod$ dont on puisse facilement calculer le caractère gradué: en effet, Kleshchev-Ram [78] montrent que pour tout $w \in \mathcal{FC}$, les sous-espaces de poids de $S(w)$ sont tous de dimension 1 et sont paramétrés par les expressions réduites de w.

Parmi cette famille de modules, on distingue une sous-famille remarquable, que l'on notera $\mathcal{H}om^+$, dont les éléments sont appelés *modules fortement homogènes*. L'image inverse de $\mathcal{H}om^+$ par la bijection précédente est un sous-ensemble $\mathcal{M}in^+ \subset \mathcal{FC}$ dont les éléments sont appelés *dominant minuscules*. Kleshchev-Ram avaient déjà observé que les dimensions de ces modules étaient donnés par la formule des équerres de Peterson-Proctor. Cette formule, introduite dans un travail non publié de Peterson et Proctor, a été généralisée par Nakada [95] dans un contexte purement combinatoire en une *formule des équerres colorée* qui peut s'écrire comme suit: si w est dominant minuscule, alors on a

$$\prod_{\beta \in \Phi_+^w} \frac{1}{\beta} = \sum_{(i_1,\ldots,i_N) \in \mathrm{Red}(w)} \frac{1}{\alpha_{i_1}} \frac{1}{\alpha_{i_1} + \alpha_{i_2}} \cdots \frac{1}{\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_N}} \tag{1.4}$$

où $\mathrm{Red}(w)$ désigne l'ensemble des expressions réduites de w et $\alpha_1, \ldots, \alpha_n$ sont des variables algébriquement indépendantes correspondant aux racines simples de $\mathfrak{g}$. Les racines positives sont vues comme des fonctions linéaires en les α_i. Si l'on évalue tous les α_i en 1, alors cette égalité entre fonctions rationnelles redonne la formule des équerres de Peterson-Proctor.

1.1.5 Catégorifications monoïdales des anneaux de coordonnées quantiques

Dans une série de travaux [66, 67, 69], Kang-Kashiwara-Kim-Oh utilisent la théorie des représentations des algèbres de Hecke carquois pour établir des résultats de catégorification monoïdale. Ces résultats sont valables lorsque $\mathfrak{g}$ est une algèbre de Kac-Moody symmétrique. Grâce aux résultats de Khovanov-Lauda et Rouquier rappelés dans la Section 1.1.4, on dispose d'un isomorphisme d'anneaux $K_0(R{-}gmod) \simeq \mathbb{C}[N]$, ce qui établit la première condition d'une catégorification monoïdale (cf. Section 1.1.3). Il reste donc à prouver que les monômes d'amas de $\mathbb{C}[N]$ correspondent toujours à des classes d'objets simples dans $R - gmod$.

Opérateurs d'entrelacement dans $R - gmod$

L'outil crucial dans les travaux de Kang-Kashiwara-Kim-Oh est la construction de *R-matrices* (ou *opérateurs d'entrelacement*) dans $R - gmod$, c'est-à-dire pour chaque couple de modules $M, N \in R - gmod$, un morphisme de modules non-trivial

$$r_{M,N} : M \circ N \longrightarrow N \circ M$$

satisfaisant l'équation de Yang-Baxter, i.e. faisant commuter le diagramme hexagonal suivant:

$$\begin{tikzcd}
 & L \circ M \circ N \arrow[dl, "r_{L,M}\circ N"'] \arrow[dr, "L\circ r_{M,N}"] & \\
M \circ L \circ N \arrow[d, "M\circ r_{L,N}"'] & & L \circ N \circ M \arrow[d, "r_{L,N}\circ M"] \\
M \circ N \circ L \arrow[dr, "r_{M,N}\circ L"'] & & N \circ L \circ M \arrow[dl, "N\circ r_{L,M}"] \\
 & N \circ M \circ L &
\end{tikzcd}$$

Dans ce diagramme la notation $r_{L,M} \circ N$ désigne le morphisme qui agit comme $r_{L,M}$ sur les deux premières composantes de $L \circ M \circ N$ et comme l'identité sur la troisième. Ces R-matrices sont construites dans [66] en utilisant un procédé d'affinisation puis de renormalisation, par analogie avec la théorie des algèbres affines quantiques. Par construction, ces R-matrices sont compatibles avec la $\mathbb{Z}$-graduation naturelle des modules de $R - gmod$ provenant de celle des algèbres $R(\beta), \beta \in Q_+$. Ces R-matrices ne sont pas des isomorphismes en général; par conséquent elles donnent lieu à des suites exactes courtes dans $R - gmod$:

$$0 \longrightarrow \text{Ker } r_{M,N} \longrightarrow M \circ N \longrightarrow \text{Im } r_{M,N} \longrightarrow 0.$$

Dans l'anneau de Grothendieck $K_0(R - gmod)$ cela donne

$$[M] \cdot [N] = [\text{Ker } r_{M,N}] + [\text{Im } r_{M,N}].$$

L'idée dans [56] est que certaines de ces égalités correspondent aux relations d'échange dans l'algèbre amassée $\mathcal{A}_q(\mathfrak{n})$. Par ailleurs les R-matrices $r_{M,N}$ donnent certains critères très utiles pour prouver que le produit de deux objets réels simples est encore simple. Ces critères font intervenir de manière essentielle la graduation naturelle des objets de $R - gmod$.

Degré des R-matrices dans $R - gmod$

Etant donnés deux modules simples M et N dans $R - gmod$, la R-matrice $r_{M,N}$ est un morphisme entre modules *gradués*; on peut donc considérer le degré de $r_{M,N}$, noté $\Lambda(M, N)$. Kang-Kashiwara-Kim-Oh [69] prouvent que si M et N sont simples et si M ou N est réel, alors $M \circ N$ est simple si

et seulement si $\Lambda(M,N) = -\Lambda(N,M)$ ([69, Lemme 3.2.3]). Dans ce cas on dit que les modules M et N *commutent*. La R-matrice $r_{M,N}$ est alors un isomorphisme de modules gradués

$$r_{M,N} : M \circ N \xrightarrow{\simeq} q^{\Lambda(N,M)} N \circ M$$

dont l'inverse est $r_{N,M}$ (à une constante multiplicative près). De plus, Kang-Kashiwara-Kim-Oh prouvent ([69, Théorème 2.2.4]) que si M et N sont des objets simples tels que M ou N est réel, alors l'image de $r_{M,N}$ est toujours simple et coïncide avec $\mathrm{hd}(M \circ N)$ ainsi qu'avec $\mathrm{soc}(N \circ M)$. De même l'image de $r_{N,M}$ est toujours simple et coïncide avec $\mathrm{hd}(N \circ M)$ ainsi qu'avec $\mathrm{soc}(M \circ N)$. En particulier $M \circ N$ et $N \circ M$ ont des têtes et des socles simples.

Graines monoïdales quantiques et leurs mutations

La notion de graine monoïdale est introduite par Kang-Kashiwara-Kim-Oh comme une version catégorifiée des graines des algèbres amassées définies Section 1.1.2 ci-dessus. Comme dans le cas des graines usuelles, on se donne un carquois Q (sans boucle et sans 2-cycle) à N sommets, mais les N variables indépendantes sont remplacées par la donnée de N objets simples $M_1, \ldots, M_N$ de $R-gmod$ tels que tous les produits $M_{i_1} \circ \cdots \circ M_{i_t}, i_1, \ldots, i_t \in \{1, \ldots, N\}$ sont simples. En particulier chaque M_i est réel. Pour chaque $k \in J_{ex}$, la mutation d'une graine monoïdale $((M_1, \ldots, M_N), Q)$ dans la direction k est constituée du carquois Q' donné par la mutation usuelle dans la direction k et des objets simples $M_1, \ldots, M_{k-1}, M'_k, M_{k+1}, \ldots, M_N$ où M'_k est un objet réel simple uniquement déterminé par k et par la graine monoïdale de départ. La relation d'échange (1.1) prend alors la forme catégorifiée suivante:

$$0 \longrightarrow \circ_{i \to k} M_i \longrightarrow M_k \circ M'_k \longrightarrow \circ_{j \leftarrow k} M_j \longrightarrow 0$$

$$0 \longrightarrow \circ_{j \leftarrow k} M_j \longrightarrow M'_k \circ M_k \longrightarrow \circ_{i \to k} M_i \longrightarrow 0.$$

L'involutivité des mutations impose certaines relations sur les poids des M_i (i.e. les $\beta_i \in Q_+$ tel que $M_i \in R(\beta_i) - gmod$). En tenant compte de la graduation naturelle des objets de $R - gmod$, Kang-Kashiwara-Kim-Oh adaptent la notion de graine monoïdale au cas quantique. Une *graine monoïdale quantique* est alors définie par la donnée d'un carquois Q, d'une matrice carrée de taille N anti-symmétrique $L = (\lambda_{ij})$, et de N objets réels simples $M_1, \ldots, M_N$ tels que $M_i \circ M_j \simeq q^{\lambda_{ij}} M_j \circ M_i$ pour tous $1 \leqslant i, j \leqslant N$; on demande en outre que la matrice L et le carquois Q soient compatibles dans le sens de Berenstein-Zelevinsky [10, Définition 3.1]. Dans le cas de $R - gmod$, la matrice L est donnée par les degrés des R-matrices entre les M_i:

$$L := (-\Lambda(M_i, M_j))_{1 \leqslant i,j \leqslant N}.$$

Les relations d'échanges issues des mutations s'expriment alors par des suites exactes courtes *graduées* de la forme

$$0 \longrightarrow q \circ_{i \to k} M_i \longrightarrow q^{\tilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \longrightarrow \circ_{j \leftarrow k} M_j \longrightarrow 0.$$

On renvoie à ([69, Définition 3.1.3]) pour une définition précise de $\tilde{\Lambda}(M_k, M'_k)$.

L'un des résultats essentiels de [69] ([69, Théorème 7.1.3]) consiste à montrer que si une graine monoïdale quantique $\mathcal{S}$ admet une mutation $\mathcal{S}'_k$ dans chaque direction $k \in J_{ex}$, alors il en est de même pour toutes les graines monoïdales $\mathcal{S}'_k, k \in J_{ex}$. Autrement dit, si on peut muter $\mathcal{S}$ une fois dans chaque direction échangeable, alors on peut en fait effectuer des suites arbitraires de mutations à partir de $\mathcal{S}$. Par conséquent, l'existence d'une telle graine monoïdale quantique $\mathcal{S}$ implique directement les résultats de catégorifications monoïdales voulus. Dans le cas de $R-gmod$,

il ne reste donc plus qu'à mettre en évidence une telle graine monoïdale. Kang-Kashiwara-Kim-Oh prouvent ([69, Théorème 11.2.2]) que les versions catégorifiées des graines $\mathcal{S}^{\mathbf{i}}$ construites par Geiss-Leclerc-Schröer [52] satisfont la condition voulue. Ceci prouve que $R-gmod$ constitue bien une catégorification monoïdale de $\mathcal{A}_q(\mathfrak{n})$ et donc que les monômes d'amas de $\mathcal{A}_q(\mathfrak{n})$ appartiennent à la base canonique duale. On renvoie à l'exposé de Kashiwara lors de l'ICM 2018 [71] pour davantage de précisions sur le rôle que jouent les catégorifications monoïdales d'algèbres amassées dans la théorie des bases cristallines, notamment via les bases globales des anneaux de coordonnée quantiques.

En utilisant les mêmes techniques, Kang-Kashiwara-Kim-Oh construisent également une catégorification monoïdale de chaque cellule unipotente quantique $\mathcal{A}_q(\mathfrak{n}(w))$ par une sous-catégorie $\mathcal{C}_w$ de $R-gmod$. Ces catégories $\mathcal{C}_w, w \in W$ peuvent être vues comme des analogues monoidaux des catégories de représentations de l'algèbre préprojective considérées par Geiss-Leclerc-Schröer [51, 52]. Lorsque $\mathfrak{g}$ est de type fini, la catégorie $R-gmod$ coincide avec $\mathcal{C}_{w_0}$ où w_0 désigne le plus long élément du groupe de Weyl de $\mathfrak{g}$.

Modules déterminantaux

Geiss-Leclerc-Schröer [52] ont construit une famille de graines remarquables dans $\mathbb{C}[N]$ ainsi que dans chaque cellule unipotente $\mathbb{C}[N(w)]$. Leur construction fait intervenir la structure d'algèbre amassée quantique de $\mathcal{A}_q(\mathfrak{n})$ (resp. $\mathcal{A}_q(\mathfrak{n}(w))$) mais on ne rappelle ici que la version classique. La construction de Geiss-Leclerc-Schröer [52] fournit pour chaque $w \in W$ une famille de graines

$$\{\mathcal{S}^{\mathbf{i}}, \mathbf{i} \in \mathrm{Red}(w)\}$$

dans $\mathbb{C}[N(w)]$, paramétrée par l'ensemble $\mathrm{Red}(w)$ des expressions réduites de w. Ces graines sont parfois appelées *graines standard* de $\mathbb{C}[N(w)]$. Cette construction repose sur des techniques de catégorification additive via des représentations des algèbres préprojectives. Kang-Kashiwara-Kim-Oh [69] adaptèrent ceci au cadre des catégorifications monoïdales en considérant les objets simples de $\mathcal{C}_w$ correspondant aux variables d'amas des graines $\mathcal{S}^{\mathbf{i}}$ via l'isomorphisme (1.2).

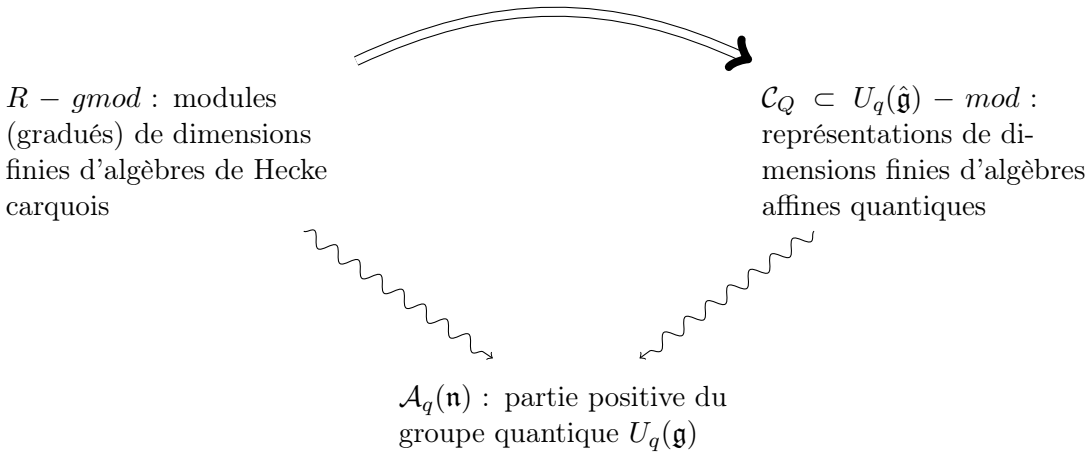
$$\begin{array}{ccccc} R-gmod & \ni & (M_1^{\mathbf{i}}, \dots M_N^{\mathbf{i}}) & & \text{modules déterminantaux de } \mathcal{S}^{\mathbf{i}} \\ \downarrow & & \downarrow & & \\ \mathbb{C}[N] & \ni & (x_1^{\mathbf{i}}, \dots, x_N^{\mathbf{i}}) & & \text{variables d'amas de } \mathcal{S}^{\mathbf{i}}. \end{array}$$

Ces objets simples de $R-gmod$ appartiennent à $\mathcal{C}_w$ (par construction de $\mathcal{C}_w$) et sont appelés *modules déterminantaux de* $\mathcal{C}_w$. Par [69, Proposition 10.2.4], les modules déterminantaux peuvent être obtenus récursivement par applications successives des foncteurs d'induction qui catégorifient la structure cristalline de la base canonique duale. On renvoie à [82] pour plus de détails sur ce dernier point.

Dualité de Schur-Weyl affine quantique généralisée

Hernandez-Lelcerc [58] ont mis en évidence un isomorphisme d'anneaux entre la partie négative $U_q^-(\mathfrak{g})$ du groupe quantique (de type fini) $U_q(\mathfrak{g})$ et une déformation quantique de l'anneau de Grothendieck d'une certaine sous-catégorie $\mathcal{C}_Q$ (dépendant d'un carquois Q) de représentations de dimension finie de l'algèbre affine quantique associée à $\mathfrak{g}$. De plus, ils ont montré que la base canonique de $U_q^-(\mathfrak{g})$ correspond à la base des objets simples dans $\mathcal{C}_Q$. Ainsi l'anneau de coordonnée quantique $\mathcal{A}_q(\mathfrak{n})$ (isomorphe à la partie positive du groupe quantique $U_q(\mathfrak{g})$) admet deux catégorifications par des catégories a priori différentes : d'une part via des catégories de représentations de dimension finie de l'algèbre affine quantique $U_q(\hat{\mathfrak{g}})$, d'autre part par des catégories de

modules sur les algèbres de Hecke carquois associées à $\mathfrak{g}$. Kang-Kashiwara-Kim-Oh [68] ont mis en évidence un foncteur qui relie ces deux catégories. Lorsque $\mathfrak{g}$ est de type A_n ou D_n, ce foncteur est exact, monoïdal, et envoie objets simples sur objets simples. Fujita [47] a également prouvé que ce foncteur est une équivalence de catégories.



1.1.6 Corps de Newton-Okounkov

Les corps de Newton-Okounkov ont été introduits par Kaveh-Khovanskii [75] et indépendamment par Lazarsfeld-Mustata [83] en s'inspirant d'idées d'Okounkov [99]. Le point de départ des travaux d'Okounkov était l'étude de certaines propriétés de log-concavité des multiplicités de représentations irréductibles d'un groupe réductif connexe G dans le G-module de l'algèbre des fonctions sur une variété projective X munie d'une action de G. La preuve d'Okounkov fait intervenir certains polytopes qui coincident dans ce cas avec des polytopes déjà connus sous le nom de polytopes de Gelfand-Tsetlin. Kaveh-Khovanskii [75] et Lazarsfeld-Mustata [83] ont repris et généralisé cette construction à un cadre plus large. Les objets obtenus sont toujours compacts convexes dans un espace vectoriel de dimension finie sur un corps algébriquement clos, mais ne sont plus des polytopes en général. On les appelle corps de Newton-Okounkov ou simplement corps d'Okounkov. Leur construction repose sur les trois objets élémentaires suivants:

- un corps algébriquement clos $\mathbf{k}$.
- une $\mathbf{k}$-algèbre $\mathcal{A}$ munie d'une $\mathbb{N}$-graduation telle que chaque composante homogène de $\mathcal{A}$ est de dimension finie en tant que $\mathbf{k}$-espace vectoriel.
- une valuation de rang rationnel maximal Ψ sur $\mathcal{A}$ à valeurs dans $\mathbb{Z}^N$ pour un certain $N \geqslant 1$.

La présente formulation de la dernière propriété est tirée du séminaire Bourbaki de Sébastien Boucksom [13]; Kaveh-Khovanskii [75] utilisent la terminologie de *valuations avec feuilles de dimension* 1. On renvoie à [13] pour une preuve du fait que ces deux définitions sont essentiellement équivalentes. Cette propriété a pour conséquence cruciale que pour chaque $n \geqslant 0$, on a

$$\dim \mathcal{A}_n = \sharp \Psi(\mathcal{A}_n \backslash \{0\})$$

permettant ainsi d'étudier l'asymptotique de $\dim \mathcal{A}_n$ lorsque $n \to \infty$. Pour toute sous-algèbre graduée $\mathcal{B}$ de $\mathcal{A}$, le corps de Newton-Okounkov de $\mathcal{B}$ est défini par

$$\Delta(\mathcal{B}) := \overline{\mathrm{Conv}\left(\bigcup_{n \geqslant 1} \frac{1}{n} \Psi(\mathcal{B}_n \backslash \{0\})\right)}.$$

La propriété essentielle des corps de Newton-Okounkov est que leur volume est intimement relié au comportement asymptotique de la fonction de Hilbert de $\mathcal{B}$ (voir par exemple [75, Théorème 2.31]). La théorie des corps de Newton-Okounkov s'est révélée être un outil puissant en géométrie algébrique notamment dans les travaux de Boucksom-Chen [14] pour apporter des preuves spectaculaires de certaines conjectures ou résultats qui utilisaient auparavant des techniques plus lourdes. Les corps de Newton-Okounkov ont aussi fait leur apparition en théorie des représentations [74, 45, 46] où ils permettent d'obtenir de nouvelles constructions des *string polytopes* définis par Berenstein-Zelevinsky [9] et Littelmann [86]. Ils sont également d'une grande utilité dans l'étude des dégénérations toriques [12, 35]. La thèse de Lara Bossinger [12] donne un excellent aperçu des différents domaines d'application de ces constructions et de leurs intéractions. Récemment, les corps de Newton-Okounkov ont également été utilisés par Rietsch-Williams [107] dans un contexte différent, plus proche de la combinatoire et de la géométrie des Grassmaniennes. Dans ce cas, les corps de Newton-Okounkov sont issus de certaines algèbres de sections globales de fibrés en droite sur la Grassmannienne $Gr_{n-k}(\mathbb{C}^n)$ (la variété des sous-espaces de codimension k dans $\mathbb{C}^n$). La valuation est définie à partir de la structure d'algèbre amassée sur l'anneau des fonctions sur $Gr_{n-k}(\mathbb{C}^n)$.

1.1.7 Cycles de Mirović-Vilonen et multiplicités équivariantes

Dans cette section, on rappelle les principales notions de la théorie des cycles de Mirković-Vilonen dans les Grassmanniennes affines d'après [92, 6]. Ces éléments seront utiles dans l'étude des multiplicités équivariantes sur $\mathbb{C}[N]$. On renvoie à [114] pour plus de détails sur les Grassmanniennes affines ainsi que sur la correspondance de Satake géométrique.

Correspondance de Satake géométrique et bases MV

Mirković-Vilonen [92] ont découvert une correspondance inattendue entre certaines catégories de faisceaux pervers sur la Grassmannienne affine Gr_G associée à un groupe réductif complexe G et la catégorie des représentations de dimension finie du dual de Langlands $G^\vee$ de G. Ils mirent en évidence un foncteur reliant ces deux catégories et prouvèrent que ce foncteur satisfait plusieurs propriétés remarquables. C'est la *correspondance de Satake géométrique*. En particulier, pour chaque couple (λ, μ) de poids de $G^\vee$ tels que λ est dominant, le sous-espace de poids μ de la représentation simple $V(\lambda)$ de plus haut poids λ de $G^\vee$ peut être interprété comme un espace d'homologie d'une certaine sous-variété de Gr_G dont les composantes irréductibles sont appelées *cycles de Mirković-Vilonen de type λ et de poids μ*. On note $\mathcal{Z}(\lambda)_\mu$ l'ensemble de ces cycles MV. Les classes des cycles MV de $\mathcal{Z}(\lambda)_\mu$ forment alors une base de $V(\lambda)_\mu$ ce qui implique en particulier

$$\dim V(\lambda)_\mu = \sharp \mathcal{Z}(\lambda)_\mu.$$

En considérant la réunion de toutes ces bases pour μ variant dans l'ensemble des poids, on obtient une base de $V(\lambda)$ appelée base MV (supérieure) de $V(\lambda)$. Cette base peut être munie d'une structure de base cristalline dans le sens de Kashiwara [70] (voir [16, 4]). On peut alors concaténer les bases MV des $V(\lambda)$ (où λ décrit l'ensemble des poids dominants) pour former une base de $\mathbb{C}[N]$ appelée base MV de $\mathbb{C}[N]$ (voir [6, Section 6.1] pour plus de détails). Cette base est notée $\{b_Z, Z \in \mathcal{Z}(\infty)\}$ où $\mathcal{Z}(\infty)$ désigne une famille de cycles MV appelés *cycles MV stables*. Les travaux de Kamnitzer [65], Anderson [2] et Baumann-Kamnitzer [5] ont montré que cette base peut être également paramétrée de manière combinatoire en termes de *polytopes MV* et que ces polytopes rendent compte de la structure cristalline de la base MV. On renvoie aussi à [3, 112] pour des généralisations des polytopes MV aux types affines.

Multiplicités équivariantes des bases de $\mathbb{C}[N]$

Dans ce paragraphe, on rappelle certains outils utilisés par Baumann-Kamnitzer-Knutson [6] pour prouver une conjecture de Muthiah [94]. Cette conjecture portait sur la W-équivariance d'une certaine application $V(\lambda) \longrightarrow \mathbb{C}(\alpha_1, \ldots, \alpha_n)$, où $L(\lambda)$ est la représentation simple de plus haut poids λ de $G^\vee$. Cette application est définie par certains outils géométriques appelés *multiplicités équivariantes* introduites par Brion [17]. La preuve de [6] fait intervenir de manière cruciale la correspondance de Satake géométrique ainsi qu'une formule de Knutson [80] pour les mesures de Duistermaat-Heckmann.

La notion de *multiplicité équivariante* d'un schéma projectif fermé fut introduite par Brion [17]. Etant donné un tel schéma X muni d'une action d'un tore T, on note X^T l'ensemble des points fixes pour cette action et on considère l'espace $H_\bullet^T(X)$ d'homologie T-équivariante de X. Les résultats de Brion [18] impliquent que l'ensemble des classes d'homologie équivariantes des points de X^T forme une base de $H_\bullet^T(X)$. On peut donc décomposer la classe d'homologie de X sur cette base:

$$[X] = \sum_{p \in X^T} \epsilon_p^T(X)[\{p\}].$$

Le coefficient $\epsilon_p^T(X)$ est une fonction rationnelle en $\alpha_1, \ldots, \alpha_n$ appelée multiplicité équivariante de X en p. Plus généralement, on peut aussi considérer la multiplicité équivariante en p de n'importe quelle sous-variété fermée T-invariante $Y \subset X$. On la note $\epsilon_p^T(Y)$. Cette fraction rationnelle est nulle si Y ne contient pas p. Lorsque $p \in Y$, il y a un essentiellement une seule situation où l'on peut connaître précisément la forme de $\epsilon_p^T(Y)$: si p est non dégénéré alors on a

$$Y \text{ lisse en } p \quad \Rightarrow \quad \epsilon_p^T(Y) = \frac{1}{\beta_{i_1} \cdots \beta_{i_r}}$$

où $r := \dim(Y)$ et $\beta_{i_1}, \ldots, \beta_{i_r}$ désignent les poids de l'action de T sur $T_xY \subset T_xX$.

Baumann-Kamnitzer-Knutson [6] ont appliqué cette notion de multiplicité équivariante à la théorie des cycles MV. Ainsi avec les notations précédentes, on considère $X := Gr_G$ la Grassmanienne affine associée à un groupe réductif simple G, munie de l'action du tore $T(\mathbb{C})$. L'ensemble des points fixes de cette action est en bijection avec l'ensemble des poids de $\mathfrak{g}$ et ses éléments dont notés $L_\mu, \mu \in P$. Pour chaque $\mu \in P$, le point L_μ appartient à tous les cycles MV de poids μ, qui jouent le rôle de Y puisqu'ils sont invariants sous l'action de T. L'un des principaux résultats de Baumann-Kamnitzer-Knutson [6] est d'interpréter la multiplicité équivariante d'un cycle MV Z comme l'image de l'élément correspondant b_Z de la base MV de $\mathbb{C}[N]$ par un morphisme d'algèbres

$$\overline{D} : \mathbb{C}[N] \longrightarrow \mathbb{Q}(\alpha_1, \ldots, \alpha_n).$$

En particulier, on obtient en utilisant les résultats de Brion [17] évoqués plus haut que pour tout Z est un cycle MV (stable) de poids $-\mu$, si Z est lisse en L_μ alors

$$\overline{D}(b_Z) = \frac{1}{\beta_1 \cdots \beta_m} \tag{1.5}$$

où $\beta_1, \ldots, \beta_m \in \Phi_+$ sont les poids de l'action de $T^\vee$ sur l'espace tangent à Z en L_μ. Ce morphisme $\overline{D}$ constitue un outil utile pour comparer différentes bases de $\mathbb{C}[N]$. Ainsi Dranowski-Kamnitzer-Morton-Ferguson [6] montrent que la base MV et la base semicanonique duale ne sont pas les mêmes en considérant certains éléments de ces bases satisfaisant certaines conditions de compatibilité (cf. [6, Definition 12.1]) mais sur lesquels $\overline{D}$ prend néanmoins des valeurs différentes.

Dans cette thèse, on s'intéressera tout particulièrement aux images par $\overline{D}$ des éléments de la base canonique duale. Comme on l'a vu dans la section 1.1.4, un tel élément peut être vu comme

la classe d'isomorphisme d'une représentation irréductible de l'algèbre de Hecke carquois associée à $\mathfrak{g}$. Un tel module M peut se décomposer comme la somme de ses sous-espaces de poids

$$M = \bigoplus_{w \in \mathrm{Seq}(\mathrm{wt}(M))} e(w) \cdot M.$$

L'évaluation de $\overline{D}$ en $[M] \in \mathbb{C}[N]$ est alors donnée par

$$\bar{D}(M) = \sum_{w=(i_1,\ldots,i_r)} \dim(e(w) \cdot M) \frac{1}{\alpha_{i_1}(\alpha_{i_1} + \alpha_{i_2}) \cdots (\alpha_{i_1} + \cdots \alpha_{i_N})}. \tag{1.6}$$

Dans la troisième partie de cette thèse, on utilisera cette expression pour calculer les multiplicités équivariantes de divers éléments de la base canonique duale. On s'intéressera notamment aux classes des modules (fortement) homogènes, ainsi qu'aux mineurs drapeaux de $\mathbb{C}[N]$. On discutera également des liens possibles entre ces deux familles déléments de $\mathbb{C}[N]$.

1.2 Résultats de la thèse

Cette section rassemble les différents résultats obtenus dans cette thèse. En premier lieu, on s'intéresse aux partitions de racines (ou mots dominants) qui paramétrisent les objets simples de $R - gmod$ comme on l'a vu dans la section 1.1.4. On montre que l'ensemble de ces partitions de racines peut être muni d'une structure de monoïde (abélien) dont la loi est compatible avec le produit de convolution dans $R - gmod$ (Théorème 1). Ceci permet de formuler une conjecture portant sur l'existence de certaines compatibilités entre certains ordres partiels provenant de la théorie des algèbres amassées d'une part et de la théorie des représentations d'autre part (Conjecture A). On prouve cette conjecture en type A_n (Théorèmes 2 et 3), puis en tout type fini simplement lacé (Théorème 3.2.7) en utilisant certains résultats récents de Kashiwara-Kim [72]. Ces compatibilités entre ordres partiels impliquent certaines relations remarquables entre g-vecteurs des variables d'amas de $\mathbb{C}[N]$ et partitions de racines des objets simples correspondant dans $R - gmod$. Des résultats similaires ont été obtenus par Kashiwara-Kim [72] dans un cadre plus général. On présente ensuite dans la Section 1.2.3 une construction de corps de Newton-Okounkov naturellement associés aux anneaux de coordonnée (quantiques) et leurs cellules unipotentes. On prouve plusieurs résultats qui décrivent des propriétés géométriques et combinatoires de ces corps (Proposition 1.2.3 et Théorèmes 5 et 6). On énonce également une conjecture reliant deux types de combinatoires a priori différents: la théorie des algèbres amassées d'une part et celle des éléments complètement commutatifs des groupes de Weyl d'autre part. Les expressions rationnelles intervenant notamment dans le Théorème 6 sont similaires à certaines formules intervenant dans un travail récent de Baumann-Kamnitzer-Knutson [6], qui porte sur la géométrie des Grassmanniennes affines et des cycles de Mirković-Vilonen. Baumann-Kamnitzer-Knutson [6] utilisent certains outils géométriques appelés multiplicités équivariantes introduits par Brion [17] et les relient à l'évaluation sur les éléments de la base MV d'un certain morphisme d'algèbres $\overline{D}$ défini sur $\mathbb{C}[N]$. A la fin de cette thèse, on se propose d'étudier les images par $\overline{D}$ de certains éléments de la base canonique duale de $\mathbb{C}[N]$ via la catégorification de cette dernière par les objets simples de $R - gmod$. On montre que lorsque $\mathfrak{g}$ est de type $A_n, n \geqslant 1$ et D_4, l'évaluation de $\overline{D}$ sur les mineurs drapeaux de $\mathbb{C}[N]$ prend une forme remarquable (Théorème 7). On met aussi en évidence certaines identités reliant les images par $\overline{D}$ des mineurs drapeaux d'une même graine standard, et on montre qu'en tout type ADE ces relations sont préservées par mutation d'une graine standard à une autre (Théorème 8). On formule également certaines conjectures qui suggèrent que ces valeurs distinguées prises par $\overline{D}$ pourrait constituer une caractérisation des monômes d'amas des graines standard (Conjectures C et D). On conclut par une possible interprétation géométrique de cette caractérisation en utilisant

la précédente construction de corps de Newton-Okounkov. Enfin on présente en annexe quelques programmes effectués à l'aide du logiciel de calcul formel SAGE qui étayent les observations et conjectures portant sur les multiplicités équivariantes des modules déterminantaux en types A_n et D_4.

1.2.1 Partitions de racines et g-vecteurs

On considère à présent une algèbre de Lie $\mathfrak{g}$ simple de type fini et on fixe un ensemble I indexant les sommets du diagramme de Dynkin de $\mathfrak{g}$. Soit x une variable d'amas quelconque dans $\mathcal{A}_q(\mathfrak{n})$ et soit M l'unique (à isomorphisme et décalage près) objet simple de $R - gmod$ tel que $[M] = x$. Considérons la décomposition amassée de x par rapport à une graine $\mathcal{S} = ((x_1, \ldots, x_N), Q)$ arbitraire fixée; on sait que parmi les monômes de Laurent en les x_i qui interviennent dans cette décomposition, il y en a un et un seul qui est maximal pour l'ordre de dominance $\preccurlyeq_{\mathcal{S}}$, donné par le g-vecteur $g_{\mathcal{S}}(x)$ de x par rapport à $\mathcal{S}$. Par ailleurs, on dispose d'une décomposition de M en un nombre fini de sous-espaces de poids

$$M = \bigoplus_{w \in \mathcal{M}} e(w) \cdot M.$$

L'ensemble $\mathcal{M}$ étant totalement ordonné, on peut considérer $\psi(M)$ le plus haut poids de M dans le sens suivant

$$\psi(M) := \max\left(w, e(w) \cdot M \neq 0\right).$$

Il est naturel de se demander si on peut relier $\psi(M)$ au g-vecteur de x par rapport à $\mathcal{S}$. Le diagramme suivant résume la situation: la commutativité du carré supérieur consiste exactement à demander que $\mathcal{C}$ soit une catégorification monoïdale de $\mathcal{A}$ (chaque ligne correspondant à l'une des deux propriétés requises). La partie gauche du diagramme n'utilise que des éléments de théorie des algèbres amassées; les flèches verticales sont issues des résultats de [43, 106, 32, 55]. A contrario, la partie droite ne dépend que de la structure de la catégorie $\mathcal{C}$ et les flèches verticales reposent seulement sur des notions classiques de théorie des représentations.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\simeq"] & & K_0(\mathcal{C}) \\
\text{monômes d'amas} \arrow[u, hook] \arrow[rr, hook] \arrow[d, squiggly] & & \text{objets simples} \arrow[u, hook] \arrow[d, squiggly] \\
\text{ordre de dominance } \preccurlyeq_{\mathcal{S}} \arrow[rr, leftrightarrow, dashed, "?"] \arrow[d, Rightarrow] & & \text{ordre naturel } \leqslant \arrow[d, Rightarrow] \\
g\text{-vecteurs} \arrow[rr, leftrightarrow, dashed, "?"] & & \text{paramètres d'objets simples}
\end{tikzcd}$$

On peut déjà noter que $\psi(M)$ est indépendant de $\mathcal{S}$, ce qui n'est pas le cas du g-vecteur $g_{\mathcal{S}}(x)$. On a donc formalisé dans cette thèse une propriété au niveau de la graine $\mathcal{S}$ qui garantit l'existence d'une telle relation: c'est la notion de *graine compatible* ([25, Définition 4.7]). Cette définition dépend d'un choix préalable de paramétrisation des objets simples dans la catégorie étudiée, c'est-à-dire une bijection ensembliste

$$\psi : \{[S], S \text{ objet simple de } \mathcal{C}\} \longrightarrow (\mathbf{M}, \odot, \leqslant)$$

de l'ensemble des classes d'isomorphisme des objets simples de $\mathcal{C}$ vers un monoïde $(\mathbf{M}, \odot)$ muni d'un ordre partiel $\leqslant$. Une graine $\mathcal{S}$ de $\mathcal{A}$ est dite compatible si la paramétrisation choisie ψ induit

une compatibilité entre l'ordre $\leqslant$ et l'ordre de dominance pour $\mathcal{S}$ au sens de F. Qin [106]. En particulier on a

$$[M] \preccurlyeq_{\mathcal{S}} [N] \text{ dans } \mathcal{A} \Rightarrow \psi([M]) \leqslant \psi([N]) \text{ dans } \mathbf{M}.$$

Dans cette thèse, on propose la conjecture suivante:

Conjecture A ([25, Conjecture 4.10])**.** *Soit $\mathcal{A}$ une algèbre amassée. On suppose que $\mathcal{A}$ admet une catégorification monoïdale artinienne $\mathcal{C}$. On suppose que les objets simples de $\mathcal{C}$ sont paramétrés par un monoïde partiellement ordonné satisfaisant un certain couple d'hypothèses A et B (voir Section 2.3.2 ci-dessous). Alors il existe une graine compatible dans $\mathcal{A}$.*

Le premier exemple de graine compatible apparaît dans les travaux de Hernandez-Leclerc [56] dans le cadre de catégorifications d'algèbres amassées de types finis A_n et D_4 par certaines catégories de représentations de dimension finie d'algèbres affines quantiques. Dans ce cas, le monoïde qui décrit les objets simples (à isomorphisme près) est l'ensemble des *monômes dominants* introduits par Frenkel-Reshetikhin [44] (la loi $\odot$ est alors simplement la multiplication usuelle de monômes) et l'ordre partiel $\leqslant$ est l'ordre de Nakajima. Les relations entre g-vecteurs et paramètres d'objets simples sont également données de manière explicite ([56, Corollaire 7.4]). Dans un travail ultérieur de Hernandez-Leclerc [59] ces relations entre g-vecteurs et polynômes de Drinfeld jouent un rôle important et ont des conséquences de nature géométrique via les résultats de Plamondon [102, 103].

Ici on s'intéresse à la catégorie $R-gmod$ pour une algèbre de Lie $\mathfrak{g}$ simple de type fini. On note I l'ensemble des sommets du diagramme de Dynkin de $\mathfrak{g}$. Par les résultats de Kleshchev-Ram [79], on dispose d'un grand nombre de choix de paramétrisations ψ des objets simples de $R-gmod$, chacune étant entièrement déterminée par le choix d'un ordre total arbitraire $<$ sur I. L'ensemble des classes d'isomorphisme d'objets simples dans $R-gmod$ est en bijection avec l'ensemble $\mathbf{M}$ des mots dominants donné par l'Equation (1.3).

Dans cette thèse, on commence par montrer que l'on peut munir $\mathbf{M}$ d'une structure naturelle de monoïde: pour tous $\mu, \mu' \in \mathbf{M}$, on note $\mu \odot \mu'$ l'unique élément de $\mathbf{M}$ tel que la décomposition du produit $[L(\mu)] \cdot [L(\mu')]$ sur la base des classes d'objets simples dans $R-gmod$ s'écrit

$$[L(\mu)] \cdot [L(\mu')] = [L(\mu \odot \mu')] + \sum_{\nu < \mu \odot \mu'} a_\nu [L(\nu)]$$

où (a_ν) est une famille presque nulle d'entiers naturels.

Notons N le nombre de racines positives de $\mathfrak{g}$ et $\{\mathbf{j}_1 < \cdots < \mathbf{j}_N\}$ les éléments de $\mathcal{GL}$ ordonnés dans l'ordre croissant. Le premier résultat de cette thèse est le suivant. Il est prouvé en type A_n dans [25] et dans sa version générale ci-dessous dans [24].

Théorème 1 ([25, Théorème 5.5],[24, Proposition 2.4])**.** *Soit $\mathfrak{g}$ une algèbre de Lie simple de type classique fini et soit $<$ un ordre total arbitraire sur I. Alors la loi $\odot$ confère à $\mathbf{M}$ une structure de monoide (abélien) et on a un isomorphisme de monoides*

$$\begin{array}{ccc} (\mathbf{M}, \odot) & \longrightarrow & (\mathbb{Z}_{\geqslant 0}^N, +) \\ {\mathbf{j}_N}^{c_N} \cdots {\mathbf{j}_1}^{c_1} & \longmapsto & (c_1, \ldots, c_N). \end{array}$$

Dans le cas où $\mathfrak{g}$ est type A_n, on met en évidence une graine compatible dans l'algèbre amassée $\mathcal{A}_q(\mathfrak{n})$. On fait le choix de l'ordre naturel $1 < \cdots < n$ sur l'ensemble des sommets du diagramme de Dynkin de $\mathfrak{g}$. La première étape consiste à calculer explicitement les mots dominants correspondant aux modules simples d'une graine $\mathcal{S}^{\mathbf{i}}$ issue de la construction de Geiss-Leclerc-Schröer.

Théorème 2 (cf. Théorème 2.5.1)**.** *Soit $\mathfrak{g}$ une algèbre de Lie de type A_n et soit $\mathcal{S}_0^n$ la graine de $\mathcal{A}_q(\mathfrak{n})$ correspondant à l'expression réduite*

$$(1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1)$$

de w_0 selon la construction de [52, 69]. Alors les modules simples de $R-gmod$ correspondant aux variables d'amas de $\mathcal{S}_0^n$ peuvent être décrits en termes de mots dominants comme suit:

$$\begin{array}{cccccc}
[L(1)] & & & & & \\
[L(12)] & [L((2)(1))] & & & & \\
[L(123)] & [L((23)(12))] & [L((3)(2)(1))] & & & \\
\vdots & \vdots & & & & \\
[L(1\dots k)] & [L((2\dots k)(1\dots k-1))] & \cdots & [L((k)\cdots(1))] & & \\
\vdots & \vdots & & & & \\
[L(1\dots n)] & [L((2\dots n)(1\dots n-1))] & \cdots & \cdots & [L((n)\cdots(1))]. &
\end{array}$$

Sur la dernière ligne figurent les modules correspondant aux variables gelées.

Ceci permet de vérifier que cette graine $\mathcal{S}_0^n$ satisfait la propriété de compatibilité désirée.

Théorème 3 (cf. Théorème 2.5.2)**.** *La graine $\mathcal{S}_0^n$ est compatible dans le sens de la Définition 2.3.7.*

En particulier, ceci prouve la Conjecture A pour la catégorie $R-gmod$ associée à une algèbre de Lie $\mathfrak{g}$ de type A_n.

1.2.2 g-vecteurs pour les objets simples de $\mathcal{C}_w$

Peu de temps après la mise en ligne de [25], Kashiwara-Kim [72] ont établi des résultats de nature similaire mais avec une plus grande généralité. Pour tout élément $w \in W$ et pour chaque graine $\mathcal{S}$ de $\mathcal{A}_q(\mathfrak{n}(w))$, Kashiwara-Kim définissent deux applications linéaires par morceaux

$$g_{\mathcal{S}}^R, g_{\mathcal{S}}^L : \{\text{simples de } \mathcal{C}_w\} \longrightarrow \mathbb{Z}^{l(w)}$$

de manière intrinsèque à la catégorie $\mathcal{C}_w$ (et donc sans faire intervenir de paramétrisation des objets simples). Kashiwara-Kim prouvent que ces applications satisfont les propriétés suivantes:

- Si $M_1, \dots, M_{l(w)}$ désignent les objets simples de $\mathcal{C}_w$ correspondant aux variables d'amas de $\mathcal{S}$, alors $g_{\mathcal{S}}^R(M_i)$ et $g_{\mathcal{S}}^L(M_i)$ coincident avec le $i$ème vecteur de la base standard de $\mathbb{Z}^{l(w)}$ pour tout $1 \leqslant i \leqslant l(w)$.
- Si $\mathcal{S}'$ est obtenue à partir de $\mathcal{S}$ par une mutation dans une direction $k \in J_{ex}$, alors $g_{\mathcal{S}'}^R$ (resp. $g_{\mathcal{S}'}^L$) est relié à $g_{\mathcal{S}}^R$ (resp. $g_{\mathcal{S}}^L$) par les relations d'échange tropicales suivantes:

  $$g_{\mathcal{S}'}^R(M) = \phi_{\mathcal{S},k}^R(g_{\mathcal{S}}^R(M)) \quad \text{et} \quad g_{\mathcal{S}'}^L(M) = \phi_{\mathcal{S},k}^L(g_{\mathcal{S}}^L(M))$$

  où M est un objet simple arbitraire de $\mathcal{C}_w$ et $\phi_{\mathcal{S},k}^R, \phi_{\mathcal{S},k}^L$ sont les applications linéaires par morceaux de $\mathbb{Z}^{l(w)}$ dans lui-même définies par

  $$\phi_{\mathcal{S},k}^R(g) := g' \quad \text{avec} \quad g'_j = \begin{cases} -g_k & \text{si } j = k, \\ g_j + [b_{ki}]_+ g_k & \text{si } j \neq k \text{ et } g_k \geqslant 0, \\ g_j + [b_{ik}]_+ g_k & \text{si } j \neq k \text{ et } g_k \leqslant 0, \end{cases}$$

  et

  $$\phi_{\mathcal{S},k}^L(g) := g'' \quad \text{avec} \quad g''_j = \begin{cases} -g_k & \text{si } j = k, \\ g_j + [b_{ki}]_+ g_k & \text{si } j \neq k \text{ et } g_k \leqslant 0, \\ g_j + [b_{ik}]_+ g_k & \text{si } j \neq k \text{ et } g_k \geqslant 0. \end{cases}$$

En particulier, si M est un objet simple de $\mathcal{C}_w$ dont la classe d'isomorphisme est une variable d'amas dans $K_0(\mathcal{C}_w) \simeq \mathcal{A}_q(\mathfrak{n}(w))$, alors $g^L_{\mathcal{S}}(M)$ est exactement le g-vecteur (au sens de Fomin-Zelevinsky) de cette variable d'amas par rapport à la graine $\mathcal{S}$. Kashiwara-Kim donnent également une formule explicite ([72, Proposition 3.14]) reliant $g^R_{\mathcal{S}}(M)$ à la décomposition cuspidale de M dans le cas où la graine de référence $\mathcal{S}$ est une des graines standard i.e. $\mathcal{S}$ est de la forme $\mathcal{S}^{\mathbf{i}}, \mathbf{i} \in \mathrm{Red}(w)$ (cf. Section 1.1.5). Dans la situation particulière considérée dans [25], i.e. $\mathfrak{g}$ de type A_n, $w = w_0$ (auquel cas $\mathcal{C}_w = R-gmod$), et $<$ est l'ordre naturel $1 < 2 < \cdots < n$, la formule de Kashiwara-Kim redonne exactement les relations obtenues dans [25].

Dans cette thèse, on utilise ces résultats de Kashiwara-Kim pour établir une généralisation du Théorème 2 valable pour toute algèbre de Lie $\mathfrak{g}$ de type A_n, D_n ou E_n, tout ordre total arbitraire $<$ sur I, et pour chaque sous-catégorie $\mathcal{C}_w, w \in W$. Comme on l'a vu dans la section 1.1.4, l'ordre $<$ induit un ordre (lexicographique) sur $\mathcal{GL}$ qui se transporte en un ordre convexe sur l'ensemble des racines positives Φ_+ associé à $\mathfrak{g}$. Etant donné $w \in W$ de longueur N, cet ordre se restreint en un ordre convexe sur Φ^w_+. Notons $\mathbf{i}_<$ l'epxression réduite de w correspondante. On peut alors décrire explicitement les modules déterminantaux correspondant aux variables d'amas de la graine $\mathcal{S}^{\mathbf{i}_<}$ en termes de partitions de racines.

Théorème 4 (cf. Theorem 2.5.1)**.** *Soit $w \in W$ de longueur N et soient $(x_1, \ldots, x_N)$ les variables d'amas de la graine $\mathcal{S}^{\mathbf{i}_<}$ dans $\mathcal{A}_q(\mathfrak{n}(w))$. Pour chaque $k \in J$, on note $\mu_k \in \mathbf{M}$ l'unique mot dominant tel que $x_k = [L(\mu_k)]$ et on écrit la factorisation canonique de μ_k sous la forme*

$$\mu_k = (\mathbf{i}_N)^{c_N} \cdots (\mathbf{i}_1)^{c_1}.$$

Alors le N-uplet $(c_1, \ldots, c_N)$ est donné par

$$c_j = \begin{cases} 1 & \text{si } j \leqslant k \text{ et } i_j = i_k \\ 0 & \text{sinon.} \end{cases}$$

1.2.3 Corps d'Okounkov et représentations d'algèbres de Hecke carquois

Les résultats de la section précédente montrent que l'on peut interpréter les paramètres des objets simples de $R - gmod$ (et plus généralement de $\mathcal{C}_w$ pour tout $w \in W$) comme des généralisations de g-vecteurs. Or, la théorie des algèbres amassées fournit plusieurs constructions géométriques intéressantes qui permettent de comprendre la structure d'une algèbre amassée à partir de la connaissance des g-vecteurs des variables d'amas. En premier lieu, la notion de *complexe d'amas* a été introduite dès le début des années 2000 par Fomin-Zelevinsky [42]: il s'agit d'un modèle polytopal purement combinatoire d'une algèbre amassée, où les sommets correspondent aux variables d'amas et les facettes aux graines. Hohlweg, Padrol, Palu, Pilaud, Plamondon, et Stella [61, 100] construisent des polytopes de nature un peu différente, en réalisant certains éventails de g-vecteurs comme éventails normaux de polytopes. Les polytopes ainsi obtenus peuvent être construits pour n'importe quelle algèbre amassée de type fini (cyclique ou acyclique) et sont appelés *associaèdres généralisés.*

Dans la situation qui nous intéresse, la structure amassée sur $\mathcal{A}_q(\mathfrak{n})$ ou $\mathcal{A}_q(\mathfrak{n}(w))$ n'est pas de type fini en général et ce type de construction ne peut donc pas s'appliquer directement a priori. Néanmoins, la théorie des représentations des algèbres de Hecke carquois fournit les outils nécessaires au contournement de cette obstruction. On considère une algèbre de Lie $\mathfrak{g}$ simple de type fini; les représentations des algèbres de Hecke carquois correspondantes ont alors la propriété remarquable d'être totalement ordonnées comme on l'a vu dans la Section 1.1.4. En particulier, une fois fixé un ordre total $<$ sur I, on dispose d'un ordre total naturel sur l'ensemble des variables d'amas de chaque graine de $\mathcal{A}_q(\mathfrak{n})$ (ou $\mathcal{A}_q(\mathfrak{n}(w))$). En s'inspirant des travaux de Rietsch-Williams [107], ceci donne une motivation naturelle pour la construction de corps d'Okounkov.

Dans ce qui suit on prendra toujours comme corps de base $\mathbf{k} = \mathbb{C}$. Comme on l'a vu plus haut, les anneaux de coordonnée unipotents quantiques $\mathcal{A}_q(\mathfrak{n}(w))$ admettent des catégorifications par certaines sous-catégories $\mathcal{C}_w$ de $R-gmod$. Par construction (voir [69]) ces catégories admettent des décompositions

$$R - gmod = \bigoplus_{\beta \in Q_+} R(\beta) - gmod \qquad \text{et} \qquad \mathcal{C}_w = \bigoplus_{\beta \in Q_+} \mathcal{C}_w \cap R(\beta) - gmod.$$

Pour chaque $w \in W$, on peut oublier la graduation dans $\mathcal{C}_w$ (ce qui revient à passer à la limite $q \to 1$ dans $\mathcal{A}_q(\mathfrak{n}(w))$) et l'isomorphisme d'algèbres $K_0(\mathcal{C}_w) \simeq \mathbb{C}[N(w)]$ permet de munir $\mathbb{C}[N(w)]$ d'une structure de $\mathbb{C}$-algèbre graduée en posant pour chaque objet simple M de $\mathcal{C}_w$:

$$[M] \in K_0(\mathcal{C}_w) \longmapsto \mathrm{ht}(\mathrm{wt}(M))$$

où $\mathrm{wt}(M)$ désigne l'unique $\beta \in Q_+$ tel que $M \in R(\beta) - gmod$. Enfin, on montre que les paramétrisations en termes de mots dominants des objets simples de $R - gmod$ (cf. Section 1.1.4) donnent naturellement lieu à des valuations de rang rationnel maximal

$$\Psi_w : \mathbb{C}[N(w)] \longrightarrow \mathbb{Z}^{l(w)}.$$

L'argument principal est donné par le Théorème 1. On peut à présent utiliser ces outils afin de construire des corps de Newton-Okounkov: ceci permet d'associer un corps convexe compact à n'importe quelle sous-algèbre graduée de $\mathbb{C}[N(w)]$. On s'intéressera tout particulièrement à deux types de telles sous-algèbres: l'algèbre $\mathbb{C}[N(w)]$ elle-même, et les sous-algèbres libres engendrées par les variables d'amas d'un même amas dans $\mathbb{C}[N(w)]$. Le corps de Newton-Okounkov $\Delta(\mathcal{A}_q(\mathfrak{n}(w)))$ est un simplexe de codimension 1 dans $\mathbb{R}^N$. Le corps $\Delta_{\mathcal{S}}$ associé à une graine $\mathcal{S} = ((x_1, \ldots, x_N), B)$ est également un simplexe, inclus dans $\Delta(\mathcal{A}_q(\mathfrak{n}(w)))$. Lorsque $\mathfrak{g}$ est de type simplement lacé, les résultats de catégorification monoïdale de Kang-Kashiwara-Kim-Oh [69] sont valables et on peut alors montrer les propriétés suivantes des simplexes $\Delta_{\mathcal{S}}$:

Proposition ([24, Proposition 5.3]). *Les points rationnels de $\Delta_{\mathcal{S}}$ correspondent aux monômes d'amas monoïdaux pour $\mathcal{S}$ dans le sens suivant:*

$$M \text{ est un monôme d'amas monoïdal pour } \mathcal{S} \Leftrightarrow \frac{1}{|M|}\Psi([M]) \in \Delta_{\mathcal{S}}.$$

De plus, tout point rationnel de $\Delta_{\mathcal{S}}$ est de la forme $\frac{1}{|[M]|}\Psi([M])$ pour un monôme d'amas monoïdal M de $\mathcal{C}_w$.

Théorème 5 ([24, Théorème 5.12]). *L'éventail normal du simplexe $\Delta_{\mathcal{S}}$ peut être relié par une correspondance explicite à l'ordre de dominance pour $\mathcal{S}$. Plus précisément, si on note $\overrightarrow{\mathcal{N}^{\mathcal{S}}}$ le cône linéaire tel que pour tout objet simple M de $\mathcal{C}_w$ on ait*

$$\Psi([M]) + \overrightarrow{\mathcal{N}^{\mathcal{S}}} = \{\Psi([N]), [N] \preccurlyeq_{\mathcal{S}} [M]\}$$

alors il existe une unique transformation universelle $T \in \mathcal{M}_N(\mathbb{Q})$ (par universelle, on entend indépendante de la graine), telle que pour chaque graine $\mathcal{S}$, le cône $T\overrightarrow{\mathcal{N}^{\mathcal{S}}}$ est une face de l'éventail normal de $\Delta_{\mathcal{S}}$.

Cet énoncé donne en particulier une interprétation géométrique de certains outils techniques introduits de manière algébrique dans [25, Section 4.2]. Enfin, on peut également s'intéresser au cas où l'algèbre amassée $\mathcal{A}_q(\mathfrak{n}(w))$ est de type fini (i.e. il n'y a qu'un nombre fini de graines distinctes). Les simplexes $\Delta_{\mathcal{S}}$ recouvrent alors $\Delta(\mathcal{A}_q(\mathfrak{n}(w)))$. Pour chaque graine $\mathcal{S}$, notons $\beta_1^{\mathcal{S}}, \ldots, \beta_N^{\mathcal{S}}$ les poids des objets simples de $\mathcal{C}_w$ correspondant aux variables d'amas de $\mathcal{S}$. On montre alors la formule suivante:

Théorème 6 (cf. Théorème 3.5.3)**.** *Supposons que $w \in W$ soit tel que $\mathcal{A}_q(\mathfrak{n}(w))$ est de type amassé fini. Alors on a*

$$\prod_{\beta \in \Phi_+^w} \frac{1}{\beta} = \sum_{\mathcal{S}} \prod_{1 \leqslant j \leqslant N} \frac{1}{\beta_j^{\mathcal{S}}}. \tag{1.7}$$

Nakada [95], a prouvé une formule appelée *formule des équerres colorées* qui prend une forme très similaire à celle de l'équation (1.7). Néanmoins, la combinatoire utilisée dans [95] est a priori très différente de la théorie des algèbres amassées. Ces formules sont valables sous une certaine condition portant sur w (w est dominant minuscule selon la terminologie de Stembridge [111]). Dans cette thèse, on propose la conjecture suivante:

Conjecture B ([24, Conjecture 6.8])**.** *Si $w \in W$ est dominant minuscule, alors l'algèbre amassée $\mathcal{A}_q(\mathfrak{n}(w))$ est de type fini.*

Cette conjecture est motivée par l'étude de plusieurs exemples, dont certains sont détaillés dans la Section 3.5. L'idée est que, bien que de nature différente de la formule des équerres colorées de Nakada [95], l'équation (1.7) ci-dessus serait valable dans une plus grande généralité que celle de Nakada. Par exemple, si $w = s_1 s_2 s_1$ est le plus grand élément du groupe de Weyl de type A_2, l'équation (1.7) s'écrit

$$\frac{1}{\alpha_1(\alpha_1+\alpha_2)\alpha_2} = \frac{1}{\alpha_1(\alpha_1+\alpha_2)^2} + \frac{1}{\alpha_2(\alpha_1+\alpha_2)^2}$$

alors que la formule de Nakada ne s'applique pas puisque w n'est pas dominant minuscule.

1.2.4 Multiplicités équivariantes des mineurs drapeaux en type simplement lacé

La formule des équerres colorée peut en fait être interprétée naturellement du point de vue de la théorie des représentations grâce à la notion de *multiplicité équivariante* introduite par Brion [17]. Baumann-Kamnitzer-Knutson [6] ont récemment défini un morphisme d'algèbres $\overline{D}$ défini sur $\mathbb{C}[N]$ et qui est étroitement relié aux multiplicités équivariantes des cycles de Mirković-Vilonen via la correspondance de Satake géométrique. La dernière partie de cette thèse porte sur l'étude des valeurs prises par $\overline{D}$ sur certains éléments de la base canonique duale. On montre qu'en types $A_n, n \geqslant 1$ et D_4, l'évaluation de $\overline{D}$ sur les mineurs drapeaux de $\mathbb{C}[N]$ prend une forme remarquable, analogue aux valeurs prises sur les classes d'isomorphisme des modules fortement homogènes des algèbres de Hecke carquois. On met également en évidence certaines identités reliant les images par $\overline{D}$ des mineurs drapeaux appartenant à une même graine standard. On montre qu'en tout type ADE, ces relations sont préservées par mutation d'une graine standard à une autre.

L'équation (1.6) permet de reformuler la formule des équerres colorée de Nakada comme suit: pour tout $w \in \mathcal{M}in^+$, l'évaluation de $\overline{D}$ sur la classe d'isomorphisme du module (fortement) homogène associé à w par la construction de Kleshchev-Ram [78] prend la forme

$$\overline{D}([S(w)]) = \frac{1}{\prod_{\beta \in \Phi_+^w} \beta}. \tag{1.8}$$

Ceci met en évidence un parallélisme remarquable entre des énoncés de nature géométrique portant sur la lissité de certains cycles MV et d'autres de nature algébrique portant sur l'homogénéité (forte) de certaines représentations d'algèbres de Hecke carquois. On peut d'ailleurs d'attendre à ce que les éléments de la base MV associés à des cycles MV lisses coïncident en fait avec les classes d'isomorphismes des modules fortement homogènes de $R-gmod$. Par ailleurs, l'étude de plusieurs

exemples (comme les types A_3 et D_4) suggère l'existence de certaines coïncidences entre les modules fortement homogènes premiers de $R-gmod$ et les modules déterminantaux qui catégorifient les mineurs drapeaux, et donne donc une motivation naturelle à l'étude des images de ces derniers par $\overline{D}$. On propose la Conjecture suivante, qui suggère que $\overline{D}$ prend des valeurs remarquables de la même forme que (1.5) et (1.8) sur tous les mineurs drapeaux de $\mathbb{C}[N]$, bien que les objets de $R-gmod$ associés ne soient pas nécessairement homogènes. Ceci suggère également d'étudier la lissité éventuelle des cycles MV correspondant aux mineurs drapeaux.

Conjecture C. *Soit $\mathfrak{g}$ une algèbre de Lie de type fini simplement lacé et soit x un mineur drapeau de $\mathbb{C}[N]$. Alors l'évaluation de $\overline{D}$ en x prend la forme*

$$\overline{D}(x) = \frac{1}{\prod_{\beta\in\Phi_+} \beta^{n_{M,\beta}}}$$

où $(n_\beta)_{\beta\in\Phi_+}$ est une famille d'entiers positifs indexée par l'ensemble des racines positives de $\mathfrak{g}$.

Le but de la dernière partie de cette thèse est de prouver le résultat suivant:

Théorème 7. *On suppose que $\mathfrak{g}$est de type $A_n, n \geqslant 1$ ou D_4. Alors la Conjecture C est vraie. De plus, pour chaque graine standard $\mathcal{S}^{\mathbf{i}} = ((x_1,\dots,x_N), Q^{\mathbf{i}})$ de $\mathbb{C}[N]$, les polynômes $P_j := \left(\overline{D}(x_j)\right)^{-1}$ vérifient les relations suivantes:*

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i})\\ i_l\cdot i_j=-1}} P_l.$$

La stratégie de la preuve peut se résumer comme suit: on sait que les graines standard de $\mathbb{C}[N]$ sont reliées entre elles par certaines mutations qui correspondent à des changements d'expression réduite de w_0. Par conséquent, on commence par montrer que l'énoncé désiré est préservé par ces mutations, avant de vérifier qu'il est effectivement vérifié pour une graine standard bien choisie.

Propagation par mutation. On considère deux graines standard $\mathcal{S}^{\mathbf{i}}$ et $\mathcal{S}^{\mathbf{i}'}$ reliées par une mutation dans la direction k. On note $x_1,\dots,x_N$ les variables d'amas de $\mathcal{S}^{\mathbf{i}}$ et x'_k la nouvelle variable produite par la mutation. On suppose que $\overline{D}(x_j)$ est de la forme $1/P_j$ pour chaque $1 \leqslant j \leqslant N$ et on veut montrer que $\overline{D}(x'_k)$ est de la forme $1/P'_k$ (où P'_k est un produit de racines positives). Le premier résultat de cette partie consiste à mettre en évidence certaines identités polynomiales reliant les P_j qui sont entièrement déterminées par $\mathbf{i}$ et qui impliquent que $\overline{D}(x'_k)$ est de la forme $1/P'_k$, ainsi qu'à prouver que ces relations se propagent par mutation, i.e. les polynômes $P_1,\dots,P_{k-1},P'_k,P_{k+1},\dots,P_N$ satisfont les relations analogues déterminées par $\mathbf{i}'$.

Si P est un produit de racines positives et β est une racine positive, on note $(\beta;P)$ la multiplicité de β dans P.

Théorème 8. *Soit $\mathfrak{g}$ une algèbre de Lie de type fini simplement lacé et soient $\mathbf{i}$ et $\mathbf{i}'$ deux expressions réduites de w_0. On suppose que les variables d'amas $x_1,\dots,x_N$ de la graine standard $\mathcal{S}^{\mathbf{i}}$ vérifient les propriétés suivantes:*

(A) Pour chaque $1 \leqslant j \leqslant N$, la fraction rationnelle $\overline{D}(x_j)$ prend la forme $1/P_j$ où P_j est un produit de racines positives.

(B) Pour chaque $1 \leqslant j \leqslant N$, on a:

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i})\\ i_l\cdot i_j=-1}} P_l.$$

(C) Pour chaque $j \in J_{ex}$ et chaque $1 \leqslant i \leqslant N$, on a $(\beta_i;P_j) - (\beta_i;P_{j_+(\mathbf{i})}) \leqslant 1$.

Alors les variables d'amas de la graine standard $\mathcal{S}^{\mathbf{i}'}$ vérifient les propriétés analogues déterminées par $\mathbf{i}'$.

La Propriété (B) est forte et a des conséquences remarquables (cf. Remarques 4.5.8 et 4.6.4). La Propriété (C) est automatiquement vérifiée en type A_n (car dans ce cas les P_j sont toujours sans facteur carré) et n'est donc utile que pour les types D_n, E_6, E_7 et E_8. Cette propriété est néanmoins cruciale pour prouver que $\overline{D}(x'_k)$ prend la forme voulue.

Condition initiale Le second résultat de cette partie consiste à montrer qu'en types $A_n, n \geqslant 1$ et D_4, les conditions requises par le précédent énoncé sont effectivement réalisées pour une graine standard bien choisie. On utilise pour cela le Théorème 4 (qui repose sur les travaux de Kang-Kashiwara-Kim-Oh [69, 72]).

Théorème 9. *On suppose que $\mathfrak{g}$ est de type $A_n, n \geqslant 1$ ou D_4. Soit $\mathbf{i}_{nat}$ l'expression réduite de w_0 correspondant à l'ordre naturel sur les sommets du diagramme de Dynkin associé à $\mathfrak{g}$. Alors les Propriétés (A), (B) et (C) sont vérifiées pour la graine standard $\mathcal{S}^{\mathbf{i}_{nat}}$.*

Lorsque $\mathfrak{g}$ est de type A_n, les modules déterminantaux qui catégorifient les mineurs drapeaux de $\mathcal{S}^{\mathbf{i}_{nat}}$ sont tous fortement homogènes et on peut alors appliquer l'Equation (1.8). En revanche, ce n'est plus le cas lorsque $\mathfrak{g}$ est de type D_4. Ainsi dans ce cas, parmi les quatre modules simples de $R - gmod$ correspondant aux variables gelées de $\mathbb{C}[N]$, trois sont fortement homogènes et le quatrième n'est pas homogène. Notons M ce module. On présente un calcul du caractère gradué de ce module, puis on en déduit (avec l'aide du logiciel de calcul formel SAGE) son image par $\overline{D}$. Par des méthodes similaires, on peut obtenir les images par $\overline{D}$ de tous les mineurs drapeaux de $\mathcal{S}^{\mathbf{i}_{nat}}$. Les calculs effectués sont présentés en Annexe.

Comme expliqué plus haut, on obtient le Théorème 7 en combinant les Théorèmes 8 et 9. De plus, on a également montré que lorsque $\mathfrak{g}$ est de type A_n ou D_4, les mineurs drapeaux de toutes les graines standard de $\mathbb{C}[N]$ vérifient les Propriétés (B) et (C).

En effectuant des mutations successives, on peut calculer sur des exemples les multiplicités équivariantes de nombreuses variables d'amas de $\mathbb{C}[N]$. Il semblerait que la propriété ci-dessus cesse d'être vérifiée dès lors que l'on considère des variables d'amas autres que les mineurs drapeaux. Ceci suggère une version plus forte de la conjecture précédente:

Conjecture D. *Soit $\mathfrak{g}$ une algèbre de Lie de type fini simplement lacé et soit w_0 le plus long élément du groupe de Weyl associé à $\mathfrak{g}$. Alors les mineurs drapeaux sont exactement les variables d'amas de $\mathbb{C}[N]$ dont l'image par $\overline{D}$ est de la forme*

$$\frac{1}{\prod_{\beta \in \Phi_+} \beta^{n_\beta}}.$$

pour une famille d'entiers positifs $(n_\beta)_{\beta \in \Phi_+}$.

En d'autres termes, les mineurs drapeaux seraient essentiellement caractérisés par la forme particulière de leurs multiplicités équivariantes.

1.2.5 Ouvertures

Les résultats obtenus dans la deuxième partie de cette thèse montrent que les fractions rationnelles de la forme (1.6) apparaissent dans des calculs d'intégrales de fonctions de la forme

$$e^{-(\beta_1 y_1 + \cdots + \beta_N y_N)}$$

sur des simplexes de Newton-Okounkov. Par conséquent, il est naturel de chercher à interpréter les multiplicités équivariantes des classes d'objets simples dans $R-gmod$ comme des intégrales sur certains polytopes. Etant donné un module simple M dans $R-gmod$, on cherche à construire un polytope $P(M)$ de codimension 1 dans un espace affine $\mathbb{R}^N$ (N étant un entier pouvant a priori dépendre de M) et une fonction

$$\boldsymbol{\beta} : \mathbb{R}^N \longrightarrow \mathbb{R}(\alpha_1, \ldots, \alpha_n)$$

tels que si Γ désigne le cône convexe de $\mathbb{R}^N$ engendré par $P(M)$ et si on pose

$$\bar{V}(P(M)) := \int_\Gamma e^{-\boldsymbol{\beta}(y_1,\ldots,y_N)} dy_1 \cdots dy_N,$$

alors on a:

$$\bar{D}(M) = \bar{V}(P(M)).$$

Ainsi, le polytope correspondant à un module fortement homogène $S(w), w \in \mathcal{M}in^+$ coïnciderait avec le simplexe de Newton-Okounkov $\Delta(w)$ issu de la construction présentée dans la Section 1.2.3. En effet si w est dominant minuscule, alors

$$\overline{D}([S(w)]) = \frac{1}{\prod_{\beta \in \Phi_+^w} \beta} = \int_{\Gamma(w)} e^{-(\beta_1 y_1 + \cdots + \beta_N y_N)} dy_1 \cdots dy_N. \tag{1.9}$$

où $\Gamma(w)$ désigne le cône convexe de $\mathbb{R}^{l(w)}$ engendré par $\Delta(w)$. Afin de pouvoir utiliser une telle construction pour l'étude des Conjectures C et D ci-dessus, il faudrait au préalable montrer que pour tout polytope P de codimension 1 dans $\mathbb{R}^N$, P est un simplexe si et seulement si $\bar{V}(P)$ est de la forme

$$\bar{V}(P) = \frac{1}{\prod_{\beta \in \Phi_P} \beta^{n_{P,\beta}}}$$

où Φ_P est un sous-ensemble fini de Q_+ et $n_{M,\beta}$ est un entier positif non nul pour chaque $\beta \in \Phi_M$. On peut alors reformuler la Conjecture D de la manière suivante:

Conjecture E. *Les monômes d'amas monoïdaux pour les graines $\mathcal{S}^{\mathbf{i}}$ de $\mathbb{C}[N]$ sont exactement les objets simples M dans $R-gmod$ tels que $P(M)$ est un simplexe.*

Chapter 2

Dominance order and monoidal categorification of cluster algebras

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We study a compatibility relationship between Qin's dominance order on a cluster algebra $\mathcal{A}$ and partial orderings arising from classifications of simple objects in a monoidal categorification $\mathcal{C}$ of $\mathcal{A}$. Our motivating example is Hernandez-Leclerc's monoidal categorification using representations of quantum affine algebras. In the framework of Kang-Kashiwara-Kim-Oh's monoidal categorification via representations of quiver Hecke algebras, we focus on the case of the category $R-gmod$ for a symmetric finite type A_n quiver Hecke algebra using Kleshchev-Ram's classification of irreducible finite-dimensional representations.

2.1 Cluster algebras and their monoidal categorifications

In this section, we recall the main results of the theory of cluster algebras from [41], [42], [43], as well as the notion of monoidal categorification from [56].

2.1.1 Cluster algebras

Cluster algebras were introduced in [41] by Fomin and Zelevinsky. They are commutative $\mathbb{Z}$-subalgebras of the field of rational functions over $\mathbb{Q}$ in a finite number of algebraically independent variables. They are defined as follows.

Let $1 \leqslant n < m$ be two nonnegative integers and let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in m independent variables. The initial data is a couple $((x_1, \ldots, x_m), B)$ called an initial seed and made out of a *cluster* i.e. m algebraically independent variables $x_1, \ldots, x_m$ generating $\mathcal{F}$ and a $m \times n$ matrix $B := (b_{ij})$ called the *exchange matrix* whose principal part (i.e. the square submatrix $(b_{ij})_{1 \leqslant i,j \leqslant n}$) is skew-symmetric.

To the exchange matrix B one can associate a quiver, whose index set is $\{1, \ldots, m\}$ with b_{ij} arrows from i to j if $b_{ij} \geqslant 0$, and $-b_{ij}$ arrows from j to i if $b_{ij} \leqslant 0$.

By construction, one can recover the exchange matrix from the data of a quiver without loops and 2-cycles in the following way:

$$b_{ij} = (\text{number of arrows from } i \text{ to } j) - (\text{number of arrows from } j \text{ to } i).$$

For any $k \in \{1, \dots, n\}$ one defines new variables :

$$x'_j := \begin{cases} \frac{1}{x_k}\left(\prod_{b_{lk}>0} x_l^{b_{lk}} + \prod_{b_{lk}<0} x_l^{-b_{lk}}\right) & \text{if } j = k, \\ x_j & \text{if } j \neq k, \end{cases} \tag{2.1}$$

as well as a new matrix B' :

$$\forall i, j \quad (B')_{ij} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & \text{if } i \neq k \text{ and } j \neq k. \end{cases}$$

Note that the principal part of the matrix B' is again skew-symmetric.

The procedure producing the seed $((x'_1, \dots, x'_m), B')$ out of the initial seed $((x_1, \dots, x_m), B)$ is called the *mutation* in the direction k of the initial seed $((x_1, \dots, x_m), B)$. This procedure is involutive, i.e. the mutation of the seed $(x'_1, \dots, x'_m), B')$ in the same direction k gives back the initial seed $((x_1, \dots, x_m), B)$. Any seed can give rise to n new seeds, each of them obtained by a mutation in the direction k for $1 \leqslant k \leqslant n$. Let $\mathbb{T}$ be the tree whose vertices correspond to the seeds and edges to mutations. There are exactly n edges adjacent to each vertex. This tree can have a finite or infinite number of vertices depending on the initial seed. Let $((x_1, \dots, x_m), B)$ be a fixed seed and t_0 the corresponding vertex in $\mathbb{T}$. For any vertex $t \in \mathbb{T}$ one denotes by $((x_1^t, \dots, x_m^t), B^t)$ the seed corresponding to the vertex t. It is obtained from the initial seed $((x_1, \dots, x_m), B)$ by applying a sequence of mutations following a path starting at t_0 and ending at t.

Definition 2.1.1. *The cluster algebra generated by the initial seed $((x_1, \dots, x_m), B)$ is the $\mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all the variables $x_1^t, \dots, x_n^t$ for all the vertices $t \in \mathbb{T}$.*

For any seed $((x_1, \dots, x_m), B)$, the variables $x_1, \dots, x_m$ are called the cluster variables, $x_1, \dots, x_n$ are the unfrozen variables and $x_{n+1}, \dots, x_m$ are the frozen variables. These last variables do not mutate and are present in every cluster.

The first main result of the theory of cluster algebras is the *Laurent phenomenon* :

Theorem 2.1.2 ([41, Theorem 3.1])**.** *Let $((x_1, \dots, x_m), B)$ be a fixed seed in a cluster algebra $\mathcal{A}$. Then for any seed $((x_1^t, \dots, x_m^t), B^t)$ in $\mathcal{A}$ and any $1 \leqslant j \leqslant n$, the cluster variable x_j^t is a Laurent polynomial in the variables $x_1, \dots, x_m$.*

Let $\mathbb{P}$ be the multiplicative group of all Laurent monomials in the frozen variables $x_{n+1}, \dots, x_m$. One can endow it with an additional structure given by

$$\prod_i x_i^{\alpha_i} \oplus \prod_i x_i^{\beta_i} := \prod_i x_i^{min(\alpha_i, \beta_i)}$$

making $\mathbb{P}$ a semifield. Any substraction-free rational expression $F(u_1, \dots, u_k)$ with integer coefficients in some variables $u_1, \dots, u_k$ can be specialized on some elements $p_1, \dots, p_k$ in $\mathbb{P}$. This will be denoted by $F_{|\mathbb{P}}(p_1, \dots, p_k)$.

The mutation relation (2.1) can be rewritten as

$$x_k x'_k = p_k^+ \prod_{1 \leqslant i \leqslant n} x_i^{[b_{ik}]_+} + p_k^- \prod_{1 \leqslant i \leqslant n} x_i^{[-b_{ik}]_+}$$

where

$$p_k^+ := \prod_{n+1 \leqslant i \leqslant m} x_i^{[b_{ik}]_+} \quad \text{and} \quad p_k^- := \prod_{n+1 \leqslant i \leqslant m} x_i^{[-b_{ik}]_+}$$

belong to the semifield $\mathbb{P}$.

Thus the frozen variables $x_{n+1}, \ldots, x_m$ play the role of coefficients and the cluster algebra $\mathcal{A}$ can be viewed as the $\mathbb{ZP}$ algebra generated by the (exchange) variables $x_1^t, \ldots, x_n^t$ for all the vertices t of the tree $\mathbb{T}$. Here $\mathbb{ZP}$ denotes the group ring of the multiplicative group of the semifield $\mathbb{P}$. This group is always torsion-free and hence the ring $\mathbb{ZP}$ is a domain.

The notion of isomorphism of cluster algebras is introduced in [42]: two cluster algebras $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{A}' \subset \mathcal{F}'$ with the same coefficient part $\mathbb{P}$ are said to be isomorphic if there exists a $\mathbb{ZP}$ algebras isomorphism $\mathcal{F} \to \mathcal{F}'$ sending a seed in $\mathcal{A}$ onto a seed in $\mathcal{A}'$. In particular the set of seeds of $\mathcal{A}$ is in bijection with the set of seeds of $\mathcal{A}'$ and $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic as algebras.

The second important result is the classification of finite type cluster algebras, i.e. the ones with a finite number of seeds.

Theorem 2.1.3 ([42, Theorem 1.4])**.** *There is a canonical bijection between isomorphism classes of cluster algebras of finite type and Cartan matrices of finite type.*

Let $\mathcal{A}$ be a cluster algebra and let us fix $((x_1, \ldots, x_m), B)$ an initial seed. In [43], Fomin and Zelevinsky define, for any $1 \leqslant j \leqslant n$:

$$y_j := \prod_{n+1 \leqslant i \leqslant m} x_i^{b_{ij}} \quad \textit{and} \quad \hat{y}_j := \prod_{1 \leqslant i \leqslant m} x_i^{b_{ij}}.$$

Theorem 2.1.4 ([43, Corollary 6.3])**.** *Let $((x_1^t, \ldots, x_n^t, x_{n+1}, \ldots, x_m), B^t)$ be any seed in $\mathcal{A}$. Then for any $1 \leqslant l \leqslant n$, the cluster variable x_l^t can be expressed in terms of the initial cluster variables $x_1, \ldots, x_m$ in the following way:*

$$x_l^t = \frac{F^{l,t}(\hat{y}_1, \ldots, \hat{y}_n)}{F^{l,t}{}_{|\mathbb{P}}(y_1, \ldots, y_n)} x_1^{g_1^{l,t}} \cdots x_n^{g_n^{l,t}}. \tag{2.2}$$

In this formula $F^{l,t}$ is a polynomial called the F-polynomial associated to the variable x_l^t and the $g_i^{l,t}$ are integers. We write for short $\mathbf{x}^{\mathbf{g}^{l,t}}$ for $x_1^{g_1^{l,t}} \cdots x_n^{g_n^{l,t}}$ and $\mathbf{g}^{l,t} = (g_1^{l,t}, \cdots, g_n^{l,t})$ is called the g-vector associated to the variable x_l^t.

The F-polynomial associated to any cluster variable satisfies several important and useful properties, which have been conjectured by Fomin-Zelevinsky in [43] and proved by Derksen-Weyman-Zelevinsky in [32] using the theory of quivers with potentials. We recall here some of these results, which we will use in Section 2.3 in the study of compatible seeds.

Theorem 2.1.5 ([32, Theorem 1.7])**.** *Let $((x_1^t, \ldots, x_n^t, x_{n+1}, \ldots, x_m), B^t)$ be any seed in $\mathcal{A}$. Let $1 \leqslant l \leqslant n$, and $F^{l,t}$ be the F-polynomial associated to the cluster variable x_l^t. Then*

(i) There is a unique monomial in $F^{l,t}$ that is strictly divisible by any other monomial in $F^{l,t}$. This monomial has coefficient 1.

(ii) The polynomial $F^{l,t}$ has constant term 1.

2.1.2 Monoidal categorification of cluster algebras

The notion of monoidal categorification of a cluster algebra was introduced by Hernandez and Leclerc in [56]. Recall that, if $\mathcal{C}$ is a monoidal category, a simple object M in $\mathcal{C}$ is said to be *real* if the tensor product $M \otimes_{\mathcal{C}} M$ is simple. It is said to be *prime* if it is not invertible in $\mathcal{C}$ cannot be decomposed as $M = M_1 \otimes_{\mathcal{C}} M_2$ with M_1 and M_2 two simple non invertible modules neither trivial nor equal to M itself. We denote by $K_0(\mathcal{C})$ the Grothendieck ring of the category $\mathcal{C}$. Recall that

for any objects A, B, C in $\mathcal{C}$, the relation $[B] = [A] + [C]$ holds in $K_0(\mathcal{C})$ if there is a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$. The ring structure on $K_0(\mathcal{C})$ is directly inherited from the monoidal structure of $\mathcal{C}$: $[M] \cdot [M'] = [M \otimes_{\mathcal{C}} M']$ for any objects M, M' in $\mathcal{C}$.

Remark 2.1.6. In the category $R-gmod$ that we will mostly be studying in this paper, all the simple objects are non-invertible. However in other categories, simple objects may be invertible. This happens for instance in categories of modules over Borel subalgebras of quantum affine algerbras in the work of Hernandez-Leclerc [60].

Definition 2.1.7 (Monoidal categorification of a cluster algebra)**.** *A monoidal category $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if the following conditions simultaneously hold:*

(i) There is a ring isomorphism

$$K_0(\mathcal{C}) \simeq \mathcal{A}.$$

(ii) Under this isomorphism, classes of simple real objects in $\mathcal{C}$ correspond to cluster monomials in $\mathcal{A}$ and classes of simple real prime objects in $\mathcal{C}$ correspond to cluster variables in $\mathcal{A}$.

Several examples of monoidal categorifications of cluster algebras appeared more recently in various contexts: on the one hand using categories of (finite dimensional) representations of quiver Hecke algebras through the works of Kang-Kashiwara-Kim-Oh [66, 67, 69]; on the other hand, via the coherent Satake category studied by Cautis-Williams in [27]. Let us point out that these examples use a slightly different notion of monoidal categorification:

Definition 2.1.8 (Monoidal categorification of a cluster algebra in the sense of [69, 27])**.** *A monoidal category $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if:*

(i) There is a ring isomorphism

$$K_0(\mathcal{C}) \simeq \mathcal{A}.$$

(ii) Under this isomorphism, any cluster monomial in $\mathcal{A}$ is the class of a simple real object in $\mathcal{C}$.

See for instance Definition 2.2.11 for a precise definition in the context of quiver Hecke algebras.

2.1.3 Example: representations of quantum affine algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of type A_n, D_n, or E_n and $\hat{\mathfrak{g}}$ the corresponding Kac-Moody algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ can be defined as a quantization of the universal enveloping algebra of $\hat{\mathfrak{g}}$ (see [33] or [30] for precise definitions). Consider the category $\mathcal{C}$ of finite dimensional $U_q(\hat{\mathfrak{g}})$-modules. In [30], Chari and Pressley proved that simple objects in this category are parametrized by their highest weights. More precisely, let I be the set vertices of the Dynkin diagram of $\mathfrak{g}$ and, for each $i \in I$ and $a \in \mathbb{C}^*$, let $Y_{i,a}$ be some indeterminate. The notion of q-character of a finite dimensional $U_q(\hat{\mathfrak{g}})$-module was introduced by Frenkel and Reshetikhin in [44] as a an injective ring homomorphism

$$\chi_q : K_0(\mathcal{C}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^*].$$

Let $\mathcal{M}$ be the set of Laurent monomials in the variables $Y_{i,a}$. For any $i \in I$ and $a \in \mathbb{C}^*$, set

$$A_{i,a} := Y_{i,aq} Y_{i,aq^{-1}} \prod_{j \neq i} Y_{j,a}^{a_{ji}} \in \mathcal{M}.$$

One defines a partial ordering (the Nakajima order) on $\mathcal{M}$ in the following way:

$$\mathfrak{m} \leqslant \mathfrak{m}' \Leftrightarrow \frac{\mathfrak{m}'}{\mathfrak{m}} \text{ is a monomial in the } A_{i,a}$$

for any monomials $\mathfrak{m}, \mathfrak{m}' \in \mathcal{M}$.

A monomial $\mathfrak{m} \in \mathcal{M}$ is called dominant if it does not contain negative powers of the variables $Y_{i,a}$. Let $\mathcal{M}^+$ denote the subset of $\mathcal{M}$ of all dominant monomials. For any simple object V of $\mathcal{C}$, the set of monomials occurring in the q-character of V has a unique maximal element μ_V for the above order, and this monomial is always dominant. Conversely, it is possible to associate a simple finite dimensional $U_q(\hat{\mathfrak{g}})$-module to any dominant monomial in the variables $Y_{i,a}$, providing a bijection between the set of simple objects in $\mathcal{C}$ and $\mathcal{M}^+$. For any dominant monomial $\mathfrak{m}$, we let $L(\mathfrak{m})$ denote the unique (up to isomorphism) simple object in $\mathcal{C}$ corresponding to $\mathfrak{m}$ via this bijection. In the case where $\mathfrak{m}$ is reduced to a single variable $Y_{i,a}$ for some $i \in I$ and $a \in \mathbb{C}^*$, the simple module $L(\mathfrak{m}) = L(Y_{i,a})$ is called a *fundamental representation.*

The Dynkin diagram of $\mathfrak{g}$ is a bipartite graph hence its vertex set I can be decomposed as $I = I_0 \sqcup I_1$ such that every edge connects a vertex of I_0 with one of I_1. Then for any $i \in I$ set:

$$\xi_i := \begin{cases} 0 & \text{if } i \in I_0, \\ 1 & \text{if } i \in I_1. \end{cases}$$

Hernandez and Leclerc introduce a subcategory $\mathcal{C}_1$ of $\mathcal{C}$ whose Grothendieck ring is generated (as a ring) by the classes of the fundamental representations $L(Y_{i,q^{\xi_i}}), L(Y_{i,q^{\xi_i+2}})(i \in I)$. One of the main results of [56] can be stated as follows:

Theorem 2.1.9 ([56, Conjecture 4.6])**.** *The category $\mathcal{C}_1$ is a monoidal categorification of a (finite type) cluster algebra of the same Lie type as the Lie algebra $\mathfrak{g}$.*

They prove this conjecture for $\mathfrak{g}$ of types A_n ($n \geqslant 1$) and D_4 ([56, Sections 10,11]). In [96], Nakajima proved this conjecture in types ADE using geometric methods involving graded quiver varieties. Note that this geometric construction is valid for any orientation of the Dynkin graph of $\mathfrak{g}$. In [57], Hernandez-Leclerc exhibit other examples of monoidal categorifications of cluster algebras via categories of representations of quantum affine algebras in types A_n and D_n ([57, Theorem 4.2, Theorem 5.6]).

2.2 Quiver Hecke algebras

The works of Kang-Kashiwara-Kim-Oh [66, 67, 69] provide many examples of monoidal categorifications of cluster algebras arising from certain categories of modules over quiver Hecke algebras. In this section, we recall the main definitions and properties of quiver Hecke algebras; then we recall the constructions of renormalized R-matrices from [66] as well as the statements of monoidal categorification from [69]. We also recall the classification of simple finite dimensional representations of quiver Hecke algebras of finite type using combinatorics of Lyndon words from [79].

2.2.1 Definition and main properties

In this subsection we recall the definitions and main properties of quiver Hecke algebras, as defined in [76] and [110].

Let $\mathfrak{g}$ be a Kac-Moody algebra, P the associated weight lattice and $\Pi = \{\alpha_i, i \in I\}$ the set of simple roots. We also define the coweight lattice as $P^\vee = Hom(P, \mathbb{Z})$ and we let $\Pi^\vee$ denote the

set of simple coroots. We also denote by A the generalized Cartan matrix, W the Weyl group of $\mathfrak{g}$, and $(\cdot,\cdot)$ a W-invariant symmetric bilinear form on P. Let $\mathbf{k}$ be a base field.

The root lattice is defined as $Q := \bigoplus_i \mathbb{Z}\alpha_i$. We also set $Q_+ := \bigoplus_i \mathbb{Z}_{\geqslant 0}\alpha_i$ and $Q_- := \bigoplus_i \mathbb{Z}_{\leqslant 0}\alpha_i$. For any $\beta \in Q$ that we write $\sum_i m_i\alpha_i$, its length is defined as $\sum_i |m_i|$. When $\beta \in Q_+$, we denote as in [76] by $Seq(\beta)$ the set of all finite sequences (called words) of the form $i_1, \dots, i_n$ (where n is the length of β) with m_i occurrences of the integer i for all i. For the sake of simplicity, we identify letters with simple roots. In particular, for any $i, j \in \{1, \dots, n\}$, (i, j) stands for (α_i, α_j) and if $\mu = i_1, \dots, i_n$ and $\nu = j_1, \dots, j_m$ are two words in $Seq(\beta)$, (μ, ν) stands for $\sum_{p,q}(i_p, j_q)$.

To define quiver Hecke algebras, we fix a nonnegative integer n and a family $\{Q_{i,j}, 1 \leqslant i, j \leqslant n\}$ of two-variables polynomials with coefficients in $\mathbf{k}$. These polynomials are required to satisfy certain properties, in particular $Q_{i,j} = 0$ if $i = j$ and $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ for any i, j (see for instance [69, Section 2.1] for more details). In the case of finite type A_n symmetric quiver Hecke algebras (which we will focus on in Sections 2.4 and 2.5), the polynomials $Q_{i,j}$ are the following (see [79]):

$$Q_{i,j}(u,v) = \begin{cases} (u-v) & \text{if } j = i+1, \\ (v-u) & \text{if } j = i-1, \\ 0 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.2.1. *For any β in Q_+ of length n, the quiver Hecke algebra $R(\beta)$ at β associated to the Kac-Moody algebra $\mathfrak{g}$ and the family $\{Q_{i,j}, 1 \leqslant i, j \leqslant n\}$ is the $\mathbf{k}$-algebra generated by operators $\{e(\nu)\}_{\nu \in Seq(\beta)}, \{x_i\}_{i \in \{1,\dots,n\}}$, and $\{\tau_k\}_{k \in \{1,\dots,n-1\}}$ satisfying the following relations :*

$$\begin{gathered} e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \\ \sum_{\nu \in Seq(\beta)} e(\nu) = 1, \\ x_i x_j = x_j x_i, \\ x_i e(\nu) = e(\nu) x_i, \\ \tau_k e(\nu) = e(s_k(\nu))\tau_k, \\ \tau_k \tau_l = \tau_l \tau_k \ \textit{if } |k-l| > 1, \\ \tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu), \\ (\tau_k x_i - x_{s_k(i)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \textit{if } i = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \textit{if } i = k+1, \nu_k = \nu_{k+1} \\ 0 & \textit{otherwise} \end{cases}, \\ (\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \frac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}} e(\nu) & \textit{if } \nu_k = \nu_{k+2}, \\ 0 & \textit{otherwise,} \end{cases} \end{gathered}$$

where for any $\nu \in Seq(\beta)$, ν_k stands for the kth letter of the word ν.

The quiver Hecke algebra $R(\beta)$ is called symmetric when the polynomials $Q_{i,j}$ are polynomials in $u - v$.

The first main property of quiver Hecke algebras is that they naturally come with a $\mathbb{Z}$-grading by setting

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = 2, \quad \deg \tau_i e(\nu) = -(\nu_i, \nu_{i+1}).$$

For any β and γ in Q_+ of respective lengths m and n, let M be a $R(\beta)$ module and N a $R(\gamma)$ module. One defines the convolution product of M and N via parabolic induction (see [76, 66]).

Set

$$e(\beta,\gamma) := \sum_{\substack{\nu\in Seq(\beta)\\ \lambda\in Seq(\gamma)}} e(\nu\lambda) \in R(\beta+\gamma).$$

It is an idempotent in $R(\beta+\gamma)$. Consider the homomorphism of $\mathbf{k}$-algebras

$$R(\beta)\otimes R(\gamma) \to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$$

given by

$$e(\nu)\otimes e(\lambda) \mapsto e(\nu\lambda), \nu\in Seq(\beta), \lambda\in Seq(\gamma)$$

$$x_k\otimes 1 \mapsto x_k e(\beta,\gamma), 1\leqslant k\leqslant m, 1\otimes x_l \mapsto x_{m+l}e(\beta,\gamma), 1\leqslant l\leqslant n$$

$$\tau_k\otimes 1 \mapsto \tau_k e(\beta,\gamma), 1\leqslant k< m, 1\otimes \tau_l \mapsto \tau_{m+l}e(\beta,\gamma), 1\leqslant l< n.$$

Then one defines

$$M\circ N := R(\beta+\gamma)\otimes_{R(\beta)\otimes R(\gamma)} M\otimes N.$$

For any $\beta\in Q_+$, let $R(\beta)-pmod$ be the category of (left) graded finite type projective $R(\beta)$ modules, $R(\beta)-gmod$ the category of left finite dimensional graded $R(\beta)$-modules, and also

$$R-pmod := \bigoplus_{\beta\in Q_+} R(\beta)-pmod, \quad R-gmod := \bigoplus_{\beta\in Q_+} R(\beta)-gmod.$$

Convolution product induces a monoidal structure on the categories $R-gmod$ and $R-pmod$. The grading on quiver Hecke algebras also yields shift functors for these categories: decompose any object M as

$$M = \bigoplus_{n\in\mathbb{Z}} M_n$$

and define qM as

$$qM = \bigoplus_{n\in\mathbb{Z}} M_{n-1}.$$

The natural $\mathbb{Z}[q^{\pm 1}]$ action

$$q\cdot[M] := [qM]$$

gives rise to $\mathbb{Z}[q^{\pm 1}]$-algebras structures on the Grothendieck rings of the categories $R-pmod$ and $R-gmod$.

The following definition introduces a notion of graded character for representations of quiver Hecke algebras.

Definition 2.2.2 ([76, 79])**.** *Let M be a finite dimensional graded $R(\beta)$-module. For any $\nu\in Seq(\beta)$, set $M_\nu := e(\nu)\cdot M$. The module M can be decomposed as*

$$M = \bigoplus_{\nu} M_\nu.$$

Define

$$ch_q(M) := \sum_{\nu}(dim_q M_\nu).\nu$$

where for any graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, $dim_q(V) := \sum_{n\in\mathbb{Z}} q^n dim V_n$. This is a formal series in words belonging to $Seq(\beta)$ with coefficients in $\mathbb{Z}[q,q^{-1}]$.

One can put a ring structure on the image set of ch_q by defining a "product" of two words called *quantum shuffle product*. For any nonnegative integer n, let $\mathfrak{S}_n$ denote the symmetric group of rank n.

Definition 2.2.3 (Quantum shuffle product)**.** *Let* $\mathbf{i} = i_1, \dots, i_r$ *and* $\mathbf{j} = j_1, \dots, j_s$ *be two words. We set* $i_{r+1} := j_1$, $i_{r+s} := j_s$ *so that we can consider the concatenation* $\mathbf{ij} = i_1 \cdots i_{r+s}$.

Define the quantum shuffle product of $\mathbf{i}$ *and* $\mathbf{j}$ *:*

$$\mathbf{i} \circ \mathbf{j} := \sum_{\sigma \in \mathfrak{S}_{r,s}} q^{-e(\sigma)} (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(r+s)})$$

where $\mathfrak{S}_{r,s}$ *denotes the subset of* $\mathfrak{S}_{r+s}$ *defined as:*

$$\mathfrak{S}_{r,s} := \{\sigma \in \mathfrak{S}_{r+s} \mid \sigma(1) < \cdots < \sigma(r) \quad \textit{and} \quad \sigma(r+1) < \cdots < \sigma(r+s)\}$$

and, for any element $\sigma \in \mathfrak{S}_{r,s}$,

$$e(\sigma) := \sum_{\substack{1 \leqslant k \leqslant r < l \leqslant r+s \\ \sigma(k) > \sigma(l)}} (i_k, i_l).$$

By linearity one can also define quantum shuffle products of two formal series in elements of $Seq(\beta)$ with coefficients in $\mathbb{Z}[q, q^{-1}]$ (for any $\beta \in Q_+$).

Proposition 2.2.4 ([76, Lemma 2.20])**.** *For any* $\beta, \gamma \in Q_+$, *and any* $M \in R(\beta) - gmod$ *and* $N \in R(\gamma) - gmod$, *we have:*

$$ch_q(M \circ N) = ch_q(M) \circ ch_q(N).$$

One can now state the main property of quiver Hecke algebras, which is to categorify the negative part of the quantum group $U_q(\mathfrak{g})$ in a way that makes correspond the basis of indecomposable objects in $R - pmod$ with the canonical basis of $U_q(\mathfrak{n})$. In the following we will mostly consider the category $R - gmod$ hence we give here the dual statements, involving the category $R - gmod$ and the quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ (the precise definition of which can be found in [52] or [69]). The first theorem was proved by Khovanov-Lauda [76] and Rouquier [110]. The second was conjectured by Khovanov-Lauda, and proved by Rouquier [110] and Varagnolo-Vasserot [113] using geometric methods.

Theorem 2.2.5 (Khovanov-Lauda, Rouquier)**.** *The map* ch_q *induces a* $\mathbb{Z}[q, q^{-1}]$*-algebra isomorphism*

$$K_0(R - gmod) \simeq \mathcal{A}_q(\mathfrak{n}).$$

Theorem 2.2.6 (Rouquier, Varagnolo-Vasserot)**.** *The map* ch_q *(see Definition 2.2.2) induces a bijection between the canonical basis of the quantum coordinate ring* $\mathcal{A}_q(\mathfrak{n})$ *and the set of isomorphism classes of self-dual simple modules in the category* $R - gmod$.

2.2.2 Renormalized R-matrices for quiver Hecke algebras

Recall from Section 2.2.1 that the weight lattice associated to the Kac-Moody algebra $\mathfrak{g}$ is given with a symmetric bilinear form $(\cdot, \cdot)$. Denoting by A the symmetrizable generalized Cartan matrix of $\mathfrak{g}$, this bilinear form is entirely determined by its values on simple roots, namely:

$$\forall i, j, \quad (\alpha_i, \alpha_j) = \mathbf{s}_i a_{ij}$$

where the $\mathbf{s}_i$ are the entries of a diagonal matrix D such that DA is symmetric.

One also defines another symmetric bilinear form $(\cdot,\cdot)_n$ on the root lattice Q as in [66]:

$$\forall i,j, \quad (\alpha_i,\alpha_j)_n := \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\beta \in Q_+$ of length m and $1 \leqslant k < m$; the following operators φ_k are introduced in [66]:

$$\forall \nu \in Seq(\beta), \varphi_k e(\nu) := \begin{cases} (\tau_k(x_k - x_{k+1}) + 1)e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise.} \end{cases}$$

These operators satisfy the braid relation, hence for any permutation σ, $\varphi_\sigma := \varphi_{i_1} \cdots \varphi_{i_l}$ does not depend on the choice of a reduced expression $\sigma = s_{i_1} \cdots s_{i_l}$.

For any $m,n \in \mathbb{Z}_{\geqslant 0}$, let $w[m,n]$ be the element of $\mathfrak{S}_{m+n}$ sending k on $k+n$ if $1 \leqslant k \leqslant m$ and on $k-m$ if $m < k \leqslant m+n$.

Consider a non-zero $R(\beta)$-module M and a non-zero $R(\gamma)$-module N. The following map is defined in [66]:

$$\begin{array}{ccc} M \otimes N & \longrightarrow & N \circ M \\ u \otimes v & \longmapsto & \varphi_{w[n,m]}(v \otimes u). \end{array}$$

It is $R(\beta) \otimes R(\gamma)$ linear and hence induces a homomorphism of $R(\beta+\gamma)$-modules

$$R_{M,N} : M \circ N \longrightarrow N \circ M.$$

The map $R_{M,N}$ satisfies the Yang-Baxter equation (see [66]).

Let z be an indeterminate, homogeneous of degree 2. For any $\beta \in Q_+$ and any non-zero module M in $R(\beta) - gmod$, one defines $M_z := \mathbf{k}[z] \otimes M$ with the following $\mathbf{k}[z] \otimes R(\beta)$-module structure:

$$\begin{aligned} e(\nu).(P \otimes m) &:= P \otimes (e(\nu)m) \\ x_k.(P \otimes m) &:= (zP) \otimes m + P \otimes (x_k m) \\ \tau_k.(P \otimes m) &:= P \otimes (\tau_k m) \end{aligned}$$

for any $\nu \in Seq(\beta)$, $P \in \mathbf{k}[z]$ and $m \in M$.

It is shown in [66] that for any $\beta,\gamma \in Q_+$ and any non-zero $R(\beta)$-module M and non-zero $R(\gamma)$-module N, the map $R_{M_z,N}$ is polynomial in z and does not vanish. Let s be the largest non-negative integer such that the image of $R_{M_z,N}$ is contained in $z^s N \circ M_z$. One defines R-matrices in the category $R - gmod$ in the following way:

Definition 2.2.7. *Let $\beta,\gamma \in Q_+$. For any non zero $R(\beta)$-module M and non zero $R(\gamma)$-module N, define a homomorphism of $R(\beta+\gamma)$-modules*

$$r_{M,N} : M \circ N \longrightarrow N \circ M$$

by setting

$$r_{M,N} := (z^{-s} R_{M_z,N})_{|z=0}$$

where s is the integer defined above.

Proposition 2.2.8 ([66])**.** *The homomorphism $r_{M,N}$ does not vanish and satisfies the Yang-Baxter equation.*

Thus the maps $r_{M,N}$ are R-matrices for the category $R-gmod$. They are called renormalized R-matrices. As in the case of categories of representations of quantum affine algebras, these R-matrices are in general not invertible and thus yield (graded) short exact sequences in the category $R-gmod$. Consequently this produces some relations in the Grothendieck ring $K_0(R-gmod) \simeq \mathcal{A}_q(\mathfrak{n})$. In the context of monoidal categorifications of (quantum) cluster algebras (see Section 2.2.3 below), the exchange relations in $\mathcal{A}_q(\mathfrak{n})$ will be identified with some of these relations.

The corresponding relations in the Grothendieck ring $K_0(R-gmod)$ will be identified with exchange relations For any non-zero modules M and N, we denote by $\Lambda(M,N)$ the homogeneous degree of the morphism $r_{M,N}$. It is given by

$$\Lambda(M,N) = -(\beta,\gamma) + 2(\beta,\gamma)_n - 2s.$$

The next statement gives a criterion for the renormalized R-matrix $r_{M,N}$ to be an isomorphism. It will be particularly useful for the proof of Theorem 2.5.1 (see for instance Corollary 2.5.6).

Lemma 2.2.9 ([69, Lemma 3.2.3]). *Let M and N be two simples in the category $R-gmod$ and assume one of them is real. Then the following are equivalent :*

(i) $\Lambda(M,N) + \Lambda(N,M) = 0$.

(ii) $r_{M,N}$ *and* $r_{N,M}$ *are inverse to each other up to a constant multiple.*

(iii) $M \circ N$ *and* $N \circ M$ *are isomorphic up to grading shift.*

(iv) $M \circ N$ *is simple in the category* $R-gmod$.

One says that M and N commute if they satisfy these properties.

2.2.3 Monoidal categorification via representations of quiver Hecke algebras

In this subsection we focus on the case where $\mathcal{C}$ is a full subcategory of $R-gmod$ stable under convolution products, subquotients, extensions, and grading shifts. $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \bigoplus_{\beta \in Q_+} \mathcal{C}_\beta$$

with $\mathcal{C}_\beta := \mathcal{C} \cap R(\beta)-gmod$ for every $\beta \in Q_+$, so that the tensor product in $\mathcal{C}$ sends $\mathcal{C}_\beta \times \mathcal{C}_\gamma$ onto $\mathcal{C}_{\beta+\gamma}$ for any $\beta, \gamma \in Q_+$.

Kang-Kashiwara-Kim-Oh [69] adapt the notion of monoidal categorification to the setting of quantum cluster algebras. In the classical setting, a monoidal seed in $\mathcal{C}$ is defined as a triple $(\{M_i\}_{1\leqslant i\leqslant n}, B, D)$ where $\{M_i\}_{1\leqslant i\leqslant n}$ is a collection of simple objects in $\mathcal{C}$ such that for any $i_1,\dots,i_t$ in $\{1,\dots,n\}$, the object $M_{i_1} \circ \cdots \circ M_{i_t}$ is simple in $\mathcal{C}$, B is an integer-valued matrix with skew-symmetric principal part and D is a diagonal matrix encoding the weights of the modules M_i (i.e. the elements $\beta_i \in Q_+$ such that $M_i \in \mathcal{C}_{\beta_i}$). Cluster mutations correspond to some (ungraded) short exact sequences in the category $\mathcal{C}$. These exact sequences come from the failure of the renormalized R-matrices (see Definition 2.2.7) to be isomorphisms. The cluster mutations being involutive imposes some relations between the entries of the matrices B and D.

In the framework of [69], one takes into account the natural grading of quiver Hecke algebras defined in Section 2.2.1: objects in $\mathcal{C}$ are graded as well. A quantum monoidal seed is the data of such a triple $(\{M_i\}, B, D)$ with the further assumption that there exist integers λ_{ij} and isomorphisms of graded modules $M_i \otimes M_j \simeq q^{\lambda_{ij}} M_j \otimes M_i$ for any $i,j \in \{1,\dots,n\}$. The matrix $L = (\lambda_{ij})_{1\leqslant i,j\leqslant n}$ is a skew-symmetric matrix and is assumed to satisfy some compatibility relations with the matrix B as in [10]. See [69, Section 6.2.1] for a precise definition.

In the quantum setting, cluster mutations correspond to some *graded* short exact sequences.

Definition 2.2.10 ([69, Definition 6.2.3]). *Let $k \in \{1, \dots, r\}$ fixed. A quantum monoidal seed $\mathcal{S} = (\{M_i\}_{1 \leqslant i \leqslant n}, L, B, D)$ admits a mutation in the direction k if there exists a simple object M'_k of $\mathcal{C}$ such that:*

a) $M'_k \in \mathcal{C}_{d'_k}$ with $d'_k := -d_k + \sum_{b_{ik}>0} b_{ik} d_i$.

b) One has the following short exact sequences in $\mathcal{C}$:

$$0 \longrightarrow qM^{\mathbf{b}'} \longrightarrow q^{m_k} M_k \otimes M'_k \longrightarrow M^{\mathbf{b}''} \longrightarrow 0$$

$$0 \longrightarrow qM^{\mathbf{b}''} \longrightarrow q^{m'_k} M'_k \otimes M_k \longrightarrow M^{\mathbf{b}'} \longrightarrow 0$$

where m_k and m'_k are some integers.

c) $\mathcal{S}'^{(k)} := (\{M_i\}_{i \neq k} \cup \{M'_k\}, L'^{(k)}, B'^{(k)}, D'^{(k)})$ is again a quantum monoidal seed in $\mathcal{C}$, where $L'^{(k)}$ and $B'^{(k)}$ are defined as in [10, Definition 3.5] and $D'^{(k)}$ is the diagonal matrix whose entries are the d_i for $i \neq k$ and d'_k for $i = k$.

Definition 2.2.11. *The category $\mathcal{C}$ is a monoidal categorification of a quantum cluster algebra $\mathcal{A}$ if:*

a) There is an isomorphism of graded rings $\mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(\mathcal{C}) \simeq \mathcal{A}$.

b) There exists a quantum monoidal seed $\mathcal{S} := (\{M_i\}, L, B, D)$ in $\mathcal{C}$ such that $[\mathcal{S}] := (\{q^{-\frac{(d_i,d_i)}{4}}[M_i]\}, L, B)$ is a quantum seed in $\mathcal{A}$.

c) The quantum monoidal seed $\mathcal{S}$ admits arbitrary sequences of mutations in all directions.

In this setting, the existence of a monoidal categorification implies that any (quantum) cluster monomial is the class of some real simple object in $\mathcal{C}$. Recall from Section 2.1.2 that this notion is slightly different from the notion of monoidal categorification initially defined by Hernandez-Leclerc [56].

The following definition provides a sufficient condition for producing quantum monoidal seeds.

Definition 2.2.12. *A pair $(\{M_i\}, B)$ is admissible if:*

(i) $\{M_i\}_{1 \leqslant i \leqslant n}$ is a family of self-dual real simple modules commuting with each other.

(ii) The matrix B is defined as above.

(iii) For each $1 \leqslant k \leqslant r$ there exists a self-dual simple module M'_k such that M'_k commutes with the M_i for $i \neq k$ and there is a short exact sequence of graded objects in $\mathcal{C}$

$$0 \longrightarrow qM^{\mathbf{b}'} \longrightarrow q^{\tilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \longrightarrow M^{\mathbf{b}''} \longrightarrow 0.$$

where $\tilde{\Lambda}(M, N)$ is defined as $\frac{1}{2}(\Lambda(M, N) + (\beta, \gamma))$ for $M \in R(\beta)-gmod$ and $N \in R(\gamma)-gmod$.

The data of an admissible pair naturally gives rise to a quantum monoidal seed in $\mathcal{C}$. More precisely, if $(\{M_i\}_{1 \leqslant i \leqslant n}, B)$ is an admissible pair in $\mathcal{C}$, M'_k as in the previous definition, then one defines a $r \times r$ skew-symmetric matrix L and a diagonal matrix D of size n by setting

$$L_{ij} := \Lambda(M_i, M_j) \quad \text{and} \quad D = Diag(d_1, \dots, d_n)$$

where d_i stands for the weight of the module M_i. Then ([69, Proposition 7.1.2]) the quadruple $\mathcal{S} := (\{M_i\}_{1 \leqslant i \leqslant n}, -L, B, D)$ is a quantum monoidal seed in $\mathcal{C}$ which admits mutations in every direction k (for $1 \leqslant k \leqslant r$).

The main result of [69] can now be stated as follows:

Let $(\{M_i\}_{1\leqslant i\leqslant n}, B)$ be an admissible pair in $\mathcal{C}$ and

$$\mathcal{S} := (\{M_i\}_{1\leqslant i\leqslant n}, -L, B, D)$$

the corresponding quantum monoidal seed. Set $[\mathcal{S}] := (\{q^{-\frac{(wt(M_i),wt(M_i))}{4}}[M_i]\}_{1\leqslant i\leqslant n}, -L, B, D)$.

Theorem 2.2.13 ([69, Theorem 7.1.3])**.** *Assume there is a $\mathbb{Q}(q^{\frac{1}{2}})$-algebras isomorphism*

$$\mathbb{Q}(q^{\frac{1}{2}}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(\mathcal{C}) \simeq \mathbb{Q}(q^{\frac{1}{2}}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{A}_{q^{\frac{1}{2}}}([\mathcal{S}]).$$

Then for each $1 \leqslant k \leqslant r$ the pair $(\{M_i\}_{i\neq k} \cup \{M'_k\}, B'^{(k)})$ is again an admissible pair in the category $\mathcal{C}$.

2.2.4 Quantum monoidal seeds for $\mathcal{C}_w$

In this subsection we recall from [69] the definition of the subcategories $\mathcal{C}_w$ of $R-gmod$ as well as the construction of admissible pairs for these categories.

For any element w of the Weyl group W associated to $\mathfrak{g}$, Geiss, Leclerc and Schröer defined algebras $\mathcal{A}_q(\mathfrak{n}(w))$ as subalgebras of the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n})$ ([52, Section 7.2]). They show ([52, Theorem 12.3]) that it is possible to put a quantum cluster algebra structure on $\mathcal{A}_q(\mathfrak{n}(w))$ for every $w \in W$. In [69], Kang-Kashiwara-Kim-Oh introduce, for each $w \in W$, a subcategory $\mathcal{C}_w$ of $R-gmod$ such that the Grothendieck ring $K_0(\mathcal{C}_w)$ is the preimage of $\mathcal{A}_q(\mathfrak{n}(w))$ under the isomorphism given by Theorem 2.2.5: $M \in \mathcal{C}_w$ if and only if $ch_q(M) \in \mathcal{A}_q(\mathfrak{n}(w))$. In [69], Kang-Kashiwara-Kim-Oh prove the following:

Theorem 2.2.14 ([69, Theorem 11.2.3])**.** *For each element w of the Weyl group W, the category $\mathcal{C}_w$ is a monoidal categorification of the quantum cluster algebra $\mathcal{A}_{q^{1/2}}(\mathfrak{n}(w))$.*

Thus the categories $\mathcal{C}_w$ provide many examples of monoidal categorifications of (quantum) cluster algebras.

Remark 2.2.15. The category $R-gmod$ corresponds $\mathcal{C}_{w_0}$ where w_0 stands for the longest element of the Weyl group of $\mathfrak{g}$. When w is the square of a well chosen Coxeter element c in W, the quantum cell $\mathcal{A}_{q^{1/2}}(\mathfrak{n}(w))$ is also categorified by the category $\mathcal{C}_1$ defined in [56]. The category $\mathcal{C}_{w_0}$ (resp. $\mathcal{C}_{c^2}$) is related to the category $\mathcal{C}_Q$ (resp. $\mathcal{C}_1$) introduced by Hernandez-Leclerc in [57] (resp. [56]) via a functor called *generalized quantum affine Schur-Weyl duality* defined in [66]. In the case of $\mathcal{C}_{w_0}$ Fujita [47] proved that this functor is an equivalence of categories.

Note that Geiss-Leclerc-Schröer defined categories $\tilde{\mathcal{C}}_w$ which provide additive categorifications of the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n}(w))$ for each $w \in W$ ([52, Theorem 12.3]). The categories $\tilde{\mathcal{C}}_w$ are defined as subcategories of the preprojective algebra of certain quivers. The categories $\mathcal{C}_w$ as defined in [69] can be seen as monoidal analogs of the categories $\tilde{\mathcal{C}}_w$ of [52] in terms of representations of quiver Hecke algebras. However, Theorem 2.2.14 provides a monoidal categorification statement and is thus of different nature than the results of [52].

In order to prove Theorem 2.2.14, Kang-Kashiwara-Kim-Oh construct an admissible pair (see Definition 2.2.12) in the category $\mathcal{C}_w$ for each $w \in W$. We now recall this construction. By the results of [69], the existence of such a pair implies Theorem 2.2.14.

First one defines *unipotent quantum minors* as some distinguished elements of $\mathcal{A}_q(\mathfrak{n})$: for any dominant weight λ in the weight lattice P and any couple (μ, ζ) of elements of $W\lambda$, the unipotent quantum minor $D(\mu, \zeta)$ is an element of $\mathcal{A}_q(\mathfrak{n})$ which is either a member of the canonical basis of $\mathcal{A}_q(\mathfrak{n})$ or zero ([69, Lemma 9.1.1]). The following statement gives a necessary and sufficient condition so that $D(\mu, \zeta)$ is non zero. First recall some notation from [69]:

Definition 2.2.16. *Let $\lambda \in P^+, \mu, \zeta \in W\lambda$. We write $\mu \lesssim \zeta$ if there exists a finite sequence $(\beta_1, \dots, \beta_l)$ such that, setting $\lambda_0 := \zeta, \lambda_k = s_{\beta_k}\lambda_{k-1} (1 \leqslant k \leqslant l)$ one has $\lambda_l = \mu$ and $\forall 1 \leqslant k \leqslant l, (\beta_k, \lambda_{k-1}) \geqslant 0$.*

Lemma 2.2.17 ([69, Lemma 9.1.4])**.** *Let $\lambda \in P^+, \mu, \zeta \in W\lambda$. Then $D(\mu, \zeta) \neq 0$ if and only if $\mu \lesssim \zeta$.*

The following statement is a direct consequence of Theorem 2.2.6 and of the previous lemma:

Corollary 2.2.18. *Let $\lambda \in P^+, \mu, \zeta \in W\lambda$ such that $\mu \lesssim \zeta$. There exists a unique self-dual simple module $M(\mu, \zeta) \in R - gmod$ whose image under the character map ch_q is $D(\mu, \zeta)$. Moreover, $M(\mu, \zeta)$ is real.*

This module is called *determinantial module* ([69, Definition 10.2.1]). Its weight is equal to $\zeta - \mu$, i.e. $M(\mu, \zeta) \in R(\zeta - \mu) - gmod$.

Remark 2.2.19. This is one of the key points that we will use to compute the dominant words of the modules corresponding to the frozen variables in $R - gmod$ in Section 2.5.

One can now construct an admissible seed for the category $\mathcal{C}_w$. Fix some element w in the Weyl group W and a reduced expression $w = s_{i_1} \cdots s_{i_r}$. For $s \in \{1, \dots, r\}$, set

$s_+ \quad := \quad \min(\{k \mid s < k \leqslant r, i_k = i_s\} \cup \{r + 1\})$

$s_- \quad := \quad \max(\{k \mid 1 \leqslant k < s, i_k = i_s\} \cup \{0\})$

For $1 \leqslant k \leqslant r$, set

$$\lambda_k := s_{i_1} \cdots s_{i_k} \omega_{i_k}.$$

For $0 \leqslant t \leqslant s \leqslant r$, set

$$D(s,t) := \begin{cases} D(\lambda_s, \lambda_t) & \text{if } 0 < t, \\ D(\lambda_s, \omega_{i_s}) & \text{if } 0 = t < s \leqslant r, \\ 1 & \text{if } t = s = 0. \end{cases}$$

Definition 2.2.20 ([69])**.** *As in Corollary 2.2.18, consider $M(s,t)$ the unique simple real module (up to shift and isomorphism) such that $ch(M(s,t)) = D(s,t)$ for any $0 \leqslant s \leqslant t \leqslant r$.*

Set $J = \{1, \dots, r\}, J_{fr} = \{k \in J \mid k_+ = r + 1\}$ and $J_{ex} = J \backslash J_{fr}$. The initial quiver is set to have $J = \{1, \dots, r\}$ as set of vertices with the following arrows:

$s \longrightarrow t \quad$ if $1 \leqslant s < t < s_+ < t_+ \leqslant r + 1$

$s \longrightarrow s_- \quad$ if $1 \leqslant s_- < s \leqslant r$

Denoting by B the corresponding exchange matrix, the main result of [69] can be stated in the following way:

Theorem 2.2.21 ([69, Theorem 11.2.2])**.** *The pair $(\{M(k,0)\}_{1 \leqslant k \leqslant r}, B)$ is admissible in the category $\mathcal{C}_w$.*

2.2.5 Irreducible representations of quiver Hecke algebras

In this subsection we recall from [79] the classification of simple finite-dimensional modules over finite type quiver Hecke algebras. The main result (Theorem 2.2.31 below) is that simple objects in the category $R - gmod$ are parametrized in a combinatorial way by *dominant words*, which are analogs of Zelevinsky's multisegments in the classification of simple representations of affine Hecke algebras of type A. As for Lie algebras, simple modules over quiver Hecke algebras are constructed as quotients of tensor products of some distinguished irreducible representations, called *cuspidal modules* in [79].

Choose a labeling of the vertices of the Dynkin diagram of $\mathfrak{g}$ by $I = \{1, \dots, n\}$. A word is a finite set of elements of I. We fix a total order on I by setting $1 < \cdots < n$. The set of all words is a totally ordered set with respect to the lexicographic order induced by $<$.

For $\mathbf{i} := (i_1, \dots, i_d)$, set $|\mathbf{i}| := \alpha_1 + \cdots + \alpha_d \in Q_+$. Recall from Section 2.2.1 that for any $\beta \in Q_+$, $Seq(\beta) = \{\mathbf{i}, |\mathbf{i}| = \beta\}$.

Definition 2.2.22. *A word is called Lyndon if it is smaller than all its proper right factors.*

Example 2.2.23. The words 123, 24, 13 are Lyndon. The word 231 is not.

The following statement is a well-known fact (see [87, Theorem 5.1.5]):

Proposition 2.2.24 (Canonical factorization)**.** *Any word μ can be written in a unique way as a concatenation of Lyndon words in the decreasing order :*

$$\mu = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r}$$

with $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(r)}$ Lyndon words satisfying $\mathbf{i}^{(1)} > \cdots > \mathbf{i}^{(r)}$ and $n_1, \dots, n_r$ nonnegative integers.

This is called the *canonical factorization* of the word μ. Recall from Section 2.2.1 (Definition 2.2.2) that for $\beta \in Q_+$, any $R(\beta)$-module M decomposes as a direct sum of vector spaces $M = \bigoplus_{\nu \in Seq(\beta)} M_\nu$ with $M_\nu := e(\nu)M$.

Definition 2.2.25. *A word μ is dominant if there is an $R(\beta)$-module M such that μ is the highest word among the words ν such that M_ν is not zero: $M = M_\mu \oplus \bigoplus_{\nu<\mu} M_\nu$ and $M_\mu \neq 0$.*

Dominant words play the same role as highest weights in the representation theory of finite dimensional semisimple Lie algebras (see [30]). The next statement provides a very useful combinatorial criterion to determine whether a word is dominant or not. In particular it shows that a dominant word can be seen as a collection (or a sum with positive coefficients) of positive roots, which is why the terminology *root partitions* is sometimes used (see [91]).

Theorem 2.2.26 ([79])**.** *(i) There is a bijection between the set of dominant Lyndon words and the set Δ_+ of positive roots of $\mathfrak{g}$, given by $\mathbf{i} \mapsto |\mathbf{i}|$.*

(ii) A word μ is dominant if and only if all the Lyndon words appearing in the canonical factorization of μ are dominant.

Example 2.2.27. In type A_4, 24 is Lyndon but not dominant Lyndon, and 123 is dominant Lyndon. The word 12312 is dominant but the word 3213 is not.

Remark 2.2.28. Dominant Lyndon words already appear in the work of Leclerc [84] as *good Lyndon words* in the study of dual canonical basis for quantum groups and quantum coordinate rings.

The next Proposition introduces the notion of *cuspidal modules*. The existence of cuspidal modules follows from results of Varagnolo-Vasserot [113] in simply-laced cases and Rouquier [110] in the general case. Cuspidal modules can be seen as analogs of fundamental representations for Lie algebras.

Proposition 2.2.29 ([79, Proposition 8.4])**.** *For any dominant Lyndon word $\mathbf{i}$ in $Seq(\beta)$, there is a unique (up to isomorphism and shift) irreducible $R(\beta)$-module of highest weight $\mathbf{i}$. We denote it by $L(\mathbf{i})$.*

In [79], Kleshchev-Ram give explicit constructions of cuspidal modules for each finite type. For instance, in type A_n, the set of positive roots is $\Delta_+ = \{\alpha_i + \cdots + \alpha_j, 1 \leqslant i \leqslant j \leqslant n\}$ and the cuspidal module $L(k \dots l)$ corresponding to the positive root $\alpha_k + \cdots + \alpha_l$ is the one dimensional vector space spanned by a vector v with action of $R(\alpha_k + \cdots + \alpha_l)$ given by:

$$x_i \cdot v = 0, \quad \tau_j \cdot v = 0, \quad e(\nu) \cdot v = \begin{cases} v & \text{if } \nu = k \dots l \\ 0 & \text{otherwise.} \end{cases}$$

Recall from Section 2.2.1 that the graded character of a finite dimensional $R(\beta)$-module M is a (finite) formal sum of elements of $Seq(\beta)$ with coefficients in $\mathbb{Z}[q, q^{-1}]$. For any such formal sum $S := \sum_\lambda P_\lambda(q)\lambda$, we let $max(S)$ denote the greatest word appearing in this sum (for the lexicographic order). In particular, for any finite dimensional $R(\alpha)$-module M, we set $max(M) := max(ch_q(M))$. The word $max(M)$ is called the *highest weight* of the module M in [79].

The next statement shows that any word (not necessarily dominant) always appears as the highest word in the quantum shuffle product of the Lyndon words appearing in its canonical factorization.

Proposition 2.2.30 ([79, Lemma 5.3])**.** *Let μ be a word, and $\mu = \mathbf{i}^{(1)} \cdots \mathbf{i}^{(r)}$ its canonical factorization. Then we have $max(\mathbf{i}^{(1)} \circ \cdots \circ \mathbf{i}^{(r)}) = \mu$.*

One can now state the main result of [79]. It shows that finite dimensional irreducible representations of finite type quiver Hecke algebras are parametrized by dominant words.

Theorem 2.2.31 ([79, Theorem 7.2])**.** *Let μ be a dominant word, and $\mu = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r}$ its canonical factorization. Set*

$$\Delta(\mu) := L(\mathbf{i}^{(1)})^{\circ n_1} \circ \cdots \circ L(\mathbf{i}^{(r)})^{\circ n_r} \langle s(\mu) \rangle$$

where $s(\mu) := \sum_{k=1}^{r} (\mathbf{i}_k . \mathbf{i}_k) n_k (n_k - 1)/4$.

Then :

(i) $\Delta(\mu)$ has an irreducible head, denoted $L(\mu)$.

(ii) The highest weight of $L(\mu)$ is μ: $max(L(\mu)) = \mu$.

(iii) The set $\{L(\mu)\}$ for μ dominant words in $Seq(\beta)$ is a complete and irredundant set of irreducible graded $R(\beta)$-modules up to isomorphism and shift.

Moreover for μ of the form $\mathbf{j}^n$ with $\mathbf{j}$ dominant Lyndon, one has $L(\mu) = L(\mathbf{j})^{\circ n}$.

Example 2.2.32. Here are some examples of characters of some simple modules:

$$\begin{aligned} ch_q(L(1)) &= (1) \\ ch_q(L(12)) &= (12) \\ ch_q(L(21)) &= (21) \\ ch_q(L(312) &= (312) + (132) \\ ch_q(L(11)) &= (q + q^{-1})(11). \end{aligned}$$

2.3 Dominance order and compatible seeds

In this section we define a partial ordering on the set of Laurent monomials in the cluster variables of a cluster algebra $\mathcal{A}$. This ordering coincides with the *dominance order* introduced by Qin in [106]. In the context of monoidal categorification of a cluster algebra, we use this dominance order to introduce the notion of *compatible seed* and state the main conjecture of this work (Conjecture 2.3.10). We end this section with a discussion of monoidal categorifications of cluster algebras via representations of quantum affine algebras following the work of Hernandez-Leclerc [56], which provides a first example where Conjecture 2.3.10 holds.

2.3.1 Partial ordering on monomials

Consider a cluster algebra $\mathcal{A}$ and choose a seed $((x_1, \ldots, x_n, x_{n+1}, \ldots, x_m), B)$, where $x_1, \ldots, x_n$ are the unfrozen variables and $x_{n+1}, \ldots, x_m$ are the frozen variables. Let $\mathcal{M}_{\mathbf{x}}$ be the monoid of all monomials in the x_i and $\mathcal{G}_{\mathbf{x}}$ the abelian group of all Laurent monomials in the x_i. Recall from Section 2.1.1 the variables $\hat{y}_j$ defined as

$$\hat{y}_j := \prod_{1 \leqslant i \leqslant m} x_i^{b_{ij}}$$

for any $1 \leqslant j \leqslant n$. In what follows we write $\mathbf{x}^\alpha$ for $\prod_i x_i^{\alpha_i}$ for any integers α_i.

One now defines a partial preorder on $\mathcal{G}_{\mathbf{x}}$ in the following way: given two Laurent monomials $\mathbf{x}^{\boldsymbol{\alpha}} = \prod_i x_i^{\alpha_i}$ and $\mathbf{x}^{\boldsymbol{\beta}} = \prod_i x_i^{\beta_i}$ in $\mathcal{G}_{\mathbf{x}}$, we set

$$\mathbf{x}^{\boldsymbol{\alpha}} \preccurlyeq \mathbf{x}^{\boldsymbol{\beta}}$$

if and only if there exist non-negative integers $\gamma_j, 1 \leqslant j \leqslant n$ such that

$$\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{x}^{\boldsymbol{\alpha}}. \prod_j \hat{y}_j{}^{\gamma_j}.$$

We denote by $\succcurlyeq$ the opposite preorder.

Assume that the initial exchange matrix B has full rank n. Then the preorder $\preccurlyeq$ becomes an order on $\mathcal{G}_{\mathbf{x}}$. This order is the same as the *dominance order* as defined by F. Qin [106, Definition 3.1.1]. Indeed, by definition, the relation $\prod_i x_i^{\alpha_i} \preccurlyeq \prod_i x_i^{\beta_i}$ is equivalent to the existence of nonnegative integers $\gamma_j, 1 \leqslant j \leqslant n$ such that

$$\forall i, \beta_i = \alpha_i + \sum_j b_{ij}\gamma_j.$$

In vector notation this can be rewritten as

$$\beta = \alpha + B\gamma$$

and thus the order $\preccurlyeq$ coincides with Qin's dominance order on multi-indices.

Following [106], one can use this dominance order to introduce the notions of *pointed elements* and *pointed sets*.

Definition 2.3.1 ([106, Definitions 3.1.4, 3.1.5])**.** *Fix a seed $((x_1, \ldots, x_n, x_{n+1}, \ldots, x_m), B)$ in $\mathcal{A}$ and assume B has full rank n.*

(i) Let P be a Laurent polynomial in the cluster variables $x_1, \ldots, x_m$. One says that P is pointed with respect to the seed $((x_1, \ldots, x_m), B)$ if among the monomials of P, there is a unique monomial which is a maximal element (for the dominance order $\preccurlyeq$) and has coefficient 1. This monomial is called the leading term of P in [106].

(ii) Let L be any set of Laurent polynomials in the cluster variables $x_1, \ldots, x_m$. One says that L is pointed with respect to the seed $((x_1, \ldots, x_m), B)$ if all the elements of L are pointed and two distinct elements of L have different leading terms.

One can associate a degree to each of the cluster variables x_i (see [106]). If P is a pointed element with respect to the seed $((x_1, \ldots, x_m), B)$, then the degree of its leading term can be seen as a generalization of the notion of g-vector.

2.3.2 Generalized parameters

Let us consider an Artinian monoidal category $\mathcal{C}$ and assume we are given a classification of simple objects in $\mathcal{C}$. That is, suppose we are given a poset $(\mathbf{M}, \leqslant)$ together with a bijection ψ between $\mathbf{M}$ and the set $\mathbf{S} := \{[V], V \text{simple in } \mathcal{C}\}$. Let $L(\mu)$ denote a representative of the isomorphism class in $K_0(\mathcal{C})$ corresponding to $\mu \in \mathbf{M}$ via this bijection.

$$\begin{array}{rccc} \psi : & \mathbf{S} & \longrightarrow & \mathbf{M} \\ & [L(\mu)] & \longmapsto & \mu. \end{array}$$

In what follows, μ will be referred to as the *parameter* of the simple object $L(\mu)$ in $\mathcal{C}$. We will also assume that the identity object is simple in $\mathcal{C}$.

From now on we assume that the category $\mathcal{C}$ satisfies the following property:

Assumption A (Decomposition property)**.** *let $\mu, \mu' \in \mathbf{M}$ and $L(\mu), L(\mu')$ the corresponding simple objects in $\mathcal{C}$; then the following equality holds in the Grothendieck ring $K_0(\mathcal{C})$*

$$[L(\mu)] \cdot [L(\mu')] = \sum_{\nu \in \mathbf{N}_{\mu,\mu'} \subset \mathbf{M}} a_\nu [L(\nu)]$$

where $\mathbf{N}_{\mu,\mu'}$ is a finite subset of $\mathbf{M}$ such that there exists a unique maximal element in $\mathbf{N}_{\mu,\mu'}$ and $\{a_\nu, \nu \in \mathbf{N}_{\mu,\mu'}\}$ is a family of nonzero integers. The maximal element of $\mathbf{N}_{\mu,\mu'}$ is denoted by $\mu \odot \mu'$.

Remark 2.3.2. In various examples of categories satisfying this property (for instance categories of modules over quiver Hecke algebras or quantum affine algebras), the integer $a_{\mu \odot \mu'}$ happens to be equal to 1 for any $\mu, \mu' \in \mathbf{M}$, but we will not need this assumption here.

In what follows we will need the additional assumption that the law $\odot$ is compatible with the partial ordering on $\mathbf{M}$ in the following sense:

Assumption B.

$$\forall \mu, \nu \in \mathbf{M}, \mu \leqslant \nu \Rightarrow \forall \lambda \in \mathbf{M}, (\lambda \odot \mu \leqslant \lambda \odot \nu \quad \textit{and} \quad \mu \odot \lambda \leqslant \nu \odot \lambda).$$

Combining Assumptions A and B leads to the following:

Lemma 2.3.3. *The law $\odot$ on $\mathbf{M}$ is associative.*

Proof. First note that for any $\mu, \mu' \in \mathbf{M}$, the set $\mathbf{N}_{\mu,\mu'}$ is finite and has a unique maximal element by Assumption A, hence this element (namely $\mu \odot \mu'$) is a greatest element in $\mathbf{N}_{\mu,\mu'}$. Now let μ,μ' and μ'' in $\mathbf{M}$ and decompose in two different ways the product $[L(\mu)][L(\mu')][L(\mu'')]$. On the one hand, Assumption A gives

$$[L(\mu)][L(\mu')][L(\mu'')] = [L(\mu)] \cdot \sum_{\nu \in \mathbf{N}_{\mu',\mu''}} a_\nu [L(\nu)]$$

with $\nu \leqslant \mu' \odot \mu''$ for every $\nu \in \mathbf{N}_{\mu',\mu''}$. For any $\nu \in \mathbf{N}_{\mu',\mu''}$, the parameters appearing in the decomposition of $[L(\mu)] \cdot [L(\nu)]$ into classes of simples are all smaller than $\mu \odot \nu$; as $\nu \leqslant \mu' \odot \mu''$, Assumption B implies $\mu \odot \nu \leqslant \mu \odot (\mu' \odot \mu'')$. Hence all the parameters involved in the decomposition of $[L(\mu)][L(\mu')][L(\mu'')]$ into classes of simples are smaller than $\mu \odot (\mu' \odot \mu'')$.

On the other hand, one can write

$$[L(\mu)][L(\mu')][L(\mu'')] = \sum_{\nu \in \mathbf{N}_{\mu,\mu'}} a_\nu [L(\nu)].[L(\mu'')]$$

and the same arguments show that all the parameters involved in the decomposition of $[L(\mu)][L(\mu')][L(\mu'')]$ into classes of simples are smaller than $(\mu \odot \mu') \odot \mu''$. In particular we get $\mu \odot (\mu' \odot \mu'') \leqslant (\mu \odot \mu') \odot \mu''$ and $\mu \odot (\mu' \odot \mu'') \geqslant (\mu \odot \mu') \odot \mu''$ and hence $\mu \odot (\mu' \odot \mu'') = (\mu \odot \mu') \odot \mu''$.

□

Thus the operation

$$\begin{array}{ccc} \mathbf{M}\times\mathbf{M} & \to & \mathbf{M} \\ (\mu,\mu') & \mapsto & \mu\odot\mu' \end{array}$$

provides $\mathbf{M}$ with a monoid structure. The neutral element 1_M is the image via ψ of the class of the identity object of $\mathcal{C}$. By Assumption B, the monoid $(\mathbf{M},\odot)$ is an ordered monoid with respect to $\leqslant$.

We now assume that $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$. Let ϕ be a ring isomorphism

$$\phi : K_0(\mathcal{C}) \xrightarrow{\simeq} \mathcal{A}.$$

As $K_0(\mathcal{C})$ is isomorphic to $\mathcal{A}$, it is in particular commutative which implies that the monoid $(\mathbf{M},\odot)$ is commutative as well. Hence it can be canonically embedded into its Grothendieck group $G(\mathbf{M})$, which is defined as follows (see [15]):

Definition 2.3.4 (Grothendieck group of $\mathbf{M}$)**.** *Elements of $G(\mathbf{M})$ are equivalence classes of couples (μ,ν) of elements of $\mathbf{M}$ with respect to the equivalence relation*

$$(\mu,\nu)\sim(\mu',\nu') \Leftrightarrow \exists\lambda\in\mathbf{M}, \mu\odot\nu'\odot\lambda=\nu\odot\mu'\odot\lambda.$$

The group $G(\mathbf{M})$ is an abelian group. We denote it by $\mathbf{G}$ in what follows.

The inverse in $\mathbf{G}$ of an element $\mu\in\mathbf{M}$ will be denoted by $\mu^{\odot-1}$. Similarly $g^{\odot-1}$ stands for the inverse in $\mathbf{G}$ of any element g of $\mathbf{G}$. We will refer to elements of $\mathbf{M}$ as *parameters* and elements of $\mathbf{G}$ as *generalized parameters.*

Proposition 2.3.5. *The ordering $\leqslant$ on $\mathbf{M}$ naturally extends to a partial ordering on $\mathbf{G}$ that we also denote by $\leqslant$.*

Proof. One defines a partial ordering on $\mathbf{M}\times\mathbf{M}$ by setting

$$(\mu,\nu)\leqslant(\mu',\nu') \Leftrightarrow \exists\lambda\in\mathbf{M}, \lambda\odot\mu\odot\nu'\leqslant\lambda\odot\mu'\odot\nu.$$

Using the assumption B, one can check that if $(\mu,\nu)\sim(\mu',\nu')$ then for any $(\mu'',\nu'')\in\mathbf{M}\times\mathbf{M}$, one has $(\mu,\nu)\leqslant(\mu'',\nu'')\Leftrightarrow(\mu',\nu')\leqslant(\mu'',\nu'')$. Thus $\leqslant$ naturally gives rise to a well-defined partial ordering on $\mathbf{G}$. □

Let us now fix a seed $((x_1,\dots,x_m),B)$ in $\mathcal{A}$ and choose for each $1\leqslant i\leqslant m$ a representative $L(\mu_i)$ of the isomorphism class $\phi^{-1}(x_i)\in\mathbf{S}$ corresponding to the cluster variable x_i . Recall that $\mathcal{M}_{\mathbf{x}}$ stands for the monoid of all monomials in the x_i and $\mathcal{G}_{\mathbf{x}}$ for the abelian group of all Laurent monomials in the x_i. For any m-tuple of integers $(\alpha_1,\dots,\alpha_m)$, we let $\boldsymbol{\mu}^{\boldsymbol{\alpha}}$ denote the element $\bigodot_{1\leqslant i\leqslant m}{\mu_i}^{\alpha_i}$ of $\mathbf{G}$. Of course $\boldsymbol{\mu}^{\boldsymbol{\alpha}}$ belongs to $\mathbf{M}$ if all the α_i are nonnegative.

Let us consider the subset $\mathcal{P}$ of $K_0(\mathcal{C})$ consisting of nonzero classes $[M]$ such that there is a unique maximal element among the parameters of the simples appearing in the Jordan-Hölder series of M. Let Ψ be the map

$$\begin{array}{cccc} \Psi: & \mathcal{P} & \longrightarrow & \mathbf{G} \\ & [M]=a_1[M_1]+\cdots+a_r[M_r] & \longmapsto & max_{\leqslant}\left(\psi([M_k]), 1\leqslant k\leqslant r\right). \end{array}$$

Here the a_k are integers and the M_i are the simples of the Jordan-Hölder series of M. Note that $\mathcal{P}$ contains 1, whose image by Ψ is the neutral element of $\mathbf{M}$. The set $\mathbf{S}$ of classes of simples in $\mathcal{C}$ is a basis of $K_0(\mathcal{C})$ hence the writing $a_1[M_1]+\cdots+a_r[M_r]$ of an element of $\mathcal{P}$ is unique up to reordering the terms and thus the map Ψ is well-defined.

Let $\tilde{\mathcal{P}}$ be the subset of $Frac(\mathcal{A})$ defined in the following way:

$$\tilde{\mathcal{P}} := \{\mathbf{x}^{\boldsymbol{\alpha}}\phi(p), \boldsymbol{\alpha} \in \mathbb{Z}^m, p \in \mathcal{P}\}.$$

In particular $\tilde{\mathcal{P}}$ contains $\mathcal{G}_{\mathbf{x}}$. Let $\tilde{\Psi}$ be the map

$$\begin{array}{rccc} \tilde{\Psi}: & \tilde{\mathcal{P}} & \longrightarrow & \mathbf{G} \\ & \mathbf{x}^{\boldsymbol{\alpha}}\phi(p) & \longmapsto & \boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p). \end{array}$$

Proposition 2.3.6. *The map $\tilde{\Psi}$ is well defined and satisfies the following properties:*

(i) $\tilde{\Psi} \circ \phi$ coincides with ψ on $\mathbf{S}$.

(ii) $\tilde{\Psi}$ defines an abelian group morphism from $\mathcal{G}_{\mathbf{x}}$ (for the natural multiplication) to $(\mathbf{G}, \odot)$.

Proof. In order to show that the map $\tilde{\Psi}$ is well-defined, we need to check that if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are two m-tuples of integers and p, q two elements of $\mathcal{P}$ such that $\mathbf{x}^{\boldsymbol{\alpha}}\phi(p) = \mathbf{x}^{\boldsymbol{\beta}}\phi(q)$, then the equality $\boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p) = \boldsymbol{\mu}^{\boldsymbol{\beta}} \odot \Psi(q)$ holds in $\mathbf{G}$. Let us write $p = a_1[M_1] + \cdots + a_r[M_r]$ and $q = b_1[N_1] + \cdots + b_s[N_s]$, with $r, s \geqslant 0$, $a_1, \ldots a_r, b_1, \ldots, b_s \in \mathbb{Z}$ and $[M_1], \ldots [M_r], [N_1], \ldots, [N_s] \in \mathbf{S}$. Let $\boldsymbol{\gamma}$ be an m-tuple of nonnegative integers such that $\mathbf{x}^{\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\beta}}$ are monomials in the x_i. One can write

$$\begin{aligned} \mathbf{x}^{\boldsymbol{\alpha}}\phi(p) = \mathbf{x}^{\boldsymbol{\beta}}\phi(q) &\Leftrightarrow \mathbf{x}^{\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\alpha}}\phi(p) = \mathbf{x}^{\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\beta}}\phi(q) \\ &\Leftrightarrow \phi\left(\prod_i [L(\mu_i)]^{\gamma_i+\alpha_i} p\right) = \phi\left(\prod_i [L(\mu_i)]^{\gamma_i+\beta_i} q\right) \\ &\Leftrightarrow [L(\boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\alpha}})].(a_1[M_1] + \cdots + a_r[M_r]) = [L(\boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\beta}})].(b_1[N_1] + \cdots + b_s[N_s]) \end{aligned}$$

as ϕ is an isomorphism. Let us set $\mu := \boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\alpha}}$ and $\nu := \boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\beta}}$; they are elements of $\mathbf{M}$. By Assumption A, for every $1 \leqslant i \leqslant r$ the product $[L(\mu)].[M_i]$ decomposes as a sum of classes of simples and $\mu \odot \psi([M_i])$ is maximal among the corresponding parameters. As $p \in \mathcal{P}$, the finite set of parameters $\psi([M_1]), \ldots \psi([M_r])$ has a unique maximal element. For simplicity let us assume it is $\psi([M_1])$. Then Assumption B implies that $\mu \odot \psi([M_1])$ is maximal among the parameters appearing in the decompositions of $[L(\mu)].[M_i], 1 \leqslant i \leqslant r$ into classes of simples. The same arguments of course hold for the right hand side $[L(\nu)].q$. As $\mathbf{S}$ is a basis of $K_0(\mathcal{C})$, one gets

$$\mu \odot \psi([M_1]) = \nu \odot \psi([N_1]).$$

Now by definition of Ψ, one has $\psi([M_1]) = \Psi(p)$ and $\psi([N_1]) = \Psi(q)$. Hence we can write in $\mathbf{G}$:

$$\boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \mu \odot \Psi(p) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \mu \odot \psi([M_1]) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \nu \odot \psi([N_1]) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \nu \odot \Psi(q) = \boldsymbol{\mu}^{\boldsymbol{\beta}} \odot \Psi(q)$$

which is the desired equality. Thus $\tilde{\Psi}$ is well-defined.

For any $\mu \in \mathbf{M}$, $\phi([L(\mu)])$ belongs to $\tilde{\mathcal{P}}$ with $\boldsymbol{\alpha}$ being zero and $p = [L(\mu)]$. Hence by definition one has

$$\tilde{\Psi}\left(\phi([L(\mu)])\right) = \Psi([L(\mu)]) = \mu = \psi([L(\mu)])$$

which proves (i).

Taking elements of $\tilde{\mathcal{P}}$ for which p is the empty sum, one gets

$$\begin{aligned} \tilde{\Psi}(\mathbf{x}^{\boldsymbol{\alpha}}) = \boldsymbol{\mu}^{\boldsymbol{\alpha}} &= \bigodot_{1 \leqslant i \leqslant m} {\mu_i}^{\alpha_i} \\ &= \bigodot_{1 \leqslant i \leqslant m} \tilde{\Psi}\left(\phi([L(\mu_i)])\right)^{\alpha_i} \text{ by (i)} \\ &= \bigodot_{1 \leqslant i \leqslant m} \tilde{\Psi}(x_i)^{\alpha_i} \end{aligned}$$

which proves (ii).

□

2.3.3 Compatible seeds

In this subsection, we introduce the notion of compatible seed, state our main conjecture (Conjecture 2.3.10) and explain some consequences.

Definition 2.3.7 (Compatible seed)**.** *Let $\mathcal{S} := ((x_1, \dots, x_m), B)$ be a seed in $\mathcal{A}$, $\mathcal{G}_{\mathbf{x}}$ the group of Laurent monomials in the cluster variables $x_1, \dots, x_m$, and $\preccurlyeq$ the corresponding dominance order on $\mathcal{G}_{\mathbf{x}}$. Let $\tilde{\Psi}$ be the map given by Proposition 2.3.6. For any $1 \leqslant j \leqslant n$, set*

$$\hat{\mu_j} := \tilde{\Psi}(\hat{y_j}) = \bigodot_{1 \leqslant i \leqslant m} \mu_i^{\odot b_{ij}}.$$

We say that the seed $\mathcal{S}$ is compatible if the restriction $\tilde{\Psi}_{|\mathcal{G}_{\mathbf{x}}} : (\mathcal{G}_{\mathbf{x}}, \preccurlyeq) \longrightarrow (\mathbf{G}, \leqslant)$ is either increasing or decreasing.

Remark 2.3.8. By construction, the restriction of $\tilde{\Psi}$ to $\mathcal{G}_{\mathbf{x}}$ is increasing if and only if for any Laurent monomials $\prod_i x_i^{\alpha_i}$ and $\prod_i x_i^{\beta_i}$ one has

$$\prod_i x_i^{\alpha_i} \preccurlyeq \prod_i x_i^{\beta_i} \Rightarrow \bigodot_i \mu_i^{\odot \alpha_i} \leqslant \bigodot_i \mu_i^{\odot \beta_i}.$$

This is equivalent to the following:

$$\forall 1 \leqslant j \leqslant n, \forall \mu \in \mathbf{M}, \mu \leqslant \hat{\mu_j} \odot \mu.$$

In the same way, $\tilde{\Psi}$ is decreasing on $\mathcal{G}_{\mathbf{x}}$ if and only if

$$\forall 1 \leqslant j \leqslant n, \forall \mu \in \mathbf{M}, \mu \geqslant \hat{\mu_j} \odot \mu.$$

In many examples of monoidal categorifications of cluster algebras, for instance via quantum affine algebras or quiver Hecke algebras, we will check this condition to prove that a seed is compatible.

Remark 2.3.9. Note that if the seed $\mathcal{S} = ((x_1, \dots, x_m), B)$ is compatible with $\tilde{\Psi}$ increasing on $\mathcal{G}_{\mathbf{x}}$, then $\tilde{\mathcal{P}}$ contains all the Laurent polynomials in the x_i that are pointed with respect to $\mathcal{S}$, i.e. $\tilde{\mathcal{P}} \supseteq \mathcal{PT}(0)$ with the notations of [106].

One can now state the main conjecture of this paper:

Conjecture 2.3.10. *Let $\mathcal{A}$ be a cluster algebra and $\mathcal{C}$ an Artinian monoidal categorification of $\mathcal{A}$. Assume there exists a poset $(\mathbf{M}, \leqslant)$ as above such that Assumptions A and B hold. Then there exists a compatible seed in $\mathcal{A}$.*

The next statements provide some useful consequences of the existence of a compatible seed. In particular we combine it with the results of [43] and [32] recalled in Section 2.1.1 to relate parameters of simple objects in $\mathcal{C}$ to some cluster algebra invariants, such as g-vectors and F-polynomials.

Let $\mathcal{C}$ be a Artinian monoidal categorification of a cluster algebra $\mathcal{A}$ and assume Conjecture 2.3.10 holds. Let $((x_1, \dots, x_n, x_{n+1}, \dots, x_m), B)$ be a compatible seed. Consider x_l^t a cluster variable in $\mathcal{A}$ belonging to another cluster and let $F^{l,t}$ be its F-polynomial. Let $X_1^{a_1^{l,t}} \cdots X_n^{a_n^{l,t}}$ be the monomial given by Theorem 2.1.5(i).

Corollary 2.3.11. *One has $F^{l,t}(\hat{y_1}, \dots, \hat{y_n}) \in \tilde{\mathcal{P}}$ and*

$$\tilde{\Psi}\left(F^{l,t}(\hat{y_1}, \dots, \hat{y_n})\right) = \begin{cases} \bigodot_j \hat{\mu_j}^{\odot a_j^{l,t}} & \text{if } \tilde{\Psi} \text{ is increasing on } \mathcal{G}_{\mathbf{x}}, \\ 1_{\mathbf{G}} & \text{in the other case.} \end{cases}$$

Here $1_{\mathbf{G}}$ denotes the neutral element of $\mathbf{G}$.

Proof. By Theorem 2.1.4, $F^{l,t}(\hat{y_1},\dots,\hat{y_n})$ is the product of a Laurent monomial in the x_i with the cluster variable $x^{l,t}$. As $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}$, $x^{l,t}$ is the image by ϕ of the class of a simple object in $\mathcal{C}$. Hence $F^{l,t}(\hat{y_1},\dots,\hat{y_n}) \in \tilde{\mathcal{P}}$.

By Theorem 2.1.5 (i) and (ii), any monomial of $F^{l,t}$ can be written as $X_1^{b_1}\cdots X_n^{b_n}$ with $0 \leqslant b_j \leqslant a_j^{l,t}$ for all $1 \leqslant j \leqslant n$. As the seed $((x_1,\dots,x_m),B)$ is compatible, evaluating on the $\hat{y_j}$ and considering the corresponding generalized parameters $\hat{\mu_j}$ yields

$$\bigodot_j \hat{\mu_j}^{a_j^{l,t}} \geqslant \bigodot_j \hat{\mu_j}^{b_j} \quad \text{if } \tilde{\Psi} \text{ is increasing on } \mathcal{G}_{\mathbf{x}}$$

and

$$\bigodot_j \hat{\mu_j}^{a_j^{l,t}} \leqslant \bigodot_j \hat{\mu_j}^{b_j} \quad \text{if } \tilde{\Psi} \text{ is decreasing } \mathcal{G}_{\mathbf{x}}.$$

Hence among all the generalized parameters appearing in the term $F^{l,t}(\hat{y_1},\dots,\hat{y_n})$, there is one which is greater than the others, namely $\bigodot_j \hat{\mu_j}^{\odot a_j^{l,t}}$ if $\tilde{\Psi}$ is increasing $\mathcal{G}_{\mathbf{x}}$, $1_{\mathbf{G}}$ in the other case.

□

The following Corollary shows that the existence of a compatible seed in $\mathcal{A}$ implies relations between the g-vector (with respect to this initial (compatible) seed) of any cluster variable in $\mathcal{A}$ and the parameter of the corresponding simple object in $\mathcal{C}$.

Let $x_{n+1}^{c_1^{l,t}}\cdots x_m^{c_{m-n}^{l,t}}$ be the monomial in the frozen variables equal to the denominator $F^{l,t}{}_{|\mathbb{P}}(y_1,\dots,y_n)$ in the right hand side of equation (2.2). Note that, in view of the definition of the semifield $\mathbb{P}$, the $c_i^{l,t}$ are negative integers, as the F-polynomial $F^{l,t}$ has constant term equal to 1 by Theorem 2.1.5(ii).

Corollary 2.3.12. *Let $\mu^{l,t}$ be the parameter of the simple module corresponding to the cluster variable $x^{l,t}$, i.e. $x^{l,t} = \phi([L(\mu^{l,t})])$. Then*

$$\mu^{l,t} = \begin{cases} \bigodot_{1\leqslant j\leqslant n} \hat{\mu_j}^{\odot a_j^{l,t}} \odot \bigodot_{1\leqslant i\leqslant m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\leqslant i\leqslant n} \mu_i^{\odot g_i^{l,t}} & \textit{if } \tilde{\Psi} \textit{ is increasing on } \mathcal{G}_{\mathbf{x}}, \\ \bigodot_{1\leqslant i\leqslant m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\leqslant i\leqslant n} \mu_i^{\odot g_i^{l,t}} & \textit{if } \tilde{\Psi} \textit{ is decreasing on } \mathcal{G}_{\mathbf{x}}. \end{cases}$$

Proof. We give the proof only in the case where the restriction of $\tilde{\Psi}$ to $\mathcal{G}_{\mathbf{x}}$ is increasing, the other case being analogous.

By Proposition 2.3.6 (i), $\tilde{\Psi}(x_l^t) = \psi([L(\mu^{l,t})]) = \mu^{l,t}$. On the other hand, one can use the previous Corollary and apply $\tilde{\Psi}$ to both sides of equation (2.2):

$$\begin{aligned} \bigodot_j \hat{\mu_j}^{\odot a_j^{l,t}} &= \tilde{\Psi}\left(F^{l,t}(\hat{y_1},\dots,\hat{y_n})\right) = \tilde{\Psi}\left(\frac{x_{n+1}^{c_1^{l,t}}\cdots x_m^{c_{m-n}^{l,t}}}{x_1^{g_1^{l,t}}\cdots x_n^{g_n^{l,t}}}.x^{l,t}\right) \\ &= \mu^{l,t} \odot \bigodot_{1\leqslant i\leqslant m-n} \mu_{n+i}^{\odot c_i^{l,t}} \odot \bigodot_{1\leqslant i\leqslant n} \mu_i^{\odot(-g_i^{l,t})} \text{by Proposition 2.3.6(ii).} \end{aligned}$$

Finally one can conclude:

$$\mu^{l,t} = \bigodot_{1\leqslant j\leqslant n} \hat{\mu_j}^{\odot a_j^{l,t}} \odot \bigodot_{1\leqslant i\leqslant m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\leqslant i\leqslant n} \mu_i^{\odot g_i^{l,t}}.$$

□

2.3.4 First example: the category $\mathcal{C}_1$

The first example of compatible seed appears in the work of Hernandez-Leclerc [56] and is one of the main motivations for this work.

Recall from Section 2.1.3 the definition of the category $\mathcal{C}_1$. This category was introduced by Hernandez-Leclerc in [56] as a (monoidal) subcategory of the category of finite dimensional representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. For $\mathfrak{g}$ of types ADE, the category $\mathcal{C}_1$ is a monoidal categorification of a cluster algebra of the same cluster type (in the classification of [42]) than the Lie type of $\mathfrak{g}$ ([56, 96]). As explained in Section 2.1.3, the simple finite dimensional $U_q(\hat{\mathfrak{g}})$-modules are parametrized by dominant monomials. The monoid $\mathbf{M}$ parametrizing simple objects in the category $\mathcal{C}_1$ is a submonoid of the set of dominant monomials involving only the variables $Y_{i,a}, i \in I, a \in q^{\mathbb{Z}}$. The monoid law $\odot$ is simply the natural multiplication of monomials. The ordering $\leqslant$ on $\mathbf{M}$ is the restriction of the Nakajima order on dominant monomials (see Section 2.1.3). Assumptions A and B are obviously satisfied for the category $\mathcal{C}_1$.

In [56], Hernandez-Leclerc give explicitly an initial seed in the category $\mathcal{C}_1$:

Theorem 2.3.13 ([56]). *Each seed has $n = |I|$ unfrozen variables and n frozen variables. These frozen variables are given by the classes of the modules $L(Y_{i,q^{\xi_i}}Y_{i,q^{\xi_i+2}})$, $i \in I$.*

Moreover, an initial seed in this cluster algebra is given by the following classes:

$$[L(Y_{i,q^{\xi_i+2}})] \qquad [L(Y_{i,q^{\xi_i}}Y_{i,q^{\xi_i+2}})] \qquad i \in I$$

together with the exchange matrix $B = (b_{ij})$ whose columns are indexed by I and rows by $I \sqcup I' = [1, n] \sqcup [n + 1, 2n]$, and whose entries are given by

$$b_{ij} := \begin{cases} (-1)^{\xi_j} a_{ij} & \text{if } i, j \in I \text{ and } i \neq j, \\ -1 & \text{if } j \in I \text{ and } i = j + n \in I', \\ -a_{kj} & \text{if } j \in J_0 \text{ and } i = k + n \in I' \text{ with } k \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Here the a_{ij} are the entries of the Cartan matrix associated to $\mathfrak{g}$.

The cluster algebra $\mathcal{A}$ is the cluster algebra (in the classification of [42]) generated by the initial seed $((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}), B)$ where B is the matrix above. Following [56], we denote by ι the unique ring isomorphism

$$\iota : \mathcal{A} \xrightarrow{\simeq} K_0(\mathcal{C}_1)$$

such that

$$\iota(x_i) = [L(Y_{i,q^{\xi_i+2}})] \qquad \iota(x_{n+i}) = [L(Y_{i,q^{\xi_i}}Y_{i,q^{\xi_i+2}})] \qquad 1 \leqslant i \leqslant n.$$

The isomorphism ι is the inverse of the isomorphism ϕ in Section 2.3.2. Using the map $\tilde{\Psi}$ associated to the seed $((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}), B)$ above, one can compute the generalized parameters $\hat{\mu_j}$ corresponding to this seed. This is done by the following statement, as a direct consequence of Theorem 2.3.13:

Proposition 2.3.14 ([56, Lemma 7.2]). *With the notations of Section 2.3.2, one has:*

$$\forall 1 \leqslant j \leqslant n, \quad \hat{\mu_j} = A^{-1}_{j,q^{\xi_j+1}}.$$

Corollary 2.3.15. *Conjecture 2.3.10 holds for the category $\mathcal{C}_1$.*

Proof. By definition of the Nakajima ordering on monomials (see Section 2.1.3), Proposition 2.3.14 implies that for any dominant monomial $\mathfrak{m}$, one has

$$\forall 1 \leqslant j \leqslant n, \mathfrak{m} \geqslant \hat{\mu_j}\mathfrak{m}.$$

By Remark 2.3.8, this implies that the map $\tilde{\Psi}$ associated to the seed $((x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}), B)$ is decreasing. Hence this seed is compatible in the sense of Definition 2.3.7 and Conjecture 2.3.10 holds. □

We conclude this section with an illustration of the relationships between g-vectors and highest weights for quantum affine algebras that occur as a consequence of the initial seed of [56] being compatible. The cluster structure on $K_0(\mathcal{C}_1)$ is a finite type cluster algebra; thus one can use the results of [42] and label the cluster variables by almost positive roots, i.e. positive roots together with the opposite of the simple roots. Let $x[\alpha]$ denote the cluster variable associated to the almost positive root α with respect to the above initial seed.

Following [42], one defines piece-wise linear involutions τ_ϵ ($\epsilon \in \{-1, 1\}$) of the root lattice Q of $\mathfrak{g}$: for any $\gamma \in Q$,

$$[\tau_\epsilon(\gamma) : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{j \neq i} a_{ij} \max(0, [\gamma, \alpha_j]) & \text{if } \epsilon_i = \epsilon \\ [\gamma : \alpha_i] & \text{if } \epsilon_i \neq \epsilon \end{cases}$$

where $[\gamma : \alpha_i]$ stands for the coefficient of α_i in the expansion of γ on simple roots.

Corollary 2.3.16 ([56, Corollary 7.4])**.** *Let α be an almost positive root. Set $\beta := \tau_-(\alpha)$. Write $\beta = \sum_i b_i \alpha_i$. The highest weight of the simple module corresponding to the cluster variable $x[\alpha]$ is given by*

$$\prod_{i \in I_0} Y_{i,1}^{b_i} \cdot \prod_{i \in I_1} Y_{i,q^3}^{b_i}.$$

It is known from [43] that in the case of a cluster algebra of finite type, the g-vector of the variable $x[\alpha]$ is given by

$$\mathbf{g}(\alpha) = E\tau_-(\alpha)$$

where E is the automorphism of the root lattice of $\mathfrak{g}$ which sends the simple root α_i onto $(-1)^{\xi_i+1}\alpha_i$.

Thus the previous corollary can be reformulated in the following way:

Corollary 2.3.17. *Let α be an almost positive root and let $\mathbf{g}(\alpha)$ be the g-vector of the cluster variable $x[\alpha]$ with respect to the above initial seed. The highest weight of the simple module corresponding to $x[\alpha]$ is given by*

$$\prod_{i \in I_0} Y_{i,1}^{-g_i} \cdot \prod_{i \in I_1} Y_{i,q^3}^{g_i}.$$

2.4 A mutation rule for parameters of simple representations of quiver Hecke algebras

In this section, we consider the category $\mathcal{C} = R - gmod$ of finite dimensional representations of symmetric quiver Hecke algebras of finite type A_n. The set $\mathbf{M}$ is the set of *dominant words* (see Section 2.1) and the order $\leqslant$ is the natural lexicographic order; it is a total ordering hence Assumption A obviously holds. Moreover, with the notations of Section 2.2.5, one has $\mu \odot \nu = \max(L(\mu) \circ L(\nu))$ for any dominant words μ and ν. We begin by describing explicitly the monoid operation $\odot$ for dominant words. In particular it can be easily computed using *canonical factorizations* of dominant words (see Proposition 2.2.24). We apply this in the context of monoidal categorifications of cluster algebras via quiver Hecke algebras following the works of Kang-Kashiwara-Kim-Oh ([66, 69]). We obtain a combinatorial rule for the transformation of dominant words under cluster mutation.

2.4.1 Convolution product of simple modules

This subsection is devoted to the description of the monoid structure $\odot$ on the monoid $\mathbf{M}$ of dominant words in the case of a symmetric quiver Hecke algebra of type A_n. First we restrict to the case where the canonical factorizations of two words μ, μ' are ordered with respect to each other. We show that in this case, Proposition 2.2.30 implies that the monoidal product $\mu \odot \mu'$ is simply the concatenation of μ and μ' (Corollary 2.4.3). Then we state the main result of this section (Proposition 2.4.4) which gives a combinatorial expression for $\mu \odot \mu'$ for any μ, μ'. Our proof involves ideas similar to the ones used in [79] for the proof of Proposition 2.2.30, but here we use the specific form of dominant Lyndon words (in bijection with positive roots) in type A_n.

Recall (see Proposition 2.2.24) that any word μ can be written in a unique way as a concatenation of Lyndon words in the decreasing order. This is called the canonical factorization of μ. Moreover, if the word μ is dominant, then all the Lyndon words involved in the canonical factorization of μ are dominant as well (Theorem 2.2.26(ii)). By Theorem 2.2.26(i), the canonical factorization of a dominant word μ can be seen as a sum of positive roots in the decreasing order. In particular, in type A_n these positive roots correspond to words of the form $k(k+1)\dots l$ with $k \leqslant l$.

We begin by recalling a technical result from [79].

Lemma 2.4.1 ([79, Lemma 5.1])**.** *Let $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(r)}, \mathbf{j}^{(1)}, \dots, \mathbf{j}^{(r)}$ be words such that for each $k \in \{1, \dots, r\}$, $\mathbf{i}^{(k)}$ and $\mathbf{j}^{(k)}$ have same length. Assume that $\mathbf{i}^{(1)} \geqslant \mathbf{j}^{(1)}, \dots, \mathbf{i}^{(r)} \geqslant \mathbf{j}^{(r)}$. Then $\max(\mathbf{i}^{(1)} \circ \dots \circ \mathbf{i}^{(r)}) \geqslant \max(\mathbf{j}^{(1)} \circ \dots \circ \mathbf{j}^{(r)})$. Moreover, this inequality is an equality if and only if all the inequalities $\mathbf{i}^{(k)} \geqslant \mathbf{j}^{(k)}$ are equalities.*

The next Proposition states that the product $\mu \odot \mu'$ of two dominant words μ and μ' coincides with the highest word in the quantum shuffle product of μ and μ'.

Proposition 2.4.2. *Let μ, ν be dominant words. Then*

$$\mu \odot \nu = \max(\mu \circ \nu).$$

Proof. By Theorem 2.2.31, we can write

$$ch_q(L(\mu)) = P(q).\mu + \sum_{\mu' < \mu} a_{\mu'}(q).\mu' \qquad \text{and} \qquad ch_q(L(\nu)) = Q(q).\nu + \sum_{\nu' < \nu} b_v(q).\nu'$$

where P, Q, a, b are Laurent polynomials in q (P and Q non zero). By definition

$$\mu \odot \nu = \max(L(\mu) \circ L(\nu)) = \max(ch_q(L(\mu) \circ L(\nu))) = \max(ch_q(L(\mu)) \circ ch_q(L(\nu))).$$

By Lemma 2.4.1, for any $\mu' < \mu$, $\max(\mu' \circ \nu) < \max(\mu \circ \nu)$. Similarly, $\max(\mu \circ \nu') < \max(\mu \circ \nu)$ and $\max(\nu' \circ \nu) < \max(\mu \circ \nu)$ for any $\mu' < \mu$ and $\nu' < \nu$. So the highest word of $ch_q(L(\mu)) \circ ch_q(L(\nu))$ can only come from the shuffle product $\mu \circ \nu$. Hence

$$\mu \odot \nu = \max(\mu \circ \nu).$$

□

More generally, the same proof shows that for any finite formal series R and S on W with coefficients in $\mathbb{Z}[q, q^{-1}]$, one has $\max(R \circ S) = \max(max(R) \circ max(S))$. The following Corollary is then a direct consequence of Proposition 2.2.30.

Corollary 2.4.3. *Let μ, μ' two dominant words. Write their canonical factorizations*

$$\mu = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r} \qquad \text{and} \qquad \mu' = (\mathbf{i}'^{(1)})^{n'_1} \cdots (\mathbf{i}'^{(r')})^{n'_{r'}}$$

with $\mathbf{i}^{(1)} > \cdots > \mathbf{i}^{(r)}$ and $\mathbf{i}'^{(1)} > \cdots > \mathbf{i}'^{(r')}$. Assume $\mathbf{i}^{(r)} \geqslant \mathbf{i}'^{(1)}$.

Then

$$\mu \odot \mu' = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r} (\mathbf{i}'^{(1)})^{n'_1} \cdots (\mathbf{i}'^{(r')})^{n'_{r'}}.$$

Proof. Setting $R := (\mathbf{i}^{(1)})^{n_1} \circ \cdots \circ (\mathbf{i}^{(r)})^{n_r}$ and $R' := (\mathbf{i}'^{(1)})^{n'_1} \circ \cdots \circ (\mathbf{i}'^{(r)})^{n'_{r'}}$, Proposition 2.2.30 implies $\mu = \max(R)$ and $\mu' = \max(R')$. Then by Proposition 2.4.2, one has:

$$\begin{aligned}
\mu \odot \mu' &= \max(\mu \circ \mu') && \text{by Proposition 2.4.2} \\
&= \max(max(R) \circ max(R')) \\
&= \max(R \circ R') && \text{by Proposition 2.4.2} \\
&= \max\left((\mathbf{i}^{(1)})^{n_1} \circ \cdots \circ (\mathbf{i}^{(r)})^{n_r} \circ (\mathbf{i}'^{(1)})^{n'_1} \circ \cdots \circ (\mathbf{i}'^{(r)})^{n'_{r'}} \right).
\end{aligned}$$

The assumption $\mathbf{i}^{(r)} \geqslant \mathbf{i}'^{(1)}$ implies that $(\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r} (\mathbf{i}'^{(1)})^{n'_1} \cdots (\mathbf{i}'^{(r)})^{n'_{r'}}$ is the canonical factorization of the concatenation $\mu\mu'$. Hence by Proposition 2.2.30, we get

$$\mu \odot \mu' = \mu\mu' = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r} (\mathbf{i}'^{(1)})^{n'_1} \cdots (\mathbf{i}'^{(r')})^{n'_{r'}}.$$

□

One can now state the main result of this section. It can be seen as a generalization of Corollary 2.4.3, as we now drop the hypothesis $\mathbf{i}^{(r)} \geqslant \mathbf{i}'^{(1)}$.

Proposition 2.4.4. *Let μ, μ' two dominant words. Write their canonical factorizations*

$$\mu = (\mathbf{i}^{(1)})^{n_1} \cdots (\mathbf{i}^{(r)})^{n_r} \qquad \text{and} \qquad \mu' = (\mathbf{i}'^{(1)})^{n'_1} \cdots (\mathbf{i}'^{(r')})^{n'_{r'}}.$$

Let $\{\mathbf{j}^{(1)}, \ldots, \mathbf{j}^{(s)}\}$ be the set of all the words $\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(r)}, \mathbf{i}'^{(1)}, \ldots, \mathbf{i}'^{(r')}$ ranged in the decreasing order. Let $m_1, \ldots, m_s, m'_1 \ldots m'_s$ the positive integers uniquely determined by

$$\mu = (\mathbf{j}^{(1)})^{m_1} \cdots (\mathbf{j}^{(s)})^{m_s} \qquad , \qquad \mu' = (\mathbf{j}^{(1)})^{m'_1} \cdots (\mathbf{j}^{(s)})^{m'_s}.$$

Then

$$\mu \odot \mu' = (\mathbf{j}^{(1)})^{m_1 + m'_1} \cdots (\mathbf{j}^{(s)})^{m_s + m'_s}.$$

Using Theorem 2.2.31, one can reformulate this statement in the following way: write the positive roots in the decreasing order

$$\begin{aligned}
\alpha_n > \alpha_{n-1} + \alpha_n > \alpha_{n-1} > \cdots > \alpha_i + \alpha_{i+1} + \cdots + \alpha_n > \cdots > \alpha_i > \cdots \\
\cdots > \alpha_1 + \cdots + \alpha_n > \alpha_1
\end{aligned} \tag{2.3}$$

and let $r_n = n(n+1)/2$ denote the number of these positive roots. Define a map

$$\begin{array}{ccc}
(\mathbf{M}, \odot) & \longrightarrow & (\mathbb{Z}_{\geqslant 0}^{r_n}, +) \\
\mu & \longmapsto & \overrightarrow{\mu}
\end{array} \tag{2.4}$$

such that the ith coordinate of the vector $\overrightarrow{\mu}$ is equal to the multiplicity of the Lyndon word corresponding to the ith positive root (in the above decreasing order) in the canonical factorization of μ.

Theorem 2.4.5. *The map* (2.4) *is an isomorphism of abelian monoids.*

Proof of Proposition 2.4.4. For simplicity we use a slight change of notation for the proof: we write

$$\mu = \mathbf{i}^{(1)} \cdots \mathbf{i}^{(r)} \qquad \textit{and} \qquad \mu' = \mathbf{i}'^{(1)} \cdots \mathbf{i}'^{(r')}$$

with $\mathbf{i}^{(1)} \geqslant \cdots \geqslant \mathbf{i}^{(r)}$ and $\mathbf{i}'^{(1)} \geqslant \cdots \geqslant \mathbf{i}'^{(r')}$ dominant Lyndon words not necessarily distinct. Let n (resp. n') be the length of μ (resp. μ'). The starting point is the word $\mu.\mu'$ which is the concatenation of the words μ and μ' and we consider permutations $\sigma \in \mathfrak{S}_{r,s}$, i.e. whose restrictions to $[|1;n|]$ and $[|n+1;n+n'|]$ are increasing (see Definition 2.2.3).

First note that the word $\mu \odot \mu'$ indeed appears in the quantum shuffle product of μ with μ': consider the permutation σ simply defined by rearranging the blocks $(\mathbf{i}^{(1)}), \ldots, (\mathbf{i}^{(r)}), (\mathbf{i}'^{(1)}), \ldots, (\mathbf{i}'^{(r')})$ of the concatenation $\mu.\mu'$ and put them in the decreasing order.

We write $\mu = h_1, \ldots, h_n$ and $\mu' = h'_1, \ldots, h'_{n'}$. The concatenation of μ and μ' is then $\mu.\mu' = h_1, \ldots, h_n, h'_1, \ldots, h'_{n'}$. As in Definition 2.2.3, we set $h_{n+1} := h'_1, \ldots, h_{n+n'} := h'_{n'}$ and thus $\mu.\mu' = h_1, \ldots, h_{n+n'}$. We set $\sigma(\mu.\mu') := h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n+n')}$ for any permutation $\sigma \in \mathfrak{S}_{r,s}$. From now on we fix $\sigma \in \mathfrak{S}_{r,s}$ and we assume that the word $\sigma(\mu.\mu')$ is greater than or equal to $\mu \odot \mu'$ (for the lexicographic order). We show that under this assumption, one necessarily has $\sigma(\mu.\mu') = \mu \odot \mu'$.

The proof is based on an induction on $r + r'$ or equivalently on the sum of the lengths of μ and μ'.

We first look at the action of σ on the Lyndon words $\mathbf{i}^{(1)}$ and $\mathbf{i}'^{(1)}$ and show that σ necessarily rearranges these two blocks so that in the word $\sigma(\mu.\mu')$ they will appear in the decreasing order. Then, considering the restriction of the action of σ on the other Lyndon words, we find ourselves considering a shuffle of parameters $\tilde{\mu}$ and $\tilde{\mu'}$, one of them being of length strictly smaller than the corresponding initial parameter.

First case: $\mathbf{i}'^{(1)} > \mathbf{i}^{(1)}$. Then the word $\mu \odot \mu'$ begins with $\mathbf{i}'^{(1)}$.

The two words $\mathbf{i}^{(1)}$ and $\mathbf{i}'^{(1)}$ are dominant Lyndon words so we write them $\mathbf{i}^{(1)} = (k, k+1, \ldots, k+d_1)$ and $\mathbf{i}'^{(1)} = (k', k'+1, \ldots, d'_1)$ with either $k' > k$ or $k' = k$ and $d'_1 > d_1$.

We first show that the assumption $\sigma(\mu.\mu') \geqslant \mu \odot \mu'$ implies $\sigma(n+1) = 1$. Indeed, if $\sigma(n+1) \geqslant d_1 + 1$ then, as the restrictions of σ to $[|1;n|]$ and $[|n+1;n+n'|]$ are increasing, we have $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$. The word $\sigma(\mu.\mu')$ then begins with $(k, \ldots k+d_1, l, \ldots)$, where l is equal to k if $\sigma(n+1) > d_1 + 1$ and to k' if $\sigma(n+1) = d_1 + 1$. If $k < k'$ then $k, \ldots, k+d_1 < \mathbf{i}'^{(1)}$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. If $k' = k$ and $d'_1 > d_1$ then

$$\begin{aligned} k, \ldots k+d_1, l \ldots &= k', \ldots, k'+d_1, k' \\ &< k', \ldots, k'+d_1, k'+d_1+1 \\ &\leqslant k', \ldots k'+d'_1 \\ &= \mathbf{i}'^{(1)} \end{aligned}$$

and the conclusion is the same.

If $\sigma(n+1) \in \{2, 3, \ldots, d_1\}$, then $\sigma(\mu.\mu')$ begins with $(k, k+1, \ldots, k+p, k', \ldots)$ where p is some integer such that $0 \leqslant p < d_1$. If $k < k'$ then $k, k+1, \ldots, k+p, k' < \mathbf{i}'^{(1)}$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. If $k' = k$ and $d'_1 > d_1$ then

$$\begin{aligned} k, k+1, \ldots, k+p, k' &= k, k+1, \ldots, k+p, k \\ &< k, k+1, \ldots, k+p, k+p+1 \\ &\leqslant k, k+1, \ldots, k+d_1 \\ &< k, k+1, \ldots, k+p, k+d'_1 \\ &= \mathbf{i}'^{(1)} \end{aligned}$$

and the conclusion is the same. Thus $\sigma(n+1) = 1$.

As the restrictions of σ to $[|1;n|]$ and $[|n+1;n+n'|]$ are increasing, $\sigma^{-1}(2)$ is either equal to 1 or to $n+2$; but the first possibility gives a word beginning with $k'k$ which is obviously strictly smaller than $\mu \odot \mu'$. Hence $\sigma(n+2) = 2$. Then by iterating this, we see that necessarily $\sigma(n+1) = 1, \ldots, \sigma(n+d'_1) = d'_1$. In other words σ sends the blocks $\mathbf{i}'^{(1)}$ on the left of the blocks $\mathbf{i}^{(1)}$, i.e. at the beginning of the word $\sigma(\mu.\mu')$.

Second case: $\mathbf{i}'^{(1)} < \mathbf{i}^{(1)}$. Then $\mu \odot \mu'$ begins with $(\mathbf{i}^{(1)})$ and with the previous notations, one has either $k' < k$ or $k' = k$ and $d'_1 < d_1$.

We show that the assumption $\sigma(\mu.\mu') \geqslant \mu \odot \nu$ implies $\sigma(n+1) > d_1$.

Indeed, if $\sigma(n+1) \in \{2, \ldots, d_1\}$, then $\sigma(\mu.\mu')$ begins with $(k, \ldots, k+p, k')$ where p is some integer such that $0 \leqslant p < d_1$. But then

$$\begin{aligned} k, k+1, \ldots, k+p, k' &\leqslant k, k+1, \ldots, k+p, k \\ &< k, k+1, \ldots, k+p, k+p+1 \\ &\leqslant k, k+1, \ldots, k+d_1 \\ &= \mathbf{i}^{(1)} \end{aligned}$$

and hence $\sigma(\mu.\mu') < \mu \odot \mu'$.

If $\sigma(n+1) = 1$, then $\sigma(\mu.\mu') \geqslant \mu \odot \mu'$ implies $k' = k$ (and hence $d'_1 < d_1$). But then it is easy to see that necessarily $\sigma(n+2) = 2, \ldots, \sigma(n+d'_1) = d'_1$, i.e. $\sigma(\mu.\mu')$ begins with $(k, \ldots, k+d'_1, \ldots)$. The letter coming after $k+d'_1$ is either the first letter of $\mathbf{i}^{(1)}$ (if $\sigma(1) = d'_1 + 1$) or the first letter of $\mathbf{i}'^{(2)}$ (if $\sigma(n+d'_1+1) = d'_1+1$); in both cases it is smaller than k and in particular smaller than $k+d'_1+1$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. Thus $\sigma(n+1) > d_1$.

In particular, $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$ (in other words, σ fixes the block $\mathbf{i}^{(1)}$, i.e. leaves it at the beginning of the resulting word.

Third case: $\mathbf{i}'^{(1)} = \mathbf{i}^{(1)}$. Then the word $\mu \odot \mu'$ begins with $(\mathbf{i}^{(1)})^2$.

We show that under the assumption $\sigma(\mu.\mu') \geqslant \mu \odot \mu'$, one has either $\sigma(n+1) = 1, \ldots, \sigma(n+d_1) = d_1$ (i.e. σ sends the block $\mathbf{i}^{(1)}$ coming from μ' to the left of the block $\mathbf{i}^{(1)}$ coming from μ) or $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$ (i.e. σ fixes the block $\mathbf{i}^{(1)}$ coming from μ).

Indeed, as the restrictions of σ to $[|1;n|]$ and $[|n+1;n+n'|]$ are increasing, $\sigma^{-1}(1)$ is either equal to 1 or to $n+1$.

If $\sigma(1) = 1$, then $\sigma(n+1)$ is necessarily strictly greater than d_1, otherwise $\sigma(\mu.\mu')$ would begin with $(k, k+1, \ldots, k+p, k, \ldots)$ (where p is some integer such that $0 \leqslant p < d_1$) and would be strictly smaller than $\mu \odot \mu'$. Hence in this case we get $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$.

If $\sigma(n+1) = 1$, then the same argument shows that $\sigma(1)$ is necessarily strictly greater than d_1, and hence we get $\sigma(n+1) = 1, \ldots, \sigma(n+d_1) = d_1$.

In conclusion, we have shown that the permutations we are seeking fix the block $\mathbf{i}^{(1)}$ if $\mathbf{i}^{(1)} > \mathbf{i}'^{(1)}$, send the block $\mathbf{i}'^{(1)}$ to the left of the block $\mathbf{i}^{(1)}$ if $\mathbf{i}^{(1)} < \mathbf{i}'^{(1)}$, and send either $\mathbf{i}^{(1)}$ or $\mathbf{i}'^{(1)}$ to the beginning of the resulting word if $\mathbf{i}^{(1)} = \mathbf{i}'^{(1)}$. The desired result follows by induction on $r + r'$.

□

2.4.2 A mutation rule for dominant words

We now use Theorem 2.4.5 (or equivalently Proposition 2.4.4) to obtain a mutation rule on the parameters of simple modules corresponding to cluster variables in the setting of [69]. We express it in a vector setting, i.e. in terms of the images of dominant words under the isomorphism (2.4). Recall that the image of any dominant word μ under the isomorphism (2.4) is the vector $\overrightarrow{\mu}$ whose

ith coordinate is equal to the multiplicity of the Lyndon word corresponding to the ith positive root (in the decreasing order (2.3)) in the canonical factorization of μ. Such vectors are of size r_n, the number of positive roots in type A_n ($r_n = n(n+1)/2$).

Example 2.4.6. In type A_2, there are three positive roots: $\alpha_2 > \alpha_1 + \alpha_2 > \alpha_1$. The word 21 will be represented by the vector ${}^t(1,0,1)$.

In type A_3 there are six positive roots: $\alpha_3 > \alpha_2 + \alpha_3 > \alpha_2 > \alpha_1 + \alpha_2 + \alpha_3 > \alpha_1 + \alpha_2 > \alpha_1$. The word 2312 will be represented by the vector ${}^t(0,1,0,0,1,0)$ and the word 321 by the vector ${}^t(1,0,1,0,0,1)$.

Let us consider a quantum monoidal seed $\mathcal{S} := (\{M_i\}_{i\in I}, B, \Lambda, D)$ in the sense of [69]. Recall that I splits into $I = J_{ex} \cup J_{fr}$ with the $\{[M_i]\}_{i\in J_{ex}}$ corresponding to unfrozen variables and the $\{[M_i]\}_{i\in J_{fr}}$ corresponding to frozen variables. For every $i \in I$, let μ_i be the parameter of the simple module M_i and $\overrightarrow{\mu_i}$ the corresponding vector.

Remark 2.4.7. 1. The abelian monoid isomorphism (2.4) naturally extends to an abelian group isomorphism between the respective Grothendieck groups of $(\mathbf{M}, \odot)$ and $(\mathbb{Z}^{r_n}_{\geqslant 0}, +)$, namely

$$(\mathbf{G}, \odot) \simeq (\mathbb{Z}^{r_n}, +). \tag{2.5}$$

Under this isomorphism, the inverse in $\mathbf{G}$ of a parameter $\mu \in \mathbf{M}$ corresponds to the opposite vector in $\mathbb{Z}^{r_n}$. For instance the vector corresponding to the generalized parameter $\hat{\mu_j}$ is

$$\overrightarrow{\hat{\mu_j}} = \sum_{1\leqslant i\leqslant n+m} b_{ij}\overrightarrow{\mu_i}.$$

2. The lexicographic order $\leqslant$ on $\mathbf{M}$ and $\mathbf{G}$ also turns into a (total) ordering on $\mathbb{Z}^{r_n}$ through the above isomorphism: a vector $\overrightarrow{\mu_1}$ is strictly greater than a vector $\overrightarrow{\mu_2}$ if and only if the first non zero component of $\overrightarrow{\mu_1} - \overrightarrow{\mu_2}$ is positive.

Let k be fixed in J_{ex} and let us look at the mutation in direction k of the seed $\mathcal{S}$. It leads to a new seed $\mathcal{S}'$ with the same variables except M_k that has turned into M'_k such that we have a short exact sequence of graded modules:

$$0 \to q \bigodot_{b_{ik}>0} M_i^{\circ b_{ik}} \to q^{\tilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \to \bigodot_{b_{ik}<0} M_i^{\circ(-b_{ik})} \to 0. \tag{2.6}$$

The next statement shows that one can deduce the parameter of the simple module M'_k from the knowledge of the parameters μ_i and the exchange matrix B of the seed $\mathcal{S}$.

Proposition 2.4.8. *Let μ'_k be the parameter of the simple module M'_k and $\overrightarrow{\mu'_k}$ be the corresponding vector. Then we have:*

$$\overrightarrow{\mu'_k} = -\overrightarrow{\mu_k} + max\left(\sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}, \sum_{b_{ik}<0} (-b_{ik})\overrightarrow{\mu_i}\right).$$

Proof. As the real simple modules M_i commute, the modules $\bigodot_{b_{ik}>0} {M_i}^{\circ b_{ik}}$ and $\bigodot_{b_{ik}<0} {M_i}^{\circ(-b_{ik})}$ are simple. Thus they correspond to some dominant words μ_+ and μ_-. Using Theorem 2.2.31 (ii), one can write

$$\mu_+ = max(L(\mu_+)) = max\left(\bigodot_{b_{ik}>0} L(\mu_i)^{\circ b_{ik}}\right) = \bigodot_{b_{ik}>0} \mu_i^{\odot b_{ik}}.$$

Under the isomorphism (2.4) we get

$$\overrightarrow{\mu_+} = \sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}.$$

Now the short exact sequence (2.6) gives the relation

$$q^{\tilde{\Lambda}(M_k,M'_k)}[M_k][M'_k] = q\prod_{b_{ik}>0}[M_i]^{b_{ik}} + \prod_{b_{ik}<0}[M_i]^{-b_{ik}}$$

in the Grothendieck ring of the category $R-gmod$. Taking the characters we get

$$q^{\tilde{\Lambda}(M_k,M'_k)}ch_q(M_k)\circ ch_q(M'_k) = qch_q(L(\mu_+)) + ch_q(L(\mu_-)).$$

Looking at the highest weight on both sides of this equality we get

$$\begin{aligned}\mu_k \odot \mu'_k &= max\left(ch_q(M_k)\circ ch_q(M'_k)\right)\\ &= max\left(max\left(ch_q(L(\mu_+))\right), max\left(ch_q(L(\mu_-))\right)\right).\end{aligned}$$

Applying isomorphism (2.4), we get

$$\overrightarrow{\mu_k} + \overrightarrow{\mu'_k} = max(\overrightarrow{\mu_+},\overrightarrow{\mu_-}) = max\left(\sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}, \sum_{b_{ik}<0}(-b_{ik})\overrightarrow{\mu_i}\right)$$

which is the desired statement in the image of isomorphism (2.5).

□

2.4.3 An example in type A_3

In this subsection we apply Proposition 2.4.8 to the example of the category $R-gmod$ for a Lie algebra of type A_3. This example provides an illustration of Theorem 2.5.2 which will be proved in general type A_n in Section 2.5. The category $R-gmod$ corresponds to $\mathcal{C}_w$ with $w = w_0$ the longest element of the Weyl group of $\mathfrak{g}$ (see Section 2.2.4). In type A_3 this element can be written as:

$$w_0 = s_1s_2s_3s_1s_2s_1.$$

Theorem 2.2.21 provides an admissible pair (in the sense of Definition 2.2.12), which gives rise to a quantum monoidal seed for this category. We denote this seed by $\mathcal{S}_0^3$. Firstly, one can see that $J_{ex} = \{1,2,3\}$ and $J_{fr} = \{4,5,6\}$ (see Section 2.2.4). The simple modules whose classes are the cluster variables of the seed $\mathcal{S}_0^3$ can be computed directly using [69, Proposition 10.2.4].

Lemma 2.4.9. *The seed $\mathcal{S}_0^3$ for the category $R-gmod$ in type A_3 is given by three unfrozen variables $[L(1)],[L(12)],[L(21)]$ and three frozen variables $[L(123)],L(2312)],[L(321)]$ together with the following exchange matrix:*

$$B_0 = \begin{pmatrix} 0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\\ 0 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}.$$

Proof. By definition of the modules $M(k,0)$ defining the underlying admissible pair of the seed $\mathcal{S}_0^3$ (see Section 2.2.4), one has

$$\begin{array}{rcl} M(1,0) & = & M(s_1\omega_1,\omega_1)) \\ M(2,0) & = & M(s_1s_2\omega_2,\omega_2) \\ M(3,0) & = & M(s_1s_2s_1\omega_1,\omega_1) \\ M(4,0) & = & M(s_3s_1s_2s_1\omega_1,\omega_1) \\ M(5,0) & = & M(s_2s_3s_1s_2s_1\omega_1,\omega_1) \\ M(6,0) & = & M(s_1s_2s_3s_1s_2s_1\omega_1,\omega_1). \end{array}$$

Using [69, Proposition 10.2.4], one gets $M(1,0) = L(1)$, $M(2,0) = hd(L(1) \circ L(2)) = L(12)$, $M(3,0) = hd(L(2) \circ L(1)) = L(21)$. The computations are similar for $M(k,0), k \in \{4,5,6\}$.

□

The (ungraded) short exact sequences corresponding to the mutations in each of the three exchange directions can be written as follows:

$$0 \to L(21) \to L(1) \circ L \to L(12) \to 0.$$

$$0 \to L(1) \circ L(2312) \to L(12) \circ M \to L(21) \circ L(123) \to 0.$$

$$0 \to L(12) \circ L(321) \to L(21) \circ N \to L(1) \circ L(2312) \to 0.$$

Let λ (resp. μ, ν) be the parameters of the simple module L (resp. M, N). We can compute these parameters using Proposition 2.4.8. For instance consider the second of the above exact sequences. Then with the notations of Remark 2.4.7, a straightforward computation gives the parameter of M as $\overrightarrow{\mu} = {}^t(0,1,0,0,0,1)$. Hence $M = L(231)$.

In the same way one can compute $L = L(2)$ and $N = L(312)$.

Let $\mathcal{S}_1$ be the seed obtained from the seed $\mathcal{S}_0$ by mutation in the first direction. One can now show that $\mathcal{S}_0$ is compatible in the sense of Definition 2.3.7 and $\mathcal{S}_1$ is not.

First we write the exchange matrix for $\mathcal{S}_1$:

$$B_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then for $\mathcal{S}_0$, the images under isomorphism (2.5) of the generalized parameters $\hat{\mu_j}$ are :

$$\overrightarrow{\hat{\mu}_1} = {}^t(0,0,1,0,1,1) \quad \overrightarrow{\hat{\mu}_2} = {}^t(0,1,-1,1,1,0) \quad \overrightarrow{\hat{\mu}_3} = {}^t(1,-1,1,0,0,0)$$

and for $\mathcal{S}_1$ we get:

$$\overrightarrow{\hat{\mu}_1} = {}^t(0,0,-1,0,1,-1) \quad \overrightarrow{\hat{\mu}_2} = {}^t(0,1,-1,-1,1,0) \quad \overrightarrow{\hat{\mu}_3} = {}^t(1,-1,2,0,-1,1).$$

Combining Remark 2.4.7(2) and Remark 2.3.8, one can see that $\mathcal{S}_0$ is compatible in the sense of Definition 2.3.7 but it is not the case for $\mathcal{S}_1$.

More generally, given an initial quantum monoidal seed $(\{M_i\}_i, B)$, Proposition 2.4.8 allows us to compute explicitly the parameters of the simple modules appearing when mutating the initial seed an arbitrary number of times in any directions.

2.5 A compatible seed for $R - gmod$ in type A

In this section we compute in type A_n the parameters of the simple modules of the monoidal seed $\mathcal{S}_0^n$ arising from the construction of [69] (see Subsection 2.2.4) for the category $R - gmod$.

2.5.1 Statements of the main theorems

In this subsection we state the two main results of this paper, Theorems 2.5.1 and 2.5.2. Recall that $\mathcal{A}_q(\mathfrak{n}) = \mathcal{A}_q(\mathfrak{n}(w_0))$ where w_0 is the longest element of the Weyl group of $\mathfrak{g}$. The category $R - gmod$ coincides with the category $\mathcal{C}_{w_0}$ (see Section 2.2.4). In type A_n we have

$$w_0 = (1 \dots n)(1 \dots (n-1)) \cdots (12)(1).$$

Recall that $r_n := n(n+1)/2$ stands for the length of w_0. In the category $R - gmod = \mathcal{C}_{w_0}$ in type A_n, Theorem 2.2.21 provides an admissible pair (in the sense of Definition 2.2.12), which gives rise to a quantum monoidal seed for this category. We denote this seed by $\mathcal{S}_0^n$.

Our first main result is the following:

Theorem 2.5.1. *The cluster variables of the seed $\mathcal{S}_0^n$ can be explicitly described in terms of parameters as follows:*

$$\begin{array}{lllll}
[L(1)] & & & & \\
[L(12)] & [L((2)(1))] & & & \\
[L(123)] & [L((23)(12))] & [L((3)(2)(1))] & & \\
\vdots & \vdots & & & \\
[L(1\dots k)] & [L((2\dots k)(1\dots k-1))] & \cdots & [L((k)\cdots(1))] & \\
\vdots & \vdots & & & \\
[L(1\dots n)] & [L((2\dots n)(1\dots n-1))] & \cdots & \cdots & [L((n)\cdots(1))].
\end{array}$$

The set of frozen variables corresponds to the last line and the set of unfrozen variables consists in the union of lines $1 \dots n-1$.

The three following sections are devoted to some intermediate steps for the proof of Theorem 2.5.1. Recall from Section 2.2.4 that J_{fr} denotes the index set of the frozen variables of the monoidal seed $\mathcal{S}_0^n$. In Section 2.5.2, we prove that $|J_{fr}| = n$. We show that the knowledge of the dominant words of the $M_j, j \in J_{fr}$ is sufficient to recover the whole seed $\mathcal{S}_0^n$. Section 2.5.3 is devoted to the computation of the weights of the $M_j, j \in J_{fr}$. These weights are determined by the construction of [69] (see Section 2.2.4). In Section 2.5.4, we use the fact that for any $j \in J_{fr}$, the module M_j necessarily commutes with any other simple module in $R - gmod$. This strongly constrains the form of the corresponding dominant word. Together with the weights obtained in Section 2.5.3, we find at most n possible dominant words, which is exactly the number of frozen variables computed in Section 2.5.2. Hence we get a bijection between these parameters and the set of modules $\{M_j, j \in J_{fr}\}$.

We complete the proof in Section 2.5.5 by determining which parameter corresponds to every simple module, which is more precise than just a global bijection. The key argument is provided by Theorem 2.4.8.

The following statement is our second main result. We deduce it from Theorem 2.5.1.

Theorem 2.5.2. *The seed $\mathcal{S}_0^n$ is compatible in the sense of Definition 2.3.7.*

In particular Conjecture 2.3.10 holds for the category $R - gmod$ in type A_n.

2.5.2 Initial seed for $R - gmod$

For $n \geqslant 1$, we consider a Lie algebra $\mathfrak{g}$ of type A_n. We let $\{\alpha_1, \dots, \alpha_n\}$ denote the simple roots and $Q_+^n := \bigoplus_{i=1..n} \mathbb{N}\alpha_i$. We also denote Δ_+^n the set of positive roots, $R - gmod^n$ the category of (graded) finite dimensional representation of the quiver Hecke algebras associated with $\mathfrak{g}$, and $\mathbf{M}^n$ the set of dominant words in bijection with the set of simple objects in $R - gmod^n$ (up to

isomorphism). There is a canonical embedding ι_n^m of $\mathbf{M}^n$ into $\mathbf{M}^m$ for any $m \geqslant n$. In particular the set of simple objects in $R - gmod^n$ is naturally included into the set of simple objects in $R - gmod^m$. We again denote ι_n^m this inclusion.

Let J_{ex}^n (resp. J_{fr}^n) denote the index set of the unfrozen variables (resp. frozen variables) of the seed $\mathcal{S}_0^n$ for every $n \geqslant 1$. We also set $J^n := J_{ex}^n \cup J_{fr}^n$. In order to prove Theorem 2.5.1, we begin by determining the sets J_{ex}^n and J_{fr}^n. We point out an inductive property of this seed: the set $\{M_i, i \in J_{ex}^n\}$ coincides with the set $\{\iota_{n-1}^n(M_i), i \in J^{n-1}\}$.

Proposition 2.5.3. *The cluster variables of the seed $\mathcal{S}_0^n$ split into the following exchange and frozen sets:*

1. *There are n frozen variables in $\mathcal{S}_0^n$, which correspond to the classes of the last n modules $M(r_{n-1}+1, 0), \ldots, M(r_n, 0)$.*
2. *The set $\{M_i, i \in J_{ex}^n\}$ coincides with the union of the sets $\{\iota_k^n(M_i), i \in J_{fr}^k\}, 1 \leqslant k \leqslant n-1$.*

Proof. For the first statement, write $w_0 = (s_1 \ldots s_n)(s_1 \ldots s_{n-1}) \cdots (s_1 s_2)(s_1) = s_{r_n} \cdots s_1$. Then for any $l \in \{1, \ldots, r_{n-1}\}$, the letter s_l is in $\{1, \ldots, n-1\}$ and this letter obviously appears again in the word w_0 as $s_{l'}$ for some $l' > l$. In other words, $l_+ \leqslant r_n$ and thus $l \in J_{ex}^n$. Conversely, if $l \in \{r_{n-1}+1, r_n\}$, then all the letters $s_{l'}, l' > l$ are distinct from s_l and thus $l_+ = r+1$. Hence one has

$$J_{ex}^n = \{1, \ldots, r_{n-1}\} \quad \text{and} \quad J_{fr}^n = \{r_{n-1}+1, \ldots, r_n\}.$$

In particular the modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$ can be written as

$$\begin{array}{rcl} M(r_{n-1}+1, 0) & = & M\left(s_1(s_2 s_1) \cdots (s_{n-1} \ldots s_1) s_n . \omega_n, \omega_n\right) \\ \vdots & \vdots & \vdots \\ M(r_n, 0) & = & M\left(s_1(s_2 s_1) \cdots (s_{n-1} \ldots s_1)(s_n \ldots s_1) . \omega_1, \omega_1\right). \end{array}$$

The second statement follows from the first one applied to the seeds $\mathcal{S}_0^k, 1 \leqslant k \leqslant n-1$. Indeed, one can write in the same way the modules $M_i, i \in J_{fr}^{n-1}$ as:

$$\begin{array}{c} M\left(s_1(s_2 s_1) \cdots (s_{n-2} \ldots s_1) s_{n-1} . \omega_{n-1}, \omega_{n-1}\right) \\ \vdots \\ M\left(s_1(s_2 s_1) \cdots (s_{n-2} \ldots s_1)(s_{n-1} \ldots s_1) . \omega_1, \omega_1\right). \end{array}$$

The images via ι_{n-1}^n of these modules respectively coincide with the modules $M(r_{n-2}+1, 0), \ldots, M(r_{n-1}, 0)$, whose classes are exactly the last $n-1$ unfrozen variables of the seed $\mathcal{S}_0^n$. Iterating this, we conclude that the set $\{M_l, l \in J_{ex}^n\}$ is the union of the sets $\{\iota_k^n(M_l), l \in J_{fr}^k\}$, $1 \leqslant k \leqslant n-1$.

□

As a direct consequence of the previous Proposition, it suffices to compute the parameters of the modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$. This is what we focus on in the next two subsections.

2.5.3 Weights of the simple modules $M(r_{n-1}+k, 0), 1 \leqslant k \leqslant n$

From now on, the integer n is fixed. We write J_{fr} for J_{fr}^n. By Proposition 2.5.3, the simple modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$ are the $M(r_{n-1}+k, 0), 1 \leqslant k \leqslant n$. For simplicity we set $M_k := M(r_{n-1}+k, 0)$ for any $1 \leqslant k \leqslant n$. This subsection is devoted to the computation of the weights of the simple modules M_k, i.e. the elements β_k such that $M_k \in R(\beta_k) - gmod$ for every $1 \leqslant k \leqslant n$. Our main tool is the definition of the modules $M(l, 0)$ from [69] (see Definition 2.2.20).

Proposition 2.5.4. *For each $1 \leqslant k \leqslant n/2$, the two modules M_k and M_{n-k+1} both belong to the subcategory $R(\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1) - mod$.*

Proof. For $1 \leqslant l \leqslant n$, we have

$$M_l = M(r_{n-1}+l,0) = M\left(s_1(s_2s_1)\cdots(s_{n-1}\dots s_1)(s_n\dots s_k)\omega_k,\omega_k\right) \text{ where } k := n-l+1.$$

One computes $\zeta_k := s_1(s_2s_1)\cdots(s_{n-1}\dots s_1)(s_n\dots s_k)\omega_k$. The weight of M_l is given by $\omega_k - \zeta_k$ (see Corollary 2.2.18).

$$\begin{aligned}
\zeta_k &= s_1(s_2s_1)\cdots(s_{n-1}\dots s_1)(\omega_k - (\alpha_n + \cdots + \alpha_k)) \\
&= s_1(s_2s_1)\cdots(s_{n-2}\dots s_1)(\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + 2\alpha_k + \alpha_{k-1})) \\
&= s_1(s_2s_1)\cdots(s_{n-3}\dots s_1)(\omega_k - (\alpha_n + 2\alpha_{n-1} + 3\alpha_{n-2} + \cdots + \\
&\qquad\qquad + \cdots + 3\alpha_k + 2\alpha_{k-1} + \alpha_{k-2})).
\end{aligned}$$

If $2k \leqslant n$ then by iterating we get

$$\begin{aligned}
\zeta_k = s_1(s_2s_1)\cdots(s_{n-k}\dots s_1)(\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + \\
+ \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1))
\end{aligned}$$

but $\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1)$ is invariant under the action of $s_1,\dots,s_{n-k}$. Hence

$$\zeta_k = \omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1).$$

If $2k > n$ then by iterating we get

$$\begin{aligned}
\zeta_k &= s_1(s_2s_1)\cdots(s_k\dots s_1)(\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + (n-k)\alpha_{k+1} \\
&\qquad\qquad + (n-k)\alpha_k + \cdots + 2\alpha_2 + \alpha_{2k-n+1})) \\
&= s_1(s_2s_1)\cdots(s_{k-1}\dots s_1).(\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + (n-k)\alpha_{k+1} \\
&\qquad\qquad + (n-k+1)\alpha_k + (n-k)\alpha_{k-1}\cdots + 2\alpha_2 + \alpha_{2k-n})) \\
&= \dots \\
&= s_1(s_2s_1)\cdots(s_{n-k}\dots s_1).(\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + (n-k)\alpha_{k+2} \\
&\qquad + (n-k+1)\alpha_{k+1} + \cdots + (n-k+1)\alpha_{n-k+1} + \cdots + 2\alpha_2 + \alpha_1))
\end{aligned}$$

and $\omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + (n-k+1)\alpha_{k+1} + \cdots (n-k+1)\alpha_{n-k+1} + \cdots + 2\alpha_2 + \alpha_1)$ is invariant under the action of $s_1,\dots,s_{n-k}$. Hence we get

$$\begin{aligned}
\zeta_k &= \begin{cases} \omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + \\ \qquad + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1) & \text{if } 2k \leqslant n, \\ \omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + (n-k+1)\alpha_k + \cdots + \\ \qquad + \cdots + (n-k+1)\alpha_{n-k+1} + \cdots + 2\alpha_2 + \alpha_1) & \text{if } 2k > n. \end{cases} \\
&= \begin{cases} \omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_l + \cdots + \\ \qquad + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1) & \text{if } k < l, \\ \omega_k - (\alpha_n + 2\alpha_{n-1} + \cdots l\alpha_k + \cdots + \\ \qquad + \cdots + l\alpha_l + \cdots + 2\alpha_2 + \alpha_1) & \text{if } k \geqslant l. \end{cases}
\end{aligned}$$

Hence for each $1 \leqslant k \leqslant n/2$, the two modules M_k and M_{n-k+1} both belong to the subcategory $R(\alpha_n + 2\alpha_{n-1} + \cdots + k\alpha_{n-k+1} + \cdots + k\alpha_k + \cdots + 2\alpha_2 + \alpha_1) - mod$.

□

Let us now fix $1 \leqslant k \leqslant n$ and consider the parameter μ_k of the simple module M_k. Let m_k be the length of μ_k. Since k and $n-k+1$ play symmetric roles, we assume from now on that $k \leqslant n/2$.

The following statement is a direct consequence of the previous proposition.

Corollary 2.5.5. *For any $1 \leqslant i \leqslant n$,*

$$(\mu_k, i) = \begin{cases} 1 & \text{if } i = k \text{ or } i = n-k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $1 \leqslant i \leqslant k-1$ or $n-k+2 \leqslant i \leqslant n$, by Proposition 2.5.4 there are $i-1$ (resp. $i, i+1$) occurrences of the letter $i-1$ (resp. $i, i+1$) in the word μ_k and hence $(\mu_k, i) = 2i-(i-1)-(i+1) = 0$.

If $k+1 \leqslant i \leqslant n-k$ then by Proposition 2.5.4 each of the letters $i-1, i, i+1$ appears k times and hence $(\mu_k, i) = 2i - i - i = 0$.

Finally if $i = k$ then by Proposition 2.5.4 there are k occurrences of the letters $k, k+1$ and $k-1$ occurrences of the letter $k-1$ which gives $(\mu_k, i) = 2k - k - (k-1) = 1$. If $i = n-k+1$ then there are k occurrences of the letters $i-1, i$ and $k-1$ occurrences of the letter $i+1$ and thus $(\mu_k, i) = 2k - k - (k-1) = 1$. □

In particular, one can compute the quantities $\Lambda(M_k, L(i))$ for any $1 \leqslant i \leqslant n$:

Corollary 2.5.6. *For any $1 \leqslant i \leqslant n$, let N_i be the number of occurrences of the letter i in the word μ_k. Let s_i and s'_i be the integers such that (see Remark 2.5.7)*

$$\Lambda(L(i), M_k) = -(\mu_k, i) + 2N_i - 2s_i \quad \text{and} \quad \Lambda(M_k, L(i)) = -(\mu_k, i) + 2N_i - 2s'_i.$$

Then one has $s_i = s'_i = N_i$ if $i \notin \{k, n-k+1\}$ and either $s_i = N_i, s'_i = N_i - 1$ or $s_i = N_i - 1, s'_i = N_i$ if $i \in \{k, n-k+1\}$.

Proof. By Corollary 2.5.5, the quantity $\Lambda(L(i), M_k)$ can be written as

$$\Lambda(L(i), M_k) = \begin{cases} 2N_i - 1 - 2s_i & \text{if } i = k \text{ or } i = n-k+1, \\ 2N_i - 2s_i & \text{otherwise.} \end{cases}$$

and similarly for $\Lambda(M_k, L(i))$ with s'_i. As M_k commutes with $L(i)$, one has by Lemma 2.2.9:

$$s_i + s'_i = \begin{cases} 2N_i - 1 & \text{if } i = k \text{ or } i = n-k+1, \\ 2N_i & \text{otherwise.} \end{cases}$$

As the integers s_i and s'_i are always smaller than N_i, one gets the desired result.

□

Remark 2.5.7. In the following we will make several computations of some $\Lambda(M, N)$ for various simple real objects M, N in $R-gmod$ in order to check commutation between these modules. For any $\beta \in Q_+$, any simple (left) $R(\beta)$-module M is cyclic, i.e. is isomorphic to $R(\beta).u$ for some $u \in M$. We will refer to any such vector u in M as a generating vector in M. Now let $\beta, \gamma \in Q_+$, M a simple $R(\beta)$-module, and N a simple $R(\gamma)$-module. As the morphism

$$\begin{array}{ccc} M \otimes N & \longrightarrow & N \circ M \\ u \otimes v & \longmapsto & \varphi_{w[n,m]}(v \otimes u) \end{array}$$

is $R(\beta) \otimes R(\gamma)$-linear, computing the map $R_{M,N}$ is equivalent to computing the action of $\varphi_{w[n,m]} \in R(\beta+\gamma)$ on the tensor product of generating vectors u and v for M and N.

Now let $u_z := 1 \otimes u \in M_z$ and let $\tilde{s}$ be the valuation of the polynomial in z given by $\varphi_{w[n,m]}.(v \otimes u_z)$. As the actions of the generators x_i, τ_k and $e(\nu)$ can only make the degree in z increase, the image of the map $R_{M_z,N}$ is contained in $z^{\tilde{s}} N \circ M_z$. Moreover, by definition of $\tilde{s}$, $\varphi_{w[n,m]}.(v \otimes u_z)$ contains a nonzero term of degree $\tilde{s}$ hence it does not belong to $z^k N \circ M_z$ for any $k > \tilde{s}$. Hence $\tilde{s}$ coincides with s in Definition 2.2.7. Thus in what follows, for any simple $R(\beta)$-module M and any simple $R(\gamma)$-module N, we will always fix some choices of generating vectors $u \in M$ and $v \in N$ and write

$$\Lambda(M, N) = -(\beta, \gamma) + 2(\beta, \gamma)_n - 2s$$

with s being the valuation of the polynomial in z given by $\varphi_{w[n,m]}.(v \otimes u_z)$.

2.5.4 Dominant words associated to frozen variables in $R-gmod$

In this subsection, we compute the dominant words associated to the frozen variables for the category $R-gmod$ in type A_n. As in the previous subsection, we fix k such that $1 \leqslant k \leqslant n/2$ and we consider $M_k = M(r_{n-1}+k, 0)$ the simple module whose isomorphism class is the kth frozen variable in the seed $\mathcal{S}_0^n$ constructed in [69]. We use the fact that M_k commutes with all the simple modules in $R-gmod$. In particular, it commutes with all the cuspidal modules $L(i), 1 \leqslant i \leqslant n$. Together with the form of the weight of M_k given by Proposition 2.5.4 and Corollay 2.5.6, this leads to only n possible dominant words. As there are exactly n frozen variables in the seed $\mathcal{S}_0^n$ (see Proposition 2.5.3), we get a bijection between the possible parameters and the frozen variables for $R-gmod$.

For every $1 \leqslant i \leqslant n$, the algebra $R(\alpha_i)$ is generated by one generator x_i and one generator $e(i)$ commuting with each other (with the notation of Section 2.2.1, the set $Seq(\alpha_i)$ is a singleton consisting in the word reduced to a single letter i). Recall from Section 2.2.5 that for every $1 \leqslant i \leqslant n$, the cuspidal module $L(i)$ is a one dimensional vector space spanned by a generating vector v_i with action of $R(\alpha_i)$ given by:

$$x_i \cdot v_i = 0, \quad e(i) \cdot v_i = v_i.$$

It is the only simple object in the category $R(\alpha_i)-mod$.

As above we let μ_k denote the parameter of the simple module M_k and m_k the length of μ_k. In what follows we will write the word μ_k as

$$\mu_k = h_1, \dots, h_{m_k}.$$

Note that in this setting the h_j are the *letters* of the word μ_k, whereas we use bold letters $\mathbf{i}_l$ to refer to *Lyndon words* in the canonical factorization of μ_k (see after Remark 2.5.11 below).

As the module M_k is simple and real, Lemma 2.2.9 shows that checking its commutation with any other simple module L is equivalent to computing the quantities $\Lambda(L, M_k)$ and $\Lambda(M_k, L)$. When $L = L(i)$ for some $i \in \{1, \dots, n\}$, these quantities are given by Corollary 2.5.6. Thus as explained in Remark 2.5.7 above, once fixed a generating vector u for M_k, we will compute the valuations s_i (resp.s_i') of the polynomial functions $\varphi_{w[m_k,1]}(u \otimes (v_i)_z) = \varphi_1 \cdots \varphi_{m_k}(u \otimes (v_i)_z)$ (resp. $\varphi_{w[1,m_k]}(v_i \otimes u_z) = \varphi_{m_k} \cdots \varphi_1(v_i \otimes u_z)$) for various choices of $i \in \{1, \dots, n\}$. We fix once for all a generating vector u (resp. $v_i, 1 \leqslant i \leqslant n$) for M_k (resp. $L(i), 1 \leqslant i \leqslant n$).

We begin by showing that there are only two possibilities for the first letter of μ_k.

Lemma 2.5.8. *Let $p = h_1$ denote the first letter of μ_k. The letter p is equal either to k or $n-k+1$.*

Proof. With the same notations as in Corollary 2.5.6, we show that $s_p \leqslant N_p - 1$.

$$\begin{aligned}
\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) &= \varphi_1 \cdots \varphi_{m_k} e(h_1, \dots, h_{m_k}, p).(u \otimes (v_p)_z) \\
&= \varphi_1 e(h_1, p, h_2, \dots, h_{m_k}) \varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) \\
&= (\tau_1(x_1 - x_2) + 1) \varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z).
\end{aligned}$$

The operator x_1 commutes with $\varphi_2, \dots, \varphi_{m_k}$ and acts trivially on u. Moreover, $x_2 \varphi_2 \cdots \varphi_{m_k} = \varphi_2 \cdots \varphi_{m_k} x_{m_k+1}$ (see for example [66, Lemma 1.3.1]) and $x_{m_k+1} \cdot (u \otimes (v_p)_z) = z(u \otimes (v_p)_z)$. Hence we get

$$\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = -z.\tau_1.\varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) + \varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z).$$

The operator $\varphi_2 \cdots \varphi_{m_k}$ acts non-trivially on $(u \otimes (v_p)_z)$ (as the renormalized R-matrix $r_{L(p),M_k}$ never vanishes). Thus $\varphi_2 \cdots \varphi_{m_k}.u$ is a non zero polynomial function, and the above equality

implies that s_p is equal to its valuation. This polynomial function has degree less than $N_p - 1$, as the only operators φ_j that can make the degree rise are the ones corresponding to an occurrence of p in μ_k.

This implies $s_p \leqslant N_p - 1$. Hence by Corollary 2.5.6, $p \in \{k, n-k+1\}$.

□

Remark 2.5.9. With the same proof, one can show that the last letter of the word μ_k is either k or $n-k+1$ as well.

Lemma 2.5.10. *(i) For each $1 \leqslant k' \leqslant k$, there is exactly one Lyndon word ending with $n-k'+1$ in the canonical factorization of μ_k.*

(ii) Moreover, denoting by $\mathbf{j}_{n-k'+1}$ the unique Lyndon word ending with $n-k'+1$ (for each $1 \leqslant k' \leqslant k$), one has $\mathbf{j}_n > \cdots > \mathbf{j}_{n-k+1}$.

Proof. We prove the first statement by induction on k'. We know that the letter n appears exactly once; thus there is a unique Lyndon word $\mathbf{j}_n$ containing (and thus ending with) n, which proves the statement for $k' = 1$. Suppose $k \geqslant 2$ and (i) holds for $1 \leqslant k' < k$. Denote by $\mathbf{j}_n, \ldots, \mathbf{j}_{n-k'+1}$ the Lyndon words respectively ending with $n, \ldots, n-k'+1$. Their first letters are all smaller than p and in particular smaller than $n-k+1$ by Lemma 2.5.8. Hence they all contain the letter $n-k'$, which makes k' occurrences of this letter. By Proposition 2.5.4, $n-k'$ has to appear $k'+1$ times in the word μ_k. Hence there is a unique Lyndon word $\mathbf{j}_{n-k'}$ containing $n-k'$ but none of the letters $n, \ldots, n-k'+1$, which means that this Lyndon word ends with $n-k'$. Thus the first statement holds by induction.

For the second statement, let $m \in \{n-k+1, \ldots, n\}$ such that $\mathbf{j}_m$ is the smallest of the $\mathbf{j}_l$; this is equivalent to saying that it is the last (among the $\mathbf{j}_l$) to appear in the canonical factorization of μ_k.

Note that the first statement implies that, for each $1 \leqslant k' \leqslant k$, the letter $n-k'+1$ appears k' times among the Lyndon words $\mathbf{j}_n, \ldots, \mathbf{j}_{n-k+1}$: once in each of $\mathbf{j}_n, \ldots, \mathbf{j}_{n-k'+1}$, and none in the others. Together with Proposition 2.5.4, this implies that the letters $n-k'+1$ ($1 \leqslant k' \leqslant k$) do not appear in any other Lyndon word of the canonical factorization of μ_k. Hence denoting by i the position of the last letter of $\mathbf{j}_m$ in the word μ_k, one has $h_i = m$ and $h_{i+1}, \ldots, h_{m_k} < m$. Thus one has

$$\begin{aligned}\varphi_{m_k} \cdots \varphi_1.(v_m \otimes u_z) &= \tau_{m_k} \cdots \tau_{i+1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &= \tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \pm z.\tau_{m_k} \cdots \tau_{i+1}\tau_i\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z).\end{aligned} \tag{2.7}$$

We denote $Q(z)$ the first term

$$\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z).$$

We show that if $m \geqslant n-k+2$, then $Q(z)$ is non zero. As the renormalized R-matrix $r_{M_k,L(m)}$ does not vanish, the action of the operator $\varphi_{i-1} \cdots \varphi_1$ on $(v_m \otimes u_z)$ is a non zero polynomial function in z of degree less than $N_m - 1$. Consider now the action of τ_{m_k} on $Q(z)$:

$$\begin{aligned}\tau_{m_k}.Q(z) &= \tau_{m_k}.\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &= \tau_{m_k}.\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.e(m\mu_k).(v_m \otimes u_z) \\ &= \tau_{m_k}.\tau_{m_k}e(s_{m_k-1} \cdots s_{i+1}s_{i-1} \cdots s_1.m\mu_k).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &= \tau_{m_k}^2.e(h_1, \ldots, h_{m_k-1}, m, h_{m_k}).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &= e(h_1, \ldots, h_{m_k-1}, m, h_{m_k}).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \quad \text{as } h_{m_k} \leqslant m-2 \\ &= \tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z)\end{aligned}$$

Similarly, all the letters in position $i+1,\dots,m_k$ are less than $n-k$ and in particular they are less than $m-2$. The same argument can be applied to $\tau_{m_k-1},\dots,\tau_{i+1}$ and thus we get

$$\tau_{i+1}\cdots\tau_{m_k}.Q(z)=\tau_{i+1}\cdots\tau_{m_k}.(\tau_{m_k}\cdots\tau_{i+1}\varphi_{i-1}\cdots\varphi_1.(v_m\otimes u_z))=\varphi_{i-1}\cdots\varphi_1.(v_m\otimes u_z)$$

which is not zero. A fortiori $Q(z)$ itself is non zero. It is thus a non zero polynomial function of degree less than N_m-1 and the equality (2.7) above shows that s'_m is necessarily equal to its valuation.

This implies $s'_m\leqslant N_m-1$, and in particular $m\in\{k,n-k+1\}$ by Corollary 2.5.6. This contradicts the hypothesis $m\geqslant n-k+2$. Thus we have shown $m=n-k+1$.

By iterating this we conclude that the Lyndon words $\mathbf{j}_n,\dots,\mathbf{j}_{n-k+1}$ appear in this order in the canonical factorization of μ_k, which is the desired statement.

□

From now on we write the canonical factorization of μ_k as $\mu_k=\mathbf{i}_0\cdots\mathbf{i}_r$ with $\mathbf{i}_0\geqslant\cdots\geqslant\mathbf{i}_r$. For each $0\leqslant j\leqslant r$ we denote by p_j the first letter of the Lyndon word $\mathbf{i}_j$.The sequence $(p_j)_{0\leqslant j\leqslant r}$ is decreasing, with $p_0=p$ and $p_r=1$ (the letter 1 appears once, necessarily in the smallest of the $\mathbf{i}_j$). We also denote by a_j the position of the letter p_j in the word μ_k.

Remark 2.5.11. Note that as an immediate consequence of the previous lemma, one has $r\geqslant k-1$.

With these notations we can make the following observation, as a straightforward consequence of Lemma 2.5.10.

Corollary 2.5.12. *The Lyndon word $\mathbf{i}_0$ ends with the letter n. In other words $\mathbf{i}_0=\mathbf{j}_n$.*

Proof. As $\mathbf{i}_0$ is greater than any other Lyndon word appearing in the canonical factorization of μ_k, in particular it is greater than $\mathbf{j}_n$. Hence by Lemma 2.5.10(ii), one can write

$$\mathbf{i}_0\geqslant\mathbf{j}_n>\cdots>\mathbf{j}_{n-k+1}. \tag{2.8}$$

Thus all of the Lyndon words $\mathbf{j}_n,\dots,\mathbf{j}_{n-k+1}$ begin with a letter smaller than $p_0=p$. By definition they end with letters greater than or equal to $n-k+1$ which is greater than p by Lemma 2.5.8 (recall that we assumed $k\leqslant n-k+1$ at the beginning of this section). Hence each of the Lyndon words $\mathbf{j}_n,\dots,\mathbf{j}_{n-k+1}$ contains the letter p which makes k occurences of p. From Lemma 2.5.8 and Proposition 2.5.4, we conclude that no other Lyndon word contains p. Thus the Lyndon word $\mathbf{i}_0$ has to be one of the $\mathbf{j}_l$ and the inequalities (2.8) impose $\mathbf{i}_0=\mathbf{j}_n$. □

Lemma 2.5.13. *For any $1\leqslant j\leqslant r$, if $p_j\neq k$, then $p_{j-1}-p_j\leqslant 1$.*

Proof. Assume there exists $j\in\{1,\dots,r\}$ such that $p_j\neq k$ and $p_{j-1}-p_j\geqslant 2$. We set $q:=p_j$ and show that $s_q\leqslant N_q-1$. Let i denote the position of q in the word μ_k. Then $h_i=q$; on the other hand all the letters in position $1,\dots,i-1$ are greater than $q+2$ (as they are greater than p_{j-1}), hence

$$\varphi_1\cdots\varphi_{m_k}.(u\otimes(v_q)_z)=\tau_1\cdots\tau_{i-1}(\tau_i(x_i-x_{i+1})+1)\varphi_{i+1}\cdots\varphi_{m_k}.(u\otimes(v_q)_z).$$

By similar arguments as in the proof of Lemma 2.5.10 (ii), this implies $s_q\leqslant N_q-1$. Hence by Corollary 2.5.6, $q\in\{k,n-k+1\}$. Now by hypothesis $q\leqslant p_{j-1}-2\leqslant p_0-2<p_0$ and $p_0\leqslant n-k+1$ by Lemma 2.5.8. In particular $q<n-k+1$. As by assumption $q=p_j\neq k$, we get the desired contradiction.

□

Proposition 2.5.14. *With the previous notations, one has:*

(i) $p_1 < p_0$.

(ii) For all $j \geqslant 1$*, if* $p_j \neq k$ *then* $p_{j+1} < p_j$.

Proof. Assume $p_1 = p_0 = p$ and let i denote the position of the letter p_1 in the word μ_k, i.e. $h_i = p_1$. First note that this implies $2 \leqslant p \leqslant n-1$ (as the letters 1 and n appear only once in the word μ_k). In particular $\mathbf{i}_0$ is of length strictly greater than 2, as $\mathbf{i}_0 = (p \dots n)$ by Corollary 2.5.12. In other words $i > 2$. One has:

$$\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = (\tau_1(x_1 - x_2) + 1)\tau_2 \cdots \tau_{i-1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z). \quad (2.9)$$

As the renormalized R-matrix $r_{L(p),M_k}$ does not vanish, the operator $\varphi_{i+1} \cdots \varphi_{m_k}$ acts as a non zero polynomial function on $(u \otimes (v_p)_z)$. We set $Q(z) := \varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z)$. Note that $deg(Q) \leqslant N_p - 2$. Let $P(z)$ denote the polynomial function given by the term

$$\tau_2 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = \tau_2 \cdots \tau_{i-1}.Q(z)$$

from the above equality and let us consider the action of the operator $\tau_{i-1} \cdots \tau_1$ on $P(z)$. Recall that $i > 2$. We first write

$$\tau_2\tau_1.P(z) = \tau_2\tau_1.(\tau_2 \cdots \tau_{i-1})Q(z) = (\tau_1\tau_2\tau_1 + 1).\tau_3 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z)$$

using the braid relation. The operator τ_1 commutes with $\tau_3 \cdots \tau_{i-1}$ as well as with $\varphi_{i+1}, \dots, \varphi_{m_k}$. As $\mathbf{i}_0$ is of length greater than 2, the action of τ_1 on $(u \otimes (v_p)_z)$ is the same as its action on the cuspidal module $L(\mathbf{i}_0)$ which is trivial. Hence we get

$$\tau_2\tau_1.P(z) = \tau_3 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z).$$

The letters $h_3, \dots, h_{i-1}$ are greater than $p+2$ hence by arguments similar to the proof of Lemma 2.5.10 (ii), we get

$$\tau_{i-1} \cdots \tau_3.(\tau_3 \cdots \tau_{i-1}).Q(z) = Q(z).$$

Finally we get

$$\tau_{i-1} \cdots \tau_1.P(z) = \varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = Q(z)$$

which is not zero. A fortiori $P(z)$ itself is a non zero polynomial function. All the other terms in equation (2.9) are either zero, either of valuation strictly greater than the valuation of $Q(z)$. This implies that s_p is equal to the valuation of $Q(z)$. In particular $s_p \leqslant N_p - 2$. This contradicts Corollary 2.5.6. Hence $p_1 < p_0$.

For the second statement assume we have $j \geqslant 1$ such that $q := p_{j+1} = p_j \neq k$ and consider j minimal for this property. For the sake of simplicity, we only deal with the case where only the two Lyndon words $\mathbf{i}_j$ and $\mathbf{i}_{j+1}$ begin with the letter q i.e. $p_{j-1} > q, p_j = p_{j+1} = q$ and $p_{j+2} \leqslant q-1$. The proof is analogous if there are several words $\mathbf{i}_{j'}$ beginning with q.

As by hypothesis $q \neq k$, Lemma 2.5.13 implies $p_{j-1} \in \{q, q+1\}$ and hence $p_{j-1} = q+1$ as $p_{j-1} > q$. Moreover by minimality of j, $q+1$ appears exactly once in the subsequence $(p_i)_{i<j}$.

Set $a := a_{j-1}$, $b := a_j$ and $c := a_{j+1}$. As above N_q denotes the number of occurrences of q in the word w_k. We write

$$\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_q)_z) = \tau_1 \cdots \tau_{b-1}(\tau_b(x_b - x_{b+1}) + 1)\tau_{b+1} \cdots \tau_{c-1}(\tau_c(x_c - x_{c+1}) + 1)Q(z) \quad (2.10)$$

where $Q(z) := \varphi_{c+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z)$. As the renormalized R-matrix $r_{L(q),M_k}$ does not vanish, Q is a nonzero polynomial function. Its degree is equal to $N_q - 2$.

Now we prove that $Q(z)$ is in fact a monomial in z. Indeed, any occurrence of q in a position $i \in \{c+1, \dots, m_k\}$ appears inside a Lyndon word $\mathbf{i}_{j'}$ (with $j' > j+1$) beginning with a letter strictly smaller than q. Then the operator τ_{i-1} commutes with any φ_h , $h > i$ and acts by zero on $(u \otimes (v_q)_z)$. Thus

$$\begin{aligned}\tau_{i-1}\varphi_i\varphi_{i+1}\cdots\varphi_{m_k}.(u\otimes(v_q)_z) &= \tau_{i-1}\left(\tau_i(x_i - x_{i+1}) + 1\right)\varphi_{i+1}\cdots\varphi_{m_k}.(u\otimes(v_q)_z)\\ &= \tau_{i-1}\tau_i(x_i - x_{i+1})\varphi_{i+1}\cdots\varphi_{m_k}.(u\otimes(v_q)_z)\\ &= z.\tau_{i-1}\tau_i\varphi_{i+1}\cdots\varphi_{m_k}.(u\otimes(v_q)_z)\end{aligned}$$

up to some sign. This is valid for any occurrence of q between the positions $c+1$ and m_k and hence

$$Q(z) = \varphi_{c+1}\cdots\varphi_{m_k}.(u\otimes(v_p)_z) = z^{N_q-2}\tau_{c+1}\cdots\tau_{m_k}.(u\otimes(v_p)_z)$$

is a monomial in z of degree $N_q - 2$.

The term $\tau_1\cdots\tau_{b-1}\tau_{b+1}\cdots\tau_{c-1}Q(z)$ coming from equation (2.10) is necessarily zero: if it was not, then equation (2.10) implies that s_q would be equal to N_q-2, which contradicts Corollary 2.5.6. There are two terms of degree N_q-1 in equation (2.10): $\tau_1\cdots\tau_{c-1}Q(z)$ and $\tau_1\cdots\tau_{b-1}\tau_{b+1}\cdots\tau_c Q(z)$. Denote them respectively by $A(z)$ and $B(z)$.

We show that the operator $\tau_{c-1}\cdots\tau_1$ acts nontrivially on $A(z)$ and trivially on $B(z)$. This implies that $A(z) + B(z)$ cannot be zero and therefore there is a nonzero term of degree $N_q - 1$ in equation (2.10).

Action of $\tau_{c-1}\cdots\tau_1$ on $A(z)$. Let us first look at the action of the operators $\tau_1, \cdots, \tau_{a-1}$ on $A(z)$; if $j = 1$ this is of course not necessary as $a = a_0 = 1$. Otherwise one has $j \geqslant 2$ and this action is easy to compute: for instance for τ_1 one has:

$$\begin{aligned}\tau_1.A(z) &= {\tau_1}^2\tau_2\cdots\tau_{c-1}Q(z)\\ &= {\tau_1}^2 e(h_1, h_c, h_2, \dots, h_{c-1}, q, h_{c+1}, \dots, h_{m_k})\tau_2\cdots\tau_{c-1}Q(z)\\ &= \tau_2\cdots\tau_{c-1}Q(z) \text{ as } h_c = q \text{ and } h_1 = p_0 > p_1 > q \text{ by minimiality of } j \geqslant 2.\end{aligned}$$

Simmilarly $h_2, \dots, h_{a-1} \geqslant q+2$ and thus one gets

$$\tau_{a-1}\cdots\tau_1.A(z) = \tau_a\cdots\tau_{c-1}.Q(z).$$

Let us now look at the action of τ_a:

$$\begin{aligned}\tau_a.(\tau_a\cdots\tau_{c-1}.Q(z)) &= {\tau_a}^2 e(h_1, \dots, h_a, h_c, h_{a+1}, \dots, h_{c-1}, q, h_{c+1}, \dots, h_{m_k})\tau_{a+1}\cdots\tau_{c-1}Q(z)\\ &= (x_a - x_{a+1})\tau_{a+1}\cdots\tau_{c-1}Q(z)\text{as } h_a = p_{j-1} = q+1 \text{ and } h_c = p_{j+1} = q\end{aligned}$$

The operator x_a commutes with $\tau_{a+1}, \cdots \tau_{c-1}$ and acts trivially on the generating vector hence one gets

$$\begin{aligned}\tau_a.(\tau_a\cdots\tau_{c-1}.Q(z)) &= -x_{a+1}\tau_{a+1}\cdots\tau_{c-1}.Q(z)\\ &= -x_{a+1}\tau_{a+1}\cdots\tau_b e(h_1, \dots, h_b, h_c, h_{b+1}, \dots, h_{c-1}, \dots)\tau_{b+1}\cdots\tau_{c-1}Q(z)\\ &= -\tau_{a+1}\cdots\tau_{b-1}x_b\tau_b e(h_1, \dots, h_b, h_c, h_{b+1}, \dots)\tau_{b+1}\cdots\tau_{c-1}.Q(z)\\ &\qquad\qquad (h_c = q,\ h_{a+1}, \dots, h_{b-1} \geqslant q+2)\\ &= -\tau_{a+1}\cdots\tau_{b-1}(\tau_b x_{b+1} + 1)\tau_{b+1}\cdots\tau_{c-1}.Q(z)(h_b = h_c = q)\\ &= -\tau_{a+1}\cdots\tau_{b-1}\tau_b\tau_{b+1}\cdots\tau_{c-1}x_c.Q(z) - \tau_{a+1}\cdots\tau_{b-1}\tau_{b+1}\cdots\tau_{c-1}.Q(z).\end{aligned}$$

The operator x_c acts trivially on $(u \otimes (v_q)_z)$ hence the first term of the right hand side in the last equality is zero. Now as $h_{a+1}, \dots, h_{b-1} \geqslant q+2$ and $h_b = q$, the action of the operator $\tau_{b-1} \cdots \tau_{a+1}$ on the surviving term is similar to the action of $\tau_{a-1} \cdots \tau_1$ computed above. Hence we get

$$\tau_{b-1} \cdots \tau_a.(\tau_a \cdots \tau_{c-1}.Q(z)) = -\tau_{b+1} \cdots \tau_{c-1}.Q(z).$$

The situation is now similar to (i): using the braid relation, one can see that the action of $\tau_{b+1}\tau_b$ on $\tau_{b+1} \cdots \tau_{c-1}.Q(z)$ will give two terms, the only non-trivial one being $\tau_{b+2} \cdots \tau_{c-1}.Q(z)$. The letters $h_{b+2}, \dots, h_{c-1}$ are all greater than $q+2$ and hence one concludes as before that

$$\tau_{c-1} \cdots \tau_{b+2}.(\tau_{b+2} \cdots \tau_{c-1}.Q(z)) = Q(z).$$

Finally we have shown that $\tau_{c-1} \cdots \tau_1$ acts by identity on $A(z)$ (up to some sign).

Action of $\tau_{c-1} \cdots \tau_1$ on $B(z)$. One again has:

$$\tau_{a-1} \cdots \tau_1.B(z) = \tau_{a-1} \cdots \tau_1.(\tau_1 \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_c Q(z)) = \tau_a \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_c Q(z).$$

But then

$$\begin{aligned} \tau_a.(\tau_a \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_c Q(z)) &= (x_a - x_{a+1})\tau_{a+1} \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_c Q(z) \\ &= \tau_{a+1} \cdots \tau_{b-1} x_b \tau_{b+1} \cdots \tau_c Q(z) \end{aligned}$$

up to some sign, then the operator x_b commutes with $\tau_{b+1}, \dots, \tau_c, \varphi_{c+1}, \dots, \varphi_{m_k}$ and acts by zero on $(u \otimes (v_q)_z)$.

Finally we have shown that the operator $\tau_1 \cdots \tau_{c-1}$ acts nontrivially on $A(z) + B(z)$ and in particular $A(z) + B(z) \neq 0$. Therefore $s_q = N_q - 1$. Now, $q < p_0 \leqslant n-k+1$ by Lemma 2.5.8 and by assumption $q \neq k$ hence $q \notin \{k, n-k+1\}$. By Corollary 2.5.6, this contradicts the inequality $s_q \leqslant N_q - 1$.

In conclusion (ii) holds.

□

Corollary 2.5.15. *The sequence (p_j) takes exactly once every value $1, \dots, k-1$ and at least once the value k.*

Proof. The last term of the sequence (p_j) is $p_r = 1$. Recall that $r \geqslant k-1$ (Remark 2.5.11). By (finite) induction on $t \in \{0, \dots, k-1\}$ one shows that $p_{r-t} = t+1$. Indeed, if $k = 1$ there is nothing to prove. If $k \geqslant 2$, assume $p_r = 1, \dots, p_{r-t} = t+1$ with $t < k-1$; then $p_{r-t} \leqslant k-1$ and Lemma 2.5.13 implies $p_{r-t-1} \leqslant p_{r-t} + 1$. If $p_{r-t-1} \neq k$ then Proposition 2.5.14 (ii) implies $p_{r-t-1} > p_{r-t}$ and thus $p_{r-t-1} = p_{r-t} + 1$ which gives $p_{r-(t+1)} = t+1$. If $p_{r-t-1} = k$ then as $p_{r-t} \leqslant k-1$, necessarily one has $p_{r-t} = t = k-1$ and $p_{r-t-1} = k$ which again gives $p_{r-(t+1)} = t+1$. This implies that the sequence (p_j) takes exactly once each value $1, \dots, k-1$ (and at least once the value k).

□

Corollary 2.5.16. *In the case $p = k$, the parameter μ_k of M_k is given by*

$$\mu_k = (k \dots n)(k-1 \dots n-1) \cdots (1 \dots n-k+1).$$

Proof. By Proposition 2.5.14 (i), the sequence (p_j) takes exactly once the value k. Together with Corollary 2.5.15, we deduce that the word μ_k has the form

$$\mu_k = (k \dots)(k-1 \dots) \cdots (1 \dots).$$

Combining this with Lemma 2.5.10 (i) and (ii), we get the desired statement.

□

One can now focus on the case $p_0 = n - k + 1$.

Proposition 2.5.17. *The sequence $(p_j)_{0 \leqslant j \leqslant r}$ takes exactly once every value between $n-k+1$ and 1. In other words, $r = n - k + 1$ and $p_j = n - k + 1 - j$ for all $1 \leqslant j \leqslant n - k + 1$.*

Proof. By Corollary 2.5.15, we already know that that the values $1, \ldots, k-1$ are taken exactly once and the value k at least once.

Values $k+1, \ldots, n-k+1$. Let $i := \max\{j, p_j > k\}$ (it exists as $p_0 = n - k + 1 > k$) and $m := p_i$.

Assume $m \geqslant k+2$. Then the commuting of M_k with $L(k)$ implies that $\mathbf{i}_{i+1}$ is the only Lyndon word beginning with k (equivalently $p_{i+1} = k, p_{i+2} = k-1, \ldots, p_r = 1$). Indeed, all the letters in position strictly smaller than a_{i+1} are greater than $k+2$ hence $\varphi_{i'} = \tau_{i'}$ for all $i' < a_{i+1}$ and

$$\tau_{a_{i+1}-1} \cdots \tau_1.(\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_k)_z)) = \varphi_{a_{i+1}} \cdots \varphi_{m_k}.(u \otimes (v_k)_z).$$

Then the same proof as for Proposition 2.5.14(i) shows that necessarily $k = p_{i+1} > p_{i+2}$.

Thus there is exactly one Lyndon word $\mathbf{i}_j$ beginning with every letter $1, \ldots, k$. The letter $m-1$ does not appear in any of the words $\mathbf{i}_j$ for $j \leqslant i$ (all these words begin with letters greater than m) and appears exactly k times in the word μ_k (as $n - k \geqslant m - 1 \geqslant k + 1$) hence it appears exactly once in each of the words $\mathbf{i}_{i+1}, \ldots, \mathbf{i}_r$. This implies that the last letters of all of these words are greater than $m-1$ and in particular so is k (last letter of $\mathbf{i}_r$), i.e. $m \leqslant k+1$ which contradicts the hypothesis.

Hence $m = k+1$ i.e. the sequence (p_j) takes all the values $n-k+1, \ldots, 1$. By Proposition 2.5.14(ii) the values $n-k+1, \ldots, k+1$ appear exactly once in the sequence (p_j).

Value k. If there are more than two Lyndon words $\mathbf{i}_j$ beginning with the letter k then the same proof as for Proposition 2.5.14(ii) (it can be applied as $m = k+1$) implies $s_k \leqslant N_k - 1$. But as the last letter of the word μ_k is k, the same proof as for Lemma 2.5.8 shows that s'_k is also smaller than $N_k - 1$. Hence both s_k and s'_k are less than $N_k - 1$ which contradicts Corollary 2.5.6.

Therefore the sequence (p_j) takes exactly once every value $n-k+1, \ldots, 1$.

□

Corollary 2.5.18. *For any $0 \leqslant j \leqslant n-k$, the Lyndon word $\mathbf{i}_j$ is $(n-k+1-j \ldots n-j)$.*

Proof. We show it by induction on j. In fact we prove the following properties:

(i) For every $0 \leqslant j \leqslant n-k$ there is exactly one Lyndon word ending with each of the letters $n-j, \ldots, n-k+1-j$.

(ii) The Lyndon word ending with $n-j$ begins with the letter $n-k+1-j$.

For $j = 0$ it follows from Corollary 2.5.12.

Assume (i) and (ii) hold until the rank j. By hypothesis the Lyndon words ending with the letters $n, n-1 \ldots, n-j$ respectively begin with the letters $n-k+1, \ldots, n-k+1-j$ and in particular do not contain $n-k+j$. As by Proposition 2.5.17 there is exactly one Lyndon word beginning with each of the letters $n-k+1, \ldots, 1$, the Lyndon words ending with letters $n-1-j, \ldots, n-k+1-j$ begin with letters less than $n-k-j$ and hence contain $n-k-j$. This gives $k-1$ Lyndon words containing the letter $n-k-j$. As this letter appears exactly k times in the word μ_k, there exists a Lyndon word that contains $n-k-j$ but is not one of the previous words, i.e. does not end with any of the letters $n, \ldots, n-j$. Hence it does not contain $n-k-j+1$ (as $n-k-j+1$ appears k times, once in each of the k words ending with $n-j, \ldots, n-k-j+1$). This means there is a unique Lyndon word ending with the letter $n-k-j$, which proves (i) at the rank $j+1$.

Now (ii) at rank j and (i) at rank $j+1$ together with Proposition 2.5.17 easily imply (ii) at rank $j+1$.

□

From Corollary 2.5.16 and Corollary 2.5.18, one concludes that among the two simple modules M_k and M_{n-k+1}, one of them has a parameter whose first letter is k, namely $(k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1)$, and the other has a parameter whose first letter is $n-k+1$, namely $(n-k+1\dots n)\cdots(1\dots k)$.

2.5.5 Proofs of main theorems

At this stage, one only has bijections between pairs of modules and pairs of dominant words: for each $1 \leqslant k \leqslant n/2$, the set of modules $\{M_k, M_{n-k+1}\}$ is in one-to-one correspondence with the set $\{(k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1),(n-k+1\dots n)\cdots(1\dots k)\}$. A priori this yields two possibilities for each k. To complete the proof of Theorem 2.5.1, we need to show that for every $1 \leqslant k \leqslant n/2$, one has

$$M_k = L\left((k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1)\right)$$

and

$$M_{n-k+1} = L\left((n-k+1\dots n)(n-k\dots n-1)\cdots(1\dots k)\right).$$

The key argument is the mutation rule for dominant words given by Proposition 2.4.8.

Proof of Theorem 2.5.1. We prove by induction on $k \in \{1,\dots,n\}$ that

$$\begin{array}{rcl} M_{r_{k-1}+1} & = & L(1\dots k) \\ M_{r_{k-1}+2} & = & L\left((2\dots k)(1\dots k-1)\right) \\ \dots & \dots & \dots \\ M_{r_k} & = & L(k\cdots 1). \end{array}$$

The result already holds for $k=1$ and $k=2$. Consider $1 \leqslant k \leqslant n$ and assume the result holds at the rank k.

Let $j \in \{r_{k-1}+2,\dots,r_k-1\}$ and let us write the (ungraded) short exact sequence corresponding to the mutation in direction j:

$$0 \to M_{j_+} \circ M_{j-1} \circ M_{j_-+1} \to M_j \circ {M_j}' \to M_{j_-} \circ M_{j+1} \circ M_{j_+-1} \to 0.$$

Let $p := j - r_{k-1}$.

By the induction hypothesis, one has

$$\begin{array}{rcl} M_j & = & L\left((p\dots k)\cdots(1\dots k-p+1)\right) \\ M_{j-1} & = & L\left((p-1\dots k)\cdots(1\dots k-p+2)\right) \\ M_{j+1} & = & L\left((p+1\dots k)\cdots(1\dots k-p)\right) \end{array}$$

The Lyndon word $(p\dots k)$ appears in the parameter of M_j hence in the parameter of $M_j \circ M_j'$. Hence by Proposition 2.4.8, it necessarily appears either in $\mu_{j_+} \odot \mu_{j-1} \odot \mu_{j_-+1}$ or in $\mu_{j_-} \odot \mu_{j+1} \odot \mu_{j_+-1}$. Obviously, it does not appear in μ_{j-1} nor in μ_{j+1}. Moreover, μ_{j_-} and μ_{j_-+1} do not contain the letter k hence $(p\dots k)$ does not appear in the canonical factorizations of these parameters either.

Now by Proposition 2.5.4, μ_{j_+} is either $(p+1\dots k+1)\cdots(1\dots k-p)$ or $(k-p+1\dots k+1)\cdots(1\dots p+1)$ and μ_{j_+-1} is either $(p\dots k+1)\cdots(1\dots k-p+2)$ or $(k-p+2\dots k+1)\cdots(1\dots p)$. The only of these words in which the Lyndon $(p\dots k)$ appears is $(p+1\dots k+1)\cdots(1\dots k-p)$ and thus $\mu_{j_+} = \mu_{r_k+p+1} = (p+1\dots k+1)\cdots(1\dots k-p)$.

One can do this for any $j \in \{r_{k-1}+2,\dots,r_k-1\}$, and the same arguments hold for $j = r_{k-1}+1$ and $j = r_k$. Thus the desired result holds at rank $k+1$.

□

One can now prove Theorem 2.5.2.

Proof of Theorem 2.5.2. We begin by describing the exchange matrix corresponding to the quiver given in [69, Definition 11.1.1]. For any $1 \leqslant k \leqslant n-1$ define the following matrices:

$$A_k := \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}, \quad B_k := \begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & \cdots 0 & 1 \end{pmatrix}$$

and

$$C_k := -\,{}^tB_{k-1}$$

of respective sizes $k \times k$, $k+1 \times k$, and $k-1 \times k$.

Now the whole exchange matrix can be written by blocks as follows:

$$\begin{pmatrix} A_1 & C_2 & 0 & \cdots & \cdots & 0 \\ B_1 & A_2 & C_3 & \ddots & & \vdots \\ 0 & B_2 & A_3 & C_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & \ddots & \ddots & C_{n-1} \\ 0 & \cdots & \cdots & 0 & B_{n-2} & A_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & B_{n-1} \end{pmatrix}.$$

Recall that for any parameter $\mu \in \mathbf{M}$, $\mu^{\odot -1}$ denotes the inverse of μ in the Grothendieck group $\mathbf{G}$ of $\mathbf{M}$. We can now compute the parameters $\hat{\mu_j}$ associated to the $\hat{y_j}$ as in Definition 2.3.7. For instance, for any $2 \leqslant j \leqslant n-2$,

$$\begin{aligned} \hat{\mu}_{r_{n-2}+j} = {} & ((j-1\ldots n-2)\cdots(1\ldots n-j))^{\odot -1} \odot ((j\ldots n-2)\cdots(1\ldots n-j-1)) \\ & \odot ((j-1\ldots n-1)\cdots(1\ldots n-j+1)) \odot ((j+1\ldots n-1)\cdots(1\ldots n-j-1))^{\odot -1} \\ & \odot ((j\ldots n)\cdots(1\ldots n-j+1))^{\odot -1} \odot ((j+1\ldots n)\cdots(1\ldots n-j)) \end{aligned}$$

which simplifies as

$$\hat{\mu}_{r_{n-2}+j} = ((j+1\ldots n)(j\ldots n-1)) \odot ((j+1\ldots n-1)(j\ldots n))^{\odot -1}.$$

Hence for any parameter $\mu \in \mathbf{M}$, one has

$$\begin{aligned} ((j+1\ldots n-1)(j\ldots n)) \odot \hat{\mu}_{r_{n-2}+j} \odot \mu &= ((j+1\ldots n)(j\ldots n-1)) \odot \mu \\ &> ((j+1\ldots n-1)(j\ldots n)) \odot \mu. \end{aligned}$$

This exactly means $\hat{\mu}_{r_{n-2}+j} \odot \mu > \mu$ in $\mathbf{G}$ for any $\mu \in \mathbf{M}$. The computations for any other index $s \in \{1, \ldots, r_{n-1}\}$ are similar.

Using Remark 2.3.8 we conclude that the seed $\mathcal{S}_0^n$ is compatible.

□

2.6 Possible further developments

In this section we mention a couple of situations where interesting consequences may arise from the study of compatible seeds in various contexts of monoidal categorifications of cluster algebras.

2.6.1 Dominant words and g-vectors

By Theorem 2.5.2, the seed $\mathcal{S}_0^n$ for the category $R-gmod$ in type A_n is compatible in the sense of Definition 2.3.7. As explained in Subsection 2.3.3, this yields some interesting combinatorial relationships between dominant words and g-vectos.

More precisely, consider as in Subsection 2.3.3 x_l^t any cluster variable in $\mathcal{A}$, M_l^t the simple module in $R-gmod$ such that $[M_l^t] = x_l^t$ and μ_l^t the dominant word associated to M_l^t. For simplicity, we will write x (resp. M, μ) for x_l^t (resp. M_l^t, μ_l^t) without ambiguity as we will focus here on this module. For any dominant Lyndon word (i.e. any positive root in type A_n) $(k \dots l)$, we let $m_{(k\dots l)}$ denote the multiplicity of the Lyndon word $(k \dots l)$ in the canonical factorization of μ. As in Section 2.1.1, we consider F and $\mathbf{g} = (g_1, \dots, g_{r_{n-1}})$ the F-polynomial and the g-vector associated to x. We also let $a_1, \dots, a_{r_{n-1}}$ denote the exponents of the unique monomial of maximal degree of F (see Theorem 2.1.5(i)), and $c_1, \dots, c_n$ the (negative) integers such that $F_{|\mathbb{P}}(y_1, \dots, y_n) = x_{r_{n-1}+1}^{c_1} \cdots x_{r_n}^{c_n}$ (see Subsection 2.3.3).

First consider the positive roots ending with the letter n. It follows from Theorem 2.5.1 that these Lyndon words do not appear in the dominant words associated to the unfrozen variables of the seed $\mathcal{S}_0^n$. Hence the g-vectors will not be involved. Moreover, for any $1 \leqslant j \leqslant n$, the Lyndon word $(j \dots n)$ appears in exactly one of the dominant words corresponding to the frozen variables of $\mathcal{S}_0^n$ namely $(j \dots n) \cdots (1 \dots n-j+1)$. If $2 \leqslant j \leqslant n-1$, then the Lyndon word $(j \dots n)$ appears in exactly two of the $\hat{\mu}_j$ namely $\hat{\mu}_{r_{n-1}+j-1}$ and $\hat{\mu}_{r_{n-1}+j}$. The relations are the following:

$$m_{(j\dots n)} = a_{r_{n-2}+j-1} - a_{r_{n-2}+j} - c_j.$$

For positive roots of the form $(j \dots n-1)$ with $2 \leqslant j \leqslant n-2$, similar arguments show that

$$m_{(j\dots n-1)} = a_{r_{n-2}+j} - a_{r_{n-2}+j+1} - c_{j+1} - a_{r_{n-2}+j-1} + a_{r_{n-2}+j+1} + a_{r_{n-3}+j-1} - a_{r_{n-3}+j} + g_{r_{n-2}+j}$$

which simplifies as

$$m_{(j\dots n-1)} = a_{r_{n-2}+j} - c_{j+1} - a_{r_{n-2}+j-1} + a_{r_{n-3}+j-1} - a_{r_{n-3}+j} + g_{r_{n-2}+j}.$$

2.6.2 The coherent Satake category

Recently, Cautis-Williams [27] exhibited a new example of monoidal categorification of cluster algebras, using the coherent Satake category. In this subsection we focus on the case of the general linear group GL_n. We begin by checking that Assumptions A and B hold in the framework of [27].

The simple objects in the coherent Satake category are parametrized (up to $\mathbb{G}_m$-equivariant shift) by couples of a coweight and a weight, modulo action of the Weyl group. Equivalently they can be parametrized by *dominant pairs*, i.e. couples of a dominant coweight $\lambda^\vee$ together with a weight μ dominant for the Levi factor of $P_{\lambda^\vee}$. Denote by $\mathcal{P}_{\lambda^\vee,\mu}$ the simple perverse coherent sheaf corresponding to a dominant pair $(\lambda^\vee, \mu) \in P^\vee \times P$. Then the following statement shows that Assumption A holds:

Proposition 2.6.1 ([27, Proposition 2.6]). *Let $\mathcal{P}_{\lambda_1^\vee,\mu_1}$ and $\mathcal{P}_{\lambda_2^\vee,\mu_2}$ be two simple objects in the coherent Satake category. Then in its Grothendieck ring $K^{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G)$ one has:*

$$[\mathcal{P}_{\lambda_1^\vee,\mu_1} * \mathcal{P}_{\lambda_2^\vee,\mu_2}] = q^s[\mathcal{P}_{\lambda_1^\vee+\lambda_2^\vee,\mu_1+\mu_2}] + \sum_{(\lambda^\vee,\mu)\in S} p_{\lambda^\vee,\mu}[\mathcal{P}_{\lambda^\vee,\mu}]$$

where s is some integer depending on $\lambda_1, \mu_1, \lambda_2, \mu_2$, $p_{\lambda^\vee,\mu} \in \mathbb{Z}[q^{\pm 1/2}]$, and S is a finite collection of dominant pairs such that for every $(\lambda^\vee, \mu) \in S$, one has either $\lambda^\vee < \lambda_1^\vee + \lambda_2^\vee$, or $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee$ and $\| \mu \|^2 \leqslant \| \mu_1 \|^2 + \| \mu_2 \|^2$ for any W-invariant quadratic form $\| \cdot \|^2$.

Taking the lexicographic order on (dominant) pairs $(\lambda^\vee, \mu) \in P^\vee \times P$, the monoid structure on the set of dominant pairs can be simply taken as

$$(\lambda_1^\vee, \mu_1) \odot (\lambda_2^\vee, \mu_2) = (\lambda_1^\vee + \lambda_2^\vee, \mu_1 + \mu_2).$$

It is then clear that Assumption B also holds.

In the case of the general linear group GL_n, Cautis-Williams explicitly describe a monoidal seed in the coherent Satake category. However, this seed is not compatible in the sense of Definition 2.3.7 above. For instance, for GL_2, this seed can be written as

$$(([\mathcal{P}_{1,0}], [\mathcal{P}_{1,1}], [\mathcal{P}_{2,0}], [\mathcal{P}_{2,1}]), B)$$

where the first two classes are the unfrozen variables and the last two are the frozen variables, and the exchange matrix B is given by:

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall from [27, Section 2.2] that $\mathcal{P}_{k,l}$ stands for $\mathcal{P}_{\omega_k^\vee, l\omega_k}$ for any $1 \leqslant k \leqslant 2$ and any $l \in \{0, 1\}$.

One can now compute the generalized parameters $\hat{\mu_1}$ and $\hat{\mu_2}$ for this seed. A straightforward computation gives $\hat{\mu_1} = (2\omega_1^\vee - \omega_2^\vee, 2\omega_1 - \omega_2)$ and $\hat{\mu_2} = (\omega_2^\vee - 2\omega_1^\vee, 0)$. The coweight $2\omega_1^\vee - \omega_2^\vee$ is exactly the coroot $\alpha_1^\vee$, and hence for any dominant pair $(\lambda^\vee, \mu)$ one has $\hat{\mu_1} \odot (\lambda^\vee, \mu) \geqslant (\lambda^\vee, \mu)$. However, the coweight part of $\hat{\mu_2}$ is obviously the opposite of $\alpha_1^\vee$ and thus $\hat{\mu_2} \odot (\lambda^\vee, \mu) \leqslant (\lambda^\vee, \mu)$ for any dominant pair $(\lambda^\vee, \mu)$. We conclude that this seed is not compatible.

It would be interesting to see if Conjecture 2.3.10 holds in the coherent Satake category of the general linear group. Note that as the ordering on dominant pairs in partial, it is not clear that one can formulate mutation rules for parameters as in Section 2.4. Indeed, we crucially used the fact that the ordering on dominant words parametrizing simple modules over quiver Hecke algebras is total. This mutation rule allows to compute explicitly as many seeds as we want from the data of an initial seed. In the case of a partial ordering, we cannot do so a priori.

Chapter 3

Newton-Okounkov bodies for categories of modules over quiver Hecke algebras

This chapter corresponds to the arXiv Preprint [24], arXiv:1911.11440v1.

We show that for a finite-type Lie algebra $\mathfrak{g}$, the representation theory of quiver Hecke algebras provides a natural framework for the construction of Newton-Okounkov bodies associated to the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n}(w))$. When $\mathfrak{g}$ is simply-laced, we use Kang-Kashiwara-Kim-Oh's monoidal categorification setting to investigate the cluster theory of these bodies. In particular, our construction yields for every seed $\mathcal{S}$ of $\mathcal{A}_q(\mathfrak{n}(w))$ a simplex $\Delta_{\mathcal{S}}$ of codimension 1 in $\mathbb{R}^{l(w)}$. We exhibit various geometric and combinatorial properties of these simplices by characterizing their rational points, their normal fans, and their volumes. The key tool is provided by the explicit description in terms of root partitions of the determinantial modules of a certain seed in $\mathcal{A}_q(\mathfrak{n}(w))$ constructed in [52, 69]. This is achieved using the recent results of Kashiwara-Kim [72]. As an application, we prove an equality of rational functions involving root partitions for cluster variables. It implies an expression of the Peterson-Proctor hook formula in terms of heights of monoidal cluster variables in $\mathcal{C}_w$, suggesting further connections between cluster theory and the combinatorics of fully-commutative elements of Weyl groups.

3.1 Quiver Hecke algebras and monoidal categorifications of quantum coordinate rings

In this section we recall some representation-theoretic background and we fix notations. We begin with some reminders about Kleshchev-Ram's classification of finite-dimensional irreducible representations of finite type quiver Hecke algebras [79]. Then we recall how quiver Hecke algebras provide a useful framework for monoidal categorifications of cluster algebras following [69] and more recently [72]. Finally we recall several technical tools from [25] that will be useful in the next sections.

3.1.1 General reminders on quiver Hecke algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra of finite type, I the set of vertices of the Dynkin diagram of $\mathfrak{g}$. We use the following standard Lie-theoretic notations: $\Pi = \{\alpha_i, i \in I\}$ stands for the set of simple roots, $Q_+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i$, and Φ_+ denotes the set of positive roots. We also let $\mathcal{M}$ denote the set of (finite) words over the alphabet I. For $\nu = h_1, \ldots, h_r \in \mathcal{M}$, we define the *weight* of ν as the element of Q_+ given by

$$\mathrm{wt}(\nu) := \sum_{i\in I} \sharp\{k, h_k = i\}\alpha_i.$$

To any $\beta \in Q_+$ one associates a $\mathbb{Z}$-graded associative algebra $R(\beta)$ defined by generators and relations. We refer to [76, 79, 66] for precise definitions. Let us only outline the fact that among the generators of $R(\beta)$, one has a family of idempotents $\{e(\nu), \nu \in \mathcal{M}$ such that $\mathrm{wt}(\nu) = \beta\}$, satisfying the relations

$$e(\mu)e(\nu) = \delta_{\mu,\nu} e(\nu).$$

This family of algebras is called quiver Hecke algebras. For any $\beta \in Q_+$, one denotes by $R(\beta)$-gmod the category of finite-dimensional graded $R(\beta)$-modules. One also sets

$$R - gmod := \bigoplus_{\beta} R(\beta) - gmod.$$

The main property of quiver Hecke algebras is that the category $R-gmod$ categorifies the quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ (which is isomorphic to the positive part of the quantum group $U_q(\mathfrak{g})$) in a way that sends the isomorphism classes of simple objects in $R-gmod$ bijectively onto the elements of the dual canonical basis of $\mathcal{A}_q(\mathfrak{n})$.

The classification of irreducible finite-dimensional representations over quiver Hecke algebras of finite type was done by Kleshchev-Ram [79]. This parametrization uses the combinatorics of Lyndon words, or root partitions. It has been generalized by Kleshchev [77] and McNamara [91] to affine type quiver Hecke algebras. Recall that $\mathcal{M}$ denotes the set of finite words over the alphabet I. Fix a total order $<$ on I; thus $\mathcal{M}$ is totally ordered for the induced lexicographic order $\leqslant$. For every $\beta \in Q_+$, any finite-dimensional $R(\beta)$-module V decomposes as

$$V = \bigoplus_{\nu, \mathrm{wt}(\nu)=\beta} e(\nu) \cdot V.$$

The subspace $e(\nu) \cdot V$ can be seen as some kind of weight space by analogy with the representation theory of semisimple finite-dimensional Lie algebras. Hence one can consider the *highest word* of V, i.e. the biggest ν (for the total order $\leqslant$) such that $e(\nu) \cdot V$ is non zero. We set

$$\mathbf{M} := \{\nu \in \mathcal{M} \mid \exists V \in R(\mathrm{wt}(\nu)) - mod, \nu \text{ is the highest word of } V\}.$$

The following statement is the main result of [79] and shows that $\mathbf{M}$ is in bijection with the set of isomorphism classes of simple modules in $R - gmod$.

Theorem 3.1.1 ([79, Theorem 7.2]). *1. There exists a finite subset $\mathcal{GL}$ of $\mathcal{M}$ in bijection with Φ_+ such that $\mathbf{M}$ is exactly the set*

$$\{\mathbf{j}_1 \cdots \mathbf{j}_k \mid \mathbf{j}_1, \ldots, \mathbf{j}_k \in \mathcal{GL}, \mathbf{j}_1 \geqslant \cdots \geqslant \mathbf{j}_k\}.$$

2. For every $\mu \in \mathbf{M}$, there is a unique (up to isomorphism) finite-dimensional simple module $L(\mu)$ of highest word μ. Moreover, write $\mu = \mathbf{j}_1 \cdots \mathbf{j}_k$; then $L(\mu)$ is given by

$$L(\mu) = hd\left(L(\mathbf{j}_1) \circ \cdots \circ L(\mathbf{j}_k)\right).$$

3. For μ of the form $\mathbf{j}^n$ with $\mathbf{j} \in \mathcal{GL}$, one has $L(\mu) = L(\mathbf{j})^{\circ n}$.

The elements of $\mathcal{GL}$ are called *good Lyndon words* (or dominant Lyndon words). Elements of $\mathbf{M}$ are called *dominant words*. The simple modules corresponding to good Lyndon words are called *cuspidal representations* (see also [91]).

Remark 3.1.2. For a dominant word $\mu \in \mathbf{M}$, the writing $\mu = \mathbf{i}_1 \cdots \mathbf{i}_k$ with $\mathbf{i}_1, \ldots, \mathbf{i}_k \in \mathcal{GL}, \mathbf{i}_1 \geqslant \cdots \geqslant \mathbf{i}_k$ is known to be essentially unique (see [79] for a precise statement). It is called the *canonical factorization* of μ.

3.1.2 Monoidal categorification of quantum coordinate rings

In their series of papers [66, 67, 69], Kang-Kashiwara-Kim-Oh constructed braiding operators for $R-gmod$ and showed that this category is a monoidal categorification (in the sense of [56]) of the cluster structure on $\mathcal{A}_q(\mathfrak{n})$. They also proved similar statements for various subcategories of $R-gmod$, denoted $\mathcal{C}_w$. This section is devoted to fixing notations and recalling the main properties of these categories. Our exposition mainly follows [72, Section 2.3].

Assume $\mathfrak{g}$ is simply-laced. Let W denote the Weyl group of $\mathfrak{g}$. For any $w \in W$, we denote by $N := l(w)$ the length of w. The quantum coordinate ring $\mathcal{A}_q(\mathfrak{n}(w))$ is a subalgebra of $\mathcal{A}_q(\mathfrak{n})$ defined in [52]. It is shown to have a (quantum) cluster algebra structure. Kang-Kashiwara-Kim-Oh [69] showed that $\mathcal{A}_q(\mathfrak{n}(w))$ admits a monoidal categorification by a subcategory $\mathcal{C}_w$ of $R-gmod$ ([69, Theorem 11.2.3]). The category $\mathcal{C}_w$ is stable under taking subquotients, extensions, and monoidal products. Following [72], we set

$$\Phi_+^w := \Phi_+ \cap w\Phi_-$$

where Φ_+ (resp. Φ_-) stands for the set of positive (resp. negative) roots of $\mathfrak{g}$. The set Φ_+^w has cardinality N and we write $\Phi_+^w = \{\beta_k, 1 \leqslant k \leqslant N\}$.

Remark 3.1.3. The set of positive roots Φ_+^w does not depend on the choice of a reduced expression for w. Moreover, $\Phi_+^w \neq \Phi_+^{w'}$ if $w \neq w'$.

There is a natural bijection between (total) convex orderings on Φ_+^w and reduced expressions of w. More precisely, consider a reduced expression $w = s_{i_1} \cdots s_{i_N}$ and let $\mathbf{w} := (i_1, \ldots, i_N)$. There is a natural corresponding convex order $\leqslant$ on Φ_+^w given by

$$\beta_1 < \cdots < \beta_N$$

where $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ for every $1 \leqslant k \leqslant N$. Let $\mathbf{i}_1 < \cdots < \mathbf{i}_N$ denote the good Lyndon words corresponding respectively to $\beta_1, \ldots, \beta_N$ via the bijection of Theorem 3.1.1(1). It is known (see [72, Section 2.3]) that the simple objects in $\mathcal{C}_w$ are exactly the $L(\mu)$ for μ dominant word of the form $(\mathbf{i}_N)^{c_N} \cdots (\mathbf{i}_1)^{c_1}, c_1, \ldots, c_N \in \mathbb{N}$.

Geiss-Leclerc-Schröer [52] constructed an initial seed $\mathcal{S}^{\mathbf{w}}$ in $\mathcal{A}_q(\mathfrak{n}(w))$ corresponding to the chosen reduced expression $\mathbf{w}$ of w. The index set J of the cluster variables of this seed is $J = \{1, \ldots, N\}$; it splits into the disjoint union $J = J_{ex} \sqcup J_{fr}$ where J_{ex} (resp. J_{fr}) denotes the index set of the unfrozen variables (resp. the frozen variables). One has

$$J_{fr} = \{k \in J, k_+ = N+1\} \qquad \text{and} \qquad J_{ex} = J \backslash J_{fr}$$

where $k_+ := \min\left(\{k | k < s \leqslant N, i_k = i_s\} \sqcup \{N+1\}\right)$.

3.1.3 Dominance order and generalized parameters

We consider a Lie algebra $\mathfrak{g}$ of type A_n, D_n or E_n and we fix and index set I of simple roots. Recall that the category $R-gmod$ associated to $\mathfrak{g}$ is a monoidal category, whose product is denoted $\circ$. We let $<$ denote an arbitrary ordering on I. The set $\mathbf{M}$ of dominant words is totally ordered for the induced lexicographic order (still denoted $<$). It is naturally endowed with a structure of abelian monoid, whose law is denoted $\odot$. By definition, for any $\mu, \nu \in \mathbf{M}$, $\mu \odot \nu$ is defined as the greatest of the dominant words corresponding to the Jordan-Hölder components of the product $L(\mu) \circ L(\nu)$. The monoid $(\mathbf{M}, \odot)$ is commutative (as $K_0(\mathcal{C})$ categorifies a cluster algebra, which is commutative); hence it is naturally embedded into its Grothendieck group $\mathbf{G}$ whose law is again denoted by $\odot$. This group is abelian and inherits a total lexicographic order that extends the one

on $\mathbf{M}$ (see [25, Definition 4.4, Proposition 4.5]). Let r be the number of positive roots and let us write $\mathbf{j}_1 < \cdots < \mathbf{j}_r$ the elements of $\mathcal{GL}$ ordered with respect to $<$. Consider the following map:

$$\begin{array}{rccc} \varphi : & \mathbf{M} & \longrightarrow & \mathbb{N}^r \\ & \mu & \longmapsto & {}^t(c_1, \ldots, c_r) \end{array}$$

if $(\mathbf{j}_r)^{c_r} \cdots (\mathbf{j}_1)^{c_1}$ is the canonical factorization of μ (the integers c_i being possibly zero). The following was proved for a type A_n underlying Lie algebra $\mathfrak{g}$ in [25]:

Proposition 3.1.4. *The map φ is a monoid isomorphism from $(\mathbf{M}, \odot)$ to $(\mathbb{N}^r, +)$.*

We delay the proof to the next section.

In [25, Section 4.2], we constructed a map

$$\tilde{\Psi} : K_0(R - gmod) \longrightarrow \mathbf{M}$$

sending the class of any simple module in $R - gmod$ onto the corresponding dominant word in $\mathbf{M}$. This map satisfies

$$\tilde{\Psi}([L(\mu)][L(\nu)]) = \mu \odot \nu$$

for any $\mu, \nu \in \mathbf{M}$. In other words one has

$$\tilde{\Psi}([M][N]) = \tilde{\Psi}([M]) \odot \tilde{\Psi}([N]) \tag{3.1}$$

for any simple objects M, N in $R - gmod$.

Now let $w \in W$ and $N := l(w)$. Fix a reduced expression $\mathbf{w}$ of w. This is equivalent to the choice of order on Φ_+^w as recalled in Section 3.1.2. As in the previous section we write $\Phi_+^w = \{\beta_1 < \cdots \beta_N\}$. By Theorem 3.1.1 there is a unique word $\mathbf{i}_k$ of $\mathcal{GL}$ with weight β_k for every $1 \leqslant k \leqslant N$. Similarly the simple objects in $\mathcal{C}_w$ are in bijection with the set

$$\mathbf{M}_w := \{(\mathbf{i}_N)^{c_N} \cdots (\mathbf{i})_1^{c_1}, c_1, \ldots, c_N \geqslant 0\}.$$

As before we let $\mathbf{G}_w$ denote the Grothendieck group of $\mathbf{M}_w$. The rings $K_0(R{-}gmod)$ (resp. $K_0(\mathcal{C}_w)$) are domains and hence are embedded into their fraction fields. The map $\tilde{\Psi}$ can be extended to these fields by setting:

$$\tilde{\Psi}\left(\frac{x}{y}\right) := \tilde{\Psi}(x) \odot \tilde{\Psi}(y)^{\odot -1}.$$

This map provides a way to study the cluster structure of $\mathcal{A}_q(\mathfrak{n}(w))$ at the level of the monoid $\mathbf{M}_w$ (or the group $\mathbf{G}_w$). Let $\mathcal{S}$ be a seed in $\mathcal{A}_q(\mathfrak{n}(w))$. Let $x_1, \ldots, x_N$ denote the cluster variables and $B = (b_{ij})_{i,j}$ the exchange matrix of $\mathcal{S}$. Following [43] we set

$$\hat{y_j} := \prod_{1 \leqslant i \leqslant N} x_i^{b_{ij}}$$

and we define (see [25, Definition 4.7]):

$$\hat{\mu_j} := \tilde{\Psi}(\hat{y_j}) = \bigodot_{1 \leqslant i \leqslant N} \mu_i^{\odot b_{ij}} \in G_w.$$

These elements are of particular interest from the perspective of monoidal categorification of cluster algebras as they can be used to define remarkable partial orderings as follows:

Definition 3.1.5 (Dominance order , [106]). *Let $\mathcal{S} = ((x_1, \ldots, x_N), B)$ be a seed in $\mathcal{A}$ and consider the elements $\hat{y}_j$ defined above. Then, given $\mathfrak{m}$ and $\mathfrak{m}'$ two Laurent monomials in the x_i, we write*

$$\mathfrak{m} \preccurlyeq \mathfrak{m}' \Leftrightarrow \exists \gamma_1, \ldots, \gamma_n \geqslant 0, \mathfrak{m}' = \mathfrak{m} \times \prod_j \hat{y_j}^{\gamma_j}.$$

In the framework of monoidal categorification of quantum coordinate rings using quiver Hecke algebras, Kashiwara-Kim [72] related this cluster-theoretic partial ordering on monomials to homogeneous degrees of renormalized R-matrices constructed in [66]. In [25], we used the $\hat{\mu_j}$ as natural analogs of $\hat{y}_j$ in terms of parameters for simple modules of quiver Hecke algebras. We introduced the notion of *compatible seed* (see [25, Definition 4.7]). The following statement is the main result of [25].

Theorem 3.1.6 ([25, Theorem 6.2]). *Take $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and let $w = w_0$ be the longest element of the corresponding Weyl group. Consider the reduced expression*

$$\mathbf{w_0} := (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1).$$

Then the seed $\mathcal{S}^{\mathbf{w_0}}$ is a compatible seed in $R - gmod$.

Remark 3.1.7. As we are working with categories of modules over a finite type quiver Hecke algebra, the set $\mathbf{S}$ is totally ordered, which implies that $\mathcal{P} = \mathcal{A}$ with the notations of [25, Section 4.2].

3.2 Seeds associated with orderings on Lyndon words

In this section we generalize several results obtained in [25]. Let $\mathfrak{g}$ be a semisimple Lie algebra of finite type, I a fixed index set of simple roots and $<$ a total ordering on I. We begin by proving Proposition 3.1.4 in the following cases: $\mathfrak{g}$ of arbitrary finite type with the choice of the natural ordering of I (see [79, Section 8]) and $\mathfrak{g}$ of classical type with any ordering. This generalizes [25, Proposition 5.1]. Then we assume $\mathfrak{g}$ is simply-laced and for any w in the Weyl group W of $\mathfrak{g}$, we consider a reduced expression $\mathbf{w}_<$ of w uniquely determined by $<$, together with the corresponding seed $\mathcal{S}^{\mathbf{w}_<}$ in $\mathcal{A}_q(\mathfrak{n}(w))$ following [52, 69]. We provide an explicit description in terms of dominant words of the simple modules corresponding to the cluster variables of $\mathcal{S}^{\mathbf{w}_<}$ (see Theorem 3.2.7). This holds for arbitrary $<$ and $\mathfrak{g}$ of finite type. It generalizes [25, Theorem 6.1] to any subcategory $\mathcal{C}_w$ (not only $R - gmod$) and any finite-type underlying Lie algebra $\mathfrak{g}$. The key tool for the proof is provided by [72, Proposition 3.14]. We state several consequences, and in particular we prove [25, Conjecture 4.10] in the cases where Proposition 3.1.4 hods.

3.2.1 The monoid structure on dominant words

This subsection is devoted to the proof of Proposition 3.1.4 for $\mathfrak{g}$ of arbitrary finite type. Let us outline the fact that for $\mathfrak{g}$ of exceptional type, we need to fix the natural order on I, as in [79, Section 8.2] (see Corollary 3.2.6 below). The cuspidal representations in $R - gmod$ (parametrized by elements of $\mathcal{GL}$) are described in [79, Sections 8.8-8.10]. For arbitrary orderings, our proof works for $\mathfrak{g}$ of any classical type.

We let $<$ denote an arbitrary total ordering on I. Recall that a Lyndon word is by definition a word which is strictly smaller than any of its proper right factors.

Let μ, ν be two words of respective lengths m, n. We denote by $\mu\nu$ the concatenation of μ and ν. By *shuffle* of μ and ν we mean a word obtained by applying a permutation σ to the letters of $\mu\nu$ such that the restrictions of σ to $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$ are increasing (see [79, Section 4.2]). We denote this word by $\sigma \cdot (\mu\nu)$.

We consider two dominant words $\mu, \nu \in \mathbf{M}$ that we write

$$\mu = \mathbf{i}_1 \cdots \mathbf{i}_r \quad , \quad \nu = \mathbf{j}_1 \cdots \mathbf{j}_s$$

where $\mathbf{i}_1, \ldots, \mathbf{i}_r, \mathbf{j}_1, \ldots, \mathbf{j}_s \in \mathcal{GL}$ with $\mathbf{i}_1 \geqslant \cdots \geqslant \mathbf{i}_r$ and $\mathbf{j}_1 \geqslant \cdots \geqslant \mathbf{j}_s$. We show that

$$\mu \odot \nu = \mathbf{l}_1 \cdots \mathbf{l}_{r+s} \tag{3.2}$$

where $\mathbf{l}_1 \geqslant \cdots \geqslant \mathbf{l}_{r+s}$ are the elements of the set $\{\mathbf{i}_1, \ldots, \mathbf{i}_r, \mathbf{j}_1, \ldots, \mathbf{j}_s\}$ ranged in the decreasing order. Equivalently, we show that the right hand side of Equation (3.2) is the biggest shuffle of μ and ν (for the lexicographic order). As it is obviously a shuffle of μ and ν, it only remains to show that it is bigger than any other shuffle of μ and ν.

We use an induction on $r + s$, i.e. we assume that $\mathbf{i}_2 \cdots \mathbf{i}_r \odot \mathbf{j}_1 \cdots \mathbf{j}_s$ (resp. $\mathbf{i}_1 \cdots \mathbf{i}_r \odot \mathbf{j}_2 \cdots \mathbf{j}_s$) is the concatenation of $\mathbf{i}_1, \ldots, \mathbf{i}_r, \mathbf{j}_2, \ldots, \mathbf{j}_s$ (resp. $\mathbf{i}_2, \ldots, \mathbf{i}_r, \mathbf{j}_1, \ldots, \mathbf{j}_s$) ranged in the decreasing order.

Let σ be a shuffle permutation of μ and ν. We show that $\sigma \cdot (\mu\nu) \leqslant \mathbf{l}_1 \cdots \mathbf{l}_{r+s}$. We assume $\mathbf{i}_1 \geqslant \mathbf{j}_1$ the other case being analogous. Note that this is equivalent to $\mathbf{l}_1 = \mathbf{i}_1$. Let us write

$$\mathbf{i}_1 = (a_1, \ldots, a_p) \quad , \quad \mathbf{j}_1 = (b_1, \ldots, b_q).$$

Lemma 3.2.1. *If $a_1 > b_1$ then σ shuffles $\mathbf{j}_1$ to the right of $\mathbf{i}_1$. Equivalently $\sigma(1) = 1, \ldots, \sigma(p) = p$.*

Proof. As the restrictions of σ to $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$ are increasing, the first letter of $\sigma \cdot (\mu\nu)$ is either a_1 or b_1. As $b_1 < a_1$, $\sigma \cdot (\mu\nu)$ cannot begin with b_1 as it would then be strictly smaller than the right hand side of Equation (3.2). Hence the first letter of $\sigma \cdot (\mu\nu)$ is a_1. Then the second letter is either a_2 or b_1. As $\mathbf{i}_1$ is Lyndon, $a_2 \geqslant a_1$ and thus $a_2 > b_1$. Hence as before the second letter of $\sigma \cdot (\mu\nu)$ has to be a_2. We conclude by a straightforward induction that $\sigma \cdot (\mu\nu)$ begins with $\mathbf{i}_1$ which proves the Lemma. □

From now on we always assume $\mathbf{i}_1 \geqslant \mathbf{j}_1$ and $a_1 = b_1$.

Lemma 3.2.2. *Assume there is only one occurrence of a_1 in $\mathbf{i}_1$ i.e. one has $a_k > a_1$ for every $k \geqslant 2$. Then σ shuffles $\mathbf{j}_1$ to the right of $\mathbf{i}_1$ if $\mathbf{i}_1 > \mathbf{j}_1$ and either to the left either to the right of $\mathbf{i}_1$ if $\mathbf{i}_1 = \mathbf{j}_1$.*

Proof. Consider k maximal in $\{1, \ldots p\}$ such that $a_1 = b_1, \ldots, a_k = b_k$. Assume $k < p$; then $a_{k+1} > b_{k+1}$ and $\mathbf{i}_1 > \mathbf{j}_1$. If σ shuffles b_1 to the first letter of $\sigma \cdot (\mu\nu)$ then the second letter is either a_1 or b_2. But $(b_1, a_1 \cdots) = (a_1, a_1 \cdots) < (a_1, a_2 \cdots)$ by assumption hence σ shuffles b_2 to the second letter of $\sigma \cdot (\mu\nu)$. Similarly we get that σ shuffles $b_1, \ldots, b_k$ to the first k letters of $\sigma \cdot (\mu\nu)$. Then the next letter is either a_1 or b_{k+1}. Both of these letters are strictly smaller than a_{k+1} which contradicts the fact that $\sigma \cdot (\mu\nu)$ is greater than the right hand side of Equation (3.2). Hence we proved that the first letter of $\sigma \cdot (\mu\nu)$ is a_1.

Then the second letter is either a_2 or b_1 but $b_1 = a_1 < a_2$ by assumption hence it has to be a_2. Another induction shows that $\sigma \cdot (\mu\nu)$ begins with $\mathbf{i}_1$.

If $k = p$ then $\mathbf{i}_1 = \mathbf{j}_1$. The same proof shows that the first letter of $\sigma \cdot (\mu\nu)$ is either a_1 and in this case $\sigma \cdot (\mu\nu)$ begins with $\mathbf{i}_1$, or b_1 and in this case $\sigma \cdot (\mu\nu)$ begins with $\mathbf{j}_1$.

□

The two previous lemmas were essentially proved in type A_n in [25, Section 5.1]. In order to deal with the remaining types, we consider a slightly more general version of Lemma 3.2.2. Recall that we assume $\mathbf{i}_1 \geqslant \mathbf{j}_1$ and $a_1 = b_1$.

Lemma 3.2.3. *Assume there are exactly two occurrences of a_1 in $\mathbf{i}_1$ and exactly one in $\mathbf{j}_1$. Then $\mathbf{j}_1$ is shuffled to the right of $\mathbf{i}_1$.*

Proof. We write $\mathbf{i}_1 = a\mathbf{k}a\mathbf{l}$, where $\mathbf{k}$ and $\mathbf{l}$ are words whose letters are all strictly greater than a, and $\mathbf{j}_1 = a\mathbf{m}$. First note that the assumptions imply that the word $\mathbf{k}$ is not empty. As $\mathbf{i}_1 \geqslant \mathbf{j}_1$, one has $\mathbf{k} \geqslant \mathbf{m}$ and thus $a\mathbf{k} \geqslant \mathbf{j}_1$. Moreover $\mathbf{k} < \mathbf{l}$ as $\mathbf{i}_1$ is Lyndon. Hence one has $\mathbf{j}_1 = a\mathbf{m} \leqslant a\mathbf{k} < a\mathbf{l}$.

Lemma 3.2.2 implies that $\mathbf{j}_1$ is shuffled either to the left of $\mathbf{i}_1$, or between $a\mathbf{k}$ and $a\mathbf{l}$, or to the right of $\mathbf{i}_1$. The first possibility is possible only if $\mathbf{m} = \mathbf{k}$. But in this case one can apply Lemma 3.2.2 in the same way with $\mathbf{j}_2$. Hence the result of the shuffle would begin either with $\mathbf{j}_1\mathbf{j}_2$ or $\mathbf{j}_1 a\mathbf{k}$. Both are strictly smaller than $a\mathbf{k}a\mathbf{l} = \mathbf{i}_1$.

Now Lemma 3.2.2 applied to $a\mathbf{l}$ and $\mathbf{j}_1$ implies that $\mathbf{j}_1$ has to be shuffled to the right of $\mathbf{i}_1$ which finishes the proof. □

Lemma 3.2.4. *Assume there are exactly two occurrences of a_1 in $\mathbf{i}_1$ as well as in $\mathbf{j}_1$. Then $\mathbf{j}_1$ is shuffled to the right of $\mathbf{i}_1$ if $\mathbf{i}_1 > \mathbf{j}_1$, and either to the left either to the right of $\mathbf{i}_1$ if $\mathbf{i}_1 = \mathbf{j}_1$.*

Proof. We write $\mathbf{i}_1 = a\mathbf{k}a\mathbf{l}$ and $\mathbf{j}_1 = a\mathbf{m}a\mathbf{n}$ where $\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}$ are words whose letters are all strictly greater than a. As in the previous Lemma, one has $\mathbf{k} < \mathbf{l}$ and $\mathbf{m} < \mathbf{n}$. Moreover one has $\mathbf{k} \geqslant \mathbf{m}$, and in case of equality one has $\mathbf{n} \leqslant \mathbf{l}$.

If $\mathbf{i}_1 > \mathbf{j}_1$ then either $\mathbf{k} < \mathbf{m}$ or $\mathbf{k} = \mathbf{m}$ and $\mathbf{n} < \mathbf{l}$. The same proof as for Lemma 3.2.3 show that $\mathbf{j}_1$ is shuffled to the right of $\mathbf{i}_1$. The case $\mathbf{i}_1 = \mathbf{j}_1$ also follows from the same arguments. □

Corollary 3.2.5. *Assume $\mathbf{i}_1$ and $\mathbf{j}_1$ contain at most two occurrences of their first letters. Then $\sigma \cdot (\mu\nu)$ is either the concatenation of $\mathbf{i}_1$ with $\mathbf{i}_2 \cdots \mathbf{i}_r \odot \mathbf{j}_1 \cdots \mathbf{j}_s$ or the concatenation of $\mathbf{j}_1$ with $\mathbf{i}_1 \cdots \mathbf{i}_r \odot \mathbf{j}_2 \cdots \mathbf{j}_s$.*

Proof. If $\mathbf{i}_1 > \mathbf{j}_1$ then $\mathbf{l}_1 = \mathbf{i}_1$ and the previous lemmas show that $\sigma \cdot (\mu\nu)$ is the concatenation of $\mathbf{i}_1$ with a shuffle of $\mathbf{i}_2 \cdots \mathbf{i}_r$ and $\mathbf{j}_1 \cdots \mathbf{j}_s$. By the induction assumption, any such shuffle is smaller than $\mathbf{l}_2 \cdots \mathbf{l}_{r+s}$. Hence $\sigma \cdot (\mu\nu) \leqslant \mathbf{i}_1\mathbf{l}_2 \cdots \mathbf{l}_{r+s} = \mathbf{l}_1 \cdots \mathbf{l}_{r+s}$. The case $\mathbf{i}_1 < \mathbf{j}_1$ is analogous.

If $\mathbf{i}_1 = \mathbf{j}_1$, then $\mathbf{l}_1 = \mathbf{l}_2 = \mathbf{i}_1$. By the previous lemmas, $\sigma \cdot (\mu\nu)$ is either the concatenation of $\mathbf{l}_1$ with a shuffle of $\mathbf{i}_2 \cdots \mathbf{i}_r$ and $\mathbf{j}_1 \cdots \mathbf{j}_s$ or the concatenation of $\mathbf{l}_1$ with a shuffle of $\mathbf{i}_1 \cdots \mathbf{i}_r$ and $\mathbf{j}_2 \cdots \mathbf{j}_s$. In both cases the conclusion is the same. □

Corollary 3.2.6. *Assume $\mathfrak{g}$ is of arbitrary finite type and $<$ is the natural ordering on I or $\mathfrak{g}$ is of classical type and $<$ is an arbitrary ordering on I. Then Equation* (3.2) *holds.*

Proof. For $\mathfrak{g}$ of types A, B, C, D, the positive roots contain at most to occurences of any simple root. A fortiori the elements of $\mathcal{GL}$ contain at most two occurrences of their first letter and this holds for any ordering $<$ on I. When $\mathfrak{g}$ is of exceptional type and $<$ is the natural ordering, the elements of $\mathcal{GL}$ are described in [79, Section 8.2] and one can see that they always contain at most two occurrences of their first letter. Hence in these cases, Corollary 3.2.5 implies that the right hand side of Equation (3.2) is the greatest shuffle of μ and ν, proving Proposition 3.1.4.

□

3.2.2 A compatible seed for $\mathcal{C}_w$

In this subsection we consider $\mathfrak{g}$ of type A_n, D_n or E_n and we prove that for any $w \in W$, there exists a reduced expression $\mathbf{w}$ of w such that the seed $\mathcal{S}^{\mathbf{w}}$ is compatible in the sense of [25, Definition 4.7].

We fix an arbitrary total order $<$ on I. We again denote by $<$ the induced lexicographic order on the set $\mathcal{M}$ as well as its restriction to $\mathcal{GL}$ (see Section 3.1.1). Via the bijection between $\mathcal{GL}$ and Φ_+, one can view $<$ as an ordering on Φ_+. For any $w \in W$, we consider the restriction of $<$ to Φ_+^w. As there is a bijection between convex orderings on Φ_+^w and reduced expressions of w (see Section 3.1.2), we consider the unique reduced expression $\mathbf{w}_<$ of w corresponding to $<$. We fix once for all the order $<$ and we write $\mathbf{w}$ instead of $\mathbf{w}_<$ if there is no ambiguity.

First we introduce a notation that will be useful in the following. For any $1 \leqslant k \leqslant N$, we set

$$J_k := \{j \leqslant k | i_j = i_k\}$$

and we write $J_k = \{j_0 = k > j_1 > \cdots > j_{r_k}\}$. In other words, $j_0 = k, j_1 = k_-, j_2 = (k_-)_-, \ldots$ with the notations of [52, Section 9.4]. The integer r_k corresponds to the position of the first occurrence of the letter i_k in the word $(i_1, \ldots, i_N)$.

The following statement is the main result of this section. It gives a description of the simple modules in $\mathcal{C}_w$ corresponding to the cluster variables of the seed $\mathcal{S}^{\mathbf{w}}$.

Theorem 3.2.7. *Let $(x_1, \ldots, x_N)$ denote the cluster variables of the seed $\mathcal{S}^{\mathbf{w}}$ and let μ_k denote the dominant word such that $x_k = [L(\mu_k)]$ for every $k \in J$. Then*

$$\mu_k = \mathbf{i}_{j_0}\mathbf{i}_{j_1} \cdots \mathbf{i}_{j_{r_k}}.$$

Proof. We write the canonical factorization of μ_k as

$$\mu_k = (\mathbf{i}_N)^{c_{N,k}} \cdots (\mathbf{i}_1)^{c_{1,k}}$$

with $c_{N,k}, \ldots, c_{1,k} \in \mathbb{N}$. We also set $c_{(N+1),N} := 0$. By [72, Proposition 3.14], the t-uple of integers $(c_{1,k} - c_{1_+,k}, \ldots, c_{N,k} - c_{N_+,k})$ is the image of $[L(\mu_k)]$ under the map $\mathbf{g}^R_{\mathcal{S}^{\mathbf{w}}_0}$ defined in [72] (see [72, Definition 3.8]). It is clear from this definition that the isomorphism class of the simple module $L(\mu_k)$ is mapped onto the kth vector e_k of the standard basis of $\mathbb{Z}^N$. Thus one has

$$e_k = (c_{1,k} - c_{1_+,k}, \ldots, c_{N,k} - c_{N_+,k}).$$

One has $c_{j,k} - c_{j_+,k} = 0$ for any $j \neq k$. In particular, one has

$$c_{k_+,k} = c_{(k_+)_+,k} = \cdots = c_{(N+1),k} := 0 \quad \text{and} \quad c_{r_k,k} = \cdots = c_{k_-,k} = c_{k,k}.$$

Moreover, $c_{k,k} - c_{k_+,k} = 1$ and hence $c_{k,k} = 1$. Finally one has

$$c_{r_k,k} = \cdots = c_{k_-,k} = c_{k,k} = 1 \quad \text{and} \quad c_{k_+,k} = c_{(k_+)_+,k} = \cdots = 0.$$

If j is any position such that the letter i_j is different from i_k then k does not appear in the sequence $(r_j, \ldots, j_-, j, j_+, \ldots, N+1)$ and hence $c_{r_j,k} = \cdots = c_{j_-,k} = c_{j,k} = c_{j_+,k} = \cdots = 0$. One concludes:

$$\mu_k = \mathbf{i}_k\mathbf{i}_{k_-} \cdots \mathbf{i}_{r_k}.$$

□

Let us point out a couple of consequences which will be useful later.

Corollary 3.2.8. *Let $k \in J$. For any integer-valued t-uple $(c_j)_{j \in J, j>k}$, one has*

$$\mu_k > \bigodot_{j<k} \mu_j^{\odot c_j}$$

in the group $\mathbf{G}$.

Proof. By Theorem 3.2.7, the highest good Lyndon word in the canonical factorization of μ_k (resp. $\mu_j, j > k$) is $\mathbf{i}_k$ (resp. $\mathbf{i}_j, j > k$). Hence by definition of the lexicographic order on $\mathbf{M}$ one has

$$\mu_k \odot \bigodot_{j<k, c_j<0} \mu_j^{\odot -c_j} > \bigodot_{j<k, c_j>0} \mu_j^{\odot c_j}$$

which implies

$$\mu_k > \bigodot_{j<k} \mu_j^{\odot c_j}$$

in the Grothendieck group $\mathbf{G}$ of $\mathbf{M}$. □

Corollary 3.2.9. *Let $\mathcal{S} = ((x_1, \dots, x_N), B)$ be any seed in $\mathcal{A}_q(\mathfrak{n}(w))$, let μ_k denote the dominant word such that $x_k = [L(\mu_k)]$ for every k, and let $\mathcal{M}_{\mathcal{S}}$ denote the matrix of the vectors $\varphi(\tilde{\Psi}(x_1)), \dots, \varphi(\tilde{\Psi}(x_N))$ in the standard basis of $\mathbb{Z}^N$. Then $\mathcal{M}_{\mathcal{S}} \in GL_N(\mathbb{Z})$.*

Proof. First consider the seed $\mathcal{S}^{\mathbf{w}}$. By Theorem 3.2.7, there is a bijection between the cluster variables of $\mathcal{S}^{\mathbf{w}}$ and good Lyndon words in $\mathbf{M}_w$: indeed, for any $1 \leqslant j \leqslant N$, there is a unique cluster variable in $\mathcal{S}^{\mathbf{w}}$ whose corresponding dominant word has a canonical factorization beginning with $\mathbf{i}_j$. Hence choosing a good permutation of the standard basis of $\mathbb{Z}^N$, the matrix $\mathcal{M}_{\mathcal{S}_0^{\mathbf{w}}}$ is (lower) unitriangular. In particular $\mathcal{M}_{\mathcal{S}_0^{\mathbf{w}}}$ is invertible with determinant equal to 1.

Now consider a mutation in any direction $k \in J_{ex}$. Set $\Psi := \varphi \circ \tilde{\Psi}$. The vector $\Psi(x'_k)$ is either equal to $-\Psi(x_k) + \sum_{b_{ik}>0} b_{ik}\Psi(x_i)$ or to $-\Psi(x_k) + \sum_{b_{ik}<0}(-b_{ik})\Psi(x_i)$. In the first case, one has

$$\begin{aligned}
\det(\Psi(x'_1), \dots, \Psi(x'_N)) &= \det(\Psi(x_1), \dots, \Psi(x_{k-1}), \Psi(x'_k), \Psi(x_{k+1}), \dots, \Psi(x_N)) \\
&= -\det(\Psi(x_1), \dots, \Psi(x_{k-1}), \Psi(x_k), \Psi(x_{k+1}), \dots, \Psi(x_N)) \\
&\qquad + \sum_{b_{ik}>0} \det(\Psi(x_1), \dots, \Psi(x_{k-1}), \Psi(x_i), \Psi(x_{k+1}), \dots, \Psi(x_N)) \\
&= -\det(\Psi(x_1), \dots, \Psi(x_N)).
\end{aligned}$$

The other case is analogous. In particular the matrix obtained after mutation is still invertible and has determinant either 1 or -1. By induction, we conclude that $\mathcal{M}_{\mathcal{S}} \in G_N(\mathbb{Z})$ for any seed $\mathcal{S}$. □

We end this section by proving that [25, Conjecture 4.10] holds in $\mathcal{C}_w$ for every $w \in W$ when the underlying Lie algebra $\mathfrak{g}$ is of finite type. The proof is similar to the proof of [25, Theorem 6.2] in the case of $\mathcal{C}_{w_0} = R - gmod$ in type A_n.

Corollary 3.2.10. *Fix any total order $<$ on I. For any $w \in W$, the seed $\mathcal{S}^{\mathbf{w}^<}$ is compatible in the sense of [25, Definition 4.7].*

Proof. As in Theorem 3.2.7, for every $1 \leqslant k \leqslant N$ we let μ_k denote the dominant word corresponding to the kth cluster variable of $\mathcal{S}^{\mathbf{w}^<}$. With the same notations as in Section 3.1.3, we consider the variables $\hat{y_j}, j \in J_{fr}$. It follows from the construction of $\mathcal{S}^{\mathbf{w}}$ ([52, 69]) that for every $j \in J_{fr}$,

$$\hat{y_j} = x_{j_+}x_{j_-}^{-1} \prod_{j<k<j_+<k_+} x_k^{-|a_{kj}|} \prod_{k<j<k_+<j_+} x_k^{|a_{kj}|}.$$

Hence

$$\hat{\mu_j} = \mu_{j_+} \odot \left(\mu_{j_-}^{\odot -1} \odot \bigodot_{k<j<k_+<j_+} \mu_k^{\odot |a_{kj}|} \odot \bigodot_{j<k<j_+<k_+} \mu_k^{\odot -|a_{kj}|} \right).$$

The expression between brackets only involves words μ_k such that $k < j_+$. Hence by Corollary 3.2.8,

$$\mu_{j_+} > \left(\mu_{j_-}^{\odot -1} \odot \bigodot_{k<j<k_+<j_+} \mu_k^{\odot |a_{kj}|} \odot \bigodot_{j<k<j_+<k_+} \mu_k^{\odot -|a_{kj}|} \right).$$

Thus one has $\hat{\mu_j} \odot \mu > \mu$ for every $\mu \in \mathbf{M}_w$ and this holds for every $j \in J_{fr}$. This implies that the seed $\mathcal{S}^{\mathbf{w}}$ is compatible (see [25, Remark 4.8]). □

Remark 3.2.11. By [25, Corollary 4.12], the seed $\mathcal{S}^{\mathbf{w}}$ being compatible implies certain relationships between dominant words and g-vectors. This relationship is provided by [72, Proposition 3.14] for any $w \in W$. It takes the form expected in [25, Section 7.1] in the case of $R - gmod$ in type A_n for the natural ordering.

Example 3.2.12. Consider $\mathfrak{g}$ of type A_2, $I = \{1, 2\}$, $w = w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$. Consider the natural ordering $1 < 2$. Then $\Phi_+ = \{\alpha_1 < \alpha_1 + \alpha_2 < \alpha_2\}$. The corresponding reduced expression of w_0 is $(1, 2, 1)$. It is known from [79, Section 8.4] that $\mathcal{GL} = \{(1) < (12) < (2)\}$. By Theorem 3.2.7, the simple modules corresponding to the cluster variables of the seed $\mathcal{S}^{(1,2,1)}$ (together with its quiver) are given by

$$\begin{tikzcd} L(1) \arrow[r] & L(12) \arrow[r] & L(21) \arrow[ll, bend right] \end{tikzcd} .$$

The matrix $\mathcal{M}_{\mathcal{S}^{(1,2,1)}}$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.3 Newton-Okounkov bodies for $\mathcal{C}_w$

It follows from Theorem 3.2.7 that for every choice of order $<$ on I and every $w \in W$, there is a natural total ordering on the set of cluster variables of $\mathcal{S}^{\mathbf{w}_<}$ (and hence of every seed in $\mathcal{A}_q(\mathfrak{n}(w))$). The corresponding lexicographic order on cluster monomials yields a monomial valuation for every seed as in [107, Section 7]. Thus it is natural to construct Newton-Okounkov bodies in this framework. It will turn out that in our setting the valuation will be naturally provided by parametrizations of simple objects in $R - gmod$ (or $\mathcal{C}_w$) and hence entirely determined by $<$. In particular it does not depend on the choice of a seed. Throughout the following sections, we consider a semisimple Lie algebra $\mathfrak{g}$ of arbitrary finite type. As in Section 3.2.2, we fix an index set I of the simple roots of $\mathfrak{g}$ and a total order $<$ on I. We also fix an element w in the Weyl group W of $\mathfrak{g}$ as well as the reduced expression $\mathbf{w}_< = (i_1, \ldots, i_N)$ corresponding to the restriction of $<$ to Φ_+^w. In all Sections 3.3 and 3.4, we will use the following notations: $\mathcal{C} := \mathcal{C}_w, \mathcal{A} := K_0(\mathcal{C}) \simeq \mathcal{A}_q(\mathfrak{n}(w)), \mathbf{M} := \mathbf{M}_w, \mathbf{G} = \mathbf{G}_w$.

In order to construct Newton-Okounkov bodies for subalgebras of $\mathcal{A}$, we begin by constructing a valuation with value in $\mathbb{Z}^N$ endowed with some total ordering, as well as a $\mathbb{N}$-graduation on $\mathcal{A}$.

3.3.1 Newton-Okounkov bodies

In this section we briefly review the general construction of Newton-Okounkov bodies, introduced by Kaveh-Khovanskii [75] and independently by Lazarsfeld-Mustata [83]. It generalizes a construction of Okounkov [99]. We refer to [13, 12] for beautiful surveys on Newton-Okounkov bodies.

Let $\mathcal{A}$ be a $\mathbb{N}$-graded commutative algebra over a base field $\mathbf{k}$. Let $\mathcal{A}_n$ denote the degree n homogeneous subspace of $\mathcal{A}$ for any $n \in \mathbb{N}$. Thus one has

$$\mathcal{A} = \bigoplus_n \mathcal{A}_n \quad , \quad \mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{n+m} \quad , \quad \mathcal{A}_0 = \mathbf{k}.$$

We assume $\mathcal{A}_n$ to be a finite-dimensional $\mathbf{k}$-vector space for every $n \in \mathbb{N}$. We also assume that $\mathcal{A}$ is a domain and that the fraction field of $\mathcal{A}$ is of finite type over $\mathbf{k}$.

Definition 3.3.1 (Valuation)**.** *A valuation on $\mathcal{A}$ is a map $\Psi : \mathcal{A} \longrightarrow \mathbb{Z}^N$ (for some $N \geqslant 1$) satisfying the following properties:*

1. $\forall f, g \in \mathcal{A}, \Psi(fg) = \Psi(f) + \Psi(g)$.
2. $\forall t \in \mathbf{k}^*, \forall f \in \mathcal{A}, \Psi(tf) = \Psi(f)$.
3. $\forall f, g \in \mathcal{A}, \Psi(f + g) \leqslant \max(\Psi(f), \Psi(g))$.

To any $\mathbb{N}$-graded subalgebra $\mathcal{B}$ of $\mathcal{A}$, one can associate a closed convex set $\Delta(\mathcal{B})$ called Newton-Okounkov body of $\mathcal{B}$.

Definition 3.3.2 (Newton-Okounkov bodies)**.** *Let $\mathcal{B}$ be any graded subalgebra of $\mathcal{A}$. Decompose it as*

$$\mathcal{B} = \bigoplus_n \mathcal{B}_n.$$

The Newton-Okounkov body associated to $\mathcal{B}$ is defined as

$$\Delta(\mathcal{B}) := \overline{ConvexHull\left(\bigcup_n \frac{1}{n}\Psi(\mathcal{B}_n\backslash\{0\})\right)}.$$

Recall that the vector spaces $\mathcal{B}_n, n \in \mathbb{N}$ are finite-dimensional. Hence it follows from [75, Proposition 2.3] that the sets $\Psi(\mathcal{B}_n\backslash\{0\})$ are finite. In order to have these bodies satisfying nice properties, one needs to make a technical assumption on the valuation Ψ, namely that it is of maximal rational rank. This means that the rank of the value group of Ψ has to be equal to the transcendence degree of $K := Frac(\mathcal{A})$. We refer to [13, Section 2.4] for more details and precise statements about maximal rational rank valuations. The crucial observation is that under this assumption, one has

$$\dim_{\mathbf{k}} \mathcal{B}_n = \sharp\Psi(\mathcal{B}_n\backslash\{0\}).$$

In other words if the valuation Ψ is of maximal rational rank, then it has *one-dimensional leaves* in the terminology of [75] (see [13, Proposition 2.23]).

3.3.2 Total order on $\mathbb{Z}^N$

As in Section 3.1.2, we write

$$\Phi_+^w = \{\beta_1 < \cdots < \beta_N\}$$

and we let $\mathbf{i}_1, \ldots, \mathbf{i}_N$ denote the corresponding good Lyndon words. Let $\mathbf{e}_k$ be the kth vector of the standard basis of $\mathbb{Z}^N$ for every $1 \leqslant k \leqslant N$. Recall from Section 3.1.3 that the isomorphism φ sends the good Lyndon word $\mathbf{i}_k$ onto $\mathbf{e}_k$. Equivalently, one has

$$\forall 1 \leqslant k \leqslant N, \varphi\left(\tilde{\Psi}\left([L(\mathbf{i}_k)]\right)\right) = \mathbf{e}_k. \tag{3.3}$$

Using the isomorphism φ, one can push forward the lexicographic order on $\mathbf{M}$ (resp. $\mathbf{G}$) onto the (reversed) lexicographic order on $\mathbb{N}^N$ (resp. $\mathbb{Z}^N$) given by:

$$(a_1, \ldots, a_N) < (b_1, \ldots, b_N) \Leftrightarrow \exists k \geqslant 1, a_N = b_N, \ldots, a_{k+1} = b_{k+1}, a_k < b_k.$$

3.3.3 The valuation

As in the proof of Corollary 3.2.9, we let $\Psi : Frac(K_0(R-gmod)) \longrightarrow \mathbb{Z}^{\Phi_+}$ denote the composition of $\tilde{\Psi}$ with φ:

$$\Psi : \ Frac(K_0(R-gmod)) \xrightarrow{\tilde{\Psi}} \mathbf{G} \xrightarrow{\varphi} \mathbb{Z}^{\Phi_+}.$$

We again denote by Ψ the restriction to $Frac(\mathcal{A})$:

$$\Psi : \ Frac(\mathcal{A}) \xrightarrow{\tilde{\Psi}} \mathbf{G} \xrightarrow{\varphi} \mathbb{Z}^N.$$

Recall that N denotes the length of w.

Lemma 3.3.3. *The map Ψ is a valuation on $\mathcal{A}$ with value group $\mathbb{Z}^N$.*

Proof. Let $x = a_1[L(\mu_1)] + \cdots + a_r[L(\mu_r)]$ and $y = b_1[L(\nu_1)] + \cdots + b_s[L(\nu_s)]$ in $\mathcal{A}$; as $\mathcal{A}$ is commutative and $\mathbf{M}$ is totally ordered, one can assume $\mu_1 > \cdots > \mu_r$ and $\nu_1 > \cdots > \nu_s$. Then one has

$$\begin{aligned}\Psi(x+y) &= \Psi(a_1[L(\mu_1)] + \cdots + a_r[L(\mu_r)] + b_1[L(\nu_1)] + \cdots + b_s[L(\nu_s)]) \\ &\leqslant \max\left(\max(\boldsymbol{\mu}_i, 1 \leqslant i \leqslant r), \max(\boldsymbol{\nu}_j, 1 \leqslant j \leqslant s)\right) \\ &= \max(\Psi(x), \Psi(y)).\end{aligned}$$

One also has:

$$\Psi(xy) = \Psi\left(\sum_{i,j} a_i b_j [L(\mu_i)][L(\nu_j)]\right).$$

For any $i \geqslant 2$ (resp. $j \geqslant 2$), $\mu_i < \mu_1$ (resp. $\nu_j < \nu 1$) hence $\mu_i \odot \nu_j < \mu_1 \odot \nu_1$ if $(i,j) \neq (1,1)$. Hence $\mu_1 \odot \nu_1$ is the highest word appearing in the decomposition on simples of the above sum. Hence

$$\begin{aligned}\Psi(xy) &= \Psi\left([L(\mu_1)][L(\nu_1)]\right)) = \varphi(\mu_1 \odot \nu_1) \\ &= \varphi(\mu_1) + \varphi(\nu_1) \qquad \text{by Proposition 3.1.4} \\ &= \Psi([L(\mu_1)]) + \Psi([L(\nu_1)]) = \Psi(x) + \Psi(y).\end{aligned}$$

The remaining axiom in Definition 3.3.1 is straightforward. Finally, recall from Proposition 3.1.4 that φ is a bijection from $\mathbf{M}$ to $\mathbb{N}^N$. In particular, $\Psi(\mathcal{A}) \supset \varphi(\mathbf{M}) = \mathbb{N}^N$ and thus the value group $\Psi(Frac(\mathcal{A})\backslash\{0\})$ is indeed the entire group $\mathbb{Z}^N$. □

Lemma 3.3.4. *The valuation Ψ is of maximal rational rank. In particular it is one-dimensional leaves in the sense of [75].*

Proof. As $\mathcal{A} = K_0(\mathcal{C})$ has a cluster algebra structure, its fraction field is just $\mathbb{Q}(x_1, \ldots, x_N)$ for any cluster $(x_1, \ldots, x_N)$. Thus it is of transcendence degree N. By construction, the rational rank of Ψ is also equal to N. Thus Ψ is of maximal rational rank. □

Remark 3.3.5. In fact the valuation Ψ essentially does the same thing as a monomial valuation: up to some automorphism of $\mathbb{Z}^N$, it can be identified with a valuation coming from the lexicographic order on cluster monomials as in [107, Definition 7.1]. Representation theory provides us with a natural choice of total order on the cluster variables of the initial seed $\mathcal{S}^{\mathbf{w}}$: denoting by μ_i the dominant word such that $x_i = [L(\mu_i)]$, we set

$$x_i \lessdot x_j \Leftrightarrow \mu_i < \mu_j.$$

This induces a total order on Laurent monomials in $x_1, \ldots, x_N$ as in [107, Definition 7.1]. We also denote it $\lessdot$. Then using Corollary 3.2.8 one can show that for any Laurent monomials $\mathbf{x^a} = \prod_i x_i^{a_i}$ and $\mathbf{x^b} = \prod_i x_i^{b_i}$ one has

$$\mathbf{x^b} \lessdot \mathbf{x^b} \Leftrightarrow \Psi(\mathbf{x^b}) < \Psi(\mathbf{x^b}).$$

3.3.4 The grading on $\mathcal{A}$

The grading on $\mathcal{A}$ will essentially be given by the following standard notion of *height* for elements of Q_+. For any $\beta \in Q_+$, write $\beta = \sum_i b_i \alpha_i$. The quantity

$$\mathrm{ht}(\beta) := \sum_i b_i$$

is called the *height* of β. For any $\beta, \gamma \in Q_+$, one has $\mathrm{ht}(\beta + \gamma) = \mathrm{ht}(\beta) + \mathrm{ht}(\gamma)$.

Lemma 3.3.6. *For any word ν in $\mathcal{M}$, the number of letters of ν is equal to $ht(wt(\nu))$.*

Therefore we will denote it using the usual notation for the length of a word namely $|\nu|$.

Remark 3.3.7. In particular, for $\beta \in Q_+$, M a simple object in $R(\beta) - gmod$ and μ the corresponding dominant word, one has $|\mu| = \mathrm{ht}(\beta)$. Consider for instance the good Lyndon words $\mathbf{i}_1, \ldots, \mathbf{i}_N$. Then for any $1 \leqslant k \leqslant N$, one has

$$|\mathbf{i}_k| = \mathrm{ht}(\beta_k).$$

Note that $|\mu \odot \nu| = |\mu| + |\nu|$ for any $\mu, \nu \in \mathbf{M}$. Hence the following definition makes $\mathcal{A}$ into a graded algebra.

Definition 3.3.8 (Grading on $\mathcal{A}$)**.** *To any simple object M in $\mathcal{C}$, we associate the length of the corresponding dominant word, i.e. the integer $|\tilde{\Psi}([M])|$.*

As $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}$, every cluster monomial is a simple object and thus is homogeneous, its degree being the length of the corresponding dominant word.

3.3.5 Newton-Okounkov bodies for $\mathcal{C}_w$

We are now ready to construct Newton-Okounkov bodies using the above grading and valuation. For any graded subalgebra $\mathcal{B}$ of $\mathcal{A} = K_0(\mathcal{C})$, we get a convex compact set $\Delta(\mathcal{B}) \subset \mathbb{R}^N$. Moreover the bodies $\Delta(\mathcal{B})(\mathcal{B} \subset \mathcal{A})$ are always contained in $\Delta(\mathcal{A})$.

Let us begin with the following statement, that will be useful in the following.

Lemma 3.3.9. *Assume $\mathcal{B}$ is a graded finitely generated subalgebra of $\mathcal{A}$. Let $b_1, \ldots, b_r$ be homogeneous generators of $\mathcal{B}$ and set $d_i := \deg b_i$ for every $1 \leqslant i \leqslant r$. Assume furthermore that the family $(\Psi(b_1), \ldots, \Psi(b_r))$ is linearly independent. Then the Newton-Okounkov body $\Delta(\mathcal{B})$ is a rational polytope. More precisely, $\Delta(\mathcal{B}) = ConvexHull(\frac{1}{d_i}\Psi(b_i), 1 \leqslant i \leqslant r)$.*

Proof. Let f be any degree d homogeneous element in $\mathcal{B}$. We prove that

$$\frac{1}{d}\Psi(f) \in \mathrm{ConvexHull}(\frac{1}{d_i}\Psi(b_i), 1 \leqslant i \leqslant r).$$

Write f as

$$f = \sum_{(i_1,\ldots,i_r), d_1 i_1 + \cdots + d_r i_r = d} a_{i_1 \ldots i_r} b_1^{i_1} \cdots b_r^{i_r}$$

and decompose each monomial $b_1^{i_1} \cdots b_r^{i_r} \in \mathcal{B} \subset \mathcal{A}$ on the basis of classes of simple objects in $\mathcal{C}$. As the vectors $\Psi(b_1), \ldots, \Psi(b_r)$ are linearly independent, one has $\Psi(b_1^{i_1} \cdots b_r^{i_r}) \neq \Psi(b_1^{j_1} \cdots b_r^{j_r})$ if $(i_1, \ldots, i_r) \neq (j_1, \ldots, j_r)$. Thus there is a unique maximal element $\boldsymbol{\mu}$ among the images $\Psi(b_1^{i_1} \cdots b_r^{i_r}), d_1 i_1 + \cdots + d_r i_r = d$. This element is then the unique maximal element in the decomposition of f on the basis of classes of simple objects in $\mathcal{C}$. Hence by definition of Ψ, one has $\Psi(f) = \boldsymbol{\mu} = \Psi(b_1^{i_1} \cdots b_r^{i_r})$ for some $(i_1, \ldots, i_r), d_1 i_1 + \cdots + d_r i_r = d$. In particular,

$$\frac{1}{d}\Psi(f) = \frac{1}{d_1 i_1 + \cdots d_r i_r}(i_1 \Psi(b_1) + \cdots + i_r \Psi(b_r)) \in \mathrm{ConvexHull}(\frac{1}{d_i}\Psi(b_i), 1 \leqslant i \leqslant r).$$

□

Recall that $\mathbf{e}_k$ stands for the vectors of the standard basis of $\mathbb{Z}^N$ (see Section 3.3.2).

Lemma 3.3.10. *The Newton-Okounkov body $\Delta(\mathcal{A})$ is given by:*

$$\Delta(\mathcal{A}) = \mathit{ConvexHull}(\frac{1}{ht(\beta_k)}\mathbf{e}_k, 1 \leqslant k \leqslant N).$$

Proof. Let x be any cluster variable in $\mathcal{A}$ and $\mu \in \mathbf{M}$ the dominant word such that $x = [L(\mu)]$. We write the canonical factorization of μ as $\mu = \mathbf{i}_1^{a_1} \cdots \mathbf{i}_N^{a_N}$ (see Theorem 3.1.1 and Remark 3.1.2). Then by definition $|\mu| = \sum_k a_k \mathrm{ht}(\beta_k)$. Hence

$$\frac{1}{|\mu|}\Psi(x) = \frac{1}{\sum_k a_k \mathrm{ht}(\beta_k)} \sum_k a_k \mathbf{e}_k$$

which implies

$$\frac{1}{|x|}\Psi(x) \in \mathrm{ConvexHull}(\frac{1}{\mathrm{ht}(\beta_k)}\mathbf{e}_k, 1 \leqslant k \leqslant N).$$

This holds for any cluster variable in $\mathcal{A}$. Now let $\mathfrak{m} = x_1^{a_1} \dots x_r^{a_r}$ be any monomial in the cluster variables (here the x_i are *any* cluster variables, not necessarily of the same cluster) and let $d_i := |x_i|, 1 \leqslant i \leqslant r$. One has

$$\frac{1}{|\mathfrak{m}|}\Psi(\mathfrak{m}) = \frac{1}{\sum_i a_i d_i} \sum_i a_i \Psi(x_i) = \frac{1}{\sum_i a_i d_i} \sum_i a_i d_i \frac{\Psi(x_i)}{d_i}.$$

Thus $\frac{1}{|\mathfrak{m}|}\Psi(\mathfrak{m})$ lies in the convex hull of the $\frac{\Psi(x_i)}{d_i}$ and hence in $\mathrm{ConvexHull}(\frac{1}{\mathrm{ht}(\beta_k)}\mathbf{e}_k, 1 \leqslant k \leqslant N)$.

As the valuation of any element f of $\mathcal{A}$ is always equal to the valuation of some monomial as above (see the proof of Lemma 3.3.9), this proves the desired statement.

□

Note in particular that $\Delta(\mathcal{A})$ is a simplex of full dimension $N-1$.

3.4 The simplices $\Delta_{\mathcal{S}}$

Throughout Sections 3.4 and 3.5, we will assume $\mathfrak{g}$ is simply-laced. We will be studying the following Newton-Okounkov bodies: for any seed $\mathcal{S} = ((x_1, \dots, x_N), B)$ in the cluster algebra $\mathcal{A}$, we consider the graded subalgebra of $\mathcal{A}$ generated by the cluster variables of $\mathcal{S}$. This is a finitely generated algebra and by Corollary 3.2.9, the images of $x_1, \dots, x_N$ under the valuation Ψ are linearly independent. Hence by Lemma 3.3.9 the corresponding Newton-Okounkov body $\Delta_{\mathcal{S}}$ is the simplex given by

$$\Delta_{\mathcal{S}} = \mathrm{ConvexHull}(\frac{1}{|x_i|}\Psi(x_i), 1 \leqslant i \leqslant N).$$

We begin by outlining the fact that for any seed $\mathcal{S}$, the simplex $\Delta_{\mathcal{S}}$ (as well as $\Delta(\mathcal{A})$) sits inside an affine hyperplane. This hyperplane is naturally defined from the representation-theoretic data introduced in Section 3.1. Then we use Theorem 3.2.7 to prove several properties of the simplices $\Delta_{\mathcal{S}}$. In particular, we exhibit a correspondence between the rational points in $\Delta_{\mathcal{S}}$ and the cluster monomials for the seed $\mathcal{S}$.

3.4.1 The hyperplane $\mathcal{H}$

Recall that $(e_k, 1 \leqslant k \leqslant N)$ stands for the standard basis of $\mathbb{Z}^N$. We also let $\langle \cdot, \cdot \rangle$ denote the standard Euclidian scalar product on $\mathbb{R}^N$. We let $\boldsymbol{\lambda}$ denote the vector whose kth component is the height of the positive root β_k:

$$\langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle := \mathrm{ht}(\beta_k)$$

for every $k \in \{1, \dots, N\}$. The following Lemma shows that $\boldsymbol{\lambda}$ encodes the grading on $\mathcal{A}$ (see Section 3.3.4).

Lemma 3.4.1. *Let M be a simple module in $\mathcal{C}$ and let $\mu \in \mathbf{M}$ the corresponding dominant word. Then one has*

$$|\mu| = \langle \boldsymbol{\lambda}, \Psi([M]) \rangle.$$

Proof. Let us write the canonical factorization of μ as $\mu = \mathbf{i}_N^{c_N} \cdots \mathbf{i}_1^{c_1}$ (see Remark 3.1.2). Then by definition one has $\Psi(\mu) = t(c_1, \ldots, c_N)$. Then using Lemma 3.3.6 we get

$$|\mu| = \mathrm{ht}(\mathrm{wt}(\mu)) = \mathrm{ht}(\sum_k c_k \beta_k) = \sum_k c_k \mathrm{ht}(\beta_k) = \sum_k c_k \langle \boldsymbol{\lambda}, \mathbf{e}_k \rangle = \langle \boldsymbol{\lambda}, \Psi([L(\mu)]) \rangle.$$

□

Let $\mathcal{H}$ denote the affine hyperplane $\{\langle \boldsymbol{\lambda}, \cdot \rangle = 1\} \subset \mathbb{R}^N$. The following observation is a straightforward consequence of Definition 3.3.8 and Lemma 3.3.10.

Lemma 3.4.2. *The simplex $\Delta(\mathcal{A})$ is contained in $\mathcal{H}$.*

As a consequence, any Newton-Okounkov body associated to a graded subalgebra of $\mathcal{A}$ will also lie in $\mathcal{H}$.

3.4.2 First properties of $\Delta_{\mathcal{S}}$

Now we state a couple of algebraic and geometric properties of the simplices $\Delta_{\mathcal{S}}$. We use Theorem 3.2.7 as well as a result of Geiss-Leclerc-Schröer ([53, Theorem 8.3]).

First we exhibit a correspondence between the rational points inside $\Delta_{\mathcal{S}}$ and the monoidal cluster monomials for this seed. By monoidal cluster monomial, we mean an object in $\mathcal{C}$ isomorphic to $\bigodot_i M_i^{d_i}$ for some nonnegative integers d_i, where M_i are the simple modules corresponding to the cluster variables of $\mathcal{S}$. The monoidal categorification statements of [69] imply that monoidal cluster monomials are always real simple objects.

Proposition 3.4.3. *Let $\mathcal{S}$ be a seed in $\mathcal{A}$ with cluster variables $x_1, \ldots, x_N$. Then for any simple object M in $\mathcal{C}$, one has*

$$M \text{ is a monoidal cluster monomial for } \mathcal{S} \Leftrightarrow \frac{1}{|M|}\Psi([M]) \in \Delta_{\mathcal{S}}.$$

Moreover, any rational point in $\Delta_{\mathcal{S}}$ is of the form $\frac{1}{|M|}\Psi([M])$ for some monoidal cluster monomial M in $\mathcal{C}$.

Proof. Let $\mu \in \mathbf{M}$ such that $M \simeq L(\mu)$. We set $d := |\mu|$ and $\boldsymbol{\mu} := \Psi([M]) \in \mathbb{Z}^N$. Fix a seed $\mathcal{S} := ((x_1, \ldots, x_N), B)$, and let μ_i denote the dominant word such that $x_i = [L(\mu_i)]$. We also set $d_i := |\mu_i|, \boldsymbol{\mu}_i := \Psi(x_i) \in \mathbb{Z}^N$.

The if part is obvious: if $[L(\mu)] = x_1^{a_1} \cdots x_N^{a_N}$, then in $\mathbf{M}$ one has $\mu = \bigodot_i \mu_i^{\odot a_i}$. Hence $\boldsymbol{\mu} = \sum_i a_i \boldsymbol{\mu}_i$ and $d = \sum_i a_i d_i$. This implies:

$$\frac{1}{d}\boldsymbol{\mu} = \frac{1}{\sum_i a_i d_i} \sum_i a_i d_i \frac{1}{d_i} \boldsymbol{\mu}_i \in \Delta_{\mathcal{S}}.$$

For the only if part, let us write $\frac{1}{d}\boldsymbol{\mu} = \sum_i t_i \frac{1}{d_i} \boldsymbol{\mu}_i$ with $t_i \geqslant 0$ for every i and $\sum_i t_i = 1$. Setting $a_i := dt_i/d_i$ for every $1 \leqslant i \leqslant N$, this can be rewritten as $\boldsymbol{\mu} = \sum_i a_i \boldsymbol{\mu}_i$ or equivalently $\boldsymbol{\mu} = \mathcal{M}_{\mathcal{S}}\, {}^t(a_1, \ldots, a_N)$. Now $\boldsymbol{\mu} \in \mathbb{Z}^N$ and $\mathcal{M}_{\mathcal{S}} \in GL_N(\mathbb{Z})$ by Corollary 3.2.9. Hence ${}^t(a_1, \ldots, a_N) \in \mathbb{Z}^N$ i.e. $\forall i, a_i \in \mathbb{Z}$ (and hence $a_i \in \mathbb{N}$ as the a_i are positive). This implies

$$\mu = \bigodot_i \mu_i^{\odot a_i}$$

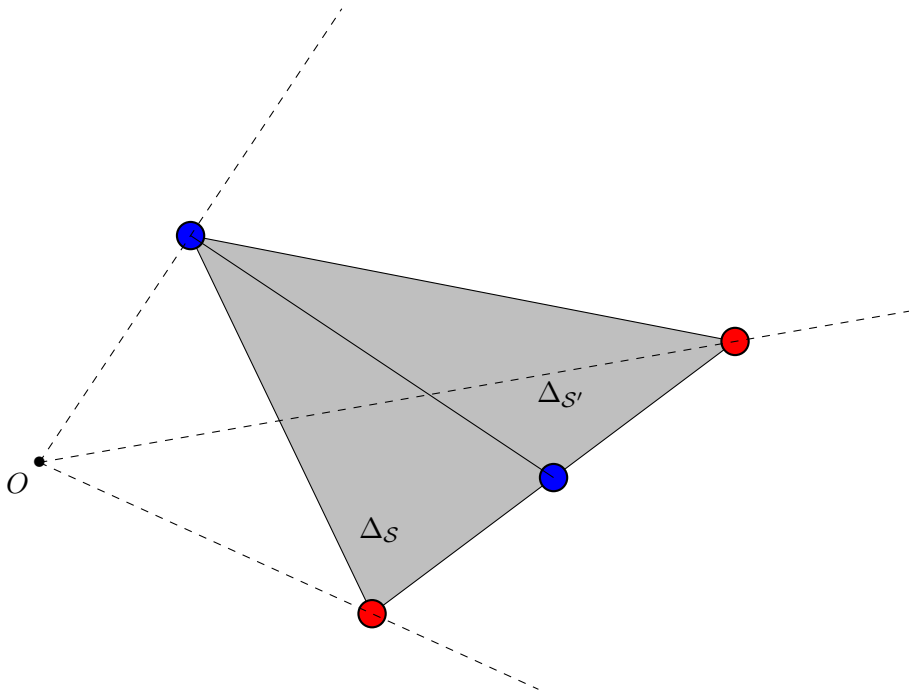

Figure 1: In type A_2 the simplex $\Delta(\mathcal{A}_q(\mathfrak{n}))$ is covered by the two simplices $\Delta_{\mathcal{S}^{(1,2,1)}}$ and $\Delta_{\mathcal{S}^{(2,1,2)}}$.

in $\mathbf{M}$ and hence $[M] = \prod_i [L(\mu_i)]^{a_i} = \prod_i x_i^{a_i}$ i.e. $[M]$ is a cluster monomial for the seed $\mathcal{S}$.

Now let $\boldsymbol{\nu} \in \mathbb{Q}^N \cap \Delta_{\mathcal{S}}$ and write as before $\boldsymbol{\nu} = \sum_i t_i \frac{1}{d_i} \boldsymbol{\mu}_i$ with $t_i \geqslant 0$ and $\sum_i t_i = 1$. Then using similar arguments, one shows that there exists a non-negative integer l such that $\boldsymbol{\mu} := l\boldsymbol{\nu} \in \mathbb{N}^N$ and $M := L(\mu)$ is a monoidal cluster monomial for the seed $\mathcal{S}$. Now $|M| = l \times \sum_i t_i = l$ and thus one has $\nu = \frac{1}{l}\boldsymbol{\nu} = \frac{1}{d}\boldsymbol{\mu}$.

□

Corollary 3.4.4. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two seeds having different sets of cluster variables. Then the simplices $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{S}'}$ have disjoint interiors.*

Proof. Let $\boldsymbol{\nu} \in \mathbb{Q}^N \cap \Delta_{\mathcal{S}} \cap \Delta_{\mathcal{S}'}$; the previous Proposition implies the existence of a monoidal cluster monomial M (resp.M') for the seed $\mathcal{S}$ (resp. $\mathcal{S}'$) such that $\frac{1}{d}\boldsymbol{\mu} = \frac{1}{d'}\boldsymbol{\mu}' = \boldsymbol{\nu}$ (with the same notations as in the previous proof). In particular one has $[M]^{d'} = [M']^d$. By [53, Theorem 8.3], this implies that any cluster variable involved in the monomial $[M]$ has to appear in $[M']$ and vice versa. As by hypothesis $\mathcal{S}$ and $\mathcal{S}'$ have different cluster variables, we conclude that at least one cluster variable of $\mathcal{S}$ does not occur in $[M]$. This obviously implies that $\frac{1}{d}\boldsymbol{\mu}$ belongs to a face of the simplex $\Delta_{\mathcal{S}}$. In other words $\boldsymbol{\nu} \notin \overset{\circ}{\Delta}_{\mathcal{S}}$. Hence $\overset{\circ}{\Delta}_{\mathcal{S}} \cap \overset{\circ}{\Delta}_{\mathcal{S}'} = \varnothing$. □

Example 3.4.5. Consider the situation of Example 3.2.12. The cluster algebra $\mathcal{A}_q(\mathfrak{n})$ has exactly two seeds, namely $\mathcal{S} = \mathcal{S}^{(1,2,1)}$ and $\mathcal{S}' = \mathcal{S}^{(2,1,2)}$. Each of them contains one unfrozen and two frozen variables. Here $\mathcal{H}$ is the plane of equation $x_1 + 2x_2 + x_3 = 1$ in $\mathbb{R}^3$. Figure 1 shows the simplex $\Delta(\mathcal{A}_q(\mathfrak{n}))$ covered by the two simplices corresponding to the seeds $\mathcal{S}$ and $\mathcal{S}'$. The blue dots correspond to the frozen variables and the red dots to the unfrozen variables.

Example 3.4.6. Consider an underlying Lie algebra $\mathfrak{g}$ of type A_3 and let $w := s_1s_2s_3s_1s_2$. Then $\Phi_+^w = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3\}$. The reduced expression of w corresponding to the restriction of the natural ordering $1 < 2 < 3$ on Φ_+^w is $(1, 2, 3, 1, 2)$. By Theorem 3.2.7, the cluster

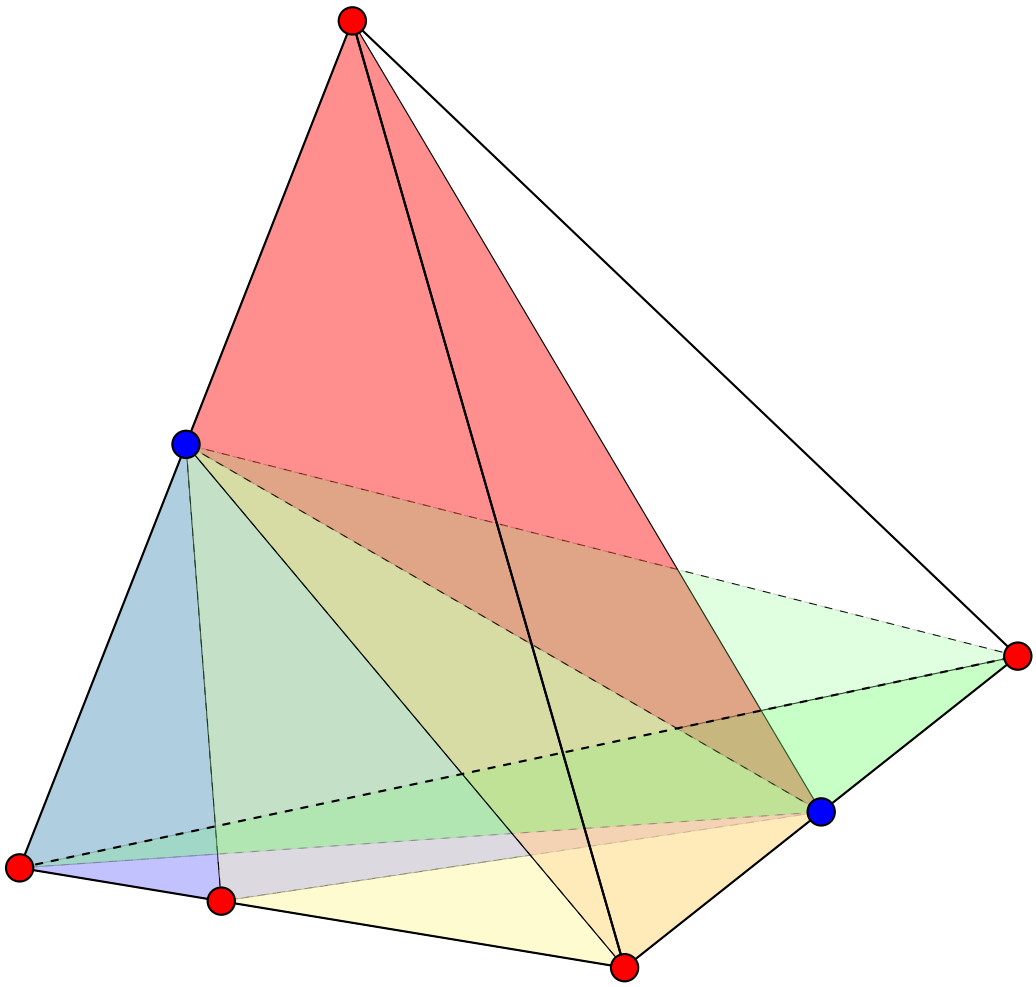

Figure 2: In type A_3 with $w := s_1s_2s_3s_1s_2$, the simplex $\Delta(\mathcal{A}_q(\mathfrak{n}(w)))$ is covered by five smaller simplices, colored in white, red, yellow, blue, and green.

variables of the seed $\mathcal{S}^{(1,2,3,1,2)}$ are given by the following dominant words:

$$(1) \qquad (12) \qquad (123) \qquad (21) \qquad (2312)$$

where the first two are unfrozen and the last three are frozen. Their respective images under Ψ are ${}^t(1,0,0,0,0)$ ${}^t(0,1,0,0,0)$, ${}^t(0,0,1,0,0)$, ${}^t(1,0,0,1,0)$ and ${}^t(0,1,0,0,1)$. The cluster algebra $\mathcal{A}_q(\mathfrak{n}(w))$ has five seeds. A straightforward computation shows that the other cluster variables of $\mathcal{A}_q(\mathfrak{n}(w))$ correspond to the dominant words $(2), (23)$ and (231) of respective images ${}^t(0,0,0,1,0)$, ${}^t(0,0,0,0,1)$ and ${}^t(1,0,0,0,1)$ under Ψ. The simplex $\Delta(\mathcal{A}_q(\mathfrak{n}(w)))$ is of full dimension 4 inside the affine hyperplane $\mathcal{H} \subset \mathbb{R}^5$ given by the equation $x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 = 1$. Note that the frozen variable $[L(123)]$ appears in every seed and its image under Ψ is the third vector of the standard basis of $\mathbb{Z}^5$ (see also Equation (3.3)). The images under Ψ of the other cluster variables have zero entry along this direction. Hence one can get a three-dimensional picture by projecting on $x_3 = 0$ as shown in Figure 2. As in the previous example, the blue dots correspond to the frozen variables and the red dots to the unfrozen variables.

3.4.3 Tropical ϵ-mutation

Tropical *epsilon*-mutations were defined in [97] as tropical exchange relations similar to the mutation rules of g-vectors or c-vectors. They involve a sign ϵ which can be chosen arbitrarily at each mutation. The usual tropical exchange relations of g-vectors and c-vectors are obtained by choosing ϵ to be the coherency sign of c-vectors. In this section we show how monoidal categorifications can provide new examples of interesting tropical ϵ-mutations. In the case of the categorifications of coordinate rings via finite type quiver Hecke algebras, the sign ϵ comes from the natural ordering on parameters of simple objects in $R-gmod$.

We fix a seed $\mathcal{S} = ((x_1, \dots, x_N), B)$ in $\mathcal{A}$ as well as a mutation direction k. Let $\mathcal{S}'$ denote the seed obtained from $\mathcal{S}$ after mutation in the direction k. The cluster variable x_k is replaced by x'_k. We let M_i (resp. M'_k) denote the simple module whose class is x_i for every $1 \leqslant i \leqslant N$ (resp. x'_k). It follows from the constructions of [69] that this cluster mutation at the level of $\mathcal{A}$ comes from a short exact sequence

$$0 \to \bigodot_{b_{ik}>0} M_i^{\circ b_{ik}} \to M_k \circ M'_k \to \bigodot_{b_{ik}<0} M_i^{\circ(-b_{ik})} \to 0 \tag{3.4}$$

in $\mathcal{C}$. We set

$$\boldsymbol{\mu}_i := \Psi(x_i), \quad i \in J \qquad \text{and} \qquad \hat{\boldsymbol{\mu}}_j := \Psi(\hat{y}_j), \quad j \in J_{ex}$$

(see Section 3.1.3 for the definition of the $\hat{y}_j$). As the exchange matrix B has full rank $\sharp J_{ex}$, one can extend it into a $N \times N$ invertible matrix $\tilde{B}$. We then set as before

$$\hat{y}_j := \prod_{1 \leqslant i \leqslant N} x_i^{b_{ij}} \quad \text{and} \quad \hat{\boldsymbol{\mu}}_j := \Psi(\hat{y}_j) \quad \text{for any } 1 \leqslant j \leqslant N.$$

We show that the vectors $\hat{\boldsymbol{\mu}}_j$ satisfy a tropical ϵ-mutation in the sense of Nakanishi [97].

As $\mathbb{Z}^N$ is equipped with a total lexicographic order, the following definition makes sense: for any $k \in J_{ex}$, we set

$$\eta_k := \begin{cases} +1 & \text{if } \hat{\boldsymbol{\mu}}_k > \overrightarrow{0} \\ -1 & \text{otherwise} \end{cases}.$$

Lemma 3.4.7. *The following equality holds in $\mathbb{Z}^N$:*

$$\boldsymbol{\mu}'_k = -\boldsymbol{\mu}_k + \sum_i [\eta_k b_{ik}]_+ \boldsymbol{\mu}_i.$$

Proof. Consider the exchange relation

$$x'_k x_k = \prod_{i, b_{ik}>0} x_i^{b_{ik}} + \prod_{i, b_{ik}<0} x_i^{-b_{ik}}.$$

By Lemma 3.3.3, Ψ is a valuation hence we get

$$\Psi(x'_k) = -\Psi(x_k) + \max\left(\sum_{b_{ik}>0} b_{ik}\Psi(x_i), \sum_{b_{ik}<0} (-b_{ik})\Psi(x_i) \right).$$

Moreover,

$$\sum_{b_{ik}>0} b_{ik}\Psi(x_i) > \sum_{b_{ik}<0} (-b_{ik})\Psi(x_i) \Leftrightarrow \sum_i b_{ik}\Psi(x_i) > \overrightarrow{0} \Leftrightarrow \Psi(\hat{y}_j) > \overrightarrow{0} \Leftrightarrow \eta_k = +1$$

Hence if $\eta_k = 1$, then $\boldsymbol{\mu}'_k = \Psi(x'_k) = -\Psi(x_k) + \sum_{b_{ik}>0} b_{ik}\Psi(x_i) = -\boldsymbol{\mu}_k + \sum_i [b_{ik}]_+ \boldsymbol{\mu}_i$. Similarly one can show that if $\eta_k = -1$ then $\boldsymbol{\mu}'_k = \Psi(x'_k) = -\Psi(x_k) + \sum_i [-b_{ik}]_+ \Psi(x_i) = -\boldsymbol{\mu}_k + \sum_i [-b_{ik}]_+ \boldsymbol{\mu}_i$. $\square$

Now we describe the mutation of the vectors $\hat{\boldsymbol{\mu}}_j$. We let $\hat{\boldsymbol{\mu}}_j{}'$ denote the analogs of $\hat{\boldsymbol{\mu}}_j$ for the seed $\mathcal{S}'$.

Lemma 3.4.8. *The vectors $\hat{\boldsymbol{\mu}}_j{}', 1 \leqslant j \leqslant N$ are given by:*

$$\hat{\boldsymbol{\mu}}_j{}' = \begin{cases} -\hat{\boldsymbol{\mu}}_k & \text{if } j = k, \\ \hat{\boldsymbol{\mu}}_j + [\eta_k b_{jk}]_+ \hat{\boldsymbol{\mu}}_k & \text{otherwise.} \end{cases}$$

Proof. By [43, Proposition 3.9], one has

$$\hat{y_j}' = \begin{cases} \hat{y_k}^{-1} & \text{if } j = k, \\ \hat{y_j}\hat{y_k}^{[b_{kj}]_+}(\hat{y_k}+1)^{-b_{kj}} & \text{otherwise.} \end{cases}$$

Hence applying the valuation Ψ yields

$$\hat{\boldsymbol{\mu}}_j{}' = \begin{cases} -\hat{\boldsymbol{\mu}}_k & \text{if } j = k, \\ \hat{\boldsymbol{\mu}}_j + [b_{kj}]_+ \hat{\boldsymbol{\mu}}_k - b_{kj}\left(\max(\hat{\boldsymbol{\mu}}_k, \overrightarrow{0})\right) & \text{otherwise.} \end{cases}$$

Consider $j \neq k$. If $\eta_k = 1$ then $\hat{\boldsymbol{\mu}}_k > \overrightarrow{0}$ and thus

$$\hat{\boldsymbol{\mu}}_j{}' = \hat{\boldsymbol{\mu}}_j + [b_{kj}]_+ \hat{\boldsymbol{\mu}}_k - b_{kj}\hat{\boldsymbol{\mu}}_k = \hat{\boldsymbol{\mu}}_j + [-b_{kj}]_+ \hat{\boldsymbol{\mu}}_k = \hat{\boldsymbol{\mu}}_j + [b_{jk}]_+ \hat{\boldsymbol{\mu}}_k.$$

Similarly if $\eta_k = -1$ then

$$\hat{\boldsymbol{\mu}}_j{}' = \hat{\boldsymbol{\mu}}_j + [b_{kj}]_+ \hat{\boldsymbol{\mu}}_k = \hat{\boldsymbol{\mu}}_j + [-b_{jk}]_+ \hat{\boldsymbol{\mu}}_k.$$

□

Remark 3.4.9. As explained in [97], tropical ϵ-mutations are of particular interest in cluster theory for describing mutation rules of g-vectors and c-vectors. This follows from a particular choice of tropical sign, namely the sign of c-vectors. This sign-coherency property has been proved by Derksen-Weyman-Zelevinsky [32] in the skew-symmetric case and by Gross-Hacking-Keel-Kontsevich [55] in the general case. In our setting, the sign in the mutation rule does not come from the sign-coherence of c-vectors. Thus the mutation rule given by Lemma 3.4.8 gives a new example of ϵ-tropical mutation where the tropical sign η_k encodes the natural ordering on dominant words.

We end this section with a couple of elementary remarks that will be useful in Section 3.4.4.

Lemma 3.4.10. *The vectors $\hat{\boldsymbol{\mu}}_j, j \in J$ form a basis of $\mathbb{R}^N$.*

Proof. Recall the matrix $M_{\mathcal{S}}$ introduced in Section 3.2. By definition, one has

$$\forall j \in J, \hat{\boldsymbol{\mu}}_j = \Psi(\hat{y_j}) = \sum_{1 \leqslant i \leqslant N} \tilde{b}_{ij}\Psi(x_i) = M_{\mathcal{S}}\tilde{B}\mathbf{e}_j$$

where $\tilde{B}$ is the extended exchange matrix of the seed $\mathcal{S}$. By Corollary 3.2.9 the matrix $M_{\mathcal{S}}$ is invertible. As $\tilde{B}$ is invertible, the composition $M_{\mathcal{S}}B$ is invertible as well and thus the family $(\hat{\boldsymbol{\mu}}_j)_{j \in J}$ is a basis. □

Remark 3.4.11. By construction the three non trivial terms in the short exact sequence (3.4) have the same weight. This implies that for any $j \in J_{ex}$, the vector $\hat{\boldsymbol{\mu}}_j$ is of weight zero. By this we mean that for $j \in J_{ex}$, $\hat{\boldsymbol{\mu}}_j$ belongs to the kernel of the following linear map:

$$\begin{array}{rccc} \mathrm{wt}: & \mathbb{R}^N & \longrightarrow & \mathbb{R}^{\sharp I} \\ \mathbf{c} = & t(c_1, \ldots, c_N) & \longmapsto & \sum_k c_k \beta_k \end{array}$$

where elements of Q_+ are identified with vectors in $\mathbb{R}^{\sharp I}$ in an obvious way. Moreover $\sharp I = \sharp J_{fr} = N - \sharp J_{ex}$ (see [72, Section 2.3]). Hence the $\hat{\boldsymbol{\mu}}_j, j \in J_{ex}$ form a basis of $\ker \mathrm{wt}$. This holds for any seed.

3.4.4 The normal fan to $\Delta_{\mathcal{S}}$

Recall that we fix a total order $<$ on I, an element $w \in W$ and the unique reduced expression $\mathbf{w}$ of w corresponding to the restriction of $<$ to Φ_+^w. Let us introduce a family of vectors $\mathbf{n}_j^{\mathcal{S}}$ for every seed $\mathcal{S}$ in $\mathcal{A}$. They are defined inductively as follows:

Definition 3.4.12. *Consider the initial seed $\mathcal{S}^{\mathbf{w}}$. We define*

$$\mathbf{n}_j^0 := \begin{cases} \mathbf{e}_j - \mathbf{e}_{j_+} - (ht(\beta_j) - ht(\beta_{j_+}))\frac{\boldsymbol{\lambda}}{||\boldsymbol{\lambda}||^2} & \text{if } j \in J_{ex}, \\ \mathbf{e}_j - ht(\beta_j)\frac{\boldsymbol{\lambda}}{||\boldsymbol{\lambda}||^2} & \text{if } j \in J_{fr}. \end{cases}$$

Then given two seeds $\mathcal{S}, \mathcal{S}'$ related to each other by a mutation in the direction $k \in J_{ex}$, the vectors $\mathbf{n}_j^{\mathcal{S}'}$ are related to the $\mathbf{n}_j^{\mathcal{S}}$ by the following tropical ϵ-mutation:

$$\mathbf{n}_j^{\mathcal{S}'} = \begin{cases} -\mathbf{n}_k^{\mathcal{S}} & \text{if } j = k, \\ \mathbf{n}_j^{\mathcal{S}} + [\eta_k b_{jk}]_+ \mathbf{n}_k^{\mathcal{S}} & \text{otherwise.} \end{cases}$$

As any seed can be reached by a finite sequence of mutations from the initial seed $\mathcal{S}^{\mathbf{w}}$, this defines in a unique way the vectors $\mathbf{n}_j^{\mathcal{S}}$ for every seed $\mathcal{S}$. The main result of this section can now be stated as follows:

Theorem 3.4.13. *For any seed $\mathcal{S}$, the vectors $\mathbf{n}_j^{\mathcal{S}}$ are the rays of the normal fan of $\Delta_{\mathcal{S}}$.*

Let $\mathcal{S}$ be an arbitrary seed in $\mathcal{A}$. The simplex $\Delta_{\mathcal{S}}$ is of full dimension $N-1$ inside $\mathcal{H}$. Hence for every vertex P_j of $\Delta_{\mathcal{S}}$, one can consider the facet F_j of $\Delta_{\mathcal{S}}$ which does not contain the vertex P_j. For every $1 \leqslant j \leqslant N$, consider the linear hyperplane of $\mathbb{R}^N$ containing the points $P_i, i \neq j$. One considers the unique vector $\mathbf{N}_j^{\mathcal{S}}$ normal to this hyperplane such that $\langle \boldsymbol{\mu}_j, \mathbf{N}_j^{\mathcal{S}} \rangle = 1$ (recall that the $\boldsymbol{\mu}_j^{\mathcal{S}}$ are linearly independent by Lemma 3.4.10).

First we show that the vectors $\mathbf{N}_j^{\mathcal{S}}$ satisfy the suitable tropical mutation rule:

Lemma 3.4.14. *The vectors $\mathbf{N}'_j, 1 \leqslant j \leqslant N$ are given by:*

$$\mathbf{N}'_j = \begin{cases} -\mathbf{N}_j & \text{if } j = k, \\ \mathbf{N}_j + [\eta_k b_{jk}]_+ \mathbf{N}_k & \text{otherwise.} \end{cases}$$

Proof. The simplices $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{S}'}$ share the facet F_k consisting of the points $P_i, i \neq k$. By definition, both $\mathbf{N}_k$ and $\mathbf{N}'_k$ are orthogonal to the linear hyperplane containing the points $P_i, i \neq k$. Hence $\mathbf{N}_k = c_k \mathbf{N}'_k$ for some (nonzero) real scalar c_k. Now,

$$1 = \langle \boldsymbol{\mu}'_k, \mathbf{N}'_k \rangle = -\langle \boldsymbol{\mu}_k, \mathbf{N}'_k \rangle + \sum_{i, sgn(b_{ik}) = \eta_k} b_{ik} \langle \boldsymbol{\mu}_i, \mathbf{N}'_k \rangle = -\langle \boldsymbol{\mu}_k, \mathbf{N}'_k \rangle = -c_k \langle \boldsymbol{\mu}_k, \mathbf{N}_k \rangle = -c_k$$

which proves the mutation relation for $\mathbf{N}_k$.

Now let $j \neq k$. One has,

$$\begin{aligned} \langle \boldsymbol{\mu}_k, \mathbf{N}'_j \rangle &= -\langle \boldsymbol{\mu}'_k, \mathbf{N}'_j \rangle + \sum_i [\eta_k b_{ik}]_+ \langle \boldsymbol{\mu}_i, \mathbf{N}'_j \rangle \quad \text{by Lemma 3.4.7,} \\ &= \sum_i [\eta_k b_{ik}]_+ \langle \boldsymbol{\mu}_i, \mathbf{N}'_j \rangle = [\eta_k b_{jk}]_+. \end{aligned}$$

Hence the vectors $\mathbf{N}_j$ and $\mathbf{N}'_j - [\eta_k b_{jk}]_+ \mathbf{N}_k$ are both orthogonal to the linear hyperplane $\mathrm{Vect}_{\mathbb{R}}(\boldsymbol{\mu}_i, i \neq j)$. Hence one can write

$$\mathbf{N}'_j = c_j \mathbf{N}_j + [\eta_k b_{jk}]_+ \mathbf{N}_k$$

for some (nonzero) real scalar c_j. Computing the scalar product of both hand sides with $\boldsymbol{\mu}_j$ gives $c_j = 1$ which finishes the proof. $\square$

Now we relate the $\mathbf{N}_j^{\mathcal{S}}$ to the $\mathbf{n}_j^{\mathcal{S}}$, beginning with the initial seed $\mathcal{S}^{\mathbf{w}}$.

Proposition 3.4.15. *Consider the seed $\mathcal{S}^{\mathbf{w}}$. Then for any $j \in J$ on has*

$$\mathbf{n}_j^0 = \mathbf{N}_j^0 - \frac{\langle \boldsymbol{\lambda}, \mathbf{N}_j^0 \rangle}{||\boldsymbol{\lambda}||^2}\boldsymbol{\lambda}.$$

Proof. For simplicity we write $\mathbf{N}_j$ (resp. $\mathbf{n}_j$) for $\mathbf{N}_j^0$ (resp. $\mathbf{n}_j^0$) throughout this proof. Recall from Section 3.2 that we set $J_k := \{j \leqslant k | i_j = i_k\} = \{j_0 = k > j_1 > \cdots > j_{r_k}\}$.

First case: $j \in J_{ex}$. First consider $k \in J \backslash J_j$. By Theorem 3.2.7, the dominant word μ_k is the concatenation in the decreasing order of the good Lyndon words $\mathbf{i}_l, l \in J_k, l \leqslant k$. By definition, $\mathbf{N}_j$ is orthogonal to every $\boldsymbol{\mu}_i, i \neq j$. In particular, it is orthogonal to $\boldsymbol{\mu}_{j_{r_k}}, \boldsymbol{\mu}_{j_{r_k-1}}, \ldots, \boldsymbol{\mu}_k$. Thus one has

$$0 = \langle \boldsymbol{\mu}_{j_{r_k}}, \mathbf{N}_j \rangle = \langle \mathbf{e}_{j_{r_k}}, \mathbf{N}_j \rangle$$

and

$$0 = \langle \boldsymbol{\mu}_{j_{r_k-1}}, \mathbf{N}_j \rangle = \langle \mathbf{e}_{j_{r_k-1}} + \boldsymbol{\mu}_{j_{r_k}}, \mathbf{N}_j \rangle = \langle \mathbf{e}_{j_{r_k-1}}, \mathbf{N}_j \rangle.$$

By a straightforward induction, this implies that the lth component of $\mathbf{N}_j$ is zero for every $l \in J_k$. This holds for every $k \notin J_j$.

Similar arguments show that the lth component of $\mathbf{N}_j$ is zero for every $l \in J_j$ with $l < j$. By definition, one has $\langle \boldsymbol{\mu}_j, \mathbf{N}_j \rangle = 1$. Hence

$$1 = \langle \boldsymbol{\mu}_j, \mathbf{N}_j \rangle = \langle \mathbf{e}_j + \boldsymbol{\mu}_{j_-}, \mathbf{N}_j \rangle = \langle \mathbf{e}_j, \mathbf{N}_j \rangle$$

and thus the jth component of $\mathbf{N}_j$ is equal to 1. Now as j is assumed to lie in J_{ex}, one has $j_+ \leqslant N$ (see Section 3.1.2). Hence one can write

$$\langle \boldsymbol{\mu}_{j_+}, \mathbf{N}_j \rangle = 0 \quad \text{with} \quad \mu_{j_+} = \mathbf{i}_{j_+}\mathbf{i}_j \cdots \mathbf{i}_{r_j}$$

by Theorem 3.2.7. Thus

$$0 = \langle \boldsymbol{\mu}_{j_+}, \mathbf{N}_j \rangle = \langle \mathbf{e}_{j_+} + \boldsymbol{\mu}_j, \mathbf{N}_j \rangle = \langle \mathbf{e}_{j_+}, \mathbf{N}_j \rangle + 1$$

by definition of $\mathbf{N}_j$. Hence the j_+th entry of $\mathbf{N}_j$ is -1. Then a straightforward induction similar to the first case shows that the lth component of $\mathbf{N}_j$ is zero if $l > j_+$.

Thus we have shown that $\mathbf{N}_j$ has exactly two non zero entries, namely the jth equal to 1 and the j_+th equal to -1. Hence

$$\mathbf{N}_j - \frac{\langle \boldsymbol{\lambda}, \mathbf{N}_j \rangle}{||\boldsymbol{\lambda}||^2}\boldsymbol{\lambda} = \mathbf{e}_j - \mathbf{e}_{j_+} - (\mathrm{ht}(\beta_j) - \mathrm{ht}(\beta_{j_+}))\frac{\boldsymbol{\lambda}}{||\boldsymbol{\lambda}||^2}.$$

Comparing with Definition 3.4.12, we conclude that the desired statement holds.

Second case: $j \in J_{fr}$. One shows as before that the kth entry of $\mathbf{N}_j$ is zero for $k \notin J_j$. In this case J_j is exactly the set of all indices of occurrences of the letter j in the chosen reduced expression of w. Writing $\langle \boldsymbol{\mu}_i, \mathbf{N}_j \rangle = 0$ for every $i \in J_j \backslash \{j\}$ implies that all the entries of $\mathbf{N}_j$ are zero except the jth. This entry is equal to $\langle \boldsymbol{\mu}_j, \mathbf{N}_j \rangle$ which is 1 by definition. Hence $\mathbf{N}_j = \mathbf{e}_j$ for every $j \in J_{fr}$. This implies

$$\mathbf{N}_j - \frac{\langle \boldsymbol{\lambda}, \mathbf{N}_j \rangle}{||\boldsymbol{\lambda}||^2}\boldsymbol{\lambda} = \mathbf{e}_j - \mathrm{ht}(\beta_j)\frac{\boldsymbol{\lambda}}{||\boldsymbol{\lambda}||^2} = \mathbf{n}_j$$

for every $j \in J_{fr}$ which finishes the proof.

□

Now we show that the statement of Proposition 3.4.15 holds for every seed:

Lemma 3.4.16. *Let $\mathcal{S}$ be any seed in $\mathcal{A}$, and let $j \in J$. Then one has*

$$\mathbf{n}_j^{\mathcal{S}} = \mathbf{N}_j^{\mathcal{S}} - \frac{\langle \boldsymbol{\lambda}, \mathbf{N}_j^{\mathcal{S}} \rangle}{||\boldsymbol{\lambda}||^2} \boldsymbol{\lambda}.$$

Proof. For the seed $\mathcal{S}^{\mathbf{w}}$, it follows from Proposition 3.4.15. As the function $\mathrm{ht}(\cdot)$ is linear, Lemma 3.4.14 shows that the $\tilde{\mathbf{N}}_j^{\mathcal{S}}$ follows the same tropical mutation rule as the $\mathbf{n}_j$. Hence by induction the equality holds for every seed. □

One can now finish the proof of Theorem 3.4.13:

Proof of Theorem 3.4.13. For any $j \in J$, the vector $\mathbf{N}_j^{\mathcal{S}}$ is orthogonal to every $\boldsymbol{\mu}_i$, $i \neq j$ hence to the vectors $\frac{\boldsymbol{\mu}_p}{\mathrm{ht}(\mu_p)} - \frac{\boldsymbol{\mu}_q}{\mathrm{ht}(\mu_q)}$ for any $p, q \neq j$. These generate the underlying linear space of F_j and hence $\mathbf{N}_j^{\mathcal{S}}$ is orthogonal to F_j. Moreover, the facet F_j is contained in $\mathcal{H}$ for any $j \in J$. Hence $\boldsymbol{\lambda}$ is orthogonal to F_j. The conclusion follows from Lemma 3.4.16. □

Let us finish this section by explaining why Theorem 3.4.13 provides an explicit geometric realization of the cluster-theoretic dominance order (see Definition 3.1.5). Fix a seed $\mathcal{S} = ((x_1, \ldots, x_N), B)$ in $\mathcal{A}$. The dominance order was introduced by F.Qin as a partial ordering on Laurent monomials in $x_1, \ldots, x_N$ in the study of common triangular bases for (quantum) cluster algebras. The vectors $\hat{\boldsymbol{\mu}}_j$ (see Section 3.4.3) were defined in [25] as a natural analog of this order at the level of parameters for simple modules in $\mathcal{C}$. More precisely let $\overrightarrow{\mathcal{N}^{\mathcal{S}}}$ denote the linear convex cone generated by the $\hat{\boldsymbol{\mu}}_j, j \in J_{ex}$. For every simple object M in $\mathcal{C}$, let $\mathcal{N}_M^{\mathcal{S}}$ denote the affine cone with origin $\Psi([M])$ and direction $\overrightarrow{\mathcal{N}^{\mathcal{S}}}$; then one can see that the simple objects in $\mathcal{C}$ whose classes are smaller than $[M]$ for $\preccurlyeq_{\mathcal{S}}$ correspond to the integral points of $\mathcal{N}_M^{\mathcal{S}}$.

Theorem 3.4.13 can now be reformulated as follows. Consider the vector subspace $\mathbf{V}$ of $\mathbb{R}^N$ generated by the $\mathbf{n}_j^0, j \in J_{ex}$. By Remark 3.4.11, the $\hat{\boldsymbol{\mu}}_j, j \in J_{ex}$ form a basis of $\ker \mathrm{wt}$ hence one can consider the unique linear map T defined as

$$\begin{array}{rccc} T: & \ker(\mathrm{wt}) & \longrightarrow & \mathbf{V} \\ & \hat{\boldsymbol{\mu}}_j & \longmapsto & \mathbf{n}_j^0 \end{array}$$

for every $j \in J_{ex}$.

Corollary 3.4.17. *The map T is a linear isomorphism and for every seed $\mathcal{S}$, the image under T of the cone $\overrightarrow{\mathcal{N}^{\mathcal{S}}}$ is the face $\mathbf{n}_j^{\mathcal{S}}, j \in J_{ex}$ of the normal fan of $\Delta_{\mathcal{S}}$.*

Proof. The vectors $\mathbf{n}_j^0, j \in J_{ex}$ form a basis of $\mathbf{V}$ hence T is an isomorphism. By Lemma 3.4.8, the vectors $\hat{\boldsymbol{\mu}}_j$ follow the same tropical mutation rule as the $\mathbf{n}_j$. Hence one has

$$\forall j \in J_{ex}, T\hat{\boldsymbol{\mu}}_j = \mathbf{n}_j$$

for every seed $\mathcal{S}$. This is the desired statement by Theorem 3.4.13. □

Remark 3.4.18. Definition 3.4.12 allows us to get an explicit description of the subspace $\mathbf{V}$. This subspace essentially describes the exchange part of the cluster algebra $\mathcal{A}$. For instance for $w = w_0$ and $\mathbf{w} = (1, 2, 1, 3, 2, 1, \ldots, n, \ldots 1)$ in type A_n, [25, Theorem 6.1] implies that $\mathrm{ht}(\beta_j) = \mathrm{ht}(\beta_{j_+})$ for every $j \in J_{ex}$. Hence for every $j \in J_{ex}$, the vector $\mathbf{n}_j^0, j \in J_{ex}$ is simply $\mathbf{e}_j - \mathbf{e}_{j_+}$. Let $M_1, \ldots, M_n$ denote the simple modules corresponding to the frozen variables in $\mathcal{A}$. Then using Theorem 3.2.7, one can check that in this case $\mathbf{V}$ is exactly the orthogonal of the vector subspace generated by $\Psi([M_1]), \ldots, \Psi([M_n])$.

3.5 Towards colored hook formulas

In this section we focus on the case where $\mathcal{A}_q(\mathfrak{n}(w))$ is a cluster algebra of finite type, i.e. there is a finite number of seeds. We obtain an equality between rational functions involving the weights of the simple modules corresponding to the cluster variables of $\mathcal{A}_q(\mathfrak{n}(w))$. As a consequence, we get a cluster-theoretic formula for the quantity $\frac{N!}{\prod_{\beta\in\Phi_+^w}\mathrm{ht}(\beta)}$. This quantity has a well-known significance in combinatorics and Lie theory: Peterson-Proctor related this quantity to the combinatorics of d-complete posets (see [104, 105]) . Under some technical assumption on w (w is assumed to be *dominant minuscule* in the terminology of [111]) they prove that this quantity is exactly the number of reduced expressions of w. This Peterson-Proctor hook formula is also related to the dimension of certain remarkable simple representations of quiver Hecke algebras constructed by Kleshchev-Ram, see [78, Theorem 3.10].

In [95], Nakada proposed a generalization of the Peterson-Proctor hook formula. Recall from Section 3.1.1 that $\alpha_1,\dots,\alpha_n$ stand for the simple roots of $\mathfrak{g}$. One considers the $\alpha_i, 1\leqslant i\leqslant n$ as formal variables and we let $\mathbb{L}$ denote the field $\mathbb{C}(\alpha_1,\dots,\alpha_n)$. For every $\beta=\sum_i a_i\alpha_i\in Q_+$, one associates a formal rational function

$$\frac{1}{\beta}:=\frac{1}{a_1\alpha_1+\cdots+a_n\alpha_n}\in\mathbb{L}.$$

Specializing the α_i to 1, the value of this rational function is exactly $\frac{1}{\mathrm{ht}(\beta)}$. Then Nakada proves the following *colored hook formula*:

Theorem 3.5.1 ([95, Corollary 7.2])**.** *Assume w is a dominant minuscule element of W in the terminology of [104, 105, 111]. Recall that N denotes the length of w. Then the following equality holds in $\mathbb{L}$:*

$$\prod_{\beta\in\Phi_+^w}\frac{1}{\beta}=\sum_{(i_1,\dots,i_N)\in MPath(w)}\frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_N}}\tag{3.5}$$

The set $\mathrm{MPath}(w)$ is a finite set in bijection with the set of all reduced expressions of w. We refer to [95, Sections 2,7] for more details. Every term of the sum in the right hand side of Equation (3.5) is equal to $1/N!$ when specializing the α_i to 1. Hence as an immediate consequence of this result, one gets that the cardinal of $\mathrm{MPath}(w)$ coincides with the Peterson-Proctor hook formula

$$\sharp\mathrm{MPath}(w)=\frac{N!}{\prod_{\beta\in\Phi_+^w}\mathrm{ht}(\beta)}.$$

Remark 3.5.2. In fact, the main result of [95] expresses the rational function $\prod_{\beta\in\Phi_+^w}\left(1+\frac{1}{\beta}\right)$ as a sum of rational functions of the form $\frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_l}}$ with $l\leqslant N$, where the tuples $(i_1,\dots,i_l)$ run over a set $\mathrm{Path}(w)$ strictly containing $\mathrm{MPath}(w)$. The equality given by Theorem 3.5.1 is obtained by considering the terms of lowest degree.

For every seed $\mathcal{S}=((x_1^{\mathcal{S}},\dots,x_N^{\mathcal{S}}),B^{\mathcal{S}})$ in $\mathcal{A}_q(\mathfrak{n}(w))$ and any $1\leqslant j\leqslant N$, consider the unique dominant word $\mu_j^{\mathcal{S}}$ such that $x_j^{\mathcal{S}}=[L(\mu_j^{\mathcal{S}})]$. We write the weight of $\mu_j^{\mathcal{S}}$ as $\mathrm{wt}(\mu_j^{\mathcal{S}})=\sum_i a_{i,j}^{\mathcal{S}}\alpha_i$ (see Section 3.1.1). Then mimicking [95], one considers the rational function

$$\frac{1}{\mathrm{wt}(\mu_j^{\mathcal{S}})}:=\frac{1}{a_{1,j}^{\mathcal{S}}\alpha_1+\cdots+a_{n,j}^{\mathcal{S}}\alpha_n}\in\mathbb{L}.$$

We can now state the main result of this section.

Theorem 3.5.3. *Assume $w \in W$ is such that the cluster algebra $\mathcal{A}_q(\mathfrak{n}(w))$ is of finite type. Then the following equality holds in $\mathbb{L}$:*

$$\prod_{\beta \in \Phi_+^w} \frac{1}{\beta} = \sum_{\mathcal{S}} \prod_{1 \leqslant j \leqslant N} \frac{1}{wt(\mu_j^{\mathcal{S}})}. \tag{3.6}$$

We fix $w \in W$ and we write as in Section 3.1.2 $\Phi_+^w = \{\beta_1 < \cdots < \beta_N\}$. We identify positive roots in Φ_+^w with elements of $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$ in a natural way and we let $\boldsymbol{\beta}$ denote the vector of $\mathbb{L}^N$ whose entries are $\beta_1, \ldots, \beta_N$. For any seed $\mathcal{S}$ and any $1 \leqslant j \leqslant N$, we also set $\beta_j^{\mathcal{S}} := \mathrm{wt}(\mu_j^{\mathcal{S}})$ and $\boldsymbol{\beta}^{\mathcal{S}} := (\beta_1^{\mathcal{S}}, \ldots, \beta_N^{\mathcal{S}}) \in \mathbb{L}^N$. We begin with the following lemma:

Lemma 3.5.4. *For any seed $\mathcal{S}$ one has:*

$$\frac{1}{\beta_1^{\mathcal{S}} \cdots \beta_N^{\mathcal{S}}} = \int_{C_{\mathcal{S}}} e^{-(\beta_1 y_1 + \cdots + \beta_N y_N)} dy_1 \cdots dy_N.$$

Proof. Let $C_{\mathcal{S}}$ be the open linear cone of $\mathbb{R}^N$ whose intersection with $\mathcal{H}$ is $\Delta_{\mathcal{S}}$. With the notations of Section 3.4.4, one has $C_{\mathcal{S}} = \bigcap_{1 \leqslant k \leqslant N} \{\langle \mathbf{N}_k^{\mathcal{S}}, \cdot \rangle \geqslant 0\}$. Let $\mathcal{N}_{\mathcal{S}}$ denote the $N \times N$ matrix whose columns are the $\mathbf{N}_k^{\mathcal{S}}, 1 \leqslant k \leqslant N$. By definition of the $\mathbf{N}_k^{\mathcal{S}}$ one has ${}^t\mathcal{N}_{\mathcal{S}}\mathcal{M}_{\mathcal{S}} = Id_N$. Hence one has

$$\frac{1}{\beta_1^{\mathcal{S}} \cdots \beta_N^{\mathcal{S}}} = \int_{\mathbb{R}_+^{*\,N}} e^{-(\beta_1^{\mathcal{S}} x_1 + \cdots + \beta_N^{\mathcal{S}} x_N)} dx_1 \cdots dx_N = \int_{C_{\mathcal{S}}} |\det(\mathcal{N}_{\mathcal{S}})| e^{-(\beta_1^{\mathcal{S}} ({}^t\mathcal{N}_{\mathcal{S}} y)_1 + \cdots + \beta_N^{\mathcal{S}} ({}^t\mathcal{N}_{\mathcal{S}} y)_N)} dy_1 \cdots dy_N.$$

By Corollary 3.2.9, $|\det(\mathcal{M}_{\mathcal{S}})| = 1$ and hence $|\det(\mathcal{N}_{\mathcal{S}})| = 1$ as well. Then for every $1 \leqslant j \leqslant N$ one has

$$({}^t\mathcal{N}_{\mathcal{S}} y)_j = \sum_i (\mathcal{N}_{\mathcal{S}})_{ij} y_i$$

and hence

$$\sum_j \beta_j^{\mathcal{S}} ({}^t\mathcal{N}_{\mathcal{S}} y)_j = \sum_i \left(\sum_j (\mathcal{N}_{\mathcal{S}})_{ij} \beta_j^{\mathcal{S}} \right) y_i = \sum_i \left(\mathcal{N}_{\mathcal{S}} \boldsymbol{\beta}^{\mathcal{S}} \right)_i y_i = \sum_i \left({}^t\mathcal{M}_{\mathcal{S}}^{-1} \boldsymbol{\beta}^{\mathcal{S}} \right)_i y_i.$$

Then it suffices to note that for any j one has

$$\left({}^t\mathcal{M}_{\mathcal{S}} \boldsymbol{\beta} \right)_j = \langle {}^t\mathcal{M}_{\mathcal{S}} \boldsymbol{\beta}, \mathbf{e}_j \rangle = \langle \boldsymbol{\beta}, \boldsymbol{\mu}_j^{\mathcal{S}} \rangle = \mathrm{wt}(\mu_j^{\mathcal{S}}) = \beta_j^{\mathcal{S}}.$$

Thus we have proven that

$$\frac{1}{\beta_1^{\mathcal{S}} \cdots \beta_N^{\mathcal{S}}} = \int_{C_{\mathcal{S}}} e^{-(\beta_1 y_1 + \cdots + \beta_N y_N)} dy_1 \cdots dy_N.$$

We conclude by performing the change of variables $\mathbb{R}_+^* \times \Delta_{\mathcal{S}} \longrightarrow C_{\mathcal{S}}$ given by $(r, \mathbf{y}) \longmapsto r\mathbf{y}$. □

Proof of Theorem 3.5.3. If $\mathcal{A}_q(\mathfrak{n}(w))$ is a cluster algebra of finite type, then all the simple objects in $\mathcal{C}_w$ are cluster monomials. Hence the union of the cones $C_{\mathcal{S}}$ for all the seeds in $\mathcal{A}_q(\mathfrak{n}(w))$ is equal to the whole positive orthant $\mathbb{R}_+^{*\,N}$ (up to some set of zero measure). Hence one has

$$\prod_{\beta \in \Phi_+^w} \frac{1}{\beta} = \int_{\mathbb{R}_+^{*\,N}} e^{-(\beta_1 x_1 + \cdots + \beta_N x_N)} dx_1 \cdots dx_N = \sum_{\mathcal{S}} \int_{C_{\mathcal{S}}} e^{-(\beta_1 x_1 + \cdots + \beta_N x_N)} dx_1 \cdots dx_N = \sum_{\mathcal{S}} \frac{1}{\beta_1^{\mathcal{S}} \cdots \beta_N^{\mathcal{S}}}.$$

□

One can also state another consequence of Lemma 3.5.4:

Corollary 3.5.5. *Let $\mathcal{S}$ be any seed in $\mathcal{A}_q(\mathfrak{n}(w))$. The volume of the simplex $\Delta_{\mathcal{S}}$ is given by*

$$Vol\,(\Delta_{\mathcal{S}}) = \frac{1}{\prod_{1\leqslant j\leqslant N}|\mu_j^{\mathcal{S}}|}.$$

Proof. Specializing the variables $\alpha_1,\dots,\alpha_n$ to 1 we get

$$\frac{1}{\prod_{1\leqslant j\leqslant N}|\mu_j^{\mathcal{S}}|} = \int_{C_{\mathcal{S}}} e^{-(\mathrm{ht}(\beta_1)y_1+\cdots+\mathrm{ht}(\beta_N)y_N)}dy_1\cdots dy_N.$$

We perform the change of variables

$$\begin{array}{ccc} \mathbb{R}_+^*\times\Delta_{\mathcal{S}} & \longrightarrow & C_{\mathcal{S}} \\ (r,\mathbf{y}) & \longmapsto & r\mathbf{y} \end{array}$$

in the right hand side. By construction, $\Delta_{\mathcal{S}}$ is included in the affine hyperplane $\mathcal{H}$ defined as $\{\mathrm{ht}(\beta_1)y_1+\cdots+\mathrm{ht}(\beta_N)y_N=1\}$. Hence we get $CVol(\Delta_{\mathcal{S}})$ where C is some constant. A straightforward computation shows that this constant is equal to 1. □

Consequently we also get the following statement:

Corollary 3.5.6. *Assume $w\in W$ is such that the cluster algebra $\mathcal{A}_q(\mathfrak{n}(w))$ is of finite type. Then one has*

$$\frac{N!}{\prod_{\beta\in\Phi_+^w} ht(\beta)} = \sum_{\mathcal{S}}\frac{N!}{\prod_{1\leqslant j\leqslant N}|\mu_j^{\mathcal{S}}|}.$$

Remark 3.5.7. In the general case, $\mathcal{A}_q(\mathfrak{n}(w))$ can be of infinite cluster type but the sum

$$\sum_{\mathcal{S}}\frac{N!}{\prod_{1\leqslant j\leqslant N}|\mu_j^{\mathcal{S}}|}$$

still makes sense (as the disjoint union of the simplices $\Delta_{\mathcal{S}}$ is always included in $\Delta(\mathcal{A}_q(\mathfrak{n}(w))))$. We don't know if this sum still takes a remarkable form in this general situation.

In order to make sense of a link between Theorem 3.5.1 and Theorem 3.5.3, one should take care of the conditions imposed on w. Theorem 3.5.1 holds under the assumption that w is dominant minuscule whereas a necessary condition for Theorem 3.5.3 to hold is that $\mathcal{A}_q(\mathfrak{n}(w))$ has to be a cluster algebra of finite type. We conjecture the following:

Conjecture 3.5.8. *If $w\in W$ is dominant minuscule, then the cluster algebra $\mathcal{A}_q(\mathfrak{n}(w))$ has a finite number of seeds.*

This would imply that the equality given by Theorem 3.5.3, although of different nature than Theorem 3.5.1, holds in a larger generality.

Let us give an example where Conjecture 3.5.8 holds. It corresponds to the example considered in [95, Section 2]. We consider a Lie algebra $\mathfrak{g}$ of type A_3 with simple roots $\Pi := \{\alpha_1,\alpha_2,\alpha_3\}$. We let s_1,s_2,s_3 denote the simple reflections of the Weyl group W of $\mathfrak{g}$. We set $w := s_2s_1s_3s_2$ and we choose the reduced expression $\mathbf{w} = (2,1,3,2)$ of w. It is straightforward to check that w is dominant minuscule using the criterion [111, Proposition 2.3]. By [52, Equation 7.6] (see also [72, Section 2.3]), the subset of positive roots Φ_+^w ordered with respect to our choice of $\mathbf{w}$ are given by

$$\Phi_+^w = \{\alpha_2<\alpha_1+\alpha_2<\alpha_2+\alpha_3<\alpha_1+\alpha_2+\alpha_3\}$$

In this case Nakada's colored hook formula is given in [95, Section 2] and can be written as

$$\frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3} = \frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} + \frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3}. \tag{3.7}$$

On the other hand, the cluster algebra $\mathcal{A}_q(\mathfrak{n}(w))$ is of rank 4 and with the notations of Section 3.1.2, $J_{ex} = \{1\}$ and $J_{fr} = \{2,3,4\}$. Thus there is only one mutation direction and hence by the involutivity of cluster mutations there are exactly two seeds in $\mathcal{A}_q(\mathfrak{n}(w))$. Each of these two seeds contains one unfrozen variable and three frozen variables. The chosen reduced expression of w corresponds to the restriction on Φ_+^w of the ordering $2 < 1 < 3$ on the A_3 Dynkin diagram. Thus Theorem 3.2.7 gives the dominant words $\mu_i^{\mathcal{S}}, i = 1,\dots,4$ for the seed $\mathcal{S} = \mathcal{S}^{\mathbf{w}}$. Here we only need their weights, which are given by

$$\mathrm{wt}(\mu_1) = \alpha_2 \qquad \mathrm{wt}(\mu_2) = \alpha_1+\alpha_2 \qquad \mathrm{wt}(\mu_3) = \alpha_2+\alpha_3 \qquad \mathrm{wt}(\mu_4) = \alpha_1+2\alpha_2+\alpha_3.$$

The exchange matrix of the seed $\mathcal{S}^{\mathbf{w}}$ is given by $B = {}^t(0,1,1,-1)$ and thus the cluster variable x_1' obtained after performing the mutation at x_1 of the seed $\mathcal{S}^{\mathbf{w}}$ is given by

$$x_1' = \frac{1}{x_1}\left(x_2x_3 + x_4\right).$$

This implies that the corresponding dominant word is μ_1' is of weight $wt(\mu_1') = \alpha_1+\alpha_2+\alpha_3$. Thus the sum in the right hand side of Equation (3.6) is

$$\frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} + \frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3}.$$

In this example one can check by a straightforward calculation that this rational function is equal to

$$\frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}$$

which is exactly the statement of Theorem 3.5.3.

Remark 3.5.9. The sums of rational functions on the right hand side of Equations (3.5) and (3.6) are a priori of a very different combinatorial natures. For instance these sums do not have the same number of terms in general. Moreover when specializing the α_i to 1, the terms in the right hand side of Equation (3.5) all take the same value $1/N!$. On the contrary the value taken by the term indexed by a seed $\mathcal{S}$ in Equation (3.6) is essentially the volume of the simplex $\Delta_{\mathcal{S}}$, which is not the same for every seed. However, it turns out (even in more complicated examples) that these two different expressions take rather similar forms. This might suggest closer connections between the combinatorics of fully-commutative elements of Weyl groups and cluster theory.

Remark 3.5.10. Rational functions of the form of Equation (3.5.3) also appeared in the recent work of Baumann-Kamnitzer-Knutson [6]. They are related with the definition of Duistermaat-Heckmann measures used to compare various bases in $\mathcal{A}_q(\mathfrak{n})$. In this framework, they prove that the Mirkovic-Vilonen basis and the dual semicanonical basis are not the same.

Chapter 4

Equivariant multiplicities of simply-laced type flag minors

This chapter corresponds to the arXiv Preprint [26], arXiv:2005.07051v1.

Let $\mathfrak{g}$ be a finite simply-laced type simple Lie algebra. Baumann-Kamnitzer-Knutson recently defined an algebra morphism $\overline{D}$ on the coordinate ring $\mathbb{C}[N]$ related to Brion's equivariant multiplicities via the geometric Satake correspondence. This map is known to take distinguished values on the elements of the MV basis corresponding to smooth MV cycles, as well as on the elements of the dual canonical basis corresponding to Kleshchev-Ram's strongly homogeneous modules over quiver Hecke algebras. In this paper we show that when $\mathfrak{g}$ is of type A_n or D_4, the map $\overline{D}$ takes similar distinguished values on the set of all flag minors of $\mathbb{C}[N]$, raising the question of the smoothness of the corresponding MV cycles. We also exhibit certain relations between the values of $\overline{D}$ on flag minors belonging to the same standard seed, and we show that in any ADE type these relations are preserved under cluster mutations from one standard seed to another. The proofs of our results partly rely on Kang-Kashiwara-Kim-Oh's monoidal categorification of the cluster structure of $\mathbb{C}[N]$ via representations of quiver Hecke algebras.

4.1 Introduction

Let $\mathfrak{g}$ be a finite simply-laced type simple Lie algebra and let $\mathfrak{n}$ denote the nilpotent subalgebra arising from a triangular decomposition of $\mathfrak{g}$. We consider the ring $\mathbb{C}[N]$ of regular functions on the algebraic group N associated to $\mathfrak{n}$. The study of good bases of $\mathbb{C}[N]$, as well as its quantized version $\mathcal{A}_q(\mathfrak{n})$, has been an intensively investigated topic since the works of Kashiwara [70] and Lusztig [88] in the early 90's. Kashiwara [70] introduced the notion of *crystal* as a combinatorial model describing the structure of the irreducible finite-dimensional representations of the quantum group $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$. He defined the lower global basis (resp. upper global basis) using the crystal structure on $U_q^-(\mathfrak{g})$ (resp. on the quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$). Lusztig [88] used certain categories of perverse sheaves on quiver varieties to define the canonical basis (resp. dual canonical basis) of $U_q^-(\mathfrak{g})$ (resp. $\mathcal{A}_q(\mathfrak{n})$). Grojnowski-Lusztig [54] and Kashiwara-Saito [73] proved that the dual canonical basis and the upper global basis coincide. Several other remarkable bases of $\mathbb{C}[N]$ have been discovered since then, such as the dual semicanonical basis introduced by Lusztig [89] or the MV basis, constructed after the discovery of the geometric Satake correspondence by Mirković-Vilonen [92].

Berenstein-Zelevinsky [8] observed that the dual canonical basis of $\mathbb{C}[N]$ had interesting multiplicative properties. This was one of the main motivations for the introduction of cluster algebras by Fomin-Zelevinsky [41]. These are defined as certain commutative subalgebras of the field of

rational functions $\mathbb{Q}(x_1, \ldots, x_N)$ where $x_1, \ldots, x_N$ are algebraically independent variables. They are generated by certain distinguished generators called *cluster variables* that are grouped into overlapping finite sets of fixed cardinality N called *clusters*. The monomials involving variables of the same cluster are called *cluster monomials*. The cluster variables can be constructed from the variables $x_1, \ldots, x_N$ by performing an inductive procedure called *mutation*. The initial data of this procedure consists in the N independent variables $x_1, \ldots, x_N$ together with a quiver Q with N vertices and without any loop or 2-cycle. Such a data is called a *seed*. For every $k \in \{1, \ldots, N\}$, one defines a new variable x'_k entirely determined by the x_j and Q, as well as a new quiver Q'_k. This yields a new seed, given by the variables $x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_N$ and the quiver Q'_k. One of the first key points of cluster theory is the involutivity of this procedure, i.e. mutating this new seed in the same direction k transforms it back into the initial seed. Thus one can iterate this by applying arbitrary sequences of mutations. Fomin-Zelevinsky [42] provided a Dynkin-type classification of the initial quivers Q for which this process produces only a finite number of distinct seeds.

It was shown by Geiss-Leclerc-Schröer [52] that the coordinate ring $\mathbb{C}[N]$ associated to a simply-laced finite type Lie algebra $\mathfrak{g}$ carries the structure of a cluster algebra. Their work strongly relies on categorification techniques using representations of preprojective algebras. The mutations arise from the study of certain T-systems called *determinantial identities* relating unipotent minors. Geiss-Leclerc-Schröer also explicitly construct a certain family of seeds called *standard seeds* parametrized by the set of reduced expressions of the longest element w_0 of the Weyl group W of $\mathfrak{g}$. The cluster variables of the standard seeds are certain special cases of unipotent minors called *flag minors*.

After the works of Geiss-Leclerc-Schröer, other categorifications of $\mathbb{C}[N]$ were constructed, but relying on categories of different natures. Unlike the *additive categorification* of [52] via representations of preprojective algebras, a new kind of categorification called *monoidal categorification* was introduced by Hernandez-Leclerc [56]. The idea is to identify a given cluster algebra $\mathcal{A}$ with Grothendieck ring of an artinian monoidal category $\mathcal{C}$ via a ring isomorphism required to send the cluster monomials of $\mathcal{A}$ onto classes of simple objects in $\mathcal{C}$. A first class of examples of such categorifications was provided in [56] for certain unipotent cells of $\mathbb{C}[N]$ associated with Coxeter elements of W. Concerning $\mathbb{C}[N]$ itself, it was proved by Hernandez-Leclerc [58] that the Grothendieck ring of a certain category of finite-dimensional representations of quantum affine algebras was isomorphic to $\mathbb{C}[N]$. A monoidal categorification of $\mathbb{C}[N]$ (as well as all its unipotent cells) was then constructed in a vast series of works due to Kang-Kashiwara-Kim-Oh [66, 67, 68, 69, 72] using the representation theory of quiver Hecke algebras.

Quiver Hecke algebras (or KLR algebras) were introduced by Khovanov-Lauda [76] and Rouquier [110] in the purpose of categorifying the negative half of the quantum group $U_q(\mathfrak{g})$. They are a family of $\mathbb{Z}$-graded associative algebras indexed by Q_+. The category $R-gmod$ of all graded finite-dimensional representations of the $R(\beta), \beta \in Q_+$ can be given a monoidal structure. Rouquier [110] and Varagnolo-Vasserot [113] proved that the set of isomorphism classes of simple objects in $R-gmod$ is in bijection with the elements of the dual canonical basis of $\mathbb{C}[N]$. In the case of a finite type Lie algebra $\mathfrak{g}$, these simple objects were classified by McNamara [90] and Kleshchev-Ram [79] in terms of root partitions (or dominant words) using the combinatorics of good Lyndon words, relying on the works of Leclerc [84]. This parametrization was shown to be compatible with the monoidal structure of $R-gmod$ (see [24]) and turns out to be convenient for studying the determinantial modules categorifying the flag minors of $\mathbb{C}[N]$. In [24], we used recent results of Kashiwara-Kim [72] to provide an explicit description in terms of root partitions of the determinantial modules corresponding to the cluster variables of the standard seed $\mathcal{S}^{\mathbf{i}}$ when $\mathbf{i} \in \mathrm{Red}(w_0)$ comes from a total ordering on an index set of simple roots. This description will be very useful in the proof of the second main result of this paper (Theorem 4.5.10).

Kleshchev-Ram [79] also exhibited a (finite) family of distinguished irreducible representations called *cuspidal representations*, satisfying several good properties. Note that the notion of cuspidality depends on a preliminary choice of total ordering an index set of simple roots. The understanding of cuspidal modules was the motivation for the construction of *homogeneous representations* over simply-laced type quiver Hecke algebras by the same authors in [78]. They are a (finite) remarkable family of simple finite-dimensional modules in $R-gmod$ parametrized by the so-called *fully-commutative* elements of W. The combinatorics of fully-commutative elements of Weyl groups is very rich and has been studied for a long time by Proctor [104, 105], Stembridge [111], Nakada [95] among others. A subfamily of homogeneous modules, called *strongly homogeneous* in the terminology of [78] will be of particular interest for us. It was already observed by Kleshchev and Ram that the dimensions of these modules are given by the Peterson-Proctor hook formula. This formula, introduced in an unpublished work of Peterson-Proctor, was generalized by Nakada [95] in a purely combinatorial context as a colored hook formula. It turns out that these colored hook formulas can be conveniently interpreted in terms of strongly homogeneous representations over quiver Hecke algebras by using certain tools developped in the recent work of Baumann-Kamnitzer-Knutson [6]. More details on the structure of homogeneous representations can be found in [36, 37].

Mirković-Vilonen [92] exhibited a spectacular equivalence between certain categories of perverse sheaves on the affine Grassmannian Gr_G associated to a simple simply-connected reductive group G and the category of finite-dimensional representations of the Langlands dual $G^\vee$ of G. Moreover, the weight subspaces of the highest weights representations of $G^\vee$ are interpreted as cohomology spaces of certain sheaves on Gr_G. This is the *geometric Satake correspondence*. Mirković-Vilonen introduced certain interesting closed irreducible subvarieties of Gr_G called *MV cycles*, that give rise to interesting bases of the finite-dimensional irreducible representations of $G^\vee$ via the geometric Satake correspondence. These bases can be glued together into a basis of $\mathbb{C}[N]$ called the *MV basis*, indexed by certain MV cycles called *stable*. The elements of this basis can also be parametrized in a more combinatorial way using *MV polytopes*. Kamnitzer [64, 65] gave an explicit description of these polytopes and showed that they carry a crystal structure in the sense of Kashiwara. These polytopes are known to be the support of certain measures called *Duistermaat-Heckmann measures* introduced by Brion-Procesi [19]. Considering certain Fourier transforms of the DH measures, Baumann-Kamnitzer-Knutson [6] defined an algebra morphism

$$\overline{D} : \mathbb{C}[N] \longrightarrow \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are algebraically independent variables representing the simple roots. One of Baumann-Kamnitzer-Knutson's main results consists in interpreting the image under $\overline{D}$ of an element of the MV basis in terms of certain geometric invariants of the corresponding MV cycle and used such connection as a key step for proving a conjecture of Muthiah [94]. These invariants, called *equivariant multiplicities*, are general tools introduced by Joseph [63] and Rossman [108] and then developped by Brion [17]. Given an algebraic variety X together with a torus T acting on X, we consider the set X^T of T-fixed points in X. For any point $p \in X^T$ and any T-invariant subvariety $Y \subset X$, the equivariant multiplicity of Y at p is a rational function denoted $\epsilon_p^T(Y)$. Brion [17] showed that if p is non-degenerate, then $\epsilon_p^T(Y)$ is always of the form

$$\epsilon_p^T(Y) = \frac{Q_{p,Y}(\beta_1, \ldots, \beta_m)}{\beta_1 \cdots \beta_m}$$

where $\beta_1, \ldots, \beta_m$ are the weights of the action of T on the tangent space of X at p and $Q_{p,Y}$ is some polynomial. These equivariant multiplicities (and in particular the polynomials $Q_{p,Y}$) are difficult to compute in general. The main general property is the following:

$$p \in Y \text{ and } Y \text{ smooth at } p \quad \Rightarrow \quad \epsilon_p^T(Y) = \frac{1}{\beta_1 \cdots \beta_m}. \tag{4.1}$$

Taking X to be the affine Grassmaniann and Y a MV cycle, Baumann-Kamnitzer-Knutson show that the equivariant multiplicity of Y at a certain point of Y coincides with the image under $\overline{D}$ of the MV basis element indexed by Y (see [6, Corollary 10.6]). In this situation, the weights involved in the denominator of the previous equality are always positive roots (seen as linear functions in $\alpha_1, \ldots, \alpha_n$).

In this paper we will mostly focus on the values of $\overline{D}$ on the elements of the dual canonical basis. Indeed, Nakada's colored hook formula can be straightforwardly interpreted as follows: if M is a strongly homogeneous module in $R - gmod$ and w is the fully-commutative element of W associated to M via the construction of Kleshchev-Ram [78], then the evaluation of $\overline{D}$ on the isomorphism class of M is of the form

$$\overline{D}([M]) = \frac{1}{\prod_{\beta \in \Phi_+^w} \beta}. \tag{4.2}$$

This outlines a remarkable similarity between geometric statements requiring certain smoothness conditions of MV cycles and algebraic ones requiring the (strong) homogeneity of certain modules over quiver Hecke algebras. On the other hand, the study of various examples (for instance when $\mathfrak{g}$ is of type A_3 or D_4) suggests the existence of certain coincidences between (prime) strongly homogeneous modules and the determinantial modules categorifying flag minors. This provides a motivation for studying the image under $\overline{D}$ of all flag minors. We propose the following Conjecture, suggesting that $\overline{D}$ surprisingly takes distinguished values similar to (4.1) and (4.2) on all flag minors of $\mathbb{C}[N]$, although the corresponding objects in $R - gmod$ may not be strongly homogeneous. In particular, this also raises the question of the smoothness of the MV cycles corresponding to flag minors.

Conjecture A. *Let $\mathfrak{g}$ be a Lie algebra of finite simply-laced type and let x be a flag minor in $\mathbb{C}[N]$. Then the evaluation of $\overline{D}$ on x is of the form*

$$\overline{D}(x) = \frac{1}{\prod_{\beta \in \Phi_+} \beta^{n_{M,\beta}}}$$

where n_β is a nonnegative integer for every positive root $\beta \in \Phi_+$.

The aim of this paper is to prove the following:

Théorème 10. *Assume $\mathfrak{g}$ is of type $A_n, n \geqslant 1$ or D_4. Then Conjecture A holds. Moreover, for any standard seed $\mathcal{S}^{\mathbf{i}} = ((x_1, \ldots, x_N), Q^{\mathbf{i}})$ of $\mathbb{C}[N]$, the polynomials $P_j := \left(\overline{D}(x_j)\right)^{-1}$ satisfy the following relations:*

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l < j < l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l.$$

The strategy of the proof is the following: we know that the standard seeds are related to each other by certain cluster mutations, corresponding to changes of reduced expressions of w_0. Thus one shall first show that the desired statement is preserved under these cluster mutations, so that it only remains to check it for one particular standard seed.

We consider two standard seeds $\mathcal{S}^{\mathbf{i}}$ and $\mathcal{S}^{\mathbf{i}'}$ related by a cluster mutation in the direction k. We denote by $x_1, \ldots, x_N$ the cluster variables of $\mathcal{S}^{\mathbf{i}}$ and x'_k the new variable produced by the mutation. We assume that $\overline{D}(x_j)$ is of the form $1/P_j$ for every $1 \leqslant j \leqslant N$ and we want to show that $\overline{D}(x'_k)$ is of the form $1/P'_k$. The first main result of this paper consists in exhibiting certain relations between the P_j entirely determined by $\mathbf{i}$ implying that $\overline{D}(x'_k)$ is of the form $1/P'_k$ (where P'_k is a product of

positive roots) and proving that these relations are preserved under mutation, i.e. the polynomials $P_1, \ldots, P_{k-1}, P'_k, P_{k+1}, \ldots, P_N$ satisfy the corresponding relations determined by $\mathbf{i}'$.

For any product of positive roots P and any positive root β, we denote by $(\beta; P)$ the multiplicity of β in P.

Théorème 11. *Let $\mathfrak{g}$ be any simply-laced type simple Lie algebra and let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced expressions of w_0. Assume that the cluster variables $x_1, \ldots, x_N$ of the standard seed $\mathcal{S}^{\mathbf{i}}$ satisfy the following properties:*

(A) For every $1 \leqslant j \leqslant N$, the rational fraction $\overline{D}(x_j)$ is of the form $1/P_j$ where P_j is a product of positive roots.

(B) For every $1 \leqslant j \leqslant N$ one has

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l.$$

(C) For every $j \in J_{ex}$ and every $1 \leqslant i \leqslant N$, one has $(\beta_i; P_j) - (\beta_i; P_{j_+(\mathbf{i})}) \leqslant 1$.

Then the cluster variables of $\mathcal{S}^{\mathbf{i}'}$ satisfy the analogous properties determined by $\mathbf{i}'$.

The Property (B) is rather strong and has interesting consequences, as explained in Remarks 4.5.8 and 4.6.4. Note that the Property (C) is only relevant in types D_n and E_6, E_7, E_8 as in type A_n the polynomials P_j are always square-free. Nonetheless, it is crucial for proving that $\overline{D}(x'_k)$ is of the desired form.

The second main result of this paper is to exhibit one particular standard seed satisfying the conditions required by the previous Theorem for types A_n and D_4. We use Kang-Kashiwara-Kim-Oh's results [69, 72] and more precisely the description of certain determinantial modules in $R - gmod$ in terms of root partitions.

Théorème 12. *Assume $\mathfrak{g}$ is of type $A_n, n \geqslant 1$ or D_4. Let $\mathbf{i}_{nat}$ denote the reduced expression of w_0 corresponding to the natural ordering on the vertices of the Dynkin diagram of $\mathfrak{g}$. Then Properties (A), (B) and (C) hold for the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$.*

As explained above, Theorem 10 follows by combining Theorems 11 and 12. Moreover, we also get that when $\mathfrak{g}$ is of type A_n or D_4, the flag minors of any standard seed of $\mathbb{C}[N]$ satisfy Properties (B) and (C).

This paper is organized as follows. We begin with some reminders on quiver Hecke algebras and their irreducible finite-dimensional representations (Section 4.2.1 and 4.2.2). We also recall the constructions of determinantial modules from the works of Geiss-Leclerc-Schröer [52] and Kang-Kashiwara-Kim-Oh [69] (Section 4.2.3). In Section 4.3, we gather the main facts on the combinatorics of fully-commutative elements of Weyl groups. We also explain (Section 4.3.3) how this combinatorics is related to the representation theory of quiver Hecke algebras via the works of Kleshchev-Ram [78]. Section 4.4 is devoted to the necessary reminders on the theory of Mirković-Vilonen cycles and equivariant multiplicities following [6, 17, 92]. In Section 4.5, we state our main results together with some motivations and explanations about the structure of the proofs. Section 4.6 contains the proof of Theorem 11. Sections 4.7 and 4.8 are respectively devoted to the proofs of Theorem 12 in types A_n and D_4. We conclude in Section 4.9 by discussing several evidences suggesting a cluster-theoretic interpretation of prime strongly homogeneous modules.

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4.2 Quiver Hecke algebras and their representations

We begin with some reminders about the representation theory of quiver Hecke algebras and in particular its applications to monoidal categorifications of cluster algebras following the works of Kang-Kashiwara-Kim-Oh [66, 69, 72].

4.2.1 Quiver Hecke algebras

We fix the notations and recall the main properties of quiver Hecke algebras.

We will always consider a semisimple Lie algebra $\mathfrak{g}$ of finite type, with a fixed labeling $I = \{1, \dots, n\}$ of the set of vertices of the associated Dynkin diagram. We let $\Pi = \{\alpha_i, i \in I\}$ denote the set of simple roots, $Q_+ := \bigoplus_{i \in I} \mathbb{N}\alpha_i$, and $\Phi_+ \subset Q_+$ the set of positive roots. We let $A = (a_{ij})$ denote the Cartan matrix associated to $\mathfrak{g}$ and we consider the symmetric bilinear form $i, j \mapsto i \cdot j$ on $\mathbb{Z}[I]$ defined by $i \cdot j := a_{i,j}$ for every $i, j \in I$. This induces a symmetric bilinear form $(\cdot, \cdot)$ on Q_+ defined by $(\alpha_i, \alpha_j) = i \cdot j$ for any $i, j \in I$. We also let $\mathcal{M}$ denote the set of all finite words over the alphabet I. For any such word $w = (h_1, \dots, h_r)$, the *weight* of w is the element of Q_+ defined as

$$\mathrm{wt}(w) := \sum_{i \in I} \sharp\{k, h_k = i\}\alpha_i \in Q_+.$$

Quiver Hecke algebras are defined as a family $\{R(\beta), \beta \in Q_+\}$ of associative $\mathbf{k}$-algebras indexed by Q_+ (where $\mathbf{k}$ is a fixed algebraically closed field of characteristic different of 2). For every $\beta \in Q_+$, the algebra $R(\beta)$ is generated by three kind of generators: there are polynomial generators $x_1, \dots, x_n$, braiding generators $\tau_1, \dots, \tau_{n-1}$, and idempotents $e(w), w \in \mathrm{Seq}(\beta)$ where $\mathrm{Seq}(\beta)$ is the finite subset of $\mathcal{M}$ given by

$$\mathrm{Seq}(\beta) := \{w \in \mathcal{M} \mid \mathrm{wt}(w) = \beta\}.$$

The idempotent generators commute with the polynomial ones and are orthogonal to each other in the sense that

$$e(w)e(w') = \delta_{w,w'}e(w).$$

It is a crucial point that the algebra $R(\beta)$ carries a natural $\mathbb{Z}$-grading given by

$$\deg e(w) = 0 \quad \deg x_k e(w) = 2 \quad \deg \tau_l e(w) = -h_l \cdot h_{l+1}$$

for every $w = (h_1, \dots, h_r), r \geqslant 1$. Thus one can consider the category $R(\beta) - gmod$ of finite dimensional graded $R(\beta)$-modules. We also define

$$R - gmod := \bigoplus_{\beta} R(\beta) - gmod.$$

The category $R - gmod$ can be endowed with a structure of a monoidal category via a monoidal product denoted $\circ$ and constructed as a parabolic induction. Therefore the Grothendieck group

$K_0(R-gmod)$ has a ring structure. There is also a grading shift functor in $R-gmod$ which yields a $\mathbb{Z}[q^{\pm 1}]$-module structure on $K_0(R-gmod)$. The following results are the main properties of quiver Hecke algebras:

Theorem 4.2.1 (Khovanov-Lauda [76], Rouquier [110]). *There is an isomorphism of $\mathbb{Z}[q^{\pm 1}]$-modules*

$$K_0(R-gmod) \xrightarrow{\simeq} \mathcal{A}_q(\mathfrak{n}).$$

Theorem 4.2.2 (Rouquier [110], Varagnolo-Vasserot [113]). *The above isomorphism induces a bijection between the set of classes of simple objects in $R-gmod$ and the dual canonical basis of $\mathcal{A}_q(\mathfrak{n})$.*

4.2.2 Irreducible finite-dimensional representations

This subsection is devoted to some reminders about Kleshchev-Ram's parametrization of simple finite-dimensional modules via root partitions (or dominant words) using the combinatorics of good Lyndon words. We also recall the notions of graded character as well as the quantum shuffle product formula. These will be useful in Section 4.8.

For a finite-type simple Lie algebra $\mathfrak{g}$, the simple objects in $R-gmod$ have been classified by Kleshchev-Ram [79] in terms of root partitions. First fix an arbitrary total order $<$ on I. It induces a lexicographic order on $\mathcal{M}$ that we still denote $<$. Any module in $R-gmod$ can be decomposed as a direct sum of vector spaces

$$M = \bigoplus_{w \in \mathrm{Seq}(\beta)} e(w) \cdot M.$$

Thus one can consider $\max(M)$ the maximal element w for $<$ such that $e(w) \cdot M \neq 0$. Then one can define the *cuspidal* representations in $R-gmod$ in the following way:

Proposition 4.2.3 ([79, Lemma 6.3]). *For every positive root $\beta \in \Phi_+$, there is a unique simple module S_β in $R(\beta)-gmod$ (up to isomorphism and grading shift) such that*

$$\max(S_\beta) = \min\left(\max(M) \mid M \text{ simple in } R(\beta)-gmod\right).$$

The module S_β is called the *cuspidal representation of weight β*. The map

$$\beta \longmapsto \max(S_\beta)$$

induces a bijection from Φ_+ to a finite subset of $\mathcal{M}$ denoted $\mathcal{GL}$ and whose elements are called *good Lyndon words*. The inverse of this bijection is given by

$$\begin{array}{ccc} \mathcal{GL} & \longrightarrow & \Phi_+ \\ \mathbf{j} & \longmapsto & \mathrm{wt}(\mathbf{j}). \end{array} \tag{4.3}$$

There is an algorithm that allows one to compute inductively the elements of $\mathcal{GL}$ (see [84, Section 4.3]). The main classification result is the following:

Theorem 4.2.4 (Kleshchev-Ram [79]). *There is a bijection between the set of isomorphism classes of simple objects in $R-gmod$ and the set*

$$\mathbf{M} := \{\mathbf{j}_1 \cdots \mathbf{j}_k \mid \mathbf{j}_1, \ldots, \mathbf{j}_k \in \mathcal{GL}, \mathbf{j}_1 \geqslant \cdots \geqslant \mathbf{j}_k\}.$$

Moreover for any $\mu = \mathbf{j}_1 \cdots \mathbf{j}_k \in \mathbf{M}$, the module $L(\mu)$ is given by

$$L(\mu) = hd\left(L(\mathbf{j}_1) \circ \cdots \circ L(\mathbf{j}_k)\right)$$

up to grading shift and one has

$$\max(L(\mu)) = \mu.$$

With the previous notations one has in particular $L(\mathbf{j}) = S_{\mathrm{wt}(\mathbf{j})}$ for every $\mathbf{j} \in \mathcal{GL}$.

This framework allows one to compute products of modules in $R - gmod$ via the *quantum shuffle product formula.* First, recall that the *graded character* of a module M in $R(\beta) - gmod$ is by definition a formal sum of words with coefficients in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$ given by

$$ch_q(M) := \sum_{w \in \mathrm{Seq}(\beta)} \dim_q(e(w) \cdot M) w.$$

Recall also that the *quantum shuffle* of two words w, w' is defined as follows: write $w = (h_1, \ldots, h_r), w' = (h_{r+1}, \ldots, h_{r+s})$. Then

$$w \circ w' := \sum_{\substack{\sigma \in \Sigma_{r+s} \\ \sigma(1) < \cdots < \sigma(r) \\ \sigma(r+1) < \cdots < \sigma(r+s)}} q^{-e(\sigma)} (h_{\sigma^{-1}(1)}, \ldots h_{\sigma^{-1}(r+s)})$$

where $e(\sigma)$ is the integer defined by

$$e(\sigma) := \sum_{\substack{k \leqslant r < l \\ \sigma(k) > \sigma(l)}} h_k \cdot h_l.$$

Then one can extend by linearity the quantum shuffle product $\circ$ to any (finite) formal sum of words. In particular given two modules M, N in $R - gmod$ one can define $ch_q(M) \circ ch_q(N)$ as

$$ch_q(M) \circ ch_q(N) = \sum_{\substack{w, w' \\ e(w) \cdot M \neq 0 \\ e(w') \cdot N \neq 0}} \dim_q(e(w) \cdot M) \dim_q(e(w') \cdot N) \quad (w \circ w').$$

Then one has the quantum shuffle product formula:

Proposition 4.2.5 ([76, Lemma 2.20])**.** *For every pair of objects M, N in $R - gmod$, one has*

$$ch_q(M \circ N) = ch_q(M) \circ ch_q(N)$$

where the symbol $\circ$ on the left hand side refers to the monoidal product in $R - gmod$.

4.2.3 Determinantial modules

In this subsection we make some reminders about determinantial modules in $R - gmod$ following [69], adapting a former construction due to Geiss-Leclerc-Schröer [52]. These constructions are also valid more generally in any $\mathcal{C}_w, w \in W$ but here we will use them only for $R - gmod = \mathcal{C}_{w_0}$ where w_0 stands for the longest element of the Weyl group W associated to $\mathfrak{g}$.

Geiss-Leclerc-Schröer [52] constructed a family of distinguished seeds for the cluster structures of $\mathbb{C}[N]$. Their construction actually involves the quantum cluster structures on the quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ but here we will always work in the classical setting. Geiss-Leclerc-Schröer consider certain elements $D(u\lambda, v\lambda)$ of $\mathbb{C}[N]$ called *(quantum)unipotent minors* parametrized by triples $(\lambda, u, v) \in P_+ \times W \times W$. These unipotent minors always belong to the dual canonical basis of $\mathbb{C}[N]$ when they are not zero (see for instance [69, Lemma 9.1.1]). These elements satisfy certain remarkable relations called *determinantial identities* (see [52, Proposition 5.4]). Note that such identities already appear in the work of Fomin-Zelevinsky [40]. A particularly interesting family of determinantial identities are obtained by considering certain special unipotent minors, obtained

by taking λ to be a fundamental weight $\omega_i, i \in I$ together with reduced Weyl groups elements of the form $u = s_{i_1} \cdots s_{i_k}$ and $v = s_{j_1} \cdots s_{j_l}$ such that $i_k = j_l = i$ and $i_p = j_p$ if $p \leqslant \min(k, l)$. Geiss-Leclerc-Schröer [52] prove that the determinantial identities relating these unipotent minors can be interpreted as exchange relations associated to cluster mutations in $\mathbb{C}[N]$.

Geiss-Leclerc-Schröer construct a family of seeds $\{\mathcal{S}^{\mathbf{i}}, \mathbf{i} \in \mathrm{Red}(w_0)\}$ in $\mathbb{C}[N]$, called *standard seeds*, indexed by the reduced expressions of w_0. For each $\mathbf{i} = (i_1, \ldots, i_N) \in \mathrm{Red}(w_0)$, the quiver of the seed $\mathcal{S}^{\mathbf{i}}$ can be constructed from $\mathbf{i}$ as explained in Section 4.6 below. The cluster variables of the seed $\mathcal{S}^{\mathbf{i}}$ are the unipotent minors

$$D(s_{i_1} \cdots s_{i_k} \omega_{i_k}, \omega_{i_k}) \quad 1 \leqslant k \leqslant N.$$

These minors are called *(quantum) flag minors*. The determinantial identities between flag minors is crucial in the study of the cluster structure of $\mathbb{C}[N]$. Note that the rank of this cluster structure (i.e. the number of cluster variables in each seed of $\mathbb{C}[N]$) is equal to the length of w_0 or equivalently to the number of positive roots of $\mathfrak{g}$. The results of [52] mainly rely on additive categorification techniques using representations of the preprojective algebra. Kang-Kashiwara-Kim-Oh [69] adapted this construction to the monoidal setting by lifting the non zero unipotent minors of $\mathbb{C}[N]$ to $R - gmod$ via the isomorphism of Theorem 4.2.1. The modules obtained this way are unique (up to isomorphism and grading shift) and are called *determinantial modules*. As the non zero unipotent minors are always elements of the dual canonical basis, it follows from Theorem 4.2.2 that the determinantial modules are simple. They are also known to be *real* in the sense of Hernandez-Lelcerc [56] (see [69, Lemma 10.2.2]).

In particular, for each $\mathbf{i} = (i_1, \ldots, i_N) \in \mathrm{Red}(w_0)$ and for every $1 \leqslant k \leqslant N$, we denote by $M_k^{\mathbf{i}}$ the determinantial module defined by

$$ch_q(M_k^{\mathbf{i}}) = D(s_{i_1} \cdots s_{i_k} \omega_{i_k}, \omega_{i_k}).$$

One of the main results due to Kang-Kashiwara-Kim-Oh ([69, Theorem 11.2.2]) is to prove that the datum the modules $M_1^{\mathbf{i}}, \ldots M_N^{\mathbf{i}}$ together with the quiver $Q^{\mathbf{i}}$ forms an monoidal seed admitting successive monoidal mutations in any exchangeable directions (in the sense of [69, Definitions 6.2.1 and 6.2.3]). This allows them to prove that $R - gmod$ is a *monoidal categorification* in the sense of Hernandez-Leclerc [56] of the cluster structure of $\mathbb{C}[N]$ (as well as analogous statements for each of the unipotent cells $\mathbb{C}[N(w)], w \in W$).

$$\begin{array}{ccccl} R - gmod & \ni & (M_1^{\mathbf{i}}, \ldots M_N^{\mathbf{i}}) & & \text{determinantial modules associated to } \mathbf{i} \\ \downarrow & & \downarrow & & \\ \mathbb{C}[N] & \ni & (x_1^{\mathbf{i}}, \ldots, x_N^{\mathbf{i}}) & & \text{flag minors of the seed } \mathcal{S}^{\mathbf{i}}. \end{array}$$

By [69, Proposition 10.2.4], the determinantial modules can be constructed inductively by successive applications of the induction functors categorifying the crystal structure of the dual canonical basis of $\mathbb{C}[N]$. In [24], we used Kashiwara-Kim's results [72] to provide a combinatorial description of the determinantial modules appearing in the seeds

$$\mathcal{S}^{\mathbf{i}}, \mathbf{i} \text{ corresponding to a total ordering on simple roots.}$$

This description uses Kleshchev-Ram's parametrization via root partitions recalled in the previous subsection. Fix a labeling I of the set of simple roots of $\mathfrak{g}$ and let $<$ be any arbitrary order on I. We still denote $<$ the induced lexicographic order $<$ on the set $\mathcal{GL}$ of Good Lyndon words (see Section 4.2.1) . It yields an order on the set of positive roots Φ_+ via the bijection (4.3). By the results of Rosso [109] this order turns out to be a convex order (see [84, Proposition 26]). Let $\mathbf{i}_<$

denote the corresponding reduced expression of w_0 and consider the seed $\mathcal{S}^{\mathbf{i}_<}$ in $\mathbb{C}[N]$. Note that different orderings on I may give the same seed. Moreover certain of the seeds $\mathcal{S}^{\mathbf{i}}$ may not come from any ordering $<$.

Example 4.2.6. Let $\mathfrak{g}$ be of type type A_3. Then there are 14 seeds in $\mathbb{C}[N]$, 6 of them being of the form $\mathcal{S}^{\mathbf{i}}$. Among these, 5 of them are of the form $\mathcal{S}^{\mathbf{i}_<}$.

We write the reduced expression $\mathbf{i}_<$ of w_0 as $\mathbf{i}_< = (i_1, \ldots, i_N)$ (with $N := l(w_0)$). The determinantial modules of the seed $\mathcal{S}^{\mathbf{i}_<}$ can be described in terms of root partitions as follows

Theorem 4.2.7 ([24, Theorem 3.7])**.** *Let $(x_1, \ldots, x_N)$ denote the cluster variables of the seed $\mathcal{S}^{\mathbf{i}_<}$ and let μ_k denote the dominant word such that $x_k = [L(\mu_k)]$ for every $k \in J$. Write the canonical factorization of μ_k as*

$$\mu_k = (\mathbf{i}_N)^{c_N} \cdots (\mathbf{i}_1)^{c_1}.$$

Then the tuple $(c_1, \ldots, c_N)$ is given by

$$c_j = \begin{cases} 1 & \text{if } j \leqslant k \text{ and } i_j = i_k \\ 0 & \text{otherwise.} \end{cases}$$

4.3 Homogeneous modules over quiver Hecke algebras

This section is devoted to some reminders about Kleshchev-Ram's construction of homogeneous simple modules over quiver Hecke algebras of finite simply-laced type. These constitute a finite class of simple objects in $R - gmod$ characterized by their being concentrated in a single degree (which explains the terminology *homogeneous*). They are classified by the rich combinatorics of fully-commutative elements of Weyl groups. We begin by recalling several known combinatorial properties of these elements after the works of Peterson-Proctor [104, 105], Stembridge [111] and Nakada [95].

4.3.1 Combinatorics of fully-commutative elements of Weyl groups

We recall known combinatorial facts about several remarkable classes of elements of Weyl groups, namely the fully-commutative, minuscule and dominant minuscule elements. Throughout this section W will stand for the Weyl group associated to a Lie algebra $\mathfrak{g}$ of finite Dynkin type (but not necessarily simply-laced). We let $\{s_i, i \in I\}$ denote the set of the simple reflections of W, where I is a finite index set. We also let P (resp. $P^\vee$) denote the weight lattice (resp. the coweight lattice) of $\mathfrak{g}$. We denote by $\alpha_i, i \in I$ (resp. $\alpha_i^\vee, i \in I$) the simple roots (resp. the simple coroots) and we let $\langle \cdot, \cdot \rangle$ denote the bilinear form on $P^\vee \times P$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = i \cdot j$ for every $i, j \in I$.

The following definition is due to Stembridge [111] generalizing definitions introduced by Peterson-Proctor relying on previous works by Proctor [104, 105].

Definition 4.3.1. *An element $w \in W$ is said to be fully-commutative if for any pair $(i, j) \in I$ such that $i \cdot j \neq 0$, there is no reduced expression of w containing a subword of the form $(i, j, i, j, \ldots)$ of length m, where m is the order in $s_i s_j$ in W.*

Note that if an element $w \in W$ admits a reduced expression satisfying this property, then it is also the case for every reduced expression of w. In the case of a simply-laced Weyl group, the notion of fully-commutative is equivalent to requiring that no reduced expression of w should contain a subword of braid form (i, j, i) with $i \cdot j = -1$. In other words, all the reduced expressions of w can be recovered from one given reduced expression by only performing changes of the form $(\ldots, k, l, \ldots) \to (\ldots, l, k, \ldots)$ for $k, l \in I$ such that $k \cdot l = 0$.

One now defines minuscule and dominant minuscule elements of W.

Definition 4.3.2. *An element $w \in W$ is said to by minuscule (resp. dominant minuscule) if there exists an integral weight $\lambda \in P$ (resp. a dominant integral weight $\lambda \in P^+$) such that for any reduced expression $(i_1, \dots, i_N)$ of w, one has*

$$\langle \alpha_{i_k}^\vee, s_{i_{k+1}} \cdots s_{i_N} \lambda \rangle = 1$$

for every $1 \leqslant k \leqslant N$.

As before one can replace "for any reduced expression" by "there exists a reduced expression" in this definition. Stembridge [111] gave the following very useful classification of minuscule and dominant minuscule elements of W. We use the following notation from [52]: if $\mathbf{i} = (i_1, \dots, i_N)$ is a reduced expression of an element $w \in W$, then for every $1 \leqslant k \leqslant N$ we set

$$k_+ := \min\left(\{k < l \leqslant N \mid i_l = i_k\} \cup \{N+1\}\right).$$

In other words k_+ is the position in $\mathbf{i}$ of the next occurrence of the letter i_k and one has $k_+ = N+1$ if and only if k is the position of the last occurrence of the letter i_k in $\mathbf{i}$.

Theorem 4.3.3 ([111, Propositions 2.3, 2.5])**.** *Let $w \in W$ and fix a reduced expression $(i_1, \dots, i_N)$ of w.*

- *We say that $w \in W$ is minuscule if and only if for every $1 \leqslant k \leqslant N$, one has*

$$k_+ \leqslant N \quad \Rightarrow \quad \sum_{k<l<k_+} i_k \cdot i_l = -2.$$

- *We say that $w \in W$ is dominant minuscule if and only if it is minuscule and in addition, one has for every $1 \leqslant k \leqslant N$,*

$$k_+ = N+1 \quad \Rightarrow \quad \sum_{l>k} i_k \cdot i_l \geqslant -1.$$

Remark 4.3.4.

1. It is proven by Stembridge [111] that minuscule (and a fortiori dominant minuscule) elements are fully-commutative.
2. If $\mathfrak{g}$ is of type A then one can show that every fully-commutative element is in fact minuscule so that these two notions coincide. But in other types, the notions of fully-commutative, minuscule and dominant minuscule elements are distinct.

Example 4.3.5. In type A_3, the elements $s_1 s_2 s_3$ and $s_2 s_1 s_3 s_2$ are dominant minuscule. The element $s_2 s_3 s_1$ is minuscule but not dominant minuscule. The element $s_3 s_2 s_1 s_2$ is not fully-commutative.

In type D_4 with 3 being the trivalent node of the Dynkin diagram, the element $s_3 s_1 s_2 s_4 s_3$ is fully-commutative but is not minuscule (and a fortiori not dominant minuscule).

We will respectively denote by $\mathcal{FC}, \mathcal{M}in, \mathcal{M}in^+$ the sets of fully-commutative, minuscule, and dominant minuscule elements of W.

4.3.2 Colored hook formulas

In this paragraph we present the most remarkable properties of dominant minuscule elements of Weyl groups, namely the Peterson-Proctor hook formula as well a generalized version established by Nakada [95] called *colored* Peterson-Proctor hook formula.

We still consider the Weyl group W of a finite type Lie algebra $\mathfrak{g}$ and we fix w a dominant minuscule element of W. We also let $N := l(w)$ denote the length of w and $\mathrm{Red}(w)$ the set of all reduced expressions of w. Peterson-Proctor proved a formula for the cardinality of $\mathrm{Red}(w)$. As in the previous parts of this thesis we let

$$\Phi_+^w := \Phi_+ \cap w\Phi_-$$

denote the set of positive roots associated to w. This set is of cardinality $l(w)$. The elements of Φ_+^w are given in the following way: choose a reduced expression $(i_1, \dots, i_N)$ of w. Then $\Phi_+^w = \{\beta_1, \dots, \beta_N\}$ with

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$$

for every $1 \leqslant k \leqslant N$. Recall that this set Φ_+^w does not depend on any choice of reduced expression of w and that $\Phi_+^w \neq \Phi_+^{w'}$ if $w \neq w'$. The Peterson-Proctor hook formula can be written as follows:

Proposition 4.3.6 (Peterson-Proctor)**.** *Let w be a dominant minuscule element of W. Then the number of reduced expressions of w is given by*

$$\sharp Red(w) = \frac{l(w)!}{\prod_{\beta \in \Phi_+^w} ht(\beta)}. \tag{4.4}$$

Nakada [95] proved a generalisation of this formula. Recall the following notations from [24, Section 6]: we consider the simple roots $\alpha_1, \dots, \alpha_n$ of $\mathfrak{g}$ as independent formal variables and for every positive root $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n \in \Phi_+$ with $a_1, \dots, a_n \in \mathbb{Z}_{\geqslant 0}$, we consider the rational function

$$\frac{1}{\beta} := \frac{1}{a_1\alpha_1 + \cdots + a_n\alpha_n} \in \mathbb{R}(\alpha_1, \dots, \alpha_n).$$

Nakada's colored hook formula can be written as follows:

Theorem 4.3.7 ([95, Corollary 7.2])**.** *Let w be a dominant minuscule element of W. Then the following equality holds in $\mathbb{R}(\alpha_1, \dots, \alpha_n)$:*

$$\prod_{\beta \in \Phi_+^w} \frac{1}{\beta} = \sum_{(i_1, \dots, i_N) \in Red(w)} \frac{1}{\alpha_{i_1}} \frac{1}{\alpha_{i_1} + \alpha_{i_2}} \cdots \frac{1}{\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_N}} \tag{4.5}$$

Specializing all the variables α_i to 1 we recover the Peterson-Proctor hook formula (4.4).

4.3.3 Homogeneous representations in $R - gmod$

We now recall a construction due to Kleshchev-Ram [78] of certain distinguished irreducible finite dimensional representations over quiver Hecke algebras. We assume that $\mathfrak{g}$ is of finite simply-laced type. Consider the category $R - gmod$ of graded finite dimensional modules over the quiver Hecke algebras associated to $\mathfrak{g}$. One of Kleshchev-Ram's motivations was the study of cuspidal modules in $R - gmod$, i.e. certain simple modules playing the role of fundamental representations in Lie theory. This led them to introduce a larger (but still finite) class of simple objects in $R - gmod$, called homogeneous modules.

Definition 4.3.8 (Kleshchev-Ram)**.** *A module in $R - gmod$ is called homogeneous if it is concentrated in a single degree with respect to the natural grading of quiver Hecke algebras.*

We let $\mathcal{H}om$ denote the set of simple homogeneous modules in $R - gmod$ up to isomorphism and grading shift. The following statement shows that the elements of $\mathcal{H}om$ are parametrized in a natural way by fully-commutative elements of the Weyl group of $\mathfrak{g}$.

Theorem 4.3.9 ([78, Theorem 3.6]). *There is a bijection*

$$\begin{array}{ccc} \mathcal{FC} & \longrightarrow & \mathcal{H}om \\ w & \longmapsto & S(w) \end{array}$$

between the set of fully-commutative elements of W and the set of isomorphism classes of simple homogeneous modules in $R-gmod$. The homogeneous module $S(w)$ admits the following decomposition into weight subspaces:

$$S(w) = \bigoplus_{(i_1,\dots,i_N)\in Red(w)} \mathbb{C}e(i_1,\dots,i_N).$$

In other words the weight spaces of $S(w)$ are all one-dimensional and they are in bijection with the set of reduced expressions of w.

The image of $\mathcal{M}in^+$ via the above bijection is a subfamily of homogeneous modules called strongly homogeneous modules.

Remark 4.3.10. It follows from the previous Theorem together with Proposition 4.3.6 that if w is dominant minuscule, then the strongly homogeneous module $S(w)$ has dimension

$$\dim S(w) = \frac{l(w)!}{\prod_{\beta\in\Phi_+^w} \mathrm{ht}(\beta)}.$$

Remark 4.3.11. In type A_n, the cuspidal modules for arbitrary orderings on Φ_+ are always homogeneous. In fact, if M is a simple module of multiplicity-free weight, i.e. M is a simple $R(\beta)$-module with $\beta \in Q_+$ of the form $\epsilon_1\alpha_1 + \cdots + \epsilon_n\alpha_n$ with $\epsilon_1,\dots,\epsilon_n \in \{0,1\}^n$, then M is always homogeneous. In the particular case of the natural ordering on the vertices of the type A_n Dynkin diagram, i.e. $1 < 2 < \cdots < n$, the cuspidal modules in $R-gmod$ turn out to be strongly homogeneous.

In type D_n, the cuspidal modules for the natural ordering are always homogeneous (see [79, Section 8.2]).

4.4 Mirković-Vilonen cycles and equivariant multiplicities

Throughout this section G denotes a simple simply-connected group, N the unipotent radical of a Borel subgroup B of G, and T a maximal torus in B. We let $\mathfrak{g}, \mathfrak{n}, \mathfrak{b}, \mathfrak{t}$ denote their respective Lie algebras. We fix a labeling $I = \{1,\dots,n\}$ of the vertices of the Dynkin diagram of $\mathfrak{g}$ and we let $\alpha_1,\dots,\alpha_n$ denote the corresponding simple roots. We also let P denote the weight lattice and W the Weyl group of $\mathfrak{g}$.

4.4.1 Geometric Satake correspondence and MV basis

Here we fix notations and recall the main features of the geometric Satake correspondence following [92, 6]. We also refer to [114] for more details about the geometry of the affine Grassmannian and the geometric Satake correspondence. Mirković-Vilonen [92] discovered an intriguing connection between certain categories of perverse sheaves on the affine Grassmannian Gr_G associated to a complex reductive group G on the one hand and categories of finite-dimensional representations of the Langlands dual $G^\vee$ of G on the other hand. They exhibit a functor relating these two categories and prove that it satisfies several remarkable properties. In particular, the weight spaces of irreducible representations of $G^\vee$ can be interpreted as the cohomology spaces of certain subvarieties of Gr_G whose irreducible components are called *Mirković-Vilonen cycles*. One can use these to define an interesting basis of the algebra $\mathbb{C}[N]$ of functions on the unipotent radical N of G called

the *MV basis*. The works of Kamnitzer [65], Anderson [2] and Baumann-Kamnitzer [5] show that this basis carries a natural crystal structure and has a combinatorial parametrization in terms of *MV polytopes*.

We consider the algebraically closed field $\mathbb{C}$ and we set

$$\mathcal{O} := \mathbb{C}[[t]] \quad \text{and} \quad \mathcal{K} := \mathbb{C}((t)).$$

We denote $G^\vee$ the Langlands dual of G. The affine Grassmannian Gr_G can be defined as

$$Gr_G := G(\mathcal{K})/G(\mathcal{O}).$$

There is a natural action of $T(\mathbb{C})$ on Gr_G whose locus of fixed points is given by a collection $\{L_\mu, \mu \in P\}$ of points in Gr_G indexed by the weight lattice of G. There is also a $G(\mathcal{O})$-action on Gr_G and for a dominant weight $\lambda \in P^+$, we let Gr_G^λ denote the orbit of L_λ under this action. The affine Grassmannian can be decomposed as

$$Gr_G = \bigsqcup_{\lambda \in P^+} Gr_G^\lambda.$$

Moreover, for every $\lambda \in P^+$ one has

$$\overline{Gr_G^\lambda} = \bigcup_{\substack{\mu \in P^+ \\ \mu \leqslant \lambda}} Gr_G^\mu.$$

where $\leqslant$ is the natural partial ordering on P defined by $\mu \leqslant \lambda \Leftrightarrow \lambda - \mu \in Q_+$ (recall the notation Q_+ from Section 4.2.1). Let us now briefly recall the definition of Mirković-Vilonen cycles. The regular dominant weight ρ induces a homomorphism $\mathbb{C}^\times \to T(\mathbb{C})$ and thus yields a $\mathbb{C}^\times$-action on Gr_G. The points $L_\mu, \mu \in P$ are exactly the fixed points of this $\mathbb{C}^\times$-action. Denoting respectively S^μ and S_-^μ the attractive and repulsive set of L_μ, we have the Bialynicki-Birula decomposition

$$Gr_G = \bigcup_{\mu \in P} S^\mu = \bigcup_{\mu \in P} S_-^\mu.$$

One can show that the sets S^μ and S_-^μ are also the orbits of L_μ under the respective actions of $N_+(\mathcal{K})$ and $N_-(\mathcal{K})$ on Gr_G. For every $\lambda \in P_+$ and $\mu \in P$, a MV-cycle of type λ and weight μ is by definition an irreducible component of the closed subvariety of Gr_G given by

$$\overline{Gr_G^\lambda \cap S_-^\mu}.$$

One denotes $\mathcal{Z}(\lambda)_\mu$ the set of all MV cycles of type λ and weight μ and

$$\mathcal{Z}(\lambda) := \bigcup_{\mu \in P} \mathcal{Z}(\lambda)_\mu.$$

It is proved by Mirković-Vilonen that the intersection $Gr_G^\lambda \cap S_-^\mu$ has pure dimension $\rho^\vee(\lambda - \mu)$. Hence a MV cycle of type λ and weight μ can also be defined as an irreducible component of dimension $\rho^\vee(\lambda - \mu)$ of $\overline{Gr_G^\lambda \cap S_-^\mu}$.

The geometric Satake correspondence [92] implies that the MV cycles carry a lot of information about the representation theory of $G^\vee$. For instance, the set of fundamental classes in $H_\bullet(\overline{Gr_G^\lambda \cap S_-^\mu})$ of all MV cycles of type λ and weight μ forms a basis of the subspace $V(\lambda)_\mu$ of weight μ in the irreducible representation $V(\lambda)$ of $G^\vee$ of highest weight λ. In particular one has

$$\dim_{\mathbb{C}} V(\lambda)_\mu = \sharp \mathcal{Z}(\lambda)_\mu.$$

Gathering these bases together we get that the set

$$\{[Z] \mid Z \in \mathcal{Z}(\lambda)\}$$

forms a basis of $V(\lambda)$ called the (upper) MV basis of $V(\lambda)$. Then there is a way to glue these bases for all $\lambda \in P^+$ to get a basis of $\mathbb{C}[N]$ called the MV basis of $\mathbb{C}[N]$ (see [6, Section 6.1] for more details). This basis is denoted $\{b_Z, Z \in \mathcal{Z}(\infty)\}$ where the parametrizing set $\mathcal{Z}(\infty)$ is given by certain MV cycles called *stable MV cycles*, whose weights belong to $-Q_+$.

4.4.2 Equivariant multiplicities of MV cycles

In this subsection we recall several notions introduced by Baumann-Kamnitzer-Knutson [6]. One of their main motivations was Muthiah's conjecture [94] stating the W-equivariance of a certain map $V(\lambda) \longrightarrow \mathbb{C}(\alpha_1, \ldots, \alpha_n)$ where $V(\lambda)$ is the irreducible representation of highest weight λ of the Langlands dual of G. This map is defined using geometric tools called *equivariant multiplicities* developped by Brion [17] relying on former constructions due to Joseph [63] and Rossmann [108]. The proof of [6] crucially involves the geometric Satake correspondence together with a formula for Duistermaat-Heckmann measures proved by Knutson [80]. Here we will mostly focus on equivariant multiplicities of elements of various bases of $\mathbb{C}[N]$ such as the Mirković-Vilonen basis or the dual canonical basis.

The notion of *equivariant multiplicity* of a closed projective scheme has been introduced by Brion [17]. Given such a scheme X together with an action of a torus T on X, we let X^T denote the set of fixed points of this action and $H_\bullet^T(X)$ denote the T-equivariant homology of X. It follows from Brion's results [17] that the set of homology classes of the points in the fixed locus X^T actually forms a basis of $H_\bullet^T(X)$. Therefore one can decompose the class of X on this basis as

$$[X] = \sum_{p \in X^T} \epsilon_p^T(X)[\{p\}].$$

The coefficient $\epsilon_p^T(X)$ is a rational function in $\alpha_1, \ldots, \alpha_n$ called the equivariant multiplicity of X at p. Let us state an important property of these equivariant multiplicities, that can be obtained as a consequence of [17, Theorem 4.2].

Proposition 4.4.1 ([17, Theorem 4.2])**.** *Let $p \in X^T$ and let $\beta_1, \ldots, \beta_m$ denote the weights of the action of T on the tangent space of X at p. Assume $p \in X^T$ is non-degenerate i.e. the weights $\beta_1, \ldots, \beta_m$ are non zero. Then for any closed T-invariant subvariety $Y \subset X$ containing p, one has*

$$Y \text{ is smooth at } p \quad \Rightarrow \quad \epsilon_p^T(Y) = \frac{1}{\beta_{i_1} \cdots \beta_{i_r}}$$

where $r := \dim(Y)$ and $\beta_{i_1}, \ldots, \beta_{i_r}$ denote the weights of the action of T on $T_xY \subset T_xX$.

Remark 4.4.2.

1. Note that by [17, Theorem 4.2 (iii)] this implication is actually an equivalence in the special case where Y is X itself.
2. If Y does not contain p, then $\epsilon_p^T(Y) = 0$ (see [17, Theorem 4.2 (i)]).
3. Brion [17] also shows that this notion of equivariant multiplicity actually coincides with a definition due to Rossmann [108]. More precisely, he proves that when X is smooth at the point p, Rossmann's equivariant multiplicity of Y at p is equal to $(\beta_1 \cdots \beta_m) \cdot \epsilon_p^T(Y)$ (see [17, Theorem 4.5]).

Baumann-Kamnitzer-Knutson [6] used this notion of equivariant multiplicity in the study of the MV basis of $\mathbb{C}[N]$ via Duistermaat-Heckmann measures. These measures were already known to be supported on the MV polytopes. One of the main results of [6] is to provide a formula allowing to relate these measures to the equivariant multiplicities of MV cycles. More precisely, with the notations of Proposition 4.4.1, we consider $X := Gr_G$ the affine Grassmaniann associated to a simple algebraic reductive group G, together with the action of the torus $T(\mathbb{C})$. As recalled in the previous section, the set of fixed points of this action is $\{L_\mu, \mu \in P\}$. For each $\mu \in P$, the point L_μ is known to belong to any MV cycle of weight μ. These MV cycles are closed irreducible subvarieties of Gr_G and are invariant under the action of T. Thus they will play the role of Y in Proposition 4.4.1.

Let us describe the formulas proved in [6]. It is known (see for instance [49, 52]) that the algebra $\mathbb{C}[N]$ can be identified with the dual (as a Hopf algebra) of $U(\mathfrak{n})$. We will denote by $(\cdot,\cdot)$ this duality. Choose a root vector $e_i \in \mathfrak{n}$ of weight α_i for each $i \in I$. For any $N \geqslant 1$ and any N-tuple $\mathbf{i} = (i_1, \dots, i_N)$ of elements of I, we set $e_{\mathbf{i}} := e_{i_1} \cdots e_{i_N} \in U(\mathfrak{n})$. We also define the following rational fraction in $\alpha_1, \dots, \alpha_n$ following [6]:

$$\overline{D}_{\mathbf{i}} := \frac{1}{\alpha_{i_1}(\alpha_{i_1} + \alpha_{i_2}) \cdots (\alpha_{i_1} + \cdots \alpha_{i_N})}.$$

Then one defines the following map:

$$\begin{array}{rccc} \overline{D}: & \mathbb{C}[N] & \longrightarrow & \mathbb{C}(\alpha_1, \dots, \alpha_n) \\ & f & \longmapsto & \sum_{\mathbf{i}} \overline{D}_{\mathbf{i}}(f, e_{\mathbf{i}}). \end{array} \tag{4.6}$$

Note that this sum is always finite as $U(\mathfrak{n})$ acts locally nilpotently on $\mathbb{C}[N]$. One of the main results of [6] is that the evaluation of $\overline{D}$ on an element b_Z of the Mirković-Vilonen basis can be related to a certain equivariant multiplicity of the MV cycle Z.

Theorem 4.4.3 ([6, Lemma 8.3, Corollary 10.6]). *1. The map $\overline{D}$ is an algebra morphism.*

2. For any $\mu \in -Q_+$ and any stable MV cycle Z of weight μ, one has

$$\overline{D}(b_Z) = \epsilon^T_{L_\mu}(Z).$$

Combining Theorem 4.4.3 with Brion's Proposition 4.4.1, we get:

Corollary 4.4.4. *Let Z be a stable MV cycle of weight $-\nu, \nu \in Q_+$. Assume Z is smooth at the point $L_{-\nu}$. Then one has*

$$\overline{D}(b_Z) = \frac{1}{\beta_{i_1} \cdots \beta_{i_r}}$$

where $\beta_{i_1}, \dots, \beta_{i_r} \in \Phi_+$ are the weights of the action of $T(\mathbb{C})$ on the tangent space of Z at $L_{-\nu}$.

Remark 4.4.5. Baumann-Kamnitzer-Knutson prove that the map $\overline{D}$ has the following geometric counterpart ([6, Proposition 8.4]). For every regular element x in $\mathfrak{t}$, the group N acts simply transitively on the subset $x + \mathfrak{n}$ of $\mathfrak{g}$. Hence one can consider the unique $n_x \in N$ such that $Ad_{n_x}(x) = x + e$. Then the algebra morphism $\overline{D}$ is dual to the morphism of varieties given by

$$\begin{array}{ccc} \mathfrak{t}^{reg} & \longrightarrow & N \\ x & \longmapsto & n_x. \end{array}$$

Remark 4.4.6. The morphism $\overline{D}$ provides a very useful tool to compare various bases of $\mathbb{C}[N]$. For instance Dranowski-Kamnitzer-Morton-Ferguson [6] show that the MV basis and the dual semicanonical basis of $\mathbb{C}[N]$ are not the same by exhibiting elements of these bases satisfying some compatibility condition (see [6, Definition 12.1]) but where $\overline{D}$ nonetheless takes different values.

4.5 Equivariant multiplicities and determinantial modules in $R-gmod$

This section contains the statements of the main results of this paper. We consider the category $R-gmod$ associated to a simply-laced type Lie algebra $\mathfrak{g}$. We begin by proving a formula for the equivariant multiplicities of strongly homogeneous modules in this category using Nakada's colored hook formula [95]. We explain why this provides a natural motivation for the study of equivariant multiplicities of the determinantial modules categorifying the flag minors in $\mathbb{C}[N]$.

4.5.1 Equivariant multiplicities and strongly homogeneous modules

In this subsection we outline a remarkable property of strongly homogeneous modules in the sense of Kleshchev-Ram [78]: the image of their isomorphism classes under the morphism $\overline{D}$ takes a distinguished form, that can be viewed as a generalized version of the Peterson-Proctor hook formula. This property will be useful in Sections 4.7 and 4.8.

We consider a Lie algebra $\mathfrak{g}$ of finite simply-laced type and we let W denote the corresponding Weyl group. Consider an element of the dual canonical basis of $\mathbb{C}[N]$. It can be written as the isomorphism class of a simple object M in $R-gmod$. Decompose M as the (finite) direct sum of its weight subspaces:

$$M = \bigoplus_{w} e(w) \cdot M.$$

Then it follows from the definition of $\overline{D}$ (see (4.6)) that its evaluation on $[M]$ is given by

$$\overline{D}([M]) = \sum_{w=(i_1,\dots,i_r)} \dim(e(w)\cdot M) \frac{1}{\alpha_{i_1}(\alpha_{i_1}+\alpha_{i_2})\cdots(\alpha_{i_1}+\cdots+\alpha_{i_N})}. \tag{4.7}$$

In what follows, we will be using this expression to compute the images under $\overline{D}$ of elements of the dual canonical basis. In particular, we can now show that $\overline{D}$ takes remarkable values on the classes of strongly homogeneous modules (see Section 4.3.3).

Proposition 4.5.1. *Let $w \in \mathcal{M}in^+ \subset W$ be a dominant minuscule element of W and let $S(w)$ denote the corresponding strongly homogeneous module in $R-gmod$. Then one has*

$$\overline{D}([S(w)]) = \prod_{\beta\in\Phi_+^w} \frac{1}{\beta}.$$

Proof. By Theorem 4.3.9, all the weight subspaces of $S(w)$ are one-dimensional and are in bijection with the reduced expressions of w. Hence it follows from Equation (4.7) that

$$\overline{D}([S(w)]) = \sum_{(i_1,\dots,i_N)\in\mathrm{Red}(w)} \frac{1}{\alpha_{i_1}} \frac{1}{\alpha_{i_1}+\alpha_{i_2}} \cdots \frac{1}{\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_N}}.$$

This is exactly Nakada's colored hook formula (4.5) and thus Theorem 4.3.7 implies

$$\overline{D}([S(w)]) = \prod_{\beta\in\Phi_+^w} \frac{1}{\beta}.$$

□

The following example shows that this property fails if one considers simple modules (even homogeneous ones) that are not strongly homogeneous.

Example 4.5.2. Consider for instance the simple module $L(231)$ in type A_3 (with the natural ordering on good Lyndon words). This module is homogeneous and corresponds to $s_2s_3s_1 \in W$ via the bijection of Theorem 4.3.9. This element is minuscule but not dominant minuscule. It has two reduced expressions, namely $(2,3,1)$ and $(2,1,3)$ and thus one has

$$\begin{aligned}\overline{D}([S(s_2s_3s_1)]) &= \frac{1}{\alpha_2(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)} + \frac{1}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)} \\ &= \frac{\alpha_1+2\alpha_2+\alpha_3}{\alpha_2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}.\end{aligned}$$

The main motivations for the present work come from the observations that there seems to be an intimate connection between the set of strongly homogeneous modules $S(w), w \in \mathcal{M}in^+$ and the set determinantial modules categorifying the flag minors of $\mathbb{C}[N]$ (see Section 4.2.3). We will go back to this in the last Section of this paper.

Example 4.5.3. Let us detail the type A_3 case as an example. There are 6 positive roots and the cluster algebra $C[N]$ has 14 seeds. There are $3! = 6$ total orderings on $I = \{1,2,3\}$. We can compute the determinantial modules of the corresponding standard seeds using Theorem 4.2.7. For instance, the choice of $1 < 2 < 3$ yields the following ordering on Φ_+:

$$\alpha_1 < \alpha_1+\alpha_2 < \alpha_1+\alpha_2+\alpha_3 < \alpha_2 < \alpha_2+\alpha_3 < \alpha_3.$$

The corresponding reduced expression of w_0 is $\mathbf{i} = (1,2,3,1,2,1)$ and the determinantial modules of the seed $\mathcal{S}^{\mathbf{i}}$ are given by

$$L(1) \qquad L(12) \qquad L(123) \qquad L(21) \qquad L(2312) \qquad L(321).$$

Similarly one can repeat this procedure for the other orderings on I. We obtain five distinct standard seeds. It turns out that this is enough to get all the flag minors. We can observe that the determinantial modules obtained this way are all strongly homogeneous and correspond to the following Weyl group elements

$$s_1, s_2, s_3, s_1s_2, s_2s_3, s_1s_2s_3, s_3s_1s_2, s_3s_2s_1, s_2s_3s_1s_2.$$

It is straightforward to check that the homogeneous modules $S(w)$ for w in this list are exactly the prime strongly homogeneous modules in $R-gmod$ (Indeed, the only dominant minuscule element of $\mathfrak{S}_3$ that is missing in this list is s_3s_1, but $S(s_3s_1) = L(31) = L(3) \circ L(1)$ is not prime). Thus the determinantial modules categorifying the flag minors coincide with the prime strongly homogeneous modules in this case.

Another interesting case is where $\mathfrak{g}$ is of type D_4, which will be studied in detail in Section 4.8. Although the determinantial modules are not necessarily homogeneous in this case, it seems that prime strongly homogeneous modules still categorify certain flag minors.

Therefore it is natural to ask the following:

Question 4.5.4. *Could the distinguished value of $\overline{D}$ be a characteristic of the flag minors, rather than a specific property of classes of strongly homogeneous modules ?*

Comparing with the geometric point of view, we know from Corollary 4.4.4 that the values of $\overline{D}$ take a remarkable form on the elements of the MV basis corresponding to MV cycles satisfying certain smoothness properties. Thus given a MV cycle Z such that b_Z is a flag minor of $\mathbb{C}[N]$, this also raises the question of the smoothness of Z at L_μ where $\mu \in -Q_+$ is the weight of Z.

4.5.2 The main results

Motivated by the discussion of the previous Section, we now state the main results of this paper. They essentially split into two main parts: one consists in exhibiting several good properties on equivariant multiplicities that propagate under certain cluster mutations, and the other one consists in checking that these properties are indeed satisfied for a particular standard seed in $\mathbb{C}[N]$. Question (4.5.4) suggests the following conjecture:

Conjecture 4.5.5. *Let $\mathfrak{g}$ be a Lie algebra of finite simply-laced type and let x be a flag minor in $\mathbb{C}[N]$. Then the equivariant multiplicity of x is of the form*

$$\overline{D}(x) = \frac{1}{\prod_{\beta\in\Phi_+} \beta^{n_{M,\beta}}}$$

where n_β is a nonnegative integer for every $\beta \in \Phi_+$.

The strategy for proving this conjecture is the following. We know that the standard seeds of $\mathbb{C}[N]$ are related to each other by certain cluster mutations, each of which correspond to a change of reduced expression of w_0 i.e. there is $k \in \{1, \dots, N\}$ such that $i_k = p, i_{k+1} = q$ and $i_{k+2} = p$ with $p \cdot q = -1$:

$$\begin{array}{ccc} \mathbf{i} = (\dots, p, q, p, \dots) & \rightsquigarrow & \mathbf{i}' = (\dots, q, p, q, \dots) \\ \mathcal{S}^{\mathbf{i}} = ((x_1, \dots, x_N), Q^{\mathbf{i}}) & \rightsquigarrow & \mathcal{S}^{\mathbf{i}'} = ((x_1, \dots x_{k-1}, x'_k, x_{k+1}, \dots, x_N), Q^{\mathbf{i}'}). \end{array}$$

The proof is composed of two steps:

- Firstly, we show that the desired result propagates under these cluster mutations. Thus we start by assuming that $\overline{D}(x_j)$ is of the form $1/P_j$ for every $1 \leqslant j \leqslant N$. We want to show that $\overline{D}(x'_k)$ is of the form $1/P'_k$. As $\overline{D}$ is an algebra morphism, one can immediately express $\overline{D}(x'_k)$ in terms of the P_j. But then it is not hard to see that $\overline{D}(x'_k)$ has no reason to take the desired form, at least without any further assumptions on the P_j. Therefore, we consider certain relations between the P_j entirely determined by $\mathbf{i}$ implying that $\overline{D}(x'_k)$ is of the form $1/P'_k$. We prove that these relations are preserved under mutation, i.e. the polynomials $P_1, \dots, P_{k-1}, P'_k, P_{k+1}, \dots, P_N$ satisfy the corresponding relations determined by $\mathbf{i}'$. Thus this procedure can be iterated to arbitrary sequences of mutations from one standard seed to another.

- Secondly, we shall exhibit a particular standard seed in $\mathbb{C}[N]$ whose cluster variables $x_j, j \in \{1, \dots, N\}$ have equivariant multiplicities of the form $1/P_j$, and check that the polynomials P_j satisfy the relations required in the previous step.

The first main result of this paper consists in proving the first step for all simply-laced types.

Theorem 4.5.6. *Let $\mathfrak{g}$ be any simply-laced type simple Lie algebra and let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced expressions of w_0 related as above. Assume that the cluster variables $x_1, \dots, x_N$ of the standard seed $\mathcal{S}^{\mathbf{i}}$ satisfy the following properties:*

(A) For every $1 \leqslant j \leqslant N$, the rational fraction $\overline{D}(x_j)$ is of the form $1/P_j$ where P_j is a product of positive roots.

(B) For every $1 \leqslant j \leqslant N$ one has

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l.$$

(C) For every $j \in J_{ex}$ and every $1 \leqslant i \leqslant N$, one has $(\beta_i; P_j) - (\beta_i; P_{j_+(\mathbf{i})}) \leqslant 1$.

Then the cluster variables of $\mathcal{S}^{\mathbf{i}'}$ satisfy the analogous properties determined by $\mathbf{i}'$.

Remark 4.5.7. Note that we allow the left hand-side of the inequality (C) to be negative. Moreover, this inequality is relevant only in types D_n and E_6, E_7, E_8 as the P_j are always multiplicity-free in type A_n (for any standard seed).

Remark 4.5.8. Let us outline a curious consequence of Property (B). The polynomial P_{j_-} as well as all the polynomials involved in the right hand side are all of the form P_l for some l strictly smaller than j. Thus if the Property (B) holds for the standard seed $\mathcal{S}^{\mathbf{i}}$, then for every j, P_j is entirely determined by β_j and by the $P_l, l < j$. By a straightforward induction, this implies that the polynomials $P_1, \ldots, P_N$ are in fact entirely determined by the data of the P_j such that $j_- = 0$, i.e. the positions of the first occurrence in $\mathbf{i}$ of each letter of I. Interestingly, these polynomials are in general easy to compute by hand, as illustrated in the next example. For instance it follows from [72, Proposition 3.14] that the corresponding determinantial modules are cuspidal for the convex ordering on Φ_+ associated to $\mathbf{i}$. In particular when $\mathbf{i}$ comes from a total ordering on I, Theorem 4.2.7 explicitly provides the good Lyndon words associated to these modules. In simply-laced types such cuspidal modules are often well understood (see for example [79, Section 8]). Hence the value of $\overline{D}$ on their isomorphism class can be computed using Equation 4.7. Thus Property (B) provides an algorithm allowing to compute the equivariant multiplicities of all the flag minors belonging to one standard seed, which was a non-trivial computation a priori.

Example 4.5.9. Let us provide an example of a direct computation of P_j for j such that $j_-(\mathbf{i}) = 0$.

Consider $\mathfrak{g}$ of type D_4 and take $\mathbf{i} = \mathbf{i}_{nat} = (1, 3, 2, 4, 3, 1, 4, 3, 2, 4, 3, 4)$ as in Section 4.8 below. Consider for instance the first occurrence of the letter 3, i.e. $j = 2$. The determinantial module $M_2^{\mathbf{i}_{nat}}$ is the cuspidal module $L(13)$ (with respect to the natural ordering). This can be deduced from Theorem 4.2.7, or can be checked by hand using for instance [69, Proposition 10.2.4]. This module is one-dimensional as a vector space (see for instance [79, Section 8.7]) and thus its weight subspace decomposition is simply $L(13) = \mathbb{C} \cdot \mathbf{v}_{13}$ with $e(13) \cdot \mathbf{v}_{13} = \mathbf{v}_{13}$. Thus it is immediate to compute its image under $\overline{D}$ using Equation 4.7 and we get $P_2 = \alpha_1(\alpha_1 + \alpha_3)$.

The second main result of this paper is to check the second step for types A_n and D_4.

Theorem 4.5.10. *Assume $\mathfrak{g}$ is of type $A_n, n \geqslant 1$ or D_4. Let $\mathbf{i}_{nat}$ denote the reduced expression of w_0 corresponding to the natural ordering on the vertices of the Dynkin diagram of $\mathfrak{g}$. Then Properties (A), (B) and (C) hold for the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$.*

In particular, this proves Conjecture 4.5.5 in types A_n and D_4. Section 4.6 is devoted to the proof of Theorem 4.5.6 and Sections 4.7 and 4.8 respectively contain the proofs of Theorem 4.5.10 in types A_n and D_4.

As we saw in Section 4.5.1, Property (A) fails for homogeneous modules that are not strongly homogeneous. Further computations in types A_n and D_4 tend to suggest that the evaluation of $\overline{D}$ on cluster variables of $\mathbb{C}[N]$ fails to take the remarkable form of (A) for the cluster variables that are not flag minors (i.e. that do not appear in any of the standard seeds $\mathcal{S}^{\mathbf{i}}, \mathbf{i} \in \mathrm{Red}(w_0)$. Therefore, we propose the following stronger version of Conjecture 4.5.5.

Conjecture 4.5.11. *Let $\mathfrak{g}$ be a Lie algebra of finite simply-laced type and let w_0 be the longest element of the associated Weyl group. Then the flag minors are exactly the cluster variables of $\mathbb{C}[N]$ whose image under $\overline{D}$ are of the form*

$$\frac{1}{\prod_{\beta \in \Phi_+} \beta^{n_\beta}}.$$

for some family of nonnegative integers $(n_\beta)_{\beta \in \Phi_+}$.

In other words, flag minors would essentially be characterized by the distinguished form of their equivariant multiplicities. Let us illustrate this by an example.

Example 4.5.12. Consider $\mathfrak{g} = \mathfrak{sl}_4$. As mentioned in Example 4.5.3, the cluster structure of $\mathbb{C}[N]$ contains 14 seeds. The simple objects of $R-gmod$ corresponding to the cluster variables are given as follows (with respect to the natural ordering $1 < 2 < 3$):

$$L(1), L(2), L(3), L(12), L(21), L(23), L(32), L(312), L(231), L(123), L(321), L(2312).$$

The three last modules correspond to the frozen variables. Among the unfrozen cluster variables, the only one that is not a flag minor (i.e. that does not appear in any standard seed) is $[L(231)]$, and as we saw in Example 4.5.2 the rational fraction $\overline{D}([L(231)])$ is not of the form (A).

4.6 Propagation under cluster mutation

This section is devoted to the proof of Theorem 4.5.6. Let $\mathfrak{g}$ be a simple Lie algebra of finite simply-laced type. We choose a labeling I of the vertices of the Dynkin diagram of $\mathfrak{g}$ and we write the associated Cartan datum as

$$i \cdot i = 2 \quad \text{and} \quad i \cdot j = -1 \Leftrightarrow i \text{ and } j \text{ are neighbours in the Dynkin diagram of } \mathfrak{g}.$$

We consider the longest element w_0 of the corresponding Weyl group W and we denote by N the length of w_0. We fix a reduced expression $\mathbf{i} = (i_1, \ldots, i_N)$ of w_0. Let $x_1, \ldots, x_N$ denote the cluster variables and $Q^{\mathbf{i}}$ denote the quiver of the standard seed $\mathcal{S}^{\mathbf{i}}$ of $\mathbb{C}[N]$. We also let $M_1, \ldots, M_N$ denote the determinantial modules corresponding to $\mathbf{i}$ i.e. the simple modules in $R-gmod$ whose isomorphism classes are $x_1, \ldots, x_N$ in $K_0(R-gmod) \simeq \mathbb{C}[N]$.

The quiver $Q^{\mathbf{i}}$ is defined as follows (see [52, 69]). First recall the following piece of notations: for any $1 \leqslant j \leqslant N$ we set

$$j_-(\mathbf{i}) := \max\left(\{l, 1 \leqslant l < j, i_l = i_j\} \sqcup \{0\}\right)$$

and

$$j_+(\mathbf{i}) := \min\left(\{l, N \geqslant l > j, i_l = i_j\} \sqcup \{N+1\}\right).$$

We outline the dependence on $\mathbf{i}$ in order to avoid confusion later in the proof, as we will be considering two different reduced expressions $\mathbf{i}$ and $\mathbf{i}'$. The index set of $Q^{\mathbf{i}}$ is $J = \{1, \ldots, N\}$, which splits into a frozen part $J_{fr}(\mathbf{i}) := \{u \in J \mid u_+(\mathbf{i}) = N+1\}$ and an unfrozen part $J_{ex}(\mathbf{i}) := J \backslash J_{fr}(\mathbf{i})$. The set of arrows of $Q^{\mathbf{i}}$ is composed of two different kinds of arrows:

- the *ordinary arrows*: there is such an arrow from the vertex u to the vertex v if and only if

$$i_u \cdot i_v = -1 \quad \text{and} \quad u < v < u_+(\mathbf{i}) < v_+(\mathbf{i}).$$

- the *horizontal arrows*: for every $u \in J_{ex}(\mathbf{i})$ there is an arrow from the vertex $u_+(\mathbf{i})$ to the vertex u.

For every $j \in J$ we let $\mathrm{in}(j)$ (resp. $\mathrm{out}(j)$) denote the set of all indices l such that there is an arrow from l to j (resp. from j to l) in $Q^{\mathbf{i}}$ and $\mathrm{inord}(j)$ (resp. $\mathrm{outord}(j)$) the set of all indices l such that there is an ordinary arrow from l to j (resp. from j to l) in $Q^{\mathbf{i}}$. We have

$$\mathrm{in}(j) = \mathrm{inord}(j) \sqcup \{j_+(\mathbf{i})\} \quad \text{and} \quad \mathrm{out}(j) = \mathrm{outord}(j) \sqcup \{j_-(\mathbf{i})\}.$$

We now consider another reduced expression $\mathbf{i}'$ of w_0 such that $\mathbf{i}'$ is obtained from $\mathbf{i}$ by performing a braid relation in W i.e. there is $k \in \{1, \dots, N\}$ such that $i_k = p, i_{k+1} = q, i_{k+2} = p$ with $p, q \in I$ such that $p \cdot q = -1$.

$$\mathbf{i} = (i_1, \dots, i_{k-1}, p, q, p, \dots) \quad \rightsquigarrow \quad \mathbf{i}' = (i_1, \dots, i_{k-1}, q, p, q, \dots).$$

We denote by $\beta'_1, \dots, \beta'_N$ the positive roots given by the reduced expression $\mathbf{i}'$ and $x'_1, \dots, x'_N$ the cluster variables of the standard seed $\mathcal{S}^{\mathbf{i}'}$. The seeds $\mathcal{S}^{\mathbf{i}}$ and $\mathcal{S}^{\mathbf{i}'}$ are related by a one-step mutation in the direction k. It is appropriate for our purpose to be slightly careful with the labeling of the vertices of $Q^{\mathbf{i}}$ and $Q^{\mathbf{i}'}$.

We let $\mathbf{s}$ denote the transposition $(k+1, k+2)$ of $\{1, \dots, N\}$ i.e. the permutation that exchanges the indices $k+1$ and $k+2$ and leaves the others fixed. The set of vertices of the quiver $Q^{\mathbf{i}'}$ is $\mathbf{s}(J)$. In other words the vertex labeled $k+1$ in $Q^{\mathbf{i}'}$ is in fact the vertex labeled $k+2$ in $Q^{\mathbf{i}}$ and vice versa. There is an arrow $i \to j$ in $Q^{\mathbf{i}'}$ if and only if there is an arrow $\mathbf{s}(i) \to \mathbf{s}(j)$ in the quiver obtained from $Q^{\mathbf{i}}$ by the usual mutation process.

$$\begin{tikzcd}
Q = Q^{\mathbf{i}} \arrow[dr, squiggly, "\text{mutation}"'] & & Q^{\mathbf{i}'} \\
& Q' \arrow[ur, squiggly, "J \mapsto \mathbf{s}(J)"'] &
\end{tikzcd}$$

We begin with a couple of elementary Lemmas as prerequisites for the proof.

Lemma 4.6.1. *The positive roots $\beta'_1, \dots, \beta'_N$ are related to the β_j as follows:*

$$\beta'_k = \beta_{k+2} \quad \beta'_{k+1} = \beta_{k+1} \quad \beta'_{k+2} = \beta_k \quad \beta'_j = \beta_j \text{ for any } j \notin \{k, k+1, k+2\}.$$

The flag minors $M'_1, \dots, M'_N$ are given by:

$$x'_{k+1} = x_{k+2} \quad x'_{k+2} = x_{k+1} \quad x'_j = x_j \text{ for any } j \notin \{k, k+1, k+2\}.$$

Proof. Recall that $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$ and similarly for the the β'_j with $\mathbf{i}'$. Thus

$$\beta'_k = s_{i'_1} \cdots s_{i'_{k-1}} \alpha_{i'_k} = s_{i_1} \cdots s_{i_{k-1}} \alpha_q = s_{i_1} \cdots s_{i_{k-1}} s_p s_q \alpha_p = \beta_{k+2}.$$

One checks the other cases in a similar way.

The flag minor M_j can be written as $D\left(s_{i_1} \cdots s_{i_j} \omega_{i_j}, \omega_{i_j}\right)$ with the notations of Section 4.2.3. Thus one has for instance

$$\begin{aligned} x'_{k+1} &= D\left(s_{i'_1} \cdots s_{i'_{k+1}} \omega_{i'_{k+1}}, \omega_{i'_{k+1}}\right) = D\left(s_{i_1} \cdots s_{i_{k-1}} s_q s_p \omega_p, \omega_p\right) \\ &= D\left(s_{i_1} \cdots s_{i_{k-1}} s_p s_q s_p \omega_p, \omega_p\right) = x_{k+2}. \end{aligned}$$

The other cases can be checked in the same way. □

Lemma 4.6.2. *One has*

$$\beta_k + \beta_{k+2} = \beta_{k+1}.$$

Proof. This is a straightforward consequence of the definition of the β_j. Indeed,

$$\beta_k + \beta_{k+2} = s_{i_1} \cdots s_{i_{k-1}} \left(\alpha_p + s_p s_q(\alpha_p)\right) = s_{i_1} \cdots s_{i_{k-1}} \left(\alpha_p + \alpha_q\right) = s_{i_1} \cdots s_{i_{k-1}} s_p(\alpha_q)$$

i.e. $\beta_k + \beta_{k+2} = \beta_{k+1}$. □

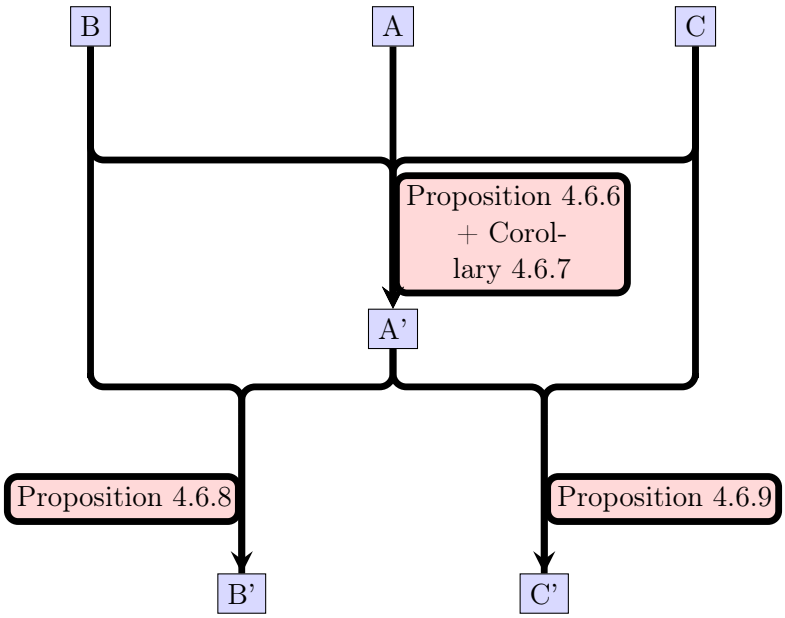

Figure 1: Structure of the proof of Theorem 4.5.6.

The structure of the proof is summarized in Figure 1. Let us briefly explain the main points. The starting assumption is that for each $j \in J$, the rational function $\overline{D}(x_j)$ is of the form $1/P_j$ where P_j is a product of positive roots. The aim is to prove that $\overline{D}(x'_k)$ is also of the form $1/P'_k$ where P'_k is a product of positive roots. It is not hard to convince oneself that this has no chance to hold without any further assumption on the $P_j, 1 \leqslant j \leqslant N$. Thus the idea is that the P_j shall satisfy certain relations entirely determined by $\mathbf{i}$ implying the desired form for $\overline{D}(x'_k)$. Moreover these relations must be preserved under mutation i.e. the polynomials $P_1, \ldots, P_{k-1}, P'_k, P_{k+1}, \ldots, P_N$ have to satisfy the analogous relations determined by $\mathbf{i}'$. Therefore we assume that the following properties are satisfied:

(A) For every $1 \leqslant j \leqslant N$, the rational fraction $\overline{D}(x_j)$ is of the form $1/P_j$ where P_j is a product of positive roots.

(B) For every $1 \leqslant j \leqslant N$ one has
$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l.$$

(C) For every $j \in J_{ex}$, one has
$$(\beta_i; P_j) - (\beta_i; P_{j_+(\mathbf{i})}) \leqslant 1$$
for every $1 \leqslant i \leqslant N$.

We prove that the flag minors of the seed $\mathcal{S}^{\mathbf{i}'}$ satisfy the analogous properties determined by $\mathbf{i}'$ and denoted (A'), (B') and (C'), as shown in Figure 1.

In what follows we will be using the following notations: for every $j \in J$ we set
$$P_{\mathrm{in}(j)} := \prod_{l \in \mathrm{in}(j)} P_l, P_{\mathrm{inord}(j)} := \prod_{l \in \mathrm{inord}(j)} P_l, P_{\mathrm{out}(j)} := \prod_{l \in \mathrm{out}(j)} P_l, P_{\mathrm{outord}(j)} := \prod_{l \in \mathrm{outord}(j)} P_l.$$

Let us begin with a straightforward consequence of Property (B) that will be useful throughout the proof.

Lemma 4.6.3. *For every $j \in J_{ex}$, one has*

$$\beta_j P_{in(j)} = \beta_{j_+(\mathbf{i})} P_{out(j)}.$$

Proof. We fix $j \in J_{ex}(\mathbf{i})$. In particular $j_+(\mathbf{i}) \leqslant N$ so we can combine Property (B) at ranks j and $j_+(\mathbf{i})$:

$$P_j P_{j_-(\mathbf{i})} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l \quad \text{and} \quad P_{j_+(\mathbf{i})} P_j = \beta_{j_+(\mathbf{i})} \prod_{\substack{l<j_+(\mathbf{i})<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l.$$

Dividing the first one by the second one we get

$$\frac{P_{j_-(\mathbf{i})}}{P_{j_+(\mathbf{i})}} = \beta_j \prod_{\substack{l<j<l_+(\mathbf{i})<j_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l \left(\beta_{j_+(\mathbf{i})} \prod_{\substack{j<l<j_+(\mathbf{i})<l_+(\mathbf{i}) \\ i_l \cdot i_j = -1}} P_l \right)^{-1} = \frac{\beta_j P_{\mathrm{inord}(j)}}{\beta_{j_+(\mathbf{i})} P_{\mathrm{outord}(j)}}.$$

By definition one has $\mathrm{in}(j) = \{j_+(\mathbf{i})\} \sqcup \mathrm{inord}(j)$ and $\mathrm{out}(j) = \{j_-(\mathbf{i})\} \sqcup \mathrm{outord}(j)$. Thus we have proven the desired statement. □

Remark 4.6.4. This statement can be rephrased in a cluster-theoretic way as follows. Recall Fomin-Zelevinsky's notations $\hat{y}_j$. They are Laurent monomials in the x_i that can be written in our case as

$$\hat{y}_j := \prod_{i \in \mathrm{in}(j)} x_i \prod_{i \in \mathrm{out}(j)} x_i^{-1}$$

for every $j \in J_{ex}(\mathbf{i})$. The algebra morphism $\overline{D}$ can be extended to the fraction field of $\mathbb{C}[N]$ so that the evaluation $\overline{D}(\hat{y}_j)$ makes sense. As $\mathrm{in}(j) = \{j_+(\mathbf{i})\} \sqcup \mathrm{inord}(j)$ and $\mathrm{out}(j) = \{j_-(\mathbf{i})\} \sqcup \mathrm{outord}(j)$, Lemma 4.6.3 can be restated as

$$\overline{D}(\hat{y}_j) = \frac{\beta_j}{\beta_{j_+(\mathbf{i})}}$$

for every $j \in J_{ex}(\mathbf{i})$.

We let $\tilde{P}_j$ denote the polynomial given by Lemma 4.6.3 i.e.

$$\tilde{P}_j := \beta_j P_{\mathrm{in}(j)} = \beta_{j_+(\mathbf{i})} P_{\mathrm{out}(j)}$$

for every $j \in J_{ex}(\mathbf{i})$. In the sequel of the proof, we denote by $(\beta; P)$ the multiplicity of β in P for every $\beta \in \Phi_+$ and $P \in \{P_1, \ldots, P_N\}$, i.e. the largest positive integer d such that $\beta^d \mid P$. Let us state another useful consequence of Property (B). The proof is straightforward by induction on j.

Lemma 4.6.5. *For every $i, j \in J_{ex}$, one has*

$$i > j \Rightarrow (\beta_i; P_j) = 0 \qquad \textit{and} \qquad (\beta_i; P_i) = 1.$$

Proof. The proof relies on an induction on j. Consider the case $j = 1$ and let h denote the letter $h = i_1$. First note that $\beta_1 = \alpha_h$. The determinantial module M_1 is the cuspidal module $L(h)$: it is one-dimensional as a $\mathbb{C}$-vector space and its weight-space decomposition is simply

$$M_1 = \mathbb{C}\mathbf{v}_h \quad \text{with } e(h) \cdot \mathbf{v}_h = \mathbf{v}_h.$$

Thus Equation (4.7) yields

$$\overline{D}(M_1) = \frac{1}{\alpha_h} = \frac{1}{\beta_1}.$$

Equivalently $P_1 = \beta_1$ and the desired statement holds for $j = 1$.

Fix $j \in \{1, \dots, N\}$ and assume the statement is true for every $j' < j$. If $i > j$ then the induction hypothesis implies

$$(\beta_i; P_{j_-(\mathbf{i})}) = 0 \quad \text{and} \quad (\beta_i; P_l) = 0 \text{ for every } l < j < l_+(\mathbf{i}).$$

Hence Property (B) implies $(\beta_i; P_j) = 0$. If $i = j$ then we use the induction hypothesis in the same way but the presence of β_j in the right hand side of Property (B) yields $(\beta_j; P_j) = 1$. □

The following statement is the heart of the proof. The idea is to combine Property (B) together with Lemma 4.6.3 at different indices and then to use Property (C). This last property plays a crucial role and is a priori not redundant with Properties (A) and (B).

Proposition 4.6.6. *Let $l \in \{1, \dots, N\}$ and assume there exists $m \in \{1, \dots, N\}$ such that*

$$i_l \cdot i_m = -1 \qquad \textit{and} \qquad m_-(\mathbf{i}) < l < m < l_+(\mathbf{i}) < m_+(\mathbf{i}).$$

Then $P_l \mid \tilde{P}_l$.

The assumptions relating l and m mean that there exists an ordinary arrow from the vertex l to the vertex m as well as from the vertex $m_-(\mathbf{i})$ to the vertex l in the quiver $Q^{\mathbf{i}}$.

Proof. Proving $P_l \mid \tilde{P}_l$ is equivalent to proving that $(\beta_i; P_l) \leqslant (\beta_i; \tilde{P}_l)$ for every $1 \leqslant i \leqslant N$. If $i > l$ then $(\beta_i; P_l) = 0$ by Lemma 4.6.5 and thus there is nothing to prove in this case. Assume $i \leqslant l$.

We deal with the case $i = l$ separatly. By Lemma 4.6.5, one has $(\beta_l; P_l) = 1$. Hence we have

$$(\beta_l; P_l) = 1 = (\beta_l; \beta_l) \leqslant (\beta_l; \tilde{P}_l)$$

which is the desired inequality.

From now on we assume $i < l$. If $(\beta_i; P_l) \leqslant (\beta_i; P_{l_+(\mathbf{i})})$ then $(\beta_i; P_l) \leqslant (\beta_i; \tilde{P}_l)$ as $P_{l_+(\mathbf{i})} \mid \tilde{P}_l$ by definition of $\tilde{P}_l$. Hence we can assume $(\beta_i; P_l) > (\beta_i; P_{l_+(\mathbf{i})})$.

Assume $(\beta_i; P_{\mathrm{inord}(l)}) = 0$. Then applying Lemma 4.6.3 at $j = l$ we can write

$$(\beta_i; P_{l_+(\mathbf{i})}) = (\beta_i; P_{l_-(\mathbf{i})}) + (\beta_i; P_{\mathrm{outord}(l)}) = (\beta_i; P_{l_-(\mathbf{i})}) + (\beta_i; P_m) + (\beta_i; R)$$

where R is the product of the polynomials attached to the other tails of ordinary arrows coming out of l in $Q^{\mathbf{i}}$. Now we use Equation (B') at $j = m$: as $i < l$, in particular $i < m$ and hence we get

$$(\beta_i; P_m) = -(\beta_i; P_{m_-(\mathbf{i})}) + \sum_{\substack{h < m < h_+(\mathbf{i}) \\ i_h \cdot i_m = -1}} (\beta_i; P_h).$$

By assumption we have $i_l \cdot i_m = -1$ and $l < m < l_+(\mathbf{i})$. Hence the previous equation can be written as

$$(\beta_i; P_m) = -(\beta_i; P_{m_-(\mathbf{i})}) + (\beta_i; P_l) + (\beta_i; Q)$$

where Q is the product of the $P_h, h \neq l, i_h \cdot i_m = -1, h < m < h_+(\mathbf{i})$. Thus we get

$$(\beta_i; P_{l_+(\mathbf{i})}) = (\beta_i; P_{l_-(\mathbf{i})}) - (\beta_i; P_{m_-(\mathbf{i})}) + (\beta_i; P_l) + (\beta_i; Q) + (\beta_i; R)$$

which can be rewritten as

$$(\beta_i; P_{m_-(\mathbf{i})}) = (\beta_i; P_{l_-(\mathbf{i})}) + (\beta_i; P_l) - (\beta_i; P_{l_+(\mathbf{i})}) + (\beta_i; Q) + (\beta_i; R).$$

Since we assumed $(\beta_i; P_l) > (\beta_i; P_{l_+(\mathbf{i})})$, this implies in particular

$$(\beta_i; P_{m_-(\mathbf{i})}) > (\beta_i; P_{l_-(\mathbf{i})}) + (\beta_i; Q) + (\beta_i; R) \geqslant 0.$$

As there is an ordinary arrow from the vertex $m_-(\mathbf{i})$ to the vertex l, one has in particular $(\beta_i; P_{\mathrm{inord}(l)}) > 0$ which is a contradiction.

Thus we have proven that $(\beta_i; P_{\mathrm{inord}(l)}) > 0$. This is where Property (C) is crucially involved. Indeed, Property (C) at rank l yields $(\beta_i; P_l) - (\beta_i; P_{l_+(\mathbf{i})}) \leqslant 1$. Therefore, we have

$$(\beta_i; P_{\mathrm{inord}(l)}) \geqslant 1 \geqslant (\beta_i; P_l) - (\beta_i; P_{l_+(\mathbf{i})}).$$

As $\mathrm{inord}(l) \sqcup \{l_+\} = \mathrm{in}(l)$, this is equivalent to $(\beta_i; P_l) \leqslant (\beta_i; P_{\mathrm{in}(l)})$ which finishes the proof. □

As a straightforward application, we can now prove that Property (A) holds for the seed $\mathcal{S}^{\mathbf{i}'}$.

Corollary 4.6.7. *One has*

$$\overline{D}(x'_k) = \frac{1}{P'_k} \quad \textit{with } P'_k \textit{ product of positive roots given by } P'_k = \frac{\tilde{P}_k}{\beta_{k+1} P_k}. \tag{A'}$$

Proof. The exchange relation at the vertex k can be written as

$$x_k x'_k = \prod_{j \in \mathrm{in}(k)} x_j + \prod_{j \in \mathrm{out}(k)} x_j.$$

Applying the algebra morphism $\overline{D}$ we get

$$\begin{aligned} \frac{\overline{D}(x'_k)}{P_k} &= \frac{1}{P_{\mathrm{in}(k)}} \left(1 + \frac{P_{\mathrm{in}(k)}}{P_{\mathrm{out}(k)}}\right) = \frac{1}{P_{\mathrm{in}(k)}} \left(1 + \frac{\beta_{k+2}}{\beta_k}\right) \text{by Lemma 4.6.3} \\ &= \frac{1}{P_{\mathrm{in}(k)}} \frac{\beta_{k+1}}{\beta_k} \text{by Lemma 4.6.2.} \end{aligned}$$

Thus we get

$$\overline{D}(x'_k) = \frac{\beta_{k+1} P_k}{\tilde{P}_k}.$$

Property (C) implies $(\beta_i; P_k) - (\beta_i; P_{k+2}) \leqslant 1$ and hence we can apply Proposition 4.6.6 with $l = k$ and $m = k+1$, which yields $P_k \mid \tilde{P}_k$. Lemma 4.6.5 implies $(\beta_{k+1}; P_k) = 0$ and $(\beta_{k+1}; P_{k+1}) = 1$. Applying Lemma 4.6.3 we get

$$(\beta_{k+1}; P_{\mathrm{in}}(k)) = (\beta_{k+1}; P_{\mathrm{out}}(k)) \geqslant (\beta_{k+1}; P_{k+1}) > 0.$$

Hence $\beta_{k+1} \mid P_{\mathrm{in}(k)}$. Finally we have $\beta_{k+1} P_k \mid \tilde{P}_k$ which proves the Corollary. □

Now we prove that Property (B) propagates under mutation.

Proposition 4.6.8. *Property (B) holds for the seed $\mathcal{S}^{\mathbf{i}'}$ i.e. one has*

$$P'_j P'_{j_-(\mathbf{i}')} = \beta'_j \prod_{\substack{l < j < l_+(\mathbf{i}') \\ i'_l \cdot i'_j = -1}} P'_l \tag{B'}$$

for every $j \in J$.

Proof. Throughout this proof, we set for every $j \in J$:

$$I(j) := \{l \in J \mid i_l \cdot i_j = -1, l < j < l_+(\mathbf{i})\} \qquad I'(j) := \{l \in J \mid i'_l \cdot i'_j = -1, l < j < l_+(\mathbf{i}')\}.$$

Consider the case $j = k$. We let r (resp. s) denote the position of the last occurrence of the letter p (resp. q) strictly before the position k. In other words $r = k_-(\mathbf{i}) = (k+1)_-(\mathbf{i}')$ and $s = (k+1)_-(\mathbf{i}) = k_-(\mathbf{i}')$. It is straightforward to check that

$$I(k+1) \sqcup \{r\} = I'(k) \sqcup \{k\}.$$

Indeed if l is such that $i_l \neq p$ then $l \in I'(k)$ if and only if $l \in I(k+1)$. If $i_l = p$, then $l \in I'(k)$ if and only if $l = r$ and $l \in I(k+1)$ if and only if $l = k$. Moreover, as all the indices in $I'(k)$ are strictly smaller than k, one has $P'_l = P_l$ for every $l \in I'(k)$. Thus we have

$$\prod_{l \in I(k+1)} P_l = \frac{P_k}{P_r} \prod_{l \in I'(k)} P_l = \frac{P_k}{P_r} \prod_{l \in I'(k)} P'_l$$

Hence

$$P'_k P'_{k_-(\mathbf{i}')} = P'_k P_s = P_s \frac{\beta_{k+2} P_{k+1} P_r}{\beta_{k+1} P_k} \quad \text{by Equation (A')}.$$

Using Property (B) at the index $k+1$ we get

$$P_{k+1} P_s = \beta_{k+1} \prod_{l \in I(k+1)} P_l$$

and thus

$$P'_k P'_{k_-(\mathbf{i}')} = \beta_{k+2} \frac{P_r}{P_k} \prod_{l \in I(k+1)} P_l = \beta_{k+2} \prod_{l \in I'(k)} P'_l = \beta'_k \prod_{l \in I'(k)} P'_l.$$

This is the desired equality.

Now consider the case $j = k+1$. Similarly we have

$$I'(k+1) \sqcup \{s\} = I(k) \sqcup \{k\}.$$

By Property (B) at the index k, we can write

$$P'_{k+1} P'_{(k+1)_-(\mathbf{i}')} = P_{k+2} P_r = \frac{P_{k+2}}{P_k} \beta_k \prod_{l \in I(k)} P_l$$

and thus

$$P'_{k+1} P'_{(k+1)_-(\mathbf{i}')} = \beta_k \frac{P_{k+2}}{P_k} \frac{P_s}{P'_k} \prod_{l \in I'(k+1)} P'_l = \beta_k \frac{P_{k+2} P_s}{P_k} \frac{\beta_{k+1} P_k}{\beta_k P_{k+2} P_s} \prod_{l \in I'(k+1)} P'_l \quad \text{by (A')}.$$

This simplifies as

$$P'_{k+1} P'_{(k+1)_-(\mathbf{i}')} = \beta_{k+1} \prod_{l \in I'(k+1)} P'_l$$

which is the desired equality as $\beta_{k+1} = \beta'_{k+1}$.

The remaining case to consider is $j = k+2$. We have

$$I'(k+2) \sqcup \{k\} = I(k+1) \sqcup \{k+1\}$$

and all the indices in $I'(k+2)$ other than $k+1$ are strictly smaller than k. Thus we have

$$\prod_{l \in I'(k+2)} P'_l = P'_{k+1} \prod_{l \in I'(k+2) \setminus \{k+1\}} P_l = \frac{P'_{k+1}}{P_k} \prod_{l \in I(k+1)} P_l = \frac{P_{k+2}}{P_k} \prod_{l \in I(k+1)} P_l.$$

Thus we have

$$P'_{k+2}P'_{(k+2)_-(\mathbf{i}')} = P_{k+1}P'_k = P_{k+1}\frac{\beta_k P_{k+2}P_s}{\beta_{k+1}P_k} = \beta_{k+1}\prod_{l\in I(k+1)} P_l\frac{\beta_k P_{k+2}}{\beta_{k+1}P_k}$$

using Property (B) at $k+1$. This yields

$$P'_{k+2}P'_{(k+2)_-(\mathbf{i}')} = \beta_k \prod_{l\in I'(k+2)} P'_l = \beta'_{k+2}\prod_{l\in I'(k+2)} P'_l$$

which finishes the proof. □

Finally we prove that Property (C) propagates under mutation. The key arguments are provided by Lemma 4.6.3 and Lemma 4.6.5.

Proposition 4.6.9. *Property (C) holds for the seed $\mathcal{S}^{\mathbf{i}'}$ i.e. one has*

$$(\beta'_i; P'_j) - (\beta'_i; P'_{j_+(\mathbf{i}')}) \leqslant 1 \tag{C'}$$

for every $1 \leqslant i, j \leqslant N$.

Proof. First note that there is nothing to prove if $j \notin \{r, s, k, k+1, k+2\}$. Moreover $P'_{k+1} = P_{k+2}$ and $(k+1)_+(\mathbf{i}') = (k+2)_+(\mathbf{i})$. Thus there is nothing to prove either for $j = k+1$ and similarly for $j = k+2$. We now focus on the cases $j = r, j = s$ and $j = k$.

Consider the case $j = r$. One has

$$(\beta'_i; P'_r) - (\beta'_i; P'_{r_+(\mathbf{i}')}) = (\beta'_i; P_r) - (\beta'_i; P'_{k+1}) = (\beta'_i; P_r) - (\beta'_i; P_{k+2}).$$

Lemma 4.6.5 implies that $(\beta_i; P_r) = 0$ for every $i > r$. Thus $(\beta'_i; P_r) = 0$ for every $i > r$ and the desired inequality holds. If $i = r$ then again Lemma 4.6.5 implies $(\beta_i; P_r) = 1$ and the conclusion is the same as $\beta_r = \beta'_r$. Assume $i < r$. By Lemma 4.6.3, we have $\beta_k P_{k+2}P_s = \beta_{k+2}P_r P_{k+1}$. This yields

$$(\beta'_i; P_r) - (\beta'_i; P_{k+2}) = (\beta_i; P_r) - (\beta_i; P_{k+2}) = (\beta_i; P_s) - (\beta_i; P_{k+1}) = (\beta_i; P_s) - (\beta_i; P_{s_+(\mathbf{i})}).$$

The desired inequality follows from Property (C) at $j = s$.

Consider the case $j = s$. One has

$$(\beta'_i; P'_s) - (\beta'_i; P'_{s_+(\mathbf{i}')}) = (\beta'_i; P_s) - (\beta'_i; P'_k).$$

Lemma 4.6.5 implies that $(\beta_i; P_s) = 0$ for every $i > s$. Thus $(\beta'_i; P_s) = 0$ for every $i > s$ and the desired inequality holds. If $i = s$ then again Lemma 4.6.5 implies $(\beta_i; P_s) = 1$ and the conclusion is the same as $\beta'_s = \beta_s$. Assume $i < s$. Then we have

$$\begin{aligned}(\beta'_i; P_s) - (\beta'_i; P'_k) &= (\beta_i; P_s) - (\beta_i; P'_k) = (\beta_i; P_s) - (\beta_i; P_s P_{k+2}) + (\beta_i; P_k)\\ &= (\beta_i; P_k) - (\beta_i; P_{k+2}) = (\beta_i; P_k) - (\beta_i; P_{k_+(\mathbf{i})}).\end{aligned}$$

Thus we can conclude using Property (C) at $j = k$.

Consider the case $j = k$. One has

$$(\beta'_i; P'_k) - (\beta'_i; P'_{k_+(\mathbf{i}')}) = (\beta'_i; P'_k) - (\beta'_i; P'_{k+2}) = (\beta'_i; P'_k) - (\beta'_i; P_{k+1}).$$

Proposition 4.6.8 implies that Equation (B') holds for the seed $\mathcal{S}^{\mathbf{i}'}$ and in particular we can apply Lemma 4.6.5 for $\mathcal{S}^{\mathbf{i}'}$. Therefore $(\beta'_i; P'_k) = 0$ if $i > k$ and $(\beta'_k; P'_k) = (\beta_{k+2}; P'_k) = 1$. As before we can focus on the case $i < k$. In particular $\beta'_i = \beta_i$. Thus we have

$$\begin{aligned}(\beta'_i; P'_k) - (\beta'_i; P_{k+1}) &= (\beta_i; P'_k) - (\beta_i; P_{k+1}) = (\beta_i; P_r P_{k+1}) - (\beta_i; P_k) - (\beta_i; P_{k+1})\\ &= (\beta_i; P_r) - (\beta_i; P_k) = (\beta_i; P_r) - (\beta_i; P_{r_+(\mathbf{i})})\end{aligned}$$

and the Property (C) at $j = r$ allows us to conclude.

This proves that Property (C') holds.

□

4.7 Initial seed in type A_n

In this section we prove Theorem 4.5.10 in the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ where $n \geqslant 1$ is fixed. We denote by $I = \{1, \dots, n\}$ the index set of the simple roots and we consider the natural order on I given by $1 < 2 < \cdots < n$. As explained in Section 4.2.3, this yields a convex order on the set Φ_+ of positive roots, corresponding to the reduced expression of w_0 given by

$$\mathbf{i}_{nat} := (1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1).$$

The aim of this section is to check that the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies Properties (A), (B) and (C). We use Kang-Kashiwara-Kim-Oh's monoidal categorification of the cluster structure of $\mathbb{C}[N]$ via representations of quiver Hecke algebras. More precisely, the cluster variables of $\mathcal{S}^{\mathbf{i}_{nat}}$ are categorified by certain determinantial modules in $R - gmod$. These were explicitly described in [25] in terms of Kleshchev-Ram's dominant words. Let us briefly remind the necessary setting.

The set $\mathcal{GL}$ of good Lyndon words is given by

$$\mathcal{GL} = \{(i, i+1, \dots, j) \mid i, j \in I, i \leqslant j\}.$$

Recall that this set is totally ordered with respect to the lexicographic order induced by the chosen order on I. The dominant words parametrizing the simple objects in $R - gmod$ according to Kleshchev-Ram's classification (see Section 4.2.1) are concatenations of elements of $\mathcal{GL}$ in the decreasing order. Dominant words thus coincide with Zelevinsky's multisegments in this case. For any dominant word, we denote by $L(\mu)$ the unique (up to isomorphism and grading shift) simple module associated to μ

For every $i \leqslant j$, we will use the notation $[i; j]$ for the positive root $\alpha_i + \cdots + \alpha_j$. The integer $j - i + 1$ is called the *height* of this positive root. For each $1 \leqslant r \leqslant n$ the occurrences of r in $\mathbf{i}_{nat}$ correspond to positive roots of height r. More precisely, for every $1 \leqslant k \leqslant n$ the kth occurrence of r in $\mathbf{i}_{nat}$ corresponds to the positive root $[k; r+k-1]$. Equivalently for every $i \leqslant j$, $[i; j]$ is the positive root corresponding to the ith occurrence of $j - i + 1$ in $\mathbf{i}_{nat}$.

Now consider the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ of $\mathbb{C}[N]$, let $x_1, \dots, x_N$ denote its cluster variables and $M_1, \dots, M_N$ denote the corresponding determinantial modules in $R - gmod$. The dominant words associated to $M_1, \dots, M_N$ were computed in [25] and are given as follows.

Proposition 4.7.1 ([25, Theorem 6.1])**.** *For every $1 \leqslant k, r \leqslant n$, the determinantial module corresponding to the kth occurrence of r in $\mathbf{i}_{nat}$ is*

$$L\left([k; r+k-1][k-1; r+k-2] \cdots [1; r]\right).$$

For every $1 \leqslant k, r \leqslant n$ we set

$$P[k, r] := \prod_{1 \leqslant l \leqslant k \leqslant m \leqslant r+k-1} [l; m].$$

We begin by proving that the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies Property (A). More precisely, we show that if j is the position of the kth occurrence of the letter r in $\mathbf{i}_{nat}$, then $P_j = P[k, r]$.

Lemma 4.7.2. *For any $1 \leqslant k, r \leqslant n$, the determinantial module $L\left([k; r+k-1] \cdots [1; r]\right)$ is strongly homogeneous and one has*

$$\overline{D}\left(L\left([k; r+k-1] \cdots [1; r]\right)\right) = \frac{1}{P[k, r]}.$$

In particular the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies Property (A).

Proof. Let $w[k,r]$ denote the element of W given by

$$w[k,r] := s_k s_{k+1} \cdots s_{r+k-1} s_{k-1} s_k \cdots s_{r+k-2} \cdots s_1 s_2 \cdots s_r.$$

It is immediate to check that for any $1 \leqslant j \leqslant n$, there is exactly one occurrence of each neighbour of j between two consecutive occurrences of j in the word $(k, k+1, \ldots, r+k-1, k-1, k, \ldots, r+k-2, \ldots, 1, 2, \ldots r)$. Moreover the last occurrence of j is either strictly inside the last segment (if $1 \leqslant j < r$) and in this case there is exactly one occurrence of a neighbour of j (namely $j+1$) after this occurrence, or it is the last letter of one of the segments (if $j \geqslant r$) in which case there is exactly one occurrence of $j-1$ and no occurrence of $j+1$ after this occurrence. Therefore using Stembridge's results [111] (see Theorem 4.3.3) we conclude that $w[k,r]$ is dominant minuscule. Hence by the construction of Kleshchev-Ram [78] (see Theorem 4.3.9) the determinantial module $L([k; r+k-1] \cdots [1; r])$ is strongly homogeneous.

By Proposition 4.5.1, we have

$$\overline{D}\left(L\left([k; r+k-1] \cdots [1; r]\right)\right) = \prod_{\beta \in \Phi_+^{w[k,r]}} \frac{1}{\beta}.$$

We prove by induction on k that for any $r \geqslant 1$,

$$\Phi_+^{w[k,r]} = \{[l; m], 1 \leqslant l \leqslant k \leqslant m \leqslant r+k-1\}.$$

If $k = 1$ then for any $r \geqslant 1$ one has

$$\Phi_+^{w[1,r]} = \Phi_+^{s_1 s_2 \cdots s_r} = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_r\} = \{[1; m], 1 \leqslant m \leqslant r\}$$

which is the desired equality for $k = 1$. Assume the result holds at rank $k-1$. It is straightforward to check that for every l, m such that $1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-2$, we have

$$s_k s_{k+1} \cdots s_{r+k-1}([l; m]) = [l; m+1].$$

Applying the induction hypothesis, this implies

$$s_k s_{k+1} \cdots s_{r+k-1}\left(\Phi_+^{w[k-1,r]}\right) = \{[l; m+1], 1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-2\}.$$

Thus we get

$$\begin{aligned}
\Phi_+^{w[k,r]} &= \{[k], [k; k+1], \ldots, [k; r+k-1]\} \sqcup s_k s_{k+1} \cdots s_{r+k-1}\left(\Phi_+^{w[k-1,r]}\right) \\
&= \{[k; m], k \leqslant m \leqslant r+k-1\} \sqcup \{[l; m+1], 1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-2\} \\
&= \{[k; m], k \leqslant m \leqslant r+k-1\} \sqcup \{[l; m], 1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-1\} \\
&= \{[l; m], 1 \leqslant l \leqslant k \leqslant m \leqslant r+k-1\}.
\end{aligned}$$

This finishes the proof. □

Corollary 4.7.3. *The standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies Property (C).*

Proof. It follows from the previous Lemma that the multiplicity of any positive root in any of the polynomials $P_j, 1 \leqslant j \leqslant N$ is always equal to zero or 1. In particular the Property (C) is trivially satisfied. □

Now we check that Property (B) is also satisfied. For each $j \in \{1, \ldots, N\}$, the polynomials involved in the right hand-side of Property (B) at rank j correspond to the last occurrences in $\mathbf{i}_{nat}$ of each neighbour of i_j (strictly) before the position j. Thus if $i_j := r$, we have to consider the last occurrences of $r+1$ and $r-1$ before j.

Lemma 4.7.4. *The standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies Property (B).*

Proof. Fix $1 \leqslant k, r \leqslant n$ and let $j \in \{1, \dots, N\}$ denote the position of the kth occurrence of r in $\mathbf{i}_{nat}$. By definition j_- corresponds to the $k-1$ th occurrence of r. Thus by Proposition 4.7.1 the associated determinantial module is

$$L\left([k-1; r+k-2][k-1; r+k-3] \cdots [1; r]\right).$$

It is not hard to see that the letter $r+1$ has appeared exactly $k-1$ times before j in $\mathbf{i}_{nat}$ as it only appears in the subwords of $\mathbf{i}_{nat}$ of the form $s, \dots, 1$ with $s > r$. Hence the last occurrence of $r+1$ before j in $\mathbf{i}_{nat}$ corresponds to the $k-1$th occurrence of $r+1$ in $\mathbf{i}_{nat}$. Thus by Proposition 4.7.1 the associated determinantial module is

$$L\left([k-1; r+k-1][k-2; r+k-2] \cdots [1; r+1]\right).$$

Similarly one can check that the letter $r-1$ appears exactly k times before j in $\mathbf{i}_{nat}$. Therefore the last occurrence of $r-1$ before j corresponds to the k occurrence of $r-1$ and Proposition 4.7.1 implies that the associated determinantial module is

$$L\left([k; r+k-2][k-1; r+k-3] \cdots [1; r-1]\right).$$

We can now rewrite Property (B) at rank j in terms of multisegments as

$$P_j P_{j_-} = [k; r+k-1] P[k, r-1] P[k-1, r+1].$$

Let us start from the left hand-side and write:

$$\begin{aligned}
P_j P_{j_-} &= P[k,r]P[k-1,r] \\
&= \prod_{1 \leqslant l \leqslant k \leqslant m \leqslant r+k-1} [l;m] \prod_{1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-2} [l;m] \\
&= \left(\prod_{1 \leqslant l \leqslant k} [l; r+k-1] \prod_{1 \leqslant l \leqslant k \leqslant m \leqslant r+k-2} [l;m] \right) \\
&\qquad \left(\left(\prod_{1 \leqslant l \leqslant k-1} [l; r+k-1] \right)^{-1} \prod_{1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-1} [l;m] \right) \\
&= [k; r+k-1] \prod_{1 \leqslant l \leqslant k \leqslant m \leqslant r+k-2} [l;m] \prod_{1 \leqslant l \leqslant k-1 \leqslant m \leqslant r+k-1} [l;m] \\
&= [k; r+k-1] P[k, r-1] P[k-1, r+1].
\end{aligned}$$

This proves that Property (B) is satisfied. □

4.8 Initial seed in type D_4

This section is devoted to the proof of Theorem 4.5.10 when $\mathfrak{g}$ is of type D_4. In this case, determinantial modules are not necessarily homogeneous. Hence we cannot always use the results of [78, 95] to compute the images under $\overline{D}$ of the flag minors. Therefore, the most difficult part is to prove that Property (A) is satisfied for a certain standard seed. Properties (B) and (C) will then be rather straightforward to check.

We fix the natural ordering on the set of vertices of its Dynkin diagram as in [79, Section 8.7], i.e. $1 < 2 < 3 < 4$ with 3 being the trivalent node. There are twelve positive roots and hence

also twelve cluster variables in every seed, with four frozen variables and eight unfrozen variables. The good Lyndon words (as well as the corresponding cuspidal representations in $R-gmod$) can be found in [79, Section 8.7] or can be directly computed using the algorithm described by Leclerc [84, Section 4.3]. Let us range them in the increasing order:

$$\mathcal{GL} = \{1 < 13 < 132 < 134 < 1342 < 13423 < 2 < 23 < 234 < 3 < 34 < 4\}.$$

Thus the corresponding convex order on the set of positive roots is given by

$$\begin{aligned}\Phi_+ = \{&\alpha_1 < \alpha_1+\alpha_3 < \alpha_1+\alpha_3+\alpha_2 < \alpha_1+\alpha_3+\alpha_4 < \alpha_1+\alpha_2+\alpha_3+\alpha_4\\ &< \alpha_1+2\alpha_3+\alpha_2+\alpha_4 < \alpha_2 < \alpha_2+\alpha_3 < \alpha_2+\alpha_3+\alpha_4 < \alpha_3 < \alpha_3+\alpha_4 < \alpha_4\}.\end{aligned}$$

The corresponding reduced expression of w_0 is $\mathbf{i}_{nat} = (1,3,2,4,3,1,4,3,2,4,3,4)$. Using [24, Theorem 3.7], we find the following dominant words for the determinantial modules of the seed $\mathcal{S}^{\mathbf{i}_{nat}}$:

$$1, 13, 132, 134, 134213, 134231, 2134, 23134213, 234132, 32134, 3423134213, 432134.$$

Among these determinantial modules, those whose dominant words are $134231, 234132, 3423134213$ and 432134 correspond to the frozen variables in $\mathbb{C}[N]$. Three of them turn out to be strongly homogeneous but one of them is not homogeneous, namely $L(3423134213)$. From now on we denote this module by M.

4.8.1 Computation of a graded character

In this subsection we determine the whole graded character of M, as an intermediate step for the computation of $\overline{D}([M])$.

As the isomorphism class of M is a frozen variable, it follows from the monoidal categorification results of Kang-Kashiwara-Kim-Oh [69] that M q-commutes with every simple module in $R-gmod$. In particular, it commutes with all the cuspidal representations in $R-gmod$. This strong property constrains the form of the graded character of this module. Actually, it is sufficient to use the fact that M q-commutes with the four cuspidal modules corresponding to the simple roots $\alpha_i, i \in \{1,\dots,4\}$. First one needs to determine the homogeneous degrees of the renormalized R-matrices $r_{M,L(i)}$ for every $i \in \{1,\dots,4\}$. We denote these degrees by $\Lambda(M, L(i))$ and $\Lambda(L(i), M)$ following [69]. The commuting of M and $L(i)$ yields an isomorphism of graded modules

$$M \circ L(i) \simeq q^{\Lambda(L(i),M)} L(i) \circ M \tag{4.8}$$

for every $i \in I$.

Lemma 4.8.1. *Let $M := L(3423134213)$. Then one has*

$$\Lambda(M, L(i)) = \Lambda(L(i), M) = 0$$

for every $1 \leqslant i \leqslant 4$,

Proof. We let $\beta := \mathrm{wt}(M) = 2(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$. It follows from the definition of $\Lambda(M,N)$ (see [66, 69]), that if $N \in R(\gamma)-gmod$ with $\gamma \in Q_+$ then

$$\Lambda(M,N) = -(\beta,\gamma) + 2(\beta,\gamma)_n - 2s_{M,N}$$

where $s_{M,N}$ is the largest non-negative integer s such that the image of $R_{M_z,N}$ is contained in $z^s(N \circ M_z)$. First consider $i \in \{1,2,4\}$. Then one has $(\alpha_i, \mathrm{wt}(M)) = 0$. Moreover, both $s_{L(i),M}$

and $s_{M,L(i)}$ are always smaller or equal to the number of occurrences of i in 3423134213 which is by definition $(\alpha_i, \mathrm{wt}(M))_n$. Hence one has

$$\Lambda(M, L(i)) \geqslant 0 \quad \text{and} \quad \Lambda(L(i), M) \geqslant 0.$$

By [69, Lemma 3.2.3], the commuting of M and $L(i)$ exactly means that the sum of these two quantities has to be zero. Hence one has $s_{L(i),M} = s_{M,L(i)} = 0$ and

$$\Lambda(M, L(i)) = \Lambda(L(i), M) = 0.$$

For $i = 3$ one can perform computations of $s_{M,L(i)}$ and $s_{L(i),M}$ in an analogous way as in [25]. □

Writing the q-commutations relations with every $L(i)$ and looking at the coefficients in front of every weight in these relations, one can compute the entire graded character of M.

Proposition 4.8.2. *The graded character of the frozen variable $L(3423134213)$ is given by:*

- *The following weights appear with q-dimension 1:*

$$3\{2,4\}313\{1,2,4\}3 \quad 3\{1,4\}323\{1,2,4\}3 \quad 3\{2,1\}343\{1,2,4\}3$$
$$3\{1,2,4\}313\{2,4\}3 \quad 3\{1,2,4\}323\{1,4\}3 \quad 3\{1,2,4\}343\{2,1\}3.$$

- *The following weights appear with q-dimension $q + q^{-1}$:*

$$3\{2,4\}3113\{2,4\}3 \quad 3\{2,1\}3443\{2,1\}3 \quad 3\{1,4\}3223\{1,4\}3 \quad 3\{1,2,4\}33\{1,2,4\}3.$$

In this statement, the notations $\{1,2,4\}$ means any of the six permutations of $1, 2$ and 4. Similarly $\{1,4\}$ means any of the words 14 or 41.

Proof. As above we set $M := L(3423134213)$. By definition, one has

$$M = \mathrm{hd}\,(L(34) \circ L(23) \circ L(1342) \circ L(13))$$

where $L(34), L(23), L(1342), L(13)$ are the cuspidal representations corresponding respectively to the positive roots $\alpha_3 + \alpha_4, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3$ for the natural ordering of the type D_4 Dynkin diagram. The graded characters of these four modules are known from Kleshchev-Ram [79, Section 8.7]. They are respectively given by $(34), (23), (1342) + (1324), (13)$. In particular the weights appearing in the graded character of M are either of the form $(34) \sqcup (23) \sqcup (1342) \sqcup (13)$ or of the form $(34) \sqcup (23) \sqcup (1324) \sqcup (13)$. We write

$$ch_q(M) = \sum_w P_w(q)(w)$$

with $P_w(q) \in \mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$ for every w. The graded character of the cuspidal module $L(i)$ is simply (i) for every $i \in I$. Thus for each $i \in I$ the graded isomorphisms (4.8) can be written in terms of quantum shuffle products of graded characters as

$$(i) \circ ch_q(M) = ch_q(M) \circ (i) \tag{4.9}$$

as the degrees of each of the corresponding R-matrix is zero by the previous lemma. We now show how these four equations strongly constrain the weights of $ch_q(M)$ as well as their coefficients in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1}]$.

All the weights of M begin with 3. First, the first letter of a weight of M is necessarily the first letter of one of the words $34, 23, 1342, 1324, 13$ and in particular it cannot be 4. If w is

a weight of M beginning with 2, we write $w = 2.w'$. Consider the equation (4.9) for $i = 2$. The word $2.2.w'$ appears in both hand sides and can only come from the shuffle of (2) with w as there are no shuffle of $34, 23, 1342, 13$ (or $34, 23, 1324, 13$) beginning with $2, 2$. Hence the quantum shuffle formula (see Proposition 4.2.5) yields

$$(1 + q^{-2})P_{2.w'}(q) = (1 + q^2)P_{2.w'}(q)$$

i.e. $P_w(q) = 0$.

If w is a weight of M beginning with 1, we write $w = 1.w'$. Let us first prove that w' cannot begin with 1. Assume so and write $w' = 1.w''$ and thus $w = 1.1.w''$. Consider the equation (4.9) for $i = 1$. The word $1.1.1.w$" appears in both hand sides and can only come from the shuffle of (1) with w as there are no shuffle of $34, 23, 1342, 13$ (or $34, 23, 1324, 13$) beginning with $1, 1, 1$. Hence the quantum shuffle product formula yields

$$(1 + q^{-2} + q^{-4})P_{1.1.w"}(q) = (1 + q^2 + q^4)P_{1.1.w"}(q)$$

i.e. $P_w(q) = 0$. This proves that w' does not begin with 1, i.e. there are no weights of M beginning with $1, 1$. Hence applying the same argument as for $i = 2$ yields $P_w(q) = P_{1.w'}(q) = 0$.

There are at least two letters between the first two occurrences of 3. Assume w is a weight of M of the form $3, i, 3.w'$ (with $i \neq 3$). Consider the equation (4.9) for $i = 3$. The word $3, i, 3, 3.w'$ appears in both hand sides and can only come from the shuffle of (3) with $3, i, 3.w' = w$ as there are no shuffle of $34, 23, 1342, 13$ (or $34, 23, 1324, 13$) beginning with $3, i, 3, 3$ and we have just proved that there are no weights of M beginning with i. Hence the quantum shuffle product formula yields

$$(q^{-1} + q^{-3})P_w(q) = (q + q^{-1})P_w(q)$$

i.e. $P_w(q) = 0$.

There is at most one occurrence of each letter $1, 2, 4$ **between the first two occurrences of** 3. Recall that there are exactly two occurences of every letter $1, 2, 4$ and four occurrences of 3 in any weight of M. We begin by showing that if w is a weight of M of the form $3.u.i, i.v$ with u not containing 3 then $P_w(q) = 0$. Indeed consider the equation (4.9) for i. The word $3.u.i, i, i.v$ appears in both hand sides and can only come from the shuffle of (i) with $3.u.i, i.v = w$. The quantum shuffle formula yields

$$(q + q^{-1} + q^{-3})P_w(q) = (q^3 + q + q^{-1})P_w(q)$$

which proves $P_w(q) = 0$ in this case.

Now assume w is a weight of M of the form $3.w'.3.w$" where w' contains two occurrences of a letter $i \in \{1, 2, 4\}$; then w" does not contain any occurrence of this letter i and contains two occurrences of 3. Write $w' = u_1.i.u_2.iu_3$ with u_1, u_2, u_3 containing neither i nor 3 and consider the equation (4.9) for i. The word $3.u_1.i.u_2.i, i.u_3.3.w$" appears in both hand sides and can only come from the shuffle of (i) with w as we have just proved that there are no weights of M of the form $3.u_1.u_2.i, i.u_3.3.w$". Hence the quantum shuffle formula yields

$$(q^{-1} + q^{-3})P_w(q) = (q^3 + q)P_w(q)$$

i.e. $P_w(q) = 0$.

This proves that the weights of M begin either with $3\{1, 2\}3$ or $3\{1, 4\}3$ or $3\{2, 4\}3$ or $3\{1, 2, 4\}3$. The exact same arguments can be applied in a symmetric way to successively prove that

1. All the weights of M end with 3.

2. There are at least two letters between the last two occurrences of 3.

3. There is at most one occurrence of each letter $1, 2, 4$ between the last two occurrences of 3.

Therefore we can write any weight w of M as

$$w = 3.u_1.3.u_2.3.u_3.3$$

with u_1 and u_3 either of the form $\{1, 2\}$ or $\{1, 4\}$ or $\{2, 4\}$ or $\{1, 2, 4\}$ and u_2 is a word containing at most two letters (necessarily $1, 2$ or 4).

If u_2 contains a letter $i \in \{1, 2, 4\}$, then u_1 and u_3 are either of the form $\{1, 2, 4\}$ or $\{j, k\}$ with $j, k \neq i$. Assume u_2 contains i and for example u_1 of the form $\{i, j\}$ ($j \neq i$). Then i does not appear in u_3 as there are only two occurrences of i in w. As u_2 is of length at most 2, i is necessarily the first or the last letter of u_2 (or both if u_2 contains only i). Assume for example u_2 begins with i and consider the equation (4.9) for i. The word $3.u_1.3.i.u_2.3.u_3.3$ appears in both hand sides and as there is no weight of M with only one single letter between the first two occurrences of 3, the quantum shuffle formula yields

$$(1 + q^{-2})P_w(q) = (q^2 + 1)P_w(q)$$

and thus $P_w(q) = 0$ which proves the desired statement. It follows in particular that u_2 cannot be composed of two distinct letters (in that case, both u_1 and u_3 could be only of the form $\{1, 2, 4\}$ but then w would contain three occurrences of a letter $1, 2$ or 4). It is then straightforward that the only remaining possible weights are the ones listed in the statement of Proposition 4.8.2.

To finish the proof of Proposition 4.8.2, it remains to compute the values of $P_w(q)$ for every weight w in this list. The starting point is that by [79, Theorem 7.2 (ii)], we know that the weight space of M corresponding to the highest weight has q-dimension 1. From this we can deduce the q-dimensions of all the other weight spaces of M.

For every distinct $i, j \in \{1, 2, 4\}$, one has $P_{3,i,j,3.w'}(q) = P_{3,j,i,3.w'}(q)$ for any w'. As there is no weight of M with three letters between the first two occurrences of 3, one can consider the equation (4.9) for i where the word $3, i, j, i, 3.w'$ appears in both hand sides. Then the quantum shuffle formula gives

$$q^{-1}P_{3,i,j,3.w'}(q) = q^{-1}P_{3,j,i,3.w'}(q)$$

which is the desired statement. Using similar arguments one can also show that for every permutation σ of the set $\{1, 2, 4\}$ one has

$$P_{3,1,2,4,3.w'}(q) = P_{3,\sigma(1),\sigma(2),\sigma(4),3.w'}(q)$$

for any w'. The symmetric statements are also valid i.e.

1. For every distinct $i, j \in \{1, 2, 4\}$, one has $P_{w".3,i,j,3}(q) = P_{w".3,j,i,3}(q)$ for any $w"$.

2. For every permutation σ of $\{1, 2, 4\}$ one has $P_{w".3,1,2,4,3}(q) = P_{w".3,\sigma(1),\sigma(2),\sigma(4),3}(q)$ for any $w"$.

In particular applying this with the highest weight 3423134213 we get that all the weights of the form $3\{2, 4\}313\{1, 2, 4\}3$ are of q-dimension 1. Then one can apply equation (4.9) for 1 and look for words of the form $3\{2, 4\}3113\{1, 2, 4\}3$. The quantum shuffle formula yields

$$\begin{aligned}(q^2 + 1)P_{3\{2,4\}313\{1,2,4\}3}(q) &+ q^{-2}P_{3\{2,4\}3113\{2,4\}3}(q)\\ &= (1 + q^{-2})P_{3\{2,4\}313\{1,2,4\}3}(q) + qP_{3\{2,4\}3113\{2,4\}3}(q).\end{aligned}$$

Knowing that $P_{3\{2,4\}313\{1,2,4\}3}(q) = 1$, this implies

$$P_{3\{2,4\}3113\{2,4\}3}(q) = q + q^{-1}.$$

Then looking for words of the form $3\{1,2,4\}3113\{2,4\}3$ in equation (4.9) for 1 and applying the quantum shuffle formula, we get

$$P_{3\{1,2,4\}313\{2,4\}3}(q) = 1.$$

Then one can look for words of the form $3\{1,2,4\}313\{1,2,4\}3$ in equation (4.9) for 1 and we get

$$P_{3\{1,2,4\}33\{1,2,4\}3}(q) = q + q^{-1}.$$

Then considering equation (4.9) for 2 and then 4, one can deduce the q-dimensions of the rest of the weight spaces listed in the statement of Proposition 4.8.2. This finishes the proof.

□

4.8.2 Computations on equivariant multiplicities

We can now finish the proof of Theorem 4.5.10 in type D_4. Evaluating at $q = 1$ the graded dimensions given in Proposition 4.8.2, we can use Equation 4.7 to deduce the equivariant multiplicity of $[M]$. This is done using the formal calculation software SAGE.

Corollary 4.8.3. *The equivariant multiplicity of the frozen variable* $[L(3423134213)] \in \mathbb{C}[N]$ *is given by:*

$$\frac{1}{\left(\prod_{J\subset\{1,2,4\}}\left(\alpha_3 + \sum_{j\in J}\alpha_j\right)\right)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)^2}.$$

Using similar arguments, one can also compute the equivariant multiplicities of the two remaining determinantial modules that are not homogeneous, namely $L(134213)$ and $L(23134213)$.

Lemma 4.8.4. *The equivariant multiplicity of* $[L(134213)] \in \mathbb{C}[N]$ *is given by:*

$$\frac{1}{{\alpha_1}^2(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_4+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

The equivariant multiplicity of $[L(23134213)] \in \mathbb{C}[N]$ *is given by:*

$$\frac{1}{\alpha_1\alpha_2(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

This proves that the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies the Property (A). The polynomials $P_j, 1 \leqslant j \leqslant 12$ are given by:

$$P_1 = \alpha_1 \qquad P_2 = \alpha_1(\alpha_1+\alpha_3)$$
$$P_3 = \alpha_1(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3) \qquad P_4 = \alpha_1(\alpha_1+\alpha_3)(\alpha_1+\alpha_3+\alpha_4)$$
$$P_5 = {\alpha_1}^2(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_4+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$
$$P_6 = \alpha_1(\alpha_1+\alpha_3)(\alpha_1+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$$
$$P_7 = \alpha_2\alpha_1(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$
$$P_8 = \alpha_1\alpha_2(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$$
$$P_9 = \alpha_2(\alpha_2+\alpha_3)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$$
$$P_{10} = \alpha_3(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$$
$$P_{11} = \left(\prod_{J\subset\{1,2,4\}}\left(\alpha_3 + \sum_{j\in J}\alpha_j\right)\right)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)^2$$
$$P_{12} = \alpha_4(\alpha_3+\alpha_4)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+2\alpha_3+\alpha_4).$$

Let us now write down the equalities required by the Property (B). For example, let us detail the cases of $j = 6$ and $j = 8$.

The positive root β_6 is $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$. Moreover, $i_6 = 1$, $6_- = 1$ and $6_+ = N + 1 = 13$ as there is no occurrence of the letter 1 after the position 6. The node 1 is monovalent in the Dynkin diagram of type D_4 and its only neighbour is 3. The last occurrence of 3 in $\mathbf{i}_{nat}$ before the position 6 is in position 5. Thus Property (B) can be written as

$$P_6P_1 = (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)P_5.$$

The positive root β_8 is $\alpha_2 + \alpha_3$. Moreover, $i_8 = 3$, $8_- = 5$ and $8_+ = 11$. The node 3 is the trivalent node. The last occurrences before the position 8 of each of its neighbours are thus 3 (for the node 2), 6 (for the node 1) and 7 (for the node 4). Hence the Property (B) can be written as

$$P_8P_5 = (\alpha_2 + \alpha_3)P_2P_6P_7.$$

The other equalities can be obtained in the same way and are listed below:

$$P_1 = \alpha_1 \qquad P_2 = (\alpha_1 + \alpha_3)P_1 \qquad P_3 = (\alpha_1 + \alpha_2 + \alpha_3)P_2$$
$$P_4 = (\alpha_1 + \alpha_3 + \alpha_4)P_2 \qquad P_5P_2 = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)P_1P_3P_4$$
$$P_6P_1 = (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)P_5 \qquad P_7P_4 = \alpha_2P_5$$
$$P_8P_5 = (\alpha_2 + \alpha_3)P_2P_6P_7 \qquad P_9P_3 = (\alpha_2 + \alpha_3 + \alpha_4)P_8$$
$$P_{10}P_7 = \alpha_3P_8 \qquad P_{11}P_8 = (\alpha_3 + \alpha_4)P_6P_9P_{10} \qquad P_{12}P_{10} = \alpha_4P_{11}$$

These equalities are straightforward to check by hand using the explicit values of the $P_j, 1 \leqslant j \leqslant 12$.

Unlike the case where $\mathfrak{g}$ is of type A_n, the Property (C) is here non trivial a priori. For instance, the positive root α_1 appears with multiplicity 2 in P_5. The required inequality is thus guaranteed by the fact that α_1 also divides $P_{5_+} = P_8$ (with multiplicity 1). Similarly, one has $(\alpha_1+\alpha_2+\alpha_3; P_8) = 2$ and the inequality (C) follows from the fact that $(\alpha_1 + \alpha_2 + \alpha_3; P_{8_+}) = (\alpha_1 + \alpha_2 + \alpha_3; P_{11}) = 1$.

Finally, we have proven that the standard seed $\mathcal{S}^{\mathbf{i}_{nat}}$ satisfies the Properties (A), (B) and (C).

4.9 Cluster theory of homogeneous modules

In this conclusive section, we discuss various evidences of the connections between the determinantial modules categorifying the flag minors of $\mathbb{C}[N]$ and the (prime) strongly homogeneous modules of $R - gmod$ in the sense of Kleshchev-Ram [78]. This leads us to propose a conjectural criterion of primeness of the homogeneous module $S(w)$ $(w \in W)$.

Recall from Section 4.3.1 the subsets $\mathcal{FC}, \mathcal{M}in, \mathcal{M}in^+$ of W. We will consider certain subsets of these sets, by intersecting them with the set of *strict* elements of W defined as follows.

Definition 4.9.1. *An element $w \in W$ is called strict if for any reduced expression $\mathbf{i} = (i_1, \ldots, i_l) \in Red(w)$, one has*

$$\forall k \in \{1, \ldots, l\}, \exists j \leqslant k < l, i_j \cdot i_l \neq 0.$$

In other words, there is no gap in any reduced expression of w. We let W_0 denote the set of all strict elements of W and we set

$$\mathcal{FC}_0 := W_0 \cap \mathcal{FC} \quad , \quad \mathcal{M}in_0 := W_0 \cap \mathcal{M}in \quad , \quad \mathcal{M}in_0^+ := W_0 \cap \mathcal{M}in^+.$$

As we saw in Example 4.5.3, the determinantial modules whose classes in $\mathbb{C}[N]$ are flag minors coincide with the prime strongly homogeneous when $\mathfrak{g} = \mathfrak{sl}_4$. We also noticed that the list of Weyl group elements parametrizing these modules is exactly $\mathcal{M}in^+ \backslash \{s_3s_1\}$. Thus it is immediate that this list is in fact the list of elements of $\mathcal{M}in_0^+$. The same observation can be checked for $\mathfrak{g} = \mathfrak{sl}_5$. We propose the following:

Conjecture 4.9.2. *Let $\mathfrak{g}$ be of type A_n. The set of cluster variables of the seeds $\mathcal{S}^{\mathbf{i}_<}$ ($<$ running over all the possible orderings on I) is exactly the set of isomorphism classes of the strongly homogeneous modules $S(w), w \in \mathcal{M}in_0^+$.*

As we saw in Section 4.8, the determinantial modules corresponding to flag minors are not necessarily homogeneous when $\mathfrak{g}$ is of type D_4. However, it is not hard in this case to list the strict dominant minuscule elements of W. We obtain the following:

$$
\begin{gathered}
s_1 \quad s_2 \quad s_3 \quad s_4 \\
s_1s_3 \quad s_3s_1 \quad s_2s_3 \quad s_3s_2 \quad s_4s_3 \quad s_3s_4 \\
s_1s_2s_3 \quad s_1s_3s_2 \quad s_4s_2s_3 \quad s_4s_3s_2 \quad s_2s_3s_1 \quad s_2s_3s_4 \quad s_1s_3s_4 \quad s_4s_3s_1 \quad s_1s_4s_3 \\
s_3s_1s_2s_3 \quad s_3s_1s_4s_3 \quad s_3s_2s_2s_3 \quad s_4s_2s_1s_3 \quad s_4s_2s_3s_1 \quad s_1s_2s_3s_4 \quad s_1s_4s_3s_2 \\
s_3s_1s_2s_3s_4 \quad s_3s_1s_4s_3s_2 \quad s_3s_2s_4s_3s_1 \\
s_4s_3s_1s_2s_3s_4 \quad s_2s_3s_1s_4s_3s_2 \quad s_1s_3s_2s_4s_3s_1.
\end{gathered}
$$

Using Theorem 4.2.7, one can compute the determinantial modules corresponding to the cluster variables of the seed $\mathcal{S}^{\mathbf{i}}$, $\mathbf{i}$ coming from a total ordering on I. Unlike the type A_n case though, this is not sufficient to get all the homogeneous modules $S(w)$, $w \in \mathcal{M}in_0^+$. Nonetheless, it is not hard to compute the determinantial modules of the remaining standard seeds by performing certain well-chosen mutations as in Section 4.6. Then one can observe that for w in the list above, the isomorphism class of the homogeneous module $S(w)$ is always a flag minor. We propose the following conjecture:

Conjecture 4.9.3. *Let $\mathfrak{g}$ be a simple Lie algebra of (finite) simply-laced type. Then for any $w \in \mathcal{M}in_0^+$, the class of $S(w)$ is a flag minor in $\mathbb{C}[N]$.*

Remark 4.9.4. Note that the assumption that w is dominant minuscule is crucial. Indeed, consider for instance $\mathfrak{g}$ of type D_4 and $w := s_3s_1s_2s_4s_3$ (3 is the trivalent node). Then w is fully-commutative but not dominant minuscule. On the other hand, one can show that the homogeneous module $S(w)$ is not real. In particular by the results of [69] it is not a cluster monomial and a fortiori not a determinantial module.

This conjecture would imply the following primeness criterion for homogeneous strongly homogeneous modules:

Conjecture 4.9.5. *The prime strongly homogeneous modules in $R-gmod$ are exactly the modules of the form $S(w), w \in \mathcal{M}in_0^+$.*

Proof. Indeed, Conjecture 4.9.3 would imply that if $w \in \mathcal{M}in_0^+$, then $S(w)$ is prime (as it categorifies a cluster variable). But conversely, it is easy to check that if w is not strict, then the module $S(w)$ can be decomposed as a convolution product of two simple modules, and thus it is not prime. Thus it would imply Conjecture 4.9.5. □

Appendices

Appendix A

Computations of equivariant multiplicities

This Appendix is devoted to the computations of the equivariant multiplicities of several cluster variables of the seed $\mathcal{S}^{\mathbf{i}_{nat}}$ of $\mathbb{C}[N]$ in type D_4. As explained in Section 4.8, the cluster structure of $\mathbb{C}[N]$ is of rank twelve, with four frozen and eight unfrozen cluster variables in every seed. We focus on a particular seed $\mathcal{S}^{\mathbf{i}_{nat}}$ arising from the construction of Kang-Kashiwara-Kim-Oh [69] adapting former results of Geiss-Leclerc-Schröer [52] to the setting of monoidal categorification. In Section 4.8, we described the cluster variables of this seed in terms of the root partitions (or dominant words) of the corresponding simple modules in $R - gmod$. Among these twelve simple modules, nine of them are strongly homogeneous in the sense of Kleshchev-Ram [78] and thus their equivariant multiplicities are given by Proposition 4.5.1. However the three remaining modules are not homogeneous and thus one has to compute their equivariant multiplicities by hand.

We begin with the computation of the equivariant multiplicity of the frozen variable $[L(3423134213)]$. The graded character of the simple module $L(3423134213)$ has been computed in Section 4.8. From this one can deduce the equivariant multiplicity $\bar{D}(L(3423134213))$ via the following SAGE program.

```
# coding: utf-8

# In[4]:

var('a1,a2,a3,a4')
f1(t) = (2*(a1+t)+a2+a4)*(a2+t)*(a4+t)*(a2+t+a4);
f2(t) = (2*(a2+t)+a1+a4)*(a1+t)*(a4+t)*(a1+t+a4);
f4(t) = (2*(a4+t)+a2+a1)*(a2+t)*(a1+t)*(a2+t+a1);
f(t) = f1(t) + f2(t) + f4(t)


# In[5]:

var('a1,a2,a3,a4')
f14(t) = (2*t+a1+a4)*(t+a2)*(t+a1+a2)*(t+a2+a4);
f12(t) = (2*t+a1+a2)*(t+a4)*(t+a1+a4)*(t+a2+a4);
f24(t) = (2*t+a2+a4)*(t+a1)*(t+a1+a2)*(t+a1+a4);
```

```
# In[10]:

var('a1,a2,a3,a4')
N11 = f24(a3)*(2*(2*a1+a2+a4+3*a3)+a2+a4)*(2*a3+a1+a2)*(2*a3+a1+a4)*(2*
    a3+a1+2*a2+a4)*(2*a3+a1+a2+2*a4)*(3*a3+a1+2*a2+a4)*(3*a3+a1+a2+2*a4)
    *(a1+2*a2+3*a3+2*a4);
N22 = f14(a3)*(2*(2*a2+a1+a4+3*a3)+a1+a4)*(2*a3+a1+a2)*(2*a3+a2+a4)*(2*
    a3+2*a1+a2+a4)*(2*a3+a1+a2+2*a4)*(3*a3+2*a1+a2+a4)*(3*a3+a1+a2+2*a4)
    *(2*a1+a2+3*a3+2*a4);
N44 = f12(a3)*(2*(2*a4+a2+a1+3*a3)+a2+a1)*(2*a3+a1+a4)*(2*a3+a2+a4)*(2*
    a3+a1+2*a2+a4)*(2*a3+2*a1+a2+a4)*(3*a3+a1+2*a2+a4)*(3*a3+2*a1+a2+a4)
    *(2*a1+2*a2+3*a3+a4);
N1 = f24(a3)*f(3*a3+a1+a2+a4)*(2*a3+a1+a2)*(2*a3+a1+a4);
N2 = f14(a3)*f(3*a3+a1+a2+a4)*(2*a3+a1+a2)*(2*a3+a2+a4);
N4 = f12(a3)*f(3*a3+a1+a2+a4)*(2*a3+a1+a4)*(2*a3+a2+a4);
N33 = f(a3)*(2*a3+a1+a2)*(2*a3+a1+a4)*(2*a3+a2+a4)*f(3*a3+a1+a2+a4);
N31 = f(a3)*(2*a3+a1+2*a2+a4)*(2*a3+a1+a2+2*a4)*(3*a3+a1+2*a2+a4)*(3*a3
    +a1+a2+2*a4)*(a1+2*a2+3*a3+2*a4)*(2*(3*a3+2*a1+a2+a4)+a2+a4);
N32 = f(a3)*(2*a3+2*a1+a2+a4)*(2*a3+a1+a2+2*a4)*(3*a3+2*a1+a2+a4)*(3*a3
    +a1+a2+2*a4)*(2*a1+a2+3*a3+2*a4)*(2*(3*a3+a1+2*a2+a4)+a1+a4);
N34 = f(a3)*(2*a3+2*a1+a2+a4)*(2*a3+a1+2*a2+a4)*(3*a3+2*a1+a2+a4)*(3*a3
    +a1+2*a2+a4)*(2*a1+2*a2+3*a3+a4)*(2*(3*a3+a1+a2+2*a4)+a1+a2);
N = (2*(N11 + N22 + N44)*(a1+a2+a3+a4) + (N31 + N32 + N34)*(2*a3+a1+a2)
    *(2*a3+a1+a4)*(2*a3+a2+a4))*(3*a3+a1+a2+a4) + ((N1 + N2 + N4)*(a1+a2
    +a3+a4) + 2*N33)*(2*a3+a1+2*a2+a4)*(2*a3+a1+a2+2*a4)*(2*a3+2*a1+a2+
    a4);
D = a3*(2*a1 + 2*a2 + 2*a4 + 3*a3)*2*(a1+a2+a4+2*a3)*(a1+a3)*(a2+a3)*(
    a4+a3)*(a3+a1+a2)*(a3+a1+a4)*(a3+a2+a4)*(a1+a2+a3+a4)*(a1+a2+a4+2*a3
    )*(2*a3+a1+a2)*(2*a3+a1+a4)*(2*a3+a2+a4)*(2*a3+a1+2*a2+a4)*(2*a3+a1+
    a2+2*a4)*(2*a3+2*a1+a2+a4)*(3*a3+a1+a2+a4)*(3*a3+a1+2*a2+a4)*(3*a3+
    a1+a2+2*a4)*(3*a3+2*a1+a2+a4)*(2*a1+2*a2+3*a3+a4)*(2*a1+a2+3*a3+2*a4
    )*(a1+2*a2+3*a3+2*a4);


# In[11]:

(N/D).factor()
```

The result is the following:

```
1/((a1 + a2 + 2*a3 + a4)^2*(a1 + a2 + a3 + a4)*(a1 + a2 + a3)*(a1 + a3
    + a4)*(a1 + a3)*(a2 + a3 + a4)*(a2 + a3)*(a3 + a4)*a3).
```

Now we do the same for the cluster variable corresponding to the class of the simple module $L(134213)$.

```
# coding: utf-8

# In[1]:
```

```
var('a1,a2,a3,a4')
f1(t) = (2*(a1+t)+a2+a4)*(a2+t)*(a4+t)*(a2+t+a4);
f2(t) = (2*(a2+t)+a1+a4)*(a1+t)*(a4+t)*(a1+t+a4);
f4(t) = (2*(a4+t)+a2+a1)*(a2+t)*(a1+t)*(a2+t+a1);
f(t) = f1(t) + f2(t) + f4(t)


# In[2]:

var('a1,a2,a3,a4')
Nc1 = f1(a1+a3);
Nc2 = f2(a1+a3);
Nc4 = f4(a1+a3);
Nc = (Nc1 + Nc2 + Nc4)*a1 + (a1+a3)*(a1+a3+a4)*(a2+a3+a1)*(a1+a2+a3+a4)
    *(4*a1+a2+2*a3+a4);
Dc = a1^2*(a1+a3)*(2*a1+a3)*(a1+a3+a4)*(a2+a3+a1)*(a1+a2+a3+a4)*(2*a1+
    a3+a4)*(a2+a3+2*a1)*(2*a1+a2+a3+a4)*(2*a1+a2+2*a3+a4);
Nc.factor()


# In[3]:

var('a1,a2,a3,a4')
(Nc/Dc).factor()
```

The result is the following:

```
1/((a1 + a2 + a3 + a4)*(a1 + a2 + a3)*(a1 + a3 + a4)*(a1 + a3)*a1^2).
```

Appendix B

Cluster variables and strict dominant minuscule words

Now we present an algorithm in order to illustrate Conjecture 4.5.11 in type A_5. First it is straightforward to list by hand all the strict dominant minuscule words in Σ_5 (i.e. the elements of $\mathcal{M}in_0^+$ with the notations of Sections 4.3.1 and 4.5). On the other hand we compute a sufficiently large number of dominant words corresponding to cluster variables starting from the initial seed $\mathcal{S}^{\mathbf{i}_{nat}}$. On the example below we performed sequences of 7 successive mutations in arbitrary (consecutively distinct) exchangeable directions. Then it remains to check that the list we obtain this way contains all the elements of $\mathcal{M}in_0^+$.

```
# coding: utf-8

# In[3]:

def root(k):
    if k==0:
        return 5
    else :
        if k==1:
            return 45
        else :
            if k==2:
                return 4
            else :
                if k==3:
                    return 345
                else :
                    if k==4:
                        return 34
                    else :
                        if k==5:
                            return 3
                        else :
                            if k==6:
                                return 2345
                            else :
```

```
                    if k==7:
                        return 234
                    else:
                        if k==8:
                            return 23
                        else :
                            if k==9:
                                return 2
                            else :
                                if k==10:
                                    return 12345
                                else :
                                    if k==11:
                                        return 1234
                                    else :
                                        if k==12:
                                            return 123
                                        else :
                                            if k==13:
                                                return 12
                                            else :
                                                if k==14:
                                                    return
                                                        1


# In[4]:

def wordpara(L):
    s=''
    for k in range(0,15):
        for i in range(0,L[k]):
            s += str(root(k))
    return s


# In[5]:

wordpara([0,0,0,0,0,0,0,0,0,0,0,0,0,1,0])


# In[6]:

wordpara([0,0,0,1,0,0,0,1,0,0,0,0,1,0,0])


# In[7]:

def first(L1,L2):
```

```
    k=0;
    while L1[k]==L2[k]:
        k=k+1
    return k


# In[8]:

first([0,0,1,2],[0,0,2,3])


# In[9]:

def compare(L1,L2):
    if first(L1,L2)==len(L1)+1:
        return 0
    else :
        if L1[first(L1,L2)] > L2[first(L1,L2)]:
            return 1
        else :
            if L1[first(L1,L2)] < L2[first(L1,L2)]:
                return -1


# In[10]:

compare([0,0,1,2],[0,0,2,3])


# In[11]:

def f(n,k,B,i,j):
    if i==k:
        return -B[i][j]
    else:
        if j==k:
            return -B[i][j]
        else:
            return B[i][j] + B[k][j]*max(0,-B[i][k]) + B[i][k]*max(0,B[
                k][j])


# In[12]:

def mutmatrix(n,k,B):
    return matrix([[f(n,k,B,i,j) for j in range(0,n*(n-1)/2)] for i in
        range(0,n*(n+1)/2)])
```

```
# In[13]:

def somme_interm_plus(n,k,L,B):
    s= vector([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0])
    for i in range(0,n*(n+1)/2):
        if B[i][k] >= 0:
            s = s + B[i][k]*L[i]
    return s


# In[14]:

def somme_interm_minus(n,k,L,B):
    s= vector([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0])
    for i in range(0,n*(n+1)/2):
        if B[i][k] <= 0:
            s = s - B[i][k]*L[i]
    return s


# In[15]:

def mutvect(n,k,L,B,i):
    if i<>k:
        return L[i]
    else:
        if compare(somme_interm_plus(n,k,L,B),somme_interm_minus(n,k,L,
           B))==1:
            return -L[k] + somme_interm_plus(n,k,L,B)
        else:
            return -L[k] + somme_interm_minus(n,k,L,B)


# In[16]:

def mutlist(n,k,L,B):
    return [mutvect(n,k,L,B,j) for j in range(0,n*(n+1)/2)]


# In[17]:

n=5;
L0=[vector([0,0,0,0,0,0,0,0,0,0,0,0,0,0,1]),vector
    ([0,0,0,0,0,0,0,0,0,0,0,0,0,1,0]),vector
    ([0,0,0,0,0,0,0,0,0,1,0,0,0,0,1]),vector
    ([0,0,0,0,0,0,0,0,0,0,0,0,1,0,0]),vector
    ([0,0,0,0,0,0,0,0,1,0,0,0,0,1,0]),vector
    ([0,0,0,0,0,1,0,0,0,1,0,0,0,0,1]),vector
    ([0,0,0,0,0,0,0,0,0,0,0,1,0,0,0]),vector
```

```
    ([0,0,0,0,0,0,0,1,0,0,0,0,1,0,0]),vector
    ([0,0,0,0,1,0,0,0,1,0,0,0,0,1,0]),vector
    ([0,0,1,0,0,1,0,0,0,1,0,0,0,0,1]),vector
    ([0,0,0,0,0,0,0,0,0,0,1,0,0,0,0]),vector
    ([0,0,0,0,0,0,1,0,0,0,0,1,0,0,0]),vector
    ([0,0,0,1,0,0,0,1,0,0,0,0,1,0,0]),vector
    ([0,1,0,0,1,0,0,0,1,0,0,0,0,1,0]),vector
    ([1,0,1,0,0,1,0,0,0,1,0,0,0,0,1])];
B0=matrix
    ([[0,-1,1,0,0,0,0,0,0,0],[1,0,-1,-1,1,0,0,0,0,0],[-1,1,0,0,-1,1,0,0,0,0],[0,1,0
```

```
# In[18]:

print B0


# In[19]:

def wordseed(S):
    s=''
    for p in range(0,10):
        s = s + wordpara(S[p]) + ','
    return s


# In[20]:

wordseed(L0)


# In[21]:

def addlist(S,v):
    if v in S:
        return S
    else:
        return S + [v]


# In[22]:

def addlistbis(S,L,v):
    if v in L:
        return S + [L]
    else:
        return S
```

```
# In[23]:

addlist(['123','231','345'],'456')


# In[24]:

def listvariable(i,j):
    stock=[];L=L0;memoL=L0;memoB=B0;memobisL=L0;memobisB=B0;memoterL=L0
        ;memoterB=B0
    for k1 in range(i,j):
        L=mutlist(5,k1,L0,B0)
        memoL=L
        memoB=mutmatrix(5,k1,B0)
        stock=addlist(stock,wordpara(L[k1]))
        for k2 in range(i,j):
            if k2 <> k1:
                L=mutlist(5,k2,memoL,memoB)
                memobisL=L
                memobisB=mutmatrix(5,k2,memoB)
                stock=addlist(stock,wordpara(L[k2]))
                for k3 in range(i,j):
                    if k3 <> k2:
                        L=mutlist(5,k3,memobisL,memobisB)
                        memoterL=L
                        memoterB=mutmatrix(5,k3,memobisB)
                        stock=addlist(stock,wordpara(L[k3]))
                        for k4 in range(i,j):
                            if k4 <> k3:
                                L = mutlist(5,k4,memoterL,memoterB)
                                stock=addlist(stock,wordpara(L[k4]))
    return stock


# In[25]:

S4 = listvariable(0,10)
print S4
len(S4)


# In[46]:

def listvariablebis(i,j):
    stock=[];L=L0;memoL=L0;memoB=B0;memobisL=L0;memobisB=B0;memoterL=L0
        ;memoterB=B0;memoquadL=L0;memoquadB=B0
    for k1 in range(i,j):
        L=mutlist(5,k1,L0,B0)
```

```
        memoL=L
        memoB=mutmatrix(5,k1,B0)
        stock=addlist(stock,wordpara(L[k1]))
        for k2 in range(i,j):
            if k2 <> k1:
                L=mutlist(5,k2,memoL,memoB)
                memobisL=L
                memobisB=mutmatrix(5,k2,memoB)
                stock=addlist(stock,wordpara(L[k2]))
                for k3 in range(i,j):
                    if k3 <> k2:
                        L=mutlist(5,k3,memobisL,memobisB)
                        memoterL=L
                        memoterB=mutmatrix(5,k3,memobisB)
                        stock=addlist(stock,wordpara(L[k3]))
                        for k4 in range(i,j):
                            if k4 <> k3:
                                L = mutlist(5,k4,memoterL,memoterB)
                                memoquadL=L
                                memoquadB=mutmatrix(5,k4,memoterB)
                                stock=addlist(stock,wordpara(L[k4]))
                                for k5 in range(i,j):
                                    if k5 <> k4:
                                        L=mutlist(5,k5,memoquadL,
                                            memoquadB)
                                        stock=addlist(stock,wordpara(L[
                                            k5]))
    return stock


# In[47]:

S5=listvariablebis(0,10)
print S5
len(S5)


# In[32]:

Ldomin=['1','2','3','4','5','12','23','34','45','21','32','43','54','
    123','234','345','2312','3423','4534','321','432','543','312','423',
    '534','1234','1243','1432','34123','42312','234123','342312','2345',
    '2354','2543','45234','53423','345234','453423','51234','451234','
    534123','3451234','4534123','542312','5234123','45234123']
len(Ldomin)


# In[27]:
```

```
len(S4)


# In[40]:

def inlist(s,L):
    res=0;k=0;
    while k < len(L):
        if s<>L[k]:
            k=k+1
        else:
            res=1;
            k=len(L)+1
    return res


# In[41]:

inlist('231',Ldomin)


# In[42]:

inlist('312',Ldomin)


# In[ ]:




# In[44]:

k=0;
for j in range(0,len(S4)):
    if inlist(S4[j],Ldomin)==1:
        k=k+1
k


# In[48]:

k=0;
for j in range(0,len(S5)):
    if inlist(S5[j],Ldomin)==1:
        k=k+1
k
```

```
# In[49]:

inlist('45',S4)


# In[50]:

inlist('45',S5)


# In[55]:

def listvariableter(i,j):
    stock=[];L=L0;memoL=L0;memoB=B0;memobisL=L0;memobisB=B0;memoterL=L0
        ;memoterB=B0;memoquadL=L0;memoquadB=B0;memoquintL=L0;memoquintB=
        B0;
    for k1 in range(i,j):
        L=mutlist(5,k1,L0,B0)
        memoL=L
        memoB=mutmatrix(5,k1,B0)
        stock=addlist(stock,wordpara(L[k1]))
        for k2 in range(i,j):
            if k2 <> k1:
                L=mutlist(5,k2,memoL,memoB)
                memobisL=L
                memobisB=mutmatrix(5,k2,memoB)
                stock=addlist(stock,wordpara(L[k2]))
                for k3 in range(i,j):
                    if k3 <> k2:
                        L=mutlist(5,k3,memobisL,memobisB)
                        memoterL=L
                        memoterB=mutmatrix(5,k3,memobisB)
                        stock=addlist(stock,wordpara(L[k3]))
                        for k4 in range(i,j):
                            if k4 <> k3:
                                L = mutlist(5,k4,memoterL,memoterB)
                                memoquadL=L
                                memoquadB=mutmatrix(5,k4,memoterB)
                                stock=addlist(stock,wordpara(L[k4]))
                                for k5 in range(i,j):
                                    if k5 <> k4:
                                        L = mutlist(5,k5,memoquadL,
                                            memoquadB)
                                        memoquintL=L
                                        memoquintB=mutmatrix(5,k5,
                                            memoquadB)
                                        stock=addlist(stock,wordpara(L[
                                            k5]))
                                        for k6 in range(i,j):
```

```
                                                    if k6<>k5:
                                                        L=mutlist(5,k6,
                                                            memoquintL,
                                                            memoquintB)
                                                        stock=addlist(stock,
                                                            wordpara(L[k6]))
    return stock


# In[56]:

S6=listvariableter(0,10)
print S6
len(S6)


# In[57]:

k=0;
for j in range(0,len(S6)):
    if inlist(S6[j],Ldomin)==1:
        k=k+1
k


# In[58]:

def listvariables(i,j):
    stock=[];L=L0;memoL=L0;memoB=B0;memobisL=L0;memobisB=B0;memoterL=L0
        ;memoterB=B0;memoquadL=L0;memoquadB=B0;memoquintL=L0;memoquintB=
        B0;memosixL=L0;memosixB=B0;
    for k1 in range(i,j):
        L=mutlist(5,k1,L0,B0)
        memoL=L
        memoB=mutmatrix(5,k1,B0)
        stock=addlist(stock,wordpara(L[k1]))
        for k2 in range(i,j):
            if k2 <> k1:
                L=mutlist(5,k2,memoL,memoB)
                memobisL=L
                memobisB=mutmatrix(5,k2,memoB)
                stock=addlist(stock,wordpara(L[k2]))
                for k3 in range(i,j):
                    if k3 <> k2:
                        L=mutlist(5,k3,memobisL,memobisB)
                        memoterL=L
                        memoterB=mutmatrix(5,k3,memobisB)
                        stock=addlist(stock,wordpara(L[k3]))
                        for k4 in range(i,j):
```

```
                                    if k4 <> k3:
                                        L = mutlist(5,k4,memoterL,memoterB)
                                        memoquadL=L
                                        memoquadB=mutmatrix(5,k4,memoterB)
                                        stock=addlist(stock,wordpara(L[k4]))
                                        for k5 in range(i,j):
                                            if k5 <> k4:
                                                L = mutlist(5,k5,memoquadL,
                                                    memoquadB)
                                                memoquintL=L
                                                memoquintB=mutmatrix(5,k5,
                                                    memoquadB)
                                                stock=addlist(stock,wordpara(L[
                                                    k5]))
                                                for k6 in range(i,j):
                                                    if k6<>k5:
                                                        L = mutlist(5,k6,
                                                            memoquintL,
                                                            memoquintB)
                                                        memosixL=L
                                                        memosixB=mutmatrix(5,k6
                                                            ,memoquintB)
                                                        stock=addlist(stock,
                                                            wordpara(L[k6]))
                                                        for k7 in range(i,j):
                                                            if k7<>k6:
                                                                L=mutlist(5,k7,
                                                                    memosixL,
                                                                    memosixB)
                                                                stock=addlist(
                                                                    stock,
                                                                    wordpara(L[
                                                                    k7]))
    return stock


# In[59]:

S7=listvariables(0,10)
print S7
len(S7)


# In[60]:

k=0;
for j in range(0,len(S7)):
    if inlist(S7[j],Ldomin)==1:
        k=k+1
```

k

 # In[ ]:

The list Ldomin turns out to be of length 47, which means there are 47 strict dominant minuscule elements in the Weyl group of type A_5. The final result of our program is 42, which means that among the 47 elements of the list Ldomin, 42 of them are obtained as cluster variables. Our program computed only the cluster variables belonging to seeds obtained after sequences of seven consecutive mutations starting from one initial seed (L_0, B_0). We believe one could check with a more powerful computer that the five remaining strict dominant minuscule words can also be obtained as cluster variables.

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