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A study of properties and computation techniques of the L^2-Alexander invariant in knot theory

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Résumé

Résumé

Cette thèse présente plusieurs propriétés, des valeurs explicites et des techniques de calcul des torsions d'Alexander L^2 des variétés de dimension 3 compactes à bord vide ou torique, notamment des extérieurs de nœuds et d'entrelacs. Les torsions d'Alexander L^2 sont des généralisations des polynômes d'Alexander tordus, où le complexe de chaînes cellulaires du revêtement universel de la variété est tordu par une représentation du groupe fondamental de la variété sur l'espace des opérateurs d'un espace de Hilbert de dimension infinie.

Les torsions d'Alexander L^2 des 3-variétés ont été définies en 2014 par J. Dubois, S. Friedl et W. Lück, et généralisent l'invariant d'Alexander L^2 des nœuds introduit par W. Li et W. Zhang en 2006. Ces torsions sont des invariants topologiques qui sont des classes de fonctions sur les réels strictement positifs. Elles existent uniquement lorsque certaines conditions techniques sont vérifiées, et sont difficiles à calculer en général. Malgré tout, nous pouvons extraire d'importantes informations de ces invariants, comme le volume simplicial de la variété ou la norme de Thurston.

Dans cette thèse, nous démontrons que l'invariant d'Alexander L^2 des nœuds détecte le nœud trivial.

Nous démontrons également une formule de chirurgie de Dehn pour les torsions d'Alexander L^2.

De même, par diverses techniques, nous calculons explicitement les torsions des extérieurs d'entrelacs toriques dans la sphère S^3 et le tore solide, ce qui nous permet de démontrer des formules générales de sommes connexes et de câblages pour les entrelacs.

Mots-clefs

Nœuds, variétés de dimension 3, déterminant de Fuglede-Kadison, CW-complexes, torsion L^2, chirurgie de Dehn, polynôme d'Alexander

A study of properties and computation techniques of the L^2-Alexander invariant in knot theory

Abstract

This manuscript presents several properties, explicit values and computation techniques of L^2-Alexander torsions for compact 3-manifolds with empty or toroidal boundary, especially for knot exteriors and link exteriors. The L^2-Alexander torsions are generalisations of the twisted Alexander polynomials, as the cellular chain complex of the universal covering of a manifold is twisted by an infinite-dimensional Hilbert representation of the fundamental group of the manifold.

The L^2-Alexander torsions of 3-manifolds were defined in 2014 by J. Dubois, S. Friedl and W. Lück, and generalize the L^2-Alexander invariant of a knot introduced by W. Li and W. Zhang in 2006. These torsions are topological invariants that are classes of maps on the positive real numbers. They only exist when certain technical conditions are satisfied, and they are hard to compute in general. Despite these difficulties, we are able to extract important information from these invariants, like the simplicial volume of the manifold or the Thurston norm.

In this thesis, we prove that the L^2-Alexander invariant of knots detects the trivial knot.

We also prove a Dehn surgery formula for the L^2-Alexander torsions.

Similarly, using various techniques, we compute explicitly the torsions of exteriors of torus links in the 3-sphere and in the solid torus, which leads us to prove general formulas for connected sums and cablings of links.

Keywords

Knots, 3-manifolds, Fuglede-Kadison determinant, CW-complexes, L^2-torsion, Dehn surgery, Alexander polynomial

Contents

Introduction

Historique

Prenez une ficelle, recollez ses deux bouts, et vous obtenez un nœud. Prenez-en une seconde, qui suivra un autre chemin dans l'espace, collez ses deux bouts, vous obtenez un autre nœud. Maintenant, comment savoir si l'on peut passer du premier nœud au second simplement en déformant la ficelle, sans la couper ? Le but de la théorie des nœuds est de répondre à cette question.

En termes plus mathématiques, un nœud K est un plongement du cercle S^1 dans la sphère S^3 ; deux nœuds sont considérés équivalents, ou isotopes, s'il existe un auto-homéomorphisme de S^3 envoyant un nœud sur l'autre.

Classifier les nœuds à isotopie près est une tâche ardue. Heureusement, de nombreux invariants de nœuds ont été découverts et étudiés durant le dernier siècle. Les théoriciens des nœuds disposent maintenant d'outils venant de nombreux domaines des mathématiques pour détecter des nœuds non isotopes, des domaines tels que la combinatoire, la topologie algébrique, la théorie des groupes, la géométrie, la théorie quantique des champs topologique, les algèbres d'opérateurs, etc.

En 1928, J.W. Alexander définit dans [Ale28] le premier invariant polynomial des nœuds. Ceci révolutionna la théorie des nœuds, car cet invariant était non seulement facile à calculer et à manipuler, mais il était également assez puissant pour distinguer la plupart des nœuds premiers. Le polynôme d'Alexander n'est cependant pas un invariant complet, pas même parmi les nœuds premiers ; en particulier, il ne détecte pas le nœud trivial.

Le polynôme d'Alexander est sans doute l'invariant de nœuds ayant le plus grand nombre de définitions différentes possibles. On peut le définir à partir de l'homologie du revêtement infini cyclique de l'extérieur du nœud comme l'avait initialement fait Alexander dans [Ale28], ou par le calcul de Fox sur une présentation du groupe du nœud à la manière de R. Fox dans [Fox54] (nous détaillons cette construction dans l'Annexe A.3). En 1962, J. Milnor prouva dans [Mil62] que le polynôme d'Alexander d'un nœud peut être obtenu à partir de la torsion de Reidemeister de l'extérieur du nœud correspondant à l'abélianisation du groupe du nœud.

En 1976, M. Atiyah développa dans [Ati76] les fondations de la théorie des invariants L^2, en définissant notamment les nombres de Betti L^2. L'idée des invariants L^2 est la suivante : la topologie algébrique compte plusieurs invariants qui mettent en jeu des espaces vectoriels de dimension finie et des applications linéaires, comme les nombres de Betti ou la caractéristique d'Euler ; en effectuant des procédés similaires à ces procédés classiques mais sur des espaces de Hilbert de dimension infinie (comme $\ell^2(G)$ où G est un groupe infini) et en considérant des algèbres d'opérateurs sur ces espaces, nous pouvons définir de nouveaux invariants, les invariants L^2.

Ainsi, dans les années 1990, A. L. Carey et V. Mathai, J. Lott, W. Lück et M. Rothen-

berg, S. P. Novikov et M. A. Shubin développèrent la théorie des torsions L^2, un analogue L^2 de la théorie des torsions de Reidemeister. Une vue d'ensemble de ces théories est présentée dans [Lüc02b].

Enfin, en 2006, W. Li et W. Zhang introduisirent dans [LZ06] *l'invariant d'Alexander* L^2 $\Delta_K^{(2)}$ *d'un nœud* K, un analogue L^2 du polynôme d'Alexander Δ_K, et prouvèrent qu'on pouvait l'exprimer en fonction d'une certaine torsion L^2 de l'extérieur du nœud tordue par l'abélianisation du groupe du nœud, à l'image de la formule de Milnor. Cet invariant de nœuds est une classe d'équivalence de fonctions sur les réels positifs. Il a été étudié en détail par F. Ben Aribi, J. Dubois, S. Friedl, W. Lück et C. Wegner, dans [BA13], [DFL14], [DW13], puis généralisé à une famille de *torsions d'Alexander* L^2 des 3-variétés compactes à bord vide ou torique, par J. Dubois, S. Friedl et W. Lück dans [DFL14].

Les torsions d'Alexander L^2

Les torsions d'Alexander L^2 sont les invariants centraux étudiés dans cette thèse. Elles englobent d'autres invariants topologiques des 3-variétés très profonds tels que le volume simplicial et la norme de Thurston, comme nous le détaillerons dans la suite.

Pour M une 3-variété compacte, on peut associer une structure de CW-complexe finie X_M à M, par exemple en considérant une triangulation de M (dont l'existence est assurée par le théorème de Moise, cf. [Moi52]). Soient $\pi = \pi_1(X_M) = \pi_1(M)$ son groupe fondamental, $t > 0$ un réel, et $\phi : \pi \to \mathbb{Z}$, $\gamma : \pi \to G$ deux morphismes de groupes tels que ϕ factorise à travers γ. On dit alors que (π, ϕ, γ) est un *triplet admissible* :

$$\begin{tikzcd} \pi_1(M) = \pi \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\ & \mathbb{Z} \end{tikzcd}$$

Le groupe $\pi = \pi_1(M)$ contient souvent beaucoup d'information sur la topologie de la variété M. Notamment W. Whitten a montré dans [Whi87] que le type d'homéomorphisme de l'extérieur d'un nœud premier est déterminé par son groupe fondamental ; F. Waldhausen avait même montré dans [Wal68] que le groupe d'un nœud quelconque pris avec un système méridien-longitude détecte le type d'isotopie du nœud. Ainsi, si $M = M_K = S^3 \setminus V(K)$ est l'extérieur d'un nœud K (avec $V(K)$ un voisinage tubulaire ouvert de K), alors le groupe $\pi = \pi_1(M_K)$ complété par un système périphérique méridien-longitude (qui engendre $\pi_1(\partial M_K)$) contient toute l'information de ce nœud. Le problème est maintenant d'extraire l'information du groupe π, et pour ce faire il est pratique d'utiliser des représentations de groupes ; ce sont les rôles que vont jouer ϕ et γ.

Le complexe de chaînes cellulaires du revêtement universel $\widetilde{X_M}$ de X_M :

$$C_*(\widetilde{X_M}, \mathbb{Z}) = \left(\ldots \to \bigoplus_i \mathbb{Z}[\pi]\tilde{e}_i^k \to \ldots \right)$$

décrit la topologie de X_M : grosso modo, les applications bord ∂ sont des morphismes de $\mathbb{Z}[\pi]$-modules qui décrivent comment les cellules e_i^k de X_M se recollent entre elles. Le morphisme d'anneau

$$\kappa(\pi, \phi, \gamma, t) : \left(\begin{array}{ccc} \mathbb{Z}[\pi] & \longrightarrow & \mathbb{R}[G] \\ \sum_{j=1}^r m_j g_j & \longmapsto & \sum_{j=1}^r m_j t^{\phi(g_j)} \gamma(g_j) \end{array} \right)$$

définit une action à droite de $\mathbb{Z}[\pi]$ sur l'espace de Hilbert

$$\ell^2(G) = \left\{ \sum_{g\in G} \lambda_g g \mid \lambda_g \in \mathbb{C}, \sum_{g\in G} |\lambda_g|^2 < \infty \right\}$$

par multiplication à droite, ce qui nous permet de considérer par produit tensoriel

$$\begin{aligned} C_*^{(2)}(X_M, \phi, \gamma, t) &= \ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} C_*(\widetilde{X_M}, \mathbb{Z}) \\ &= \left(\dots \xrightarrow{\partial_{k+1}^{(2)}} \bigoplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{\partial_k^{(2)}} \dots \right) \end{aligned}$$

le $\mathcal{N}(G)$-complexe de chaînes cellulaires de X_M associé, où $\mathcal{N}(G)$ est l'algèbre de von Neumann des opérateurs G-équivariants agissant sur $\ell^2(G)$.

Pour construire un analogue L^2 de la torsion de Reidemeister d'un complexe de chaînes, il nous faut une version en dimension infinie du déterminant d'un opérateur ; c'est le déterminant de Fuglede-Kadison $\det_{\mathcal{N}(G)}$ qui jouera ce rôle pour les opérateurs G-équivariants agissant sur les espaces $\ell^2(G)^m$. Ce déterminant fut introduit par B. Fuglede et R. Kadison dans [FK52] pour les opérateurs inversibles, et fut étendu ensuite aux opérateurs plus généraux (voir [Lüc02b, Section 3.2] et le compendium [dlH13]). La définition du déterminant de Fuglede-Kadison est relativement technique (voir Définition 1.49), nous ne la détaillerons donc pas dans cette introduction ; mentionnons seulement qu'il peut être construit comme un produit infini avec multiplicités des valeurs spectrales de l'opérateur considéré, à l'aide d'une intégrale sur une certaine mesure sur le spectre de l'opérateur (qui n'est en général pas fini, ni même discret). Par conséquent, le déterminant de Fuglede-Kadison est difficile à manipuler et à calculer explicitement ; néanmoins, les propriétés classiques qu'il partage avec le déterminant usuel (comme la multiplicativité pour la composition des opérateurs dans certains cas) nous permettent parfois d'en déterminer la valeur.

Si le $\mathcal{N}(G)$-complexe de chaînes $C_*^{(2)}(X_M, \phi, \gamma, t)$ est *faiblement acyclique*, ce qui veut dire qu'il forme une suite exacte faible (dans le sens où on considère l'adhérence de l'image d'un opérateur bord et non son image) et s'il est *à classe de déterminant*, ce qui veut dire que tous les opérateurs $\partial_k^{(2)}$ sont de déterminant de Fuglede-Kadison $\det_{\mathcal{N}(G)}$ strictement positif (ces deux définitions sont détaillées à la Section 1.3.5), alors on peut définir la *torsion d'Alexander L^2 du triplet (X_M, ϕ, γ) en $t > 0$* comme :

$$T^{(2)}(X_M, \phi, \gamma)(t) = T^{(2)}\left(C_*^{(2)}(X_M, \phi, \gamma, t)\right) = \prod_k \det{}_{\mathcal{N}(G)} \left(\partial_k^{(2)}\right)^{(-1)^k} \in \mathbb{R}_{>0}$$

Pour deux fonctions $f, g \in \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{>0})$, nous noterons $f \doteq g$ s'il existe un entier $m \in \mathbb{Z}$ tel que $g = (t \mapsto t^m) \cdot f$. À cette relation d'équivalence $\doteq$ près, la torsion d'Alexander L^2 $(t \mapsto T^{(2)}(X_M, \phi, \gamma)(t))$ de X_M ne dépend ni de l'ordre, ni de l'orientation, ni des choix des relevés des cellules de X_M ; en effet, un tel changement de choix sur les cellules revient à composer les opérateurs bord par des opérateurs de permutation ou de dilatation, dont le déterminant de Fuglede-Kadison est connu et vaut t^m pour un certain $m \in \mathbb{Z}$.

Mieux, nous montrons ensuite que si X et Y sont deux CW-complexes équivalents par homotopie simple, alors ils ont les mêmes torsions d'Alexander L^2. Plus précisément :

Théorème 0.1. *(Theorem 2.12)*

Pour $f : X \to Y$ une équivalence d'homotopie simple entre deux CW-complexes finis induisant l'isomorphisme de groupes fondamentaux $f_* : \pi_1(X) \to \pi_1(Y)$, un triplet

(Y, ϕ, γ) est admissible si et seulement si $(X, \phi \circ f_*, \gamma \circ f_*)$ l'est, le $\mathcal{N}(G)$-complexe de chaînes $C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t)$ est faiblement acyclique et à classe de déterminant si et seulement si $C_*^{(2)}(Y, \phi, \gamma, t)$ l'est, et dans ce cas

$$T^{(2)}(X, \phi \circ f_*, \gamma \circ f_*)(t) \doteq T^{(2)}(Y, \phi, \gamma)(t).$$

Nous montrons ce résultat en incluant $C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t)$ et $C_*^{(2)}(Y, \phi, \gamma, t)$ dans une suite exacte courte de $\mathcal{N}(G)$-complexes de chaînes dans le cas où f est une expansion élémentaire de CW-complexes puis en utilisant la propriété de multiplicativité de la torsion L^2 détaillée dans [Lüc02b, Theorem 3.35 (1)]. Le théorème découle alors du fait que toute homotopie simple est une suite finie d'expansions élementaires et de rétractions élémentaires (voir [Coh73, Section 4]).

Comme deux structures de CW-complexes sur une 3-variété compacte M sont équivalentes par homotopie simple (voir [Cha74]), deux 3-variétés homéomorphes M et M' sont équivalentes par homotopie simple et ont donc mêmes torsions d'Alexander L^2. Les torsions d'Alexander L^2 sont donc des invariants topologiques des 3-variétés compactes.

Si M est une 3-variété compacte irréductible à bord vide ou torique, W. Lück et T. Schick ont montré dans [LS99] que la torsion L^2 classique de M n'est autre qu'une reformulation de son volume simplicial :

$$T^{(2)}(M) = T^{(2)}(M, 0, id)(1) = \exp\left(\frac{\mathrm{vol}(M)}{6\pi}\right).$$

Le théorème de rigidité de Mostow-Prasad-Marden (voir par exemple [AFW12, Theorem 1.10]) assure qu'une structure hyperbolique complète de volume fini sur une 3-variété M est unique à isométrie près et ne dépend que de la topologie de M ; ainsi, tout invariant géométrique construit à partir de la structure hyperbolique d'une variété, tel que le volume, est en fait un invariant topologique. Le volume hyperbolique d'une 3-variété hyperbolique (et plus généralement le volume simplicial d'une variété irréductible) est ainsi un invariant profond et puissant. Par exemple, la plupart des nœuds premiers hyperboliques sont distingués par le volume hyperbolique de leur complément.

Si l'on inclut maintenant la déformation abélienne des complexes de chaînes donnée par t^ϕ, les torsions d'Alexander L^2 de M détectent également la norme de Thurston $x_M(\phi)$ pour tout morphisme γ si M est une variété fibrée sur le cercle (cf. [DFL14, Theorem 8.2]). On peut comparer cette dernière propriété au fait que le degré du polynôme d'Alexander d'un nœud fibré est le double du genre du nœud. Pour M plus générale, la torsion d'Alexander L^2 de M $T^{(2)}(M, \phi, \gamma)$ détecte aussi la norme de Thurston $x_M(\phi)$, mais pour un certain γ à valeurs dans un groupe G virtuellement abélien (cf. [DFL14, Theorem 10.1]).

Vérifier que $C_*^{(2)}(M, \phi, \gamma, t)$ est faiblement acyclique et à classe de déterminant est souvent difficile. On peut relier ces questions à de vastes conjectures de la théorie des invariants L^2 comme la conjecture d'Atiyah forte. Si jamais ces conditions sont vérifiées, calculer exactement les déterminants de Fuglede-Kadison des opérateurs de bord est encore un problème ardu. Néanmoins nous pouvons parfois déceler des propriétés de la variété M, comme la valeur de son volume simplicial ou de sa norme de Thurston, en considérant ses torsions d'Alexander L^2 mais sans les calculer explicitement.

Ainsi, dans le Chapitre 2, nous prouvons que les torsions d'Alexander L^2 distinguent le nœud trivial des autres nœuds.

L'invariant d'Alexander L^2 détecte le nœud trivial

Quand la 3-variété M est l'extérieur $M_K = S^3 \setminus V(K)$ d'un nœud K, l'étude des torsions d'Alexander L^2 de $M = M_K$ se simplifie de deux façons.

D'une part, le groupe fondamental $G_K = \pi_1(M_K)$ de M_K a pour abélianisé G_K^{ab} le groupe cyclique infini $\mathbb{Z}$, et comme tout morphisme de groupe $\phi : G_K \to \mathbb{Z}$ factorise par l'abélianisation $\alpha_K : G_K \to G_K^{ab} \cong \mathbb{Z}$, ϕ s'écrit donc comme un multiple entier de α_K. Il provient naturellement des définitions que $T^{(2)}(M, r\phi, \gamma)(t)$ et $T^{(2)}(M, \phi, \gamma)(t^r)$ sont égales pour tous ϕ, γ, $t > 0$ et $r \in \mathbb{Z}$ (voir Proposition 2.7). Quand $M = M_K$ est un extérieur de nœud, il nous suffit donc de considérer le cas $\phi = \alpha_K$ pour connaître toutes les torsions d'Alexander L^2 de M_K.

D'autre part, M_K est une variété compacte irréductible à bord torique et de groupe fondamental infini (et est en particulier un espace d'Eilenberg-Maclane $K(G_K, 1)$), ce qui implique qu'elle est équivalente par homotopie simple à tout CW-complexe de dimension 2 W_P construit à partir d'une présentation de groupe P de défaut 1 du groupe G_K (la démonstration de ce résultat est détaillée dans la Section 2.1.3). Le Théorème 0.1 assure donc que l'on peut calculer les torsions d'Alexander L^2 de M_K à l'aide de celles de W_P.

Tout ceci justifie donc l'étude non plus des torsions d'Alexander L^2 générales de M_K, mais de celle de l'invariant d'Alexander L^2 $\Delta_K^{(2}(t)$ de K égal à

$$\Delta_K^{(2)}(t) = T^{(2)}(M_K, \alpha_K, id)(t) \cdot \max(1, t) = T^{(2)}(W_P, \alpha_K, id)(t) \cdot \max(1, t)$$

Nous nous restreignons dans la suite au cas où $\gamma = id$ pour simplifier les notations et pour suivre au plus près l'étude originelle de Li-Zhang dans [LZ06]. Il est également possible de définir et d'étudier un invariant d'Alexander L^2 tordu

$$\Delta_{K,\gamma}^{(2)}(t) = T^{(2)}(M_K, \alpha_K, \gamma)(t) \cdot \max(1, t)$$

mais nous n'en avons pas besoin pour établir des propriétés fortes comme le fait que l'invariant d'Alexander L^2 détecte le nœud trivial.

Considérer l'invariant d'Alexander L^2 est pratique car $C^{(2)}(W_P, \alpha_K, id, t)$ s'écrit simplement à l'aide du calcul de Fox sur la présentation P du groupe G_K, ce qui rejoint la construction originelle de $\Delta_K^{(2)}(t)$ par le calcul de Fox que nous détaillons ci-après. Cette construction de l'invariant d'Alexander L^2 $\Delta_K^{(2)}(t)$ offre une méthode alternative efficace pour calculer les torsions d'Alexander L^2 de l'extérieur d'un nœud.

Considérons K un nœud et $P = \langle g_1, \ldots, g_k | r_1, \ldots, r_{k-1} \rangle$ une présentation de Wirtinger de son groupe G_K. Rappelons qu'une présentation de Wirtinger est une présentation du groupe du nœud K construite à partir d'un diagramme planaire D qui représente K, avec k croisements et k arcs ; chaque générateur g_i de la présentation représente un lacet méridien d'un arc et chaque relation r_j de la forme $g_i g_k g_i^{-1} g_l^{-1}$ représente le croisement associé. Tous les générateurs sont conjugués et une des k relations est redondante. Ainsi l'abélianisation de G_K vérifie

$$\alpha_K : \begin{pmatrix} G_K \longrightarrow \mathbb{Z} \\ g_i \longmapsto 1 \end{pmatrix}.$$

Soit W_P le CW-complexe de dimension 2 construit à partir de la présentation P, avec une 0-cellule, une 1-cellule pour chaque générateur g_i et une 2-cellule pour chaque relateur r_j, recollée sur les g_i suivant le mot libre r_j. La matrice de Fox de la présentation P

$$F_P = \left(\left(\frac{\partial r_j}{\partial g_i} \right) \right)_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k-1} \in M_{k,k-1}(\mathbb{C}[G_K])$$

décrit alors le morphisme de $\mathbb{Z}[G_K]$-modules qu'est l'opérateur bord

$$C_2(\widetilde{W_P}, \mathbb{Z}) \to C_1(\widetilde{W_P}, \mathbb{Z}).$$

Comme W_P est équivalent par homotopie simple à M_K, la matrice F_P nous donne des informations sur la topologie de M_K.

Considérons alors, pour $t > 0$,

$$R_{\kappa(G_K,\alpha_K,id,t)(F_{P,1})} : \ell^2(G_K)^{k-1} \to \ell^2(G_K)^{k-1}$$

l'opérateur "multiplication à droite" induit par la représentation $t^{\alpha_K} : G_K \to \mathbb{R}^*$ et par $F_{P,1} \in M_{k-1,k-1}(\mathbb{C}[G_K])$, la matrice de Fox privée de sa première ligne. Si cet opérateur est injectif, une condition technique similaire à l'acyclité faible mentionnée dans la définition des torsions d'Alexander L^2, on peut alors définir l'invariant d'Alexander L^2 du nœud K comme le déterminant de Fuglede-Kadison de l'opérateur précédent :

$$\Delta_K^{(2)}(t) = \det{}_{\mathcal{N}(G)} \left(R_{\kappa(G_K,\alpha_K,id,t)(F_{P,1})} \right).$$

À multiplication par $(t \mapsto t^m)$, $m \in \mathbb{Z}$ près, $\Delta_K^{(2)}$ ne dépend pas de la présentation de Wirtinger P choisie (voir Proposition 2.21), comme nous le montrons à l'aide des *mouvements de Tietze forts* introduits par Wada dans [Wad94] pour décrire les polynômes d'Alexander tordus. L'invariant d'Alexander L^2 peut également être défini pour une présentation quelconque de G_K de défaut 1 (voir Théorème 2.28), par les considérations d'homotopie simple et d'asphéricité déjà mentionnées. Cette dernière propriété avait été précédemment établie dans [DW13].

En $t = 1$, l'invariant d'Alexander L^2 de K coïncide avec la torsion L^2 usuelle de la variété M_K, et vaut donc, par le théorème de W. Lück et T. Schick (cf. [LS99]) :

$$\Delta_K^{(2)}(1) = \exp\left(\frac{\mathrm{vol}(K)}{6\pi}\right).$$

L'objet principal du Chapitre 2 est la preuve du résultat suivant :

Théorème principal 1. *(Théorème 2.40)*
Soit K un nœud dans S^3. L'invariant d'Alexander L^2 de K est trivial, i.e. $\left(t \mapsto \Delta_K^{(2)}(t)\right) = (t \mapsto 1)$, si et seulement si K est le nœud trivial.

Nous prouvons ce théorème en utilisant la propriété classique (voir par exemple [MM01, Lemma 5.5]) suivante : pour un nœud K donné, ou bien son extérieur M_K possède un volume simplicial non nul, ou bien M_K est une variété graphée, et dans ce second cas K est un *nœud torique itéré*, i.e. K peut être obtenu par sommes connexes et câblages à partir du nœud trivial. Dans le premier cas, le théorème de W. Lück et T. Schick que nous avons mentionné précédemment implique que $\left(t \mapsto \Delta_K^{(2)}(t)\right) \neq (t \mapsto 1)$, et nous traitons le second cas par induction à l'aide des formules suivantes de câblages et de sommes connexes :

Théorème 0.2. *(Théorèmes 2.33 et 2.36)*
(1) L'invariant d'Alexander L^2 est multiplicatif pour la somme connexe des nœuds.
(2) L'invariant d'Alexander L^2 vérifie la formule de câblage suivante :
si S est le (p,q)-câblage du nœud compagnon C, alors

$$\Delta_S^{(2)}(t) = \Delta_C^{(2)}(t^p) \max(1,t)^{(|p|-1)(|q|-1)}.$$

Pour prouver ces formules-ci, nous calculons des présentations appropriées des groupes de nœuds composés ou câbles à l'aide du théorème de Seifert van Kampen (voir l'Annexe A.1) ; nous remarquons ensuite que les matrices de Fox associées sont presque triangulaires par blocs, ce qui nous permet de décomposer le déterminant de Fuglede-Kadison des opérateurs associés comme un produit de déterminants déjà connus. Ces résultats font l'objet de la publication [BA13].

Par un raisonnement similaire à celui de la preuve du Théorème Principal 1, nous établissons aussi que parmi tous les nœuds, l'invariant d'Alexander L^2 caractérise la paire des nœuds de trèfle gauche et droit. Plus précisément :

Théorème 0.3. *(Théorème 2.41)*
Soit K un nœud dans S^3. L'invariant d'Alexander L^2 de K est de la forme

$$\left(t \mapsto \Delta_K^{(2)}(t)\right) = (t \mapsto \max(1,t)^2)$$

si et seulement si K est le nœud de trèfle gauche ou droit.

Manipuler des matrices de Fox et des morphismes d'abélianisation nécessite de travailler réellement au niveau combinatoire de la présentation du groupe du nœud, en conservant précisément l'origine topologique des générateurs de la présentation. Ainsi, selon l'orientation du nœud K, un générateur g représentant le même méridien géométrique enlacera positivement ou négativement K et sera donc envoyé sur 1 ou -1 par l'abélianisation α_K. Ces subtilités sont fondamentales dans les calculs de l'invariant d'Alexander L^2 de l'image miroir (voir Théorème 2.30) ou de l'inverse d'un nœud (voir Corollaire 2.37).

Enfin, nous concluons le Chapitre 2 en simplifiant l'écriture des torsions d'Alexander L^2 pour un entrelacs L à c composantes : comme le morphisme $\phi : G_L \to \mathbb{Z}$ factorise par l'abélianisation $\alpha_L : G_L \to \mathbb{Z}^c$, on peut écrire ϕ comme $(n_1, \ldots, n_c) \circ \alpha_L$ où $n_1, \ldots, n_c \in \mathbb{Z}$. Les torsions d'Alexander L^2 s'expriment donc naturellement comme des invariants d'entrelacs décorés par un entier sur chaque composante.

Propriétés de recollement

Selon un théorème de W. B. Lickorish et A. H. Wallace (voir [Rol90, Theorem 9I1]) toute 3-variété connexe compacte orientable sans bord peut être obtenue comme *remplissage de Dehn* sur l'extérieur d'un entrelacs dans S^3, i.e. un recollement de tores solides le long des composantes de bord de l'extérieur de l'entrelacs, suivant certaines pentes rationnelles qui sont les *coefficients de la chirurgie de Dehn*. Les chirurgies de Dehn sur les entrelacs fournissent donc un outil pratique de description des 3-variétés.

En comparant certaines présentations de groupe de l'entrelacs de Whitehead et des nœuds twists, on remarque une relation entre leurs torsions d'Alexander L^2 (voir Théorème 3.9), qui illustre le fait que les nœuds twists sont obtenus par $1/n$-chirurgie sur une composante de l'entrelacs de Whitehead.

On observe également que les torsions d'Alexander L^2 d'un entrelacs L et du sous-entrelacs L' obtenu en supprimant une composante de L, vérifient une relation similaire à celle que Torres établit dans [Tor53] pour le polynôme d'Alexander classique (voir Théorème 3.8). Or L' peut être vu comme le résultat d'une ∞-chirurgie sur une composante de L.

Les torsions d'Alexander L^2 vérifient en fait une formule générale de chirurgie de Dehn :

Théorème 0.4. *(Théorème 3.6)*

Soient N une 3-variété obtenue par remplissage de Dehn sur une variété M, $Q : \pi_1(M) \twoheadrightarrow \pi_1(N)$ le morphisme de groupe surjectif et $\iota : c^{\mathbb{Z}} \cong \pi_1(S^1 \times D^2) \to \pi_1(N)$ le morphisme de groupe induits par ce remplissage.

Pour un triplet admissible (π_N, ϕ, γ) tel que $\gamma(\iota(c))$ est d'ordre infini, et pour tout $t > 0$, si $C_*^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)$ est faiblement acyclique et à classe de déterminant, alors $C_*^{(2)}(N, \phi, \gamma)(t)$ l'est aussi et dans ce cas

$$T^{(2)}(N, \phi, \gamma)(t) \doteq \frac{T^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)}{\max(1, t)^{|\phi(\iota(c))|}}.$$

Dans le Chapitre 3 nous présentons cette formule, ses variations, sa preuve et les conséquences que nous en tirons en théorie des nœuds et entrelacs et en géométrie des 3-variétés plus générales.

La formule de chirurgie du Théorème 0.4 est un cas particulier d'une formule de type Mayer-Vietoris : si un CW-complexe X est obtenu comme union de deux CW-complexes A et B le long de leur intersection V, alors les torsions d'Alexander L^2 des quatre CW-complexes sont reliées, à condition de considérer quatre paires (ϕ, γ) compatibles avec les morphismes de groupes induits par les inclusions de V, A, B dans X, que l'on peut voir dans le diagramme suivant :

$$\begin{tikzcd}
 & A \arrow[dr, "J_A"] & \\
V \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & X \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_A"] & & & \\
\pi_1(V) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(X) \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\
 & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}
\end{tikzcd}$$

C'est ici que l'on peut voir l'intérêt du paramètre γ dans la définition de la torsion d'Alexander L^2 : comment relier sinon un complexe de chaînes $C_*^{(2)}(A, \phi, id)(t)$ de $\mathcal{N}(\pi_1(A))$-modules avec un complexe de chaînes $C_*^{(2)}(X, \phi', id)(t)$ de $\mathcal{N}(\pi_1(X))$-modules en général ? La formule de Mayer-Vietoris pour les torsions d'Alexander L^2 s'écrit :

Théorème principal 2. *(Théorème 3.1)*
Si les trois $\mathcal{N}(G)$-complexes de chaînes cellulaires

$$C_*^{(2)}(V, \phi \circ i, \gamma \circ i, t), C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t), C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t)$$

sont faiblement acycliques et à classe de déterminant, alors $C_*^{(2)}(X, \phi, \gamma, t)$ l'est également, et

$$T^{(2)}(X, \phi, \gamma)(t) \cdot T^{(2)}(V, \phi \circ i, \gamma \circ i)(t) = T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A)(t) \cdot T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B)(t).$$

Nous démontrons cette formule en incluant les quatre $\mathcal{N}(G)$-complexes de chaînes de V, A, B et X dans une suite exacte courte, puis en utilisant la propriété de multiplicativité de la torsion L^2 détaillée dans [Lüc02b, Theorem 3.35 (1)]. Le point délicat est de relier entre elles les bases cellulaires des quatre $\mathcal{N}(G)$-complexes de chaînes, ce qui implique que les opérateurs de bord ont une forme matricielle triangulaire qui garantit l'existence de la suite exacte courte.

La formule de chirurgie du Théorème 0.4 est ainsi une conséquence de cette formule de Mayer-Vietoris, obtenue en calculant au préalable les torsions d'Alexander L^2 du tore (Théorème 3.5) et du tore solide (Théorème 3.4).

La formule de chirurgie nous permet de calculer certaines torsions d'Alexander L^2, quand le morphisme γ correspond à un remplissage de Dehn. En particulier nous pouvons calculer plusieurs exemples de fonctions $(t \mapsto T^{(2)}(M_L, (n_1, n_2) \circ \alpha_L, \gamma)(t))$ quand L est l'entrelacs de Whitehead (voir Théorème 3.9 et Proposition 3.13).

Dans le Chapitre 4 nous généralisons les formules du Chapitre 2 de sommes connexes et de câblages en passant des nœuds aux entrelacs. Pour ce faire nous calculons notamment les torsions d'Alexander L^2 de tous les entrelacs dont l'extérieur est une variété de Seifert. Rappelons qu'une 3-variété compacte orientable est dite de Seifert si elle admet un feuilletage par des cercles (ceci équivaut à la définition classique de [Hat00, p. 13] par le théorème d'Epstein énoncé dans [Eps72]).

Considérons S^3 alternativement comme la sphère unité de $\mathbb{C}^2$ et le compactifié de $\mathbb{R}^3$ avec un point ∞. On définit

- $T(m,n) = \{(z_1, z_2) \in S^3 \subset \mathbb{C}^2 | z_1^m = z_2^n\}$ l'entrelacs torique de type (m,n) (à $e = \mathrm{pgcd}(m,n)$ composantes),
- $H_v = \{(z_1, 0) \in S^3\}$ le cercle représenté par une droite verticale dans $\mathbb{R}^3$ passant par ∞ (en assimilant S^3 à $\mathbb{R}^3 \cup \{\infty\}$),
- $H_h = \{(0, z_2) \in S^3\}$ le cercle unité dans le plan horizontal de $\mathbb{R}^3$.

Ceci nous permet de décrire tous les entrelacs L dans S^3 dont l'extérieur est une variété de Seifert (voir [Bud06, Proposition 3.3]) : un tel L est de la forme $T(m,n)$, $T(m,n) \cup H_v$ ou $T(m,n) \cup H_v \cup H_h$ (nous excluons les entrelacs toriques de la forme $T(m,0)$ pour $|m| \geqslant 2$ car leur extérieur n'est pas une variété irréductible). Nous calculons les torsions d'Alexander L^2 des entrelacs de cette forme :

Théorème 0.5. *(Théorèmes 4.12, 4.11 et 4.10)*

- *Si $L = T(m,n) = T(ep, eq)$ avec p, q premiers entre eux (et non nuls quand $e \geqslant 2$), alors*
$$T^{(2)}(M_L, (n_1, \ldots, n_e) \circ \alpha_L, id)(t) = \max(1,t)^{(e|pq|-|p|-|q|)|n_1+\ldots+n_e|}.$$
- *Si $L = T(ep, eq) \cup H_v$ avec p, q premiers entre eux et p non nul, alors*
$$T^{(2)}(M_L, (n_1, \ldots, n_e, n_{e+1}) \circ \alpha_L, id)(t) = \max(1,t)^{(e|p|-1)|q(n_1+\ldots+n_e)+n_{e+1}|}.$$
- *Si $L = T(ep, eq) \cup H_v \cup H_h$ avec p, q premiers entre eux, alors*
$$T^{(2)}(M_L, (n_1, \ldots, n_e, n_{e+1}, n_{e+2}) \circ \alpha_L, id)(t) = \max(1,t)^{e|pq(n_1+\ldots+n_e)+pn_{e+1}+qn_{e+2}|}.$$

Pour démontrer ces formules, nous utilisons des outils variés. Pour les entrelacs peu complexes (comme les nœuds toriques), nous calculons explicitement une présentation de défaut 1 du groupe de l'entrelacs G_L et nous calculons la torsion d'Alexander L^2 $T^{(2)}(M_L, (n_1, \ldots, n_e) \circ \alpha_L, \gamma)(t)$ à l'aide du calcul de Fox et des mêmes considérations d'homotopie simple que pour l'invariant d'Alexander L^2 des nœuds. Nous pouvons aussi identifier un extérieur d'entrelacs par homéomorphisme à un autre déjà calculé, il suffit alors d'expliciter la correspondance entre les morphismes ϕ. Enfin, nous utilisons la formule de Mayer-Vietoris du Théorème Principal 2 en exprimant l'extérieur d'un entrelacs

compliqué comme recollement d'extérieurs d'entrelacs plus simples le long de tores. Notamment nous utilisons le fait qu'un entrelacs torique $T(ep, eq)$ est un (e, epq)-câblage sur un nœud torique $T(p, q)$ (voir Annexe A.2).

La formule de Mayer-Vietoris nous permet plus généralement d'exprimer des torsions d'Alexander L^2 d'une variété M irréductible en fonction des torsions d'Alexander L^2 des composantes M_i de sa décomposition JSJ (voir Proposition 4.1). Une 3-variété irréductible se scinde en effet selon une famille minimale de tores incompressibles disjoints en une réunion de variétés hyperboliques ou de Seifert, ce qu'on appelle sa décomposition JSJ. Les torsions d'Alexander L^2 d'une variété de Seifert générale M sont de la forme $\max(1, t)^n$, où n est en fait la norme de Thurston $x_M(\phi)$ associée (voir [DFL14, Theorem 8.5]) ; quand la variété M est un extérieur d'entrelacs, le Théorème 0.5 nous offre donc une méthode pour calculer explicitement la norme de Thurston de M.

Enfin, en utilisant les formules du Théorème 0.5 et la formule de recollement JSJ, nous établissons des formules de somme connexe et de câblages pour les torsions d'Alexander L^2 des entrelacs, dans les Théorèmes 4.13 et 4.7.

Chapter 1

Preliminaries, Context, Notations and Tools

In this chapter we recall definitions and present several objects that will be often used in this manuscript.

1.1 Topology, combinatorics, algebra and group theory

1.1.1 Basic knot theory

Knots and links

Here we follow mostly [BZH14]. We choose an orientation for the 3-sphere S^3.

A *knot* in S^3 is a (topological) embedding (i.e. an homeomorphism onto its image) of a circle S^1 into S^3. All knots will be assumed oriented.

Definition 1.1. Two knots $K\colon S^1 \hookrightarrow S^3$ and $K'\colon S^1 \hookrightarrow S^3$ are *ambient isotopic* if there is an orientation-preserving homeomorphism

$$\begin{aligned} H\colon S^3 \times [0;1] &\to S^3 \times [0;1] \\ (y,t) &\mapsto (h_t(y),t) \end{aligned}$$

such that $h_0 = Id_{S^3}$ and $h_1 \circ K = K'$. We call H an *ambient isotopy connecting K and K'*.

We will only consider *tame knots*, i.e. knots that are ambient isotopic to a piece-wise linear embedding of S^1 into S^3.

A knot K will mean alternatively an embedding, a class of embeddings up to ambient isotopy, the image of an embedding (which is a 1-dimensional sub-manifold of S^3) or the class of images of embeddings up to ambient isotopy.

Let K be an oriented knot in S^3, and $V(K)$ an open tubular neighbourhood of K. The *exterior of* K is denoted $M_K = S^3 \setminus V(K)$, it is a compact 3-manifold with toroidal boundary. For $V(K)$ thin enough, M_K does not depend on the chosen $V(K)$. The orientation of M_K comes from the one of S^3, and does not depend on the orientation of K. The boundary torus ∂M_K is oriented with the convention that vectors normal to the boundary point outside of M_K.

Since K is oriented, there is, up to isotopy, a unique pair of simple closed curves μ_K and λ_K on the 2-torus $\partial M_K = \partial V(K)$ such that μ_K bounds a disk in $V(K)$ and λ_K is homologous to K in $V(K)$. We choose an orientation for these two curves such that the

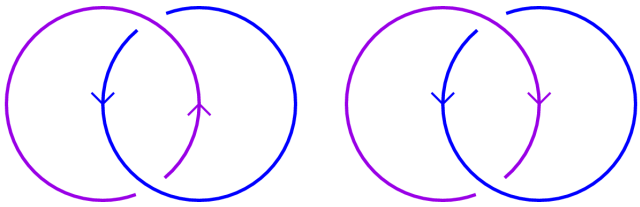


Figure 1.1 – The two Hopf links

linking number (see Section 1.1.4) between μ_K and K and the intersection number between μ_K and λ_K are both $+1$. The pair (μ_K, λ_K) is called a *preferred meridian-longitude pair for K* . Any such μ_K is called a *meridian curve*. Here we have used the notations and definitions of [Tsa88].

A link with $c \in \mathbb{N}$ components, or *c- link* is an embedding of a disjoint union of c circles $\sqcup_{i=1}^{c} S^1$ into S^3; we will assume that all links have ordered oriented components (we note for instance $L_1 \cup \ldots \cup L_c$). We consider links up to ambient isotopies in S^3 that preserve the order and the orientation of the components, unless precised otherwise. We only consider tame links as well.

Example 1.2. The natural unit circle of a plane in R^3 is called the *trivial knot O*. The disjoint union of m such circles included in m parallel planes is called the *trivial m-link*.

Example 1.3. The two Hopf links of Figure 1.1 are ambient isotopic as unoriented links but not as oriented links.

Any component L_i of a link L can be seen as a knot in S^3, and thus we can choose a preferred meridian-longitude pair $(\mu_{L_i}, \lambda_{L_i})$ for each L_i.

A *split link* is a link $L \subset S^3$ such that there exists a 2-sphere $\Sigma \subset S^3$, $L = L' \sqcup L''$ with L' and L'' sub-links, and L' and L'' are contained in different connected components of $S^3 \setminus \Sigma$. Most of the time we will assume that links are non-split.

Knot invariants and diagrams

Definition 1.4. Let F be a correspondence from the class of knots to any other class.

F is a *knot invariant* if

$$(K \text{ is ambient isotopic to } K') \Rightarrow (F(K) = F(K'))$$

F is a *complete knot invariant* if

$$(K \text{ is ambient isotopic to } K') \Leftrightarrow (F(K) = F(K'))$$

Similar definitions hold for links.

The following theorem allows us to define knot invariants from topological invariants defined on the knot exteriors.

Theorem 1.5 (Gordon-Luecke, [GL89])**.** *Two knots K and K' in S^3 are (orientation-preserving) ambient isotopic if and only if M_K and $M_{K'}$ are homeomorphic by an orientation-preserving homeomorphism. Any such homeomorphism sends a meridian to a meridian.*

One such invariant is the knot group, defined in Section 1.1.4.

Remark 1.6. For links, the result is a little different: two links L and L' are (order and orientation-preserving) ambient isotopic if and only if M_L and $M_{L'}$ are homeomorphic by an orientation-preserving homeomorphism that sends a meridian curve to a meridian curve (with respect of the orderings).

For instance, the two 4-component links of Figure 4.1 and Figure 4.3 are not ambient isotopic, but their exteriors are homeomorphic. The assumption about meridians is essential.

Regular diagrams offer an other way of clarifying this class of embeddings considered up to ambient isotopies without losing information.

Definition 1.7. A *regular diagram* is an immersion of a finite number of oriented circles in the plane such that multiple points are only double points with non-tangent intersection (the *crossings*), and are granted with «under-over » information at each crossing.

If L is a link in S^3, then we can choose a point ∞ in $S^3 \setminus L$, and this gives an embedding of L in $\mathbb{R}^3 \cong S^3 \setminus \infty$. Let L_∞ denote this embedding . A *regular diagram of a link* L is a projection $D = p(L_\infty)$ of L_∞ (with any chosen ∞) on an affine plane of $\mathbb{R}^3$ that is itself a regular diagram.

A *planar isotopy* between two regular diagrams D and D' is an orientation-preserving homeomorphism

$$\begin{aligned} H\colon \mathbb{R}^2 \times [0;1] &\to \mathbb{R}^2 \times [0;1] \\ (y,t) &\mapsto (h_t(y), t) \end{aligned}$$

such that $h_0 = Id_{\mathbb{R}^2}$ and $h_1(D) = D'$.

For example, a regular diagram of an embedding of the trefoil knot is given in Figure 1.2.

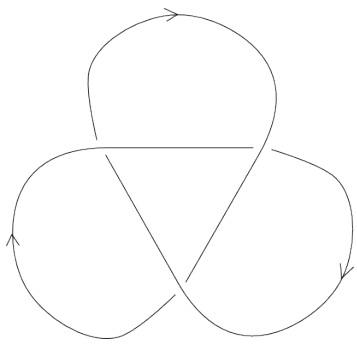


Figure 1.2 – A canonical trefoil knot diagram

Inverse and mirror image

Definition 1.8. Let K be a knot in S^3. The knot K comes with a chosen orientation. Its *inverse knot* $-K$ is K with the opposite orientation. Let $L = L_1 \cup \ldots \cup L_c$ be a c-link in S^3. The *inverse of* L is denoted $-L$ and defined as

$$-L = (-L_1) \cup \ldots \cup (-L_c).$$

Remark 1.9. Let (m_K, l_K) denote the preferred meridian-longitude system of the knot K. Then $(-m_K, -l_K)$ is a preferred meridian-longitude system for $-K$.

Definition 1.10. Let L be a link in S^3. The *mirror image of* L is the image of L by any planar reflection in $\mathbb{R}^3$. It is written L^*. The link L^* does not depend on the plane of reflection up to isotopy.

In particular, if D is a regular diagram of L in the plane, let D^* be the diagram obtained from D by swapping all under-crossings for over-crossings and vice-versa. Then the link corresponding to D^* is L^* and the plane of reflection is implicitly parallel to the plane of D.

For example, the two Hopf links of Figure 1.1 are mirror images of each other.

Remark 1.11. Let (m_K, l_K) denote the preferred meridian-longitude system of the knot K and σ the planar reflection that sends K to K^*. Then $(-\sigma(m_K), \sigma(l_K))$ is a preferred meridian-longitude system for K^*.

1.1.2 Algebraic topology

In this section we fix notations and recall classical results about fundamental groups, covering spaces and CW-complexes. We mostly follow the presentation given by V. Turaev in [Tur01].

Fundamental group

Let X be a topological space. If $c, d\colon [0,1] \to X$ are (continuous) paths in X such that $c(1) = d(0)$, then let $c * d$ denote the *concatenation* of paths, where c is followed at twice the speed, and then d at twice the speed. We denote $[c]_X$ (or $[c]$ if there is no ambiguity) the homotopy class (of paths in X with ends $c(0)$ and $c(1)$) of the path c. The *fundamental group of* X with respect to the base point $P \in X$ is the set of classes $[\gamma]$ where γ is a loop in X of base point P, the group operation being $*$; we denote it by $\pi_1(X,P)$ or $\pi_1(X)$ (we will omit the base point P in the notation when it is not relevant).

If $f\colon X \to Y$ is a continuous map that preserves base points, then the induced group homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ is defined by $f_*([\gamma]_X) := [f \circ \gamma]_Y$ where γ is a loop in X.

Let us recall the Seifert van Kampen theorem, which is useful for computing presentations of fundamental groups.

Theorem 1.12. *[Hat02, p. 43] Let X be a path-connected topological space and let $A, B \subset X$ be open path-connected subspaces of X such that $A \cup B = X$ and such that $V = A \cap B$ is path connected and non-empty. Let $P \in V$ be the basepoint for the four spaces V, A, B, X. The topological inclusions induce the following group homomorphisms:*

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_1"] & \\
\pi_1(V) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(X) \\
 & \pi_1(B) \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Furthermore, $\pi_1(X)$ is isomorphic to the amalgamated product $\pi_1(A) *_{\pi_1(V)} \pi_1(B)$. If

- $P_V = \langle c_i | u_j \rangle$ *is a group presentation of* $\pi_1(V)$
- $P_A = \langle a_k | r_l \rangle$ *is a group presentation of* $\pi_1(A)$
- $P_B = \langle b_m | s_p \rangle$ *is a group presentation of* $\pi_1(B)$

then $P = \langle a_k, b_m | r_l, s_p, i_1(c_i) = i_2(c_i) \rangle$ *is a group presentation of* $\pi_1(X)$.

Remark 1.13. It follows from [MKS04, Theorem 4.3] that j_1 and j_2 are both injective if and only if the map $i_1(c_i) \mapsto i_2(c_i)$ from $i_1(\pi_1(V))$ to $i_2(\pi_1(V))$ is an isomorphism.

In particular, if i_1 and i_2 are both injective, then j_1 and j_2 are both injective.

CW-complexes

Let D^k denote the **closed** k-ball in $\mathbb{R}^k$, $\mathrm{Int}(D^k)$ its interior and $\partial D^k = S^{k-1}$ its boundary.

For X, X' two topological spaces such that $X \subset X'$, we say that X' *is obtained from* X *by adjoining* k*-cells* if there exists a continuous map $f = \sqcup_i f_i \colon \sqcup_i D^k \to X'$ such that

- $f_{\sqcup_i \mathrm{Int}(D^k)} \colon \sqcup_i \mathrm{Int}(D^k) \to X' \setminus X$ is an homeomorphism
- a subset $U \subset X'$ is open in X' if and only if $U \cap X$ is open in X and $f_i^{-1}(U) \subset (D^k)_i$ is open in D^k, for all i.

The map $f_i \colon (D^k)_i \to X'$ is called a *characteristic map*, and the set $e_i^k = f_i(\mathrm{Int}(D^k))$ is called an *open* k*-cell*.

A topological space X is a *CW-complex* if there exists an increasing sequence of closed subspaces $X^0 \subset X^1 \subset \dots$ such that

- X^0 is a discrete space
- $X = \bigcup_{k \geqslant 0} X^k$
- X^{k+1} is obtained form X^k by adjoining k-cells
- $U \subset X$ is open in X if and only if $U \cap X^k$ is open in X^k for each k.

The subspace X^k is called the k*-skeleton of* X. A CW-complex X is *finite* if it is formed by a finite number of cells. Remark that X is finite if and only if X is compact. We implicitly orient and order the cells of a CW-complex X. A continuous map $f \colon X \to Y$ is called *cellular* if $f(X^k) = Y^k$ for every k.

In this thesis, we will often require the stronger property that f maps every k-cell to a k-cell.

If X is finite, then its *Euler characteristic* $\chi(X)$ is defined as

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k n_k \ \in \mathbb{Z}$$

where n_k is the number of k-cells. The integer $\chi(X)$ is a topological invariant of X, and does not depend on its cellular decomposition.

Coverings

For X, Y two CW-complexes, a surjective continuous map $p\colon Y \twoheadrightarrow X$ is called a *covering map* if each point $x \in X$ has an open neighbourhood $U \subset X$ such that $p^{-1}(U)$ is a union of disjoint open subsets of Y, each of which is mapped homeomorphically onto U by p.

Let Z be a connected CW-complex and $f\colon Z \to Y$ be a continuous map. For $z \in Z$ and $y \in p^{-1}(f(z)) \subset Y$, if $f_*\pi_1(Z, z) \subset p_*\pi_1(Y, y)$ (notably if Z is simply connected), then there exists a unique continuous map $\widetilde{f}\colon Z \to Y$ such that $\widetilde{f}(z) = y$ and $p \circ \widetilde{f} = f$. This is called the Unique Lifting Property.

Universal covering

Any connected CW-complex X admits a unique *universal covering* $p_X\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ has trivial fundamental group.

The *universal cover* $\widetilde{X}$ of X is defined as

$$\widetilde{X} = \{[c] \mid c\colon [0,1] \to X,\ c(0) = P\},$$

the set of homotopy classes of paths of X starting at the base point P. The natural base point of $\widetilde{X}$ is $\widetilde{P}$, the homotopy class of the constant path at P.

The corresponding covering map is

$$p_X\colon \begin{pmatrix} \widetilde{X} & \to & X \\ [c] & \mapsto & c(1) \end{pmatrix}.$$

One can define a topology on $\widetilde{X}$, and prove that p_X is a covering map. Details can be found in [Hat02, Pages 64-65].

CW-complex structure on the universal covering

The CW-structure of $\widetilde{X}$ is defined in the following way. For each k-cell e of X, $e = f(\mathrm{Int}(D^k))$ with f the corresponding characteristic map. Choose $d \in \mathrm{Int}(D^k)$ and $\widetilde{d} \in p_X^{-1}(f(d))$. There exists a unique lift $\widetilde{f}\colon D^k \to \widetilde{X}$ of f such that $\widetilde{f}(d) = \widetilde{d}$ by the Unique Lifting Property. The set $\widetilde{e} := \widetilde{f}(\mathrm{Int}(D^k))$ is an open k-cell of $\widetilde{X}$ with characteristic map $\widetilde{f}$, and $\widetilde{e}$ is homeomorphic to e by p. The k-skeleton of $\widetilde{X}$ is $\widetilde{X}^k = p_X^{-1}(X^k)$.

There is a $\pi_1(X)$-action (on the left) on $\widetilde{X}$, defined as follows: If $[\gamma] \in \pi_1(X)$ and $[c] \in \widetilde{X}$, then $[\gamma].[c] := [\gamma * c] \in \widetilde{X}$.

Remark that $p_X([\gamma].[c]) = p_X([c])$ since the endpoint does not change. Thus p_X is invariant by the action of $\pi_1(X)$ and this action sends a k-cell of $\widetilde{X}$ to an other k-cell. Moreover, the action is free and transitive. For details we refer to [Tur01, Chapter 5].

Boundary operator and cellular chain complex

If $f\colon S^n \to S^n$ is a continuous map, its *degree* is the integer $d = \deg(f)$ such that $f_*\colon \mathbb{Z}[S] \cong H_n(S^n) \to H_n(S^n) \cong \mathbb{Z}[S]$ satisfies $f_*([S]) = d[S]$.

Take a k-cell e_i^k of X, with characteristic map $f_i^k\colon D^k \to X$, $e_i^k = f_i^k(Int(D^k))$.

Let $f_{i,j}^k$ be the composition of maps:

$$f_{i,j}^k\colon S^{k-1} = \partial D^k \xrightarrow{f_i^k|_{\partial D^k}} X^{k-1} \xrightarrow{r} X^{k-1}/X^{k-2} = \bigvee_j S_j^{k-1} \xrightarrow{r_j} S_j^{k-1}$$

where r is the topological quotient map that collapses X^{k-2} to a point, which naturally makes X^{k-1}/X^{k-2} a wedge of spheres S_j^{k-1} indexed by the $(k-1)$-cells $\{e_j^{k-1}\}_j$ of X, and r_j is the topological quotient map that collapses every $(k-1)$-sphere but the j-th one. The orientations on the cells induce orientations on $S^{k-1} = \partial D^k$ and S_j^{k-1}.

The *k-th cellular chain group of X* is $C_k(X,\mathbb{Z}) = \bigoplus_i \mathbb{Z}e_i^k$, the free abelian group generated by the set of oriented k-cells of X.

The *boundary homomorphism* $\partial_{X,k}\colon C_k(X,\mathbb{Z}) \to C_{k-1}(X,\mathbb{Z})$, abbreviated by ∂, is defined by:

$$\partial(e_i^k) = \sum_j \deg(f_{i,j}^k) e_j^{k-1}.$$

The complex $C(X,\mathbb{Z}) = (\ldots \to C_k(X,\mathbb{Z}) \xrightarrow{\partial} C_{k-1}(X,\mathbb{Z}) \to \ldots)$ is a chain complex, and is called the *cellular chain complex of X*.

Cellular chain complex of the universal covering

Let X be a finite connected CW-complex and $p_X : \widetilde{X} \to X$ its universal covering. We orient the cells of X and then the cells of $\widetilde{X}$ such that p_X restricted to each cell is orientation preserving. The group $\pi = \pi_1(X)$ acts on $\widetilde{X}$ (on the left) and thus on $C_k(\widetilde{X})$ as well. If we extend this action linearly to an action of $\mathbb{Z}[\pi]$, $C_k(\widetilde{X})$ becomes a left $\mathbb{Z}[\pi]$-module.

The boundary homomorphism $\partial\colon C_k(\widetilde{X}) \to C_{k-1}(\widetilde{X})$ is linear over $\mathbb{Z}[\pi]$. To see this we only need to prove that for any k-cell E of $\widetilde{X}$ and for any $g \in \pi_1(X)$, $\partial(g \cdot E) = g \cdot \partial E$. Let E_j denote the $(k-1)$-cells of $\widetilde{X}$, with j indexed by a possibly infinite set J. Let $f_E\colon D^k \to \widetilde{X}$ be the characteristic map of E. Take any $j \in J$, and let $l \in J$ be such that $E_l = g \cdot E_j$. Then the following diagram

$$\begin{array}{ccccccc}
S^{k-1} = \partial D^k & \xrightarrow{f_E|_{\partial D^k}} & \widetilde{X}^{k-1} & \xrightarrow{r} & \widetilde{X}^{k-1}/\widetilde{X}^{k-2} = \bigvee_j S_j^{k-1} & \xrightarrow{r_j} & S^{k-1} \\
\updownarrow = & & \downarrow g\cdot & & \downarrow g\cdot & & \updownarrow = \\
S^{k-1} = \partial D^k & \xrightarrow{f_{g\cdot E}|_{\partial D^k}} & \widetilde{X}^{k-1} & \xrightarrow{r} & \widetilde{X}^{k-1}/\widetilde{X}^{k-2} = \bigvee_j S_j^{k-1} & \xrightarrow{r_l} & S^{k-1}
\end{array}$$

is commutative, since $g \cdot f_E = f_{g\cdot E}$ by the Unique Lifting Property and since g maps every m-cell to a m-cell for every integer m. This proves that $\deg(f_{E,j}) = \deg(f_{g.E,l})$ (where $f_{E,j}$ is the first long horizontal composite map in the previous diagram and $f_{g.E,l}$ is the second one) and therefore that $\partial(g \cdot E) = g \cdot \partial E$.

Let $\{e_i^k\}$ be the set of oriented k-cells of X ordered in an arbitrary way, and choose for each e_i^k a k-cell $\widetilde{e}_i^k$ in $\widetilde{X}$ in the pre-image of e_i^k. Then the set $\{\widetilde{e}_i^k\}$ is a $\mathbb{Z}[\pi]$-basis of $C_k(\widetilde{X})$, i.e. $C_k(\widetilde{X}) = \bigoplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k$.

Thus, the cellular chain complex $C(\widetilde{X})$ is a free based chain complex of left $\mathbb{Z}[\pi]$-modules.

1.1.3 Case of a pair of CW-complexes: universal coverings, cellular chain complexes

Note that contrary to the rest of this chapter, the content of this section does not follow the classical notations and conventions.

Let V, X be compact connected topological spaces endowed with structures of finite CW-complexes such that the inclusion $V \overset{I}{\hookrightarrow} X$ maps every k-cell to a k-cell. Let P be a base point in V. Let $Q = I(P)$ be the base point in X. Let $\widetilde{V} \overset{p_V}{\twoheadrightarrow} V$ and $\widetilde{X} \overset{p_X}{\twoheadrightarrow} X$ be the universal coverings. Choose $\widetilde{P} \in \widetilde{V}$ the natural lift of P and $\widetilde{Q} \in \widetilde{X}$ the natural lift of Q.

The space $\widetilde{V}$ is a simply connected CW-complex, therefore by the Unique Lifting Property the continuous map $I \circ p_V : \widetilde{V} \to X$ lifts to an unique map $\widetilde{I} : \widetilde{V} \to \widetilde{X}$ such that $\widetilde{I}(\widetilde{P}) = \widetilde{Q}$.

Remark 1.14. Here $\widetilde{V}$ is the actual universal covering of V, not the lift $p_X^{-1}(I(V))$ as the reader may be used to. In particular, $\widetilde{I} : \widetilde{V} \to \widetilde{X}$ will **not** be injective in general.

Let $\pi_V = \pi_1(V, P)$ and $\pi_X = \pi_1(X, Q)$. If $g = [\alpha] \in \pi_V$ where α is a loop in V, then $I_*(g) = [I \circ \alpha] \in \pi_X$.

Let $\widetilde{e}$ denote a k-cell of $\widetilde{V}$, and $e = p_V(\widetilde{e})$ the corresponding k-cell of V. The image $f = I(e)$ is a k-cell of X and its lift $\widetilde{f} = \widetilde{I}(\widetilde{e})$ is a k-cell of $\widetilde{X}$, since it is a connected component of the pre-image by p_X of the cell f. Thus $\widetilde{I} : \widetilde{V} \to \widetilde{X}$ maps every k-cell to a k-cell (but is not necessarily injective).

Recall that the universal cover $\widetilde{V}$ is the set of homotopy classes $[\alpha]$ of paths α in V starting at the base point P. Since the map $([\alpha]_V \mapsto [I \circ \alpha]_X)$ is a lift of $I \circ p_V$ that maps $\widetilde{P}$ (the class of the constant path at P) to $\widetilde{Q}$ (the class of the constant path at $Q = I(P)$), the map $([\alpha]_V \mapsto [I \circ \alpha]_X)$ is equal to $\widetilde{I}$ by the Unique Lifting Property. Therefore $\widetilde{I}([\alpha]_V) = [I \circ \alpha]_X$. Besides, if $g = [\gamma]_V \in \pi_1(V)$ where γ is a loop in (V, P), and $R = [\alpha]_V \in \widetilde{V}$, then

$$\widetilde{I}(g \cdot R) = \widetilde{I}([\gamma]_V \cdot [\alpha]_V) = \widetilde{I}([\gamma * \alpha]_V) = [I \circ (\gamma * \alpha)]_X = [I \circ \gamma]_X \cdot [I \circ \alpha]_X = I_*(g) \cdot \widetilde{I}(R).$$

Now let us prove that $\widetilde{I}$ commutes with the boundary operators. Let $\widetilde{e}$ be a k-cell of $\widetilde{V}$ as above. Let us prove that $\partial(\widetilde{I}(\widetilde{e})) = \widetilde{I}(\partial\widetilde{e})$. Let $\{\widetilde{e_j}\}_{j \in J}$ be the set of $(k-1)$-cells of $\widetilde{V}$. Then $\{\widetilde{I}(\widetilde{e_j})\}$ is a subset of the set of $(k-1)$-cells of $\widetilde{X}$ but contains all the boundary cells of $\widetilde{I}(\widetilde{e})$. With the same notations as above, the following diagram

$$\begin{array}{ccccccc}
S^{k-1} = \partial D^k & \xrightarrow{f_{\widetilde{e}}|_{\partial D^k}} & \widetilde{V}^{k-1} & \xrightarrow{r} & \widetilde{V}^{k-1}/\widetilde{V}^{k-2} = \bigvee_j S_j^{k-1} & \xrightarrow{r_{\widetilde{e_j}}} & S^{k-1} \\
\updownarrow = & & \downarrow \widetilde{I} & & \downarrow \widetilde{I} & & \updownarrow = \\
S^{k-1} = \partial D^k & \xrightarrow{f_{\widetilde{I}(\widetilde{e})}|_{\partial D^k}} & \widetilde{X}^{k-1} & \xrightarrow{r} & \widetilde{X}^{k-1}/\widetilde{X}^{k-2} = \bigvee_j S_j^{k-1} & \xrightarrow{r_{\widetilde{I}(\widetilde{e_j})}} & S^{k-1}
\end{array}$$

is commutative, therefore $\deg(f_{\widetilde{I}(\widetilde{e}),\widetilde{I}(\widetilde{e_j})}) = \deg(f_{\widetilde{e},\widetilde{e_j}})$. Hence $\partial(\widetilde{I}(\widetilde{e})) = \widetilde{I}(\partial\widetilde{e})$. Be careful that different cells $\widetilde{e_j} \neq \widetilde{e_l}$ can have the same image $\widetilde{I}(\widetilde{e_j}) = \widetilde{I}(\widetilde{e_l})$.

The map $\widetilde{I}$ commutes with the group actions and the boundary operators, therefore if $\partial_{\widetilde{V},k}(\widetilde{e}) = \sum_j m_j(h_j \cdot \widetilde{e_j})$, where $m_j \in \mathbb{Z}$, then $\partial_{\widetilde{X},k}(\widetilde{I}(\widetilde{e})) = \sum_j m_j(I_*(h_j) \cdot \widetilde{I}(\widetilde{e_j}))$. If we equip the cellular chain complexes of $\widetilde{V}$ and $\widetilde{X}$ of bases as finite free $\mathbb{Z}[\pi_V]$-module and $\mathbb{Z}[\pi_X]$-module $\left\{\widetilde{e_j^k}\right\}$ and $\left\{\widetilde{I}(\widetilde{e_j^k})\right\} \cup \left\{\widetilde{f_j^k}\right\}$ respectively, then as matrices over $\mathbb{Z}[\pi_V]$ and $\mathbb{Z}[\pi_X]$ the boundary operators satisfy:

$$\partial_{\widetilde{X},k} : \left(\bigoplus_j \mathbb{Z}[\pi_X]\widetilde{I}(\widetilde{e_j^k}) \oplus \bigoplus_j \mathbb{Z}[\pi_X]\widetilde{f_j^k} \longrightarrow \bigoplus_j \mathbb{Z}[\pi_X]\widetilde{I}(\widetilde{e_j^k}) \oplus \bigoplus_j \mathbb{Z}[\pi_X]\widetilde{f_j^k} \right)$$

$$\partial_{\widetilde{X},k} = \begin{pmatrix} I_*\left(\partial_{\widetilde{V},k}\right) : \left(\bigoplus_j \mathbb{Z}[\pi_V]\widetilde{e_j^k} \to \bigoplus_j \mathbb{Z}[\pi_V]\widetilde{e_j^k}\right) & * \\ 0 & * \end{pmatrix}.$$

The fact that the matricial forms of the boundary operators of $\widetilde{X}$ are naturally block trigonal, with one diagonal block being obtained from the boundary operators of $\widetilde{V}$, is the main point of this section. We will use this result several times in Section 3.1.

1.1.4 Groups and knot theory

Knot group

Let K be a knot in S^3. The *knot group* $G_K = \pi_1(M_K)$ of K is the fundamental group of its exterior $M_K = S^3 \setminus V(K)$. Recall that if G is a group, G' is its *commutator subgroup* and the *abelianization of* G denotes both the quotient group $G^{ab} = G/G'$ and the quotient group homomorphism $\alpha_G \colon G \to G^{ab}$. The abelianization of a knot group G_K is the infinite cyclic group. There are therefore exactly two surjective group homomorphisms from G_K to $\mathbb{Z}$. We will denote $\alpha_K \colon G_K \to \mathbb{Z}$ the one that sends homotopy classes of meridian curves to 1. Note that this choice depends on the orientation of K.

This generalises to links. Let $L = L_1 \cup \ldots \cup L_c$ be a c-link in S^3. The *link group* of L is $G_L = \pi_1(S^3 \setminus V(L))$. The abelianization α_L is

$$\begin{aligned} \alpha_L \colon G_L &\to \mathbb{Z}^c \\ [\gamma] &\mapsto (\mathrm{lk}(\gamma, L_1), \ldots, \mathrm{lk}(\gamma, L_c)) \end{aligned}$$

where $\mathrm{lk}(\gamma,\delta)$ is the *linking number* between two simple oriented closed curves γ and δ in S^3, which can be defined as the number of positive crossings of γ under δ minus the number of negative crossings of γ under δ. Other equivalent definitions of the linking number can be found in [Rol90, Section 5D].

Group presentations with generators and relations

When considering a group presentation $P = \langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle$, it is usual to identify the combinatoric $(k+l)$-tuple and the generated group. In this manuscript, we distinguish between the two and we denote $Gr(P)$ the quotient of the free group $\mathbb{F}[g_1, \ldots, g_k]$ by its normal subgroup generated by the free words $r_1, \ldots, r_l$.

Sometimes we will write a relator r as a free word, sometimes as $r = 1$ or $r = r'$ an equality between free words in the generators, whatever is clearer at the moment.

We will say that a group G *admits the presentation* $P = \langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle$ when G is isomorphic to $Gr(P)$, and we will assume that this isomorphism is implicit, or equivalently that we implicitly know which elements of G are associated to $g_1, \ldots, g_k$.

For instance, the well-known Wirtinger process takes a regular diagram D of a knot K and gives a group presentation P of the knot group G_K, and the generators of P all implicitly correspond to homotopy classes of meridian curves in G_K; therefore they are all sent to the same image 1 by the abelianization homomorphism α_K. This process is recalled in the next section.

The *deficiency of a presentation* $P = \langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle$ is the integer $\mathrm{def}(P) = k - l$. The *deficiency of a group* G $\mathrm{def}(G)$ is the maximum deficiency of a group presentation of G. It is easy to add redundant relators and leave the underlying group unchanged, but it is much harder to find a minimal number of relators for one given group and one given set of generators.

The following theorem give us the deficiency of link groups.

Theorem 1.15. *[Hil12, Theorem 1.2] A link $L \in S^3$ is split if and only if def$(G_L) \geqslant 2$.*
A link $L \in S^3$ is non-split if and only if def$(G_L) = 1$.

In this manuscript, we will mostly be interested in groups and group presentations of deficiency one.

P = < a,b,c | ab=ca , bc = ab , ca=bc >

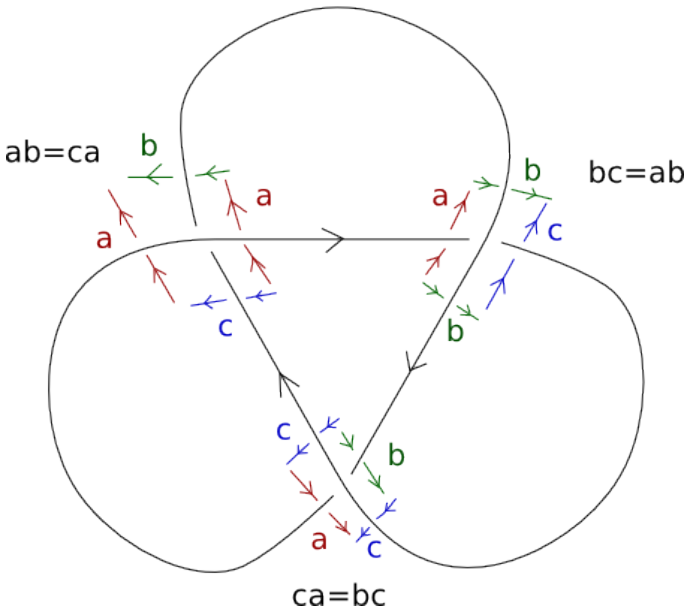


Figure 1.3 – A Wirtinger presentation for the trefoil knot

Wirtinger presentations for knots and links

Let D be a regular diagram of a knot K. We construct a group presentation P of the knot group G_K in the following way. For every (oriented) arc e_i on D we define a generator g_i, which is the homotopy class in $G_K = \pi_1(S^3 \setminus V(K))$ of the meridian curve circling e_i positively (with the base point of the fundamental group being the eye of the reader looking at the diagram from above, this means the curve goes under e_i from right to left when e_i goes from bottom to top); then for every crossing c_j on D we define a relator r_j of the form $g_a g_b = g_c g_a$ where g_a, g_b, g_c are the generators associated to the three arcs meeting at the crossing c_j. See the example of the trefoil knot in Figure 1.3 for clarity.

As we can see in the example of Figure 1.3, any relator is a consequence of all the others. This is actually true in general (see [BZH14, Corollary 3.6]). We will therefore consider P to be a *Wirtinger presentation* of G_K if P is obtained by the previous process on a regular diagram of K and by taking out one relator. If the diagram has k arcs and k crossings, P has therefore k generators and $k-1$ relators, and thus is of deficiency one. One such presentation for the example of Figure 1.3 would be $\langle a, b, c | ab = ca, bc = ab \rangle$.

Note that we can do the same process with any link L. Two generators g_i, g_j of a Wirtinger presentation are conjugates in G_L if and only if they are homotopy classes of meridian curves associated to the same component of L.

Remark 1.16. Be careful that sometimes the link diagram has more arcs than crossings, for example if it is a naturally embedded circle (1 arc, no crossings) or the natural diagram of the trivial m-link, with m disjoint naturally embedded circles in the plane (m arcs, no crossings). Fortunately, since by Theorem 1.15 a link group has deficiency one if and only if it is split, we can define the Wirtinger presentation as a deficiency one presentation obtained from the diagram D of a non-split link L, either by taking the whole presentation

if it already has deficiency one (like in the case of the circle for L the trivial knot), or by taking out one relator from the presentation of deficiency zero (for all the other cases, as above).

Note finally that the same diagram can generate many different Wirtinger presentations, since we did not choose any order on the generators nor any rule on writing the relators. These ambiguities will be described by the Strong Tietze moves on group presentations (we recall their definition in the proof of Proposition 2.21).

Fox calculus

Let $P = \langle g_1, \dots, g_k \,|\, r_1, \dots r_l \rangle$ be a presentation of a finitely presented group G. If w is an element of the free group $\mathbb{F}[g_1, \dots, g_k]$ on the generators g_i, we let $\overline{w}$ denote the element of G that is the image of w by the composition of the quotient homomorphism (quotient by the normal subgroup $\langle r_j \rangle$ generated by $r_1, \dots, r_l$) and the implicit group isomorphism between this quotient $Gr(P)$ and G. To simplify the notations in the sequel, we will often write an element of G a instead of $\overline{a}$ when there is no ambiguity. We write the corresponding ring morphisms similarly: if $w \in \mathbb{C}\left[\mathbb{F}[g_1, \dots, g_k]\right]$ then its quotient image is written $\overline{w} \in \mathbb{C}[G]$.

The *Fox derivatives associated to the presentation* P are the linear maps $\dfrac{\partial}{\partial g_i} : \mathbb{C}\left[\mathbb{F}[g_1, \dots, g_k]\right] \longrightarrow \mathbb{C}\left[\mathbb{F}[g_1, \dots, g_k]\right]$ for $i = 1, \dots, k$, defined by induction as follows:

$\dfrac{\partial}{\partial g_i}(1) = 0$, $\dfrac{\partial}{\partial g_i}(g_j) = \delta_{i,j}$, $\dfrac{\partial}{\partial g_i}(g_j^{-1}) = -\delta_{i,j} g_j^{-1}$ (where $\delta_{i,j}$ is 1 when $i = j$ and 0 when $i \neq j$) and for all $u, v \in \mathbb{F}[g_1, \dots, g_n]$, $\dfrac{\partial}{\partial g_i}(uv) = \dfrac{\partial}{\partial g_i}(u) + u\dfrac{\partial}{\partial g_i}(v)$.

The matrix $F_P = \left(\overline{\left(\dfrac{\partial r_j}{\partial g_i}\right)}\right)_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l} \in M_{k,l}(\mathbb{C}[G])$ is called the *Fox matrix* of the presentation P.

For $i = 1, \dots, k$, $F_{P,i} \in M_{k-1,l}(\mathbb{C}[G])$ is defined as the matrix obtained from F_P by deleting its i-th row.

We will sometimes use the following notation, to «remember the coordinates»:

$$F_P = \begin{array}{c} \\ x_1 \\ \vdots \\ x_k \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & \dots & r_l \end{array} \\ \left(\begin{array}{ccc} & & \\ & \left(\overline{\left(\dfrac{\partial r_j}{\partial g_i}\right)}\right)_{i,j} & \\ & & \end{array} \right) \end{array}$$

Remark 1.17. As a quick consequence of the definition, $\dfrac{\partial u^{-1}}{\partial g} = -u^{-1}\dfrac{\partial u}{\partial g}$ for g a generator and u a free word in the generators.

Remark 1.18. We will often use the following fact: if r is a relator of P and $r = uv^{-1}$, where r, u, v are free words in the generators, then $\dfrac{\partial r}{\partial g} = \dfrac{\partial u}{\partial g} - r\dfrac{\partial v}{\partial g}$ and thus $\overline{\dfrac{\partial r}{\partial g}} = \overline{\dfrac{\partial u}{\partial g}} - \overline{\dfrac{\partial v}{\partial g}}$.

This is why, in the following sections, we will sometimes use the convention that if a relator is written $u = v$, its Fox derivative (seen in the quotient G, not in the free group) is $\dfrac{\partial u}{\partial g} - \dfrac{\partial v}{\partial g}$.

1.1.5 Torus knots and torus links

Let p and q be relatively prime integers and let V be a solid torus with a preferred meridian-longitude system (and thus an oriented core). The knot $T(p,q)$ on the boundary ∂V of V will denote the knot that wraps around V q times in the meridional direction and p times in the longitudinal direction; it will be called the (p,q)-*torus knot*.

Note that this follows the conventions of [Rol90] but not the ones of [BZH14] and [Cro04], where the roles of p and q are reversed.

One can also define $T(p,q)$ as the knot on the boundary ∂V of the solid torus obtained in the following way: let c be the oriented core of V and m an oriented meridian such that the linking number of c and m is $+1$.

- if $p = 0$, then $T(0, \pm 1)$ is a meridian of ∂V.
- if $p > 0$, then $T(p,q)$ is obtained by taking p parallel strands following the core c and twisting them $|q|$ times by an angle of $\frac{2\pi \text{ sign}(q)}{p}$ in the direction of m.
- if $p < 0$, then $T(p,q)$ is obtained by taking $|p|$ parallel strands following c^{-1} and twisting them $|q|$ times by an angle of $\frac{2\pi \text{ sign}(q)}{p}$ in the direction of m^{-1}.

For example, the right trefoil knot of Figure 1.2 is the torus knot $T(2,3)$.

Note that $T(p,q)$ and $T(q,p)$ are ambient isotopic in S^3, as are $T(p,q)$ and its inverse $T(-p,-q)$. The mirror image of $T(p,q)$ is $T(p,-q)$ and is not ambient isotopic to $T(p,q)$.

These definitions extend to the case when $e = \gcd(p,q) \geqslant 2$. Then $T(p,q) = T(ea,eb)$ is a e-component link called the (p,q)-*torus link*. Each of its components is a torus knot $T(a,b)$.

Torus knots and torus links will be studied in more detail in Section 4.2.

Remark 1.19. Torus knots are the only knots whose group has a non-trivial center. The classical group presentation of $T(p,q)$ is $\langle x, y | x^p = y^q \rangle$, and the center is infinite cyclic generated by x^p.

1.1.6 Connected sum of knots

Let K_1 and K_2 be knots in S^3. Their *connected sum* K is the knot obtained by removing one small arc on K_1 and K_2 and joining the four vertices by two other arcs respecting the orientations and forming a single knot; it is denoted $K = K_1 \sharp K_2$. Up to ambient isotopy, the knot K does not depend on the choice of the two small arcs. The connected sum of two knots can be seen in the following Figure 1.4 for closed projections and «vertical» projections, i.e. projections in S^3 where the knots pass by the point at infinity (from seeing S^3 as the one-point compactification of $\mathbb{R}^3$); in this case, obtaining K consists in knotting K_1 on the vertical strand, then knotting K_2 a little further. We say that K_1 and K_2 are *factors* of K. A knot is *prime* if its only factors are itself or the trivial knot.

Let G_1, G_2 and G be the fundamental groups of the knots K_1, K_2 and K in $\mathbb{R}^3$ respectively. In Section 2.3.4 we will use the following technical lemma (see [BZH14, Proposition 7.10]):

Lemma 1.20. *The groups G_1, G_2 admit Wirtinger presentations $P_1 = \langle x_1, \ldots, x_k | r_1, \ldots, r_{k-1} \rangle$, $P_2 = \langle y_1, \ldots, y_l | s_1, \ldots, s_{l-1} \rangle$ such that*

$$P = \langle x_1, \ldots, x_k, y_1, \ldots, y_l | r_1, \ldots, r_{k-1}, s_1, \ldots, s_{l-1}, x_k y_l^{-1} \rangle$$

is a Wirtinger presentation of G.

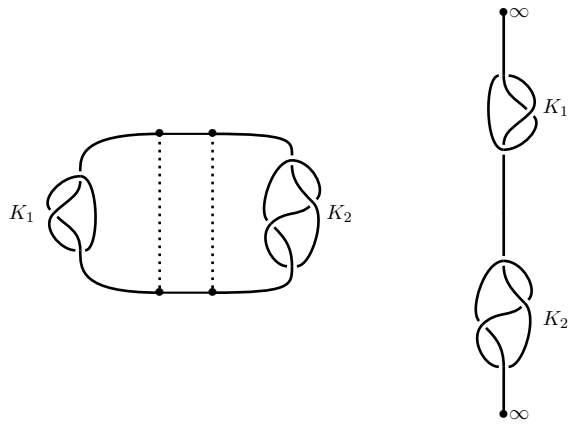


Figure 1.4 – The connected sum of the trefoil knot and the figure-eight knot

The following proposition is a consequence of Remark 1.13, and will be useful for induction properties in Section 2.3.4.

Proposition 1.21. *If K is the connected sum of the knots K_1 and K_2, and G, G_1, G_2 are their respective groups, then there exist injective group homomorphisms $G_1 \hookrightarrow G$ and $G_2 \hookrightarrow G$.*

We can extend this definition and the previous statements to links, the connected sum being done on one component of each of the two links. A *prime link* is a non-split link whose only factors are itself and the trivial knot.

For example, the keychain link of Figure 1.7 is a connected sum of several Hopf links.

We will study connected sum of links in more detail in Section 4.2.4.

1.1.7 Satellites

Let C be a non-trivial knot in S^3 (it will be called the *companion knot*).

We consider P a knot inside an open solid torus T_P, T_P being also naturally embedded in S^3 (P will be called the *pattern knot*). We choose an orientation for the core of T_P. We assume that P meets every meridional disk of T_P. We let $n_P \in \mathbb{Z}$ denote the linking number between P and a preferred meridian curve of ∂T_P (assumed to be positively oriented with the orientation of the core of T_P); n_P is walled the *winding number of P*. Note that preferred longitude curves of T_P have zero linking number with the core of T_P and follow the same direction.

Let T_C be an open tubular neighbourhood of C (its core having the same orientation as C). Remark that a preferred longitude curve of T_C has zero linking number with C. Thus the homotopy class in G_C of such a curve is sent to zero by the abelianization α_C.

Let $h_{PC} \colon T_P \to T_C$ be an orientation-preserving homeomorphism between the two solid tori. We also assume that h_{PC} sends a preferred meridian-longitude pair of T_P to a preferred meridian-longitude pair of T_C.

Then $S_{C,P} := h_{PC}(P)$ is a knot in S^3 and is called *the satellite knot of companion C and pattern P*.

If P is a torus knot $T(p,q)$ (naturally defined on the boundary of a solid sub-torus of T_P) with $p \neq 0$, then we say that $S_{C,P}$ is a *cable knot*, or *the (p,q)-cable of C*. In this case $n_P = p$. Figure 1.5 gives an example of $S_{C,P}$ when C is the trefoil knot and P is the torus knot pattern $T(2,-1)$. The orientations are not marked but should be clear.

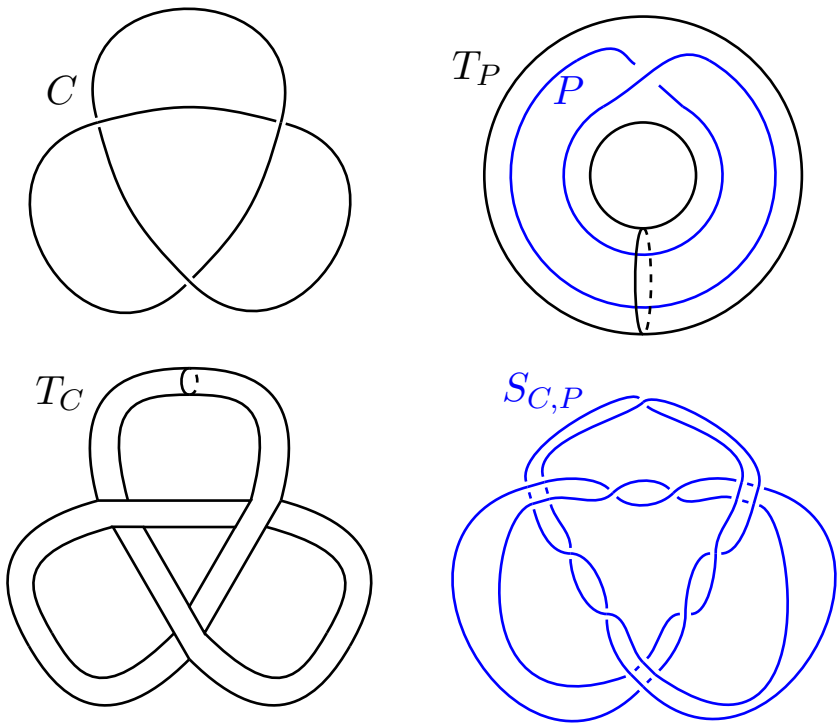


Figure 1.5 – The $(2,-1)$-cabling of the trefoil knot

Remark that if K is a knot in S^3, then the $(1,0)$-cable of K is K and the $(-1,0)$-cable of K is its inverse $-K$.

If P is the pattern in Figure 1.6, called the *Whitehead double pattern*, then $n_P = 0$ and $S_{C,P}$ is called the *Whitehead double of* C.

Remark 1.22. This construction generalises to links in three cases:

- if C is a link, do the satellite construction by using one of its components as companion.
- if P is a c-link in T_P, define $S_{C,P}$ as $h_{PC}(P)$ as before, it will be a c-link.
- if C and P are links, mix the two previous constructions.

We will study these cases in more detail in Chapter 4.

We give a detailed proof of the following proposition in Section A.1 of the Annex. Note that this result can be found in a different flavour in [BZH14, Section 4.12] and [Rol90, Theorem 4D9].

Proposition 1.23. *Let us consider the (p,q)-cable knot S of companion C.*

(1) There exists $P_C = \langle a_1, \dots, a_k | r_1, \dots, r_{k-1} \rangle$ a Wirtinger presentation of G_C such that

$$P_S = \langle a_1, \dots, a_k, x, \lambda | r_1, \dots, r_{k-1}, x^p a_k^{-q} \lambda^{-p}, \lambda^{-1} W(a_i) \rangle$$

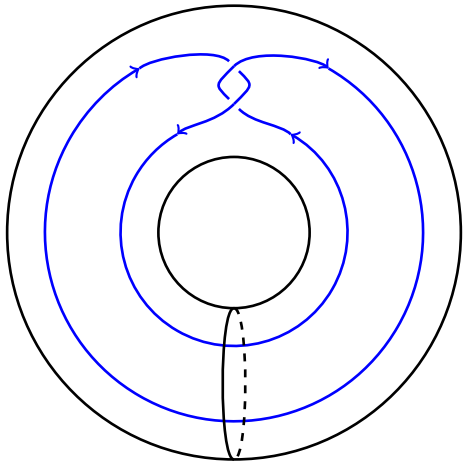


Figure 1.6 – The Whitehead double pattern

is a presentation of G_S, with x and λ the homotopy classes of the core and a longitude of T_C, and $W(a_i)$ a word in the $a_1, \ldots, a_k$.

(2) Furthermore, $\alpha_S(x) = q$, $\alpha_S(\lambda) = 0$ and $\alpha_S(a_i) = p$, for $i = 1, \ldots, k$.

The following proposition is a consequence of Remark 1.13, and will be useful for induction properties.

Proposition 1.24. *If S is the satellite knot obtained from the companion C and the pattern P, then there exist injective group homomorphisms $G_C \hookrightarrow G_S$ and $\pi_1(T_P \setminus P) \hookrightarrow G_S$.*

Proof. Apply Remark 1.13 with $A = S^3 \setminus T_C$, $B = h_{PC}(\overline{T_P} \setminus P)$, $X = A \cup B = S^3 \setminus S$ and $V = A \cap B = \partial T_C$. Since C is not trivial, it follows from [Rol90, Theorem 4B2] that $V \hookrightarrow A$ induces an injective group homomorphism in the fundamental groups. Similarly, it follows from assumptions on P and [Rol90, Exercise 4D3] that $V \hookrightarrow B$ induces an injective group homomorphism in the fundamental groups. □

1.2 Geometry

1.2.1 Genus and Thurston norm

Genus of knots and links

Compact connected oriented surfaces are determined up to homeomorphism by two non-negative integers: the number b of boundary (circular) components and the genus g, that counts the numbers of «handles». Such a surface will be written $S_{g,b}$.

Example 1.25. The 2-sphere S^2 is $S_{0,0}$, the disk is $S_{0,1}$, the annulus is $S_{0,2}$. The 2-torus $S^1 \times S^1$ is $S_{1,0}$.

Recall that the Euler characteristic of $S_{g,b}$ is $\chi(S_{g,b}) = 2 - 2g - b$.

Let L be a c-component (oriented) link in S^3. A *Seifert surface* for L is a compact oriented surface S embedded in S^3 whose boundary ∂S is equal to L as an oriented 1-manifold. The minimal genus g of a Seifert surface spanning a link L is called *the genus of the link* L and is written $g(L)$.

Remark 1.26. The genus detects the trivial knot, i.e. a knot K has genus 0 if and only if K is the trivial knot (see [BZH14, Section 2.B]).

Thurston norm

Let M be a 3-manifold and let $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}\,(\pi_1(M),\mathbb{Z})$ be a non-trivial 1-cohomology class.

Definition 1.27. The *Thurston norm of the class* ϕ is denoted $x_M(\phi)$ and is defined as

$$x_M(\phi) = \min_{\Sigma} \left\{ \chi_-(\Sigma) = \sum_{i=1}^{k} \max(-\chi(\Sigma_i), 0) \right\}$$

where $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_k$ is a surface (properly embedded in M) dual to ϕ (meaning that the homology class $[\Sigma] \in H_2(M, \partial M)$ is the Poincaré dual of $\phi \in H^1(M)$).

Observe that x_M is a *semi-norm* on $H^1(M;\mathbb{Z})$ and not a norm in general ($x_M(\phi)$ can sometimes vanish for non zero ϕ).

Example 1.28. If K is a knot in S^3, and $M = M_K = S^3 \setminus V(K)$, then for $\phi = \alpha_K \colon G_K \to \mathbb{Z}$ the abelianization, $x_M(\alpha_K) = 2g(K) - 1$.

For a c-link L the situation is more complex, since there are as many possible classes $\phi \in H^1(M_L, \mathbb{Z})$ as there are linear maps from $\mathbb{Z}^c$ to $\mathbb{Z}$.

Example 1.29. (see [Thu86, Section 2, Example 1]) If $L = L_1 \cup L_2$ is the Whitehead link, then for $\phi = ((n_1, n_2) \circ \alpha_L) \in \mathrm{Hom}(G_L, \mathbb{Z})$ (with $n_1, n_2 \in \mathbb{Z}$), $x_{M_L}(\phi) = |n_1| + |n_2|$ and x_{M_L} is a norm.

1.2.2 Hyperbolic knots and links

A 3-manifold M is called *hyperbolic* if its interior admits a complete Riemannian metric whose sectional curvature is constant equal to -1. We refer to [Rat06] for details.

Hyperbolic structures of finite volume on 3-manifolds are rigid in the following sense:

Theorem 1.30. *(Mostow-Prasad-Marden Rigidity theorem)*
Let M and N be finite-volume hyperbolic 3-manifolds. Any isomorphism $\pi_1(M) \to \pi_1(N)$ is induced by a unique isometry $M \to N$.

This statement can be found in [AFW12, Theorem 1.10], it summarizes theorems of [Mos68], [Mos73], [Pra73] and [Mar74].

This fundamental result implies that the hyperbolic structure on a finite volume hyperbolic 3-manifold is determined by its topology and is unique up to isometry. In particular, any invariants which are defined in terms of the hyperbolic structure of such a manifold, such as the volume, are actually topological invariants of the 3-manifold.

A knot K in S^3 is *hyperbolic* if its complement $S^3 \setminus K$ admits a hyperbolic structure of finite volume. The volume of this hyperbolic structure is called the *volume of the knot* K and is written $\mathrm{vol}(K)$.

Unless we say otherwise, in the remainder of the manuscript, a hyperbolic 3-manifold is always understood to have finite volume.

We can then see that a knot $K \subset S^3$ is hyperbolic if and only if $M_K = S^3 \setminus V(K)$ is hyperbolic, since the interior of M_K is homeomorphic to $S^3 \setminus K$.

The volume of a knot is thus a invariant of knots up to isotopy. The volume is a very deep and powerful knot invariant, deep because computing an exact value of the volume is hard (fortunately, Jeffrey Weeks' *SnapPea* computer program can approximate it as far as one could want) and powerful because it distinguishes most of the prime hyperbolic knots.

Example 1.31. The figure-eight knot 4_1 is hyperbolic. Its complement can be seen as a gluing of two regular ideal hyperbolic tetrahedra, thus its volume is twice the volume of such a tetrahedra:

$$\mathrm{vol}(4_1) \approx 2.0298832.$$

It has the smallest hyperbolic volume among hyperbolic knots. Details of the hyperbolic structure and the computation of this volume can be found in [Thu91].

Remark 1.32. W. Thurston and T. Jorgensen proved that the set of the volumes of orientable hyperbolic 3-manifolds is a well-ordered subset of $\mathbb{R}_{>0}$, see [BP92, Corollary E.7.5]. In particular, the set of volumes of 3-manifolds with one toroidal boundary component has a minimal element, which is $\mathrm{vol}(4_1)$, as Cao and Meyerhoff proved in [CM01].

These definitions immediately generalise to links.

1.2.3 JSJ decompositions of 3-manifolds

For this section we refer to [Hat00, Chapter 1] and [AFW12, Section 1].

All 2-manifolds and all 3-manifolds will be considered connected, orientable, compact and with possibly non-empty boundary, unless specified otherwise.

Prime decomposition

Let M be a 3-manifold. A 2-manifold $S \subset M$ is *properly embedded in* M if $S \cap \partial M = \partial S$ in a transverse intersection. This means that if S has any boundary, it must be exactly the only part of S intersecting ∂M.

If $S \subset M$ is a properly embedded 2-sphere (thus $S \cap \partial M = \varnothing$), then let $M|S$ be the 3-manifold obtained by deleting a small tubular (open) neighbourhood $V(S)$ from M. Let us assume $M|S$ has two connected components M_1' and M_2', and let M_i be the 3-manifold obtained from M_i' by filling the boundary sphere corresponding to S by a 3-ball. In this case we say that M is *the connected sum of* M_1 *and* M_2 and we write $M = M_1 \sharp M_2$.

A theorem of Alexander states that every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball. Therefore S^3 is a neutral element for the operation $\sharp$. Moreover, $\sharp$ is commutative and associative (up to homeomorphism).

A 3-manifold M is called *prime* if $M = P \sharp Q$ implies $P = S^3$ or $Q = S^3$, and *irreducible* if every 2-sphere $S^2 \subset M$ bounds a 3-ball $B^3 \subset M$.

These two properties are not exactly equivalent: if M is irreducible, then M is prime, but $S^1 \times S^2$ is prime and not irreducible. This is the only counter-example: if $M \neq S^1 \times S^2$, then M is prime if and only if M is irreducible.

Theorem 1.33. *(Sphere decomposition) Let M be a 3-manifold. There is a decomposition $M = P_1 \sharp \ldots \sharp P_n$ with each P_i prime, and this decomposition is unique up to order, and insertion or deletion of S^3's.*

Example 1.34. Let L be a c-component link in S^3, $V(L)$ an open tubular neighbourhood and $M_L = S^3 \setminus V(L)$ the exterior of L. M_L is compact, connected and orientable. Furthermore, M_L is irreducible if and only if L is not a split link. In particular, if $L = K$ is a knot, then M_K is irreducible.

Tori decomposition

Once we cannot properly split a 3-manifold M along spheres, we can find deeper geometric information by splitting M along tori.

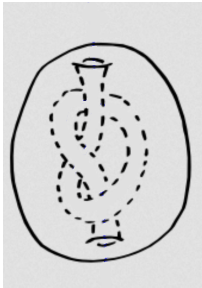
A properly embedded connected surface $S \subset M^3$ is called 2-*sided* if $V(S) \setminus S$ has two connected components ($V(S)$ is still an open neighbourhood of S). A connected 2-sided surface S not equal to S^2 nor D^2, $S \subset M$ is *incompressible* if for each disk $D \subset M$ with $D \cap S = \partial D$ there is a disk $D' \subset S$ with $\partial D' = \partial D$.

Note that a connected 2-sided surface $S \notin \{S^2, D^2\}$ is incompressible if and only if the group homomorphism $\pi_1(S) \to \pi_1(M)$ induced by the inclusion is injective. This algebraic characterization will often be useful.

The fact that we exclude the sphere and the disk means an incompressible surface necessarily has infinite fundamental group. It can be an annulus, a disk with many holes, a torus, a punctured torus, a g-genus surface, punctured or not, etc. We will be especially interested in incompressible tori.

If M is an irreducible 3-manifold, a 2-sided torus $T^2 \subset M^3$ is compressible if and only if T^2 bounds a solid torus in M or lies in a 3-ball in M. Moreover, every torus $T^2 \subset S^3$ bounds a solid torus on one side or the other (see [Hat00, Exercise 1.1-3]).

Example 1.35. Here is a 2-torus that lies in a 3-ball B^3 but does not bound a solid torus in B^3.



But if we add a point at infinity to B^3 and turn it into S^3, then this torus bounds a solid torus in the form of a trefoil knot. This picture comes from
https://themathingpot.wordpress.com/category/low-dimensional-topology/

Note that if $S \subset M$ is a finite collection of disjoint, properly embedded incompressible surfaces, then M is irreducible if and only if $M|S$ is irreducible and any surface $T \subset M|S$ is incompressible in $M|S$ if and only if it is incompressible in M.

An irreducible 3-manifold M is called *atoroidal* if every incompressible torus in M is isotopy equivalent to a subsurface of ∂M, or *boundary parallel*.

We will say that a (compact oriented) 3-manifold M is a *Seifert manifold*, or a *Seifert-fibered manifold* if M admits a foliation by circles. This is equivalent to the classical definition (of [Hat00, p.13]) by Epstein's theorem (see [Eps72]). We will not study geometric structures of Seifert manifolds in detail but we give this definition to help the reader picture simple examples.

The following results come from [AFW12, Section 1.4].

Theorem 1.36. *(JSJ decomposition, [AFW12, Theorem 1.7])*

For an irreducible 3-manifold M with empty or toroidal boundary there exists a collection $T \subset M$ of disjoint incompressible tori (the JSJ tori) such that each connected component of $M|T$ (called a JSJ piece) is either atoroidal or a Seifert manifold, and a minimal (with respect to inclusions of such collections) such collection T is unique up to isotopy.

By the Hyperbolization Theorem, every irreducible atoroidal 3-manifold M with empty or toroidal boundary and infinite fundamental group is hyperbolic of finite volume. Besides, by the Elliptization Theorem, every closed 3-manifold with finite fundamental group is spherical. Since every spherical 3-manifold is Seifert, we can re-formulate the JSJ decomposition theorem as:

Corollary 1.37. *(Geometrization Theorem, [AFW12, Theorem 1.14])*
For an irreducible 3-manifold M there exists a collection $T \subset M$ of disjoint incompressible tori such that each connected component of $M|T$ is either a Seifert manifold or a hyperbolic manifold of finite volume.

Remark 1.38. The JSJ tori must be incompressible, in particular no JSJ component can be a solid torus. It follows from Remark 1.13 that if $i\colon M_i \hookrightarrow M$ is a JSJ piece of M, then i is π_1-injective (since the JSJ tori are incompressible). This will be useful for simplifying JSJ formulas for L^2-Alexander torsions in Chapter 4.

Definition 1.39. Let M be a irreducible 3-manifold. The *JSJ-decomposition of M*, denoted $\mathrm{JSJ}(M)$, is the partition of M

$$\mathrm{JSJ}(M) := (T_1, \dots, T_a, S_1, \dots, S_b, H_1, \dots, H_c)$$

where $(T_1, \dots, T_a)$ is a minimal collection of disjoint incompressible sub-tori of M, the S_i are the Seifert manifolds and the H_j are the hyperbolic ones.

The *volume of M*, or *simplicial volume of M*, is defined as the sum of the hyperbolic volumes of the H_j and noted $\mathrm{vol}(M)$.

The manifold M is a *graph manifold* if its JSJ decomposition does not contain any hyperbolic pieces.

Recall that the simplicial volume is proportional to the Gromov norm (see for instance [BP92, Theorem C.4.2]).

JSJ-decomposition of link exteriors

We study the JSJ-decompositions of knot and link exteriors, following [Bud06].

Remark 1.40. Thanks to Thurston's work, we know that a knot K is either torus, satellite or hyperbolic. More precisely,

- K is a torus knot when there is an incompressible annulus in $S^3 \setminus K$, drawn with its core z in Figure A.4 in the Annex.
- K is a satellite knot when there exists an incompressible torus not isotopic to the boundary in $S^3 \setminus K$. This is the torus T_C in Section 1.1.7.
- when K is of neither of the two previous types, Thurston proved that its complement admits a complete hyperbolic structure of finite volume, i.e. K is hyperbolic.

Example 1.41. The $(k+1)$-component keychain link $L = KC_k$, drawn in Figure 1.7, has a Seifert-fibered exterior M_L. In this case, $\mathrm{JSJ}(M_L) = (M_L)$. We will list all the links that have a Seifert-fibered exterior in Section 4.2.1.

Proposition 1.42. *Let K be the connected sum of r non trivial knots $K_1, \dots, K_r$. We denote*

$$\mathrm{JSJ}(M_{K_i}) = (T_1^{(i)}, \dots, T_{a_i}^{(i)}, S_1^{(i)}, \dots, S_{b_i}^{(i)}, H_1^{(i)}, \dots, H_{c_i}^{(i)}).$$

There exist r irreductible tori $T_1, \dots, T_r$ that separate M_K into $r+1$ manifolds homeomorphic to $M_{KC_r}, M_{K_1}, \dots, M_{K_r}$ (with homeomorphisms onto M_K denoted respectively $f_0, f_1, \dots f_r$), and the JSJ-decomposition of M_K is:

$$\mathrm{JSJ}(M_K) = \begin{pmatrix} T_1, \dots, T_k, f_1(T_1^{(1)}), \dots, f_1(T_{a_1}^{(1)}), \dots, f_r(T_1^{(r)}), \dots, f_r(T_{a_r}^{(r)}), \\ f_0(KC_r), f_1(S_1^{(1)}), \dots, f_1(S_{b_1}^{(1)}), \dots, f_r(S_1^{(r)}), \dots, f_r(S_{b_r}^{(r)}), \\ f_1(H_1^{(1)}), \dots, f_1(H_{c_1}^{(1)}), \dots, f_r(H_1^{(r)}), \dots, f_r(H_{c_r}^{(r)}) \end{pmatrix}$$

In particular, $\mathrm{vol}(M_K) = \mathrm{vol}(M_{K_1}) + \ldots + \mathrm{vol}(M_{K_r})$.

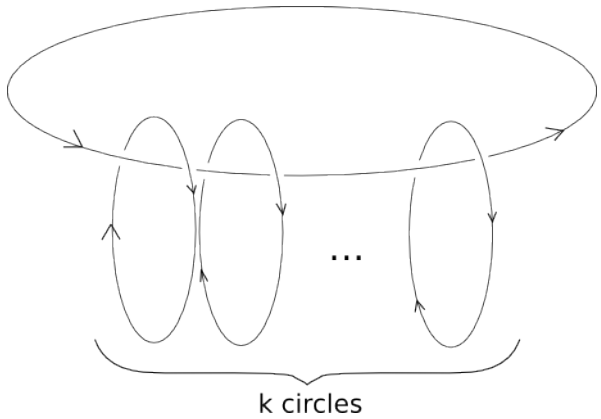


Figure 1.7 – The k-keychain link, a $(k+1)$-component link

Proposition 1.43. *Let $K = S_{C,P}$ be a satellite knot of non trivial companion C and pattern $P \subset T_P$, where T_P is a solid torus. Let L_P denote the intuitive 2-component link such that M_{L_P} is homeomorphic to $T_P \setminus V(P)$. We denote*

$$\mathrm{JSJ}(M_{L_P}) = (T'_1, \dots, T'_{a'}, S'_1, \dots, S'_{b'}, H'_1, \dots, H'_{c'}).$$

Let $h\colon M_{L_P} \to T_C \setminus V(K)$ denote the previous homeomorphism. We denote

$$\mathrm{JSJ}(M_C) = (T_1, \dots, T_a, S_1, \dots, S_b, H_1, \dots, H_c).$$

Then ∂T_C is a separating torus for M_K and

$$\mathrm{JSJ}(M_K) = \begin{pmatrix} \partial T_C, T_1, \dots, T_a, h(T'_1), \dots, h(T'_{a'}), \\ S_1, \dots, S_b, h(S'_1), \dots, h(S'_{b'}), \\ H_1, \dots, H_c, h(H'_1), \dots, h(H'_{c'}) \end{pmatrix}.$$

In particular, $\mathrm{vol}(S_{C,P}) = \mathrm{vol}(C) + \mathrm{vol}(L_P)$.

These results generalise to connected sums on links and satellite operations with links as companion or pattern. One only has to be careful of the torus component on which the operation is made. Particular examples will be studied in Chapter 4.

1.3 L^2-invariants

1.3.1 L^2-invariants

Let G be a countable discrete group (a knot group, for example). In the following, every algebra will be a $\mathbb{C}$-algebra, unless specified otherwise.

Consider the vector space $\mathbb{C}[G]= \bigoplus_{g\in G}\mathbb{C}g$ (which is also an algebra) and its scalar product:

$$\left\langle \sum_{g\in G}\lambda_g g, \sum_{g\in G}\mu_g g\right\rangle := \sum_{g\in G}\lambda_g\overline{\mu_g}.$$

The completion of $\mathbb{C}[G]$ is

$$\ell^2(G) := \left\{\sum_{g\in G}\lambda_g g \mid \lambda_g\in\mathbb{C}, \sum_{g\in G}|\lambda_g|^2<\infty\right\},$$

the Hilbert space of square-summable complex functions on the group G.

We denote $\mathcal{B}(\ell^2(G))$ the algebra of operators on $\ell^2(G)$ that are continuous (or equivalently, bounded) for the operator norm.

To any $h\in G$ we associate a *left-multiplication* $L_h\colon \ell^2(G)\to\ell^2(G)$ defined by

$$L_h\left(\sum_{g\in G}\lambda_g g\right) := \sum_{g\in G}\lambda_g(hg) = \sum_{g\in G}\lambda_{h^{-1}g}g$$

and a *right-multiplication* $R_h\colon \ell^2(G)\to\ell^2(G)$ defined by

$$R_h\left(\sum_{g\in G}\lambda_g g\right) := \sum_{g\in G}\lambda_g(gh) = \sum_{g\in G}\lambda_{gh^{-1}}g.$$

Both L_h and R_h are isometries, and therefore belong to $\mathcal{B}(\ell^2(G))$.

We will use the same notation for right-multiplications by elements of the complex group algebra $\mathbb{C}[G]$:

$$R_{\sum_{i=1}^k\lambda_i g_i} := \sum_{i=1}^k\lambda_i R_{g_i}\in\mathcal{B}(\ell^2(G)).$$

We will also use this notation to define a right-multiplication by a matrix A with coefficients in $\mathbb{C}[G]$, p rows and q columns, in the following way:

If $A=(a_{i,j})_{1\leqslant i\leqslant p,1\leqslant j\leqslant q}\in M_{p,q}(\mathbb{C}[G])$, then the operator R_A is defined as:

$$R_A := \left(R_{a_{i,j}}\right)_{1\leqslant i\leqslant p,1\leqslant j\leqslant q}\in\mathcal{B}(\ell^2(G)^q;\ell^2(G)^p).$$

Here $\ell^2(G)^p$ will denote the direct sum of p copies of $\ell^2(G)$, endowed with a natural Hilbert structure.

We denote $\mathcal{N}(G)$ the algebraic commutant of $\{L_g; g\in G\}$ in $\mathcal{B}(\ell^2(G))$. It will be called the *von Neumann algebra of the group* G.

Let us remark that $R_g\in\mathcal{N}(G)$ for all g in G.

The *trace* $tr_{\mathcal{N}(G)}(\phi)$ of an element ϕ of $\mathcal{N}(G)$ is defined as

$$tr_{\mathcal{N}(G)}(\phi) := \langle\phi(e), e\rangle$$

where e is the neutral element of G. This induces a trace on the $M_{n,n}(\mathcal{N}(G))$ for $n \geq 1$ by summing up the traces of the diagonal elements. We will denote this new trace $tr_{\mathcal{N}(G)}$ as well.

We will call a *finitely generated Hilbert $\mathcal{N}(G)$-module* any Hilbert space V on which there is a left G-action by isometries, and such that there exists a positive integer m and an embedding ϕ of V into $\bigoplus_{i=1}^m \ell^2(G)$ (an embedding meaning here a linear isometrical injective G-equivariant map, where the left G-action on $\bigoplus_{i=1}^m \ell^2(G)$ is by left-multiplication coordinate by coordinate).

The *von Neumann dimension* $\dim_{\mathcal{N}(G)}(V)$ of such a finitely generated Hilbert $\mathcal{N}(G)$-module V is defined as the trace of the projection:

$$\dim_{\mathcal{N}(G)}(V) := \mathrm{tr}_{\mathcal{N}(G)}(\mathrm{pr}_{\phi(V)}) \in \mathbb{R}_{\geqslant 0},$$

where

$$\mathrm{pr}_{\phi(V)} \colon \bigoplus_{i=1}^m \ell^2(G) \to \bigoplus_{i=1}^m \ell^2(G)$$

is the orthogonal projection onto $\phi(V)$. The von Neumann dimension does not depend on the embedding of V into the finite direct sum of copies of $\ell^2(G)$.

For U and V two finitely generated Hilbert $\mathcal{N}(G)$-modules, we will call $f \colon U \to V$ a *morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules* if f is a linear G-equivariant map, bounded for the respective scalar products of U and V.

Remark 1.44. If G is finite of order $|G|$, then $\ell^2(G) = \mathbb{C}[G] = \mathbb{C}^{|G|}$, $\mathcal{N}(G)$ is isomorphic to $\oplus_k M_{i_k}(\mathbb{C})$ with $\sum_k i_k = |G|$, L_g and R_g are permutation matrices for any $g \in G$ (they are the left and right regular representations of G) and $\mathrm{tr}_{\mathcal{N}(G)} = \dfrac{1}{|G|}\mathrm{tr}_{M_{|G|}(\mathbb{C})}$.

Let us now write a little about induction. Let $i \colon H \hookrightarrow G$ be an injective group homomorphism. To simplify notations, we will also call i the inducted algebra homomorphism on $\mathbb{C}[H]$ and matrices over $\mathbb{C}[H]$, and the isometric injection on $\ell^2(H)$. Let M be a finitely generated Hilbert $\mathcal{N}(H)$-module. Then, according to [Lüc02b, Section 1.1.5], we can construct an induction covariant functor i_* from the category (finitely generated Hilbert $\mathcal{N}(H)$-modules, morphisms of finitely generated Hilbert $\mathcal{N}(H)$-modules) to (finitely generated Hilbert $\mathcal{N}(G)$-modules, morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules), such that $i_*(\ell^2(H)) = \ell^2(G)$.

The following properties of this induction functor will be used in this manuscript:

Proposition 1.45. *(1) Let $w \in \mathbb{C}[H]$ and $R_w \colon \ell^2(H) \to \ell^2(H)$ be the corresponding right multiplication. Then $i_* R_w = R_{i(w)}$. A similar result stands for matrices over $\mathbb{C}[H]$.*

(2) Let $f \colon M \to N$ be a morphism of finitely generated Hilbert $\mathcal{N}(H)$-modules. If f is injective (resp. surjective), then $i_* f \colon i_* M \to i_* N$ is also injective (resp. surjective).

(3) If M is a finitely generated Hilbert $\mathcal{N}(H)$-module, then

$$dim_{\mathcal{N}(G)}(i_* M) = dim_{\mathcal{N}(H)}(M).$$

Remark 1.46. For any $\phi \in \mathcal{N}(H)$, $i_*\phi$ is in $\mathcal{N}(G)$, because commuting with the left multiplications is the same as being equivariant for the group action.

1.3.2 The Fuglede-Kadison determinant

Let G be a finitely generated group and U, V be two finitely generated Hilbert $\mathcal{N}(G)$-modules. Let $f\colon U \to V$ be a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. The *spectral density of* f is the map $\lambda \in \mathbb{R}_{\geqslant 0} \mapsto F(f)(\lambda)$ defined by:

$$F(f)(\lambda) := \sup\{\dim_{\mathcal{N}(G)}(L) | L \in \mathcal{L}(f, \lambda)\}$$

where $\mathcal{L}(f,\lambda)$ is the set of finitely generated Hilbert $\mathcal{N}(G)$-sub-modules of U on which the restriction of f has a norm smaller than or equal to λ.

Let us remark that $F(f)(\lambda)$ is monotonous and right-continuous, and thus defines a measure $dF(f)$ on the Borel set of $\mathbb{R}_{\geqslant 0}$ solely determined by the equation

$$dF(f)(]a, b]) = F(f)(b) - F(f)(a)$$

for all $a < b$.

Remark 1.47. Observe that $\mathcal{L}(f, 0)$ is the set of finitely generated Hilbert $\mathcal{N}(G)$-sub-modules of $\mathrm{Ker}(f)$, and $F(f)(0) = \dim_{\mathcal{N}(G)}(\mathrm{Ker}(f))$.

For any $\lambda \geqslant \|f\|$, $\mathcal{L}(f, \lambda)$ is the set of finitely generated Hilbert $\mathcal{N}(G)$-sub-modules of U, and $F(f)(\lambda) = \dim_{\mathcal{N}(G)}(U)$.

Remark 1.48. For all λ, $F(f)(\lambda) = F(f^*f)(\lambda^2) = F(|f|)(\lambda)$ where $f^*f\colon U \to U$ is a positive operator and $|f|$ is its square root.

We can thus think with positive operators and observe that $dF(f)$ measures the «density of eigenvalues». If λ is atomic then $dF(f)(\lambda)$ is the von Neumann dimension of the eigenspace associated to λ.

Definition 1.49. The *Fuglede-Kadison determinant of* f is defined by:

$$\det_{\mathcal{N}(G)}(f) := \exp\left(\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda)\right)$$

if $\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty$; if not, $\det_{\mathcal{N}(G)}(f) = 0$.

When $\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty$, we say that f *is of determinant class.*

Remark 1.50. If G is finite and f is positive invertible, then finitely generated Hilbert $\mathcal{N}(G)$-modules are finitely dimensional complex vector spaces and $\det_{\mathcal{N}(G)}(f) = |\det(f)|^{\frac{1}{|G|}}$.

Here are several properties of the determinant we will use in the rest of this paper (see [Lüc02b] for more details and proofs).

Proposition 1.51. *(1)* $\det_{\mathcal{N}(G)}(0\colon U \to V) = 1$.

(2) For every nonzero complex number λ, $\det_{\mathcal{N}(G)}(\lambda Id_U) = |\lambda|^{\dim_{\mathcal{N}(G)}(U)}$.

(3) For all f, g *morphisms of finitely generated Hilbert* $\mathcal{N}(G)$*-modules,*

$$\det{}_{\mathcal{N}(G)}\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\right) = \det{}_{\mathcal{N}(G)}(f) \cdot \det{}_{\mathcal{N}(G)}(g).$$

(4) Let $f\colon U \to V$ *and* $g\colon V \to W$ *be morphisms of finitely generated Hilbert* $\mathcal{N}(G)$*-modules. Assume that* f *has dense image and* g *is injective. Then*

$$\det{}_{\mathcal{N}(G)}(g \circ f) = \det{}_{\mathcal{N}(G)}(g) \cdot \det{}_{\mathcal{N}(G)}(f).$$

(5) Let $f_1\colon U_1 \to V_1$, $f_2\colon U_2 \to V_2$, $f_3\colon U_2 \to V_1$ be morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules. If f_1 has dense image and f_2 is injective, then

$$\det_{\mathcal{N}(G)}\left(\begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix}\right) = \det_{\mathcal{N}(G)}(f_1)\cdot\det_{\mathcal{N}(G)}(f_2).$$

(6) Let $i\colon H \hookrightarrow G$ be an injective group homomorphism. Let M and N be two finitely generated Hilbert $\mathcal{N}(H)$-modules and $f\colon M \to N$ be a map of finitely generated Hilbert $\mathcal{N}(H)$-modules. Then

$$\det_{\mathcal{N}(G)}(i_*(f)) = \det_{\mathcal{N}(H)}(f).$$

(7) Let f be a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. Then

$$\det_{\mathcal{N}(G)}(f) = \det_{\mathcal{N}(G)}(f^*) = \sqrt{\det_{\mathcal{N}(G)}(f^*f)} = \sqrt{\det_{\mathcal{N}(G)}(ff^*)}.$$

Remark 1.52. If $f\colon U \to V$ is a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules that have the same von Neumann dimension, then f is injective if and only if f has dense image (see [Lüc02b, Lemma 1.13]).

Therefore, when dealing with «square» operators, the property «has dense image» can be replaced by «is injective» in the assumptions of Proposition 1.51 (4) and (5).

Proposition 1.53. *Let $g \in G$ be of infinite order, let $t \in \mathbb{C}$, then $Id - tR_g$ is injective and*

$$\det_{\mathcal{N}(G)}(Id - tR_g) = \max(1, |t|).$$

The proof of this proposition can be found in [LZ06, Proposition 3.2, Remark 3.3]. The value of the determinant can also be computed as a direct consequence of [Lüc02b, Example 3.22].

Remark 1.54. Note that for $t > 0$ and any integer k, $\max(1, t^k) = t^{\frac{k-|k|}{2}}\max(1,t)^{|k|}$. This shall be used often and implicitly in the following proofs.

1.3.3 Combinatorial computations

In this section we follow [Lüc02b, Section 3.7]. We want to give a more combinatorial approach to von Neumann dimensions and Fuglede-Kadison determinants. In general it is very hard to compute the spectral density function of some morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules. However in the geometric situation these morphisms are induced by matrices over the group ring $\mathbb{R}[G]$, for which we can compute algorithms to approximate the L^2-invariants.

Let $A \in M(n, m, \mathbb{C}[G])$ and $R_A\colon \ell^2(G)^m \to \ell^2(G)^n$ the induced G-equivariant operator. Let $C > 0$ such that $C \geqslant \|R_A\|_\infty$, i.e. C is greater than the operator norm of R_A.

The *characteristic sequence of the matrix A and the positive real number C satisfying* $C \geqslant \|R_A\|_\infty$ is the sequence $(c(A,C)_p)_{p\in\mathbb{N}}$ where

$$c(A,C)_p := \mathrm{tr}_{\mathcal{N}(G)}\left(\left(Id_{\ell^2(G)^m} - \frac{1}{C^2}R_A^*R_A\right)^p\right) \in \mathbb{R}.$$

Remark that $\mathrm{tr}_{\mathcal{N}(G)}(\sum_i \lambda_i R_{g_i}) = \sum_{g_i=1}\lambda_i$.

Proposition 1.55. *(Combinatorial computations, [Lüc02b, Theorem 3.172])*
With A, C as above,

(1) $(c(A,C)_p)_{p\in\mathbb{N}}$ *is a monotone decreasing sequence of non-negative real numbers;*

(2) We have

$$\dim_{\mathcal{N}(G)}(Ker(R_A)) = \lim_{p\to\infty} c(A,C)_p;$$

In particular, R_A *is injective if and only if* $c(A,C)_p \underset{p\to\infty}{\longrightarrow} 0$.

(3) The operator R_A *is of determinant class if and only if the sum of non-negative real numbers*

$$\sum_{p=1}^{\infty} \frac{1}{p}\Big(c(A,C)_p - \dim_{\mathcal{N}(G)}(Ker(R_A))\Big)$$

converges, and in this case

$$\det{}_{\mathcal{N}(G)}(R_A) = C^{n-\dim_{\mathcal{N}(G)}(Ker(R_A))} \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\Big(c(A,C)_p - \dim_{\mathcal{N}(G)}(Ker(R_A))\Big)\right).$$

Each term of the characteristic sequence $c(A,C)_p$ can be computed by an algorithm as long as the word problem for G has a solution. This provides a way of approximating $\dim_{\mathcal{N}(G)}(\mathrm{Ker}(R_A))$ and $\det_{\mathcal{N}(G)}(R_A)$ numerically.

1.3.4 Elementary operations on L^2-matrices

Similarly as the transvection, dilatation and permutation operators in classical finite-dimensional linear algebra, $B(\ell^2(G)^p)$ contains interesting particular classes of operators.

Dilatations

Dilatations are diagonal operators, of the form $\begin{pmatrix} T_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & T_p \end{pmatrix}$ and they act on the rows by left multiplication and on the columns by right multiplication.

The typical example to remember is:

$$\begin{pmatrix} T & 0 \\ 0 & Id \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} TA & TB \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} T & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} AT & B \\ CT & D \end{pmatrix}$$

(Here, A, B, C, D, T, T_i all lie in $B(\ell^2(G))$).

The operator $\begin{pmatrix} T_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & T_p \end{pmatrix}$ is injective (resp. invertible) if and only if each T_i is injective (resp. invertible), and by Proposition 1.51 (3), its Fuglede-Kadison determinant is $\prod_{i=1}^{p} \det_{\mathcal{N}(G)}(T_i)$.

Permutations

A permutation operator in $B(\ell^2(G)^p)$ is simply R_{M_σ} where $\sigma \in \mathcal{S}_p$ is a permutation on p elements and M_σ is the associated matrix in $M_p(\mathbb{C})$ that permutes the coordinates of $\mathbb{C}^p$ by σ. A permutation operator contains thus exactly one coefficient $Id_{\ell^2(G)}$ by row and by column, and zeroes everywhere else.

Since $M_\sigma \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = \begin{pmatrix} a_{\sigma(1)} \\ \vdots \\ a_{\sigma(p)} \end{pmatrix}$ and $\begin{pmatrix} a_1 & \dots & a_p \end{pmatrix} \cdot M_\sigma = \begin{pmatrix} a_{\sigma^{-1}(1)} & \dots & a_{\sigma^{-1}(p)} \end{pmatrix}$, then for $T \in B(\ell^2(G)^p)$, $R_{M_\sigma} \circ T$ is T with its rows permuted by σ and $T \circ R_{M_\sigma}$ is T with its columns permuted by σ^{-1}.

The operator R_{M_σ} is always unitary, since $R^*_{M_\sigma} = R_{M_{\sigma^{-1}}} = R^{-1}_{M_\sigma}$, therefore R_{M_σ} is always invertible and $\det_{\mathcal{N}(G)}(R_{M_\sigma}) = 1$ by Proposition 1.51 (7).

Transvections

A transvection operator $T \in B(\ell^2(G)^p)$, as a $p-p$-matrix over $B(\ell^2(G))$, has diagonal coefficients equal to Id and zeroes everywhere else except maybe at a single coefficient $(i,j), i \neq j$, where the coefficient in question is an operator $S \in B(\ell^2(G))$.

Every transvection operator is invertible, the inverse operator simply swaps S for $-S$, and the Fuglede-Kadison determinant is always 1 by Proposition 1.51 (5).

The typical example to remember is:

$$\begin{pmatrix} Id & S \\ 0 & Id \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + SC & B + SD \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} Id & S \\ 0 & Id \end{pmatrix} = \begin{pmatrix} A & B + AS \\ C & D + CS \end{pmatrix}.$$

1.3.5 L^2-torsion

We follow the notations of [Lüc02b] and [DFL14].

A *finite Hilbert $\mathcal{N}(G)$-chain complex* C_* is a sequence of morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$C_* = 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

such that $\partial_p \circ \partial_{p+1} = 0$ for all p.

The *p-th L^2-homology of C_**, denoted $H^{(2)}_p(C_*)$, is defined as:

$$H^{(2)}_p(C_*) := \mathrm{Ker}(\partial_p)/\overline{\mathrm{Im}(\partial_{p+1})}$$

is a finitely generated Hilbert $\mathcal{N}(G)$-module. Its von Neumann dimension

$$b^{(2)}_p(C_*) := \dim_{\mathcal{N}(G)}(H^{(2)}_p(C_*))$$

is called the *p-th Betti number of C_** and is denoted $b^{(2)}_p(C_*)$.

We say that C_* is *weakly acyclic* if its L^2-homology is trivial, i.e. if all its L^2-Betti numbers vanish. We say that C_* is of *determinant class* if all the operators ∂_p are of determinant class.

Proposition 1.56. *Let $i\colon H \hookrightarrow G$ be an injective group homomorphism and C_* a finite Hilbert $\mathcal{N}(H)$-chain complex. Then $i_*(C_*)$ is a finite Hilbert $\mathcal{N}(G)$-chain complex and for all $p \in \mathbb{Z}$*

$$H_p^{(2)}(i_*(C_*)) = i_*(H_p^{(2)}(C_*))$$

$$b_p^{(2)}(i_*(C_*)) = b_p^{(2)}(C_*).$$

Definition 1.57. Let C_* be a finite Hilbert $\mathcal{N}(G)$-chain complex as above. If C_* is weakly acyclic and of determinant class, define its *L^2-torsion* by

$$T^{(2)}(C_*) := \prod_{i=1}^{n} \det{}_{\mathcal{N}(G)}(\partial_i)^{(-1)^i} \in \mathbb{R}_{>0}.$$

Note that in [Lüc02b, Definition 3.29] weak acyclicity is not assumed, it is in fact not necessary to define $T^{(2)}(C_*)$. However we will often require the weak acyclicity assumption in the various formulas of this manuscript, that is why we chose to assume it in the definition.

Besides, when C_* is simply 2-dimensional, $T^{(2)}(C_*) = \dfrac{\det_{\mathcal{N}(G)}(\partial_2)}{\det_{\mathcal{N}(G)}(\partial_1)}$ can be naturally defined to be zero if ∂_1 is of determinant class but ∂_2 is not. This is why the L^2-Alexander invariant of knots defined in Section 2.2.1 can theoretically be zero for some knots K and some values t, since we do not need to assume that the operators are of determinant class.

Finally let us point out that we could define $T^{(2)}(C_*) = 0$ when C_* is either not weakly acyclic or not of determinant class, as done in [DFL14]. We will not use this convention in this manuscript, because even if it allows us to state the results more simply, we prefer to keep track of the precise cases where weak acyclicity or determinant class are important properties.

The following proposition will be useful for computations of L^2-torsions. Compare with [DFL14, Lemma 3.1].

Proposition 1.58. *Let*

$$C_* = 0 \to \ell^2(G)^k \xrightarrow{\partial_2} \ell^2(G)^{k+l} \xrightarrow{\partial_1} \ell^2(G)^l \to 0$$

be a 2-dimensional finite Hilbert $\mathcal{N}(G)$-chain complex and let $J \subset \{1, \dots, k+l\}$ be a subset of $\{1, \dots, k+l\}$ of size l.

For $i = 1, 2$, ∂_i is naturally written as a matrix with coefficients operators in $B(\ell^2(G))$. We denote $\partial_1(J)\colon \ell^2(G)^l \to \ell^2(G)^l$ the operator composed of the columns of ∂_1 indexed by J, and $\partial_2(J)\colon \ell^2(G)^k \to \ell^2(G)^k$ the operator obtained from ∂_2 by deleting the rows indexed by J.

If $\partial_2(J)$ and $\partial_1(J)$ are injective and of determinant class, then C_* is weakly acyclic and of determinant class, and

$$T^{(2)}(C_*) = \frac{\det_{\mathcal{N}(G)}(\partial_2)}{\det_{\mathcal{N}(G)}(\partial_1)} = \frac{\det_{\mathcal{N}(G)}(\partial_2(J))}{\det_{\mathcal{N}(G)}(\partial_1(J))}.$$

Proof. Let us first assume that $J = \{1, \dots, l\}$. Then $\partial_2 = \begin{pmatrix} B' \\ B \end{pmatrix}$, $\partial_2(J) = B$, $\partial_1 = \begin{pmatrix} A \\ A' \end{pmatrix}$, $\partial_2 = A$, where A and B are square matrices.

Let

$$A_* = 0 \to \ell^2(G)^l \xrightarrow{A} \ell^2(G)^l$$

and

$$B_* = \ell^2(G)^k \xrightarrow{B} \ell^2(G)^k \to 0$$

be two 2-dimensional finite Hilbert $\mathcal{N}(G)$-chain complexes.

Then there exists an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to A_* \xrightarrow{\iota_*} C_* \xrightarrow{q_*} B_* \to 0$$

(meaning that $0 \to A_p \xrightarrow{\iota_p} C_p \xrightarrow{q_p} B_p \to 0$ is exact at each p and that ι_* and q_* commute with the boundary operators) detailed below:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \ell^2(G)^k \arrow[r, "Id"'] \arrow[d, "\begin{pmatrix} B' \\ B \end{pmatrix}"] & \ell^2(G)^k \arrow[r] \arrow[d, "B"] & 0 \\
0 \arrow[r] & \ell^2(G)^l \arrow[r, "\begin{pmatrix} Id \\ 0 \end{pmatrix}"'] \arrow[d, "A"'] & \ell^2(G)^{l+k} \arrow[r, "\begin{pmatrix} 0 & Id \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} A & A' \end{pmatrix}"] & \ell^2(G)^k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ell^2(G)^l \arrow[r, "Id"'] \arrow[d] & \ell^2(G)^l \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We assume A and B are injective. The square operator A is injective, thus A has dense image and $\begin{pmatrix} A & A' \end{pmatrix}$ has dense image too. Since B is injective, $\begin{pmatrix} B' \\ B \end{pmatrix}$ is injective too.

This implies that the long exact homology sequence

$$LHS_* = LHS_*(A_*, C_*, B_*)$$

($LHS_*(A_*,C_*,B_*)$ is defined in [Lüc02b, Theorem 1.21]) is a sequence where all the finitely generated Hilbert $\mathcal{N}(G)$-modules are equal to zero, except maybe $H_1^{(2)}(C_*)$; however, since

the sequence is exact, $H_1^{(2)}(C_*)$ is zero as well and C_* is weakly acyclic. Hence LHS_* is trivial, of determinant class and of L^2-torsion equal to 1.

We assume that A and B are of determinant class, thus A_*, B_* and LHS_* are of determinant class. Therefore, by the multiplicativity of the L^2-torsion (see [Lüc02b, Theorem 3.35 (1)]), C_* is of determinant class and

$$T^{(2)}(C_*) \cdot T^{(2)}(A_*, C_*, B_*) = T^{(2)}(A_*) \cdot T^{(2)}(B_*)$$

where

$$T^{(2)}(A_*, C_*, B_*) = \prod_{p=0}^{2} \left(\frac{\det_{\mathcal{N}(G)}(\iota_p)}{\det_{\mathcal{N}(G)}(q_p)} \right)^{(-1)^p}$$

is the L^2-torsion of the short exact sequence ($T^{(2)}(A_*,C_*,B_*)$ is defined in [Lüc02b, (3.34)]), well defined and equal to 1 since all the ι_p and q_p are of determinant 1.

Hence $T^{(2)}(C_*) = \dfrac{\det_{\mathcal{N}(G)}(B)}{\det_{\mathcal{N}(G)}(A)}$.

If J is a more general subset of $\{1, \ldots, k+l\}$, the situation is almost identical, the operators are simply multiplied by appropriate permutation matrices, which does not change their injectivity, the fact that they are of determinant class and the values of their Fuglede-Kadison determinants. Hence the formula is proven.

□

Chapter 2

The L^2-Alexander torsion detects the trivial knot

In this chapter we define the central object of this manuscript, the L^2-Alexander torsion $T^{(2)}(M, \phi, \gamma)$ associated to a 3-manifold M with boundary and two group homomorphisms ϕ, γ. One can see the L^2-Alexander torsion as a infinite-dimensional version of the twisted Alexander invariants. This is the object of the first section.

In the second section we give two equivalent definitions of the L^2-Alexander invariant for knots of [LZ06]. The first definition comes from Fox's free differential calculus on the knot group, and the second is built from the L^2-Alexander torsion of the knot exterior associated to the abelianization. One can see the L^2-Alexander invariant $\Delta_K^{(2)}$ as a convenient way of computing the L^2-Alexander torsion $T^{(2)}(M_K, \alpha_K, id)$.

In the third section we state several properties of the L^2-Alexander invariant that help us eventually prove the main result of this chapter in the fourth section, i.e. the fact that the L^2-Alexander invariant detects the trivial knot. This result was first announced in [BA13]. In the fifth section, we use similar tools to prove that the L^2-Alexander invariant detects the trefoil knot.

In the last section we extend the notations and some useful properties from knots to links .

2.1 The L^2-Alexander torsion

2.1.1 The L^2-Alexander torsion of a CW-complex

We follow the definitions and notations of [DFL14].

Let π be a group, $\phi\colon \pi \to \mathbb{Z}$ a homomorphism, and $\gamma\colon \pi \to G$ an homomorphism. We say that (π,ϕ,γ) *forms an admissible triple* if $\phi\colon \pi \to \mathbb{Z}$ factors through γ (i.e. there is a group homomorphism $\psi\colon G \to \mathbb{Z}$ such that $\phi = \psi \circ \gamma$).

$$\begin{tikzcd} \pi \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\ & \mathbb{Z} \end{tikzcd}$$

Let X be a CW-complex, then we say that $(X, \phi\colon \pi_1(X) \to \mathbb{Z}, \gamma\colon \pi_1(X) \to G)$ *forms an admissible triple* if $(\pi_1(X), \phi, \gamma)$ forms one.

Let (X,ϕ,γ) be an admissible triple, $\pi = \pi_1(X)$ and $t > 0$. We define a ring homomorphism

$$\kappa(\pi,\phi,\gamma,t)\colon \begin{pmatrix} \mathbb{Z}[\pi] & \longrightarrow & \mathbb{R}[G] \\ \sum_{j=1}^r m_j g_j & \longmapsto & \sum_{j=1}^r m_j t^{\phi(g_j)}\gamma(g_j) \end{pmatrix}$$

and we also denote $\kappa(\pi,\phi,\gamma,t)$ its induction over the $M_{p,q}(\mathbb{Z}[\pi])$.

Assume X is compact. The cellular chain complex of $\widetilde{X}$

$$C_*(\widetilde{X},\mathbb{Z}) = \left(\ldots \to \bigoplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k \to \ldots\right)$$

is a chain complex of (left) $\mathbb{Z}[\pi]$-modules and contains all the topological information on how the cells are glued with one another. Here the $\widetilde{e}_i^k$ are lifts of the cells e_i^k of X. The group π acts on the right on $\ell^2(G)$ by $g \mapsto R_{\kappa(\pi,\phi,\gamma,t)(g)}$, an action which induces a structure of right $\mathbb{Z}[\pi]$-module on $\ell^2(G)$.

Let

$$\begin{aligned} C_*^{(2)}(X,\phi,\gamma,t) &= \ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} C_*(\widetilde{X},\mathbb{Z}) \\ &= \left(\ldots \to \bigoplus_i \left(\ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} \mathbb{Z}[\pi]\widetilde{e}_i^k\right) \to \ldots\right) \end{aligned}$$

denote the finite Hilbert $\mathcal{N}(G)$-chain complex obtained by tensor product; we will call $C_*^{(2)}(X,\phi,\gamma,t)$ *a $\mathcal{N}(G)$-cellular chain complex of X*.

We will denote the modules $\ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} \mathbb{Z}[\pi]\widetilde{e}$ as $\ell^2(G)\widetilde{e}$ to simplify notations, the implicit isometric isomorphism of finitely generated $\mathcal{N}(G)$-Hilbert modules being

$$\begin{aligned} \ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} \mathbb{Z}[\pi]\widetilde{e} &\longleftrightarrow \ell^2(G) \\ \sum_{i=1}^s a_i \otimes (w_i\widetilde{e}) = \left(\sum_{i=1}^s R_{\kappa(\pi,\phi,\gamma,t)(w_i)}(a_i)\right) \otimes \widetilde{e} &\longleftrightarrow \sum_{i=1}^s R_{\kappa(\pi,\phi,\gamma,t)(w_i)}(a_i) \end{aligned}$$

(where $w_i \in \mathbb{Z}[\pi]$, $a_i \in \ell^2(G)$).

The boundary operators $\partial_k^{(2)}$ are as follows: if $\widetilde{e}$ is a k-cell of $\widetilde{X}$ and if

$$\partial_{\widetilde{X}}(\widetilde{e}) = \sum_{j=1}^r w_j \cdot \widetilde{e}_j^{k-1}$$

(where $w_j \in \mathbb{Z}[\pi]$ is a word that can be trivial), then for $a \in \ell^2(G)$,

$$\begin{aligned} \partial_k^{(2)}(a \otimes \widetilde{e}) &= a \otimes \left(\partial_{\widetilde{X}}(\widetilde{e})\right) \\ &= a \otimes \left(\sum_{j=1}^r w_j \cdot \widetilde{e}_j^{k-1}\right) \\ &= \sum_{j=1}^r R_{\kappa(\pi,\phi,\gamma,t)(w_j)}(a) \otimes \widetilde{e}_j^{k-1} \end{aligned}$$

This is why we can naturally write $\partial_k^{(2)}|_{\ell^2(G)\widetilde{e}}\colon \ell^2(G)\widetilde{e} \to \oplus_{j=1}^r \ell^2(G)\widetilde{e}_j^{k-1}$ as $\begin{pmatrix} R_{\kappa(\pi,\phi,\gamma,t)(w_1)} \\ \vdots \\ R_{\kappa(\pi,\phi,\gamma,t)(w_r)} \end{pmatrix}$ and the whole $\partial_k^{(2)}$ as the concatenation of those columns.

We denote the $\mathcal{N}(G)$-cellular chain complex of X associated to the admissible triple (π, ϕ, γ) and the parameter $t > 0$ in the following way:

$$\begin{aligned} C_*^{(2)}(X, \phi, \gamma, t) &= \ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} C_*(\widetilde{X}, \mathbb{Z}) \\ &= \left(\dots \xrightarrow{\partial_{k+1}^{(2)}} \bigoplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{\partial_k^{(2)}} \dots \right). \end{aligned}$$

Definition 2.1. If $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class, then we call

$$T^{(2)}(X, \phi, \gamma)(t) = T^{(2)}\left(C_*^{(2)}(X, \phi, \gamma, t)\right)$$

the *L^2-Alexander torsion of* (X, ϕ, γ) *at* $t > 0$.

As for the classical twisted Alexander invariants, we try to numerically extract some of the topological information of X contained in $\pi = \pi_1(X)$ and in $C_*(\widetilde{X}, \mathbb{Z})$, by twisting $C_*(\widetilde{X}, \mathbb{Z})$ by an infinite-dimensional representation of π on $\ell^2(G)$.

We want to study the map $t \mapsto T_*^{(2)}(X,\phi,\gamma)(t)$, defined on a subset $\mathcal{D}_X$ of $\mathbb{R}_{>0}$, the set of the t such that $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class.

We have to check that this map is well defined. Actually, it depends on the CW-structure of X in the following way: we implicitly chose an ordering and an orientation of the cells of X, and then a particular lift in $\widetilde{X}$ for each cell; this endowed $\bigoplus_i \ell^2(G)\widetilde{e}_i^k$ with a basis as a free left $\mathcal{N}(G)$-module and a natural isometric isomorphism with a power of $\ell^2(G)$ (defined above). Let us see how other choices would impact the value of the L^2-Alexander torsion.

- Reversing the orientation of a cell $\widetilde{e}_i^k$ would multiply the i-th row of $\partial_{k+1}^{(2)}$ and the i-th column of $\partial_k^{(2)}$ by -1, which is a multiplication with a dilatation operator, invertible of determinant 1, which does not change $T^{(2)}(X, \phi, \gamma, t)$.
- Changing the ordering of the cells $\widetilde{e}_i^k$ would similarly multiply the operators $\partial_{k+1}^{(2)}$ and $\partial_k^{(2)}$ by permutation operators, which does not change $T^{(2)}(X, \phi, \gamma, t)$.
- However, changing a lift $\widetilde{e}_i^k$ by $g \cdot \widetilde{e}_i^k$, $g \in \pi$, would multiply the operators $\partial_{k+1}^{(2)}$ and $\partial_k^{(2)}$ by dilatation operators, the i-th diagonal term being $t^{\phi(g)} R_{\gamma(g)}$, which multiplies $T^{(2)}(X, \phi, \gamma, t)$ by a term t^m, $m \in \mathbb{Z}$.

Hence the equivalence class of $(t \mapsto T^{(2)}(X, \phi, \gamma, t))$ up to multiplication by the $(t \mapsto t^m), m \in \mathbb{Z}$, is a well-defined invariant of (X, ϕ, γ). For two maps $f, g \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, we write

$$f \doteq g \iff \exists m \in \mathbb{Z}, \forall t > 0, f(t) = t^m g(t)$$

and extend immediately this notation $\doteq$ to maps defined only on a subset $\mathcal{D}$ of $\mathbb{R}_{>0}$.

For X a CW-complex, its *L^2-torsion* $T^{(2)}(X)$ is defined as

$$T^{(2)}(X) := T^{(2)}(X, 0, id)(1)$$

when $C_*^{(2)}(X, 0, id, 1)$ is weakly acyclic and of determinant class. We refer to [Lüc02b, Section 3.4] for details.

We call

$$b_p^{(2)}(X) := b_p^{(2)}(C_*^{(2)}(X, 0, id, 1))$$

the *p-th L^2-Betti number of X* (see [Lüc02b, Definition 1.30]).

The L^2-Betti numbers $b_p^{(2)}(X)$ of a CW-complex with infinite fundamental group have very little relation with the classical Betti numbers, except for the following formula:

Proposition 2.2. *[Lüc02b, Theorem 1.35 (2)] Let X be a finite CW-complex. We have*

$$\chi(X) = \sum_p (-1)^p \, b_p^{(2)}(X).$$

Remark 2.3. Since we need $C_*^{(2)}(X, \phi, \gamma, t)$ to be weakly acyclic to compute the L^2-Alexander torsion of (X, ϕ, γ), it is necessary that $\chi(X) = 0$. Indeed, if $\chi(X) \neq 0$, then the von Neumann dimensions of the modules $C_p^{(2)}(X, \phi, \gamma, t)$, i.e. the number of cells of X of each dimension, do not give a zero alternating sum (by Proposition 2.2), thus $C_*^{(2)}(X, \phi, \gamma, t)$ cannot possibly be weakly acyclic. Like Reidemeister torsion, the L^2-Alexander torsion is an invariant we turn to only when the Euler characteristic vanishes, and more precisely when all L^2-Betti numbers (of $C_*^{(2)}(X, \phi, \gamma, t)$) vanish.

Compact connected orientable closed 3-manifolds have zero Euler characteristic by Poincaré duality. If there is boundary, it needs to have zero total Euler characteristic, like a finite union of tori. This is the case we consider in this manuscript: we study L^2-Alexander torsions for 3-manifolds M only when M is closed or compact with toroidal boundary.

The following astonishing theorem of W. Lück and T. Schick (see [LS99]) states that the L^2-torsion of a 3-manifold gives precisely the simplicial volume of this manifold.

Theorem 2.4 ([Lüc02b], Theorem 4.3)**.** *Let M be a compact connected orientable prime 3-manifold with infinite fundamental group and empty or incompressible toroidal boundary.*

According to the JSJ-decomposition, M splits along disjoint incompressible tori into pieces that are Seifert manifolds or hyperbolic manifolds. The hyperbolic pieces $M_1, \dots, M_h$ have all finite hyperbolic volume.

Then $C_*^{(2)}(M, 0, id, 1)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(M) = \exp\left(\frac{vol(M)}{6\pi}\right)$$

where vol is the simplicial volume.

2.1.2 Basic properties of the L^2-Alexander torsion

We review several basic properties of the L^2-Alexander torsion. The first one is an application of induction formulas of Proposition 1.45.

Proposition 2.5. *[DFL14, Lemma 5.1] Let X be a compact CW-complex, $\pi = \pi_1(X)$, $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ group homomorphisms and $\iota\colon G \hookrightarrow H$ an injective group homomorphism such that $(X, \phi, \iota \circ \gamma)$ is an admissible triple. Let $t > 0$.*

Then $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(X, \phi, \iota \circ \gamma, t)$ is weakly acyclic and of determinant class, and in this case

$$T^{(2)}(X, \phi, \iota \circ \gamma)(t) \doteq T^{(2)}(X, \phi, \gamma)(t).$$

Proof. Assume that $(X, \phi, \iota \circ \gamma)$ is an admissible triple, and let $t > 0$.

By the definitions of the previous section, let

$$C_*^{(2)}(\widetilde{X}, \mathbb{Z}) = \ldots \to \oplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k \xrightarrow{\partial_{\widetilde{X},k}} \oplus_j \mathbb{Z}[\pi]\widetilde{e}_j^{k-1} \to \ldots$$

denote the cellular chain complex of $\widetilde{X}$. Consider

$$C_* = C_*^{(2)}(X, \phi, \gamma, t) = \ldots \to \oplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{S_k} \oplus_j \ell^2(G)\widetilde{e}_j^{k-1} \to \ldots$$

and

$$D_* = C_*^{(2)}(X, \phi, \iota \circ \gamma, t) = \ldots \to \oplus_i \ell^2(H)\widetilde{e}_i^k \xrightarrow{T_k} \oplus_j \ell^2(H)\widetilde{e}_j^{k-1} \to \ldots$$

where $S_k = R_{\kappa(\pi,\phi,\gamma,t)(\partial_{\widetilde{X},k})}$ and $T_k = R_{\kappa(\pi,\phi,\iota\circ\gamma,t)(\partial_{\widetilde{X},k})}$.

Let $\iota\colon \mathbb{C}[G] \hookrightarrow \mathbb{C}[H]$ also denote the induction of ι to group algebras. Then

$$\kappa(\pi, \phi, \iota \circ \gamma, t) = \iota \circ \kappa(\pi, \phi, \gamma, t)$$

and thus $T_k = \iota_*(S_k)$ for all k, by Proposition 1.45. Hence $D_* = \iota_*(C_*)$, and by Proposition 1.56, C_* and D_* have the same Betti numbers. Therefore C_* is weakly acyclic if and only if D_* is weakly acyclic.

Finally, by Proposition 1.51(6), C_* is of determinant class if and only if D_* is of determinant class, and in this case they have same L^2-torsion. The proposition follows. □

Remark 2.6. By Proposition 2.5, we can always assume that γ is surjective, by considering $\iota\colon \gamma(\pi) \hookrightarrow G$.

The admissible homomorphisms γ will be exactly the quotients of π between π itself and the one induced by ϕ, the maximal one.

The following result follows immediately from the definitions.

Proposition 2.7. *[DFL14, Lemma 5.2] Let X be a compact CW-complex, $\pi = \pi_1(X)$, $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ group homomorphisms such that (X, ϕ, γ) is an admissible triple. Let $t > 0$ and $r \in \mathbb{Z}$. Then $C_*^{(2)}(X, r\phi, \gamma, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and in this case*

$$T^{(2)}(X, r\phi, \gamma)(t) \doteq T^{(2)}(X, \phi, \gamma)(t^r).$$

The following proposition states that the L^2-Alexander torsion is symmetric as a map on $\mathbb{R}_{>0}$.

Proposition 2.8. *[DFL14, Theorem 5.4] Let (N, ϕ, γ) be an admissible triple and let τ be a representative of $(t \mapsto T^{(2)}(N, \phi, \gamma)(t))$. Then there exists an integer n such that $n = x_N(\phi)$ mod 2 and such that for all $t > 0$,*

$$\tau(t^{-1}) = t^n.\tau(t).$$

Finally we explain why taking $t \in \mathbb{C}^*$ offers exactly the same information as taking $t > 0$:

Proposition 2.9. *Let $(\pi, \phi, \gamma\colon \pi \to G)$ be an admissible triple, $A \in M(m, n, \mathbb{R}[\pi])$ a matrix, $t > 0$ and $\theta \in \mathbb{R}$. Let R_θ denote the G-equivariant operator*

$$R_\theta = R_{\kappa(\pi,\phi,\gamma,te^{i\theta})(A)}\colon \ell^2(G)^m \to \ell^2(G)^n$$

and let $C \geqslant \|R_\theta\|_\infty$. Then

- *the characteristic sequence* $(c(R_\theta, C)_p)_{p\in\mathbb{N}}$,
- *the dimension of the kernel* $\dim_{\mathcal{N}(G)}(Ker(R_\theta))$,
- *the dimension of the closure of the image* $\dim_{\mathcal{N}(G)}(\overline{Im}(R_\theta))$,
- *the fact that* R_θ *is of determinant class or not,*
- *the determinant* $\det_{\mathcal{N}(G)}(R_\theta)$ *if* R_θ *is of determinant class,*

do not depend on θ.

Proof. Let us first prove that $(c(R_\theta, C)_p)_{p\in\mathbb{N}}$ does not depend on θ. Let $p \in \mathbb{N}$. Let $\mathcal{A}_\theta \subset \mathcal{N}(G)$ denote the real vector space generated by the family of operators $e^{i\theta\phi(g)}R_{\gamma(g)}, g \in \pi$.

By definition of $\kappa(\pi, \phi, \gamma, te^{i\theta})$, all nm coefficients of R_θ are in $\mathcal{A}_\theta$. Since

$$\left(e^{i\theta\phi(g)}R_{\gamma(g)}\right)^* = \overline{e^{i\theta\phi(g)}}R^*_{\gamma(g)} = e^{-i\theta\phi(g)}R_{\gamma(g^{-1})} = e^{i\theta\phi(g^{-1})}R_{\gamma(g^{-1})},$$

$\mathcal{A}_\theta$ is stable by $*$. Moreover $\mathcal{A}_\theta$ is stable by composition, since ϕ is a group homomorphism.

Therefore the operator

$$S = \left(Id_{\ell^2(G)^m} - \frac{1}{C^2}R^*_\theta R_\theta\right)^p$$

is in A_θ, thus S is of the form

$$S = \sum_j \lambda_j e^{i\theta\phi(g_j)}R_{\gamma(g_j)},$$

where $\lambda_j \in \mathbb{R}$; hence

$$c(R_\theta, C)_p = \mathrm{tr}_{\mathcal{N}(G)}(S) = \sum_{\gamma(g_j)=1} \lambda_j e^{i\theta\phi(g_j)} = \sum_{\gamma(g_j)=1} \lambda_j,$$

since $\gamma(g_j) = 1$ implies $\phi(g_j) = 0$. Consequently $c(R_\theta, C)_p$ does not depend on θ.

The characteristic sequence $(c(R_\theta, C)_p)_{p\in\mathbb{N}}$ converges to the dimension of the kernel $\dim_{\mathcal{N}(G)}(\mathrm{Ker}(R_\theta))$ by Proposition 1.55 (2), thus this dimension does not depend on θ.

Since $\dim_{\mathcal{N}(G)}(\overline{\mathrm{Im}}(R_\theta)) = m - \dim_{\mathcal{N}(G)}(\mathrm{Ker}(R_\theta))$ (see [Lüc02b, Theorem 1.12 (2)]), the dimension of the closure of the image of R_θ does not depend on θ.

Finally, by Proposition 1.55 (3), whether R_θ is of determinant class or not and the value of its Fuglede-Kadison determinant depend only on C, on $(c(R_\theta, C)_p)_{p\in\mathbb{N}}$ and on $\dim_{\mathcal{N}(G)}(\mathrm{Ker}(R_\theta))$, thus do not depend on θ.

□

As a consequence we prove that the L^2-Alexander torsions are rigid regarding the unitary part of the parameter t. This result was proven in [LZ06, Section 6] for the L^2-Alexander invariant on the unit circle.

Corollary 2.10. *Let* X *be a compact CW-complex,* $\pi = \pi_1(X)$, $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ *group homomorphisms such that* (X, ϕ, γ) *is an admissible triple. Let* $t \in \mathbb{C}^*$. *Then* $C^{(2)}_*(X, \phi, \gamma, t)$ *is weakly acyclic and of determinant class if and only if* $C^{(2)}_*(X, \phi, \gamma, |t|)$ *is weakly acyclic and of determinant class, and in this case*

$$T^{(2)}(X, \phi, \gamma)(t) = T^{(2)}(X, \phi, \gamma)(|t|).$$

Proof. Let

$$C_*^{(2)}(\widetilde{X},\mathbb{Z}) = \ldots \to \oplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k \xrightarrow{\partial_{\widetilde{X},k}} \oplus_j \mathbb{Z}[\pi]\widetilde{e}_j^{k-1} \to \ldots$$

denote the cellular chain complex of $\widetilde{X}$. Consider the corresponding $\mathcal{N}(G)$-cellular chain complexes:

$$C_* = C_*^{(2)}(X,\phi,\gamma,t) = \ldots \to \oplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{S_k} \oplus_j \ell^2(G)\widetilde{e}_j^{k-1} \to \ldots$$

and

$$D_* = C_*^{(2)}(X,\phi,\gamma,|t|) = \ldots \to \oplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{T_k} \oplus_j \ell^2(G)\widetilde{e}_j^{k-1} \to \ldots$$

where $S_k = R_{\kappa(\pi,\phi,\gamma,t)}\left(\partial_{\widetilde{X},k}\right)$ and $T_k = R_{\kappa(\pi,\phi,\gamma,|t|)}\left(\partial_{\widetilde{X},k}\right)$.

We apply Proposition 2.9; here, $A = \partial_{\widetilde{X},k}$ and $e^{i\theta} = \dfrac{t}{|t|}$. As a consequence, the pairs $(\mathrm{Ker}(S_k), \mathrm{Ker}(T_k))$, $(\overline{\mathrm{Im}}(S_k), \overline{\mathrm{Im}}(T_k))$ are both pairs of finitely generated Hilbert $\mathcal{N}(G)$-modules of same dimension. Consequently, the L^2-Betti numbers of C_* and D_* are the same, and C_* and D_* are either both weakly acyclic or both not weakly acyclic. Likewise, for each k, S_k and T_k are of determinant class at the same time and have the same Fuglede-Kadison determinant, therefore $C_*^{(2)}(X,\phi,\gamma,t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(X,\phi,\gamma,|t|)$ is weakly acyclic and of determinant class, and in this case

$$T^{(2)}(X,\phi,\gamma)(t) = T^{(2)}(X,\phi,\gamma)(|t|).$$

□

2.1.3 Computing L^2-Alexander torsion with group presentations

Let M be a connected compact orientable irreducible 3-manifold with boundary ∂M a non-empty finite union of tori, and infinite fundamental group $G_M = \pi_1(M)$. The typical example of such an M is a (non-split) link exterior.

We will explain how we can compute L^2-torsions of M using deficiency one group presentations of G_M. Notably we will prove that L^2-Alexander torsions are invariants under simple homotopy in Theorem 2.12.

2-CW-complexes and asphericity

Recall that a CW-complex Y is *aspherical* if all its homotopy groups π_i vanish for $i \geqslant 2$. By considering L^2-Betti numbers, we prove that any M as above is aspherical.

The manifold M admits a triangulation (by works of Bing and Moise, see for example [Moi52]), and this triangulation defines a finite CW-structure on M. Now for any 3-simplex c having one of its four faces f in the boundary ∂M, we elementary retract M by taking out the 2-cell f and the 3-cell $\mathrm{Int}(c)$, the interior of c; we obtain an elementary retract of M with one less 3-cell. We can continue this process of «pushing the boundary inside» and finally we obtain a 2-CW complex W which is homotopically equivalent to M. Thus $\pi_1(W) = \pi_1(M) = G_M$.

The 3-manifold M has toroidal boundary, therefore $\chi(M) = 0$ (as a consequence of Poincaré duality and the fact that a solid torus has zero Euler characteristic). Each step of the process does not change the Euler characteristic (since we take out one 2-cell and one 3-cell), therefore $\chi(W) = 0$. Similarly, since M is connected, W is connected as well.

The first L^2-Betti number of a finite CW-complex depends only of its fundamental group (see [Hil02, Section 2.2]), therefore $\beta_1^{(2)}(W) = \beta_1^{(2)}(G_M) = \beta_1^{(2)}(M)$. Besides, since

M is prime with toroidal boundary and infinite fundamental group, $\beta_1^{(2)}(M) = 0$ by [Lüc02b, Theorem 4.1]. Therefore $\beta_1^{(2)}(W) = 0$.

By [Hil02, Theorem 2.4], since W is a connected 2-CW complex with

$$\chi(W) = 0 = -\beta_1^{(2)}(\pi_1(W)),$$

we conclude that W is aspherical.

Since M is homotopically equivalent to W, M is aspherical as well.

The 2-CW-complex constructed from a group presentation

Definition 2.11. If $P = \langle g_1, \dots, g_k | r_1, \dots r_l \rangle$ is a presentation of a group G, we can build a 2-dimensional CW-complex W_P having:

- one 0-cell e_0.
- k 1-cells $\gamma_1, \dots, \gamma_k$ with boundary sent to e_0 (k circles passing by e_0 so to speak).
- l 2-cells $\rho_1, \dots, \rho_l$ with the boundary of ρ_i sent to $\gamma_{i_1} * \dots * \gamma_{i_m}$ (note that we can see the cells γ_i as loops of base point e_0) if r_i is written $g_{i_1} \dots g_{i_m}$.

It follows from the Seifert-van Kampen theorem that $\pi_1(W_P)$ is isomorphic to G.

Now, let $P = \langle g_1, \dots, g_k | r_1, \dots r_{k-1} \rangle$ be a deficiency one group presentation of G_M. Then $\pi_1(W_P) = G_M$ and by construction $\chi(W_P) = 0$.

Since W_P is a connected 2-CW complex satisfying $\chi(W_P) = 0 = -\beta_1^{(2)}(\pi_1(W_P))$, W_P is aspherical by [Hil02, Theorem 2.4].

Eilenberg MacLane spaces and homotopy equivalence

The CW-complexes M and W_P are aspherical and have the same fundamental group G_M, therefore they are Eilenberg-MacLane spaces $K(G_M, 1)$ (see [Hat02, Section 4.2] for more details on the $K(G,1)$), thus they are homotopically equivalent by [Hat02, Proposition 4.30].

We have proven that any connected compact prime orientable 3-manifold M with non-empty toroidal boundary and infinite fundamental group G_M is homotopically equivalent to any 2-dimensional CW-complex W_P constructed from a deficiency one group presentation P of G_M.

Whitehead group and simple homotopy equivalence

An homotopy equivalence $f\colon X \to Y$ between two CW-complexes is called *simple* if it can be decomposed into a finite sequence of elementary expansions and elementary collapses. In this case we say that X and Y are *simple homotopy equivalent*. We refer to [Tur01, Section 8] and [Coh73, Section 4] for the details, and we will give a description of these elementary operations in the proof of Theorem 2.12.

Note that by construction the two CW-complexes M and W of the previous sections are simple homotopy equivalent. A sufficient condition for M and W_P (the 2-CW-complex constructed from the presentation P) to be simple homotopy equivalent is if G_M has trivial Whitehead group, according to [Coh73, (22.2)]. One can read [Coh73] for a detailed definition of the Whitehead group of a group and the Whitehead torsion of a homotopy

equivalence, but we will only need to consider the following cases when the Whitehead group vanish:

By [Wal73], if M is the exterior of a link in S^3, then the Whitehead group of G_M vanishes, and thus M and W_P are simple homotopy equivalent.

More generally, by [AFW12, (D.9)]:

- if M is a non-spherical compact orientable irreducible 3-manifold, then the Whitehead group of $\pi_1(M)$ vanishes.
- if M is a 3-manifold such that $\pi_1(M)$ is torsion-free, then the Whitehead group of $\pi_1(M)$ vanishes.

In particular, if M is irreducible and $\pi_1(M)$ is infinite, then $\pi_1(M)$ is torsion-free (see [AFW12, (C.2)]) and thus the Whitehead group of $\pi_1(M)$ vanishes.

Consequences on L^2-Alexander torsions

The following theorem states that the L^2-Alexander torsions are invariant by simple homotopy equivalence (compare with [Tur01, Corollary 9.2] and [Lüc02b, Theorem 3.96 (1)]).

Theorem 2.12. *Let $f\colon X \to Y$ be a simple homotopy equivalence between two finite CW-complexes inducing the group isomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$. The triple (Y, ϕ, γ) is an admissible triple if and only if $(X, \phi \circ f_*, \gamma \circ f_*)$ is one, the $\mathcal{N}(G)$-cellular chain complex $C_*^{(2)}(X, \phi\circ f_*, \gamma\circ f_*, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(Y, \phi, \gamma, t)$ is, and in this case one has*

$$T^{(2)}(X, \phi \circ f_*, \gamma \circ f_*)(t) \doteq T^{(2)}(Y, \phi, \gamma)(t).$$

Proof. Since any simple homotopy is a finite composition of elementary collapses or elementary expansions, we can prove Theorem 2.12 in the case when $f = I\colon X \hookrightarrow Y$ is an elementary expansion (therefore an inclusion that maps every k-cell to a k-cell).

The CW-complex Y is obtained from X by adjoining a $(k-1)$-cell e^{k-1} and then adjoining a k-cell e^k such that the boundary of e^k is the union of the closures of e^{k-1} and of some other $(k-1)$-cells of X. As in Section 1.1.3, the cellular chain complexes of the universal coverings $\widetilde{X}$ and $\widetilde{Y}$ are parts of the following commutative diagram:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
C_{k+1}(\widetilde{X}) \arrow[r, "\widetilde{I}"] \arrow[d, "\partial_{\widetilde{X},k+1}"] & C_{k+1}(\widetilde{Y}) \arrow[r, "0"] \arrow[d, "\begin{pmatrix} \widetilde{I}(\partial_{\widetilde{X},k+1}) \\ 0 \end{pmatrix}"] & 0 \arrow[d, "0"] \\
C_k(\widetilde{X}) \arrow[r, "\begin{pmatrix} \widetilde{I} \\ 0 \end{pmatrix}"] \arrow[d, "\partial_{\widetilde{X},k}"] & C_k(\widetilde{Y}) \arrow[r, "\begin{pmatrix} 0 & \dots & 0 & Id \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \widetilde{I}(\partial_{\widetilde{X},k}) & * \\ 0 & h \end{pmatrix}"] & \mathbb{Z}[\pi_Y]\widetilde{e}^k \arrow[d, "h"] \\
C_{k-1}(\widetilde{X}) \arrow[r, "\begin{pmatrix} \widetilde{I} \\ 0 \end{pmatrix}"] \arrow[d, "\partial_{\widetilde{X},k-1}"] & C_{k-1}(\widetilde{Y}) \arrow[r, "\begin{pmatrix} 0 & \dots & 0 & Id \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \widetilde{I}(\partial_{\widetilde{X},k-1}) & * \end{pmatrix}"] & \mathbb{Z}[\pi_Y]\widetilde{e}^{k-1} \arrow[d, "0"] \\
C_{k-2}(\widetilde{X}) \arrow[r, "\widetilde{I}"] \arrow[d] & C_{k-2}(\widetilde{Y}) \arrow[r, "0"] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}
\qquad (*)
$$

where $\widetilde{e}^k$ and $\widetilde{e}^{k-1}$ are lifts of e^k and e^{k-1} in $\widetilde{Y}$ and $h \in \pi_Y$ is such that $h \cdot \widetilde{e}^{k-1}$ is part of the boundary of $\widetilde{e}^k$.

Observe that the $\mathcal{N}(G)$-cellular chain complexes $C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t)$ and $C_*^{(2)}(Y, \phi, \gamma, t)$ are part of the exact sequence

$$0 \to C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t) \overset{\iota_*}{\hookrightarrow} C_*^{(2)}(Y, \phi, \gamma, t) \overset{q_*}{\twoheadrightarrow} D_* \to 0$$

where D_* is the L^2 cellular chain complex which reduces to

$$\dots \to 0 \to \ell^2(G)\widetilde{e}^k \xrightarrow{t^{\phi(h)} R_\gamma(h)} \ell^2(G)\widetilde{e}^{k-1} \to 0 \to \dots$$

at dimensions k and $k-1$. Indeed, this sequence is of the form:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
C^{(2)}_{k+1}(X,\phi\circ f_*,\gamma\circ f_*,t) \arrow[r, "Id"] \arrow[d, "A"] & C^{(2)}_{k+1}(Y,\phi,\gamma,t) \arrow[r, "0"] \arrow[d, "\begin{pmatrix} A \\ 0 \end{pmatrix}"] & 0 \arrow[d, "0"] \\
C^{(2)}_{k}(X,\phi\circ f_*,\gamma\circ f_*,t) \arrow[r, "\begin{pmatrix} Id \\ 0 \end{pmatrix}"] \arrow[d, "B"] & C^{(2)}_{k}(Y,\phi,\gamma,t) \arrow[r, "\begin{pmatrix} 0 & \dots & 0 & Id \end{pmatrix}"] \arrow[d, "\begin{pmatrix} B & * \\ 0 & t^{\phi(h)}R_\gamma(h) \end{pmatrix}"] & \ell^2(G)\tilde{e}^k \arrow[d, "t^{\phi(h)}R_\gamma(h)"] \\
C^{(2)}_{k-1}(X,\phi\circ f_*,\gamma\circ f_*,t) \arrow[r, "\begin{pmatrix} Id \\ 0 \end{pmatrix}"] \arrow[d, "C"] & C^{(2)}_{k-1}(Y,\phi,\gamma,t) \arrow[r, "\begin{pmatrix} 0 & \dots & 0 & Id \end{pmatrix}"] \arrow[d, "\begin{pmatrix} C & * \end{pmatrix}"] & \ell^2(G)\tilde{e}^{k-1} \arrow[d, "0"] \\
C^{(2)}_{k-2}(X,\phi\circ f_*,\gamma\circ f_*,t) \arrow[r, "Id"] \arrow[d] & C^{(2)}_{k-2}(Y,\phi,\gamma,t) \arrow[r, "0"] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

Exactness and commutativity are immediate consequences of the ones of $(*)$.

Assume that either $C^{(2)}_*(X,\phi\circ f_*,\gamma\circ f_*,t)$ or $C^{(2)}_*(Y,\phi,\gamma,t)$ is weakly acyclic and of determinant class.

The finite Hilbert $\mathcal{N}(G)$-chain complex D_* is composed of a single operator $t^{\phi(h)}R_\gamma(h)$, which is invertible with Fuglede-Kadison determinant equal to $|t|^{\phi(h)}$, thus D_* is weakly acyclic and therefore two of the three

$$C^{(2)}_*(X,\phi\circ f_*,\gamma\circ f_*,t),\ C^{(2)}_*(Y,\phi,\gamma,t),\ D_*$$

are weakly acyclic; as a consequence the three are and the long exact homology sequence

$$LHS_* = LHS_*(C^{(2)}_*(X,\phi\circ f_*,\gamma\circ f_*,t), C^{(2)}_*(Y,\phi,\gamma,t), D_*)$$

is trivial.

The chain complex LHS_* is of determinant class, and since D_* is too, three of the four chain complexes

$$C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t),\ C_*^{(2)}(Y, \phi, \gamma, t),\ D_*,\ LHS_*$$

are of determinant class, therefore, according to [Lüc02b, Theorem 3.35 (1)] all four are and one has:

$$T^{(2)}(X, \phi \circ f_*, \gamma \circ f_*)(t) \cdot T^{(2)}(D_*)) = T^{(2)}(Y, \phi, \gamma)(t) \cdot T$$

where $T = T^{(2)}\left(C_*^{(2)}(X, \phi \circ f_*, \gamma \circ f_*, t), C_*^{(2)}(Y, \phi, \gamma, t), D_*\right) = 1$ (since all the horizontal maps have determinant 1).

The formula follows from the fact that $T^{(2)}(D_*(\phi, \gamma, t))) = |t|^{\pm\phi(h)}$. □

Let N be a compact smooth 3-manifold. It follows from theorems due to Chapman and Cohen (see [Cha74], [Coh73]) that any two CW-structures on N are simple homotopy equivalent. Therefore, for any admissible triple $(\pi_1(N), \phi, \gamma)$, $T^{(2)}(N, \phi, \gamma)$ is a well defined invariant of the manifold N up to homeomorphism.

Conclusion

Let M be a connected orientable **compact** smooth 3-manifold of fundamental group $G_M = \pi_1(M)$. Let $\phi\colon G_M \to \mathbb{Z}$ and $\gamma\colon G_M \to G$ be group homomorphisms such that (G_M, ϕ, γ) forms an admissible triple. Let $t > 0$. Then both the fact that $C_*^{(2)}(M, \phi, \gamma)(t)$ is weakly acyclic and of determinant class and the value of $T^{(2)}(M, \phi, \gamma)(t)$ in this case **do not depend on the CW-structure chosen on** M.

If one of the following conditions is satisfied:

- M is a sub-manifold of S^3 (for instance M is the exterior of a link L in S^3),
- M is non-spherical and irreducible,
- G_M is torsion-free (for instance M is irreducible and G_M is infinite),

then the Whitehead group of $G_M = \pi_1(M)$ vanishes.

Assume that G_M has trivial Whitehead group and also that M is **irreducible** and ∂M is a **non-empty** finite union of **tori**. Then for all group presentations P of G_M with deficiency one, both the fact that $C_*^{(2)}(X, \phi, \gamma)(t)$ is weakly acyclic and of determinant class and the value of $T^{(2)}(X, \phi, \gamma)(t)$ **are the same for** $X = M$ **and for** $X = W_P$ the 2-dimensional CW-complex constructed from P (see Definition 2.11).

2.1.4 Weak acyclicity and determinant class

To compute L^2-Alexander torsions, we need to assume technical conditions on the $\mathcal{N}(G)$-cellular chain complexes: *weak acyclicity* and *determinant class*. These technical conditions are often hard to check. Fortunately, there are particular cases where these technical conditions are satisfied; this is the object of the following section.

Determinant class

Let M be a connected orientable **compact** smooth 3-manifold of fundamental group $G_M = \pi_1(M)$. Let $\phi\colon G_M \to \mathbb{Z}$ and $\gamma\colon G_M \to G$ be group homomorphisms such that (G_M, ϕ, γ) forms an admissible triple. Let $t > 0$. Then G_M is in the class $\mathcal{G}$ of sofic groups, referenced in [DFL14, Section 2.6].

The following result was proven by G. Elek E. Szabó in [ES05].

Proposition 2.13. *([ES05]) If G is a sofic group (for instance if G is the fundamental group of a compact 3-manifold) then the following hold.*

1. *The operator $R_A \in B(\ell^2(G)^n)$ is of determinant class for all matrices $A \in M_n(\mathbb{Q}[G])$.*
2. *The operator $R_A \in B(\ell^2(G)^n)$ is of Fuglede-Kadison determinant*
$$\det{}_{\mathcal{N}(G)}(R_A) \geqslant 1$$
for all matrices $A \in M_n(\mathbb{Z}[G])$.
3. *The operator $R_A \in B(\ell^2(G)^n)$ is of Fuglede-Kadison determinant*
$$\det{}_{\mathcal{N}(G)}(R_A) = 1$$
for all matrices $A \in GL_n(\mathbb{Z}[G])$ that are invertible over $\mathbb{Z}[G]$.

Remark 2.14. Let M denote a compact 3-manifold and $(M, \phi, \gamma\colon \pi_1(M) \to G)$ an admissible triple such that G is sofic. It follows from Proposition 2.13(1) that $C_*^{(2)}(M, \phi, \gamma)(t)$ is of determinant class for all $t \in \mathbb{Q}_{>0}$.

Weak acyclicity

By Proposition 1.58, assuming we have determinant class, weak acyclicity of a 2-dimensional finite Hilbert $\mathcal{N}(G)$-chain complex follows from the injectivity of two operators of the form R_A, $A \in M_k(\mathbb{R}[G])$, $k \in \mathbb{N}^*$.

Injectivity of such operators can be related to the strong Atiyah conjecture, see [Lüc02b, Section 10]:

Definition 2.15. Given a group G, let $\mathrm{FIN}(G)$ be the set of finite subgroups of G. Denote by $\frac{1}{\mathrm{FIN}(G)}\mathbb{Z}$ the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $\{\frac{1}{|H|} | H \in \mathrm{FIN}(G)\}$. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. A group G satisfies the *strong Atiyah conjecture for* $\mathbb{K}$ if for any matrix $A \in M(m, n, \mathbb{K}G)$ the von Neumann dimension of the kernel of the induced operator R_A satisfies $\dim_{\mathcal{N}(G)} \mathrm{Ker}(R_A) \in \frac{1}{\mathrm{FIN}(G)}\mathbb{Z}$.

In particular, if G is torsion-free (for example if G is a link group) then $\mathrm{FIN}(G)$ contains only the trivial subgroup, thus $\frac{1}{\mathrm{FIN}(G)}\mathbb{Z}$ is simply $\mathbb{Z}$.

If G is torsion-free and satisfies the strong Atiyah conjecture for $\mathbb{K}$, then for all $w \in \mathbb{K}G$, the von Neumann dimension of $R_w\colon \ell^2(G) \to \ell^2(G)$ is an integer, thus is zero or one, therefore
$$R_w \text{ is injective} \iff R_w \neq 0 \iff w \neq 0.$$

The following proposition follows from [Lüc02a, Lemma 2.2] and the computation of the L^2-Betti numbers of 3-manifolds in [Lüc02b, Theorem 4.1].

Proposition 2.16. *([Lüc02a, Lemma 2.2], [Lüc02b, Theorem 4.1])*

Let G be a finitely presented torsion-free infinite group, such that for every CW-complex X of fundamental group $\pi_1(X) = G$, the L^2-Betti numbers $b_p^{(2)}(X)$ of X are all integers. Then G satisfies the strong Atiyah conjecture for $\mathbb{Q}$.

Suppose that M is the connected sum of compact connected orientable prime 3-manifolds all of which having infinite fundamental group. If $G = \pi_1(M)$ is infinite and torsion-free, then G satisfies the strong Atiyah conjecture for $\mathbb{Q}$.

Finally, the following proposition establishes that a certain class of groups satisfies the strong Atiyah conjecture for $\mathbb{C}$.

Proposition 2.17. *([Lin93, Theorem 1.5], [Sch00, Theorem 1.9])*

Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed union and extensions with virtually abelian quotients.

If $G \in \mathcal{C}$ is torsion-free, then G satisfies the strong Atiyah conjecture for $\mathbb{C}$.

Let K be a fibered knot and let S denote its associated fiber surface (see Section 5.1 and [BZH14, Chapter 5] for details). The group G_K of K is a semi-direct product of the infinite cyclic group $\mathbb{Z}$ by the free group $\pi_1(F) = G'_K$, thus G_K is an extension of a free group with a virtually abelian quotient, therefore G_K satisfies the strong Atiyah conjecture for $\mathbb{C}$.

2.2 Definition and invariance

We now review two definitions of the L^2-Alexander invariant of a knot, introduced in [LZ06].

2.2.1 Definition from Fox calculus

Let K be a knot in S^3, G_K its group, and $P = \langle g_1, \dots, g_k \,|\, r_1, \dots r_{k-1} \rangle$ a Wirtinger presentation of G_K. Let $\alpha_K \colon G_K \to \mathbb{Z}, g_i \mapsto 1$ denote the abelianization of G_K.

For $t \in \mathbb{C}^*$, we define the algebra homomorphism:

$$\psi_{K,t} \colon \left(\begin{array}{c} \mathbb{C}[G_K] \longrightarrow \mathbb{C}[G_K] \\ \sum_{g \in G_K} c_g \cdot g \longmapsto \sum_{g \in G_K} c_g \cdot t^{\alpha_K(g)} \cdot g \end{array} \right)$$

and we also call $\psi_{K,t}$ its induction to any matrix ring with coefficients in $\mathbb{C}[G_K]$. Think of it as a way of «tensoring by the abelianization representation».

Let P be any presentation of G_K with deficiency one, not necessarily Wirtinger; we say that (P, t) has *Property $\mathcal{I}$* if $R_{\psi_{K,t}(F_{P,1})} \colon \ell^2(G_K)^{k-1} \to \ell^2(G_K)^{k-1}$ is injective.

We let $\mathcal{D}_P$ denote the set of all $t \in \mathbb{C}^*$ such that (P, t) has Property $\mathcal{I}$.

Definition 2.18. Let K be a knot, let P be a Wirtinger presentation of its knot group G_K, and let $t \in \mathbb{C}^*$.

If (P, t) has Property $\mathcal{I}$ then *the L^2-Alexander invariant of K for the presentation P at t* is:

$$\Delta^{(2)}_{K,P}(t) := \det_{\mathcal{N}(G_K)} \left(R_{\psi_{K,t}(F_{P,1})} \right) \in [0, \infty[.$$

The *L^2-Alexander invariant of K associated to the presentation P* is the map

$$\Delta^{(2)}_{K,P} = \left(t \mapsto \Delta^{(2)}_{K,P}(t) \right) \in \mathcal{F}(\mathcal{D}_P; \mathbb{R}_{\geqslant 0}).$$

Remark 2.19. We do not require the operator $R_{\psi_{K,t}(F_{P,1})}$ to be of determinant class in the definition, thus $\Delta^{(2)}_{K,P}(t)$ can theoretically be zero. In practice, in the following of this thesis, when the number t is such that $R_{\psi_{K,t}(F_{P,1})}$ is injective, this operator is often also of determinant class. In particular, for $t \in \mathbb{Q}_{>0}$ and $k = 2$, it follows from Section 2.1.4 that $R_{\psi_{K,t}(F_{P,1})}$ is both injective and of determinant class.

Example 2.20. Let us compute the invariant for the trivial knot O.

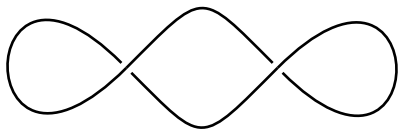


Figure 2.1 – A diagram for the unknot

The «doubly twisted rubber band» knot diagram of Figure 2.1 gives the Wirtinger presentation $P = \langle g, h | gh^{-1} \rangle$ of the unknot group G_O (which is isomorphic to $\mathbb{Z}$), and the associated Fox matrix is $F_P = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore for all $t > 0$, $R_{\psi_{O,t}(F_{P,1})} = -Id \colon \ell^2(G_O) \to \ell^2(G_O)$ has Property $\mathcal{I}$ and $\Delta^{(2)}_{O,P}(t) = 1$, from Proposition 1.51 (2). Thus, the invariant for the trivial knot is the constant map equal to 1.

Proposition 2.21. *Let K be a knot in S^3. Let P and Q be two Wirtinger presentations of the knot group G_K. Then $\mathcal{D}_P = \mathcal{D}_Q$ and there exists an integer $m \in \mathbb{Z}$ such that $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t) \cdot |t|^m$ for all t in $\mathcal{D}_P$.*

The proof of this proposition is somewhat technical. It is based on a study of Tietze transformations between Wirtinger presentations and of how the respective associated operators are consequently modified by these transformations. We include the following detailed proof for the sake of completeness (compare with [Wad94, Section 5] and [LZ06, Proposition 3.4]).

Proof. Let P and Q be two Wirtinger presentations with deficiency one of the same knot group G_K. This means that P and Q were constructed respectively from two diagrams D and D' of the same knot K. Therefore D' is obtained from D by a finite sequence of planar isotopies and Reidemeister moves (see for example [BZH14, Proposition 1.17]). As explained in [Wad94, Lemma 6], this means that Q can be obtained from P by a finite sequence of certain Tietze transformations (and their inverses), called *Strong Tietze moves*, which are the following:

- I_a. To replace one of the relators r_i by its inverse r_i^{-1}
- I_b. To replace one of the relators r_i by its conjugate wr_iw^{-1} where w is a word in the generators.
- I_c. To replace one of the relators r_i by its product r_ir_k with a different relator ($k \neq i$)

- II_W. To add a new generator x and a new relator $x = w$ where w is of the form $x_j x_i x_j^{-1}$ or $x_j^{-1} x_i x_j$ with x_i and x_j some previous generators.
- III. To apply a permutation on the generators.

Note that the transformation III simply describes the ambiguity in ordering the generators during the Wirtinger process. Moreover, we use the transformation II_W and not the transformation II of [Wad94, Section 1] because II_W is sufficient to describe the modifications caused by Reidemeister moves; this helps us ensure the following fact: if a sequence of such Tietze moves transforms the Wirtinger presentation P into the Wirtinger presentation Q, then all intermediate presentations are not necessarily Wirtinger presentations but they all have the fundamental property that *their generators are all conjugates of one another.*

To prove Proposition 2.21, it suffices to prove that if Q is obtained from P by a single previous transformation, then $\mathcal{D}_P = \mathcal{D}_Q$ and there is an integer m such that $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t) \cdot |t|^m$ for all t in $\mathcal{D}_P$.

1. If Q is obtained from P by a I_a move, for example the j-th relator r is changed to r^{-1}, then by construction the respective free groups in the generators and quotient maps are the same (notably $Gr(P) = Gr(Q)$). Remark that this will also be the case for moves of type I_b, I_c and III. Since for any generator x
$$\overline{\frac{\partial}{\partial x}(r^{-1})} = \overline{\left(-r^{-1}\frac{\partial}{\partial x}(r)\right)} = -\overline{\frac{\partial}{\partial x}(r)},$$
we deduce that $\psi_{K,t}(F_{Q,1})$ is simply $\psi_{K,t}(F_{P,1})$ with the j-th column multiplied by -1. Therefore $R_{\psi_{K,t}(F_{Q,1})} = R_{\psi_{K,t}(F_{P,1})} \cdot D$ where D is the dilatation operator (defined in Section 1.3.4) with all diagonal coefficients Id except the j-th which is $-Id$. Thus $R_{\psi_{K,t}(F_{Q,1})}$ and $R_{\psi_{K,t}(F_{P,1})}$ are both injective for the same values of t, i.e. $\mathcal{D}_P = \mathcal{D}_Q$, and furthermore $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t)$ (by Proposition 1.51 (2) and (4)).

2. If Q is obtained from P by a I_b move, for example the j-th relator r is changed to wrw^{-1} with w a word in the generators, then since for any generator x
$$\begin{aligned}\overline{\frac{\partial}{\partial x}(wrw^{-1})} &= \overline{\frac{\partial}{\partial x}(w) + w\frac{\partial}{\partial x}(r) + wr\frac{\partial}{\partial x}(w^{-1})} \\ &= \overline{\frac{\partial}{\partial x}(w) + w\frac{\partial}{\partial x}(r) + wr\left(-w^{-1}\frac{\partial}{\partial x}(w)\right)} \\ &= \overline{w} \cdot \overline{\frac{\partial}{\partial x}(r)},\end{aligned}$$
we deduce that $\psi_{K,t}(F_{Q,1})$ is simply $\psi_{K,t}(F_{P,1})$ with the j-th column multiplied on the left by $\psi_{K,t}(\overline{w}) = t^m\overline{w}$ where m is an integer. Therefore $R_{\psi_{K,t}(F_{Q,1})} = R_{\psi_{K,t}(F_{P,1})} \cdot D$ where D is the dilatation operator with all diagonal coefficients Id except the j-th which is $R_{t^m w}$. The operator D is invertible and of Fuglede-Kadison determinant $|t|^m$. Therefore $\psi_{K,t}(F_{Q,1})$ and $\psi_{K,t}(F_{P,1})$ are both injective for the same values of t, i.e. $\mathcal{D}_P = \mathcal{D}_Q$, and furthermore $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t) \cdot |t|^m$ (by Proposition 1.51 (2) and (4)).

3. If Q is obtained from P by a I_c move, for example the j-th relator r is changed to rr' with r' the l-th relator, then since for any generator x
$$\overline{\frac{\partial}{\partial x}(rr')} = \overline{\frac{\partial}{\partial x}(r) + r\frac{\partial}{\partial x}(r')} = \overline{\frac{\partial}{\partial x}(r)} + \overline{\frac{\partial}{\partial x}(r')},$$

we deduce that $\psi_{K,t}(F_{Q,1})$ is simply $\psi_{K,t}(F_{P,1})$ where the l-th column was added to the j-th one. Therefore $R_{\psi_{K,t}(F_{Q,1})} = R_{\psi_{K,t}(F_{P,1})} \cdot T$ where T is the transvection operator with only nonzero nondiagonal coefficient is Id at the (l,j) position. Proposition 1.51 (2) and (4) let us conclude that $\mathcal{D}_P = \mathcal{D}_Q$ (composing by an invertible transvection operator does not change the injectivity) and that $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t)$ (since a transvection operator has Fuglede-Kadison determinant 1).

4. Suppose that Q is obtained from P by a II_W move, then write $P = \langle g_1, \dots, g_k \,|\, r_1, \dots r_{k-1} \rangle$ and $Q = \langle g_1, \dots, g_k, h \,|\, r_1, \dots r_{k-1}, wh^{-1} \rangle$ where w is a word in the g_i. Here $Gr(P)$ and $Gr(Q)$ are naturally isomorphic via

$$Gr(P) = \mathbb{F}[g_i]/\langle r_j \rangle \hookrightarrow \mathbb{F}[g_i, h]/\langle r_j \rangle \twoheadrightarrow \mathbb{F}[g_i, h]/\langle r_j, wh^{-1} \rangle = Gr(Q)$$

(where $\langle r_j \rangle$ is the normal generated subgroup), therefore we dare an abuse of notation by writing

$$F_Q = \begin{array}{c} \\ g_1 \\ \vdots \\ g_k \\ h \end{array} \begin{array}{c} \begin{array}{cccc} r_1 & \dots & r_{k-1} & wh^{-1} \end{array} \\ \begin{pmatrix} & & & * \\ & F_P & & \vdots \\ & & & * \\ 0 & \dots & 0 & -1 \end{pmatrix} \end{array}$$

where the $*$ are elements of $\mathbb{Z}[G_K]$. Thus $R_{\psi_{K,t}(F_{Q,1})}$ is injective if and only if $R_{\psi_{K,t}(F_{P,1})}$ is injective, i.e. $\mathcal{D}_P = \mathcal{D}_Q$. Hence, by Proposition 1.51 (2) and (5), $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t)$ for all $t \in \mathcal{D}_P$.

5. Suppose that Q is obtained from P by a III move. A permutation is a finite product of transpositions, therefore we can assume that the III move is a transposition τ.

   Let us assume that τ leaves the first generator fixed. In this case the Fox matrix $F_{Q,1}$ is $F_{P,1}$ with two of its rows swapped, i.e. $F_{Q,1}$ is equal to $F_{P,1}$ multiplied by a permutation matrix S. Since the associated operator $R_{\psi_{K,t}(S)} = R_S$ is unitary, it is invertible and has Fuglede-Kadison determinant 1. Thus $R_{\psi_{K,t}(F_{Q,1})}$ is injective if and only if $R_{\psi_{K,t}(F_{P,1})}$ is injective, i.e. $\mathcal{D}_P = \mathcal{D}_Q$. Hence, by Proposition 1.51 (5), $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t)$ for all $t \in \mathcal{D}_P$.

   Now let us assume that τ swaps the first and second generators. We denote $F_P = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{pmatrix}$ and $F_Q = \begin{pmatrix} L_2 \\ L_1 \\ \vdots \\ L_k \end{pmatrix}$ where $L_i = \left(\overline{\frac{\partial r_j}{\partial g_i}} \right)_{1 \leqslant j \leqslant k}$ denotes the i-th row of F_P. Let us remind the reader that the generators g_i are conjugates of one another, therefore they have the same image 1 by the abelianization α_K, which means that $\psi_{K,t}(g_i - 1) = tg_i - 1$ for each i.

   The fundamental formula of Fox calculus (see for instance [BZH14, Proposition 9.8]) implies that the following formula stands in $\mathbb{C}[G_K]$:

$$\sum_{i=1}^{k} L_i \cdot (\overline{g_i} - 1) = 0. \qquad (*)$$

Let

$$A = \begin{pmatrix} R_{tg_2-1} & & & 0 \\ & R_{tg_3-1} & & \\ & & \ddots & \\ 0 & & & R_{tg_k-1} \end{pmatrix},$$

$$B = \begin{pmatrix} R_{tg_1-1} & & & 0 \\ & R_{tg_3-1} & & \\ & & \ddots & \\ 0 & & & R_{tg_k-1} \end{pmatrix},$$

$$C = \begin{pmatrix} -Id & -Id & \dots & -Id \\ & Id & & 0 \\ & & \ddots & \\ 0 & & & Id \end{pmatrix}.$$

We recognize a transvection matrix in C, which is thus invertible and of determinant 1. Proposition 1.53 tells us that A and B are injective and that their Fuglede-Kadison determinant is $\max(1, |t|)^{k-1}$.

Formula $(*)$ implies the following equality for operators in $B(\ell^2(G_K)^{k-1})$:

$$C \circ A \circ R_{\psi_{K,t}(F_{P,1})} = \begin{pmatrix} -R_{\psi_{K,t}(L_1(g_1-1))} - R_{\psi_{K,t}(L_3(g_3-1))} - \dots - R_{\psi_{K,t}(L_k(g_k-1))} \\ R_{\psi_{K,t}(L_3(g_3-1))} \\ \vdots \\ R_{\psi_{K,t}(L_k(g_k-1))} \end{pmatrix}$$

$$= \begin{pmatrix} R_{\psi_{K,t}(L_2(g_2-1))} \\ R_{\psi_{K,t}(L_3(g_3-1))} \\ \vdots \\ R_{\psi_{K,t}(L_k(g_k-1))} \end{pmatrix} = B \circ R_{\psi_{K,t}(F_{Q,1})}.$$

Since C, A and B are injective, $R_{\psi_{K,t}(F_{P,1})}$ is injective if and only if $R_{\psi_{K,t}(F_{Q,1})}$ is injective, i.e. $\mathcal{D}_P = \mathcal{D}_Q$. Finally, by Proposition 1.51 (4) and the values of the determinants of A, B, C, we conclude that $\Delta^{(2)}_{K,Q}(t) = \Delta^{(2)}_{K,P}(t)$ for all $t \in \mathcal{D}_P$.

Any permutation can be decomposed as a finite product of transpositions swapping the first and second elements and transpositions leaving the first element fixed. Therefore the case of the III move is treated, and the proposition is proven.

□

Definition 2.22. Let K be a knot. Let P be any Wirtinger presentation of its knot group G_K and let $\mathcal{D}_K$ be the set of $t \in \mathbb{C}^*$ such that (P, t) has Property $\mathcal{I}$ (according to the previous proposition, this does not depend on P). The *L^2-Alexander invariant of K* is the class of functions $(t \mapsto \Delta^{(2)}_K(t))$ defined as the equivalence class of $\left(t \mapsto \Delta^{(2)}_{K,P}(t)\right)$ up to multiplication by $(t \mapsto |t|^{\mathbb{Z}})$ on the functions from $\mathcal{D}_K$ to $\mathbb{R}_{\geqslant 0}$.

It is a knot invariant by the previous proposition.

Remark 2.23. Until now we know of no knots K such that $\mathcal{D}_K \neq \mathbb{C}^*$. However we know that $\mathcal{D}_K$ always contains at least the entire unit circle, thanks to Theorem 2.27.

Remark 2.24. Let us remark that we can take $F_{P,i}$ for any $i \neq 1$ instead of $F_{P,1}$ in the definition of the invariant, since it simply corresponds to an other Wirtinger presentation where the generators are permuted.

The following result is proven for the unit circle in [LZ06, Section 6] and can be easily extended to $\mathbb{C}^*$. Compare with Corollary 2.10.

Proposition 2.25. *([LZ06, Section 6])*

1. *Let K be a knot and P a Wirtinger presentation of G_K, and let $t \in \mathbb{C}^*$. Then (P, t) has Property $\mathcal{I}$ if and only if $(P, |t|)$ has Property $\mathcal{I}$.*
2. *Let K be a knot and $t \in \mathbb{C}^*$, such that there is a Wirtinger presentation P with (P, t) having Property $\mathcal{I}$. Then $\Delta_K^{(2)}(t) = \Delta_K^{(2)}(|t|)$.*

We will now always assume $t > 0$. The L^2-Alexander invariant is thus a class of maps from (a subset $\mathcal{D}_K$ of) $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geqslant 0}$ (up to multiplication by $(t \mapsto t^m), m \in \mathbb{Z}$).

2.2.2 Definition from the L^2-torsion

Let K be a knot in S^3, $M_K = S^3 \setminus V(K)$ its exterior, $G_K = \pi_1(M_K)$ the group of K, $\alpha_K \colon G_K \to \mathbb{Z}$ the abelianization, and $P = \langle g_1, \dots, g_k | r_1, \dots r_{k-1} \rangle$ a deficiency one presentation of G_K.

Any homomorphism $\phi \colon G_K \to \mathbb{Z}$ factors through the abelianization, thus $\phi = r\alpha_K$, for some $r \in \mathbb{Z}$. Therefore, up to using Proposition 2.7, we can assume that $\phi = \alpha_K$.

Section 2.1.3 tells us that M_K is simple homotopy equivalent to the 2-dimensional CW-complex W_P constructed from P, therefore:

$$T^{(2)}(M_K, \alpha_K, id)(t) \doteq T^{(2)}(W_P, \alpha_K, id)(t).$$

The finite Hilbert $\mathcal{N}(G)$-chain complex $C^{(2)}(W_P, \alpha_K, id)(t)$ is:

$$C^{(2)}(W_P, \alpha_K, id)(t) = \dots \to 0 \to \ell^2(G_K)^{k-1} \xrightarrow{\partial_2} \ell^2(G_K)^k \xrightarrow{\partial_1} \ell^2(G_K) \to 0 \to \dots$$

where

$$\partial_2 = R_{\psi_{K,t}(F_P)} = R_{\kappa(G_K, \alpha_K, id, t)(F_P)}$$

and

$$\partial_1 = \left(t^{\alpha_K(g_1)} R_{g_1} - Id \ ; \quad \dots \ ; \quad t^{\alpha_K(g_k)} R_{g_k} - Id \right).$$

Let i be such that $g_i \neq 1 \in G_K$. Then $\left(t^{\alpha_K(g_i)} R_{g_i} - Id \right)$ is injective and of determinant class by Proposition 1.53 and the fact that G_K is torsion-free. Therefore, by Proposition 1.58, $C^{(2)}(W_P, \alpha_K, id)(t)$ is weakly acyclic if the operator $R_{\psi_{K,t}(F_{P,i})} \colon \ell^2(G_K)^{k-1} \to \ell^2(G_K)^{k-1}$ is injective.

If this is the case (i.e. if $(\widetilde{P}, t)$ has Property $\mathcal{I}$, where $\widetilde{P}$ is P where the generators are permuted), then the L^2-torsion of $C^{(2)}(W_P, \alpha_K, id)(t)$ is defined and is non-zero if and only if $R_{\psi_{K,t}(F_{P,i})}$ is of determinant class. In this case the L^2-Alexander torsion is equal to:

$$T^{(2)}(W_P, \alpha_K, id)(t) \doteq \frac{\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,i})})}{\det_{\mathcal{N}(G_K)}(t^{\alpha_K(g_i)} R_{g_i} - Id)} \doteq \frac{\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,i})})}{\max(1, t)^{|\alpha_K(g_i)|}}.$$

If Q is an other deficiency one presentation of G_K, then W_Q is simple homotopy equivalent to M_P, thus to W_P. Therefore $T^{(2)}(M_K, \alpha_K, id)(t) \doteq T^{(2)}(W_P, \alpha_K, id)(t)$ does not depend on the chosen presentation P.

If K and K' are ambient isotopic, then M_K and $M_{K'}$ are homeomorphic by an homeomorphism $h\colon M_K \to M_{K'}$ that preserves meridians by Theorem 1.5, therefore M_K and $M_{K'}$ are simple homotopy equivalent, and if we denote by $\kappa\colon G_K \to G_{K'}$ the group isomorphism induced by h, then $\alpha_{K'} \circ \kappa = \alpha_K$. Thus

$$T^{(2)}(M_K, \alpha_K, id)(t) \doteq T^{(2)}(M_{K'}, \alpha_{K'}, id)(t)$$

by Proposition 2.5, since κ is injective.

Hence, we could define the L^2-Alexander invariant of K as

$$T^{(2)}(M_K, \alpha_K, id)(t) \cdot \max(1, t),$$

it would be a knot invariant equal to $\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,1})})$ when $R_{\psi_{K,t}(F_{P,1})}$ is injective; this corresponds to the L^2-Alexander invariant $\Delta_K^{(2)}(t)$ defined in the previous section.

Note that the reasoning leading to the formula

$$T^{(2)}(W_P, \alpha_K, id)(t) \doteq \frac{\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,i})})}{\max(1, t)^{|\alpha_K(g_i)|}}$$

gives an alternative proof of Theorem 2.28.

We can now sum up this result in the following terms:

Proposition 2.26. *With the previous notations, if $P = \langle g_1, \dots, g_k | r_1, \dots r_{k-1} \rangle$ is a deficiency one presentation of G_K, if (P, t) has Property $\mathcal{I}$ and $g_1 \neq 1$, then*

$$T^{(2)}(W_P, \alpha_K, id)(t) \doteq \frac{\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,1})})}{\max(1, t)^{|\alpha_K(g_1)|}} \doteq \frac{\Delta_K^{(2)}(t)}{\max(1, t)}.$$

2.3 Formulas

2.3.1 The simplicial volume appears at $t = 1$

Theorem 2.4 of W. Lück and T. Shick states that the L^2-torsion of a 3-manifold gives precisely the simplicial volume of this manifold. In the case of knots, using the double language of L^2-Alexander invariant and L^2-Alexander torsions, we can write it as follows:

Theorem 2.27 ([Lüc02b], Theorem 4.6)**.** *If K is a non-trivial knot then the 3-manifold M_K is irreducible and, according to the JSJ-decomposition, splits along disjoint incompressible tori into pieces that are Seifert manifolds or hyperbolic manifolds. The hyperbolic pieces $M_1, \dots, M_h$ have all finite hyperbolic volume. Then for any deficiency one presentation P of the knot group G_K, $(P, 1)$ has Property $\mathcal{I}$ and*

$$\Delta_K^{(2)}(1) = \exp\left(\frac{1}{6\pi}\sum_{i=1}^{h} vol(M_i)\right) = \exp\left(\frac{1}{6\pi} vol(M_K)\right)$$

where vol is the simplicial volume.

2.3.2 Computing with any presentation

The following result helps us compute the L^2-Alexander invariant of a knot using any deficiency one presentation. Compare with the end of Section 2.2.2.

Theorem 2.28 ([DW13], Theorem 3.5 and Proposition 6.2)**.**
(1) Let K be a knot, G_K its group, and $P = \langle g_1, \dots, g_k \mid r_1, \dots r_{k-1} \rangle$ any deficiency one presentation of G_K. If $t > 0$ is such that (P, t) has Property $\mathcal{I}$, then $\dfrac{\det_{\mathcal{N}(G_K)}(R_{\psi_{K,t}(F_{P,1})})}{\max(1,t)^{|\alpha_K(g_1)|-1}}$ does not depend on P, and is equal to $\Delta^{(2)}_{K,P}(t)$ when P is Wirtinger. Thus we will also call this quantity $\Delta^{(2)}_{K,P}(t)$.

(2) If K is the (p,q)-torus knot, then for any $t > 0$, $\Delta^{(2)}_K(t)$ is defined (i.e. $\mathcal{D}_K = \mathbb{R}_{>0}$) and equals $\max(1,t)^{(|p|-1)(|q|-1)}$.

Remark 2.29. This theorem implies that the L^2-Alexander invariant is not a complete knot invariant. For example $T(2,7)$ and $T(3,4)$ are distinct torus knots but they both have $(t \mapsto \max(1,t)^6)$ as their L^2-Alexander invariant.

Furthermore, since these two knots have different Alexander polynomials, we conclude that the L^2-Alexander invariant is not a «strictly stronger» knot invariant than the Alexander polynomial (the Alexander polynomial is not strictly stronger either, since the L^2-Alexander invariant detects the trivial knot and the Alexander polynomial does not).

2.3.3 Mirror image formula

We compute the L^2-Alexander invariant of the mirror image of a knot. Compare this to the classical property for the Alexander polynomial (see for example [Cro04, Theorem 7.1.4 (d)] and the Annex A.3).

Theorem 2.30. *Let K be a knot in S^3 and K^* its mirror image. Let P be a Wirtinger presentation of G_K and let $t > 0$. Suppose (P, t) has Property $\mathcal{I}$.*

Then G_{K^*} admits a group presentation P^* naturally obtained from P, (P^*, t^{-1}) has Property $\mathcal{I}$ and $\Delta^{(2)}_{K^*}(t^{-1}) \doteq \Delta^{(2)}_K(t)$.

Proof. Take a diagram D of K and its image D' by a reflection by a line ℓ not intersecting D. Then D' is a diagram for the image of K by a planar reflection in $\mathbb{R}^3$ for a plane generated by the line ℓ and the normal at the plane of D. Thus D' is a diagram of K^*. Take a base point in $\mathbb{R}^3$ above the common plane of the diagrams D and D'.

Each crossing of D corresponds to a crossing of D' as in Figure 2.2.

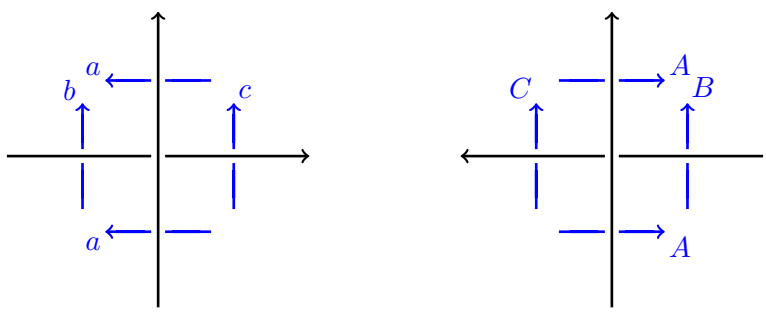


Figure 2.2 – A crossing of D, its mirror image in D', and the associated meridians

Let $P = \langle a_i | r_j \rangle$ be a Wirtinger presentation of $G_K = \pi_1(S^3 \setminus V(K))$ associated to D. Its relators are of the form $aba^{-1}c^{-1}$. As in Figure 2.2, for each generator a_i of P, define A_i a (negatively-oriented) meridian curve of D', and for $r_j = aba^{-1}c^{-1}$, define $R_j = ABA^{-1}C^{-1}$. Then $P^* = \langle A_i | R_j \rangle$ is a presentation for $G_{K^*} = \pi_1(S^3 \setminus K^*)$. Note that $\alpha_{K^*}(A_i) = -1$ for all i.

Let $\phi\colon G_K \to G_{K^*}$ denote the natural group isomorphism sending a_i to A_i and its induction on the associated complex group algebras. Then

$$\begin{array}{ccc} \mathbb{C}[G_K] & \xrightarrow{\psi_{K,t}} & \mathbb{C}[G_K] \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{C}[G_{K^*}] & \xrightarrow{\psi_{K^*,t^{-1}}} & \mathbb{C}[G_{K^*}] \end{array}$$

is a commutative diagram, since $\psi_{K^*,t^{-1}}(A_i) = tA_i$ for all i.

Suppose (P,t) has Property $\mathcal{I}$, thus $R_{\psi_{K,t}(F_{P,1})}$ is injective. Therefore, by Proposition 1.45 (1), the commutativity of the previous diagram, and Proposition 1.45 (2), in this order,

$$(\phi)_*(R_{\psi_{K,t}(F_{P,1})}) = R_{\phi(\psi_{K,t}(F_{P,1}))} = R_{\psi_{K^*,t^{-1}}(\phi(F_{P,1}))} = R_{\psi_{K^*,t^{-1}}(F_{P^*,1})}$$

is injective. Thus (P^*, t^{-1}) has Property $\mathcal{I}$.

By Theorem 2.28, since P^* has deficiency one,

$$\Delta^{(2)}_{K^*}(t^{-1}) \doteq \frac{\det_{\mathcal{N}(G_{K^*})}(R_{\psi_{K^*,t^{-1}}(F_{P^*,1})})}{\max(1,t)^{|\alpha_{K^*}(A_1)|-1}} = \det_{\mathcal{N}(G_{K^*})}\left((\phi)_*(R_{\psi_{K,t}(F_{P,1})})\right),$$

and by Proposition 1.51 (6) we conclude that $\Delta^{(2)}_{K^*}(t^{-1}) \doteq \Delta^{(2)}_K(t)$. □

Remark 2.31. Proposition 2.8 implies that the L^2-Alexander invariant cannot distinguish a knot from its mirror image.

2.3.4 Connected sum formula

Let K_1 and K_2 be knots in S^3 and $K = K_1 \sharp K_2$ their connected sum. We prove that the L^2-Alexander invariant of K can be computed from those of its factors. This multiplicativity of the invariant can be compared to the classical property of the Alexander polynomial of a composite knot, see for example [BZH14, Proposition 8.14] and Annex A.3.

Lemma 2.32. *Let K be the connected sum of K_1 and K_2, with G, G_1 and G_2 their respective groups.*

Then for $j = 1, 2$ and for all $t > 0$ we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}[G_j] & \xrightarrow{\psi_{K_j,t}} & \mathbb{C}[G_j] \\ \downarrow i_j & & \downarrow i_j \\ \mathbb{C}[G] & \xrightarrow{\psi_{K,t}} & \mathbb{C}[G] \end{array}$$

where $i_j\colon G_j \hookrightarrow G$ denotes both the group inclusion of Proposition 1.21 and its induction on the complex group algebras.

Proof. Let us take P_1, P_2 and P like in Lemma 1.20, and $t > 0$. We have

$$P_1 = \langle x_1, \dots, x_k | r_1, \dots, r_{k-1} \rangle,$$

$$P_2 = \langle y_1, \dots, y_l | s_1, \dots, s_{l-1} \rangle,$$

$$P = \langle x_1, \dots, x_k, y_1, \dots, y_l | r_1, \dots, r_{k-1}, s_1, \dots, s_{l-1}, x_k y_l^{-1} \rangle.$$

These three presentations are Wirtinger presentations, therefore the x_i are sent to 1 by α_{K_1} as elements of G_1 and by α_K as elements of G, and the same can be said for the generators y_j.

It follows that the diagram is commutative for any $g \in \mathbb{C}[G_j]$ where g is a generator of P_1 or P_2. The result follows from the fact that the $\psi_{.,t}$ and i_j are algebra homomorphisms and that the previous g generate the two group algebras. □

Theorem 2.33. *Let K be the connected sum of K_1 and K_2, with G, G_1 and G_2 their respective groups, and P, P_1, P_2 the presentations given by Lemma 1.20.*

Let t be any positive number. If we assume that (P_1, t) and (P_2, t) have Property $\mathcal{I}$, then (P, t) has Property $\mathcal{I}$ and $\Delta_K^{(2)}(t) \doteq \Delta_{K_1}^{(2)}(t)\Delta_{K_2}^{(2)}(t)$.

Proof. Let P_1, P_2 and P be like in Lemma 1.20, and $t > 0$. We have two injective group homomorphisms $i_1 \colon G_1 \hookrightarrow G$ and $i_2 \colon G_2 \hookrightarrow G$ by Proposition 1.21.

The values of P, P_1, P_2 imply that $R_{\psi_{K,t}(F_P)}$ is written:

$$\begin{array}{c|c}
 & \begin{array}{ccc|ccc|c} r_1 & \dots & r_{k-1} & s_1 & \dots & s_{l-1} & x_k y_l^{-1} \end{array} \\
\begin{array}{c} x_1 \\ \vdots \\ x_{k-1} \\ x_k \\ y_1 \\ \vdots \\ y_{l-1} \\ y_l \end{array} &
\left(\begin{array}{ccc|ccc|c}
 & & & 0 & \dots & 0 & 0 \\
 & R_{\psi_{K,t}(i_1(F_{P_1,k}))} & & \vdots & & \vdots & \vdots \\
 & & & 0 & \dots & 0 & 0 \\
\hline
 & * & & 0 & \dots & 0 & Id \\
\hline
0 & \dots & 0 & & & & 0 \\
\vdots & & \vdots & & R_{\psi_{K,t}(i_2(F_{P_2,l}))} & & \vdots \\
0 & \dots & 0 & & & & 0 \\
\hline
0 & \dots & 0 & & * & & -Id
\end{array}\right)
\end{array}$$

Since (P_1, t) has Property $\mathcal{I}$, $R_{\psi_{K_1,t}(F_{P_1,k})}$ is injective (by Remark 2.24). Therefore, by Proposition 1.45 (1), Lemma 2.32 and Proposition 1.45 (2), in this order,

$$(i_1)_*(R_{\psi_{K_1,t}(F_{P_1,k})}) = R_{i_1(\psi_{K_1,t}(F_{P_1,k}))} = R_{\psi_{K,t}(i_1(F_{P_1,k}))}$$

is injective. Similarly, $R_{\psi_{K,t}(i_2(F_{P_2,l}))}$ is injective. Finally, $-Id_{\ell^2(G)}$ is clearly injective.

Therefore the block trigonal matrix $R_{\psi_{K,t}(F_{P,k})}$ is injective, thus, by Remark 2.24, (P, t) has Property $\mathcal{I}$.

Hence by Proposition 1.51 (5) and (2),

$$\det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(F_{P,k})}\right) = \det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(i_1(F_{P_1,k}))}\right) \cdot \det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(i_2(F_{P_2,l}))}\right).$$

Finally,

$$\det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(i_1(F_{P_1,k}))}\right) = \det{}_{\mathcal{N}(G)}\left((i_1)_*(R_{\psi_{K_1,t}(F_{P_1,k})})\right) = \det{}_{\mathcal{N}(G_1)}\left(R_{\psi_{K_1,t}(F_{P_1,k})}\right)$$

by Lemma 2.32 and Proposition 1.51 (6). We use a similar argument for the second term, and thus

$$\Delta_K^{(2)}(t) = \Delta_{K_1}^{(2)}(t)\Delta_{K_2}^{(2)}(t).$$

□

2.3.5 Cabling formula

Let C be a knot in S^3 and S denote its (p,q)-cable, where p,q are relatively prime integers. We prove that the L^2-Alexander invariant of S can be computed from the one of its companion C. This results mirrors the classical satellite formula for the Alexander polynomial (see for example [BZH14, Theorem 8.23]).

Lemma 2.34. *Let S be the (p,q)-cable of C, and let G_S, G_C be their respective groups. Then for all $t>0$ we have the commutative diagram*

$$\begin{array}{ccc} \mathbb{C}[G_C] & \xrightarrow{\psi_{C,t^p}} & \mathbb{C}[G_C] \\ \downarrow i_C & & \downarrow i_C \\ \mathbb{C}[G_S] & \xrightarrow{\psi_{S,t}} & \mathbb{C}[G_S] \end{array}$$

where $i_C\colon G_C \hookrightarrow G_S$ denotes both the group inclusion of Proposition 1.24 and its induction on the complex group algebras.

Proof. Let us take $P_C = \langle a_1,\dots,a_k | r_1,\dots,r_{k-1}\rangle$ and

$$P_S = \langle a_1,\dots,a_k,x,\lambda | r_1,\dots,r_{k-1}, x^p a_k^{-q}\lambda^{-p}, \lambda^{-1}W(a_i)\rangle$$

like in Proposition 1.23. Let $t>0$.

Proposition 1.23 (2) tells us that every a_i is sent to 1 by α_C as an element of G_C and is sent to p by α_S as an element of G_S.

Therefore the diagram is commutative for any $a_i \in \mathbb{C}[G_C]$ where a_i is a generator of P_C. The lemma follows from the fact that ψ_{C,t^p}, $\psi_{S,t}$ and i_C are algebra homomorphisms and that the a_i generate $\mathbb{C}[G_C]$. □

Lemma 2.35. *Let G be a discrete countable group, let $g\in G$ of infinite order, let p be a positive integer and let $t>0$. Then $Id + tR_g + \ldots + t^{(p-1)}R_{g^{p-1}}$ is injective and*

$$\det{}_{\mathcal{N}(G)}\left(Id + tR_g + \ldots + t^{(p-1)}R_{g^{p-1}}\right) = \max(1,t)^{p-1}.$$

Proof. Let $R = Id + tR_g + \ldots + t^{(p-1)}R_{g^{p-1}}$. We have $(Id - tR_g)\circ R = Id - t^pR_{g^p}$. By Proposition 1.53, $Id - t^pR_{g^p}$ is injective, therefore R is injective.

Both $Id - tR_g$ and R are injective, therefore, by Proposition 1.51 (4),

$$\det{}_{\mathcal{N}(G)}\left(Id - t^pR_{g^p}\right) = \det{}_{\mathcal{N}(G)}\left(Id - tR_g\right)\cdot \det{}_{\mathcal{N}(G)}\left(R\right).$$

Thus, by Proposition 1.53, $\max(1,t^p) = \max(1,t)\cdot \det_{\mathcal{N}(G)}(R)$ and the lemma follows. □

Theorem 2.36. *Let S be the (p,q)-cable knot of companion knot C, G_S, G_C their respective groups, and t any positive real number.*

If there exists P_w a Wirtinger presentation of G_C such that (P_w, t^p) has Property $\mathcal{I}$, then there is a presentation P_S of G_S such that (P_S, t) has Property $\mathcal{I}$, and

$$\Delta_S^{(2)}(t) \doteq \Delta_C^{(2)}(t^p)\cdot \max(1,t)^{(|p|-1)(|q|-1)} = \Delta_C^{(2)}(t^p)\Delta_{T(p,q)}^{(2)}(t).$$

Proof. Let $P_C = \langle a_1,\dots,a_k | r_1,\dots,r_{k-1}\rangle$ and

$$P_S = \langle a_1,\dots,a_k,x,\lambda | r_1,\dots,r_{k-1}, x^p a_k^{-q}\lambda^{-p}, \lambda^{-1}W(a_i)\rangle$$

be like in Proposition 1.23.

Observe that P_C is a Wirtinger presentation of G_C, as is P_w, therefore (P_C, t^p) also has Property $\mathcal{I}$, by Proposition 2.21.

Besides, P_S is a presentation of deficiency one, thus by Theorem 2.28, $\Delta_S^{(2)}(u)$ will be equal to $\Delta_{S,P_S}^{(2)}(u)$ for any $u > 0$ such that (P_S, u) has Property $\mathcal{I}$.

Recall from Proposition 1.23 (2) that $\alpha_S(a_i) = p, \alpha_S(x) = q$ and $\alpha_S(\lambda) = 0$.

The values of P_S and P_C imply that $R_{\psi_{S,t}(F_{P_S})}$ is written:

$$\begin{array}{c} \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \\ x \\ \lambda \end{array} \begin{array}{c} \begin{array}{ccc|c|c} r_1 & \dots & r_{k-1} & x^p a_k^{-q}\lambda^{-p} & \lambda W(a_i)^{-1} \end{array} \\ \left(\begin{array}{ccc|c|c} & & & 0 & * \\ & R_{\psi_{S,t}(i_C(F_{P_C,k}))} & & \vdots & \vdots \\ & & & 0 & * \\ \hline * & \dots & * & * & * \\ \hline 0 & \dots & 0 & T & 0 \\ \hline 0 & \dots & 0 & * & Id \end{array}\right) \end{array}$$

where $T = Id + t^q R_x + \ldots + t^{q(p-1)} R_{x^{p-1}}$ if p is positive, and

$$T = -t^{-q} R_{x^{-1}} - \ldots - t^{-q|p|} R_{x^p} = \left(-t^{-q|p|} R_{x^p}\right) \circ (Id + t^q R_x + \ldots + t^{q(|p|-1)} R_{x^{|p|-1}})$$

if p is negative. In both cases T is injective, by Lemma 2.35 and the fact that $\left(-t^{-q|p|} R_{x^p}\right)$ is invertible.

We know (P_C, t^p) has Property $\mathcal{I}$, thus $R_{\psi_{C,t^p}(F_{P_C,k})}$ is injective, by Remark 2.24. We have the injective group homomorphism $i_C \colon G_C \hookrightarrow G_S$ by Proposition 1.24. Therefore, by Proposition 1.45 (1), Lemma 2.34 and Proposition 1.45 (2), in this order,

$$(i_C)_*(R_{\psi_{C,t^p}(F_{P_C,k})}) = R_{i_C(\psi_{C,t^p}(F_{P_C,k}))} = R_{\psi_{S,t}(i_C(F_{P_C,k}))}$$

is injective.

Finally $Id_{\ell^2(G)}$ is clearly injective.

Thus the block trigonal square matrix $R_{\psi_{S,t}(F_{P_S,k})}$ is injective, hence,by Remark 2.24, (P_S, t) has Property $\mathcal{I}$. Therefore, by Proposition 1.51 (5) and (2),

$$\det{}_{\mathcal{N}(G_S)}\left(R_{\psi_{S,t}(F_{P_S,k})}\right) = \det{}_{\mathcal{N}(G_S)}\left(R_{\psi_{S,t}(i_C(F_{P_C,k}))}\right) \cdot \det{}_{\mathcal{N}(G_S)}(T).$$

However we have

$$\det{}_{\mathcal{N}(G_S)}\left(R_{\psi_{S,t}(i_C(F_{P_C,k}))}\right) = \det{}_{\mathcal{N}(G_S)}\left((i_C)_*(R_{\psi_{C,t^p}(F_{P_C,k})})\right) = \det{}_{\mathcal{N}(G_C)}\left(R_{\psi_{C,t^p}(F_{P_C,k})}\right)$$

by Lemma 2.34 and Proposition 1.51 (6).

Besides, from Lemma 2.35, we have

$$\det{}_{\mathcal{N}(G_S)}\left(Id + t^q R_x + \ldots + t^{q(|p|-1)} R_{x^{|p|-1}}\right) = \max(1, t^q)^{|p|-1},$$

therefore, by the fact that $\det_{\mathcal{N}(G_S)}\left(-t^{-q|p|} R_{x^p}\right) \in t^{\mathbb{Z}}$ and Proposition 1.51 (4), $\det_{\mathcal{N}(G_S)}(T)$ is equal to $\max(1, t^q)^{|p|-1}$ up to $t^{\mathbb{Z}}$.

Note that $\max(1, t^q)^{|p|-1} = \max(1, t)^{|q|(|p|-1)}$ up to $t^{\mathbb{Z}}$ from Remark 1.54.

Finally, Theorem 2.28 tells us that

$$\Delta_S^{(2)}(t) \doteq \frac{\det{}_{\mathcal{N}(G_S)}(R_{\psi_{S,t}(F_{P_S,k})})}{\max(1, t)^{|\alpha_S(a_k)|-1}} = \frac{\det{}_{\mathcal{N}(G_C)}\left(R_{\psi_{C,t^p}(F_{P_C,k})}\right) \cdot \max(1, t)^{|q|(|p|-1)}}{\max(1, t)^{|p|-1}}.$$

Thus we have proven the formula

$$\Delta_S^{(2)}(t) \doteq \Delta_C^{(2)}(t^p) \cdot \max(1,t)^{(|p|-1)(|q|-1)}.$$

□

Corollary 2.37. *Let K be a knot, $-K$ its inverse knot, and P and P_- Wirtinger presentations of their respective groups. Then for all positive real numbers t, (P,t) has Property $\mathcal{I}$ if and only if (P_-, t^{-1}) has Property $\mathcal{I}$, and in this case*

$$\Delta_{-K}^{(2)}(t^{-1}) \doteq \Delta_K^{(2)}(t).$$

Proof. Observe that $-K$ is a $(-1,0)$-cable of K, and apply Theorem 2.36. □

2.3.6 The class of iterated torus knots

Let us call $\mathcal{IT}$ the class of iterated torus knots, i.e. the class of knots in S^3 generated by the trivial knot, the connected sum operation, and all cabling operations.

Theorem 2.38. *Let K be a knot in $\mathcal{IT}$, and P a Wirtinger presentation of the group G_K of K. Then for all $t > 0$, (P,t) has Property $\mathcal{I}$ and*

$$\left(t \mapsto \Delta_K^{(2)}(t)\right) \doteq (t \mapsto \max(1,t)^{n_K})$$

where $n_K = 2g(K)$.

Proof.
1. From Example 2.20, the result is true for the unknot O for which $n_O = 0 = 2g(O)$.
2. If the result is true for K_1 and K_2 in $\mathcal{IT}$, then, using Theorem 2.33,

$$\left(t \mapsto \Delta_{K_1\sharp K_2}^{(2)}(t)\right) \doteq (t \mapsto \max(1,t)^{n_{K_1\sharp K_2}})$$

   where $n_{K_1\sharp K_2} = n_{K_1} + n_{K_2}$. Besides, $g(K) = g(K_1) + g(K_2)$, see for example [Rol90, Theorem 5.14], thus the result holds for $K_1\sharp K_2$.
3. If the result is true for $C \in \mathcal{IT}$ and S is the (p,q)-cable of C, then

$$\left(t \mapsto \Delta_S^{(2)}(t)\right) \doteq (t \mapsto \max(1,t)^{n_S})$$

   where $n_S = |p| \cdot n_C + (|p|-1)(|q|-1)$, by Theorem 2.36. Furthermore, it follows from the main formula of [Shi89] that $g(S) = |p|g(C) + \dfrac{(|p|-1)(|q|-1)}{2}$, therefore the result is proven for S.

We have proven the theorem by induction on the class $\mathcal{IT}$. □

From [MM01, Lemma 5.5], $\mathcal{IT}$ is exactly the class of knots whose exterior has zero simplicial volume, i.e. the knots whose exterior has no hyperbolic JSJ pieces (see Section 1.2.3).

Remark 2.39. Theorem 2.38 can be restated as follows.
For a knot K, the following properties are equivalent:

- $\forall t > 0, \Delta_K^{(2)}(t) = \max(1,t)^{2g(K)}$;

- $\exists n \in \mathbb{Z}, \forall t > 0, \Delta_K^{(2)}(t) = \max(1,t)^n$;
- $\Delta_K^{(2)}(1) = 1$;
- $\mathrm{vol}(K) = 0$;
- $K \in \mathcal{IT}$.

2.4 Detection of the unknot

Let us state the main result of this chapter:

Theorem 2.40. *Let K be a knot in S^3. The L^2-Alexander invariant of K is trivial, i.e. $\left(t \mapsto \Delta_K^{(2)}(t)\right) \doteq (t \mapsto 1)$, if and only if K is the trivial knot.*

Proof. First, let K_0 be an arbitrary knot. If the exterior of K_0 has hyperbolic pieces in its JSJ decomposition, then $\Delta_{K_0}^{(2)}(1) \neq 1$, by Theorem 2.27. Therefore, let us assume $\widetilde{K}$ is a knot whose exterior does not have hyperbolic pieces and such that $\Delta_{\widetilde{K}}^{(2)} = (t \mapsto 1)$. Let us prove that $\widetilde{K}$ is the unknot.

From [MM01, Lemma 5.5] we know that $\widetilde{K} \in \mathcal{IT}$. Thus, by Theorem 2.38,

$$\Delta_{\widetilde{K}}^{(2)} \doteq (t \mapsto \max(1,t)^n)$$

where $n = 2g(\widetilde{K})$. Therefore $\widetilde{K}$ has genus zero, and is thus the unknot.

Thus, if $\widetilde{K}$ is a knot whose exterior does not have hyperbolic pieces and such that $\Delta_{\widetilde{K}}^{(2)} = (t \mapsto 1)$, then $\widetilde{K}$ is the unknot. The theorem follows.

□

2.5 Detection of the trefoils

Since a knot and its mirror image have the same L^2-Alexander invariant, this invariant cannot distinguish the two trefoils from one another. However, it can single them out from the other knots.

Theorem 2.41. *Let K be a knot in S^3. Its L^2-Alexander invariant satisfies*

$$\left(t \mapsto \Delta_K^{(2)}(t)\right) \doteq (t \mapsto \max(1,t)^2)$$

if and only if K is the left or right trefoil knot.

Proof. It follows from Theorem 2.28(2) that the L^2-Alexander invariant of the two trefoils is equal to $(t \mapsto \max(1,t)^2)$.

Now let K be a knot in S^3 such that $\left(t \mapsto \Delta_K^{(2)}(t)\right) \doteq (t \mapsto \max(1,t)^2)$.

It follows from Remark 2.39 that $K \in \mathcal{IT}$ and its genus $g(K)$ is equal to 1. Since the genus is additive under connected sum and $g(K) = 1$, K is a prime knot. Thus K is a (p,q)-cable on a (possibly trivial) knot C.

By [Shi89], $1 = g(K) = |p|g(C) + \dfrac{(|p|-1)(|q|-1)}{2}$. Two cases are possible:

- if $g(C) = 0$, then (p,q) is equal to $(\pm 2, \pm 3)$ or $(\pm 3, \pm 2)$, thus K is either the torus knot $T(2,3)$ or the torus knot $T(2,-3)$, i.e. a right or left trefoil knot.

- if $g(C) = 1$, then $p = \pm 1$ and $C = \pm K$, thus the cabling operation is trivial.

□

Remark 2.42. The L^2-Alexander invariant cannot single out any torus knot of genus bigger than the trefoils. For example, the knots having L^2-Alexander invariant equal to

$$\left(t \mapsto \max(1,t)^4\right)$$

are the iterated torus knots of genus 2

$$T(2,\pm 5),\ T(2,\pm 3)\sharp T(2,\pm 3),\ S_{(2,\pm 1)}(T(2,\pm 3)).$$

2.6 General L^2-Alexander torsions for knots and links

To conclude this chapter we will extend the L^2-Alexander invariant for knots to more general L^2-Alexander torsions for knot and link exteriors, as in Section 2.2.2.

2.6.1 A twisted L^2-Alexander invariant for knots

If $(M_K, \alpha_K, \gamma\colon G_K \to G)$ is an admissible triple, we can define $\Delta^{(2)}_{K,\gamma}$ as a class of maps from a subset $\mathcal{D}_{K,\gamma}$ of $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geqslant 0}$ up to multiplication by the maps $(t \mapsto t^m), m \in \mathbb{Z}$, in the two equivalent ways:

1. $\Delta^{(2)}_{K,\gamma}(t) \doteq \det_{\mathcal{N}(G)}\left(R_{\kappa(G_K,\alpha_K,\gamma,t)(F_{P,1})}\right)$ for P a Wirtinger presentation of G_K, if the operator $R_{\kappa(G_K,\alpha_K,\gamma,t)(F_{P,1})}$ is injective.
2. $\Delta^{(2)}_{K,\gamma}(t) \doteq T^{(2)}(M_K,\alpha_K,\gamma)(t)\cdot \max(1,t)$ if $C^{(2)}_*(M_K,\alpha_K,\gamma,t)$ is weakly acyclic and of determinant class.

The equivalence between these two points of view comes from the same arguments as in Section 2.2.2.

In particular, if γ is the abelianization α_K, we can compute the L^2-Alexander invariant with coefficient α_K from the value of the Alexander polynomial:

Proposition 2.43. *([DFL14, Proposition 7.2]) Let K be a knot and $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ be a representative of the Alexander polynomial of K. We write*

$$\Delta_K(z) = C\cdot z^m \cdot \prod_{i=1}^{k}(z-a_i)$$

where $C \in \mathbb{Z}^*, m \in \mathbb{Z}, a_1,\dots,a_k \in \mathbb{C}^*$. Then $C^{(2)}_*(M_K,\alpha_K,\alpha_K,t)$ is weakly acyclic and of determinant class for all $t > 0$, and

$$\Delta^{(2)}_{K,\alpha_K}(t) \doteq |C|\cdot \prod_{i=1}^{k}\max(|a_i|,t).$$

For a given knot K, many twisted L^2-Alexander invariants $\Delta^{(2)}_{K,\gamma}(t)$ may exist. We can sort them by the groups G such that $G_K \twoheadrightarrow G \twoheadrightarrow \mathbb{Z}$, i.e. the quotients of G_K by normal subgroups of G_K contained in the commutator subgroup G'_K.

We **cannot** hope for a relation between these invariants of the magnitude of:

$$\left(G_K \overset{\gamma}{\twoheadrightarrow} G \overset{\gamma'}{\twoheadrightarrow} G' \twoheadrightarrow \mathbb{Z}\right) \Longrightarrow \left(\Delta^{(2)}_K(t) \geqslant \Delta^{(2)}_{K,\gamma}(t) \geqslant \Delta^{(2)}_{K,\gamma'\circ\gamma}(t) \geqslant \Delta^{(2)}_{K,\alpha_K}(t)\right),$$

since there are knots K with $\Delta^{(2)}_{K,\alpha_K}(1) < \Delta^{(2)}_K(1)$, like Whitehead doubles of an hyperbolic knot, and knots K with $\Delta^{(2)}_{K,\alpha_K}(1) > \Delta^{(2)}_K(1)$, like the figure-eight knot (see Annex B).

2.6.2 The L^2-Alexander torsion for link exteriors

Let $L = L_1 \cup \ldots \cup L_c$ be a c-link in S^3, $M_L = S^3 \setminus V(L)$ its exterior and $G_L = \pi_1(M_L)$ its group.

Let $\alpha_L \colon G_L \to \mathbb{Z}^c$ denote the abelianization that sends homotopy classes of meridian curves of the component L_i to $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is at the i-th place.

Any group homomorphism $\phi \colon G_L \to \mathbb{Z}$ factors through the abelianization α_L, therefore depends only on the linear map $\mathbb{Z}^c \to \mathbb{Z}$, which will be denoted by its natural matrix $(n_1, \ldots, n_c)$ where the n_i are in $\mathbb{Z}$.

The following theorem generalises Corollary 2.37.

Theorem 2.44. *(Reversing the orientation of a component of a link)*

Let $L = L_1 \cup \ldots \cup L_c$ and $L' = L_1 \cup \ldots L_{i-1} \cup (-L_i) \cup L_{i+1} \cup \ldots \cup L_c$ the link obtained by reversing the orientation on the i-th component of L. Then G_L and $G_{L'}$ are equal and $C_*^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(M_{L'}, (n_1, \ldots, n_{i-1}, -n_i, n_{i+1}, \ldots n_c) \circ \alpha_{L'}, \gamma, t)$ is, and in this case:

$$T^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t) \doteq T^{(2)}(M_{L'}, (n_1, \ldots, n_{i-1}, -n_i, n_{i+1}, \ldots n_c) \circ \alpha_{L'}, \gamma)(t).$$

Proof. The manifolds M_L and $M_{L'}$ are equal as oriented 3-manifolds, the difference is in the orientation of meridian curves, which depends on the orientation of the links L and L'. Therefore G_L and $G_{L'}$ are equal as groups of homotopy classes of loops in the *same* topological spaces, but α_L and $\alpha_{L'}$ are not the same group homomorphisms from $G_L = G_{L'}$ to $\mathbb{Z}^c$.

If we call $\mu_j \in G_L$ the homotopy class of a meridian curve of L_j for all $j \in \{1, \ldots, c\}$, then $\alpha_L(\mu_j) = \alpha_{L'}(\mu_j)$ for all $j \neq i$ and $\alpha_L(\mu_i) = 1 = -\alpha_{L'}(\mu_i)$.

Therefore, for all integers $n_1, \ldots, n_c$,

$$((n_1, \ldots, n_c) \circ \alpha_L) = ((n_1, \ldots, n_{i-1}, -n_i, n_{i+1}, \ldots n_c) \circ \alpha_{L'})$$

as group homomorphisms from G_L to $\mathbb{Z}$. The result follows. □

Theorem 2.45. *(Reordering the components of a link)*

Let $L = L_1 \cup \ldots \cup L_c$ and $L' = L_{\sigma(1)} \cup \ldots \cup L_{\sigma(c)}$ the link obtained by re-ordering the components of L by a permutation σ. Then G_L and $G_{L'}$ are equal and $C_*^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(M_{L'}, (n_{\sigma(1)}, \ldots, n_{\sigma(c)}) \circ \alpha_{L'}, \gamma, t)$ is, and in this case:

$$T^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t) \doteq T^{(2)}(M_{L'}, (n_{\sigma(1)}, \ldots, n_{\sigma(c)}) \circ \alpha_{L'}, \gamma)(t).$$

Proof. The manifolds M_L and $M_{L'}$ are equal as oriented 3-manifolds, and G_L and $G_{L'}$ are equal as groups of homotopy classes of loops in the same topological spaces, but α_L and $\alpha_{L'}$ are not the same group homomorphisms from $G_L = G_{L'}$ to $\mathbb{Z}^c$, because they depend on the ordering of the components of the link.

If $\mu \in G_L$ is the homotopy class of a meridian curve of the component L_i, then $\alpha_L(\mu) = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in i-th place; in L' the component L_i is now the $\sigma^{-1}(i)$-th one, and thus $\alpha_{L'}(\mu) = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in $\sigma^{-1}(i)$-th place.

Therefore, for all integers $n_1, \ldots, n_c$,

$$((n_1, \ldots, n_c) \circ \alpha_L)(\mu) = n_i = ((n_{\sigma(1)}, \ldots, n_{\sigma(c)}) \circ \alpha_{L'})(\mu).$$

Since this is true for every i, we have

$$((n_1, \ldots, n_c) \circ \alpha_L) = ((n_{\sigma(1)}, \ldots, n_{\sigma(c)}) \circ \alpha_{L'})$$

as group homomorphisms from G_L to $\mathbb{Z}$. The result follows. □

The following theorem generalises Theorem 2.30.

Theorem 2.46. *(Mirror image of a link)*
Let $L = L_1 \cup \ldots \cup L_c$ and L^* the mirror image of L. Then there is a natural group isomorphism $\kappa\colon G_L \to G_{L^*}$ and for all $t > 0$, $C_*^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(M_{L^*}, (-n_1, \ldots, -n_c) \circ \alpha_{L^*}, \gamma \circ (\kappa^{-1}), t)$ is, and in this case

$$T^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t) \doteq T^{(2)}\left(M_{L^*}, (-n_1, \ldots, -n_c) \circ \alpha_{L^*}, \gamma \circ (\kappa^{-1})\right)(t).$$

In particular

$$T^{(2)}(M_L, (n_1, \ldots, n_c) \circ \alpha_L, id)(t) \doteq T^{(2)}(M_{L^*}, (-n_1, \ldots, -n_c) \circ \alpha_{L^*}, id))(t).$$

Proof. The manifolds M_L and M_{L^*} are homeomorphic as unoriented 3-manifolds by the restriction of a planar reflection p in $\mathbb{R}^3$. Therefore there is a natural group isomorphism $\kappa = p_*\colon G_L \to G_{L'}$ such that $\alpha_{L^*} \circ \kappa = -\alpha_L$. Indeed, L and L^* are seen as links in S^3 with S^3 having the same orientation, but the planar reflection p sends a meridian curve of L to its mirror image, which reverses its orientation as a curve in the exterior of a link. The first part of the theorem follows from Theorem 2.12.

The second formula follows from Proposition 2.5, since κ^{-1} is injective. □

Remark 2.47. As a consequence of Proposition 2.7 and Proposition 2.8, the L^2-Alexander torsion cannot distinguish a link from its mirror image.

Remark 2.48. If L is a split link, then $M_L = S^3 \setminus V(L)$ is not irreducible, and the link group $G_L = \pi_1(M_L)$ has deficiency at least two. The manifold M_L is not aspherical (see [Rol90, Exercise 4E4]) therefore we cannot use Section 2.1.3 to compute its L^2-Alexander torsions from group presentations of G_L.

This is why we will exclude the cases of split link exteriors.

Chapter 3

Surgery formulas

In this chapter we prove a Mayer-Vietoris formula for the L^2-Alexander torsion. As a consequence, we deduce a general Dehn Surgery formula. We then review three particular cases of Dehn surgeries on exteriors of links in S^3: forgetting one component of the link (∞-surgery), filling the exterior of the Whitehead link to obtain the exteriors of the twist knots ($1/n$-surgeries) and finally 0-surgery that involves considering exteriors of links in $S^2 \times S^1$.

3.1 Mayer-Vietoris formula for the L^2-Alexander torsion

We prove a Mayer-Vietoris type formula for the L^2-Alexander torsions.

Let X, A, B, V be compact connected topological spaces, such that $X = A \cup B$ and $V = A \cap B$. Assume that these four spaces are endowed with structures of finite CW-complexes such that the inclusions $V \overset{I_A}{\hookrightarrow} A$, $V \overset{I_B}{\hookrightarrow} B$, $A \overset{J_A}{\hookrightarrow} X$, $B \overset{J_B}{\hookrightarrow} X$ and $V \overset{I}{\hookrightarrow} X$ all map a k-cell to a k-cell (which means that the CW-structure of X is constructed from those of A and B), and such that $I = J_A \circ I_A = J_B \circ I_B$. Let P be a base point in V.

Let $P_A = I_A(P)$, $P_B = I_B(P)$, $Q = I(P)$ denote the base points of A, B and X.

We call $\pi_V \overset{i_A}{\to} \pi_A$, $\pi_V \overset{i_B}{\to} \pi_B$, $\pi_A \overset{j_A}{\to} \pi_X$, $\pi_B \overset{j_B}{\to} \pi_X$ and $\pi_V \overset{i}{\to} \pi_X$ the group homomorphisms induced by I_A, I_B, J_A, J_B, I. Remark that $i = j_A \circ i_A = j_B \circ i_B$.

These numerous maps are all written on a diagram below for clarity.

$$\begin{tikzcd}
 & A \arrow[dr, "J_A"] & \\
V \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & X \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_A"] & & & \\
\pi_1(V) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(X) \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\
 & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}
\end{tikzcd}$$

Theorem 3.1. *Let $\phi\colon \pi_X \longrightarrow \mathbb{Z}$ and $\gamma\colon \pi_X \to G$ such that (π_X, ϕ, γ) is an admissible triple. Let $t > 0$. If the three $\mathcal{N}(G)$-cellular chain complexes*

$$C_*^{(2)}(V, \phi \circ i, \gamma \circ i, t),\ C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t),\ C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t)$$

are weakly acyclic and of determinant class, then $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class as well, and

$$T^{(2)}(X, \phi, \gamma)(t) \cdot T^{(2)}(V, \phi \circ i, \gamma \circ i)(t) = T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A)(t) \cdot T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B)(t).$$

Proof. Let

- e_i^k be the cells of V,
- $\epsilon_i^k = I_A(e_i^k)$ and $\alpha_i^k \in A \setminus I_A(V)$ be the cells of A,
- $\eta_i^k = I_B(e_i^k)$ and $\beta_i^k \in B \setminus I_B(V)$ be the cells of B,
- $E_i^k = I(e_i^k)$, $A_i^k = J_A(\alpha_i^k) \in J_A(A) \setminus I(V)$, $B_i^k = J_B(\beta_i^k) \in J_B(B) \setminus I(V)$ be the cells of X.

Let $\widetilde{e}_i^k$ be choices of lifts of the e_i^k in $\widetilde{V}$. Let $\widetilde{\epsilon}_i^k = \widetilde{I_A}(\widetilde{e}_i^k)$ be the corresponding lifts of ϵ_i^k in $\widetilde{A}$, $\widetilde{\eta}_i^k = \widetilde{I_B}(\widetilde{e}_i^k)$ the corresponding lifts of η_i^k in $\widetilde{B}$, and $\widetilde{E}_i^k = \widetilde{I}(\widetilde{e}_i^k)$ the corresponding lifts of E_i^k in $\widetilde{X}$. Choose some lifts $\widetilde{\alpha}_i^k$ of α_i^k in $\widetilde{A}$, some lifts $\widetilde{\beta}_i^k$ of β_i^k in $\widetilde{B}$, and let $\widetilde{A}_i^k = \widetilde{J_A}(\widetilde{\alpha}_i^k)$ and $\widetilde{B}_i^k = \widetilde{J_B}(\widetilde{\beta}_i^k)$ be the corresponding lifts of A_i^k and B_i^k in $\widetilde{X}$.

Consequently the maps $\widetilde{I_A}$, $\widetilde{I_B}$, $\widetilde{J_A}$, $\widetilde{J_B}$ and $\widetilde{I}$ all send a k-cell to a k-cell and commute with the boundary operators and the actions of the fundamental groups on the universal covers. We draw them on the following diagram for the reader's convenience.

$$\begin{tikzcd}
 & \widetilde{A} \arrow[dr, "\widetilde{J_A}"] & \\
\widetilde{V} \arrow[ur, "\widetilde{I_A}"] \arrow[rr, "\widetilde{I}"] \arrow[dr, "\widetilde{I_B}"'] & & \widetilde{X} \\
 & \widetilde{B} \arrow[ur, "\widetilde{J_B}"'] &
\end{tikzcd}$$

The cellular chain complexes of the four universal covers are written:

$$C_*(\widetilde{V}, \mathbb{Z}) = \dots \xrightarrow{\partial_{V,k+1}} \bigoplus_i \mathbb{Z}[\pi_V]\widetilde{e}_i^k \xrightarrow{\partial_{V,k}} \dots$$

$$C_*(\widetilde{A}, \mathbb{Z}) = \dots \xrightarrow{\partial_{A,k+1}} \bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\epsilon}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\alpha}_i^k \xrightarrow{\partial_{A,k}} \dots$$

$$C_*(\widetilde{B}, \mathbb{Z}) = \dots \xrightarrow{\partial_{B,k+1}} \bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\eta}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\beta}_i^k \xrightarrow{\partial_{B,k}} \dots$$

$$C_*(\widetilde{X}, \mathbb{Z}) = \dots \xrightarrow{\partial_{X,k+1}} \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{E}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{A}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{B}_i^k \xrightarrow{\partial_{X,k}} \dots$$

The boundary maps of these four cellular chain complexes are of the following forms:

$$\partial_{V,k}\colon \left(\bigoplus_i \mathbb{Z}[\pi_V]\widetilde{e}_i^k \longrightarrow \bigoplus_i \mathbb{Z}[\pi_V]\widetilde{e}_i^{k-1} \right)$$

$$\partial_{A,k}\colon \left(\bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\epsilon}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\alpha}_i^k \longrightarrow \bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\epsilon}_i^{k-1} \oplus \bigoplus_i \mathbb{Z}[\pi_A]\widetilde{\alpha}_i^{k-1} \right)$$

$$\partial_{A,k} = \begin{pmatrix} i_A(\partial_{V,k}) & U_A \\ 0 & D_A \end{pmatrix}$$

$$\partial_{B,k}\colon \left(\bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\eta}_i^k \oplus \bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\beta}_i^k \longrightarrow \bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\eta}_i^{k-1} \oplus \bigoplus_i \mathbb{Z}[\pi_B]\widetilde{\beta}_i^{k-1} \right)$$

$$\partial_{B,k} = \begin{pmatrix} i_B(\partial_{V,k}) & U_B \\ 0 & D_B \end{pmatrix}$$

$$\partial_{X,k}\colon \begin{pmatrix} \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{E}_i^k & & \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{E}_i^{k-1} \\ \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{A}_i^k & \longrightarrow & \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{A}_i^{k-1} \\ \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{B}_i^k & & \oplus \bigoplus_i \mathbb{Z}[\pi_X]\widetilde{B}_i^k \end{pmatrix}$$

$$\partial_{X,k} = \begin{pmatrix} i(\partial_{V,k}) & j_A(U_A) & j_B(U_B) \\ 0 & j_A(D_A) & 0 \\ 0 & 0 & j_B(D_B) \end{pmatrix}.$$

We observe that these boundary operators have block trigonal matricial forms with immediate correspondences, as in Section 1.1.3. Of course this follows from the compatible choices of cellular bases we made for the four cellular chain complexes.

Now let us look at the $\mathcal{N}(G)$-cellular chain complexes twisted by the actions of ϕ, γ and t. As in Section 2.1.1, we write

- $\ell^2(G)\widetilde{e}$ for $\ell^2(G) \otimes_{\kappa(\pi_V,\phi\circ i,\gamma\circ i,t)} \mathbb{Z}[\pi_V]\widetilde{e}$,
- $\partial^{(2)}_{V,k}$ for $R_{\kappa(\pi_V,\phi\circ i,\gamma\circ i,t)(\partial_{V,k})}$.

We compute the four $\mathcal{N}(G)$-cellular chain complexes V_*, A_*, B_* and C_* associated to V, A, B and X and also the sum $A_* \oplus B_*$ since, as we will see, $(V_*, A_* \oplus B_*, C_*)$ forms an exact sequence. The five $\mathcal{N}(G)$-cellular chain complexes and their boundary operators are of the following forms:

$$V_* = C^{(2)}_*(V, \phi\circ i, \gamma\circ i, t) = \dots \xrightarrow{\partial^{(2)}_{V,k+1}} \bigoplus_i \ell^2(G)\widetilde{e}_i^k \xrightarrow{\partial^{(2)}_{V,k}} \dots$$

$$\partial^{(2)}_{V,k} = R_{\kappa(\pi_V,\phi\circ i,\gamma\circ i,t)(\partial_{V,k})} = R,$$

$$A_* = C^{(2)}_*(A, \phi\circ j_A, \gamma\circ j_A, t) = \dots \xrightarrow{\partial^{(2)}_{A,k+1}} \bigoplus_i \ell^2(G)\widetilde{\epsilon}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{\alpha}_i^k \xrightarrow{\partial^{(2)}_{A,k}} \dots$$

$$\begin{aligned} \partial^{(2)}_{A,k} = R_{\kappa(\pi_A,\phi\circ j_A,\gamma\circ j_A,t)(\partial_{A,k})} &= \begin{pmatrix} R_{\kappa(\pi_A,\phi\circ j_A,\gamma\circ j_A,t)(i_A(\partial_{V,k}))} & R_{\kappa(\pi_A,\phi\circ j_A,\gamma\circ j_A,t)(U_A)} \\ 0 & R_{\kappa(\pi_A,\phi\circ j_A,\gamma\circ j_A,t)(D_A)} \end{pmatrix} \\ &= \begin{pmatrix} R & R_{UA} \\ 0 & R_{DA} \end{pmatrix}, \end{aligned}$$

$$B_* = C^{(2)}_*(B, \phi\circ j_B, \gamma\circ j_B, t) = \dots \xrightarrow{\partial^{(2)}_{B,k+1}} \bigoplus_i \ell^2(G)\widetilde{\eta}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{\beta}_i^k \xrightarrow{\partial^{(2)}_{B,k}} \dots$$

$$\begin{aligned} \partial^{(2)}_{B,k} = R_{\kappa(\pi_B,\phi\circ j_B,\gamma\circ j_B,t)(\partial_{B,k})} &= \begin{pmatrix} R_{\kappa(\pi_B,\phi\circ j_B,\gamma\circ j_B,t)(i_B(\partial_{V,k}))} & R_{\kappa(\pi_B,\phi\circ j_B,\gamma\circ j_B,t)(U_B)} \\ 0 & R_{\kappa(\pi_B,\phi\circ j_B,\gamma\circ j_B,t)(D_B)} \end{pmatrix} \\ &= \begin{pmatrix} R & R_{UB} \\ 0 & R_{DB} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} A_* \oplus B_* &= C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t) \oplus C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t) \\ &= \dots \xrightarrow{\partial^{(2)}_{A \sqcup B, k+1}} \bigoplus_i \ell^2(G)\widetilde{\epsilon}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{\alpha}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{\eta}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{\beta}_i^k \xrightarrow{\partial^{(2)}_{A \sqcup B, k}} \dots \end{aligned}$$

$$\partial^{(2)}_{A \sqcup B, k} = \begin{pmatrix} \partial^{(2)}_{A,k} & 0 \\ 0 & \partial^{(2)}_{B,k} \end{pmatrix} = \begin{pmatrix} R & R_{UA} & 0 & 0 \\ 0 & R_{DA} & 0 & 0 \\ 0 & 0 & R & R_{UB} \\ 0 & 0 & 0 & R_{DB} \end{pmatrix},$$

$$C_* = C_*^{(2)}(X, \phi, \gamma, t) = \dots \xrightarrow{\partial^{(2)}_{X,k+1}} \bigoplus_i \ell^2(G)\widetilde{E}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{A}_i^k \oplus \bigoplus_i \ell^2(G)\widetilde{B}_i^k \xrightarrow{\partial^{(2)}_{X,k}} \dots$$

$$\begin{aligned} \partial^{(2)}_{X,k} = R_{\kappa(\pi_X, \phi, \gamma, t)(\partial_{X,k})} &= \begin{pmatrix} R_{\kappa(\pi_X, \phi, \gamma, t)(i(\partial_{V,k}))} & R_{\kappa(\pi_X, \phi, \gamma, t)(j_A(U_A))} & R_{\kappa(\pi_X, \phi, \gamma, t)(j_B(U_B))} \\ 0 & R_{\kappa(\pi_X, \phi, \gamma, t)(j_A(D_A))} & 0 \\ 0 & 0 & R_{\kappa(\pi_X, \phi, \gamma, t)(j_B(D_B))} \end{pmatrix} \\ &= \begin{pmatrix} R & R_{UA} & R_{UB} \\ 0 & R_{DA} & 0 \\ 0 & 0 & R_{DB} \end{pmatrix}. \end{aligned}$$

From the forms of these $\mathcal{N}(G)$-cellular chain complexes, we deduce that there is an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to V_* \xrightarrow{\iota_*} A_* \oplus B_* \xrightarrow{\rho_*} C_* \to 0$$

where $\iota_k = \begin{pmatrix} -Id \\ 0 \\ Id \\ 0 \end{pmatrix}$ and $\rho_k = \begin{pmatrix} Id & 0 & Id & 0 \\ 0 & Id & 0 & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}$ for every k (the size of each square submatrice Id depends of the number of cells of each kind at each k, and the zero submatrices may not be square). The exactness is immediate.

Moreover, if we let n_k denote the number of cells e_i^k, then it follows from Proposition 1.51 (2), (3) and (7) that

$$\det{}_{\mathcal{N}(G)}(\iota_k) = \sqrt{\det{}_{\mathcal{N}(G)}(\iota_k^* \iota_k)} = \sqrt{2^{n_k}} = \sqrt{\det{}_{\mathcal{N}(G)}(\rho_k \rho_k^*)} = \det{}_{\mathcal{N}(G)}(\rho_k),$$

therefore the L^2-torsion $T^{(2)}(V_*, A_* \oplus B_*, C_*)$ of the exact sequence $(V_*, A_* \oplus B_*, C_*)$ (defined in [Lüc02b, (3.34)]) is equal to 1.

We assumed that $A_* = C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t)$ and $B_* = C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t)$ were weakly acyclic and of determinant class, therefore $A_* \oplus B_*$ is as well.

Since $V_* = C_*^{(2)}(V, \phi \circ i, \gamma \circ i, t)$ and $A_* \oplus B_*$ are weakly acyclic, the long homology sequence

$$LHS_* = LHS_*(V_*, A_* \oplus B_*, C_*)$$

(defined in [Lüc02b, Theorem 1.21]) is trivial and thus C_* is weakly acyclic as well. Furthermore, as a trivial finite $\mathcal{N}(G)$-Hilbert chain complex, LHS_* is of determinant class and $T^{(2)}(LHS_*) = 1$.

We assumed that V_* was of determinant class, and we proved that $A_* \oplus B_*$ and LHS_* were of determinant class as well. It follows from the multiplicativity of the L^2-torsion (see [Lüc02b, Theorem 3.35 (1)]) that C_* is of determinant class and

$$T^{(2)}(C_*) \cdot T^{(2)}(V_*) \cdot T^{(2)}(LHS_*) = T^{(2)}(A_* \oplus B_*).$$

Since $T^{(2)}(LHS_*) = 1$, and since $T^{(2)}(A_* \oplus B_*) = T^{(2)}(A_*) \cdot T^{(2)}(B_*)$ by Proposition 1.51 (3), we conclude that

$$T^{(2)}(X, \phi, \gamma)(t) \cdot T^{(2)}(V, \phi \circ i, \gamma \circ i)(t) \doteq T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t) \cdot T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t).$$

□

3.2 General formula for Dehn surgery

We are going to apply the Mayer-Vietoris formula we just proved to the case of Dehn surgery, where we glue a solid torus on a toroidal boundary component of a 3-manifold. First we recall the definition of Dehn surgery.

3.2.1 Dehn Surgery

We follow [Rol90, Section 9F].

Let M be a 3-manifold and let $T_1, \ldots, T_n$ be 2-tori that are connected components of ∂M. For each $i = 1, \ldots, n$, specify a simple closed curve J_i on each T_i. Let

$$M' = M \cup_h \Big((S^1 \times D^2) \sqcup \ldots \sqcup (S^1 \times D^2) \Big)$$

where h is an union of homeomorphisms $h_i \colon S^1 \times S^1 \to T_i$, each of which take a meridian curve m_i of $\partial(S^1 \times D^2)$ to the curve J_i.

Up to homeomorphism M' does not depend on the choice of h. We say that M' is obtained by *Dehn Filling on M*.

Dehn surgery refers to the more general process of drilling out links in M and then filling them along certain curves. We will mostly be interested in the Dehn filling process.

When M is the exterior of an oriented link $L = L_1 \cup \ldots \cup \ldots L_n \cup L_{n+1} \cup \ldots \cup L_c$ in S^3 and $T_i = \partial V(L_i)$ for $i = 1, \ldots, n$, each L_i has a preferred meridian-longitude pair (μ_i, λ_i). We only need to specify the homotopy class of J_i in T_i, described by two relatively prime integers p_i, q_i:

$$[J_i] = p_i[\mu_i] + q_i[\lambda_i].$$

We call $p_i/q_i \in \mathbb{Q} \cup \{\infty\}$ the *surgery coefficient* associated with the component L_i.

Example 3.2. A p/q-surgery on the trivial knot yields the lens space $L(p,q)$.

In particular, a 0-surgery on the trivial knot yields $S^2 \times S^1$, and a $\pm 1/n$-surgery, $n \in \mathbb{N}$ on the trivial knot yields S^3.

A ∞-surgery on any knot yields S^3 (this corresponds to the trivial filling of the knotted tunnel).

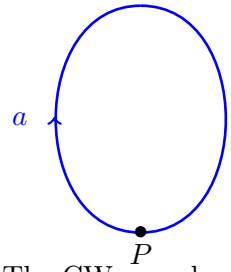


Figure 3.1 – The CW-complex structure on S^1

3.2.2 L^2-Alexander torsion of the solid torus

Let X be the circle S^1 with the CW-structure of one 0-cell P (equal to the base point) and one 1-cell a, as in Figure 3.1. Let us write $\pi = \pi_1(X) = \langle c | \rangle$ multiplicatively. Let $\widetilde{P}$ and $\widetilde{a}$ be lifts of P and a in $\widetilde{X}$.

Then the cellular chain complex of $\widetilde{X}$ is

$$C_*(\widetilde{X}) = 0 \to \mathbb{Z}[\pi]\widetilde{a} \xrightarrow{\partial_1} \mathbb{Z}[\pi]\widetilde{P} \to 0$$

where

$$\partial_1(\widetilde{a}) = c^{m+1}\widetilde{P} - c^m\widetilde{P} = c^m(c-1)\widetilde{P}$$

with $m \in \mathbb{Z}$ depending on the lifts.

Let $\phi\colon \pi \to \mathbb{Z}$ and $\gamma\colon \pi \to G$ such that (π, ϕ, γ) is an admissible triple.

The $\mathcal{N}(G)$-cellular chain complex of $X = S^1$ is

$$C_*^{(2)}(S^1, \phi, \gamma, t) = 0 \to \ell^2[G]\widetilde{a} \xrightarrow{\partial_1^{(2)}} \ell^2[G]\widetilde{P} \to 0$$

where $\partial_1^{(2)} = t^{m\phi(c)} R_{\gamma(c)^m} \circ (t^{\phi(c)} R_{\gamma(c)} - Id)$.

By Proposition 1.53, the operator $\partial_1^{(2)}$ is injective and of determinant class if and only if $\gamma(c)$ is of infinite order in G, and in this case:

Proposition 3.3. *For c a generator of $\pi_1(S^1)$, if $\gamma(c)$ is of infinite order in G, then $C_*^{(2)}(S^1, \phi, \gamma, t)$ is weakly acyclic and of determinant class for all t, and its L^2-Alexander torsion is*

$$T^{(2)}(S^1, \phi, \gamma)(t) \doteq \frac{1}{\max(1,t)^{|\phi(c)|}}.$$

This result was first proven in [Lüc02b]. Since the solid torus $S^1 \times D^2$ is simple homotopy equivalent to the circle S^1 (by two successive elementary expansions), it follows from Theorem 2.12 that

Proposition 3.4. *For c a generator of $\pi_1(S^1 \times D^2)$, if $\gamma(c)$ is of infinite order in G, then $C_*^{(2)}(S^1 \times D^2, \phi, \gamma, t)$ is weakly acyclic and of determinant class for all t, and its L^2-Alexander torsion is*

$$T^{(2)}(S^1 \times D^2, \phi, \gamma)(t) \doteq \frac{1}{\max(1,t)^{|\phi(c)|}}.$$

3.2.3 L^2-Alexander torsion of the torus

The following proposition states that the L^2-Alexander torsions of the torus $S^1 \times S^1$ are trivial; this result was proven in [Lüc02b] and [DFL14, Lemma 5.6], and we detail the proof for the reader's convenience.

Proposition 3.5. *Let $\pi = \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ denote the fundamental group of the torus, and let α, β denote a pair of generators of π. Let $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ be group homomorphisms such that (π, ϕ, γ) is an admissible triple. Let $t > 0$.*

If $\gamma(\pi)$ is infinite, i.e. if $\gamma(\alpha)$ or $\gamma(\beta)$ is of infinite order in G, then $C_*^{(2)}(S^1 \times S^1, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and its L^2-Alexander torsion is

$$T^{(2)}(S^1 \times S^1, \phi, \gamma)(t) \doteq 1.$$

Proof. Let $X = S^1 \times S^1$ be the torus endowed with its classical CW structure:

- one 0-cell P, base point of the fundamental group,
- two 1-cells a and b, that are loops of base point P,
- one 2-cell R, attached on the 1-skeleton following the path $a * b * a^{-1} * b^{-1}$.

The cells are drawn on Figure 3.2 for clarity.

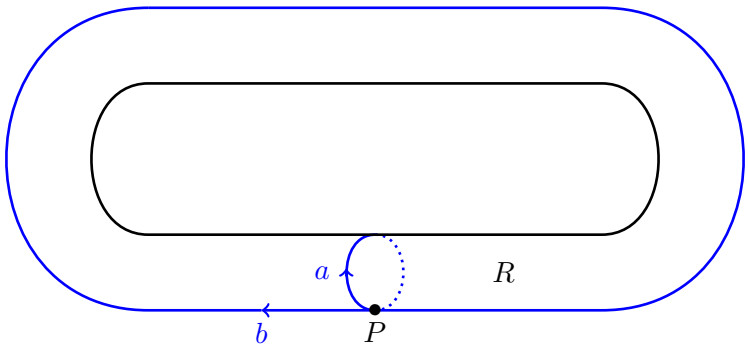


Figure 3.2 – The CW-complex structure on $S^1 \times S^1$

Let α, β denote the homotopy classes of a and b in $\pi = \pi_1(X)$. The group π admits the presentation $\langle \alpha, \beta | \alpha\beta = \beta\alpha \rangle$ and is naturally isomorphic to $\mathbb{Z}^2$. Let $\phi\colon \pi \to \mathbb{Z}$ be a group homomorphism; this homomorphism is exactly determined by the integers $n_1 = \phi(\alpha)$ and $n_2 = \phi(\beta)$. Let $\gamma\colon \pi \to G$ be a group homomorphism such that ϕ factors through γ. We assume γ is surjective, according to Remark 2.6.

The cellular chain complex of $\widetilde{X}$ is

$$C_*(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}[\pi]\widetilde{R} \xrightarrow{\begin{pmatrix} \beta - 1 \\ 1 - \alpha \end{pmatrix}} \mathbb{Z}[\pi]\widetilde{a} \oplus \mathbb{Z}[\pi]\widetilde{b} \xrightarrow{\begin{pmatrix} \alpha - 1 & ; & \beta - 1 \end{pmatrix}} \mathbb{Z}[\pi]\widetilde{P}$$

for appropriate lifts of the cells.

Consequently, the $\mathcal{N}(G)$-cellular chain complex $C_*^{(2)}(X, \phi, \gamma, t)$ associated to (X, ϕ, γ, t) is:

$$\ell^2(G)\widetilde{R} \xrightarrow{\begin{pmatrix} t^{n_2} R_{\gamma(\beta)} - Id \\ Id - t^{n_1} R_{\gamma(\alpha)} \end{pmatrix}} \ell^2(G)\widetilde{a} \oplus \ell^2(G)\widetilde{b} \xrightarrow{\begin{pmatrix} t^{n_1} R_{\gamma(\alpha)} - Id & ; & t^{n_2} R_{\gamma(\beta)} - Id \end{pmatrix}} \ell^2(G)\widetilde{P}$$

Assume that $\gamma(\pi)$ is a finite group; this implies that $n_1 = n_2 = 0$. Let $w = \sum_{h\in G} h \in \mathbb{Z}[G]$. Then since any R_h, $h \in G$, acts by permutation of the coordinates on $\ell^2(G)$, w is in the kernel of $\begin{pmatrix} t^{n_2}R_{\gamma(\beta)} - Id \\ Id - t^{n_1}R_{\gamma(\alpha)} \end{pmatrix} = \begin{pmatrix} R_{\gamma(\beta)} - Id \\ Id - R_{\gamma(\alpha)} \end{pmatrix}$ and thus the finite Hilbert chain complex $C_*^{(2)}(X, \phi, \gamma, t)$ is not weakly acyclic.

However, if $\gamma(\pi)$ is infinite, which is equivalent to the fact that either $\gamma(\alpha)$ or $\gamma(\beta)$ is of infinite order in G, then assume for instance that $\gamma(\alpha)$ is of infinite order in G. Then, by applying Proposition 1.58 and Proposition 1.53, we conclude that $C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(X, \phi, \gamma)(t) \doteq \frac{\det_{\mathcal{N}(G)}(Id - t^{n_1}R_{\gamma(\alpha)})}{\det_{\mathcal{N}(G)}(t^{n_1}R_{\gamma(\alpha)} - Id)} = \frac{\max(1,t)^{n_1}}{\max(1,t)^{n_1}} = 1.$$

□

3.2.4 The Dehn Surgery formula

Let M be a 3-manifold with non-empty toroidal boundary, B a solid torus, T a boundary part of M, and J a simple closed curve on T. Let N be the manifold obtained by doing a surgery on the curve J the boundary part T of M.

Thus $N = M \cup B$ and $T = M \cap B$. Let J' be a simple closed curve on T such that the classes of J and J' form a system of generators of $\pi_1(T) \cong \mathbb{Z}^2$. We can assume that J and J' intersect on a single point P, which will be the base point for all the following fundamental groups.

We choose a CW-structure on M and T such that P is a 0-cell and J and J' are 1-cells. For constructing the CW-structure of B we choose a 2-cell D bounded by J, and a 3-cell ρ glued in the usual way to close the solid torus. Thus J' and the core of B have the same homotopy class in $\pi_1(B)$. We can thus see J as a meridian of B and J' as a longitude of B. Finally we give N the CW-structure composed of those of M, T and B.

Let $\pi_M = \pi_1(M)$, $\pi_N = \pi_1(N)$ and c the homotopy class of the core of B in $\pi_1(B)$. Then the inclusion $J_M \colon M \subset N$ induces a quotient group homomorphism $Q \colon \pi \twoheadrightarrow \pi_N$ (whose kernel is normally generated by $[J]$), and the inclusion $J_B \colon B \subset N$ induces a group homomorphism $\iota \colon c^{\mathbb{Z}} \to \pi_N$.

$$\begin{tikzcd}
 & M \arrow[dr, "J_M"] & \\
T \arrow[ur, "I_M"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & N \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(M) \arrow[dr, "Q"] & & & \\
\pi_1(T) \arrow[ur, "i_M"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(N) \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\
 & \pi_1(B) \cong c^{\mathbb{Z}} \arrow[ur, "\iota"'] & & \mathbb{Z}
\end{tikzcd}$$

Theorem 3.6. *Let $\phi \colon \pi_N \to \mathbb{Z}$ and $\gamma \colon \pi_N \to G$ be group homomorphisms such that (π_N, ϕ, γ) forms an admissible triple. For all $t > 0$, if $\gamma(\iota(c))$ is of infinite order in G and if $C_*^{(2)}(M, \phi \circ Q, \gamma \circ Q, t)$ is weakly acyclic and of determinant class, then $C_*^{(2)}(N, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and*

$$T^{(2)}(N, \phi, \gamma)(t) \doteq \frac{T^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)}{\max(1,t)^{|\phi(\iota(c))|}}.$$

Proof. Since $\gamma(\iota(c))$ is of infinite order in G, by Proposition 3.4, $C_*^{(2)}(B, \phi \circ \iota, \gamma \circ \iota, t)$ is weakly acylic and of determinant class, and $T^{(2)}(B, \phi \circ \iota, \gamma \circ \iota)(t) \doteq \dfrac{1}{\max(1,t)^{|\phi(\iota(c))|}}$.

Likewise, $\gamma(i(\pi_1(T))) = \gamma(\iota(\pi_1(B)))$ is an infinite subgroup of G, thus, by Proposition 3.5, $C_*^{(2)}(T, \phi \circ i, \gamma \circ i, t)$ is weakly acylic and of determinant class, and

$$T^{(2)}(T, \phi \circ i, \gamma \circ i)(t) = 1.$$

Finally, since $C_*^{(2)}(M, \phi \circ Q, \gamma \circ Q, t)$ is assumed weakly acyclic and of determinant class, it follows from Theorem 3.1 that $C_*^{(2)}(N, \phi, \gamma)(t)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(N, \phi, \gamma)(t) \doteq \frac{T^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)}{\max(1,t)^{|\phi(\iota(c))|}}.$$

□

3.3 Dehn surgery of link exteriors

Let M be the exterior of an oriented link $L = L_1 \cup \ldots \cup L_c$ in S^3 and $T = \partial V(L_c)$. Let (μ, λ) be a preferred meridian-longitude pair for T. We describe a simple closed curve J on T by its homotopy class, which is characterised by two relatively prime integers p, q:

$$[J] = p[\mu] + q[\lambda]$$

Let $r, s \in \mathbb{Z}$ be relatively prime integers such that

$$\det\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = 1$$

and let J' be a curve in T such that

$$[J'] = r[\mu] + s[\lambda].$$

We can assume that J and J' intersect on a single point P.

Let N denote the manifold obtained by Dehn surgery on L_c with coefficient p/q, and B the filling solid torus. Then $Q\colon \pi_1(M) \twoheadrightarrow \pi_1(N)$ is the quotient group homomorphism that adds the relation $[\mu]^p[\lambda]^q = 1$. We have trivialised the curve $[J]$.

Theorem 3.6 can thus be re-written as:

Theorem 3.7. *Let $\phi\colon \pi_N \to \mathbb{Z}$ and $\gamma\colon \pi_N \to G$ be group homomorphisms such that (π_N, ϕ, γ) forms an admissible triple. For all $t > 0$, if $(\gamma \circ Q)([\mu]^r[\lambda]^s)$ is of infinite order in G and if $C_*^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)$ is weakly acyclic and of determinant class, then $C_*^{(2)}(N, \phi, \gamma)(t)$ is weakly acyclic and of determinant class, and*

$$T^{(2)}(N, \phi, \gamma)(t) \doteq \frac{T^{(2)}(M, \phi \circ Q, \gamma \circ Q)(t)}{\max(1,t)^{|r(\phi \circ Q)([\mu]) + s(\phi \circ Q)([\lambda])|}}.$$

3.3.1 ∞-surgery: erasing one component of a link

Let $L = L_1 \cup \ldots \cup L_{c-1} \cup L_c$ a c-component link, and $L' = L_1 \cup \ldots \cup L_{c-1}$ the link obtained by forgetting the last component, or alternatively by applying a trivial Dehn filling of the last component.

Then the natural injection $i\colon M_L \hookrightarrow M_{L'}$ passes to fundamental groups as a surjective homomorphism $Q = i_*\colon G_L \to G_{L'}$, which is the same as the quotient homomorphism by the normal subgroup generated by any meridian of L_c. Let (μ_c, λ_c) be a preferred meridian-longitude system of L_c.

Here the surgery coefficients are $p = 1, q = 0, r = 0, s = 1$.

Note that if L is brunnian, then L' is trivial split. We need to assume that neither L nor L' is split, or equivalently that M_L and $M_{L'}$ are both irreducible.

Theorem 3.8. *Let $\phi\colon \pi_1(M_{L'}) \to \mathbb{Z}$ and $\gamma\colon \pi_1(M_{L'}) \to G$ be group homomorphisms such that $(\pi_1(M_{L'}), \phi, \gamma)$ forms an admissible triple.*

We can denote $\phi = (n_1, \ldots, n_{c-1}) \circ \alpha_{L'}$ and thus $\phi \circ Q = (n_1, \ldots, n_{c-1}, 0) \circ \alpha_L$ for some non zero vector $(n_1, \ldots, n_{c-1}) \in \mathbb{Z}^{c-1}$.

For all $t > 0$, if $(\gamma \circ Q)([\lambda])$ is of infinite order in G and if $C_*^{(2)}(M_L, (n_1, \ldots, n_{c-1}, 0) \circ \alpha_L, \gamma \circ Q)(t)$ is weakly acyclic and of determinant class, then $C_*^{(2)}(M_{L'}, (n_1, \ldots, n_{c-1}) \circ \alpha_{L'}, \gamma)(t)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(M_{L'}, (n_1, \ldots, n_{c-1}) \circ \alpha_{L'}, \gamma)(t) \doteq \frac{T^{(2)}(M_L, (n_1, \ldots, n_{c-1}, 0) \circ \alpha_L, \gamma \circ Q)}{\max(1, t)^{|lk(L_1, L_c)n_1 + \ldots + lk(L_{c-1}, L_c)n_{c-1}|}}.$$

It seems one can prove this result with a purely diagrammatic reasoning, by studying Wirtinger presentations and the consequences on the Fox matrices of removing one component of the link.

Proof. We apply Theorem 3.7 and we use the fact that here

$$\begin{aligned} r(\phi \circ Q)([\mu]) + s(\phi \circ Q)([\lambda]) &= (\phi \circ Q)([\lambda_c]) \\ &= (n_1, \ldots, n_{c-1}, 0) \circ \alpha_L([\lambda_c]) \\ &= \mathrm{lk}(L_1, L_c)n_1 + \ldots + \mathrm{lk}(L_{c-1}, L_c)n_{c-1}, \end{aligned}$$

the last equality following from the results of Section 1.1.4. □

3.3.2 $1/n$-surgery: Twist knots and the Whitehead link

Let L be the Whitehead link in S^3, and M_L its exterior. We draw it as in Figure 3.3 with components L_1 and L_2. Note that L is actually ambient isotopic to the link obtained by reordering the components, therefore doing a given surgery on L_1 or L_2 yields the same manifold up to homeomorphism. We will do a $1/n$-surgery on the component L_2.

The following theorem relates a particular L^2-Alexander torsion of the Whitehead link, where ϕ sends the second component to zero and γ is an epimorphism to a knot group, to the L^2-Alexander torsion of this knot group. The possible knots in question are the *twist knots* K_n, described by the diagram of Figure 3.4.

Note that $n \in \mathbb{Z}$ can be positive or negative, that $K_0 = O$ is the trivial knot, $K_1 = 3_1$ is the trefoil knot, $K_{-1} = 4_1$ is the figure-eight knot, $K_2 = 5_2$, $K_{-2} = 6_1$, etc.

All non-trivial twist knots have genus 1.

All twist knots except for O and 3_1 are hyperbolic.

Let (α, β) be a preferred meridian-longitude system for L_2 as in Figure 3.3. Note that α is called λ in Figure A.7 in the Annex, since there it represents the longitude of the

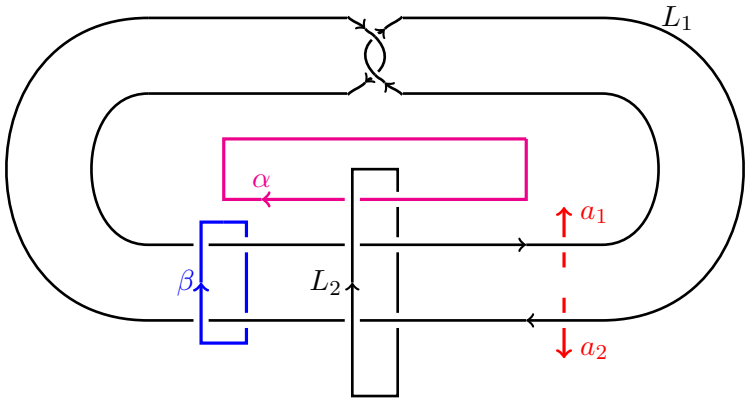


Figure 3.3 – The Whitehead link

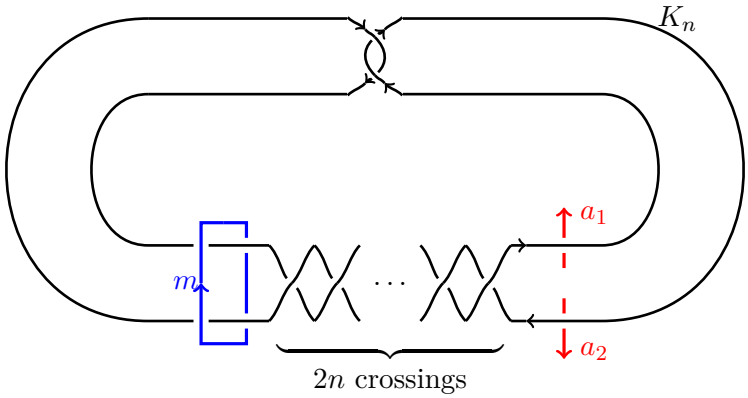


Figure 3.4 – The twist knot K_n

solid torus $S^3 \setminus V(L_2)$ and β is called μ because it represents a meridian of this same solid torus. In this section, however, α is the meridian and β the longitude of the component we are doing the surgery on, that is to say L_2.

Here we do $1/n$-surgery on L_2, which means that $(p, q) = (1, n)$, and thus $(r, s) = (0, 1)$ is a possible choice of coefficients for the curve J', which means we can assume $J' = \beta$.

Let N be the manifold obtained by this surgery on M_L. Let $J_M \colon M_L \hookrightarrow N$ be the associated natural inclusion, which extends to an inclusion $S^3 \setminus V(L_2) \hookrightarrow S^3$ since $1/n$-surgery on the trivial knot in S^3 yields S^3. The image of L_1 by this inclusion is K_n, as Figures 3.3 and 3.4 illustrate. Thus $N = M_{K_n} = S^3 \setminus V(K_n)$. The inclusion J_M induces an epimorphism $Q_n \colon \pi_1(M_L) \twoheadrightarrow \pi_1(M_{K_n})$ whose kernel is the normal subgroup generated by $[J] = [\alpha][\beta]^n$.

As a conclusion the following diagram is commutative.

$$\begin{tikzcd}
\pi_1(M_L) \arrow[r, "Q_n"] \arrow[d, "\alpha_L"] & \pi_1(M_{K_n}) \arrow[r, "\gamma"] \arrow[d, "\alpha_{K_n}"] & G \\
\mathbb{Z}^2 \arrow[r, "(1,0)"'] & \mathbb{Z} \arrow[ur, dotted] &
\end{tikzcd}$$

Theorem 3.9. *Let $\gamma \colon \pi_1(M_{K_n}) \to G$ be a group homomorphism such that $(\pi_1(M_{K_n}), \alpha_{K_n}, \gamma)$ forms an admissible triple.*

For all $t > 0$, if $\gamma(m)$ is of infinite order in G and if $C_*^{(2)}(M_L, (1,0) \circ \alpha_L, \gamma \circ Q_n)(t)$ is weakly acyclic and of determinant class, then $C_*^{(2)}(M_{K_n}, \alpha_{K_n}, \gamma)(t)$ is weakly acyclic and

of determinant class, and

$$T^{(2)}(M_{K_n}, \alpha_{K_n}, \gamma)(t) \doteq T^{(2)}(M_L, (1,0) \circ \alpha_L, \gamma \circ Q_n)(t).$$

Proof. We apply Theorem 3.7. Here M is M_L, N is M_{K_n}, ϕ is α_{K_n}, $(p,q,r,s) = (1,n,0,1)$, $[\mu]^r[\lambda]^s = [\beta]$, Q is equal to Q_n, and thus

$$(\gamma \circ Q_n)([\alpha]^r[\beta]^s) = (\gamma \circ Q_n)([\beta]) = \gamma(m).$$

Observe also that

$$(\alpha_{K_n} \circ Q_n) = (1,0) \cdot \alpha_L.$$

As a consequence, the assumptions of Theorem 3.9 match exactly with the ones of Theorem 3.7, therefore one has

$$T^{(2)}(M_{K_n}, \alpha_{K_n}, \gamma)(t) \doteq \frac{T^{(2)}(M_L, (1,0) \circ \alpha_L, \gamma \circ Q_n)(t)}{\max(1,t)^{((1,0)\circ\alpha_L)([\beta])}}.$$

Since $(\alpha_{K_n} \circ Q_n)([\beta]) = (1,0) \cdot \alpha_L([\beta]) = (1,0) \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, we conclude that the denominator is equal to 1 and the theorem follows. □

Remark 3.10. We can also prove Theorem 3.9 by using Fox calculus on two particular group presentations P_L and P_{K_n} of G_L and G_{K_n}:

$$P_L = \langle a_1, \alpha, \beta | \beta = [a_1, \alpha][a_1^{-1}, \alpha], \alpha\beta = \beta\alpha \rangle$$

$$P_{K_n} = \langle a_1, \alpha, \beta | \beta = [a_1, \alpha][a_1^{-1}, \alpha], \alpha\beta^n = 1 \rangle$$

(here $[a,b] = aba^{-1}b^{-1}$) The presentation P_{K_n} is interesting for its brevity.

Remark 3.11. Results of W. Thurston and T. Jorgensen demonstrate that if one does p/q-Dehn filling on a hyperbolic link complement, with $p^2 + q^2$ large enough the resulting manifold will also be hyperbolic with volume approaching the volume of the original link complement by smaller values as $p^2 + q^2 \to \infty$.

In particular, as $n \to \infty$, by Theorem 3.9,

$$\begin{aligned} T^{(2)}(M_L, (1,0) \circ \alpha_L, Q_n)(1) &= T^{(2)}(M_{K_n}, \alpha_{K_n}, id)(1) \\ &= \exp\left(\frac{\mathrm{vol}(K_n)}{6\pi}\right) \\ &\underset{n\to\infty}{\longrightarrow} \exp\left(\frac{\mathrm{vol}(L)}{6\pi}\right) \\ &= T^{(2)}(M_L, (1,0) \circ \alpha_L, id)(1) \end{aligned}$$

It is now natural to wonder if there exists a similar convergence of the L^2-Alexander torsions for $t \neq 1$.

Question 3.12. Do we have

$$T^{(2)}(M_L, (1,0) \circ \alpha_L, Q_n)(t) \underset{n\to\infty}{\longrightarrow} T^{(2)}(M_L, (1,0) \circ \alpha_L, id)(t)$$

for every $t > 0$? for every $t \in \mathbb{Q}_{>0}$?

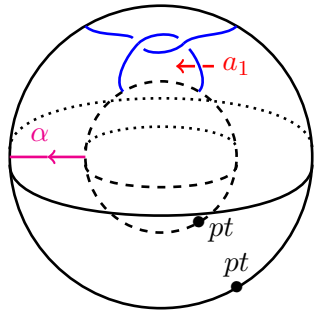


Figure 3.5 – 0-surgery on the Whitehead link: a knot inside $S^2 \times S^1$

3.3.3 0-surgery: the Whitehead link into $S^2 \times S^1$

Let L be the Whitehead link as in the previous section. Let M_0 be the manifold obtained by a 0-surgery on the component L_2 in M_L. Since L_2 is unknotted in S^3 and since 0-surgery on the trivial knot in S^3 yields $S^2 \times S^1$, we can see M_0 as the exterior of a non trivial knot in $S^2 \times S^1$ (the image of L_1 in $S^2 \times S^1$). See Figure 3.5 for clarity.

According to the previous section,

$$P_L = \langle a_1, \alpha, \beta | \beta = [a_1, \alpha][a_1^{-1}, \alpha], \alpha\beta = \beta\alpha \rangle$$

is a group presentation for $G_L = \pi_1(M_L)$. Thus $G_0 = \pi_1(M_0)$ admits the presentation

$$\langle a_1, \alpha, \beta | \beta = [a_1, \alpha][a_1^{-1}, \alpha], \alpha\beta = \beta\alpha, \beta \rangle$$

since 0-surgery kills the curve $J = \alpha^0\beta^1 = \beta$. The previous presentation simplifies to the following presentation of $G_0 = \pi_1(M_0)$:

$$\begin{aligned} P_0 &= \langle a_1, \alpha | [a_1, \alpha] = [\alpha, a_1^{-1}] \rangle \\ &= \langle a_1, \alpha | a_1 \alpha a_1^{-1} \alpha^{-1} = \alpha a_1^{-1} \alpha^{-1} a_1 \rangle \\ &= \left\langle a_1, \alpha | a_1 \left(\alpha a_1^{-1} \alpha^{-1}\right) = \left(\alpha a_1^{-1} \alpha^{-1}\right) a_1 \right\rangle \end{aligned}$$

We will prove that M_0 is irreducible by using properties of G_0.

Let us assume M_0 is not irreducible and find a contradiction. Since M_0 is not irreducible and $M_0 \neq S^1 \times S^2$, M_0 is not prime, thus $M_0 = N \sharp N'$ with N, N' compact connected oriented manifolds with empty or toroidal boundary both different from S^3. Assume for instance that N is closed and N' has the single toroidal boundary component of M_0. By the Poincaré conjecture, since $N \neq S^3$, $G = \pi_1(N)$ is not trivial. Besides, since $\partial N' = S^1 \times S^1$ has non trivial H_1, so does N'; indeed, by the homology exact sequence (and Poincaré duality)

$$\ldots \to H_1(N') \cong H_2(N'; \partial N') \to H_1(\partial N') \cong \mathbb{Z}^2 \to H_1(N') \to \ldots$$

we can see that $H_1(N')$ cannot be zero. Thus $H = \pi_1(N')$ is not trivial. Hence, by the Seifert van Kampen theorem, $G_0 = G * H$ is a free product. However, by a Corollary of the Grushko-Neumann theorem (see [MKS04, p.192]), the rank of a group (i.e. its minimal number of generators) is additive under free product. Since G and H are not trivial, they have rank at least 1, but since G_0 admits the presentation P_0, it has rank at most two. Thus G and H both have rank 1, and are therefore cyclic, of the form $\mathbb{Z}$ or $\mathbb{Z}/m$. However, we can see from P_0 that the abelianization of G_0 is $\mathbb{Z}^2$; since the abelianization of a free

product is the direct product of the abelianizations of the free factors, we conclude that $G \cong H \cong \mathbb{Z}$. But this implies that G_0 is the free group on two generators, thus since a_1 and $\alpha a_1 \alpha^{-1}$ commute in G_0, they are powers of the same element w by [MKS04, Corollary 4.1.6], which contradicts the fact that, as conjugates, they are sent to the same nonzero element in the abelianization $\mathbb{Z}^2$ of G_0. We have our contradiction. Thus M_0 is irreducible.

By Section 2.1.3, since M_0 is a compact connected oriented irreducible 3-manifold with nonempty toroidal boundary and infinite fundamental group, then M_0 is aspherical and G_0 has trivial Whitehead group, therefore we can compute the L^2-Alexander torsion of M_0 from the deficiency one presentation P_0, that we rewrite

$$P_0 = \langle b, c | bcbc^{-1} = cbc^{-1}b \rangle$$

for clarity (a_1 becomes b and α becomes c). The Fox matrix is

$$\begin{pmatrix} 1 - c - cbc^{-1} + bc \\ -1 + cbc^{-1} + b - bcbc^{-1} \end{pmatrix} = \begin{pmatrix} 1 - c - cbc^{-1} + bc \\ (b-1)(1 - cbc^{-1}) \end{pmatrix}$$

Let $(G_0, \phi\colon G_0 \to \mathbb{Z}, \gamma\colon G_0 \to G)$ be an admissible triple. From the form of the Fox matrix, the L^2-Alexander torsion of M_0 exists and is equal to
$T^{(2)}(M_0, \phi, \gamma)(t) = \det_{\mathcal{N}(G)} \left(Id - t^{\phi(cbc^{-1})} R_{\gamma(cbc^{-1})} \right) = \max(1,t)^{|\phi(cbc^{-1})|} = \max(1,t)^{|\phi(b)|}$
as long as $\gamma(b)$ is of infinite order in G.

From this we conclude that M_0 is of zero volume, thus Seifert-fibered, and the value of its Thurston norm follows from Proposition 4.2.

Now we can use the Surgery formula of Theorem 3.7 to obtain information on certain L^2-Alexander torsions of the Whitehead link.

The abelianization α_L sends a_1 to $(1,0)$ and α to $(0,1)$ in $\mathbb{Z}^2$. Let $\alpha_{M_0}\colon G_0 \to \mathbb{Z}^2$ denote the abelianization that sends $b = a_1$ to $(1,0)$ and $c = \alpha$ to $(0,1)$. Here, the curve J' of the previous sections is α^{-1}, since $(p,q,r,s) = (0,1,-1,0)$, and it follows from Theorem 3.7 that for all integers n_1, n_2 not both zero, if $\gamma(c^{-1})$ has infinite order in G, then

$$T^{(2)}(M_0, (n_1,n_2) \circ \alpha_{M_0}, \gamma)(t) = \frac{T^{(2)}(M_L, (n_1,n_2) \circ \alpha_L, \gamma \circ Q_0)(t)}{\max(1,t)^{|n_2|}}$$

where $Q_0\colon G_L \to G_0$ is the quotient group epimorphism by β.

Finally we conclude that:

Proposition 3.13. *For all $t > 0$, if $\gamma\colon G_0 \to G$ is such that $(\gamma \circ Q_0)(\alpha)$ and $(\gamma \circ Q_0)(a_1)$ are of infinite order in G and if $C_*^{(2)}(M_L, (n_1,n_2) \circ \alpha_L, \gamma \circ Q_0, t)$ is weakly acyclic and of determinant class, then*

$$T^{(2)}(M_L, (n_1,n_2) \circ \alpha_L, \gamma \circ Q_0)(t) = \max(1,t)^{|n_1|+|n_2|}.$$

We studied the example of the Whitehead link in this section, but the reasoning can be extended to 0-surgeries on other knots and links.

3.3.4 Consequences for closed manifolds

If M is a smooth compact connected oriented irreducible 3-manifold with non-empty toroidal boundary (and thus infinite fundamental group) then by the Conclusion of Section 2.1.3, we can compute the L^2-Alexander torsion of M from any deficiency one presentation of $G_M = \pi_1(M)$.

We can no longer do this if M is closed. However, the fundamental theorem of W. B. Lickorish and A. H. Wallace (see for instance [Rol90, Theorem 9I1]) states that:

Theorem 3.14. *If M is a connected orientable closed manifold, then M can be obtained by surgery on a link L in S^3.*

Moreover, one may always find such a surgery presentation in which the surgery coefficients are all ± 1 and the individual components of the link are unknotted.

Using Theorem 3.7, we can thus compute L^2-Alexander torsions of M from particular L^2-Alexander torsions of link exteriors $S^3 \setminus V(L)$, for which we can simplify computations using deficiency one group presentations.

Therefore we can hope to compute $T^{(2)}(M, \phi, \gamma)(t)$ for closed M from Fuglede-Kadison determinants of operators with a small number of terms, and thus extract relevant geometrical information.

Remark 3.15. One has to be careful about the abelianization of G_M if M is such a closed manifold.

For example, if K is a knot in S^3, then $G_K^{ab} \cong \mathbb{Z}$. Therefore, if M is obtained by p/q-surgery on K, then $G_M^{ab} = \mathbb{Z}/p$ is finite every time except in the case of a 0-surgery. But if G_M^{ab} is finite, any group homomorphism $\phi\colon G_M \to \mathbb{Z}$ factors through G_M^{ab} and thus is trivial, therefore

$$T^{(2)}(M, \phi, \gamma)(t) = T^{(2)}(M, 0, \gamma)(1) = T^{(2)}(M_K, \phi \circ Q, \gamma \circ Q)(t) = T^{(2)}(M_K, 0, \gamma \circ Q)(1)$$

and we lose any information of the L^2-Alexander torsions of the knot for $t \neq 1$. Note that for $\gamma = id$, this still offers us the hope of computing the volume of a closed manifold M from Fuglede-Kadison determinants of operators on a small number of terms.

Chapter 4

Link operations, cablings and JSJ decompositions

In this chapter we compute the L^2-Alexander torsions for all link exteriors that are Seifert-fibered, like exteriors of torus links. As a consquence of the Mayer-Vietoris formula of the previous chapter, we also prove a formula for the L^2-Alexander torsions of 3-manifolds that are obtained as gluings of simpler manifolds along tori; often this tori are incompressible and correspond to the JSJ decomposition of the 3-manifold.

These various computations allow us to determine the L^2-Alexander torsions of a connected sum of links and of a general multi-component cabling of a link by a torus link.

4.1 Toroidal gluings and L^2-Alexander torsions

The following result appeared first in [DFL14, Theorem 5.5]. We will illustrate how it can be seen as a consequence of Theorem 3.1.

Proposition 4.1. *Let N be a 3-manifold and $\phi \in Hom(\pi_1(N); \mathbb{Z})$.*

Let $T_1, \dots, T_k$ be disjoint tori in M and $N_1, \dots, N_l$ the connected components of $M|(T_1 \cup \dots \cup T_k)$.

For $i = 1, \dots, l$, we denote by $\iota_i \colon N_i \to N$ and $\tau_j \colon T_j \to N$ the inclusions.

Let $\gamma \colon \pi_1(N) \to G$ be a homomorphism such that $(\pi_1(N), \phi, \gamma)$ is an admissible triple and the restriction $\gamma \circ (\tau_j)_*$ to each $\pi_1(T_j)$ has infinite image. Let $t > 0$.

If $C_*^{(2)}(N_i, \phi \circ (\iota_i)_*, \gamma \circ (\iota_i)_*, t)$ is weakly acyclic and of determinant class for all N_i, then $C_*^{(2)}(N, \phi, \gamma, t)$ is weakly acyclic and of determinant class and

$$T^{(2)}(N, \phi, \gamma)(t) \doteq \prod_{i=1}^{l} T^{(2)}(N_i, \phi \circ (\iota_i)_*, \gamma \circ (\iota_i)_*).$$

Furthermore, if N is irreducible, then for $T_1, \dots, T_k$ the associated collection of JSJ tori and $N_1, \dots, N_l$ the JSJ pieces, the homomorphisms $(\iota_i)_*$ and the $(\tau_j)_*$ are injective and

$$T^{(2)}(N, \phi, id)(t) \doteq \prod_{i=1}^{l} T^{(2)}(N_i, \phi \circ (\iota_i)_*, id).$$

Proof. Let us first assume that $k = 1$ and $l = 2$. We apply Theorem 3.1 with $A = N_1, B = N_2, V = T_1, X = N$. If we assume that $(\gamma \circ (\tau_1)_*)(\pi_1(T_1))$ is infinite, then by Theorem 3.5, $C_*^{(2)}(T_1, \phi \circ (\tau_1)_*, \gamma \circ (\tau_1)_*, t)$ is weakly acyclic, of determinant class, and of L^2-torsion equal to 1.

Besides, we assumed that $C_*^{(2)}(N_i, \phi \circ (\iota_i)_*, \gamma \circ (\iota_i)_*, t)$ is weakly acyclic and of determinant class for $i = 1, 2$. The first part of the theorem follows from Theorem 3.1.

For bigger k and l one just applies the previous reasoning by induction on k, tori by tori. Note that rigorously speaking, the base points of the fundamental groups change at each step but this does not change the final formula.

For the second part of the theorem, the key property is that the tori T_j are incompressible in N, thus the homomorphisms $(\tau_j)_*$ are injective. Therefore the homomorphisms $(\iota_i)_*$ are injective as well by Remark 1.13. The formula follows from Proposition 2.5. □

This formula is a great help for computing the L^2-Alexander torsions of an irreducible 3-manifold, assuming we know the L^2-Alexander torsions of its JSJ pieces. When such a piece M is Seifert-fibered, we can use the following Proposition 4.2 to compute its L^2-Alexander torsions, which depend only on the Thurston norm $x_M(\phi)$ of the homomorphism ϕ.

Proposition 4.2. *([DFL14, Theorem 8.5]) Let $(M, \phi, \gamma \colon \pi_1(M) \to G)$ be an admissible triple with M a Seifert-fibered 3-manifold not equal to $S^1 \times S^2$ nor the solid torus $S^1 \times D^2$, and such that the image of any regular fiber under γ is an element of infinite order in G. Then for all $t > 0$, $C_*^{(2)}(M, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and*

$$T^{(2)}(M, \phi, \gamma) \doteq \max(1, t)^{|x_M(\phi)|}.$$

As an immediate consequence of the two previous results, we can compute the L^2-Alexander torsions of graph manifolds.

Proposition 4.3. *([DFL14, Theorem 8.6]) Let $(N, \phi, \gamma \colon \pi_1(N) \to G)$ be an admissible triple with N a graph manifold not equal to $S^1 \times S^2$ nor the solid torus $S^1 \times D^2$, and such that given any JSJ component N_i of N, with $\iota_i \colon N_i \hookrightarrow N$, the image of any regular fiber of N_i under $\gamma \circ (\iota_i)_* \colon \pi_1(N_i) \to G$ is an element of infinite order in G. Then for all $t > 0$, $C_*^{(2)}(N, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and*

$$T^{(2)}(N, \phi, \gamma) \doteq \max(1, t)^{|x_N(\phi)|} = \max(1, t)^{\left|\sum_{i=1}^{l} x_{N_i}(\phi \circ (\iota_i)_*)\right|}.$$

Computing Thurston norms is a difficult problem in general. Thus in this chapter, for all Seifert-fibered link exteriors, we will use convenient tools such as explicit group presentations and Fox calculus to compute the values of the L^2-Alexander torsions. This will allow us to prove formulas for connected sum and cabling of links, which will generalise the ones for knots of Theorem 2.33 and Theorem 2.36.

Remark 4.4. In the following sections, we will only consider gluings of link exteriors along a toroidal boundary with a slope zero, in the sense that a preferred meridian of a component of the first link will be glued with a preferred longitude of a component of the second link, and vice-versa. This is the case for connected sums of links, satellite operations like cabling, and also ∞-surgery (removing one component of the link).

However, Proposition 4.1 allows us to compute L^2-Alexander torsions of manifolds obtained by toroidal gluings of any slope p/q. Changing the slopes p_j/q_j changes the inclusions $N_i \hookrightarrow N$ and thus changes $\phi \circ (\iota_i)_*$ and $\gamma \circ (\iota_i)_*$. The surgery formulas of Chapter 3 can be seen as a particular example of this, where a solid torus (i.e. the exterior of a trivial knot) is glued to a 3-manifold.

4.2 Seifert link exteriors, connected sums and cablings

4.2.1 Links with Seifert-fibered exterior

Let us consider S^3 both as the unit sphere of $\mathbb{C}^2$ and as the one-point compactification of $\mathbb{R}^3$ by the point ∞. We define

- $T(m,n) = \{(z_1,z_2) \in S^3 \subset \mathbb{C}^2 | z_1^m = z_2^n\}$ the torus link of type (m,n) (with $e = \gcd(m,n)$ components),
- $H_v = \{(z_1,0) \in S^3\}$ the trivial knot drawn as the vertical line passing through ∞ in $\mathbb{R}^3$,
- $H_h = \{(0,z_2) \in S^3\}$ the trivial knot drawn in $\mathbb{R}^3$ as the unit circle of an horizontal plane (normal to H_v in its origin).

This allows us to describe the links L in S^3 whose exterior is a Seifert manifold:

Proposition 4.5. *(see [Bud06, Proposition 3.3])*
Let L be a non-split link in S^3. Its exterior M_L is Seifert-fibered if and only if L is one of the following links:

- *a torus link $T(m,n) = T(ep,eq)$ with p,q relatively prime (and both nonzero if $e \geqslant 2$),*
- *a link $T(ep,eq) \cup H_v$ with p,q relatively prime and $p \neq 0$,*
- *a link $T(ep,eq) \cup H_v \cup H_h$ with p,q relatively prime.*

We exclude the torus links of the form $T(m,0)$ with $|m| \geqslant 2$ since they are split.

4.2.2 The strategy

We want to compute the L^2-Alexander torsions of all links listed in Proposition 4.5. We will need various tools for this: Fox calculus, Mayer-Vietoris formulas, toroidal gluing formulas, explicit homeomorphisms between link exteriors, ∞-surgery, etc. For the reader's convenience we outline the several steps of our strategy:

1. We compute the torsions for the keychain links $T(e,0) \cup H_v$ with Fox calculus.
2. We deduce the torsions for a connected sum of links thanks to the gluing formula.
3. We compute the torsions for the links $T(e,ek) \cup H_v$ by identifying their exterior with the exterior of the keychain link $T(e,0) \cup H_v$.
4. We compute the torsions for the links $T(p,q) \cup H_v \cup H_h$ with the Mayer-Vietoris formula.
5. We deduce the torsions for the links $T(ep,eq) \cup H_v \cup H_h$ thanks to the gluing formula.
6. We apply two successive ∞-surgeries and deduce the torsions for the links $T(ep,eq) \cup H_v$ and $T(ep,eq)$.
7. We deduce general cabling formulas for links, thanks to the gluing formula.

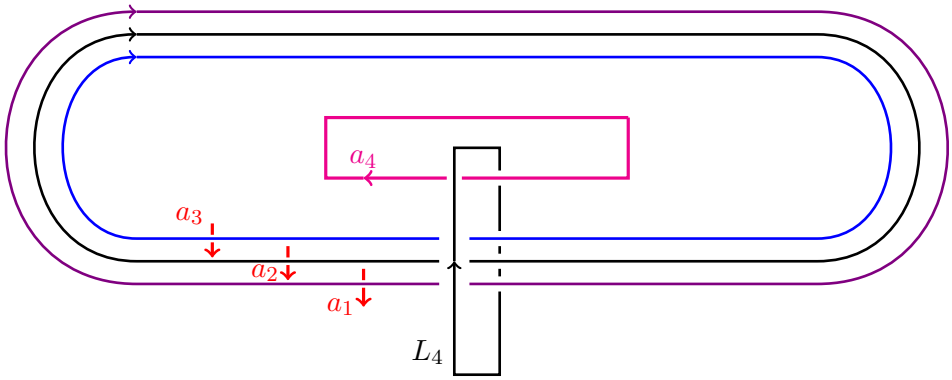


Figure 4.1 – The keychain link, for $e = 3$

4.2.3 Keychain links

Let $e \geqslant 1$. Let L be the $(e+1)$-component link $T(e,0) \cup H_v$ drawn in Figure 4.1. Let us call $L_1, \dots L_e$ the e parallel components of $T(e,0)$ and $L_{e+1} = H_v$ the one that circles them all.

The link group $G_L = \pi_1(M_L)$ is isomorphic to $\mathbb{F}[g_1, \dots g_e] \times \mathbb{Z}$ and one of its presentations is

$$P = \langle a_1, \dots a_e, a_{e+1} | a_1 a_{e+1} = a_{e+1} a_1, \dots, a_e a_{e+1} = a_{e+1} a_e \rangle.$$

We can prove this fact either by the Wirtinger process from the diagram of Figure 4.1 (for general e) or by e successive applications of the Seifert van Kampen theorem, since L is a connected sum of $(e-1)$ Hopf links $T(1,0) \cup H_v$ (the connected sum being always made on the same component H_v). The abelianization $\alpha_L \colon G_L \to \mathbb{Z}^{e+1}$ sends a_i, the meridian of L_i, to the i-th vector of the natural base of $\mathbb{Z}^{e+1}$.

Theorem 4.6. *Let $e \geqslant 1$. The L^2-Alexander torsion for the exterior of the $(e+1)$-component keychain link L exists for all admissible triples $(G_L, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma)$ such that $\gamma(a_{e+1})$ has infinite order in G and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma)(t) \doteq \max(1,t)^{(e-1)|n_{e+1}|}.$$

Proof. Let $n_1, \dots, n_{e+1} \in \mathbb{Z}$. Let $\gamma \colon G_L \to G$ such that $(G_L, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma)$ is an admissible triple.

The Fox matrix associated to P is

$$F_P = \begin{array}{c} \\ a_1 \\ \vdots \\ a_e \\ a_{e+1} \end{array} \begin{array}{c} \begin{array}{ccc} a_1 a_{e+1} = a_{e+1} a_1 & \dots & a_e a_{e+1} = a_{e+1} a_e \end{array} \\ \begin{pmatrix} 1 - a_{e+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 - a_{e+1} \\ a_1 - 1 & \dots & a_e - 1 \end{pmatrix} \end{array}.$$

Thus, the $\mathcal{N}(G)$-cellular chain complex $C_*^{(2)}(W_P, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma, t)$ associated to the presentation P is

$$C_*^{(2)}(W_P, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma, t) = \ell^2(G)^e \xrightarrow{\partial_2^{(2)}} \ell^2(G)^{e+1} \xrightarrow{\partial_1^{(2)}} \ell^2(G)$$

where

$$\partial_2^{(2)} = \begin{array}{c} \\ a_1 \\ \vdots \\ a_e \\ a_{e+1} \end{array} \begin{array}{c} \begin{array}{ccc} a_1a_{e+1}=a_{e+1}a_1 & \dots & a_ea_{e+1}=a_{e+1}a_e \end{array} \\ \begin{pmatrix} Id - t^{n_{e+1}}R_{\gamma(a_{e+1})} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Id - t^{n_{e+1}}R_{\gamma(a_{e+1})} \\ t^{n_1}R_{\gamma(a_1)} - Id & \dots & t^{n_e}R_{\gamma(a_e)} - Id \end{pmatrix} \end{array}$$

and

$$\partial_1^{(2)} = \left(t^{n_1}R_{\gamma(a_1)} - Id \quad ; \quad \dots \quad ; \quad t^{n_{e+1}}R_{\gamma(a_{e+1})} - Id\right).$$

If we assume that $\gamma(a_{e+1})$ has infinite order in G, then by Proposition 1.53, Proposition 1.51 and Proposition 1.58, $C_*^{(2)}(W_P, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned} T^{(2)}(W_P, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma)(t) &= \frac{\det_{\mathcal{N}(G)}\left(Id - t^{n_{e+1}}R_{\gamma(a_{e+1})}\right)^e}{\det_{\mathcal{N}(G)}\left(t^{n_{e+1}}R_{\gamma(a_{e+1})} - Id\right)} \\ &= \max(1,t)^{(e-1)|n_{e+1}|}. \end{aligned}$$

The result follows from the fact that M_L and W_P are simple homotopy equivalent. □

4.2.4 Connected sum for links

Let $L = L_1 \cup \dots \cup L_{c+1}$ and $L' = L'_1 \cup \dots \cup L'_{d+1}$ be two non-split links in S^3 such that $L \cup L'$ is split. Let L'' be the $(c+d+1)$-component link obtained by deleting small parts of L_{c+1} and of L'_{d+1} and then connecting them to form a single component (in a way that respects the orientations of L_{c+1} and of L'_{d+1}). The link L'' is *the connected sum of L and L' along the components L_{c+1} and L'_{d+1}*, and we order its components in the following way:

$$\begin{aligned} L'' &= L''_1 \cup \dots \cup L''_c \cup L''_{c+1} \cup \dots \cup L''_{c+d} \cup L''_{c+d+1} \\ &= L_1 \cup \dots \cup L_c \cup L'_1 \cup \dots \cup L'_d \cup (L_{c+1} \sharp L'_{d+1}). \end{aligned}$$

The manifold $M_{L''}$ is the toroidal gluing of M_L, $M_{L'}$ and a 3-component keychain link $KC = T(2,0) \cup H_v = KC_1 \cup KC_2 \cup KC_3$, where L_{c+1} is glued with KC_1, L'_{d+1} is glued with KC_2, and the boundary of KC_3 becomes the boundary of L''_{c+d+1}. For details and examples we refer to [Bud06].

Let $n_1, \dots, n_{c+d+1} \in \mathbb{Z}$ and let $\gamma\colon G_{L''} \to G$ such that $(G_{L''}, (n_1, \dots, n_{c+d+1}) \circ \alpha_{L''}, \gamma)$ is an admissible triple. Let $t > 0$.

Let $J\colon M_L \hookrightarrow M_{L''}$ and $J'\colon M_{L'} \hookrightarrow M_{L''}$ denote the inclusions associated with the toroidal gluing and j, j' the induced injective group homomorphisms on the fundamental groups (see the following diagram for clarity).

$$\begin{tikzcd}[column sep=small] \partial(V(L_{c+1})) \arrow[r] \arrow[dr] & M_L \arrow[dr, "J"] & & \\ & M_{KC} \arrow[r] & M_{L''} \\ \partial(V(L'_{d+1})) \arrow[r] \arrow[ur] & M_{L'} \arrow[ur, "J'"'] & & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathbb{Z}^2 \arrow[r] \arrow[dr] & G_L \arrow[dr, "j"] & & & \\ & G_{KC} \arrow[r] & G_{L''} \arrow[rr, "\gamma"] \arrow[dr, "\alpha_{L''}"'] & & G \arrow[d, dotted] \\ \mathbb{Z}^2 \arrow[r] \arrow[ur] & G_{L'} \arrow[ur, "j'"'] & & \mathbb{Z}^{c+d+1} \arrow[r, "(n_1,\dots,n_{c+d+1})"'] & \mathbb{Z} \end{tikzcd}$$

We can see that

$$(n_1, \dots, n_{c+d+1}) \circ \alpha_{L''} \circ j = (n_1, \dots, n_c, n_{c+d+1}) \circ \alpha_L$$

and that

$$(n_1, \dots, n_{c+d+1}) \circ \alpha_{L''} \circ j' = (n_{c+1}, \dots, n_{c+d}, n_{c+d+1}) \circ \alpha_{L'}$$

by checking these identities on each of the meridians of L''.

Let m''_{c+d+1} a preferred meridian of L''_{c+d+1}. Then $m''_{c+d+1} = j(m_{c+1}) = j'(m'_{d+1})$ where m_{c+1} is a preferred meridian of L_{c+1} and m'_{d+1} a preferred meridian of L'_{d+1}.

Theorem 4.7. *Assume that*

- $C^{(2)}_*(M_L, (n_1, \dots, n_c, n_{c+d+1}) \circ \alpha_L, \gamma \circ j, t)$ *is weakly acyclic and of determinant class,*
- $C^{(2)}_*(M_{L'}, (n_{c+1}, \dots, n_{c+d}, n_{c+d+1}) \circ \alpha_{L'}, \gamma \circ j', t)$ *is weakly acyclic and of determinant class,*
- $\gamma(m''_{c+d+1})$ *is of infinite order in* G,

then $C^{(2)}_*(M_{L''}, (n_1, \dots, n_{c+d+1}) \circ \alpha_{L''}, \gamma, t)$ *is weakly acyclic and of determinant class, and*

$$T^{(2)}(M_{L''}, (n_1, \dots, n_{c+d+1}) \circ \alpha_{L''}, \gamma)(t) \doteq T \cdot T' \cdot \max(1, t)^{|n_{c+d+1}|}$$

where

$$T = T^{(2)}(M_L, (n_1, \dots, n_c, n_{c+d+1}) \circ \alpha_L, \gamma \circ j)(t)$$

and

$$T' = T^{(2)}(M_{L'}, (n_{c+1}, \dots, n_{c+d}, n_{c+d+1}) \circ \alpha_{L'}, \gamma \circ j')(t).$$

This theorem generalizes Theorem 2.33 (where $c = 0, d = 0, n_1 = 1$).

Proof. We can prove this theorem in two different ways.

The first way is a generalisation of the proof of Theorem 2.33, for links and general coefficients n_i. We can write Wirtinger presentations

$$P = \langle a_1, \dots a_k | r_1, \dots r_{k-1} \rangle$$

for G_L and

$$P' = \langle b_1, \dots b_l | s_1, \dots s_{l-1} \rangle$$

for $G_{L'}$ such that a_k represents m_{c+1} and b_l represents m'_{d+1}. Then

$$P'' = \langle a_1, \dots a_k, b_1, \dots b_l | r_1, \dots r_{k-1}, s_1, \dots s_{l-1}, a_k = b_l \rangle$$

is a (Wirtinger) group presentation of $G_{L''}$, by the Seifert-van Kampen theorem.

One can see this with the help of Figure 4.2, where $c = 2$, $d = 1$, and the components L_3 and L'_2 are drawn in $\mathbb{R}^3$ as passing through the point at infinity. The base point pt for the fundamental groups is marked, one can see that U is homotopy equivalent to M_L, U' is homotopy equivalent to $M_{L'}$ and $U \cap U'$ retracts to a circle generated by a meridian μ, that circles the interval $(A; B)$. We can choose Wirtinger presentations of G_L and $G_{L'}$ such that μ is sent to a_k and b_l, therefore the presentation P'' is the concatenation of P and P' with the relator $a_k = b_l$ added.

From the form of P'', one can see that the Fox matrix $F_{P''}$ is almost block diagonal of blocks $F_{P,k}$ and $F_{P',l}$, like in the proof of Theorem 2.33. The assumptions of the theorem

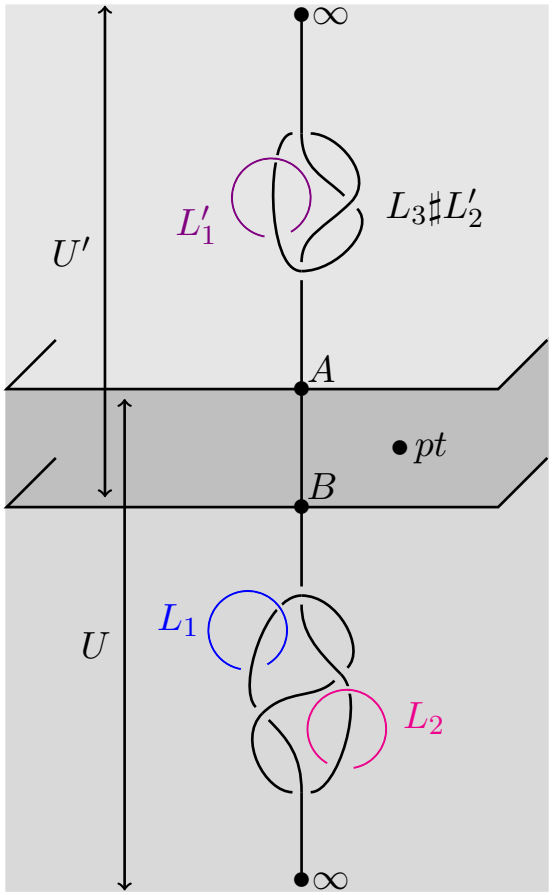


Figure 4.2 – The Seifert-van Kampen partition

allow us then to compute $T^{(2)}(M_{L''}, (n_1, \ldots, n_{c+d+1}) \circ \alpha_{L''}, \gamma)(t)$ from the Fox matrices and the formula follows.

The second way consists in using Proposition 4.1 and Theorem 4.6. Since we assume that $\gamma(m''_{c+d+1})$ is of infinite order in G, it follows that the tori $\partial(V(KC_1))$ and $\partial(V(KC_2))$ have infinite image under γ, because their preferred longitudes are homotopic to a preferred meridian of KC_3 which is sent to m''_{c+d+1}. The formula follows then from Proposition 4.1 and Theorem 4.6. □

4.2.5 The link $T(e, ek) \cup H_v$

We consider the link $L = T(e, ek) \cup H_v = L_1 \cup \ldots \cup L_e \cup L_{e+1}$. An example is drawn in Figure 4.3 for $e = 3, k = 2$. We compute the L^2-Alexander torsions of its exterior.

Let λ denote a meridian of H_v and μ a preferred longitude of H_v. Remark that λ, μ are respectively a longitude and a meridian of the torus on which $T(e, ek)$ is drawn. Let b_i denote the meridians of the components of $T(e, ek)$, as in Figure 4.3.

Theorem 4.8. *The L^2-Alexander torsion for the exterior of the $(e+1)$-component link $L = T(e, ek) \cup H_v$ exists for all admissible triples $(G_L, (n_1, \ldots, n_e, n_{e+1}) \circ \alpha_L, \gamma)$ such that $\gamma(\lambda\mu^k)$ has infinite order in G and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1, \ldots, n_e, n_{e+1}) \circ \alpha_L, \gamma)(t) \doteq \max(1, t)^{(e-1)|n_{e+1} + k(n_1 + \ldots + n_e)|}.$$

Proof. Let $KC = T(e, 0) \cup H'_v = KC_1 \cup \ldots \cup KC_e \cup KC_{e+1}$ be the $(e+1)$-component keychain link, see Figure 4.1.

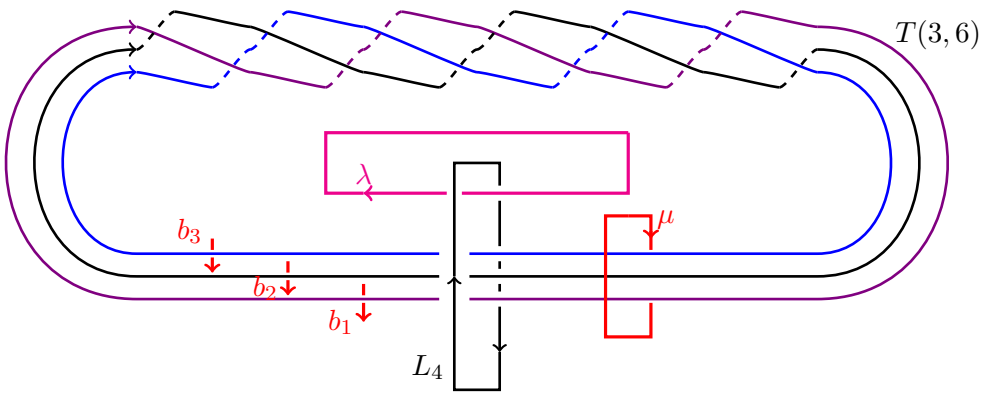


Figure 4.3 – The link $T(3,6) \cup H_v$

Then the exteriors M_L and M_{KC} are homeomorphic, by a sequence of k twists of the solid torus $S^3 \setminus V(KC_{e+1}) \cong S^3 \setminus V(L_{e+1})$.

The induced group isomorphism $\psi\colon G_{KC} \to G_L$ relates the generators written in the two figures in the following way:

$$\begin{array}{c}
\mathbb{Z}^{e+1} \xleftarrow{\alpha_{KC}} G_{KC} \xrightarrow{\psi} G_L \xrightarrow{\alpha_L} \mathbb{Z}^{e+1} \\
(1,\dots,0,0) \longleftarrow a_1 \longleftrightarrow b_1 \longmapsto (1,\dots,0,0) \\
\vdots \\
(0,\dots,1,0) \longleftarrow a_e \longleftrightarrow b_e \longmapsto (0,\dots,1,0) \\
(0,\dots,0,1) \longleftarrow a_{e+1} \longleftrightarrow \lambda\mu^k \longmapsto (k,\dots,k,1)
\end{array}$$

Thus, for all integers $n_1,\dots,n_e,n_{e+1}$,

$$(n_1,\dots,n_e,n_{e+1}) \circ \alpha_L \circ \psi = (n_1,\dots,n_e,n_{e+1}+kn_1+\ldots+kn_e) \circ \alpha_{KC}.$$

Let ϕ denote $(n_1,\dots,n_e,n_{e+1}) \circ \alpha_L$. Since $(G_{KC}, \phi\circ\psi, \gamma\circ\psi)$ is an admissible triple and since $\gamma(\psi(a_{e+1})) = \gamma(\lambda\mu^k)$ has infinite order in G, it follows from Theorem 4.6 that $C_*^{(2)}(M_{KC}, \phi\circ\psi, \gamma\circ\psi, t)$ is weakly acyclic and of determinant class and

$$T^{(2)}(M_{KC}, \phi\circ\psi, \gamma\circ\psi)(t) \doteq \max(1,t)^{(e-1)|\phi(\psi(a_{e+1}))|} = \max(1,t)^{(e-1)|n_{e+1}+kn_1+\ldots+kn_e|}.$$

Since M_L and M_{KC} are homeomorphic, they are simple homotopy equivalent and the result follows from Theorem 2.12.

□

Note that we could also have proven this theorem by direct computations of the Fox matrix of the presentation

$$P = \langle b_1,\dots,b_e,\lambda,\mu | \mu^k b_1 \mu^{-k} = \lambda^{-1} b_1 \lambda, \dots, \mu^k b_e \mu^{-k} = \lambda^{-1} b_e \lambda, b_e \dots b_2 b_1 = \mu \rangle$$

of the link group $G_L = \pi_1(S^3 \setminus V(L))$.

4.2.6 The link $T(p,q) \cup H_v \cup H_h$

We consider the 3-component link $L = T(p,q) \cup H_v \cup H_h$ where $p \neq 0$ and p,q are relatively prime. An example for $p=3, q=4$ is drawn in Figure 4.4.

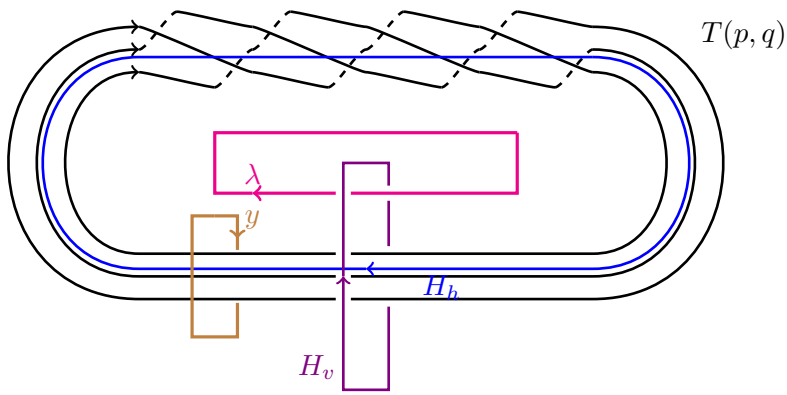


Figure 4.4 – The link $T(3,4) \cup H_v \cup H_h$

Tubular neighbourhoods of H_h and H_v have a torus T as a common boundary. The manifold $M_H = S^3 \setminus (V(H_v) \cup V(H_h))$ is homeomorphic to a thickened torus $T \times [-1;1]$. We identify $T \cong T \times \{0\}$ to the torus on which the torus knot $T(p,q)$ is drawn.

The space $Z = T \setminus V(T(p,q))$ is homeomorphic to an annulus. Let δ be a curve that generates $\pi_1(Z)$. The curve δ is thus locally parallel to the torus knot $T(p,q)$. See Figure 4.5 for clarity.

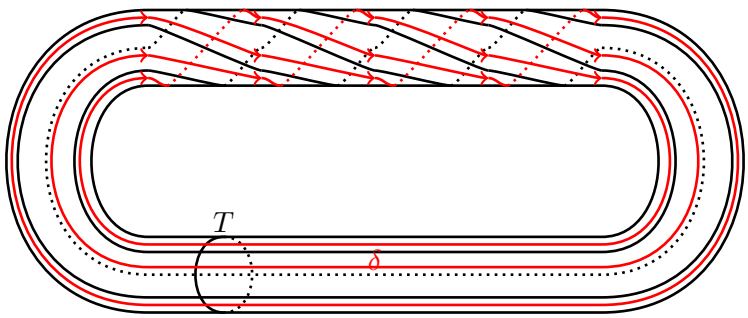


Figure 4.5 – The generator δ of $\pi_1(Z)$

Theorem 4.9. *The L^2-Alexander torsion for the exterior of the link $L = T(p,q) \cup H_v \cup H_h$ exists for all admissible triples $(G_L, (n_1,n_2,n_3) \circ \alpha_L, \gamma)$ such that the homotopy class of the curve δ is sent by γ to an element of infinite order, and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1,n_2,n_3) \circ \alpha_L, \gamma)(t) \doteq \max(1,t)^{|pqn_1+pn_2+qn_3|}.$$

Proof. The torus T separates M_H in two thickened tori $N_1 = V(H_v) \setminus H_v$ and $N_2 = V(H_h) \setminus H_h$.

Let $X = M_L$, $A = N_1 \cup T \setminus V(T(p,q))$, $B = N_2 \cup T \setminus V(T(p,q))$ and $Z = T \setminus V(T(p,q))$, so that $X = A \cup B$ and $Z = A \cap B$, and X, A, B, Z are path connected. We pick a base point $pt \in Z$ for all the following fundamental groups.

The space Z is an annulus, and its group $\pi_Z = \pi_1(Z)$ is isomorphic to $\mathbb{Z}$ and is generated by an element δ that runs between the p strands of $T(p,q)$.

The space A is homeomorphic to a thickened torus, by filling the missing surface lines of $V(T(p,q))$. Let (y,λ) be a preferred meridian-longitude system of A, as in Figure 4.4. Note that λ acts as a meridian of the unknot H_v. The group $\pi_A = \pi_1(A)$ has the presentation $\langle y, \lambda | y\lambda = \lambda y \rangle$ and is isomorphic to $\mathbb{Z}^2$.

Similarly, the space B is homeomorphic to a thickened torus, by filling the missing surface lines of $V(T(p,q))$. Let (m,c) be a preferred meridian-longitude system of A. Note that m acts as a meridian of the unknot H_h. The group $\pi_B = \pi_1(B)$ has the presentation $\langle m, c | mc = cm \rangle$ and is isomorphic to $\mathbb{Z}^2$.

The element z is sent to $\lambda^p y^q$ in π_A and to $c^p m^q$ in π_B. Thus the group $G_L = \pi_1(X)$ admits the presentation

$$\langle y, \lambda, c, m | \lambda^p y^q = c^p m^q, y\lambda = \lambda y, mc = cm \rangle$$

by the Seifert van Kampen theorem.

$$\begin{tikzcd}
 & A \arrow[dr, "J_A"] & \\
Z \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & X \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_A"] & & & \\
\pi_1(Z) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & G_L \arrow[rr, "\gamma"] \arrow[dr, "\alpha_L"'] & & G \arrow[d, dotted] \\
 & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}^3 \arrow[r, "(n_1,n_2,n_3)"'] & \mathbb{Z}
\end{tikzcd}$$

Let ϕ denote the homomorphism $(n_1, n_2, n_3) \circ \alpha_L$. We assume that the homotopy class of δ is sent by γ to an element of infinite order, i.e. $\gamma \circ i(\delta) = \gamma(c^p m^q) = \gamma(\lambda^p y^q)$ has infinite order in G. Therefore $\gamma(\pi_A)$ and $\gamma(\pi_B)$ are infinite and it follows from Theorems 3.4 and 3.5 that the three $\mathcal{N}(G)$-cellular chain complexes

$$C_*^{(2)}(Z, \phi \circ i, \gamma \circ i, t), C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t), C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t)$$

are weakly acyclic and of determinant class, and

$$T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A)(t) \doteq T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B)(t) \doteq 1,$$

$$T^{(2)}(Z, \phi \circ i, \gamma \circ i)(t) \doteq \max(1,t)^{-|\phi(c^p m^q)|}.$$

Hence, by Theorem 3.1, $C_*^{(2)}(M_L, (n_1, n_2, n_3) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class as well, and

$$T^{(2)}(M_L, (n_1, n_2, n_3) \circ \alpha_L, \gamma)(t) \doteq \max(1,t)^{|\phi(c^p m^q)|} = \max(1,t)^{|pqn_1 + pn_2 + qn_3|}.$$

□

4.2.7 The link $T(ep, eq) \cup H_v \cup H_h$

We can now compute the L^2-Alexander torsions for a general link $L = T(ep, eq) \cup H_v \cup H_h$ by using the fact that the torus link $T(ep, eq)$ is a (e, epq)-cable on the torus knot $K = T(p,q)$ (see Annex A.2).

In Figure 4.6, we draw a torus link $T(e, epq)$ inside a solid torus $S^3 \setminus V(H_K)$, the link $T(p,q) \cup H_v \cup H_h$, and the torus link $T(ep, eq)$ which is a (e, epq)-cable on $T(p,q)$ (we did not draw H_v and H_h in the third part in order to make the figure easier to read). Here $p = 2, q = 1, e = 2$.

One can see the torus T_K (drawn with red dotted lines) that separates $M_{T(4,2)}$ in the disjoint union of the exterior of the torus knot $T(2,1)$ in S^3 and the exterior of the torus link $T(2,4) \cup H_K$. This torus T_K is the boundary of a tubular neighbourhood of $K = T(p,q)$. A preferred longitude l_K of K is drawn on the figure. We identify $S^3 \setminus V(H_K)$ to the solid torus $V(K)$; the component H_K looks like a preferred meridian of K.

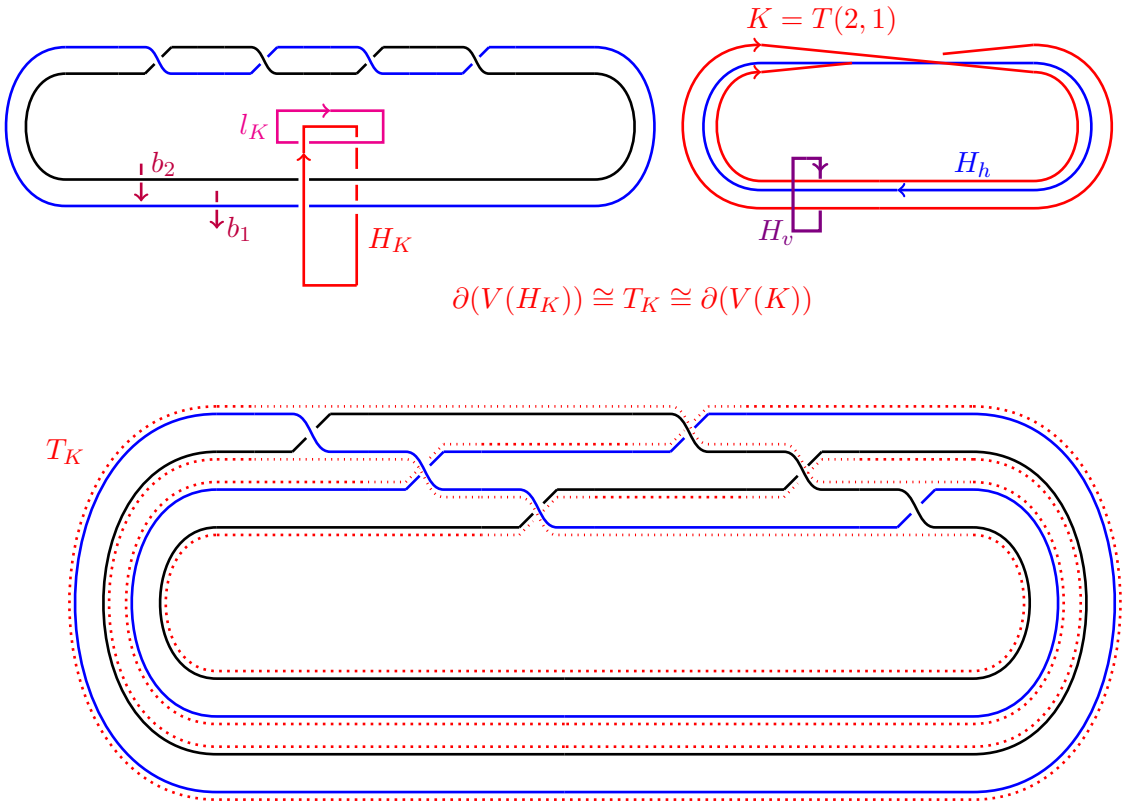


Figure 4.6 – The torus link $T(4,2)$ as a $(2,4)$ cable on $T(2,1)$

As a consequence, the torus link exterior $S^3 \setminus V(T(ep,eq))$ is the JSJ gluing of $S^3 \setminus V(T(p,q))$ and $S^1 \times D^2 \setminus V(T(e,epq))$.

Let $M = M_L$ denote the exterior of $L = T(ep,eq) \cup H_v \cup H_h$, $A = S^3 \setminus V(K \cup H_v \cup H_h)$ and $B = S^3 \setminus V(T(e,epq) \cup H_K)$ (in Figure 4.6, A is the exterior of the drawing up right and B of the one up left). We see that M is the toroidal gluing of A and B along their intersection $T_K = A \cap B$. The following diagrams are commutative:

$$\begin{tikzcd}
 & A \arrow[dr, "J_A"] & \\
T_K \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & M \\
 & B \arrow[ur, "J_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(A) \arrow[dr, "j_A"] & & & \\
\pi_1(T_K) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(M) = G_L \arrow[rr, "\gamma"] \arrow[dr, "\alpha_L"'] & & G \arrow[d, dotted] \\
 & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}^{e+2} \arrow[r] & \mathbb{Z} \\
 & & & (n_1, \dots, n_{e+2}) &
\end{tikzcd}$$

As in the previous section, let T be the torus on which K is drawn, and δ a simple closed curve that generates the fundamental group of $T \setminus V(K)$. The curve δ is once again locally parallel to the components of $T(ep,eq)$.

Theorem 4.10. *Let $e \geqslant 2$. The L^2-Alexander torsion for the exterior of the link $L = T(ep,eq) \cup H_v \cup H_h$ exists for all admissible triples $(G_L, (n_1, \dots, n_e, n_{e+1}, n_{e+2}) \circ \alpha_L, \gamma)$ such that the homotopy class of the curve δ is sent by γ to an element of infinite order, and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1, \dots, n_e, n_{e+1}, n_{e+2}) \circ \alpha_L, \gamma)(t) \doteq \max(1,t)^{e|pq(n_1+\dots+n_e)+pn_{e+1}+qn_{e+2}|}.$$

Proof. Let $t > 0$. Let ϕ denote the homomorphism $(n_1, \dots, n_{e+2}) \circ \alpha_L$. We assume that

the homotopy class of δ is sent by γ to an element of infinite order. First, as the cabling torus T_K is the boundary of a tubular neighbourhood $V(K)$ of $K = T(p,q)$ and contains such a curve δ, the torus T_K has thus infinite image under γ, therefore $C_*^{(2)}(T_K, \phi\circ i, \gamma\circ i, t)$ is weakly acyclic and of determinant class and its L^2-torsion is 1, by Theorem 3.5.

Secondly, the curve $\lambda\mu^k = \lambda\mu^{pq}$ of Theorem 4.8 is ambient isotopic to δ, thus it is sent by γ to an element of infinite order (in Figure 4.6, λ is written l_K, and μ is parallel to H_K), therefore $C_*^{(2)}(B, \phi\circ j_B, \gamma\circ j_B, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned}
&T^{(2)}(B, \phi\circ j_B, \gamma\circ j_B)(t)\\
&= T^{(2)}\left(S^3 \setminus V(T(e,epq)\cup H_K), (n_1,\ldots,n_{e+2})\circ\alpha_L\circ j_B, \gamma\circ j_B\right)(t)\\
&= T^{(2)}\left(S^3 \setminus V(T(e,epq)\cup H_K), (n_1,\ldots,n_e, pn_{e+1}+qn_{e+2})\circ\alpha_{T(e,epq)\cup H_K}, \gamma\circ j_B\right)(t)\\
&\doteq \max(1,t)^{(e-1)|pn_{e+1}+qn_{e+2}+pq(n_1+\ldots+n_e)|}.
\end{aligned}$$

Finally, the last piece of the toroidal gluing is $A = M_H \setminus V(K)$, which corresponds to the case $e = 1$ of the previous section; from the assumption on δ, it follows from Theorem 4.9 that $C_*^{(2)}(A, \phi\circ j_A, \gamma\circ j_A, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned}
&T^{(2)}(A, \phi\circ j_A, \gamma\circ j_A)(t)\\
&= T^{(2)}\left(S^3 \setminus V(T(p,q)\cup H_v\cup H_h), (n_1,\ldots,n_{e+2})\circ\alpha_L\circ j_A, \gamma\circ j_A\right)(t)\\
&= T^{(2)}\left(S^3 \setminus V(T(p,q)\cup H_v\cup H_h), (n_1+\ldots+n_e, n_{e+1}, n_{e+2})\circ\alpha_{T(p,q)\cup H_v\cup H_h}, \gamma\circ j_A\right)(t)\\
&\doteq \max(1,t)^{|pn_{e+1}+qn_{e+2}+pq(n_1+\ldots+n_e)|}.
\end{aligned}$$

It then follows from Proposition 4.1 that $C_*^{(2)}(M_L, \phi, \gamma, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned}
&T^{(2)}(M_L, (n_1,\ldots,n_e,n_{e+1},n_{e+2})\circ\alpha_L, \gamma)(t)\\
&= T^{(2)}(A, \phi\circ j_A, \gamma\circ j_A)(t)\cdot T^{(2)}(B, \phi\circ j_B, \gamma\circ j_B)(t)\\
&\doteq \max(1,t)^{(e-1)|pn_{e+1}+qn_{e+2}+pq(n_1+\ldots+n_e)|}\cdot\max(1,t)^{|pn_{e+1}+qn_{e+2}+pq(n_1+\ldots+n_e)|}\\
&= \max(1,t)^{e|pn_{e+1}+qn_{e+2}+pq(n_1+\ldots+n_e)|}.
\end{aligned}$$

□

4.2.8 The link $T(ep,eq)\cup H_v$

The link $L = L_1\cup\ldots\cup L_e\cup H_v = T(ep,eq)\cup H_v$ is obtained from $L' = T(ep,eq)\cup H_v\cup H_h$ by deleting the component H_h, therefore M_L is obtained from $M_{L'}$ by a ∞-surgery on the boundary component of H_h. This helps us compute the L^2-Alexander torsions of L. Let λ_h be the homotopy class of H_h in M_L and δ the simple closed curve locally parallel to the strands of $T(ep,eq)$ as in the previous section. The epimorphism $Q\colon G_{L'}\to G_L$ corresponds to the trivialization of the curve λ_h.

Theorem 4.11. *The L^2-Alexander torsion for the exterior of the link $L = T(ep,eq)\cup H_v$ exists for all admissible triples $(G_L, (n_1,\ldots,n_{e+1})\circ\alpha_L, \gamma)$ such that $\gamma(\delta)$ and $\gamma(\lambda_h)$ have infinite order in G and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1,\ldots,n_{e+1})\circ\alpha_L, \gamma)(t) = \max(1,t)^{(e|p|-1)|n_{e+1}+q(n_1+\ldots+n_e)|}.$$

Proof. We will use Theorem 3.8. Here λ_h corresponds to the curve λ in the assumptions of Theorem 3.8. Since $\gamma(\delta)$ has infinite order in G, it follows from Theorem 4.10 that $C_*^{(2)}(M_{L'}, (n_1, \dots, n_{e+1}, 0) \circ \alpha_{L'}, \gamma \circ Q, t)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(M_{L'}, (n_1, \dots, n_{e+1}, 0) \circ \alpha_{L'}, \gamma \circ Q)(t) \doteq \max(1, t)^{e|pq(n_1+\dots+n_e)+pn_{e+1}|}.$$

Since $\gamma(\lambda_h)$ has infinite order in G, it follows from Theorem 3.8 that $C_*^{(2)}(M_L, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned} T^{(2)}(M_L, (n_1, \dots, n_{e+1}) \circ \alpha_L, \gamma)(t) &\doteq \frac{T^{(2)}(M_{L'}, (n_1, \dots, n_{e+1}, 0) \circ \alpha_{L'}, \gamma \circ Q)(t)}{\max(1, t)^{|\mathrm{lk}(L_1, H_h)n_1+\dots+\mathrm{lk}(L_e, H_h)n_e+\mathrm{lk}(H_v, H_h)n_{e+1}|}} \\ &\doteq \frac{\max(1, t)^{e|pq(n_1+\dots+n_e)+pn_{e+1}|}}{\max(1, t)^{|qn_1+\dots+qn_e+n_{e+1}|}} \\ &= \max(1, t)^{(e|p|-1)|n_{e+1}+q(n_1+\dots+n_e)|}. \end{aligned}$$

□

Note that we could also have proven this theorem by direct computations of the Fox matrix of the presentation P of the link group G_L with generators

$$m, l, x, y, \lambda', \lambda, \mu, b_1 \dots b_e$$

and relators

$$b_1(\lambda\mu^{pq}) = (\lambda\mu^{pq})b_1, \ \dots \ , \ b_{e-1}(\lambda\mu^{pq}) = (\lambda\mu^{pq})b_{e-1},$$

$$b_e \dots b_1 = \mu = m = m(x, y), \ x^p = \lambda'^p y^q, \ \lambda = l = l(x, y), \ \lambda' y = y\lambda',$$

where $m(x, y)$ and $l(x, y)$ are words in x, y.

4.2.9 The torus link $T(ep, eq)$

Now we can compute L^2-Alexander torsions for general torus links of the form $L = T(ep, eq)$, where $e \geqslant 2$ is an integer and p, q are relatively prime integers. The link $T(ep, eq)$ is obtained by ∞-surgery from $T(ep, eq) \cup H_v$ on the component H_v. The epimorphism $Q\colon G_{T(ep,eq)\cup H_v} \to G_{T(ep,eq)}$ corresponds to the trivialization of the curve λ_v.

Let δ and λ_h be as in the previous sections, and let λ_v denote the homotopy class of H_v in $G_{T(ep,eq)}$. Note that the fundamental group of the torus T (on which $T(ep, eq)$ is drawn) is generated by classes of curves homotopic to λ_h and λ_v. Thus the equality

$$\delta = \lambda_h^p \lambda_v^q$$

stands in $G_{T(ep,eq)}$. This equality and the fact that $\lambda_h\lambda_v = \lambda_v\lambda_h$ imply that, for any homomorphism $\gamma\colon G_{T(ep,eq)} \to G$, if two elements of $\{\gamma(\delta), \gamma(\lambda_h), \gamma(\lambda_v)\}$ are of infinite order, then the third is of infinite order as well.

Theorem 4.12. *The L^2-Alexander torsion for the exterior of the torus link $L = T(ep, eq)$ exists for all admissible triples $(G_L, (n_1, \dots, n_e) \circ \alpha_L, \gamma)$ such that two of the three elements $\gamma(\delta), \gamma(\lambda_h), \gamma(\lambda_v)$ have infinite order in G, and for all $t > 0$. One has:*

$$T^{(2)}(M_L, (n_1, \dots, n_e) \circ \alpha_L, \gamma)(t) \doteq \max(1, t)^{|n_1+\dots+n_e|(e|p||q|-|p|-|q|)}.$$

This theorem generalises Theorem 2.28 (2).

Proof. We will use Theorem 3.8. Here λ_v corresponds to the curve λ in the assumptions of Theorem 3.8. Since $\gamma(\delta)$ and $\gamma(\lambda_h)$ have infinite order in G, it follows from Theorem 4.11 that $C_*^{(2)}(M_{T(ep,eq)\cup H_v}, (n_1,\dots,n_e,0)\circ\alpha_{T(ep,eq)\cup H_v}, \gamma\circ Q, t)$ is weakly acyclic and of determinant class, and

$$T^{(2)}(M_{T(ep,eq)\cup H_v}, (n_1,\dots,n_e,0)\circ\alpha_{T(ep,eq)\cup H_v}, \gamma\circ Q)(t) \doteq \max(1,t)^{(e|p|-1)|q||n_1+\dots+n_e|}.$$

Since $\gamma(\lambda_v)$ has infinite order in G, it follows from Theorem 3.8 that $C_*^{(2)}(M_L, (n_1,\dots,n_e)\circ\alpha_L, \gamma, t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned} T^{(2)}(M_L, (n_1,\dots,n_e)\circ\alpha_L, \gamma)(t) &\doteq \frac{T^{(2)}(M_{T(ep,eq)\cup H_v}, (n_1,\dots,n_e,0)\circ\alpha_{T(ep,eq)\cup H_v}, \gamma\circ Q)(t)}{\max(1,t)^{|\mathrm{lk}(L_1,H_v)n_1+\dots+\mathrm{lk}(L_e,H_v)n_e|}} \\ &\doteq \frac{\max(1,t)^{(e|p|-1)|q||n_1+\dots+n_e|}}{\max(1,t)^{|pn_1+\dots+pn_e|}} \\ &= \max(1,t)^{(e|p||q|-|p|-|q|)|n_1+\dots+n_e|}. \end{aligned}$$

□

Note that we could also have proven this theorem by direct computations of the Fox matrix of the presentation P of the link group $G_L = \pi_1(S^3\setminus V(T(ep,eq)))$ with generators

$$m, l, x, y, \lambda, \mu, b_1 \dots b_e$$

and relators

$$b_1(\lambda\mu^{pq}) = (\lambda\mu^{pq})b_1,\ \dots\ ,\ b_{e-1}(\lambda\mu^{pq}) = (\lambda\mu^{pq})b_{e-1},$$
$$b_e\dots b_1 = \mu = m = m(x,y),\ x^p = y^q,\ \lambda = l = l(x,y),$$

where $m(x,y)$ and $l(x,y)$ are words in x, y.

4.2.10 General cabling formulas

We can now prove a general cabling formula for L^2-Alexander torsions as a consequence of Theorem 4.11.

Let $L = L_1\cup\dots\cup L_{c+1}$ a link in S^3, and $L' = L_1\cup\dots\cup L_c\cup L'_{c+1}\cup\dots\cup L'_{c+e}$ the link obtained by cabling the component L_{c+1} by the torus link $T(ep,eq)$ with p,q two relatively prime integers.

Then $M = M_{L'} = S^3\setminus V(L')$ is obtained by a toroidal gluing of $A = M_L = S^3\setminus V(L)$ and $B = (S^1\times D^2)\setminus V(T(ep,eq)) \cong S^3\setminus V(T(ep,eq)\cup H_v)$ between the components L_{c+1} and H_v.

Let $n_1,\dots,n_{c+e}\in\mathbb{Z}$. Let $\gamma\colon G_{L'}\to G$ be a group homomorphism such that $(G_{L'}, (n_1,\dots,n_{c+e})\circ\alpha_{L'}, \gamma)$ is an admissible triple. Let $t>0$.

The following diagrams should help the reader picture the various maps we consider.

$$\begin{tikzcd} & A \arrow[dr, "J_A"] & \\ T \arrow[ur, "I_A"] \arrow[rr, "I"] \arrow[dr, "I_B"'] & & M \\ & B \arrow[ur, "J_B"'] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(A) = G_L \arrow[dr, "j_A"] & & & \\ \pi_1(T) \arrow[ur, "i_A"] \arrow[rr, "i"] \arrow[dr, "i_B"'] & & \pi_1(M) = G_{L'} \arrow[rr, "\gamma"] \arrow[dr, "\alpha_{L'}"'] & & G \arrow[d, dotted] \\ & \pi_1(B) \arrow[ur, "j_B"'] & & \mathbb{Z}^{e+1} \arrow[r] & \mathbb{Z} \\ & & & (n_1,\dots,n_{c+e}) & \end{tikzcd}$$

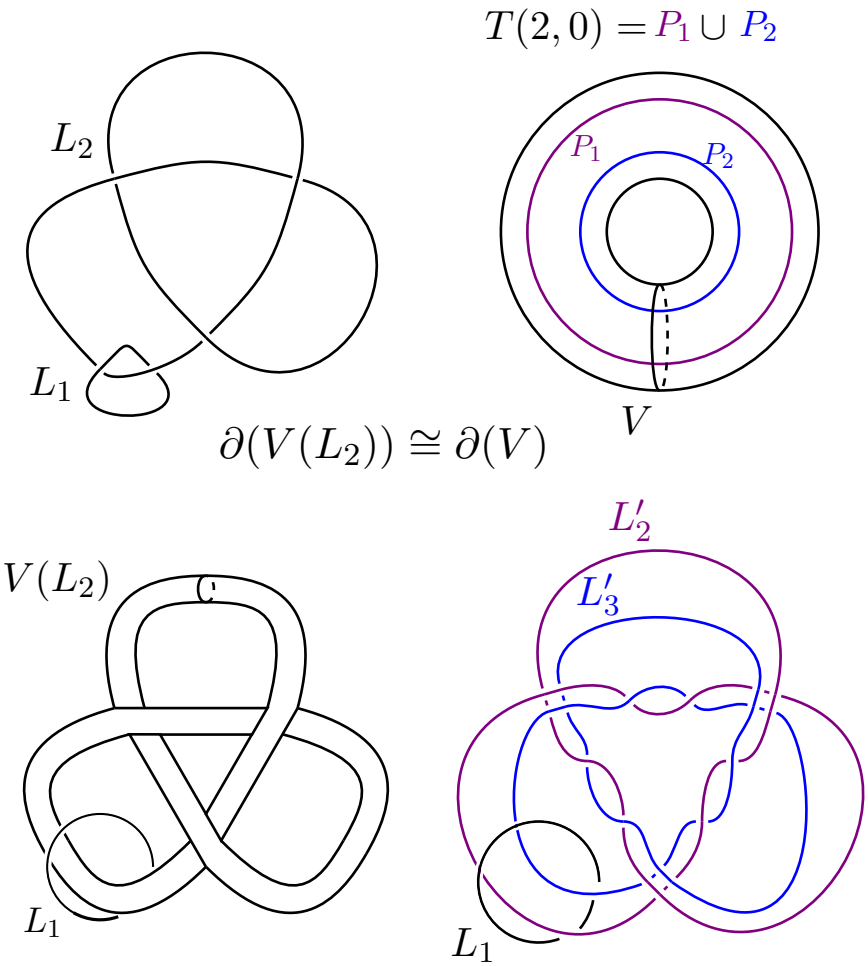


Figure 4.7 – The $(2,0)$-cabling on the second component of $L = L_1 \cup L_2$

Let

$$N = n_{c+1} + \ldots + n_{c+e}$$

and

$$\ell = \sum_{i=1}^{c} \mathrm{lk}(L_i, L_{c+1}) n_i.$$

To clarify the notations, let us consider the example in Figure 4.7. The link L has two components ($c = 1$), L_1 which is unknotted and L_2 which is a trefoil, with linking number $\mathrm{lk}(L_1, L_2) = 1$. We do a $(2,0)$-cabling on L_2 (thus $e = 2, p = 1, q = 0$), and the resulting link L' has 3 components. We glue the tori $\partial(V(L_2))$ and $\partial(V)$ such that a meridian of L_2 is identified with m_V the meridian of V that circles both components of $T(2,0)$. Here $N = n_2 + n_3$ and $\ell = n_1$.

Theorem 4.13. *Assume that*

- $C_*^{(2)}(M_L, (n_1, \ldots, n_c, pN) \circ \alpha_L, \gamma \circ j_A, t)$ *is weakly acyclic and of determinant class*
- $C_*^{(2)}(M_{T(ep,eq) \cup H_v}, (n_{c+1}, \ldots, n_{c+e}, \ell) \circ \alpha_{T(ep,eq) \cup H_v}, \gamma \circ j_B, t)$ *is weakly acyclic and of determinant class*

- $T = \partial(V) \cong \partial(V(L_{c+1}))$ *has infinite image under* γ

then $C_*^{(2)}(M_{L'}, (n_1, \ldots, n_{c+e}) \circ \alpha_{L'}, \gamma, t)$ *is weakly acyclic and of determinant class and*

$$\begin{aligned} &T^{(2)}(M_{L'}, (n_1, \ldots, n_{c+e}) \circ \alpha_{L'}, \gamma)(t) \\ &\doteq T^{(2)}(M_L, (n_1, \ldots, n_c, pN) \circ \alpha_L, \gamma \circ j_A)(t) \max(1, t)^{(e|p|-1)|\ell+qN|}. \end{aligned}$$

Proof. First let us prove that

$$(n_1, \ldots, n_{c+e}) \circ \alpha_{L'} \circ j_A = (n_1, \ldots, n_c, pN) \circ \alpha_L$$

and that

$$(n_1, \ldots, n_{c+e}) \circ \alpha_{L'} \circ j_B = (n_{c+1}, \ldots, n_{c+e}, \ell) \circ \alpha_{T(ep,eq) \cup H_v}.$$

The group $G_L = \pi_1(A)$ is generated by $m_1, \ldots m_{c+1}$, preferred meridians of $L_1, \ldots L_{c+1}$ in M_L. We have

$$((n_1, \ldots, n_{c+e}) \circ \alpha_{L'} \circ j_A)(m_i) = ((n_1, \ldots, n_c, pN) \circ \alpha_L)(m_i)$$

for $i = 1, \ldots, c$ since $L_1, \ldots, L_c$ are the c first components of L'. The identity is also true for $i = c+1$, since $j_A(m_{c+1})$ circles the e components $L'_{c+1}, \ldots, L'_{c+e}$ p times and is unlinked with $L_1, \ldots, L_c$.

The group $G_B = \pi_1(B) = \pi_1(S^1 \times D^2 \setminus V(T(ep, eq)))$ is generated by $b_1, \ldots, b_e$ (preferred meridians of the components of $T(ep, eq)$) and λ a longitude of the solid torus $S^1 \times D^2 \cong \partial V(L_{c+1})$. Note that $j_B(\lambda) = j_A(l_{c+1})$ in $M_{L'}$ where l_{c+1} is a preferred longitude of L_{c+1} in M_L. The identity

$$(n_1, \ldots, n_{c+e}) \circ \alpha_{L'} \circ j_B = (n_{c+1}, \ldots, n_{c+e}, \ell) \circ \alpha_{T(ep,eq) \cup H_v}$$

is true on each of the generators b_i, $i = 1 \ldots, e$ (both terms of the equality are immediately equal to n_{c+i}), and for λ the second term is equal to ℓ, and the first term is equal to

$$\begin{aligned} ((n_1, \ldots, n_{c+e}) \circ \alpha_{L'})(j_B(\lambda)) &= ((n_1, \ldots, n_{c+e}) \circ \alpha_{L'})(j_A(l_{c+1})) \\ &= ((n_1, \ldots, n_c, pN) \circ \alpha_L)(l_{c+1}) \\ &= n_1 \mathrm{lk}(L_1, L_{c+1}) + \ldots + n_c \mathrm{lk}(L_c, L_{c+1}) + 0 = \ell. \end{aligned}$$

We have proven that the three different coefficients ϕ of the statement of the result were indeed compatible. Now, since the cabling torus $T = \partial(V(L_{c+1}))$ has infinite image under γ, the result follows from Proposition 4.1 and Theorem 4.11.

□

Note that Theorem 4.13 generalizes Theorem 2.36, where $c = 0, e = 1, \ell = 0$ and $N = n_1 = 1$.

Chapter 5

The L^2-Alexander invariant for fibered knots

In this chapter we prove that the L^2-Alexander invariant of a fibered knot is eventually monomial as a function on the positive real numbers; our proof is based on Fox calculus and elementary operations on operators. This provides an alternative proof of [DFL14, Theorem 8.2] in the case of knot exteriors.

As a consequence, we prove that the L^2-Alexander invariant detects the figure-eight knot.

5.1 Fibered knots and fibered manifolds

Let M be a 3-manifold and let $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ be a 1-cohomology class.

Definition 5.1. The homomorphism ϕ is *fibered* if there exists a surface bundle $p\colon M \to S^1$ and an integer $r \in \mathbb{Z}$ such that the induced homomorphism $p_*\colon \pi_1(M) \to \mathbb{Z}$ is equal to $r\phi$.

A knot K in S^3 is *fibered* if the abelianization $\alpha_K\colon G_K \to \mathbb{Z}$ is fibered.

Proposition 5.2. *([BZH14, Corollary 5.4])*

The exterior $M_K = S^3 \setminus V(K)$ of a fibered knot K of genus $g = g(K)$ is obtained from the product space $\Sigma \times [0;1]$, where $\Sigma = S_{g,1}$ is a compact surface of genus g with connected nonempty boundary, by the identification

$$(x,0) = (h(x),1),\ x \in \Sigma$$

where $h\colon \Sigma \to \Sigma$ is an orientation preserving homeomorphism:

$$M_K = (\Sigma \times [0;1])/h.$$

The homeomorphism h is called the monodromy associated to K.

Furthermore, $G_K = \pi_1(M_K)$ is a semidirect product $G_K = \mathbb{Z} \ltimes G'_K$ where $G'_K \cong \pi_1(\Sigma) \cong \mathbb{F}[a_1,\dots,a_{2g}]$ is the free group on $2g$ generators, and G_K admits the group presentation

$$P = \langle T, a_1, \dots, a_{2g} | Ta_1T^{-1} = h_*(a_1), \dots, Ta_{2g}T^{-1} = h_*(a_{2g})\rangle,$$

where $h_*\colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ is the isomorphism induced by h.

5.2 Eventual monomiality

The following technical property follows from Proposition 2.13 and [CFM97, Theorem 1.10(e)], and establishes that the Fuglede-Kadison determinant of certain operators depending on a parameter $t > 0$ is monomial in t for small and great values of t. Compare with [DFL14, Proposition 8.4].

Proposition 5.3. *Let G be a sofic group, let $\alpha\colon G \to \mathbb{Z}$ denote a surjective group homomorphism and $H = Ker(\alpha)$ its kernel. Let $T \in G$ such that $\alpha(T) = 1$ and let $W \in GL_n(\mathbb{Z}[H])$ be a matrix invertible over $\mathbb{Z}[H]$.*

Let $t > 0$, let $A = R_W = R_{\kappa(G,\alpha,id,t)(W)}$ and let $\widetilde{R_T}\colon \ell^2(G)^n \to \ell^2(G)^n$ denote the diagonal operator with R_T at each coefficient.

Let $N = \max\left(\|A\|_\infty, (\|A^{-1}\|_\infty)^{-1}\right)$.

1. *If $t \in]0; \frac{1}{N}[$, then the operator $t\widetilde{R_T} - A$ is invertible and*

$$\det{}_{\mathcal{N}(G)}\left(t\widetilde{R_T} - A\right) = 1.$$

2. *If $t > N$, then the operator $t\widetilde{R_T} - A$ is invertible and*

$$\det{}_{\mathcal{N}(G)}\left(t\widetilde{R_T} - A\right) = t^n.$$

Proof. Let $t \in]0; \frac{1}{N}[$. The operator $A = R_W$ is invertible and $\det_{\mathcal{N}(G)}(A) = 1$, as a consequence of Proposition 2.13 (3). Since

$$t\widetilde{R_T} - A = (-A) \circ (Id - tA^{-1}\widetilde{R_T}),$$

it follows from Proposition 1.51 that we simply have to prove that $Id - tA^{-1}\widetilde{R_T}$ is invertible of Fuglede-Kadison determinant equal to 1.

For all $u \in [0; 1]$, the operator

$$S_u := Id - utA^{-1}\widetilde{R_T}$$

is invertible, since

$$\|utA^{-1}\widetilde{R_T}\|_\infty \leqslant t\|A^{-1}\|_\infty < \frac{1}{N}\|A^{-1}\|_\infty \leqslant 1.$$

In particular, $S_0 = Id$ and $S_1 = Id - tA^{-1}\widetilde{R_T}$.

It then follows from [CFM97, Theorem 1.10(e)] that

$$\det{}_{\mathcal{N}(G)}(S_1) = \frac{\det{}_{\mathcal{N}(G)}(S_1)}{\det{}_{\mathcal{N}(G)}(S_0)} = \exp\left(\mathrm{Re}\left(\int_0^1 \mathrm{tr}_{\mathcal{N}(G)}\left(S_u^{-1} \circ \frac{\partial S_v}{\partial v}|_{v=u}\right) du\right)\right).$$

We compute

$$\frac{\partial S_v}{\partial v} = -tA^{-1}\widetilde{R_T}$$

and

$$S_u^{-1} = Id + utA^{-1}\widetilde{R_T} + \left(utA^{-1}\widetilde{R_T}\right)^2 + \dots$$

Thus the operator

$$S_u^{-1} \circ \frac{\partial S_v}{\partial v}|_{v=u} = -tA^{-1}\widetilde{R_T} - ut^2\left(A^{-1}\widetilde{R_T}\right)^2 - \dots$$

is an infinite sum of terms R_g where $\alpha(g) \geqslant 1$ (since the terms of A^{-1} come from H), and its Von Neumann trace $\mathrm{tr}_{\mathcal{N}(G)}$ is therefore zero (since none of these g can be the neutral element of G).

Consequently $\det_{\mathcal{N}(G)}(S_1) = 1$ and the first part of the proposition is proven.

To prove the second part of the proposition, let $t > N$ and observe that

$$t\widetilde{R_T} - A = \left(t\widetilde{R_T}\right) \circ \left(Id - \frac{1}{t}\widetilde{R_T}^{-1} A\right);$$

since $t\widetilde{R_T}$ is invertible of Fuglede-Kadison determinant t^n, as a consequence of Proposition 1.51, it suffices to prove that the operator $Id - \frac{1}{t}\widetilde{R_T}^{-1} A$ is invertible of Fuglede-Kadison determinant 1. This follows from the fact that $t > \|A\|_\infty$, from [CFM97, Theorem 1.10(e)] and from the definition of A, similarly as above. □

5.3 The L^2-Alexander torsions for fibered manifolds

Let N be a 3-manifold, and let $\phi\colon \pi_1(N) \to \mathbb{Z}$. We assume that there exists a surface bundle $p\colon N \to S^1$ whose fiber is a surface Σ and such that there exists $r \in \mathbb{Z}$, $p_* = r\phi$. In this case the manifold N is called *fibered*, and can be obtained as:

$$N = (\Sigma \times [0;1])/h$$

where $h\colon \Sigma \to \Sigma$ is a self diffeomorphism called the *monodromy* and $(x, 0)$ is identified with $(h(x), 1)$ for all $x \in \Sigma$. It follows from the works of Thurston that when N is an hyperbolic manifold, the monodromy h is *pseudo-Anosov*, i.e. there exist two transverse measured foliations on Σ and a real number $\lambda > 1$ such that h is isotopic to a diffeomorphism that preserves the two foliations and multiply their transverse measures by $\lambda > 1$ and $\lambda^{-1} < 1$ respectively. The number $\lambda > 1$ is called the *dilatation factor* associated to the monodromy h. See [FLP+12] for details.

The following theorem establishes that the L^2-Alexander torsions $\left(t \mapsto T^{(2)}(N, \phi, \gamma)(t)\right)$ are monomial for small and great values of t.

Theorem 5.4. *([DFL14, Theorem 8.2]) Let $(N, \phi, \gamma\colon \pi_1(N) \to G)$ be an admissible triple such that $\phi \in H^1(N, \mathbb{Z})$ is fibered and such that G is sofic. Then there exists a representative τ of $T^{(2)}(N, \phi, \gamma)$ such that for a certain $T > 0$, we have $\tau(t) = 1$ if $0 < t < \frac{1}{T}$ and $\tau(t) = t^{x_N(\phi)}$ if $t > T$.*

This is true for T equal to the dilatation factor of the monodromy associated to ϕ.

5.4 The L^2-Alexander invariant for fibered knots

The following theorem establishes that the L^2-Alexander invariant of a knot is monomial in t for extremal values of t, and that the span between the degrees of the two monomials is equal to twice the genus of the knot. Compare with the monicity property of the Alexander polynomial for fibered knots (see Theorem A.7). This theorem is a variant of Theorem 5.4 for knot exteriors, in the sense that the bound N for monomiality is computed from operator norms associated to the monodromy, and is not necessarily equal to the dilatation factor of this monodromy as in Theorem 5.4.

Theorem 5.5. *Let K be a fibered knot of genus g, and*

$$P = \langle T, a_1, \ldots, a_{2g} | Ta_1T^{-1} = h_*(a_1), \ldots, Ta_{2g}T^{-1} = h_*(a_{2g})\rangle$$

the presentation of its group G_K associated to the fibration.

There exists a real number $N > 1$ and a representative $(t \mapsto \delta(t))$ of $\Delta_K^{(2)}$ such that for all $t \in]0; \frac{1}{N}[\cup]N; +\infty[$, (P, t) has Property $\mathcal{I}$ and

$$\delta(t) = \begin{cases} 1 & \text{if } t < \frac{1}{N}, \\ t^{2g} & \text{if } t > N. \end{cases}$$

Example 5.6. If $K = 4_1$ is the figure-eight knot, then there exists a $T > 1$ such that

$$\Delta_K^{(2)}(t) = \begin{cases} 1 & \text{if } t < \frac{1}{T}, \\ \exp\left(\dfrac{\mathrm{vol}(4_1)}{6\pi}\right) \approx 1.113 & \text{if } t = 1, \\ t^2 & \text{if } t > T. \end{cases}$$

The smallest known T satisfying the above is

$$T = \frac{3 + \sqrt{5}}{2} \approx 2.618,$$

the dilatation factor associated to the monodromy.

Proof. Let K be a fibered knot of genus g, Σ the associated fibre, a once-punctured surface of genus g. The group $\pi_1(\Sigma)$ is a free group on $2g$ elements $a_1, \dots a_{2g}$, and G_K has the presentation $P = \langle T, a_1, \dots, a_{2g} | T a_i T^{-1} = W_i(a_j) \rangle$ where $W_i(a_j) = h_*(a_j)$ is a word in the letters a_j (see Proposition 5.2).

The abelianization $\alpha_K \colon G_K \to \mathbb{Z}$ sends T to 1 and the a_i to 0.

Since the presentation P is of deficiency one, we can compute the L^2-Alexander invariant of K from the Fox matrix F_P, using Theorem 2.28.

The Fox matrix F_P is written

$$F_P = \begin{pmatrix} 1 - Ta_1T^{-1} & 1 - Ta_2T^{-1} & \dots & 1 - Ta_nT^{-1} \\ T - w_{1,1} & -w_{1,2} & \dots & -w_{1,n} \\ -w_{2,1} & T - w_{2,2} & \dots & -w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n,1} & -w_{n,2} & \dots & T - w_{n,n} \end{pmatrix},$$

where $n = 2g$ and $w_{i,j} = \dfrac{\partial W_j(a_1, \dots, a_{2g})}{\partial a_i} \in \mathbb{Z}[G_K]$ is a linear combination of words on the generators $a_1, \dots, a_{2g}$; these words are all sent to zero by the abelianization α_K.

Let W denote the jacobian matrix

$$W = \begin{pmatrix} w_{1,1} & \dots & w_{1,n} \\ \vdots & \ddots & \vdots \\ w_{n,1} & \dots & w_{n,n} \end{pmatrix} = \left(\frac{\partial(h_*(a_j))}{\partial a_i}\right)_{1 \leqslant i,j \leqslant n}$$

and $A = R_{\psi_{K,t}(W)}$ the associated operator.

For all $t > 0$, the operator $R_{\psi_{K,t}(F_{P,1})}$ is of the form

$$R_{\psi_{K,t}(F_{P,1})} = t\widetilde{R_T} - A = \begin{pmatrix} tR_T - A_{1,1} & -A_{1,2} & \dots & -A_{1,n} \\ -A_{2,1} & tR_T - A_{2,2} & \dots & -A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{n,1} & -A_{n,2} & \dots & tR_T - A_{n,n} \end{pmatrix},$$

where $n = 2g$ and $A_{i,j} = R_{\psi_{K,t}(w_{i,j})}$ for all i, j, and $\widetilde{R_T}$ is the diagonal operator with R_T at each coefficient.

Since $h_*\colon \mathbb{F}[a_1, \dots, a_n] \to \mathbb{F}[a_1, \dots, a_n]$ is an isomorphism, $(h_*(a_1), \dots, h_*(a_n))$ is a basis for the free group $\mathbb{F}[a_1, \dots, a_n]$, it follows from [Bir73] that the jacobian matrix

$$W = \left(\frac{\partial(h_*(a_j))}{\partial a_i}\right)_{1 \leqslant i,j \leqslant n}$$

is invertible over $\mathbb{Z}[\mathbb{F}[a_1, \dots, a_n]]$. Let $N = \max\left(\|A\|_\infty, (\|A^{-1}\|_\infty)^{-1}\right)$. It follows from Proposition 5.3 that if $t \in]0; \frac{1}{N}[\cup]N; +\infty[$, then the operator $R_{\psi_{K,t}(F_{P,1})} = t\widetilde{R_T} - A$ is invertible and

1. If $t \in]0; \frac{1}{N}[$, then
$$\det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(F_{P,1})}\right) = 1.$$
2. If $t > N$, then
$$\det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(F_{P,1})}\right) = t^n.$$

It follows from Theorem 2.28 that for all $t \in]0; \frac{1}{N}[\cup]N; +\infty[$, (P, t) has Property $\mathcal{I}$ and:

$$\Delta_K^{(2)}(t) \doteq \frac{\det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(F_{P,1})}\right)}{\max(1,t)^{|\alpha_K(T)|-1}} = \det{}_{\mathcal{N}(G)}\left(R_{\psi_{K,t}(F_{P,1})}\right) = \begin{cases} 1 & \text{if } t < \frac{1}{N}, \\ t^{2g} & \text{if } t > N. \end{cases}$$

□

5.5 The L^2-Alexander invariant detects the figure-eight knot

Theorem 5.7. *Let K be a knot in S^3 such that its L^2-Alexander invariant $\Delta_K^{(2)}$ is of the form:*

$$\Delta_K^{(2)}(t) = \begin{cases} 1 & \text{if } t < \frac{1}{T} \\ \exp\left(\dfrac{vol(4_1)}{6\pi}\right) \approx 1.113 & \text{if } t = 1 \\ t^2 & \text{if } t > T \end{cases}$$

for a certain $T \in [1, +\infty[$. Then K is the figure-eight knot 4_1.

Proof. Let K be a knot satisfying the assumptions of the theorem. It follows from Theorem 2.27 that $\mathrm{vol}(M_K) = \mathrm{vol}(4_1)$.

The JSJ-decomposition of M_K contains Seifert-fibered and hyperbolic pieces; these pieces are all sub-manifolds of S^3 whose boundary is a non-empty finite union of tori.

A compact hyperbolic 3-manifold with toroidal boundary and at least three boundary components has volume at least three times the volume of a regular ideal terahedron (see [Ada88]), thus it has volume greater than 3. Moreover, a compact hyperbolic 3-manifold with toroidal boundary and two boundary components has volume at least 3.66... (the volume of the Whitehead link) by [Ago10]. Since the simplicial volume $\mathrm{vol}(M_K)$ of M_K is equal to $\mathrm{vol}(4_1) = 2.029...$, it is smaller than 3 and since $\mathrm{vol}(M_K)$ is equal to the sum of the simplicial volumes of the JSJ-pieces of M_K, we conclude that all the hyperbolic pieces in the JSJ decomposition of M_K have exactly one boundary component (which is a torus).

A compact hyperbolic 3-manifold with one toroidal boundary component has volume at least $\mathrm{vol}(4_1) = 2.029...$; among these manifolds, only the exterior of the figure-eight

knot M_{4_1} and its sibling M' (which can be described as the $(-5/1)$-Dehn filling on the Whitehead link) have volume equal to this number (see [CM01]).

This implies that the JSJ decomposition of M_K has exactly one hyperbolic piece N, which is homeomorphic to M_{4_1} or M'; the other pieces are Seifert-fibered manifolds that we denote by S_j.

The manifold N is a compact sub-manifold of S^3 with boundary a single torus, thus N is the exterior $M_{K'}$ of a knot K' (see for example [Bud06, Proposition 2.2]). Since the first homology group of the manifold M' is

$$H_1(M';\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z},$$

the manifold M' cannot be the exterior of a knot in S^3 and therefore $N = M_{4_1}$.

Since the manifold M_K has a JSJ decomposition composed of M_{4_1} and Seifert-fibered manifolds S_j, it follows from [Bud06, Theorem 4.18] and Section 2.3.6 that K is obtained from 4_1 by a finite number of

- cablings,
- connected sums with an iterated torus knot.

The knot 4_1 is fibered, all iterated torus knots are fibered, the connected sum of two fibered knots is fibered and all cablings of a fibered knot are fibered (see for example [Rol90, p. 326] and [Sta78]). Thus the knot K is fibered.

It follows from Theorem 5.5 and the assumptions on K that $g(K) = 1$. Since the only hyperbolic fibered knot of genus 1 is 4_1 (see [BZH14, Proposition 5.14]) we conclude that K is the figure-eight knot 4_1.

□

Chapter 6

Open questions and future prospects

6.1 Approximate values of the L^2-Alexander torsion

In Annex B.1 we present a method to compute upper bounds $(\Delta_N(t))_{N\geqslant 1}$ of the L^2-Alexander invariant $\Delta_K^{(2)}(t)$ for a 2-generator hyperbolic knot K. Since the computing time of $\Delta_N(t)$ is exponential in N, we cannot compute many of these upper bounds. We thus wish to better the algorithm and reduce its complexity, in order to find more bounds for L^2-Alexander invariants of knots.

Similar computations of upper bounds of Fuglede-Kadison determinants could offer approximations for more general L^2-Alexander torsions $T^{(2)}(M, \phi, \gamma)$.

6.2 Regularity properties

All the upper approximations of $\Delta_K^{(2)}$ we computed with the algorithm described in Annex B.1 turn out to be convex functions. Since convexity is preserved by pointwise convergence, we state the following conjecture:

Conjecture 6.1. *Let K be a knot in S^3 and I an open interval contained in $\mathcal{D}_K$. The L^2-Alexander invariant $\left(t \mapsto \Delta_K^{(2)}(t)\right)$ is a convex function on I, therefore it is continuous on I.*

Continuity properties of the L^2-Alexander invariant are related to continuity properties of the Fuglede-Kadison determinant on injective operators. The Fuglede-Kadison determinant is known to be continuous on invertible operators (see [CFM97, Theorem 1.10 (d)]), but its regularity properties on certain classes of noninvertible injective operators remain unknown. W. Lück constructed an explicit example of a sequence of operators converging for the operator norm and on which the Fuglede-Kadison determinant is not continuous.

Note that we cannot hope the L^2-Alexander invariant $\left(t \mapsto \Delta_K^{(2)}(t)\right)$ to be differentiable everywhere on $\mathbb{R}_{>0}$, since it is known to be of the form $(t \mapsto \max(1, t)^n)$, $n \in \mathbb{N}$, for iterated torus knots.

6.3 Virtually fibered manifolds

It follows from Theorem 5.4 that the L^2-Alexander torsions associated to a fibered manifold M and a fibered cohomology class ϕ are eventually monomial.

Moreover, the L^2-Alexander torsions satisfy the following covering formula:

Proposition 6.2. *[DFL14, Lemma 5.3]*

Let $(N, \phi, \gamma\colon \pi = \pi_1(N) \to G)$ be an admissible triple . Let $p\colon \widehat{N} \to N$ be a finite d-sheeted regular cover such that Ker$(\gamma) \subset \widehat{\pi} = \pi_1(\widehat{N})$. Let $\iota = p_*\colon \widehat{\pi} \to \pi$ denote the group inclusion induced by the covering. Then

$$T^{(2)}(\widehat{N}, \phi \circ \iota, \gamma \circ \iota)(t) \doteq \left(T^{(2)}(N, \phi, \gamma)(t)\right)^d.$$

It is now natural to wonder if certain L^2-Alexander torsions of a virtually fibered manifold M are eventually monomial.

The main difficulty in combining Theorem 5.4 and Proposition 6.2 is the following. For (M, ϕ, γ) an admissible triple with M a virtually fibered manifold, and $p\colon \widehat{M} \to M$ a finite regular cover such that $\widehat{M}$ is fibered, the homomorphism $\phi \circ p_*$ is not necessarily fibered.

Computations of approximations of the L^2-Alexander invariant $\left(t \mapsto \Delta_K^{(2)}(t)\right)$ of twist knots (see Annex B.2) let us observe that the first approximations of the maps $\Delta_K^{(2)}$ behave similarly for different twist knots K.

The twist knots K are not fibered, except for $O, 3_1, 4_1$, but are all virtually fibered (see [Lei02] and [Wal05]) and are all of genus one (except for O). From the observations we just mentioned and the fact that twist knots are virtually fibered, we conjecture that all non-trivial twist knots have an eventually monomial L^2-Alexander invariant like 3_1 and 4_1.

Conjecture 6.3. *Let K be a non-trivial twist knot. The L^2-Alexander invariant $\Delta_K^{(2)}(t)$ is defined for all $t > 0$ and is equal to*

$$\Delta_K^{(2)}(t) = \begin{cases} 1 & \text{if } t < \frac{1}{T}, \\ \exp\left(\dfrac{vol(K)}{6\pi}\right) & \text{if } t = 1, \\ t^2 & \text{if } t > T. \end{cases}$$

where $T > 1$ is a real number depending on K.

6.4 Asymptotic properties

It follows from results of W. Thurston and T. Jorgensen that if one does p/q-Dehn filling on a hyperbolic link complement M_L, with $p^2 + q^2$ large enough the resulting manifold $M_{p/q}$ will also be hyperbolic with volume approaching the volume of the original link complement vol(L) by inferior values, as $p^2 + q^2 \to \infty$.

In particular, as $p^2 + q^2 \to \infty$, by Theorem 3.7,

$$\begin{aligned} T^{(2)}(M_L, 0, Q_{p/q})(1) &= T^{(2)}(M_{p/q}, 0, id)(1) \\ &= \exp\left(\frac{vol(M_{p/q})}{6\pi}\right) \\ &\underset{p^2+q^2\to\infty}{\longrightarrow} \exp\left(\frac{vol(M_L)}{6\pi}\right) \\ &= T^{(2)}(M_L, 0, id)(1) \end{aligned}$$

where $Q_{p/q}\colon \pi_1(M_L) \to \pi_1(M_{p/q})$ is the group epimorphism induced by the Dehn filling.

It is now natural to wonder if there exists a similar convergence of the L^2-Alexander torsions for $t \neq 1$.

Let us denote $L = L_1 \cup \ldots \cup L_{c+1}$ such that the previous p/q-Dehn filling is applied on the component L_{c+1}. The result manifold $M_{p/q}$ is the exterior of a c-link in the lens space $L(p,q)$, whose components are the images of $L_1, \ldots, L_c$. The abelianization of the fundamental group of $M_{p/q}$ is

$$\alpha_{p/q}\colon \pi_1(M_{p/q}) \to \mathbb{Z}^c \oplus \mathbb{Z}/p\mathbb{Z}.$$

Let $Q\colon \mathbb{Z}^c \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}^c$ the quotient homomorphism that trivialises the torsion elements. Any group homomorphism $\phi\colon \pi_1(M_{p/q}) \to \mathbb{Z}$ factors through $Q \circ \alpha_{p/q}$ and is thus written

$$\phi = (n_1, \ldots, n_c) \circ Q \circ \alpha_{p/q}$$

where $n_1, \ldots, n_c \in \mathbb{Z}$. Consequently the following diagram is commutative:

$$\begin{tikzcd}
G_L = \pi_1(M_L) \arrow[r, "Q_{p/q}"] \arrow[dd, "\alpha_L"] & \pi_1(M_{p/q}) \arrow[d, "\alpha_{p/q}"'] \arrow[ddr, "\phi"] & \\
& \mathbb{Z}^c \oplus \mathbb{Z}/p\mathbb{Z} \arrow[d, "Q"'] & \\
\mathbb{Z}^{c+1} \arrow[r, "(Id\ \ 0)"'] & \mathbb{Z}^c \arrow[r, "(n_1, \ldots, n_c)"'] & \mathbb{Z}
\end{tikzcd}$$

Question 6.4. Do we have

$$T^{(2)}(M_L, (n_1, \ldots, n_c, 0) \circ \alpha_L, Q_{p/q})(t) \underset{p^2+q^2 \to \infty}{\longrightarrow} T^{(2)}(M_L, (n_1, \ldots, n_c, 0) \circ \alpha_L, id)(t)$$

for all $t > 0$ and for all $n_1, \ldots, n_c \in \mathbb{Z}$?

Appendix A

Knots and groups

A.1 Group presentation for a cable knot

The aim of this section is to give a detailed proof of the following technical result (see Proposition 1.23):

Proposition A.1. *Let us consider the (p,q)-cable knot S of companion C.*

(1) There exists $P_C = \langle a_1, \dots, a_k | r_1, \dots, r_{k-1} \rangle$ a Wirtinger presentation of G_C such that

$$P_S = \langle a_1, \dots, a_k, x, \lambda | r_1, \dots, r_{k-1}, x^p a_k^{-q} \lambda^{-p}, \lambda^{-1} W(a_i) \rangle$$

is a presentation of G_S, with x and λ the homotopy classes of the core and a longitude of T_C, and $W(a_i)$ a word in the $a_1, \dots, a_k$.

(2) Furthermore, $\alpha_S(x) = q$, $\alpha_S(\lambda) = 0$ and $\alpha_S(a_i) = p$, for $i = 1, \dots, k$.

Compare with [BZH14, Exercise 9.4]. In order to prove this result we will develop a general way of computing a group presentation for the exterior of a link in a solid torus, a method interesting in its own right.

A.1.1 Group of a torus knot pattern

Let T_{int} be an open solid torus and T_{ext} an open tubular neighbourhood of T_{int}, thus a second solid torus. We will draw the torus knot $K = T(p,q)$ on the boundary of T_{int}. Let us take pt any point on $\partial T_{int} \smallsetminus K$. It will be the base point for all the following fundamental groups. Figure A.1 (where $p = 3$ and $q = 4$) should clarify the notations.

We want to prove the following result:

Lemma A.2. *The presentation $P_{p,q} = \langle x, y, \lambda | x^p = \lambda^p y^q, \lambda y = y \lambda \rangle$ is a presentation of $\widetilde{G}_{p,q} = \pi_1(T_{ext} \smallsetminus K)$. Furthermore, the elements of $\widetilde{G}_{p,q}$ represented by λ and y are the homotopy classes of a longitude curve and a meridian curve of $T_{ext} \setminus \overline{T_{int}}$, and x is the homotopy class of the core of T_{int}.*

The following proof has been inspired by the computation of the classical presentation of torus knot groups (see for example [Rol90, Section 3.C]).

Proof. We will use the Seifert-van Kampen theorem (see Theorem 1.12).

We denote $U_1 = T_{ext} \smallsetminus (T_{int} \sqcup K)$, $U_2 = \overline{T_{int}} \smallsetminus K$, $W = T_{ext} \smallsetminus K$, $V = \partial T_{int} \smallsetminus K$ and G_1, G_2, G, G_0 their respective fundamental groups (for the same base point pt in V).

The space U_1 can be deformed to $T_{ext} \smallsetminus T_{int}$ (by «filling up K»), and so it is homotopically equivalent to a 2-torus. Thus $\langle y, \lambda | y\lambda = \lambda y \rangle$ is a presentation of G_1, where y

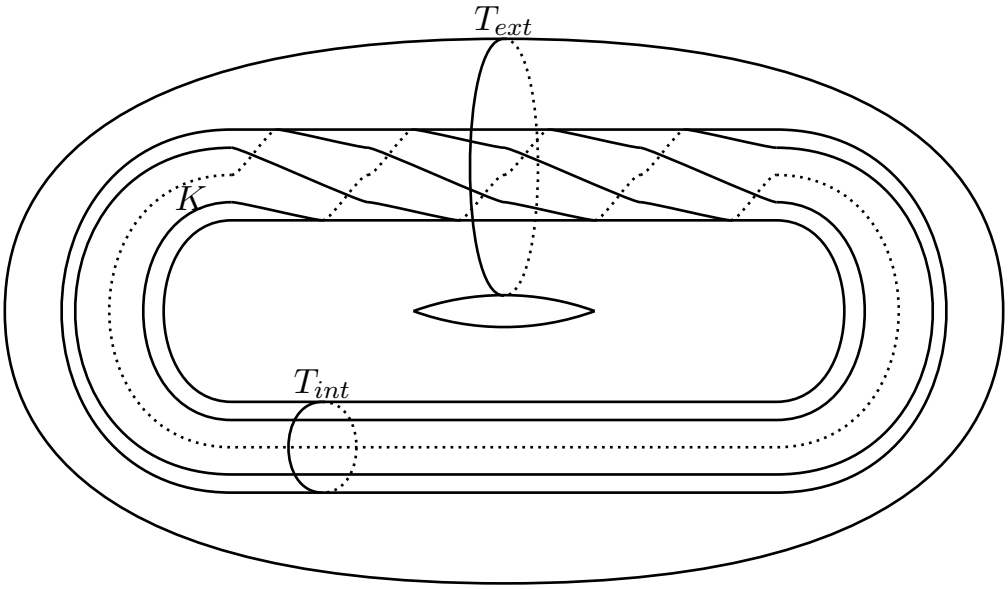


Figure A.1 – The inside and outside tori T_{int} and T_{ext} and the (p, q)-torus knot K

and λ are the homotopy classes of a natural meridian-longitude system of $T_{ext} \setminus T_{int}$, see Figure A.2.

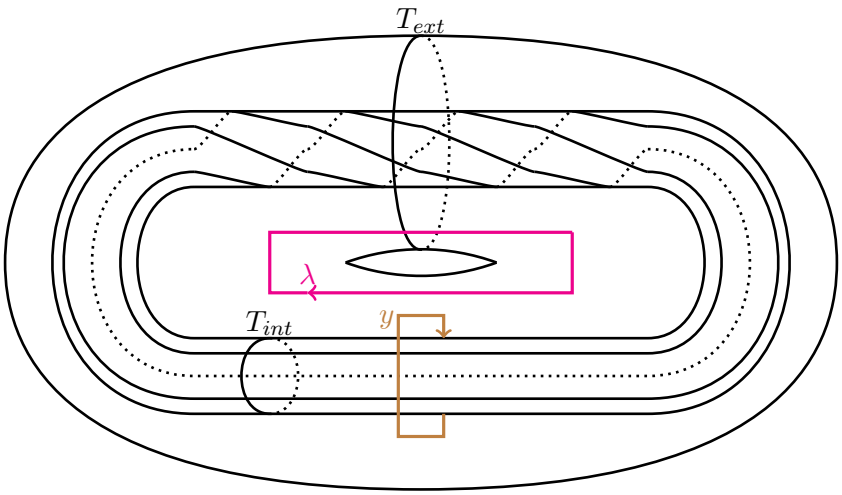


Figure A.2 – A natural meridian-longitude system

The space U_2 can be deformed to T_{int} by a similar process, therefore G_2 admits the presentation $\langle x|-\rangle$, where x is the homotopy class of the core of T_{int}, see Figure A.3.

The space V is homeomorphic to an annulus, thus G_0 admits the presentation $\langle z|-\rangle$ where the generator z is drawn on Figure A.4. Note that z follows the direction of the strands, that is the same as the one of the core if $p > 0$ and the opposite if $p < 0$.

The inclusions $V \subset U_1$ and $V \subset U_2$ induce homotopy maps that send z to x^p and $y^q\lambda^p$ respectively. We hope the figures make this point clearer.

Thus, by the Seifert-van Kampen theorem, $G = \widetilde{G}_{p,q}$ admits the presentation

$$P_{p,q} = \langle x, y, \lambda | x^p = \lambda^p y^q, \lambda y = y\lambda \rangle.$$

□

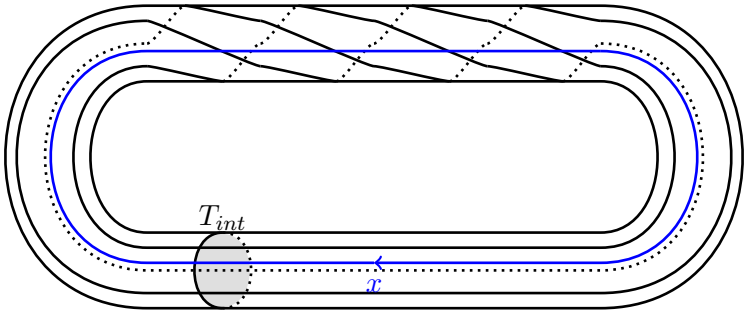


Figure A.3 – The generator x, core of T_{int}

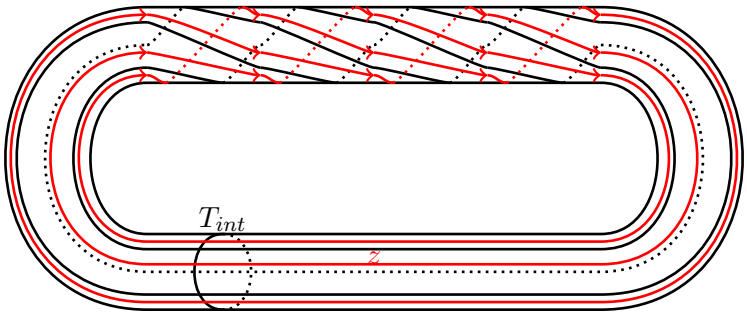


Figure A.4 – The generator z of G_0

A.1.2 A meridian-longitude system in the group presentation of the pattern

In this section we will explain how to obtain in general a group presentation for $G_{P\subset T_P}=\pi_1(T_P\setminus P)$ containing the homotopy classes of a preferred meridian-longitude pair of T_P as generators. This will not help us to prove Proposition 1.23, but this illustrates that the hypotheses of Lemma A.4 are not as restrictive as we could have thought.

The method will use Wirtinger presentations, and thus is not the same as the one used in Lemma A.2, but it will work for any pattern P.

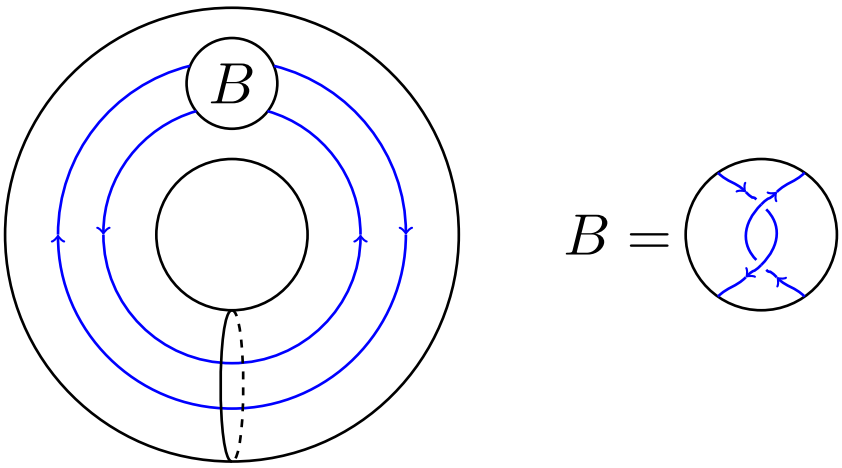


Figure A.5 – The pattern seen as one (m,m)-tangle B and m parallel strands

First, notice that we can draw P as m parallel strands (not necessarily going in the same direction) and a (m,m)-tangle B. See Figure A.5, where we took $m=2$ and P the

Whitehead double pattern.

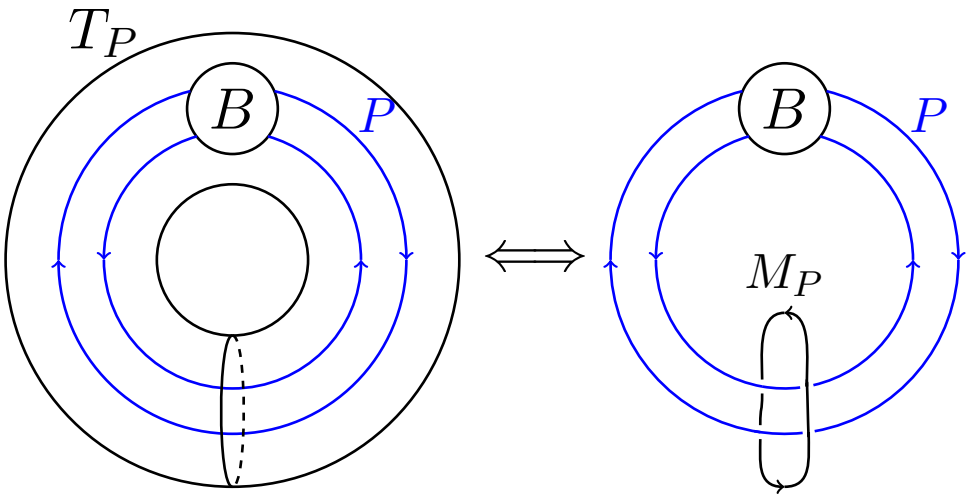


Figure A.6 – The knot P inside T_P is the same as the 2-link $P \cup M_P$ inside S^3

To compute a presentation of $G_{P \subset T_P} = \pi_1(T_P \setminus P)$, we remark that this group is naturally isomorphic to $G_{P \cup M_P} = \pi_1(S^3 \setminus (P \cup M_P))$ where M_P is a meridian curve of T_P, see Figure A.6.

Now we can compute a Wirtinger presentation of $G_{P \cup M_P}$ by the well-known process of the same name (see for example [BZH14, Section 3.B]).

The Wirtinger generators are:

- λ the generator for the arc of M_P that passes over the m strands, which corresponds naturally to a longitude loop of T_P.
- $\lambda_1, \ldots, \lambda_{m-1}$ the other generators of M_P, listed from the outside to the inside.
- $a_1, \ldots, a_m$ and $a'_1, \ldots, a'_m$ the generators for the m strands of P, listed from the outside to the inside, such that $a'_i = \lambda a_i \lambda^{-1}$.
- $b_1, \ldots, b_k$ the generators for the arcs strictly inside the tangle B.

Figure A.7 pictures them partially (as always, the base point is assumed to be above the diagram).

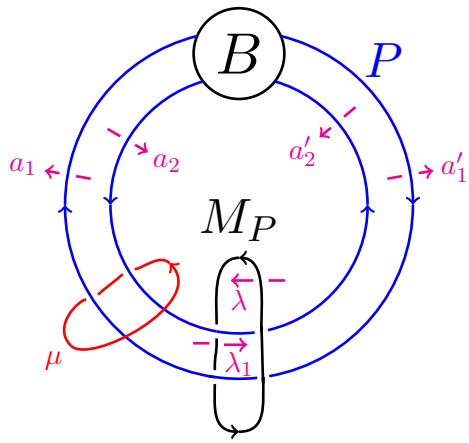


Figure A.7 – The Wirtinger generators

Note that we can assume that the a_i and the a'_i are all distinct, since we can add a first Reidemeister move twist at each of the $2m$ points of entrance of P into B.

The relators are:

- $r_1, \dots, r_{m+k-1}$, which are words in the a_i, a'_i and b_j, corresponding to the crossings inside B.
- $a'_i = \lambda a_i \lambda^{-1}$ for the crossings where M_P passes over P.
- $\lambda_1 = a_1^{e_1} \lambda a_1^{-e_1}, \lambda_2 = a_2^{e_2} \lambda_1 a_2^{-e_2}, \dots, \lambda = a_m^{e_m} \lambda_{m-1} a_m^{-e_m}$ for the crossings where M_P passes under P (here $e_i = \pm 1$ depends on the orientation of the i-th strand).

Thus $G_{P \cup M_P}$ admits the Wirtinger presentation

$$Q = \langle a_i, a'_i, b_j, \lambda_\alpha, \lambda | r_l, a'_i = \lambda a_i \lambda^{-1}, \lambda_1 = a_1^{e_1} \lambda a_1^{-e_1}, \dots, \lambda = a_m^{e_m} \lambda_{m-1} a_m^{-e_m} \rangle,$$

where $i = 1, \dots, m$, $j = 1, \dots k$, $\alpha = 1, \dots, m-1$ and $l = 1, \dots, m+k-1$.

A preferred longitude of T_P is among the generators of Q, as λ. We also want a meridian μ. As shown in Figure A.7, μ is equal to $a_m^{e_m} \dots a_1^{e_1}$. We can thus write

$$Q_1 = \left\langle a_i, a'_i, b_j, \lambda_\alpha, \lambda, \mu \,\middle|\, \begin{array}{c} r_l, a'_i = \lambda a_i \lambda^{-1}, \lambda_1 = a_1^{e_1} \lambda a_1^{-e_1}, \dots, \lambda = a_m^{e_m} \lambda_{m-1} a_m^{-e_m}, \\ \mu = a_m^{e_m} \dots a_1^{e_1} \end{array} \right\rangle$$

an other presentation of $G_{P \cup M_P}$, that has the form we wanted.

Now we can simplify this presentation and get rid of the generators λ_α.

By substituting λ_α with $a_\alpha^{e_\alpha} \lambda_{\alpha-1} a_\alpha^{-e_\alpha}$ from $\alpha = 1$ to $m-1$ (with the convention $\lambda_0 = \lambda$), we obtain the simplified presentation

$$Q_2 = \langle a_i, a'_i, b_j, \lambda, \mu | r_l, a'_i = \lambda a_i \lambda^{-1}, \lambda = (a_m^{e_m} \dots a_1^{e_1}) \lambda (a_1^{-e_1} \dots a_m^{-e_m}), \mu = a_m^{e_m} \dots a_1^{e_1} \rangle$$

that is equivalent to

$$Q_3 = \langle a_i, a'_i, b_j, \lambda, \mu | r_l, a'_i = \lambda a_i \lambda^{-1}, \lambda \mu = \mu \lambda, \mu = a_m^{e_m} \dots a_1^{e_1} \rangle.$$

In conclusion, the group of the pattern knot P inside its solid torus T_P admits a group presentation of the form of Q_3. This presentation is simple in the sense that the generators a_i, a'_i, b_j and the relators r_l can all be read of the diagram of P. Moreover, Q_3 contains a preferred meridian-longitude pair of T_P in its generators.

Remark A.3. This method gives us the (simplified) presentation

$$\langle b, \lambda, \mu | \lambda \mu \lambda^{-1} \mu^{-1}, b \lambda b \lambda^{-1} b^{-1} \lambda \mu b^{-1} \lambda^{-1} \rangle$$

for the Whitehead link.

A.1.3 Group presentation of a satellite knot

The following lemma gives us a group presentation of the satellite knot group when we know a presentation of the pattern group with a preferred meridian-longitude pair of the pattern torus among its generators and any presentation of the companion group.

Lemma A.4. *Let T be a tubular neighbourhood of T_C distinct from it. We will take pt any point in $T \setminus T_C$, it will be the base point for all the following fundamental groups. Notice that $G_{P\subset T_P} = \pi_1(T \setminus S_{C,P})$ is isomorphic to $\pi_1(T_P \setminus P,\ pt')$ where $pt' = h_{PC}^{-1}(pt)$.*

Suppose there exists $P_{P\subset T_P} = \langle b_1, \ldots, b_{l-1}, \lambda, \mu | s_1, \ldots, s_l\rangle$ a presentation of $G_{P\subset T_P}$ where λ and μ are the homotopy classes of a longitude curve and a meridian curve of T_P.

Then there exists a presentation $P_C = \langle a_1, \ldots, a_k | r_1, \ldots, r_{k-1}\rangle$ of G_C and a presentation

$$P_S = \langle a_1, \ldots, a_k, b_1, \ldots, b_{l-1}, \lambda, \mu | r_1, \ldots, r_{k-1}, s_1, \ldots, s_{l-1}, \lambda^{-1}W(a_i), a_k^{-1}\mu\rangle$$

of $G_S = \pi_1(S^3 \setminus S_{C,P})$, with $W(a_i)$ a word in the $a_i, i = 1, \ldots, k$.

Proof. We will use the Seifert-van Kampen theorem with the base point pt. We denote $W = S^3 \setminus S_{C,P}$, $U_C = S^3 \setminus \overline{T_C}$, $U_P = T \setminus S_{C,P}$, $V = T \setminus \overline{T_C}$, and $G_S, G_C, G_{P\subset T_P}, G_0$ their respective fundamental groups.

The drawings of Figure A.8 are meant to represent an angular fraction of the C-shaped sets, a fraction that contains the «essence of the pattern P» and also the base point pt. They are here to make perfectly clear what W, U_C, U_P, V are.

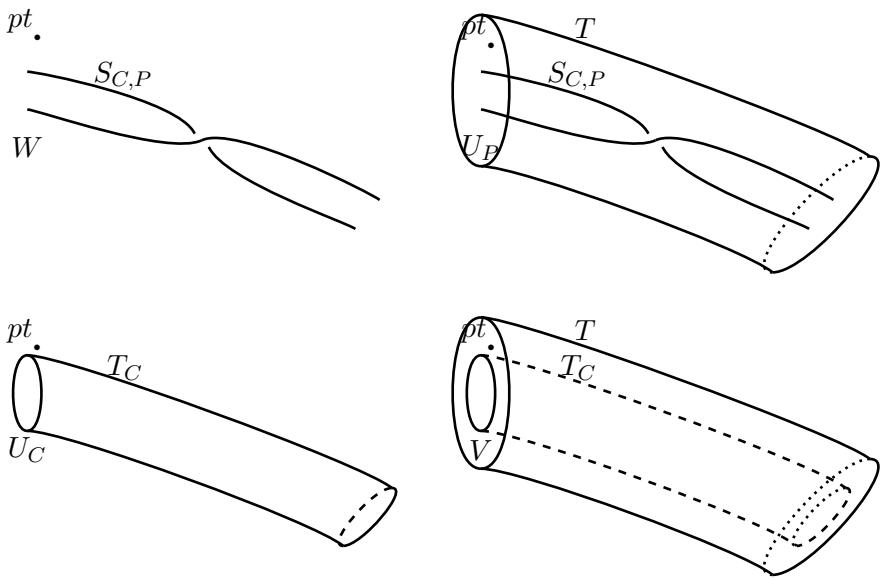


Figure A.8 – The four open sets for the Seifert-van Kampen theorem

We take a Wirtinger presentation $P_C = \langle a_1, \ldots, a_k | r_1, \ldots, r_{k-1}\rangle$ of $G_C = \pi_1(S^3 \setminus C) = \pi_1(S^3 \setminus \overline{T_C}) = \pi_1(U_C)$ associated to a planar regular diagram projection of C.

We then consider P inside T_P. The open set $U_P = T \setminus S_{C,P}$ is homotopy equivalent to $T_C \setminus S_{C,P}$, which is the image of $T_P \setminus P$ by the homeomorphism h_{PC}. Thus $\pi_1(U_P) = G_{P\subset T_P}$. Let λ denote a longitude of T_P and the corresponding element of $G_{P\subset T_P}$.

The space V is homotopy equivalent to a 2-torus, thus G_0 admits the group presentation $\langle \lambda_0, \mu_0 | \lambda_0\mu_0\lambda_0^{-1}\mu_0^{-1}\rangle$, where (μ_0, λ_0) is the homotopy class of a preferred meridian-longitude pair.

The inclusion $V \subset U_C$ maps μ_0 to any meridian of G_C, for instance a_k, and λ_0 to $W(a_i)$ a word in the a_i such that $W(a_i)$ is a longitude loop of the knot C.

The inclusion $V \subset U_P$ maps μ_0 to μ (a meridian of ∂T_P that passes around the m strands), and λ_0 to λ.

Hence, by the Seifert-van Kampen theorem,

$$P = \langle a_1, \dots, a_k, b_1, \dots, b_{l-1}, \lambda, \mu | r_1, \dots, r_{k-1}, s_1, \dots, s_{l-1}, \lambda^{-1}W(a_i), a_k^{-1}\mu \rangle$$

is a presentation of $G_S = \pi_1(W) = \pi_1(S^3 \setminus S_{C,P})$.

□

A.1.4 Details of the proof

Let us prove (1) of the Proposition 1.23.

Let us consider the cable knot S of companion C and pattern $T(p,q)$. There exists $P_C = \langle a_1, \dots, a_k | r_1, \dots, r_{k-1} \rangle$ a Wirtinger presentation of $G_C = \pi_1(S^3 \setminus C)$.

Lemma A.2 and Lemma A.4 give us the following presentation of G_S:

$$P = \langle a_1, \dots, a_k, x, y, \lambda | r_1, \dots, r_{k-1}, x^p y^{-q} \lambda^{-p}, y\lambda y^{-1}\lambda^{-1}, \lambda^{-1}W(a_i), a_k^{-1}y \rangle$$

with b_1 being x and μ being y.

Then we can suppress the relation $y\lambda = \lambda y$ because it is equivalent to the relation $a_k W(a_i) = W(a_i)a_k$ which is already true in G_C because a_k is the homotopy class of a meridian curve of the knot C and $W(a_i)$ is a corresponding longitude loop. Furthermore, we can replace y by a_k in the relators and delete the generator y and the relator $a_k^{-1}y$.

Therefore

$$P_S = \langle a_1, \dots, a_k, x, \lambda | r_1, \dots, r_{k-1}, x^p a_k^{-q} \lambda^{-p}, \lambda^{-1}W(a_i) \rangle$$

is a presentation of $G_S = \pi_1(S^3 \setminus S)$, with $W(a_i)$ a word in the $a_i, i = 1, \dots, k$.

Furthermore, λ is a longitude loop of C and x is the homotopy class of the core of T_C, since it is the image of the core of T_P by h_{PC}.

Now let us prove (2):

Since λ is a longitude loop of C, its linking number with C is zero, thus its linking number with S is zero (it is multiplied by p at each crossing during the cabling process), thus $\alpha_S(\lambda) = 0$.

All the a_i have the same abelianization as a_k, and $a_k = y$ is a meridian of ∂T_C and therefore circles p strands. Thus $\alpha_S(y) = p$.

Finally, the relation $x^p y^{-q} \lambda^{-p}$ in G_S implies that $\alpha_S(x) = q$, which concludes the proof of Proposition 1.23.

A.2 A torus link is a cable on a torus knot

The following proposition is somewhat known among knot theorists but we could not find a written proof of it in the literature. We deeply thank Peter Feller for offering us a proof where pictures are not strictly necessary; we adapted his proof to our setting, thus any inconsistency should be assumed to originate from us.

Proposition A.5. *Let $e \geqslant 2$ and p, q be two relatively prime integers. The torus link $T(ep, eq)$ is a (e, epq)-cable on the torus knot $T(p,q)$.*

Proof. To simplify notation we will assume that $p > 0$ and $q > 0$. Let V be an unknotted solid torus naturally embedded in S^3, c its core and (m, l) a preferred meridian longitude system of $T = \partial V$. The torus T will denote the torus on which both $T(p,q)$ and $T(ep, eq)$ will be drawn.

The torus knot $T(p,q)$ is obtained by taking p parallel strands on T following c and twisting them q times by an angle of $\frac{2\pi}{p}$ following m.

The torus link $T(ep,eq)$ is obtained by taking ep parallel strands on T following c and twisting them eq times by an angle of $\frac{2\pi}{ep}$ following m, which is the same as twisting them q times by an angle of $\frac{2\pi}{p}$. Thus we see that $T(ep,eq)$ is obtained by taking $T(p,q)$ and replacing it with e parallel copies of itself lying next to each other on T.

Therefore $T(ep,eq)$ is obtained as a certain (e,ek)-cable on $T(p,q)$, with $k \in \mathbb{Z}$. We want to prove that $k = pq$.

We follow the notation of Section 1.1.7 on satellite operations; here $T(p,q)$ is the companion C, $T(e,ek)$ is the pattern P and $T(ep,eq)$ is the corresponding cable knot. Any two components L_1, L_2 of the torus link pattern $T(e,ek)$ inside the solid torus T_P have linking number k, since each sequence of e twists of angle $\frac{2\pi}{e}$ is a full turn and adds 1 to this linking number.

Since the cabling operation sends a preferred meridian-longitude system of ∂T_P to a preferred meridian-longitude system of $\partial T_C = \partial V(T(p,q))$, the images L_1' and L_2' of L_1 and L_2 by this operation, which are components of $T(ep,eq)$, still have linking number k.

Thus k is equal to the linking number of $L_1' = T(p,q)$ and L_2' a parallel copy of $T(p,q)$ lying next to it on T. This linking number does not change if we slightly isotope L_2' outside of T, and one can then see that L_2' wraps q times in the meridional direction around the p strands of L_1', thus the linking number is equal to pq. □

A.3 The Alexander polynomial

For the classical results of this section we refer for example to [BZH14], [Cro04] and [Rol90].

Let K be a knot in S^3, G_K its knot group, and $P = \langle g_1, \dots, g_k \mid r_1, \dots r_{k-1} \rangle$ a Wirtinger presentation of G_K.

Recall that

$$\alpha_K \colon \begin{pmatrix} G_K \longrightarrow \mathbb{Z} \\ g_i \longmapsto 1 \end{pmatrix}$$

is the abelianization of G_K. We extend it to a ring homomorphism t^{α_K} defined as:

$$t^{\alpha_K} \colon \begin{pmatrix} \mathbb{Z}[G_K] \longrightarrow \mathbb{Z}[t,t^{-1}] \\ \sum_i n_i g_i \longmapsto \sum_i n_i t^{\alpha_K(g_i)} \end{pmatrix}$$

For $Q, R \in \mathbb{Z}[t,t^{-1}]$, we denote $Q \sim R$ if $\exists m \in \mathbb{Z}, R(t) = \pm t^m Q(t)$. This means that the Laurent polynomials Q, R are equal up to multiplication with an invertible element of the ring $\mathbb{Z}[t,t^{-1}]$.

The Alexander polynomial of K is the Laurent polynomial

$$\Delta_K(t) := \det\left((t^{\alpha_K})(F_{P,1})\right)$$

and does not depend of the Wirtinger presentation P up to the equivalence relation $\sim$. The Alexander polynomial $\Delta_K(t)$ will denote either the equivalence class for $\sim$ or a particular Laurent polynomial in this equivalence class.

Example A.6. The trivial knot has Alexander polynomial $\Delta_O(t) = 1$.

The trefoil knot has Alexander polynomial $\Delta_{3_1}(t) = 1 - t + t^2$.

The figure-eight knot has Alexander polynomial $\Delta_{4_1}(t) = 1 - 3t + t^2$.

Theorem A.7. *([BZH14, Propositions 8.12, 8.14, 8.16, 8.23, Example 9.15])*

- *For any knot K, $\Delta_K(1) = \pm 1$.*
- *For any knot K, $\Delta_K(t^{-1}) = \Delta_K(t)$.*
- *For any knots K, K', $\Delta_{K \sharp K'}(t) = \Delta_K(t)\Delta_{K'}(t)$.*
- *For any knot K, $\Delta_{-K}(t) = \Delta_{K^*}(t) = \Delta_K(t)$.*
- *For any satellite knot $S_{C,P}$ of companion C and pattern P with winding number n_P,*

$$\Delta_{S_{C,P}}(t) = \Delta_C(t^{n_P})\Delta_P(t).$$

- *For any torus knot $T(p,q)$,*

$$\Delta_{T(p,q)}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}.$$

- *For any fibered knot K, $\Delta_K(t)$ is monic and the span of its exponents is equal to $2g(K)$.*

Remark A.8. As an immediate consequence of these properties, the Alexander polynomial of any Whitehead double of a knot is 1, thus the Alexander polynomial does not detect the trivial knot.

Appendix B

Databases for knots and links

B.1 Combinatorial computation and approximation

Let K be an hyperbolic knot in S^3. Using the results mentioned in Section 1.3.3, we can compute approximations of the L^2-Alexander invariant $\left(t \mapsto \Delta_K^{(2)}(t)\right)$ of K.

We assume that the knot group G_K admits a presentation $P = \langle a, b | r \rangle$ with two generators and one relation r (which is a free word in a, b). The Fox matrix of P is

$$F_P = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{\partial r}{\partial a} \\ \dfrac{\partial r}{\partial b} \end{pmatrix},$$

where $v, w \in \mathbb{Z}[G_K]$.

For $t > 0$, let $A_t = R_{\psi_{K,t}(v)} \colon \ell^2(G) \to \ell^2(G)$. It follows from Theorem 2.28 that if A_t is injective, then (P, t) has Property $\mathcal{I}$ and

$$\Delta_K^{(2)}(t) \doteq \frac{\det_{\mathcal{N}(G)}(A_t)}{\max(1,t)^{|\alpha_K(b)|-1}}.$$

For $t > 0$, let $f(t)$ denote $\det_{\mathcal{N}(G)}(A_t)$. It follows from Proposition 1.55 that, for $C \geqslant \|A_t\|_\infty$, one has:

$$f(t) = \det_{\mathcal{N}(G)}(A_t) = C \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} c(A_t, C)_n\right),$$

where $c(A_t, C)_n = \mathrm{tr}_{\mathcal{N}(G)}\left(\left(Id - \dfrac{1}{C^2} A_t^* A_t\right)^n\right)$.

The map $(t \mapsto f(t)) \in \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{\geqslant 0})$ is thus the pointwise limit of the maps

$$t \mapsto f_N(t) := C \exp\left(-\frac{1}{2}\sum_{n=1}^{N} \frac{1}{n} c(A_t, C)_n\right).$$

Since $c(A_t, C)_n$ is polynomial in t and nonnegative for all n (see Proposition 1.55), $(f_N)_{N \geqslant 1}$ is a decreasing sequence of smooth functions that converges pointwise to f. The map f is thus upper-semi-continuous, and the L^2-Alexander invariant

$$\Delta_K^{(2)} = \left(t \mapsto \frac{f(t)}{\max(1,t)^{|\alpha_K(b)|-1}}\right)$$

is upper-semi-continuous as well.

To compute $c(A_t, C)_n$, we start from the operator $A_t = \sum_{i=1}^{s} m_i t^{\alpha_K(g_i)} R_{g_i}$, where the $m_i \in \mathbb{Z}$. We then compute

$$B_t := \left(Id - \frac{1}{C^2} A_t^* A_t \right) = \sum_{j=1}^{c} \mu_j(t) R_{g_j}$$

where the $\mu_j(t)$ are polynomials in t and the g_j are distinct. We observe that

$$c(A_t, C)_n = \mathrm{tr}_{\mathcal{N}(G)}(B_t^n) = \sum_{g_{j_1} \cdots g_{j_n} = 1} \mu_{j_1}(t) \dots \mu_{j_n}(t),$$

thus we can compute $c(A_t, C)_n$ as long as we know how to solve the word problem on G_K.

A convenient way to determine whether or not an element $g = g_{j_1} \dots g_{j_n} \in G_K$ is trivial is to use a linear faithful representation $\rho \colon G_K \hookrightarrow GL_2(\mathbb{C})$ and to compare the matrix $\rho(g)$ to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. When K is hyperbolic, there exists such a faithful representation ρ, which is built from the complete hyperbolic structure on $S^3 \setminus K$; the computer program *Snappea* computes the value of this representation ρ on the generators of G_K.

To summarize, our method of computing upper approximations of $\Delta_K^{(2)}$, for K a 2-generator hyperbolic knot, is as follows:

1. Compute a 2-generator presentation $P = \langle a, b | r \rangle$ of G_K.
2. Compute the Fox derivative $v = \dfrac{\partial r}{\partial a} \in \mathbb{Z}[G_K]$ and the operator $A_t = R_{\psi_{K,t}(v)}$.
3. Choose an interval $I = [t_0; t_1] \subset \mathbb{R}_{>0}$ and a constant $C > 0$ such that $C \geqslant \|A_t\|_\infty$ for all $t \in I$.
4. Compute the operator $B_t = Id - \dfrac{1}{C^2} A_t^* A_t$ as a sum of $\mu_j(t) R_{g_j}$, $j = 1, \dots, c$.
5. Fix an integer $N \geqslant 1$, and an integer $n \in [1; N]$.
6. Consider a faithful linear representation $\rho \colon G_K \hookrightarrow GL_2(\mathbb{C})$ and compute $\rho(g_j)$ for $j = 1, \dots, c$ using *Snappea*.
7. For all $n \in \{1; \dots; N\}$, determine which of the c^n words $g = g_{j_1} \dots g_{j_n}$ are trivial by computing $\rho(g)$ and comparing it to the identity matrix.
8. For all $n \in \{1; \dots; N\}$, compute $c(A_t, C)_n$ as the sum of the $\mu_{j_1}(t) \dots \mu_{j_n}(t)$ associated to such words $g = g_{j_1} \dots g_{j_n} = 1$.
9. Compute the map $t \mapsto f_N(t) = C \exp\left(-\frac{1}{2} \sum_{n=1}^{N} \frac{1}{n} c(A_t, C)_n \right)$ on the interval I.
10. Compute the sequence $\left(t \mapsto \dfrac{f_N(t)}{\max(1, t)^{|\alpha_K(b)|-1}} \right)_{N \geqslant 1}$, which gives an upper approximation of the L^2-Alexander invariant $\Delta_K^{(2)}$ on the interval I.

The longest parts of the algorithm are parts 7. and 8., the computing time being exponential in n. As a consequence, we can only compute $f_N(t)$ for N up to 6 or 7, depending on the size of the operator A_t.

We deeply thank Élie de Panafieu for his tremendous help in writing the algorithm in SAGE.

Observe that our algorithm and our conclusions are exactly the same if we choose C no longer a constant, but a polynomial in t such that $C(t) \geqslant \|A_t\|_\infty$ for all $t \in I$.

Examples of computations for the figure-eight knot are drawn in Figures B.1 and B.2.

B.2 Values of the invariant for particular knots and links

In this section, the knot and link diagrams are obtained from the websites *KnotInfo* and *LinkInfo* created by Chuck Livingston and Jae Choon Cha.

B.2.1 Torus knots and torus links

We use the results of Chapter 4. For L a c-component link in S^3 such that $M_L = S^3 \setminus V(L)$ is Seifert-fibered, it follows from the various theorems of Chapter 4 that $T^{(2)}(M_L, (n_1, \dots, n_c) \circ \alpha_L, id)(t)$ is well-defined for all $t > 0$ and all $n_1, \dots, n_c \in \mathbb{Z}$.

- The L^2-Alexander invariant of the torus knot $K = T(p, q)$ is well-defined everywhere and is equal to
$$\Delta_K^{(2)}(t) = \max(1, t)^{(|p|-1)(|q|-1)}.$$
In particular
$$\Delta_{3_1}^{(2)}(t) = \Delta_{T(2,3)}^{(2)}(t) = \max(1, t)^2,$$
$$\Delta_{5_1}^{(2)}(t) = \Delta_{T(2,5)}^{(2)}(t) = \max(1, t)^4,$$
$$\Delta_{7_1}^{(2)}(t) = \Delta_{T(2,7)}^{(2)}(t) = \max(1, t)^6.$$

- The L^2-Alexander torsions with $\gamma = id$ for the torus link $L = T(m, n) = T(ep, eq)$ (where p, q are relatively prime) is well-defined for all $t > 0$ and $\phi\colon G_L \to \mathbb{Z}$ and is equal to
$$T^{(2)}(M_L, (n_1, \dots, n_e) \circ \alpha_L, id)(t) = \max(1, t)^{(e|pq|-|p|-|q|)|n_1+\dots+n_e|}.$$

B.2.2 Twist knots

According to Remark 3.10, the group G_{K_n} of the twist knot K_n (obtained by $1/n$-surgery on the Whitehead link) admits the following presentation:
$$P_{K_n} = \langle a_1, \alpha, \beta | [a_1, \alpha][a_1^{-1}, \alpha] = \beta, \alpha\beta^n = 1\rangle$$
(here $[a, b] = aba^{-1}b^{-1}$).

The abelianisation $\alpha_{K_n}\colon G_{K_n} \to \mathbb{Z}$ acts as follows:
$$\alpha_{K_n}\colon a_1 \mapsto 0,\ \alpha, \beta \mapsto 1.$$

The fox matrix of the presentation P_{K_n} is
$$F_{P_{K_n}} = \begin{pmatrix} 1 - a_1\alpha a_1^{-1} - [a_1, \alpha]a_1^{-1} + [a_1, \alpha]a_1^{-1}\alpha & 0 \\ a_1 - [a_1, \alpha] + [a_1, \alpha]a_1^{-1} - \beta & 1 \\ -1 & * \end{pmatrix},$$

where $*$ denotes $\alpha(1+\beta+\dots\beta^{n-1})$ if $n>0$ and $-(1+\beta+\dots\beta^{|n|-1})$ if $n<0$.

The operator $R_{\psi_{K_n,t}(F_{P_{K_n},3})}$ thus has the following form:

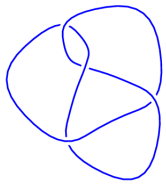
$$R_{\psi_{K_n,t}(F_{P_{K_n},3})} = \begin{pmatrix} Id - R_{a_1\alpha a_1^{-1}} - \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}} + \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}\alpha} & 0 \\ tR_{a_1} - R_{[a_1,\alpha]} + \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}} - R_\beta & Id \end{pmatrix}.$$

Since $\alpha_K(\beta)=0$, it follows from Theorem 2.28 that (P_{K_n},t) has Property $\mathcal{I}$ if and only if $Id - R_{a_1\alpha a_1^{-1}} - \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}} + \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}\alpha}$ is injective, which is true for all $t\in\mathbb{Q}_{>0}$ since knot groups satisfy the Atiyah conjecture for $\mathbb{Q}$ (see Section 2.1.4). In this case

$$\Delta^{(2)}_{K_n}(t) \doteq \det{}_{\mathcal{N}(G_{K_n})}\left(Id - R_{a_1\alpha a_1^{-1}} - \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}} + \frac{1}{t}R_{[a_1,\alpha]a_1^{-1}\alpha}\right)\cdot\max(1,t).$$

Observe that, as n changes, the form of the operator stays the same, but the underlying group G_{K_n} changes. As a consequence, computations of upper approximations of $\Delta^{(2)}_{K_n}(t)$ using this operator display similar behaviours for all twist knots K_n (see Figure B.2 for $K_{-1}=4_1$).

B.2.3 The figure-eight knot 4_1



K	$Vol(K)$	$g(K)$	Fibered	$\Delta_K(t)$	$\Delta^{(2)}_{K,\alpha_K}(t)$	$\Delta^{(2)}_K(t)$
4_1	2.029	1	**yes**	$1-3t+t^2$	$\max(0.38,t)\max(2.61,t)$	$\begin{cases} 1 \text{ if } t<0.38 \\ 1.113 \text{ if } t=1 \\ t^2 \text{ if } t>2.618 \end{cases}$

The knot group G_K satisfies the strong Atiyah Conjecture on $\mathbb{C}$ (since K is fibered, see Section 2.1.4). Since G_K is 2-generator, we conclude that $\Delta^{(2)}_K(t)$ is defined for all $t>0$ and non-zero for all $t\in\mathbb{Q}_{>0}$.

Wirtinger presentation

A Wirtinger presentation of G_K reduces (by a finite sequence of Strong Tietze moves) to the presentation

$$P = \langle x,y | xyXyx = yxYxy\rangle,$$

where $X=x^{-1}$ and $Y=y^{-1}$.

The abelianization acts as:

$$\alpha_K\colon x,y\mapsto 1.$$

The associated operator is

$$A_t = R_{\psi_{K,t}(F_{P,2})} = Id - tR_y - tR_{xyX} - tR_{yxY} + t^2R_{xyXy},$$

and the L^2-Alexander invariant is

$$\Delta^{(2)}_{4_1}(t) \doteq \det{}_{\mathcal{N}(G_K)}(A_t).$$

We compute upper approximations of $\Delta^{(2)}_{4_1}(t)$ from the operator A_t. We choose $C(t) = 1 + 3t + t^2$ and the interval $I = [0.001; 4]$. The approximations up to $N = 6$ are drawn in blue in Figure B.1. The known exact values of $\Delta^{(2)}_{4_1}(t)$ (1 on $]0; 0.38[$, 1.113 in 1 and t^2 on $]2.618; 4[$, see Example 5.6) are drawn in red.

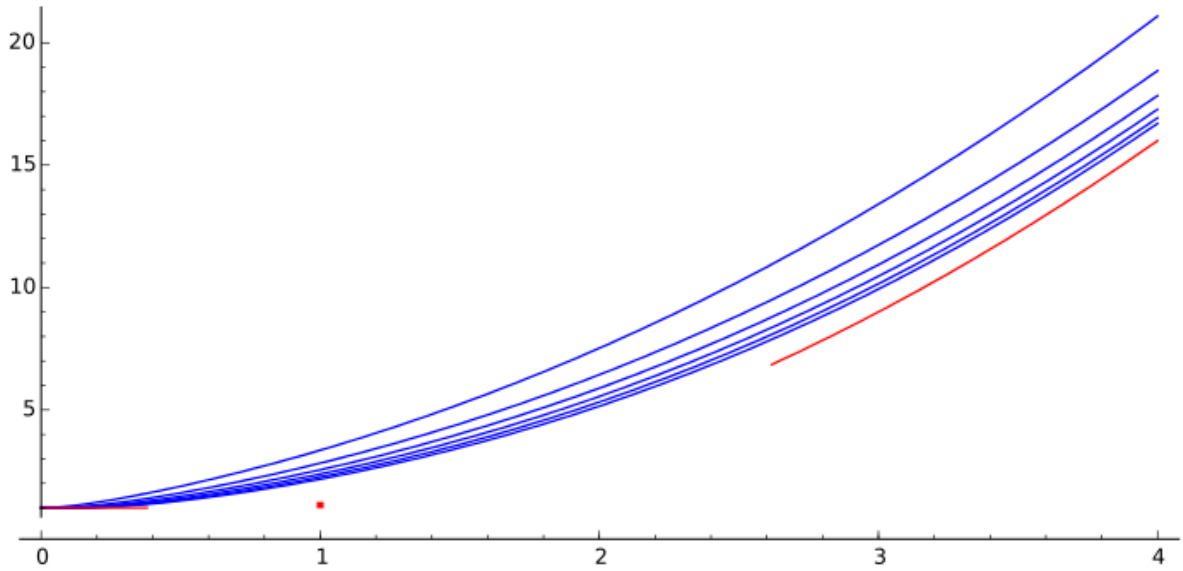


Figure B.1 – Upper approximation for $\Delta^{(2)}_{4_1}(t)$ from the Wirtinger presentation

Twist knot presentation

The figure-eight knot 4_1 is the twist knot K_{-1}. Its group admits the presentation

$$P_{K_{-1}} = \langle a_1, \alpha, \beta | [a_1, \alpha][a_1^{-1}, \alpha] = \beta, \alpha\beta^{-1} = 1 \rangle.$$

The abelianization acts as:

$$\alpha_K \colon a_1 \mapsto 1, \ \alpha, \beta \mapsto 0.$$

The associated operator is

$$A'_t = Id - R_{a_1 \alpha a_1^{-1}} - \frac{1}{t} R_{[a_1,\alpha]a_1^{-1}} + \frac{1}{t} R_{[a_1,\alpha]a_1^{-1}\alpha},$$

and the L^2-Alexander invariant is

$$\Delta^{(2)}_{4_1}(t) \doteq \det{}_{\mathcal{N}(G_K)}(A'_t) \cdot \max(1, t).$$

We compute upper approximations of $\Delta^{(2)}_{4_1}(t)$ from approximations $f_N(t)$ of $\det_{\mathcal{N}(G_K)}(tA'_t)$. We choose $C(t) = 2+2t$ and the interval $I = [0.001; 4]$. The approximations $(f_N(t) \cdot \max(1, t))$ up to $N = 7$ are drawn in blue in Figure B.2. The known exact values of $\Delta^{(2)}_{4_1}(t)$ (1 on $]0; 0.38[$, 1.113 in 1 and t^2 on $]2.618; 4[$, see Example 5.6) are drawn in red.

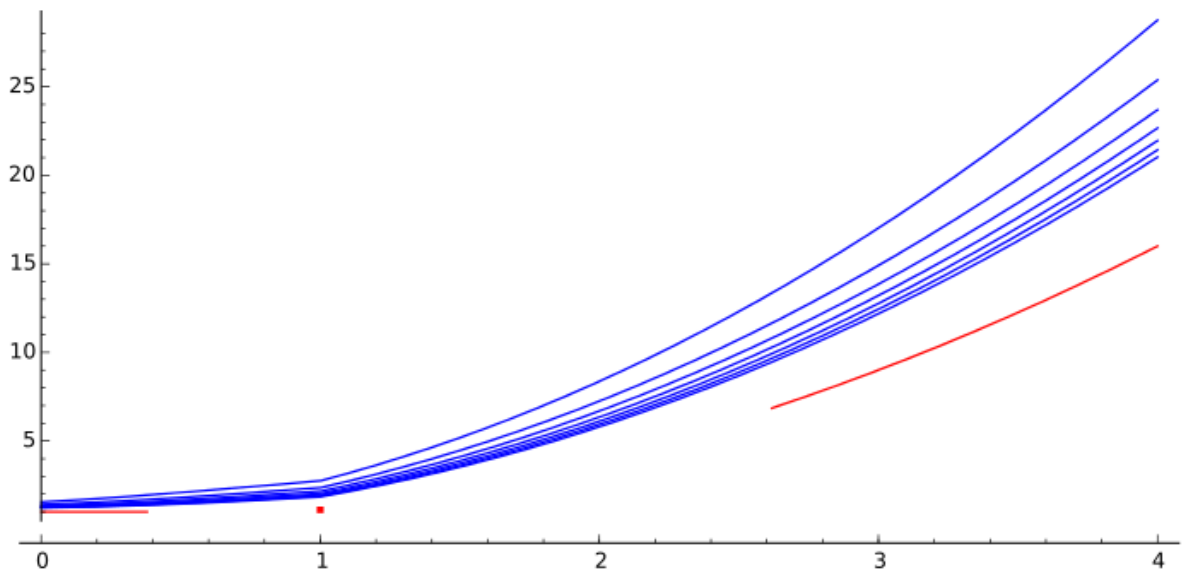


Figure B.2 – Upper approximation for $\Delta^{(2)}_{4_1}(t)$ from the twist knot presentation

Fibered presentation

The group presentation of G_K coming from the fibered structure is:

$$P_f = \langle T, a, b | TaT^{-1} = aba, TbT^{-1} = ab \rangle$$

The abelianization acts as:

$$\alpha_K \colon T \mapsto 1, \ a, b \mapsto 0.$$

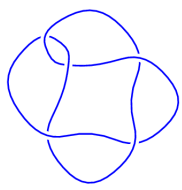
The associated operator is

$$A''_t = \begin{pmatrix} tR_T - Id - R_{ab} & -Id \\ -R_a & tR_T - R_a \end{pmatrix},$$

and the L^2-Alexander invariant is

$$\Delta^{(2)}_{4_1}(t) \doteq \det{}_{\mathcal{N}(G_K)}(A''_t).$$

B.2.4 The three-twist knot 5_2



K	$Vol(K)$	$g(K)$	Fibered	$\Delta_K(t)$	$\Delta^{(2)}_{K,\alpha_K}(t)$	$\Delta^{(2)}_K(t)$
5_2	2.828	1	no	$2 - 3t + 2t^2$	$2\max(1,t)^2$	1.161 for $t = 1$

The knot group G_K satisfies the strong Atiyah Conjecture on $\mathbb{Q}$ (see Section 2.1.4). Since G_K is 2-generator, we conclude that $\Delta^{(2)}_K(t)$ is defined and non-zero for all $t \in \mathbb{Q}_{>0}$.

The three-twist knot 5_2 is the twist knot K_2. Its group admits the presentation

$$P_{K_2} = \langle a_1, \alpha, \beta | [a_1, \alpha][a_1^{-1}, \alpha] = \beta, \alpha\beta^2 = 1 \rangle.$$

The abelianization acts as:

$$\alpha_K \colon a_1 \mapsto 1, \ \alpha, \beta \mapsto 0.$$

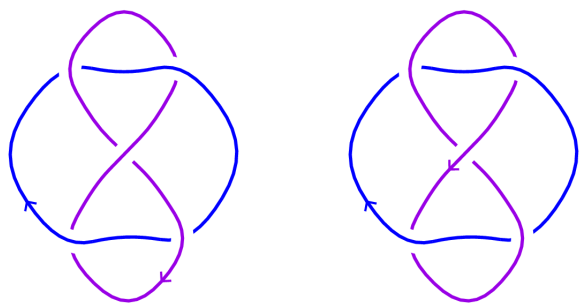
The associated operator is

$$A_t = Id - R_{a_1 \alpha a_1^{-1}} - \frac{1}{t} R_{[a_1,\alpha]a_1^{-1}} + \frac{1}{t} R_{[a_1,\alpha]a_1^{-1}\alpha},$$

and the L^2-Alexander invariant is

$$\Delta^{(2)}_{5_2}(t) \doteq \det{}_{\mathcal{N}(G_K)}(A_t) \cdot \max(1, t).$$

B.2.5 The Whitehead link $L5a1$



L	$\mathrm{lk}(L)$	$\mathrm{vol}(L)$	$x_{M_L}((n_1, n_2) \circ \alpha_L)$	$T^{(2)}(L, (\alpha_1, \alpha_2))(t)$
$L5a1\{0\}$ $L5a1\{1\}$	0	3.663	$\lvert n_1\rvert + \lvert n_2\rvert$	For $n_1, n_2 \neq 0$: $\begin{cases} 1 \text{ if } t < \frac{1}{T} \\ 1.214 \text{ if } t = 1 \\ t^{\lvert n_1\rvert + \lvert n_2\rvert} \text{ if } t > T \end{cases}$

The link group G_L satisfies the strong Atiyah conjecture on $\mathbb{Q}$ (see Section 2.1.4). Since G_L is 2-generator, we conclude that $T^{(2)}(L, (\alpha_1, \alpha_2))(t)$ is well-defined for all $t \in \mathbb{Q}_{>0}$ and $n_1, n_2 \in \mathbb{Z}$, as a quotient of Fuglede-Kadison determinants of injective operators of determinant class.

It follows from Remark A.3 that the link group G_L admits the presentation

$$P_L = \langle a_1, \alpha, \beta | [a_1, \alpha][a_1^{-1}, \alpha] = \beta, \alpha\beta = \beta\alpha \rangle.$$

The abelianization $\alpha_L \colon G_L \to \mathbb{Z}^2$ acts as follows:

$$\alpha_L \colon a_1 \mapsto (1, 0), \ \alpha \mapsto (0, 1), \ \beta \mapsto (0, 0).$$

The fox matrix of the presentation P_L is

$$F_{P_L} = \begin{pmatrix} 1 - a_1\alpha a_1^{-1} - [a_1, \alpha]a_1^{-1} + [a_1, \alpha]a_1^{-1}\alpha & 0 \\ a_1 - [a_1, \alpha] + [a_1, \alpha]a_1^{-1} - \beta & 1 - \beta \\ -1 & \alpha - 1 \end{pmatrix}.$$

Let W_{P_L} denote the 2-dimensional CW-complex constructed from the group presentation P_L (see Definition 2.11). For $n_1, n_2 \in \mathbb{Z}$ and $t > 0$, the $\mathcal{N}(G_L)$-cellular chain complex $C_*^{(2)}(W_{P_L}, (n_1, n_2) \circ \alpha_L, id)(t)$ is of the following form:

$$C_*^{(2)}(W_{P_L}, (n_1, n_2) \circ \alpha_L, id)(t) = \left(\ell^2(G_L)^2 \xrightarrow{\partial_2^{(2)}} \ell^2(G_L)^3 \xrightarrow{\partial_1^{(2)}} \ell^2(G_L) \right),$$

where

$$\partial_2^{(2)} = \begin{pmatrix} Id - t^{n_2} R_{a_1 \alpha a_1^{-1}} - t^{-n_1} R_{[a_1,\alpha]a_1^{-1}} + t^{n_2-n_1} R_{[a_1,\alpha]a_1^{-1}\alpha} & 0 \\ t^{n_1} R_{a_1} - R_{[a_1,\alpha]} + t^{-n_1} R_{[a_1,\alpha]a_1^{-1}} - R_\beta & Id - R_\beta \\ -Id & t^{n_2} R_\alpha - Id \end{pmatrix}$$

and

$$\partial_1^{(2)} = \left(t^{n_1} R_{a_1} - Id \quad ; \; t^{n_2} R_\alpha - Id \quad ; \; R_\beta - Id \right).$$

If $t \in \mathbb{Q}_{>0}$, then the operators $Id - t^{n_2} R_{a_1 \alpha a_1^{-1}} - t^{-n_1} R_{[a_1,\alpha]a_1^{-1}} + t^{n_2-n_1} R_{[a_1,\alpha]a_1^{-1}\alpha}$, $Id - R_\beta$ and $R_\beta - Id$ are injective and of determinant class. It follows from Proposition 1.58, Proposition 1.51 (5), Proposition 1.53, the fact that M_L and W_{P_L} are simple homotopy equivalent (see Section 2.1.3) and Theorem 2.12 that $C_*^{(2)}(M_L, (n_1, n_2) \circ \alpha_L, id)(t)$ is weakly acyclic and of determinant class, and

$$\begin{aligned} &T^{(2)}(M_L, (n_1, n_2) \circ \alpha_L, id)(t) \\ &= \det{}_{\mathcal{N}(G_L)} \left(Id - t^{n_2} R_{a_1 \alpha a_1^{-1}} - t^{-n_1} R_{[a_1,\alpha]a_1^{-1}} + t^{n_2-n_1} R_{[a_1,\alpha]a_1^{-1}\alpha} \right). \end{aligned}$$

Assume that n_1 and n_2 are non-zero, for example that they are both positive. In this case, one has:

$$\begin{aligned} &Id - t^{n_2} R_{a_1 \alpha a_1^{-1}} - t^{-n_1} R_{[a_1,\alpha]a_1^{-1}} + t^{n_2-n_1} R_{[a_1,\alpha]a_1^{-1}\alpha} \\ &= t^{-n_1} \left(t^{n_1} Id - t^{n_1+n_2} R_{a_1 \alpha a_1^{-1}} - R_{[a_1,\alpha]a_1^{-1}} + t^{n_2} R_{[a_1,\alpha]a_1^{-1}\alpha} \right) \\ &= (-t^{-n_1} R_b) \circ \left(Id - t^{n_1} R_c - t^{n_2} R_d + t^{n_1+n_2} R_f \right) \end{aligned}$$

where $c, d, f \in G_L$ are words in a_1, α. It follows from [CFM97, Theorem 1.10(e)] and the same kind of argument as in the proof of Proposition 5.3 that there exists a $T > 1$ such that

$$\det{}_{\mathcal{N}(G_L)} \left(Id - t^{n_1} R_c - t^{n_2} R_d + t^{n_1+n_2} R_f \right) = \begin{cases} 1 \text{ if } t < \frac{1}{T} \\ t^{n_1+n_2} \text{ if } t > T \end{cases}$$

The same reasoning applies for general non-zero n_1, n_2, so that there exists a $T > 1$ (potentially depending on n_1, n_2) such that:

$$T^{(2)}(M_L, (n_1, n_2) \circ \alpha_L, id)(t) = \begin{cases} 1 \text{ if } t < \frac{1}{T} \\ t^{|n_1|+|n_2|} \text{ if } t > T \end{cases}$$

B.2.6 Whitehead doubles

Let K be a Whitehead double of a non-trivial knot C; then the knot K is never fibered, has genus one and has trivial Alexander polynomial: $\Delta_K(t) = 1$. We observe that the L^2-Alexander invariant of K is not trivial but is the same for an infinite number of knots knots.

Let L denote the Whitehead link. It follows from Proposition 1.24 that there exist two injective homomorphisms $i_C\colon G_C \hookrightarrow G_K$ and $i_L\colon G_L \hookrightarrow G_K$.

The groups G_C, G_L, G_K satisfy the strong Atiyah conjecture for $\mathbb{Q}$ and admit a presentation with two generators, therefore the Alexander torsions for M_K, M_L, M_C for $\gamma = id$ are well-defined for all $t \in \mathbb{Q}_{>0}$ (see Section 2.1.4).

It follows from Proposition 4.1 that

$$\begin{aligned} T^{(2)}(M_K, \alpha_K, id)(t) &= T^{(2)}(M_C, 0, i_C)(1) \cdot T^{(2)}(M_L, (1,0) \circ \alpha_L, i_L)(t) \\ &= \exp\left(\frac{\mathrm{vol}(C)}{6\pi}\right) \cdot T^{(2)}(M_L, (1,0) \circ \alpha_L, id)(t). \end{aligned}$$

As a consequence, the L^2-Alexander invariant of K is:

$$\begin{aligned} \Delta_K^{(2)}(t) &= T^{(2)}(M_K, \alpha_K, id)(t) \cdot \max(1,t) \\ &= \exp\left(\frac{\mathrm{vol}(C)}{6\pi}\right) \cdot T^{(2)}(M_L, (1,0) \circ \alpha_L, id)(t) \cdot \max(1,t); \end{aligned}$$

remark that it depends only on $\mathrm{vol}(C)$. In particular, **all Whitehead doubles have proportional L^2-Alexander invariant**.

B.3 Tables

In the following table we list the knots K up to seven crossings, their simplicial volume $\mathrm{vol}(K)$, their genus $g(K)$, whether or not they are fibered, their Alexander polynomial $\Delta_K(t)$ and their twisted L^2-Alexander invariant $\Delta^{(2)}_{K,\alpha_K}(t)$ for $\gamma = \alpha_K$ (this one is computed from the value of the Alexander polynomial, see Proposition 2.43).

K	$\mathrm{vol}(K)$	$g(K)$	Fibered	$\Delta_K(t)$	$\Delta^{(2)}_{K,\alpha_K}(t)$
0_1	0	0	**yes**	1	1
3_1	0	1	**yes**	$1 - t + t^2$	$\max(1,t)^2$
4_1	2.029	1	**yes**	$1 - 3t + t^2$	$\max(0.38,t)\max(2.61,t)$
5_1	0	2	**yes**	$1 - t + t^2 - t^3 + t^4$	$\max(1,t)^4$
5_2	2.828	1	no	$2 - 3t + 2t^2$	$2\max(1,t)^2$
6_1	3.163	1	no	$2 - 5t + 2t^2$	$2\max(0.5,t)\max(2,t)$
6_2	4.400	2	**yes**	$1 - 3t + 3t^2 - 3t^3 + t^4$	$\max(0.46,t)\max(1,t)^2\max(2.15,t)$
6_3	5.693	2	**yes**	$1 - 3t + 5t^2 - 3t^3 + t^4$	$\max(0.58,t)^2\max(1.72,t)^2$
7_1	0	3	**yes**	$1 - t + t^2 - t^3 + t^4 - t^5 + t^6$	$\max(1,t)^6$
7_2	3.331	1	no	$3 - 5t + 3t^2$	$3\max(1,t)^2$
7_3	4.592	2	no	$2 - 3t + 3t^2 - 3t^3 + 2t^4$	$2\max(1,t)^4$
7_4	5.137	1	no	$4 - 7t + 4t^2$	$4\max(1,t)^2$
7_5	6.443	2	no	$2 - 4t + 5t^2 - 4t^3 + 2t^4$	$2\max(1,t)^4$
7_6	7.084	2	**yes**	$1 - 5t + 7t^2 - 5t^3 + t^4$	$\max(0.3,t)\max(1,t)^2\max(3.31,t)$
7_7	7.643	2	**yes**	$1 - 5t + 9t^2 - 5t^3 + t^4$	$\max(0.42,t)^2\max(2.36,t)^2$

In the following table we list the same knots up to seven crossings, the set $\mathcal{D}_K$ on which the L^2-Alexander invariant $\Delta_K^{(2)}(t)$ is known to be defined, and the known values of this invariant. The bound T for fibered knots either comes from Theorem 5.4 or Theorem

5.5, and can be computed. Observe that the set $\mathcal{D}_K$ always contains $\mathbb{Q}_{>0}$, and is equal to the whole set $\mathbb{R}_{>0}$ if K is fibered, since all knots up to seven crossings are 2-bridge and thus are 2-generator knots (see Section 2.1.4).

K	$\mathrm{vol}(K)$	$g(K)$	Fibered	Known $\mathcal{D}_K$	$\Delta_K^{(2)}(t)$
0_1	0	0	**yes**	$\mathbb{R}_{>0}$	1
3_1	0	1	**yes**	$\mathbb{R}_{>0}$	$\max(1,t)^2$
4_1	2.029	1	**yes**	$\mathbb{R}_{>0}$	$\begin{cases} 1 & \text{if } t < 0.38 \\ 1.113 & \text{if } t = 1 \\ t^2 & \text{if } t > 2.61 \end{cases}$
5_1	0	2	**yes**	$\mathbb{R}_{>0}$	$\max(1,t)^4$
5_2	2.828	1	no	$\mathbb{Q}_{>0}$	1.161 for $t = 1$
6_1	3.163	1	no	$\mathbb{Q}_{>0}$	1.182 for $t = 1$
6_2	4.400	2	**yes**	$\mathbb{R}_{>0}$	$\begin{cases} 1 \text{ if } t < 1/T \\ 1.263 \text{ if } t = 1 \\ t^4 \text{ if } t > T \end{cases}$
6_3	5.693	2	**yes**	$\mathbb{R}_{>0}$	$\begin{cases} 1 \text{ if } t < 1/T \\ 1.352 \text{ if } t = 1 \\ t^4 \text{ if } t > T \end{cases}$
7_1	0	3	**yes**	$\mathbb{R}_{>0}$	$\max(1,t)^6$
7_2	3.331	1	no	$\mathbb{Q}_{>0}$	1.193 for $t = 1$
7_3	4.592	2	no	$\mathbb{Q}_{>0}$	1.275 for $t = 1$
7_4	5.137	1	no	$\mathbb{Q}_{>0}$	1.313 for $t = 1$
7_5	6.443	2	no	$\mathbb{Q}_{>0}$	1.407 for $t = 1$
7_6	7.084	2	**yes**	$\mathbb{R}_{>0}$	$\begin{cases} 1 \text{ if } t < 1/T \\ 1.456 \text{ if } t = 1 \\ t^4 \text{ if } t > T \end{cases}$
7_7	7.643	2	**yes**	$\mathbb{R}_{>0}$	$\begin{cases} 1 \text{ if } t < 1/T \\ 1.500 \text{ if } t = 1 \\ t^4 \text{ if } t > T \end{cases}$

The following table lists all 2-component links $L = L_1 \cup L_2$ up to five crossings, their linking number $\mathrm{lk}(L)$, their simplicial volume $\mathrm{vol}(L)$, their associated Thurston norm $x_{M_L}((n_1,n_2)\circ\alpha_L)$, and their L^2-Alexander torsions for $\gamma = id$.

L	$\mathrm{lk}(L)$	$\mathrm{vol}(L)$	$x_{M_L}((n_1,n_2)\circ\alpha_L)$	$T^{(2)}(M_L,(n_1,n_2)\circ\alpha_L,id)(t)$
$L2a1\{0\}$	-1	0	0	1
$L2a1\{1\}$	$+1$	0	0	1
$L4a1\{0\}$	-2	0	$\lvert n_1 - n_2\rvert$	$\max(1,t)^{\lvert n_1-n_2\rvert}$
$L4a1\{1\}$	$+2$	0	$\lvert n_1 + n_2\rvert$	$\max(1,t)^{\lvert n_1+n_2\rvert}$
$L5a1\{0\}$ $L5a1\{1\}$	0	3.663	$\lvert n_1\rvert + \lvert n_2\rvert$	For $n_1, n_2 \neq 0$: $\begin{cases} 1 \text{ if } t < \frac{1}{T} \\ 1.214 \text{ if } t = 1 \\ t^{\lvert n_1\rvert+\lvert n_2\rvert} \text{ if } t > T \end{cases}$

List of Figures

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Index

Notations

$C(X,\mathbb{Z})$ The cellular chain complex of the CW-complex X. 29

$C_k(X,\mathbb{Z})$ The k-th cellular chain group of the CW-complex X. 29

(X,ϕ,γ) An admissible triple of a CW-complex X and two group homomorphisms ϕ,γ on $\pi_1(X)$. 54

(π,ϕ,γ) An admissible triple of a group π and two group homomorphisms ϕ,γ. 53

$(n_1,\ldots,n_c)\circ\alpha_L$ The typical form of a homomorphism from the group of a link L to $\mathbb{Z}$. 81

$-K$ The inverse of the knot K. 25

$-L$ The inverse of the link L. 25

A_t A G-equivariant operator on $\ell^2(G)$. 135

B_t A positive G-equivariant operator on $\ell^2(G)$ associated to A_t. 136

$C(\widetilde{X})$ The cellular chain complex of left $\mathbb{Z}[\pi_1(X)]$-modules for $\widetilde{X}$. 29

$C_*^{(2)}(X,\phi,\gamma,t)$ A $\mathcal{N}(G)$-cellular chain complex of the CW-complex X. 54

$C_k(\widetilde{X})$ The k-th cellular left $\mathbb{Z}[\pi_1(X)]$-module of $\widetilde{X}$. 29

D^* The mirror image of the regular diagram D. 26

D^k The k-dimensional closed ball in $\mathbb{R}^k$. 27

$F(f)(\lambda)$ The spectral density of the operator f at $\lambda\geqslant 0$. 45

F_P The Fox matrix associated to the presentation P. 33

$F_{P,i}$ The Fox matrix of P without its i-th row. 33

G' The commutator subgroup of G. 31

G^{ab} The abelianization of G. 31

G_K The group of the knot K. 31

G_L The group of the link L. 31

G_M The fundamental group of the 3-manifold M. 59

$G_{P\subset T_P}$ The fundamental group of the solid torus T_P minus the pattern knot P. 127

$Gr(P)$ The group naturally associated to the presentation P. 31

H_h The trivial knot drawn in $\mathbb{R}^3$ as the unit circle of an horizontal plane. 101

$H_p^{(2)}(C_*)$ The p-th L^2-homology of C_*. 48

H_v The trivial knot drawn as the vertical line passing through ∞ in $\mathbb{R}^3$. 101

J A curve on a boundary torus of a 3-manifold associated to a Dehn surgery. 90

J' A curve on a boundary torus of a 3-manifold transverse to the curve J. 90

J_i A curve on a boundary torus of a 3-manifold associated to a Dehn surgery. 87

$K(G,1)$ The Eilenberg-MacLane space associated to the group G. 60

KC_k The $(k+1)$-component keychain link. 41

$K_1 \sharp K_2$ The connected sum of the knots K_1 and K_2. 34

K_n The twist knot obtained by $1/n$-surgery on the Whitehead link. 92

$L(p,q)$ The (p,q)-lens space, with fundamental group $\mathbb{Z}/p\mathbb{Z}$. 87

$LHS_*(A_*,C_*,B_*)$ The long exact homology sequence associated to (A_*, C_*, B_*). 50

L^* The mirror image of the link L. 26

$L_1 \cup \ldots \cup L_c$ The c-component link with components the knots $L_1, \ldots, L_c$. 24

L_h The left-multiplication operator by the group element h. 43

$M|S$ The 3-manifold M minus a tubular neighbourhood of the surface S. 39

M' The sibling of the figure-eight knot exterior. 120

$M_1 \sharp M_2$ The connected sum of the 3-manifolds M_1 and M_2. 39

M_K The exterior of the knot K. 23

M_P A meridian curve of the solid torus T_P. 128

M_σ The permutation matrix associated to the permutation σ. 48

O The trivial knot. 24

P_L A particular presentation of the group of the Whitehead link. 94

P_{K_n} A particular presentation of the group of the twist knot K_n. 94

Q A quotient group homomorphism induced by a Dehn surgery. 90

Q_0 The quotient group homomorphism induced by 0-surgery on the Whitehead link. 96

Q_n The quotient group homomorphism induced by $1/n$-surgery on the Whitehead link. 93

R_A The right-multiplication operator by the matrix A of group ring elements. 43

R_h The right-multiplication operator by the group element h. 43

S^1 The circle. 23

S^3 The three-dimensional sphere. 23

S^{k-1} The $(k-1)$-dimensional sphere. 27

$S_{C,P}$ The satellite knot of companion C and pattern P. 35

$S_{g,b}$ The compact connected oriented surface of genus g and with b boundary components. 37

$T(ea,eb)$ The (ea, eb)-torus link with e components. 34

$T(p,q)$ The (p, q)-torus knot. 34

$T^{(2)}(A_*,C_*,B_*)$ The L^2-torsion of the short exact sequence (A_*, C_*, B_*). 51

$T^{(2)}(C_*)$ The L^2-torsion of the finite Hilbert $\mathcal{N}(G)$-chain complex C_*. 49

$T^{(2)}(X)$ The L^2-torsion of the CW-complex X. 55

$T_*^{(2)}(X,\phi,\gamma)(t)$ The L^2-Alexander torsion of the triple (π, ϕ, γ) at $t > 0$. 55

T_C An open tubular neighbourhood of the companion knot C. 35

T_P The open solid torus associated to the pattern knot P. 35

T_{ext} An open tubular neighbourhood of the solid torus T_{int}. 125

T_{int} An open solid torus in S^3. 125

$V(K)$ An open tubular neighbourhood of the knot K. 23

$V(S)$ An open tubular neighbourhood of the surface S. 39

W_P The 2-dimensional CW-complex associated to the group presentation P. 60

X^k The k-skeleton of the CW-complex X. 27

$[\gamma].[c]$ The action of $[\gamma] \in \pi_1(X)$ on $[c] \in \widetilde{X}$. 28

$[c]$ The homotopy class of the path c. 26

$[c]_X$ The homotopy class of the path c in X. 26

$\Delta^{(2)}_{K,P}$ The L^2-Alexander invariant of the knot K associated to the presentation P. 67

$\Delta^{(2)}_K$ The L^2-Alexander invariant of the knot K. 70

$\Delta_K(t)$ The Alexander polynomial of the knot K. 132

$\Delta^{(2)}_{K,\gamma}$ The L^2-Alexander invariant of the knot K twisted by the homomorphism γ. 80

$\mathbb{F}[g_1, \ldots, g_k]$ The free group on the generators g_i. 31

$\mathbb{K}$ A subfield of $\mathbb{C}$. 65

Σ A fiber surface associated to a fibered knot. 115

α_G The abelianization homomorphism of G. 31

α_K The abelianization homomorphism of the knot group G_K. 31

α_L The abelianization homomorphism of the link group G_L. 31

$\chi(X)$ The Euler characteristic of the CW-complex X. 27

$\deg(f)$ The degree of the continuous map f. 28

δ A curve parallel to the torus knot $T(p,q)$. 107

$\delta_{i,j}$ The integer that is 1 if $i=j$ and 0 if $i \neq j$. 33

$\det_{\mathcal{N}(G)}(f)$ The Fuglede-Kadison determinant of the operator f. 45

$\dfrac{\partial}{\partial g_i}$ The Fox derivative associated to the generator g_i. 33

$\dim_{\mathcal{N}(G)}(V)$ The von Neumann dimension of the finitely generated Hilbert $\mathcal{N}(G)$-module V. 44

$\doteq$ Equality to a function on $\mathbb{R}_{>0}$ up to multiplication by a monomial function. 55

$\ell^2(G)\widetilde{e}$ The Hilbert space $\ell^2(G) \otimes_{\kappa(\pi,\phi,\gamma,t)} \mathbb{Z}[\pi]\widetilde{e}$. 54

$\ell^2(G)$ The Hilbert space of square-summable complex functions on the group G. 43

$\ell^2(G)^p$ The Hilbert direct sum of p copies of $\ell^2(G)$. 43

∞ The point at infinity in S^3 as the compactification of $\mathbb{R}^3$. 25

$\kappa(\pi,\phi,\gamma,t)$ The ring homomorphism associated to the triple (π,ϕ,γ) and $t>0$. 54

λ_K The preferred longitude of the knot K. 23

$(c(A,C)_p)_{p\in\mathbb{N}}$ The characteristic sequence of the matrix A and the number C. 46

$\mathbb{C}[G]$ The algebra of the group G. 43

$\mathcal{B}(\ell^2(G))$ The algebra of bounded operators on $\ell^2(G)$. 43

$\mathcal{D}_K$ The set of the $t>0$ such that (P,t) has Property $\mathcal{I}$ for any presentation P of the group of K. 70

$\mathcal{D}_P$ The set of the $t>0$ such that (P,t) has Property $\mathcal{I}$. 66

$\mathcal{D}_X$ The set of the $t>0$ such that $C_*^{(2)}(X,\phi,\gamma,t)$ is weakly acyclic and of determinant class. 55

$\mathcal{IT}$ The class of iterated torus knots. 78

$\mathcal{L}(f,\lambda)$ The set of sub-modules on which f has an operator norm smaller than or equal to λ. 45

$\mathcal{N}(G)$ The von Neumann algebra of the group G. 43

$\mathrm{FIN}(G)$ The set of finite subgroups of G. 65

$g(L)$ The genus of the link L. 37

h The monodromy map on the surface Σ associated to a fibration. 115

h_{PC} The preferred homeomorphism betwen T_P and T_C. 35

i_* The induction functor associated to the injective group homomorphism i. 44

$i_*(C_*)$ The finite Hilbert $\mathcal{N}(G)$-chain complex induced by C_* and i_*. 49

n_P The winding number of the pattern knot P. 35

p_X The universal covering map from $\widetilde{X}$ to X. 28

p_i/q_i A rational coefficient associated to a Dehn surgery. 87

pt A base point common to several topological spaces. 130

pt' The base point sent to pt by the homeomorphism h_{PC}. 130

t^{α_K} The ring homomorphism from the group ring of the knot K to the ring of Laurent polynomials. 132

$tr_{\mathcal{N}(G)}(\phi)$ The trace of the operator $\phi \in \mathcal{N}(G)$. 43

$x_M(\phi)$ The Thurston norm of the class $\phi \in H^1(M;\mathbb{Z})$. 38

$|f|$ The positive operator associated to the operator f. 45