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présentée par

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Sur la catégorification des invariants quantiques $\mathfrak{sl}_n$: étude algébrique et diagrammatique

dirigée par Christian BLANCHET et Catharina STROPPEL

Soutenue le 5 décembres 2013 devant le jury composé de :

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Résumé

En 2000, Khovanov ouvre la voie au programme de catégorification en théorie des nœuds, définissant une homologie dont la caractéristique d'Euler redonne le polynôme de Jones. De nombreuses perspectives apparaissent alors, dont en particulier les relations entre catégorifications pour les nœuds et pour les groupes quantiques, de même que les extensions vers les 3-variétés, forment les deux axes de cette thèse.

Dans la recherche d'extensions aux 3-variétés, les outils classiques que sont la chirurgie ou les scindements de Heegaard nous invitent à nous intéresser aux surfaces (de recollement), et la nouvelle preuve de la formule de Frohman-Gelca que nous présentons s'intègre dans ce contexte.

La réinterprétation de l'homologie de Khovanov avec les cobordismes, proposée par Bar-Natan, s'étend aux surfaces épaissies, mais cette extension impose des relations supplémentaires. Nous montrons que ce n'est pas le cas si l'on considère des modèles désorientés tels qu'introduits par Clark-Morrison-Walker et Blanchet.

Un autre point de vue sur l'homologie de Khovanov provient de la théorie des représentations, d'où est originaire la définition du polynôme de Jones donnée par Reshetikhin et Turaev. Nous montrons à l'aide de mousses et d'une antidualité de Howe catégorique, que l'homologie de Khovanov s'intègre dans le cadre de la théorie des représentations d'ordre supérieur.

Cette réinterprétation du polynôme de Jones et de l'homologie de Khovanov suggère également des extensions pour d'autres cadres topologiques. Ainsi, l'analogue affine nous fournit une version des modules d'écheveau pour l'anneau qui puise sa source dans la théorie des représentations.

Mots-clefs

Catégorification, groupes quantiques, homologie de Khovanov, invariants de Reshetikhin-Turaev, modules d'écheveau, nœuds, polynôme de Jones.

On categorification of quantum $\mathfrak{sl}_n$ invariants: algebraic and diagrammatic study

Abstract

Khovanov set up in 2000 the basis of the categorification program in knot theory, introducing an homology whose Euler characteristics recovers the Jones polynomial. This opened many perspectives, and in particular the relations between categorifications in knots and in quantum groups, as well as possible extensions towards other 3-manifolds are central questions of this thesis.

Looking for extensions to general 3-manifolds, classical tools such as surgery or Heegaard splittings invite us to look at (gluing) surfaces, and we present a new proof of Frohman-Gelca formula which embeds in that context.

Bar-Natan's interpretations of Khovanov homology in terms of cobordisms are also suitable for extensions towards thickened surfaces, although requiring new relations in the category. We show however that they are not necessary if one considers disoriented models as introduced by Clark-Morrison-Walker and Blanchet.

Another point of view on Khovanov homology can be provided by representation theory, from which Reshetikhin-Turaev's presentation of the Jones polynomial is originated. We show, using foams and categorical skew-Howe duality, that Khovanov homology can be understood in the context of higher representation theory.

This reinterpretation of Jones polynomial and Khovanov homology also suggests extensions towards other topological settings. For instance, the affine analogue gives us a representation-theory flavored reinterpretation of skein modules of an annulus.

Keywords

Categorification, Jones polynomial, Khovanov homology, knots, quantum groups, Reshetikhin-Turaev invariants, skein module.

Contents

Introduction

Prenons une corde, et nouons-la avant d'en coller les deux bouts : nous avons un nœud. Prenons-en une autre, nouons-la encore, et collons à nouveau les bouts : c'est un autre nœud ! Ces deux nœuds sont-ils les mêmes ? Autrement dit, est-ce qu'on peut, en tirant sur certaines parties de l'une ou l'autre ficelle (mais sans les couper !), faire en sorte que les deux ficelles soient identiques ? Par exemple, un "nœud magique" dont on aurait recollé les deux extrémités peut être dénoué et va donc être le même qu'un simple anneau de ficelle, ce qui n'est pas du tout le cas pour un nœud de huit – et c'est heureux pour les grimpeurs et les marins !

Cette question de la comparaison de deux noeuds semble élémentaire, et dans les cas suffisamment simples, il suffit généralement de quelques minutes pour se convaincre de la réponse. Pourtant, la recherche d'un invariant universel, c'est-à-dire d'un outil qui permettrait à coup sûr de différencier deux noeuds, a constitué et constitue encore un domaine de recherche actif. Cette thèse s'inscrit dans ce contexte, et développe en particulier le cadre des invariants dits quantiques.

Histoire

De la physique aux mathématiques

La classification des nœuds à déformation près est un problème ancien, que l'on peut faire remonter à la fin du 18ème siècle avec les travaux de Lord Kelvin. La question de la structure de la matière émergeait alors, et Kelvin soutenait que les atomes étaient de petits nœuds dans un espace dont la matière fondamentale serait l'éther. La volonté de comprendre alors la structure de la matière a naturellement initié le développement mathématique de la théorie des nœuds.

Après que la théorie de l'éther se fut avérée erronée, l'intérêt suscité par les problèmes mathématiques reliés à l'étude des nœuds perdura, et celle-ci s'institua comme un domaine d'étude à part entière, notamment en relation avec l'émergence de la topologie algébrique sous l'influence de Poincaré, permettant un formalisme nouveau dans la géométrie.

Des fondements de la théorie des nœuds dans son acception moderne, nous retiendrons principalement les deux grands noms de James Waddell Alexander II [Jam01] et de Kurt Reidemeister [BBF72]. Alexander, né en 1888, effectue sa carrière aux États-Unis. D'abord très actif dans les fondations de la topologie algébrique telles que développées par Poincaré [Poi95] et par Brouwer, il se tourne vers la théorie des nœuds à la fin des années 20. De son travail, perdure notamment la découverte en 1928 du polynôme qui porte son nom et qui a été jusque dans les années 80 la principale source d'études en théorie des nœuds. Il rejoint l'*Institute for Advanced Studies (IAS)* en 1933, où il côtoiera Einstein, Von Neumann et Weyl. Reidemeister, de quelques années le cadet d'Alexander (il est né en 1893), effectue le début de sa carrière en Allemagne, où il rédige une thèse en théorie des nombres, avant de rejoindre Wirtinger à Vienne, lequel l'initie aux méthodes de calculs des groupes de nœuds. De retour en Allemagne, ses engagements politiques et philosophiques lui valent d'être muté en 1934 de Königsberg à Marburg. Il effectue un séjour de deux ans à l'*IAS* à la fin de la guerre, date à laquelle Alexander avait déjà rompu de nombreux contacts avec les mathématiques. Professeur émérite à l'Université de Göttingen, il décède en 1971, la même année qu'Alexander. Son livre *Knotentheorie* [Rei32], écrit en 1932, a fait référence pendant près d'un demi-siècle, et son nom reste aujourd'hui associé à l'ensemble des mouvements élémentaires qui permettent, avec l'isotopie planaire, de relier deux diagrammes d'un même nœud, et dont la vérification constitue le point central de toute preuve d'invariance !

Jones et les groupes quantiques, et le retour en physique ?

Le polynôme d'Alexander et les méthodes issues de la topologie algébrique classique ont dominé pendant une grande partie du vingtième siècle, jusqu'à ce qu'en 1984, Jones [Jon85] découvre un nouveau polynôme de nature très différente du polynôme d'Alexander, issu de la théorie des algèbres de Von Neumann. Ces travaux, entre autres, vaudront à Jones l'attribution d'une médaille Fields en 1990.

C'est aussi dans les années 1980 que la notion de groupe quantique, initiée par l'école de Faddeev á partir de motivations physiques, e développe rapidement, principalement en Union Soviétique, sous l'impulsion de Drinfeld et Jimbo. Citons aussi Lusztig, Kirillov, ou encore Kac, qui ont largement étudié ces déformations d'algèbres avant 1990. Reshetikhin et Turaev [RT90, RT91], au fait à la fois des découvertes de Jones et du développement de l'algèbre quantique, reformulent largement la théorie de Jones, donnant à son polynôme la définition qu'on lui connaît actuellement, comme morphisme de représentations du groupe quantique formé sur $\mathfrak{sl}_2$. Ce formalisme, largement généralisé en utilisant la théorie des catégories, s'étend pour former des invariants de 3-variétés, qui, avec les travaux de Kirby, ont posé les bases de la topologie quantique.

Dans le même temps, Witten [Wit94], qui obtiendra une médaille Fields la même année que Jones, rapproche considérablement la physique quantique et la topologie quantique, et donne le départ de nombreuses années d'études de la théorie quantique des champs dans les deux domaines. Ces travaux, avec ceux d'Atiyah, ont été à la base de l'idée de Théorie Quantique des Champs Topologiques (TQFT, voir notamment [BHMV95] et [Koc04]), qui s'est avérée être un outil axiomatique extrêmement utile dans l'étude de la théorie des nœuds et des 3-variétés.

La 3ème dimension... et au-delà ?

En 2000, Khovanov [Kho00] ouvre de nouvelles perspectives en initiant le programme de catégorification en théorie des nœuds. Le propos de la catégorification est de se représenter un objet usuel (invariant numérique, polynomial, groupe,...) comme l'ombre d'une notion d'ordre supérieur. La question est alors de parvenir à dévoiler toute la nouvelle structure (homologie, catégorie,...) et de mettre en évidence le processus (caractéristique d'Euler, groupe de Grothendiek, trace,...) qui retrouve l'objet original comme ombre du nouveau.

L'exemple le plus connu nous ramène quelques décennies en arrière, avec le développement de l'homologie pour les variétés : les nombres de faces de chaque dimension d'une variété (sous la forme de complexe simplicial dans le cadre le plus simple) peuvent apparaître comme les dimensions d'espaces vectoriels engendrés par les faces. La somme alternée de ces nombres se relève alors en définissant des différentielles entre ces espaces, et en prenant l'homologie du complexe ainsi formé. Enfin, la caractéristique d'Euler de la variété se retrouve naturellement en prenant la caractéristique d'Euler de l'homologie ainsi définie.

Outre le fait que les théories homologiques que l'on peut définir avec ce type de constructions s'avèrent généralement des invariants plus riches que les traces qu'elles catégorifient, elles présentent le grand intérêt de prendre aussi en compte les morphismes entre les objets qu'elles décrivent. Ainsi, dans le cas de théories homologiques de variétés, les homéomorphismes (avec structure adéquate) définissent des morphismes entre les homologies associées à la source et à la cible de l'homéomorphisme.

Les constructions développées par Khovanov se sont d'abord appuyées sur une reformulation du polynôme de Jones à l'aide du crochet de Kauffman, qui en donne une version combinatoire. Les développements qui ont suivi [Kho02, Kho06, BN05] ont mis en avant l'aspect géométrique et les possibles extensions aux morphismes entre nœuds en allant vers la dimension supérieure, et aussi en dimension inférieure en autorisant les noeuds à être coupés en enchevêtrements. La version formelle développée par Bar-Natan va aussi permettre des extensions dans des variétés plus générales, ce qui sera l'une des pistes d'étude de cette thèse.

Du côté algébrique, de nombreux travaux ont contribué au développement des outils reliés aux idées de Khovanov, parmi lesquels on peut citer des études de la catégorie $\mathcal{O}$ [BFK99, FKS06, BS08, Str09] menant à une catégorification de $U_q(\mathfrak{sl}_2)$. Ces catégorifications ont aussi reçu des descriptions diagrammatiques [KL09, KL11a, KLMS12], plus abordables, et que l'on utilisera largement dans cette thèse. C'est aussi à partir de ces algèbres de diagramme que Webster [Web10a, Web10b] a développé des reformulations générales d'invariants de Reshetikhin-Turaev.

Les invariants initiés par Khovanov pour le cas $\mathfrak{sl}_2$ ont aussi été étendus à $\mathfrak{sl}_n$, ainsi qu'au polynôme de Homflypt [KR08a, KR08b, KR07, MSV09, MSV11]. De la même manière que le polynôme de Jones a pu mener à la découverte d'invariants des variétés de dimension 3, le développement d'invariants catégoriques pour les 3-variétés, c'est-à-dire de TQFT en dimension $3+1$, est un objectif important. Une étape difficile dans ce processus tient au fait que les invariants quantiques des variétés de dimension 3 sont généralement construits à partir de groupes quantiques considérés aux racines de l'unité, et dont la catégorification constitue une tâche difficile, même si de récents travaux permettent d'espérer des avancées importantes dans les prochaines années[Kho05, KQ12, EQ13].

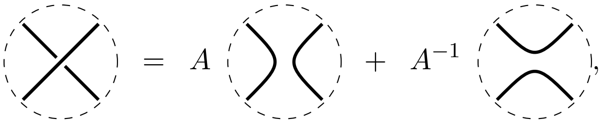
S'articulant autour du thème de la catégorification des invariants quantiques des variétés en dimension 3, le travail de thèse présenté ici repose en grande partie sur l'étude de l'homologie de Khovanov [Kho00]. La reformulation en termes topologiques due à Bar-Natan [BN05] admet des extensions très naturelles dans le cadre des surfaces épaissies, et la compréhension du module d'écheveau de Bar-Natan dans lequel ces constructions prennent place occupe la première partie de ce travail, individuel et en collaboration avec H. Russell.

Par ailleurs, le récent développement de l'antidualité de Howe et la réinterprétation des invariants de Reshetikhin-Turaev ont ouvert la voie à une catégorification d'origine algébrique entièrement encodée à travers un 2-foncteur entre une catégorification du groupe quantique $U_q(\mathfrak{sl}_m)$ et des catégories de mousses modifiées en suivant des idées de Blanchet [Bla10]. La deuxième partie du manuscrit présente ce travail effectué en collaboration avec A. Lauda et D. Rose [LQR12], suivi d'une extension affine.

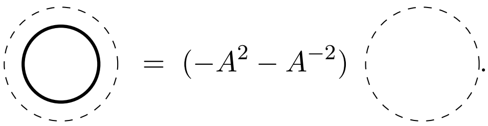
Nous nous proposons maintenant de présenter plus en détails le contexte de chacune de ces parties.

Modules d'écheveaux et généralisations

La reformulation du polynôme de Jones à l'aide du crochet de Kauffman fournit un processus de calcul récursif d'une grande simplicité, qui évite la technicité du passage par la théorie des représentations de $U_q(\mathfrak{sl}_2)$, le pont reliant le monde diagrammatique à l'aspect algébrique nous étant fourni par l'algèbre de Temperley-Lieb. Le calcul du polynôme de Jones peut alors se faire en partant d'un diagramme du nœud grâce au processus récursif qui consiste à successivement remplacer chacun des croisements par une somme formelle :



où A est une indéterminée. Une fois que tous les croisements ont élé lissés, les courbes que nous obtenons ne sont plus que des unions de cercles plongés dans le plan, que l'on va pouvoir remplacer par des polynômes de Laurent en A à l'aide de la normalisation locale :



Les formulations les plus simples de l'homologie de Khovanov sont basées sur une remontée à la fois de la relation de lissage des croisements et de la normalisation. L'idée maîtresse est de voir le paramètre A comme l'ombre d'une graduation sur les modules que l'on veut considérer, et donc d'associer au cercle un module de rang 2 engendré par des éléments en degrés 2 et -2. Le lissage se voit alors remplacé par la construction pas à pas d'un complexe, la difficulté étant de trouver les bonnes applications différentielles.

Une réinterprétation géométrique proposée par Bar-Natan consiste à faire cette construction du côté topologique, où l'idée de la catégorification consistant à rajouter une dimension prend tout son sens : les nœuds étant de dimension 1, il nous faut considérer des surfaces, que l'on quotientera par un certain nombre de relations permettant à la fois de mimer la relation de normalisation et de construire des complexes. La catégorie de cobordismes dans laquelle cette construction prend place est aussi parfois appelée *module d'écheveau de Bar-Natan*.

Ce module d'écheveau, entièrement géométrique, se prête particulièrement bien à des extensions aux nœuds non plus regardés dans $\mathbb{R}^3$ mais dans des surfaces épaissies. Ce cadre apparaît comme une extension naturelle du travail de Khovanov, mais aussi comme un premier pas vers de possibles définitions d'invariants catégoriques de 3-variétés. Rozansky, par exemple, a étudié une extension dans la variété $\mathbb{S}^2 \times \mathbb{S}^1$ [Roz10a]. Si sa construction est très liée à la structure particulière de cette variété et ne peut pas forcément être étendue telle quelle, le fait que $\mathbb{S}^2 \times \mathbb{S}^1$ soit la plus simple des variétés construites par chirurgie de la sphère suggère d'utiliser des méthodes de chirurgie ou de scindements de Heegaard pour étendre les invariants. Dans les deux cas, les recollements de 3-variétés se feraient le long d'une surface, et l'étude de la structure intermédiaire, et en particulier avec son produit, prend donc tout son sens.

Si, comme nous l'avons dit, la version de Bar-Natan de l'homologie de Khovanov fournit des extensions pour les surfaces épaissies, la liberté plus riche donnée par la variété ambiante dans laquelle les cobordismes sont plongés met à mal les constructions habituelles, et en particulier les TQFT, qui procurent un pont entre l'aspect géométrique et les versions algébriques. La recherche de solutions à ces questions a été notamment menée par Turaev et Turner [TT06], Carter et Saito [CS09], ainsi qu'Asaeda, Przytycki et Sikora (APS) [APS04].

La première remarque consiste à observer que la décatégorification des théories APS donne un résultat plus complexe que le simple élément d'écheveau associé au nœud de départ. Des idées extraites de cette remarque, mises en relation avec une description de la structure produit du module d'écheveau due à Frohman et Gelca [FG00], ont été le point de départ du travail collaboratif présenté dans le chapitre 1.

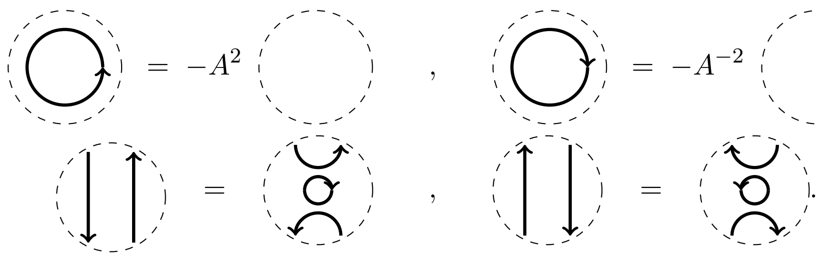
La relation de lissage permettant de définir le module d'écheveau est aussi la base d'une structure produit dans le cas des surfaces. Deux courbes dessinées sur une surface peuvent être superposées l'une sur l'autre, produisant de nouveaux croisements, qui peuvent à nouveau être lissés. L'algèbre ainsi obtenue est souvent appelée *algèbre de Jones-Kauffman*.

Le cas du tore est probablement le plus simple des cas intéressants : la structure produit n'est pas triviale (ce qui est le cas pour le plan ou l'anneau, où il ne s'agit que de l'union disjointe), mais des bases du module comme des générateurs de l'algèbre sont simples à donner. Pour autant, la base la plus simple de la structure de module, donnée par des unions parallèles de courbes de base de l'homologie, n'est pas robuste sous l'effet de la structure produit.

L'idée centrale de l'article de Frohman et Gelca [FG00] est justement d'exhiber une base différente de la structure de module dont le produit des éléments est donné par une formule dite *Produit-En-Somme* (*Product-To-Sum*) extrêmement simple. Cette nouvelle base est mystérieusement définie à l'aide des polynômes de Tchebychev du premier type, dont la formule de récurrence occupe le centre d'une preuve assez technique.

Le propos de la première section de cette thèse est de reprendre entièrement cette preuve à l'aide d'un module diagrammatique inspiré par la décatégorification de la théorie APS. Cette méthode permet de faire apparaître clairement les étapes distinctes de cette preuve, et en particulier met en évidence le rôle des polynômes de Tchebychev comme processus de diagonalisation de la base naturelle, après s'être affranchi des lemmes techniques concernant le produit

Nous introduisons pour cela un module d'écheveau orienté défini, dans le cas du tore, comme le $\mathbb{Z}[q, q^{-1}]$-module engendré par les courbes closes orientées soumises aux relations locales :

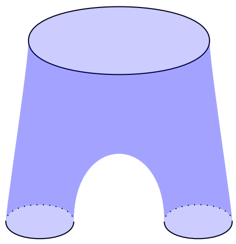


La structure produit se définit aussi dans ce cadre comme le lissage des croisements apparus dans la superposition, avec respect de l'orientation locale. Deux étapes distinctes permettent alors de redémontrer la formule de Frohman et Gelca : d'abord, une étude du lien entre modules d'écheveau habituels et orientés, principalement technique, puis une comparaison des bases naturelles des deux, différentes mais qui s'avèrent être reliées très naturellement par les polynômes de Tchebychev.

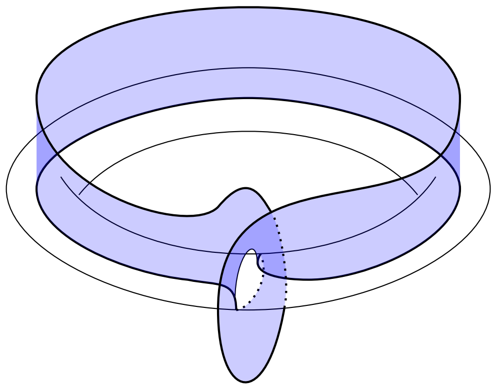
Outre le rôle intermédiaire que ce module joue dans la preuve de la formule de Produit-En-Somme, il s'avère aussi être par lui-même un module d'écheveau, c'est-à-dire le réceptacle naturel d'un invariant d'une classe de nœuds dits *noeuds en bande parallèles* (*boundary parallel band knots*).

Par ailleurs, ce module se prête à une généralisation dans le cadre des surfaces épointées, dont nous étudions certaines propriétés. Ces généralisations semblent être un lieu naturel où chercher des analogues de la formule de Frohman-Gelca, et nous dessinons les premières pistes vers de tels résultats, qui seraient à mettre en relation avec les travaux récents de Bonahon et Wong [BW12a], [BW12b], lesquels reposent sur la formule de Frohman-Gelca dans le tore épointé utilisée aux racines de l'unité.

Une deuxième remarque à propos des modèles développés par Asaeda-Przytycki-Sikora, remarque à mettre en relation avec les travaux de Turaev-Turner et de Carter-Saito, concerne la présence de selles particulières dans les cobordismes au-dessus de certaines surfaces, et en particulier du tore. En effet, si une selle ne peut se refermer au-dessus du plan que comme un pantalon (deux jambes et une taille) :



il est possible dans certaines surfaces de ne faire que deux trous, en bordant des courbes de types d'homologie différents, comme dessiné ci-dessous dans le cas du tore.



Ce type de selle est l'un des obstacles majeurs dans la recherche de TQFT pour les surfaces épaissies, la solution d'Asaeda, Przytycki et Sikora consistant à leur assigner un morphisme nul, avec pour conséquence la nullité de plusieurs autres applications. Il est intéressant de noter que ces selles forment des surfaces non-orientables, ce qui nous amène à nous interroger sur l'existence des mêmes difficultés dans le cadre de catégories de cobordismes enrichis d'orientations locales. Ce type de cobordismes a été introduit par Clark-Morrison-Walker, Caprau et Blanchet, pour donner des solutions à un défaut de fonctorialité dans l'homologie de Khovanov.

En effet, et c'est bien là l'un des intérêts centraux de la catégorification, il est possible de considérer les nœuds comme objets d'une catégorie dont les morphismes sont des surfaces, et le processus de Khovanov permet d'associer à ces surfaces un morphisme entre les complexes correspondants aux nœuds qu'elles bordent [Kho06, BN05]. Cette construction passe par la décomposition de chaque surface en une succession de mouvements élémentaires. Cette décomposition n'étant pas unique, la question de l'indépendance du résultat au choix de décomposition se pose. Il apparaît que cette invariance ne tient, dans le modèle original de Khovanov, qu'à un signe près.

Ce défaut de fonctorialité a été résolu par Clark-Morrison-Walker [CMW09] et Caprau [Cap10, Cap09, Cap07] en travaillant avec les entiers de Gaus. Le modèle que Blanchet [Bla10] a ensuite proposé permet de garder l'anneau de base $\mathbb{Z}$. Les deux théories introduisent une nouvelle classe de cobordismes désorientés. Il est alors possible de réinterpréter le module de Bar-Natan [BN05] dans le cadre désorienté, et d'étendre ces constructions dans le cas de n'importe quelle surface.

Les relations reliant alors la selle non-orientable à d'autres morphismes structurels de la catégorie de cobordismes se réinterprètent alors dans le cadre désorienté, et nous montrons dans le chapitre 2 qu'elles sont très différentes du cas orienté. En particulier, on peut choisir de quotienter la catégorie de cobordismes désorientés considérée par les surfaces non-orientables (ce qui est la manière la plus simple d'y associer ensuite un morphisme nul !), sans pour autant modifier la structure du reste de la catégorie. Ce résultat souligne encore l'intérêt de ces catégories désorientées, et invite aussi à se consacrer à l'étude de TQFT qui s'appliqueraient dans ce cadre étendu, étude qui nécessiterait en premier lieu une bonne axiomatisation de la structure de ces catégories.

La différence entre les modèles de Clark-Morrison-Walker et de Blanchet que nous étudions dans le chapitre 2 semble à première vue relativement mystérieuse. La partie "intéressante" des cobordismes est essentiellement la même, et chacun des deux modèles introduit un élément supplémentaire (une racine de -1 ou des 2-faces) pour permettre un traitement plus fin des changements d'orientation sur la surface. Pour autant, nous montrons ici que ces deux modèles n'ont pas la même stabilité dans la recherche d'extensions naturelles à des catégories de nœuds généralisés : seul le modèle de Blanchet s'étend aisément comme invariant des graphes noués, alors que l'on butte sur les bien-nommées *confusions* lorsqu'on veut définir un analogue pour le modèle CMW.

Ayant étendu une définition de l'homologie de Khovanov à la classe de nœuds généralisés que forment les toiles nouées, se pose encore la question de la fonctorialité. Nous donnons à la fin du chapitre 2 une liste complète de mouvements (*movie moves*) à vérifier pour prouver la fonctorialité

Ces derniers résultats d'extensions sont un premier pas vers la définition d'invariants catégoriques $\mathfrak{sl}_n$ pour les toiles nouées. Si les méthodes de preuve utilisées dans la première partie de cette thèse sont efficaces tant que n est suffisamment petit, puisque le nombre de cas à considérer reste raisonnable, leur utilisation dans le cadre général serait certainement moins adaptée. Dans ce cadre, le traitement algébrique abstrait que fournit l'anti-dualité de Howe que nous étudions dans la seconde partie de ce travail, et les preuves du chapitre 4 qui concernent les invariants non catégorifiés, suggèrent un programme de preuves algébriques unifiées pour les catégorifications des invariants de Jones-Reshetikhin-Turaev. Le développement des outils nécessaires à de telles preuves constituera un axe de recherches futures dans la continuité du travail présenté ici.

Anti-dualité de Howe et homologies de Khovanov

La construction originale du polynôme de Jones repose sur une utilisation de la théorie des représentations du groupe quantique $U_q(\mathfrak{sl}_2)$. Après l'apparition des premières catégorifications de ces invariants développées par Khovanov, la question de relier les versions géométriques de celles-ci à des catégorifications de groupes quantiques s'est naturellement posée.

Le processus de catégorification d'un groupe quantique consiste essentiellement à donner une dimension supplémentaire à sa catégorie de représentations, ou une sous-catégorie de celle-ci. On obtient ainsi une 2-catégorie, dans laquelle l'idée sous-jacente est de conserver les générateurs de l'algèbre comme 1-morphismes, et de remplacer chaque relation initiale par un isomorphisme entre les 1-morphismes considérés. Ces isomorphismes sont réalisés par des 2-morphismes explicites, la structure que l'on donne à la 2-catégorie permettant de s'assurer de leur inversibilité. Dans le même temps, on cherche aussi à contrôler les dimensions des espaces de 2-morphismes entre les 1-morphismes, de manière à être capable de retrouver sous forme de trace de cette 2-catégorie la catégorie de représentations initialement considérée. Une version particulièrement agréable de catégorification de $U_q(\mathfrak{sl}_2)$ a été introduite par Khovanov et Lauda [KL09, KL11a, KL10]. Basée sur des espaces de diagrammes, elle autorise des méthodes de calcul simples et se manipule aisément.

Cette idée de mettre en relation, en substance, groupes quantiques catégorifiés et homologie de Khovanov s'est avérée jusqu'ici assez technique [BFK99, MS09, Str05, FKS06, Web10a, Web10b]. La mise en relation que nous proposons dans le chapitre 3 prend sa source dans une théorie d'*antidualité de Howe* dûe à Cautis, Kamnitzer et Licata [Cau12, CKL10a], qui consiste à faire agir simultanément $U_q(\mathfrak{sl}_n)$ et $U_q(\mathfrak{sl}_m)$ sur les puissances extérieures quantiques du produit tensoriel de leurs représentations vectorielles. Ces deux actions s'avèrent commuter, et surtout, être en dualité. La preuve de ce résultat par Cautis, Kamnitzer et Licata a permis à Cautis, Kamnitzer et Morrison [CKM12], de résoudre des conjectures sur la présentation de catégories d'endomorphismes pour des représentations de $U_q(\mathfrak{sl}_n)$ (reliées à des généralisations au cas $\mathfrak{sl}_n$ de modules d'écheveau).

Cette théorie a aussi mené à une reconstruction des invariants de Reshetikhin-Turaev [Cau12], qui permet d'identifier les morphismes de représentations de $U_q(\mathfrak{sl}_n)$ apparaissant par exemple dans la construction du polynôme de Jones avec l'action du groupe quantique $U_q(\mathfrak{sl}_m)$. Le point central consiste alors à identifier le tressage comme l'action du groupe de Weyl quantique.

Ce processus d'antidualité, basé sur deux actions conjointes de $U_q(\mathfrak{sl}_n)$ et $U_q(\mathfrak{sl}_m)$, ne donne pas le même statut à ces deux groupes : le premier traduit le fait que l'on s'intéresse à des invariants $\mathfrak{sl}_n$, et a donc vocation à être conservé. Le second par contre tient du paramètre : la valeur de m, que l'on peut faire varier, est reliée au nombre de brins de certains diagrammes du nœud que l'on considère. Il apparaît donc comme naturel de chercher dans quel mesure ce groupe peut être modifié tout en conservant un processus similaire menant à des invariants de type Reshetikhin-Turaev. La première de ces extensions, que nous étudions dans le chapitre 4, consiste à remplacer $U_q(\mathfrak{sl}_m)$ par son extension affine $U_q(\widehat{\mathfrak{sl}_m})$. Le processus construit alors des invariants de nœuds et de toiles nouées considérées sur un anneau.

Notons que la construction de l'extension affine du processus utilise de manière assez poussée l'action du groupe de Weyl quantique déjà évoquée, ce qui nous oblige à porter une plus grande attention aux normalisations que dans le cas linéaire. Nous présentons dans ce même chapitre 4 une version renormalisée de l'anti-dualité de Howe dont le tressage sorti de son contexte permet de définir un module skein beaucoup plus souple que la version qui en provient initialement. Les preuves d'invariances présentées ici (qui consistent essentiellement à redéfinir et ré-étudier les invariants MOY [MOY98]) le sont dans un contexte général et avec des méthodes qui laissent espérer que les questions abordées dans le chapitre 2 pour le seul cas $\mathfrak{sl}_2$ admettront un traitement général dans le cas $\mathfrak{sl}_n$.

Le processus d'antidualité de Howe présente l'avantage d'admettre une catégorification beaucoup plus naturelle que la construction originale de Jones. La difficulté à construire un processus catégorique mimant la construction de Jones, et notamment à identifier le tressage, est ici considérablement simplifiée par le fait que le pendant dual du tressage qui apparaît, à savoir l'action du groupe de Weyl quantique, a été catégorifié par Chuang et Rouquier [CR08] par les *complexes de Rickard*. Il apparaît en fait que chacune des pièces du puzzle permettant la reconstruction des invariants de Reshetikhin-Turaev admet déjà une catégorification bien comprise : tout le propos du chapitre 3, qui reprend *in extenso* un article écrit en commun avec Aaron Lauda et David Rose (*University of Southern California*), est de comprendre comment relier les catégorifications des différentes pièces, pour les invariants $\mathfrak{sl}_2$ et $\mathfrak{sl}_3$.

Ce travail consiste principalement à définir un 2-foncteur entre le groupe quantique $U_q(\mathfrak{sl}_m)$ catégorifié et une version rigide d'une catégorie de mousses, qui ne sont a priori rien d'autre, dans le cas $\mathfrak{sl}_2$, que des cobordismes tels qu'ils apparaissent chez Bar-Natan. Notons cependant que cette présentation ne permet en réalité pas d'obtenir un foncteur, et qu'il faut pour atteindre cet objectif travailler avec les versions étendues de cobordismes telles qu'étudiées dans le chapitre 2 et leurs analogues $\mathfrak{sl}_3$.

La construction de ce 2-foncteur consiste à prouver que la définition naturelle respecte bien les relations de $\mathcal{U}(\mathfrak{sl}_m)$, la catégorification de $U_q(\mathfrak{sl}_m)$, c'est-à-dire que chacune de ces relations correspond à une relation de mousses. Ce travail produit plusieurs résultats auxiliaires, en particulier le fait que les relations de mousses $\mathfrak{sl}_2$ et $\mathfrak{sl}_3$ proviennent essentiellement du groupe quantique catégorifié. Si ces mousses étaient déjà bien connues auparavant, cela donne par contre l'espoir que le même procédé étendu à $U_q(\mathfrak{sl}_n)$ pourra donner la description complète de la catégorie des mousses $\mathfrak{sl}_n$ qui fait encore défaut.

Par ailleurs, la catégorification du phénomène d'antidualité de Howe que nous proposons nous permet ensuite de reconstruire l'homologie de Khovanov d'un nœud ou entrelacs, qui se comprend comme un complexe d'une catégorie construite sur le groupe quantique catégorifié $\mathcal{U}(\mathfrak{sl}_m)$, redonnant via le 2-foncteur que nous avons défini la version de Bar-Natan de l'homologie de Khovanov. Les résultats que l'on pourrait obtenir dans l'étude de l'homologie de Khovanov, ainsi que les extensions à l'homologie de Khovanov-Rozansky, voire aux catégorifications du polynôme de Homflypt, sont les principales pistes d'études qui se dégagent de ce travail.

Introduction (English)

Take a rope, and tie it, before gluing both ends together: this produces a knot. Take another one, tie it again and glue again both ends: here is another knot! Are both knots the same? In other words, is it possible, by pulling some parts of the strings (but without cutting them!), to make both strings identical? For example, a "magical knot" whose ends have been glued together can be untied, and therefore is the same as a simple ring of string, while this cannot be done for a figure-eight knot – which is good news for climbers and sailmen.

The problem of comparing two knots seems elementary, and in simple cases, a piece of rope and a few minutes are enough to form a convincing opinion on the answer. However, the search for a universal invariant, that is a tool which would in all cases distinguish two knots, has been and remains a very active field of research. This PhD thesis is immersed in that context, with a particular interest for the field of so-called quantum invariants.

History

From physics to mathematics

The classification of knots up to deformation is an ancient problem whose origins go back to the end of the 18^{th} century, with Lord Kelvin's work. At that time, the question of understanding the structure of matter was emerging, and Lord Kelvin supported a model where atoms were thought to be small knots in some space whose fundamental constituent would be ether. These ideas naturally set up the basis of knot theory, with the final objective to understand the structure of matter.

After the theory of ether was proved to be inaccurate, the interest for knots-related mathematical problems survived, setting up knot theory as an independent field of research, with strong relations with the newly developed algebraic topology of Poincaré, which provided a new formalism in geometry.

Among the founders of knot theory as we understand it today, we want to mention the two major names of James Waddell Alexander II [Jam01] and Kurt Reidemeister [BBF72]. Alexander, born in 1888, achieved his career in the US. First involved in the foundations of algebraic topology as initiated by Poincaré [Poi95] and Brouwer [Bro11], he started working in knot theory in the late 20's. His discovery of an invariant now known as the Alexander polynomial has given to knot theory its major tool until the 80's, and is still at the basis of a dynamic research. Alexander joined the *Institute for Advanced Studies (IAS)* in 1933, where he met Einstein, Von Neumann and Weyl. Reidemeister, born a few years after him (in 1893), began his career in Germany, where he wrote a dissertation on number theory, before joining Wirtinger in Vienna, who introduced him to the methods for computing knot groups. He then returned to Germany, but he was transferred from Königsberg to Marburg because of his political and philosophical ideas. At the end of the war, he joined the *IAS* for two years, but probably did not meet Alexander there, who had already distanced himself from the mathematical world. Appointed emeritus professor at the University of Göttingen, he died in 1971, the same year as Alexander. The book *Knotentheorie* [Rei32] he wrote in 1932 has been a reference for half a century, and his name is today closely tied to the set of moves that allow, together with planar isotopies, to relate any two diagrams of a same knot ; checking them is the central point of any invariance proof!

Jones and the quantum groups : back to physics?

The Alexander polynomial and the methods of classical algebraic topology prevailed during most of the twentieth century, until Jones [Jon85] discovered in 1984 a new polynomial invariant, fundamentally different from Alexander's, with roots in the theory of Von Neumann algebras. It is based on these works, among others, that Jones was awarded the Fields medal in 1990.

At the same time the theory of quantum groups, motivated by physics and initiated by the Faddeev school, developed quickly mathematically in the Soviet Union, under the impulsion of Drinfeld and Jimbo. We can also cite Lusztig, Kirillov, or Kac, who largely studied these deformations of algebras before 1990. Reshetikihin and Turaev [RT90, RT91], aware both of Jones' discovery and the development of quantum algebra, largely reformulated Jones' theory, giving to his polynomial the definition we now use, as a morphism of representations of the quantum group associated to $\mathfrak{sl}_2$. This formalism, which can be generalized using category theory, also allows extensions toward invariants of 3-manifolds, which, together with Kirby's works, set up the basis of quantum topology.

During the same period, Witten [Wit94], who was awarded a Fields medal the same year as Jones, brings quantum physics and quantum topology considerably closer, and initiates many years of joint studies from both perspective about quantum field theories. Witten's work, together with Atiyah's, has introduced the idea of Topological Quantum Field Theories (TQFT, see in particular [BHMV95] and [Koc04]), which turned out to be an extremely useful axiomatic tool in the study of knot theory and of 3-manifolds.

The 3^{rd} dimension... and beyond?

In 2000, Khovanov [Kho00] opened new perspectives by initiating the categorification program in knot theory. The purpose of categorification is to consider a usual object (numerical invariant, polynomial, group,...) as the shadow of a notion of higher order. The question is then to unveil the whole new structure (homology, category,...) and to highlight the process (Euler characteristics, Grothendieck group, trace) retrieving the original object as a shadow of the new one.

The most famous example brings us a few decades back, with the emergence of homology for manifolds: the number of faces of each dimension of a manifold (as a simplicial complex in the easiest case) can be considered as dimensions of vector spaces generated by the faces. The alternate sum of these numbers then corresponds to defining differentials between these spaces, and taking the homological of the complex formed. Finally, the Euler characteristics of the manifold is naturally the Euler characteristics of the homology so defined.

Beside the fact that the homological theories defined with this type of constructions usually prove to be richer invariants than the traces they categorify, they also have the great advantage of dealing with morphisms between the objects they describe. For example, in the case of homology theories for manifolds, the homeomorphisms (with adequate structure) define morphisms between the homologies associated to the image and coimage of the homeomorphism.

The constructions Khovanov introduced first lied on a reformulation of the Jones polynomial using the Kauffman bracket, which gives a combinatorial presentation of it. Further developments [Kho02, Kho06, BN05] enlightened the geometric aspect of this construction, and possible extensions toward both higher dimension, with cobordisms between knots, and lower dimension, by allowing knots to be cut into tangles. The formal version developed by Bar-Natan also allowed for extensions in more general manifolds, which will be one of the fields of study of the present PhD thesis.

On the algebraic side, many works contributed to the development of tools related to Khovanov's ideas, among which we can cite studies of category $\mathcal{O}$ [BFK99, FKS06, BS08, Str09] driving to a categorification of $U_q(\mathfrak{sl}_2)$. These categorifications also have diagrammatic descriptions [KL09, KL11a, KLMS12], easier to deal with, and which will be largely used in this thesis. It is also with help of related diagrammatic algebras that Webster [Web10a, Web10b] developed a general reformulation of Reshetikhin-Turaev invariants.

The invariants initiated by Khovanov in the $\mathfrak{sl}_2$-case also have been extended to $\mathfrak{sl}_n$, and to the Homflypt polynomial [KR08a, KR08b, KR07, MSV09, MSV11]. Just as the Jones polynomial led to the discovery of invariants for manifolds of dimension 3, the development of categorical invariants for 3-manifolds, that is of $3+1$-TQFT, is a major goal. A difficult step in this process concerns the fact that quantum invariants of manifolds of dimension 3 are usually built from quantum groups at roots of unity, whose categorification is a hard task. Recent work from Khovanov, Qi and Elias [Kho05, KQ12, EQ13] however raises hope for some important progress in this field in the next few years.

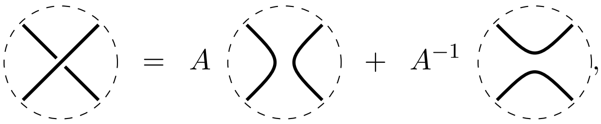
Around the central thema of categorification of quantum invariants of manifolds of dimension 3, the work of the PhD thesis presented here is mostly based on a study of Khovanov homology [Kho00]. The topologically flavored reformulation due to Bar-Natan [BN05] admits very natural extensions in the context of thickened surfaces, and the understanding of Bar-Natan's skein module in which these constructions take place is the main question addressed in the first part of this work, both individual and in collaboration with H. Russell.

Furthermore, the recent development of the skew-Howe duality, and the reinterpretation of Reshetikhin-Turaev invariants led to an algebra-based categorification entirely encoded into a 2-functor between a categorification of the quantum group $U_q(\mathfrak{sl}_m)$ and categories of foams modified following ideas of Blanchet [Bla10]. The second part of this dissertation presents this collaborative work with A. Lauda and D. Rose [LQR12], followed by an affine extension.

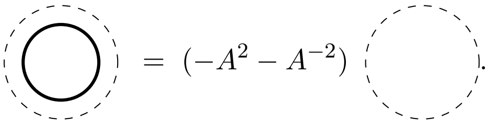
We now intend to further present the context in which these two parts are embedded.

Skein modules and generalizations

The reformulation of Jones polynomial in terms of Kauffman bracket builds a recursive process of great simplicity, which avoids the technical question of the $U_q(\mathfrak{sl}_2)$ representation theory, the bridge linking the diagrammatic world to the algebraic aspect being established by the Temperley-Lieb algebras. The computation of the Jones polynomial can then be achieved starting from a diagram of a knot with the recursive process consisting in replacing each crossing by a formal sum:



where A is an indeterminate. Once all crossings have been smoothed, the curves we obtain are simply unions of circles embedded in the plane, which we can replace by Laurent polynomials in the variable A using the local normalization:



The easiest formulations of Khovanov homology are based on an upgrade of both the smoothing relations for crossings and of the normalization. The key idea consists in seeing the parameter A as the shadow of a grading on the modules we want to consider, and therefore to associate to a circle a rank 2 module generated by elements in degrees 2 and -2. The smoothing can then be seen as a step-by-step building of a complex, the difficulty lying in the determination of the adequate differential maps.

A geometric reinterpretation due to Bar-Natan makes this construction stand on the topological side, where the idea of categorification consisting in adding a dimension becomes apparent: since knots are of dimension 1, one needs to consider surfaces, modulo some relations allowing to both mimic the normalization relation and build complexes. The cobordism category in which this construction takes place is sometimes called *Bar-Natan's skein module*.

This skein module, entirely geometric, is particularly convenient for extensions to knots embedded not anymore in $\mathbb{R}^3$, but in thickened surfaces. This context appears to be a natural extension of Khovanov's work, but also a possible first step toward the definition of categorical invariants of 3-manifolds. For example, Rozansky studied an extension in the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ [Roz10a]. If his construction is very closely linked to the particular structure of this manifold, and therefore cannot necessarily be extended as it is, the fact that $\mathbb{S}^2 \times \mathbb{S}^1$ is the easiest manifold built by surgery on the sphere suggests to use surgery or Heegaard splitting methods to extend these invariants. In both cases, the process of gluing 3-manifolds would appear along a surface, and the study of this intermediate structure, and in particular its product, is a very natural question in that context.

If, as we said, Bar-Natan's version of Khovanov homology furnishes extensions for thickened surfaces, the richer freedom given par the ambient manifold where the cobordisms are embedded undermines the usual constructions, and in particular the TQFT, which typically give a bridge between the geometric aspect and the algebraic versions. The search for solutions to these questions was notably led by Turaev and Turner [TT06], Carter and Saito [CS09], and Asaeda, Przytycki and Sikora (APS) [APS04].

We can first notice that the decategorification of the Asaeda-Przytycki-Sikora's theories gives a more complex result as the single skein elements associated to the knot considered at first. Ideas extracted from this remark, linked with a description of the product structure of the skein module due to Frohman and Gelca [FG00], have been the starting point of the collaborative work exposed in Chapter 1.

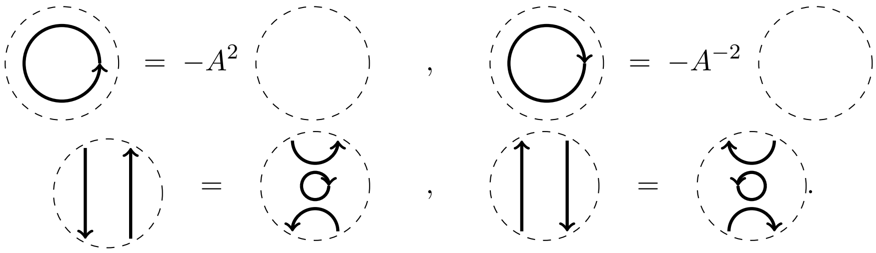
The smoothing relation allowing to define the skein module is also the cornerstone of a product structure in the case of surfaces. Two curves drawn on a surface can be superposed one on top of the other, producing new crossings, which can be smoothed as well. This builds an algebra, often called *Jones-Kauffman algebra*.

The torus case is most certainly the easiest interesting one: the product structure is not as trivial as it would be for the plane or the annulus, since it corresponds to a disjoint union, but basis of the module as well as generators of the algebra are easy to find. However, the easiest basis of the module structure, which is given by parallel unions of base curves in homology, is not strong enough for the product structure.

The central idea of Frohman and Gelca's paper [FG00] is precisely to exhibit a different basis of the module structure, whose product of elements is given by the so-called *Product-To-Sum* formula, which is extremely simple. This new basis is mysteriously defined in terms of Chebyshev's polynomials of the first kind, whose recurrence formula is the heart of a rather technical proof.

The purpose of the first section of this dissertation is to entirely revisit this proof with the help of a diagrammatic module inspired by the decategorification of APS-theory. This method clearly enlightens the different steps of this proof, and in particular makes very apparent the role of Chebyshev's polynomials as diagonalization process of the natural basis, after one has isolated the difficulty in technical lemmas concerning the product.

We introduce for this purpose an oriented skein module defined, in the torus case, as the $\mathbb{Z}[q, q^{-1}]$-module generated by oriented closed curves modulo local relations:

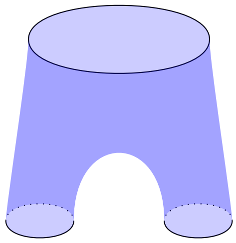


The structure product can also be defined in that context as the smoothing of crossings created by superposition, with respect to local orientation. Two different steps allow us to reprove Frohman and Gelca's formula. First, we study the link between usual skein modules and the oriented one, which is mostly based on technical lemmas. Then, a comparison of a natural basis of the skein module and the one we introduce, reveal them to be related very naturally by Chebyshev's polynomials, from which we can deduce Frohman and Gelca's formula.

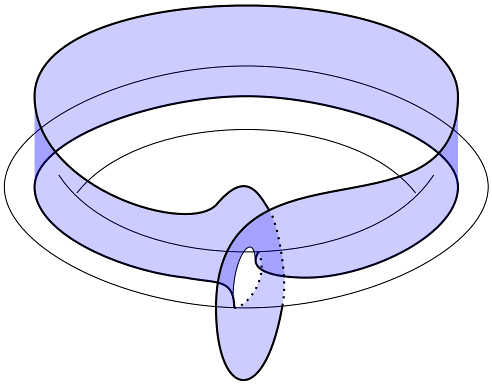
Beside the role of intermediary played by this module in the Product-To-Sum formal, it also appears to be by itself a skein module, that is the natural receptacle of an invariant of a class of knots said *boundary parallel band knots*.

Furthermore, this module easily generalizes for punctured surfaces, and we study the properties of such extensions. These generalizations seem to be a natural setting to derive analogues of Frohman-Gelca's formula, and we construct the basis for what could lead to such results. The formulas we could obtain from such work would then be to relate to recent works by Bonahon and Wong [BW12a, BW12b], lying on a Frohman-Gelca's formula for the punctured torus considered at roots of unity.

A second remark concerning the models developed by Asaeda, Przytycki and Sikora, remark that should be related to results of Turaev-Turner and Carter-Saito, is about the appearance of particular saddles in the cobordisms over some surfaces, and in particular over the torus. Indeed, if a saddle over a plane can only be closed as a pair of pants (two legs, one waist):



it is possible over some surfaces to have only two holes, by bounding curves of different homology type, as drawn below in the torus case.



This kind of saddles is one of the major obstacles in the search of TQFT for thickened surfaces. The solution of Asaeda, Przytycki and Sikora consists in assigning to it a zero morphism, with as consequence the nullity of many other maps. It is interesting to note that these saddles form non-orientable surfaces, which drives us to wonder about the existence of the same difficulties in the context of categories of cobordisms enriched with local orientations. Such cobordisms were introduced by Clark-Morrison-Walker, Caprau and Blanchet, in order to give solutions to a lack of functoriality in Khovanov homology.

Indeed, and this is one of the central interests of categorification, it is possible to consider knots as objects of a category whose morphisms are surfaces, and Khovanov's process allows to associate to these surfaces a morphism between the complexes corresponding to the knots they bound [Kho06, BN05]. This assignment process demands a decomposition of each surface into a succession of elementary moves. Since this decomposition is not unique, the independence of the result from the choice of decomposition requests more investigations. It turns out that this invariance holds, in the original Khovanov's model, only up to sign.

This functoriality defect has been solved by Clark, Morrison and Walker [CMW09] and Caprau [Cap10, Cap09, Cap07] by working with Gauss integers. The model that Blanchet [Bla10] presented then allowed to keep $\mathbb{Z}$ as base ring. Both theories introduce a new class of disoriented cobordisms. It is then possible to translate Bar-Natan's skein module [BN05] into the disoriented context, and to extend these constructions in the case of any surface.

The relation that then link the unorientable saddle to other structural morphisms of the cobordism category can be translated in the disoriented case, and it is shown in Chapter 2 that they really differ from the oriented case. In particular, one can choose to mod out the disoriented cobordism category by the unorientable surfaces (which is the easiest way to assign to them zero morphisms!), without causing any modification in the structure of the remaining part of the category. This result highlights again the interest of these disoriented categories, and invites to further study the TQFT that could be applied in this extend context, a study that would first request an adequate axiomatisation of the structure of these categories.

The difference between Clark-Morrison-Walker's and Blanchet's models, which we study in Chapter 2 is at first sight relatively mysterious. The "interesting" part of the cobordisms is essentially the same, and both models introduce an extra element (namely, a square of -1 or some 2-facets) in order to allow a better treatment of the changes in the orientation of the surface. However, we show here that the two models have different stability in the search of natural extensions to generalized knot categories: only Blanchet's model easily extends to an invariant of knotted graphs, while we run into the well-named *confusions* when we attempt to define an analogue in Clark-Morrison-Walker's model.

After having defined this extended Khovanov homology for the generalized class of knots formed from knotted webs, the question of the functoriality arises. We give at the end of Chapter 2 a complete list of movie-moves one should check for proving functoriality.

These last extension results are the first step toward the definition of categorified $\mathfrak{sl}_n$ invariants for knotted webs. If the methods used in the proofs of the first part of this thesis are efficient as long as n is small enough, since the number of cases to consider remains reasonable, their use in the general case would certainly be less adapted. In that context, the abstract algebraic treatment furnished by skew-Howe duality, which we study in the second part of this work, and the proofs of Chapter 4 concerning the non-categorified invariants, suggest a program of unified algebraic proofs for categorifications of the Jones-Reshetikhin-Turaev invariants. The development of the tools to be used in such proofs will constitute an axis for future research, in the continuity of the work presented here.

Skew-Howe duality and Khovanov homologies

The original construction of the Jones polynomial is achieved by using the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$. After Khovanov defined the first categorifications of these invariants, the question of relating geometric versions of those to categorifications of quantum groups naturally emerged.

The categorification process for a quantum group essentially consists in giving an extra dimension to its category of representations, or a subcategory of it. We therefore obtain a 2-category, in which the underlying idea is to keep generators of the algebra as 1-morphisms, and to replace any of the initial relations by an isomorphism between the 1-morphisms we consider. These isomorphisms are realized by explicit 2-morphisms, and the structure we give to the 2-category ensures that they are indeed invertible. At the same time, we also want to control the dimensions of the spaces of 2-morphisms between 1-morphisms, in order to be able to recover as a trace of this 2-category the original category of representations. A particularly convenient version of this categorification has been introduced by Khovanov and Lauda [KL09, KL11a, KL10]. Based on spaces of diagrams, it allows simple computation methods and is easy to manipulate.

This idea of relating, in essence, categorified quantum groups and Khovanov homology appeared up to now rather technical [BFK99, MS09, Str05, FKS06, Web10a, Web10b] . The relation we propose in Chapter 3 arises from a theory of *skew-Howe duality* due to Cautis, Kamnitzer and Licata [Cau12, CKL10a], consisting in two quantum groups $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$ acting simultaneously on the exterior powers of the tensor product of their vector representations. It turns out that these two actions commute, and most importantly are dual. The proof of this result by Cautis, Kamnitzer and Licata allowed Cautis, Kamnitzer and Morrison [CKM12], to solve conjectures on presentations of endomorphisms categories for $U_q(\mathfrak{sl}_n)$ representations (related to $\mathfrak{sl}_n$-generalized skein modules).

This theory also led to a reconstruction of Reshetikhin-Turaev invariants [Cau12], which allows to identify the morphisms of $U_q(\mathfrak{sl}_n)$ representations that appear for example in the construction of Jones polynomial with the action of the quantum group $U_q(\mathfrak{sl}_m)$. The key point then consists in identifying the braiding as the quantum Weyl group action.

The skew-duality process, basd on two joint actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$, do not give the same status to the two groups: the first one carries the information on the $\mathfrak{sl}_n$ invariants we look at, and therefore is intended to be kept. On the other hand, the second one is closer to a parameter: the value of m, which can vary, is related to the number of strands of certain diagrams of the knot we consider. It therefore appears as a natural idea to look for possible modifications of this group, while keeping a similar process which would lead to invariants of Reshetikhin-Turaev type. The first extension, which we study in Chapter 4, is based on a replacement of $U_q(\mathfrak{sl}_m)$ by its affine extension $U_q(\widehat{\mathfrak{sl}_m})$. The process then leads to invariants of knots and knotted webs considered in an annulus.

Notice that the construction of the affine extension of the process uses in details the action of the quantum Weyl group already mentioned, which forces us to care more precisely about normalizations than in the linear case. We present in the same Chapter 4 a renormalized version of skew-Howe duality whose braiding considered generally allows to define a more flexible version of the skein module than the one that usually arises from it. The invariance proofs (which essentially consist in redefining and study again the MOY invariants [MOY98]) are presented here in a general context and with methods that let hope that the questions addressed in Chapter 2 for the single $\mathfrak{sl}_2$ case will receive a general answer in the $\mathfrak{sl}_n$ case.

The skew-Howe duality process has the major advantage of suggesting a much more natural categorification than the original Jones construction. The difficulty in building a categorical process that mimic Jones' construction, and in particular the identification of the braiding, is here considerably simplified by the fact that the dual analogue of the braiding which appears, namely the quantum Weyl group action, has been categorified by Chuang and Rouquier [CR08] using the *Rickard complex*. It actually appears that each piece of the puzzle in the reconstruction of Reshetikhin-Turaev invariants already admits a well-known categorification: the whole work in Chapter 3, which is the precise transcription of a paper written in collaboration with Aaron Lauda and David Rose (*University of Southern California)*, stands on the understanding of the way to relate the categorification of all pieces, which is achieved for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ invariants.

The core of this work consists in defining a 2-functor from the categorification of the quantum group $U_q(\mathfrak{sl}_m)$ and a rigid version of a foam category, which are nothing more, in the $\mathfrak{sl}_2$ case, than the cobordisms as they appear in Bar-Natan's construction. Observe nonetheless that this presentation actually wouldn't lead to a functor, and that we need, for achieving our goal, to work with the extended versions of cobordisms as studied in Chapter 2, and their $\mathfrak{sl}_3$ analogues.

The construction of this 2-functor is done by proving that the natural definition does respect all relations in $\mathcal{U}(\mathfrak{sl}_m)$, the categorification of $U_q(\mathfrak{sl}_m)$, that is proving that any of these relations corresponds to a foam relation. This work leads to some auxiliary results, in particular the fact that all $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foam relations can be essentially obtained from the categorified quantum group. If these foams were already well-known, this suggests nevertheless that the same process extended to $U_q(\mathfrak{sl}_n)$ with any value of n could give a complete description of the $\mathfrak{sl}_n$ foam category which we still miss.

Furthermore, the categorification of the skew-Howe duality phenomenon that we propose allows us to rebuild the Khovanov homology of a knot or a tangle, which corresponds to a complex in a category built over the categorified quantum group $\mathcal{U}(\mathfrak{sl}_m)$, giving back, via the 2-functor we defined, Bar-Natan's version of Khovanov's homology. The results we could obtain by studying Khovanov homology as well as extensions toward Khovanov-Rozansky homologies, or even categorifications of Homflypt polynomial, are the major tracks for future studies that emerge from this work.

Summary of results

Frohman-Gelca formula and extensions

Note: This chapter is a joint work with **Heather M. Russell** from the *University of Southern California*. This project is based on Frohman and Gelca's paper [FG00], and uses objects defined or studied by Sallenave [Sal00], as well as many methods due to Przytycki [Prz98].

The main goal of Chapter 1 is to reprove Frohman-Gelca's *Product-To-Sum* formula, by enlightening the role of Chebyshev's polynomials.

We consider the torus $\mathbb{S}^1 \times \mathbb{S}^1$, and $Sk(\mathbb{S}^1 \times \mathbb{S}^1)$ its skein module, which can be endowed by superposition with its Jones-Kauffman algebra structure. A basis for the module is given by all possible disjoint unions of toric curves (p, q) (with $GCD(p, q) = 1$) and the empty curve, over the ring $R = \mathbb{Z}[A, A^{-1}]$. This basis however is very inconvenient for multiplication, since the product of two basis elements is a possibly huge sum of other basis elements. Frohman and Gelca's purpose was to exhibit a basis that behaves better under multiplication [FG00].

They introduce for this the *Chebyshev's polynomials of the first kind*, defined by : $T_0 = 2$, $T_1 = X$, $T_n = X \cdot T_{n-1} - T_{n-2}$. Since the skein module is an algebra, we can apply polynomials to its elements, and they denote, for $GCD(p, q) = 1$, $(np, nq)_T = T_n((p, q))$.

The elements one can form this way turn out to give another basis of the skein module, and furthermore, Frohman and Gelca prove a Product-to-Sum formula:

$$(a, b)_T * (c, d)_T = A^{-\begin{vmatrix} a & c \\ b & d \end{vmatrix}} (a - c, b - d)_T + A^{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} (a + c, b + d).$$

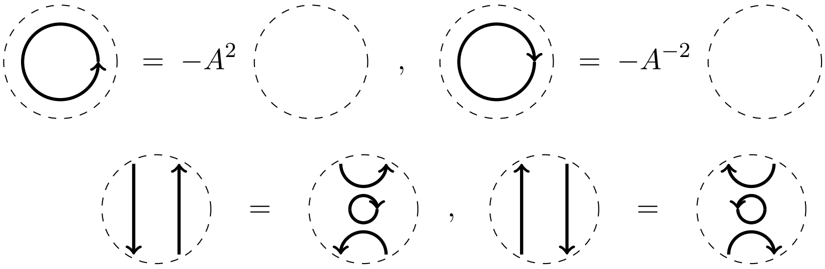
This proof is achieved by introducing another module ${}_A\mathbb{T}$, and by using the iterative definition of the Chebychev's polynomials. ${}_A\mathbb{T}$, called the *quantum torus*, can be defined as $< L^{\pm 1}, M^{\pm 1} \,|ML = A^{-2}LM >_R$, and if we define Θ to be the map of algebra given by: $\Theta(L) = L^{-1}$, $\Theta(M) = M^{-1}$ and $\Theta(A) = A$, we denote ${}_A\mathbb{T}^\Theta$ the eigenspace of Θ for the eigenvalue 1.

Then, Frohman and Gelca exhibit a basis of ${}_A\mathbb{T}$ given by elements $e_{m,n} = A^{-mn}L^m M^n$ for $(m, n) \in \mathbb{Z}^2$, and a related basis of ${}_A\mathbb{T}^\Theta$ given by elements $c_{(m,n)} = e_{(m,n)} + e_{(-m,-n)}$.

The central theorem of [FG00], in which the Product-To-Sum formula is the key ingredient, can be stated as:

Theorem 1.1.4. *$(np, nq)_T \to c_{(np,nq)}$ induces an isomorphism of algebras between $Sk(\mathbb{S}^1 \times \mathbb{S}^1)$ and ${}_A\mathbb{T}^\Theta$.*

In order to better understand the role of the Chebyshev's polynomials in this proof, we introduce a third module, $\mathcal{A}$, close to ${}_A\mathbb{T}$ for its algebraic description, but diagrammatically strongly related to Sk. $\mathcal{A}$ can be defined as the R-module generated by oriented embedded curves on $\mathbb{S}^1 \times \mathbb{S}^1$ up to isotopy and the relations depicted below.



(0.0.1)

It can also be endowed with an algebra structure by assigning to $\alpha * \beta$, for α and β two curves on the torus, the result of superposing α over β and smoothing all created crossings with according coefficients.

We have a natural basis of this module, given in terms of elements $\gamma_{(np,nq)}$ which are represented by n disjoint unions of the toric oriented curve (p,q) (with $GCD(p,q)=1$).

Proposition 1.1.6. *$\{\gamma_{(np,nq)}\} \cup \{\emptyset\}$ is a basis of $\mathcal{A}$.*

This allows us to relate $\mathcal{A}$ and ${}_A\mathbb{T}$.

Proposition 1.1.9. *$\mathcal{A}$ is isomorphic to ${}_A\mathbb{T}$.*

On the other hand, we can also relate $\mathcal{A}$ and $Sk(\mathbb{S}^1 \times \mathbb{S}^1)$, as stated below. The proof of this theorem is mostly technical, and does not request any use of the Chebyshev's polynomials.

Theorem 1.1.11. *If Θ is the endomorphism of $\mathcal{A}$ defined on generators by reversion of orientation, we have: $Sk(\mathbb{S}^1 \times \mathbb{S}^1) \simeq \mathcal{A}^\Theta$ as algebras.*

We now have three different descriptions of the same object:

$$ {}_A\mathbb{T}^\Theta \ \simeq \ \mathcal{A}^\Theta \ \simeq \ Sk(\mathbb{S}^1 \times \mathbb{S}^1). $$

The key idea for understanding the Product-to-Sum formula is then to relate the natural basis of the right and the middle modules, and deduce relations for the product from this comparison. Basis for these modules are given:

– for $Sk(\mathbb{S}^1 \times \mathbb{S}^1)$, by multicurves $c_{(np,nq)}$, which is hard to use for multiplication!

– for $\mathcal{A}^\Theta$, by terms of type $(\gamma_{np,nq} + \gamma_{-np,-nq})$, respecting:

$$\begin{aligned} (\gamma_{a,b} + \gamma_{-a,-b}) \quad &* \quad (\gamma_{c,d} + \gamma_{-c,-d}) \\ &= \quad A^{-\begin{vmatrix} a & c \\ b & d \end{vmatrix}} (\gamma_{a-c,b-d} + \gamma_{-a+c,-b+d}) + A^{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} (\gamma_{a+c,b+d} + \gamma_{-a-c,-b-d}). \end{aligned}$$

This basis therefore behaves very well under multiplication.

Noticing that the image of $c_{(np,nq)}$ in $\mathcal{A}$ is $(\gamma_{np,nq} + \gamma_{-np,-nq})^n$, the next remark is the central ingredient for recovering Frohman-Gelca's formula:

Remark 1.1.12. *$T_n(x + x^{-1}) = x^n + x^{-n}$.*

This completes the first purpose of our study.

We now turn to studying and extending $\mathcal{A}$. It can be understood as an invariant of the next particular class of knots.

Definition 1.1.17. *Consider $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ foliated by $\mathbb{S}^1 \times \mathbb{S}^1 \times \{i\}$ for $i \in \mathbb{I}$. A* boundary parallel band knot *in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ is a smoothly embedded annulus with normal vector everywhere transverse to the foliation. A* boundary parallel band link *is analogously defined.*

Definition 1.1.18. *Two boundary parallel band links are equivalent if they are related by a continuous deformation through boundary parallel band links.*

Proposition 1.1.22. *The map that assigns to a boundary-parallel band knot its smoothing in $\mathcal{A}$ is an invariant of boundary-parallel band knots.*

Furthermore, denoting $\mathcal{C}$ Bar-Natan's category of cobordisms as it appears in Asaeda, Przytycki and Sikora's extension of Khovanov homology [APS04], we can identify (and this was the starting point of this study):

Proposition 1. *$K^0(\mathcal{C}) \simeq \mathcal{A}$ as $\mathbb{Z}$-modules.*

$\mathcal{A}$ can also be generalized to the case of orientable surfaces S of higher or lower genus, if these surfaces contain at least one puncture. We have a description of it as follows.

Proposition 1.3.3. *There is an isomorphism $\pi\colon {}_A\mathbb{T} \otimes_R {}_A\mathbb{T} \cdots \otimes_R {}_A\mathbb{T} \mapsto \mathcal{A}(S \setminus \{p\})$.*

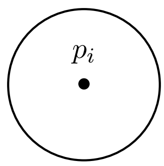
Just as in the torus case, this module also provides invariants for boundary-parallel band knots in the corresponding thickened punctured surface.

We then investigate the relations between $\mathcal{A}(S)$ and $Sk(S)$, and present a generalization of Chebyshev's polynomial in the multi-punctured case.

Following Sallenave [Sal00] , we can define the map $\varphi\colon\ Sk \mapsto \mathcal{A}$ by $\varphi(c) = \sum or(c)$, meaning that we take the sum of all possible orientations of the curve c.

Proposition 1.4.1. *φ is a map of algebras.*

We define $Sk_h(S) = Sk(S)/ker(\varphi)$, and, in the case where there are multiple punctures $\{p_1, \ldots, p_r\}$, we can define an augmented version of it, by adding in $Sk(S)$ the relation:



$$= \quad z_i^+ + z_i^-.$$

We extend φ to this new module by sending z_i^+ to a counterclockwise oriented circle around the i-th puncture (which we will denote a_i^+), and z_i^- to the same circle oppositely oriented (denoted a_i^-), and we can then define the augmented version of Sk_h, which we will denote $\widetilde{Sk_h}$, as the quotient of the augmented skein module by the kernel of φ.

We now define a generalized Chebyshev's element. Let c be a multicurve which is a disjoint union of curves $c_1, \cdots, c_n$ so that all the curves are parallel in S, but not necessarily in $S \setminus \{p_1, \cdots, p_r\}$. Each of the curves c_i is sent by φ to $c_i^+ + c_i^-$, which can be chosen so that all c_i^+ coincide in $\mathcal{A}(S)$.

Proposition 1.4.4. *$c_T := \varphi^{-1}(c_1^+ \cdots c_n^+ + c_1^- \cdots c_n^-)$ exists.*

This gives a generalized Frohman-Gelca formula:

Theorem 1.4.5. *Let c and d be two multicurves of the previous type. Let $|c \cdot d|$ denote the algebraic intersection number of c and d, and let $c + d$ be the multicurve obtained by* *smoothing all crossings in $c * d$ with the A coefficient. Similarly, $c - d$ is the multicurve obtained by performing all opposite smoothings. Then:*

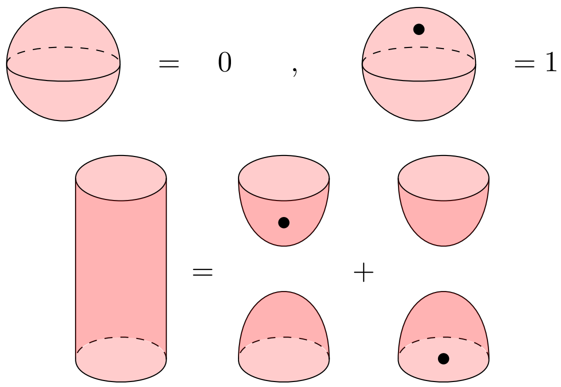
$$c_T * d_T = A^{-|c \cdot d|}(c-d)_T + A^{|c \cdot d|}(c+d)_T.$$

A complete understanding of $Sk_h(S)$ and $\widetilde{Sk_h}(S)$ remains to be achieved in general.

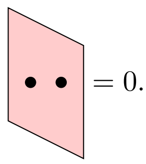
Disoriented Bar-Natan skein modules

Khovanov homology [Kho00] can be presented in a very topological framework, using graded categories of cobordisms modulo some relations [BN05], which we will generically call Bar-Natan's skein modules. We intend here to review and compare different versions of them.

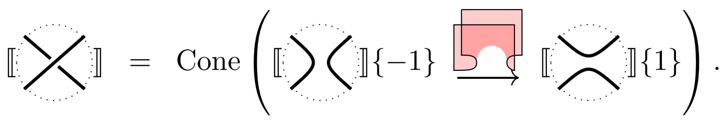
The original one is formed of surfaces embedded in $\mathbb{R}^3$, carrying dots, up to isotopies and the relations below.



Furthermore, we request that any closed connected foam with two dots is zero:



Khovanov's construction can then be summarized by the recursive process consisting in replacing every crossing by a piece of complex, which we consider in a category of complexes over Bar-Natan's category:



We then have the following fundamental theorem:

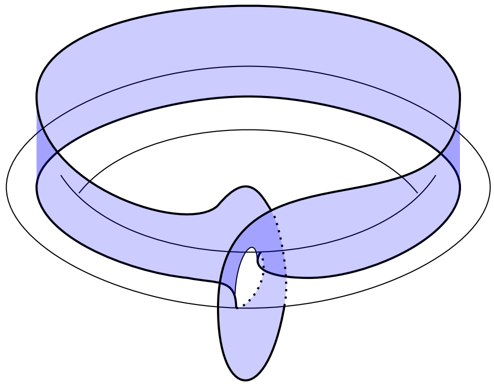
Theorem 2.1.1. *([Kho00, Theorem 1], [BN05, Theorem 1]) The homotopy class of the complex associated to the diagram of a knot does not depend, up to convenient grading shifts, on the particular diagram used for building it. Furthermore, its graded Euler characteristic is the Jones polynomial of the knot.*

A bridge between Bar-Natan's construction and Khovanov's original one is provided by TQFT, which are functors between the cobordism category and a category of modules.

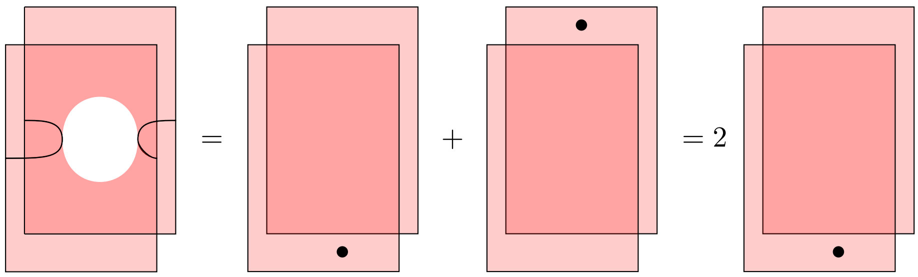
Bar-Natan's model can be extended to the case of knots embedded in a thickened surface $S \times \mathbb{I}$ (we restrict to the orientable case), drawing diagrams on the base surface. The same theorem as before remains valid.

Theorem 2.1.2. *([BN05, Theorem 1 and section 11.6]) The homotopy class of the complex associated to a diagram of a knot is independent, up to a convenient grading shift, of the chosen diagram. Furthermore, this complex decategorifies to the image of the knot in the skein module of the surface.*

However, TQFT are much harder to understand in this generalized case. One of the reasons is the existence of a new kind of saddle, unorientable, which relates two non-homotopic connected curves. An example for the torus is depicted below.



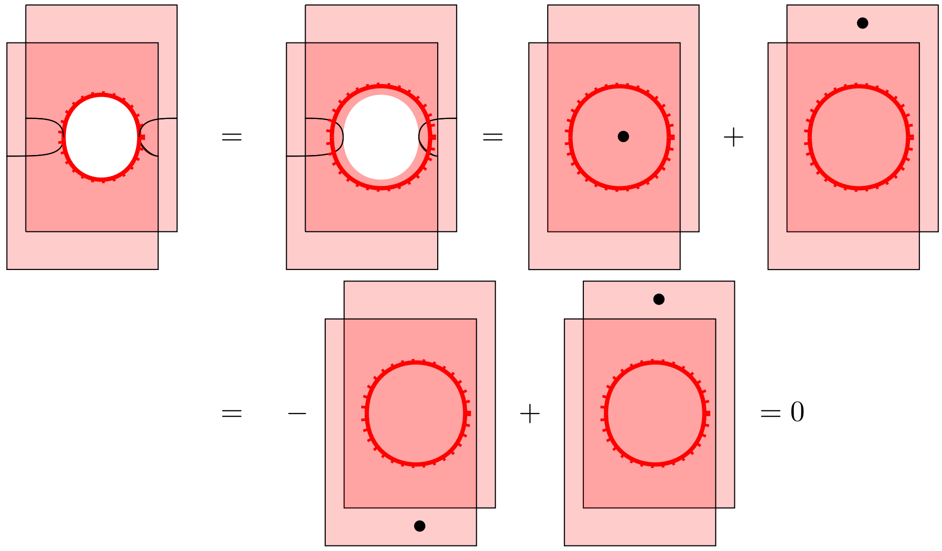
Composing this kind of saddle with its mirror image and applying to it a neck-cutting relation gives the equality of the composition with two dotted identities.



This is a key equation in the study of possible TQFT, and Turaev-Turner [TT06] and Carter-Saito [CS09] showed that the only reasonable extension for Khovanov homology is probably the one given by Asaeda, Przytycki and Sikora [APS04], where the new kind of saddle is sent to zero. Pulling back this assignment as a quotient of Bar-Natan's skein module, we observe that killing the unorientable saddle imposes via the key relation that some dotted identities are zero, and considerably weakens the module-comodule structure of the non-trivial circles (see [CS09]).

We now wish to understand better how the same question can be dealt with in the context of enriched Bar-Natan's categories of cobordisms with disorientation structure that arise in [CMW09, Cap07, Bla10], and which solve the functoriality issue in Khovanov homology.

With Clark-Morrison-Walker's model, the key relation takes a different form, as depicted below: the right hand side is now zero, and all terms in the equation are unorientable.



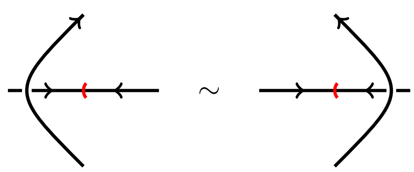
This suggests to define a category $Cob^{or}_{CMW}(S)$ as the category of cobordisms with disorientation lines $Cob_{CMW}(S)$ mod out by unorientable cobordisms. The key difference with the usual Bar-Natan's case is that this process now does not modify at all the structure of the orientable part of the category, as stated below.

Proposition 2.2.1. *If two orientable cobordisms are equal in $Cob^{or}_{CMW}(S)$, this was already the case in $Cob_{CMW}(S)$.*

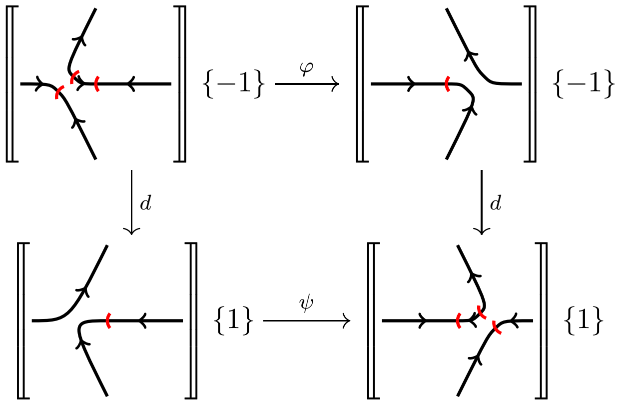
The exact same process can also be achieved in Blanchet's version of disoriented cobordisms, namely $\mathfrak{sl}_2$ foams. These results highlight again the better stability of the models with disorientations, compared to the unoriented one.

We then want to compare both Clark-Morrison-Walker and Blanchet's model, and their stability under extensions. Indeed, these new models contain new kind of objects, as diagrams with disorientation points or $\mathfrak{sl}_2$-webs, but are at first designed to deal only with knot diagrams. A natural development is therefore to try to define the same invariants for knotted generalized objects.

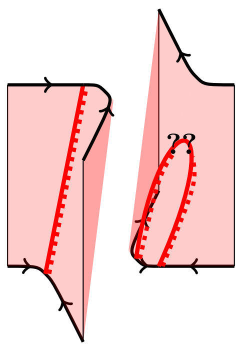
Looking to invariance properties of the complexes we want to define, we need generalized Reidemeister moves for these new classes (see [Kau89] or [Car12] for the web case). In [CMW09], Clark, Morrison and Walker briefly mentioned a difficulty in achieving the proof of these moves, and in particular, the following one seems to give an obstruction to extension.



We would need an homotopy equivalence between smoothings of both diagrams below.



There are no easy solutions to this problem: although one would like φ and ψ to have as underlying unoriented surfaces only identities, there is no way to match the tags on such surfaces.



We show that the situation works better in Blanchet's case, since, using his model, we can state the following theorem.

Theorem 2.3.1. *Khovanov homology is an invariant of knotted $\mathfrak{sl}_2$-webs.*

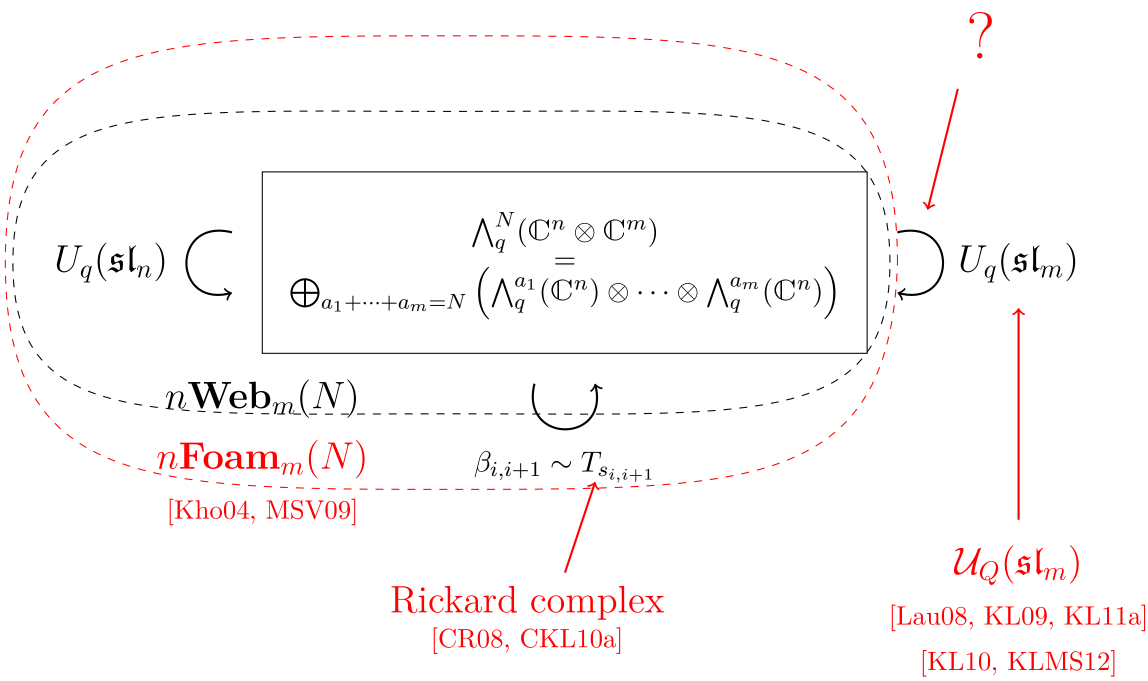
The natural question then is to check that this extension preserves the functoriality of the functor. Indeed, just as in the knot case, the proof of invariance is achieved by assigning to elementary foam moves some homotopy equivalences, so that we obtain a putative definition of a functor. We therefore need to achieve two steps: first, obtaining a complete set of moves for foams ; and then, checking them in all of their possible realizations. We will here only obtain the first step, and the second one will remain as future work.

Lemma 2.3.2. *Two movies of $\mathfrak{sl}_2$-foams describe isotopic foams if and only if they are related through a sequence of the following moves:*

– *usual movie-moves MM_1 - MM_{15} [CS98],*
– *Roseman-type moves for foams [Car12] specialized in the $\mathfrak{sl}_2$-case, that is, we do not consider all moves that contain a singular vertex bounding 6 facets,*
– *moves depicted during the proof: Zipper zig-zag, TwistsiwT, Sliding zipper, Sauron, Saddle zipper 1 and 2, Cup zipper 1 and 2 and Twist zipper 1 and 2.*

Khovanov homology is a skew-Howe 2-representation of categorified quantum $\mathfrak{sl}_m$

Note: Chapter 3 is the result of a joint work with **Aaron D. Lauda** and **David E. V. Rose**, from the *University of Southern California*, also available in the preprint [LQR12].



The above picture explains the general idea of skew-Howe duality [CKL10a, CKM12], its relations with webs and the embedding of our work in this context. $(U_q(\mathfrak{sl}_n), U_q(\mathfrak{sl}_m))$ form a Howe pair for the representation $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$ induced by vector representations, which means that the actionsof these two quantum groups on $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$ commute and are each other commutant. In particular, the endomorphisms of the $U_q(\mathfrak{sl}_n)$ representation, which are usually depicted by webs, correspond to the action of $U_q(\mathfrak{sl}_m)$, which allowed Cautis, Kamnitzer and Morrison to solve important conjectures about the presentation of spider categories (the category formed by webs).

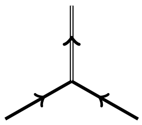
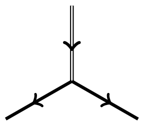
Furthermore, one can also identify the braiding on the $U_q(\mathfrak{sl}_n)$ side (depicted $\beta_{i,i+1}$ in the picture) as the quantum Weyl group action on the right side, which receives an explicit expression as an infinite sum in $U_q(\mathfrak{sl}_m)$, acting as a finite sum on any finite-dimensional representation [CKL10b, KT09] (this is $T_{s_{i,i+1}}$ in the picture). This allows for a reconstruction of Reshetikhin-Turaev invariants for knots and links [Cau12].

A major interest of this reconstruction is that it admits very natural categorifications. Indeed, almost all pieces of the puzzle admit a categorification (the red part of the picture): webs can be categorified by foams, $U_q(\mathfrak{sl}_m)$ is the Grothendieck group of $\mathcal{U}_q(\mathfrak{sl}_m)$, and Chuang and Rouquier showed that the Rickard complex generalizes the quantum Weyl group action. The only missing part is the categorified action of the categorified quantum group on foams: our paper defines it for $n = 2$ and $n = 3$.

We need for our purpose to work in rigid versions of Blanchet's foam category.

Definition 3.3.1. $2\mathbf{BFoam}_m(N)$ *is the* 2*-category defined as follows:*

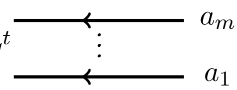
- *Objects are sequences* $(a_1, \ldots, a_m)$ *labeling points in the interval* $[0,1]$ *with* $a_i \in \{0,1,2\}$ *and* $N = \sum_{i=1}^{m} a_i$*, together with a zero object.*
- 1*-morphisms are formal direct sums of* $\mathbb{Z}$*-graded enhanced* $\mathfrak{sl}_2$ *webs - directed planar graphs with boundary with two types of edges -* 1*-labeled edges*  *and* 2*-labeled edges*  *- where all vertices are trivalent and take the following two forms:*

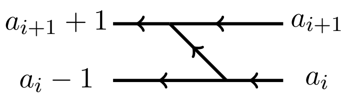
   *or*  .

  1*- (respectively* 2*-) labeled edges are directed out from points labeled by* 1 *(respectively* 2*) in the domain and directed into such labeled points in the codomain. No edges are attached to points labeled by* 0*.*
- 2*-morphisms are formal matrices of* $\Bbbk$*-linear combinations of degree-zero* $\mathfrak{sl}_2$ *foams - surfaces with oriented singular seams which locally look like the product of the letter* Y *with an interval - considered up to isotopy (relative to the boundary) and local relations.*

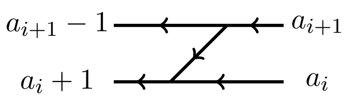
Then, we define the 2-functor on a subset of strands, which is proved to be enough, as follows. The pictures are generic ones, and all facets with labels lower than 0 or greater than 2 lead to the zero object. 0-labeled facets can be erased.

Proposition 3.3.3. *For each* $N > 0$ *there is a* 2*-representation* $\Phi_2 \colon \mathcal{U}_Q(\mathfrak{sl}_m) \to 2\mathbf{BFoam}_m(N)$ *given on* 1*-morphisms by*

$\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

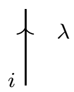
$\mathcal{E}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

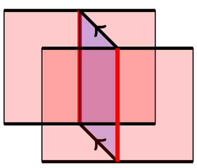
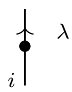
and

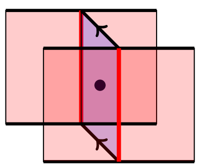
$\mathcal{F}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

when the boundary values lie in $\{0,1,2\}$ *and to the zero* 1*-morphism otherwise. The labelings of the edges incident upon the boundary are given by the boundary labels; edges incident upon boundary points labeled by zero should be deleted. Note that we have not depicted* $m-2$ *horizontal strands in each of the latter two formulae.*

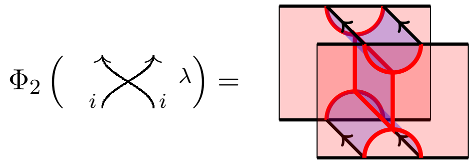
On single strand 2*-morphisms, it is defined by:*

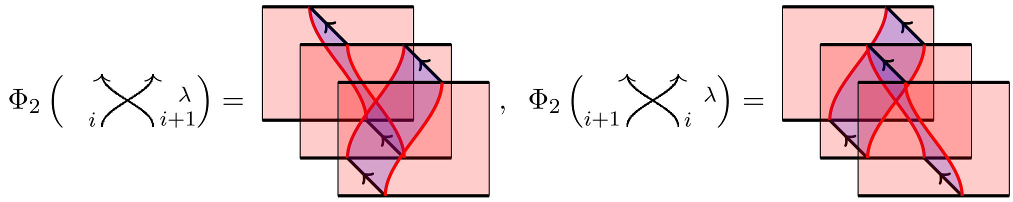
$\Phi_2\Bigg($ 

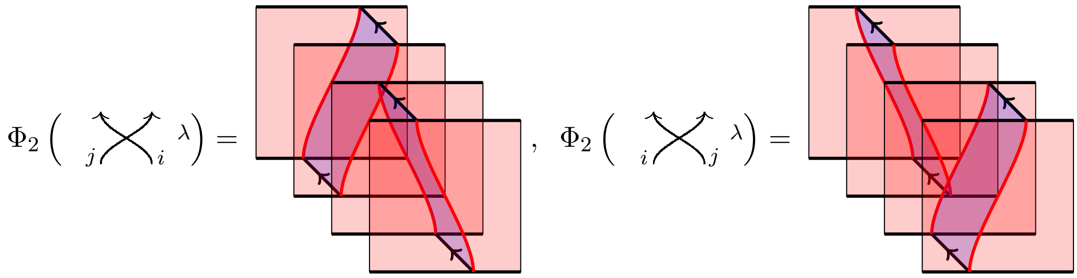
$\Bigg) =$  , $\Phi_2\Bigg($ 

$\Bigg) =$ 

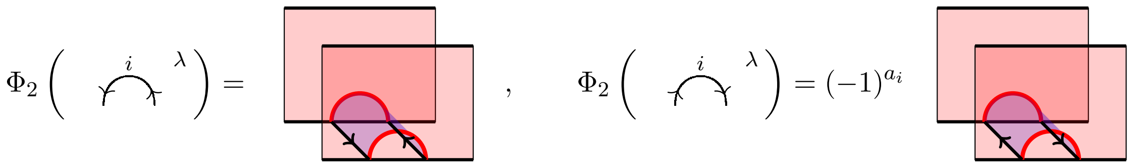
on crossings by:

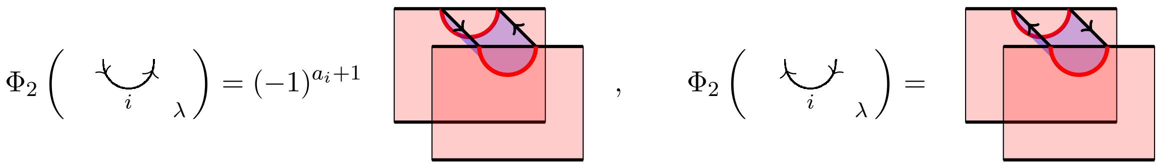






where $j - i > 1$, and on caps and cups by:





where in the above diagrams the i^{th} sheet is always in the front.

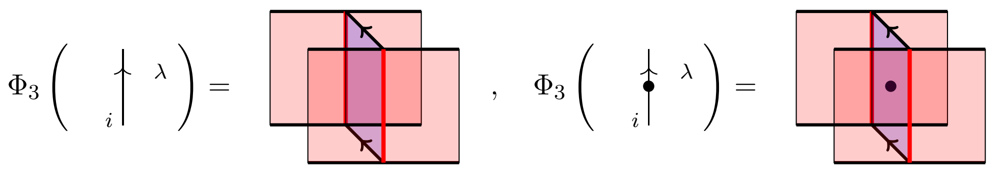
Similarly, we can define a rigid $\mathfrak{sl}_3$ foam 2-category from usual $\mathfrak{sl}_3$-foams.

Definition 3.3.6. $3\mathbf{Foam}_m(N)$ *is the 2-category defined as follows:*

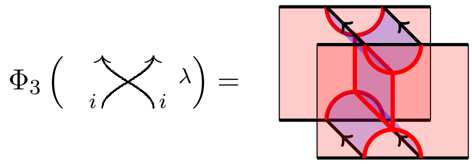
- *Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2,3\}$ and $N = \sum_{i=1}^m a_i$ together with a zero object.*
- *1-morphisms are formal direct sums of $\mathbb{Z}$-graded $\mathfrak{sl}_3$ webs mapping between the points labeled by 1 and 2 as in* $3\mathbf{Foam}$.
- *2-morphisms are formal matrices of $\Bbbk$-linear combinations of degree-zero $\mathfrak{sl}_3$ foams mapping between such webs.*

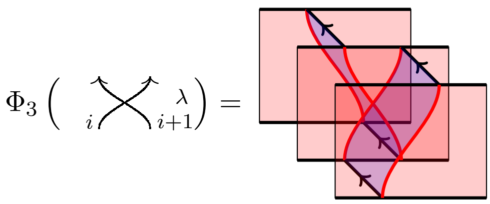
Using this definition, we can also define the 2-functor, with extra rescalings.

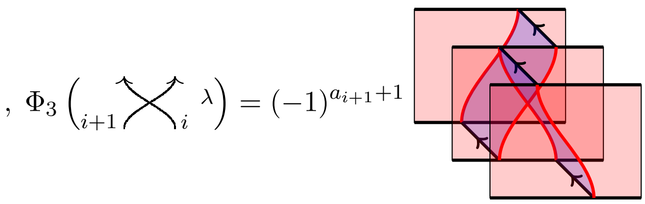
Proposition 3.3.8. *For each $N > 0$ there is a 2-representation*

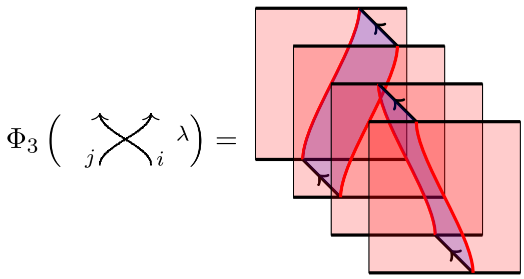


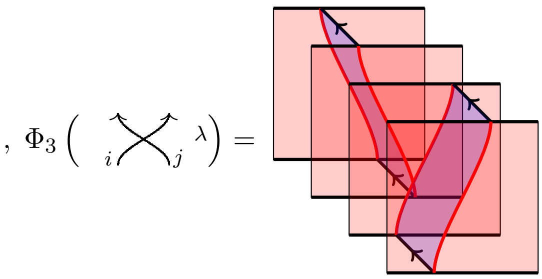
on crossings by:



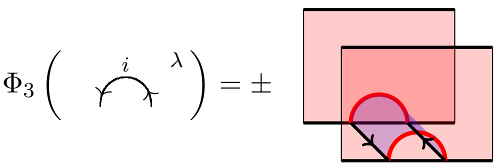


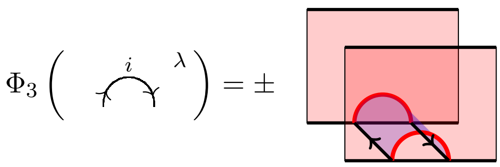


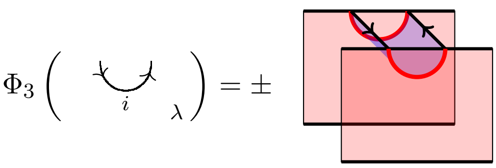


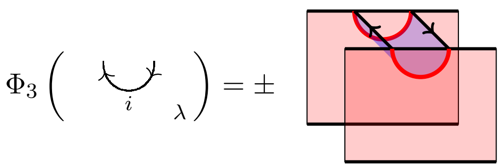


where $j - i > 1$, and on caps and cups by:









where the $\pm$ signs above depend on (the image of) the weight λ and are given by:

Counterclockwise cap	*Sign*	*Clockwise cap*	*Sign*
$N_i = 3,\ \lambda_i = 1$	-	$N_i = 3,\ \lambda_i = -1$	-
$N_i = 2,\ \lambda_i = 0$	+	$N_i = 2,\ \lambda_i = 0$	-
$N_i = 4,\ \lambda_i = 0$	-	$N_i = 4,\ \lambda_i = 0$	-
$N_i = 1,\ \lambda_i = -1$	+	$N_i = 1,\ \lambda_i = 1$	+
$N_i = 3,\ \lambda_i = -1$	+	$N_i = 3,\ \lambda_i = 1$	-
$N_i = 5,\ \lambda_i = -1$	-	$N_i = 5,\ \lambda_i = 1$	-
$N_i = 2,\ \lambda_i = -2$	+	$N_i = 2,\ \lambda_i = 2$	+
$N_i = 4,\ \lambda_i = -2$	+	$N_i = 4,\ \lambda_i = 2$	-
$N_i = 3,\ \lambda_i = -3$	+	$N_i = 3,\ \lambda_i = 3$	+
Counterclockwise cup	*Sign*	*Clockwise cup*	*Sign*
$N_i = 3,\ \lambda_i = 1$	+	$N_i = 3,\ \lambda_i = -1$	+
$N_i = 2,\ \lambda_i = 0$	+	$N_i = 2,\ \lambda_i = 0$	+
$N_i = 4,\ \lambda_i = 0$	-	$N_i = 4,\ \lambda_i = 0$	+
$N_i = 1,\ \lambda_i = -1$	+	$N_i = 1,\ \lambda_i = 1$	+
$N_i = 3,\ \lambda_i = -1$	-	$N_i = 3,\ \lambda_i = 1$	+
$N_i = 5,\ \lambda_i = -1$	-	$N_i = 5,\ \lambda_i = 1$	-
$N_i = 2,\ \lambda_i = -2$	-	$N_i = 2,\ \lambda_i = 2$	+
$N_i = 4,\ \lambda_i = -2$	-	$N_i = 4,\ \lambda_i = 2$	-
$N_i = 3,\ \lambda_i = -3$	-	$N_i = 3,\ \lambda_i = 3$	-

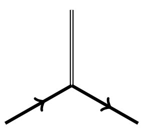
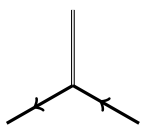
where $N_i = a_i + a_{i+1}$.

Again, the depiction is generic: 0 and 3 labeled facets are to be erased, 1 and 2-labeled ones are identical, and anything else is zero.

There is a way to better understand the sign, by introducing a Blanchet's version of $\mathfrak{sl}_3$-foams, as follows.

Definition 3.3.9. $3\mathbf{BFoam}_m(N)$ *is the 2-category defined as follows:*

- *Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2,3\}$ and $N = \sum_{i=1}^{m} a_i$ together with a zero object.*
- *1-morphisms are formal direct sums of $\mathbb{Z}$-graded enhanced $\mathfrak{sl}_3$ webs - trivalent planar graphs with boundary with edges of two types: directed edges*  *and undirected "3-labeled" edges* *where vertices involving only the directed edges are as in* $3\mathbf{Foam}$ *and vertices involving the 3-labeled edges take the form*

 *or*  .

Oriented edges are directed out from points labeled by 1 and into points labeled by 2 in the domain and vice-versa in the codomain and 3-labeled edges are attached to points labeled by 3 in both the domain and codomain. As in $3\mathbf{Foam}_m(N)$, no edges are attached to points labeled by 0.

– *2-morphisms are $\mathfrak{sl}_3$ foams between such webs where we allow additional 3-labeled facets incident upon the 3-labeled strands of the webs and attached to the remainder of the foam along singular seams which are allowed to intersect the traditional singular seams; the 3-labeled facets are not allowed to carry dots. We impose local relations on these foams.*

In this category, the generic definition of the 2-functor as it appears in Proposition 3.3.3 remains valid, which suggests that this category is the most natural one from the quantum group perspective, and opens the way for generalizations.

Many known constructions in knot theory can be recovered from the 2-functor. First, we can recover $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ versions of Khovanov homology.

We can define for any tangle τ, using the foamation functor and the braiding, a complex $[\![\tau]\!]_2$ in $2\mathbf{BFoam}_{r+2l}(N)$, for l sufficiently large.

Proposition 3.4.2. *The complex $[\![\tau]\!]_2$ assigned to a tangle diagram τ, viewed in the homotopy category of complexes of $2\mathbf{BFoam}_m(N)$, gives an invariant of framed tangles.*

Given a link L, we can renormalize it into an invariant $[\![L]\!]_2^r$, which is a complex of webs mapping between the sequence $(\underline{0},\underline{2}) := (0,\dots,0,2,\dots,2)$ and itself. Applying the functor

$$\mathrm{HOM}(\mathrm{id}_{(\underline{0},\underline{2})}, -) := \bigoplus_{t\in\mathbb{Z}} \mathrm{Hom}(q^{-t}\mathrm{id}_{(\underline{0},\underline{2})}, -)$$

to this complex (where $\mathrm{id}_{(\underline{0},\underline{2})}$ is the identity web), and setting some parameters to zero (triply-dotted bubbles) gives a complex of finite-dimensional graded vector spaces, which we denote $Kh_2(L)$. As the notation indicates, we have the following result.

Proposition 3.4.3. *The (co)homology of the complex $Kh_2(L)$ is the Khovanov homology of the link L.*

The $\mathfrak{sl}_3$-case works exactly the same.

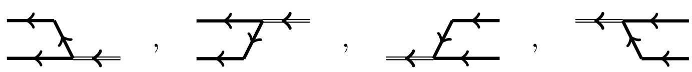
Proposition 3.4.4. *We can assign to a tangle τ a complex $[\![\tau]\!]_3$, which induces an invariant of framed tangles in the homotopy category of complexes of $3\mathbf{Foam}_m(N)$.*

Renormalizing this invariant gives an invariant independent of framing which is (essentially) the same as Morrison-Nieh's [MN08] extension of Khovanov's $\mathfrak{sl}_3$ link homology [Kho04] to tangles, after setting the 3-, 4-, and 5-dotted spheres equal to zero.

Cautis [Cau12] defined categorified Jones-Wenzl projectors (*clasps*) with a process that applies to any categorification of the skew-Howe duality phenomenon. We can thus apply our 2-functor to obtain complexes $\mathsf{P}^+_{\mathsf{m}}$, and we have the following result, which should be viewed as the Blanchet foam analog of [CK12b, Theorem 3.2].

Proposition 3.4.5. *The complex $\mathsf{P}^+_{\mathsf{m}}$ satisfies the following properties:*

1. *$\mathsf{P}^+_{\mathsf{m}}$ is supported only in positive homological degree.*
2. *The identity web $\mathrm{id}_{(1,\dots,1)}$ appears only in homological degree zero.*
3. *$\mathsf{P}^+_{\mathsf{m}}$ annihilates the webs*



in $2\mathbf{BFoam}_m(N)$, up to homotopy.

This shows that $\mathsf{P}_{\mathsf{m}}^{+}$ categorifies the analog of the Jones-Wenzl projector p_m in the category of Blanchet webs.

In the $\mathfrak{sl}_3$ case, there also exists a version of $\mathsf{P}_{\mathsf{s}}^{+}$, and we have:

Proposition 3.4.6. *The complex $\mathsf{P}_{\mathsf{s}}^{+}$ is the categorified clasp $\widetilde{P}_{\mathsf{s}}$ constructed in [Ros11].*

These tools allow to extend the invariants to non-minuscules representations.

Finally, we show that, just as Cautis, Kamnitzer and Morrison [CKM12] were able to obtain a full presentation of the spider category as image of the $U_q(\mathfrak{sl}_m)$ relations, we can recover the whole structure of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foam categories from the $\mathcal{U}_q(\mathfrak{sl}_m)$ relations. This result strongly suggests interesting extensions in the general $\mathfrak{sl}_n$ case, where the structure of the foam categories is not yet completely understood.

Skein modules from skew Howe duality and affine extensions

Skew-Howe duality involves two commuting actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$ on the exterior power of the tensor product of their vector representations. $U_q(\mathfrak{sl}_n)$ encodes the fact that we look at $\mathfrak{sl}_n$ invariants, and we would like to keep this unchanged. But $U_q(\mathfrak{sl}_m)$ could be changed. We intend here to show that we can replace it by its affine extension in order to obtain a representation-theory-flavored presentation of the annular skein module.

Before turning to the affine case, we begin by studying more carefully the possible rescalings of the quantum Weyl group action as it appears in [Lus93], since these renormalizations play an important role in the affine proofs.

Lusztig defines two versions of it, which depend on a parameter $e = \pm 1$. To s_i the elementary transposition corresponding to the root α_i of $U_q(\mathfrak{sl}_m)$, is associated the map $T''_{i,e} \in \widetilde{U_q(\mathfrak{sl}_m)}$:

$$T''_{i,e}\mathbf{1}_\lambda := \sum_{a-b+c=-\lambda_i} (-1)^b q^{e(-ac+b)} E_i^{(a)} F_i^{(b)} E_i^{(c)} \mathbf{1}_\lambda.$$

Similarly, there is a map $T'_{i,e}$:

$$T'_{i,e}\mathbf{1}_\lambda := \sum_{a-b+c=\lambda_i} (-1)^b q^{e(-ac+b)} F_i^{(a)} E_i^{(b)} F_i^{(c)} \mathbf{1}_\lambda.$$

We define a mixed version of these braidings, which uses the fact that N being fixed, any $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \dots, \lambda_{m-1})$ corresponds to (at most) one sequence $(a_1, \dots, a_m)$ so that $\lambda_i = a_{i+1} - a_i$ and $\sum a_i = N$ (this hides the fact we actually have a $\mathfrak{gl}_m$ action).

$$(4.1.5) \qquad \begin{aligned} T_{i,e}\mathbf{1}_\lambda &= (-1)^{-a_{i+1}} q^{-ea_{i+1}} T''_{i,e}\mathbf{1}_\lambda = (-1)^{-a_i} q^{-ea_i} T'_{i,e}\mathbf{1}_\lambda, \\ T_{i,e}^{-1}\mathbf{1}_\lambda &= (-1)^{a_i} q^{ea_i} {T''_{i,e}}^{-1}\mathbf{1}_\lambda = (-1)^{a_{i+1}} q^{ea_{i+1}} {T'_{i,e}}^{-1}\mathbf{1}_\lambda. \end{aligned}$$

This definition turns out to allow much more freedom than in the usual case, since we can deduce a skein module from it, while preserving most of the interesting relations proved in [Lus93].

Proposition 4.1.5. *For all n, smoothings defined by Relations (4.1.5) define a skein module for knotted $\mathfrak{sl}_n$ webs, that is, the application defined by smoothing rules on diagrams of knotted webs is independent of the choice of a diagram for a given web-tangle.*

The major use of $U_q(\mathfrak{sl}_m)$ relations for proving skein relations suggests that the same tool could be very useful at the categorified level as well, once all categorified relations we need will be proved.

We now consider our first goal of extending the skew-Howe duality process to the affine case. We will usually work with the algebra $U'_q(\widehat{\mathfrak{sl}_m})$ generated by $K_i^{\pm 1}$, E_i and F_i for $i \in \{0, \dots, m-1\}$. The elements E_i, F_i and $K_i^{\pm 1}$ are subject to $\mathfrak{sl}_m$ relations, understood modulo m, so that the quantum Serre relations hold for pairs (E_0, E_1), (F_0, F_1), (E_0, E_{m-1})

and (F_0, F_{m-1}):

$$\begin{array}{rcl} K_iK_i^{-1} = K_i^{-1}K_i = 1 & , & K_iK_j = K_jK_i, \\ K_iE_jK_i^{-1} = q^{a_{ij}}E_j & , & K_iF_jK_i^{-1} = q^{-a_{ij}}F_j, \\ E_iF_j - F_jE_i & = & \delta_{ij}\dfrac{K_i - K_i^{-1}}{q-q^{-1}}, \\ E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 & = & 0 \text{ if } j = i \pm 1, \\ F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 & = & 0 \text{ if } j = i \pm 1, \\ E_iE_j = E_jE_i & , & F_iF_j = F_jF_i \text{ if } |i-j| > 1. \end{array}$$

We will also use an idempotented version of $U'_q(\widehat{\mathfrak{sl}_m})$, that we denote $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$, generated by $\mathbf{1}_\lambda$, $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$ with the obvious generalization of the relations of the $\mathfrak{sl}_m$ case. Weights here are m-tuples $\lambda = (\lambda_0, \dots, \lambda_{m-1})$, which in our case, with N fixed, will be related to the sequences $(a_1, \dots, a_m)$ by $\lambda_i = a_{i+1} - a_i$ for $i \neq 0$ and $\lambda_0 = a_1 - a_m$.

In order to extend our $U_q(\mathfrak{sl}_m)$ representation into a $U'_q(\widehat{\mathfrak{sl}_m})$ one, we define, for a complex number, $\rho_a \colon \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \mapsto \dot{\mathbf{U}}'_q(\mathfrak{sl}_m)$ by:

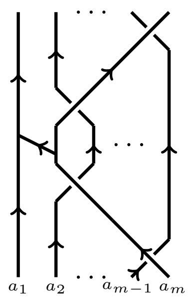
$$\rho_a(E_0\mathbf{1}_\lambda) = aq^{-e(a_1+a_m)}C_{T''_{m-1,e}} \cdots C_{T''_{2,e}}(F_1)\mathbf{1}_\lambda$$

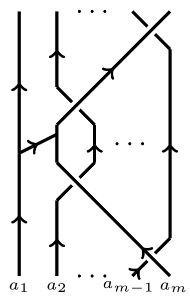
and

$$\rho_a(F_0\mathbf{1}_\lambda) = a^{-1}q^{e(a_1+a_m)}C_{T''_{m-1,e}} \cdots C_{T''_{2,e}}(E_1)\mathbf{1}_\lambda.$$

ρ_a is extended to other generators E_i $(i \neq 0)$ by sending $E_i \in U_q(\widehat{\mathfrak{sl}_m})$ to $E_i \in U_q(\mathfrak{sl}_m)$, and similarly for F_i.

In terms of graphical calculus, this corresponds to:

$$E_0\mathbf{1}_\lambda \quad \mapsto \quad aq^{-e(a_1+a_m)}$$ 

$$F_0\mathbf{1}_\lambda \quad \mapsto \quad a^{-1}q^{e(a_1+a_m)}$$  .

We can then prove that this indeed defines a representation.

Proposition 4.2.1. *Letting $E_0\mathbf{1}_\lambda$ and $F_0\mathbf{1}_\lambda$, elements of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$, act as defined in Relation 4.2.2 (that is, using ρ_a), extends the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$ to $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$.*

For the purpose of skein modules, it is useful to study a version using $T_{i,e}$. We denote $\widetilde{\rho}_a$, for a complex number, the analogue of ρ_a:

$$\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) = aq^{-e(a_1+a_m)}C_{T_{m-1,e}}\cdots C_{T_{2,e}}(F_1)\mathbf{1}_\lambda$$

and

$$\widetilde{\rho}_a(F_0\mathbf{1}_\lambda) = a^{-1}q^{e(a_1+a_m)}C_{T_{m-1,e}}\cdots C_{T_{2,e}}(E_1)\mathbf{1}_\lambda.$$

ρ_a and $\widetilde{\rho}_a$ are defined similarly for other generators.

The proof of Proposition 4.2.1 remains valid if we replace the braidings $T''_{i,e}$'s by the ($\mathfrak{gl}_m$) rescaled $T_{i,e}$'s.

Proposition 4.2.2. *Using the map* $\widetilde{\rho}_a\colon \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \mapsto \dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ *extends the action of* $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ *to an action of* $U'_q(\widehat{\mathfrak{sl}_m})$.

Using the previous proposition combined with the result on the skein module, we obtain the following result.

Proposition 4.2.3. *If* $e=-1$*, we have:*

$$\begin{aligned}\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) &= aq^{a_1+a_m-\frac{m-2}{2}}[[\dots[F_1,F_2]_{q^{\frac{1}{2}}}\dots,F_{m-2}]_{q^{\frac{1}{2}}},F_{m-1}]_{q^{\frac{1}{2}}}\\ &= aq^{a_1+a_m-\frac{m-2}{2}}[F_1,[F_2,\dots,[F_{m-2},F_{m-1}]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}},\end{aligned}$$

and if $e=1$*, we have:*

$$\begin{aligned}\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) &= (-1)^{m-2}aq^{-a_1-a_m+\frac{m-2}{2}}[F_{m-1},[F_{m-2},\dots,[F_2,F_1]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}}\\ &= (-1)^{m-2}aq^{-a_1-a_m+\frac{m-2}{2}}[[\dots[F_{m-1},F_{m-2}]_{q^{\frac{1}{2}}}\dots,F_2]_{q^{\frac{1}{2}}},F_1]_{q^{\frac{1}{2}}}.\end{aligned}$$

This ensures that we recover formulas for evaluation representations related to the ones that appear in [CP95].

We then want to understand how taking the affine extension translates in the knot part. It appears that we can draw (ladder) webs on an annulus rather than on a square, and we can thus define a category $n\mathbf{AWeb}_m$ extending the usual ladder web category. This is related to $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$ as shown in the next proposition.

Proposition 4.2.4. *For all* m, n, N, *there is a functor* $\Phi : \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \mapsto n\mathbf{AWeb}_m(N)$ *defined on weights as above and on morphisms by* $E_i\mathbf{1}_\lambda \mapsto \mathsf{E}_i\mathbf{1}_\lambda$ *and* $F_i\mathbf{1}_\lambda \mapsto \mathsf{F}_i\mathbf{1}_\lambda$.

The evaluation representations then naturally produce invariants for these webs.

Proposition 4.2.5. *The evaluation representation produce annular web invariants for ladder-type webs. In other words, if* $X \in \widetilde{\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})}$ *is such that* $\Phi(X)=w$*,* w *a web-tangle, then the morphism of* $U_q(\mathfrak{sl}_n)$ *representations given by* X *is an invariant of the web-tangle.*

However, these invariants are not very rich, as explained in the following proposition.

Proposition 4.2.6. *The evaluation representation with* $a=-q^{e(n+1)}$ *recovers the* $\mathfrak{sl}_n$ *skein module of the filled cylinder. In other words, for each fixed value of* N *we have the following commutative diagram:*

$$\begin{tikzcd} \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \arrow[r, "\Phi"] \arrow[d, "ev"] & n\mathbf{AWeb} \arrow[d, "Filling"] \\ \dot{\mathbf{U}}_q(\mathfrak{sl}_m) \arrow[r, "\Phi"] & n\mathbf{Web} \end{tikzcd}$$

We can obtain richer invariants by taking the affinization of the evaluation representations, but they are still not as rich as the skein module.

Another analysis can drive us to extend only the relation between webs and $U_q(\mathfrak{sl}_m)$ rather than the whole skew-Howe process. We then obtain an algebraic description of $n\mathbf{AWeb}_m$ as follows.

Proposition 4.2.8. *For λ a dominant $\mathfrak{sl}_m$ weight, the map*

$$\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n/I_\lambda \mapsto n\mathbf{AWeb}_m(a_1^\lambda, \cdots, a_m^\lambda)$$

is an isomorphism. The pre-image of a knotted ladder in $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n/I_\lambda$ is therefore an invariant of the knot.

However, the algebras that we consider here are strongly related to q-Schur algebras, and this suggests to look closer to usual representations of those, as presented in [DG07], [MT13] and [Ugl00].

We adapt from [MT13] tools that could help us achieving such work, and we give the basic ideas of a possible process of proof.

We conclude with remarks on extensions to the categorified setting.

Part I

Skein module and categorification

Chapter 1

Frohman-Gelca formula and extensions

Note: This chapter is the result of a joint work with **Heather M. Russell**, from the *University of Southern California*. The text presented here is the latest version of a joint paper still in progress.

Introduction

The reinterpretation of the Jones polynomial in terms of Kauffman's bracket consists of a diagrammatic process assigning a Laurent polynomial in the variable A to a diagram of a knot. This process is built on two easy local rules, that say how to replace each crossing by a formal sum of two different smoothings, and allow to replace each closed circle by the elementary Laurent polynomial $-A - A^{-1}$. At the end, one thus obtains a formal element written as a polynomial multiplied by the empty diagram. The polynomial part recovers Jones' invariant.

Since the process is entirely built over local rules applied to a diagram, one could as well perform it in situations where the knot diagram is not anymore drawn on a disk but on any (orientable) surface. In terms of knots, this means that we are looking at knots embedded in the thickening of the base surface. The diagram of such a knot is then given by the projection onto the surface. Because of the richer structure of the surface, all diagrams cannot be reduced to the empty one, and we therefore define the skein module to be the natural place where the process ends. There are even extensions to the case of any 3-dimensional manifolds [Prz91], but the surface case is particularly interesting since it can be endowed very naturally with a structure of an algebra, called the *Jones-Kauffman algebra*.

Although this algebra seems at first very easy, as we can deduce from the homotopical data of the surface a nice basis of the module, it appears that exhibiting a presentation of the algebra by generators and relations is a rather difficult task. Several results have been achieved by works of Bullock, Frohman, Gelca, Kania-Bartoszynska, Przytycki... But most presentations don't help very much in finding a nice basis of the module that behaves well under the product: the decomposition in this basis of the product between two basis elements cannot be computed with an easy formula.

Frohman and Gelca [FG00] investigated the torus case, introducing a new basis of the skein module where the result of the product of two basis elements is a sum of only two other basis elements. This rather miraculous formula relies on an extensive use of

Chebyshev's polynomial of the first kind. In the present work, we give a new proof of this formula, focusing on enlightening the role of the Chebyshev's polynomial as change of basis. This is achieved by using an *oriented* skein module, previously introduced by Sallenave [Sal00], which clearly splits the proof between a technical part which does not involve Chebyshev polynomials, and a change of basis argument where they are the key ingredient.

The notion of an oriented skein module turns out to be not only a useful tool in this proof but also an invariant by itself for a particular class of knots, called oriented boundary-parallel band knots. The same module can also be extended to other surfaces at the expense of adding at least one puncture, and provides therefore an invariant for this class of knots over orientable punctured surfaces. It also seems to be a natural place to look for extensions of Frohman-Gelca formula, and we investigate in this work the relations between the usual skein module and the oriented one.

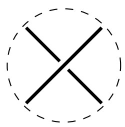
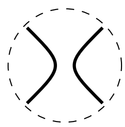
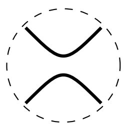
Acknowledgements: We wish to thank Charlie Frohman for his inspiring suggestions, and Francis Bonahon for interesting discussions.

1.1 Jones-Kauffman algebra and the non-commutative torus

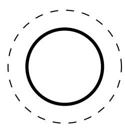
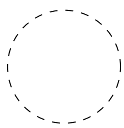
1.1.1 Jones-Kauffman algebra

1.1.1.1 Definitions

Jones [Jon85] introduced in 1984 a new polynomial invariant for knots, whose origins come from Von Neumann algebras. Kauffman [Kau90] developed an easy combinatorial process for computing this polynomial, which we now refer to as the *Kauffman bracket*. It is a map that assigns to a knot a Laurent polynomial in the variable A, and that one can define recursively. Starting from a particular diagram of a knot, that is, a regular projection of a knot in $\mathbb{R}^3$ onto $\mathbb{R}^2$ with up-down information at each singular point (thus forming crossings), the following rule tells us how to deal with crossings.

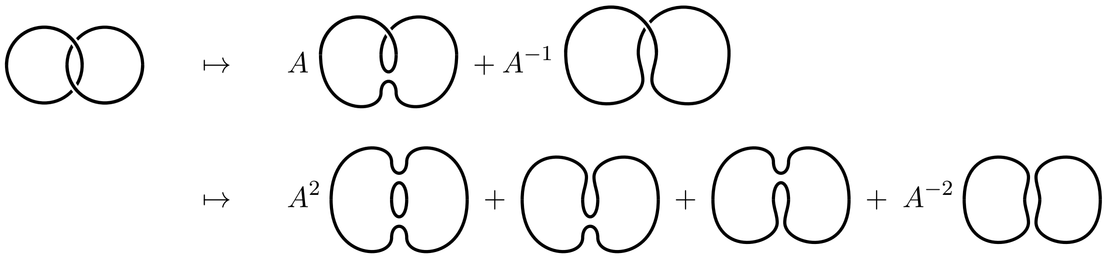
 $= A$  $+ A^{-1}$  (1.1.1)

Smoothing this way all crossings, one obtains a formal sum with coefficients in the ring $R = \mathbb{Z}[A, A^{-1}]$ of unions of closed curves in the plane, that is circles. The following normalization rule then allows to evaluate all of these circles.

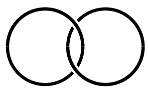
 $= (-A^2 - A^{-2})$ 

Therefore, starting from a diagram D drawn on the plane, one defines the Kauffman bracket of D as the Laurent polynomial which remains at the end, once one has deleted all curves (this appears as the coefficient of the empty curve). This polynomial (with extra rescalings) turns out not to depend on the choice of a particular diagram for a given knot, and so provides a knot invariant, which is a combinatorial version of the Jones polynomial. Below is drawn an example of a computation for the Hopf link.

We start by smoothing all crossings.



And we can then evaluate the circles.



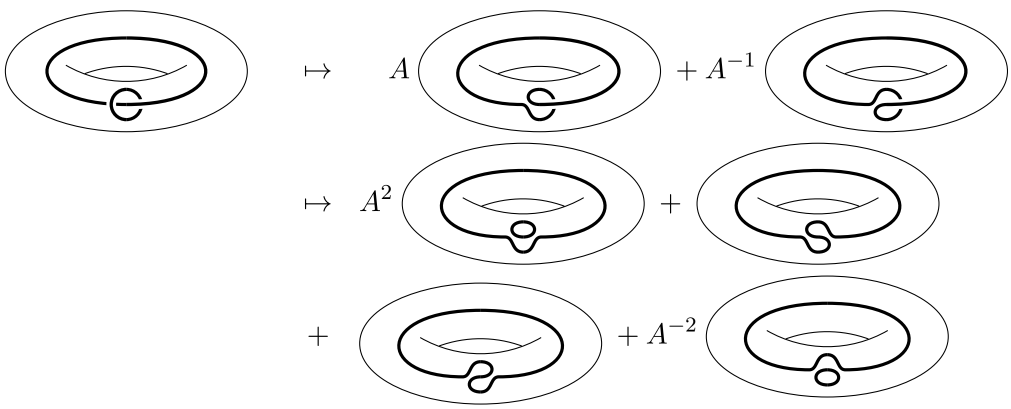
$$\mapsto A^2(-A^2 - A^{-2})^2 + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2$$
$$= A^6 + A^2 + A^{-2} + A^{-6}.$$

Instead of looking at knots in $\mathbb{R}^3$ that project onto $\mathbb{R}^2$, one could as well look at knot over other surfaces. We have an easy generalization of the notion of knots and links in the surface (we will always assume surfaces to be orientable here), as follows.

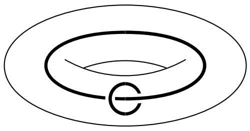
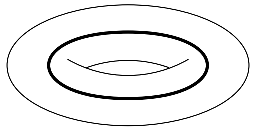
Definition 1.1.1. Let S be a surface. A knot in the thickening of S is the embedding of a circle $\mathbb{S}^1$ in $S \times \mathbb{I}$ (where $\mathbb{I} = [0, 1]$) considered up to global isotopy of $S \times \mathbb{I}$. Similarly, a link in $S \times \mathbb{I}$ is the embedding of a disjoint union $\sqcup_k \mathbb{S}^1$ in $S \times \mathbb{I}$.

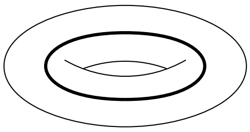
Just as the usual case, where we can see our knots in $\mathbb{R}^2 \times \mathbb{I}$ with projection onto $\mathbb{R}^2$, the projection $S \times \mathbb{I} \mapsto S$ draws (in generic cases) a diagram on S, and all diagrams of a given knot are related by Reidemeister moves and isotopies of S.

The Kauffman bracket could be extending to our new setting: we have diagrams with crossings, to which we can apply the two local rules defining Kauffman's process. Below is drawn an example. We first smooth all crossings.



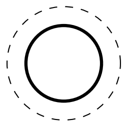
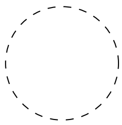
And we then evaluate all possible null-homotopic circles.

 $\mapsto \quad (A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2}((-A^2 - A^{-2}))$ 

$= (-A^4 - A^{-4})$ 

As we see, the situation in the surface case is more complicated than in the plane case: we cannot always replace all curves by polynomials, since there is no way to evaluate homotopically non-trivial curves. This observation is the key idea that led to the definition of the *skein module* (see for instance [Prz91]).

Definition 1.1.2. Let S be a surface. Then the *skein module* of S is the R-module generated by unoriented closed embedded curves on S up to isotopy subject to the following local relation.

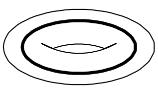
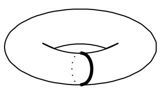
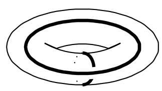
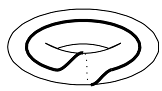
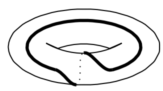
 $= \ (-A^2 - A^{-2})$ 

Note that the previous relation states that circles bounding a disk can be deleted, but does not say anything for other kind of circles.

This module is the natural place where to consider the result of Kauffman's process, and we can then define the Kauffman bracket to be the map that, to a diagram drawn on S, associates the skein element corresponding to smoothing all crossings according to Relation (1.1.1).

1.1.1.2 Product

The smoothing process that defines Kauffman's bracket may also be used for endowing $Sk(S)$ with an algebra structure: the product is linear and defined on curves α and β as the result of the smoothing of all crossings in any non-singular superposition of α over β.

 $*$  $=$  $= A$  $+A^{-1}$ 

More precisely, if $\alpha\colon \sqcup_k \mathbb{S}^1 \mapsto S \times \mathbb{I}$ and $\beta\colon \sqcup_l \mathbb{S}^1 \mapsto S \times \mathbb{I}$, we can first lift β into $\widetilde{\beta}$ as the composition of β and the isomorphism $S \times \mathbb{I} \mapsto S \times [1,2]$ sending $(x,t) \mapsto (x, t+1)$. Then we can consider $\alpha * \beta$ to be the smoothing result of the following surface link:

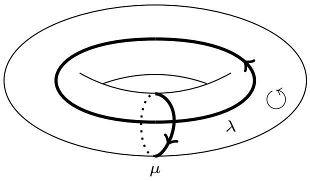
$$\sqcup_{k+l} \mathbb{S}^1 \mapsto S \times [0,2] \mapsto S \times [0,1],$$

where the first map is $x \mapsto \alpha(x)$ if x is in one of the first k copies of $\mathbb{S}^1$, $x \mapsto \widetilde{\beta}(x)$ otherwise, and the second one is $(x,t) \mapsto (x, \frac{t}{2})$.

Note that the skein module of the plane can also be considered with the same product. However, when performing a multiplication, the top element can be pushed away so that it does not lie over the bottom one, and so the product is just the disjoint union of diagrams. Similarly, if one multiplies two copies of the same curve (on an orientable surface), we can realize it as a disjoint union.

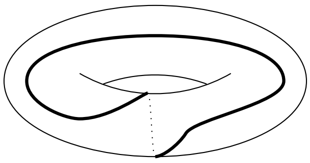
1.1.1.3 Conventions for toric curves

In the first part of this paper, we will focus on the torus case. We consider $\mathbb{S}^1 \times \mathbb{S}^1$ with its longitude λ and its meridian μ, both oriented, and an orientation on $\mathbb{S}^1 \times \mathbb{S}^1$, so that on a neighborhood of the intersection between the longitude and the meridian, their germs in this order are a negative basis of the tangent space.



Connected embedded curve on $\mathbb{S}^1 \times \mathbb{S}^1$ are classified up to isotopy by pairs (p, q) with $GCD(p, q) = 1$ and $(p, q) \in \mathbb{N}^* \times \mathbb{Z} \cup \{0\} \times \mathbb{N}^*$. A pair (p, q) corresponds to the only connected embedded curve in $\mathbb{S}^1 \times \mathbb{S}^1$ whose homology is the same as $p\lambda + q\mu$ (for an adequate choice of orientations).

Below is drawn a curve of type $(1, -1)$.



If we denote $c_{(np,nq)}$ the multicurve formed from n parallel copies of the (p, q) simple toric curve (sometimes we will simply write (np, nq)), we have:

Remark 1.1.3. $\{c_{(np,nq)} \mid GCD(p, q) = 1,\ (p, q) \in \mathbb{Z}^{+*} \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}^{+*},\ n \in \mathbb{Z}^{+*}\}$ and the empty curve form a basis of the skein module of the torus.

1.1.2 Non-commutative torus

We now introduce, following [FG00] (see also [BG02, Chapter I.2] for more details), an algebraic object that will turn out to be related to the skein module of the torus. Let ${}_A\mathbb{T} =< L^{\pm 1}, M^{\pm 1} \mid ML = A^{-2}LM >_R$ be the *quantum torus*. Let Θ be the map of algebra defined by: $\Theta(L) = L^{-1}$, $\Theta(M) = M^{-1}$ and $\Theta(A) = A$. Denote ${}_A\mathbb{T}^\Theta$ the eigenspace of Θ for the eigenvalue 1.

Following [FG00], define $e_{m,n} = A^{-mn}L^mM^n$ for $(m, n) \in \mathbb{Z}^2$. This defines a basis of the R-module ${}_A\mathbb{T}$. Then, for $(m, n) \in \mathbb{N}^* \times \mathbb{Z} \cup \{0\} \times \mathbb{N}$, denote $c_{(m,n)} = e_{(m,n)} + e_{(-m,-n)}$. Note that $\Theta(e_{(m,n)}) = e_{(-m,-n)}$, so that $c_{(m,n)} \in {}_A\mathbb{T}^\Theta$. Actually, these elements define a basis of the R-module ${}_A\mathbb{T}^\Theta$.

We have the next multiplication formulas:

$$
\begin{aligned}
e_{(m,n)} \cdot e_{(m',n')} &= A^{\begin{vmatrix} m & m' \\ n & n' \end{vmatrix}} e_{(m+m',n+n')} \\
c_{(m,n)} \cdot c_{(m',n')} &= A^{\begin{vmatrix} m & m' \\ n & n' \end{vmatrix}} c_{(m+m',n+n')} + A^{-\begin{vmatrix} m & m' \\ n & n' \end{vmatrix}} c_{(m-m',n-n')}
\end{aligned}
$$

1.1.3 Isomorphism

Define the *Chebyshev polynomials of the first kind* by: $T_0 = 2$, $T_1 = X$, $T_n = X \cdot T_{n-1} - T_{n-2}$. These polynomials only differ from the more usual *Chebyshev polynomials of the second kind* by the fact that the first term T_0 is 2 instead of 1.

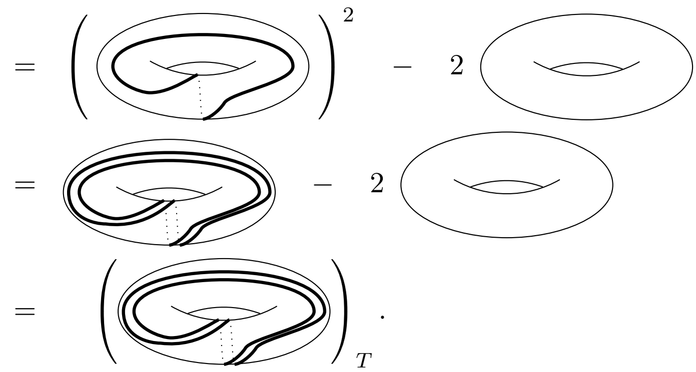
Let (p, q) be relatively prime integers representing a toric curve, that is a basis element of the skein module of the torus. Since $Sk(\mathbb{S}^1 \times \mathbb{S}^1)$ is an algebra, we can evaluate the polynomials T_n on any elements in it. In other words, for all polynomial $P(X) = \sum_k a_k X^k$ and all element $y \in Sk(\mathbb{S}^1 \times \mathbb{S}^1)$, we can consider the element $P(y) = \sum_k a_k y^k$, the product being induced by superposition and smoothing of generating curves. Note that the constant term evaluates on the skein element given by the empty curve.

Define $(np, nq)_T = T_n((p, q)) \in Sk(\mathbb{S}^1 \times \mathbb{S}^1)$, and more generally:

$$(a, b)_T = T_{GCD(a,b)}\left(\left(\frac{a}{GCD(a, b)}, \frac{b}{GCD(a, b)}\right)\right) \in Sk(\mathbb{S}^1 \times \mathbb{S}^1).$$

For example,

$$(2, -2)_T \quad = \quad T_2((1, -1)) = (1, -1)^2 - 2$$



Then, Frohman and Gelca prove [FG00]:

Theorem 1.1.4. $(np, nq)_T \to c_{(np,nq)}$ induces an isomorphism of algebras between $Sk(S)$ and ${}_A\mathbb{T}^\Theta$.

A central piece of this theorem is the product-to-sum formula:

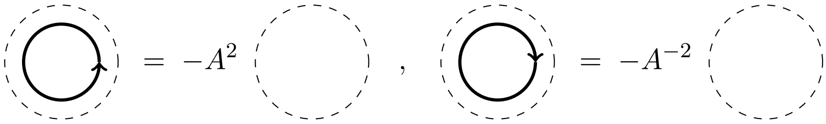
$$(a, b)_T * (c, d)_T = A^{-\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(a - c, b - d)_T + A^{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(a + c, b + d). \tag{1.1.2}$$

Indeed, it is easy to establish isomorphisms between the two modules, but the above formula is the key tool that helps understanding the product on the Jones-Kauffman algebra, and therefore check that the chosen isomorphism respects the algebra structure.

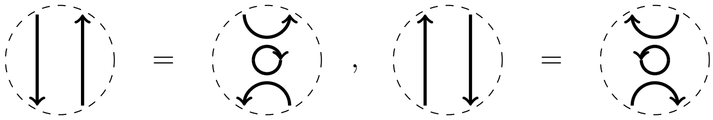
The first purpose of the present paper consists on reproving Formula 1.1.2 and Theorem 1.1.4. Indeed, the original proof of Formula 1.1.2 [FG00] is based on a recurrence, which makes the appearance of the Chebyshev polynomials somewhat mysterious. We therefor intend here to focus on enlightening the particular role of these polynomials. This will be made possible by introducing an intermediate algebra $\mathcal{A}$, which we define and study in Section 1.1.4, before we re-prove in Section 1.1.5 Frohman-Gelca's Formula 1.1.2.

1.1.4 Diagrammatic presentation of ${}_A\mathbb{T}$

We introduce a new module $\mathcal{A}$, defined to be the R-module generated by oriented embedded curves on $\mathbb{S}^1 \times \mathbb{S}^1$ up to isotopy, and subjects to the relations below.

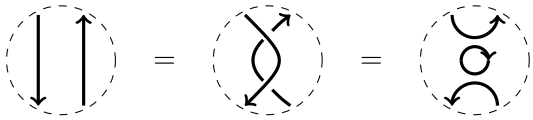


(1.1.3)



(1.1.4)

Note that the last relation corresponds to an oriented smoothing of a Reidemeister II move:



This relation will therefore ensure that the second Reidemeister move holds, in particular in Section 1.1.6, but it also has the interest of helping reduce parallel non-trivial curves on the torus with opposite orientations: reading the relation from left to right, we can pair such opposite curves and obtain two homotopically trivial circles, which we delete using Relations (1.1.3).

Any element in $\mathcal{A}$ is a sum of Laurent polynomials in the variable A times oriented curves. Take such an oriented curve. Then, by the two first relations, we can delete all trivial components, with as only effect a change of the coefficient. We are left with n copies of some (p, q) curve with possibly different orientations. By the last relation, we can pair adjacent copies with opposite orientations, producing two trivial circles, that we delete. So, by this process, any curve may be reduced to k copies of parallel and similarly oriented non-trivial curves, which gives us a system of generators.

Remark 1.1.5. The topology of the torus plays a role here, as we shall see in Section 3: for the torus, non-trivial curves can only be parallel copies of the same connected non-trivial curve.

Denoting $\gamma_{(np,nq)}$ the element given by drawing n times the oriented curve (p, q), $\{\gamma_{(np,nq)}\}$ is a generating set for the R-module $\mathcal{A}$.

Proposition 1.1.6. $\{\gamma_{(np,nq)}\} \cup \{\emptyset\}$ is a basis of $\mathcal{A}$.

Proof. Since no generator has a trivial component, the only possible relation between generators is the one shown in Figure 1.1. Since generators have parallel and identically oriented curves, it follows that $a = b$ and $c = d$. In other words, the two oppositely oriented edges must come from the same curve.

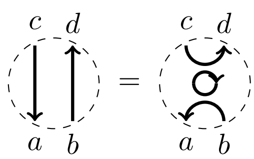
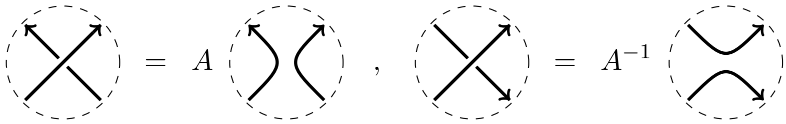


Figure 1.1: A relation on a generating element

After performing this move, one curve has split into three: the obviously nulhomologous one pictured and two others at least one of which must be nontrivial. If both are nontrivial, they must be parallel since they are disjoint. Thus they cobound a cylinder on the torus, and that original curve would have to have been trivial. Therefore, performing this relation on a generator can only produce a copy of the original generator with two oppositely oriented hence canceling trivial curves. ☐

Corollary 1.1.7. $\mathcal{A}$ is a torsion-free module.

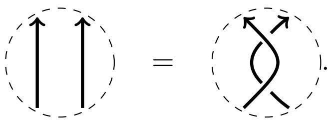
We define on $\mathcal{A}$ a multiplication by superposition of the curves and the smoothing process depicted below.



(1.1.5)

Proposition 1.1.8. The multiplication process is well defined.

Proof. Isotopies of curves before superposition correspond after superposition to Reidemeister 2 moves. The translation of a Reidemeister 2 move through the multiplication process is either Relation (1.1.4) or an identity:



☐

Proposition 1.1.9. $\mathcal{A}$ is isomorphic to ${}_A\mathbb{T}$.

Proof. Associate to $e_{(np,nq)}$ the generator $\gamma_{(np,nq)}$. We should check that this module isomorphism is as well an algebra isomorphism. For this, take two elements $\gamma_{(a,b)}$ and $\gamma_{(c,d)}$. Up to an homeomorphism and isotopies, we can assume that $\gamma_{(a,b)}$ is $\gamma_{(m,0)}$ and $\gamma_{(c,d)}$ is $\gamma_{(np,nq)}$ presented as in Figure 1.2.

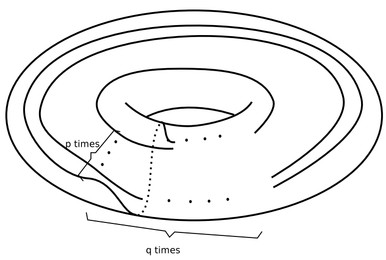


Figure 1.2: (p,q) curve

Then, performing the multiplication gives us mnp crossings, and the smoothing is $A^{mnq}\gamma_{(m+np,nq)}$, and mnq is the determinant of $(m,0)$ and (np,nq), which remains unchanged under homeomorphisms.

So both basis follow the same multiplication rules.

☐

1.1.5 Retrieving Frohman-Gelca formula

Denote Θ the R-module map on $\mathcal{A}$ that reverses the orientation of the generators. On a general curve, this corresponds to reversing the orientation of the non-trivial components. Denote $\mathcal{A}^\Theta$ the stable part of $\mathcal{A}$ under Θ.

Remark 1.1.10. $\Theta(\gamma_{(np,nq)}) = \gamma_{(-np,-nq)}$. We can then check that Θ is a map of algebras, and thus that $\mathcal{A}^\Theta$ is a sub-algebra of $\mathcal{A}$.

Consider the map $\psi : Sk(\mathbb{S}^1 \times \mathbb{S}^1) \to \mathcal{A}^\Theta$ that to a simple multicurve γ associates the sum of all possible orientations of this curve. We can prove that this very natural morphism furnishes us an isomorphism respecting the algebra structure, as stated below.

Theorem 1.1.11. $Sk(\mathbb{S}^1 \times \mathbb{S}^1) \simeq \mathcal{A}^\Theta$ as algebras.

Proof. ψ gives a well-defined map of modules, and this is an isomorphism: the image of the usual base of $Sk(S)$ is a base of $\mathcal{A}$. It remains to check that this gives a map of algebras.

Considering γ_1 and γ_2 two curves that we can assume to be essential and connected, we want to compute $\psi(\gamma_1 * \gamma_2) - \psi(\gamma_1)\psi(\gamma_2)$. Each of the crossings of $\gamma_1 * \gamma_2$ produces two terms, giving rise to eight local possible oriented curves: see Figure 1.3. The total number of possible orientations of the smoothings may be less than 8 times the number of crossings, since all possible local orientations for each smoothing do not give rise to a consistent orientation of the whole resulting curve. We will say that some terms don't make sense globally, but they will cancel pairwise anyway (such a case appears where there is more than one crossing between two connected components).

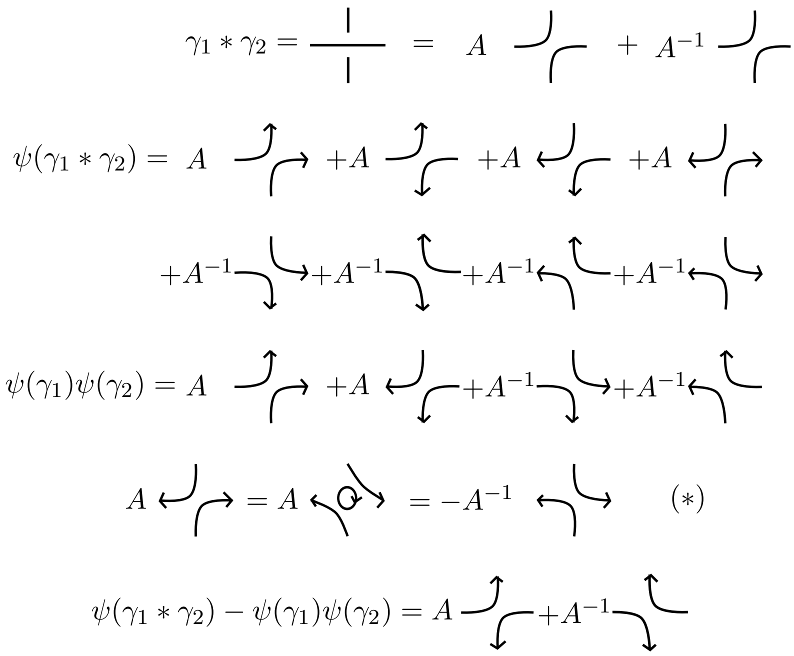
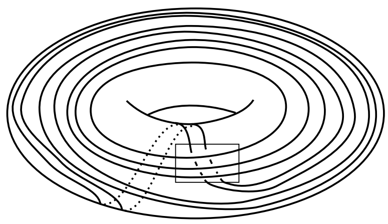


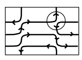
Figure 1.3: $\psi(\gamma_1 * \gamma_2) - \psi(\gamma_1)\psi(\gamma_2)$

So, starting from one (globally oriented) factor of $\psi(\gamma_1 * \gamma_2)$, the local orientations of each smoothed crossing comes at most from one orientation of $\gamma_1 * \gamma_2$: we say we reform all possible crossings with agreeing orientation. Some of the smoothings have a local orientation that doesn't come from a smoothing of an oriented crossing between γ_1 and γ_2. We then say that this crossing is *non-reformable*. One of the possibility for such a case if when the smoothed crossing is locally as Equality ($*$) of Figure 1.3. A sub-factor containing such a local orientation can be paired up it with the other corresponding factor of Equality ($*$), and they cancel.

Furthermore, observe that isotopies on each of the two terms induce R_{II} moves after multiplication, and that R_{II} moves are compatible with ψ. We can so assume that both curves we multiply are in generic position.

We now turn to the last possible non-reformable local orientation. Assume that a local smoothing is oriented as in the last line of Figure 1.3, for example . Then, if we keep going down, we will meet another non-reformable crossing (otherwise, orientations on the top and the bottom don't match although this would be the same strand). This non-reformable crossing is subject to the equality ($*$): we can cancel this element by changing this other crossing. See for example Figure 1.4 for an illustration. $\square$



gives for example A^2 

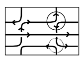
"coming from" A^2 

Figure 1.4: $\psi(\gamma_1 * \gamma_2) - \psi(\gamma_1)\psi(\gamma_2)$

This gives us two isomorphisms: ${}_A\mathbb{T}^\Theta \simeq \mathcal{A}^\Theta \simeq Sk(S)$.

Let us look at natural basis in each of the modules:

– The natural basis in $Sk(T)$ is given by multicurves $c_{(np,nq)}$. This basis behaves terribly under multiplication!
– The natural basis in $\mathcal{A}^\Theta$ is given by terms of type $(\gamma_{np,nq} + \gamma_{-np,-nq})$. We easily see that we have:

$$\begin{aligned}(\gamma_{a,b} + \gamma_{-a,-b}) * (\gamma_{c,d} + \gamma_{-c,-d}) \quad &= \quad A^{-\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(\gamma_{a-c,b-d} + \gamma_{-a+c,-b+d}) \\ &\quad + A^{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(\gamma_{a+c,b+d} + \gamma_{-a-c,-b-d}). \quad (1.1.6)\end{aligned}$$

This basis therefore behaves very well under multiplication.

Now, the image of $c_{(np,nq)}$ in $\mathcal{A}$ is $(\gamma_{np,nq} + \gamma_{-np,-nq})^n$. We just need to conclude the following interesting property of the Chebyshev polynomials:

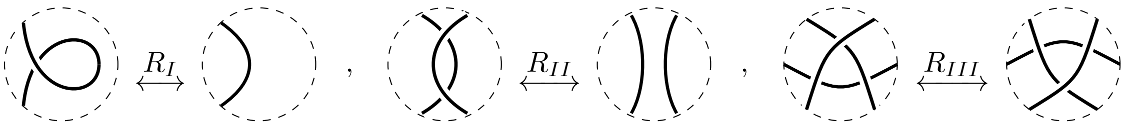
Remark 1.1.12. $T_n(x + x^{-1}) = x^n + x^{-n}$.

This tells us that $\gamma_{np,nq} + \gamma_{-np,-nq} = T_n(\gamma_{p,q} + \gamma_{-p,-q})$, and we can pull Formula (1.1.6) back in $Sk(T)$, which recovers Frohman and Gelca's results from [FG00]. In this proof, the Chebyshev polynomials give the way to pass from one basis to another, while the technicality lies only in the isomorphism between $Sk(T)$ and $\mathcal{A}^{\Theta}$, that doesn't use Chebyshev polynomials.

1.1.6 $\mathcal{A}$ as skein module?

The module $\mathcal{A}$ has been introduced as a diagrammatic analogue of the quantum torus ${}_A\mathbb{T}$, and seems to be closely related to the usual skein module. A natural question then is to know whether the image of a knot in this module defines an invariant.

Recall that two diagrams represent the same knot if and only if there are related by a sequence of planar isotopies and elementary Reidemeister moves, which are depicted below:



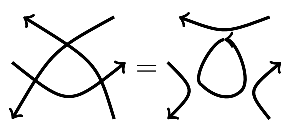
Since all objects considered in the module $\mathcal{A}$ are oriented, we consider oriented knots in the thickening of $\mathbb{S}^1 \times \mathbb{S}^1$, or equivalently oriented diagrams on $\mathbb{S}^1 \times \mathbb{S}^1$. We can apply to it the smoothing process 1.1.5, and we obtain from the knot an element of $\mathcal{A}$, which we want to see as invariant under the choice of the diagram of the knot. We thus have to check invariance under oriented versions of the Reidemeister moves depicted above. Invariance under Reidemeister II move is obvious from the definition.

Lemma 1.1.13. The class of the smoothing is invariant under Reidemeister III move.

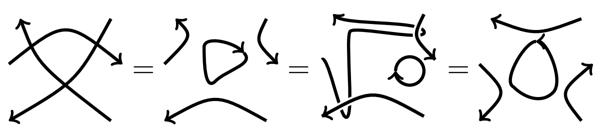
Corollary 1.1.14. The product in $\mathcal{A}$ is associative.

Proof. **(of the lemma)** Since writhe is preserved by Reidmeister III, crossing information does not matter. Hence we only need to consider the various ways of orienting the strands. Up to symmetry, there are two of them.

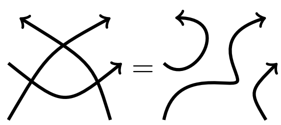
In the first case, the first position smooths as:



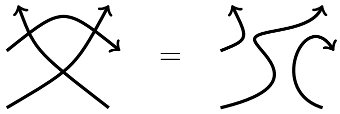
and the second one gives:



from which we deduce the equality. The second case is given by:

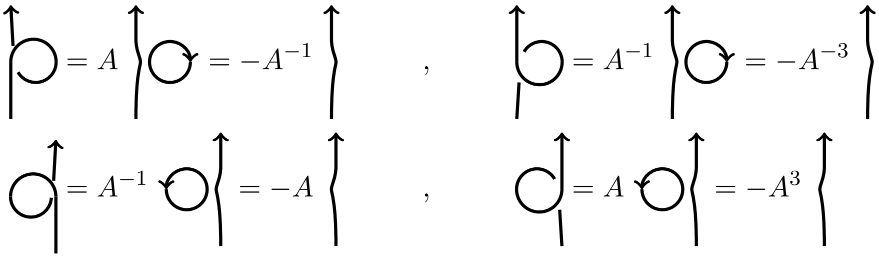


and



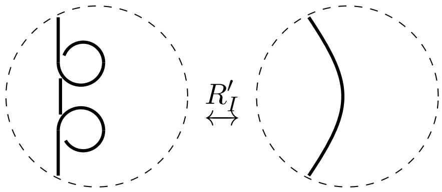
whose smoothings are isotopic. □

For R_I, the situation becomes wilder: see Relations 1.1.7 below.



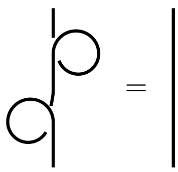
(1.1.7)

We have no invariance under R_I, but the framed version of R_I (see Relation 1.1.8 below) fails as well. Recall that, if one looks at framed knots represented by usual diagrams (and blackboard framing), two diagrams represent the same knot if and only if they are related by a sequence of planar isotopies, Reidemeister moves R_{II}, R_{III}, and the following framed Reidemeister I move (which replaces the move R_I):



(1.1.8)

It seems that the only easy relation we can find that preserve the skein element is depicted in Relation 1.1.9, but this is actually implied by Reidemeister II and Reidemeister III moves.



(1.1.9)

Remark 1.1.15. This module with A specialized to a fourth root of the unity provides an invariant of knots.

Question 1.1.16. Is there any enriched class of knots for which diagram equivalence would be described by Reidemeister II, III and weakened versions of Reidemeister I we could deduce from Relations 1.1.7?

An answer to this question is given by considering blackboard-framed band knots with adequate equivalence relations.

1.1.7 Boundary parallel band knot

Definition 1.1.17. Consider $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ foliated by $\mathbb{S}^1 \times \mathbb{S}^1 \times \{i\}$ for $i \in \mathbb{I}$. A *boundary parallel band knot* in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ is a smoothly embedded annulus with normal vector everywhere transverse to the foliation. A *boundary parallel band link* is analogously defined.

See Figure 1.5 for an example.

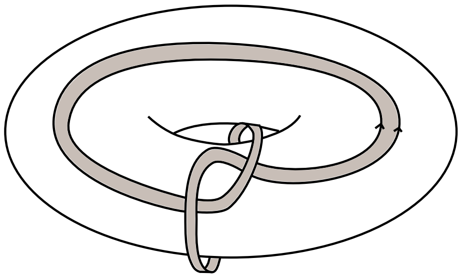


Figure 1.5: Band knot

Definition 1.1.18. Two boundary parallel band links are equivalent if they are related by a continuous deformation through boundary parallel band links.

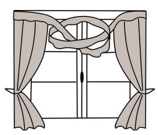
Lemma 1.1.19. [1] R'_I move on knots diagrams is not an equivalence of boundary-parallel band links, while R_{II} and R_{III} are.

Proof. The second part of the statement is obvious.

For the first part, we begin by defining a tool that will distinguish both sides of the move R'_I. Let γ be a boundary-parallel band knot, and suppose we look at a part of it by restricting to a thickened disk. Fix an orthonormal coordinate system $(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ such that the normal vectors of the band are transverse to the space generated by $(\overrightarrow{i}, \overrightarrow{j})$ (take $\overrightarrow{k}$ to be the $\mathbb{I}$ direction). The intersection of the curve γ with the thickened disk can be assumed to have a smooth core, given by s a smooth function from $\mathbb{I}$ to the thickened disk. Then, $t \to \frac{(Ts(t)\cdot\overrightarrow{i}, Ts(t)\cdot\overrightarrow{j})}{||(Ts(t)\cdot\overrightarrow{i}, Ts(t)\cdot\overrightarrow{j})||}$ gives a function $\mathbb{I} \to \mathbb{S}^1$, where $Ts(t)$ is the tangent vector to the core of γ at $s(t)$ and the dot denotes the scalar product (this is called the *Gauss map*[2]). Closing trivially the strand, we get a continuous function $\mathbb{S}^1 \to \mathbb{S}^1$. For usual band knots, when performing isotopies, the tangent vector lies in the plane orthogonal to the "band direction" – that may move. For blackboard-framed knots, this direction is kept to be transverse to $< \overrightarrow{i}, \overrightarrow{j} >$ at all time, and thus, isotopies of blackboard-framed links induce homotopies on functions of $\mathbb{S}^1$.

We conclude by noticing that the functions associated to the two sides of R'_I move are not in the same homotopy class. $\square$

1. Here is the idea behind this lemma. A boundary parallel band knot is equivalent to looking at a knot as a (knotted) curtain rod on which a curtain hangs. The natural equivalence relation among knotted curtains is the one that lets at all time the curtain hanging, with respect to the gravity. Of course, there is no way to untie a rod with a Reidemeister I move without twisting the curtain around the rod. But there is also no way to untie a rod with a framed Reidemeister I' move without letting the curtain turn around the rod at some point...



2. Thanks to Stéphane Baseilhac for pointing this out.

Remark 1.1.20. Note that the Gauss map above has been defined above for closed curves on a disk. More generally, this definition extends to closed curves drawn on the torus, since one can choose two orthonormal tangent vector fields over the torus. If $T(\mathbb{S}^1 \times \mathbb{S}^1)$ denotes the tangent space to the torus, and $\mathbb{S}^1 \times \mathbb{S}^1 \mapsto T(\mathbb{S}^1 \times \mathbb{S}^1)\colon x \mapsto (\overrightarrow{i}(x), \overrightarrow{j}(x))$ is such a choice, and if c is a curve (the projection of the core of a boundary-parallel band knot) given by a smooth function $s\colon \mathbb{S}^1 \mapsto \mathbb{S}^1 \times \mathbb{S}^1$, its Gauss map is given by:

$$\mathbb{S}^1 \mapsto \mathbb{S}^1 \colon t \mapsto \frac{(Ts(t) \cdot \overrightarrow{i}(s(t)), Ts(t) \cdot \overrightarrow{j}(s(t)))}{||(Ts(t) \cdot \overrightarrow{i}(s(t)), Ts(t) \cdot \overrightarrow{j}(s(t)))||}.$$

Not that the same construction cannot be achieved for general surfaces, since there does not exist non-vanishing tangent vector fields: this will be one of the reasons for considering punctured surfaces later.

Lemma 1.1.21. Equivalence relations on diagrams of boundary-parallel band links are generated by isotopies, R_{II} and R_{III}.

Proof. An equivalence of boundary-parallel band link diagrams is in particular an equivalence of link diagrams, so may be described by isotopies and Reidemeister moves. R_{II} and R_{III} moves come from isotopies, and locally preserve the Gauss functions of the strands they involved. Since an R_I move does not preserve it, any R_I move must be performed with at least another one. Since R_I moves are local and concern a single strand, they can be transported along this strand, at the cost of R_{II} and R_{III} moves for each crossing. So, we can reduce to a local strand on which we perform some R_I moves. A framing argument and the same argument as previously lead us to identify this as a succession of weakened R_I moves of Relation 1.1.9, which are actually equivalent to a sequence of R_{II} and R_{III} moves. [There must be an equal number of left and right moves, and an equal number of positive and negative crossings. These moves commute (up to other Reidemeister moves), and so may be paired up.] □

Putting together all pieces, we can now state:

Proposition 1.1.22. The map that assigns to a boundary-parallel band knot the class of its smoothing in $\mathcal{A}$ is an invariant of boundary-parallel band knots.

The idea used locally in the previous lemmas may be extended to the whole torus: fix a tangent field of norm 1 vectors in $\mathbb{S}^1 \times \mathbb{S}^1$ (this exists for the torus), denoted x. The $\mathbb{I}$-direction gives us a normal field z. Then, take y to be the vector product of z and x. So, at all time, we can perform the scalar product of $Ts(t)$ and x, and $Ts(t)$ and y, and to each strand of a link is associated an element of the mapping class group of $\mathbb{S}^1$.

Remark 1.1.23. The Gauss map gives us an information on the framing that is different from the framing number: each of the four R_I moves is uniquely determined by its framing and its Gauss map.

For surfaces other than the torus, such vector fields don't exist. This, together with the problems explained in section 1.3, will lead us to punctured surfaces, that admit such vector fields.

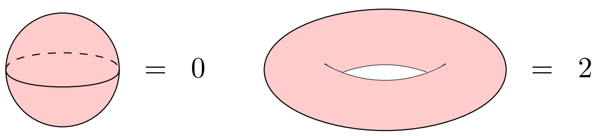
1.2 Categorification

The skein module of a surface extends the planar notion of Kauffman bracket, which gives to the Jones polynomial a diagrammatic description (up to a rescaling). The Kauffman bracket has been categorified by Khovanov [Kho00], who defined a process for assigning to a diagram a complex whose homology is an invariant of the knot, and whose (graded) Euler characteristic is the Kauffman bracket of the knot. This algebraically-defined homology admits a geometric re-interpretation [BN05], which produces complexes in a category of cobordisms. The primary tool for relating both processes is given by *Topological Quantum Field Theories* (TQFT), which are functors from a category of cobordisms to a category of modules over a ring. These geometric theories can be extended very easily to the case of surfaces, which is not true for the algebraic process... In particular, finding interesting TQFT for surfaces turns out to be a difficult task [APS04, TT06, CS09, Kai09].

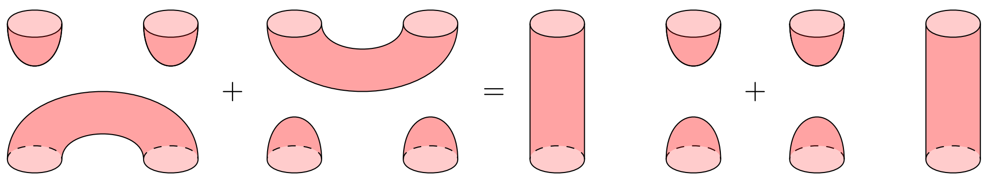
We quickly review in this section the solution provided by Asaeda, Przytycki and Sikora, and relate it to the module $\mathcal{A}$ we introduced in the previous section.

Let us denote $Cob(\mathbb{S}^1 \times \mathbb{S}^1)$ Bar-Natan's category of cobordisms, whose:
– objects are (formal $\mathbb{Z}$-graded sums of) unoriented curves drawn on $\mathbb{S}^1 \times \mathbb{S}^1$.
– morphisms between two curves γ_1 and γ_2 on $\mathbb{S}^1 \times \mathbb{S}^1$ are cobordisms between them, that is, embedded surfaces Σ in $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ so that $\partial\Sigma = (\partial\Sigma) \cap (\mathbb{S}^1 \times \mathbb{S}^1 \times \{0\}) \cup (\partial\Sigma) \cap (\mathbb{S}^1 \times \mathbb{S}^1 \times \{1\}) = \gamma_1 \cup \gamma_2$, subject to relations. Morphisms between formal sums are given by matrices of cobordisms.

The relations on cobordisms are:



(1.2.1)



(1.2.2)

This category also has a grading: the degree of a cobordism F is $2\chi(F)$ where χ denotes the Euler characteristic. Objects in the category can be shifted in degree, which will be denoted with $\{\cdot\}$.

Asaeda, Przytycki and Sikora presented in [APS04] a TQFT that can be applied to Bar-Natan's formal complex [BN05] in the case of knots in thickened surfaces.

On the level of objects, there are two types of curves to deal with. To a trivial circle is assigned the usual Khovanov's algebra $\mathbb{A} =< 1, X > /(X^2)$ [Kho00], 1 having degree -2 and X degree 2. To a non-trivial connected curve is assigned the rank-2 module formally generated by the two possible orientations of the curve. A curve γ has two possible orientations, and we denote the generators γ^+ for the preferred orientation, and γ^- for the other one. The map sends disjoint union to tensor product, which allows to define it for any object. Note that we can also make the part concerning trivial curves fit into the definition in terms of orientation, by saying that a non-essential circle with orientation induced by the one on the torus is sent to 1 and it is sent to X if it comes with the other orientation. For any curve, we can thus assign the $\mathbb{Z}$-module generated by all possible orientations of this curve.

The usual Khovanov's TQFT [Kho00, BN05] involves circles on a plane, which are therefore non-essential, and cobordisms between them, generated by birth, death, a pair of pant and its mirror image. The definition of the TQFT is as follows:

– the birth of a circle is sent to the unity of the Frobenius algebra: $\eta\colon \mathbb{Z} \mapsto \mathbb{A}$, $1 \mapsto 1$.
– the death of a circle is sent to the counity: $\varepsilon\colon \mathbb{A} \mapsto \mathbb{Z}$, $1 \mapsto 0$, $X \mapsto 1$.
– a pair of pants relating two circles to a single one is sent to the multiplication of $\mathbb{A}$.
– a pair of pants in the other direction is sent to the comultiplication: $\Delta\colon \mathbb{A} \mapsto \mathbb{A} \otimes \mathbb{A}$, $1 \mapsto 1 \otimes X + X \otimes 1$, $X \mapsto X \otimes X$.

The TQFT defined by Asaeda, Przytycki and Sikora uses the same maps as long as only non-essential circles are involved. For other generating cobordisms, a Frobenius module-comodule structure is defined on $< \gamma^+, \gamma^- >$ as follows:

$$\begin{array}{rcl} 1 \cdot \gamma^{\pm} = \gamma^{\pm} & , & X \cdot \gamma^{\pm} = 0 \\ \gamma^{\pm} & \to & \gamma^{\pm} \otimes X. \end{array}$$

There are also evaluation and coevaluation maps:

$$\begin{array}{rcl} \gamma^+ \cdot \gamma^- = \gamma^- \cdot \gamma^+ = X & , & 0 \; otherwise \\ 1 \to \gamma^+ \otimes \gamma^- + \gamma^- \otimes \gamma^+ & , & X \to 0. \end{array}$$

These maps are the images of elementary cobordisms, as follows:

– the module structure on $< \gamma^+, \gamma^- >$ corresponds to merging a trivial circle and a non-trivial one,
– the comodule structure is the mirror process of creating a trivial circle from a non-trivial one.
– coevaluation and evaluation correspond to bounding two parallel non-trivial curves with an annulus.

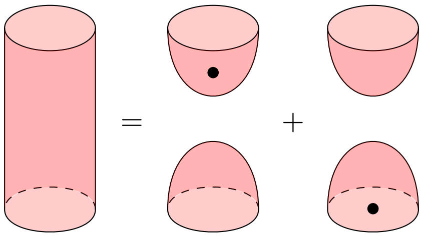
All elementary cobordisms that can appear are not treated by the previous analysis, and the ones left undefined are sent to zero maps. In particular, there is over the torus a saddle that bounds only two curves, which are of different homotopy type. This saddle, which is unorientable will be sent to the zero map (see Chapter 2 for more details on this saddle). Note that in the case of the torus, there exists no way to have a saddle with 3 non-trivial boundary components (it would be assigned the zero map in the case of higher-genus surfaces).

We now want to investigate the decategorification of APS model, in order to relate it to our module $\mathcal{A}$. There are two ways we could adopt. The first one would be to consider a category of $\mathbb{Z}$-modules with extra structure built from elementary pieces used in the definition of APS version of extended Khovanov homology. We rather choose the second one, consisting, in the spirit of Bar-Natan [BN05] and related to works of Boerner[Boe08a, Boe08b], to work in a category of cobordisms suitably modified to mimic the algebraic expected behavior.

Let us denote $\mathcal{C}$ the graded category whose objects are $\mathbb{Z}$-graded formal sums of oriented curves, and whose morphisms are the same as in $Cob(\mathbb{S}^1 \times \mathbb{S}^1)$ (we do not care about the orientation).

A process to build this category $\mathcal{C}$ from the $Cob(\mathbb{S}^1 \times \mathbb{S}^1)$ can be given by adding in $Cob(\mathbb{S}^1 \times \mathbb{S}^1)$ two orthogonal idempotents over each essential curve, and define $\mathcal{C}$ to be the karoubian completion of the augmented category.

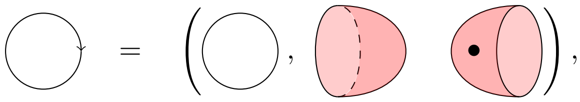
Note that it is not necessary to add idempotents for trivial circles: we already have the following decomposition:



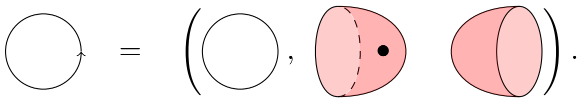
(1.2.3)

A dot in the above formula stands for a handle times $\frac{1}{2}$ (see Chapters 2 and 3 for more details).

We thus associate, for trivial circles,



and



Furthermore, we mod out $\mathcal{C}$ by killing two elementary cobordisms that are assigned zero under APS process: dotted identities over an essential curve, and non-orientable saddle from an essential curve to another one.

This way, APS's TQFT becomes an isomorphism between the cobordism category and a category of free $\mathbb{Z}$-modules.

We can then identify [3]:

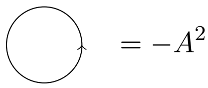
Proposition 1.2.1. *$K^0(\mathcal{C}) \simeq \mathcal{A}$ as $\mathbb{Z}$-modules.*

Proof. Define:

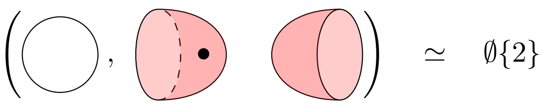
$$\varphi \quad : \quad K^0\mathcal{C} \mapsto \mathcal{A} \quad : \quad \mathcal{A} \quad : \quad [\gamma\{k\}] \mapsto (-1)^{|\gamma|}A^k\gamma$$

where γ is an oriented curve (they generate $\mathcal{C}$) with $|\gamma|$ connected components.

Check first that this map is well-defined. The relation:



derives in $K^0(\mathcal{C})$ from:



via a death cobordism with a dot.

The other circle relation can be dealt with in a similar way.

All Reidemeister-II-like relations are of one of the forms listed in Figure 1.6. In each case, there is an obvious isomorphism.

So φ is well-defined. We can easily see that it is surjective.

3. This identification has been the primary motivation for defining the module $\mathcal{A}$.

Let us now check the injectivity.

A first step is to investigate all elementary morphisms in $\mathcal{C}$: the only remaining elementary morphisms are a death or a birth of a trivial circle and their dotted versions, together with (dotted) identities. Any of the morphisms splits with help of the idempotents, so that the induced relations they give all exist in $\mathcal{A}$.

Then, since the category $\mathcal{C}$ is isomorphic to a category of free $\mathbb{Z}$-modules, any exact sequence splits, and therefore $K^0(\mathcal{C})$ is defined considering only isomorphisms.

□

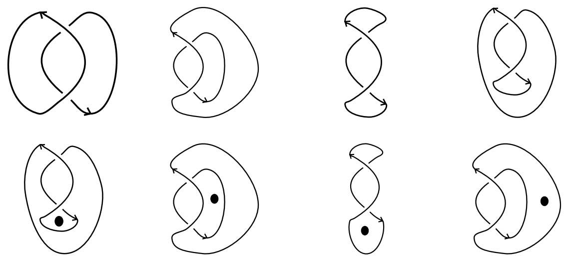


Figure 1.6: Reidemeister move.

1.3 Other surfaces

All defining relations of $\mathcal{A}$ are local, and therefore could be used for defining an extension of $\mathcal{A}$ to any surface. In addition to creating an interesting invariant for boundary parallel band knots, this idea seems also quite natural when looking for extensions of Frohman-Gelca's product-to-sum formula. Indeed, skein modules for general surfaces are known to be quite complicated objects. Although basis or presentations by generators and relations can be found in many interesting cases (see for example [BP00]), these presentations do not lead to basis of the module that behave well under the multiplicative structure.

The guiding line for this part will be to both understand the structure of extensions of $\mathcal{A}$ and looking for generalized Frohman-Gelca's formulas for (weakened) skein modules.

The generalization of $\mathcal{A}$ we will consider in here has already been introduced as a deformation of the first homology group of parallelized surfaces by Sallenave [Sal00]. Related q-deformations of the first homology group of a 3-manifold have also been studied by Przytycki [Prz98].

1.3.1 Definition of the module and multiplication

Unfortunately, it turns out that for surfaces other than the torus, the very same defining process as in the case of the torus would lead to torsion. See Figure 1.7 for examples.

A reason for this problem to appear is that the only orientable surface that admits a non-vanishing tangent vector field is the torus (this argument has been a central piece of the invariance statement in the torus case, since this is the object that allows to compute the Gauss map; see also [Sal00] where it plays a central role). A natural way to fix this problem is to consider, instead of closed surfaces, punctured surfaces. The problems of Figure 1.7 then do not exist anymore: there is only "one way" to delete trivial circles on punctured surfaces.

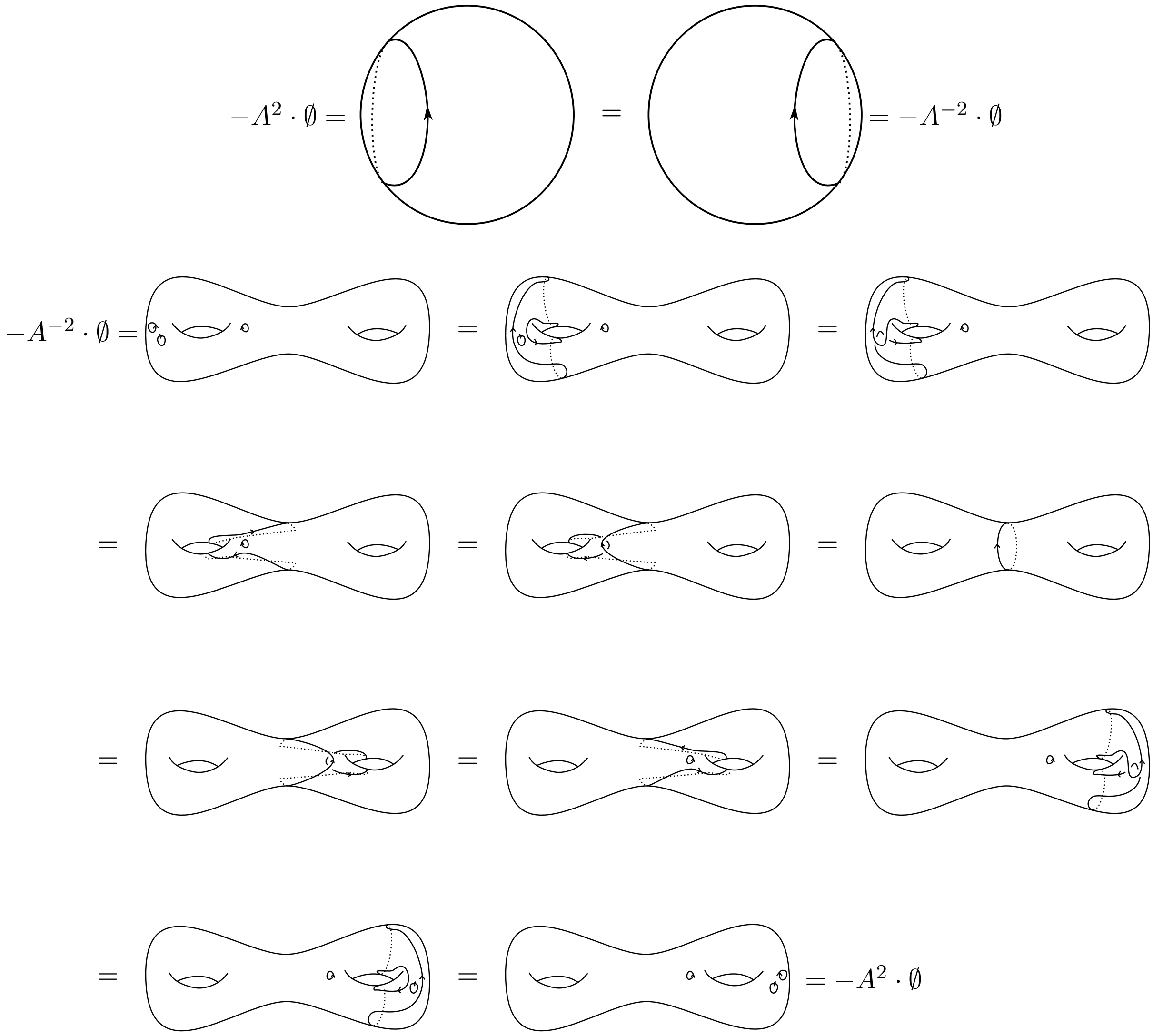


Figure 1.7: Some problems

We will thus work with an oriented surface S with a point p marked on it, and we define the module $\mathcal{A}$ as follows.

Definition 1.3.1. Let $\mathcal{A}$ be generated by oriented curves on $S \setminus \{p\}$ up to isotopy, subjet to Relations (1.1.3) and (1.1.4).

1.3.2 Presentation of the module and multiplicative structure

Remark 1.3.2. In the previous subsection, we defined for general punctured surfaces the algebra $\mathcal{A}$. In the first section, we defined for the non-punctured torus a similar algebra. In fact, we have an isomorphism: ${}_A\mathbb{T} \simeq \mathcal{A}(\mathbb{S}^1 \times \mathbb{S}^1) \simeq \mathcal{A}(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{p\})$, simply by forgetting the puncture.

Let $S \setminus \{p\}$ be a general punctured surface, which can be presented as in Figure 1.8: it is seen as a union of tori with a disk removed, glued on a punctured sphere with the same numbers of disks removed. This gives a morphism: ${}_A\mathbb{T} \otimes_R {}_A\mathbb{T} \cdots \otimes_R {}_A\mathbb{T} \to$

$\mathcal{A}(S \setminus \{p\})$, simply by identifying each torus with one of the tori with a disk removed in the decomposition of $S \setminus \{p\}$. Let us denote π this map. We place a puncture on the tori for understanding, but the place of this puncture is of no influence.

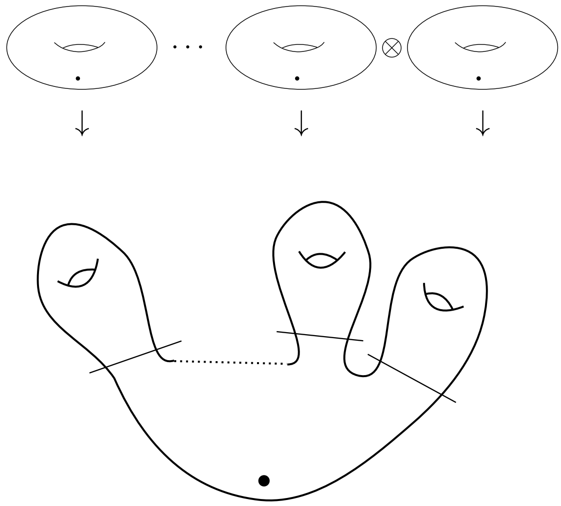


Figure 1.8: Presentation of a surface

Proposition 1.3.3. π is an isomorphism.

Proof. Let us begin with the surjectivity. Let c be a separating circle on $S \setminus \{p\}$. Then, the (algebraic) intersection number between c and any curve γ drawn on $S \setminus \{p\}$ is zero. So, we can pair up curves with opposite orientation, and then, using Relation 1.1.4, separate the classes of γ and c in $\mathcal{A}(S \setminus \{p\})$. This in particular applies for each of the circles gluing the tori with the bottom sphere, which allows to decompose any curve into a union of curves in each of the punctured tori.

For the injectivity: if $\pi(\gamma_1 \otimes \cdots \otimes \gamma_n)$ and $\pi(\gamma_1' \otimes \cdots \otimes \gamma_n')$ are related through a sequence of surfaces, these surfaces, by a similar argument than previously, can be cut along the separating circles, and this relation, up to the action of R, is then transported on the level of each of the factors of ${}_A\mathbb{T} \otimes \cdots \otimes {}_A\mathbb{T}$. $\square$

Remark 1.3.4. Two elements of $\mathcal{A}$ can be, as explained, cut into pieces lying in the handles. Then, the multiplication process is the same one in $\mathcal{A}$ and in ${}_A\mathbb{T} \otimes {}_A\mathbb{T} \otimes \cdots \otimes {}_A\mathbb{T}$, which ensures that π is a morphism of algebras.

Corollary 1.3.5. $\mathcal{A}$ is torsion-free.

Recall that Frohman and Gelca [FG00] introduce a basis of ${}_A\mathbb{T}$ corresponding in the diagrammatic world to parallel copies of the basis curves on the torus. This basis induces a basis of $\mathcal{A}$ as an R-module, through the isomorphism π. The image of a tensor product of basis element in the tensor product ${}_A\mathbb{T} \otimes \cdots {}_A\mathbb{T}$ is in S a disjoint union of parallel copies of elementary curves. We can then easily deduce the following multiplication formula, which mimics the first step in proving Frohman-Gelca formula:

$$\begin{aligned}\pi(e_{(p_1,q_1)} \otimes \cdots \otimes e_{(p_n,q_n)}) \quad \cdot \quad & \pi(e_{(p_1',q_1')} \otimes \cdots \otimes e_{(p_n',q_n')}) = \\ & A^{\sum(det((p_i,q_i),(p_i',q_i'))} \pi(e_{(p_1+p_1',q_1+q_1')} \otimes \cdots \otimes e_{(p_n+p_n',q_n+q_n')}).\end{aligned}$$

The next step, which will be the core of section 1.4, consists in relating the module $\mathcal{A}$ to the original skein module of the punctured surface, and see what product formulas we may deduce from the previous one. Before we turn to that part, let us explain in the next paragraph how the module $\mathcal{A}$ we defined is also an invariant in some sense.

1.3.3 Skein module

As in the first section, since the punctured surface S may be equipped with a tangent vector field with no zeros, we can define a Gauss map and perform the same process that led us to Proposition 1.1.22. $\mathcal{A}$ appears to be the skein module of boundary-parallel band knots with blackboard framing, that is, we can state:

Proposition 1.3.6. The image of a boundary parallel band knot in $\mathcal{A}$ is an invariant of the knot.

1.4 Relations between $\mathcal{A}$ and the Jones-Kauffman algebra

1.4.1 Symmetric part of $\mathcal{A}$ and Jones-Kauffman algebra

Let Sk denote the skein module of the punctured surface $S \setminus \{p\}$, with its Jones-Kauffman algebra structure.

Following Sallenave, define $\varphi : Sk \to \mathcal{A}$ by $\varphi(c) = \sum or(c)$, meaning that we take the sum of all possible orientations of the curve c. This maps extends linearly to the skein module.

Proposition 1.4.1. φ is a map of algebras.

[This is very similar to the proof of Theorem 1.1.11 (the same cancellation argument works as well here). Through different objects, this result was also proven by Sallenave.]

Let us now turn toward extensions of Frohman-Gelca formulas. Multiplication of parallel curves with the same orientation behave very well in $\mathcal{A}$, while it is known that multiplication of Chebyshev polynomials of simple curves create in other cases than the non-punctured torus one more than 2 terms. Looking to places where a product-to-sum formula could remain valid, it is interesting to define a weakened form of the skein module as $Sk_h = Sk/ker(\varphi)$.

The question of finding a basis of that module is most generally quite difficult, but we can investigate what role the Chebyshev polynomials could play here, and in what extend this does provide a natural where to extend Frohman-Gelca's formula.

1.4.2 Particular case: Punctured torus and miraculous cancellations

Let T_p be the punctured torus $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{p\}$. Denote a the generator of the center of the skein module, which consists on a circle around the puncture. Then, $Sk(T_p) \simeq Sk(T)[a]$.

In [BW12a], Bonahon and Wong study the representations of the Jones-Kauffman algebra at roots of unity, and extensively use the fact that, with A being an appropriate root of the unity, Frohman-Gelca's product-to-sum formula may be extended to the product of some Chebyshev polynomials of skein elements. For general A, this behavior doesn't extend, since the product of skein elements will produce polynomials in a that would arrange when specializing $a = -A^2 - A^{-2}$ to reform Frohman-Gelca's formula.

The following proposition is to be understood similarly: we can have the same miraculous cancellation in $Sk(T_p)$ while keeping A to be generic, but at the cost of modding out Sk by the kernel of φ.

Proposition 1.4.2. If one chooses a preferred longitude and meridian of the punctured torus $T \setminus \{p\}$, then the set of elements $\{(np, nq)_T | GCD(p,q) = 1, (p,q) \in \mathbb{Z}^{+*} \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}^{+*}\}$ and the empty curve generate $Sk_h T \setminus p$ and follow Frohman-Gelca's multiplication rule:

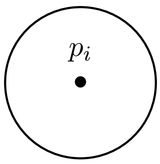
$$(a,b)_T * (c,d)_T = A^{-\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(a-c, b-d)_T + A^{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}(a+c, b+d). \tag{1.4.1}$$

Proof. It is easy to see that $Sk_h(T_p) \simeq Sk(T)$, since $Sk(T) \simeq Sk_h(T) \simeq \mathcal{A}^{\Theta}(T) \simeq \mathcal{A}^{\Theta}(T_p)$. The proposition then is nothing but the usual Frohman-Gelca's formula. $\square$

1.4.3 Augmented module for generalized Chebyshev polynomials for punctures

Note that, so far, we have focused on once-punctured surfaces. However, we can generalize the definition of $\mathcal{A}$ to multiply punctured surfaces. In the case of the torus, this has the interest of keeping a basis close to the one of the torus.

Let S be a surface with punctures $\{p_1, \cdots, p_r\}$. Let us augment Sk by adding the following relations:



$$= \quad z_i^+ + z_i^-.$$

We extend φ to this new module by sending z_i^+ to a counterclockwise oriented circle around the i-th puncture (which we will denote a_i^+), and z_i^- to the same circle oppositely oriented (denoted a_i^-). Note that filling the punctures causes $a_i^+ = -A^2$, $a_i^- = -A^{-2}$, which can be pulled back to the skein module as $z_i^+ = -A^2$ and $z_i^- = -A^{-2}$.

We can then define the augmented version of Sk_h, which we will denote $\widetilde{Sk_h}$, as the quotient of the augmented skein module by the kernel of φ.

Recall that usual Chebyshev polynomials are defined as:

$$T_{n+2}(X) = XT_{n+1}(X) - T_n(X), \quad T_0(X) = 2, \quad T_1(X) = X.$$

Let c be a multicurve that we see as an element of Sk_h, and which can be written as n copies of the same simple connected curve c'. Then define $c_T = T_n(c')$. Frohman-Gelca's formula directly extend as:

Proposition 1.4.3. Let c and d be two multicurves, each of them being made of parallel copies of one connected curve (not necessarily the same one for c and d). Let $|c \cdot d|$ denote the algebraic intersection number of c and d, and let $c + d$ be the multicurve obtained by smoothing all crossings in $c * d$ with the A coefficient. Similarly, $c - d$ is the multicurve obtained by performing all opposite smoothings. Then:

$$c_T * d_T = A^{-|c \cdot d|}(c-d)_T + A^{|c \cdot d|}(c+d)_T.$$

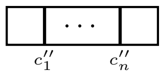
The proof lies on a direct use of the map φ.

The above proposition holds true both in $Sk_h$and in $\widetilde{Sk_h}$. We can slightly extend this formula by considering the puncture, under the condition of working in $\widetilde{Sk_h}$.

Let c be a multicurve which is a disjoint union of curves $c_1, \cdots, c_n$ so that all the curves are parallel in S, but not necessarily in $S \setminus \{p_1, \cdots, p_r\}$. Each of the curves c_i is sent under φ on $c_i^+ + c_i^-$, which can be chosen so that all c_i^+ coincide in $\mathcal{A}(S)$.

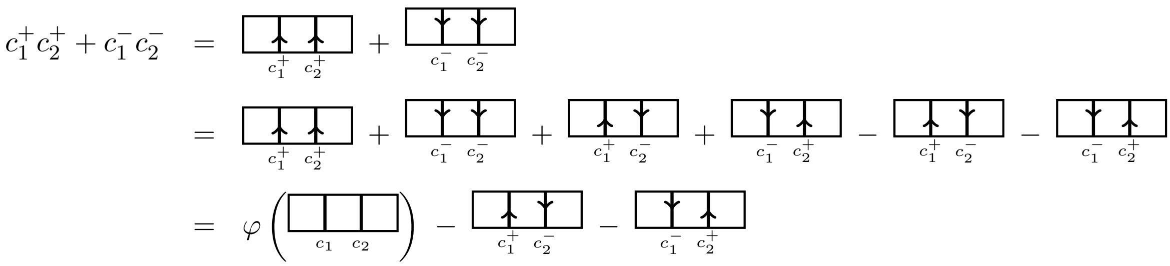
Proposition 1.4.4. $c_T := \varphi^{-1}(c_1^+ \cdots c_n^+ + c_1^- \cdots c_n^-)$ exists.

Proof. $c_1, \ldots, c_n$ primarily live on $S \setminus \{p_1, \ldots, p_r\}$. For clarity, let us denote $c_1', \ldots, c_n'$ their images in S. These curves are all isotopic in S and without intersection. They can be isotoped to n parallel copies of a same curve, which we write as $c_1'', \ldots, c_n''$. Then, we can consider a rectangle R'' on S so that the intersection of this rectangle with $c_1'', \ldots, c_n''$ looks like:

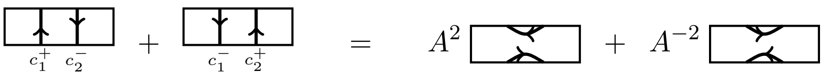


R'' corresponds for the c_i''s to a (deformed) rectangle R', and for the c_i's to another (deformed) rectangle R that can be assumed not to contain any puncture up to an isotopy. We will now work in the area delimited by the rectangle.

If $n = 1$, the result is obvious. If $n = 2$, then

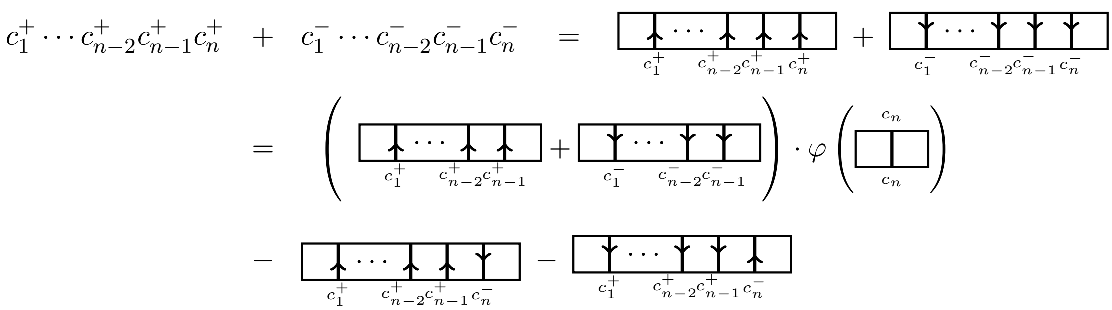


Then we have again:

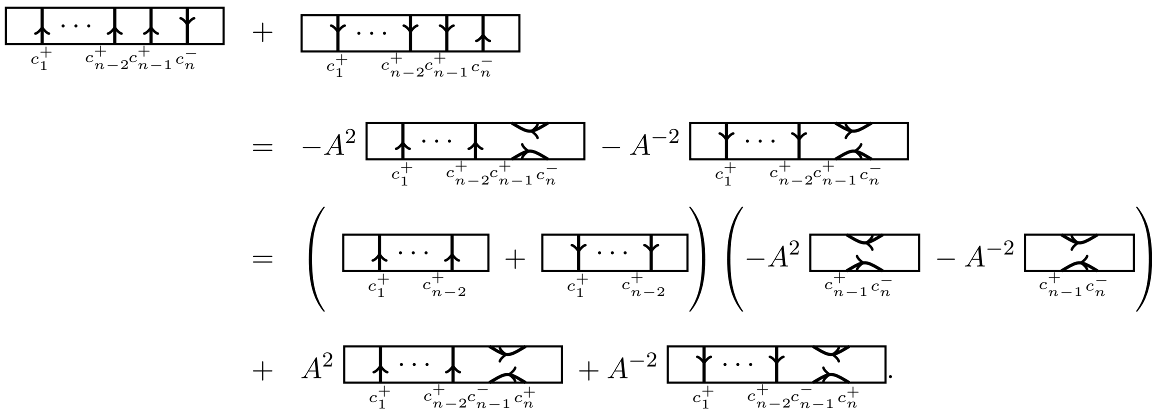


Now, both curves are of homotopy type zero in the unpunctured surface, and therefore can be decomposed as a product of a power of A and elementary circles around the punctures. This element can be pulled back to the augmented algebra.

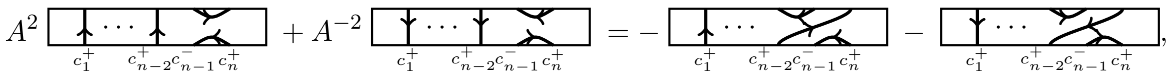
Let us now look at the general case.



The first term is dealt with by iteration. The other term equals:



The first term above is a product of an element in $Im(\varphi)$ by iteration and an element lying in the algebra generated by $A^{\pm 1}$ and $z_i^{\pm 1}$. Finally, the remaining term equals:



which is also covered by the iteration, since the right-most three strands actually form a curve of the same homotopy type than the other $c_i's$.

□

We can deduce from the previous proposition the following multiplication formula.

Theorem 1.4.5. Let c and d be two multicurves of the previous type. Let $|c \cdot d|$ denote the algebraic intersection number of c and d, and let $c + d$ be the multicurve obtained by smoothing all crossings in $c * d$ with the A coefficient. Similarly, $c - d$ is the multicurve obtained by performing all opposite smoothings. Then:

$$c_T * d_T = A^{-|c \cdot d|}(c-d)_T + A^{|c \cdot d|}(c+d)_T.$$

In the case of the punctured torus, curves of the above type generate the module $\widetilde{Sk_h}$, but they do not form a free family. The question of finding a minimal family of generators remains to be investigated.

1.4.4 General surfaces

The complete case of general surfaces remains to be understood. Indeed, we can use Chebyshev's polynomial or the punctured generalization of them to produce particular elements whose multiplication is easy to understand in the relevant quotient of the skein module. However, it is a hard task to completely understand the kernel of the map $Sk \to \mathcal{A}$, and thus to exhibit a basis of Sk_h.

Such questions will provide a guideline for future work.

1.4.5 Remarks on categorification

APS model does not in general decategorify to our module. Indeed, the saddles bounding three non-trivial curves are specified in their model to be zero, while there are some relations in our module. The search for a TQFT which would assign a non-trivial map to these saddles will remain as future work. This is also related to the analysis led in Chapter 2, where we investigate ways to understand TQFT over surfaces.

1.5 A topological interpretation of $\mathcal{A}$

$\mathcal{A}$ can be thought at in a topological way, which is very much related to tools used by Przytycki [Prz98]. The key idea is to look at bounding surfaces and to relate boundary curves using the Euler characteristics of the surface. We present here this reinterpretation as an extension of our work. It would be interesting as future work to see if one can use any of these ideas to prove more about the presentation of the module $\mathcal{A}$.

Definition 1.5.1. Let γ be an oriented multi-curve. F an oriented closed 2-dimensional sub-manifold of $S \setminus \{p\}$ is said to be bounding γ if its boundary is γ with the same orientation. We generalize this definition to union of sub-manifolds: F a union of oriented closed sub-manifolds is said to be bounding an oriented multi-curve γ if any component γ_i appears exactly once in the boundary of the components of F, with adequate orientation. We call this last union of surfaces a *bounding multi-surface*.

Remark 1.5.2. If an oriented multi-curve γ bounds (not multi) surfaces F and G as in the previous definition, then $F = G$. Indeed, γ splits S into a collection of regions one of which contains p. F and G consist of sub-collections of those regions such that (1) each component of γ is part of the boundary of exactly one region and (2) no regions contain p. This choice is unique, so $F = G$.

A bounding multi-surface can be seen as a disjoint union of its components lying over $S \setminus \{p\}$, with the projection inducing some multiple local coverings. Each of the sub-surfaces of a multi-surface can be seen through the projection as a sub-manifold of $S \setminus \{p\}$, and thus we can compare its orientation with the one of $S \setminus \{p\}$.

Definition 1.5.3. For F a multi-surface over $S \setminus \{p\}$, let F^+ be the part whose orientation is induced by the one of S, and F^- the other part ; then, we define its *oriented Euler characteristic* to be $\chi_{or}(F) = \chi(F^+) - \chi(F^-)$, where χ is the usual Euler characteristic.

The next theorem shows that χ_{or} only depends on the boundary.

Theorem 1.5.4. Let γ be a multi-curve on $S \setminus \{p\}$. Let F and G be multi-surfaces bounding γ. Then $\chi_{or}(F) = \chi_{or}(G)$.

Proof. We will first reduce F and G into a "minimal position". If, locally, there are two parts of F superposed with opposite orientation, say F_1 and F_2, then replace F_1 and F_2 by $\Delta(F_1, F_2) = \overline{F_1 \setminus F_2} \cup \overline{F_2 \setminus F_1}$. $\Delta(F_1, F_2)$ splits into two disjoint parts with opposite orientations, and the difference of the Euler characteristics is preserved. Indeed, notice that $\overline{F_1 \setminus F_2} \cap \overline{F_2 \setminus F_1} = \emptyset$ (because any intersection of these would have to occur on their boundaries, and in this case, it would imply that F_1 and F_2 have a common boundary component, which violates the definition of bounding), and so, assuming that F_1 is compatible with the orientation of S:

$$\begin{aligned}
\chi^+(\Delta(F_1, F_2)) - \chi^-(\Delta(F_1, F_2)) &= \chi(\overline{F_1 \setminus F_2}) - \chi(\overline{F_2 \setminus F_1}) \\
\chi(F_1) - \chi(F_2) &= \chi(\overline{F_1 \setminus F_2}) + \chi(F_1 \cap F_2) - \chi(\overline{F_1 \setminus F_2} \cap (F_1 \cap F_2)) \\
&\quad - \chi(\overline{F_2 \setminus F_1}) - \chi(F_2 \cap F_1) + \chi(\overline{F_2 \setminus F_1} \cap (F_1 \cap F_2))
\end{aligned}$$

and this gives the equality since $\overline{F_i \setminus F_j} \cap (F_i \cap F_j) = F_i \cap \partial F_j$ is a union of circles (of Euler characteristic zero).

This process makes F and G look locally like a superposition of sheets all with the same orientation.

Then, consider $H = F \sqcup (-G)$: the minus sign means that we reverse the orientation on G. $\chi_{or}(H) = \chi_{or}(F) - \chi_{or}(G)$. We want to prove that $\chi_{or}(H) = 0$. On each neighborhood of components of γ, there are two bounding surfaces. If they are one on each side, we can glue both sides over γ without any change of oriented Euler characteristic (since the common part is a union of circles, of Euler characteristic zero). This produces a local embedding of the union. If both surfaces are on the same side of the curve, then they have opposite orientations (since we considered $-G$). Consider the connected components F_i and G_j realizing the last case along some component of γ. Then, replace $F_i \cup G_j$ by the symmetric difference of the two components. Since the orientations of F_i and G_j do not agree, we have:

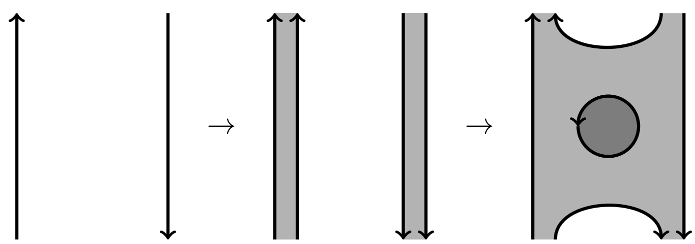
$$\begin{aligned}
\overline{F_i \setminus G_j} \cap \overline{G_j \setminus F_i} &= (F_i \setminus G_j \cup (\partial G_j \cap \mathring{F}_i)) \cap (G_j \setminus F_i \cup (\partial F_i \cap \mathring{G}_j)) \\
&= ((F_i \setminus G_j) \cap (G_j \setminus F_i)) \cup ((\partial G_j \cap \mathring{F}_i) \cap (G_j \setminus F_i)) \\
&\quad \cup ((F_i \setminus G_j) \cap (\partial F_i \cap \mathring{G}_j)) \cup ((\partial G_j \cap \mathring{F}_i) \cap (\partial F_i \cap \mathring{G}_j)) \\
&= \emptyset
\end{aligned}$$

The same computation as before shows that $\chi_{or}(F_i \sqcup G_j) = \chi_{or}(\Delta(F_i, G_j))$. We reduce $F \cup (-G)$ to a superposition of surfaces with the same local orientations. Any component of γ is deleted by the previous process: either we glue the local surfaces together, or we perform a symmetric difference. This is so a union of closed surfaces with no boundary embedded in $S \setminus \{p\}$. The only solution is the empty surface. So $\chi_{or}(H) = 0$ and $\chi_{or}(F) = \chi_{or}(G)$. □

Proposition 1.5.5. If F is a multi-surface bounding $\gamma_2 \sqcup (-\gamma_1)$, then $\gamma_2 = (-A^2)^{\chi_{or}(F)}\gamma_1$ in $\mathcal{A}$. Conversely, any relation in Relations (1.1.3) and (1.1.4) can be converted into a bounding multi-surface, and any isotopy can be converted into a sequence of one or two bounding multi-surfaces.

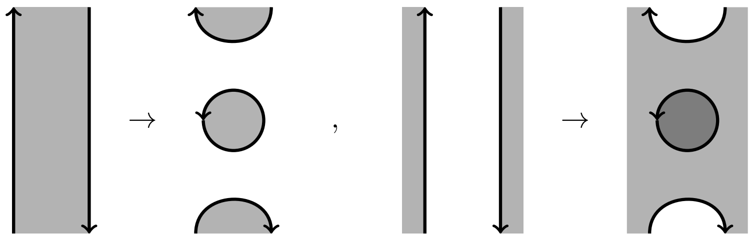
The following proof is very similar to [Prz98], Lemma 1.3, (ii).

Proof. Let us prove the second part of the statement first. For addition or deletion of trivial circles, this corresponds to filling each circle with the adequate oriented disk. Focus now on the Reidemeister II relation. Consider a multi-curve γ, on which we want to perform the relation. Take γ' a parallel. There is an annulus (or a union of annuli) between $-\gamma$ and γ'. Then perform on γ' the relation. There is a way to induce a surface between $-\gamma$ and γ', depicted below. This surface is of oriented Euler characteristic zero.



Remark 1.5.6. We can even give a better description for inducing a single surface from a sequence of relations between two curves.

Assume γ is already part of the boundary of a multi-surface, and we want to perform a Reidemeister II relation on it. If the two local strands on which we are performing the relation are parts of the boundary of the same surface, then the process for obtaining a surface bounding the result of the move is depicted below. We observe that the Euler characteristic remains unchanged.



If the two curves bound different surfaces, we describe a process for combining the surfaces without changing χ_{or}. The result then follows by the previous argument. Say F and G are these two surfaces. If F and G have the same orientations, consider instead the multi-surface given, for one sheet, by $F \cup G$ (this union is seen as a union of sub-manifold of $S \setminus \{p\}$: the common part is then counted only once), and for the other sheet by $F \cap G$, which doesn't change the whole Euler characteristic. If F and G have opposite orientations (say F has the one of S), consider the new surface $\Delta(F, G) = \overline{F \setminus G} \cup \overline{G \setminus F}$. As explained before, this preserves χ_{or}.

Let us now consider two isotopic multicurves. If they don't intersect, their difference cobounds a multi-surface of Euler characteristic zero. If they do intersect, we can consider a parallel to both of them which doesn't intersect any of them.

For the first part of the statement, consider now a given surface bounding a curve γ, embedded in $S \setminus \{p\}$. This can be presented as a surface with g handles and l boundary circles. Choose a particular circle and move it over the handles by two Reidemeister II moves, producing two trivial disks per handle. We are reduced to a sphere with l disks removed. Two boundary circles can be reduced to one by a Reidemeister II move, producing again a trivial disk. Then, we are left with a set of trivial disks, which are produced by addition of a trivial circle. See Figure 1.9 for an illustration of the process. $\square$

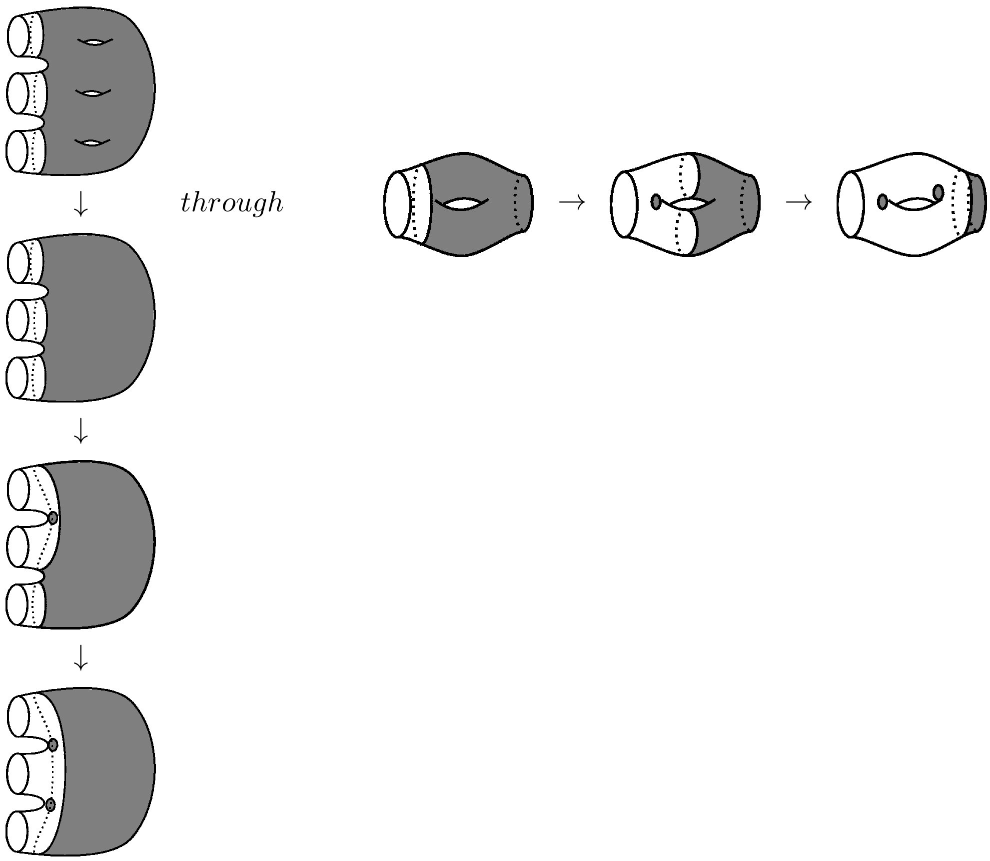


Figure 1.9: A surface induces a relation

Chapter 2

Disoriented Bar-Natan skein modules

Introduction

In 2000, Khovanov [Kho00] introduced a categorification of the Jones polynomial of knots and links. This algebraically flavored construction replaces the defining relation of the Kauffman bracket by explicit morphisms between modules, building up a complex of $\mathbb{Z}$-modules. Computing the graded Euler characteristic of this complex then recovers the Jones polynomial. A few years later, Bar-Natan [BN05] presented a geometric reinterpretation living in a category of cobordisms. A bridge between the two models is then given by *Topological Quantum Field Theories*. Although both of these theories are *a priori* developed for knots and links in $\mathbb{R}^3$, Bar-Natan noticed that his construction naturally extends to the case of knots embedded in thickened surfaces. TQFT however are less natural than in the previous case, mostly because the identity morphism over an essential curve in a surface doesn't factors through the empty curve.

TQFT for general surfaces have been investigated by Turaev and Turner [TT06], Carter and Saito [CS09] and Kaiser [Kai09], and extensions of Khovanov homology can be found in a paper by Asaeda, Przytycki and Sikora [APS04]. A major difficulty in the surface case is to deal with a new kind of saddle, that maps an essential curve to another non-homotopic essential curve. It appears that the easiest way to do so is to kill this saddle by assigning to it a zero map, but this drives us to also weaken the structure of Frobenius module assigned to essential curves. Translated in Bar-Natan's category, a similar idea would be to simply kill all unorientable surfaces, which strongly modifies the relations on morphisms.

Khovanov homology is known to provide an invariant of knots whose extension for cobordisms is functorial only up to signs. Fully functorial versions of it have been developed by Clark, Morrison and Walker [CMW09], Caprau [Cap07, Cap08] and Blanchet [Bla10]. These theories are also built on (generalized) cobordism categories, and therefore admit natural extensions to the case of surfaces. Our purpose here is to investigate these extended cobordisms category. In particular, we will show that the analogue in functorial models of the singular saddles is governed by a very different relation, that allows to mod out the cobordism categories of CMW and Blanchet's models by unwanted surfaces without causing any degeneration.

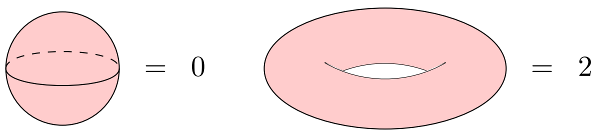
The generalized classes of curves and cobordisms that appear in both CMW and Blanchet categories suggest to perform the same construction on knots formed with generalized curves. This drives us to study in Clark-Morrison-Walker's case locally oriented knots with disorientation points, while in Blanchet's case this produces knotted foams, which make the $\mathfrak{sl}_2$ world closer to the general $\mathfrak{sl}_n$-one than usual. We use this idea of extension to compare both categories, and we show here that we can define from Blanchet's functorial version of Khovanov homology an invariant of knotted $\mathfrak{sl}_2$ foams, the only remaining questions being to know whether this extension may cause any loss of functoriality.

Acknowledgments: I especially want to thank J. Scott Carter for his help with the movie-moves section.

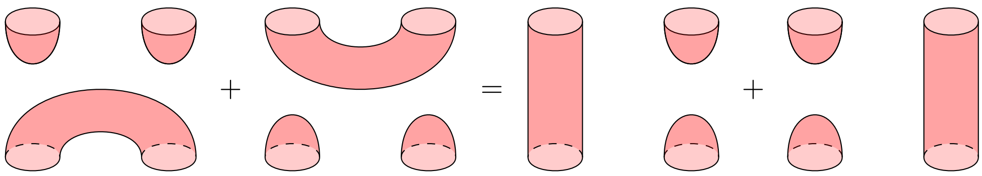
2.1 Unoriented model

2.1.1 Bar-Natan's category

Bar-Natan introduced in [BN05] a category of cobordisms, in which we can build a geometric version of Khovanov homology. Let us denote it *Cob*. Objects of this category are (formal $\mathbb{Z}$-graded sums of) unoriented curves embedded in a plane, that is, unions of circles. Morphisms between curves γ_1 and γ_2 are cobordisms between the two curves, that is surfaces Σ embedded in $\mathbb{R}^2 \times \mathbb{I}$ (where $\mathbb{I} = [0,1]$), with $\partial\Sigma = \partial\Sigma \cap (\mathbb{R}^2 \times \{0\}) \cup \partial\Sigma \cap (\mathbb{R}^2 \times \{1\}) = \gamma_1 \cup \gamma_2$. Morphisms between sums of curves are matrices of cobordisms. These cobordisms are subject to isotopy and local relations:

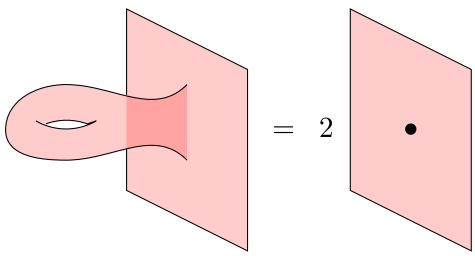


(2.1.1)



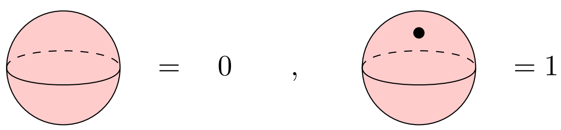
(2.1.2)

Dots can be introduced as a short-hand of handles, with the following equality:

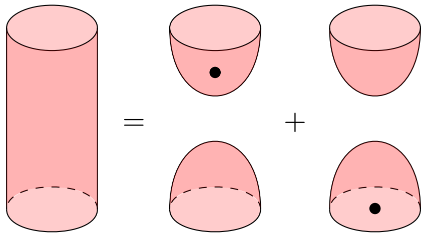


(2.1.3)

The previous relations may then be reformulated as :

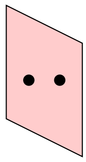


(2.1.4)



(2.1.5)

Furthermore, we request that any closed connected foam with two dots is zero:

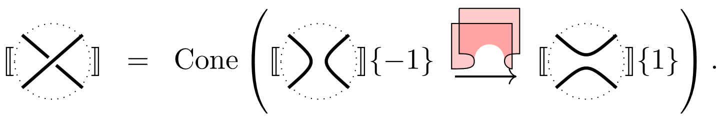


$= 0.$

The category *Cob* is a graded category, with $deg(\Sigma) = 2\chi(\Sigma) - 4\sharp dots$, where χ denotes the Euler characteristic. Grading shifts on objects will be denoted by $\{\ \cdot\ \}$, with the convention: $deg(\Sigma\colon \gamma_1\{m_1\} \mapsto \gamma_2\{m_2\}) = deg(\Sigma) - m_1 + m_2$.

Khovanov [Kho00, Kho02, Kho06] defined about ten years ago a categorical model for Jones polynomial, which assigns to a knot diagram a complex of $\mathbb{Z}$-modules whose graded Euler characteristics recovers the Jones polynomial. Although this model was at first defined with an algebraic flavor, Bar-Natan [BN05] developed a geometric re-interpretation of these theories that takes place in the cobordism category.

Starting from a diagram of a knot, the basic idea is to smooth all crossings successively with the following process:



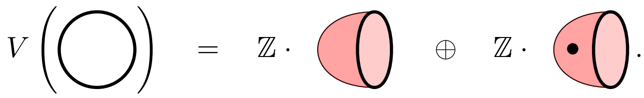
The process above is recursively defined using the notion of the cone of a morphism, which is nothing but the natural way to transform two complexes with a map between them into a single complex. The order we adopt for successively smoothing the crossing does not play any significant role here.

At the end of the process, one obtains a complex (we often refer to it as *formal*, by opposition to Khovanov's complex of very concrete modules) living in the category of complexes built over *Cob*. Then:

Theorem 2.1.1. [Kho00, Theorem 1][BN05, Theorem 1] The homotopy class of the complex associated to the diagram of a knot does not depend, up to convenient grading shifts, on the particular diagram used for building it. Furthermore, its graded Euler characteristic is the Jones polynomial of the knot.

2.1.2 TQFT and universal construction

A convenient way to relate this geometric construction to Khovanov's original one is the use of a TQFT, which provides a graded functor between *Cob* and the category of modules. We can do it using a *universal construction* process in the spirit of [BHMV95]. This consists in assigning to a curve γ the (graded) vector space $V(\gamma) = Hom(\emptyset, \gamma)$. $V(\gamma)$ appears to be finite dimensional vector space, and the translation of cobordisms into linear maps is direct. We can for example describe the space associated to a circle:



The first element above corresponds to the generator 1 of Khovanov's Frobenius algebra $A = \mathbb{Z}[X]/(X^2)$ [Kho00], the second one to X. The whole structure of Frobenius algebra can be identified among cobordisms, a birth of a circle being the unity, a death the counity, and the two pairs of pants the multiplication and the comultiplication.

Applying this TQFT to Bar-Natan's formal complex, the objects are translated into graded modules, and differential cobordisms become explicit differential maps between these modules, and we obtain Khovanov's original complex.

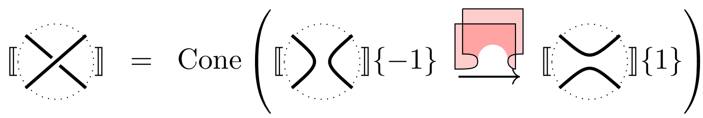
2.1.3 Extension to surfaces

Although Bar-Natan's construction is generally understood for curves lying in the plane $\mathbb{R}^2$ (or the sphere $\mathbb{S}^2$) and morphisms in its thickening $\mathbb{R}^2 \times \mathbb{I}$ (or $\mathbb{S}^2 \times \mathbb{I}$), this work naturally extends to other surfaces.

More specifically, we can define a version of the category of cobordisms $Cob(S)$ for any surface S (we restrict here to the case of orientable closed surfaces). Objects of this category are ($\mathbb{Z}$-graded formal sums of) unoriented curves embedded in S. Morphisms between two curves γ_1 and γ_2 are cobordisms embedded in $S \times \mathbb{I}$, with all boundary contained in $S \times \{0\}$ and $S \times \{1\}$ and recovering γ_1 and γ_2. Relations for morphisms are isotopy relative boundary, and the same local relations (2.1.1, 2.1.4, 2.1.5) as before.

Similarly, a morphism Σ in $Cob(S)$ has a degree given by $2\chi(\Sigma) - 4\sharp dots$.

The notion of knots generalizes as well to thickened surfaces $S \times \mathbb{I}$ as embeddings of circles into $S \times \mathbb{I}$. The case of surfaces is particularly convenient, since the natural projection $S \times \mathbb{I} \mapsto S$ associates (in generic cases) a diagram on S to a knot. The analogue of the Jones polynomial for such situations is the skein module (see the beginning of Chapter 1 for a study of the torus case), but extending Khovanov's original homology theory appears to be a quite challenging task [APS04]. However, since Bar-Natan's cobordism category extends to the case of surfaces, one can just mimic the geometric construction and, from a diagram of a knot drawn on S, solve recursively each crossings with the same rule as before:

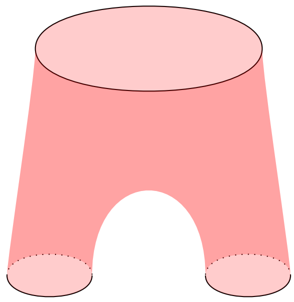


We build this way a formal Khovanov's complex associated to a knot on S, that lives in the category of complexes built over $Cob(S)$. Since all proofs of invariance in [BN05] are local, we then have the following invariance result:

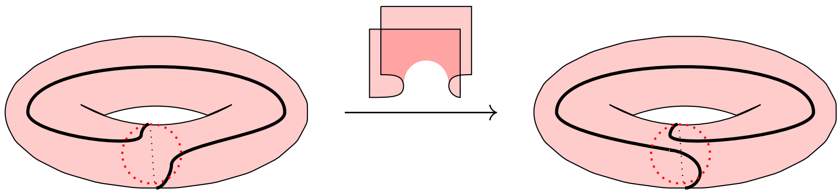
Theorem 2.1.2. [BN05, Theorem 1 and section 11.6] The homotopy class of the complex associated to a diagram of a knot is independent, up to a convenient grading shift, of the chosen diagram. Furthermore, this complex decategorifies to the image of the knot in the skein module of the surface.

Once we have this geometric model, we could hope that applying a suitable generalization of a TQFT produces an algebraic counterpart. However, it turns out that defining TQFT in the surface case is not an easy work, and in particular the universal construction process cannot be applied anymore: any curve drawn on S is not bounding, and therefore we can have $Hom(\emptyset, \gamma) = \emptyset$. Turaev and Turner [TT06] investigated abstract solutions to this problem, and Carter and Saito [CS09] extended this analysis to a larger model, distinguishing essential and non-essential curves on S, giving rise to the notion of Frobenius modules. We will see with Relation (2.1.6) one of the major difficulties in defining TQFT over surfaces.

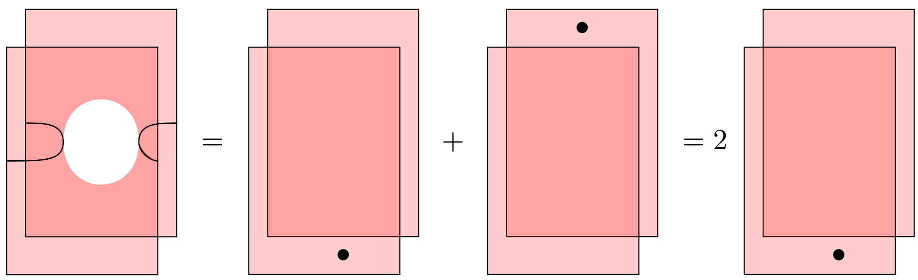
Let us now study a bit more the category $Cob(S)$. Local generators of the cobordism category are just as in the plane case given by identity cobordisms, birth, death, and a local saddle. Over a plane, the only possible closure of a saddle is as a pair of pants (and its upside-down version) :



However, a new kind of saddle may appear for more general surfaces. Consider for example the morphism given by :



In the previous picture, the local saddle is applied over the dotted red circle, and we glue to it identities elsewhere. This saddle has only two boundary components, and is unorientable. Let us denote it ν. Composing it with its mirror image, we can apply the neck-cutting relation 2.1.5 and obtain the following equality :



(2.1.6)

The last equality comes from the fact that the surface we obtain is connected : the second term in the middle is actually equal to the first one.

When trying to extend Khovanov's homology to surfaces (or trying to define TQFT), one quickly realizes that the major difficulty is to deal with this unorientable saddle. The solution chosen by Asaeda, Przytycki and Sikora is mostly to assign a zero map to this saddle. Turaev-Turner and Carter-Saito works mostly show that there are no smooth way to deal with this unorientable saddle and to keep a good module-comodule structure for essential circles.

Similarly, if one wants to mimic the universal construction process, the above relation appears to be very restrictive. The easiest way to deal with it would be to mod out $Cob(S)$ by unorientable surfaces. But this produces new relations between morphisms in the category and makes it less manageable.

The key point of this work will be to obtain a better relation in a slightly different category.

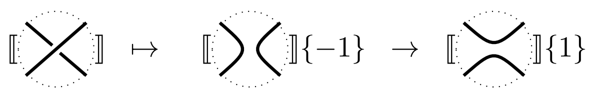
2.2 Disoriented model

Khovanov homology is known to be functorial only up to sign. A model fixing the sign issue has been introduced by Clark, Morrison and Walker in [CMW09], that works over Gaussian integers, $\mathbb{Z}[i]$. Other models are due to Caprau [Cap07, Cap08] and Blanchet [Bla10], the latter one working over $\mathbb{Z}$. These models are built over categories of cobordisms very close to Bar-Natan's one, where cobordisms carry extra combinatorial or topological date allowing to bound locally oriented curves with possibly conflicting orientation.

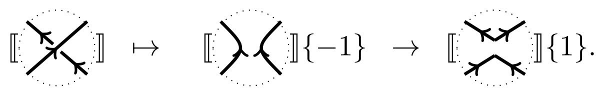
We intend here to understand the role of the special unorientable saddle in analogues of the category $Cob(S)$ for the functorial models. By opposition to the previous unoriented model, we generically refer to these new models as disoriented.

2.2.1 Cobordisms with disorientation lines

If we start with an oriented diagram of a knot and smooth the crossings, one of the two resolutions has non-matching local orientations. Indeed,



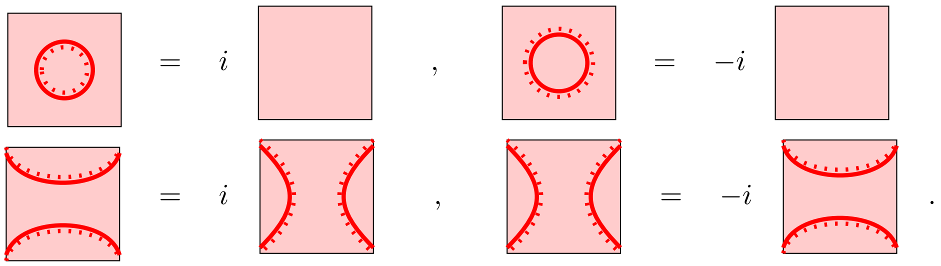
would become:



We see in the last smoothing above a conflict of orientations. CMW idea is to keep track of this conflict by considering locally oriented curves with disorientation points when local orientations don't match. These disorientation points are marked with a fringe. When smoothing a crossing, the convention for the fringe is the next one: .

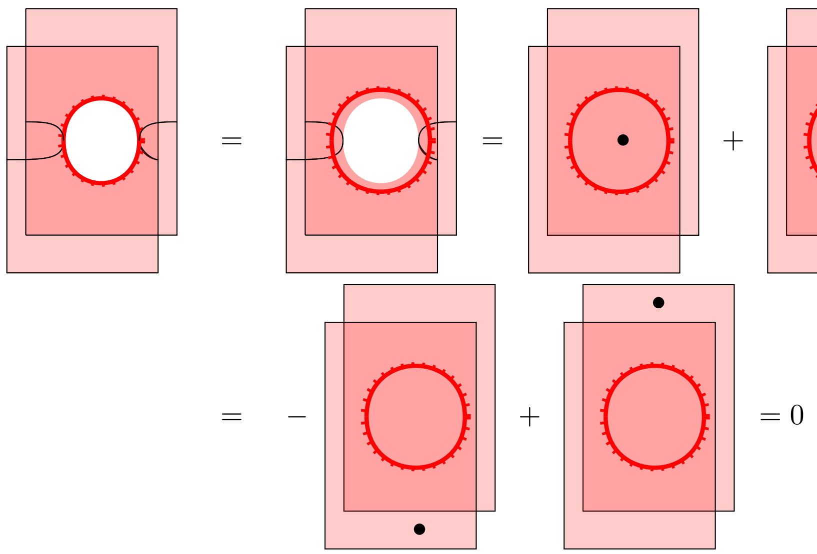
We therefore define a new category, denoted $Cob_{CMW}(S)$, whose objects are now ($\mathbb{Z}[i]$[1]-graded formal sums of) locally oriented curves with fringes embedded in S. Morphisms are as before matrices of cobordisms, but we assume that they are locally oriented. Whenever two local orientations do not match, this produces a disorientation line marked with a fringe, that agrees with the one on the boundaries. The cobordisms are considered up to the same local relations as in the unoriented case, when the orientation agrees. If we add to these relations the following ones for disorientation lines, we obtain a complete set of relations for closed surfaces in the planar case (and actually for orientable closed surfaces as well):

1. Note the ground ring is now the Gaussian integers.



(2.2.1)

Just as before, cobordisms are generated by identity cobordisms, birth, death, and two kind of saddles: pairs of pants, and unorientable saddles. This last saddle, although globally unorientable, is now locally oriented, with a disorientation line, and therefore Relation (2.1.6) will have a new expression:



(2.2.2)

This new equation indicates that now, the value we may want to give to unorientable saddles is largely independent of the value of the multiplication by a dot. This opens two perspectives:

– the first idea is to investigate all possible TQFT we could obtain without imposing these unorientable saddles to be zero. This mostly consists in starting over all computations from [CS09] in any of CMW or Blanchet's cobordism category. Although this is theoretically possible, explicit computations are extremely intricate. One can exhibit some possible solutions, but proving them to be well-defined requests a lot of topological and combinatorial analysis of the categories we start from, and the algebraic translation of this is not yet completely clear. Furthermore, saddles in Khovanov homology (which appear as differentials in the complexes associated to knots) usually come with degree -2 (up to some changing conventions), and the Frobenius algebra associated to one circle has generators in degrees 2 and -2. This usually fits together because we have on one side a tensor product of two copies of the Frobenius algebras, and on the other side only one copy. The map an unorientable saddle would induce would only involve one copy on each side, and therefore it cannot be a graded map, unless we distinguish the module on which we send each isotopy class of curve.

– another solution, consists in still assigning a zero map to an unorientable saddle, and investigate if this still requests extra relations to be introduced. We will rather focus on this perspective in the next section.

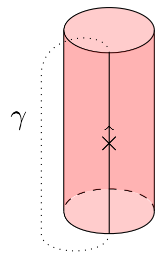
2.2.2 Orientable disoriented Bar-Natan's category

A cobordism in $Cob_{CMW}(S)$ has an underlying unoriented surface, that can be orientable or not. Let us define $Cob^{or}_{CMW}(S)$ as $Cob_{CMW}(S)$ where unorientable cobordisms are sent to zero. We will see that this process induces no new relation between orientable cobordisms.

Proposition 2.2.1. If two orientable cobordisms are equal in $Cob^{or}_{CMW}(S)$, this was already the case in $Cob_{CMW}(S)$.

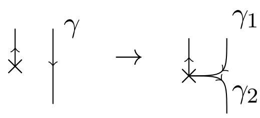
Proof. Let S be a connected unorientable dotted surface with disorientation lines, possibly with boundary. We want to see whether there are orientable representatives in the equivalence class of S under local relations (2.1.4), (2.1.5) and (2.2.1).

Relations (2.1.4) and (2.2.1) do not change the orientability, so we can focus on (2.1.5). Assume that there exists a compression disk where we can perform the neck-cutting relation, so that the resulting surface (denoted $NC(S)$, where there is possibly only one connected component) we obtain is orientable. Since S is unorientable and $NC(S)$ is orientable, there is a path γ in S starting on the annulus A where we perform the relation, and coming back after having crossed an odd number of disorientation lines.

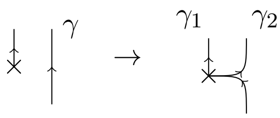


(2.2.3)

We can assume that this path γ goes only once through A. Indeed, if this is not the case, we locally have :

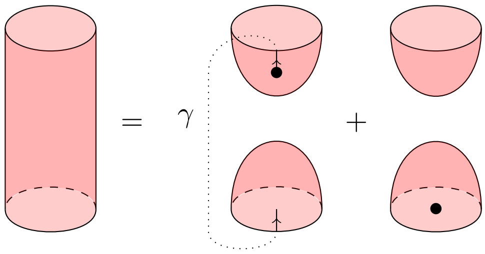


or



Then, either γ_1 is a disorientation path, and we focus on it, or γ_2 is. Applying this reduction process as many times as necessary, we end up with a path going one or zero time through A. Zero corresponds to the case where $NC(S)$ is unorientable, which contradicts our assumption.

So, applying the neck-cutting relation, we get a sum :



The image of γ after neck-cutting transports the dot of the left term to the place where the one of the right term lies. This is done by crossing an odd number of disorientation lines, and therefore it produces a (-1) coefficient, and the sum is actually zero.

Therefore, sending S to zero doesn't produce any new relation on orientable surfaces. □

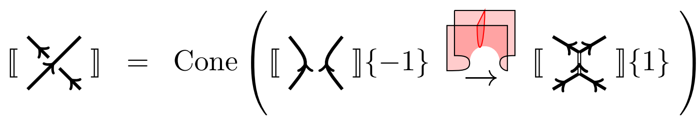
Remark 2.2.2. The previous proposition is the key difference between the unoriented and the disoriented cases: Relation (2.1.6) shows that Proposition 2.2.1 no longer holds in the unoriented category.

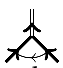
We can define a formal Khovanov's complex in this new category just as before. The previous proposition ensures that this category is not empty and has the same local relations as before. All squares were skew-commutative before killing all unorientable saddles, and this remains true after killing them.

The Khovanov complex of a knot is still a homotopy invariant: all of Bar-Natan's geometric proof are transported to the new quotient category.

2.2.3 Blanchet's $\mathfrak{sl}_2$-foams

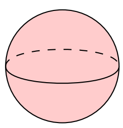
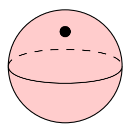
Blanchet introduced in [Bla10] a functorial model that avoids the use of Gaussian integers, and that lies on the use of foams in the spirit of Khovanov [Kho04] and Mackaay-Stosic-Vaz [MSV09]. The difference with the previous model lies in the smoothing of crossings of the diagram of a knot :

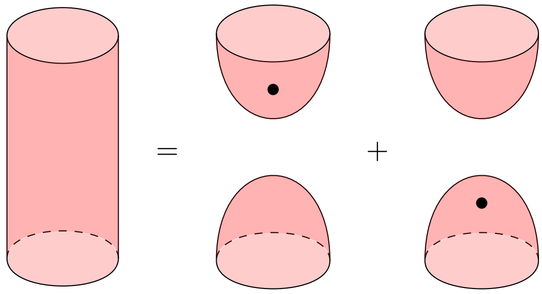


We briefly sketch here Blanchet's model, but refer to [Bla10] for further details. Objects of the base category are now $\mathfrak{sl}_2$-webs, that is, oriented trivalent graphs with 1- and 2-labeled edges, so that locally we have:  or . There is a local ordering between 1-labeled edges in the neighborhood of each vertex.

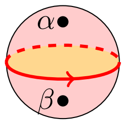
Morphisms are $\mathfrak{sl}_2$-foams, which are unions of oriented 1- and 2-labeled facets along seams. The neighborhood of each seam is isomorphic to the letter Y times $\mathbb{I}$ or times $\mathbb{S}^1$. The 1-labeled facets have local orderings near each seam. We can represent this local ordering by orienting the seams and use the right-hand rule (see also Chapter 3 for more details about these foams).

We can perform exactly the same work with the same results : a set of local relations is given by :

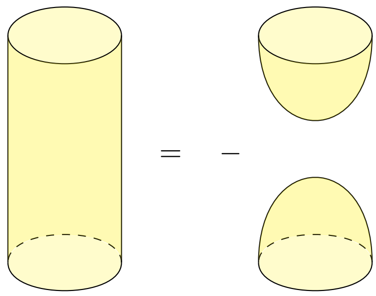
 $= 0$,  $= 1$ (2.2.4)



(2.2.5)



$$= \begin{cases} 1 & \text{if } (\alpha,\beta) = (1,0) \\ -1 & \text{if } (\alpha,\beta) = (0,1) \\ 0 & \text{else} \end{cases} \tag{2.2.6}$$

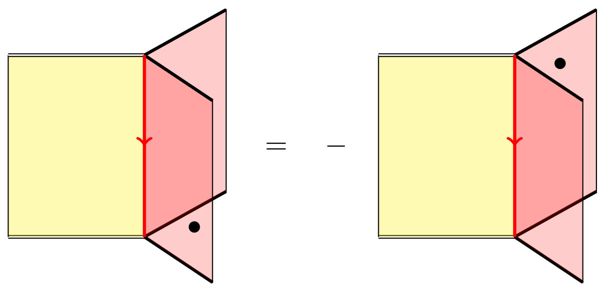


(2.2.7)

In these pictures, the yellow surfaces are the 2-labeled facets.

The previous relations allow for the evaluation of any closed foams. Other local relations may be deduced from those by a closure process: two foams are equal if they have the same boundary and if any of their closures are equal.

As before, the only relation we have to focus on is the neck-cutting relation for 1-facets. The very same argument as before will work as well. Indeed, disorientation lines are here replaced by seams, and we have the dot-migration relation:



(2.2.8)

2.3 CMW Vs Blanchet: Dealing with confusions

We have seen that both CMW and Blanchet models, in addition to solve the functoriality problem for Khovanov homology, also provide a nicer category of cobordisms for surfaces for searching extensions of the homology process. CMW model, except the fact it requests the introduction of Gaussian integers, seems more easily manageable than Blanchet's one, since the objects are almost the same as in $Cob(S)$, while Blanchet introduces additional 2-strands that rigidify the curves.

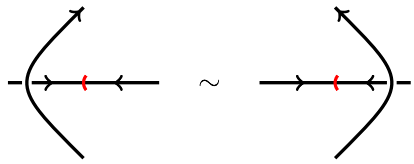
However, a natural attempt arising from this work is to extend the definition of the skein module and of Khovanov homology to knots built from any object of the category, that is, knotted locally oriented curves (resp. knotted $\mathfrak{sl}_2$-webs).

2.3.1 Confusions

2.3.1.1 Clark-Morrison-Walker's model

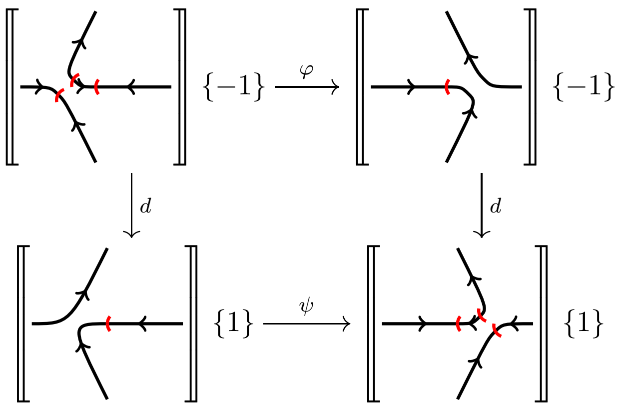
We extend the class of knots to a class of knotted locally oriented curves (for CMW model). These are embeddings of curves with local orientations and possible fringes in $\mathbb{R}^2 \times \mathbb{I}$ (we could also deal with thickened surfaces) with a generic projection on $\mathbb{R}^2$. We request in the genericity condition that all crossings are away of disorientation points, so that we can define on it the same smoothing process as before. One can also consider framed objects by thickening the curves with blackboard framing. The result of the bracket process is thus a complex in the suitable category of cobordisms.

Proving that these complexes are invariant of our generalized class of knots requests an extension to these new classes of the Reidemeister moves (see [Kau89] or [Car12] for the web case). In [CMW09], Clark, Morrison and Walker briefly mention a difficulty in achieving such a work. In particular, the following move seems hard to deal with:

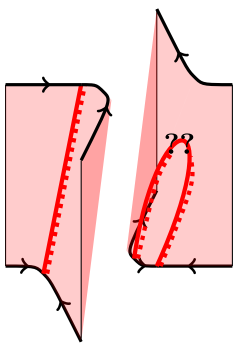


(2.3.1)

Smoothing both diagrams above, we are looking for an homotopy equivalence of the following form:



There are no easy solutions to this problem: although one would like φ and ψ to have as underlying unoriented surfaces only identities, there is no way to match the tags on such surfaces.

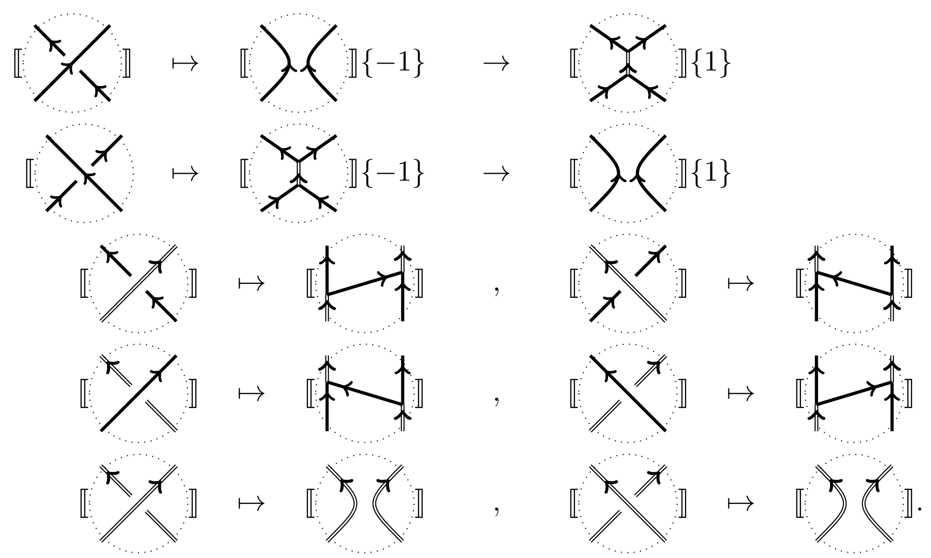


This leads Clark, Morrison and Walker to introduce the notion of *confusions*, which are changes of tags along a disorientation line... These confusions, although they appear naturally from this perspective, but also from a combinatorial point of view, turn out to deserve their name, and dealing with them is at least complicated, if not impossible.

2.3.1.2 Blanchet's model

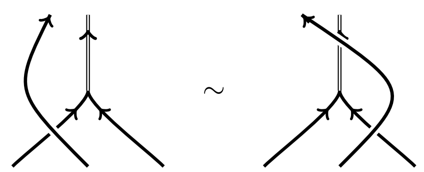
We now want to perform the same analysis using Blanchet's model. The key difference lies on the fact that tags, which are only local information, are replaced by 2-strands, which on the contrary cannot just vanish: they will here play a non-trivial role. This remark is to be related with the process described in Chapters 3 and 4, where we show that recovering a skein module from the skew-Howe duality process seems more natural if one uses Blanchet-inspired *enhanced* foam categories.

The generalized class of knots will now consist in knotted $\mathfrak{sl}_2$ webs, which are embeddings of webs in $\mathbb{R}^2 \times \mathbb{I}$ with generic projection on the base plane. Crossings should be generically away of the trivalent points. These crossings can now involve different type of strands, and therefore we should define the smoothing process for all possible cases, which can be done as follows [2]:



We adopt here, in the spirit of Rozansky [Roz10b], a $\frac{1}{2}\mathbb{Z}$ grading for the homological grading [3]. We then consider that all homological gradings above are symmetric. In other words, one-term complexes are all in homological degree zero and 2-term complexes are in degrees $-\frac{1}{2}$ and $\frac{1}{2}$. Whenever necessary, shifts in homological grading will be denoted by $[\,\cdot\,]$.

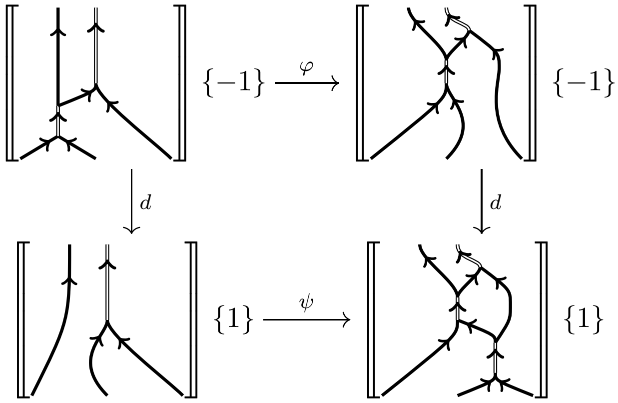
Blanchet's version of the equivalence move given in Relation 2.3.1 would be as depicted below, where we can observe the role of the 2-strand.



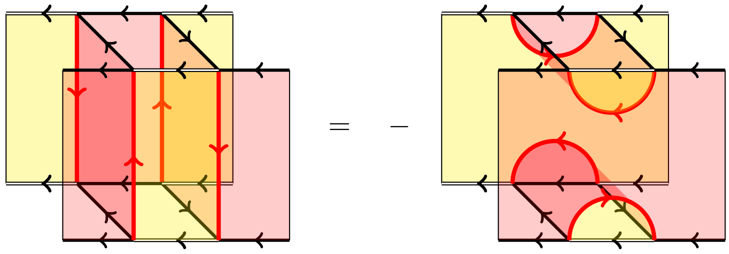
2. This is a rescaling of the one defined in Chapter 4. We also switch from a q-grading to an A-grading, which makes the situation more symmetric...

3. Note that Rozansky usually uses also a grading with half integers for the quantum grading. Here, we choose to turn to an A-grading rather than to a $q-$grading, so the two processes are very similar.

Smoothing both diagrams above, we want to find an homotopy equivalence between the following complexes.

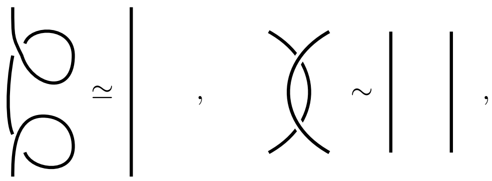


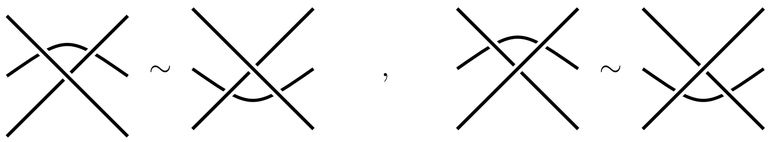
This time, φ can be chosen to be an isotopy, and ψ will be a realization of an equivalence of $\mathfrak{sl}_2$-webs via:

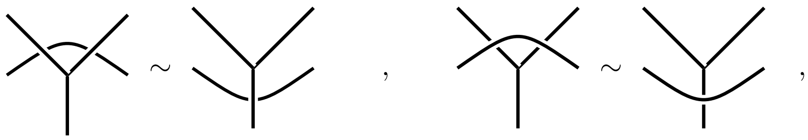


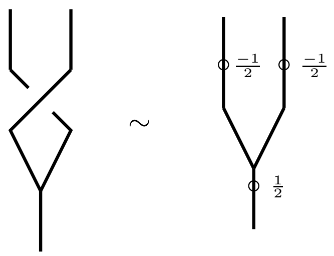
In the $\mathfrak{sl}_2$ case that we consider, we can even prove by hand that the homology we build is an invariant of knotted $\mathfrak{sl}_2$-webs.

Two diagrams of a knotted framed web describe the same web if they are related by a sequence of Reidemeister-Kauffman moves (see [Kau89] and Chapter 4 for further discussion):


(2.3.2)


(2.3.3)


(2.3.4)



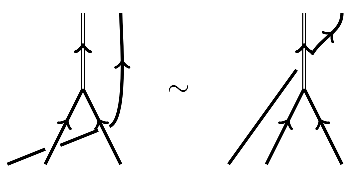
(2.3.5)

The small circles here denote a twist on a framed strand, the framing being assumed to be the blackboard one otherwise.

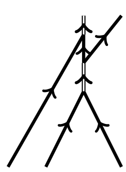
Theorem 2.3.1. Khovanov homology is an invariant of knotted $\mathfrak{sl}_2$-webs.

Proof. The proof consists in checking all Reidemeister-Kauffman web moves.

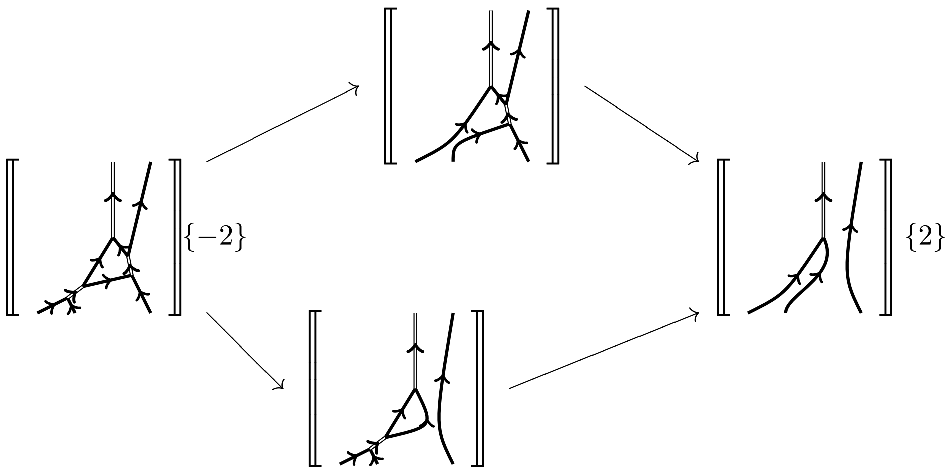
All moves concerning only 1-labeled strands are proved in Blanchet's paper [Bla10]. We only have to deal with usual Reidemeister moves involving 2-labeled strands and with web-moves. The first part is left to the reader. Let us focus first on the following move:



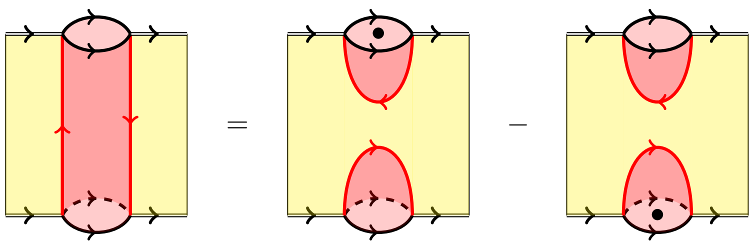
The second part of the move smooths as:



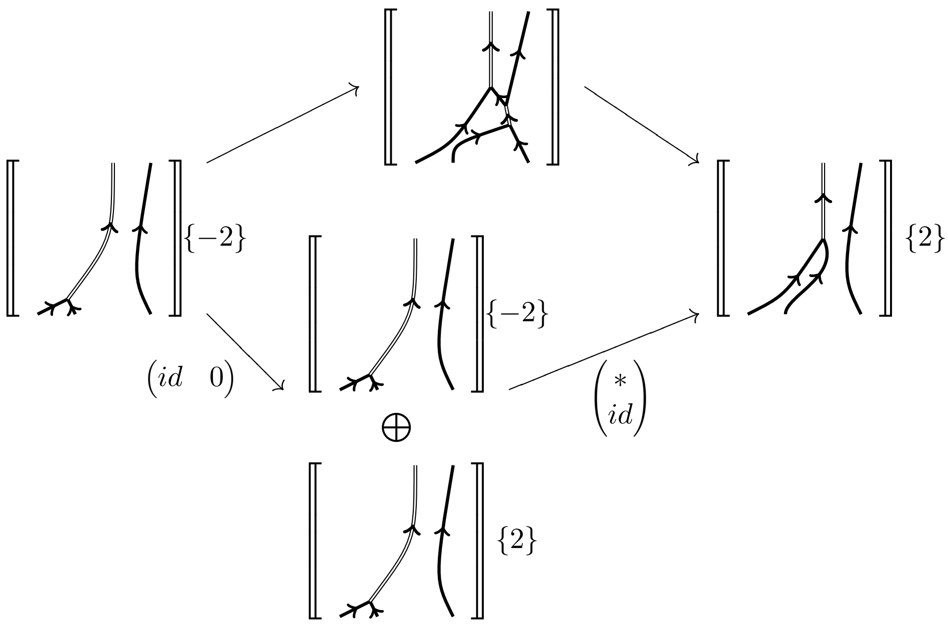
The first part gives the following complex (where we didn't pictured the differentials, which are all signed local singular saddles glued with identities elsewhere):



The bottom element contains a digon, so we can perform a reduction process explained by Bar-Natan [BN07]. The idea is to split the identity of a digon into a sum of two orthogonal idempotents:

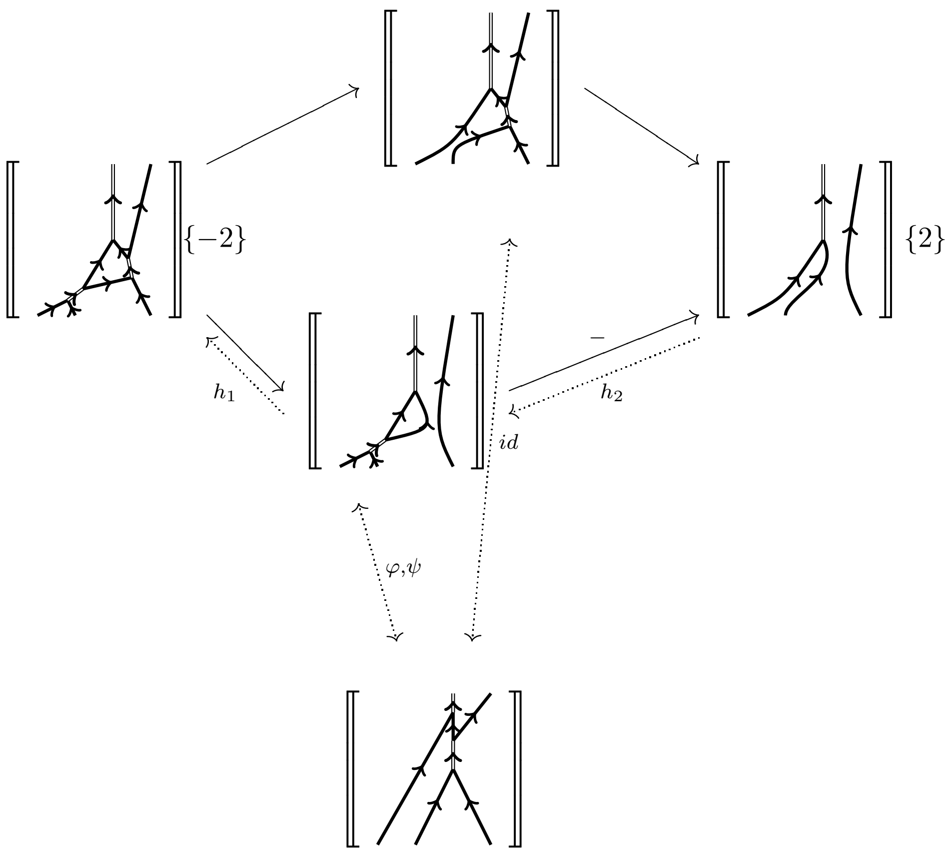


Applying also an isomorphism on the left term, we obtain the following isomorphic complex:

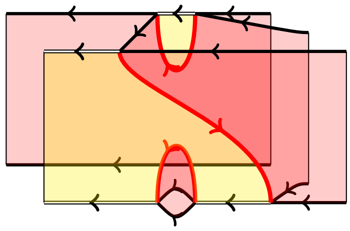


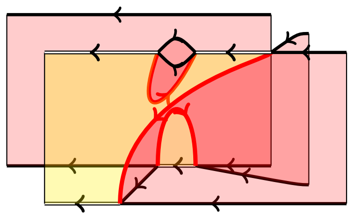
which is homotopy equivalent to the 1-term complex of the second part of the move.

When we will be looking at functoriality, it will be useful however to have an explicit homotopy equivalence, which we give here:

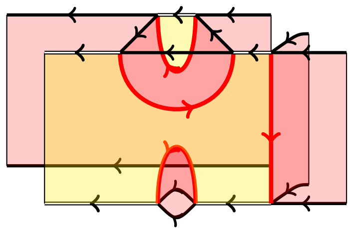


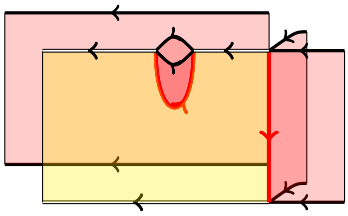
where the morphisms are given by:

$\varphi \quad =$ 

$\psi \quad =$ 

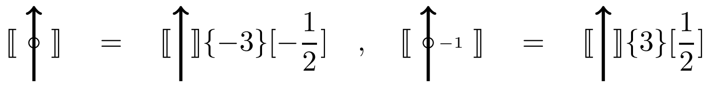
and the homotopy equivalences are:

$h_1 \quad =$ 

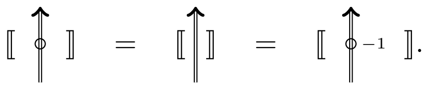
$h_2 \quad =$ 

Performing a global reversion of the orientation, or switching the crossings leads to similar situations. The case where the single strand is 2-labeled instead of 1-labeled is easy to deal with. Other orientations of the same moves can be deduced from this one, as explained in Chapter 4.

Finally, we should deal with the framing moves. This is done by checking that:



and



The extension to twists with half-integer indices in the move we consider causes no problem: the sum of coefficients concerning the 1-labeled strands is always an integer, and we can neglect the twists on 2-labeled strands.

□

2.3.2 Functoriality

Blanchet's version of Khovanov homology is known to be functorial for knots. However, just as from the invariance proof for knots we can define an invariant of knot cobordisms, we could here ask for functoriality for $\mathfrak{sl}_2$ webs. Foams bounding knotted webs can be described by elementary foams given by the extended set of Reidemeister moves, isotopies, and deaths, births, saddles and foam versions of them. All of these elementary foams have a translation as morphisms of complexes, but since the decomposition is not unique, we have to check that it doesn't depend on a choice of it.

The first step will thus consist in finding a complete set of movie moves for $\mathfrak{sl}_2$-moves, that describe all relations to be checked. A partial answer is due to Carter [Car12], who describes a set of Roseman-type moves. We give here a description of a complete set of movie moves for the $\mathfrak{sl}_2$-case.[4]

2.3.2.1 Foams, movies and movie moves

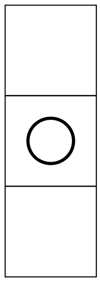
We first consider foams in a broad sense, as a union of surfaces with singularities, the neighborhood of a singularity being isomorphic to an interval or a circle times the shape of the letter Y: $Y \times \mathbb{I}$ or $Y \times \mathbb{S}^1$. Usual foams can also have singular points with 6 local facets, but we forbid them here since they cannot appear in the $\mathfrak{sl}_2$-case.

Let F be such a foam, considered embedded in $\mathbb{R}^4$: $f\colon F \mapsto \mathbb{R}^4$. We have a projection $p\colon \mathbb{R}^4 \mapsto \mathbb{R}^3$, so that in generic case $p \circ f(F)$ gives a *diagram* of F in $\mathbb{R}^3$. We also consider

4. A conjectural set of movie moves for general foams has been very kindly sent to us by J. Scott Carter. The set we provide here is a very particular case of the general one he considers. We wish to thank him warmly for his great help!

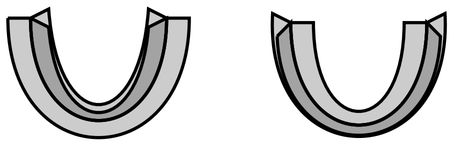
on $\mathbb{R}^3$ a height function $h\colon \mathbb{R}^3 \mapsto \mathbb{R}$, which will decompose the diagram of the foam in a movie.

The diagrams that we consider are requested to be generic with respect to both p and h. Given a diagram, we draw its move by cutting the foam horizontally so that between each slice, there is only one singularity. For example, a bubble could be drawn as:



Singular sets we will be interested in are:

– edges (3-valent edges, the center of the letter Y). The restriction of h to each connected component is requested to be a Morse function. During an isotopy, edges draw 2-dimensional sets. Singularities for edges are 0 and 1-handles for h, that can be drawn locally as:

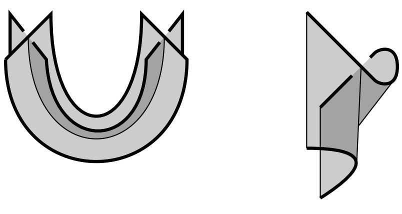


,

and their upside-down versions.

– 0-, 1- and 2-handles (for h) on facets, which are 0-dimensional sets, and draw during an isotopy a set of dimension 1. These singularities correspond locally to a birth, a saddle or a death cobordism, and draw during an isotopy 1-dimensional sets.

– double points are induced by the projection p, and correspond at the level of knots to crossings. Back to $\mathbb{R}^4$, they correspond to the *double decker set*. h restricted to this set is requested to be a Morse function, and we have 0 and 1-handles, and also images of singular points in the double decker set, which correspond to twists. Below are drawn some examples.



Double points draw during an isotopy a 2-dimensional set, and handles and twists draw 1-dimensional sets on them.

Lemma 2.3.2. Two movies of $\mathfrak{sl}_2$-foams describe isotopic foams if and only if they are related through a sequence of the following moves:

– usual movie-moves MM_1 - MM_{15} [CS98],

– Roseman-type moves for foams [Car12] specialized in the $\mathfrak{sl}_2$-case, that is, we do not consider all moves that contain a singular vertex bounding 6 facets,

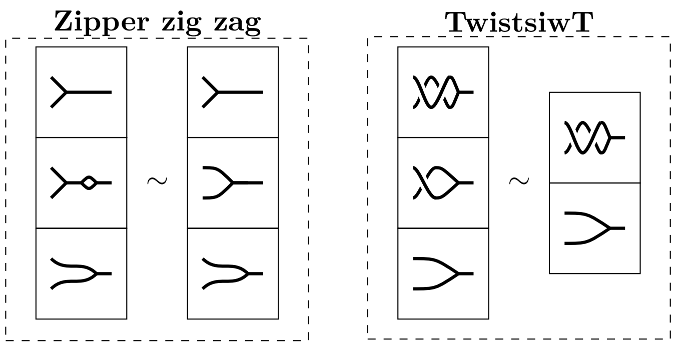
– moves depicted during the proof: Zipper zig-zag, TwistsiwT, Sliding zipper, Sauron, Saddle zipper 1 and 2, Cup zipper 1 and 2 and Twist zipper 1 and 2.

Proof. We have to study all possible non-generic situations. We already know:

– usual movie moves (for knots): they deal with everything that does not concern edges.
– Roseman-type moves for foams (from [Car12]): does not deal with the height function h.

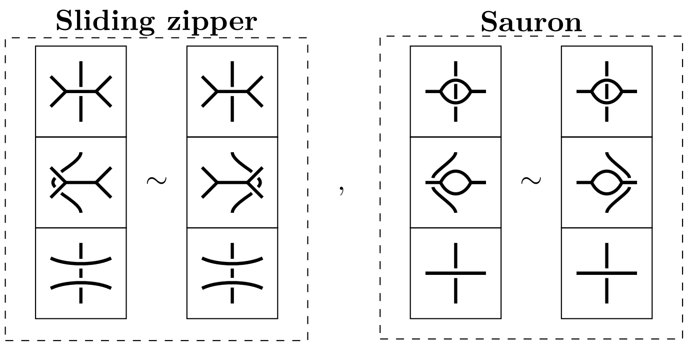
We therefore miss critical points of the edge set, and transverse intersections given by a handle on an edge and something else, or an edge and something that uses the height function h.

Critical points of 1-dimensional sets The only new data we have here corresponds to 0- or 1-saddles on the edges of the foam. Critical points for these sets give the following moves:

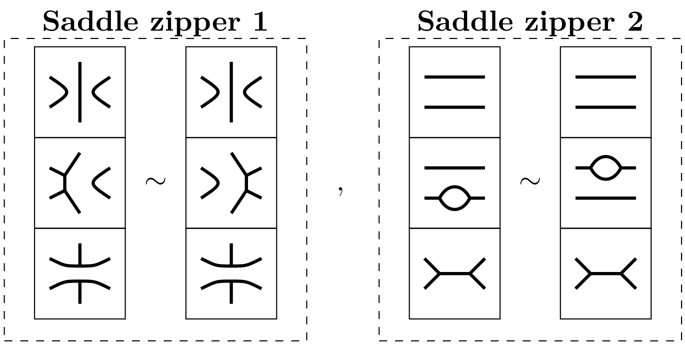


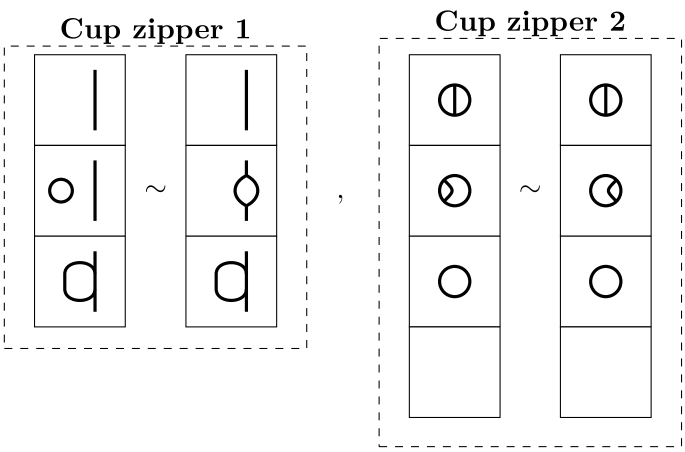
There is no new data of dimension 2, so we do not need to investigate this case.

Let us now turn to transverse intersections between two stratas of the foam. We first look at transverse intersections in a 4-space (3-space plus time) of 1-dimensional objects induced from the 0-data of the foam with 3-dimensional objects induced by the components of dimension 2 of the foam. The only new situation we have to look at is the one induced by transverse intersection between 0- and 1-handles on an edge and a sheet. In that case, we obtain the following moves:

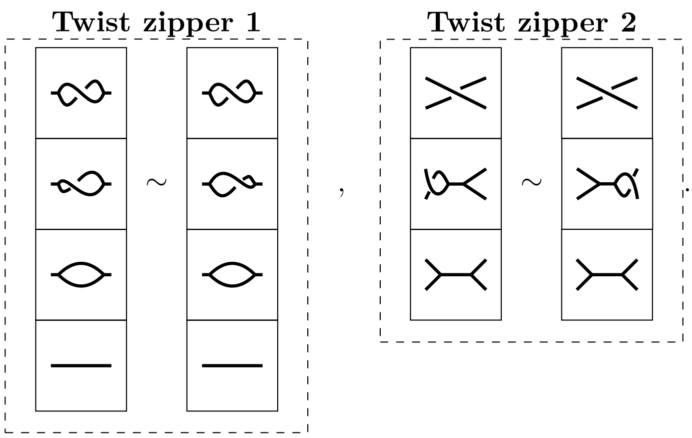


Observe now that some of the data we look at can be constrained on the same sheet. In that case, the intersections we look at live in a 3-dimensional space drawn by the sheet during isotopy, and the transverse case occurs for dimension 2 (edge during isotopy) plus 1 (0, 1 or 2-handle on the sheet). Up to usual symmetries, we obtain the following set of moves:





The only case left is obtained by considering transverse intersections of two 1-dimensional sets constrained on a 2-dimensional one, the only missing situation being given by a twist and a zipper, as depicted below:

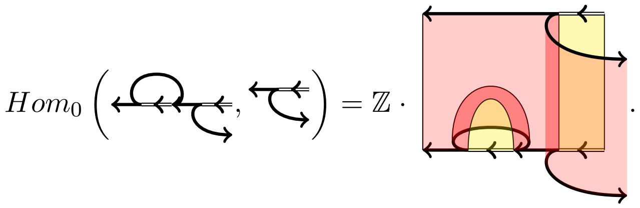


We can now extend Carter's argument concerning the assumptions on general position [Car12, end of Section 6] to conclude the proof. □

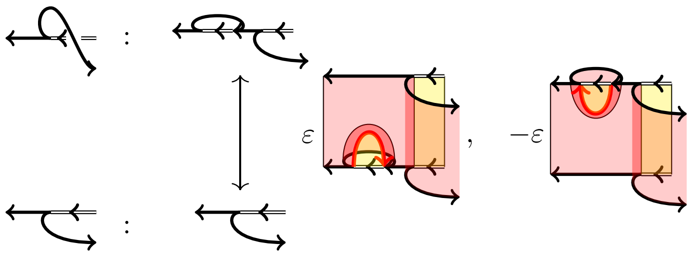
2.3.2.2 Lack of functoriality

We show here that it is impossible to extend Blanchet's version of Khovanov homology into a functorial invariant of knotted webs directly extracted from the previous setting: the move *Twist zipper 1* has no solution.

First, notice that:

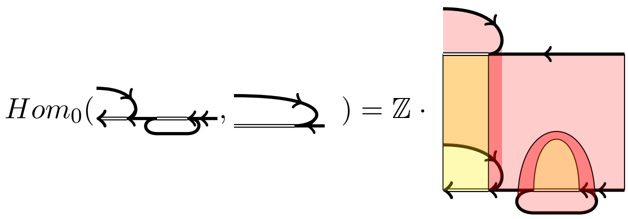


Therefore, the only choice we can make in the following equivalence concerns ε that can be 1 or -1.

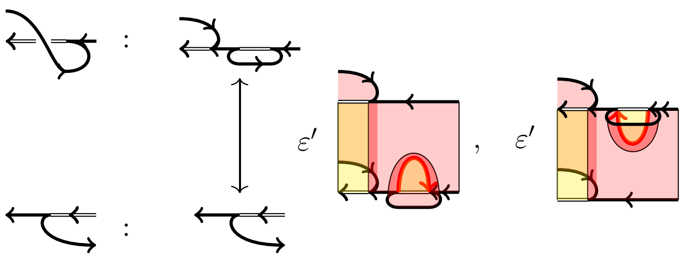


Note that each cobordism comes with a different sign: this is a consequence of the bubble relation 3.3.12.

Similarly,

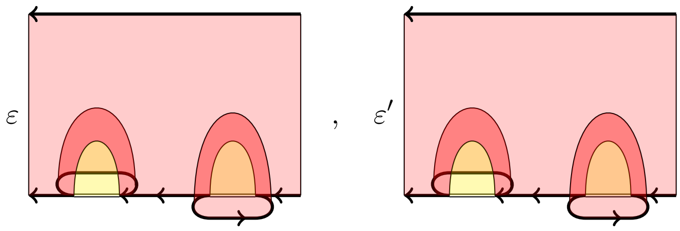


and we have:

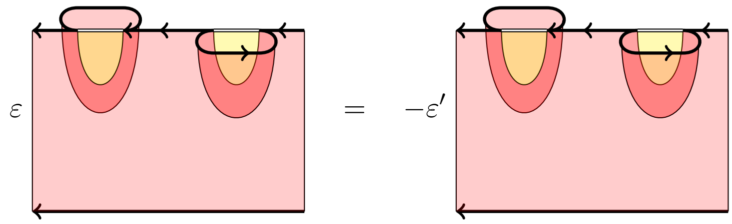


Note that this time, the two cobordisms come with the same sign. This is a consequence of Formula 3.3.12, where the orientation of the seams differs from the previous one.

Then, the LHS and the RHS of the move *Twist zipper 1* give the two following cobordisms:



from which we conclude that $\varepsilon = \varepsilon'$. But the same move read in the other direction (from top to bottom) requires the equality of the following cobordisms:



which imposes $\varepsilon = -\varepsilon'$. We have a contradiction.

This tells us that Blanchet's version of Khovanov homology is not functorial for usual web-tangles. However, we may want to impose some more structure in order to reduce the set of moves. A first observation is that if we consider foams with both an orientation of seams and a local ordering of the facets along a seam (note that the right-hand rule do not work in a 4-dimensional space), this data is not preserved by the twist move: we will first discuss the moves that preserve this structure.

A second class of webs we may want to study consists on framed web-tangles (note that we also use them in Chapter 4): in this case, the Reidemeister I move is replaced by its framed analogue R_I', and we will investigate the situation for the twist.

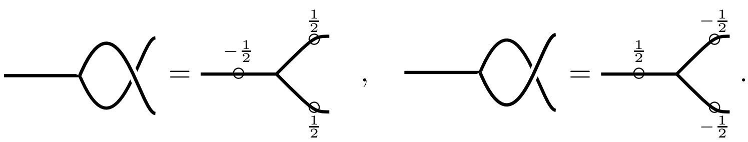
2.3.2.3 Moves and movie-moves for framed web-tangles

Among Reidemeister and Reidemeister-Kauffman moves for webs, all but Reidemeister I and twist move preserve the framing. Note that both of these moves commute with all other ones, so that they can be driven up to the end of any sequence.

We intend to study both an analogue of Reidemeister moves for framed web-tangles and for foams with oriented seams and local ordering among facets.

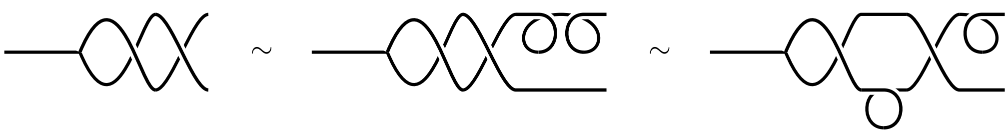
Lemma 2.3.3. Any two twists for a same vertex can be reduced in terms of framed moves.

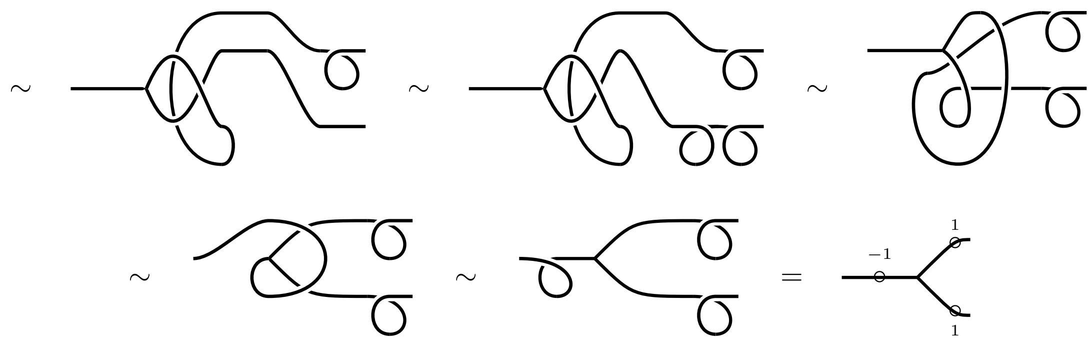
Proof. Recall that in a framed version, we have:



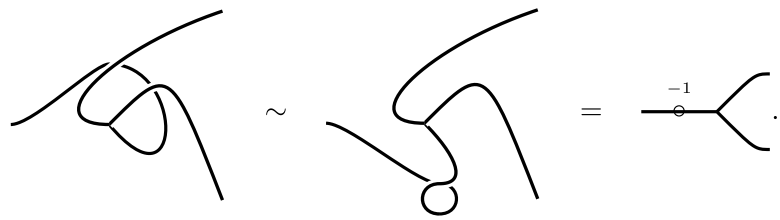
We can then investigate all possible pairs of twists on a vertex, and reduce them to a composition of framed Reidemeister I moves, and other framed Reidemeister and Reidemeister-Kauffman web-moves. We present two cases, the other ones can be solved similarly.

We first consider a case where the two twists occur on the same pair of strands.





Then we deal with an example of two twists on two different pairs.



□

Two isotopic web-tangles with local ordering around the vertices have in particular isotopic underlying web-tangles in the first broad sense, so are related through a sequence of web-moves. The previous lemma tells us that we can actually replace any pair of twist moves around a vertex by a composition of other elementary moves, which means that we can assume that there is at most on twist per vertex. However, since a single twist does not preserve the local ordering, this means that we can actually eliminate all twists. This drives us to the following proposition:

Proposition 2.3.4. Any two isotopic web-tangles with local orderings are related through a sequence of moves which are either usual Reidemeister moves or the framed Reidemeister-Kauffman web-move.

Then, it is a direct analysis to see that all movie-moves for this class of foams are exactly the ones investigated previously, deleting among them the ones that involve a twist.

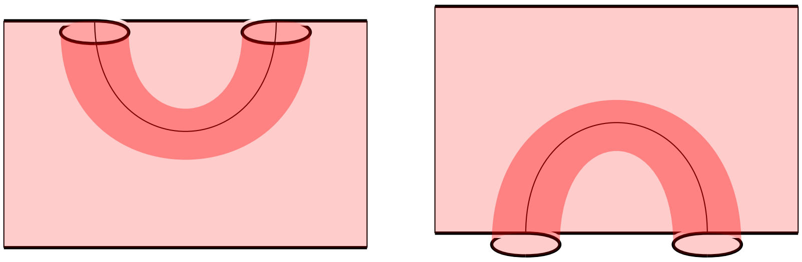
The same story can be performed in the framed case: Lemma 2.3.3 tells us as before that we can assume that there is at most on twist per vertex. However, since vertex twists are the only moves to cause half framing twists, any edge adjacent to a vertex where we perform a twist must actually be involved in two twists, so there must be one twist on each vertex of the corresponding component. This in fact corresponds to flipping the whole connected component of the web-tangle, which reverses the framing and is not permitted: thus, any two isotopic web-tangles are related through a sequence of usual Reidemeister moves and the only remaining framed web-move. Similar framing arguments as in the knot case then drive us to replace R_I by its framed analogue R_I' as drawn in Equation 2.3.2, and we have:

Proposition 2.3.5. Any two isotopic framed web-tangles are related through a sequence of moves which are either framed Reidemeister moves or the framed Reidemeister-Kauffman web-move.

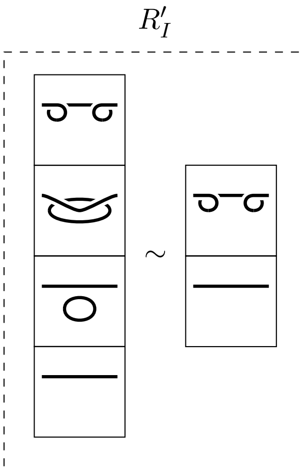
Let us now turn toward movie moves in the framed case.

The foams we are looking at are supposed to have a framed structure with respect to the height function. In other words, we want any slice (regular or singular) to be blackboard-framed. The previous lemma tells us that elementary foams diagrams are birth and death of a circle, saddles, zippers, framed Reidemeister moves and the only Reidemeister-Kauffman move.

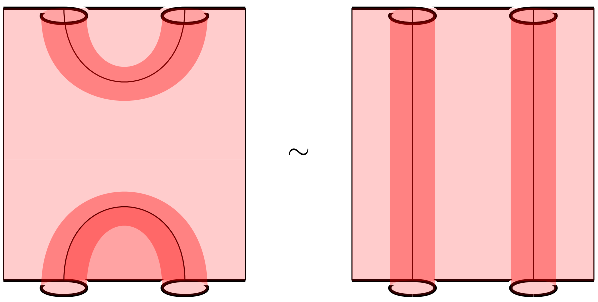
Moves are all usual moves that relate all of these elementary foams, except R_I' which is new and requests some more investigation. Let us depict a possible version of a foam corresponding to realizing such a move[5].



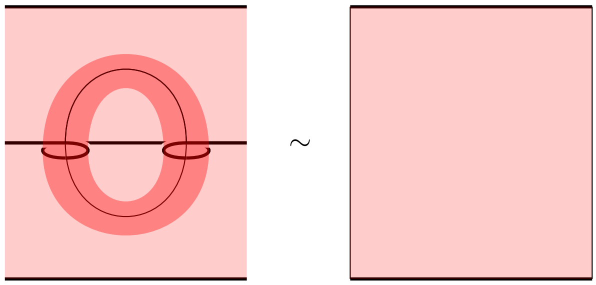
We observe that this foam is actually generated by other elementary moves, namely: a cup, followed by an R_{II} move, and a saddle.



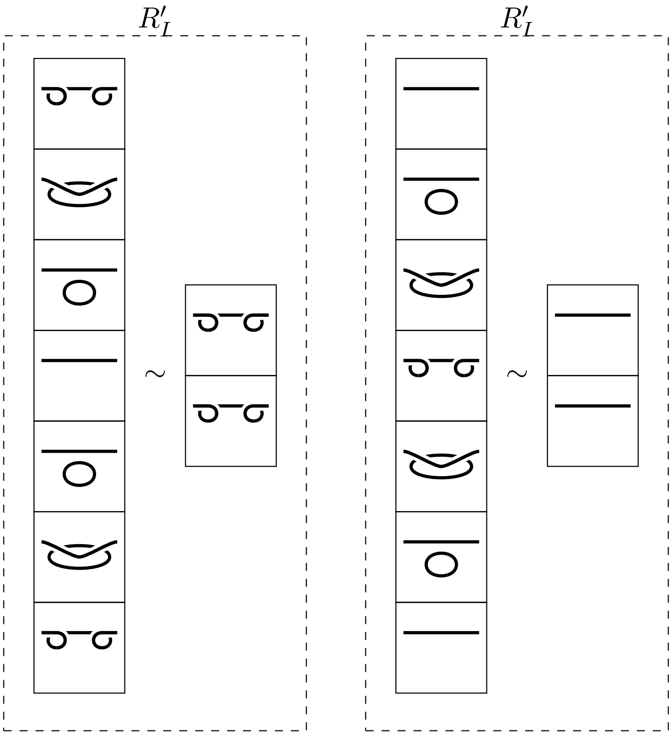
Therefore, all the functoriality requirement is already encoded in all moves from Lemma 2.3.2, and we only need to request the invertibility of the framed R_I'-move:



5. Note that the corresponding movie-move modifies the topology of the knot. The elementary cobordism that would be drawn by untying a rope is different, but doesn't preserve at all time the framing: see Chapter 1 for more details. We actually only care about finding an homotopy equivalence between smoothings for R_I' ; we will show that this choice of foam satisfies this criterion.



The above equivalences induce the following movie-moves:

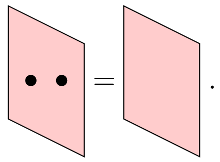


For checking functoriality, one should *a priori* compute the cobordisms associated to all new movie-moves, that is:

– Usual movie moves MM_1 to MM_{15} with at least one 2-labeled strand (since the ones with only 1-labeled strands are checked by Blanchet [Bla10]).
– Carter's Roseman-type move from [Car12] that do not involve twists.
– New movie-moves induced by the height function which do not involve twists: Zipper zig zag, Sliding zipper, Sauron, Saddle zipper 1 and 2, Cup zipper 1 and 2.
– New movie-moves for R'_I, as drawn above.

This would request to draw all orientations in all possible cases of these moves (with 1 and 2-strands), and compute the complex associated to both ends as well as the morphism induced by the succession of moves.

An argument used by Blanchet in the original proof helps reducing the computation in the usual knot case, and can be partially adapted to our extended situation. Recall that there exists a filtered version of Khovanov homology [Lee05, Ras10] which can be built with the exact same rules, but with a new formula for the cancellation of two dots:

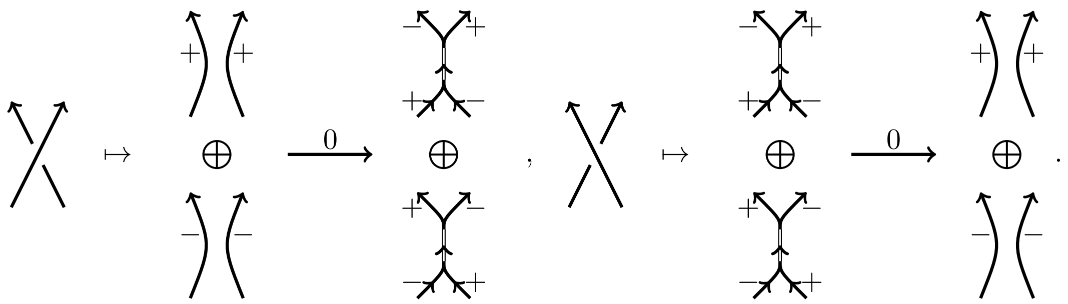


This also induces an homological invariant of knots, and furthermore there is a spectral sequence whose second page is Khovanov homology and which converges to Lee-Rasmussen filtered version of Khovanov homology.

Let us also recall about Bar-Natan's simplicity argument [BN05]: any of the movie-moves can be proven to induce an equality up to a global sign just by studying the dimensions of the degree-zero morphisms of its top and bottom objects. The same argument extends to our generalized case [6], and we can therefore restrict to check that all signs fit.

The Frobenius algebra induced by Lee-Rasmussen rules is isomorphic to $A' = \mathbb{Z}[X]/(X^2 - 1)$, 1 having degree 2 and X degree -2. This algebra can also be decomposed as $A' = \mathbb{Z}\cdot\frac{1+X}{2} \oplus \mathbb{Z}\cdot\frac{1-X}{2}$. $p_+ = \frac{1+X}{2}$ and $p_- = \frac{1-X}{2}$ are orthogonal idempotents, with $p_+ + p_- = id$. If one works in the karoubian completion of the category of $\mathfrak{sl}_2$-foams, identity morphisms with each 1-labeled facet decorated with p_+ or p_- will give projectors. Note that along a seam, the decoration of each of the 1-labeled facets must be different: otherwise, the dot migration relation imposes that this piece of foam is zero.

Locally, when smoothing a crossing, we can identify which objects in the karoubian completion may survive in homology or not. We draw below the surviving subcomplex of each of the $1-1$ smoothings (other ones are not affected):



In the above pictures, the objects are to be understood as the corresponding webs together with the identity marked with the idempotents p_+ or p_- depending on the sign written near each 1-labeled edge.

The above pictures show that the only (possibly) surviving objects are the ones induced by a marked knot diagram: any of the possible smoothings can be seen as a smoothing of a knot diagram marked with idempotents. In the case of knots, there are only two solutions: the edge can be marked either by $+$ or by $-$: these are the two generators of Lee-Rasmussen version of Khovanov homology.

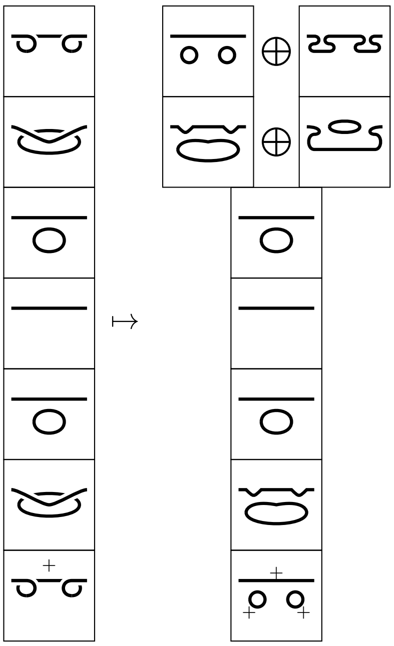
Choosing one generator (which can be done by cuping any circle and apply to it p_+), one can then explicitly compute locally the action of each of the generating foams, and check all signs of the movie-moves. This checking turns out to be enough to conclude:

6. Another argument is to say that over $\mathbb{Z}/2\mathbb{Z}$, Blanchet's version of Khovanov homology is exactly Khovanov's original one. In the $\mathbb{Z}/2\mathbb{Z}$ case, the 2-labeled strands can be erased without any change, and all movie-moves become either identities or usual movie-moves, which are known to hold over $\mathbb{Z}/2\mathbb{Z}$ (this is Bar-Natan's simplicity argument for knots). They can thus be pulled back up to sign in the original Blanchet's category.

a signed equality between two morphisms for Khovanov homology induces through the spectral sequence the same signed equality for Lee-Rasmussen version.

A part of the previous argument can be extended to our case: Lee-Rasmussen version of Khovanov homology extends at no cost to the knotted webs case, and the same spectral sequence as before exists. We can thus check all movie-moves in the karoubian completion of Lee-Rasmussen foam category. However, an argument based on checking only the image of a generator will not work the same way anymore: while there is a canonical way to assign locally a generator to a tangle, it is harder to do it for web-tangles. We rather need to choose any projector which we know to induce a non-zero space in at least some cases, and check explicitly that the movie move can be verified with the right sign over this object.

We illustrate the process by checking the framed Reidemeister I move. We will look at the move from bottom to top, and we compute the cobordisms over the surviving part associated to p_+.



$$\frac{1}{2}p_+ \otimes (1 \otimes X + X \otimes 1) + (\frac{1}{2}p_+ \otimes p_- \otimes p_-)$$
$$= (p_+ \otimes p_+ \otimes p_+ - p_+ \otimes p_- \otimes p_-) + (\frac{1}{2}p_+ \otimes p_- \otimes p_-)$$
$$\frac{1}{2}p_+ \otimes 1 + \frac{1}{2}p_+ \otimes 1$$
$$\frac{1}{2}p_+ \otimes 1$$
$$\frac{1}{2}p_+$$
$$p_+ \otimes p_+$$
$$p_+ \otimes p_+$$
$$p_+ \otimes p_+ \otimes p_+$$

Since $p_+ \otimes p_- \otimes p_-$ and $\frac{1}{2}p_+ \otimes p_- \otimes p_-$ die in homology, we are only left with an identity cobordism.

A complete check of all cases for these moves by brute force is unfortunately extremely long. However, it may be interesting to adapt algorithms computing Khovanov's homology to the $\mathfrak{sl}_2$-webs case [Lew12], which would probably provide an answer in a reasonable amount of time.

2.3.3 Comments and perspectives

The generalized question of precisely defining a $\mathfrak{sl}_n$ Khovanov-Rozansky homology for a category of webs and foams won't be completed here. However, some steps will be given in Chapters 3 and 4.

Two difficulties arise in this question. The first one is to check all Reidemeister relations and their extensions. This question is answered at the uncategorified level in Chapter 4, where we use key relations for the braidings from [Lus93] to prove the general $\mathfrak{sl}_n$ case. We could try to categorify these key relations and therefore look for a proof of all $\mathfrak{sl}_n$ cases together. Such categorified relations could be proven in terms of extended thick calculus [KLMS12, Sto11]. This, together with the extension of the skew-Howe construction of Khovanov-Rozansky homology still in progress [LQR] would provide a nice general proof of the invariance under Reidemeister-Kauffman web moves.

A second question, (which we have left as a conjecture above in the $\mathfrak{sl}_2$-case) concerns the functoriality. We provided in the $\mathfrak{sl}_2$ case a set of movie moves, which should be extended to the general $\mathfrak{sl}_n$-case where there is one more elementary foam. A direct proof for any value of n seems to be very difficult. We would rather suggest to try other approaches to solve this problem... perhaps using skew Howe duality?

Part II

Skew Howe duality

Chapter 3

Khovanov homology is a skew Howe 2-representation of categorified quantum $\mathfrak{sl}_m$

Note: This chapter is the result of a joint work with **Aaron D. Lauda** and **David E. V. Rose**, from the *University of Southern California*. We present here *in extenso* the version of the preprint [LQR12].

3.1 Introduction

3.1.1 Categorified knot invariants and quantum groups

One of the original motivations for categorifying quantum groups was to provide a representation theoretic explanation for the existence of Khovanov homology and other link homologies categorifying quantum link invariants. Just as the Jones polynomial is described representation theoretically by the quantum group $U_q(\mathfrak{sl}_2)$ and tensor powers of its two dimensional representation, the categorification of the Jones polynomial via Khovanov homology should be described in terms of the 2-representation theory of the categorified quantum group associated to $U_q(\mathfrak{sl}_2)$.

Currently, the link between categorified quantum groups and Khovanov homology follows the indirect path through Webster's work on categorified tensor products [Web10a, Web10b]. This connection utilizes an isomorphism relating Webster's categorifications of tensor products with categories associated to blocks of parabolic graded category $\mathcal{O}$. Categorifications associated with category $\mathcal{O}$ were initiated by Bernstein, Frenkel, and Khovanov [BFK99] and were further developed in [Str05, FKS06]. The relation to the familiar picture-world [BN02, BN05] of Khovanov homology then relies on several technical results of Stroppel [Str09, Str05] relating the knot homologies constructed using category $\mathcal{O}$ to Khovanov's more elementary construction [Kho00, Kho02]. More generally, for link homology theories associated with fundamental $\mathfrak{sl}_n$ representations, Webster describes an isomorphism relating his construction to Sussan's category $\mathcal{O}$ based link homology theory [Sus07] , which is related via Koszul duality to a theory defined by Mazorchuk and Stroppel [MS09]. When $n = 3$, the latter of these link homologies can then be identified [MS09] with Khovanov's more elementary construction [Kho04] of $\mathfrak{sl}_3$ link homology defined using singular cobordisms called foams.

Alternatively, there is an algebro-geometric construction of Khovanov homology and related $\mathfrak{sl}_n$ link homologies due to Cautis and Kamnitzer [CK08a, CK08b]. These knot homologies arise from derived categories of coherent sheaves on algebraic varieties associated to orbits in the affine Grassmannian. In the $\mathfrak{sl}_2$ case this knot homology agrees with Khovanov homology [CK08a, Theorem 8.2] and these geometric categories can be understood as 2-representations of categorified quantum groups [CL11, CKL10b]. These link homologies are related to those of Seidel-Smith [SS06] and Manolescu [Man07] by mirror symmetry.

In this article, we provide a direct construction of foam based $\mathfrak{sl}_n$ link homology theories for $n = 2$ or $n = 3$ intrinsically in terms of categorified quantum groups. We show that all of the components involved in these knot homologies are already present within the structure of categorified quantum groups including the relations in foam categories and the complexes defining the braiding. Utilizing Cautis-Rozansky categorified clasps [Cau12, Roz10b] we also obtain categorified projectors lifting Jones-Wenzl idempotents and their $\mathfrak{sl}_3$ analogs purely from the higher relations of categorified quantum groups. In the $\mathfrak{sl}_2$ case this work reveals the importance of a modified class of foams introduced by Christian Blanchet [Bla10], suggesting that this version of the foam category is most natural from the perspective of categorified quantum groups. In the $\mathfrak{sl}_3$ case these results suggest a similar modified version of the $\mathfrak{sl}_3$ foam category.

3.1.2 Categorified representation theory

Recall that in categorified representation theory, $\mathbb{C}(q)$-vector spaces V with decompositions into weight spaces $V = \oplus_\lambda V_\lambda$, are replaced by graded categories $\mathcal{V} = \oplus_\lambda \mathcal{V}_\lambda$, and instead of linear maps between spaces, Chevalley generators act by functors $\mathcal{E}_i \mathbf{1}_\lambda \colon \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha_i}$, $\mathcal{F}_i \mathbf{1}_\lambda \colon \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_i}$ satisfying quantum Serre relations up to isomorphism of functors. The higher structure of categorified representation theory appears at the level of natural transformations between these functors. In most instances when $U_q(\mathfrak{sl}_n)$ admits a categorical action of this form, the natural transformations that appear between functors are predictable and can be systematically described. A key part of this structure is that $\mathcal{F}$ is a left and right adjoint for $\mathcal{E}$ and that the endomorphisms of $\mathcal{E}^a$ are acted upon by the so called KLR algebras developed in [CR08, KL09, KL11a, Rou08].

In [Lau08, KL10] it was suggested that the full structure of categorical representations of $U_q(\mathfrak{sl}_n)$ is described by a 2-functor from an additive 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$. This 2-category categorifies Lusztig's modified version $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ [Lus90] of the quantum group $U_q(\mathfrak{sl}_n)$. The objects of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ are indexed by the weight lattice of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, 1-morphisms correspond to the elements of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, and the 2-morphisms govern the natural transformations that appear in categorical representations. However, the 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ has additional relations on 2-morphisms beyond specified adjoints and KLR relations. We refer to the collection of relations on 2-morphisms as "higher relations" because they can be viewed as replacements for the quantum Serre relations. Indeed, these higher relations give rise to explicit isomorphisms lifting the defining relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, while simultaneously controlling the Grothendieck group of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$, allowing for a $\mathbb{Z}[q,q^{-1}]$-algebra isomorphism between its split Grothendieck ring $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ and the integral version of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$. Under this isomorphism, the images of indecomposable 1-morphisms from $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ map to the canonical basis of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ [Lau08, Web12].

Here we show that these higher relations also encode the information needed to construct *all* $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ knot homology theories in a framework where computations are accessible.

3.1.3 Braidings via skew Howe duality.

The key insight for our elementary construction of knot homologies from categorified quantum groups is the fundamental observation of Cautis, Kamnitzer and Licata that the R-matrix describing the braiding in an m-fold tensor product of fundamental representations of $U_q(\mathfrak{sl}_n)$ in Reshetikhin-Turaev link invariants can be obtained from a deformed Weyl group action associated with $U_q(\mathfrak{sl}_m)$ [CKL10a].

Recall that the Weyl group W of a simply-laced Kac-Moody algebra $\mathfrak{g}$ is a finite Coxeter group associated to the root system of $\mathfrak{g}$. Passing from $U(\mathfrak{g})$ to $U_q(\mathfrak{g})$, the Weyl group deforms to a braid group of type $\mathfrak{g}$, which acts on $U_q(\mathfrak{g})$-modules. In the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$, the Weyl group $W = \mathfrak{S}_2$ deforms to the braid group B_2 giving a reflection isomorphism $T\colon V_\lambda \to V_{-\lambda}$ between weight spaces of a $U_q(\mathfrak{sl}_2)$-module. This action can be expressed in a completion of the idempotented quantum algebra $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ by the power series

$$T1_\lambda = \sum_{s\geq 0}(-q)^s F^{(\lambda+s)}E^{(s)}1_\lambda \quad \lambda \geq 0, \qquad\qquad T1_\lambda = \sum_{s\geq 0}(-q)^s E^{(-\lambda+s)}F^{(s)}1_\lambda \quad \lambda \leq 0. \tag{3.1.1}$$

On any finite-dimensional representation, $T1_\lambda$ can be expressed as a finite sum. When $\mathfrak{g} = \mathfrak{sl}_m$ and $W = \mathfrak{S}_m$, there are analogous maps $T_i 1_\lambda$ for each $1 \leq i \leq m-1$ satisfying the braid relations.

Cautis, Kamnitzer, and Licata related the braiding of fundamental $U_q(\mathfrak{sl}_n)$ representations to the Weyl group action using a version of Howe duality for exterior algebras they called skew Howe duality [CKL10a]. The key idea is to study quantum exterior powers. Denote by $\mathbb{C}_q^n$ the standard $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module with basis denoted $x_1, \dots, x_n$. The quantum exterior algebra is the $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module defined as

$$\textstyle\bigwedge_q^\bullet(\mathbb{C}_q^n) = \mathbb{C}(q)\langle x_1, \dots, x_n\rangle/(x_i^2,\ x_ix_j + qx_jx_i \ \text{ for } i<j).$$

By assigning degree one to each x_i the quantum exterior algebra is a graded $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module whose homogenous subspace of degree N is denoted by $\bigwedge_q^N(\mathbb{C}_q^n)$.

The space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ admits commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ which constitute a Howe pair. For example, when $m=2$ the space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$ decomposes into $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ weight spaces as

$$\textstyle\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2) \cong \bigwedge_q^N(\mathbb{C}_q^n \oplus \mathbb{C}_q^n) \cong \bigoplus_{a+b=N} \bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n),$$

where the weight of a summand $\bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n)$ is $\lambda = b-a$. The action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ is given by maps

$$\begin{aligned} E1_\lambda &\colon \textstyle\bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \to \bigwedge_q^{a-1}(\mathbb{C}_q^n) \otimes \bigwedge_q^{b+1}(\mathbb{C}_q^n), \\ F1_\lambda &\colon \textstyle\bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \to \bigwedge_q^{a+1}(\mathbb{C}_q^n) \otimes \bigwedge_q^{b-1}(\mathbb{C}_q^n). \end{aligned} \tag{3.1.2}$$

For more details on quantum skew Howe duality see [CKM12, Cau12].

The Weyl group action gives an isomorphism between the λth and $-\lambda$th weight spaces of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$.

$$\begin{tikzcd}
\cdots & \bigwedge_q^{b+1}(\mathbb{C}_q^n)\otimes \bigwedge_q^{a-1}(\mathbb{C}_q^n) \arrow[r, bend left, "E"] & \bigwedge_q^{b}(\mathbb{C}_q^n)\otimes \bigwedge_q^{a}(\mathbb{C}_q^n) \arrow[l, bend left, "F"] \arrow[rr, bend left, "T"] & \cdots & \bigwedge_q^{a}(\mathbb{C}_q^n)\otimes \bigwedge_q^{b}(\mathbb{C}_q^n) \arrow[ll, bend right, "T"'] \arrow[r, bend left, "E"] & \bigwedge_q^{a-1}(\mathbb{C}_q^n)\otimes \bigwedge_q^{b+1}(\mathbb{C}_q^n) \arrow[l, bend left, "F"] & \cdots \\
& -\lambda-2 & -\lambda & & \lambda & \lambda+2 &
\end{tikzcd}$$

Since $\mathbb{C}_q^n$ is the defining representation of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, the quantum exterior powers $\bigwedge_q^a(\mathbb{C}_q^n) = V_{\omega_a}$ correspond to fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-representations, where ω_a for $1 \leq a \leq n-1$ are the fundamental weights of $\mathfrak{sl}_n$. The deformed reflection isomorphism

$$V_{\omega_a} \otimes V_{\omega_b} \cong \textstyle\bigwedge_q^a(\mathbb{C}_q^n) \otimes \bigwedge_q^b(\mathbb{C}_q^n) \xrightarrow{\quad T \quad} \bigwedge_q^b(\mathbb{C}_q^n) \otimes \bigwedge_q^a(\mathbb{C}_q^n) \cong V_{\omega_b} \otimes V_{\omega_a}.$$

gives a braiding of fundamental representations that agrees with the R-matrix in the Reshetikhin-Turaev construction [CKL10a] (up to a power of $\pm q$). The key advantage of this realization of the R-matrix in terms of skew Howe duality is that it suggests a procedure for categorification.

3.1.4 Knot homology from categorical skew Howe duality

Following the ideas of Chuang and Rouquier [CR08] (see also [CK12a]), one can define a categorification of the reflection isomorphism $T1_\lambda \colon V_\lambda \to V_{-\lambda}$ using the 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ categorifying $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$. Passing to the category of complexes $Kom(\dot{\mathcal{U}}_Q(\mathfrak{sl}_2))$, it is possible to define a complex $\mathcal{T}\mathbf{1}_\lambda$ of 1-morphisms

$$\mathcal{E}^{(-\lambda)}\mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{E}^{(-\lambda+1)}\mathcal{F}\mathbf{1}_\lambda\{1\} \xrightarrow{\quad d_2 \quad} \mathcal{E}^{(-\lambda+2)}\mathcal{F}^{(2)}\mathbf{1}_\lambda\{2\} \longrightarrow \quad \cdots \quad \longrightarrow \mathcal{E}^{(-\lambda+k)}\mathcal{F}^{(k)}\mathbf{1}_\lambda\{k\} \xrightarrow{d_{k+1}} \quad \cdots \tag{3.1.3}$$

for $\lambda \leq 0$ and a similar complex for $\lambda \geq 0$ (compare with (3.1.1)). The differentials in this complex can be explicitly defined using the 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$. Verification that $d^2 = 0$ follows from the relations in the 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$; the enhanced graphical calculus from [KLMS12] is useful for this computation.

Given a 2-representation $\mathcal{V}$ of the 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ with weight decomposition into abelian categories $\mathcal{V}_\lambda$, the functor of tensoring with the complex $\mathcal{T}\mathbf{1}_\lambda$ gives rise to derived equivalences $\mathcal{T}\mathbf{1}_\lambda \colon D(\mathcal{V}_\lambda) \to D(\mathcal{V}_{-\lambda})$. The resulting derived equivalences are highly non-trivial and have led to the resolution of several important conjectures [CR08, CKL10b, CKL09]. Our interest in these equivalences stems from their application to knot homology theory. Given a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^2)$ with commuting categorical actions of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ and $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$, the categorified braid group action gives a categorification of the R-matrix. More generally, one can categorify the braid group action on an m-fold tensor product of $U_q(\mathfrak{sl}_n)$ representation using the categorified braid group action coming from the deformed Weyl group action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ [CK12a].

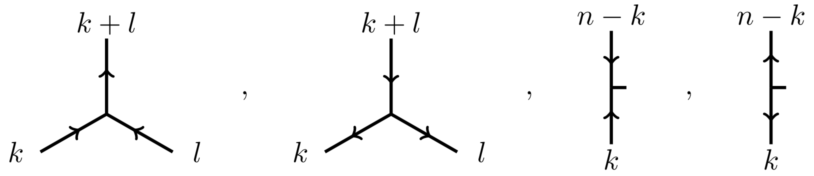
In fact, Cautis and Kamnitzer's algebro-geometric construction of Khovanov homology [CK08a] and $\mathfrak{sl}_n$ link homology [CK08b] can be understood in this framework. Their invariants arise from a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ defined from derived categories of coherent sheaves on varieties related to orbits in the affine Grassmannian [Cau12, Theorem 2.6].

3.1.5 Reinterpreting $\mathfrak{sl}_n$ skein theory using skew Howe duality.

While categorifications of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ defined via derived categories of coherent sheaves are far from elementary, it turns out that this story has a more combinatorial description. In the decategorified case, the usual skein theory description of $\mathfrak{sl}_n$ link invariants in terms of MOY-calculus [MOY98] can also be understood in terms of skew Howe duality.

Recall that an $\mathfrak{sl}_n$ web is a graphical presentation of intertwiners between tensor products of fundamental representations of $U_q(\mathfrak{sl}_n)$. When $n = 2$, the calculus of $\mathfrak{sl}_2$ webs is described by the Temperley-Lieb algebra; Kuperberg described the $n = 3$ case using a graphical calculus of oriented trivalent graphs [Kup96] which depict the morphisms in a combinatorially defined pivotal category called the $\mathfrak{sl}_3$ spider. These descriptions have recently been generalized by Cautis, Kamnitzer, and Morrison [CKM12] to general n, building on earlier work of Kim [Kim03] and Morrison [Mor07]. We briefly summarize this construction, referring the reader to their work for the details.

The category $n\mathbf{Web}$ is the pivotal category whose objects are sequences in the symbols $\{1^\pm, \ldots, (n-1)^\pm\}$. Morphisms are oriented graphs with edges labeled by $\{1, \ldots, n-1\}$ generated by the following:

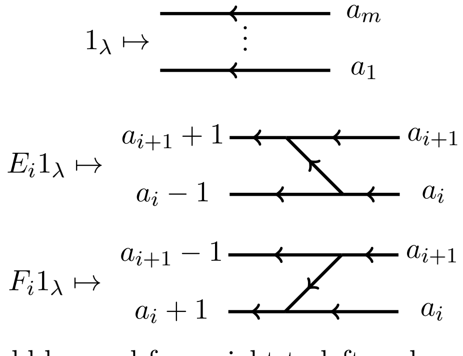


(3.1.4)

where a strand labeled by k is directed out from the label k^+ and into the label k^- in the domain, and vice versa in the codomain. These graphs, called $\mathfrak{sl}_n$ webs, are considered up to isotopy (relative to their boundary) and local relations. The category $n\mathbf{Web}$ can be identified with the full subcategory of $U_q(\mathfrak{sl}_n)$ representations generated (as a pivotal category) by the fundamental representations by identifying the symbol k^+ with $\bigwedge_q^k(\mathbb{C}_q^n)$ and identifying k^- with its dual. Sequences correspond to tensor products of the relevant representations.

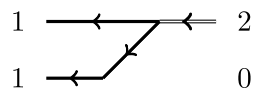
The connection to skew Howe duality is given by considering a related family of m-sheeted web categories. Let $n\mathbf{Web}_m(N)$ denote the category whose objects are sequences $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ with $0 \le a_i \le n$ and $\displaystyle\sum_{i=1}^m a_i = N$. Note that here we allow the symbols 0 and n in the object sequences, but none of the dual symbols k^-. As above, these labels should be interpreted as representations $\bigwedge_q^k(\mathbb{C}_q^n)$ for $0 \le k \le n$ with $\bigwedge_q^0(\mathbb{C}_q^n) = \bigwedge_q^n(\mathbb{C}_q^n) = \mathbb{C}(q)$ corresponding to the trivial representation. Morphisms in $n\mathbf{Web}_m(N)$ are given by $\mathfrak{sl}_n$ webs mapping between the symbols $a_i \neq 0, n$ in each sequence.

Via skew Howe duality, the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ gives morphisms between tensor products of fundamental representations. This map has a graphical interpretation described in [CKM12] using "ladder diagrams" to represent webs:



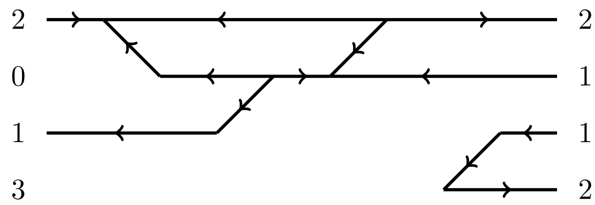
where these diagrams should be read from right to left and we omit $m-2$ lines in each of the latter two diagrams (compare with (3.1.2)). The sequences on the right are determined by the $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \ldots, \lambda_{m-1})$ by $\lambda_i = a_{i+1} - a_i$; edges connected to the label 0 should be deleted and those connected to the label n should be truncated to the "tags" depicted in the last two diagrams in equation (3.1.4).

In this paper, we categorify Cautis, Kamnitzer, and Morrison's construction for the cases $n = 2$ and $n = 3$. In fact, for the $\mathfrak{sl}_2$ case we work with related categories $2\mathbf{BWeb}_m(N)$ when we allow strands labeled by 2 and no longer require the tag morphisms [1]. For example in $2\mathbf{BWeb}_2(2)$ we have the morphism:



where  depicts a 1-labeled edge and  depicts a 2-labeled edge.

In the $\mathfrak{sl}_3$ case, we continue to work with $3\mathbf{Web}_m(N)$, although this category has a simpler description than the one given above. Since $\bigwedge_q^2(\mathbb{C}_q^3)$ can be canonically identified with the dual of $\bigwedge_q^1(\mathbb{C}_q^3)$ we can replace 2-labeled edges with 1-labeled edges oriented in the opposite direction and do away with the tag morphisms. For example, the diagram:

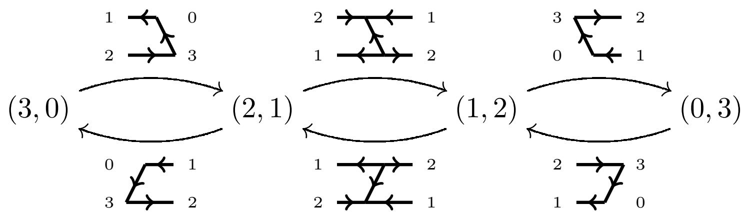


depicts a morphism in $3\mathbf{Web}_4(6)$. We will later also consider a (categorified) version of this category in which we retain 3-labeled edges.

We will now exhibit the power of the skew Howe approach to diagrammatic representation theory (which hints at the utility of its categorified counterpart) in an example. The decomposition of $\bigwedge_q^3(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$ into $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ weight spaces gives:

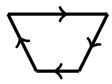
$$\begin{tikzcd} \bigwedge_q^3(\mathbb{C}_q^3)\otimes\bigwedge_q^0(\mathbb{C}_q^3) \arrow[r, bend left, "E1_{-3}"] & \bigwedge_q^2(\mathbb{C}_q^3)\otimes\bigwedge_q^1(\mathbb{C}_q^3) \arrow[l, bend left, "F1_{-1}"] \arrow[r, bend left, "E1_{-1}"] & \bigwedge_q^1(\mathbb{C}_q^3)\otimes\bigwedge_q^2(\mathbb{C}_q^3) \arrow[l, bend left, "F1_{1}"] \arrow[r, bend left, "E1_{1}"] & \bigwedge_q^0(\mathbb{C}_q^3)\otimes\bigwedge_q^3(\mathbb{C}_q^3) \arrow[l, bend left, "F1_{3}"] \end{tikzcd}$$

or diagrammatically:

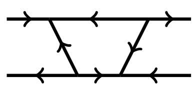
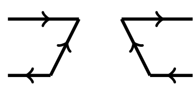
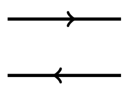


where we again read the webs from right to left in the above.

The local relations for $\mathfrak{sl}_3$ webs from [Kup96] can be deduced from the fact that the above is an $\mathfrak{sl}_2$ representation. Indeed, the $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ relation $EF1_3 = FE1_3 + [3]1_3$ gives the "circle" relation:

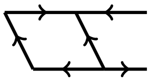
 $= [3]$

and the "square" relation:

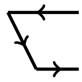
 $=$  $+$ 

1. Here the "B" stands for Blanchet; this category is related to a decategorification of his work [Bla10].

follows from the relation $EF1_1 = FE1_1 + [1]1_1$. The above diagrammatics extends to a description of the action by the integral version ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_2)$ on $\bigwedge_q^3(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$ where divided powers $E^{(k)} := E^k/[k]!$ act by ladder web diagrams with diagonal lines labeled by k. In the above example, the divided power relation $E^2 1_{-1} = [2]E^{(2)}1_{-1}$ gives the remaining "bigon" relation

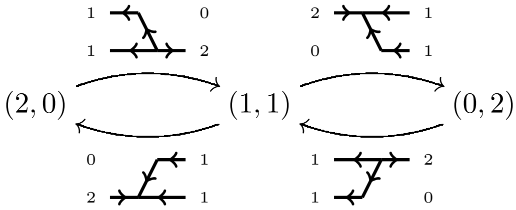


$= [2]$

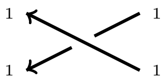


from [Kup96].

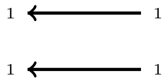
The skein theoretic definition of the braiding can also be constructed from skew Howe duality using the deformed Weyl group action. For example, the braiding for edges labeled by the standard $\mathfrak{sl}_3$ representation can be recovered from the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ on $\bigwedge_q^2(\mathbb{C}_q^3 \otimes \mathbb{C}_q^2)$. The decomposition into weight spaces is given diagrammatically by:



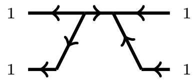
and the Weyl group action (3.1.1) on the 0-weight space gives the braiding:



$=$



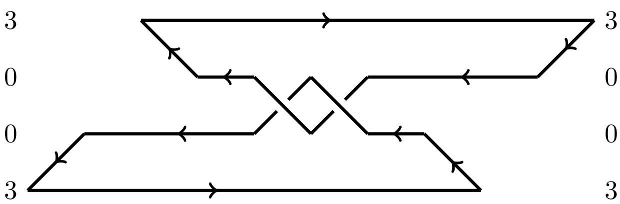
$- \; q$



since $T1_0 = 1_0 - qFE1_0$. Up to a power of q, this recovers the formula for the positive crossing from [Kup96]; the negative crossing can be recovered by considering $T^{-1}1_0 = 1_0 - q^{-1}EF1_0$.

In a similar manner, one can recover the $\mathfrak{sl}_2$ skein theory (i.e. the Kauffman bracket) from the action of ${}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $2\mathbf{Web}_m(N)$. In fact, Cautis, Kamnitzer, and Morrison use this approach to deduce the $\mathfrak{sl}_n$ web relations for $n \geq 4$. One can use their setup to give a combinatorial description of $\mathfrak{sl}_n$ link invariants labeled by any fundamental representation of $U_q(\mathfrak{sl}_n)$.

Moreover, one may realize the invariant of a link (or tangle) as the image of an element in $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ under the (appropriate) skew Howe map. For example, the $\mathfrak{sl}_3$ invariant of the Hopf link



is the element in

$$\mathrm{End}_{\mathfrak{sl}_3}(\textstyle\bigwedge_q^3(\mathbb{C}_q^3) \otimes \bigwedge_q^0(\mathbb{C}_q^3) \otimes \bigwedge_q^0(\mathbb{C}_q^3) \otimes \bigwedge_q^3(\mathbb{C}_q^3)) \cong \mathbb{C}(q)$$

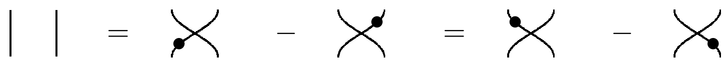
given by the action of the element $F_1E_3T_2^2E_1F_3 1_{(-3,0,3)} \in \dot{\mathbf{U}}_q(\mathfrak{sl}_4)$ on $\bigwedge_q^6(\mathbb{C}_q^3 \otimes \mathbb{C}_q^4)$.

3.1.6 Foamation functors for knot homologies

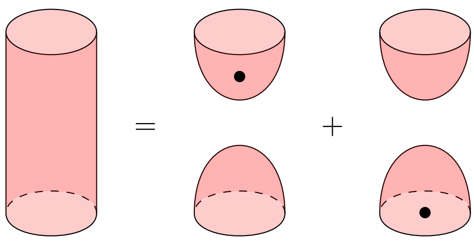
The observations from the previous section suggest an approach to obtaining diagrammatic $\mathfrak{sl}_n$ link homologies using categorical skew Howe duality. In his work categorifying the $\mathfrak{sl}_3$ polynomial, Khovanov utilized certain singular web cobordisms called foams [Kho04]. In [MSV09] these singular surfaces were generalized to the $\mathfrak{sl}_n$ case to supply a diagrammatic counterpart of Khovanov-Rozansky homology [KR08a, KR08b, MV08]. These foams also appear to be connected with category $\mathcal{O}$ [MS09] and with Soergel bimodules [Vaz10]. However, unlike Khovanov's construction for $\mathfrak{sl}_3$, there is no known finite set of relations on $\mathfrak{sl}_n$ foams for $n > 3$ that guarantee any closed foam can be evaluated to an element of the ground ring. For general $\mathfrak{sl}_n$, matrix factorizations become the primary computation tool [Yon11, Yon08, Wu09], and the only way to evaluate a closed foam is through the mysterious Kapustin-Li formula [MSV09, DM12]. For foams this formula was discovered by Khovanov and Rozansky [KR07] generalizing work of the physicists Vafa [Vaf91], Kapustin, and Li [KL03]. It arises from the topological Landau-Ginzburg model associated to components of the foam. A purely combinatorial foam construction of $\mathfrak{sl}_n$ link homology remains an important open problem.

Foams can be viewed as a categorification of webs. Indeed, this point of view motivates our approach to constructing $\mathfrak{sl}_n$ link homologies for $n = 2$ and $n = 3$. In section 3.3 we describe 2-categories of m-sheeted $\mathfrak{sl}_n$ foams categorifying the above web categories. We define 2-functors $\Phi_n \colon \dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to n\mathbf{Foam}_m(N)$ for $n = 2$ and $n = 3$. The existence of such functors was predicted by Khovanov and previously defined by Mackaay in the $n = 3$ case working in the restrictive setting of $\mathbb{Z}/2\mathbb{Z}$ coefficients in [Mac09] where he called them "foamation" 2-functors.

Here we reinterpret Mackaay's work (and extend it to the $\mathfrak{sl}_2$ case) using skew Howe duality, defining foamation functors for $n = 2, 3$ with integer coefficients. Working over $\mathbb{Z}$, it is not obvious for $n = 2$ how to connect categorified quantum groups with the Bar-Natan's foam description of Khovanov homology. For example, one of the most basic relations for categorified quantum groups $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ is the nilHecke relation



which should correspond to the so called "neck-cutting" relation



via the foamation 2-functor. However, the signs under this assignment do not match. One can try to rescale the foamation functors, but one quickly finds that there is no way to fix the signs under this assignment.

The difficulty in matching the neck-cutting relation with the nilHecke relation is closely related to the solution of another famous problem related to Khovanov homology. As originally defined, Khovanov homology is a projective functorial invariant, meaning that to a cobordism $f \colon T \to T'$ between two tangles one can assign a map $Kh(f) \colon Kh(T) \to Kh(T')$ between the respective homologies well defined only up to a ± 1 sign [Kho06, Kho02, BN05, Jac04].

Clark, Morrison and Walker [CMW09], and independently Caprau [Cap07, Cap08], showed that the functoriality of Khovanov homology could be fixed by considering modified foam categories. From the representation-theoretic point of view, these foam categories keep track of the fact that the defining representation of $U_q(\mathfrak{sl}_2)$ is non-canonically isomorphic to its dual. Keeping track of this information gives rise to a fully functorial tangle invariant. For both of these fixes to Khovanov homology one must work with foams defined over the Gaussian integers $\mathbb{Z}[i]$.

Christian Blanchet proposed yet another construction fixing the functoriality of Khovanov homology [Bla10]. He works with an enhanced version of the foam category where one labels facets by elements of the set $\{1, 2\}$. The 2-labeled facets are the primary difference from the previous two constructions. The presence of these 2-labeled facets introduces additional signs that are not present in the CMW or Caprau approaches to functoriality. Blanchet's approach gives rise to a functorial version of Khovanov homology defined over the integers [Bla10].

These modified foam categories are quite natural from the representation theoretic viewpoint. In the skew Howe framework, foams naturally provide a representation of $\mathcal{U}_Q(\mathfrak{gl}_n)$. Seen from this perspective, the foams introduced by Blanchet keep track of the difference between the trivial representation $\bigwedge_q^0(\mathbb{C}_q^2)$ and the determinant representation $\bigwedge_q^2(\mathbb{C}_q^2)$. As $U_q(\mathfrak{sl}_2)$ representations there is of course no difference between these two representations, but it appears that Blanchet's approach has additional information that contributes additional signs coming from the 2-labeled facets corresponding to the determinant representation $\bigwedge_q^2(\mathbb{C}_q^2)$.

In this article, we construct foamation functors into both the CMW foam categories as well as the foam categories of Blanchet. To define the functors into the CMW foam categories one must continue working with complex coefficients, while Blanchet's foam categories naturally admit foamation functors defined over the integers. This suggests that Blanchet's approach is the most natural from the perspective of categorified representation theory. It is also interesting to note that the $\mathfrak{sl}_2$ knot homology most closely related to categorified quantum groups is integral and functorial.

It turns out that in the $n = 3$ case it is possible to modify Mackaay's definition of the foamation functors to work over $\mathbb{Z}$, although this requires rather complicated and unnatural sign assignments. Motivated by the $\mathfrak{sl}_2$ case, we consider a modified $\mathfrak{sl}_3$ foam category that incorporates additional 3-labeled facets. To distinguish these foams from the usual $\mathfrak{sl}_3$ foams we call them Blanchet $\mathfrak{sl}_3$ foams. We show that there are 2-functors into Blanchet $\mathfrak{sl}_3$ foams with much more natural sign assignments for the generating 2-morphisms in $\mathcal{U}_Q(\mathfrak{sl}_m)$. There is also a natural construction of a forgetful 2-functor into the usual $\mathfrak{sl}_3$ foams defined intrinsically in terms of the topology of the Blanchet foams. Taking the composite of these 2-functors provides an explanation for the complicated signs occurring in the standard $\mathfrak{sl}_3$ foamation functors.

Checking the relations for the 2-category $\mathcal{U}_Q(\mathfrak{sl}_m)$ needed to define foamation functors is a laborious task. Here we utilize recent results of the first author with Cautis showing that in a 2-representation with finitely many nonzero weight spaces many of the relations come for free [CL11].

An independent construction of the integral foamation functors into the usual $\mathfrak{sl}_3$ foam 2-category was given by Mackaay, Pan, and Tubbenhauer in a recent update to their work in [MPT12]. They utilize the foamation functors for a different application related to a generalization of Khovanov's arc algebra to the $\mathfrak{sl}_3$ setting.

3.1.7 Comparing knot homologies

A careful analysis of Cautis' arguments in [Cau12] reveals that the skew Howe duality approach also supplies a mechanism for equating different constructions of $\mathfrak{sl}_n$ link homologies. Indeed, given any 2-representation of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ whose objects are indexed by the nonzero weights in $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$, and whose endomorphisms of the highest weight object are one dimensional in degree zero and zero dimensional otherwise, one obtains a unique knot homology theory that is formally determined by the relations imposed by the 2-representation. In Section 3.4.2, we show that Khovanov's $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ link homology theories fit into this framework. We also sketch a proof of the $\mathfrak{sl}_3$ case of Conjecture 6.4 from [CK08b] relating Khovanov-Rozansky link homology to the geometrically defined Cautis-Kamnitzer link homology, contingent on results to appear in [Cau13].

3.1.8 Cautis-Rozansky categorified clasps

Categorifying $\mathfrak{sl}_n$ link invariants labeled by arbitrary (non-fundamental) representations appears to be a much more difficult problem [Web10b]. In the $n = 2$ case, there are several approaches to defining categorifications of the coloured Jones polynomial by categorifying Jones-Wenzl projectors. The approach of Cooper-Krushkal uses foam based methods [CK12b], while another approach of Frenkel, Stroppel, and Sussan uses Lie theoretic methods [FSS12] based on category $\mathcal{O}$ for $\mathfrak{gl}_n$. These two approaches are compared and related via Koszul duality in [SS11]. Rozansky defined yet another approach to categorifying Jones-Wenzl projectors using complexes in Bar-Natan's foam category [Roz10b]. These complexes are presented as the stable limit of the complexes assigned to k-twist torus braids as $k \to \infty$, or "infinite twists". This construction also agrees with the Cooper-Krushkal $\mathfrak{sl}_2$ projectors.

There are analogs of Jones-Wenzl projectors for $\mathfrak{sl}_n$. Given a tensor product of fundamental $U_q(\mathfrak{sl}_n)$ representations, there is a corresponding idempotent

$$P\colon V_{\omega_{i_1}} \otimes V_{\omega_{i_2}} \otimes \cdots \otimes V_{\omega_{i_m}} \to V_{\sum_k i_k},$$

called a *clasp* following Kuperberg's terminology from the $\mathfrak{sl}_3$ case. For $n = 3$ these clasps were categorified by the third author using an $\mathfrak{sl}_3$ foam based construction and a generalization of Rozansky's infinite twist approach to projectors [Ros11].

A related, but more general, approach using infinite twists was independently considered by Cautis who showed that $\mathfrak{sl}_n$ clasps can be categorified explicitly using the higher structure of categorified quantum groups [Cau12]. His approach utilizes an infinite twist construction together with the categorified braid group action described above. Given a reduced decomposition of $w = s_{i_1} \dots s_{i_k}$ of the longest braid word w in the Weyl group for $\mathfrak{sl}_m$, Cautis defines a complex $\mathcal{T}_w \mathbf{1}_\lambda := \mathcal{T}_{i_1} \dots \mathcal{T}_{i_k} \mathbf{1}_\lambda$ in $Kom(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m))$. He shows that the infinite twist $\lim_{\ell\to\infty} \mathcal{T}_w^{2\ell} \mathbf{1}_\lambda$ converges and categorifies the clasp P in any appropriate 2-representation.

Cautis' categorified clasps are formulated explicitly using the 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. This is advantageous in that it allows for explicit computations not accessible within Webster's formalism. Given appropriate families of 2-representations of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ with nonzero weight spaces matching the vector space $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ where N and m vary, Cautis' framework gives rise to $\mathfrak{sl}_n$ knot homology theories and categorifications of $\mathfrak{sl}_n$ clasps. Cautis describes such 2-representations using derived categories of coherent sheaves. In section 3.4.1 we show that foamation functors allow Cautis' categorified clasps to be utilized in the foam setting. In the $\mathfrak{sl}_2$ case this gives categorified projectors which can be viewed as an extension of the Cooper-Krushkal and Rozansky projectors to the functorial foam categories of Clark-Morrison-Walker and Blanchet. In the $\mathfrak{sl}_3$ case the resulting projectors agree with those from [Ros11].

3.1.9 Recovering relations from categorified quantum groups

Foams can be thought of as a categorification of webs. This perspective suggests that new insights into foam categories can be achieved through categorical skew Howe duality. In [CKM12], the authors use skew Howe duality to deduce the $\mathfrak{sl}_n$ web relations. In section 3.4.3, we show that this holds at the categorified level as well, namely that relations for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foams can be deduced from the categorified quantum group.

This suggests that one may gain further insight to $\mathfrak{sl}_n$ foams for $n \geq 4$ using categorical skew Howe duality. In a follow-up paper, we will give a foam-based construction of $\mathfrak{sl}_n$ link homologies for $n \geq 4$ which avoids the use of the Kapustin-Li formula [LQR].

Note that the relations we derive use graded parameters that are usually set to zero in the literature. These relations are similar to the ones of [MV07] in the $\mathfrak{sl}_3$ case, but in the $\mathfrak{sl}_2$ case we obtain relations that slightly extend both Blanchet's [Bla10] and Clark-Morrison-Walker [CMW09] models.

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3.2 Categorified quantum groups

In this section we recall the relevant background information on categorified quantum groups and higher representation theory.

3.2.1 The 2-category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$

Fix a base field $\Bbbk$. We will always work over this field which is not assumed to be of characteristic 0, nor algebraically closed.

3.2.1.1 The Cartan datum

Let $I = \{1, 2, \dots, m-1\}$ consist of the set of vertices of the Dynkin diagram of type A_{m-1}

$$\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\circ}\!\!-\!\!\!-\!\!\!-\quad \cdots \quad -\!\!\!-\!\!\!-\!\!\overset{m-1}{\circ}$$

enumerated from left to right. Let $X = \mathbb{Z}^{m-1}$ denote the weight lattice for $\mathfrak{sl}_m$ and $\{\alpha_i\}_{i\,\in I} \subset X$ and $\{\Lambda_i\}_{i \in I} \subset X$ denote the collection of simple roots and fundamental weights, respectively. There is a symmetric bilinear form on X defined by $(\alpha_i, \alpha_j) = a_{ij}$ where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

is the (symmetric) Cartan matrix associated to $\mathfrak{sl}_m$. For $i \in I$ denote the simple coroots by $h_i \in X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Write $\langle \cdot, \cdot \rangle \colon X^\vee \times X \to \mathbb{Z}$ for the canonical pairing $\langle i, \lambda \rangle := \langle h_i, \lambda \rangle = 2\frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $i \in I$ and $\lambda \in X$ that satisfies $\langle h_i, \Lambda_i \rangle = \delta_{i,j}$. Any weight $\lambda \in X$ can be written as $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{m-1})$, where $\lambda_i = \langle h_i, \lambda \rangle$.

We let $X^+ \subset X$ denote the dominant weights, which are those of the form $\sum_i \lambda_i \Lambda_i$ with $\lambda_i \geq 0$. Finally, let $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = [n][n-1]\dots[1]$.

3.2.1.2 The algebra $\mathbf{U}_q(\mathfrak{sl}_m)$

The algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ is the $\mathbb{Q}(q)$-algebra with unit generated by the elements E_i, F_i and $K_i^{\pm 1}$ for $i = 1, 2, \dots, m-1$, with the defining relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \tag{3.2.1}$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \tag{3.2.2}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{3.2.3}$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \ \text{ if } j = i \pm 1, \tag{3.2.4}$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \ \text{ if } j = i \pm 1, \tag{3.2.5}$$

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \ \text{ if } |i - j| > 1. \tag{3.2.6}$$

Recall that $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ is the modified version of $\mathbf{U}_q(\mathfrak{sl}_m)$ where the unit is replaced by a collection of orthogonal idempotents 1_λ indexed by the weight lattice X of $\mathfrak{sl}_m$,

$$1_\lambda 1_{\lambda'} = \delta_{\lambda\lambda'} 1_\lambda, \tag{3.2.7}$$

such that if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{m-1})$, then

$$K_i 1_\lambda = 1_\lambda K_i = q^{\lambda_i} 1_\lambda, \quad E_i 1_\lambda = 1_{\lambda + \alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda - \alpha_i} F_i, \tag{3.2.8}$$

where

$$\lambda + \alpha_i \quad = \quad \begin{cases} (\lambda_1 + 2, \lambda_2 - 1, \lambda_3, \dots, \lambda_{m-2}, \lambda_{m-1}) & \text{if } i = 1 \\ (\lambda_1, \lambda_2, \dots, \lambda_{m-2}, \lambda_{m-1} - 1, \lambda_{m-1} + 2) & \text{if } i = m-1 \\ (\lambda_1, \dots, \lambda_{i-1} - 1, \lambda_i + 2, \lambda_{i+1} - 1, \dots, \lambda_{m-1}) & \text{otherwise.} \end{cases} \tag{3.2.9}$$

Let $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$; the $\mathcal{A}$-algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ is the integral form of $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ generated by products of divided powers $E_i^{(a)} 1_\lambda := \frac{E_i^a}{[a]!} 1_\lambda$, $F_i^{(a)} 1_\lambda := \frac{F_i^a}{[a]!} 1_\lambda$ for $\lambda \in X$ and $i = 1, 2, \dots, m-1$.

3.2.1.3 Choice of scalars Q

Associated to the Cartan datum for $\mathfrak{sl}_m$ we also fix a choice of scalars Q consisting of:
– t_{ij} for all $i, j \in I$,
such that
– $t_{ii} = 1$ for all $i \in I$ and $t_{ij} \in \Bbbk^\times$ for $i \neq j$,
– $t_{ij} = t_{ji}$ when $a_{ij} = 0$.

3.2.1.4 The definition

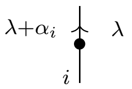
We now recall the general version of the 2-category categorifying $\dot{\mathbf{U}}(\mathfrak{sl}_m)$ given in [CL11]. There a 2-category $\mathcal{U}_Q(\mathfrak{g})$ was defined associated to any root datum and choice of scalars Q. This 2-category is a modest generalization of the 2-category originally defined in [KL10] for the choice of scalars Q where all $t_{ij} = 1$. It follows from [KL11a, pg. 15] and [KL10, KL11b] that $\mathcal{U}_Q(\mathfrak{sl}_m)$ is independent of the choice of scalars Q up to isomorphism. Here we present the general definition; in later sections we will choose a convenient choice of scalars.

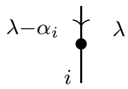
Definition 3.2.1. The 2-category $\mathcal{U}_Q(\mathfrak{sl}_m)$ is the graded additive $\Bbbk$-linear 2-category consisting of:

- objects λ for $\lambda \in X$.
- 1-morphisms are formal direct sums of (shifts of) compositions of

$$\mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i = \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbf{1}_\lambda = \mathcal{E}_i\mathbf{1}_\lambda, \quad \text{and} \quad \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i = \mathbf{1}_{\lambda-\alpha_i}\mathcal{F}_i\mathbf{1}_\lambda = \mathcal{F}_i\mathbf{1}_\lambda$$

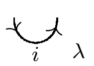
  for $i \in I$ and $\lambda \in X$.
- 2-morphisms are $\Bbbk$-vector spaces spanned by compositions of (decorated) tangle-like diagrams illustrated below.

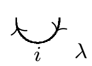

$: \mathcal{E}_i\mathbf{1}_\lambda \to \mathcal{E}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_i)\}$

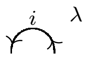

$: \mathcal{F}_i\mathbf{1}_\lambda \to \mathcal{F}_i\mathbf{1}_\lambda\{(\alpha_i,\alpha_i)\}$

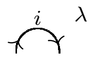

$: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_\lambda \to \mathcal{E}_j\mathcal{E}_i\mathbf{1}_\lambda\{-(\alpha_i,\alpha_j)\}$


$: \mathcal{F}_i\mathcal{F}_j\mathbf{1}_\lambda \to \mathcal{F}_j\mathcal{F}_i\mathbf{1}_\lambda\{-(\alpha_i,\alpha_j)\}$


$: \mathbf{1}_\lambda \to \mathcal{F}_i\mathcal{E}_i\mathbf{1}_\lambda\{1+(\lambda,\alpha_i)\}$


$: \mathbf{1}_\lambda \to \mathcal{E}_i\mathcal{F}_i\mathbf{1}_\lambda\{1-(\lambda,\alpha_i)\}$

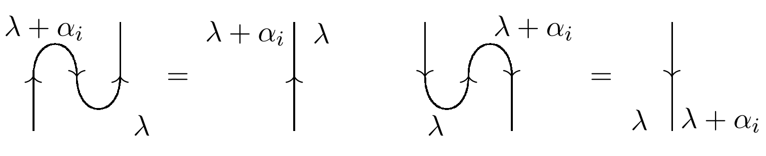

$: \mathcal{F}_i\mathcal{E}_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\{1+(\lambda,\alpha_i)\}$

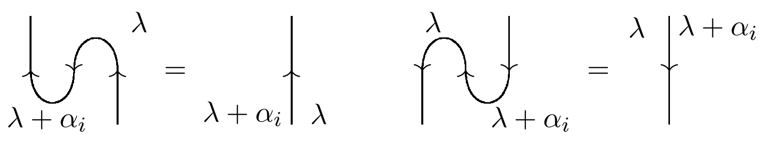

$: \mathcal{E}_i\mathcal{F}_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\{1-(\lambda,\alpha_i)\}$

Here we follow the grading conventions in [CL11] which are opposite to those from [KL10] but line up nicely with the gradings on foams used later in the paper. In this 2-category (and those throughout the paper) we read diagrams from right to left and bottom to top. The identity 2-morphism of the 1-morphism $\mathcal{E}_i\mathbf{1}_\lambda$ is represented by an upward oriented line labeled by i and the identity 2-morphism of $\mathcal{F}_i\mathbf{1}_\lambda$ is represented by a downward such line.

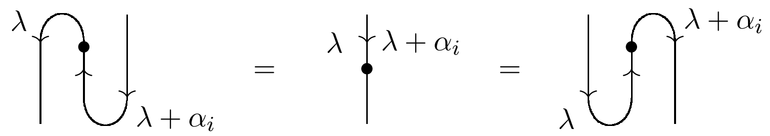
The 2-morphisms satisfy the following relations:

1. The 1-morphisms $\mathcal{E}_i\mathbf{1}_\lambda$ and $\mathcal{F}_i\mathbf{1}_\lambda$ are biadjoint (up to a specified degree shift). These conditions are expressed diagrammatically as

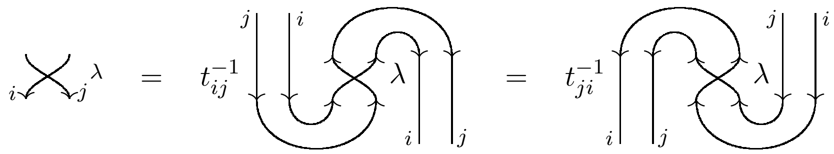

(3.2.10)


(3.2.11)

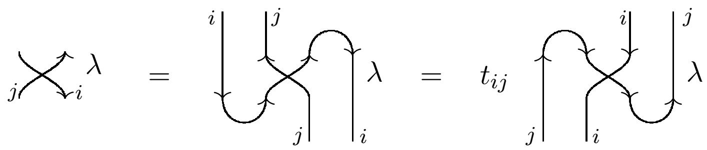
2. The 2-morphisms are Q-cyclic with respect to this biadjoint structure.

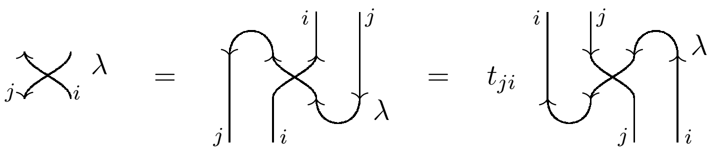

(3.2.12)

The Q-cyclic relations for crossings are given by


(3.2.13)

The Q-cyclic condition for sideways crossings is given by the equalities:

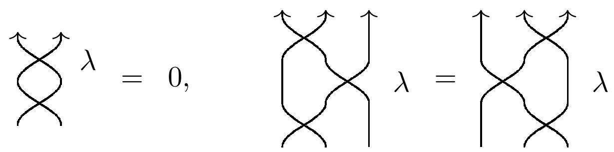

(3.2.14)

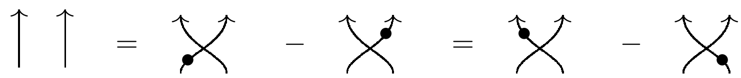

(3.2.15)

where the second equality in (3.2.14) and (3.2.15) follow from (3.2.13).

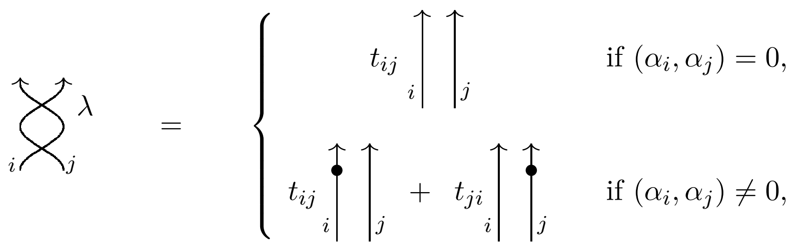
3. The $\mathcal{E}$'s carry an action of the KLR algebra associated to Q. The KLR algebra $R = R_Q$ associated to Q is defined by finite $\Bbbk$-linear combinations of braid–like diagrams in the plane, where each strand is labeled by a vertex $i \in I$. Strands can intersect and can carry dots but triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations:

   i) If all strands are labeled by the same $i \in I$ then the nilHecke algebra axioms hold


(3.2.16)


(3.2.17)

   ii) For $i \neq j$



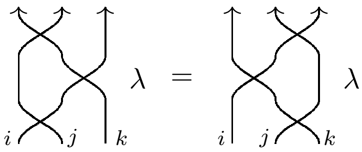
(3.2.18)

iii) For $i \neq j$ the dot sliding relations

(3.2.19)

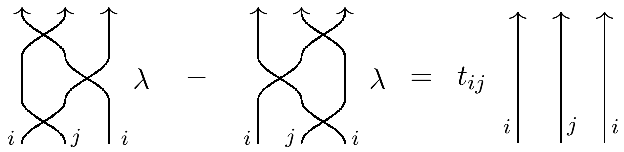
hold.

iv) Unless $i = k$ and $(\alpha_i, \alpha_j) < 0$ the relation



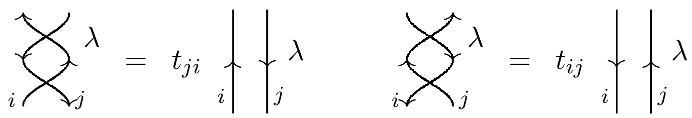
(3.2.20)

holds. Otherwise, $(\alpha_i, \alpha_j) = -1$ and



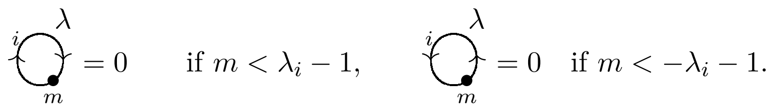
(3.2.21)

4. When $i \neq j$ one has the mixed relations relating $\mathcal{E}_i\mathcal{F}_j$ and $\mathcal{F}_j\mathcal{E}_i$:



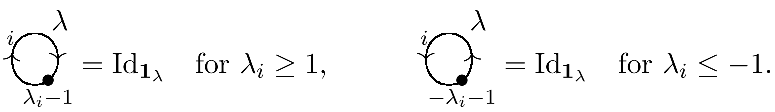
(3.2.22)

5. Negative degree bubbles are zero. That is, for all $m \in \mathbb{Z}_+$ one has

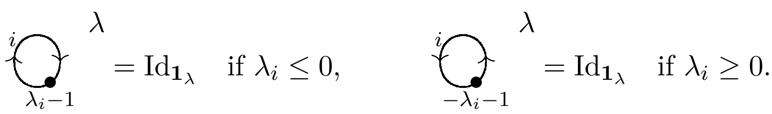


(3.2.23)

On the other hand, a dotted bubble of degree zero is just the identity 2-morphism [2]:

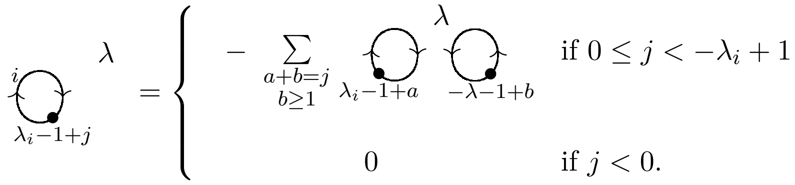


6. For any $i \in I$ one has the extended $\mathfrak{sl}_2$-relations. In order to describe certain extended $\mathfrak{sl}_2$ relations it is convenient to use a shorthand notation from [Lau08] called fake bubbles. These are diagrams for dotted bubbles where the labels of the number of dots is negative, but the total degree of the dotted bubble taken with these negative dots is still positive. They allow us to write these extended $\mathfrak{sl}_2$ relations more uniformly (i.e. independent on whether the weight λ_i is positive or negative).
– Degree zero fake bubbles are equal to the identity 2-morphisms



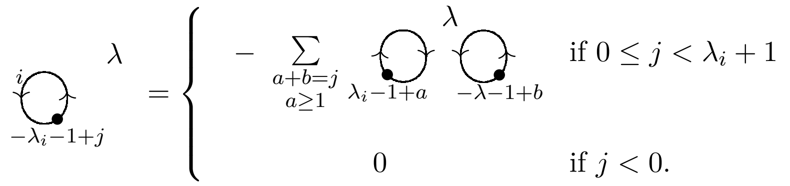
2. One can define the 2-category so that degree zero bubbles are multiplication by arbitrary scalars at the cost of modifying some of the other relations, see for example [Lau12, MSV10]. However, it is shown in [CL11] that the resulting 2-categories are all isomorphic.

– Higher degree fake bubbles for $\lambda_i < 0$ are defined inductively as



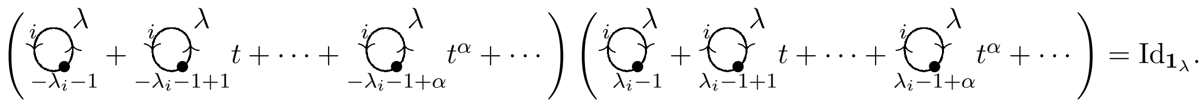
(3.2.24)

– Higher degree fake bubbles for $\lambda_i > 0$ are defined inductively as



(3.2.25)

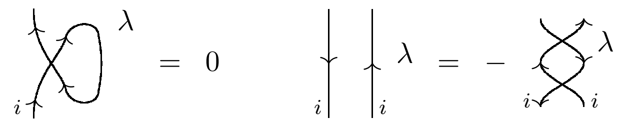
These equations arise from the homogeneous terms in t of the 'infinite Grassmannian' equation



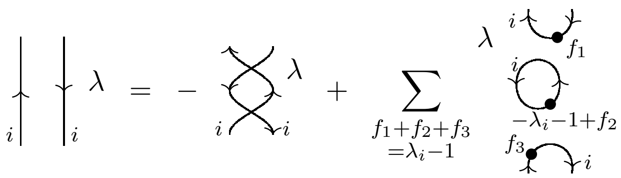
(3.2.26)

Now we can define the extended $\mathfrak{sl}_2$ relations. Note that in [CL11] additional curl relations were provided that can be derived from those above. Here we provide a minimal set of relations.

If $\lambda_i > 0$ then we have:

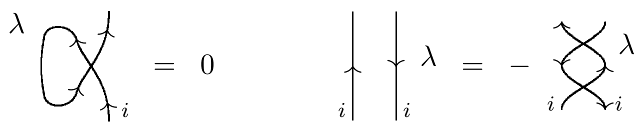


(3.2.27)

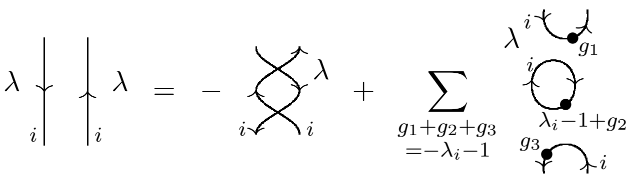


(3.2.28)

If $\lambda_i < 0$ then we have:

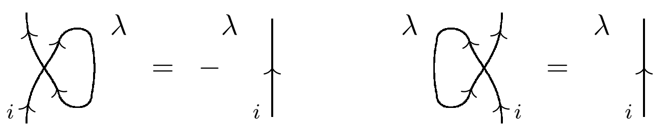


(3.2.29)

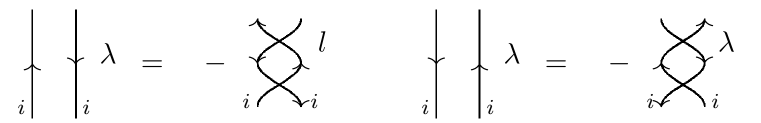


(3.2.30)

If $\lambda_i = 0$ then we have:



(3.2.31)



(3.2.32)

3.2.1.5 Karoubi completions

Recall that an idempotent $e\colon b \to b$ in a category $\mathcal{C}$ is a morphism such that $e^2 = e$. The idempotent is said to split if there exist morphisms $b \xrightarrow{g} b' \xrightarrow{h} b$ such that $e = hg$ and $gh = \mathrm{id}_{b'}$. The Karoubi envelope $Kar(\mathcal{C})$ (also called the idempotent completion or Cauchy completion) of a category $\mathcal{C}$ is a minimal enlargement of the category $\mathcal{C}$ in which all idempotents split. More precisely, the category $Kar(\mathcal{C})$ has

– objects of $Kar(\mathcal{C})$: pairs (b, e) where $e\colon b \to b$ is an idempotent of $\mathcal{C}$.

– morphisms: $(e, f, e')\colon (b, e) \to (b', e')$ where $f\colon b \to b'$ in $\mathcal{C}$ making the diagram

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, "f"] & b' \arrow[d, "e'"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \tag{3.2.33}$$

commute, i.e. $ef = f = fe'$.

– identity 1-morphisms: $(e, e, e)\colon (b, e) \to (b, e)$.

When $\mathcal{C}$ is an additive category we write $(b, e) \in Kar(\mathcal{C})$ as im e and we have $b \cong \text{im } e \oplus \text{im } (1 - e)$ in $Kar(\mathcal{C})$.

The Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) := Kar(\mathcal{U}_Q(\mathfrak{sl}_m))$ of the 2-category $\mathcal{U}_Q(\mathfrak{sl}_m)$ is the 2-category with the same objects as $\mathcal{U}_Q(\mathfrak{sl}_m)$ whose Hom categories are given by

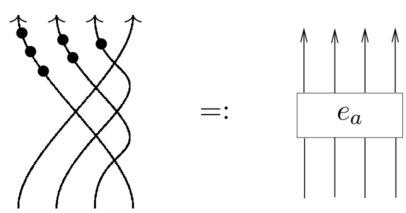
$$\dot{\mathcal{U}}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'}) := Kar\big(\mathcal{U}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'})\big).$$

In particular, all idempotent 2-morphisms split in $\dot{\mathcal{U}}_Q(\mathbf{1}_\lambda, \mathbf{1}_{\lambda'})$. It was shown in [KL10] that there is an isomorphism of $\mathcal{A}$-algebras

$$\gamma\colon K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)) \longrightarrow_{\mathcal{A}} \dot{\mathbf{U}}(\mathfrak{sl}_m) \tag{3.2.34}$$

between the split Grothendieck ring $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_m))$ and the integral form ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ of the idempotent modified quantum enveloping algebra. Recent results of Webster have generalized this statement to arbitrary type [Web10a]. Furthermore, the images of the indecomposable 1-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ in $K_0(\mathcal{U}_Q(\mathfrak{sl}_m))$ agree with the Lusztig canonical basis in ${}_{\mathcal{A}}\dot{\mathbf{U}}(\mathfrak{sl}_m)$ [Web12].

Typically the passage from a diagrammatically defined category to its Karoubi envelope results in the loss of a completely diagrammatic description of the resulting category. However, the Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ of the 2-category $\mathcal{U}_Q(\mathfrak{sl}_2)$ still admits a completely diagrammatic description [KLMS12]. In this case, one defines idempotent 2-morphisms $e_a\colon \mathcal{E}^a\mathbf{1}_\lambda \to \mathcal{E}^a\mathbf{1}_\lambda$ given by the composite of any reduced presentation of the longest braid word on a strands together with a specific pattern of dots starting with $a - 1$ dots on the left-most strand, $a - 2$ on the next strand, and ending with no dots on the last of the a strands. An example is shown below for $a = 4$.



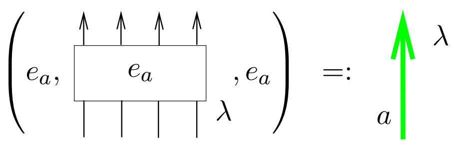
It is convenient to introduce a box notation for this composite 2-morphism.

The divided power $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ is defined in the Karoubi envelope $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ as the pair

$$\mathcal{E}^{(a)}\mathbf{1}_\lambda := (\mathcal{E}^a\mathbf{1}_\lambda\{\frac{a(a-1)}{2}\}, e_a)$$

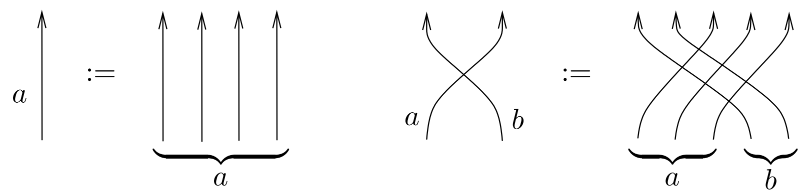
where the grading shift is necessary to get an isomorphism $\mathcal{E}^a\mathbf{1}_\lambda \cong \oplus_{[a]!}\mathcal{E}^{(a)}\mathbf{1}_\lambda$. The divided power $\mathbf{1}_\lambda\mathcal{F}^{(a)}$ is then defined as the adjoint of $\mathcal{E}^{(a)}\mathbf{1}_\lambda$. It was shown in [KLMS12] that splitting the idempotents e_a by adding $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ and $\mathcal{F}^{(b)}\mathbf{1}_\lambda$ gives rise to explicit decompositions of arbitrary 1-morphisms into indecomposable 1-morphisms using only the relations from $\mathcal{U}_Q(\mathfrak{sl}_2)$. This allows for a strengthening of the categorification result to the case when we define $\mathcal{U}_Q(\mathfrak{sl}_2)$ by taking $\mathbb{Z}$-linear combinations of 2-morphisms, rather than $\Bbbk$-linear combinations for a field $\Bbbk$.

It is possible to represent the 1-morphisms $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ by introducing an augmented graphical calculus of thickened strands. For example, the identity 2-morphism for $\mathcal{E}^{(a)}\mathbf{1}_\lambda$ is given by the triple

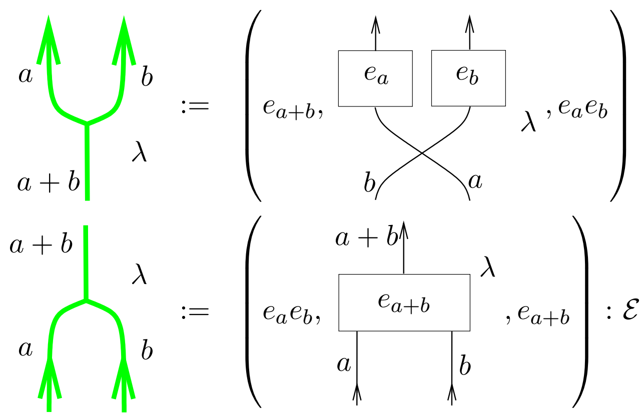
$(e_a, e_a, e_a) \quad =$  (3.2.35)

where we think of the label a on the right as describing the thickness of the strand. A downward oriented line of thickness b conveniently describes the 1-morphism $\mathcal{F}^{(b)}\mathbf{1}_\lambda$ in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$.

One can introduce further notation to describe natural 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$. For example, using the shorthand



there are 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ given by



$$: \mathcal{E}^{(a+b)}\mathbf{1}_\lambda\{t\} \to \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda\{t-ab\}$$

$$: \mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda\{t\} \to \mathcal{E}^{(a+b)}\mathbf{1}_\lambda\{t-ab\}.$$

To compute the degree of the above diagrams one must account for the shift in the definition of divided powers. For example, in the first diagram the degree shift in the divided power for $\mathcal{E}^{(a+b)}\mathbf{1}_\lambda$ is $\frac{(a+b)(a+b-1)}{2}$, while the degree shift in the composite $\mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbf{1}_\lambda$ is

$\frac{a(a-1)}{2} + \frac{b(b-1)}{2}$, so that the net difference is $\frac{2ab}{2} = ab$. Both of the above diagrams in the thick calculus have degree $-ab$.

For general m there is no completely diagrammatic description of the Karoubi envelope of $\mathcal{U}_Q(\mathfrak{sl}_m)$. In this case one lacks a set of diagrammatic relations needed to decompose arbitrary 1-morphisms into indecomposables, though explicit isomorphisms giving higher Serre relations were defined by Stošić [Sto11]. It will nevertheless be convenient to introduce a version of the 2-category $\mathcal{U}_Q(\mathfrak{sl}_m)$ where we have split the idempotents needed to define divided powers, but where we have not passed to the full Karoubi completion. Diagrammatically this 2-category can be defined using thick strands carrying two labels, one indicating the thickness of the strand, and one indicated the label $i \in I$ of the a strands. Since the thick strands are defined in terms of idempotents in thin strands, all the 2-morphisms can be studied using only the relations from $\mathcal{U}_Q(\mathfrak{sl}_m)$.

Definition 3.2.2. Let $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ denote the full sub-2-category of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ with the same objects $\lambda \in X$ as $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ and with 1-morphisms generated as a graded additive $\Bbbk$-linear category by the 1-morphisms $\mathcal{E}_i\mathbf{1}_\lambda := (\mathcal{E}_i\mathbf{1}_\lambda, \mathrm{id}_{\mathcal{E}_i\mathbf{1}_\lambda})$ and $\mathcal{E}_i^{(a)}\mathbf{1}_\lambda := (\mathcal{E}_i^a\mathbf{1}_\lambda\{\frac{a(a-1)}{2}\}, e_a)$ and their adjoints.

3.2.1.6 2-representations

Let $\mathcal{U}_Q$ denote any one of the 2-categories $\mathcal{U}_Q(\mathfrak{sl}_m)$, $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$, or $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$.

Definition 3.2.3. A 2-representation of $\mathcal{U}_Q$ is a graded additive $\Bbbk$-linear 2-functor $\mathcal{U}_Q \to \mathcal{K}$ for some graded, additive 2-category $\mathcal{K}$.

When all of the Hom categories $\mathcal{K}(x,y)$ between objects x and y of $\mathcal{K}$ are idempotent complete, in other words $Kar(\mathcal{K}) \cong \mathcal{K}$, then any graded additive $\Bbbk$-linear 2-functor $\mathcal{U}_Q(\mathfrak{g}) \to \mathcal{K}$ extends uniquely to a 2-representation of $\dot{\mathcal{U}}_Q(\mathfrak{g})$.

Remark 3.2.4. For each $i \in I$ there is a sub 2-category $\mathcal{U}_Q(\mathfrak{sl}_2)_i$ of $\mathcal{U}_Q(\mathfrak{sl}_m)$ where we restrict to diagrams where all strands are labeled i. For general 2-representations $\mathcal{F}\colon \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ it may happen that $\mathcal{K}$ is not Karoubi complete. However, there are many instances when the images of divided powers $\mathcal{E}_i^{(a)}\mathbf{1}_\lambda$ and $\mathcal{F}_i^{(b)}\mathbf{1}_\lambda$ exist in $\mathcal{K}$. In this case, the composite 2-functors $\mathcal{F}_i\colon \mathcal{U}_Q(\mathfrak{sl}_2)_i \to \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ extend to give 2-representations from the Karoubi envelope of the $\mathfrak{sl}_2$ subcategories $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)_i \to \mathcal{K}$. In this case, the 2-representation $\mathcal{F}$ extends to a 2-representation $\mathcal{F}\colon \check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$.

3.2.1.7 Minimal relations and defining 2-functors

In [CL11], it is shown that a 2-representation of $\mathcal{U}_Q(\mathfrak{sl}_m)$ can be specified by defining a 2-category satisfying a small number of axioms. The following is a slightly stronger statement of the main theorem from that work.

Theorem 3.2.5 ([CL11] Theorem 1.1)**.** A map $\mathcal{R}$ from the set of weights X of $\mathfrak{sl}_m$ to the objects of graded additive $\Bbbk$-linear 2-category $\mathcal{K}$ extends to a 2-representation $\mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ provided the following conditions are satisfied:

1. The object $\mathcal{R}(\lambda + r\alpha_i)$ is (isomorphic to) the zero object for $r \gg 0$ or $r \ll 0$.
2. $\mathrm{Hom}_{\mathcal{K}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda\{l\})$ is zero if $l < 0$ and one-dimensional if $l = 0$, where $\mathbb{1}_\lambda$ denotes the identity endomorphism of $\mathcal{R}(\lambda)$. Moreover, the space of 2-morphisms between any two 1-morphisms in $\mathcal{K}$ is finite dimensional.

3. There exist 1-morphisms $\mathsf{E}_i\mathbb{1}_\lambda : \mathcal{R}(\lambda) \to \mathcal{R}(\lambda+\alpha_i)$ in $\mathcal{K}$ which possess both right and left adjoints.
4. Defining 1-morphisms $\mathsf{F}_i\mathbb{1}_\lambda : \mathcal{R}(\lambda) \to \mathcal{R}(\lambda-\alpha_i)$ for all $\lambda \in X$ via

$$\mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i} := (\mathsf{E}_i\mathbb{1}_\lambda)_R\{-\lambda_i - 1\}$$

we have the following isomorphisms in $\mathcal{K}$:

$$\mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathbb{1}_{\lambda-\alpha_i}\mathsf{F}_i\mathbb{1}_\lambda \oplus \left(\bigoplus_{[-\langle i,\lambda\rangle]} \mathbb{1}_\lambda\right) \text{ if } \langle i,\lambda\rangle \leq 0$$

$$\mathsf{E}_i\mathbb{1}_{\lambda-\alpha_i}\mathsf{F}_i\mathbb{1}_\lambda \cong \mathsf{F}_i\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \oplus \left(\bigoplus_{[\langle i,\lambda\rangle]} \mathbb{1}_\lambda\right) \text{ if } \langle i,\lambda\rangle \geq 0.$$

5. The E's carry an action of the KLR algebra associated to Q.
6. If $i \neq j \in I$ then $\mathsf{F}_j\mathbb{1}_{\lambda+\alpha_i}\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathbb{1}_{\lambda-\alpha_j}\mathsf{F}_j\mathbb{1}_\lambda$ in $\mathcal{K}$.

In the above, we set

$$\bigoplus_{f(q)} M := \bigoplus_{i=-l}^{k} (M\{i\})^{\oplus r_i}.$$

when $f(q) = \sum_{i=-l}^{k} r_i q^i$ is a Laurent polynomial with $r_i \geq 0$.

3.2.2 Categorified Weyl group action

The Weyl group for $\mathfrak{sl}_m$ is the symmetric group $\mathfrak{S}_m$ generated by transpositions s_i associated to the roots $\alpha_1, \dots, \alpha_{m-1}$. The action of the Weyl group on the weights lifts to a braid group action on representations of the associated quantum group $U_q(\mathfrak{sl}_m)$ (see for example [Lus93, KT09, CKL10a]).

The action of a simple transposition is described by an element of the completion $\widetilde{U_q(\mathfrak{sl}_m)}$ of $U_q(\mathfrak{sl}_m)$. This ring is defined as a quotient of the ring of series $\sum_{k=1}^{\infty} X_k$ of elements of $U_q(\mathfrak{sl}_m)$ acting on each irreducible representation V_λ of highest weight λ by zero but for finitely many terms X_k (see [KT09]). To s_i, we associate the braiding map $T_i \in \widetilde{U_q(\mathfrak{sl}_m)}$:

$$T_i 1_\lambda := \sum_{s\geq 0} (-q)^s E_i^{(-\lambda_i+s)} F_i^{(s)} 1_\lambda \tag{3.2.36}$$

if $\lambda_i \leq 0$,

$$T_i 1_\lambda := \sum_{s\geq 0} (-q)^s F_i^{(\lambda_i+s)} E_i^{(s)} 1_\lambda \tag{3.2.37}$$

if $\lambda_i \geq 0$. This definition differs from the one given in [Lus93, Section 5.2.1] but is equivalent up to rescaling, see [Cau12, Remark 6.4]. With this definition, $T_i = \sum_{\lambda\in X} T_i 1_\lambda$ gives an endomorphism of any finite-dimensional representation. Note that if v is a weight vector of weight λ, $T_i(v)$ is a weight vector of weight $s_i(\lambda)$.

For $\mathfrak{sl}_2$ the deformed Weyl group action on a $U_q(\mathfrak{sl}_2)$-representation V gives a reflection isomorphism from the λ weight space of V to the $-\lambda$ weight space. This reflection isomorphism was categorified by Chuang and Rouquier in the context of abelian categories [CR08]. Their work is closely related to a variant of the 2-category $\mathcal{U}_Q(\mathfrak{sl}_2)$ where the nilHecke algebra is replaced by the affine Hecke algebra and there is no grading. Cautis, Kamnitzer and Licata later developed analogous complexes in the context of $\mathcal{U}_Q(\mathfrak{sl}_2)$ and generalized Chuang and Rouquier's results to triangulated categories [CKL09].

To categorify the reflection isomorphism $T_i 1_\lambda$ it is clear from (3.2.36) and (3.2.37) that we will need to work in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ so that we have lifts of divided powers. Also, the minus signs in the definition of the braid group generators suggests that we will have to pass to the 2-category $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$ of bounded complexes over the 2-category $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ whose objects are weights $\lambda \in X$, 1-morphisms are bounded complexes of 1-morphisms in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$, and 2-morphisms are chain maps constructed from the 2-morphisms in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$.

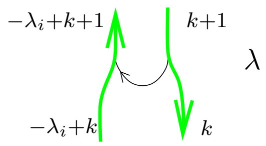
The braid group generator $T_i 1_\lambda$ lifts to a complex $\mathcal{T}_i \mathbf{1}_\lambda$ in $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$ of the form[3]:

$$\mathcal{T}_i \mathbf{1}_\lambda = \mathcal{E}_i^{(-\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{E}_i^{(-\lambda_i+1)} \mathcal{F}_i \mathbf{1}_\lambda\{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{E}_i^{(-\lambda_i+s)} \mathcal{F}_i^{(s)} \mathbf{1}_\lambda\{s\} \xrightarrow{d_{s+1}} \cdots \tag{3.2.38}$$

when $\lambda_i \leq 0$ and

$$\mathcal{T}_i \mathbf{1}_\lambda = \mathcal{F}_i^{(\lambda_i)} \mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{F}_i^{(\lambda_i+1)} \mathcal{E}_i \mathbf{1}_\lambda\{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{F}_i^{(\lambda_i+s)} \mathcal{E}_i^{(s)} \mathbf{1}_\lambda\{s\} \xrightarrow{d_{s+1}} \cdots \tag{3.2.39}$$

when $\lambda_i \geq 0$, where in the above formulae the leftmost term is in homological degree zero. The differential d_k that appears in the first complex is conveniently expressed in the extended graphical calculus from [KLMS12] as

$d_k =$ 

where all strands are colored by the index $i \in I$ and the labels indicate the thickness of strands. The differential in the second complex is defined similarly. Using the extended calculus it is easy to see that $d^2 = 0$. Results of Cautis and Kamnitzer show the images of the complexes $\mathcal{T}_i \mathbf{1}_\lambda$ under any integrable 2-representation $\check{\mathcal{U}}(\mathfrak{sl}_m) \to \mathcal{K}$ satisfy braid relations up to homotopy in $Kom(\mathcal{K})$ [CK12a, Section 6].

The complexes $\mathcal{T}_i \mathbf{1}_\lambda$ are invertible, up to homotopy, with inverses given by taking the left adjoint of the complex $\mathcal{T}_i \mathbf{1}_\lambda$ in the 2-category $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$. More explicitly, the inverses are given by

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{E}_i^{(s)} \mathcal{F}_i^{(-\lambda_i+s)}\{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{E}_i \mathcal{F}_i^{(-\lambda_i+1)}\{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{F}_i^{(-\lambda_i)}$$

when $\lambda_i \leq 0$ and

$$\mathbf{1}_\lambda \mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda \mathcal{F}_i^{(s)} \mathcal{E}_i^{(\lambda_i+s)}\{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda \mathcal{F}_i \mathcal{E}_i^{(\lambda_i+1)}\{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda \mathcal{E}_i^{(\lambda_i)}$$

when $\lambda_i \geq 0$, where in these formulae the rightmost term is in homological degree zero.

Given a 2-representation $\mathcal{F} \colon \check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$, the braiding $T_i 1_\lambda$ is lifted to a complex $\mathcal{T}_i \mathbf{1}_\lambda$ that gives an equivalence between $\mathcal{F}(\lambda)$ and $\mathcal{F}(s_i(\lambda))$ in the 2-category of complexes

3. Note that we take the mirror of Cautis' definition of these complexes in [Cau12], in order to better fit with usual definition of Khovanov homology. This also reverses the decategorification process so that a shift by k will decategorify to q^k, while it decategorifies to q^{-k} in [Cau12].

$Kom(\mathcal{K})$ over $\mathcal{K}$. For $\mathfrak{sl}_2$ there are no interesting braid relations to check. The content of a categorification of the reflection isomorphism is that the complex $\mathcal{T}_i \mathbf{1}_\lambda$ has a homotopy inverse, so that a 2-representation $\check{\mathcal{U}}_Q(\mathfrak{sl}_2) \to \mathcal{K}$ induces an equivalence in the category of complexes over $\mathcal{K}$ [CR08, Theorem 6.4]. The resulting equivalences are highly nontrivial and have been applied to a variety of contexts ranging from the representation theory of the symmetric group [CR08] to coherent sheaves on cotangent bundles [CKL10b, CKL09]. Cautis and Kamnitzer later showed that given an integrable 2-representation $\mathcal{U}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ the complexes $\mathcal{T}_i \mathbf{1}_\lambda$ defined for each $i \in I$ satisfy the braid relations [CK12a, Section 6], see also [Cau12, Section 4.1]. This is a crucial observation for Cautis's construction of knot homology theories from the 2-category $\mathcal{U}_Q(\mathfrak{sl}_m)$.

3.3 Foams and foamation

We now aim to define families of foamation 2-functors from $\mathcal{U}_Q(\mathfrak{sl}_\mathfrak{m})$ to certain 2-categories of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foams. We use the particular choice of scalars Q given by $t_{i,i+1} = 1$, $t_{i,i-1} = -1$, and $t_{i,j} = 1$ when $a_{i,j} = 0$.

3.3.1 $\mathfrak{sl}_2$ foam 2-categories

In this section, we define a family of 2-functors from $\mathcal{U}_Q(\mathfrak{sl}_\mathfrak{m})$ to suitable categories of $\mathfrak{sl}_2$ foams. We first review Bar-Natan's cobordism-based construction of ($\mathfrak{sl}_2$) Khovanov homology [BN05] as well as a functorial enhancement of this theory due to Blanchet [Bla10] which encodes additional representation-theoretic information. We will define our foamation 2-functors into a family of related 2-categories which are natural to consider from the perspective of skew Howe duality. We also construct such 2-functors into the Clark-Morrison-Walker functorial formulation of Khovanov homology [CMW09].

3.3.1.1 Standard $\mathfrak{sl}_2$ foams

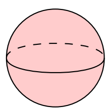
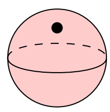
In [BN05], Bar Natan gave a construction of Khovanov homology as a quotient of the cobordism category of planar tangles and surfaces. This work gives a categorification of (a version of) the category $2\mathbf{Web}$. We summarize this construction, which can be understood as a 2-category defined as follows:

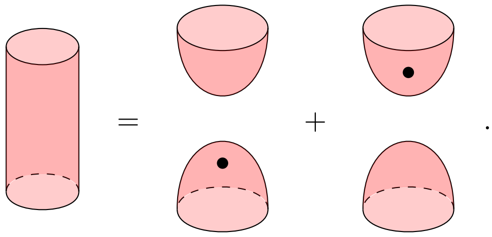
– Objects are sequences of points in the interval $[0, 1]$, together with a zero object.
– 1-morphisms are formal direct sums of $\mathbb{Z}$-graded planar tangles with boundary corresponding to the sequences of points in the domain and codomain.
– 2-morphisms are formal matrices of $\Bbbk$-linear combinations of degree-zero dotted cobordisms between such planar curves, modulo isotopy (relative to the boundary) and local relations.

If we denote the $\mathbb{Z}$-grading of a planar tangle by the monomial q^t for $t \in \mathbb{Z}$, then the degree of a cobordism $C : q^{t_1}T_1 \to q^{t_2}T_2$ is given by the formula

$$\deg(C) = \chi(C) - 2\#D - \frac{\#\partial}{2} + t_2 - t_1 \tag{3.3.1}$$

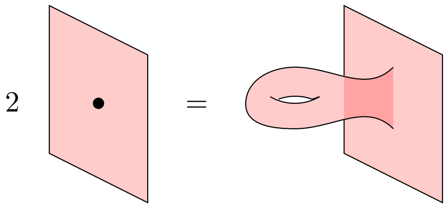
where $\#D$ is the number of dots and $\#\partial$ is the number of boundary points in either T_1 or T_2 (they agree!). The local relations are then given as follows:

 $= \quad 0 \quad ,$  $= \quad 1$ (3.3.2)



(3.3.3)

The neck-cutting relation (3.3.3) gives the formula:



(3.3.4)

which allows for a completely topological description of the 2-category when 2 is invertible in $\Bbbk$.

As mentioned in the introduction, the $+$ sign in the neck-cutting relation prevents us from defining a 2-functor from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to this 2-category since it is incompatible with the sign in the nilHecke relation. We hence consider related versions of this construction.

3.3.1.2 Enhanced foams

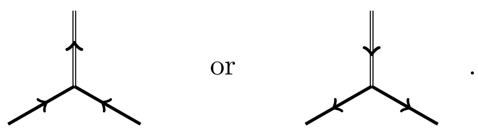
Bar-Natan formulates Khovanov homology in the homotopy category of complexes in the above 2-category, giving an invariant which is functorial only up to a (± 1)-sign under tangle cobordism. This functoriality issue was fixed by Clark, Morrison, and Walker [CMW09] working in a related 2-category of disoriented curves and cobordisms defined over the Gaussian integers [4] (see also the work of Caprau [Cap07], [Cap08], [Cap09] for a related construction). Blanchet [Bla10] later gave another functorial construction of Khovanov homology in a related 2-category defined over the integers.

It turns out that in addition to fixing functoriality, these later constructions also fix the incompatibility of the neck-cutting and nilHecke relations. We will work in Blanchet's enhanced foam model since it is more natural to consider from the perspective of skew Howe duality and it avoids the introduction of complex coefficients. We return to the Clark-Morrison-Walker (CMW) construction in the following section.

We begin by defining a family of 2-categories related to Blanchet's construction which should be viewed as categorifications of the categories $2\mathbf{BWeb}_m(N)$.

Definition 3.3.1. $2\mathbf{BFoam}_m(N)$ is the 2-category defined as follows:

- Objects are sequences $(a_1, \dots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2\}$ and $N = \sum_{i=1}^m a_i$, together with a zero object.
- 1-morphisms are formal direct sums of $\mathbb{Z}$-graded enhanced $\mathfrak{sl}_2$ webs - directed planar graphs with boundary with two types of edges - 1-labeled edges  and 2-labeled edges  - where all vertices are trivalent and take the following two forms:



(3.3.5)

4. Actually, they work over the ring $\mathbb{Z}[\frac{1}{2}, i]$.

1- (respectively 2-) labeled edges are directed out from points labeled by 1 (respectively 2) in the domain and directed into such labeled points in the codomain. No edges are attached to points labeled by 0.

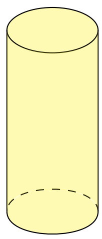
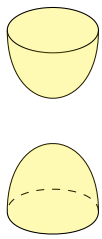
– 2-morphisms are formal matrices of $\Bbbk$-linear combinations of degree-zero $\mathfrak{sl}_2$ foams - surfaces with oriented singular seams which locally look like the product of the letter Y with an interval - considered up to isotopy (relative to the boundary) and local relations.

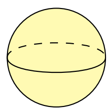
There are two types of facets of an $\mathfrak{sl}_2$ foam, 1-labeled and 2-labeled, depending on which type of edge they are incident upon when the foam is expressed as a composition of elementary foams. The degree of a foam $F : q^{t_1}W_1 \to q^{t_2}W_2$ is given by the degree of the cobordism resulting from deleting all the 2-labeled facets and edges and forgetting the orientation of the 1-labeled edges.

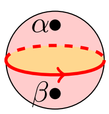
As in $\mathcal{U}_Q(\mathfrak{sl}_m)$, we shall read diagrammatic depictions of webs and foams from right to left and from bottom to top. The orientation of a singular seam gives a cyclic ordering of the facets incident upon the seam via the right hand rule. By convention, a seam travels down through the first vertex in (3.3.5) and up through the second; this corresponds to the cyclic orientation of web vertices from [Bla10].

The relations for $\mathfrak{sl}_2$ foams come from a non-local, universal construction detailed in [Bla10]; however, we can exhibit a complete set of local relations giving an equivalent description of Blanchet's work. In what follows, the 2-labeled facets are depicted in yellow and 1-labeled facets are drawn in red.

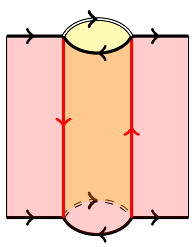
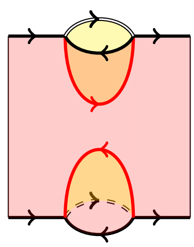
We impose the relations (3.3.2) and (3.3.3) for 1-labeled facets, as well as the following relations involving 2-labeled facets:

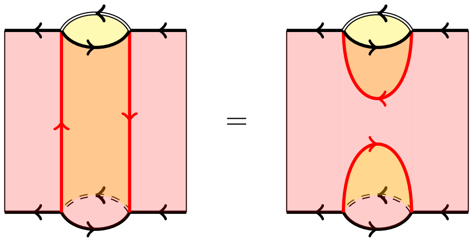
 $= \; -$  (3.3.6)

 $= \; -1$ (3.3.7)

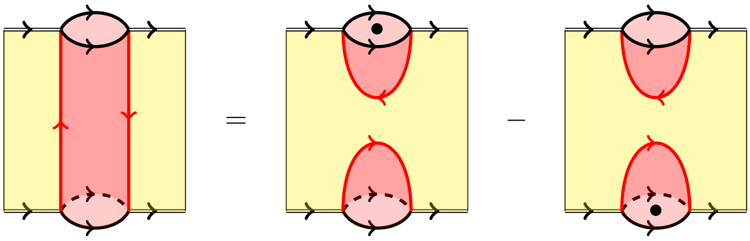


$$= \begin{cases} 1 & \text{if } (\alpha,\beta) = (1,0) \\ -1 & \text{if } (\alpha,\beta) = (0,1) \\ 0 & \text{if } (\alpha,\beta) = (0,0) \text{ or } (1,1) \end{cases} \tag{3.3.8}$$

 $= \; -$  (3.3.9)

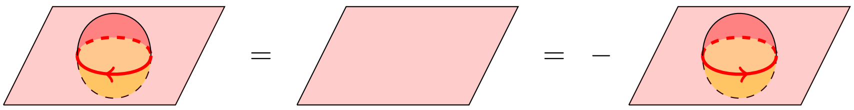


(3.3.10)

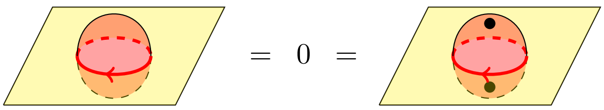


(3.3.11)

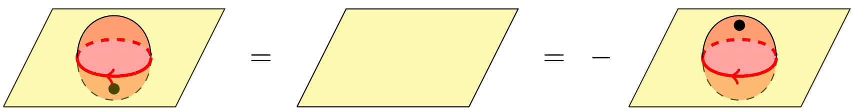
Relations (3.3.2), (3.3.3), (3.3.6), (3.3.7), and (3.3.8) allow for the evaluation of any closed $\mathfrak{sl}_2$ foam. Additionally, note that these relations imply that we can reverse the direction of any closed, singular seam at the cost of multiplying by -1. Equations (3.3.9), (3.3.10), and (3.3.11) guarantee that if a linear combination of foams evaluates to zero whenever it is “closed off” to give a closed foam, then that linear combination is zero; the latter is a non-local relation from [Bla10]. The equivalence of this relation to the collection of local relations given above can be proved in a manner similar to the proof of [MN08, Lemma 3.5] using the above local relations. The proof utilizes several relations that follow from the above local relations, allowing a web with a digon or square face to be expressed in terms of webs with fewer faces:



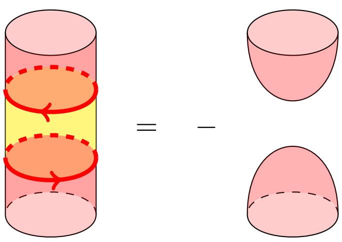
(3.3.12)



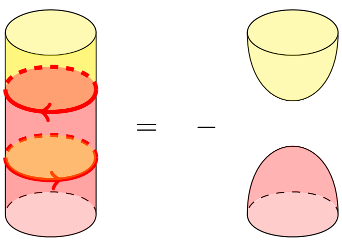
(3.3.13)



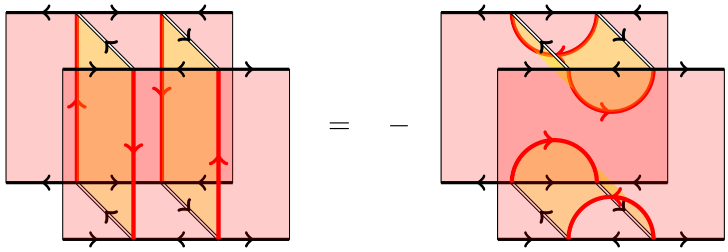
(3.3.14)



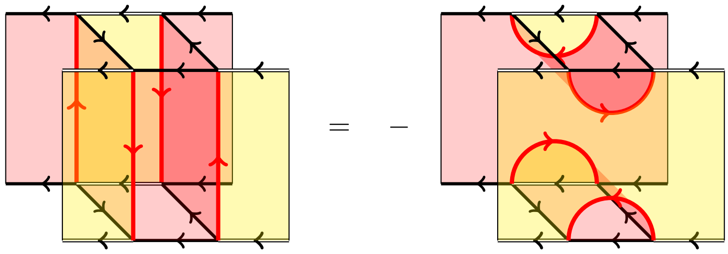
(3.3.15)



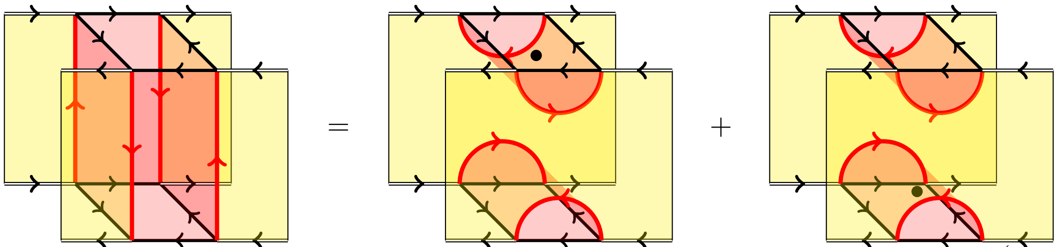
(3.3.16)



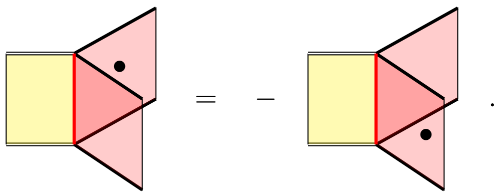
(3.3.17)



(3.3.18)



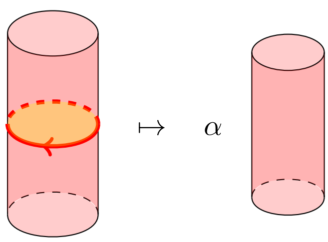
(3.3.19)



(3.3.20)

The neck-cutting relation (3.3.6) implies that the topology of the 2-labeled facets plays a limited role. One may hence ask if there is a way to coherently deleted such facets and obtain a forgetful 2-functor from $2\mathbf{BFoam}_m(N)$ to (an appropriate version of) the Bar-Natan 2-category.

Such a 2-functor would act via

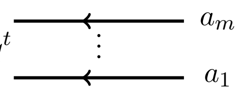


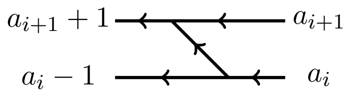
for some scalar α; equation (3.3.12) shows that composing the left-hand foam with a cap produces a cap, while pre-composing with a cup gives -1 multiplied by a cup. It is therefore impossible to define such a 2-functor which acts as the identity on foams which contain no 2-labeled facets.

3.3.1.3 Foamation

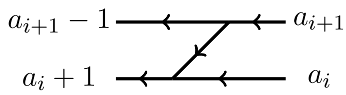
We now define $\mathfrak{sl}_2$ foamation 2-functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to 2\mathbf{BFoam}_m(N)$ categorifying the skew Howe map to webs discussed in the introduction. As in the decategorified case, we define the 2-functor on objects by sending an $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \dots, \lambda_{m-1})$ to the sequence $(a_1, \dots, a_m)$ with $a_i \in \{0,1,2\}$, $\lambda_i = a_{i+1} - a_i$, and $\sum_{i=1}^m a_i = N$ provided it exists and to the zero object otherwise.

The map is given on 1-morphisms by

$\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

$\mathcal{E}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

and

$\mathcal{F}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

when the boundary values lie in $\{0,1,2\}$ and to the zero 1-morphism otherwise. The labelings of the edges incident upon the boundary are given by the boundary labels; edges incident upon boundary points labeled by zero should be deleted. Note that we have not depicted $m-2$ horizontal strands in each of the latter two formulae.

We will make use of a preparatory lemma to deduce the existence of the foamation functors. Let the images of $\mathbf{1}_\lambda$, $\mathcal{E}_i\mathbf{1}_\lambda$, and $\mathcal{F}_i\mathbf{1}_\lambda$ given above be denoted $\mathbb{1}_\lambda$, $\mathsf{E}_i\mathbb{1}_\lambda$, and $\mathsf{F}_i\mathbb{1}_\lambda$.

Lemma 3.3.2. There are isomorphisms

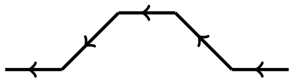
$$\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \oplus \left(\bigoplus_{[-\langle i,\lambda\rangle]} \mathbb{1}_\lambda \right) \text{ if } \langle i,\lambda\rangle \leq 0$$

$$\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \cong \mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \oplus \left(\bigoplus_{[\langle i,\lambda\rangle]} \mathbb{1}_\lambda \right) \text{ if } \langle i,\lambda\rangle \geq 0,$$

and $\mathsf{F}_j\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_j\mathbb{1}_\lambda$ for $i \neq j \in I$ in $2\mathbf{Foam}_m(N)$.

Proof. We'll prove only the first relation since the proof of the second is analogous and the third is straightforward. The condition on weights implies that λ maps to a sequence with $a_{i+1} \leq a_i$.

If $a_i = 0$, $a_{i+1} = 0$, both sides of the equation map to the zero foam. If $a_i = 1$, $a_{i+1} = 0$, then the web:

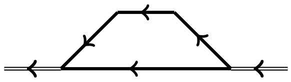


is isomorphic to $\mathbb{1}_\lambda$ via the foam realizing the web isotopy.

If $a_i = 1 = a_{i+1}$, then the relevant webs are isotopic:

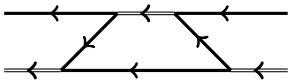
hence isomorphic.

If $a_i = 2$, $a_{i+1} = 0$, then the web:



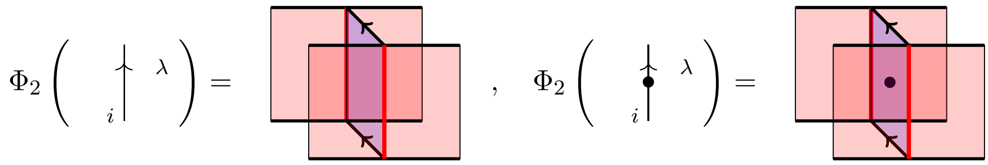
is isomorphic to $q^{-1}\mathbb{1}_\lambda \oplus q\mathbb{1}_\lambda$ using equations (3.3.11), (3.3.13), and (3.3.14). This confirms the relation since $\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \mapsto 0$ for such a weight.

If $a_i = 2$, $a_{i+1} = 1$, then the web:

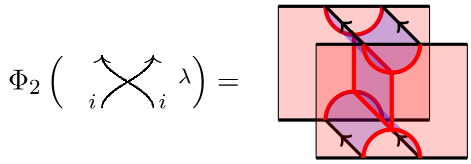


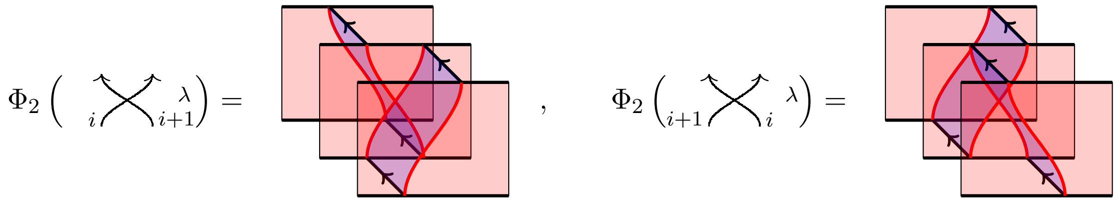
is isomorphic to $\mathbb{1}_\lambda$ using equations (3.3.16) and (3.3.18). Finally, if $a_i = a_{i+1} = 2$, both sides of the equation map to zero. $\square$

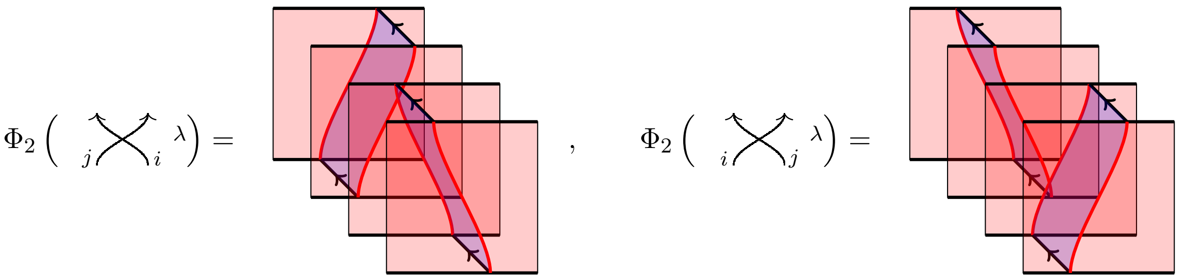
Proposition 3.3.3. For each $N > 0$ there is a 2-representation $\Phi_2 \colon \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathbf{2BFoam}_m(N)$ defined on single strand 2-morphisms by:



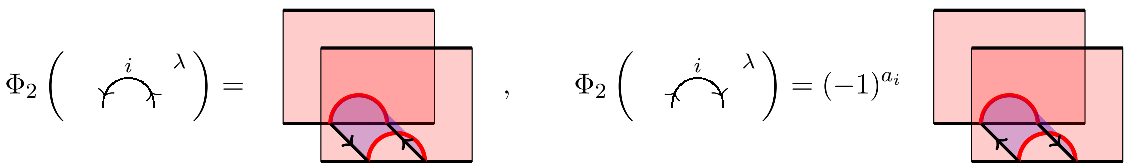
on crossings by:

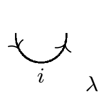


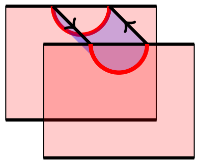
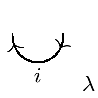


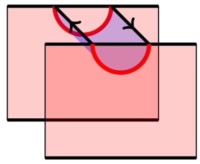


where $j - i > 1$, and on caps and cups by:



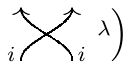
$\Phi_2\Big($ 

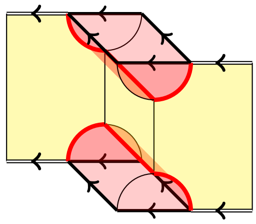
$\Big) = (-1)^{a_i+1}$  , $\Phi_2\Big($ 

$\Big) =$ 

where in the above diagrams the i^{th} sheet is always in the front.

The foams drawn above are general depictions of the images. To obtain the specific image foam we delete any facets incident upon deleted web edges and re-color facets appropriately (in particular, the blue colored facets in the above are used only to make the pictures more readable). The singular seams may degenerate in such examples, e.g.

$\Phi_2\Big($ 

$\Big) =$ 

when λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$.

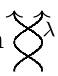
Proof. While it is not difficult to verify all relations by hand, we apply Theorem 3.2.5 to $2\mathbf{BFoam}_m(N)$ to reduce the number of relations that need to be verified. For each m and N, the non-zero objects of this 2-category are indexed by the non-zero $\mathfrak{sl}_m$ weight spaces of the finite-dimensional $U_q(\mathfrak{sl}_m)$-module $\wedge_q^N(\mathbb{C}^2 \otimes \mathbb{C}^m)$, so condition (1) is satisfied. Furthermore, it is clear from the definitions that $\mathcal{E}_i \mathbb{1}_\lambda$ has $\mathcal{F}_i \mathbb{1}_{\lambda+\alpha_i}$ as a left and right adjoint, up to a grading shift. Lemma 3.3.2 establishes conditions (4) and (6), thus, it suffices to show that conditions (2) and (5) are satisfied.

We first check condition (2). Given a foam in $\mathrm{Hom}(\mathbb{1}_\lambda, q^t \mathbb{1}_\lambda)$, we can apply the neck-cutting relations (3.3.3) and (3.3.6) in the neighborhood of each boundary component to express the foam as a linear combination of foams which are the disjoint unions of closed foams, 2-labeled sheets, and 1-labeled sheets, which may carry dots. Relations (3.3.2), (3.3.3), (3.3.6), (3.3.7), and (3.3.8) give that any closed foam is equal to an element of $\Bbbk[$ 

$]$. Equation (3.3.1) then shows that $\mathrm{Hom}(\mathbb{1}_\lambda, q^t \mathbb{1}_\lambda)$ is zero for $t < 0$ and 1-dimensional for $t = 0$.

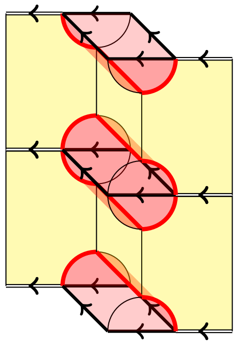
Using the neck-cutting relations, we can express any foam mapping between fixed webs W_1 and W_2 as a linear combination of foams in which every 2-labeled facet is a disk incident upon the boundary; such facets are determined by the collection of singular seams incident upon the web vertices. The union of the 1-labeled facets gives a (dotted) cobordism between the 1-labeled edges of W_1 and W_2. Using the neck-cutting relations and (3.3.11) we can assume that this cobordism consists of (dotted) disks. Since there are only finitely many ways to connect the vertices of the boundary webs with singular seams lying on the cobordism (up to isotopy), it follows from (3.3.1) that $\mathrm{Hom}(q^{t_1} W_1, q^{t_2} W_2)$ is finite dimensional for all values of t_1 and t_2.

Finally, we check condition (5), i.e. that the KLR relations are satisfied.

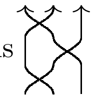
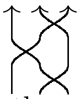
– Relation (3.2.16): the 2-morphism 

automatically maps to the zero foam unless

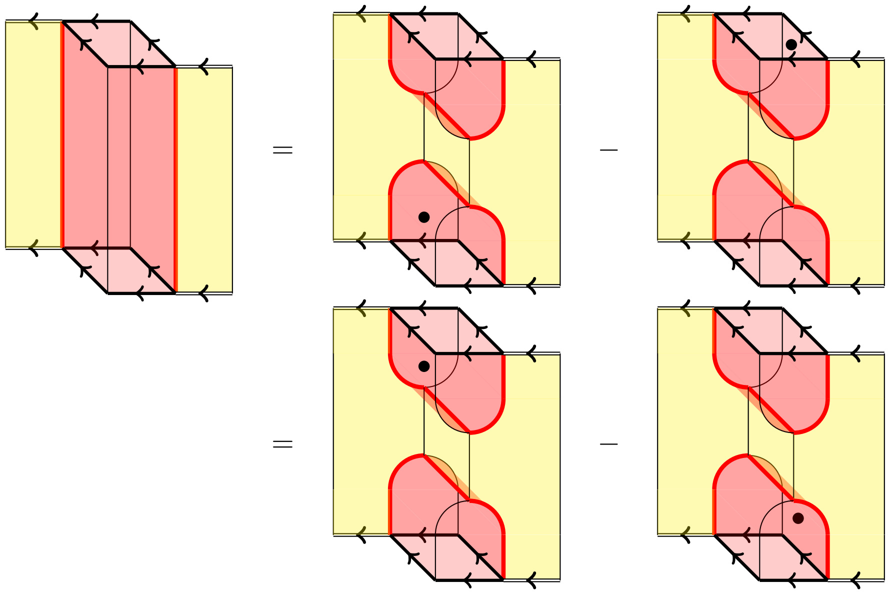
λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$. In this case, we compute the image:



$= 0$

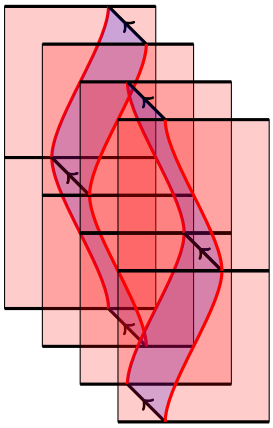
which follows from equation (3.3.13). The images of the 2-morphisms  λ and  λ are both zero since either λ or $\lambda + 3\alpha_i$ maps to the zero object, confirming the relation.

– Relation (3.2.17): As before, the only non-trivial case is when λ maps to a sequence with $a_i = 2$ and $a_{i+1} = 0$. In this case, we must have the equalities:

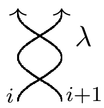


both of which follow from (3.3.11).

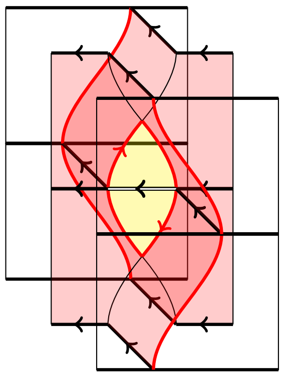
– Relation (3.2.18): The equality $(\alpha_i, \alpha_j) = 0$ corresponds to $|i - j| \geq 2$ in which case the image of the relation is realized via an isotopy. For example, when $i < j$ we see that



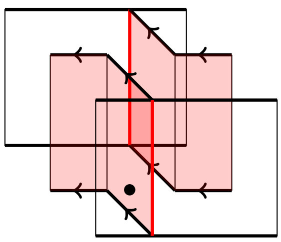
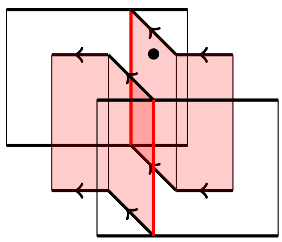
is isotopic to the identity foam for any values of the a_l's.
If $i \neq j$ and $(\alpha_i, \alpha_j) \neq 0$ we must have $j = i \pm 1$. We begin with the case $j = i + 1$. The image of the left-hand side is zero unless $a_{i+1} = 1$ since the intermediate objects in the relation map to the zero object in the image; similarly, the right-hand image is zero unless $a_{i+1} = 1, 2$.
When $a_{i+1} = 1$ we have

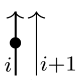


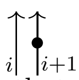
$\mapsto$



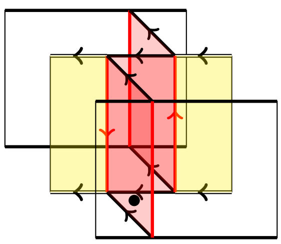
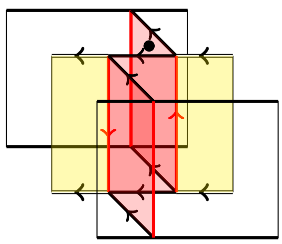
where we have omitted the shading on the front and back sheets for clarity. Applying (3.3.3) on both sides of the singular seam and evaluating the resulting theta-foams using (3.3.8), this gives

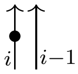
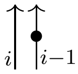
 $-$ 

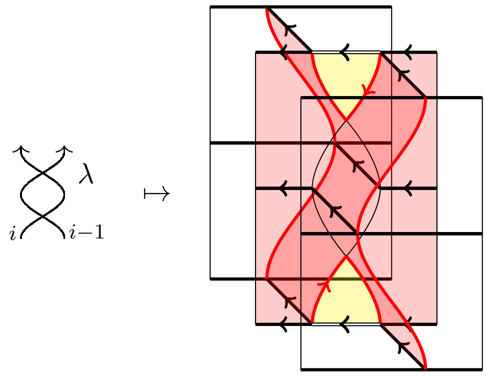
which is the image of 

$-$ 

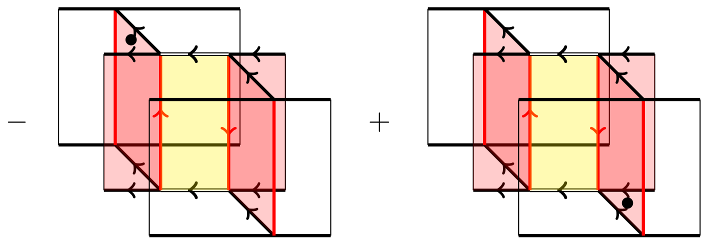
.
When $a_{i+1} = 2$, we must confirm that

 $-$ 

is the zero foam. This follows from the dot-sliding relation (3.3.20).
When $j = i - 1$, it similarly suffices to confirm the relation when $a_i = 0, 1$. For $a_i = 0$, the images of  and  are isotopic, so both sides of the relation map to zero.
For $a_i = 1$, we compute



which equals

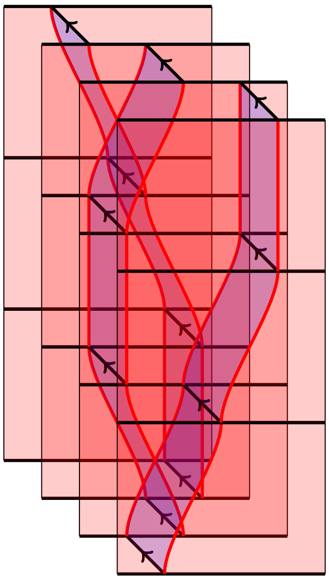


by equations (3.3.11) (turned sideways!) and (3.3.20).

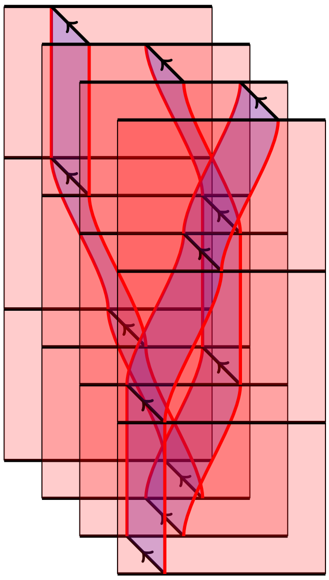
– Relation (3.2.19) follows by sliding a dot along a facet, i.e. via isotopy.

– Relation (3.2.20): For all choices of i, j, and k this relation holds via isotopy (or since both sides map to zero). This is obvious in the case that two of the three values (α_i, α_j), (α_i, α_k) and (α_j, α_k) are zero. In the other cases a computation is necessary; note that we can assume $i \neq j \neq k$ since otherwise both sides of the equation automatically map to zero (an intermediate weight must map to the zero object).
Suppose that $j = i + 1$ and $k = i + 2$, then we compute both sides of the relation to

be



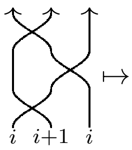
and

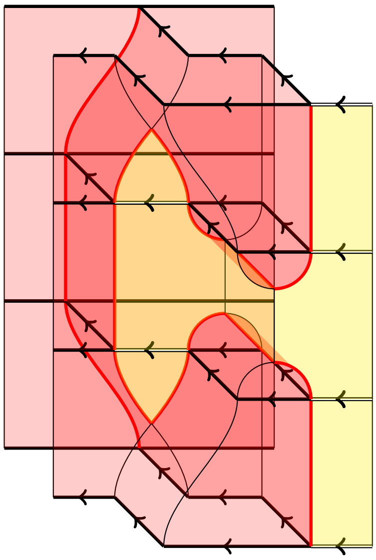


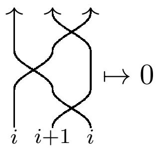
(3.3.21)

which are equal up to isotopy for any value of the a_l's. The other cases follow similarly.

– Relation (3.2.21): We must have $j = i \pm 1$ and we'll only compute for $j = i + 1$ since the other case is analogous. Note that all 2-morphisms involved automatically map to zero if λ is sent to a sequence with $a_{i+1} = 2$ or with $a_i = 0, 1$, so we'll compute for the remaining values.
When $a_i = 2$ and $a_{i+1} = 0$ we have

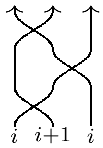


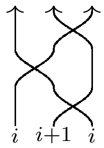


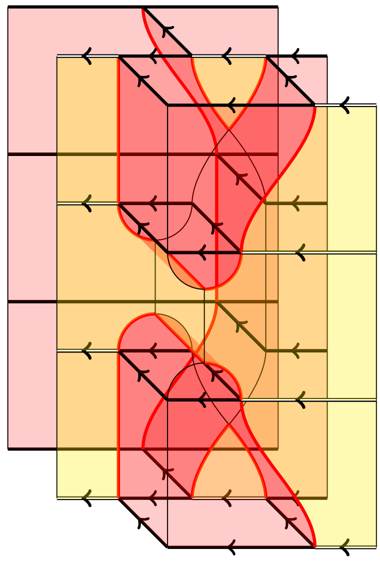


which gives the relation since the former is isotopic to the identity.

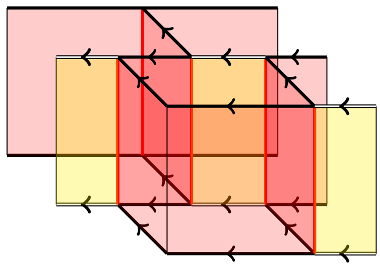
Finally, when $a_i = 2$ and $a_{i+1} = 1$ we compute that



$\mapsto 0,$ 

$\mapsto$  ;

applying (3.3.18) to the above gives the foam

$-$ 

which confirms the relation.

□

Note that the scalings of the images of the cap and cup 2-morphisms play no role in the proof of the proposition. They are determined by the proof of Theorem 3.2.5.

3.3.1.4 Clark-Morrison-Walker foams

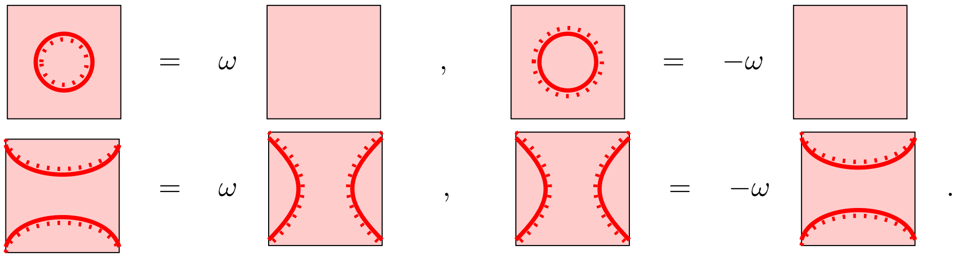
In the original construction of functorial Khovanov homology [CMW09], Clark-Morrison-Walker use a variation of Bar-Natan's 2-category involving disoriented surfaces defined over the Gaussian integers. We can define foamation 2-functors to a family of 2-categories related to their construction. We will assume some familiarity with their work.

We fix once and for all ω to be a primitive fourth root of the unity.

Definition 3.3.4. **CMWFoam**$_m(N)$ is the 2-category defined as follows:

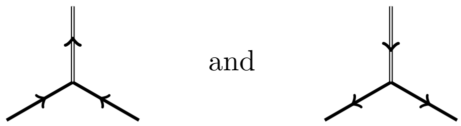
- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2\}$ and $N = \sum_{i=1}^m a_i$, together with a zero object.
- 1-morphisms are formal direct sums of $\mathbb{Z}$-graded disoriented planar tangles directed out from 1-labeled points in the domain and into such points in the codomain.
- 2-morphisms are formal matrices of $\Bbbk[\omega]$-linear combinations of degree-zero dotted disoriented cobordisms between such disoriented planar tangles, modulo isotopy and local relations.

The disorientations are represented by fringed seams; the local relations are given by (3.3.2) and (3.3.3) in regions where no seams are present and the following local seam relations:



(3.3.22)

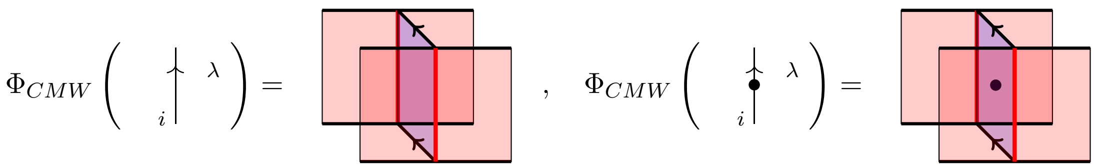
By adjusting some coefficients in the formulation of Proposition 3.3.3 and appropriately interpreting the image foams as 2-morphisms in $\mathbf{CMWFoam}_m(N)$, we obtain the desired 2-functor. The interpretation is as follows:



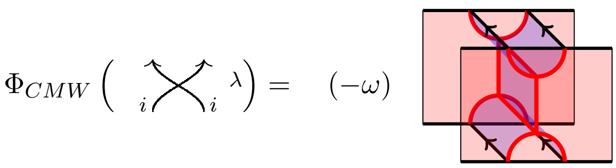
should be read as the disoriented tangles

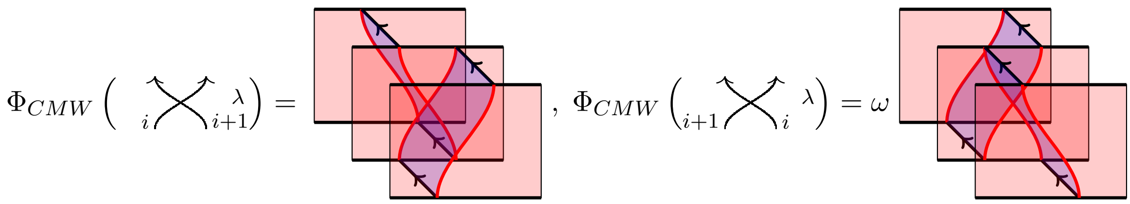
and the 2-labeled sheets should be deleted from the image foams, retaining the seams and adding fringes aligned with the disorientation "tags" on the tangles.

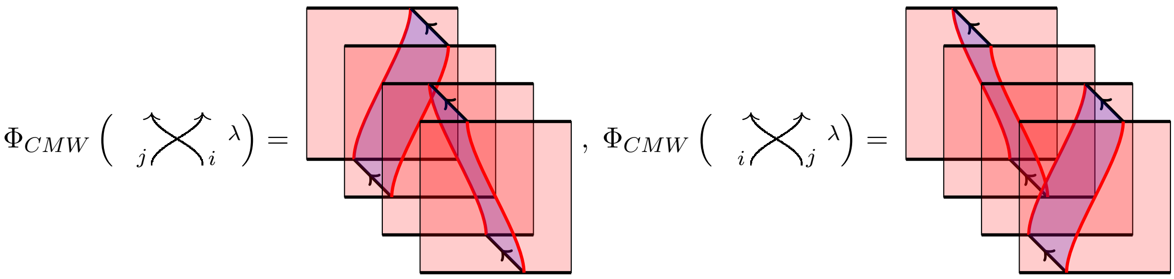
Proposition 3.3.5. For each $N > 0$ there is a 2-representation $\Phi_{CMW} \colon \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathbf{CMWFoam}_m(N)$ defined on single strand 2-morphisms by:



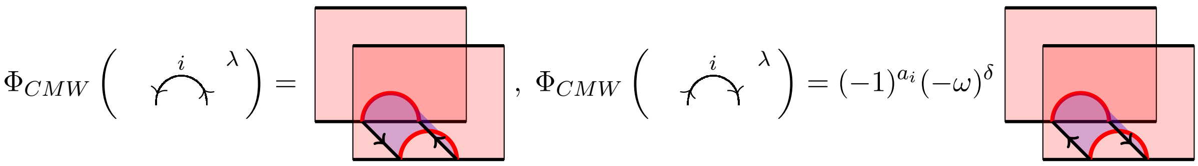
on crossings by:

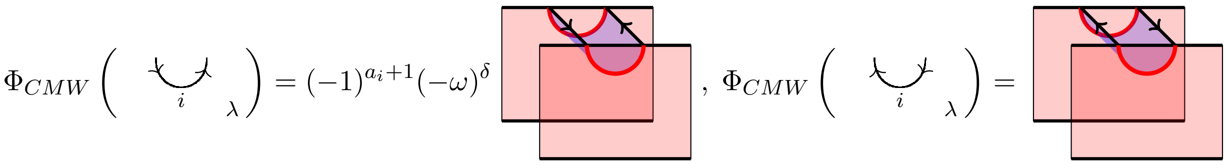






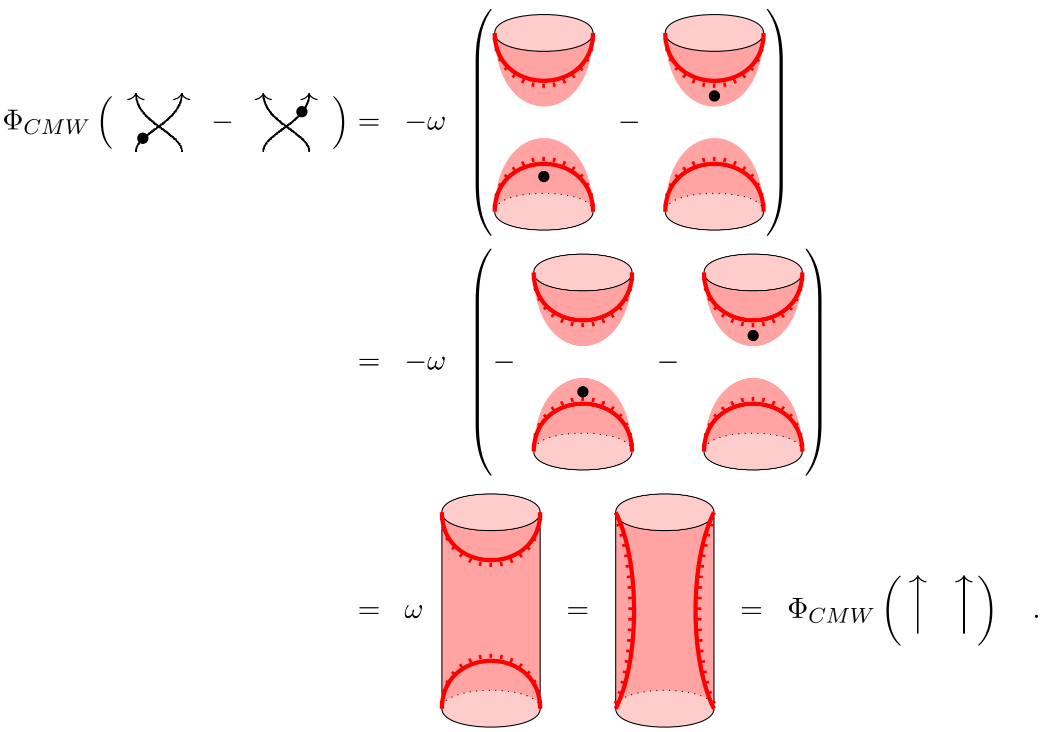
where $j - i > 1$, and on caps and cups by:





where in the above diagrams the i^{th} sheet is always in the front, and $\delta = 1$ if λ_i is even and $\delta = 0$ otherwise.

The proof is the same as for Proposition 3.3.3: we apply Theorem 3.2.5. Most of the work involves checking the KLR relations and is straightforward, so we omit almost all of the details. The following calculation confirms the nilHecke relation (3.2.17), which we include to show the importance of the disorientation seams:



In the above pictures we have applied isotopies to the disoriented tangles and cobordisms for clarity. This relation determines the scaling of the (i,i) crossing in the definition above. The KLR relations also fix the scaling on the composition of an $(i,i+1)$ and an $(i+1,i)$ crossing; we choose to rescale the $(i+1,i)$ crossing. The scalings of the other 2-morphisms again follow from the proof of Theorem 3.2.5.

At present time, we don't have a good explanation for the rescalings in the above 2-functor. For this reason, we believe that $2\mathbf{BFoam}_m(N)$ is a more natural setting for the foamation 2-functors; in particular, we'll see in Proposition 3.3.10 that the definition of the foamation functor from Proposition 3.3.3 generalizes mutatis mutandis to give a foamation 2-functor to an enhanced version of $\mathfrak{sl}_3$ foams.

3.3.2 $\mathfrak{sl}_3$ foam 2-categories

In this section, we recall the definition of the $\mathfrak{sl}_3$ foam 2-category and prove the existence of the $\mathfrak{sl}_3$ foamation 2-functor. We then define an enhanced $\mathfrak{sl}_3$ foam 2-category similar to Blanchet's $\mathfrak{sl}_2$ foam category which appears naturally in the categorical skew Howe context. Finally, we give a functor from enhanced foams to standard foams, contrasting the $\mathfrak{sl}_2$ case.

3.3.2.1 Standard $\mathfrak{sl}_3$ foams

In [Kho04], Khovanov gives a foam based categorification of the $\mathfrak{sl}_3$ link invariant. This construction was generalized by Mackaay-Vaz in [MV07] and Morrison-Nieh in [MN08] in the spirit of Bar-Natan's $\mathfrak{sl}_2$ cobordism 2-category [BN05], giving a categorification of Kuperberg's $\mathfrak{sl}_3$ spider [Kup96]. Mackaay and Vaz showed [MV08] that these foam based constructions of $\mathfrak{sl}_3$ link homologies coincide with the $n = 3$ case of Khovanov and Rozansky's $\mathfrak{sl}_n$ link homologies defined via matrix factorizations [KR08a].

We now recall the definition of this 2-category, which we denote $3\mathbf{Foam}$, using a hybrid of the above approaches:

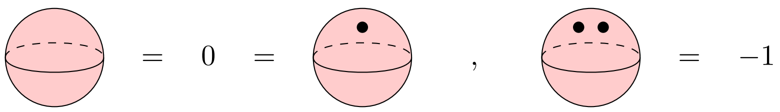
- Objects are sequences of points in the interval $[0,1]$ labeled by 1 or 2, together with a zero object.
- 1-morphisms are formal direct sums of $\mathbb{Z}$-graded $\mathfrak{sl}_3$ webs - directed, trivalent planar graphs with boundary in which each (interior) vertex is a sink or a source - where an edge is directed out from a point labeled by 1 and into a point labeled by 2 in the domain and vice-versa in the codomain.
- 2-morphisms are formal matrices of $\Bbbk$-linear combinations of degree-zero $\mathfrak{sl}_3$ foams - dotted surfaces with oriented singular seams which locally look like the product of the letter Y with an interval - considered up to isotopy (relative to the boundary) and local relations.

Denoting the $\mathbb{Z}$-grading of a web by the monomial q^t for $t \in \mathbb{Z}$, the degree of a foam $F : q^{t_1}W_1 \to q^{t_2}W_2$ is given by the formula

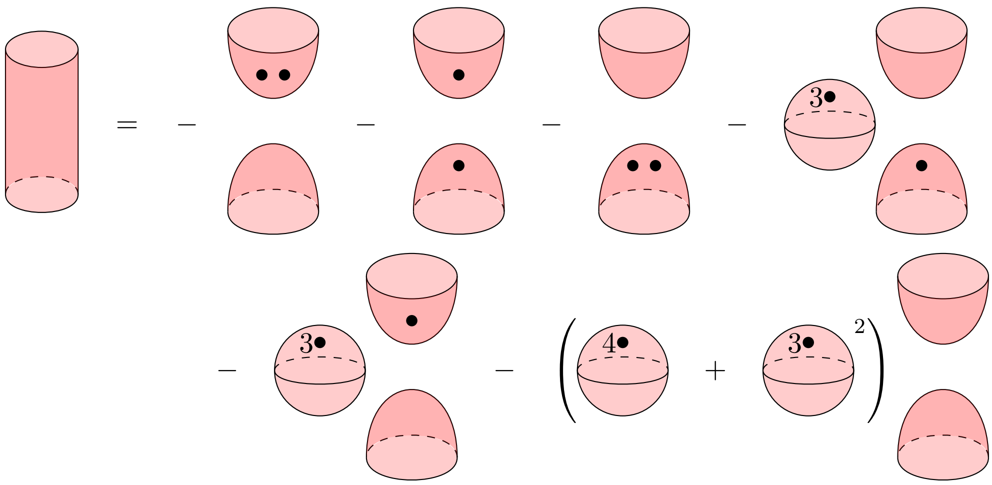
$$\deg(F) = 2\chi(F) - \#\partial + \frac{\#V}{2} + t_2 - t_1$$

where $\chi(F)$ is the Euler characteristic of the foam F, $\#\partial$ is the number of boundary points in either W_1 or W_2 (they agree!), and $\#V$ is the total number of trivalent vertices in $W_1 \coprod W_2$. A dot should be viewed as a puncture for the sake of computing an $\mathfrak{sl}_3$ foam's Euler characteristic. We shall depict $\mathfrak{sl}_3$ foams using the colors red and blue, for clarity; unlike the $\mathfrak{sl}_2$ case, these colors have no meaning as all facets are treated equally.

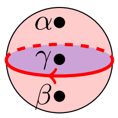
The local $\mathfrak{sl}_3$ foam relations are as follows (where a number next to a dot denotes the number of dots present):



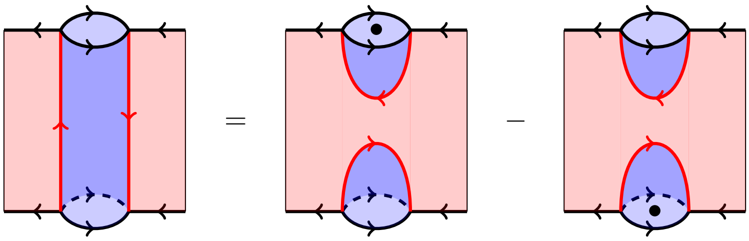
(3.3.23)



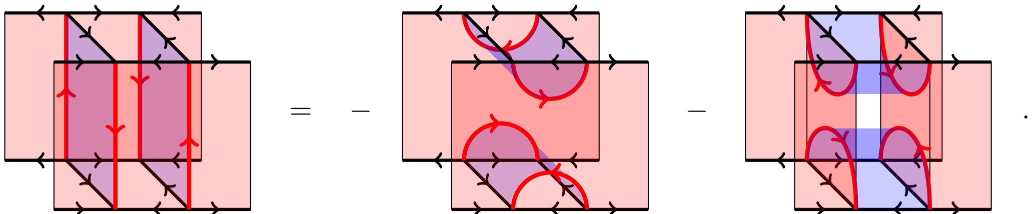
(3.3.24)



$$= \begin{cases} 1 & \text{if } (\alpha,\beta,\gamma) = (0,1,2) \text{ or a cyclic permutation} \\ -1 & \text{if } (\alpha,\beta,\gamma) = (0,2,1) \text{ or a cyclic permutation} \\ 0 & \text{all other triples with } \alpha,\beta,\gamma \le 2 \end{cases} \tag{3.3.25}$$



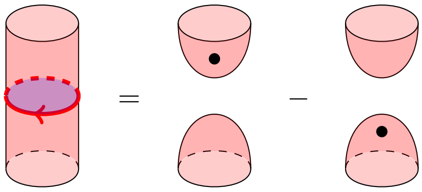
(3.3.26)

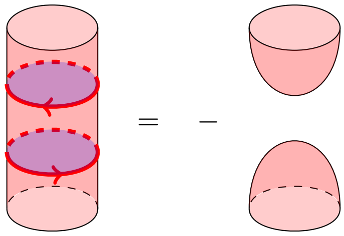


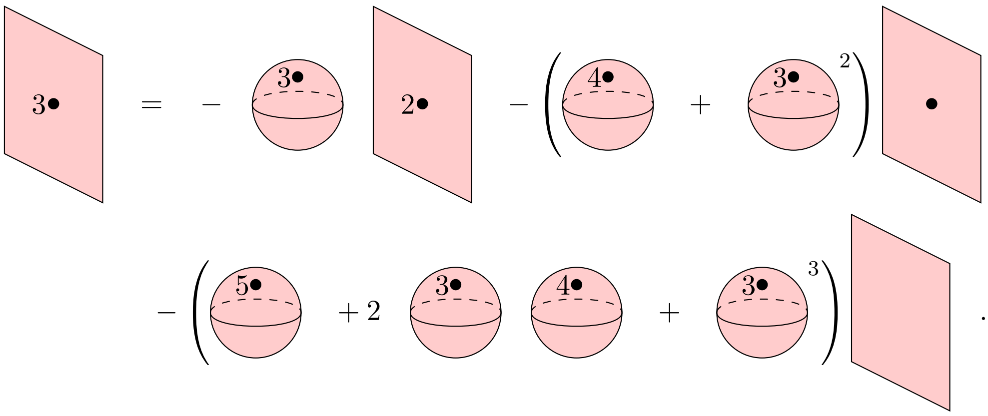
(3.3.27)

Note that the local foam relations are all degree-homogeneous. The direction of the singular seams keeps track of a cyclic ordering of the incident facets; by convention, we take this ordering to be given by the right-hand rule. This convention is opposite to that used in [Kho04] and [MV07], hence one would expect to see opposite foam relations above; however, we also reverse the relation between singular seams and trivalent vertices (seams are directed up through source vertices and down through sink vertices) which corresponds to taking different generating morphisms for the 2-category. It is easy to see that the above 2-category is isomorphic to the standard $\mathfrak{sl}_3$ 2-category. Our conventions are chosen to align with those in the definition of the 2-category $2\mathbf{BFoam}_m(N)$.

Using the neck-cutting relation (3.3.24) and the theta-foam relation (3.3.25), we can derive the following local relations:

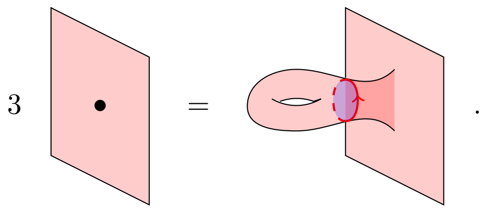

(3.3.28)


(3.3.29)



(3.3.30)

The values of the 3-, 4-, and 5-dotted spheres should be viewed as (graded) parameters which are typically set equal to zero in the literature, e.g. in [Kho04] and [MN08], although this is not required for our considerations. In the case that the 3-dotted sphere is zero, Morrison-Nieh show the relation



which allows for a completely topological description of this 2-category when 3 is invertible in $\Bbbk$.

Note that the set of relations above does not explicitly correspond with that from either [MV07] or [MN08]. The relations (3.3.23), (3.3.24), (3.3.25), together with a non-local relation constitute the relations from [MV07] (although in that work the authors introduce parameters a, b, and c in place of the dotted-sphere parameters above). In [MN08], Morrison and Nieh show that the relations (3.3.26) and (3.3.27) imply the non-local relation. Our chosen set of relations above almost agree with those of Morrison-Nieh (when the dotted surface parameters equal zero): they impose the relation that reversing the orientation of a singular seam negates the foam instead of specifying the values of the theta-foams; this seam reversal relation follows from (3.3.24) and (3.3.25).

As in the $\mathfrak{sl}_2$ case, we are interested in a related family of 2-categories which is natural to consider from the perspective of categorical skew Howe duality.

Definition 3.3.6. $3\mathbf{Foam}_m(N)$ is the 2-category defined as follows:

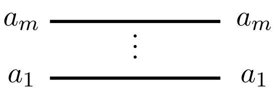
- Objects are sequences $(a_1, \dots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2,3\}$ and $N = \sum_{i=1}^m a_i$ together with a zero object.
- 1-morphisms are formal direct sums of $\mathbb{Z}$-graded $\mathfrak{sl}_3$ webs mapping between the points labeled by 1 and 2 as in $3\mathbf{Foam}$.
- 2-morphisms are formal matrices of $\Bbbk$-linear combinations of degree-zero $\mathfrak{sl}_3$ foams mapping between such webs.

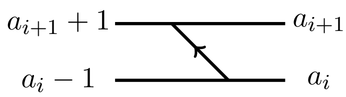
Note that the objects in $3\mathbf{Foam}_m(N)$ correspond with the direct summands appearing in the decomposition of $\bigwedge_q^N \left(\mathbb{C}^3 \otimes \mathbb{C}^m\right)$ into $\mathfrak{sl}_m$ weight spaces and 1-morphisms correspond to $\mathfrak{sl}_3$ intertwiners between such summands. For each m and N, there is an obvious 2-functor $3\mathbf{Foam}_m(N) \to 3\mathbf{Foam}$ which forgets the 0's and 3's.

3.3.2.2 Foamation

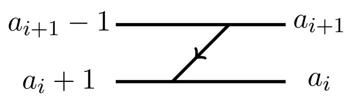
We now define $\mathfrak{sl}_3$ foamation 2-functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{Foam}_m(N)$. As in the $\mathfrak{sl}_2$ case, the 2-functor is defined on objects by sending an $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \dots, \lambda_{m-1})$ to the sequence $(a_1, \dots, a_m)$ with $a_i \in \{0,1,2,3\}$, $\lambda_i = a_{i+1} - a_i$, and $\sum_{i=1}^m a_i = N$ provided such a sequence exists and to the zero object otherwise.

The map on 1-morphisms is again given by:

$\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

$\mathcal{E}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

and

$\mathcal{F}_i\mathbf{1}_\lambda\{t\} \mapsto q^t$ 

when the boundary values lie in $\{0,1,2,3\}$ and to the zero 1-morphism otherwise. Note that the orientation of the undirected strands (and whether they become deleted) is determined by these boundary values and that we have not depicted $m-2$ horizontal strands in each of the latter two formulae.

We will make use of a preparatory lemma to deduce the existence of the foamation 2-functors. Let the images of $\mathbf{1}_\lambda$, $\mathcal{E}_i\mathbf{1}_\lambda$, and $\mathcal{F}_i\mathbf{1}_\lambda$ given above be denoted $\mathbb{1}_\lambda$, $\mathsf{E}_i\mathbb{1}_\lambda$, and $\mathsf{F}_i\mathbb{1}_\lambda$.

Lemma 3.3.7. There are isomorphisms

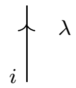
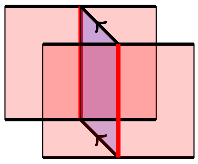
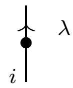
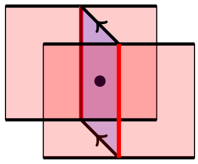
$$\mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \oplus_{[-\langle i,\lambda\rangle]} \mathbb{1}_\lambda \text{ if } \langle i,\lambda\rangle \leq 0$$

$$\mathsf{E}_i\mathsf{F}_i\mathbb{1}_\lambda \cong \mathsf{F}_i\mathsf{E}_i\mathbb{1}_\lambda \oplus_{[\langle i,\lambda\rangle]} \mathbb{1}_\lambda \text{ if } \langle i,\lambda\rangle \geq 0,$$

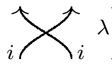
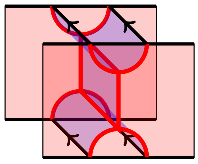
and $\mathsf{F}_j\mathsf{E}_i\mathbb{1}_\lambda \cong \mathsf{E}_i\mathsf{F}_j\mathbb{1}_\lambda$ for $i \neq j \in I$ in $3\mathbf{Foam}_m(N)$.

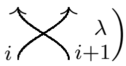
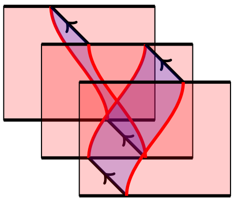
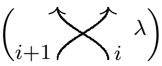
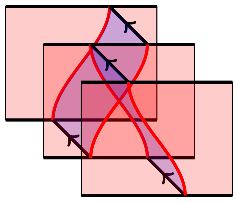
Proof. The proof is similar to the proof of Lemma 3.3.2. $\square$

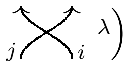
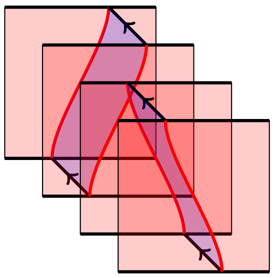
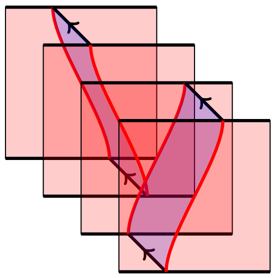
Proposition 3.3.8. For each $N > 0$ there is a 2-representation $\Phi_3 \colon \mathcal{U}_Q(\mathfrak{sl}_m) \to \mathbf{3Foam}_m(N)$ defined on single strand morphisms by:

$\Phi_3\Big($  $\Big) =$  , $\Phi_3\Big($  $\Big) =$ 

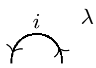
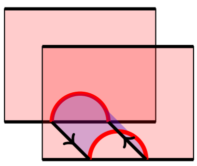
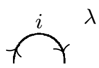
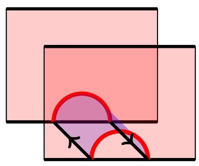
on crossings by:

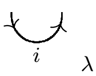
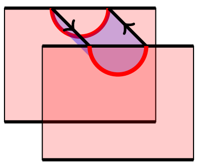
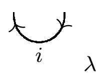
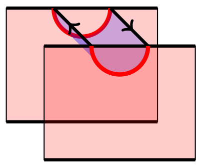
$\Phi_3\Big($  $\Big) =$ 

$\Phi_3\Big($  $\Big) =$  , $\Phi_3\Big($  $\Big) = (-1)^{a_{i+1}+1}$ 

$\Phi_3\Big($  $\Big) =$  , $\Phi_3\Big($  $\Big) =$ 

where $j - i > 1$, and on caps and cups by:

$\Phi_3\Big($  $\Big) = \pm$  , $\Phi_3\Big($  $\Big) = \pm$ 

$\Phi_3\Big($  $\Big) = \pm$  , $\Phi_3\Big($  $\Big) = \pm$ 

where the $\pm$ signs above depend on (the image of) the weight λ and are given by [5]:

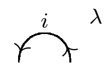
Counterclockwise cap	Sign	Clockwise cap	Sign
$N_i = 3,\ \lambda_i = 1$	-	$N_i = 3,\ \lambda_i = -1$	-
$N_i = 2,\ \lambda_i = 0$	+	$N_i = 2,\ \lambda_i = 0$	-
$N_i = 4,\ \lambda_i = 0$	-	$N_i = 4,\ \lambda_i = 0$	-
$N_i = 1,\ \lambda_i = -1$	+	$N_i = 1,\ \lambda_i = 1$	+
$N_i = 3,\ \lambda_i = -1$	+	$N_i = 3,\ \lambda_i = 1$	-
$N_i = 5,\ \lambda_i = -1$	-	$N_i = 5,\ \lambda_i = 1$	-
$N_i = 2,\ \lambda_i = -2$	+	$N_i = 2,\ \lambda_i = 2$	+
$N_i = 4,\ \lambda_i = -2$	+	$N_i = 4,\ \lambda_i = 2$	-
$N_i = 3,\ \lambda_i = -3$	+	$N_i = 3,\ \lambda_i = 3$	+

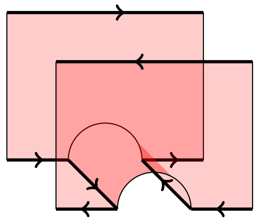
Counterclockwise cup	Sign	Clockwise cup	Sign
$N_i = 3,\ \lambda_i = 1$	+	$N_i = 3,\ \lambda_i = -1$	+
$N_i = 2,\ \lambda_i = 0$	+	$N_i = 2,\ \lambda_i = 0$	+
$N_i = 4,\ \lambda_i = 0$	-	$N_i = 4,\ \lambda_i = 0$	+
$N_i = 1,\ \lambda_i = -1$	+	$N_i = 1,\ \lambda_i = 1$	+
$N_i = 3,\ \lambda_i = -1$	-	$N_i = 3,\ \lambda_i = 1$	+
$N_i = 5,\ \lambda_i = -1$	-	$N_i = 5,\ \lambda_i = 1$	-
$N_i = 2,\ \lambda_i = -2$	-	$N_i = 2,\ \lambda_i = 2$	+
$N_i = 4,\ \lambda_i = -2$	-	$N_i = 4,\ \lambda_i = 2$	-
$N_i = 3,\ \lambda_i = -3$	-	$N_i = 3,\ \lambda_i = 3$	-

(3.3.31)

where $N_i = a_i + a_{i+1}$.

Again, the foams drawn above are general depictions of the image. To obtain the specific image of a generating 2-morphism, we delete any facets incident upon web strands which are deleted. The singular seams in such pictures may degenerate in such situations, e.g. in the case that λ maps to a sequence with $a_i = 1, a_{i+1} = 2$ we have

$$\Phi_3\Bigg($$ 

$$\Bigg) = -$$ 

which is a saddle cobordism.

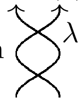
Proof. As with the proof of Proposition 3.3.3, we apply Theorem 3.2.5. Conditions (1) and (3) follow as before and Lemma 3.3.7 gives conditions (4) and (6).

Working in the setting where the 3-, 4-, and 5-dotted spheres are set equal to zero, it is shown in [MN08] that the vector space $\mathrm{Hom}(\mathbb{1}_\lambda, q^t\mathbb{1}_\lambda)$ is zero for $t < 0$ and one-dimensional for $t = 0$ (provided $\mathbb{1}_\lambda$ is non-zero) and that for any webs W_1 and W_2, the vector space $\mathrm{Hom}(q^{t_1}W_1, q^{t_2}W_2)$ is finite dimensional. The same arguments show that this

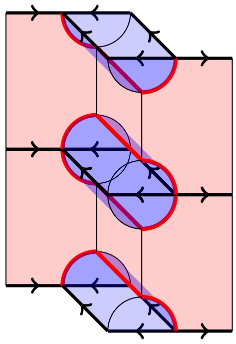
5. In fact, we will see that the $\pm$ signs involved in the definition of the 2-functor on caps and cups play no role in the proof of this proposition; they are determined by the proof of Theorem 3.2.5. In the next section, we will give a topological interpretation of this system of signs.

holds when these dotted spheres are not set equal to zero (since they have negative Euler characteristic). This confirms condition (2).

We thus conclude by checking condition (5), the KLR relations:

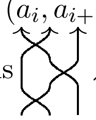
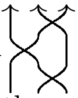
– Relation (3.2.16): the 2-morphism 

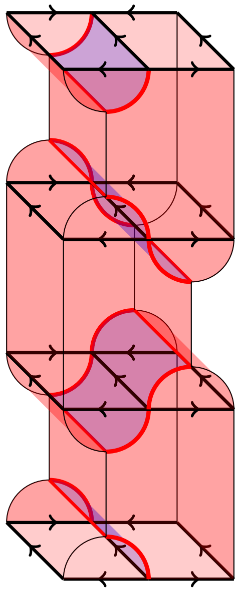
maps to a foam which can only possibly be non-zero for λ mapping to sequences with $a_i = 2, 3$ and $a_{i+1} = 0, 1$. When $(a_i, a_{i+1}) = (2, 0)$ we compute the image:



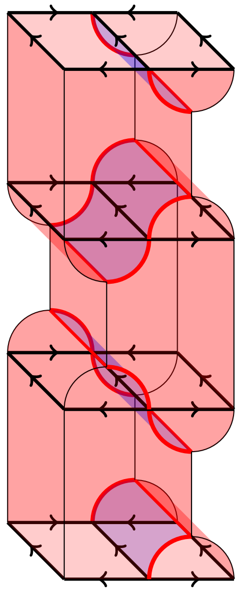
$= 0$

which follows from neck-cutting in a neighborhood of the center singular seam. The computation for the remainder of the values of (a_i, a_{i+1}) follows similarly.

We next compute the images of the 2-morphisms  λ and  λ, noting that the images can only be non-zero for λ mapping to sequences with $a_i = 3$ and $a_{i+1} = 0$. The above 2-morphisms map to the foams



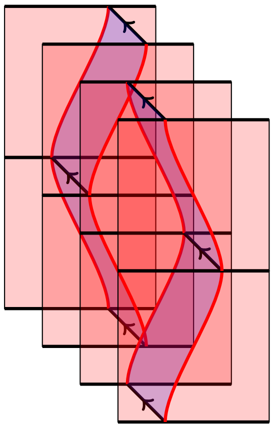
and



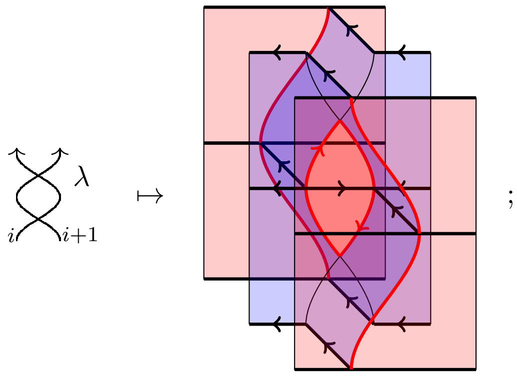
respectively; these are equal by equation (3.3.29).

– Relation (3.2.17): the images of this relation are simply a restatement of equation (3.3.26).

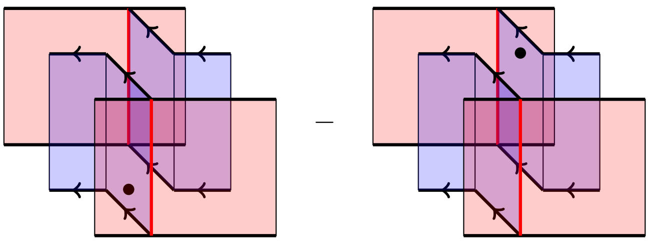
– Relation (3.2.18): The equality $(\alpha_i, \alpha_j) = 0$ corresponds to $|i - j| \geq 2$ in which case the image of the relation is realized via isotopy. For example, when $i < j$ we see that

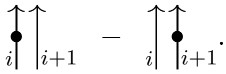


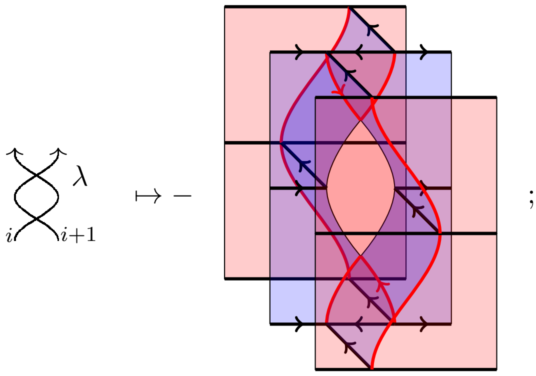
is isotopic to the identity foam.
When $i \neq j$ and $(\alpha_i, \alpha_j) \neq 0$ we must have $j = i \pm 1$. We'll compute the image of this relationship in the case $j = i+1$ (the other case is similar). The image of the left-hand side is zero when $a_{i+1} = 0, 3$ since the intermediate objects in the relation map to the zero object in the image; the same is true on the right-hand side when $a_{i+1} = 0$. When $a_{i+1} = 3$, both 2-morphisms involved in the expression on the right-hand side map to the same foam, so the relation is satisfied since $t_{i,i+1} = -t_{i+1,i}$. When $a_{i+1} = 1$ we have



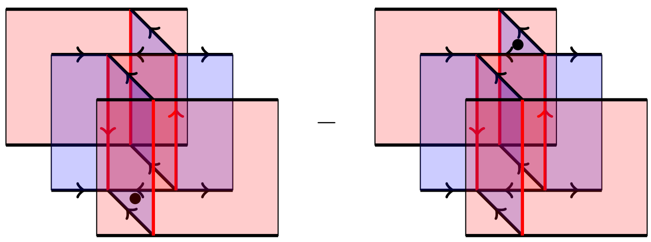
after applying (3.3.28) to the center singular seam this gives the foam

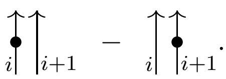


which is the image of  Finally, when $a_{i+1} = 2$



and applying (3.3.26) (turned sideways!) to the region between the semi-circular seams gives the foam

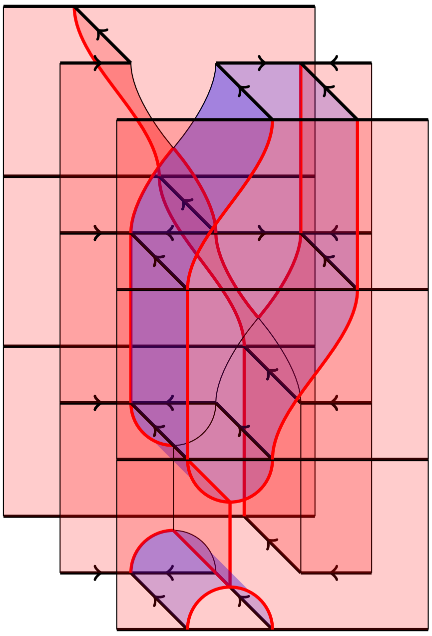


which again is the image of 

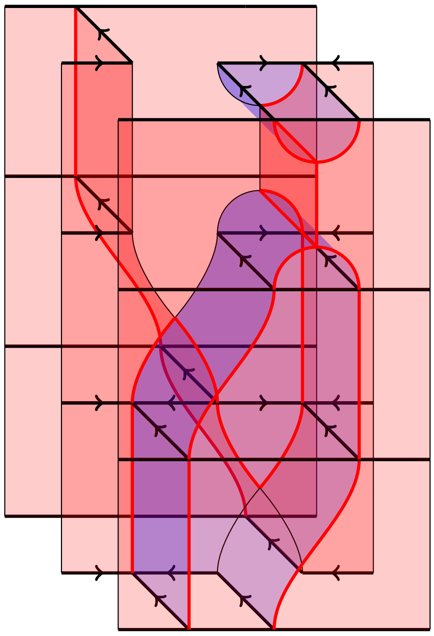
- Relation (3.2.19): These hold by sliding a dot along a facet, i.e. via isotopy.

- Relation (3.2.20): For all choices of i, j, and k this relation holds via isotopy. This is obvious whenever one of the strands carries a label which is at least 2 bigger or smaller than both other labels. In the remaining cases a computation is necessary; we'll exhibit this only for two cases, since the remaining cases follow similarly.
  First, suppose that $j = i$ and $k = i+1$, then both sides of the relation automatically map to zero unless λ maps to a sequence with $a_{i+1} = 1$. We hence compute the

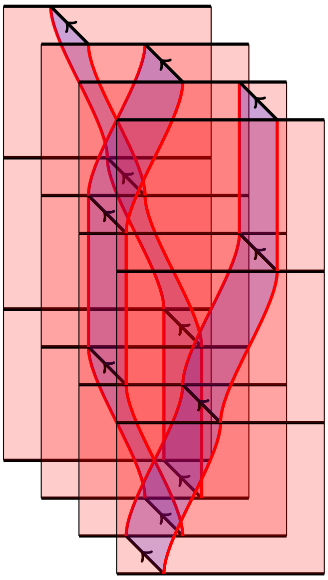
image (of both sides of the relation) in this case, finding them to be



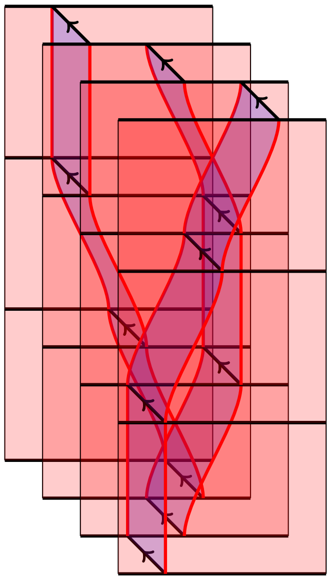
and



which are related via isotopy (no matter which values of a_i and a_{i+2} we choose). Next, suppose that $j = i + 1$ and $k = i + 2$, then we compute both sides of the relation to be



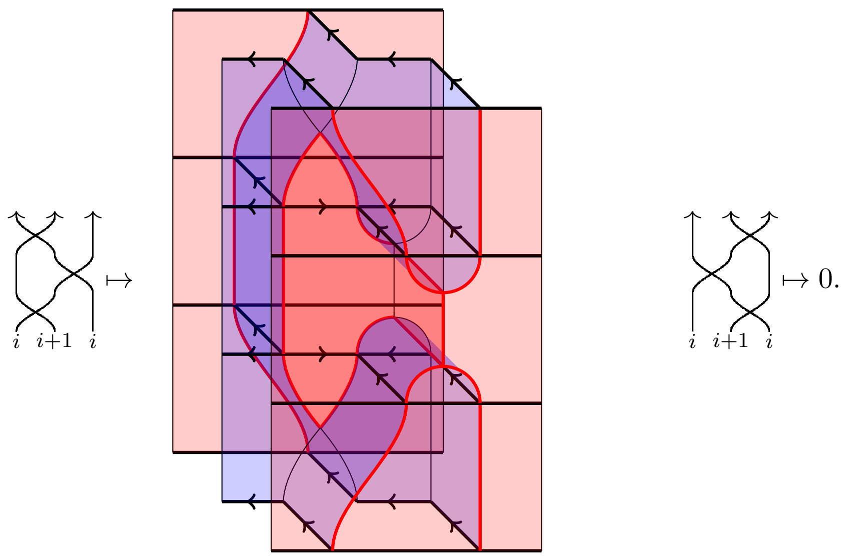
and



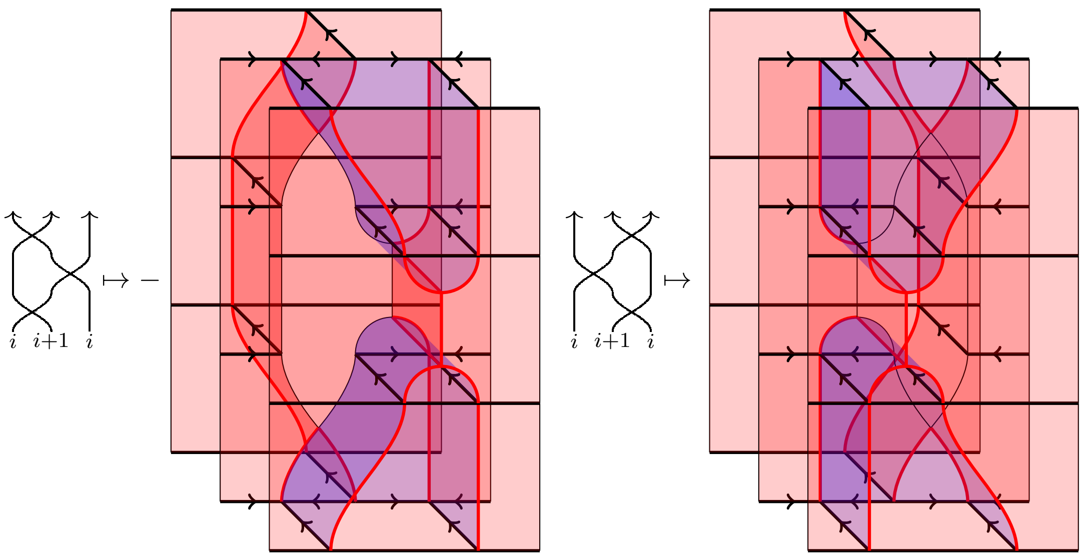
which are equal up to isotopy.

– Relation (3.2.21): We must have $j = i \pm 1$ and we'll only compute for $j = i + 1$ since the other case is analogous. Note that all 2-morphisms involved automatically map to zero if λ is sent to a sequence with $a_{i+1} = 3$ or with $a_i = 0, 1$, so we'll compute for the remaining values.

First, let $a_{i+1} = 0$, then

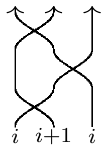


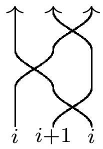
This confirms the relation since the former is isotopic to the identity foam when $a_i = 2, 3$, noting that in these cases either the leftmost or rightmost front facet is deleted.
Next, let $a_{i+1} = 1$, then we compute

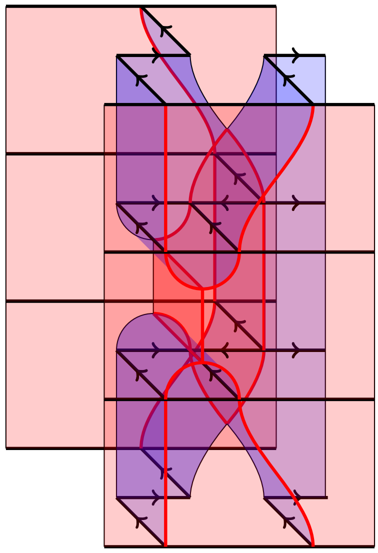


the relation then follows from equation (3.3.27) when $a_i = 2, 3$, again noting that either the leftmost or rightmost front facet is deleted for both foams.

Finally, if $a_{i+1} = 2$, then



$\mapsto 0$


$\mapsto -$

;

again, this confirms the relation since the latter is the identity foam when $a_i = 2, 3$.
□

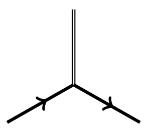
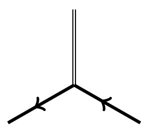
3.3.2.3 Enhanced $\mathfrak{sl}_3$ foams

We now aim to better explain the signs in (3.3.31) which give the scalings of the cap and cup 2-morphisms. We take inspiration from Blanchet's $\mathfrak{sl}_2$ foam construction in which the edges of webs labeled by 2 and the corresponding facets of foams play a special role (and in particular are not deleted).

We hence define an $\mathfrak{sl}_3$ foam 2-category in which we retain 3-labeled edges and the corresponding 3-labeled facets. Although such a construction is not suggested at the decategorified level (as it was in the $\mathfrak{sl}_2$ case) we will see that the foamation functor is much more natural to define in this context and that an appropriately defined functor which forgets the 3-labeled data gives a topological interpretation of the scalings. We believe that such n-labeled facets will play a role in extending the work in this paper to the $n \geq 4$ case; this will be the subject of a follow-up paper [LQR].

Definition 3.3.9. 3**BFoam**$_m(N)$ is the 2-category defined as follows:

- Objects are sequences $(a_1, \ldots, a_m)$ labeling points in the interval $[0,1]$ with $a_i \in \{0,1,2,3\}$ and $N = \sum_{i=1}^m a_i$ together with a zero object.
- 1-morphisms are formal direct sums of $\mathbb{Z}$-graded enhanced $\mathfrak{sl}_3$ webs - trivalent planar graphs with boundary with edges of two types: directed edges  and undirected "3-labeled" edges where vertices involving only the directed edges are as in 3**Foam** and vertices involving the 3-labeled edges take the form

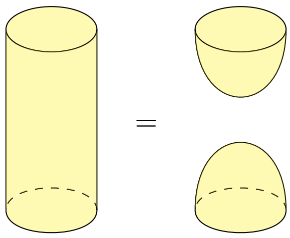
 or  .

Oriented edges are directed out from points labeled by 1 and into points labeled by 2 in the domain and vice-versa in the codomain and 3-labeled edges are attached to points labeled by 3 in both the domain and codomain. As in $3\mathbf{Foam}_m(N)$, no edges are attached to points labeled by 0.

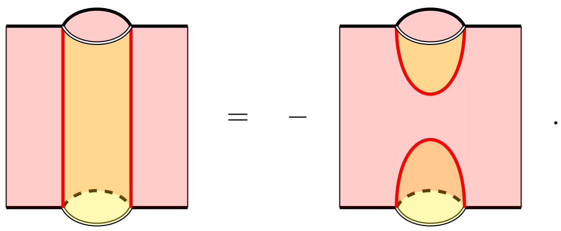
– 2-morphisms are $\mathfrak{sl}_3$ foams between such webs where we allow additional 3-labeled facets incident upon the 3-labeled strands of the webs and attached to the remainder of the foam along singular seams which are allowed to intersect the traditional singular seams; the 3-labeled facets are not allowed to carry dots. We impose local relations on these foams.

We shall refer to the 2-morphisms in this category as "Blanchet" $\mathfrak{sl}_3$ foams and depict the 3-labeled facets in yellow. The traditional facets of these foams will continue to be depicted in both red and blue.

The local relations are given by the relations in $3\mathbf{Foam}$ in regions where 3-labeled facets are not present with additional relations for the 3-labeled facets. The seams where a 3-labeled facet meet the traditional facets are allowed to move freely on the foam (relative to the points where such seams meet the web vertices depicted above). We impose a strong neck-cutting relation for these facets:



and the condition that we may delete any 3-labeled facet F not incident upon the boundary at the cost of multiplying by $(-1)^{\chi(F)}$. Finally, we have the relation



(3.3.32)

An Euler characteristic argument shows that these relations are consistent. Using such foams, we have the following result.

Proposition 3.3.10. The definition in Proposition 3.3.3 describes a family of 2-functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N)$.

As before, we view the definition as showing the general image of each generating 2-morphism; edges connected to points labeled by 0 and facets incident upon them are understood to be deleted. The proof of this proposition follows as in the proof of Propositions 3.3.3 and 3.3.8.

The relations for the 3-labeled facets allow us to delete any facet which does not meet the boundary; however, this is not enough to define a forgetful functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$. We can give such a rule by taking into account the boundary data.

Given a Blanchet $\mathfrak{sl}_3$ foam F, define $\chi_3(F)$ to be the Euler characteristic of its 3-labeled facets and let $n_u(F)$ denote the number of 3-labeled edges in the codomain of F. Let $\Psi(F) = (-1)^{\chi_3(F)-n_u(F)}\bar{F}$ where $\bar{F}$ is the $\mathfrak{sl}_3$ foam obtained from F by deleting the 3-labeled facets (and the 3-labeled edges from the boundary webs). Similarly, define

$n_b(F)$ to be the number of 3-labeled edges in the domain of F and $n_l(F)$ and $n_r(F)$ to be the number of points labeled by 3 in the codomain and domain (respectively) of the webs between which F maps; of course, these later two denote the number of 3-labeled vertical intervals on the left and right boundary of F.

Proposition 3.3.11. The assignment $F \mapsto \Psi(F)$ defines a 2-functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$ where objects are sent to themselves and enhanced webs are sent to the webs obtained by deleting the 3-labeled edges.

Proof. It suffices to show that Ψ is compatible with horizontal and vertical composition of foams. To this end, consider foams F_1, F_2, and F_3 which can be composed as indicated by the following schematic:

	F_2
F_3	F_1

.

We have

$$\begin{aligned}\chi_3(F_1 \cup F_2) - n_u(F_1 \cup F_2) &= \chi_3(F_1 \cup F_2) - n_u(F_2)\\ &= \chi_3(F_1) - n_u(F_1) + \chi_3(F_2) - n_u(F_2)\end{aligned}$$

and

$$\begin{aligned}\chi_3(F_1 \cup F_3) - n_u(F_1 \cup F_3) &= \chi_3(F_1) - n_l(F_1) + \chi_3(F_3) - n_u(F_1) + n_l(F_1) - n_u(F_3)\\ &= \chi_3(F_1) - n_u(F_1) + \chi_3(F_3) - n_u(F_3)\end{aligned}$$

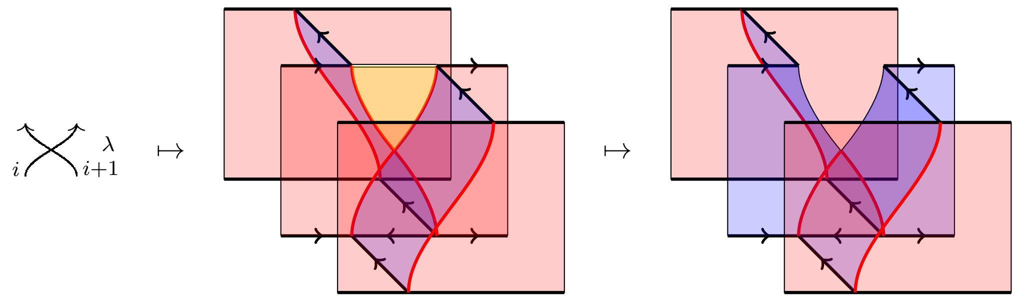
which gives the result. $\square$

One can now consider the composition of the 2-functors defined in Propositions 3.3.10 and 3.3.11.

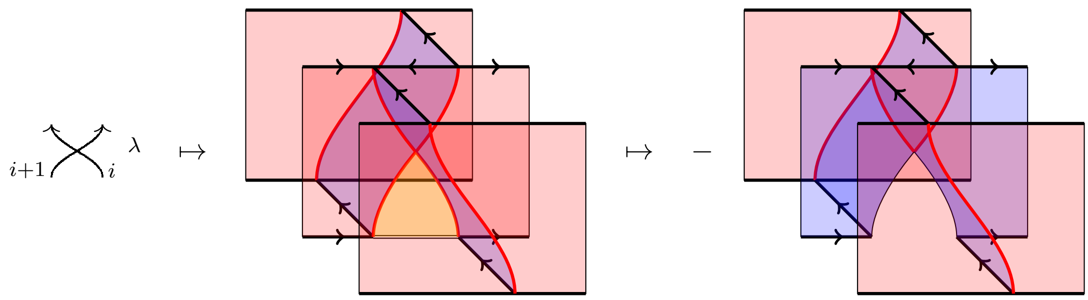
Proposition 3.3.12. The composition $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$ gives the 2-functor from Proposition 3.3.8.

Proof. The proof follows from a routine, yet tedious, calculation. We'll exhibit a few of the more interesting cases:

– Let λ map to a sequence where $a_{i+1} = 2$, then

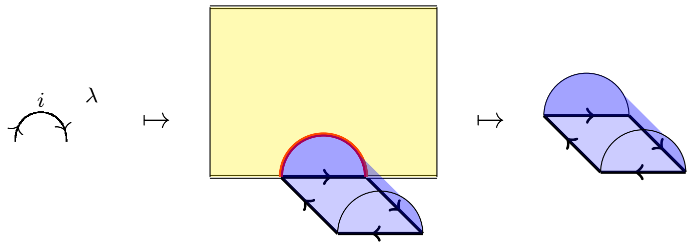


and

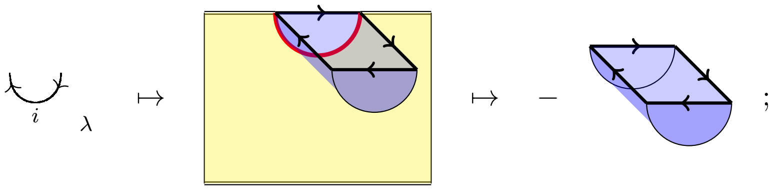


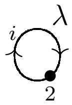
.

– Let λ map to a sequence with $a_i = 0$ and $a_{i+1} = 3$ (i.e. $N_i = 3$ and $\lambda_i = 3$), then



and



note that this guarantees that the relevant degree zero bubble  is sent to the identity 2-morphism in $3\mathbf{Foam}_m(N)$.

□

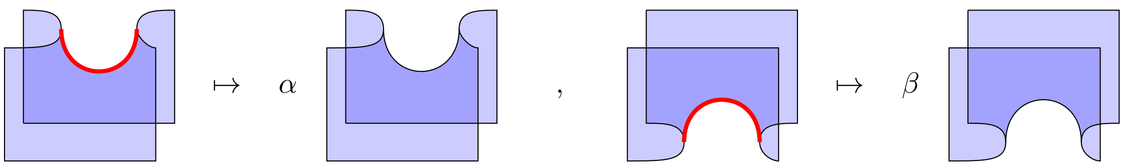
3.3.2.4 Clark-Morrison-Walker $\mathfrak{sl}_3$ foams

One may notice that the topology of a 3-labeled facet is relatively unimportant; the signs obtained by removing any 3-labeled facet (not incident upon the boundary webs) depend only on the facet's boundary seams. One may then ask why one needs these facets at all: couldn't we instead introduce a Clark-Morrison-Walker (CMW) version of $\mathfrak{sl}_3$ foams?

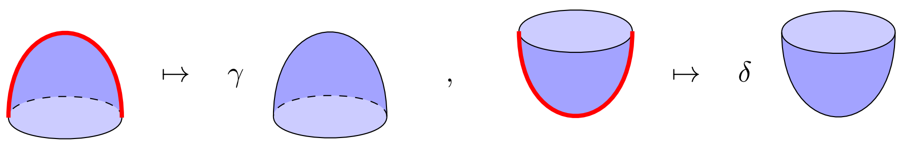
Indeed, we can define such a theory by removing all 3-labeled facets and edges of webs from the definitions in the previous subsection, keeping only the new "CMW seams" and imposing the relation that one may remove a closed seam at the cost of multiplying by -1. It is easy to see that we obtain a family of 2-functors from $\mathcal{U}_Q(\mathfrak{sl}_m)$ to the 2-category of CMW $\mathfrak{sl}_3$ foams. Note that the CMW seams in such a theory do not need fringes, unlike the $\mathfrak{sl}_2$ case.

However, when one tries to define a forgetful 2-functor to the category of (traditional) $\mathfrak{sl}_3$ foams as before, it surprisingly appears that the rigidity obtained from the interaction of the 3-labeled facets with the 3-labeled edges plays a non-trivial role. Indeed, there is no hope to define such a 2-functor as we now demonstrate.

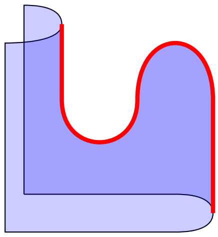
Assume a forgetful 2-functor exists. We need maps



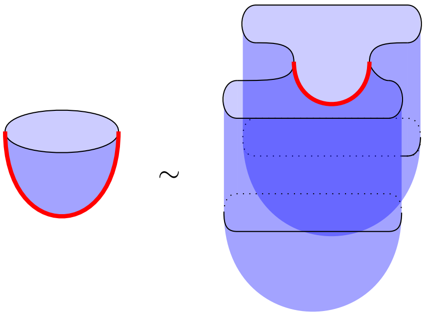
with $\alpha\beta = -1$ and



with $\gamma\delta = -1$. Since the foam



is isotopic to the identity, we must have $\alpha\gamma = 1$ and since



we must have $\delta = \alpha$. This then gives

$$1 = \alpha\gamma = \delta\gamma = -1,$$

a contradiction.

Note that this argument remains valid even if we enhance the CMW seams with fringes (repeat the above argument with all seam directed into the page). The above argument no longer remains valid if the seams are given an orientation; however, similar arguments prevent a definition of CMW $\mathfrak{sl}_3$ foams possessing a forgetful functor to $\mathfrak{sl}_3$ foams.

3.3.3 Extended foamation 2-functors

In order to construct categorified link invariants, we will need to consider the images of the Rickard complexes under the foamation 2-functors and hence the images of divided powers. Recall that the later are 1-morphisms which lie in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ but not in $\mathcal{U}_Q(\mathfrak{sl}_m)$.

The foamation 2-functors $\mathcal{U}_Q(\mathfrak{sl}_m) \to n\mathbf{(B)Foam}_m(N)$ extend to 2-functors

$$\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to Kar(n\mathbf{(B)Foam}_m(N)).$$

We would like to consider these extended 2-functors; however, the Karoubi envelope of the foam 2-categories is not easy to work with. Indeed, the indecomposable 2-morphisms in these 2-categories are closely related to (dual) canonical bases.

It turns out, however, that the foam 2-categories are "closer" to their Karoubi envelopes than the 2-categories $\mathcal{U}_Q(\mathfrak{sl}_m)$ are to $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. Indeed we shall see that the images of the divided powers already exist in the foam 2-categories.

Proposition 3.3.13. The 2-functors defined in Propositions 3.3.3, 3.3.8, and 3.3.10 extend to 2-functors from $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$ to the relevant foam 2-categories.

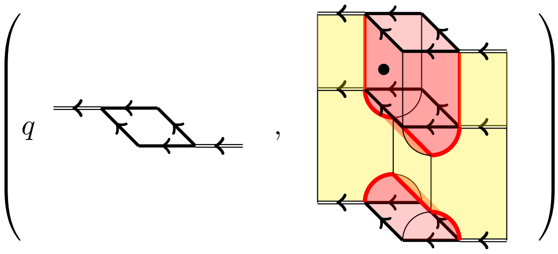
Proof. Recall that any category $\mathcal{C}$ embeds fully faithfully into $Kar(\mathcal{C})$ by sending an object c to the pair (c, id_c) and a morphism $f : c \to d$ to the triple $(\mathrm{id}_c, f, \mathrm{id}_d)$, see [Bor94, Chapter 6.5]. It thus suffices to show that when we restrict the 2-functor $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to$

$Kar(n(\mathbf{B})\mathbf{Foam}_m(N))$ to $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$, the images of all 1-morphisms lie in the subcategory $n(\mathbf{B})\mathbf{Foam}_m(N) \subset Kar(n(\mathbf{B})\mathbf{Foam}_m(N))$ (up to isomorphism).

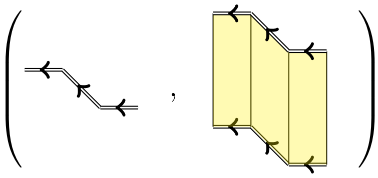
We begin with the $\mathfrak{sl}_2$ case. Since $\mathcal{E}_i^k\mathbf{1}_\lambda \mapsto 0$ and $\mathcal{F}_i^k\mathbf{1}_\lambda \mapsto 0$ for all $k \geq 3$, we need only to consider the 1-morphisms $\mathcal{E}_i^{(2)}\mathbf{1}_\lambda$ and $\mathcal{F}_i^{(2)}\mathbf{1}_\lambda$. Note that the 1-morphism

$$\mathcal{E}_i^{(2)}\mathbf{1}_\lambda = (\mathcal{E}_i^2\mathbf{1}_\lambda\{1\},$$  $$\lambda)$$

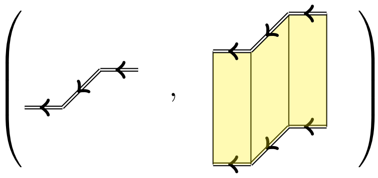
is mapped to zero unless $a_i = 2$ and $a_{i+1} = 0$. In this case, the above is mapped to



which is isomorphic to



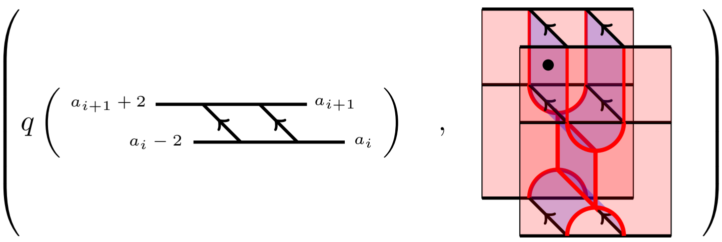
in $Kar(2\mathbf{BFoam}_m(N))$ using equation (3.3.11). Similarly, we find that the image of $\mathcal{F}_i^{(2)}\mathbf{1}_\lambda$ is isomorphic to



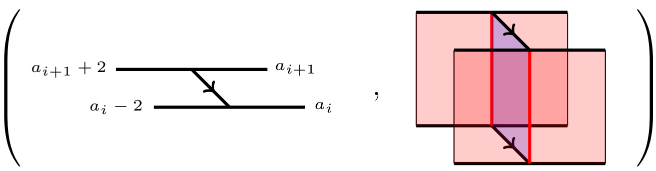
when $a_i = 0$ and $a_{i+1} = 2$ (the only case when $\mathcal{F}_i^2\mathbf{1}_\lambda$ is not mapped to zero).

In the $\mathfrak{sl}_3$ case, it suffices to consider the 2-functor $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{BFoam}_m(N)$, since the 2-functor $\mathcal{U}_Q(\mathfrak{sl}_m) \to 3\mathbf{Foam}_m(N)$ is obtained via composition with the forgetful functor. We find that $\mathcal{E}_i^k\mathbf{1}_\lambda$ and $\mathcal{F}_i^k\mathbf{1}_\lambda$ are both sent to zero for $k \geq 4$, so it suffices to consider the 1-morphisms $\mathcal{E}_i^{(2)}\mathbf{1}_\lambda$, $\mathcal{E}_i^{(3)}\mathbf{1}_\lambda$, $\mathcal{F}_i^{(2)}\mathbf{1}_\lambda$, and $\mathcal{F}_i^{(3)}\mathbf{1}_\lambda$.

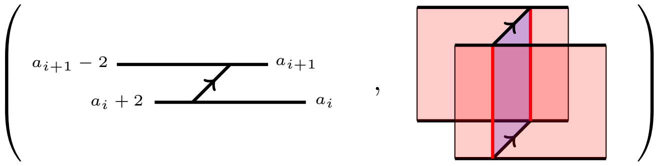
We find that $\mathcal{E}_i^{(2)}\mathbf{1}_\lambda$ is (again) mapped to the 1-morphism



which is isomorphic in $Kar(3\mathbf{BFoam}_m(N))$ to

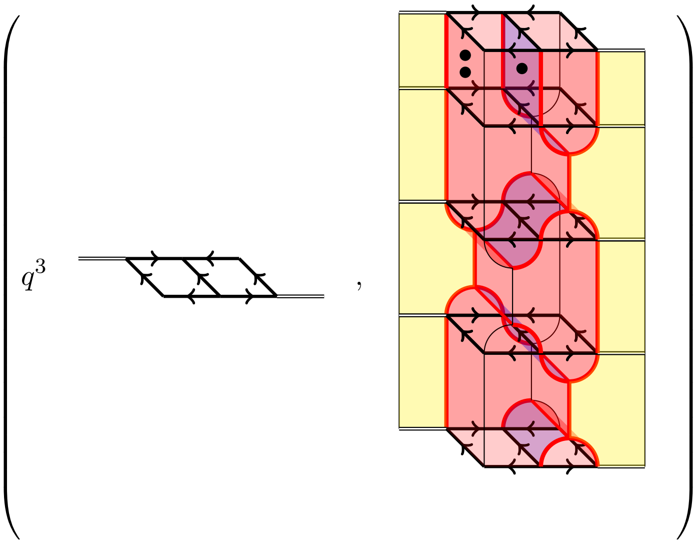


for any value of the a_l's using the $\mathfrak{sl}_3$ foam relations. Similarly, the image of $\mathcal{F}_i^{(2)}\mathbf{1}_\lambda$ is isomorphic to



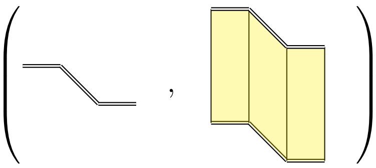
The directed lines in the latter two 1-morphisms above can be viewed as "2-labeled" edges directed in the opposite direction; from this perspective, the boundary labels appear more appropriate.

The only case where $\mathcal{E}_i^3\mathbf{1}_\lambda$ is not sent to zero is for $a_i = 3$ and $a_{i+1} = 0$. and in this case $\mathcal{E}_i^{(3)}\mathbf{1}_\lambda$ is mapped to

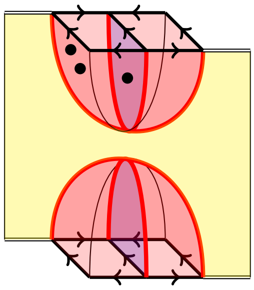


(3.3.33)

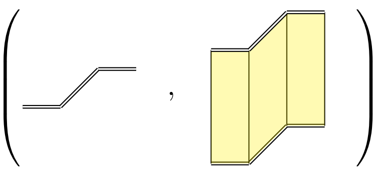
which is isomorphic to



in $Kar(3\mathbf{BFoam}_m(N))$. The isomorphism above is evident after noticing that the foam in (3.3.33) is equal to the following foam:

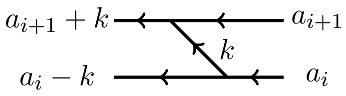


using (3.3.29) and (3.3.32). Finally, the image of $\mathcal{F}_i^{(3)}\mathbf{1}_\lambda$ is isomorphic to

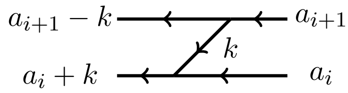


when $a_i = 0$ and $a_{i+1} = 3$ (i.e. the only non-zero case). $\square$

We can summarize the 2-functors $\check{\mathcal{U}}_Q(\mathfrak{sl}_m) \to n\mathbf{(B)Foam}_m(N)$ using ladders with labelings on the diagonal edges:

$\mathcal{E}_i^{(k)}\mathbf{1}_\lambda \mapsto$ 

and

$\mathcal{F}_i^{(k)}\mathbf{1}_\lambda \mapsto$ 

where 2-labeled edges in $3\mathbf{Foam}_m(N)$ and $3\mathbf{BFoam}_m(N)$ should be viewed as 1-labeled edges oriented in the opposite direction and 3-labeled edges should be deleted in $3\mathbf{Foam}_m(N)$ and un-oriented in $3\mathbf{BFoam}_m(N)$. For example, the image of $\mathcal{E}^{(3)}\mathbf{1}_{-3}$ in $3\mathbf{Foam}_2(N)$ is the "empty web" between the sequences $(3,0)$ and $(0,3)$.

3.4 Applications

Many known constructions in link homology follow from the 2-functors defined in the previous section. Indeed, we will re-construct Khovanov homology, $\mathfrak{sl}_3$ link homology, and categorified highest weight projectors in these theories using the categorified quantum Weyl group action. The skew Howe perspective also provides a framework for showing that Cautis and Kamnitzer's algebro-geometric formulation of $\mathfrak{sl}_3$ link homology is isomorphic to Khovanov's $\mathfrak{sl}_3$ link homology (the latter is known to be the same as $\mathfrak{sl}_3$ Khovanov-Rozanksy homology and the category $\mathcal{O}$ $\mathfrak{sl}_3$ link invariant). We also explain how foam relations follow as consequences of the relations in the categorified quantum group.

3.4.1 Link homology via skew Howe duality

In this section, we show that all of the ingredients needed to define $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ link homology theories can be recovered from the foamation functors. We also show how the invariant of any link can be given as the image of a complex in $\check{\mathcal{U}}_Q(\mathfrak{sl}_m)$. This suggests that the graphical calculus in the categorified quantum group can be used to explore properties of categorified link invariants.

3.4.1.1 Categorified braidings

In [CKL10a, Theorem 4.3], Cautis-Kamnitzer-Licata show that the action of the quantum Weyl group elements $T_i 1_\lambda$ on the skew Howe representation $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ gives the braiding on the category of finite-dimensional $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ representations, up to a factor of $\pm q^r$. In the language of webs, this says that the value of a crossing is given by the image of the corresponding quantum Weyl group element.

The same holds true at the categorified level (after extending the foamation 2-functors to 2-categories of complexes), i.e. the complexes assigned to crossings in $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ link homology can be recovered, up to shifts in quantum and homological degree, as the images of the Rickard complexes. Recall these are given by

$$\mathcal{T}_i\mathbf{1}_\lambda = \mathcal{E}_i^{(-\lambda_i)}\mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{E}_i^{(-\lambda_i+1)}\mathcal{F}_i\mathbf{1}_\lambda\{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{E}_i^{(-\lambda_i+s)}\mathcal{F}_i^{(s)}\mathbf{1}_\lambda\{s\} \xrightarrow{d_{s+1}} \cdots$$

when $\lambda_i \leq 0$ and

$$\mathcal{T}_i\mathbf{1}_\lambda = \mathcal{F}_i^{(\lambda_i)}\mathbf{1}_\lambda \xrightarrow{d_1} \mathcal{F}_i^{(\lambda_i+1)}\mathcal{E}_i\mathbf{1}_\lambda\{1\} \xrightarrow{d_2} \cdots \xrightarrow{d_s} \mathcal{F}_i^{(\lambda_i+s)}\mathcal{E}_i^{(s)}\mathbf{1}_\lambda\{s\} \xrightarrow{d_{s+1}} \cdots$$

when $\lambda_i \geq 0$, where in the above formulae the leftmost term is in homological degree zero. The above complexes are isomorphic when $\lambda_i = 0$.

The complexes $\mathcal{T}_i\mathbf{1}_\lambda$ are invertible, up to homotopy, with inverses given by

$$\mathbf{1}_\lambda\mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda\mathcal{E}_i^{(s)}\mathcal{F}_i^{(-\lambda_i+s)}\{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda\mathcal{E}_i\mathcal{F}_i^{(-\lambda_i+1)}\{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda\mathcal{F}_i^{(-\lambda_i)}$$

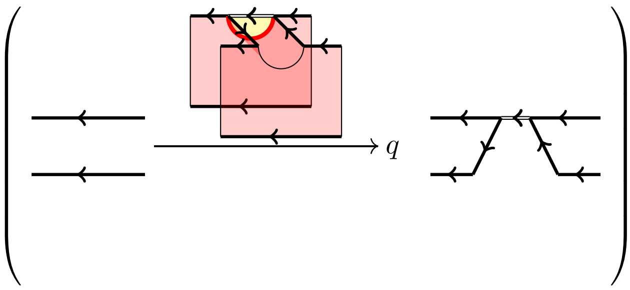
when $\lambda_i \leq 0$ and

$$\mathbf{1}_\lambda\mathcal{T}_i^{-1} = \cdots \xrightarrow{d_{s+1}^*} \mathbf{1}_\lambda\mathcal{F}_i^{(s)}\mathcal{E}_i^{(\lambda_i+s)}\{-s\} \xrightarrow{d_s^*} \cdots \xrightarrow{d_2^*} \mathbf{1}_\lambda\mathcal{F}_i\mathcal{E}_i^{(\lambda_i+1)}\{-1\} \xrightarrow{d_1^*} \mathbf{1}_\lambda\mathcal{E}_i^{(\lambda_i)}$$

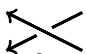
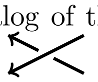
when $\lambda_i \geq 0$, where in these formulae the rightmost term is in homological degree zero. Note that these complexes are obtained by taking the adjoints of the above (in the category of complexes).

We begin with the $\mathfrak{sl}_2$ case. When λ maps to a sequence with $a_i = 1 = a_{i+1}$,

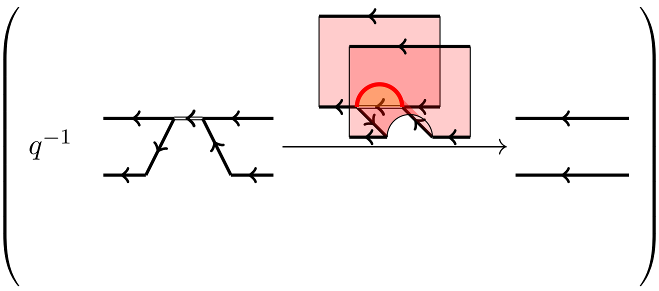
$$\Phi_2(\mathcal{T}_i\mathbf{1}_\lambda) =$$



(3.4.1)

which gives the value of the positive $(1,1)$ crossing . This complex is the Blanchet foam analog of the formula for the crossing given in [Kho00] and [BN05]. The negative crossing  is given by

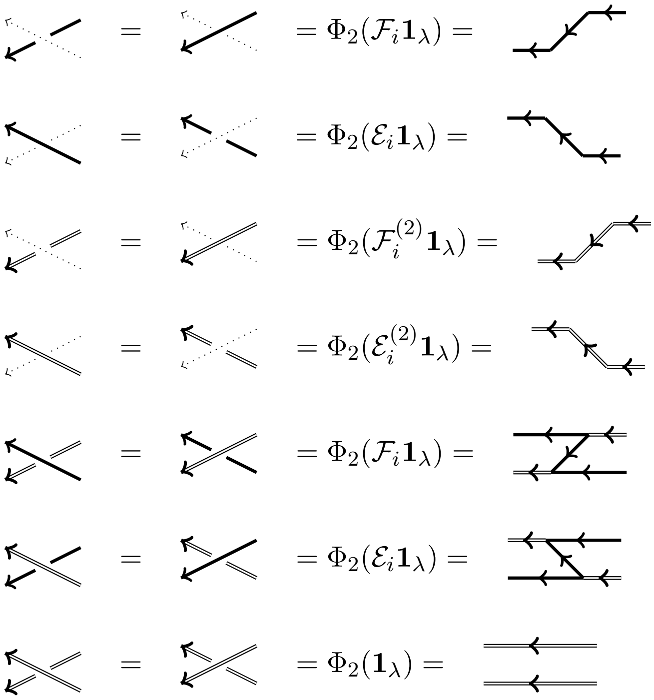
$$\Phi_2(\mathbf{1}_\lambda\mathcal{T}_i^{-1}) =$$



. (3.4.2)

In both (3.4.1) and (3.4.2), the identity web appears in homological degree zero.

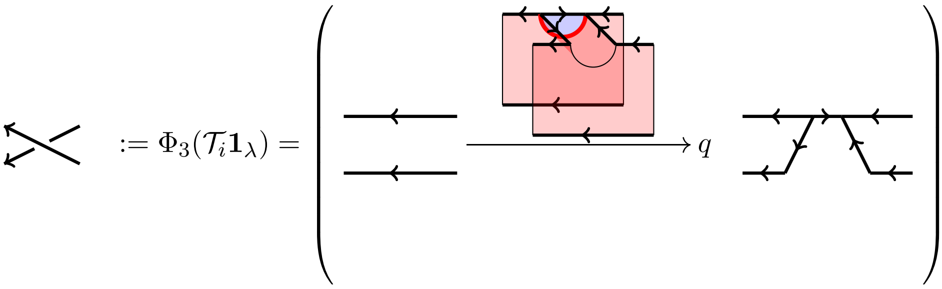
In order to give a construction of the link invariant via the foamation 2-functors, we will also need the formulae for the braidings involving 0's and 2's. Defining the positive crossings to be the images of the $\mathcal{T}_i\mathbf{1}_\lambda$ in the appropriate weights and the negative crossings to be the images of the $\mathbf{1}_\lambda \mathcal{T}_i^{-1}$, this gives the formulae



(3.4.3)

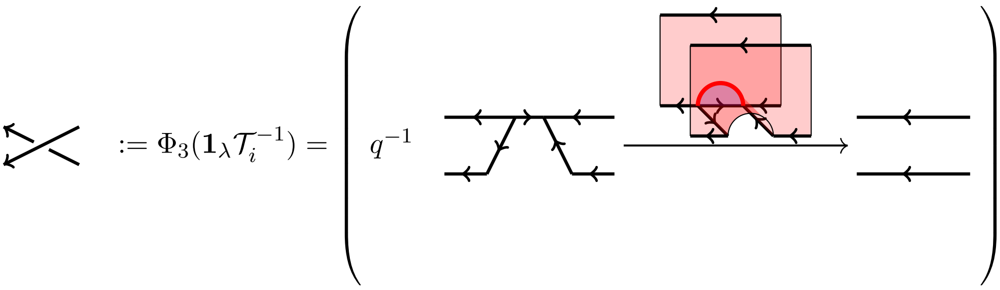
where the dotted strands are meant to indicate a "0-labeled" edge, i.e. an edge that is not actually present. The braiding on two such 0-labeled edges is simply the empty web mapping between the appropriate labels.

In the $\mathfrak{sl}_3$ case, we give the formulae for the braidings in $3\mathbf{BFoam}_m(N)$, since those in $3\mathbf{Foam}_m(N)$ can be recovered from these via the forgetful 2-functor. We'll first compute the braidings for the traditional $\mathfrak{sl}_3$ edges. The $(1,1)$ crossings are again given as



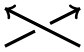
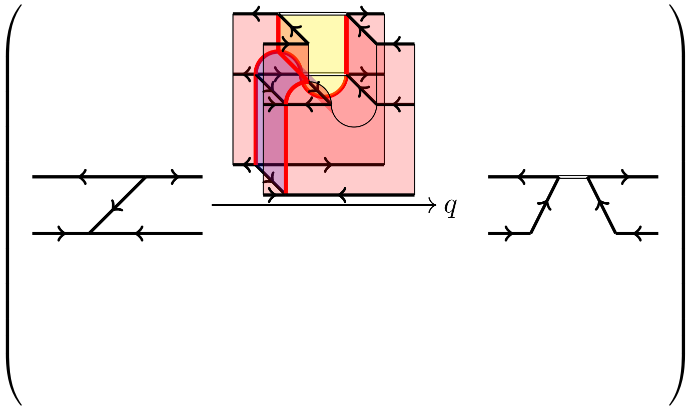
(3.4.4)

and

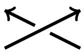
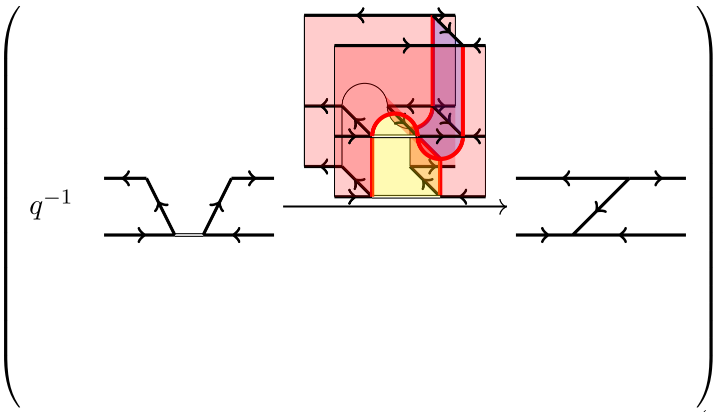


(3.4.5)

where the identity webs are in homological degree zero in both of the above formulae. Similarly, the $(1,2)$ braidings are given by

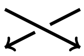
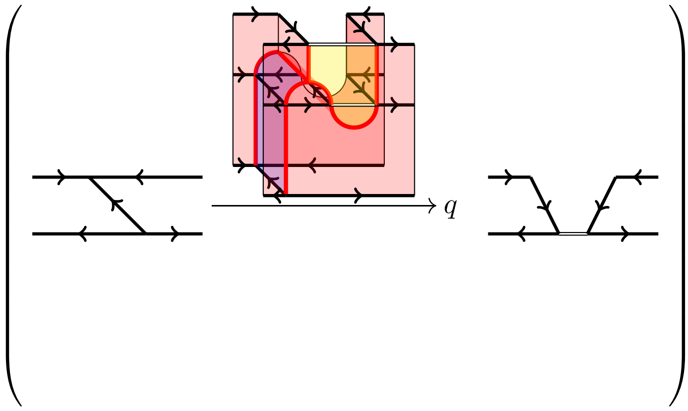
 $:= \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda) =$  (3.4.6)

and

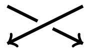
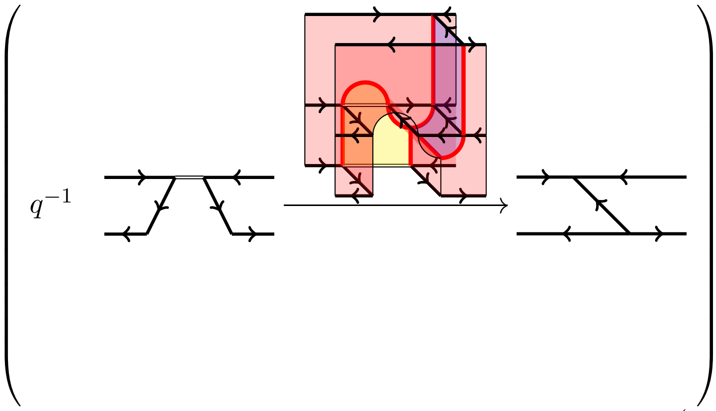
 $:= \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1}) =$  ;

(3.4.7)

note that (3.4.6) is a positive $(1,2)$ braiding and (3.4.7) is a negative $(1,2)$ braiding, although topologically the former is a left-handed crossing and the latter is right-handed. The $(2,1)$ braidings are given by

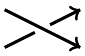
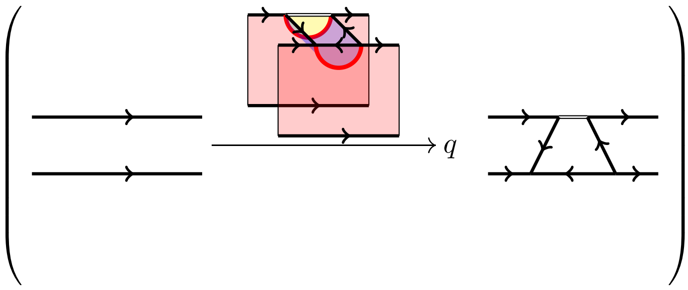
 $:= \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda) =$  (3.4.8)

and

 $:= \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1}) =$ 

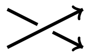
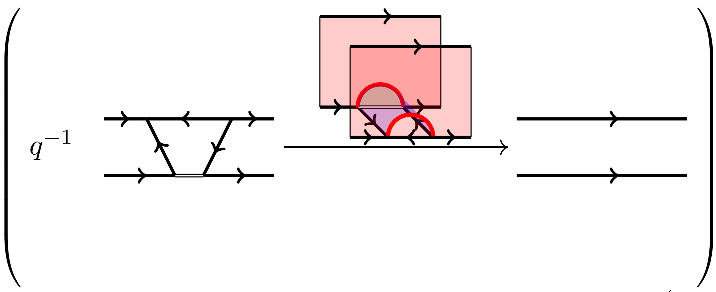
(3.4.9)

and the $(2,2)$ braidings are given by

 $:= \Phi_3(\mathcal{T}_i \mathbf{1}_\lambda) =$ 

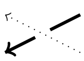
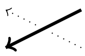
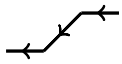
(3.4.10)

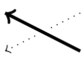
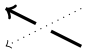
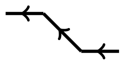
and

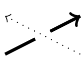
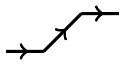
 $:= \Phi_3(\mathbf{1}_\lambda \mathcal{T}_i^{-1}) =$ 

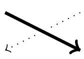
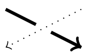
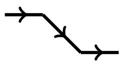
(3.4.11)

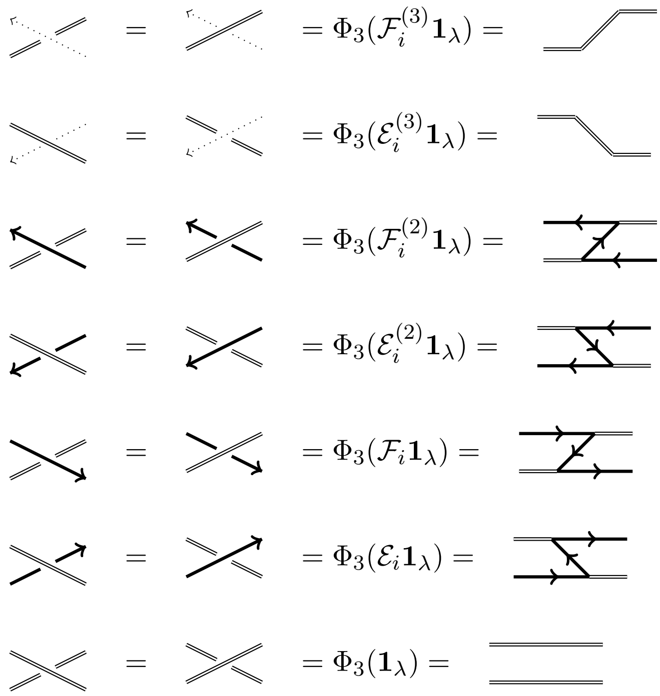
As in the $\mathfrak{sl}_2$ case, we will also need the formulae for the braidings between sequences involving 0's and 3's in order to construct the link invariant. Defining the positive crossings to be the images of the $\mathcal{T}_i \mathbf{1}_\lambda$ (for appropriate λ) and the negative crossings to be the images of the $\mathbf{1}_\lambda \mathcal{T}_i^{-1}$, this gives the formulae

 $=$  $= \Phi_3(\mathcal{F}_i \mathbf{1}_\lambda) =$ 

 $=$  $= \Phi_3(\mathcal{E}_i \mathbf{1}_\lambda) =$ 

 $=$  $= \Phi_3(\mathcal{F}_i^{(2)} \mathbf{1}_\lambda) =$ 

 $=$  $= \Phi_3(\mathcal{E}_i^{(2)} \mathbf{1}_\lambda) =$ 



(3.4.12)

where again the dotted strands are meant to indicate "0-labeled" (non-)edges. The braiding on two such edges is the empty web.

Let $n = 2, 3$; it will be useful to note that any object $\mathbf{a} = (a_1, \cdots, a_m)$ in $n\mathbf{(B)Foam}_m(N)$ can be identified with a canonical object which corresponds to the same $\mathfrak{sl}_n$ representation as $\mathbf{a}$ (up to isomorphism). Given an object $\mathbf{a}$ in $n\mathbf{(B)Foam}_m(N)$, denote by $\bar{\mathbf{a}}$ the associated "reduced sequence" defined to be the same sequence as $\mathbf{a}$ with all values 0 and n deleted. For example, if $\mathbf{a} = (1, 3, 0, 2, 0, 1)$ and $n = 3$, then $\bar{\mathbf{a}} = (1, 2, 1)$.

Definition 3.4.1. Given an object $\mathbf{a}$ of $n\mathbf{(B)Foam}_m(N)$, the associated *canonical sequence* is the unique object $\mathbf{a}'$ in $n\mathbf{(B)Foam}_m(N)$ such that $\bar{\mathbf{a}'} = \bar{\mathbf{a}}$ and

$$\mathbf{a}' = (0, \ldots, 0, a'_k, a'_{k+1}, \ldots, a'_{k+r}, n, \ldots, n)$$

with $0 < a'_{k+s} < n$ for $0 \leq s \leq r$.

The trivial braidings (3.4.3) and (3.4.12) can be used to give an equivalence between an object $\mathbf{a}$ in $n\mathbf{(B)Foam}_m(N)$ and its canonical sequence $\mathbf{a}'$ (this is the analog of [Cau12, Corollaries 7.3 and 7.8] in the web and foam setting). Let the web $\mathbf{a} \xrightarrow{\beta_{\mathbf{a}}} \mathbf{a}'$ be given by the (composition of) braidings involving 0- and n-labeled edges and let the web $\mathbf{a}' \xrightarrow{\beta_{\mathbf{a}}^{-1}} \mathbf{a}$ be given using the inverses of the above braidings. Since the images of the Rickard complexes braid in any (integrable) 2-representation [CK12a], the above maps are uniquely defined up to coherent isomorphism. Fix once and for all choices of $\beta_{\mathbf{a}}$ and $\beta_{\mathbf{a}}^{-1}$ for each object $\mathbf{a}$ in each of the foam 2-categories.

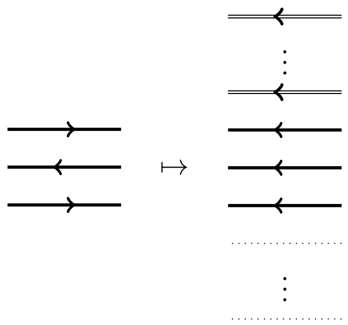
3.4.1.2 The $\mathfrak{sl}_2$ tangle invariant

The webs that appear in the image of the foamation 2-functors are all in ladder form; hence, we require a method for assigning a complex of ladders in the foam 2-categories to each tangle. A process which transforms any web to a ladder is detailed in [CKM12]; however, an adaptation of a construction from [Cau12] is more useful for our purposes.

Let τ be an oriented (r,t)-tangle diagram, i.e. a tangle diagram with r endpoints on the right and t endpoints on the left, which we assume to be in Morse position with respect to the horizontal axis. We now describe a method for assigning to this diagram a complex $[\![\tau]\!]_2$ in $2\mathbf{BFoam}_{r+2l}(N)$, for l sufficiently large.

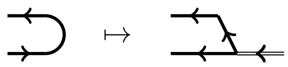
We assign to each basic tangle a complex of 1-morphisms mapping between canonical sequences; the complex assigned to a tangle will then be the horizontal composition (in the 2-category of complexes) of the basic complexes.

A tangle involving no crossings, cups, or caps is mapped to the identity web of the sequence $(0,\ldots,0,1,\ldots,1,2,\ldots,2)$ where the number of 1's is equal to the number of strands in the tangle. For example, we have

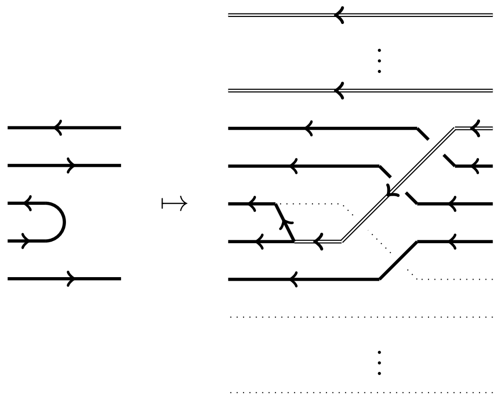


where the bottom dotted web edges are zero-labeled, i.e. not actually present.

We'd like to map the cup as follows

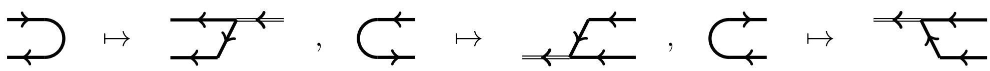


however, the domain on this web will not be a canonical sequence (especially when other strands of the tangle are present). We will hence pre-compose with the relevant web $\beta_{\mathbf{a}}^{-1}$. For example,



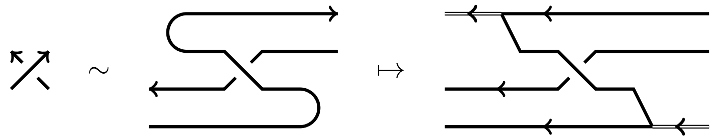
where again we have depicted the 0-labeled edges.

We similarly define the map on the remaining cup and caps to be given by



where we pre- or post-compose with the appropriate braiding maps β^{-1} and β as necessary to ensure that the webs map between canonical sequences.

We assign the complexes (3.4.1) and (3.4.2) to the positive and negative left-oriented crossings. This assignment determines the value of the invariant on the remainder of the crossings (up to isomorphism) since they can be obtained from the left-oriented crossings by composing with caps and cups, e.g. we have

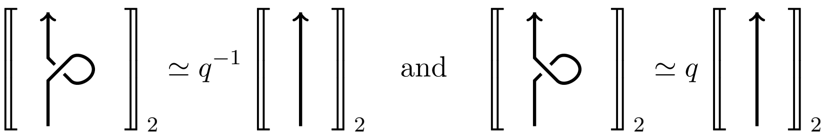


where the latter is understood to represent the complex assigned to a left-oriented crossing horizontally composed in the category of complexes with the indicated webs (and the necessary braiding maps β^{-1} and β so that the webs in the complex map between canonical sequences). Formulae for the other crossings can be obtained similarly.

Proposition 3.4.2. The complex $[\![\tau]\!]_2$ assigned to a tangle diagram τ, viewed in the homotopy category of complexes of $2\mathbf{BFoam}_m(N)$, gives an invariant of framed tangles.

Proof. It suffices to check the tangle Reidemeister moves (see [Kas95] or [CK08a]); this is a standard computation following the argument detailed in [BN05] adapted to the Blanchet foam setting. Alternatively, one can simplify the computation using the proof of [Cau12, Proposition 7.9], where it is shown that (most of) the desired relations hold already in the categorified quantum group. □

One can check that (locally)



so renormalizing the invariant using the writhe $w(\tau)$ of the tangle:

$$[\![\tau]\!]_2^r := q^{w(\tau)}[\![\tau]\!]_2$$

gives an invariant independent of framing.

Given a link L, the invariant $[\![L]\!]_2^r$ is a complex of webs mapping between the sequence $(\underline{0}, \underline{2}) := (0, \dots, 0, 2, \dots, 2)$ and itself. Applying the functor

$$\mathrm{HOM}(\mathrm{id}_{(\underline{0},\underline{2})}, -) := \bigoplus_{t \in \mathbb{Z}} \mathrm{Hom}(q^{-t}\mathrm{id}_{(\underline{0},\underline{2})}, -)$$

to this complex (where $\mathrm{id}_{(\underline{0},\underline{2})}$ is the identity web) and setting the parameter  $= 0$ gives a complex of finite-dimensional graded vector spaces, which we denote $Kh_2(L)$. As the notation indicates, we have the following result.

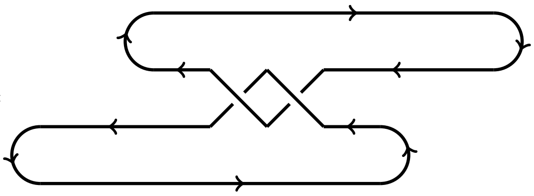
Proposition 3.4.3. The (co)homology of the complex $Kh_2(L)$ is the Khovanov homology of the link L.

Proof. Let D be a diagram of the link L. The complex $[\![D]\!]_2^r$ consists of $\mathfrak{sl}_2$ webs with no 1-labeled boundary, i.e. these webs consist of 1-labeled circles joined to each other (and to the boundary) by 2-labeled edges. Such a web in $[\![D]\!]_2^r$ contributes a direct summand of dimension $2^{\#\text{ of circles}}$ to the complex $Kh_2(D)$. Indeed, if W is such a web then $\mathrm{HOM}(\mathbb{1}_{(\underline{0},\underline{2})}, W)$ is a free $\Bbbk[$  $]$-module with basis given by 1-labeled cups with one or no dots, intersecting 2-labeled sheets transversely.

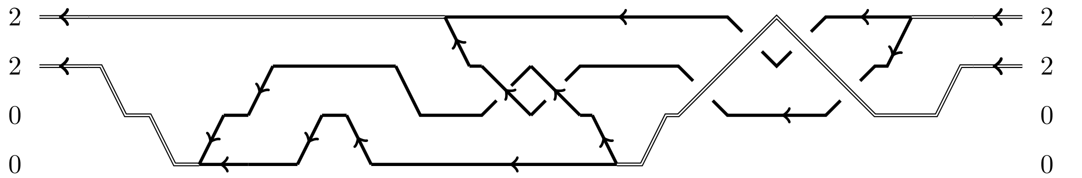
The complex $Kh_2(D)$ is hence obtained from a cube of resolutions in which the nodes of the cube are exactly those appearing in the construction of Khovanov homology. One can check that the maps labeling the edges of this cube are, up to a $\pm$-sign, the maps m and Δ from [Kho00]. Since the squares in this cube of resolutions anti-commute (by construction), an argument from [ORS07] shows that this complex is isomorphic to the complex assigned to D in [Kho00]. $\square$

3.4.1.3 An explicit example

The invariant of the Hopf link:

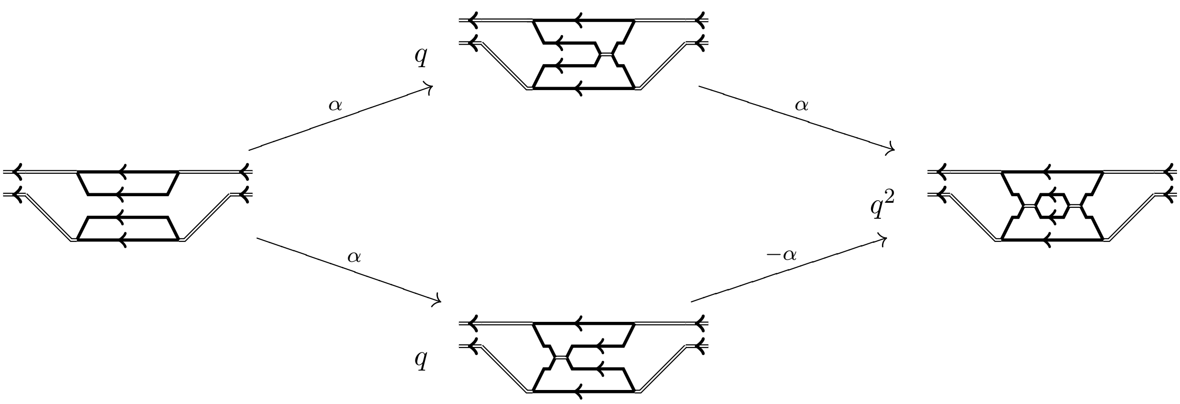
$L =$ 

can be constructed in $2\mathbf{BFoam}_{2l}(N)$ for any $l \geq 2$; we'll take the minimal case $l = 2$. By the procedure detailed above, we have that $[\![L]\!]_2$ is given by the complex:



(3.4.13)

which is shorthand for the complex obtained from the following cube of resolutions (after applying some web isomorphisms):



(3.4.14)

where the foams α in the complex are those depicted in (3.4.1) horizontally composed with the relevant identity foams.

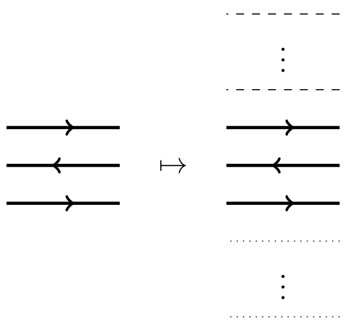
Note that the complex (3.4.13) is the image under Φ_2 of the complex:

$$\mathcal{E}_2^{(2)}\mathcal{E}_1^{(2)}\mathcal{F}_1\mathcal{F}_2\mathcal{F}_1\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{T}_2\mathcal{T}_2\mathcal{E}_1\mathcal{F}_1^{(2)}\mathcal{T}_2^{-1}\mathcal{T}_3^{-1}\mathcal{T}_3\mathcal{T}_2\mathcal{F}_3\mathcal{F}_2^{(2)}\mathbf{1}_{(0,2,0)}$$

in $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_4))$. Indeed, for any tangle τ, we can realize the complex $[\![\tau]\!]_2$ as the image of a complex in the categorified quantum group by pulling back the various pieces assigned to elementary tangles to $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$. One may then use the graphical calculus of the categorified quantum group to perform calculations in link homology, see e.g. [Cau12, Section 10].

3.4.1.4 The $\mathfrak{sl}_3$ tangle invariant

We define the $\mathfrak{sl}_3$ tangle invariant $[\![\tau]\!]_3$ in $3\mathbf{Foam}_m(N)$ in a similar manner as above [6]. An oriented tangle (diagram) with no caps, cups, or crossings determines a sequence $\mathbf{s}$ of 1's and 2's (corresponding to the strands directed to the left and right respectively) and we map such a tangle to the identity web of the sequence $(\underline{0}, \mathbf{s}, \underline{3})$, e.g.



where the dotted and dashed lines denote web edges which are not actually present, i.e. 0- and 3-labeled edges.

The invariant is defined on cups by

and on caps by

where, as in the $\mathfrak{sl}_2$ case, we pre- and post-compose with the relevant braidings so that the webs map between canonical sequences. These braidings are given by deleting the 3-labeled edges from those given in (3.4.12).

We define the invariant on left-oriented crossings by equations (3.4.4) and (3.4.5). We'd like to define the image of the remainder of the crossings using the images of the braidings (3.4.6) - (3.4.11) under the forgetful functor $3\mathbf{BFoam}_m(N) \to 3\mathbf{Foam}_m(N)$; however, this assignment would not be invariant under planar isotopy as the complexes differ by factors of $q^{\pm 1}$. It is possible to rescale the Rickard complexes $\mathcal{T}_i \mathbf{1}_\lambda$ depending on the weight λ to fix this issue, but this introduces unwanted scalings on the trivial braidings (3.4.12). We instead follow our $\mathfrak{sl}_2$ approach and define the remainder of the crossings in terms of the left-oriented crossings and caps and cups.

Proposition 3.4.4. The complex $[\![\tau]\!]_3$ assigned to a tangle diagram τ, viewed in the homotopy category of complexes of $3\mathbf{Foam}_m(N)$, is an invariant of framed tangles.

Renormalizing this invariant via $[\![\tau]\!]_3^r = q^{2w(\tau)}[\![\tau]\!]_3$ gives an invariant independent of framing which is (essentially) the same as Morrison-Nieh's [MN08] extension of Khovanov's $\mathfrak{sl}_3$ link homology [Kho04] to tangles, after setting the 3-, 4-, and 5-dotted spheres equal to zero.

3.4.1.5 Categorified clasps

In [Cau12], Cautis showed that given any categorification of the skew Howe representations $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$, one obtains a categorification of $\mathfrak{sl}_n$ clasps, the $\mathfrak{sl}_n$ analogs

6. We could define this invariant in $3\mathbf{BFoam}_m(N)$ as well; however, the invariant in $3\mathbf{Foam}_m(N)$ is (essentially) the $\mathfrak{sl}_3$ invariant found in the literature.

of the Jones-Wenzl projectors, using the higher representation theory of the categorified quantum group. He conjectured that these representations could be categorified in the foam setting and that this construction would give the categorified Jones-Wenzl projectors from [CK12b] and [Roz10b] and the categorified $\mathfrak{sl}_3$ projectors from [Ros11]. Although the foam categories only categorify the intertwiners between such representations (and not the representations themselves), Cautis' methods indeed give a uniform construction of categorified clasps in the $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ foam 2-categories. We now recall the details of this construction.

Fix a reduced expression $w = s_{i_1} \dots s_{i_k}$ for the longest word w in the Weyl group for $\mathfrak{sl}_m$ and consider the complex $\mathcal{T}_w \mathbf{1}_\lambda := \mathcal{T}_{i_1} \dots \mathcal{T}_{i_k} \mathbf{1}_\lambda$ in $Kom(\check{\mathcal{U}}_Q(\mathfrak{sl}_m))$; this complex gives the invariant assigned to a "half-twist" tangle. Cautis shows that the images of the complexes $\mathcal{T}_w^{2k} \mathbf{1}_\lambda$ in any integrable 2-representation stabilize as $k \to \infty$. Denote the image of $\mathcal{T}_w \mathbf{1}_\lambda$ in such a 2-representation by $\mathsf{T}_w \mathbb{1}_\lambda$ and let $\mathsf{T}_w^\infty \mathbb{1}_\lambda := \lim_{k\to\infty} \mathsf{T}_w^{2k} \mathbf{1}_\lambda$.The complexes $\mathsf{T}_w^\infty \mathbb{1}_\lambda$ are idempotents (with respect to horizontal composition of complexes) and give categorified clasps in any 2-representation categorifying $\bigwedge_q^N (\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$.

We first consider the $\mathfrak{sl}_2$ case. Let $\mathsf{P}_\mathsf{m}^+ := \mathsf{T}_\mathsf{w}^\infty \mathbb{1}_{(0,\dots,0)}$ in $2\mathbf{BFoam}_m(N)$ (which is a complex of webs mapping from the sequence $(1, \dots, 1)$ to itself), then we have the following result, which should be viewed as the Blanchet foam analog of [CK12b, Theorem 3.2].

Proposition 3.4.5. The complex P_m^+ satisfies the following properties:

1. P_m^+ is supported only in positive homological degree.
2. The identity web $\mathrm{id}_{(1,\dots,1)}$ appears only in homological degree zero.
3. P_m^+ annihilates the webs

in $2\mathbf{BFoam}_m(N)$, up to homotopy.

Proof. Properties (1) and (2) follow via inspection. Property (3) follows from arguments in [Cau12, Section 5] or by adapting arguments from [Ros11] to the $\mathfrak{sl}_2$ foam setting. □

It follows that P_m^+ categorifies the analog of the Jones-Wenzl projector p_m in the category of Blanchet webs. Using the foamation 2-functor Φ_{CMW}, the above procedure also gives a construction of the categorified Jones-Wenzl projectors from [CK12b] and [Roz10b] in the CMW foam setting.

In the $\mathfrak{sl}_3$ case, let $\mathbf{s}$ denote a sequence of 1's and 2's of length m; let $\#\mathbf{s}_1$ denote the number of 1's and $\#\mathbf{s}_2$ the number of 2's in $\mathbf{s}$. Define $\mathsf{P}_\mathbf{s}^+ := \mathsf{T}_\mathsf{w}^\infty \mathbb{1}_\lambda$ in $3\mathbf{Foam}_m(m + \#\mathbf{s}_2)$ where λ maps to $\mathbf{s}$ under Φ_3.

Proposition 3.4.6. The complex $\mathsf{P}_\mathbf{s}^+$ is the categorified clasp $\widetilde{P}_\mathbf{s}$ constructed in [Ros11].

There is nothing to prove here; the categorified $\mathfrak{sl}_3$ clasps in [Ros11] are constructed precisely as the limit of the complexes $\mathsf{T}_w^{2k} \mathbb{1}_\lambda$ as $k \to \infty$. Note that the $+$'s and $-$'s in the sequences in that work correspond to our 1's and 2's, respectively.

Having constructed categorified clasps, we can extend our $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ tangle invariants to give categorified invariants of framed tangles in which each component is labeled by an irreducible representation. This construction is detailed in many places in the literature, in particular in [CK12b], [Ros11], and [Cau12], so we will be brief. Given a framed tangle τ with components labeled by irreducible representations, choose for each component a tensor product of fundamental representations having the corresponding irreducible as a highest weight subrepresentation. Assign to the tangle the complex assigned to a cabling of the tangle (we use here the fact that τ is framed) with the categorified projector inserted along the cabling. The number of strands in the cabling of each component (and the direction of such strands in the $\mathfrak{sl}_3$ case) as well as which projector P^+ is inserted is given by the relevant tensor product of fundamental representations; this corresponds to a sequence of 1's in the $\mathfrak{sl}_2$ case and a sequence of 1's and 2's in the $\mathfrak{sl}_3$ case.

One can show (see [CK12b], [Ros11], or [Cau12]) that the above invariant doesn't depend on the choice of where the projector is inserted or which tensor product of fundamentals is used (up to equivalence in the case that the tangle is not a link) and gives a categorification of the Reshetikhin-Turaev invariant of framed tangles.

3.4.2 Comparing knot homologies

Let $\Phi\colon \dot{\mathcal{U}}_Q(\mathfrak{sl}_m) \to \mathcal{K}$ be any 2-representation giving a categorification of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. Cautis shows that for N and m sufficiently large, this 2-representation assigns to any framed, oriented link K, a complex of 1-morphisms $\Psi(K) \in \mathrm{End}(Kom(\Phi(\lambda)))$ where λ is the highest weight in $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ [Cau12, Section 7.5]. His framework does not require the full structure of a 2-representation of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$, but rather the weaker data encoded in what he calls a categorical 2-representation. This weaker action is more like the data described in Theorem 3.2.5 without requiring the KLR action. The KLR relations greatly simplify the resulting complexes; in particular, they imply analogs of the higher Serre relation [Sto11] and commutativity relations for divided powers [KLMS12]. Using these relations, the complex $\Psi(K)$ associated to a link K can be reduced to a complex that only involves direct sums of the identity 1-morphism $\mathbb{1}_\lambda$ of $\Phi(\lambda)$, with various grading shifts. One does not actually need to know that the 2-representation $\mathcal{K}$ categorifies $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$; it suffices that the nonzero weight spaces of $\bigwedge_q^N(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ parameterize the nonzero objects in $\mathcal{K}$.

Applying the functor $\mathrm{HOM}(\mathbb{1}_\lambda, -)$ to the complex $\Psi(K)$ maps it to a complex of graded vector spaces. The number of $\mathbb{1}_\lambda$ summands and their grading shifts are formally determined by the categorified quantum group, hence so are the vector spaces appearing in the complex. The differentials depend only on the map $\mathrm{HOM}(\mathbf{1}_\lambda, \mathbf{1}_\lambda) \to \mathrm{HOM}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$, so it follows that this map completely determines the link homology theory.

When the graded algebra $A := \mathrm{HOM}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$ is 1-dimensional in degree zero and 0 in all other degrees, only one such map exists, hence all constructions of $\mathfrak{sl}_n$ link homology satisfying this condition are equivalent. After quotienting by the 3-dotted sphere in the $\mathfrak{sl}_2$ case and the 3-, 4-, and 5-dotted spheres in the $\mathfrak{sl}_3$ case, the foam 2-categories satisfy this condition.

This observation gives a method for showing that Cautis-Kamnitzer link homology is equivalent to Khovanov-Rozansky homology. Using constructions from previous work [CKL09, CKL11], Cautis describes (weak) categorical 2-representations on derived categories $\mathcal{K}_{Gr,m}$ of coherent sheaves on varieties arising as orbits in the affine Grassmannian, as well as on coherent sheaves on Nakajima quiver varieties $\mathcal{K}_{Q,m}$. Both of these categorical 2-representations are conjectured by Cautis, Kamnitzer, and Licata to extend to 2-representations of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. By the results of [CL11] it suffices to prove that the KLR algebras act; this was done in the $m = 2$ case in [CKL10b] and will be generalized to symmetric Kac-Moody algebras (in particular for arbitrary m) in [Cau13]. Moreover, in this setting the algebra A satisfies the 1-dimensionality condition, so this will show that the link homology theory from [CK08b] fits into the framework described above.

The results from this paper will hence show that the foam based constructions of $\mathfrak{sl}_n$

link homology agree with the Cautis-Kamnitzer construction for $n = 2, 3$. This re-proves Theorem 8.2 from [CK08a] and pairs with the results from [MV08] to give the $n = 3$ case of Conjecture 6.4 from [CK08b] equating $\mathfrak{sl}_3$ Cautis-Kamnitzer and Khovanov-Rozansky link homology. In the sequel to this paper, will we establish the analogous results for general n.

3.4.3 Deriving foam relations from categorified quantum groups

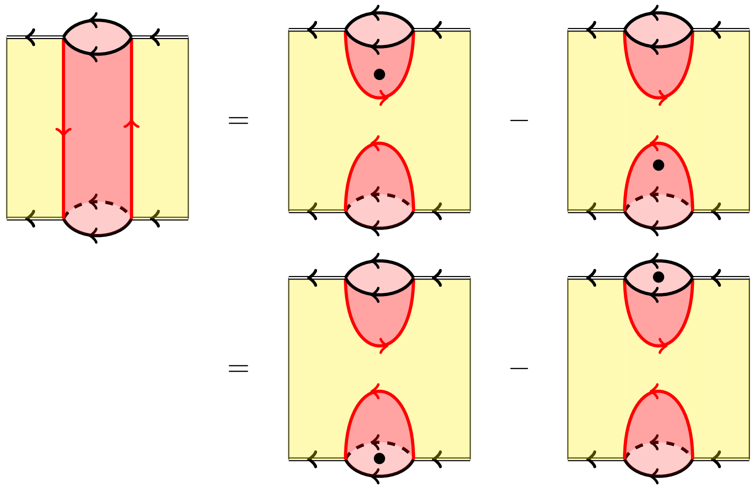
In [CKM12], Cautis, Kamnitzer, and Morrison showed that the relations on $\mathfrak{sl}_n$ webs could be derived via skew Howe duality from the relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. Here we categorify this result in the $n = 2, 3$ case to show that many foam relations can be deduced from the assignments defining the foamation 2-functors Φ_2 and Φ_3. The main result of this section is that all $\mathfrak{sl}_3$ foam relations, all CMW $\mathfrak{sl}_2$ foam relations (assuming a strong form of locality), and many Blanchet $\mathfrak{sl}_2$ foam relations follow from relations in the categorified quantum group $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$. In a follow-up paper, we study foam categories for arbitrary n using this framework [LQR].

3.4.3.1 Blanchet $\mathfrak{sl}_2$ foam relations

Since the Blanchet foams arising as images under our 2-functors must contain both 1- and 2-labeled facets (unless they are identity foams) and always bound webs whose edges are oriented leftward, we cannot expect to recover all defining relations from the relations in $\mathcal{U}_Q(\mathfrak{sl}_m)$. For example, we have no hope of recovering the 1- and 2-labeled neck-cutting relations or closed foam relations.

There are nevertheless numerous foam relations arising from the quantum group relations, which we list below. Note that some of the relations we obtain actually slightly generalize Blanchet's original relations, using 2- and 3-dotted enhanced spheres as graded parameters.

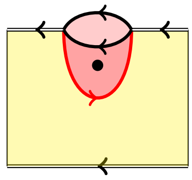
– The nilHecke relation (3.2.17) implies the enhanced neck-cutting relation:



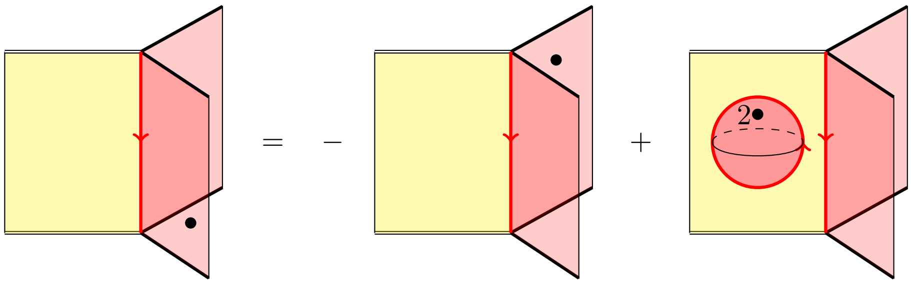
The first is relation (3.3.11) (note the orientation of the seams) and the second is equivalent to this using an isotopy of the 1-labeled tube.

– Degree-zero bubbles in weight ± 2 imply the "blister" relation (3.3.14). The LHS of the blister relation (3.3.13) follows from the non-dotted bubble in weight ± 2.

– Composing the second enhanced neck-cutting relation with



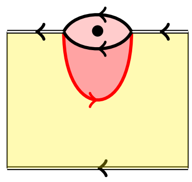
and using the previous blister relations, we obtain the following generalization of the dot-migration relation (3.3.20):



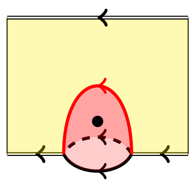
(3.4.15)

Relation (3.2.18) with $j = i + 1$ and $a_{i+1} = 2$ implies that twice-dotted blisters can migrate between 2-labeled facets; this allows us to view them as graded parameters.

– Composing the enhanced neck-cutting relation with

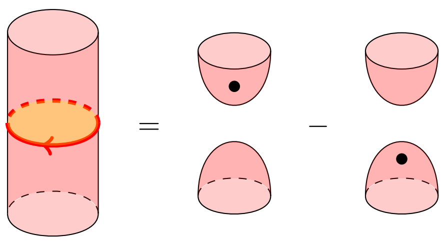


and



gives the RHS of (3.3.13), using (3.3.14). We do not obtain the analog of relation (3.3.13) with two dots on the same facet, since the dot-sliding relation has an additional term.

– Relation (3.2.18) with $j = i + 1$ and $a_{i+1} = 1$ implies the relation:

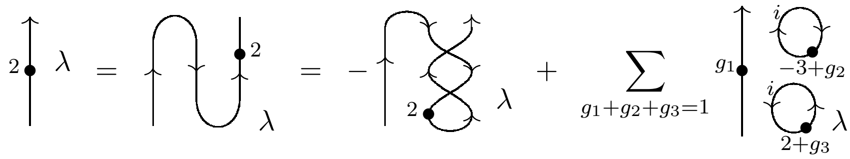


which can be viewed as another enhanced version of neck-cutting.

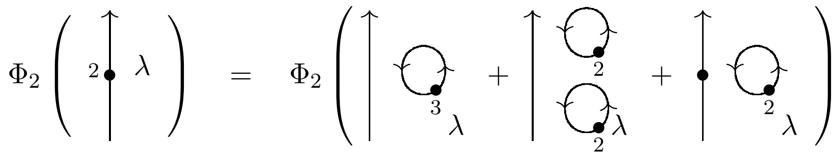
– Degree zero bubbles in weight ± 1 with $m = 2$ and $N = 3$ imply relation (3.3.16).

– Relation (3.2.21) implies the foam relation (3.3.18).

Finally, we comment on the behavior of a twice-dotted foam facet. When $\lambda = -2$, we compute



The term on the left in the last part is sent to zero under the foamation functor. We obtain the following:

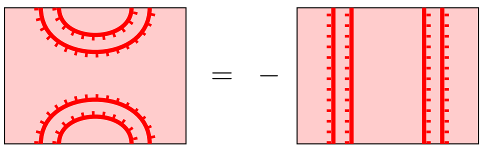


providing a way to decompose twice-dotted foam facets using the image of 2- and 3-dotted bubbles.

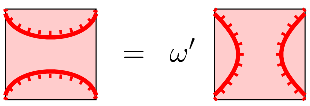
3.4.3.2 CMW $\mathfrak{sl}_2$ foam relations

The local relations for disorientation seams, together with the neck-cutting relation and evaluations of (dotted) spheres constitute the only CMW foam relations. Since the CMW seam relations (3.3.22) involve complex coefficients, we cannot expect to derive them from the categorical skew Howe action of $\mathcal{U}_Q(\mathfrak{sl}_m)$. We hence must impose the additional requirement that some relations can be performed completely locally (which in practice says that some relations have a “square root”). We will show that with this additional assumption we can derive a slightly more general CMW foam 2-category in which both the 2- and 3-dotted spheres are (graded) parameters.

– Seam relations: Considering (3.2.21) with $a_i = 2$ and $a_{i+1} = 1$, we find that the first term of the relation is mapped to zero and the remaining foams give

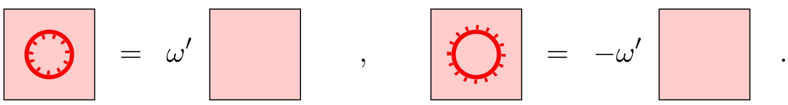


(up to isotopy). Assuming this relation can be expressed locally, this requires the relation:

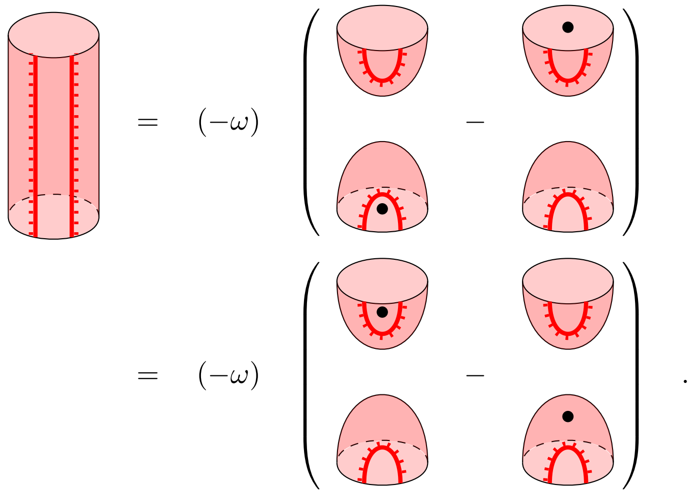


with ω' a primitive fourth root of the unity (a priori, this is not required to equal the fourth root ω from the section 3.3.1.4).

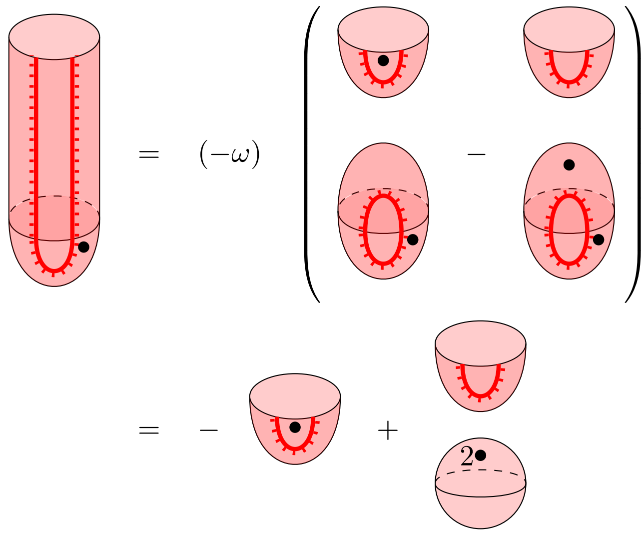
Having fixed a value for ω', the values of degree-zero bubbles in weight ± 1 with $m = 2$ and $N = 3$ give that a circular seam squares to give -1 (in both cases). Again assuming complete locality, this gives that a circle can be removed from a foam at the cost of multiplying by a primitive fourth root of unity. Using the above, we determine:



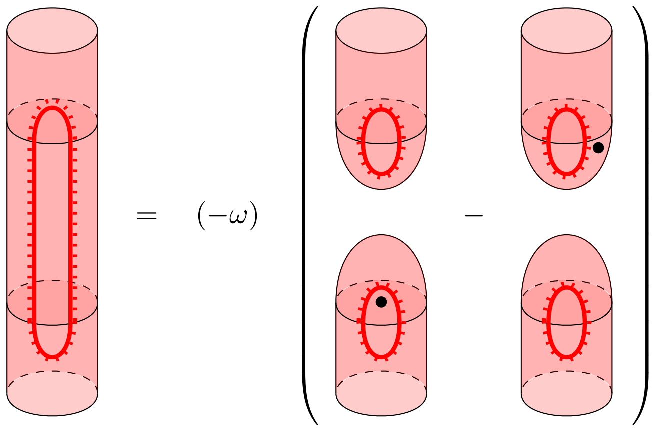
- Closed foam relations: Since negative degree bubbles are zero, we deduce that a non-dotted sphere is zero by considering the image of (non-dotted) bubbles in weight ± 2 with $N = 2$ and $m = 2$. The values of once-dotted bubbles in weight ± 2 give the value of a once-dotted sphere, depending on the value of ω. Choosing $\omega' = \omega$ (which we do for the duration), we obtain that a once-dotted sphere has value 1. After we deduce a neck-cutting relation, we will be able to evaluate n-dotted spheres with $n \geq 4$ in terms of spheres with fewer dots.
- Neck-cutting: The nilHecke relation (3.2.17) gives us two neck-cutting relations:



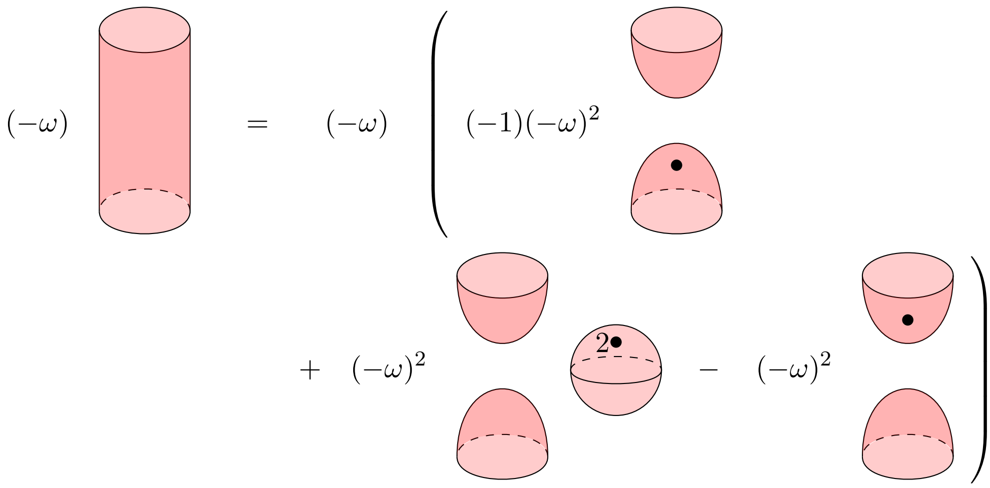
Using the seam and closed foam relations, we can recover a deformed version of the neck-cutting relation from equation (3.3.3). Caping with a dotted disk containing a disorientation seam, we have:



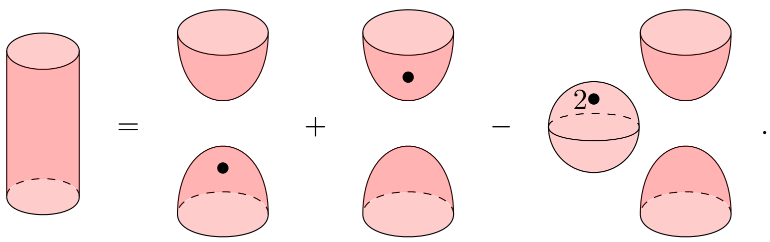
which gives a relation for sliding a dot through a seam. We then compute



which gives



i.e. the following deformation of the neck-cutting relation:



Specializing 

= 0, we recover the foam 2-category from [CMW09].

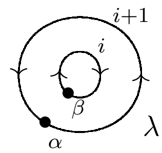
3.4.3.3 $\mathfrak{sl}_3$ foam relations

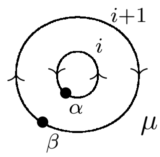
In the $\mathfrak{sl}_3$ setting, all foam relations are consequences of the relations in $\mathcal{U}_Q(\mathfrak{sl}_m)$:

– Dotted spheres: The values in relation (3.3.23) are recovered by the value of 

in weight 3 with $m = 2$ and $N = 3$ for $\alpha = 0, 1, 2$.

- Neck-cutting: The image of equation (3.2.28) in weight 3 and with $m = 2$ and $N = 3$ gives the neck-cutting relation (3.3.24) (note that the cup gives a -1 coefficient and the cap gives $+1$). One can obtain the simpler neck-cutting relations found in the literature by quotienting the categorified quantum group by the relevant bubbles (or equivalently passing to the quotient of the foam category where we set the 3- and 4-dotted spheres equal to zero).
- Equation (3.3.26) is a consequence of the nilHecke relation (3.2.17).
- Equation (3.3.27) is a consequence of equation (3.2.21).
- Θ-foams: For $\alpha + \beta \leq 3$, the values of:

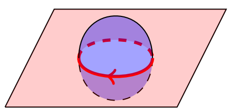


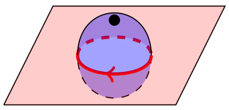
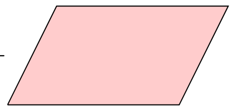
and


(3.4.16)

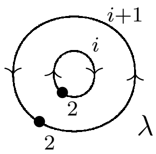
when λ maps to a sequence with $a_i = 0$, $a_{i+1} = 3$ and $a_{i+2} = 0$ and μ maps to a sequence with $a_i = 3$, $a_{i+1} = 0$ and $a_{i+2} = 3$ give the values in relation (3.3.25) when $\alpha + \beta \leq 3$ and $\gamma = 0$. In fact, these values, together with the remainder of the foam relations, determine the values of all theta-foams.

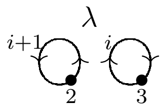
Using the values of theta-foams we have already determined, we can deduce the blister relations:

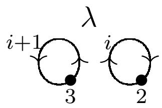

$= 0$

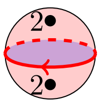

$= -$


from the neck-cutting and dotted sphere relations. The equality

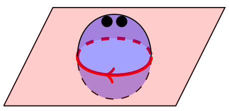
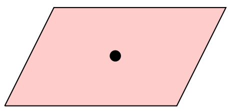
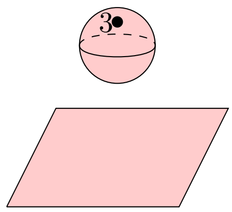


$=$


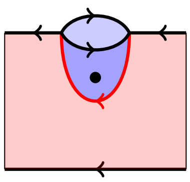
$-$


implies that 

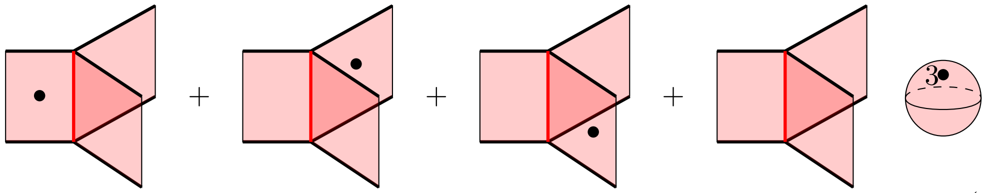
$= 0$; the neck-cutting relation then gives the additional blister relation:


$=$

$+$


Composing (3.3.26) with the foam



then gives the dot migration relation:



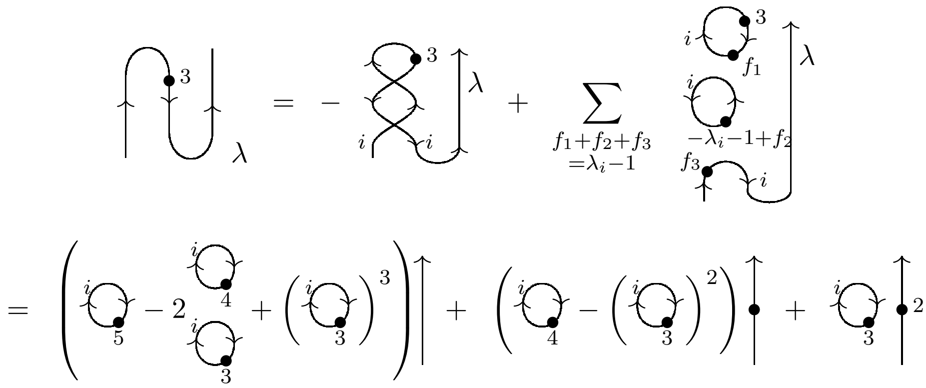
$= 0$

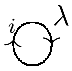
(3.4.17)

(compare to [Kho04, Figure 17]). Using this relation, in conjunction with (3.3.30), we can evaluate the remaining theta-foams from equation (3.3.25).

Note that we may also recover many of the relations which follow as consequences of the defining relations:

- Equation (3.3.28) is a consequence of equation (3.2.18).
- Using (3.2.28) with $\lambda_i = 1$ and $N_i = 3$, we compute



  which gives equation (3.3.30).
- Equation (3.3.29) follows from the degree-zero bubble  $= \mathrm{id}$ when $a_i = 1$ and $a_{i+1} = 2$.

Chapter 4

Skein modules from skew Howe duality and affine extensions

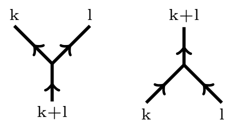
Note: An extended version of this chapter is now available online [Que13].

Introduction

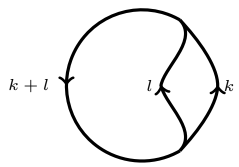
Webs and skew-Howe duality

Cautis, Kamnitzer and Licata [CKL10a, Cau12] recently introduced a reformulation of the $\mathfrak{sl}_n$ Reshetikhin-Turaev invariants for knots and links based on the *quantum skew Howe duality*. This duality phenomenon involves two commuting actions of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$ on the quantum exterior algebra $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$, where n corresponds to the $\mathfrak{sl}_n$-invariants we look at, and m governs the braiding of m-fold tensor products of $\mathfrak{sl}_n$-representations. In this framework, braidings arise from the so-called *quantum Weyl group action* [Lus93, KT09] on $U_q(\mathfrak{sl}_m)$.

This new process is naturally related to the concept of *webs*, which emerge from the study of $\mathfrak{sl}_n$ knot invariants and describe intertwiners of $\mathfrak{sl}_n$-representations (see [Kup96, Kim03, Mor07] for detailed studies of the spider categories they form). For each n, $\mathfrak{sl}_n$ webs are trivalent oriented graphs with edges labeled with integers in $1, \dots, n$. At each vertex, the sum of the indices of the incoming edges equals the sum of the indices of the outgoing edges:



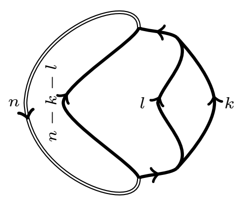
Here is an example of a web:



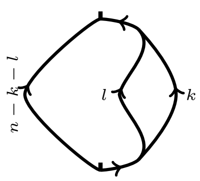
These diagrams are to be understood up to some local relations (see [CKM12] for example), which are a diagrammatic analogue of relations between morphisms of $U_q(\mathfrak{sl}_n)$-representations that are at the origin of the definition of webs. Note that there are more refined notions of $\mathfrak{sl}_n$-webs, in particular concerning what to do with n-labeled strands.

Indeed, a k-labeled strand corresponds to the k-th exterior power $\bigwedge_q^k(\mathbb{C}^n)$ of the standard $U_q(\mathfrak{sl}_n)$ representation $\mathbb{C}^n$. The 0-th power is the trivial representation, and it appears natural not to depict it in webs. Similarly, the maximal exterior power $\bigwedge_q^n(\mathbb{C}^n)$ is just the trivial representation, and it is usually forgotten as well (which comes with a correspondence between an edge labeled by k and the same edge with opposite orientation labeled by $n-k$, see for example [MS09]). However, it appears that these maximal exterior powers play non-trivial roles in some places, in particular when looking at categorification questions. An heuristic interpretation of this could be the fact that this representation corresponds to the determinant representation, which is indeed a trivial $\mathfrak{sl}_n$-representation, but is not a trivial $\mathfrak{gl}_n$-representation. This non-triviality has been encoded by *tags* in some places [Mor07, CKM12], and applications to categorified knot invariants using these tags in the $\mathfrak{sl}_2$ case can be found in a work by Clark, Morrison and Walker [CMW09]. One can also choose to keep all the n-edges (which we will then depict doubled). Although the difference is at first sight minimal on the level of webs, it seems to play an important role at the categorified level, as suggested by Blanchet's work [Bla10] and developed in [LQR12]. We sometime refer to these webs as *enhanced*.

For example, the enhanced web:



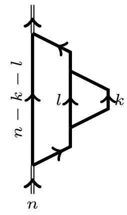
is represented in the tagged version of webs as



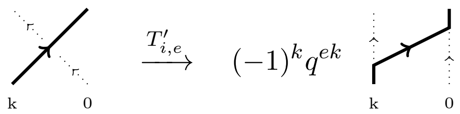
The only difference between the two pictures above lies in the way to deal with the strands decorated with the maximum exterior power of the fundamental representation: while we keep them completely in the first case, they only appear locally in the second case.

The Jones polynomial and its $\mathfrak{sl}_n$ analogues naturally take place in the spider categories. Their reformulation in terms of quantum skew Howe duality proved to be a very powerful tool for understanding these categories, and led Cautis, Kamnitzer and Morrison [CKM12] to solve conjectures on generators and relations for categories of representations $Rep_q(\mathfrak{sl}_n)$. Furthermore, this process admits a very natural categorification [Cau12], linking [LQR12] topological categorifications based on skein theory [Kho00, Kho02, Kho04, BN05] and categorified quantum groups [KL09, KL11a, KL10, KLMS12].

However, the skew Howe duality process is quite rigid, allowing to deal only with *ladder webs*, which are a particular class of webs with only upward oriented edges. This is a generalization to the web case of the notion of upward-oriented tangles, with the additional requirement that webs are presented in a rigid structure where strands are either vertical (the uprights of the ladder) or elementary horizontal pieces (the rungs of the ladder). For example, a ladder version of the previous web would be:



Furthermore, the relation established by Cautis-Kamnitzer-Licata between the braiding (or R-matrix) and the quantum Weyl group action holds up to coefficients $\pm q^k$. For example, the definition of the braidings as it appears in [Lus93] gives for crossings involving a 0-labeled strand:

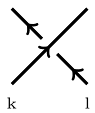


while we would like this crossing to be smoothed without creating any coefficient, since we do not want to consider 0-labeled strands in the skein context. Similarly, definitions provided in [Cau12] wouldn't produce any coefficients, but the use of tags in some of the Reidemeister-like web moves produces difficulties.

In this paper, we find an appropriate rescaling of the Weyl group action that removes the rigidity in the diagrammatic formulation of link (and knotted web) invariants.

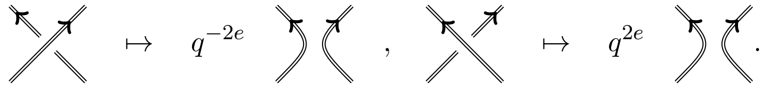
Obtaining a skein module

We give in this paper a detailed explanation of the skew-Howe duality process, focusing on obtaining $\mathfrak{sl}_n$ skein modules from this rather rigid context, for any value of n. One of the problems that usually appears when looking at a local crossing in a skein context is that it can be understood in different ways. For example,



can be translated as a positive (k, l) crossing, or (if we look at it from left to right) as a negative (l, k) crossing with the l strand in reverse direction. These both crossings would give rise to different smoothings in their ladder transcriptions.

The refinement we introduce in this paper is based on both a convenient rescaling of Lusztig's definition of the braidings [Lus93] with a $\mathfrak{gl}_m$-information, and keeping the whole *enhanced* information of webs, following ideas of Blanchet [Bla10]. It is interesting to note that the original construction of Murakami-Ohtsuki-Yamada [MOY98] was actually also keeping n-labeled strands and is consistent with this presentation. In the $\mathfrak{sl}_2$ case, this leads to a rather unusual presentation of the skein module, since 2-labeled crossings produce when smoothed some non-trivial coefficients, as shown below:



In the $\mathfrak{sl}_3$ case, we similarly keep 1-, 2- and 3-labeled strands, which produce again different coefficients in the smoothings.

The main result is then that there exists a version of the skew-Howe duality process from which the definition of the braiding can be used locally to define an invariant of framed web-tangles. A good understanding of the behavior of the braidings back in the representation-theory world is of great help in the proof of the invariance under Kauffman's web-moves and considerably simplify them, and also clarifies the categorification of these results.

Affine extensions

The skew-Howe duality process is based on two commuting actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$ on the module $\bigwedge_q^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. $U_q(\mathfrak{sl}_n)$ corresponds to the quantum invariant we are looking at, and we want to keep it unchanged, but $U_q(\mathfrak{sl}_m)$ appears more as a parameter, and we may want to consider extensions of it. A first step is to replace $U_q(\mathfrak{sl}_m)$ by its affine version $U_q(\widehat{\mathfrak{sl}_m})$.

Classical representation-theoretic tools tell us that we can extend the action of $U_q(\mathfrak{sl}_m)$ to a $U_q(\widehat{\mathfrak{sl}_m})$ one, keeping by construction the commutation property with the $U_q(\mathfrak{sl}_n)$ action. These extensions can be achieved by the process of *evaluation representations* [CP95]. This naturally provides knotted-web invariants for the cylinder, and the only question is to relate these invariants to the usual skein module associated to the surface. We show that the evaluation representations with a particular choice of the parameter give the skein module of the filled cylinder, that can be refined by passing to the affinization of the representations. These constructions therefore provide a very natural extension of Jones' construction in the case of web-tangles drawn on the cylinder.

We also investigate better descriptions of the annular skein module in terms of $U_q(\mathfrak{sl}_m)$, and we give ideas that could lead to identify it as a sub-algebra of intertwiners for an explicit $U_q(\mathfrak{sl}_n)$ representation, which would give to it the same kind of representation-theory flavored interpretation than we have in the linear case.

Many proofs use the fact that relations for $U_q(\mathfrak{sl}_m)$ and $U_q(\widehat{\mathfrak{sl}_m})$ locally have the same form. Thus, just as at the uncategorified level, the categorification of the skew Howe process provided in [LQR12] admits a direct extension to the affine case.

Note that a recent paper by Mackaay and Thiel [MT13] presents a categorification of affine q-Schur algebras. Although their paper does not directly deal with annular knots, it would be interesting to understand its implications in terms of categorified invariants of annular web-tangles and the links with categorified quantum skew-Howe duality.

Acknowledgments: I would like to thank my advisors Christian Blanchet and Catharina Stroppel for their constant support, and Aaron Lauda for his great help. Many thanks also to David Rose, Peng Shan, Pedro Vaz and Emmanuel Wagner for all useful discussions we had, Marco Mackaay for pointing out the interest of the affine Hecke algebra, and especially to Mathieu Mansuy for teaching me everything I know about affine algebras.

4.1 Skew Howe duality and skein modules

4.1.1 Skew Howe duality

4.1.1.1 Context

We first give a short description of the skew Howe duality phenomenon for usual $\mathfrak{sl}_n$ as explained in [CKL10a] and [CKM12].

We look at the quantum group $U_q(\mathfrak{sl}_m)$ as the $\mathbb{C}[q, q^{-1}]$-algebra generated by the

Chevalley elements E_i, F_i, $K_i^{\pm 1}$, for $1 \leq i \leq n-1$, subject to the relations:

$$\begin{array}{rcl}
K_iK_i^{-1} = K_i^{-1}K_i = 1 & , & K_iK_j = K_jK_i, \\
K_iE_jK_i^{-1} = q^{a_{ij}}E_j & , & K_iF_jK_i^{-1} = q^{-a_{ij}}F_j, \\
E_iF_j - F_jE_i & = & \delta_{ij}\dfrac{K_i - K_i^{-1}}{q-q^{-1}}, \\
E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 & = & 0 \text{ if } j = i \pm 1, \\
F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 & = & 0 \text{ if } j = i \pm 1, \\
E_iE_j = E_jE_i & , & F_iF_j = F_jF_i \text{ if } |i-j| > 1.
\end{array}$$

The idempotented version of $U_q(\mathfrak{sl}_m)$ will be denoted $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. Generators are $\mathbf{1}_\lambda$, $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$, for all weights λ. The unit is then replaced by a collection of orthogonal idempotents $\mathbf{1}_\lambda$ indexed by the weight lattice of $\mathfrak{sl}_m$,

$$\mathbf{1}_\lambda\mathbf{1}_{\lambda'} = \delta_{\lambda\lambda'}\mathbf{1}_\lambda,$$

such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$, then

$$K_i\mathbf{1}_\lambda = \mathbf{1}_\lambda K_i = q^{\lambda_i}\mathbf{1}_\lambda, \quad E_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i}E_i, \quad F_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\alpha_i}F_i,$$

where

$$\lambda + \alpha_i \quad = \quad \begin{cases} (\lambda_1+2, \lambda_2-1, \lambda_3, \ldots, \lambda_{m-2}, \lambda_{m-1}) & \text{if } i = 1 \\ (\lambda_1, \lambda_2, \ldots, \lambda_{m-3}, \lambda_{m-2}-1, \lambda_{m-1}+2) & \text{if } i = m-1 \\ (\lambda_1, \ldots, \lambda_{i-1}-1, \lambda_i+2, \lambda_{i+1}-1, \ldots, \lambda_{m-1}) & \text{otherwise.} \end{cases}$$

$U_q(\mathfrak{sl}_m)$ can be endowed with the structure of a Hopf algebra, with coproduct $\Delta\colon U_q(\mathfrak{sl}_m) \mapsto U_q(\mathfrak{sl}_m) \otimes U_q(\mathfrak{sl}_m)$ given on Chevalley generators by:

$$\begin{array}{rcl}
\Delta(E_i) & = & 1 \otimes E_i + E_i \otimes K_i \\
\Delta(F_i) & = & K_i^{-1} \otimes F_i + F_i \otimes 1 \\
\Delta(K_i^{\pm 1}) & = & K_i^{\pm 1} \otimes K_i^{\pm 1}.
\end{array} \tag{4.1.1}$$

Define $\bigwedge_q(\mathbb{C}^r)$ as the algebra generated by r variables:

$$\bigwedge_q(\mathbb{C}^r) = \mathbb{C}[q,q^{-1}] < X_1, \ldots, X_r > /(X_i^2,\ X_iX_j + q^{-1}X_jX_i\ for\ i < j).$$

This algebra can be given a $U_q(\mathfrak{sl}_r)$ action, extending the natural representation [1]. More precisely:

$$\begin{array}{rcl}
E_iX_i = X_{i+1} & , & E_iX_j = 0 \quad \text{if} \quad j \neq i, \\
F_iX_{i+1} = X_i & , & F_iX_j = 0 \quad \text{if} \quad j \neq i+1, \\
K_iX_i = q^{-1}X_i & , & K_iX_{i+1} = qX_{i+1}, \quad K_iX_j = X_j \quad \text{otherwise.}
\end{array}$$

We now consider $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$, where, following [CKM12], the generating variables can be denoted z_{ij} with $1 \leq i \leq n$, $1 \leq j \leq m$, subject to skew-commutation relations.

There are two isomorphisms:

$$\bigwedge_q(\mathbb{C}^n)^{\otimes m} \leftarrow \bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m) \rightarrow \bigwedge_q(\mathbb{C}^m)^{\otimes n}.$$

1. Actually, we choose here a non-standard form (dual) for the natural representation in order to obtain the same conventions as in [LQR12].

We can thus endow this module with actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$, which Cautis, Kamnitzer and Licata have proved to commute, calling this *quantum skew Howe duality*. Furthermore, $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$ form a Howe pair, which is a key argument in [CKM12].[2] The actions of the two quantum groups can be deduced, for $U_q(\mathfrak{sl}_m)$ for example, from the expressions on the variables z_{ij}:

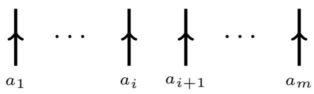
$$\begin{array}{lll} E_j z_{ij} = z_{i,j+1} \quad , & E_k z_{ij} = 0 \quad \text{if} \quad k \neq j, & \\ F_j z_{i,j+1} = z_{i,j} \quad , & F_k z_{ij} = 0 \quad \text{if} \quad k \neq j+1, & \\ K_j z_{ij} = q^{-1} z_{ij} \quad , & K_j z_{i,j+1} = q z_{i,j+1}, \quad K_k z_{ij} = z_{ij} & \text{otherwise.} \end{array}$$

We can assign degree one to each generating variable. Given an integer N, the subspace of degree N decomposes as an $\mathfrak{sl}_n$-representation:

$$\bigwedge\nolimits_q^N (\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{a_1+\cdots+a_m=N} \bigwedge\nolimits_q^{a_1} (\mathbb{C}^n) \otimes \cdots \bigwedge\nolimits_q^{a_m} (\mathbb{C}^n).$$

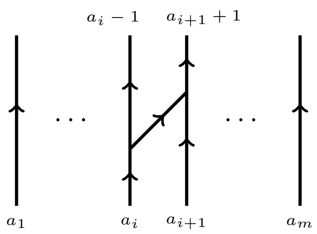
Each direct summand is an m-fold tensor product of minuscule $\mathfrak{sl}_n$-representations, but is not stable under the action of $U_q(\mathfrak{sl}_m)$. However, it appears (tracking it from the explicit definition of the actions) that each subspace $\bigwedge_q^{a_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge_q^{a_m}(\mathbb{C}^n)$ is a $U_q(\mathfrak{sl}_m)$ weight space of weight $(a_2 - a_1, a_3 - a_2, \cdots, a_m - a_{m-1})$. In particular, if $m = 2$, the subspaces are of the form $\bigwedge_q^k(\mathbb{C}^n) \otimes \bigwedge_q^l(\mathbb{C}^n)$, which are both $U_q(\mathfrak{sl}_n)$-modules and $U_q(\mathfrak{sl}_2)$ weight spaces of weight $l - k$. The action of $U_q(\mathfrak{sl}_m)$ can be explicitly tracked (see Table (4.1.2) for the case where $m = n = N = 2$), and we see that $E_i\colon \bigwedge_q^{a_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge_q^{a_i}(\mathbb{C}^n) \otimes \bigwedge_q^{a_{i+1}}(\mathbb{C}^n) \cdots \otimes \bigwedge_q^{a_m}(\mathbb{C}^n) \mapsto \bigwedge_q^{a_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge_q^{a_i-1}(\mathbb{C}^n) \otimes \bigwedge_q^{a_{i+1}+1}(\mathbb{C}^n) \cdots \otimes \bigwedge_q^{a_m}(\mathbb{C}^n)$.

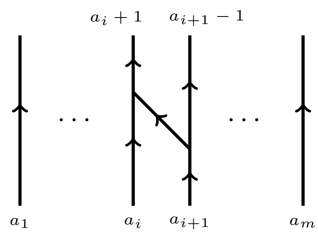
This $U_q(\mathfrak{sl}_m)$ action can be depicted by some particular diagrams called *ladders*. To a direct summand $\bigwedge_q^{a_1}(\mathbb{C}^n) \otimes \cdots \bigwedge_q^{a_m}(\mathbb{C}^n)$ we assign a sequence $(a_1, \cdots, a_m)$ depicted as weighted upward strands:

$(a_1, \cdots, a_m) \mapsto$ 

Strands labeled by zero will be erased, and we will sometime depict the n-labeled strands doubled.

We represent the action of E_i and F_i as follows:

$E_i \mapsto$ 

and
$F_i \mapsto$ 

where we only depicted the strands 1, i, $i+1$ and m: straight strands with indices a_j, $j \neq 1, i, i+1, m$ have to be added in place of the dots. The diagrams are to be read from bottom to top.

We can define the notion of ladder as any morphism obtained by composition of identities and elementary morphisms given by the images of E_i and F_i.

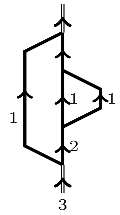
2. Note that proving the commutation is an easy computation, while it is much harder to prove that both algebras are each other commutant.

The above diagrams have an interpretation in terms of webs. Recall that $\mathfrak{sl}_n$ webs are trivalent oriented graphs with edges indexed by integers $1, \cdots, n$, so that at each vertex, the sum of outgoing labels equals the sum of ingoing labels (in the literature, the n-strands are usually erased or only kept as *tags* on the other strands, which we will not do here). These graphs are considered modulo some local relations (see for example [CKM12] for a more precise description). The webs may be understood as $\mathfrak{sl}_n$ analogues of the skein module in the $\mathfrak{sl}_2$ case (see also [MOY98] or [MS09] for more details).

An interesting fact is that all of the web relations in the spider category can be recovered from the relations in $U_q(\mathfrak{sl}_m)$ via its action on webs [3]. We refer for this and for a complete description of the spider category to [CKM12].

We give below an example of the translation process, which gives a ladder whose closure is the one depicted in the introduction (with $k = l = 1$) with $n = 3$, $m = 3$ and $N = 3$.

$$E_1F_2E_2F_1\mathbf{1}_{(3,-3)} \quad \mapsto$$

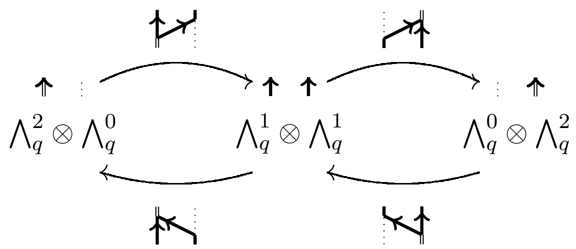


In the case where $m = n = N = 2$, the $U_q(\mathfrak{sl}_m)$ action can be explicitly given:

Sub factor	Generator	Image under E	Image under F
$\bigwedge_q^2 \otimes \bigwedge_q^0$	$z_{11} \otimes z_{21}$	$z_{11} \otimes z_{22} + q^{-1} z_{12} \otimes z_{21}$	0
$\bigwedge_q^1 \otimes \bigwedge_q^1$	$z_{11} \otimes z_{12}$	0	0
	$z_{21} \otimes z_{22}$	0	0
	$z_{11} \otimes z_{22}$	$qz_{12} \otimes z_{22}$	$qz_{11} \otimes z_{21}$
	$z_{12} \otimes z_{21}$	$z_{12} \otimes z_{22}$	$z_{11} \otimes z_{21}$
$\bigwedge_q^0 \otimes \bigwedge_q^2$	$z_{12} \otimes z_{22}$	0	$q^{-1} z_{12} \otimes z_{21} + z_{11} \otimes z_{22}$

(4.1.2)

The previous table corresponds in the diagrammatic world to the next situation:



where in the above pictures, 0-strands are depicted dotted and 2-strands are doubled.

4.1.1.2 Quantum Weyl group action

The action of the Weyl group $\mathfrak{S}_m$ of $\mathfrak{sl}_m$ on the weights q-deforms to give rise to a braid group action on representations of $U_q(\mathfrak{sl}_m)$. This phenomenon is referred to as the *quantum Weyl group action* (see [Lus93], [KT09], [CKL10a]).

3. This statement holds in the case where n-strands are only kept as tags. We will therefore add some relations on the n-strands later.

Generators of the braid group action are elements of the completion $\widetilde{U_q(\mathfrak{sl}_m)}$ of $U_q(\mathfrak{sl}_m)$. This ring is defined (see [KT09] for example) as a quotient of the ring of series $\sum_{k=1}^{\infty} X_k$ of elements of $U_q(\mathfrak{sl}_m)$, acting on each irreducible representation $V(\lambda)$ of highest weight λ by zero but for finitely many terms X_k. We then consider the quotient of this ring by the two-sided ideal of elements acting by zero on all $V(\lambda)$.

Following [Lus93], to s_i the elementary transposition corresponding to the root α_i, we associate the map $T''_{i,e} \in \widetilde{U_q(\mathfrak{sl}_m)}$:

$$T''_{i,e}\mathbf{1}_\lambda := \sum_{a-b+c=-\lambda_i} (-1)^b q^{e(-ac+b)} E_i^{(a)} F_i^{(b)} E_i^{(c)} \mathbf{1}_\lambda, \tag{4.1.3}$$

where $e = \pm 1$.

With this definition, $T''_{i,e}$ gives an endomorphism of any finite-dimensional representation. Note that if v is a weight vector of weight λ, $T_i(v)$ is a weight vector of weight $s_i(\lambda)$.

Taking $m = 2$ for simplicity, we have $T''^{\pm}_e \in \widetilde{U_q(\mathfrak{sl}_m)}$, acting on $\bigwedge_q^N(\mathbb{C}^n \otimes C^2)$. This stabilizes the whole representation, and gives a morphism of $U_q(\mathfrak{sl}_n)$ representations, from $\bigwedge_q^k(\mathbb{C}^n) \otimes \bigwedge_q^l(\mathbb{C}^n)$ to $\bigwedge_q^l(\mathbb{C}^n) \otimes \bigwedge_q^k(\mathbb{C}^n)$. It is shown in [CKL10a] that this $U_q(\mathfrak{sl}_n)$ endomorphism recovers the braiding. This is the starting point of a reinterpretation of Reshetikhin-Turaev invariants in terms of skew-Howe duality [Cau12], which admits natural categorifications [Cau12, LQR12].

The name *quantum Weyl group* is used by different authors with slightly different significations. The first one, where we use the notation $T''_{i,e}$, consists on considering morphisms of representations, acting on the category of finite-dimensional modules. We can also use it to build automorphism of the quantum group itself, by conjugation. Following [KT09], we denote the latter $C_{T''_{i,e}} : X \mapsto T''_{i,e} X T''^{\,-1}_{i,e}$. We will use both versions in this paper. We will need some results concerning the behavior of these elements for later use.

For $w = s_{i_1} \cdots s_{i_n}$ element of the Weyl group written in reduced form, where s_i are simple reflexions, we define $T''_{w,e} = T''_{i_1,e} \cdots T''_{i_k,e}$.

Proposition 4.1.1. [CP95, Theorem 8.1.2], [Lus93, Section 37.1.3], [KT09]

$$C_{T''_{i,e}}(E_i \mathbf{1}_\lambda) = -q^{-e\lambda_i} F_i \mathbf{1}_{s_i(\lambda)}$$
$$C_{T''_{i,e}}(\mathbf{1}_\lambda F_i) = -q^{e\lambda_i} \mathbf{1}_{s_i(\lambda)} E_i.$$

For $w \in W$ such that $w(\alpha_i) = \alpha_j$, $C_{T''_{w,e}}(E_i) = E_j$.

Other intertwiners, defined in [Lus93], may also be of interest:

$$T'_{i,e}\mathbf{1}_\lambda := \sum_{a-b+c=\lambda_i} (-1)^b q^{e(-ac+b)} F_i^{(a)} E_i^{(b)} F_i^{(c)} \mathbf{1}_\lambda. \tag{4.1.4}$$

We have an analogue of Proposition 4.1.1:

Proposition 4.1.2.

$$C_{T'_{i,e}}(\mathbf{1}_\lambda E_i) = -q^{-e\lambda_i} \mathbf{1}_{s_i(\lambda)} F_i$$
$$C_{T'_{i,e}}(F_i \mathbf{1}_\lambda) = -q^{e\lambda_i} E_i \mathbf{1}_{s_i(\lambda)}.$$

For $w \in W$ such that $w(\alpha_i) = \alpha_j$, $C_{T'_{w,e}}(E_i) = E_j$.

The relation between the actions of both definitions is given by:

Proposition 4.1.3. [Lus93, Sections 5.2.3, 37.1.2] $T''_{i,e}$ and $T'_{i,-e}$ are inverse of each other. $T''_{i,e}\mathbf{1}_\lambda$ and $(-1)^{\lambda_i} q^{e\lambda_i} T'_{i,e}\mathbf{1}_\lambda$ act the same way on any integrable module.

4.1.1.3 Skew Howe duality and quantum invariants for knots

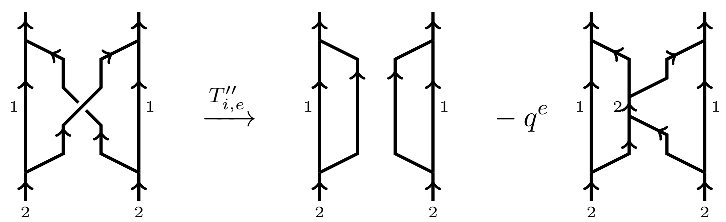
As we have seen, the skew-Howe duality process gives us different pieces of the Jones (or Reshetikhin-Turaev) invariants:

- minuscule representations $\bigwedge_q^k(\mathbb{C}^n)$ of $U_q(\mathfrak{sl}_n)$. This means that we are looking at knot invariants where we decorate the strands with minuscule representations. In particular, this does not deal with the colored Jones polynomial or its $\mathfrak{sl}_n$ generalizations, where the strands of the knot can be decorated with any finite-dimensional representations. Paths using Jones-Wenzl projectors, and their categorifications in the categorified case, can be given to relate the general invariants to the ones we study here [FSS12, SS11, CK12b, Ros11].
- elementary morphisms between tensor products of these representations, given as images of E_i and $F_i \in U_q(\mathfrak{sl}_m)$. These morphisms involve minuscule representations, but do not directly deal with duals, which in the language of knots means that we are looking at upward tangles (or their generalization for webs). The bridge with general knots or links is established in our case in [CKM12] (see also [MS09]).
- braiding between minuscule representations, understood in terms of the quantum Weyl group action of $U_q(\mathfrak{sl}_m)$. Again, this is given in the framework of ladders, and relaxing this structure will be one of the goals of the next section.

4.1.2 Skein modules

4.1.2.1 Braidings for skein modules

Let us now turn toward knots, or ladder analogues of them. The previous diagrammatic process gives us an algebraic interpretation of ladder webs, as well as a definition of the braiding for the tensor product of two minuscule representations. This braiding corresponds in the diagrammatic world to a crossing between two adjacent strands in a ladder, the explicit formulas for $T'_{i,e}$ or $T''_{i,e}$ giving a way to smooth it and replace it by a sum of ladders without crossing. Let us therefore define a knotted ladder (or a knotted web, or web-tangle in ladder position) to be a vertical composition of images of E_i and F_i and crossings between two adjacent uprights in the ladder.



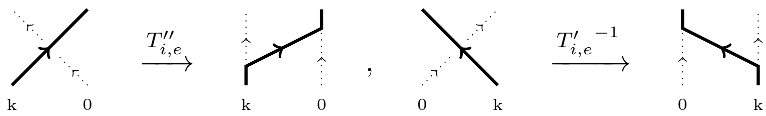
Thus, smoothing all crossings in a ladder (knotted) web, one obtains a formal sum of non-knotted ladders that one can see as an element of a skein module.

To obtain a more powerful skein module allowing less rigidity, we want to forget the 0-labeled strands. Indeed, in ladder position, even if the 0-labeled strands are not depicted, one knows where they are. If we want to start from any diagram and use the same smoothing rules as in the ladder case, we cannot know where 0-strands should be and we want to make sure that crossings involving 0-labeled strands do not play any role.

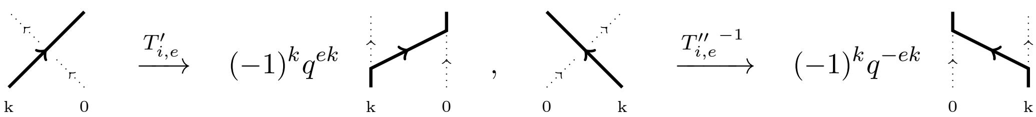
The goal of this section is to obtain a skew-Howe duality process with a conveniently rescaled braiding, so that using the smoothing rules deduced from this braiding to solve the crossings of any knotted web gives a well-defined map that to a knotted web assigns a formal sum of ladders (which is therefore requested to be invariant under the generalization for webs to Reidemeister moves). The $\mathbb{Z}[q, q^{-1}]$-module generated by webs (without crossings) and the smoothing map define what we will call a *skein module*.

Using [Lus93, Proposition 5.2.2], we have:

Proposition 4.1.4.



Note that if we use $T''_{i,e}$ in positive degree and $T'_{i,e}$ in negative degree, the situation is different: see Proposition 4.1.3. We would have:



It appears that we cannot choose one of the two solutions and apply it in all cases. A natural idea would be to use a braiding mixing both definitions... which may produce some gaps if we still want to have some instances of Propositions 4.1.1 and 4.1.2 (which will prove useful later).

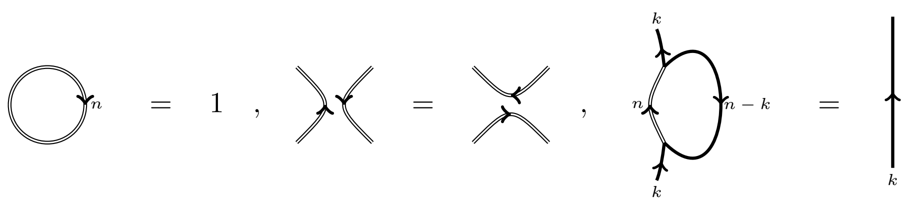
In order to avoid these distortions, we introduce additional rescalings that utilize $U_q(\mathfrak{gl}_m)$ data that is naturally encoded in the representation we are looking at (namely, the sequence $(a_1, \cdots, a_m)$, which is determined by the $\mathfrak{sl}_m$ weight and the choice of an integer N). Note that the following definitions are rather symmetric in the $T'_{i,e}$'s and $T''_{i,e}$'s.

$$\begin{aligned} T_{i,e}\mathbf{1}_\lambda &= (-1)^{-a_{i+1}} q^{-ea_{i+1}} T''_{i,e}\mathbf{1}_\lambda = (-1)^{-a_i} q^{-ea_i} T'_{i,e}\mathbf{1}_\lambda, \\ T_{i,e}^{-1}\mathbf{1}_\lambda &= (-1)^{a_i} q^{ea_i} {T''_{i,e}}^{-1}\mathbf{1}_\lambda = (-1)^{a_{i+1}} q^{ea_{i+1}} {T'_{i,e}}^{-1}\mathbf{1}_\lambda. \end{aligned} \tag{4.1.5}$$

It is easy to see from Proposition 4.1.3 that both definitions agree, and that this definition still provides a braiding. We can check that we still have $C_{T_{1,e}T_{2,e}}(E_1) = E_2$ as endomorphisms of a given representation appearing in the skew Howe context.

All webs we considered so far were obtained from ladder webs, and therefore have their boundary split in two parts, with all strands oriented inside for the bottom part, and outside for the upper part. General webs are more general than this particular situation, but it is shown in [CKM12] that general webs can be related to the particular class of webs obtained from ladders using the tool of pivotal categories. This involves in particular duals of the objects we have been looking at.

The use of tags makes the situation somewhat simpler (but harder to fit in a skein module formulation!), but the next relations (and the ones obtained by symmetry on the next ones) are particular realizations of the ones given in [CKM12] in the case where we keep the n-labeled strands.



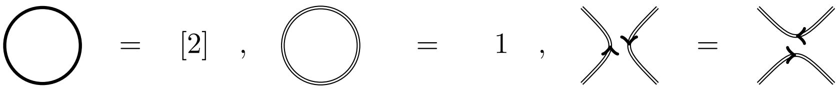
In the above pictures, the n-th strands are depicted doubled for underlying their particular role.

4.1.2.2 $\mathfrak{sl}_2$ case

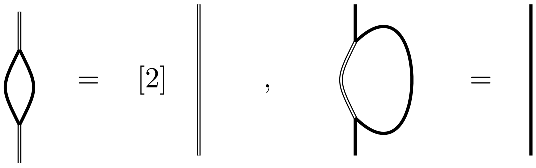
Let us now give a complete description of the $\mathfrak{sl}_2$ case.

In [Bla10], Blanchet introduces $\mathfrak{sl}_2$ webs to be oriented trivalent graphs with two kind of edges (1 and 2-labeled, we draw the latter doubled), and vertices having two inner 1-labeled strands and an outer 2-labeled one, or one inner 2-labeled strand and two outer 1-labeled ones.

Define the $\mathfrak{sl}_2$ skein module to be the quotient of (linear combinations of) $\mathfrak{sl}_2$ webs by the next relations:



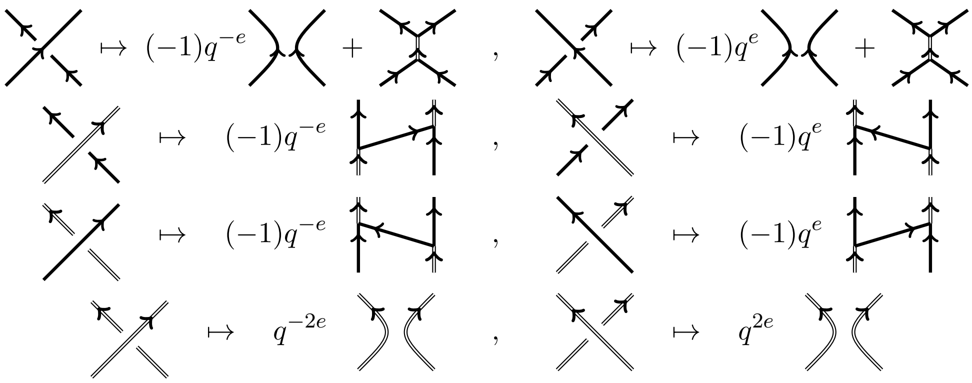
(4.1.6)



(4.1.7)

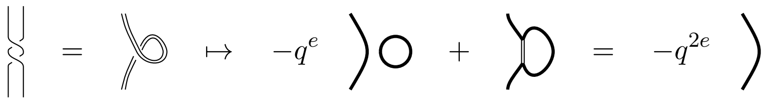
For non-oriented webs above, the depicted relations hold for any compatible orientation.

The definition of the braidings then give the following smoothing rules:



We could check, following [Kau89], that the previous relations 4.1.6, 4.1.7, and the above smoothing relations define a framed skein module. Checking directly all formulas is rather long and tedious, and we note that using the description in terms of the $U_q(\mathfrak{sl}_m)$-action gives us an efficient way to considerably simplify the proof, in the general case. Indeed, most formulas we want to check are consequences of $U_q(\mathfrak{sl}_m)$-relations.

The previous skein module provides invariants of framed webs. Here are the effects of adding a negative twist on a 1-strand (depicted in a ribbon version in the two left parts of the equation below):

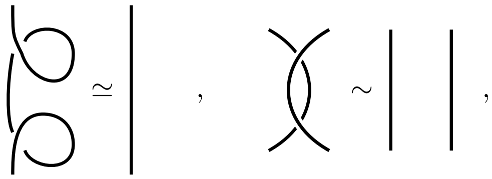


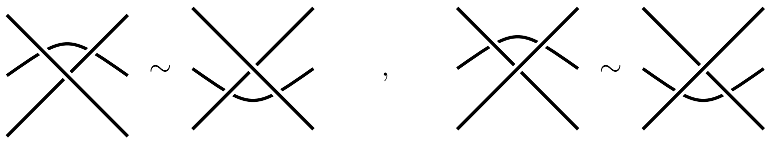
The same computation for a 2-labeled strand gives a q^{2e} coefficient. We may introduce twists with half-integers, assigning to them in the negative case the multiplication by q^e, and in the positive case the multiplication by q^{-e}, up to fourth roots of the unity. We fix the value to be $(-1)^{\frac{k}{2}}q^{\frac{-ekn+ek(k-1)}{2}}$ where k stands for the labeling of the strand, and where we have chosen a favorite primitive fourth root of the unity $(-1)^{\frac{1}{2}}$.

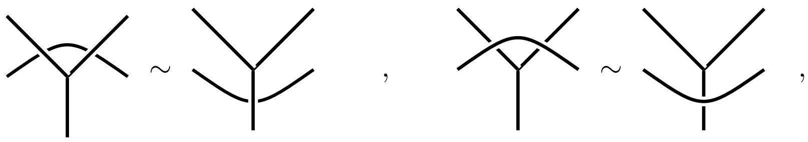
4.1.2.3 $\mathfrak{sl}_n$ case

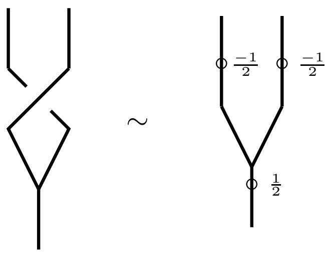
The previous process applies as well to any value of n, and produces a skein module in the following sense: Cautis, Kamnitzer and Morrison [CKM12] prove that all web relations come from $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ relations, so we just need to extend it to crossings and prove the invariance under Reidemeister moves. This is the purpose of Prop 4.1.5.

Recall from [Kau89] (see also [Car12, Theorem 2]) the relations we need to check (in a framed version, where a numbered circle on a strand stands for twists):


(4.1.8)


(4.1.9)

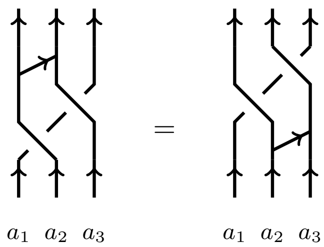

(4.1.10)



(4.1.11)

Proposition 4.1.5. For all n, smoothings defined by Relations (4.1.5) define a skein module for knotted $\mathfrak{sl}_n$ webs, that is, the application defined by smoothing rules on diagrams of knotted webs is independent of the choice of a diagram for a given web-tangle.

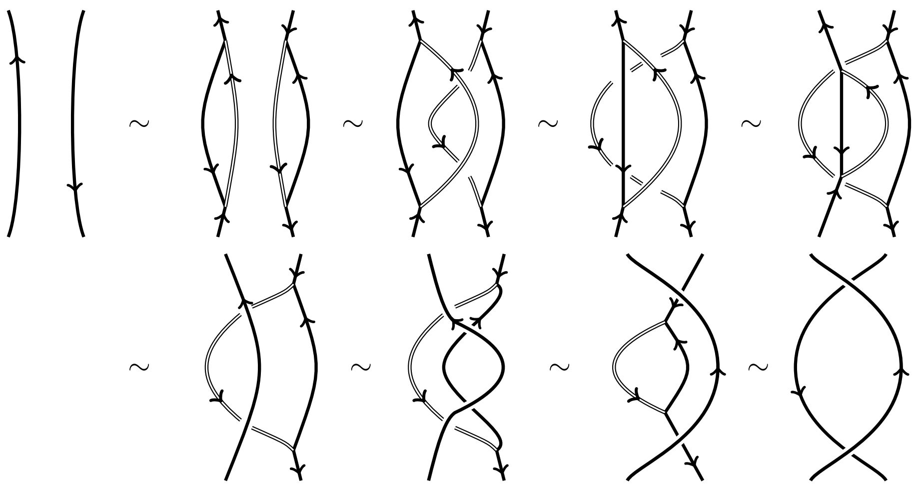
Proof. **Braid-like relations:** Braid-like Reidemeister II relation is direct, and braid-like Reidemeister III relations are consequences of the braiding relation (see [CMW09] for a presentation of all 6 braid-like Reidemeister III relations).
Framing: The framed Reidemeister I relation is easy to check. Furthermore, we can deal locally with the framing as we did in the $\mathfrak{sl}_2$-case. Details will be given in Lemma 2.
Braid-like web relation 4.1.10:[4] Braid-like relations 4.1.10 are consequences of the next equality, or similar ones:



4. Note that similar relations are studied at the categorified level in [MSV11].

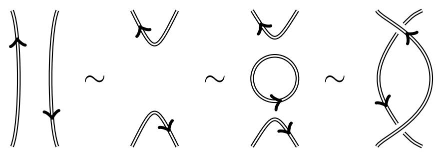
The previous relation is a diagrammatically depicted consequence of $C_{T_{1,e}T_{2,e}}(E_1) = E_2$, a relation from Propositions 4.1.1 and 4.1.2, which still holds after rescalings. For obtaining the general case, one needs a straightforward generalization of the previous relation: $C_{T_{1,e}T_{2,e}}(E_1^{(k)}) = E_2^{(k)}$.

Star relations using duality: Following [CMW09], it suffices to have the Reidemeister II relation with opposite orientations to deduce the Reidemeister III star relation. We obtain the Reidemeister II case from the braid-like one as follows:

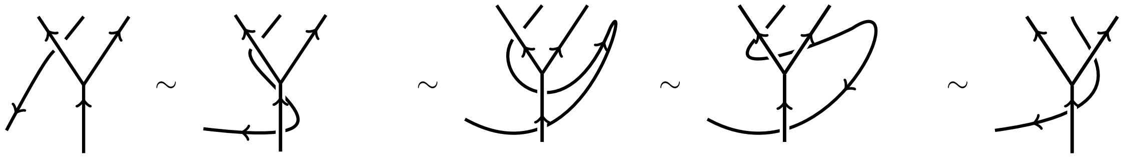


(4.1.12)

We used in the previous computation the star Reidemeister II relation in the n-n case, which is easy to prove:

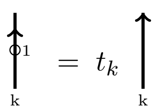


Then, we want to obtain the missing forms of Relation 4.1.10. We proceed as follows in one case, the other ones being similar:



Relation 4.1.11 and framing: The last relation, for which we need a better understanding of the framing, will be proved separately in Lemma 3 below. ☐

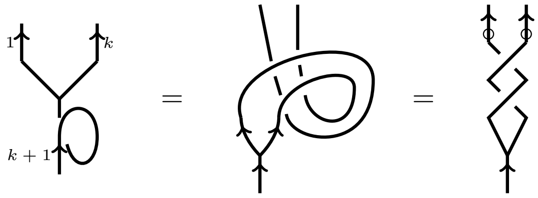
It is easy to see that a positive framing on a k-strand is equivalent to multiplication by a polynomial in q and q^{-1}. Note t_k this polynomial.



Lemma 2. $t_k = (-1)^k q^{-ekn} q^{ek(k-1)}$.

Note that this formula explains the choice for the half twists.

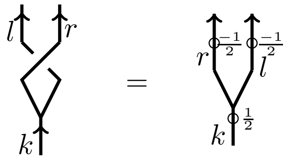
Proof. We claim that $t_{k+1}t_k^{-1} = -q^{2ek-en}$. Indeed, the following relation holds from already proven Kauffman relation:



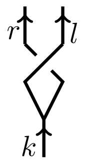
The equality of the LHS and RHS parts implies the recurrence relation. The general solution follows then from the computation of the value for a 1-strand. An explicit computation for this gives $t_1 = -q^{-en}$. □

We are now ready to prove the last relation with a recurrence:

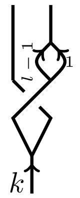
Lemma 3.



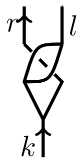
Proof. The computation is easy for $r = 1$ or $l = 1$. Then we use:



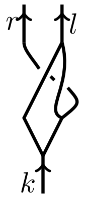
$= \frac{1}{[l]}$



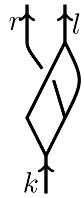
$= \frac{1}{[l]}$



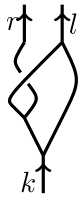
$= \frac{1}{[l]}$



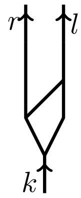
$= \frac{a_{r+1}^{\frac{1}{2}} a_r^{\frac{-1}{2}} a_1^{\frac{-1}{2}}}{[l]}$



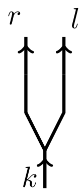
$= \frac{a_{r+1}^{\frac{1}{2}} a_r^{\frac{-1}{2}} a_1^{\frac{-1}{2}}}{[l]}$



$= \frac{a_{r+1}^{\frac{1}{2}} a_r^{\frac{-1}{2}} a_1^{\frac{-1}{2}} a_{k-1}^{\frac{1}{2}} a_{l-1}^{\frac{-1}{2}} a_r^{\frac{-1}{2}}}{[l]}$

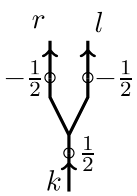


$= a_{r+1}^{\frac{1}{2}} a_r^{\frac{-1}{2}} a_1^{\frac{-1}{2}} a_{k-1}^{\frac{1}{2}} a_{l-1}^{\frac{-1}{2}} a_r^{\frac{-1}{2}}$



.

An explicit computation of the coefficient shows that the previous term equals:



,

which completes the proof. □

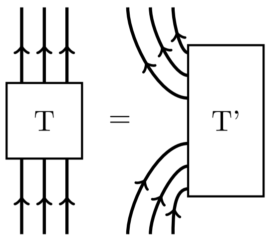
We therefore obtain a well-defined skein module providing an invariant of knotted webs.

Note that in the $\mathfrak{sl}_3$ case, 2-strands are usually translated into 1-strands by reversing the orientation. In this case, smoothings of crossings would be defined only up to a power of q, and understanding a skew-Howe based way to fix this power seems difficult. We choose here not to apply this duality process and keep distinct 1- and 2-strands with their own orientations, and more generally to keep all strands numbered $1 \dots n$ in the $\mathfrak{sl}_n$ case, with their orientation.

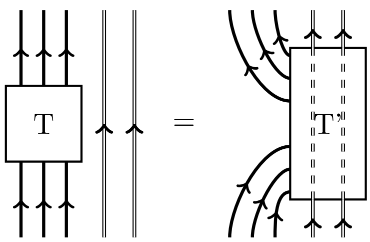
So, we have seen that to a web-tangle in ladder position, we can assign a $U_q(\mathfrak{sl}_n)$ morphism of tensor product of minuscule representations. The diagrammatic form of this morphism corresponds to the image of the same web-tangle in the $\mathfrak{sl}_n$ skein module. If we start with a non-ladder web, we can assign to it its skein element, but the skew-Howe process does not directly apply. Cautis, Kamnitzer and Morrison [CKM12] explain a process for turning upward webs to ladder form, which we summarize in the following section.

4.1.2.4 Turning a knot to a ladder

Let us now consider a tangle T (possibly a web-tangle) with only upward boundaries, which we will call *upward web*. Following Cautis-Kamnitzer-Morrison, we can present it as:

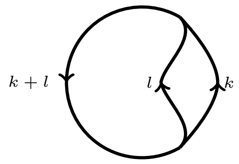


The left part of the RHS is easily presentable as a ladder. The tangle T' is assumed to be presented as a horizontal grid diagram generated by caps, cups and crossings (plus 3-valent vertices for webs). We request to have all crossings vertical, which is possible up to some isotopy. So, two caps or cups cannot lie one over the other one, and we determine the number of n-strands we will have to add as the number of elementary pieces that contain a downward strand: we will then put a strand on the right of this place. Let this number be denoted α. This being done, start over, but adjoining on the right of T α upward n-strands placed at the right place.

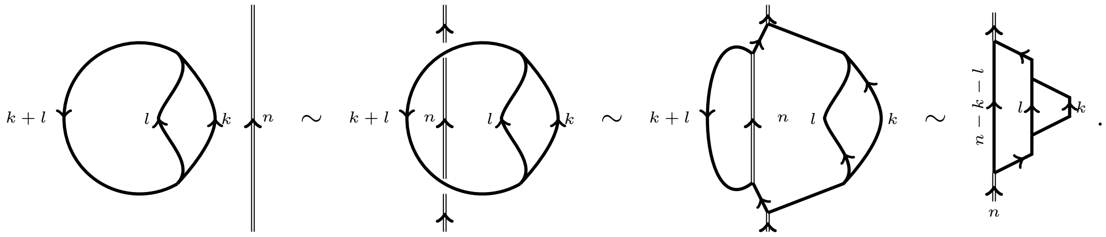


By performing some moves and simplifications near the downward strands, we get a ladder L. These local changes Cautis-Kamnitzer-Morrison perform are smoothings and simplifications of some Reidemeister moves, and so the image of the tangle is equal in the previous skein module to T with α disjoint n-strands added to it.

For example, if we start from the elementary web we considered in the introduction:



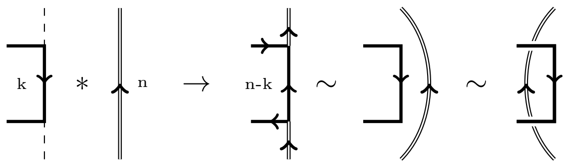
we can add on the right one n-labeled strand and perform Reidemeister-Kauffman moves:



The last isomorphism above is a digon removal, which can be found in [CKM12] for example.

So, from any upward web-tangle union n-strands, we can obtain by a succession of Reidemeister moves and equivalences a ladder diagram. The morphism of representation we compute by the skew Howe process has then a diagrammatic depiction equivalent to the skein element associated to the web-tangle we started from union the n-strands.

Note now that instead of pulling n-strands from far away, we could have performed a Jones-Kauffman product. Recall that the skein module may be endowed with an algebra structure by defining $\alpha * \beta$ to be the smoothing of the superposition of diagrams of α over β. This superposition is usually assumed to be a knotted web diagram, meaning that the only singularities are crossings. However, we can allow a singular case:



where the dashed line on the left above indicates the place we want to put the n strand: this allows not to perform any simplification on the diagram. This re-interpretation of the process will show useful when we turn to the annular case, where we have no free space where to put the n-strands before pulling them on the place they are needed.

We have seen here only the case where all the boundary of the tangle is upward. First, notice that this is enough for dealing with knots. However, as explained in [CKM12], any tangle is actually isomorphic to such an upward tangle.

4.2 Affine extensions

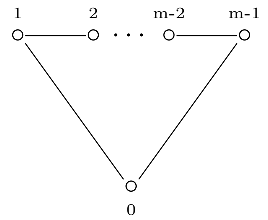
We have seen how the skew-Howe duality process, that involves two commuting actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_m)$, helps redefining Reshetikhin-Turaev $\mathfrak{sl}_n$ invariants for knots and links, that extend to invariants of knotted webs. The first quantum group controls the invariant we are looking at, and we therefore want to keep it unchanged. But the second one plays the role of a parameter related to the topology of the space we are working in. We can thus attempt to change it to try to modify the topology of this space.

One of the easiest extension we can perform starting from $U_q(\mathfrak{sl}_m)$ is to pass to its affine version $U_q(\widehat{\mathfrak{sl}_m})$, and we will show that the topological analogue of this is to close the square the knots where drawn into an annulus.

We begin by defining different versions of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_m})$ that we will use here, before turning toward easy representations of it. We then relate this extension to knots, and study the invariants we can deduce from it.

4.2.1 Affine $\mathfrak{sl}_m$

$U_q(\widehat{\mathfrak{sl}_m})$ is the quantum affine algebra corresponding to $U_q(\mathfrak{sl}_m)$, that is the Kac-Moody algebras described by the following Dynkin diagram:



(4.2.1)

Following [HK02], we consider the algebra $U_q(\widehat{\mathfrak{sl}_m})$ as generated by Chevalley generators E_i, F_i and $K_i^{\pm 1}$ for $0 \leq i \leq m-1$, and extra generators $K_d^{\pm 1}$ corresponding to the null root. The elements E_i, F_i and $K_i^{\pm 1}$ are subject to $\mathfrak{sl}_m$ relations, where we identify m and 0, so that the quantum Serre relations hold for pairs (E_0, E_1), (F_0, F_1), (E_0, E_{m-1}) and (F_0, F_{m-1}):

$$\begin{aligned}
K_iK_i^{-1} = K_i^{-1}K_i = 1 \quad &, \quad K_iK_j = K_jK_i, \\
K_iE_jK_i^{-1} = q^{a_{ij}}E_j \quad &, \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j, \\
E_iF_j - F_jE_i \quad &= \quad \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}}, \\
E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 \quad &= \quad 0 \text{ if } j = i \pm 1, \\
F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 \quad &= \quad 0 \text{ if } j = i \pm 1, \\
E_iE_j = E_jE_i \quad &, \quad F_iF_j = F_jF_i \text{ if } |i-j| > 1.
\end{aligned}$$

Furthermore, K_d and K_d^{-1} are subject to the following relations:

$$\begin{aligned}
K_dK_i &= K_iK_d \quad \forall i \in \{0, \dots, m-1\} \\
K_dE_iK_d^{-1} &= q^{\delta_{0,i}}E_i \quad \forall i \in \{0, \dots, m-1\} \\
K_dF_iK_d^{-1} &= q^{-\delta_{0,i}}F_i \quad \forall i \in \{0, \dots, m-1\}.
\end{aligned}$$

If we restrict to the sub-algebra generated by E_i, F_i and $K_i^{\pm 1}$, we produce a quantum group usually denoted $U'_q(\widehat{\mathfrak{sl}_m})$. A key difference between the two versions is that the second one has finite dimensional irreducible modules, while the first one admits no non-trivial finite dimensional representations.

We will also use an idempotented version of $U'q(\widehat{\mathfrak{sl}_m})$, that we denote $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$, generated by $\mathbf{1}_\lambda$, $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$ with the obvious generalization of the relations of the $\mathfrak{sl}_m$ case. Weights here are m-tuples $\lambda = (\lambda_0, \dots, \lambda_{m-1})$, which in our case, with N fixed, will be related to the sequences $(a_1, \dots, a_m)$ by $\lambda_i = a_{i+1} - a_i$ for $i \neq 0$ and $\lambda_0 = a_1 - a_m$. Note that we have $\sum \lambda_i = 0$.

4.2.2 Evaluation representations

$U_q(\mathfrak{sl}_m)$-representations may be extended to representations of $U'_q(\widehat{\mathfrak{sl}_m})$. The complete formulas (that require a step through $U_q(\mathfrak{gl}_m)$) can be found in [CP95, p. 400]. These formulas seem at first a bit mysterious, and rather than directly using them, we choose to define differently extensions based on the use of the braidings $T''_{i,e}$ and $T_{i,e}$. We will denote by ρ_a and $\widetilde{\rho}_a$ the two morphisms used in these definitions. We will then investigate *a posteriori* the relation with usual formulas as written in [CP95] in Proposition 4.2.3.

First, recall that $C_{T''_{i,e}}$ is the automorphism of $U_q(\mathfrak{sl}_m)$ (or its idempotented version) given by conjugation by $T''_{i,e}$ (that is, $X \in U_q(\mathfrak{sl}_m) \mapsto T''_{i,e} X {T''_{i,e}}^{-1}$). We assume here that whenever we refer to a weight λ, a number N is fixed, and we freely refer to the associated m-tuple $(a_1, \dots, a_m)$.

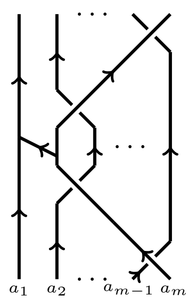
For a complex number, define $\rho_a \colon \dot{\mathbf{U}}_q(\widehat{\mathfrak{sl}_m}) \mapsto \dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ by:

$$\begin{aligned} \rho_a(E_0 \mathbf{1}_\lambda) &= a q^{-e(a_1+a_m)} C_{T''_{m-1,e}} \cdots C_{T''_{2,e}}(F_1)\mathbf{1}_\lambda, \\ \rho_a(F_0 \mathbf{1}_\lambda) &= a^{-1} q^{e(a_1+a_m)} C_{T''_{m-1,e}} \cdots C_{T''_{2,e}}(E_1)\mathbf{1}_\lambda. \end{aligned}$$

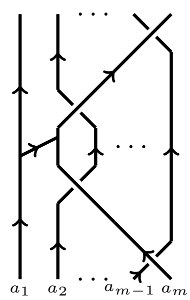
ρ_a is extended to other generators E_i ($i \neq 0$) by sending $E_i \in U_q(\widehat{\mathfrak{sl}_m})$ to $E_i \in U_q(\mathfrak{sl}_m)$, and similarly for F_i.

In terms of the graphical calculus previously defined, this can be drawn as:

$E_0 \mathbf{1}_\lambda \quad \mapsto \quad a q^{-e(a_1+a_m)}$



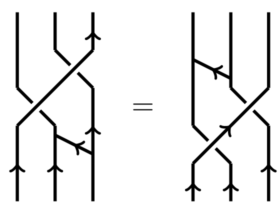
$, \quad F_0 \mathbf{1}_\lambda \quad \mapsto \quad a^{-1} q^{e(a_1+a_m)}$



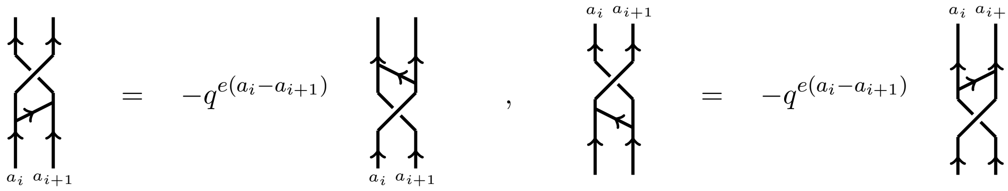
. (4.2.2)

The above depiction corresponds to letting $\dot{\mathbf{U}}_q(\widehat{\mathfrak{sl}_m})$ act on a tensor product of minuscule representations of $U_q(\mathfrak{sl}_m)$ via the map ρ_a. This diagrammatic definition gives us an easy way to reprove (in Proposition 4.2.1) in a diagrammatic fashion the results of [CP95, Prop 12.2.10] applied to these particular representations $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$. However, since all fundamental $U_q(\mathfrak{sl}_m)$ representations $\bigwedge_q^k(\mathbb{C}^m)$ appear as summands of a $\bigwedge^N(\mathbb{C}^n \otimes \mathbb{C}^m)$, using the coproduct, we can recover from this construction all evaluation representation associated to finite-dimensional representations.

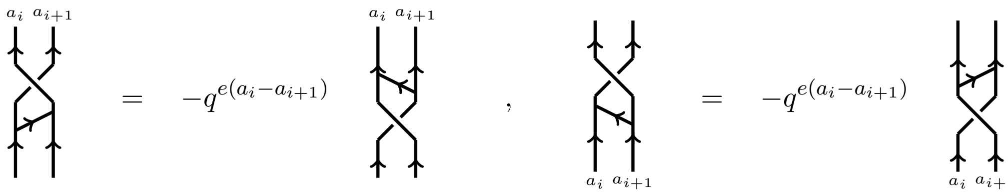
The relations in Propositions 4.1.1 and 4.1.2 together with the fact that the $T''_{i,e}$'s provide a braiding give the next useful diagrammatic relations:



(4.2.3)



(4.2.4)



(4.2.5)

Proposition 4.2.1. Letting $E_0\mathbf{1}_\lambda$ and $F_0\mathbf{1}_\lambda$, elements of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$, act as defined in Relation 4.2.2 (that is, using ρ_a), extends the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on $\bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m)$ to $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$.

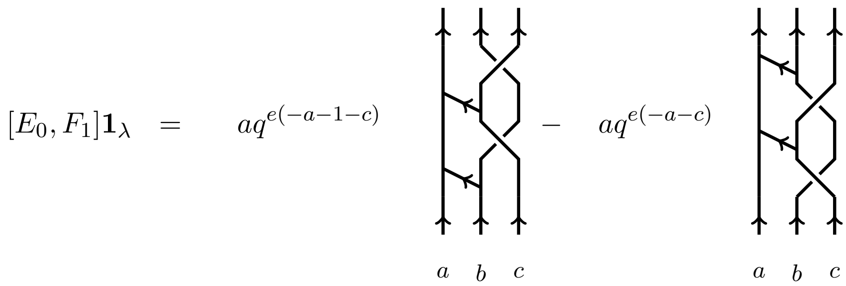
Proof. The proof consists on checking the defining relations of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$ that involve E_0 or F_0, that is:

$$\begin{aligned}
[E_0, F_0]\mathbf{1}_\lambda &= [\lambda_0]\mathbf{1}_\lambda \\
[E_0, F_j]\mathbf{1}_\lambda = 0 \ \ , \ \ [F_0, E_j]\mathbf{1}_\lambda &= 0 \\
E_0^2E_j\mathbf{1}_\lambda - (q+q^{-1})E_0E_jE_0\mathbf{1}_\lambda + E_jE_0^2\mathbf{1}_\lambda &= 0 \text{ if } j=1 \text{ or } j=m-1 \\
E_j^2E_0\mathbf{1}_\lambda - (q+q^{-1})E_jE_0E_j\mathbf{1}_\lambda + E_0E_j^2\mathbf{1}_\lambda &= 0 \text{ if } j=1 \text{ or } j=m-1 \\
F_0^2F_j\mathbf{1}_\lambda - (q+q^{-1})F_0F_jF_0\mathbf{1}_\lambda + F_jF_0^2\mathbf{1}_\lambda &= 0 \text{ if } j=1 \text{ or } j=m-1 \\
F_j^2F_0\mathbf{1}_\lambda - (q+q^{-1})F_jF_0F_j\mathbf{1}_\lambda + F_0F_j^2\mathbf{1}_\lambda &= 0 \text{ if } j=1 \text{ or } j=m-1 \\
E_0E_j = E_jE_0 \ \ , \ \ F_0F_j &= F_jF_0 \text{ if } j \neq 1 \text{ and } j \neq m-1.
\end{aligned}$$

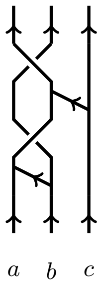
By symmetry between the $E's$ and the F's, it is enough to check only relations involving E_0 and E_j or F_j, plus the first one. The proof mostly relies on a straightforward use of Relations 4.2.3, 4.2.4, 4.2.5, and the braiding relation. We present some of them below, where we identify an element of $\dot{\mathbf{U}}'_q(\mathfrak{sl}_m)$ with its image under the representation.

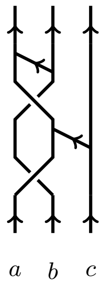
$$\begin{aligned}
[E_0, F_0]\mathbf{1}_\lambda &= E_0F_0\mathbf{1}_\lambda - F_0E_0\mathbf{1}_\lambda = q^0[C_{T''_{w,e}}(F_1), C_{T''_{w,e}}(E_1)]\mathbf{1}_\lambda = C_{T''_{w,e}}([F_1, E_1]\mathbf{1}_{w\cdot\lambda}) \\
&= C_{T''_{w,e}}(-[(w^{-1}\cdot\lambda)_1])\mathbf{1}_\lambda = -(-[\lambda_0])\mathbf{1}_\lambda = [\lambda_0]\mathbf{1}_\lambda.
\end{aligned}$$

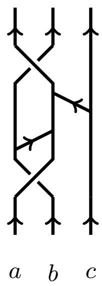
Under the braiding relation, we can reduce $[E_0, F_1]$ to:

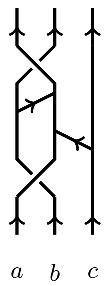


which equals

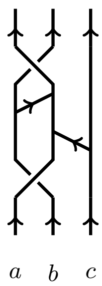
$= \quad aq^{e(-a-1-c)}$ 

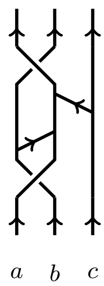
$- \quad aq^{e(-a-c)}$ 

$= \quad aq^{e(-a-1-c)}\left(-q^{e(-a-1+b-1)}\right)$ 

$- \quad aq^{e(-a-c)}\left(-q^{e(-a-2+b-1)}\right)$ 

We then obtain:

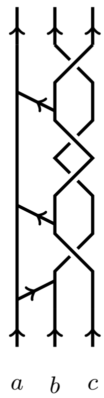
$-\,aq^{-2ea+eb-ec-3e} \Bigg($ 

$-$ 

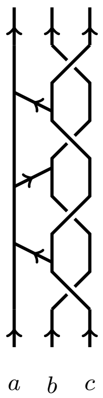
$\Bigg) = 0$

using the $U_q(\mathfrak{sl}_m)$ relation $[E_1, F_2] = 0$. We now turn to:

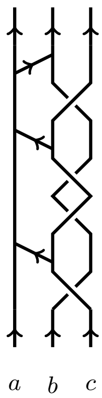
$$E_0^2E_1 \quad - \quad (q+q^{-1})E_0E_1E_0 \; + \; E_1E_0^2 = \quad a^2q^{e(-2a-2c+2)}$$



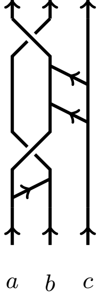
$$-a^2(q+q^{-1})q^{e(-2a-2c+1)}$$



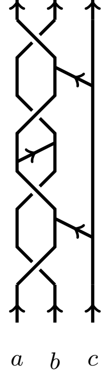
$$+\,a^2q^{e(-2a-2c)}$$



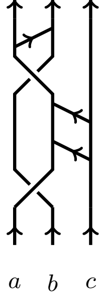
$$= \; a^2q^{e(-2a-2c+2)}$$

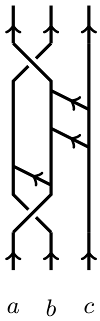


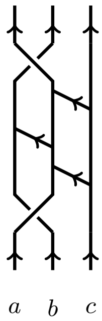
$$-\,(q+q^{-1})a^2q^{e(-2a-2c+1)}$$

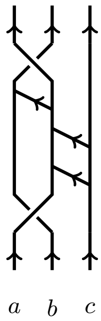


$$+\,a^2q^{e(-2a-2c)}$$



$= -a^2 q^{e(-a-b-2c+2)}$ 

$+a^2(q+q^{-1})q^{e(-a-b-2c+2)}$ 

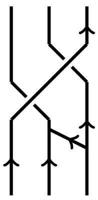
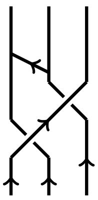
$-a^2 q^{e(-a-b-2c+2)}$ 

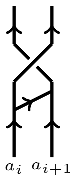
The latter equals 0 since $F_2^2F_1 - (q+q^{-1})F_2F_1F_2 + F_1F_2^2 = 0$.

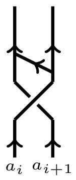
□

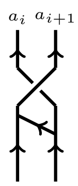
Note that the usual definition of the evaluation representations requires to have a $U_q(\mathfrak{gl}_m)$ action. In our case, the $U_q(\mathfrak{sl}_m)$ action, with a choice of N made, is actually a disguised $U_q(\mathfrak{gl}_m)$-representation, and the factor $q^{\pm(a_1+a_m)}$ utilizes this $\mathfrak{gl}_n$-information.

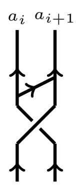
It will be useful when we turn to skein modules to rescale the braidings and replace the $T''_{i,e}$'s by the $T_{i,e}$'s. Doing so, we have analogues of Relations 4.2.3, 4.2.4 and 4.2.5:

 $=$ 

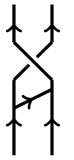


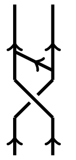
$= -q^{e(a_i-a_{i+1}+1)}$ 

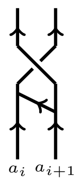
, 

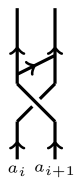
$= -q^{e(a_i-a_{i+1}-1)}$ 

(4.2.6)



$= -q^{e(a_i-a_{i+1}+1)}$ 

, 

$= -q^{e(a_i-a_{i+1}-1)}$ 

(4.2.7)

We denote $\widetilde{\rho}_a$, for a complex number, the analogue of ρ_a:

$$\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) = aq^{-e(a_1+a_m)}C_{T_{m-1,e}}\cdots C_{T_{2,e}}(F_1)\mathbf{1}_\lambda,$$

$$\widetilde{\rho}_a(F_0\mathbf{1}_\lambda) = a^{-1}q^{e(a_1+a_m)}C_{T_{m-1,e}}\cdots C_{T_{2,e}}(E_1)\mathbf{1}_\lambda.$$

ρ_a and $\widetilde{\rho}_a$ are defined similarly for other generators.

The proof of Proposition 4.2.1 remains valid if we replace the braidings $T''_{i,e}$'s by the $(\mathfrak{gl}_m)$ rescaled $T_{i,e}$'s.

Proposition 4.2.2. Using the map $\widetilde{\rho}_a \colon \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \mapsto \dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ extends the action of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ to an action of $U'_q(\widehat{\mathfrak{sl}_m})$.

Now, using the fact that the rescaled braidings $T''_{i,e}$ allow some liberty, we can give *a posteriori* a formula very close to Chari-Pressley original evaluation representations[CP95].

Assume that we have chosen a square root $q^{\frac{1}{2}}$ of q, and define the bracket $[\,\cdot\,,\,\cdot\,]_{q^{\frac{1}{2}}}$ by $[u,v]_{q^{\frac{1}{2}}} = q^{\frac{1}{2}}uv - q^{-\frac{1}{2}}vu$.

Proposition 4.2.3. If $e=-1$, we have:

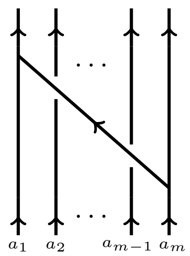
$$\begin{aligned}
\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) &= aq^{a_1+a_m-\frac{m-2}{2}}[[\dots[F_1,F_2]_{q^{\frac{1}{2}}}\dots,F_{m-2}]_{q^{\frac{1}{2}}},F_{m-1}]_{q^{\frac{1}{2}}} \\
&= aq^{a_1+a_m-\frac{m-2}{2}}[F_1,[F_2,\dots,[F_{m-2},F_{m-1}]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}},
\end{aligned}$$

and if $e=1$, we have:

$$\begin{aligned}
\widetilde{\rho}_a(E_0\mathbf{1}_\lambda) &= (-1)^{m-2}aq^{-a_1-a_m+\frac{m-2}{2}}[F_{m-1},[F_{m-2},\dots,[F_2,F_1]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}} \\
&= (-1)^{m-2}aq^{-a_1-a_m+\frac{m-2}{2}}[[\dots[F_{m-1},F_{m-2}]_{q^{\frac{1}{2}}}\dots,F_2]_{q^{\frac{1}{2}}},F_1]_{q^{\frac{1}{2}}}.
\end{aligned}$$

Proof. The core of the identification is to understand the bracket process in the diagrammatic definition. By Reidemeister or Kauffman-type moves, we can rewrite $\widetilde{\rho}_a(E_0\mathbf{1}_\lambda)$ as (which doesn't make sense in $U_q(\mathfrak{sl}_m)$ anymore):

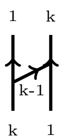
$$E_0\mathbf{1}_\lambda \quad \mapsto \quad aq^{-e(a_1+a_m)}$$



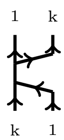
(4.2.8)

Then, we can smooth the rightmost crossing using:

$$= \quad -q^e$$

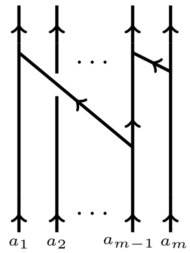
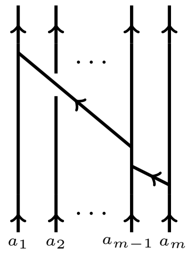


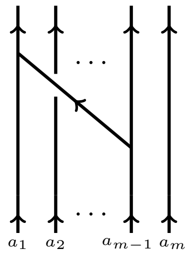
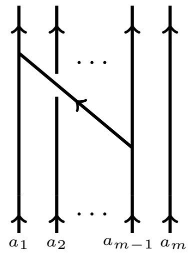
$$+$$



. (4.2.9)

This gives:

$$E_0\mathbf{1}_\lambda \quad \mapsto \quad -q^e a q^{-e(a_1+a_m)}$$

$$+\, a q^{-e(a_1+a_m)}$$


$$= \quad -q^e a q^{-e(a_1+a_m)} F_{m-1}$$

$$+\, a q^{-e(a_1+a_m)}$$

$$F_{m-1}$$

Depending on the value of e, this gives one or the other bracket, and we can iterate by smoothing successive crossings. This proves the first part of each equality. For the second part, we perform the same process, but we start with the left-most crossing. $\square$

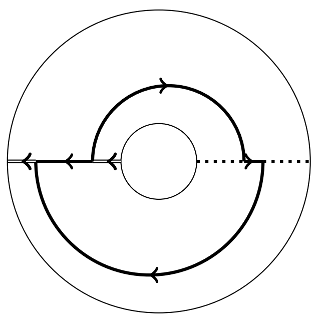
Because of the similarity of this definition with the usual one, we will from now on refer to the representations induced by $\widetilde{\rho}_a$ as *evaluation representations*.

This process helps us extending $\bigwedge_q(\mathbb{C}^n\otimes\mathbb{C}^m)$ to a $U_q'(\widehat{\mathfrak{sl}_m})$ representation. The $U_q'(\widehat{\mathfrak{sl}_m})$ and $U_q(\mathfrak{sl}_n)$ actions still commute, but this certainly cannot provide new information, since the new representation is entirely built on the old one. An extension of this representation will be studied later.

Since the usual skew Howe duality context is closely related to skein modules (or spider categories), it is natural to wonder if we can understand a skein analogue for the evaluation representations. It appears that closing the Dynkin diagram of $\mathfrak{sl}_n$ corresponds to gluing two opposite sides of the box on which one usually looks at tangles: this produces an annulus.

4.2.3 Annular knots

Represent $\bigwedge_q^N(\mathbb{C}^n\otimes\mathbb{C}^m) \simeq \bigoplus_{a_1+\cdots+a_m=N}\bigwedge_q^{a_1}(\mathbb{C}^n)\otimes\cdots\otimes\bigwedge_q^{a_m}(\mathbb{C}^n)$ by a sequence $(a_1,\cdots,a_m)$, as before, but now drawn on a circle instead of drawing it on a segment, so that a_1 and a_m lie next to each other. E_0 is a map: $\bigwedge_q^{a_1}(\mathbb{C}^n)\otimes\cdots\otimes\bigwedge_q^{a_m}(\mathbb{C}^n) \to \bigwedge_q^{a_1+1}(\mathbb{C}^n)\otimes\cdots\otimes\bigwedge_q^{a_m-1}(\mathbb{C}^n)$. This idea gives a diagrammatic presentation of the affine extension of the skew Howe duality phenomenon. See below the diagram corresponding to E_0E_1 acting on a sequence $(2,0)$, with $m=2$, $n=2$, $N=2$.

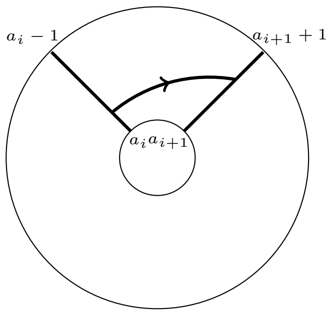


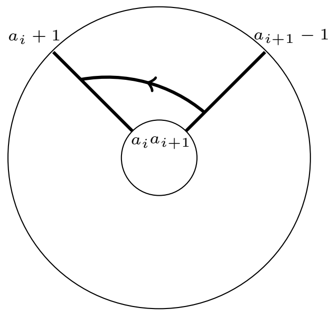
Annular webs can be defined in a very similar fashion as in the plane case. One just embeds trivalent graphs in the annulus instead of the plane, and all relations being local, they look the same for webs considered in any surface.

All web relations and all $\widehat{\mathfrak{sl}_m}$ relations being "local relations", it is easy to observe that web relations are implied by $\widehat{\mathfrak{sl}_m}$. However, if we consider the evaluation representation for extending the $\mathfrak{sl}_m$ action, there are other relations than web relations: for example, the above diagram corresponds to a scalar action, while in the skein module, this would rather correspond to a generator. We shall first identify here a skein module that corresponds to these representations, before seeking for a situation closer to the skein module of the annulus.

Define $n\mathbf{AWeb}_m$ to be the category with objects, sequences $(a_1, \cdots, a_m)$ regularly drawn on a circle, with $0 \leq a_i \leq n$, together with a zero object. Points labeled by zero can be erased, but, as in [Bla10] and [LQR12], we will keep the n-strands as well. Morphisms are formal sums over $\mathbb{Z}[q^{\pm 1}]$ of webs drawn on a cylinder.

$n\mathbf{AWeb}_m(N)$ will be the full subcategory with objects such that $\sum a_i = N$. 1-morphisms are generated up to isotopy by 1-morphisms [5]:

$\mathsf{E}_i\mathbf{1}_\lambda \quad =$ 

, $\quad \mathsf{F}_i\mathbf{1}_\lambda \quad =$ 

In the above pictures, we have drawn only strands i and $i+1$.

Given m, n, N, we can define on weights a map Φ that sends $\lambda = (\lambda_0, \cdots, \lambda_{m-1})$ to a sequence $(a_1, \cdots, a_m)$ such that $a_{i+1} - a_i = \lambda_i$ and $a_1 - a_{m-1} = \lambda_0$, with $\sum a_i = N$. If such a solution doesn't exist, the weight is sent to the zero object.

Proposition 4.2.4. For all m, n, N, there is a functor $\Phi : \dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \mapsto n\mathbf{AWeb}_m(N)$ defined on weights as above and on morphisms by $E_i\mathbf{1}_\lambda \mapsto \mathsf{E}_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda \mapsto \mathsf{F}_i\mathbf{1}_\lambda$.

$\widehat{\mathfrak{sl}_m}$ relations are locally $\mathfrak{sl}_m$ relations: the proof is straightforward. [Cau12, Prop 7.4] receives a direct translation:

Proposition 4.2.5. The evaluation representation produce annular web invariants for ladder-type webs. In other words, if $X \in \widetilde{\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})}$ is such that $\Phi(X) = w$, w a web-tangle, then the morphism of $U_q(\mathfrak{sl}_n)$ representations given by X is an invariant of the web-tangle.

$\widetilde{\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})}$ above denotes the completion of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$ given by the quotient of the ring of series of elements of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$ acting by evaluation representation on each irreducible

5. Note that this a priori doesn't recover the complete braid group of the annulus. We miss in the braid group associated to $U_q(\widehat{\mathfrak{sl}_m})$ the element given by sending each point on one boundary of the annulus to the one corresponding to its right (for example) neighbor in the other boundary of the annulus. In case of ladders, this can be artificially solved by adding a 0-strand.

$\dot{\mathbf{U}}'_q(\mathfrak{sl}_m)$ representation V_λ by zero but for finitely many terms, mod out by the two-sided ideal of elements acting by zero on all V_λ.

As explained previously, the above process produces an invariant of annular web-tangles. However, it appears to come with only little information about the topology of the annulus, and we can indeed explicitly identify this invariant. It may be useful for this purpose to see the annulus as a cylinder rather than flatten on a plane: this way, we can fill it.

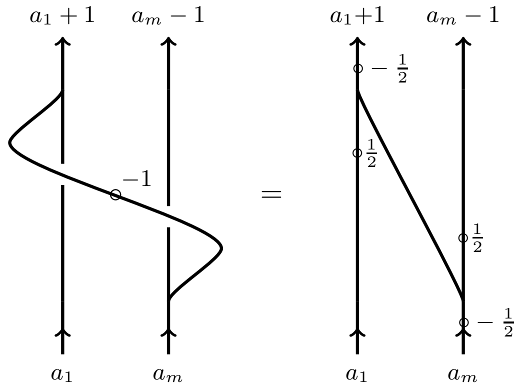
Proposition 4.2.6. The evaluation representation with $a = -q^{e(n+1)}$ recovers the $\mathfrak{sl}_n$ skein module of the filled cylinder. In other words, for each fixed value of N we have the following commutative diagram:

$$\begin{tikzcd}
\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m}) \arrow[r, "\Phi"] \arrow[d, "ev"] & n\mathbf{AWeb} \arrow[d, "Filling"] \\
\dot{\mathbf{U}}_q(\mathfrak{sl}_m) \arrow[r, "\Phi"] & n\mathbf{Web}
\end{tikzcd}$$

Proof. We just have to check that the action of $E_0\mathbf{1}_\lambda$ and $F_0\mathbf{1}_\lambda$ corresponds to $\mathsf{E}_0\mathbf{1}_\lambda$ and $\mathsf{F}_0\mathbf{1}_\lambda$ seen in the skein module of the filled cylinder.

For E_0 for example, we have the following situation:

$$E_0\mathbf{1}_\lambda \quad \mapsto \tag{4.2.10}$$



The negative twist on the LHS of the above equation comes from the fact that the strand goes along the cylinder with framing parallel to the cylinder. When filling the cylinder and projecting it on the back side, this produces a twist.

We have depicted here only the leftmost and the rightmost strands. This whole process is to be understood in front of the other strands. Then, a succession of Reidemeister II moves presents this piece of tangle as the elements defining the evaluation representation, presented in 4.2.2.

In the previous computation, the twists produce a coefficient $t_{a_1}^{\frac{1}{2}} t_{a_1+1}^{-\frac{1}{2}} t_{a_m}^{-\frac{1}{2}} t_{a_{m-1}}^{\frac{1}{2}} = -q^{-e(a_1+a_m)+en+e}$, while the evaluation representation provides $aq^{-e(a_1+a_m)}$. Choosing $a = -q^{e(n+1)}$ adjusts the coefficients. Checking the results for F_0 is similar. $\square$

So, it appears that extending the skew Howe duality phenomenon to the affine case by evaluation representation gives a coherent process, but is too weak for recovering the skein module of the annulus. We miss the fact that acting by $E_1E_2..E_{m-1}E_0$, although it does not change anything on the weight, has no reason to be something trivial in the skein module. This is a well-known phenomenon in the study of $\widehat{\mathfrak{sl}_m}$: if we want to understand the non-triviality of this action, we have to keep the whole data coming from the Dynkin diagram and not only generators E_i, F_i, $K_i^{\pm}$. We should therefore work in the whole $U_q(\widehat{\mathfrak{sl}_m})$ and not only with $U'_q(\widehat{\mathfrak{sl}_m})$.

Nonetheless, we want to keep working with an analogue of Howe duality, and it would be convenient to have a process built on these particular representations. It turns out that there is an easy way to do it, called *affinization*, as explained for example in [HK02, p.233].

4.2.4 Affinization

Following [HK02], we now consider $\mathbb{C}(q)[z,z^{-1}] \otimes_{\mathbb{C}(q)} \bigwedge_q^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. The $U_q(\mathfrak{sl}_n)$ action can be extended by acting by identity on the z-part, and the previous $U'_q(\widehat{\mathfrak{sl}_m})$ action may be extended to an $U_q(\widehat{\mathfrak{sl}_m})$ action by the following rules:

$$\begin{aligned}
E_0(z^m \otimes v) &= z^{m+1} \otimes (E_0 v),\\
E_i(z^m \otimes v) &= z^m \otimes (E_i v) \text{ for } i \neq 0,\\
F_0(z^m \otimes v) &= z^{m-1} \otimes (F_0 v),\\
F_i(z^m \otimes v) &= z^m \otimes (F_i v) \text{ for } i \neq 0,\\
K_i(z^m \otimes v) &= z^m \otimes (K_i v),\\
K_d(z^m \otimes v) &= q^m z^m \otimes v.
\end{aligned}$$

Here, K_d is the derivation element, that was neglected in the previous subsection.

We have the same decomposition as before:

$$\bigoplus_{a_1+\cdots+a_m=N} \mathbb{C}(q)[z,z^{-1}] \otimes_{\mathbb{C}(q)} \bigwedge\nolimits_q^{a_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge\nolimits_q^{a_m}(\mathbb{C}^n),$$

and more precisely:

$$\bigoplus_{a_1+\cdots+a_m=N,k} \mathbb{C}(q) \cdot z^k \otimes \bigwedge\nolimits_q^{a_1}(\mathbb{C}^n) \otimes \cdots \otimes \bigwedge\nolimits_q^{a_m}(\mathbb{C}^n).$$

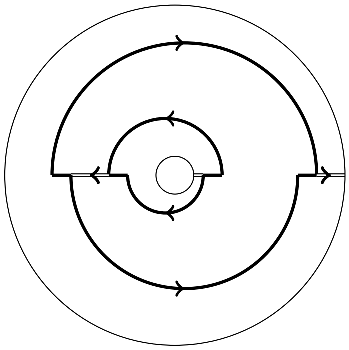
The $U_q(\widehat{\mathfrak{sl}_m})$-weight of one sub-factor is $(a_1-a_m, a_2-a_1, \cdots, a_m-a_{m-1})+k\delta$ where δ is the null root (that was neglected in the previous subsections). Note that these representations are of level 0.

We obtain here a new process: given a ladder, we can turn it into an element of $\dot{\mathbf{U}}_q(\widehat{\mathfrak{sl}_m})$ acting on $\bigwedge_q^N(\mathbb{C}^n \otimes \mathbb{C}^m)$, and extend this into an action on $\mathbb{C}(q)[z^{\pm 1}] \otimes_{C(q)} \bigwedge_q^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. If we restrict to knots drawn on a cylinder, that is ladders with a boundary sequence with only 0's and n's, we obtain an element of $Hom(\mathbb{C}(q)[z^{\pm 1}] \otimes_{\mathbb{C}(q)} \mathbb{C}[q^{\pm 1}]) = \mathbb{C}(q)[z^{\pm 1}]$.

Since all relations are local, [Cau12, Prop 7.4] receives a direct translation:

Proposition 4.2.7. The previous process defines a web tangle invariant.

So, to an annular web-tangle presented in a ladder form, we can assign a morphism of $U_q(\mathfrak{sl}_n)$ representations that is an invariant of the web-tangle. This morphism may be expressed in a diagrammatic way, producing a skein element. Unfortunately, this process is not well-defined: $E_1F_0F_1E_0\mathbf{1}_\lambda$ acting for example on a 2-strands sequence $(0,2)$ acts as $[2]^2 \cdot \mathbf{1}_\lambda$, while the skein element corresponding to both these elements are not equal.



This would suggest to look for richer $U_q(\widehat{\mathfrak{sl}_m})$ representations extending the skew-Howe duality phenomenon.

The previous invariant contains two pieces of information: the same as the evaluation representation, that is, the skein element associated to the web tangle in the skein module of the filled annulus, and an information given by the action on the z-part. This traces a kind of algebraic linking number with the core of the annulus. The problem is that this algebraic number doesn't detect the possible non-triviality of an algebraically non-linked web, as explained above: the representations we have been working with are still too weak to give a full representation-theoretic counterpart of the annular webs.

In the $\mathfrak{sl}_2$-case, the unoriented skein module is well understood, isomorphic to $\mathbb{Z}[q^{\pm 1}][z]$, with z the generator given by a circle around the hole. Note that, if we try to compare the obtained invariant with the usual unoriented $\mathfrak{sl}_2$ skein module of the annulus, the first difference comes from the fact that the 2-labeled strands don't play any role in the unoriented version, while when wrapped around the hole, it produces $z^{\pm 2}$ in the oriented version. A second difference comes from the default explained above. So, modulo z^2 and some renormalizations (since z in the unoriented case corresponds in spirit to $[2]z$ in the oriented case), we can understand the relation between both versions.

4.2.5 Forgetting about $\mathfrak{sl}_n$...

Note that the default in obtaining an algebraic object that would mimic the behavior of the skein module comes from the fact that we consider particular $U_q(\mathfrak{sl}_n)$ representations that are not powerful enough to detect all the topological data. A kind of virtual analogue would be to only keep the $U_q(\mathfrak{sl}_m)$-part in the duality, mod out by information extracted from the usual case, and extend only this to the annular case.

Recall from [CKM12] that we can understand the quotient of $U_q(\mathfrak{sl}_m)$ which corresponds to classes of $\mathfrak{sl}_n$-webs. Fix N (from the ladders we are looking at) and a dominant weight λ (corresponding to a sequence $(a_1^\lambda, \cdots, a_m^\lambda)$), and denote I_λ the ideal of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ generated by all weights which do not lie in the Weyl orbit of any weight μ so that λ dominates μ. Furthermore, denote $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)^n$ the quotient of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ by the set of weights whose associated sequence $(a_1, \cdots, a_m)$ either does not exist or has at least one coefficient $a_i < 0$ or $a_i > n$.

Then, Cautis, Kamnitzer and Morrison tell us that the morphism $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)^n/I_\lambda \mapsto n\mathbf{Web}_m(a_1^\lambda, \cdots, a_m^\lambda)$ is an isomorphism, where $n\mathbf{Web}_m(a_1^\lambda, \cdots, a_m^\lambda)$ is the algebra of ladders that can be reached starting from the sequence $(a_0^\lambda, \cdots, a_m^\lambda)$. In other words, these are the ladders $W\colon (a_1^1, \cdots, a_m^1) \mapsto (a_1^2, \cdots, a_m^2)$, so that the set of ladders between $(a_1^\lambda, \cdots, a_m^\lambda)$ and $(a_1^1, \cdots, a_m^1)$ is non-empty.

Let us now denote $n\mathbf{AWeb}_m(a_1, \cdots, a_m)$ the algebra of annular ladder webs built from a sequence $(a_1, \cdots, a_m)$. Each I_λ extends in the affine case to a module $I_{\widehat{\lambda}}$ simply given by assigning to any $\mu \in I_\lambda$ the affine weight $\widehat{\mu}$ so that if $\mu = (a_2^\mu - a_1^\mu, \cdots, a_m^\mu - a_{m-1}^\mu)$, $\widehat{\mu} = (a_1^\mu - a_m^\mu, a_2^\mu - a_1^\mu, \cdots, a_m^\mu - a_{m-1}^\mu)$. We can then consider the quotient $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n$ of $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})$ by weights whose associated sequence has indices lower than 0 or bigger than n, and we have the quotient $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n/I_{\widehat{\lambda}}$. The next result then gives us somehow the result we hoped to find with an explicit $U_q(\mathfrak{sl}_n)$-rep... but just by looking on the dual side of the picture!

Proposition 4.2.8. For λ a dominant $\mathfrak{sl}_m$ weight, the map:

$$\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n/I_\lambda \mapsto n\mathbf{AWeb}_m(a_1^\lambda, \cdots, a_m^\lambda)$$

is an isomorphism. The pre-image of a knotted ladder in $\dot{\mathbf{U}}'_q(\widehat{\mathfrak{sl}_m})^n/I_\lambda$ is therefore an invariant of the knot.

Proof. First, note that since E_0 acts on weights as $F_1 \cdots F_n$, the weights $\widehat{\mu}$ are those that cannot be reached from $\widehat{\lambda}$. On objects, the statement is therefore obvious.

The surjectivity on morphisms comes from the definition.

For morphisms, the injectivity argument is the same as before: since all relations are local (and elementary relations involve at most three strands), either the generators E_0 and F_0 are not involved and we can assume we work with $\mathfrak{sl}_m$, or they are but (for $m \geq 3$) there exists i so that E_i and F_i are not involved. There is then an inclusion of $U_q(\mathfrak{sl}_m)$ in $U_q(\widehat{\mathfrak{sl}_m})$ that does not involve E_i and F_i and we can assume we work in this one. ☐

4.2.6 ... to better recover it?

The objects that appear in the previous paragraph are closely related to q-Schur algebras and affine versions of them. We refer to [DG07] for a clear presentation of the context in which they appear.

Following [DG07], we will consider in what follows an extension [6] of $U'_q(\widehat{\mathfrak{sl}_m})$ with two extra generators R and R^{-1} subject to relations (indices are to be understood modulo m):

$$\begin{aligned} RR^{-1} = \quad & R^{-1}R \quad = 0 \\ R^{-1}K_{i+1}R = K_i \quad , \quad & R^{-1}K_{i+1}^{-1}R = K_i^{-1} \\ R^{-1}E_{i+1}R = E_i \quad , \quad & R^{-1}F_{i+1}R = F_i. \end{aligned} \tag{4.2.11}$$

This algebra will be denoted $\widehat{U}_q(\widehat{\mathfrak{sl}_m})$, and its idempotented version $\widehat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})$. Note that the previous relations in particular give us that $E_0 = R^{-1}E_1R$ and $F_0 = R^{-1}F_1R$.

Doty and Green suggest us to replace the fundamental representation $\mathbb{C}^m$ of $U_q(\mathfrak{sl}_m)$ by an infinite-dimensional version $V_\infty = \mathbb{C}^\infty =< X_i, i \in \mathbb{Z} >$ with action:

$$\begin{aligned} E_iX_j = X_{j+1} \text{ if } i = j \; mod(m) \quad &, \quad E_iX_j = 0 \text{ otherwise,} \\ F_iK_{j+1} = X_j \text{ if } i = j \; mod(m) \quad &, \quad F_iX_j = 0 \text{ otherwise,} \\ K_iX_j = q^{-1}X_j \text{ and } K_iX_{j+1} = qX_{j+1} \text{ if } i = j \; mod(m) \quad &, \quad K_iX_j = X_j \text{ otherwise,} \\ RX_j \quad &= \quad X_{j+1}. \end{aligned}$$

As in the linear case, we can endow $\mathbb{C}^n \otimes V_\infty$ with two commutative actions of $U_q(\widehat{\mathfrak{sl}_n})$ and $\widehat{U}_q(\widehat{\mathfrak{sl}_m})$. We now wish to consider the quantum exterior power of this tensor product

6. This extension has the nice property that is gives us the missing generators of the braid group of the annulus that was previously discussed.

of representations and perform the same type of process as before. However, this quantum exterior power is more complicated to define in the affine case than in the linear one.

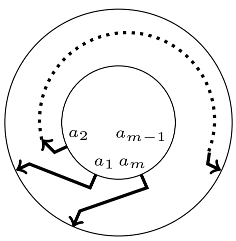
We refer to [Ste95, TU98, Ugl00] and references therein for details about these representations and tools one could use to study them. We intend here to sketch a process allowing us to relate m-uprights annular ladders to $\mathfrak{sl}_n$-representation theory, but the question of completely understanding this relation, and also relating higher-level analogous phenomenon to knot theory remains open.

Denote $V_m =< X_1, \dots, X_m >$ that we see as a subspace of V_∞. V_m is the vector representation of $U_q(\mathfrak{sl}_m)$, (but is not a $U_q(\widehat{\mathfrak{sl}_m})$ module), and its affinization $V_m \otimes \mathbb{C}[z^{\pm 1}]$ is isomorphic to V_∞. The precise definition of the quantum exterior power of V_∞ can be found in [Ugl00, Section 2.1]. We have in particular: $X_{i+km} \wedge X_{i+lm} + X_{i+lm} \wedge X_{i+km} = 0$.

Uglov's process consists in making $\bigwedge_q \left(\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}] \otimes V_m\right)$ into a $U_q'(\widehat{\mathfrak{sl}_n}) \otimes U_q'(\widehat{\mathfrak{sl}_m})$ module. In particular, this implies that $\bigwedge_q (\mathbb{C}^n \otimes V_\infty)$ has two commuting actions of $U_q(\mathfrak{sl}_n)$ and $U_q'(\widehat{\mathfrak{sl}_m})$, and it is not hard to see that the latter can be extended into a $\widehat{U}_q(\widehat{\mathfrak{sl}_m})$ action.

We now want to relate annular webs, the algebra $\hat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})$ and the morphisms of the $U_q(\mathfrak{sl}_n)$ representation $\bigwedge_q (\mathbb{C}^n \otimes V_\infty)$.

Let us slightly extend the definition of $n\mathbf{AWeb}_m(N)$ into $n\widehat{\mathbf{A}}\mathbf{Web}_m(N)$ by adding to it the image of R as the next elementary annular ladder web:



,

and similarly for R^{-1}.

It is a direct extension of the usual case that we have an isomorphism $\oplus_N \hat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})^n \mapsto n\widehat{\mathbf{A}}\mathbf{Web}_m$.

Usually, when we relate $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$ endomorphisms, the fact that we can kill in $U_q(\mathfrak{sl}_m)$ all weights corresponding to sequences $(a_1, \dots, a_m)$ with an $a_i > n$ comes from the fact that exterior powers of $\mathbb{C}^n$ of degree more than n are zero. Here, we can decompose:

$$\begin{aligned}\bigwedge\nolimits_q^N (\mathbb{C}^n \otimes V_\infty) \;&\simeq\; \bigwedge\nolimits_q^N \left(\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}] \otimes V_m\right) \simeq \bigwedge\nolimits_q^N \left(\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}] \otimes (\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C})\right) \\ &\simeq \bigoplus_{a_1+\cdots+a_m=N} \bigwedge\nolimits_q^{a_1} (\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}]) \otimes \cdots \otimes \bigwedge\nolimits_q^{a_m} (\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}]).\end{aligned}$$

However, the relation $X_{i+km} \wedge X_{i+lm} + X_{i+lm} \wedge X_{i+km} = 0$ does not ensure that $\bigwedge_q^N (\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}]) \simeq \mathbb{C}$ nor that it is zero after.

If we denote $\{z_{i,j}, i \in \{1, \dots, n\}, j \in \mathbb{Z}\}$ a basis of $\mathbb{C}^n \otimes V_\infty$, we can artificially mod out the exterior power by the ideal generated by the extra relation $z_{i,j} \wedge z_{i,j+km} = 0$ under tensor product and both actions of the two quantum groups. Let us denote ${}^a\bigwedge_q (\mathbb{C}^n \otimes V_\infty)$ this quotient.

First, we want to ensure that this extra quotient does not make this representation trivial. For this, note that we have a map:

$$\mathbb{C}^n \otimes \mathbb{C}[z, z^{-1}] \otimes V_m \xrightarrow{z=1} \mathbb{C}^n \otimes V_m$$

that preserves the actions of $U_q(\mathfrak{sl}_n)$ and $\widehat{U}_q(\widehat{\mathfrak{sl}_m})$. It extends to the exterior power, so that we have the following commutative diagram:

$$\begin{tikzcd}
\bigwedge_q \left(\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}] \otimes \mathbb{C}^m\right) \arrow[rr, two heads, "z=1"] \arrow[dr, hook] & & \bigwedge_q \left(\mathbb{C}^n \otimes \mathbb{C}^m\right) \\
& {}^a\bigwedge_q \left(\mathbb{C}^n \otimes \mathbb{C}[z^{\pm 1}] \otimes \mathbb{C}^m\right) \arrow[ur, two heads, "z=1"'] &
\end{tikzcd}$$

This gives the following successive relations:

$$n\widehat{\mathbf{A}}\mathbf{Web}_m \simeq \oplus_N \hat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})^n \mapsto End_{U_q(\mathfrak{sl}_n)}\left({}^a\bigwedge_q(\mathbb{C}^n \otimes V_\infty)\right).$$

We want to prove that the last map is actually an inclusion of algebras. A very useful tool for this is provided by [MT13, Proposition 3.14], which requests a little adaptation.

Assume that $N < m$ (we can restrict to that case by adding 0-labeled strands), and denote $\mathbb{1}_r$ the idempotent corresponding the sequence $(a_1, \dots, a_m) = (1, \dots, 1, 0, \dots, 0)$ containing r times the number 1. Then, $\mathbb{1}_r \hat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})\mathbb{1}_r$ is isomorphic to the affine Hecke algebra $\widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$ (see [MT13, DG07]). We can present it as generated by $b_1, \dots, b_r$ and T_ρ, T_ρ^{-1} subject to the following relations:

$$\begin{aligned}
b_i^2 &= (q+q^{-1})b_i && \text{for } i = 1, \dots, r \\
b_ib_j &= b_jb_i && \text{for distant } i,j = 1, \dots, r \\
b_ib_{i+1}b_i + b_{i+1} &= b_{i+1}b_ib_{i+1} + b_i && \text{for } i = 1, \dots, r \\
T_\rho b_i T_\rho^{-1} &= b_{i+1} && \text{for } i = 1, \dots, r.
\end{aligned} \tag{4.2.12}$$

The last equations is to be understood with indices modulo r. Note that we can obtain the generator b_r as $b_r = T_\rho b_{r-1} T_\rho^{-1}$.

Mackaay and Thiel then state, where $\widehat{S}(m,N)$ is the affine Schur algebra:

Proposition 4.2.9. [MT13, Proposition 3.14] Let $N < m$. Suppose that A is a $\mathbb{Q}(q)$ algebra and

$$f \colon \widehat{S}(m,N) \mapsto A$$

is a surjective $\mathbb{Q}(q)$-algebra homomorphism which is an embedding when restricted to $1_r\widehat{S}(m,N)\mathbb{1}_r \simeq \widehat{\mathcal{H}}_{\widehat{A}_{r-1}}$. Then f is a $\mathbb{Q}(q)$-algebra isomorphism

$$A \simeq \widehat{S}(m,N).$$

The idea is that it is enough to check the injectivity on the Hecke algebra for deducing it for the whole Schur algebra. There is a slight difference in the context we want to consider: all versions of $\hat{\dot{\mathbf{U}}}_q(\widehat{\mathfrak{sl}_m})$, Schur and Hecke algebras have weights killed whenever the corresponding sequence $(a_1, \dots, a_m)$ has one term bigger than n.

However, the exact same proof can be achieved in this case: the proof consists in factorizing elements of the Hecke algebra through other weights. In our case, we have to consider that if the weight is killed, then the associated element will be killed. We can thus state:

Proposition 4.2.10. Let $N < m$. Suppose that A is a $\mathbb{Q}(q)$ algebra and

$$f\colon \widehat{S}(m,N)^n \mapsto A$$

is a surjective $\mathbb{Q}(q)$-algebra homomorphism which is an embedding when restricted to $1_r\widehat{S}(m,N)^n\mathbb{1}_r \simeq \widehat{\mathcal{H}}^2_{\widehat{A}_{r-1}}$. Then f is a $\mathbb{Q}(q)$-algebra isomorphism

$$A \simeq \widehat{S}(m,N)^n.$$

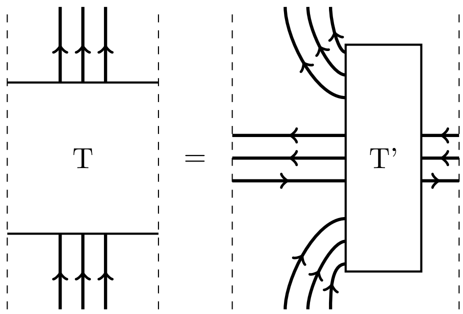
We conjecture that there should be a way to represent annular webs as intertwiners of $U_q(\mathfrak{sl}_n)$-representations built on $\bigwedge_q(\mathbb{C}^n \otimes V_\infty)$, and that Mackaay and Thiel's proposition could be a powerful simplifying tool in proving such a result. This would give to annular webs a representation-theory-based interpretation, in the same spirit that usual webs originate from the intertwiners of $U_q(\mathfrak{sl}_n)$ minuscule representations.

Possible relations between infinite q-wedges, the phenomenon studied in [Ugl00] and knot theory would also be of great interest.

4.2.7 Turning an annular knot to a ladder

So far, we saw that to a ladder annular web-tangle, we can assign a $U_q(\mathfrak{sl}_n)$ morphism of tensor product of minuscule representations (possibly tensorized with $\mathbb{C}(q)[z,z^{-1}]$), whose diagrammatic depiction equals the skein element associated to the web-tangle. This holds for any annular web-tangle isotopic to a ladder, but we can extend the process used in the case of usual webs for turning webs to ladder webs.

We can present any upward annular web-tangle in a similar form as in [CKM12]:



The above pictures are to be understood on an annulus.

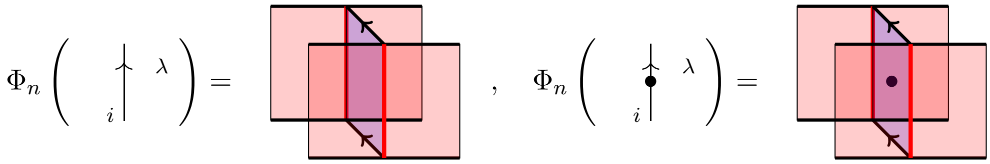
Then, taking the Jones-Kauffman product with a set of well-placed n-strands, we can apply to T' the same process as usual and turn T to a ladder form. Note that taking the Jones-Kauffman product with n-strands is an invertible process, by taking the product with the same number of n-strands oriented downward and pairing the obtained couples of oppositely oriented n-strands.

4.3 Categorification

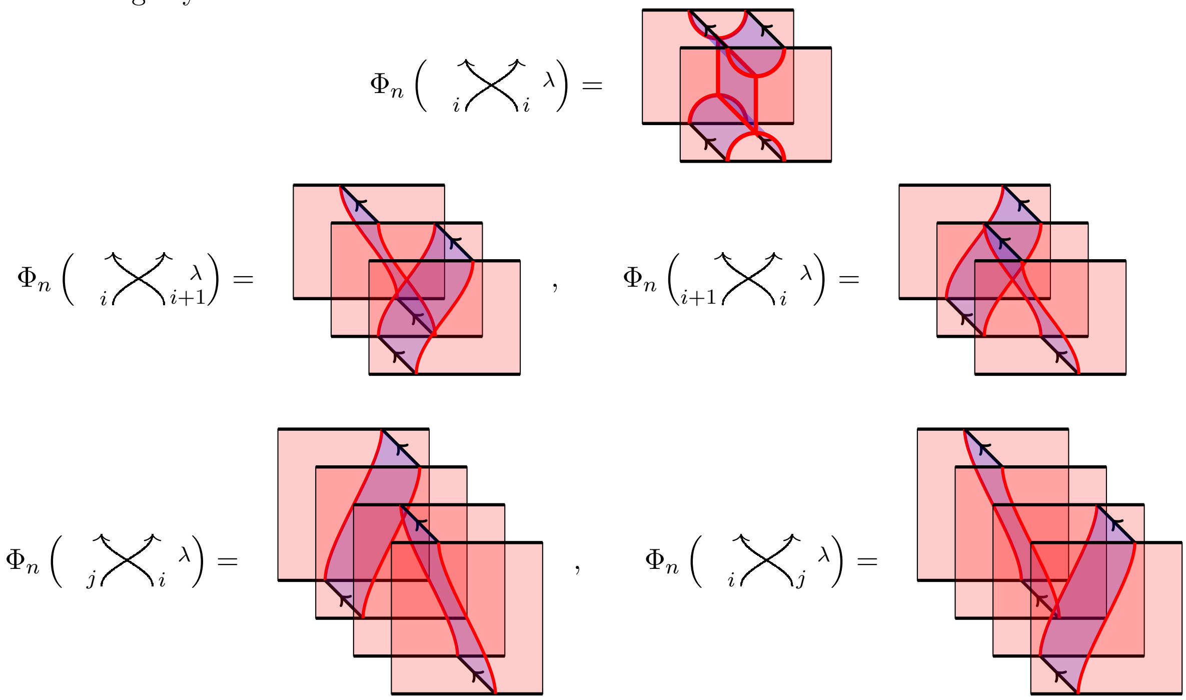
From the Dynkin diagram 4.2.1, one can build a categorified quantum group $\mathcal{U}_Q(\widehat{\mathfrak{sl}_m})$ following [CL11], which generalizes works by Khovanov and Lauda [KL09, KL11a, KL10, Lau08].

Similarly, it is a straightforward generalization of $n\mathbf{BFoam}_m(N)$ to consider foams on an annulus $n\mathbf{ABFoam}_m(N)$: local generators are the same ones as in the disk case, but instead of embedding the foams in the thickening of a disk, we embed them in the thickening of an annulus. Then, the main result from [LQR12] generalizes at no cost:

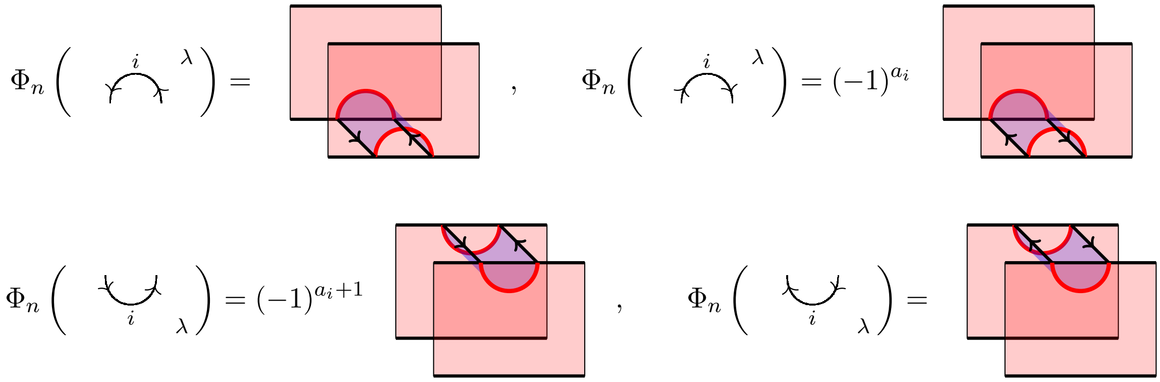
Proposition 4.3.1. For $n = 2, 3$, for each $N > 0$ there is a 2-representation $\Phi_n : \mathcal{U}_Q(\widehat{\mathfrak{sl}_m}) \to n\mathbf{ABFoam}_m(N)$ defined on single strand 2-morphisms by:



on crossings by:



where $j - i > 1$, and on caps and cups by:



where in the above diagrams the i^{th} sheet is always in the front.

Then, following [LQR12], we can build from any annular knot, turned into an annular entangled ladder, form a complex over the category $\mathcal{U}_Q(\mathfrak{sl}_m)$. Applying to it Φ_n (for $n = 2$ or $n = 3$), we obtain extensions to the annulus of Khovanov's homology [Kho00, Kho02, Kho04] built in the spirit of Bar-Natan [BN05]. Again, the proof of the invariance relies on checking Reidemeister moves, which are local and therefore directly extend from the usual case to the affine one.

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