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Étude semi-locale des variétés normalement hyperboliques et applications à la diffusion

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À ma mère,

إلى أُمّي،

« لي بَلَع البحر ما بِغصّ بالساقية »
وطول ما إِنتِ حدّي، ما بحَياتي رَح غصّ
لأنُّ هَمّي كِلّو بتحِملي
علّمتيني المسؤليّة والشغل والأخلاق والعاطفة والتضحية
دعمِك و إِيمانِك ما إِلُن حدود، وحُبّي إِلِك كَمان.

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A semi-local study on normally hyperbolic manifolds with applications to diffusion

Abstract

Let $(M,\,\Omega)$ be a smooth symplectic manifold and $f : M \to M$ be a symplectic diffeomorphism of class C^l $(l \geq 3)$. Let N be a smooth compact submanifold of M which is normally hyperbolic for f and such that the dimension of its unstable bundle is equal to that of the stable one. We suppose that N is boundaryless, controllable and that its stable and unstable bundles are trivial.

First, we consider a C^1-submanifold Δ of M whose dimension is equal to the dimension of a fiber of the unstable bundle of $T_N M$. We suppose that Δ transversely intersects the stable manifold of N. Then, we prove the basic λ-lemma that states that for all $\varepsilon > 0$, and for $n \in \mathbb{N}$ large enough, there exists $x_n \in N$ such that $f^n(\Delta)$ is ε-close, in the C^1 topology, to the strongly unstable manifold of x_n. As an application of the basic λ-lemma, we prove the existence of shadowing orbits for a finite family of invariant minimal sets (for which we do not assume any regularity) with heteroclinic connections. As a particular case, we recover classical results on the existence of diffusion orbits (Arnold's example).

Then, we state and prove the fibered λ-lemma which is a generalization of the basic λ-lemma to C^2-submanifolds whose dimension is between that of the unstable leaves and that of the unstable manifold. As an application of the fibered λ-lemma, we prove the transitivity of transversal heteroclinic connections for systems having the strong torsion property and some additional assumptions.

Under the same assumptions, we derive an explicit construction of correctly aligned windows for proving the existence of shadowing orbits along a chain of invariant tori contained in a normally hyperbolic manifold. Moreover, we prove that the diffusion time splits into three characteristic parameters: the ergodization time associated with each torus of the chain, the "straightening" time given by the fibered λ-lemma, and the torsion time on each torus.

Finally, we construct a simple class of *a priori* stable nearly integrable systems on $\mathbb{A}^3$ for which we prove the existence of "asymptotically dense projected orbits", that is, orbits at fixed energy whose projection on the energy level passes within an arbitrarily small distance from each point of the projected energy level, when the size of the perturbation tends to 0.

Keywords

Dynamical systems, Hamiltonian systems, Nearly integrable Hamiltonian systems, Normally hyperbolic manifolds, λ-lemmas, Arnold's diffusion.

Étude semi-locale des variétés normalement hyperboliques et applications à la diffusion

Résumé

Soient (M, Ω) une variété symplectique C^∞ et $f : M \to M$ un difféormorphisme symplectique de classe C^l ($l \geq 3$). Soit N une sous-variété C^∞ de M, compacte et sans bord, qui est normalement hyperbolique pour f. Supposons que N est contrôlable et que ses fibrés stable et instable sont triviaux et de même dimension.

Dans un premier temps, on considère une C^1 sous-variété Δ de M dont la dimension est égale à celle d'une feuille instable de N. On suppose que Δ coupe transversalement la variété stable de N. Alors, on prouve le λ-lemme basique qui dit que pour tout $\varepsilon > 0$, et pour tout n entier assez grand, il existe un point x_n dans N tel que $f^n(\Delta)$ est ε-proche, en topologie C^1 à la variété fortement instable de x_n. Nous utilisons ensuite le λ-lemme pour montrer l'existence d'orbites qui diffusent le long d'une famille finie d'ensembles invariants, contenus dans une variété normalement hyperbolique, sur lesquels la dynamique est minimale et qui possèdent des connexions hétéroclines. Comme cas particulier, on retrouve l'exemple d'Arnold.

Dans la deuxième partie, on prouve le λ-lemme fibré qui est une généralisation du λ-lemme basique aux sous-variétés C^2 dont la dimension varie entre celle des feuilles instables et celle de la variété instable. Nous utilisons ensuite ce λ-lemme fibré pour prouver la transitivité des connexions hétéroclines transverses pour des systèmes vérifiant une propriété de torsion forte et des hypothèses supplémentaires.

Sous ces mêmes conditions, on établit une construction explicite de fenêtres correctement alignées le long d'une chaîne de tores contenus dans une variété normalement hyperbolique. On en déduit l'existence d'orbites qui dérivent le long de la chaîne, et on prouve que le temps de diffusion dépend de trois paramètres caractéristiques : le temps de redressement (donné par le λ-lemme fibré), le temps d'ergodisation de chaque tore et le temps de torsion.

Enfin, on construit une classe simple de systèmes presque intégrables, *a priori* stables, définis sur $\mathbb{A}^3$ pour lesquels on prouve l'existence d'orbites asymptotiquement denses en projection, c'est-à-dire, d'orbites à énergie fixée dont la projection sur le niveau d'énergie passe arbitrairement près de tout point du niveau d'énergie projeté, quand la taille de la perturbation tend vers 0.

Mots-clefs

Systèmes dynamiques, Systèmes hamiltoniens, Systèmes presque-intégrables, Variétés normalement hyperboliques, λ-lemmes, Diffusion d'Arnold.

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Introduction

English version

Since the pioneering paper [Arn64] of Arnold, numerous studies were dedicated to proving the existence of diffusion orbits along chains of *partially hyperbolic tori* having heteroclinic connections. It took almost 40 years to understand that this point of view does not reflect the full structure of the problem, and that the most important object to have in mind is the common center manifold of these tori, which turns out to be a *normally hyperbolic cylinder* with many homoclinic connections. In this setting, the initial proof of existence of diffusion orbits can be significantly simplified and many new results become available.

Maybe the simplest non trivial case occurs when the homoclinic intersection of the cylinders contains a submanifold which is itself diffeomorphic to a cylinder. This was introduced by Moeckel in an abstract setting and since then, studies were devoted to the implementation of Moeckel's idea in the proper context of the semi-local analysis of normally hyperbolic cylinders. In this setting, under proper homoclinic assumptions, the main result is a fibered analogue to the classical Birkhoff-Smale theorem for a fixed point and the existence of a skew-product in which simultaneouly appear the dynamics on the cylinder and the dynamics of the so-called scattering maps. This is under development in several published or unpublished recent studies (Marco, Gelfreich, Nassiri, Pujals...). The main obstacle in generalizing this set of ideas is the fact that "typical" cylinders in the problem of Arnold diffusion do not satisfy the previous homoclinic assumptions. In this setting, the previous Birkoff-Smale result cannot be global and should yield a polysystem (or IFS) formed by the dynamics on the manifold and the homoclinic "correspondence", which is no longer a map. To properly prove such a result, one needs new ingredients of λ-lemma type.

Roughly speaking, a λ-lemma (also called Inclination Lemma) for normally hyperbolic manifolds asserts that, given a smooth manifold M and a diffeomorphism $f : M \longrightarrow M$ and a normally hyperbolic submanifold N of M, if Γ is a submanifold that transversely intersects the *stable* manifold of N, then, under iteration by f, Γ "approaches" the foliation of the *unstable* manifold of N in a suitable topology.

On a different note, in [PS70], the authors proved that a diffeomorphism f is conjugate, in the vicinity of a normally hyperbolic manifold, to Df restricted to its normal bundle, where the conjugacy is a homeomorphism. This may imply λ-lemma type results (in the C^0-topology), however, the fact that the conjugacy is a homeomorphism (and thus does not guarantee the preservation of the transversality) deprives the result of geometrical applications.

In the following years, there has been various sparks of interest to generalize this result to regularities of higher order, namely to give improved results for the famous Smooth Conjugacy Sternberg-Chen Theorem (see [BK01] and the references therein). These conjugacies make it easier to depict the dynamical behavior near normally hyperbolic manifolds and yield the λ-lemmas. However, in order to get smoother conjugacies, one has to settle for rigid assumptions. We are therefore interested in studying the problem from a different perspective to see if these restrictions are crucial, or if, on the contrary, one can avoid them at a small fee.

In this thesis, we prove two λ-lemmas. Let f, M, N and Γ be as above. Let n_s be the dimension of the stable bundle, n_u the dimension of the unstable bundle and n_0 the dimension of N. By transversality of Γ and the stable manifold of N, the dimension of Γ is $n_u + r$ where $0 \leq r \leq n_0$. When $r = 0$, we prove the convergence of Γ to a suitable unstable leaf, in the C^1-topology, without any of the restrictions needed for the conjugacy method. Then, we let r vary between 1 and n_0. We prove that for all r, under iteration, Γ approaches the unstable manifold of N in the C^0-topology. Moreover, we prove that, under iteration, the norm of the component in the s-direction of its unit tangent vectors tends to zero. The C^1-convergence in the central direction is a more complex matter. If f admits a suitable normal form in the vicinity of N, we prove that, under iteration, Γ approaches the unstable manifold of N in the C^1-topology.

Then we use these λ-lemmas to get three types of diffusion results :

1. Existence of diffusion orbits along transition chains,

2. Estimate of diffusion times using the windowing method,

3. Construction of nearly integrable Hamiltonian systems on the annulus $\mathbb{A}^3$ having an "asymptotically dense projected orbit".

These three types of results will be discussed below. Before, let us indicate that we limit ourselves to the symplectic case and we assume that our normally hyperbolic manifolds have trivial stable and unstable bundles having the same dimension (this will in particular give us easy regularity conditions for the lamination of the invariant manifolds). This will be no restriction to us since all the applications that we have in mind will fall into this category (diffusion orbits, Easton's windows,...). Moreover, we adopt a very basic point of view and explicitly describe the iterates of our transverse manifolds by means of a "controlled" straightening coordinate system instead of using methods pertaining more to functional analysis (fixed point theorems for instance). In particular, this will enable us to directly use our various computations for the construction of windows and for estimating the transition times.

λ-lemmas and diffusion

In his famous note [Arn64], Arnold gave the first example of a three-degree-of-freedom system where diffusion orbits shadowing whiskered tori were constructed. More precisely, the system admits orbits for which the action undergoes a drift of length independent of the size of the perturbation. Arnold's example was chosen so that the Lagrangian invariant tori in the unperturbed system break down under the perturbation and give rise to partially hyperbolic tori in the perturbed system.

The diffusion mechanism is then based on the existence of a transition chain, that is, a family of invariant dynamically minimal tori with heteroclinic connections. One gets the orbits shadowing the extremal tori of this chain by an "obstruction argument" satisfied by each torus of the chain. This obstruction argument was first proved in the paper [Mar96], as a corollary of a partially hyperbolic λ-lemma. The proof was then improved in [FM00].

The λ-lemmas proved in this thesis turn out to be new tools for proving the obstruction argument as well as several generalizations. These λ-lemmas deal with invariant objects contained in a normally hyperbolic manifold. This is not a genuine restriction, since one can in general embed partially hyperbolic tori into their central manifolds, which, as a rule, are normally hyperbolic. In that respect, our result generalizes the results of [Mar96], [Cre00] and [FM00], and enables us to significatively simplify the previous proofs. Moreover, our λ-lemmas can be applied to more general systems than that of Arnold ([DDLLS06], [DH11], [GR07], [GR09],...).

We first state and prove a "basic λ-lemma" for normally hyperbolic invariant manifolds. The basic λ-lemma will enable us to prove the existence of drifting orbits along a chain of invariant minimal sets contained in a normally hyperbolic manifold *without any assumption on the geometrical nature of the invariant sets* (in particular, they do not need to be submanifolds). As an easy particular case, we recover Arnold's example. This also applies to prove the existence of shadowing orbits along chains of primary and secondary tori, which appear in the works of Delshams, De La Llave and Seara.

Moreover, in the last chapter of the thesis, we use the basic λ-lemma to construct "simple" examples of nearly integrable Hamiltonian systems with asymptotically *dense* projected orbits.

Our second result (the "fibered λ-lemma") will generalize the basic λ-lemma to submanifolds whose dimension is between that of the unstable leaves and that of the unstable manifold. We will apply this result to the case of chains of invariant tori contained in a normally hyperbolic manifold, such that two consecutive elements of the chain admit *transverse* heteroclinic connections. Under additional assumptions (of *strong torsion* and normal form) for the diffeomorphism, we prove the *transitivity* of the transversality of heteroclinic connections.

Windows and diffusion times

In this thesis, we develop another geometrical technique to prove the existence of diffusion orbits and, more importantly, to give quantitative estimates of their speed of drift. We use Easton's windowing method (introduced in [EM79] and [Eas81] and revisited in [GR07] and [GR09] in a topological point of view) to detect these trajectories, and the fibered λ-lemma that provides explicit estimates of the "straightening" time needed to compute diffusion times.

We will deal with transition chains contained in a normally hyperbolic manifold $N \subset \mathbb{T} \times \mathbb{R}$ and where f is an integrable twist map. Moreover, we will apply a version of the fibered λ-lemma that ensures the C^1-convergence of the manifold transversely intersecting $W^s(N)$, which requires a specific normal form near N. We will work in this specific context because we aim at giving a simple construction where we can completely describe the parameters that come into play when computing diffusion times on the one hand, and where the diffusion time is uniform with respect to the chain in the sense that the estimates do not deteriorate with iterations (like in the case of [Mar96] for example) on the other hand. We think of our construction as a first step to computing diffusion times in a more general context.

A window in a manifold is a diffeomorphic copy of a multidimensional rectangle with "horizontal" and "vertical" directions. Two windows are said to be correctly aligned if each horizontal of one is *transverse* to each vertical of the other at a point that is *interior* to the rectangles. Given a finite family of windows having connecting diffeomorphisms, that is, the image of each window under the connecting diffeomorphism is correctly aligned with the next one, Easton's Shadowing Lemma (see Theorem 4.2.3) yields the existence of an orbit that runs through these windows in the alignement's order. Therefore, given a transition chain in a dynamical system, one can prove the existence of an orbit wandering arbitrarily close to the chain, by constructing windows arbitrarily close to each torus, that correctly align under suitable powers (abused naming for the number of iterations) of the diffeomorphism. The time needed to drift along this chain is the sum of these powers.

Our construction of the windows is reminiscent of that in [GR07] and [GR09]. In these papers, the authors proved the existence of shadowing orbits for transition chains alternating with Birkhoff zones of instability. We simplify the previous constructions along the transition chain in our context in the following ways. We will need less windows to shadow our chain: in [GR07], two windows were constructed around each heteroclinic point, and additional two windows near each torus. We will only need two windows near each torus. We first construct a *static* window arbitrarily close to each torus. We then construct the *mobile* window such that the *static* one is correctly aligned with the *mobile* one, and the image of the latter under a suitable power of the diffeomorphism is correctly aligned with the *static* window of the next torus. Moreover, while Gidea and Robinson's approach (in [GR09]) is topological, ours is more geometrical. It relies on the differential structure and on transversality. While in many examples, the proof of the transversality is not straightforward, the fibered λ-lemma provides, in addition to the explicit time estimates (the power of the diffeomorphism), the straightening of the horizontals which yields the transversality property needed for the alignment. The fibered λ-lemma also replaces the criteria of topological linearization of Pugh-Shub needed in [GR07] to align two of each set of windows.

In order to prove the correct alignment of the windows, we will need to carry the *mobile* windows around each torus and estimate the time needed to shift from a neighborhood of the stable manifold of the torus to a neighborhood of the unstable manifold of the same torus. This will be possible thanks to ergodization results due to Dumas, Bourgain, Berti, Bolle, Biasco,... Roughly speaking, for δ small, the δ-ergodization time of a torus is the rate at which the orbit of a point fills the torus within δ, when subjected to a nonresonant or near nonresonant rotation. We will deal with Diophantine rotations because in this case we have "optimal" ergodization times. This will be no restriction to us since Diophantine tori will densely fill the normally hyperbolic manifolds in the examples that we have in mind.

As for the diffusion times, we will give quantitative estimates (in an abstract setting) of the time needed for the orbit to drift along the chain. We will prove that the diffusion time is governed by three characteristic parameters: the "straightening" time (given by the fibered λ-lemma) needed to ensure the transversality criteria for the alignment in the normal directions, the ergodization time on each torus, and the torsion time that completes the transversality property in the central direction.

The next step (which could not be achieved here because of the lack of time) is to apply these abstract statements to particular examples such as [LM05].

Asymptotically dense orbits

Throughout the first parts of the thesis (chapters 2, 3 and 4), we will deal with diffeomorphisms. Here, we will deal with Hamiltonian flows.

The basic λ-lemma will enable us to prove a diffusion result which is based on a recent work by Marco on generic properties for classical systems on $\mathbb{T}^2$. We will construct "simple examples" of global diffusion involving nonintegrable averaged systems at double resonances.

In [Marc], Marco proved that classical systems on $\mathbb{T}^2$ with a *generic* potential U admit *chains of annuli*. From the latter, we deduce the existence of *chains of cylinders* at fixed energy for systems on $\mathbb{A}^3$ of the form

$$\mathcal{H}_n(\theta, r) = \frac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3),\ (\theta, r) \in \mathbb{A}^3,$$

where $\|.\|$ is the Euclidean norm.

Thanks to the properties of the annuli, we deduce that the cylinders are normally hyperbolic manifolds and that the chains of cylinders project in the space of actions asymptotically (that is when n tends to the infinity) close to simple resonances. Moreover, the cylinders are foliated with invariant tori with homoclinic connections.

Note that the systems $\mathcal{H}_n$ are direct products of the Hamiltonian $\frac{1}{2}r_1^2$ with the Hamiltonian $\frac{1}{2}(r_2^2+r_3^2)+\frac{1}{n}U(\theta_2,\theta_3)$. Therefore, they do not admit diffusion orbits because of the preservation of the energy in each factor. In order to get the diffusion orbits, we need to choose a perturbation f_n that creates “the splitting of the separatrices". More precisely, we construct a sequence of functions (f_n) whose support is contained in the complement of these cylinders, so that the cylinders are also normally hyperbolic for the system $\mathcal{H}_n + f_n$. Moreover, the perturbation is chosen in such a way that nearby invariant tori that foliate each cylinder admit heteroclinic connections.

One remarkable property is that the basic λ-lemma yields the existence of orbits that shadow the tori in each cylinder, and thus the full chains of cylinders since two successive cylinders are heteroclinically connected. We then deduce the existence of orbits whose projection on the action space asymptotically fill the projected energy level, when the perturbation tends to 0.

Content of the thesis

The thesis is organized as follows. Chapter 1 introduces the basic language and notation used in the thesis. Chapter 2 and Chapter 3 are concerned with the λ-lemmas and some of their applications to diffusion. Chapter 4 is dedicated to the windowing method and the diffusion times. In Chapter 5, we give examples of nearly integrable Hamiltonian systems with asymptotically dense projected orbits.

Chapter 1

We start this chapter with a reminder on normally hyperbolic manifolds and we introduce the straightening neighborhood in which almost all our theorems will be stated. Then we state some results regarding ergodization times for the rotations on $\mathbb{T}^n$. In the third section of Chapter 1, we give a brief reminder on nearly integrable Hamiltonian systems, a notion that will only show up in the end of Chapter 2 and in Chapter 5. We end this chapter with describing the assumptions for chapters 2, 3 and 4. Chapter 5 will deal with specific examples and will have its own assumptions.

Chapter 2

We start this chapter with stating and proving the basic λ-lemma. Two versions will be given. The first one, Theorem 1, is a simplified version stated in the straightening neighborhood. Theorem 2 is the second version of the basic λ-lemma stated in a more general context. We now give a rough statement of the basic λ-lemma.

Theorem 1 and Theorem 2. Let (M, Ω) be a smooth symplectic manifold and $f : M \to M$ be a symplectic diffeomorphism of class C^l $(l \geq 3)$. Let N be a smooth compact submanifold of M

which is normally hyperbolic for f. We suppose that N has trivial stable and unstable bundles of the same dimension. We consider a C^1-submanifold Δ of M whose dimension is equal to the dimension of a fiber of the unstable bundle of $T_N M$. We suppose that Δ transversely intersects the stable manifold of N. Then, we prove that for all $\varepsilon > 0$, and for $n \in \mathbb{N}$ large enough, there exists $x_n \in N$ such that $f^n(\Delta)$ is ε-close, in the C^1-topology, to the strongly unstable manifold of x_n.

We then apply the basic λ-lemma to prove a diffusion result, Corollary 2.4.2 in Section 2.4, that gives the existence of a shadowing orbit along a finite family of invariant dynamically minimal sets contained in a normally hyperbolic manifold and having heteroclinic connections.

Corollary. Let f, M and N be as in Theorem 2. Let $(A_k)_{1 \leq k \leq n}$ be a transition chain in N such that, for all $k = 1, \dots, n-1$, there exist $a_k \in A_k$, $b_{k+1} \in A_{k+1}$ and $c_k \in W^{uu}(a_k) \cap W^{ss}(b_{k+1})$ such that $W^{uu}(a_k)$ and $W^s(N)$ transversely intersect at c_k. Then,

$$W^u(A_n) \subset \overline{W^u(A_1)}.$$

We then show that Arnold's example is a particular case of this application.

Chapter 3

This chapter is dedicated to the fibered λ-lemma and one of its applications. We first state and prove Theorem 3 which is a generalization of the basic λ-lemma to C^2-submanifolds whose dimension is between that of the unstable leaves and that of the unstable manifold. In case we have a specific normal form in the vicinity of N, we get a version of the fibered λ-lemma that ensures the C^1-convergence of the manifold transversely intersecting $W^s(N)$, that is, Corollary 3.1.3. As an application of the fibered λ-lemma, we prove the following corollary (Corollary 3.3.2) that gives the transitivity of transversal heteroclinic connections for systems having the strong torsion property that we will introduce.

Corollary. We keep the assumptions of Theorem 3. Moreover, we suppose that we have the normal form needed to apply the suitable version of the fibered λ-lemma, and that f has the strong torsion property. Let $(T_k)_{1 \leq k \leq n}$ be a finite family of tori forming a transition chain in N such that for all $k = 1, \dots, n-1$, $W^u(T_k) \pitchfork W^s(T_{k+1})$. Then, $W^u(T_1) \pitchfork W^s(T_n)$.

Chapter 4

In this chapter, we use the windowing method and the fibered λ-lemma of Chapter 3 to estimate the time needed for an orbit to drift along a transition chain contained in a normally hyperbolic manifold. More precisely, we prove the following result.

Theorem 4. We consider a system that satisfies the assumptions of the fibered λ-lemma with the required normal form. We suppose that $N \subset \mathbb{T} \times \mathbb{R}$ and that $f_{|_N}$ is an integrable twist map. Moreover, we suppose that N contains a transition chain of Diophantine circles. Then, for all $\varepsilon > 0$, there exist an orbit $(x_i)_{1 \leq i \leq n}$ and a sequence of positive integers $(k_i)_{1 \leq i \leq n-1}$ such that

$$\begin{aligned} &d(x_i, T_i) < \varepsilon, \text{ for all } i = 1, \dots, n, \\ &x_{i+1} = f^{k_i}(x_i), \text{ for all } i = 1, \dots, n-1, \end{aligned}$$

where k_i splits into three characteristic parameters. Namely:

$$k_i \leq n_0 + \max(m_i, p_i) + Q_i,$$

where n_0 is a unifom integer along the chain, m_i is the straightening time given by the fibered λ-lemma, p_i is the torsion time and Q_i is the ergodization time of the rotation over T_{i+1}.

Chapter 5

In this Chapter, we use results due to Marco regarding generic properties for classical systems on $\mathbb{T}^2$ along with the basic λ-lemma to give examples of nearly integrable Hamiltonian systems which admit asymptotically dense projected orbits. More precisely, given an integer $\kappa \geq 2$, we introduce a class of nearly integrable systems on $\mathbb{A}^3$ of the form

$$H_n(\theta, r) = \frac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3) + f_n(\theta, r),$$

where $U \in C^\kappa(\mathbb{T}^2, \mathbb{R})$ is a *generic* potential function and f_n a C^κ additional perturbation such that $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \leq \frac{1}{n}$, so that H_n is a perturbation of the completely integrable system $h(r) = \frac{1}{2}\|r\|^2$. We prove the following diffusion result.

Theorem 5. Let $\Pi : \mathbb{A}^3 \to \mathbb{R}^3$ be the canonical projection. Then, for each $\delta > 0$, there exists n_0 such that for $n \geq n_0$, the system H_n admits an orbit Γ_n at energy $\frac{1}{2}$ whose projection $\Pi(\Gamma_n)$ is δ–dense in $\Pi(H_n^{-1}(\frac{1}{2}))$, in the sense that the δ–neighborhood of $\Pi(\Gamma_n)$ in $\mathbb{R}^3$ covers $\Pi(H_n^{-1}(\frac{1}{2}))$.

Introduction

Version française

Après l'article fondateur d'Arnold [Arn64], beaucoup de travaux ont été consacrés à prouver l'existence d'orbites de diffusion le long de chaînes de *tores partiellement hyperboliques* possédant des connexions hétéroclines. Il aura fallu 40 ans pour comprendre que ce point de vue ne reflète pas la nature profonde du problème et que l'objet le plus important à étudier est la variété centrale commune à ces tores ; cette variété se trouve être un *cylindre normalement hyperbolique* qui possède beaucoup de connexions homoclines. Dès lors, la preuve initiale de l'existence d'orbites de diffusion peut être simplifiée de manière significative et beaucoup d'autres résultats apparaissent à portée de main.

Sans doute le cas le plus simple non trivial apparaît-il quand les connexions homoclines des cylindres contiennent une sous-variété elle-même difféomorphe à un cylindre. Ce phénomène a été étudié en premier par Moeckel dans un cadre abstrait et, depuis, plusieurs recherches ont été menées pour transposer cette étude dans le cadre de l'analyse semi-locale des cylindres normalement hyperboliques. Dans ce cadre, et avec des hypothèses de connexions homoclines adaptées, le résultat principal est un analogue fibré du théorème classique de Birkhoff-Smale pour un point fixe ainsi que l'existence d'un produit semi-direct dans lequel apparaissent simultanément la dynamique sur le cylindre et la dynamique des *scattering maps*. Ces études sont encore en développement dans de nombreux travaux récents, publiés ou non (Marco, Gelfreich, Nassiri, Pujals...). L'obstacle principal pour généraliser l'ensemble de ces idées est le fait que les cylindres « typiques » dans le problème de diffusion d'Arnold ne satisfont pas les hypothèses précédentes de connexions homoclines. Alors, le résultat précédent de Birkhoff-Smale ne peut plus être global et conduit à un polysystème (ou IFS) formé par la dynamique sur la variété et la « correspondance » homocline qui n'est plus une application. Pour montrer rigoureusement un tel résultat, on a besoin de nouveaux ingrédients du type λ*-lemme*.

De manière informelle, un λ-lemme (aussi appelé lemme d'inclinaison) pour une variété normalement hyperbolique dit que, étant donnés une variété différentielle M, un difféomorphisme $f : M \to M$ et une sous-variété N de M normalement hyperbolique pour f, si Γ est une sous-variété qui coupe transversalement la variété *stable* de N, alors les images de Γ par les itérées successives de f « approchent » le feuilletage de la variété *instable* de N dans une topologie convenable.

Dans un autre registre, dans [PS70], les auteurs montrent qu'un difféomorphisme f est conjugué, au voisinage de la variété normalement hyperbolique, à sa linéarisée Df en restriction au fibré normal. Ici, la conjugaison est un homéomorphisme. Ceci peut impliquer des résultats de type λ-lemme en topologie C^0. Cependant, le fait que la conjugaison soit un homéomorphisme (et donc ne garantisse pas la préservation de la transversalité) prive le résultat d'applications géométriques.

Dans les années suivantes, des études nombreuses et variées ont été menées en vue de généraliser ce résultat dans des classes de régularité plus élevées, et plus précisément d'améliorer le célèbre théorème de conjugaison de Chen-Sternberg (voir [BK01] et les références citées). Ces conjugaisons permettent de décrire plus facilement la dynamique au voisinage de la variété normalement hyperbolique et entraînent les λ-lemmes. Cependant, pour obtenir des conjugaisons plus lisses, il faut imposer des hypothèses plus contraignantes. Notre approche consiste à étudier le problème sous un autre angle afin de déterminer si ces hypothèses plus contraignantes sont effectivement cruciales ou si, au contraire, on peut les éviter à peu de frais.

Dans cette thèse, nous prouvons deux nouveaux λ-lemmes. Considérons f, M, N et Γ comme précédemment, et notons n_s la dimension du fibré stable de N, n_u celle du fibré instable et n_0 celle de N. Comme Γ et la variété stable de N sont transverses, il existe un entier r tel que $0 \leq r \leq n_0$ et tel que la dimension de Γ soit égale à $n_u + r$. Quand $r = 0$, nous montrons que la suite des images successives de Γ par f converge, en topologie C^1, vers une feuille instable particulière et ce sans aucune des hypothèses demandées par la méthode de conjugaison. Si maintenant r prend sa valeur entre 1 et n_0, nous montrons que la suite des images successives de Γ par f approche la variété instable de N en topologie C^0, et nous prouvons de plus que la norme de la composante dans la direction stable des vecteurs unitaires tangents tend vers 0. La convergence C^1 dans la direction centrale est une question plus compliquée et nécessite de faire une hypothèse supplémentaire : nous montrons que si f admet une forme normale convenable au voisinage de N, la suite des images successives de Γ par f converge vers la variété instable de N en topologie C^1.

Nous utilisons ensuite ces λ-lemmes pour obtenir trois types de résultats de diffusion :

1. Existence d'orbites de diffusion le long de chaînes de transition,

2. Estimation de temps de diffusion par la méthode des fenêtres,

3. Construction de systèmes hamiltoniens presque intégrables sur l'anneau $\mathbb{A}^3$ possédant une orbite « asymptotiquement dense en projection ».

Ces trois types de résultats seront présentés ci-après. Signalons maintenant que nous nous limitons au cas symplectique et que nous supposons que la variété normalement hyperbolique a des fibrés stables et instables triviaux de même dimension (cela nous permettra en particulier d'obtenir facilement des conditions de régularité pour les laminations des variétés invariantes). Ces hypothèses ne sont pas restrictives pour nous ; en effet, les cadres dans lesquels nous souhaitons appliquer ces résultats pour obtenir des résultats de diffusion sont tous dans cette catégorie. De plus, nous adoptons un point de vue très basique et décrivons explicitement les itérées de la sous-variété transverse Γ au moyen d'un système de coordonnées redressantes « contrôlé » au lieu d'utiliser des méthodes plus abstraites d'analyse fonctionnelle comme des théorèmes de points fixes. Cela nous permet en particulier d'utiliser directement nos divers calculs pour la construction des fenêtres et l'estimation des temps de transition.

λ-lemmes, chaînes de transition et orbites de diffusion.

Dans son célèbre article [Arn64], Arnold a construit le premier exemple d'une perturbation d'un système hamiltonien intégrable à trois degrés de liberté dans lequel apparaissent des orbites de diffusion. Plus précisément, le système perturbé possède des orbites dont l'action subit une variation de taille indépendante de la taille de la perturbation. L'exemple d'Arnold a été construit de sorte que les tores lagrangiens invariants du système initial ne résistent pas à la perturbation et donnent naissance à des tores partiellement hyperboliques dans le système perturbé.

Le mécanisme de diffusion est basé sur l'existence d'une *chaîne de transition* de tores invariants, c'est-à-dire une famille de tores invariants minimaux, partiellement hyperboliques qui possèdent des connexions hétéroclines entre eux. On obtient alors des orbites reliant les tores extrémaux de cette chaîne au moyen d'un « argument d'obstruction » satisfait par chaque tore de la chaîne. Cet argument d'obstruction a été prouvé en premier dans l'article [Mar96], comme corollaire d'un λ-lemme partiel. La preuve a ensuite été améliorée dans [FM00].

Les λ-lemmes établis dans cette thèse permettent de retrouver l'argument d'obstruction d'Arnold, mais surtout d'en donner plusieurs généralisations. Ces λ-lemmes concernent des objets invariants contenus dans une variété normalement hyperbolique ce qui n'est pas une réelle restriction car les tores partiellement hyperboliques peuvent en général être plongés dans leurs variétés centrales, qui, par définition, sont des variétés normalement hyperboliques. En ce sens, nos résultats généralisent ceux de [Mar96], [Cre00] et [FM00] et nous permettent d'en simplifier considérablement les preuves. Enfin, nos λ-lemmes s'appliquent à des systèmes plus généraux que ceux d'Arnold ([DDLLS06], [DH11], [GR07], [GR09],...).

Notre premier λ-lemme, que nous appelons « λ-lemme basique » nous permet de prouver l'existence d'orbites de diffusion le long d'une chaîne d'ensembles invariants minimaux contenus dans une variété normalement hyperbolique, *sans aucune hypothèse sur la nature géométrique de ces ensembles invariants* (en particulier, on ne demande pas qu'il s'agisse de sous-variétés). L'exemple donné par Arnold devient un cas particulier élémentaire de notre résultat. Nous retrouvons ainsi aussi l'existence d'une orbite de diffusion le long des chaînes de tores primaires et secondaires, qui apparaissent dans les travaux de Delshams, De La Llave et Seara.

Notre deuxième λ-lemme, que nous appelons « λ-lemme fibré » est une généralisation du λ-lemme basique aux sous-variétés transverses de dimension variant entre celle des feuilles instables et celle de la variété instable. On applique ensuite ce résultat au cas d'une chaîne de tores invariants (contenus dans une variété normalement hyperbolique) tels que deux tores consécutifs de la chaîne possèdent une intersection hétérocline *transverse*. Nous montrons, sous certaines hypothèses supplémentaires sur le difféomorphisme f (une propriété de torsion forte, et l'existence d'une forme normale au voisinage de variété normalement hyperbolique), la *transitivité* des connexions hétéroclines *transverses*.

Fenêtres et temps de diffusion.

Nous développons une autre technique géométrique permettant de prouver l'existence d'orbites de diffusion, mais surtout de donner des estimées quantitatives de leur vitesse de diffusion. Cette technique repose d'une part sur la méthode des fenêtres d'Easton (introduite dans [EM79] and [Eas81] et revisitée dans [GR07] et [GR09] avec un point de vue topologique) qui permet de détecter les trajectoires des orbites et d'autre part sur notre λ-lemme fibré qui, lui, permet d'expliciter les estimées des temps de « redressement » nécessaires pour calculer les temps de diffusion.

Notre but ici est de donner une construction simple dans laquelle on peut complètement décrire les paramètres qui entrent en jeu quand on veut calculer des temps de diffusion d'une part, et d'autre part telle que le temps de diffusion soit uniforme par rapport à la chaîne de transition, au sens où les estimées ne se détériorent pas avec les itérations (comme dans le cas de [Mar96], par exemple). Nous envisageons cette construction comme une première étape pour calculer les temps de diffusion dans un cadre plus général.

Nous travaillons donc avec une chaîne de transition contenue dans une variété normalement hyperbolique $N \subset \mathbb{T}\times\mathbb{R}$ et avec un difféomorphisme f qui est un twist intégrable. Nous supposons de plus l'existence d'une forme normale pour f au voisinage de N, ce qui nous permet d'appliquer une version de notre λ-lemme fibré qui assure la convergence C^1 des itérées de la variété qui coupe transversalement la variété stable $W^s(N)$.

Expliquons brièvement la méthode des fenêtres. Une fenêtre est l'image par un difféomorphisme d'un rectangle multidimensionnel qui possède des directions « horizontales » et « verticales ». On dit que deux fenêtres sont correctement alignées si chaque horizontale de l'une est transverse à toutes les verticales de l'autre en un point *intérieur* aux deux fenêtres. Étant donnée une famille finie de fenêtres reliées par des « difféomorphismes de connexion » (c'est à dire que l'image d'une fenêtre par un tel difféomorphisme est correctement alignée avec la fenêtre suivante), le Shadowing Lemma d'Easton assure l'existence d'une orbite qui traverse ces fenêtres dans l'ordre de leur alignement. Alors, si on a une une chaîne de transition de tores invariants, on peut prouver l'existence d'une orbite passant arbitrairement près de cette chaîne en construisant des fenêtres arbitrairement proches de chaque tore, qui s'alignent correctement grâce à une puissance convenable du difféomorphisme (on utilise abusivement le terme puissance pour désigner le nombre d'itérations). Le temps nécessaire à une orbite pour diffuser le long de cette chaîne est alors la somme des puissances utilisées.

Notre construction de fenêtres est inspirée de celles de [GR07] et [GR09]. Dans ces deux articles, les auteurs prouvent l'existence d'orbites de diffusion qui longent alternativement des chaînes de transition et des zones d'instabilité de Birkhoff. Notre approche simplifie les constructions précédentes le long de nos chaînes de transition au sens où nous avons besoin de construire moins de fenêtres pour longer notre chaîne. En effet, dans [GR07], deux premières fenêtres sont construites autour de chaque point hétérocline, et deux autres près de chaque tore invariant, tandis que notre méthode ne fait intervenir que deux fenêtres près de chaque tore. Nous construisons premièrement une fenêtre *statique* arbitrairement proche de chaque tore. Ensuite nous construisons une fenêtre *mobile* de sorte que la fenêtre statique soit correctement alignée avec la fenêtre mobile et que l'image de cette dernière par une puissance convenable du difféomorphisme soit correctement alignée avec la fenêtre statique du tore suivant. D'autre part, alors que l'approche de Gidea et Robinson est topologique, la notre est plus géométrique au sens où elle est basée sur la structure différentielle et la transversalité. En particulier, alors que dans beaucoup d'exemples, la preuve de la transversalité n'est pas immédiate, notre λ-lemme fibré donne (en plus des estimées explicites de temps de redressement) le redressement des horizontales qui conduit immédiatement à la propriété de transversalité souhaitée. Enfin, le λ-lemme fibré permet de s'affranchir du critère de linéarisation topologique de Pugh et Shub nécessaire dans l'approche de [GR07] pour aligner deux de chaque set de fenêtres.

Afin de prouver l'alignement correct des fenêtres, nous avons besoin de transporter les fenêtres mobiles autour de chaque tore et d'estimer le temps nécessaire pour passer d'un voisinage de la variété stable du tore à un voisinage de la variété instable du même tore. Ceci est possible grâce à des résultats d'ergodisation dûs à Dumas, Bourgain, Berti, Bolle, Biasco... De manière informelle, pour δ assez petit, le temps de δ-ergodisation d'un tore soumis à une rotation résonnante ou « presque résonnante » est le nombre d'itérées nécessaires pour qu'une orbite soit δ-dense dans le tore. Nous travaillerons avec des rotations diophantiennes, car dans ce cas nous obtenons des temps d'ergodisation « optimaux » ; celà ne sera pas une réelle restriction pour notre étude car, dans tous les exemples que nous avons à l'esprit, les tores diophantiens remplissent densément la variété normalement hyperbolique.

Enfin, notre méthode permet de donner des estimées quantitatives (dans un cadre abstrait) des temps nécessaires à une orbite pour diffuser le long d'une chaîne de transition. Nous prouverons que le temps de diffusion dépend de trois paramètres caractéristiques : le « temps de redressement » (donné par le λ-lemme fibré) qui est le temps nécessaire pour assurer le critère de transversalité pour l'alignement des fenêtres dans les directions normales, le temps d'ergodisation de chaque tore et le temps de torsion qui complète la propriété de transversalité dans la direction centrale.

L'étape suivante (qui n'a pas pu être achevée par manque de temps) est d'appliquer ces résultats abstraits à des exemples particuliers comme ceux de [LM05].

Orbites asymptotiquement denses

Alors que dans les problèmes précédents que nous avons étudiés (et qui correspondent aux chapitres 2, 3 et 4) nous avons travaillé avec des difféomorphismes, ici nous travaillons avec des flots hamiltoniens.

Le λ-lemme basique va nous permettre de démontrer un résultat de diffusion basé sur un travail récent de Marco sur les propriétés génériques des systèmes classiques sur $\mathbb{T}^2$. On va construire des exemples simples de diffusion globale impliquant des systèmes non intégrables moyennés aux résonnances doubles.

Dans [Marc], Marco a démontré que les systèmes classiques sur $\mathbb{T}^2$ avec un potentiel *générique* U possède des *chaînes d'anneaux*. De ces derniers, on déduit l'existence de *chaînes de cylindres* à énergie fixée pour des systèmes sur $\mathbb{A}^3$ de la forme

$$\mathcal{H}_n(\theta, r) = \frac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3), \ (\theta, r) \in \mathbb{A}^3,$$

où $\|.\|$ est la norme euclidienne.

Grâce aux propriétés des anneaux, on déduit que les cylindres sont des variétés normalement hyperboliques et que les chaînes de cylindres se projettent sur l'espace des actions asymptotiquement (c'est-à-dire quand n tend vers l'infini) proche des résonnances simples. De plus, les cylindres sont feuilletés en tores invariants avec des connexions homoclines.

Notons que les systèmes $\mathcal{H}_n$ sont des produits directs du hamiltonian $\frac{1}{2}r_1^2$ avec le hamiltonien $\frac{1}{2}(r_2^2+r_3^2)+\frac{1}{n}U(\theta_2, \theta_3)$. Alors, ils n'admettent pas d'orbite de diffusion à cause de la conservation de l'énergie dans chaque terme. Pour obtenir des orbites de diffusion, on choisit une perturbation f_n qui crée le « splitting des séparatrices » . Plus précisément, on construit une suite de fonctions (f_n) à supports dans le complémentaire des cylindres, de sorte que les cylindres soient également normalement hyperboliques pour le système $\mathcal{H}_n+f_n$. De plus, la perturbation est choisie de sorte que deux tores voisins admettent des connexions hétéroclines.

Une propriété remarquable est que le λ-lemme basique entraîne l'existence d'orbites qui longent les tores dans chaque cylindre, et donc qui longent les chaînes de cylindres vu que deux cylindres consécutifs admettent des connexions hétéroclines. On déduit alors l'existence d'orbites dont la projection sur l'espace d'actions remplit asymptotiquement le niveau d'énergie projetté, quand la perturbation tend vers 0.

Plan de la thèse

La thèse est organisée comme suit. Le chapitre 1 introduit le langage basique et les notations de la thèse. Dans les chapitres 2 et 3, nous énonçons et démontrons nos deux λ-lemmes pour les variétés normalement hyperboliques ainsi que leurs applications à la diffusion. Le chapitre 4 est consacré à la méthode des fenêtres et au calcul de temps de diffusion. Dans le chapitre 5, nous construisons des exemples de systèmes hamiltoniens presque intégrables sur $\mathbb{A}^3$ qui admettent des orbites asymptotiquement denses en projection.

Chapitre 1

Nous commençons ce chapitre par un rappel sur les variétés normalement hyperboliques et nous introduisons un « voisinage de redressement » dans lequel nous énoncerons plusieurs de nos théorèmes. Ensuite nous énonçons des résultats sur les temps d'ergodisation pour les rotations sur le tore $\mathbb{T}^n$. Dans la troisième section du chapitre 1, nous rappelons brièvement la notion de système hamiltonien presque intégrable qui n'interviendra qu'à la fin du chapitre 2 et dans le chapitre 5. Nous terminons ce chapitre en décrivant les hypothèses sous lesquelles nous travaillons dans les chapitres 2, 3 et 4. Le chapitre 5 concerne des exemples spécifiques et possède ses propres hypothèses.

Chapitre 2

Nous commençons ce chapitre en énonçant et démontrant le λ-lemme basique dont nous donnerons deux versions. La première, le Théorème 1, est une version plus simple établie dans le voisinage de redressement de la variété normalement hyperbolique. La deuxième version, le Théorème 2, est plus générale. De manière informelle, le λ-lemme s'énonce comme suit.

Théorème 1 et Théorème 2. Soit $(M,\, \Omega)$ une variété symplectique C^∞ et $f : M \to M$ un difféormorphisme symplectique de classe C^l ($l \geq 3$). Soit N une sous-variété compacte C^∞ de M qui est normalement hyperbolique pour f. Supposons que les fibrés stable et instable de N sont de même dimension. Considérons une C^1 sous-variété Δ de M dont la dimension est égale à celle d'une feuille instable de N. On suppose enfin que Δ coupe transversalement la variété stable de N. Alors, on prouve que pour tout $\varepsilon > 0$, et pour tout n entier assez grand, il existe un point x_n dans N tel que $f^n(\Delta)$ est ε-proche, en topologie C^1 à la variété fortement instable de x_n.

Nous utilisons ensuite le λ-lemme pour prouver un premier résultat de diffusion, le corollaire 2.4.2 de la section 2.4, qui donne l'existence d'une orbite de diffusion ; cette orbite longe une famille finie d'ensembles invariants, contenus dans la variété normalement hyperbolique, sur lesquels la dynamique est minimale et qui possèdent des connexions hétéroclines.

Corollaire Soient f, M et N comme dans le Théorème 2 et soient $(A_k)_{1\leq k\leq n}$ une chaîne de transition dans N telle que, pour tout $k = 1, \dots, n-1$, il existe $a_k \in A_k$, $b_{k+1} \in A_{k+1}$ et $c_k \in W^{uu}(a_k) \cap W^{ss}(b_{k+1})$ tels que $W^{uu}(a_k)$ et $W^s(N)$ se coupent transversalement en c_k. Alors,

$$W^u(A_n) \subset \overline{W^u(A_1)}.$$

Nous montrons enfin que l'exemple d'Arnold est un cas particulier de ce corollaire.

Chapitre 3

Ce chapitre est consacré au λ-lemme fibré et à une de ses applications. Nous commençons par énoncer et prouver le Théorème 3 qui est une généralisation du λ-lemme basique aux sous-variétés C^2 dont la dimension varie entre celle des feuilles instables et celle de la variété instable. Dans le cas où on a une forme normale spécifique au voisinage de N, nous parvenons à augmenter partiellement la régularité de la convergence, c'est le corollaire 3.1.3. Nous utilisons ensuite ce λ-lemme fibré pour prouver la transitivité des connexions hétéroclines transverses pour des systèmes vérifiant une propriété de torsion forte (que nous introduisons), c'est le corollaire 3.3.2.

Corollaire. On se place dans le cas où on a une forme normale spécifique au voisinage de N permettant d'augmenter partiellement la régularité de la convergence et où f possède la propriété de torsion forte. Soit $(T_k)_{1\leq k\leq n}$ une famille finie de tores formant une chaîne de transition dans N tels que pour $k = 1, \ldots, n-1$, $W^u(T_k) \pitchfork W^s(T_{k+1})$. Alors, $W^u(T_1) \pitchfork W^s(T_n)$.

Chapitre 4

Dans ce chapitre, nous utilisons la méthode des fenêtres et le λ-lemme fibré du chapitre 3 pour estimer le temps nécessaire à une orbite pour diffuser le long d'une chaîne de transition contenue dans une variété normalement hyperbolique. Plus précisément, nous prouvons le résultat suivant.

Théorème 4. Considérons un système satisfaisant les hypothèses du λ-lemme fibré obtenu avec l'existence d'une forme normale spécifique au voisinage de N (corollaire 3.1.3). Supposons que $N \subset \mathbb{T}\times\mathbb{R}$ et que $f_{|_N}$ est un twist intégrable. On suppose de plus que N contient une chaîne de transition de cercles diophantiens. Alors pour tout $\varepsilon > 0$, il existe une orbite $(x_i)_{1\leq i\leq n}$ et une suite d'entiers positifs $(k_i)_{1\leq i\leq n-1}$ tels que

$$\begin{aligned} &d(x_i, T_i) < \varepsilon, \text{ pour tout } i = 1, \ldots, n, \\ &x_{i+1} = f^{k_i}(x_i), \text{ pour tout } i = 1, \ldots, n-1. \end{aligned}$$

On a de plus la majoration suivante

$$k_i \leq n_0 + \max(m_i, q_i) + Q_i,$$

où n_0 est un entier indépendant de i, m_i le temps de redressement donné par le λ-lemme fibré, p_i le temps de torsion et Q_i le temps d'ergodisation de la rotation sur T_{i+1}.

Chapitre 5

Dans ce chapitre, on utilise les résultats récents de Marco concernant les propriétés génériques des systèmes classiques sur le tore $\mathbb{T}^2$ ainsi que le λ-lemme basique pour donner des exemples de systèmes hamiltoniens presque intégrables qui admettent des orbites asymptotiquement denses en projection. Plus précisément, étant donné un entier $\kappa \geq 2$, on considère la classe de systèmes presque intégrables sur l'anneau $\mathbb{A}^3$ de la forme suivante

$$H_n(\theta, r) = \frac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3) + f_n(\theta, r),$$

où $U \in C^\kappa(\mathbb{T}^2, \mathbb{R})$ est un potentiel *générique* et f_n une perturbation ajoutée C^κ satisfaisant $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \leq \frac{1}{n}$; ainsi H_n est une pertubation du système complètement intégrable $h(r) = \frac{1}{2}\|r\|^2$.

On prouve le résultat de diffusion suivant.

Theorem 5. Soit $\Pi : \mathbb{A}^3 \to \mathbb{R}^3$ la projection canonique. Alors pour tout $\delta > 0$, il existe un entier n_0 tel que pour $n \geq n_0$, le système H_n admet une orbite Γ_n d'énergie $\frac{1}{2}$ dont la projection $\Pi(\Gamma_n)$ est δ–dense dans $\Pi(H_n^{-1}(\frac{1}{2}))$, au sens où le δ–voisinage de $\Pi(\Gamma_n)$ dans $\mathbb{R}^3$ recouvre $\Pi(H_n^{-1}(\frac{1}{2}))$.

Chapter 1

Framework: reminders, definitions and notation

1.1 Normally hyperbolic invariant manifolds

We begin with a reminder on normally hyperbolic manifolds in a general context, and then specialize to the symplectic case where we can use a "controlled" straightening neighborhood in which it is easy to depict the geometry of the invariant foliations induced by the normal hyperbolicity.

The normally hyperbolic invariant manifolds we consider will be compact for technical simplicity, but the non-compactness could easily be replaced with uniform lower bounds for the first and second derivatives of our diffeomorphisms, and the constants of hyperbolicity (see (1.1) below). Moreover, let us point out that eventhough we state our definitions and results for diffeomorphisms, analogous results hold for continuous time Hamiltonian systems (using the usual suspension way).

1.1.1 General definitions

Let M be a smooth n-dimensional manifold ($n \geq 3$) and $f : M \to M$ be a C^l-diffeomorphism ($l \geq 1$) which leaves a smooth boundaryless compact submanifold N of M invariant. Given a Riemannian metric $\| \,.\, \|$ on M and a subbundle E of $T_N M$ invariant under Df, we set:

$$\mathrm{norm}\,(Df_{|E}) = \sup\{\|Df(a)_{|E_a}\|; a \in N\}, \quad \mathrm{conorm}\,(Df_{|E}) = (\mathrm{norm}\,(Df^{-1}_{|E}))^{-1}.$$

Definition 1.1.1. *Let $q \leq l$ ($q \in \mathbb{N}^*$). The manifold N is q-normally hyperbolic for f if the tangent bundle of M restricted to N splits into three continuous subbundles $T_N M = TN \oplus E^s \oplus E^u$ invariant under Df, such that*

$$\mathrm{norm}\,(Df_{|E^s}) < (\mathrm{conorm}\,(Df_{|TN}))^q \leq 1 \leq (\mathrm{norm}\,(Df_{|TN}))^q < \mathrm{conorm}\,(Df_{|E^u}). \tag{1.1}$$

This says that the behavior of f normal to N dominates the tangent behavior of f^q and is hyperbolic.

Now we state the local stable/unstable manifolds theorem. We do not mean to give the most general possible results, we rather limit ourselves to those which are strictly necessary for our purposes. For a more elaborate study on invariant manifolds, we refer to [HPS77], [Cha04] and [BB].

Theorem [HPS77]. *Let f, M and N be as above. We suppose that N is q-normally hyperbolic for f. Then if d is the distance associated with the Riemannian metric on M, the following properties hold true:*

1. Existence, characterization and smoothness. *There exists a neighborhood $\mathcal{O}$ of N in M such that the sets:*

$$W^s_{loc}(N) = \Big\{y \in \mathcal{O}\,; f^n(y) \in \mathcal{O},\, \forall n \in \mathbb{N}\Big\} \text{ and } W^u_{loc}(N) = \Big\{y \in \mathcal{O}\,;\, f^{-n}(y) \in \mathcal{O},\, \forall n \in \mathbb{N}\Big\}$$

are C^q-manifolds that satisfy

- $\forall\, y \in W^s_{loc}(N), \forall\, \rho \in \left]\,\mathrm{norm}\,(Df_{|_{E^s}});\, \mathrm{conorm}\,(Df_{|_{TN}})\,\right[,\ \lim_{n\to\infty} \rho^{-n} d(f^n(y), N) = 0,$
- $\forall\, y \in W^u_{loc}(N), \forall\, \rho \in \left]\,\mathrm{norm}\,(Df_{|_{TN}});\, \mathrm{conorm}\,(Df_{|_{E^u}})\,\right[,\ \lim_{n\to\infty} \rho^{-n} d(f^{-n}(y), N) = 0.$

Moreover, $W^u_{loc}(N)$ and $W^s_{loc}(N)$ are tangent to $TN \oplus E^u$ and $TN \oplus E^s$ respectively at each point of N.

2. Lamination. *There exist two f-invariant laminations of $W^u_{loc}(N)$ and $W^s_{loc}(N)$, the leaves of which are unstable and stable leaves $W^{uu}_{loc}(x)$ and $W^{ss}_{loc}(x)$ associated with the points of N, defined as follows:*

$$W^{ss}_{loc}(x) = \Big\{y \in \mathcal{O}\,; \lim_{n\to\infty} d\left(f^n(y), f^n(x)\right) = 0\Big\} \text{ and}$$

$$W^{uu}_{loc}(x) = \Big\{y \in \mathcal{O}\,; \lim_{n\to\infty} d\left(f^{-n}(y), f^{-n}(x)\right) = 0\Big\}.$$

These leaves are C^q and tangent to the fibers E^u_x and E^s_x at each point x of N.

Note that one gets the global stable (resp. unstable) manifolds by taking the union of the inverse (resp. direct) images of the local ones as follows:

$$W^s(N) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(W^s_{loc}(N)\right) \quad \text{and} \quad W^u(N) = \bigcup_{n\in\mathbb{N}} f^{n}\left(W^u_{loc}(N)\right).$$

The same holds for the leaves:

$$W^{ss}(x) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(W^{ss}_{loc}\left(f^n(x)\right)\right) \quad \text{and} \quad W^{uu}(x) = \bigcup_{n\in\mathbb{N}} f^{n}\left(W^{uu}_{loc}\left(f^{-n}(x)\right)\right).$$

These are immersed C^q-submanifolds of M. In the rest of the thesis, we will drop the subscript *loc* from the notation. The local and the global invariant manifolds will be denoted by $W^{s,u}(N)$ since the context will always be clear. The same holds for the global and local leaves.

Definition 1.1.2. *Let N be a q-normally hyperbolic manifold for f $(q \leq l)$. We say that N is controllable if the following inequalities hold true*

$$\mathrm{norm}\,(Df_{|_{E^s}}).\mathrm{norm}\,(Df_{|_{TN}}) < 1, \quad \textit{and} \quad \mathrm{conorm}\,(Df_{|_{TN}}).\mathrm{conorm}\,(Df_{|_{E^u}}) > 1. \tag{1.2}$$

This will be no restriction to us, since all the invariant manifolds that we will get in the perturbative setting will be controllable.

We set $n_s := \dim(E^s)$, $n_u := \dim(E^u)$ and $n_0 := \dim(N)$, so that $n_0 + n_s + n_u = n$.

1.1.2 Symplectic Geometry and normal hyperbolicity

Under symplecticity assumptions, the stable and unstable leaves are regular with respect to the points in N. More precisely, we have the following proposition which will enable us in the next section to introduce a *straightening* coordinate system in the vicinity of normally hyperbolic manifolds.

Proposition A. [Marco]. *Let (M, Ω) be a smooth symplectic manifold and let f be a C^l symplectic diffeomorphism of M ($l \geq 2$). We suppose that N is a controllable q-normally hyperbolic manifold for f ($q \leq l$) and that $n_s = n_u$. Then*

- *N is symplectic,*
- *$W^u(N)$ and $W^s(N)$ are coisotropic,*
- *for all $x \in N$, $W^{uu}(x)$ and $W^{ss}(x)$ are isotropic and they coincide with the leaves of the characteristic foliations of $W^u(N)$ and $W^s(N)$.*

The proof of this proposition can be found in [Mara]. Since the leaves of the characteristic foliations coincide with the leaves $W^{uu}(x)$ and $W^{ss}(x)$, the latter are C^{q-1} with respect to x. We get then the regularity we need for Proposition B below.

1.1.3 Straightening neighborhood

Under the assumptions of Proposition A, one can find in the vicinity of a normally hyperbolic manifold a neighborhood in which the invariant manifolds and the leaves are straightened, making it easier to depict the behavior of f. More precisely, we have the following proposition.

Proposition B. [Tubular neighborhood and straightening]. *Let M, N and f be as in Proposition A with $l \geq 3$. Let $p := n_s = n_u$. We suppose that N is 3-normally hyperbolic for f and that its stable and unstable bundles are trivial. Then, there exist a neighborhood U of N in M and a C^2-diffeomorphism $\varphi : U \to V := N \times B^p \times B^p$, where B^p is an open ball centered at 0 in $\mathbb{R}^p$, such that for all $x \in N$:*

1. $\varphi(x) = (x, 0, 0)$,
2. $\widetilde{W}^s(N) := \varphi(W^s(N) \cap U) = \{(x, s, u) \in V\,;\, u = 0\}$,
3. $\widetilde{W}^u(N) := \varphi(W^u(N) \cap U) = \{(x, s, u) \in V\,;\, s = 0\}$,
4. $\widetilde{W}^{ss}(x) := \varphi(W^{ss}(x) \cap U) = \{(x, s, 0)\,;\, s \in B^p\}$,
5. $\widetilde{W}^{uu}(x) := \varphi(W^{uu}(x) \cap U) = \{(x, 0, u)\,;\, u \in B^p\}$.

The proof is straightforward once Proposition A is known. We will not prove Proposition B, we will content ourselves with the following few remarks. Near N, one can always find a tubular neighborhood. The straightening of the invariant manifolds is an immediate consequence of the graph property. We refer to [LMS03] and [HPS77] for details. When f is symplectic, the strongly stable/unstable leaves are straightened the same way.

The first B^p and the second B^p in $N \times B^p \times B^p$ do not play the same role since the first one is the stable direction while the second one is the unstable direction. In order to distinguish them

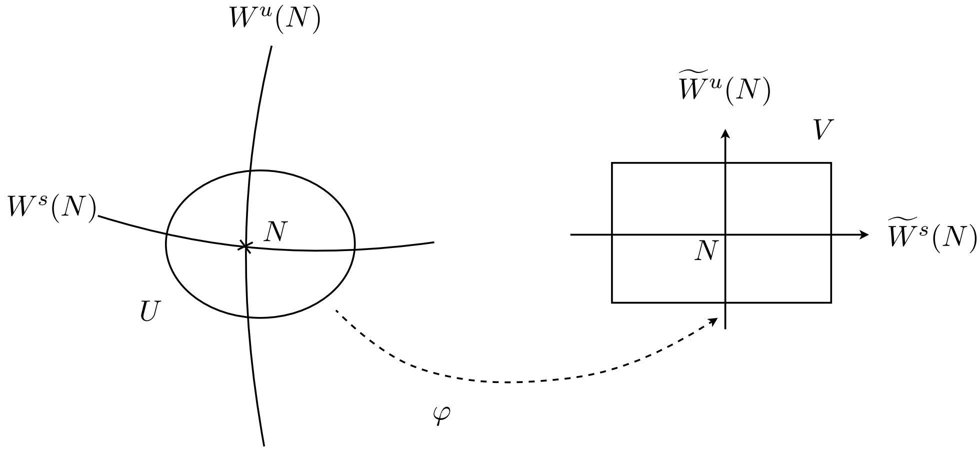

Figure 1.1: The straightening neighborhood

from one another when we want to use them separately, we will add u and s in the notation as follows

$$N \times B_s^p \times B_u^p. \tag{1.3}$$

We use the same convention for $N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$. In the rest of the thesis, for notational symplicity, we will identify N with $\varphi(N) = N \times \{0\} \times \{0\}$. This will not lead to confusion since the context will always be clear enough.

1.1.4 Partially hyperbolic tori

Let M be a smooth $2m$-dimensional symplectic manifold endowed with a Riemannian metric with associated distance d. Let k be an integer, $1 \leq k \leq m-1$. Let $f : M \to M$ be a C^l symplectic diffeomorphism ($l \geq 1$) which leaves a smooth k-dimensional torus $\mathcal{T}$ invariant. We say that $\mathcal{T}$ is *hyperbolic* when there exist a neighborhood $\mathcal{U}$ of $\mathcal{T}$ and a constant $\alpha > 0$ such that the sets

$$W_{loc}^s(\mathcal{T}) = \Big\{ y \in \mathcal{U}\,; \exists\, C > 0, d(f^n(y), \mathcal{T}) \leq Ce^{-\alpha n}, \forall\, n \in \mathbb{N} \Big\}$$

$$W_{loc}^u(\mathcal{T}) = \Big\{ y \in \mathcal{U}\,; \exists\, C > 0, d(f^{-n}(y), \mathcal{T}) \leq Ce^{-\alpha n}, \forall\, n \in \mathbb{N} \Big\}$$

are two Lagrangian submanifolds of $\mathcal{U}$. The set $W^s(\mathcal{T}) = \cup_{n\in\mathbb{N}} f^{-n} \left(W_{loc}^s(\mathcal{T})\right)$ and the set $W^u(\mathcal{T}) = \cup_{n\in\mathbb{N}} f^{n} \left(W_{loc}^u(\mathcal{T})\right)$ are the stable and unstable manifolds of $\mathcal{T}$. They are m-dimensional immersed Lagrangian submanifolds of M.

The only case we will deal with is when $\mathcal{T}$ is located inside a normally hyperbolic submanifold N of M. For more details on hyperbolic tori, we refer to [LMS03].

1.2 Ergodization times for the rotations of $\mathbb{T}^n$

In this section, we state some results regarding the ergodization times for rotations (possibly resonant but only at a "high order") of the torus $\mathbb{T}^n$ ($n \geq 1$).

Let $n \geq 1$ and let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. For $0 < \alpha \leq \frac{1}{2}$, we define the open ball centered on θ in $\mathbb{T}^n$ and of radius α by $B(\theta, \alpha) := \{\varphi \in \mathbb{T}^n \,;\, d(\theta, \varphi) < \alpha\}$.

Let $r \in \mathbb{R}^n$ and let $F_r : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be the map given by $F_r(\theta) = \theta + r \mod \mathbb{Z}^n$. The orbit of θ under F_r is the sequence $(F_r^k(\theta))_{k\in\mathbb{N}}$, where F_r^0 is the identity map. When it exists, for $0 < \alpha \leq \frac{1}{2}$, we define the "$\alpha$-ergodization" time of $\mathbb{T}^n$ for r, as the smallest natural number $q = q_r(\alpha)$ such that, for any initial condition θ in $\mathbb{T}^n$, the finite piece of orbit $\{\theta, F_r(\theta), \ldots, F_r^q(\theta)\}$ is a α-dense subset of $\mathbb{T}^n$. More precisely, we have the following definition.

Definition 1.2.1. **[Ergodization time.]** *Let $\alpha \in \,]0; \frac{1}{2}]$ and $r \in \mathbb{R}^n$. The α-ergodization time of $\mathbb{T}^n$ for r, when it exists, is the smallest natural number $q = q_r(\alpha)$ such that, for any initial condition θ in $\mathbb{T}^n$,*

$$\bigcup_{0\leq k\leq q} B\left(F_r^k(\theta), \alpha\right) = \mathbb{T}^n.$$

Note that the property of α-ergodizing $\mathbb{T}^n$ after time q is independent of the initial condition θ in $\mathbb{T}^n$. More precisely, if r α-ergodizes $\mathbb{T}^n$ after time q with initial condition θ, then $\bigcup_{0\leq k\leq q} B\left(F_r^k(\theta), \alpha\right) = \mathbb{T}^n$. Let $\varphi \in \mathbb{T}^n$. Thanks to the particular form of F_r, it is easy to see that

$$\begin{aligned} \bigcup_{0\leq k\leq q} B\left(F_r^k(\varphi), \alpha\right) &= \bigcup_{0\leq k\leq q} B\left(F_r^k(\theta), \alpha\right) + \theta - \varphi \mod \mathbb{Z}^n \\ &= \mathbb{T}^n. \end{aligned}$$

Let $r \in \mathbb{R}^n$. If r is nonresonant, that is, if for all $p \in \mathbb{Z}^n \setminus \{0\}$ and for all $l \in \mathbb{Z}$, $p.r - l \neq 0$, the orbits are dense in $\mathbb{T}^n$ and therefore, for any $0 < \alpha \leq \frac{1}{2}$, the α-ergodization time for r exists. When r is resonant at a "sufficiently high order", namely if $p.r - l \neq 0$ for all $l \in \mathbb{Z}$ and for all $p \in \mathbb{Z}^n$ with $0 < |p| \leq K(\alpha)$ for some large enough $K(\alpha)$, the α-ergodization time for r exists too. For more details, we refer to [Dum91], [BGW98] and [BBB03] and the references therein.

Let $|\,.\,|_{\mathbb{Z}}$ be the distance to $\mathbb{Z}$, that is, $|x|_{\mathbb{Z}} = \inf\{|x - s|;\ s \in \mathbb{Z}\}$. For $r \in \mathbb{R}^n$ and $K \in \,]1; +\infty[$, we set

$$\Psi_r(K) := \max\left\{|p.r|_{\mathbb{Z}}^{-1}\,;\, p \in \mathbb{Z}^n \setminus \{0\}\,,\, |p| \leq K\right\}.$$

We now state a theorem due to Berti, Biasco and Bolle ([BBB03]).

Theorem 1.2.2. [Berti-Biasco-Bolle.] *There exist two positive constants C and $M = M(n)$ such that, for all $r \in \mathbb{R}^n$, for all $\alpha \in \,]0; \frac{1}{2}]$,*

$$q_r\left(\alpha\right) \leq C\Psi_r\left(M\alpha^{-1}\right).$$

This is, in fact, a consequence of Theorem 4.2 in [BBB03], where the result was proved for the continuous case. The constant C appears when adapting this result to the discrete case. The authors in [BBB03] give an explicit value of the constant M which will satisfy $M \geq 1$. However, both constants will be of no importance to us.

In Chapter 4 (Lemma 4.3.1), we prove this result for the case $n = 1$. The proof we give significatively simplifies the previous proofs on ergodization times for the rotations of $\mathbb{T}$ since we do not use the continued fractions. Instead, we apply Dirichlet's box principle and get a short and simple proof of the result.

If r is nonresonant, with no additional information, the ergodization time is of course highly dependent on r, and may, in general, be arbitrarily long. In practice, we require r to be not only nonresonant, but also "far from" low-order resonances by means of Diophantine condition, which ensures a *fast* ergodization of the torus.

Definition 1.2.3. **[Diophantine vectors.]** *Let $\tau \geq n$ be a real number and let $r \in \mathbb{R}^n$. We say that r is Diophantine with exponent τ if there exists $c > 0$ such that for all $p \in \mathbb{Z}^n \setminus \{0\}$, $|p.r|_{\mathbb{Z}} \geq c|p|^{-\tau}$.*

We also use the language (c, τ)-Diophantine. The Diophantine vectors of exponent $\tau \geq n$ are nonresonant and dense in $\mathbb{R}^n$. We have the following easy corollary that gives the ergodization time for Diophantine vectors.

Corollary 1.2.4. *Let $\tau \geq n$ and $c > 0$, and let r be a (c, τ)-Diophantine vector in $\mathbb{R}^n$. Then there exist two positive constants C and $M = M(n)$ such that*

$$q_r(\alpha) \leq Cc^{-1}M^{\tau}\alpha^{-\tau}.$$

The proof follows from Theorem 1.2.2. Note that this estimate was first proved in Theorem D of [BGW98]. In Section 4.5, we recover this estimate in the case $n = 1$.

1.3 Nearly integrable Hamiltonian systems

Given an integer $n \geq 1$, we denote by $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ the cotangent bundle of the torus $\mathbb{T}^n$, that we endow with its usual angle-action coordinates (θ, r) and its Liouville symplectic form $\Omega = \sum_{i=1}^n dr_i \wedge d\theta_i$. Let H be a C^2 function defined on $\mathbb{A}^n$. The Hamiltonian vector field X^H associated with H is defined by

$$\begin{cases} \dot{\theta} = & \partial_r H(\theta, r) \\ \dot{r} = & -\partial_\theta H(\theta, r), \end{cases}$$

and the Hamiltonian flow Φ_t^H is the flow generated by X^H. Observe that for each t, Φ_t^H is a diffeomorphism on $\mathbb{A}^n$. One says that H is a Hamiltonian on $\mathbb{A}^n$.

We say that H is completely integrable if H depends only on the action variable r, that is, $H(\theta, r) = h(r)$ where $h : \mathbb{R}^n \longrightarrow \mathbb{R}$. In this case, the action variables of the solutions are trivially constant for all times, and each Lagrangian torus $\mathbb{T}^n \times \{r_0\}$ is invariant under the Hamiltonian flow. Moreover, the dynamics on such a torus is quasi-periodic, that is, for all t, $\Phi_t^H(\theta, r_0) = (\theta + t\omega(r_0)\,[\mathbb{Z}^n], r_0)$, where $\omega(r_0) = \partial_r h(r_0)$.

Nearly integrable Hamiltonian systems are small perturbations of completely integrable Hamiltonian systems. More precisely, they are Hamiltonian systems of the following form:

$$H(\theta, r) = h(r) + \varepsilon f(\theta, r) \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter and h, f are C^2 functions.

When $\varepsilon > 0$, the action variables are no longer constants of motion and it is very interesting to investigate their evolution with time. Under suitable nondegeneracy and regularity conditions, the KAM (Kolmogorov-Arnold-Moser) theory asserts that, for each r_0 such that $\omega(r_0)$ is Diophantine, there exist $\varepsilon_0(r_0) > 0$ and a family of Lagrangian tori $(\mathcal{T}(\varepsilon))_{0 \leq \varepsilon \leq \varepsilon_0}$ with $\mathcal{T}(0) = \mathbb{T}^n \times \{r_0\}$ such that the flow generated by H leaves $\mathcal{T}(\varepsilon)$ invariant and is conjugate on $\mathcal{T}(\varepsilon)$ to a quasi-periodic flow with frequency $\omega(r_0)$. In this case, the action variables of the solutions are *almost* constant and their variation is a $O(\sqrt{\varepsilon})$ for any t. For more details, we refer to [Kol54], [Bos86] and [Pös01]. The theory gives no information on the complement of the set of KAM tori, except that it has a relative measure of order $\sqrt{\varepsilon}$ and that when $n = 2$, the 2-dimensional invariant tori disconnect the 3-dimensional energy level leaving *all solutions* stable for *all time*.

When $n \geq 3$, it is possible to find solutions for which the variation of the actions is of order independent of the size of the perturbation. In [Arn64], Arnold gave the first example of a three-degree-of-freedom system where such a drift occurs no matter how small the perturbation is. This phenomenon is usually referred to as Arnold diffusion.

By Nekhoroshev's theorem, the *instability* of action variables is an exponentially slow phenomenon, or equivalently, the action variables are stable for an exponentially large time. More precisely, under assumptions of regularity of the system and (quantitative) non-degeneracy of h, for any perturbation and any solution, the variation of the actions is a $O(\varepsilon^b)$ for $|t| \leq T(\varepsilon)$, where $T(\varepsilon)$ is a $O\left(\exp\left(\varepsilon^{-a}\right)\right)$ with constants $0 < a, b \leq 1/2$. In [BM11], the authors proved that, under convexity assumptions, a can be chosen such that $a = \frac{1}{2(n-1)} - \delta$, with δ as small as desired.

In this thesis, we are mainly concerned with diffusion orbits and diffusion times.

1.4 Standing assumptions and convention

Here we state the assumptions needed for chapters 2, 3 and 4. Chapter 5 will deal with Hamiltonian flows (instead of diffeomorphisms) and will have its own assumptions.

Standing assumptions for chapters 2, 3 and 4. We assume that

- (M, Ω) is a smooth symplectic Riemannian manifold,
- $f : M \to M$ is a symplectic diffeomorphism of class C^l $(l \geq 3)$,
- N is a smooth compact submanifold of M and that it is boundaryless,
- N is a controllable 3-normally hyperbolic manifold for f,
- $n_s = n_u = p$,
- N has trivial stable and unstable bundles.

Convention. Let d be the distance associated with the Riemannian metric on M. We will equip the neighborhood V defined in Proposition B with the distance given by the sup of $d_{|N}$ and the Euclidian distance on $\mathbb{R}^{2p}$. It is equivalent to the image under φ of d since V is relatively compact.

We will use the usual operator norms for the linear applications defined on Banach spaces that we will deal with throughout the thesis. We will equip the product spaces with the sup norm and the subsets with the induced norm. For notational simplicity, we will denote all our norms by the same symbol $\|.\|$; the context will always be clear enough to avoid ambiguities.

To prove our results, we will use compositions of linear applications defined on the tangent spaces of some suitable manifolds. They will be normed algebras for the induced norm.

Chapter 2

A “basic" λ-lemma and an application to diffusion

In this Chapter, we prove a “basic" λ-lemma for normally hyperbolic manifolds. Given a normally hyperbolic invariant manifold N for a diffeomorphism f satisfying the assumptions in Section 1.4, we consider a submanifold Δ that transversely intersects the stable manifold of N and whose dimension is equal to the dimension of a fiber of the unstable bundle. We prove that under iteration by f, this submanifold is as close as desired (in the C^1 compact open topology) to a suitable unstable leaf. In Section 2.1.1, we will use the straightening neighborhood given in Section 1.1.3 to set out a simplified version of the basic λ-lemma (Theorem 1) and to properly define the notion of C^1-convergence. Then we state the λ-lemma in a more general context (Theorem 2). We devote Sections 2.2 and 2.3 to the proofs of these theorems.

The basic λ-lemma will enable us to prove a diffusion result in Section 2.4. More precisely, we apply the λ-lemma to prove the existence of drifting orbits along a transition chain of invariant minimal sets contained in a normally hyperbolic manifold, without any assumption on the nature of the invariant sets (in particular, they do not need to be submanifolds). As an easy particular case, we recover Arnold's example.

In the sequel, we consider f, M and N as in Section 1.4.

2.1 A basic λ-lemma for normally hyperbolic manifolds

Let Δ be a C^1-submanifold of M of dimension p which transversely intersects $W^s(N)$ at some point a. We will state two versions of the basic λ-lemma.

2.1.1 Theorem 1: in the straightening neighborhood

In this section, we state the basic λ-lemma in the straightening neighborhood. Let us start with fixing the notation. We keep the notation of Proposition B. We will restrict our diffeomorphism φ to the open set $\mathcal{U} := U \cap f^{-1}(U)$, so that $F = \varphi \circ f \circ \varphi^{-1}$ is well defined on $\mathcal{V} := \varphi(\mathcal{U}) \subset V$, with values in V. A point in V will be written as a triple (x, s, u) and F as (F_x, F_s, F_u), according to the splitting $V = N \times B_s^p \times B_u^p$. Up to iterating Δ if necessary (and resetting the counters), we can suppose that $a \in \mathcal{U}$ without loss of generality, since we are interested in the behavior of Δ after a large number of iterations.

We introduce the projection $\Pi_N : \widetilde{W}^s(N) \longrightarrow N$ that sends each $(x, s, 0)$ to $(x, 0, 0)$. Let $P := \varphi(a) = (x, s, 0)$ be the intersection point of $\varphi(\Delta \cap \mathcal{U})$ and $\widetilde{W}^s(N)$. We set $P_0 := \Pi_N(P)$.

For $n \geq 1$, we denote by $P^n = F^n(P)$, and $P_0^n := \Pi_N(P^n) = F^n_{|N}(P_0)$, which is the point in N such that $P^n \in \widetilde{W}^{ss}(P_0^n)$ (see Figure 2.2). We denote by $\widetilde{\Delta}$ the connected component of $\varphi(\Delta \cap \mathcal{U})$ in $\mathcal{V}$ containing P. For all $n \in \mathbb{N}$, we denote by $\widetilde{\Delta}^{n+1}$ the connected component of $F(\widetilde{\Delta}^n) \cap \mathcal{V}$ containing P^n (where $\widetilde{\Delta}^0 = \widetilde{\Delta}$).

Definition 2.1.1. **[The graph property].** *Let Λ be a C^1-submanifold of $N \times \mathbb{R}^p_s \times \mathbb{R}^p_u$. Let B be an open ball in $\mathbb{R}^p_u$. We say that Λ has the* graph property over B, *or equivalently that Λ is a* graph over B, *if there exists a C^1-map $\varpi : B \to N \times \mathbb{R}^p_s$, such that $\Lambda = \{(\varpi(u), u); u \in B\}$.*

For δ small enough, we set $B_\delta := \{u \in B^p_u\,;\, \|u\| < \delta\}$ and $D_\delta := \{(x, s, u) \in \mathcal{V}\,;\, u \in B_\delta\}$. For $n \in \mathbb{N}$, we introduce the constant map

$$\begin{array}{rccc} \ell_n : & B_\delta & \longrightarrow & N \times B^p_s \\ & u & \longmapsto & (P_0^n, 0) \end{array}$$

so that clearly $\widetilde{W}^{uu}(P_0^n) \cap D_\delta$ is the graph of ℓ_n, for all $n \in \mathbb{N}$.

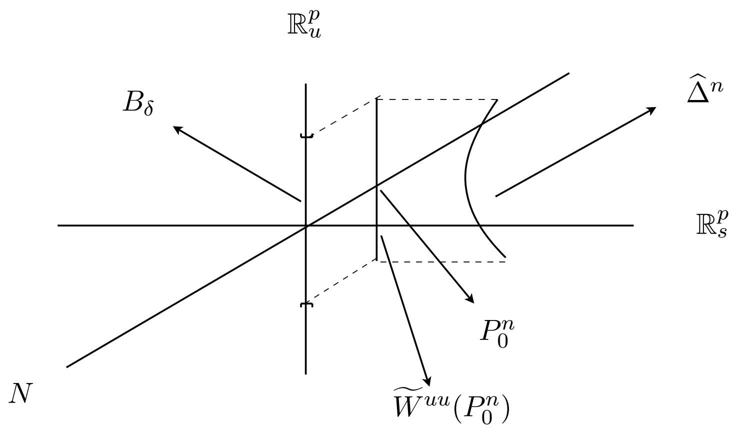

Figure 2.1: Graphs

The basic λ-lemma in V takes the following form.

Theorem 1. *For all $n \in \mathbb{N}$, let $\widetilde{\Delta}^n$ and ℓ_n be as above. Then, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, there exists a C^1-map $\xi_n : B_\delta \to N \times B^p_s$, such that $\widehat{\Delta}^n := \widetilde{\Delta}^n \cap D_\delta$ is the graph of ξ_n. Moreover,*

$$\lim_{n \to \infty} d_{C^1}(\xi_n, \ell_n) = 0,$$

where $d_{C^1}(\xi_n, \ell_n) = \sup_{u \in B_\delta} \big(d(\xi_n(u), \ell_n(u)) + \|\xi_n'(u) - \ell_n'(u)\| \big)$.

We will need 4 steps to prove Theorem 1 in Section 2.2. We will first show how, under iteration, arbitrary tangent vectors in $T_{P_0}\widetilde{\Delta}$ are straightened. We will then use the transversality of $\widetilde{\Delta}$ to $\widetilde{W}^s(N)$ to prove that some suitable part of $\widetilde{\Delta}$ (close to P) is a graph over a ball in $\mathbb{R}^p_u$. In the third step, we will show how this graph property is preserved under iteration *over the same domain in* $\mathbb{R}^p_u$. We will finally prove that tangent vectors along these graphs are straightened and a simple application of the Mean Value Theorem ends the proof of Theorem 1.

We end this section of the chapter with the definition of a notion of “closeness" for graphs which will be useful in the sequel.

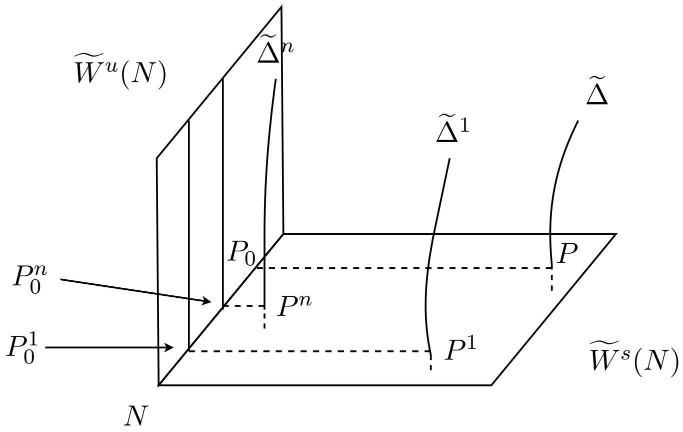

Figure 2.2: Straightening of $\widetilde{\Delta}$

Definition 2.1.2. *We keep the notation of Theorem 1. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that $\widehat{\Delta}^n$ and $\widetilde{W}^{uu}(P_0^n) \cap D_\delta$ are C^1 ε-close if $d_{C^1}(\xi_n, \ell_n) < \varepsilon$.*

2.1.2 Theorem 2: in an arbitrary compact subset of M

In this section, we introduce a new notion of graphs and convergence in the C^1 compact open topology (in a fixed relatively compact set in M).

Definition 2.1.3. *Let $\mathcal{U}$ and $\overline{U}$ be two neighborhoods of N in M such that $\overline{U} \subset \mathcal{U}$. We suppose that there exists a C^2-diffeomorphism $\varphi : \mathcal{U} \longrightarrow N \times \mathbb{R}^p_s \times \mathbb{R}^p_u$. Let $m \in \mathbb{N}$ be fixed. We set $\psi_{(m,\overline{U})} := f^m \circ \varphi^{-1}|_{\varphi(\overline{U})}$. Let Q_1 be a C^1-submanifold of M contained in $W^u(N) \cap f^m(\overline{U})$ and Q_2 be a C^1-submanifold of M contained in $f^m(\overline{U})$. We say that Q_2 is a $(m, \overline{U})$-graph over Q_1 if $\psi^{-1}_{(m,\overline{U})}(Q_2)$ is a graph over $\Pi_3\left(\psi^{-1}_{(m,\overline{U})}(Q_1)\right)$ in the sense of Definition 2.1.1, where Π_3 denotes the projection on the third variable.*

If $\psi^{-1}_{(m,\overline{U})}(Q_2) = \operatorname{graph}\xi = \operatorname{graph}(X, S) = \{(X(u), S(u), u); u \in \Pi_3(\psi_1^{-1}(Q_1))\}$, we define the following distance

$$d_{(C^1,m,\overline{U})}\big(Q_1, Q_2\big) := \sup_{u\in\Pi_3\left(\psi^{-1}_{(m,\overline{U})}(Q_1)\right)} d\left(\psi_{(m,\overline{U})}\left(X(u), S(u), u\right), \psi_{(m,\overline{U})}(X(0), 0, u)\right) +$$

$$\sup_{\substack{u\in\Pi_3(\psi^{-1}_{(m,\overline{U})}(Q_1))\\ v_1\in B_{\mathbb{R}^p_u}}} \left\| D\psi_{(m,\overline{U})}\left(X(u), S(u), u\right).\big(X'(u).v_1, S'(u).v_1, v_1\big) - D\psi_{(m,\overline{U})}\left(X(0), 0, u\right).(0, v_1)\right\|$$

where $B_{\mathbb{R}^p_u}$ is the unit ball in $\mathbb{R}^p_u$.

We now state the global version of the λ-lemma.

Theorem 2. **[Basic λ-lemma]**. *Let f, M and N be as in Section 1.4. Let Δ be a p-dimensional C^1-submanifold transversely intersecting $W^s(N)$ at some point a, and let $\Delta^k = f^k(\Delta)$, for $k \geq 1$. Let a_0 be the point in N such that $a \in W^{ss}(a_0)$ and set $a_0^k := f^k(a_0)$.*

Then, there exist two neighborhoods $\mathcal{U}$ and $\overline{U}$ of N in M satisfying $\overline{U} \subset \mathcal{U}$, and a C^2-diffeomorphism $\varphi : \mathcal{U} \longrightarrow N \times \mathbb{R}^p_s \times \mathbb{R}^p_u$, such that $\forall m \in \mathbb{N}$, $\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}$; $\forall k \geq k_0$, there exists

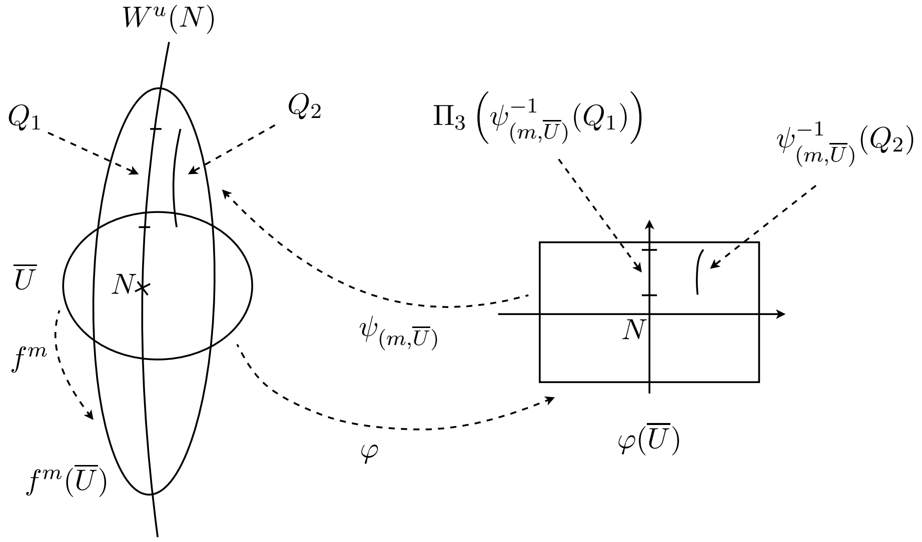

Figure 2.3: The $(m,\overline{U})$-graph property

a C^1-submanifold $\overline{\Delta}^k$ in $f^k(\Delta)\cap f^m(\overline{U})$ such that $\overline{\Delta}^k$ is a $(m,\overline{U})$-graph over $W^{uu}(a_0^k)\cap f^m(\overline{U})$. Moreover,

$$d_{(C^1,m,\overline{U})}\left(\overline{\Delta}^k, W^{uu}(a_0^k)\cap f^m(\overline{U})\right) < \varepsilon.$$

We devote Section 2.3 to the proof of Theorem 2. It will be a direct consequence of the proof of Theorem 1.

Comments. Theorem 2 actually states the straightening property in any relatively compact set K (with a non-empty interior) in M intersecting all the unstable leaves of the submanifold N. More precisely, let K be such a set. The sequence $(f^m(\overline{U})\cap W^u(N))_{m\in\mathbb{N}}$ is clearly an exhaustion of $W^u(N)$ by relatively compact sets. By definition of the unstable manifold, there exists an integer m_0 such that $W^u(N)\cap K\subset f^{m_0}(\overline{U})$. Then, one can easily prove that for all $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that for all $k\geq k_0$, there exists a submanifold $\underline{\Delta}^k$ in $f^k(\Delta)\cap K$ such that $\underline{\Delta}^k$ is a $(m_0,\overline{U})$-graph over $W^{uu}(a_0^k)\cap K$. Moreover,

$$d_{(C^1,m_0,\overline{U})}\left(\underline{\Delta}^k, W^{uu}(a_0^k)\cap K\right) < \varepsilon.$$

Note that the convergence given by the basic λ-lemma is stronger than the Hausdorff one, for Δ and for its tangent space as well.

2.2 Proof of Theorem 1

In this section, we prove the basic λ-lemma. The largest part is dedicated to the proof of Theorem 1. Theorem 2 is a simple corollary which uses the notion of the C^1-distance introduced in Section 2.1.2.

2.2.1 General assumptions for Theorem 1

Here we keep the notation of Proposition B and of Section 2.1.1 and we limit ourselves to the behavior of F in $\mathcal{V}$. Recall that $\mathcal{V}\subset V=N\times B_s^p\times B_u^p$, where $B_{s,u}^p$ is an open ball centered at

0 in $\mathbb{R}^p$. Let $B^p_{s,u}$ be of radius ς.

Since $\widetilde{W}^{s,u}(N)$ are invariant under F, then

$$\forall\ x \in N, \forall\ s \in B^p_s,\ F_u(x,s,0) = 0, \tag{2.1}$$

$$\forall\ x \in N, \forall\ u \in B^p_u,\ F_s(x,0,u) = 0. \tag{2.2}$$

In addition, $\forall\ x \in N,\ F(x,0,0) = (F_x(x,0,0),0,0)$. Since the strongly stable and unstable *foliations* are invariant, then for all $(x,s,u) \in \mathcal{V}$,

$$F_x(x,0,u) = F_x(x,s,0) = F_x(x,0,0). \tag{2.3}$$

Therefore, for $X = (x,0,0) \in N \times \{(0,0)\}$, the derivative $DF(X)$ at X can be represented as a diagonal matrix:

$$DF(X) = \begin{pmatrix} \partial_x F_x(X) & 0 & 0 \\ 0 & \partial_s F_s(X) & 0 \\ 0 & 0 & \partial_u F_u(X) \end{pmatrix}. \tag{2.4}$$

The manifold $N \times \{(0,0)\}$ being normally hyperbolic for F, one can find a real number $\lambda \in]0;1[$ such that $\forall\ x \in N$,

$$\|\partial_s F_s(x,0,0)\| < \lambda,\ \|(\partial_u F_u)^{-1}(x,0,0)\| < \lambda,\ \|\partial_s F_s(x,0,0)\|.\|(\partial_x F_x(x,0,0))^{-1}\| < \lambda$$
$$\text{and } \|\partial_x F_x(x,0,0)\|.\|(\partial_u F_u(x,0,0))^{-1}\| < \lambda.$$

Let $Y = (x,s,0)$ be in $\widetilde{W}^s(N)$. Using Equations (2.1) and (2.3), one easily sees that $DF(Y)$ takes the following form:

$$DF(Y) = \begin{pmatrix} \partial_x F_x(Y) & 0 & \partial_u F_x(Y) \\ \partial_x F_s(Y) & \partial_s F_s(Y) & \partial_u F_s(Y) \\ 0 & 0 & \partial_u F_u(Y) \end{pmatrix}. \tag{2.5}$$

One has an analogous property for the points of $\widetilde{W}^u(N)$.

We need to shrink $\mathcal{V}$ in order to have some estimates useful later on. Note first that $\mathcal{V}$ can be chosen so that $\partial_x F_x(Z)$ and $\partial_u F_u(Z)$ are invertible for all $Z \in \mathcal{V}$.

Let $\overline{\lambda}$ be in $]\lambda;1[$. For simplicity, we choose $\overline{\lambda} = \frac{1+\lambda}{2}$. However, all the calculations in this proof can be adjusted so that they are compatible with any value of $\overline{\lambda} \in]\lambda;1[$. Recall that $B^p_{s,u}$ is of radius ς.

Proposition 2.2.1. *For ς small enough, there exist real positive constants C_1 and C_2 such that for all $Z = (x,s,u) \in \mathcal{V}$, the following inequalities hold true*

1. $\|s\| < \frac{5-5\lambda}{2C_2(11+\lambda)}$,
2. $\|DF(Z)\| \leq C_1$ *and* $\|D^2F(Z)\| \leq C_2$,
3. $\|\partial_s F_s(Z)\| < \overline{\lambda}$ *and* $\|[\partial_u F_u(Z)]^{-1}\| < \overline{\lambda}$,

4. $\|\partial_x F_x(Z)\|.\|[\partial_u F_u(Z)]^{-1}\| < \overline{\lambda}$,

5. $\max\big(\|\partial_s F_x(Z)\|, \|\partial_x F_s(Z)\|\big) < \frac{5-5\lambda}{2(11+\lambda)}$.

Proof. The proof is immediate because F is at least C^2 and $\mathcal{V}$ is relatively compact. Note that the last item is immediate thanks to the form of DF in (2.4). □

2.2.2 Linear straightening of $T_{P^m}\widetilde{\Delta}^m$

The following proposition states the straightening of the tangent space of $\widetilde{\Delta}$ at its base point, under iteration by F.

Proposition 2.2.2. *For all $m \in \mathbb{N}$, the tangent space $T_{P^m}\widetilde{\Delta}^m$ is the graph of a linear map $L_m = (B_m, C_m) : \mathbb{R}^p_u \longrightarrow T_{P_0^m}N \times \mathbb{R}^p_s$, whose norm satisfies:*

$$\lim_{m\to\infty} \|L_m\| = 0.$$

Proof. We start with a quick study of the dynamics in $\widetilde{W}^s(N)$. Recall that $P = (x, s, 0)$ is the intersection point of $\widetilde{\Delta}$ and $\widetilde{W}^s(N)$. Note first that by Proposition 2.2.1, $\|\partial_s F_s(P^i)\| < \overline{\lambda}$, for all $i \geq 0$. For $i \geq 1$, we set $s_i := F_s(P^{i-1})$. Then by the Mean Value Theorem, one gets $\|s_i\| \leq \overline{\lambda}\|s_{i-1}\|$, and thus under iteration $\|s_i\| \leq \overline{\lambda}^i\|s\|$, that tends to 0 with an exponential speed.

• We will see now where the graph property appears. By transversality of $\widetilde{\Delta}$ and $\widetilde{W}^s(N)$, and since $\dim\widetilde{\Delta} = p$, $T_P\widetilde{\Delta}$ is the graph of a linear map defined on $\mathbb{R}^p_u$, with values in $T_{P_0}N \times \mathbb{R}^p_s$. More precisely, recall that $\Pi_3 : N \times B^p_s \times B^p_u \longrightarrow B^p_u$ is the projection over the third variable. By transversality, $D\Pi_{3|_{\widetilde{\Delta}}}(P)$ is an isomorphism between $T_P\widetilde{\Delta}$ and $\mathbb{R}^p_u$. Therefore, there exist two linear maps B and C on $\mathbb{R}^p_u$, such that $T_P\widetilde{\Delta}$ is the image of the map

$$(B, C, I) : \mathbb{R}^p_u \longrightarrow T_{P_0}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u,$$

where $I : \mathbb{R}^p_u \to \mathbb{R}^p_u$ is the identity map.

• Let us now see how the property of $T_P\widetilde{\Delta}$ being a graph of a linear map persists under iteration. We will proceed by induction. However, since the calculations are similar for all the iterates, we will content ourselves with detailing the proof for the first iteration.

The image of $T_P\widetilde{\Delta}$ under $DF(P)$ is $T_{F(P)}F(\widetilde{\Delta})$. For notational convenience, we will identify our linear maps with the matrices below (in the suitable algebras of linear applications) and the partial derivatives with the blocks in the matrices. For instance, $T_{F(P)}F(\widetilde{\Delta})$ is identified with the image of the linear map

$$DF(P).\begin{pmatrix} B \\ C \\ I \end{pmatrix} : \mathbb{R}^p_u \longrightarrow T_{P_0^1}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u.$$

Since P lies in $\widetilde{W}^s(N)$, this is nothing but the image of the following map

$$\begin{pmatrix} \partial_x F_x(P) & 0 & \partial_u F_x(P) \\ \partial_x F_s(P) & \partial_s F_s(P) & \partial_u F_s(P) \\ 0 & 0 & \partial_u F_u(P) \end{pmatrix} \begin{pmatrix} B \\ C \\ I \end{pmatrix} = \begin{pmatrix} \partial_x F_x(P).B + \partial_u F_x(P) \\ \partial_x F_s(P).B + \partial_s F_s(P).C + \partial_u F_s(P) \\ \partial_u F_u(P) \end{pmatrix}.$$

Since $\partial_u F_u(P) : \mathbb{R}^p_u \longrightarrow \mathbb{R}^p_u$ is invertible, $T_{F(P)}F(\widetilde{\Delta})$, that is, $T_{P^1}\widetilde{\Delta}^1$ coincides with the image of

$$\begin{pmatrix} \partial_x F_x(P).B + \partial_u F_x(P) \\ \partial_x F_s(P).B + \partial_s F_s(P).C + \partial_u F_s(P) \\ \partial_u F_u(P) \end{pmatrix} .(\partial_u F_u(P))^{-1} =$$

$$\begin{pmatrix} \partial_x F_x(P).B.(\partial_u F_u(P))^{-1} + \partial_u F_x(P).(\partial_u F_u(P))^{-1} \\ \partial_x F_s(P).B.(\partial_u F_u(P))^{-1} + \partial_s F_s(P).C.(\partial_u F_u(P))^{-1} + \partial_u F_s(P).(\partial_u F_u(P))^{-1} \\ I \end{pmatrix}.$$

This shows that $T_{P^1}\widetilde{\Delta}^1$ is also a graph. It is the image of the linear map

$$(B_1, C_1, I) : \mathbb{R}^p_u \longrightarrow T_{P_0^1}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u,$$

where we have set

$$B_1 = \partial_x F_x(P).B.(\partial_u F_u(P))^{-1} + \partial_u F_x(P).(\partial_u F_u(P))^{-1},$$

and

$$C_1 = \partial_x F_s(P).B.(\partial_u F_u(P))^{-1} + \partial_s F_s(P).C.(\partial_u F_u(P))^{-1} + \partial_u F_s(P).(\partial_u F_u(P))^{-1}.$$

Pursuing the induction, one gets B_i and C_i ($i > 1$), by applying $DF(P^{i-1})$ to $T_{P^{i-1}}\widetilde{\Delta}^{i-1}$ (which is the image of (B_{i-1}, C_{i-1}, I)), and then normalizing by $(\partial_u F_u(P^{i-1}))^{-1}$. We set $b_i = \|B_i\|$ and $c_i = \|C_i\|$, for $i \in \mathbb{N}$, where $B_0 = B$ and $C_0 = C$.

• To end the proof, it is enough now to prove that (b_i) and (c_i) converge to 0. We begin with (b_i). We fix an arbitrary $\varepsilon > 0$. Proposition 2.2.1 yields, for all $i \in \mathbb{N}$,

$$\|\partial_x F_x(P^i)\|.\|(\partial_u F_u(P^i))^{-1}\| < \overline{\lambda},$$

so that, since $\|(\partial_u F_u(P^i))^{-1}\| < 1$,

$$b_{i+1} \quad \leq \quad \|\partial_x F_x(P^i)\|.b_i.\|(\partial_u F_u(P^i))^{-1}\| + \|\partial_u F_x(P^i))\|.\|(\partial_u F_u(P^i))^{-1}\| \leq \overline{\lambda} b_i + \beta_i,$$

where we have set $\beta_i := \|\partial_u F_x(P^i))\|$. Therefore, for $n \in \mathbb{N}^*$,

$$b_n \leq \overline{\lambda}^n b_0 + \sum_{i=0}^{n-1} \overline{\lambda}^i \beta_{n-1-i}.$$

Note that we are not interested in giving the optimal expression for the convergence. Since $\overline{\lambda} < 1$, then for n large enough, $\overline{\lambda}^n b_0 \leq \frac{\varepsilon}{2}$. On the other hand, by the Mean Value Theorem, β_i satisfies:

$$\beta_i \leq C_2 \overline{\lambda}^i \|s\|$$

since $\|\partial_u F_x(P_0^i))\| = 0$. As a consequence of Proposition 2.2.1, it is easy to see that $C_2\|s\| \leq 1$. Therefore

$$\sum_{i=0}^{n-1} \overline{\lambda}^i \beta_{n-1-i} \leq \sum_{i=0}^{n-1} \overline{\lambda}^i \overline{\lambda}^{n-1-i} \leq \sum_{i=0}^{n-1} \overline{\lambda}^{n-1} = n.\overline{\lambda}^{n-1}.$$

Since $\overline{\lambda} < 1$, then for n large enough, $n.\overline{\lambda}^{n-1} \leq \frac{\varepsilon}{2}$. Then, for n large enough, $b_n \leq \varepsilon$.

Note that one can also prove that the series $\sum b_i$ is convergent. This will be needed for the convergence of (c_i).

Let us now study the convergence of the sequence (c_i). For $i \geq 0$,

$$\begin{aligned} c_{i+1} \; & \leq \|\partial_x F_s(P^i)\|.b_i.\|(\partial_u F_u(P^i))^{-1}\| + \|\partial_s F_s(P^i)\|.c_i.\|(\partial_u F_u(P^i))^{-1}\| \\ & +\|\partial_u F_s(P^i)\|.\|(\partial_u F_u(P^i))^{-1}\|. \end{aligned}$$

It is easy to see, using the Mean Value Theorem, that $\|\partial_x F_s(P^i)\| < C_2\|s_i\| < 1$, for all i. As we did for (b_i), we define $\gamma_i := \|\partial_u F_s(P^i)\|$ and get $\gamma_i \leq \overline{\lambda}^i$, following the same steps as for β_i. Therefore,

$$c_{i+1} \leq b_i + \overline{\lambda} c_i + \overline{\lambda}^i,$$

and, for $n \geq 1$,

$$c_n \leq \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} b_i + \overline{\lambda}^n c_0 + \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} \overline{\lambda}^i.$$

Since $\overline{\lambda} < 1$, for n large enough, one gets $\overline{\lambda}^n c_0 \leq \frac{\varepsilon}{3}$. On the other hand, for n large enough, $\sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)}.\overline{\lambda}^i = n\overline{\lambda}^{n-1} \leq \frac{\varepsilon}{3}$. Finally, let $s_{n-1} := \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} b_i$. Observe that s_n is the general term of the Cauchy product of the series of general terms b_i and $\overline{\lambda}^i$ respectively. These series are both convergent, so is their Cauchy product. Then (s_n) converges to 0. More precisely, for n large enough, one has $s_{n-1} \leq \frac{\varepsilon}{3}$. This ends the proof of Proposition 2.2.2. $\square$

2.2.3 The graph property for $\widetilde{\Delta}$

We have seen above that, because of the transversality, $D\Pi_3(P)$ restricted to $T_P\widetilde{\Delta}$ is an isomorphism between $T_P\widetilde{\Delta}$ and $\mathbb{R}^p_u$. Then, by the Inverse Function Theorem, there exist a neighborhood $\mathcal{O}_1$ of P in $\widetilde{\Delta}$ and a neighborhood $\mathcal{O}_2$ of 0 in $\mathbb{R}^p_u$ such that $\Pi_{3|_{\widetilde{\Delta}}}$ is a diffeomorphism from $\mathcal{O}_1$ onto $\mathcal{O}_2$. More precisely, there exists a real number $\widetilde{\delta} > 0$ such that, if we set $B_{\widetilde{\delta}} := \{u \in B^p_u; \|u\| < \widetilde{\delta}\}$ and $D_{\widetilde{\delta}} := \{(x,s,u) \in \mathcal{V}; u \in B_{\widetilde{\delta}}\}$, then there exists a C^1-map $\xi : B_{\widetilde{\delta}} \to N \times B^p_s$, such that $\widetilde{\Delta} \cap D_{\widetilde{\delta}}$ is the graph of ξ (in the sense of Definition 2.1.1). We set $\xi = (X, S)$.

2.2.4 The graph property for the iterates $\widetilde{\Delta}^n$ over a fixed strip

We set $\widetilde{\nu} := \|\xi'\| = \max(\|X'\|, \|S'\|) = \sup_{u \in B_\delta}(\|\xi'(u)\|)$ and $\nu := \max(1, \widetilde{\nu})$. We will see later on why we choose ν (and not just $\widetilde{\nu}$) to bound the norm of all the derivatives of the graph maps. Let us set

$$\varepsilon_\nu = \frac{1-\lambda}{12\nu(1+\lambda)} = \frac{1-\overline{\lambda}}{12\nu\overline{\lambda}}. \tag{2.6}$$

The reason behind this choice will be clarified later on. By uniform continuity, and due to the form of DF on $\widetilde{W^s}(N)$ (Equation (2.5)), there exists $\eta > 0$, such that for all $(x,s,u) \in \mathcal{V}$, if $\|u\| < \eta$, then

$$\|\partial_x F_u(x,s,u)\| < \varepsilon_\nu \;\text{ and }\; \|\partial_s F_u(x,s,u)\| < \varepsilon_\nu. \tag{2.7}$$

We then set

$$\delta := \min\left(1, \widetilde{\delta}, \eta, \frac{1-\overline{\lambda}}{3C_2(2\nu+1)^2}\right). \tag{2.8}$$

Proposition 2.2.3. *Let δ and ν be as above. Then, for all $n \in \mathbb{N}$, there exists a C^1-map $\xi_n : B_\delta \to N \times B_s^p$ such that $\widehat{\Delta}^n := \widetilde{\Delta}^n \cap D_\delta$ is the graph of ξ_n. Moreover, if for all $n \in \mathbb{N}$, $\xi_n = (X_n, S_n)$, then $\|\xi_n'\| := \max(\|X_n'\|, \|S_n'\|) = \sup_{u \in B_\delta}(\|\xi_n'(u)\|)$ satisfies $\|\xi_n'\| \leq \nu$.*

The rest of this subsection is devoted to the proof of Proposition 2.2.3. We will proceed by induction. We first prove these statements for the first iteration, by using intermediate lemmas which will be very useful for the estimates later on. All the computations will be independent of n, which will easily yield the proof of the inductive step.

Note that when $n = 0$, the statement follows from Section 2.2.3 and the definition of ν. Therefore, we have to prove that if for $n \in \mathbb{N}$, $\widehat{\Delta}^n = \operatorname{graph} \xi_n = \{(X_n(u), S_n(u), u); u \in B_\delta\}$ with $\|\xi_n'\| \leq \nu$, then $F(\widehat{\Delta}^n)$ is also the graph of a map ξ_{n+1} over an open set in $\mathbb{R}_u^p$ strictly containing B_δ. We then set

$$\widehat{\Delta}^{n+1} = F(\widehat{\Delta}^n) \cap D_\delta = \widetilde{\Delta}^{n+1} \cap D_\delta = \operatorname{graph} \xi_{n+1} = \{(X_{n+1}(u), S_{n+1}(u), u); u \in B_\delta\}.$$

Note that we will keep the same notation for ξ_{n+1} and its restriction to B_δ. We also have to prove that $\|\xi_{n+1}'\| < \nu$.

To simplify this step and to keep our formulas legible, we will actually prove that Proposition 2.2.3 holds true when $n = 1$. Since all the computations will be independent of n, one can easily see that the statements are valid for an arbitrary n.

By applying F to $\widehat{\Delta} = \{(X(u), S(u), u); u \in B_\delta\}$, one gets

$$F(\widehat{\Delta}) = \big\{\big(F_x(X(u), S(u), u), F_s(X(u), S(u), u), F_u(X(u), S(u), u)\big); u \in B_\delta\big\}.$$

Let $G(u) := F_u(X(u), S(u), u) = h$. We will prove that G is a homeomorphism onto its image B_δ' and that the latter strictly contains B_δ. Then, it is easy to see that $F(\widehat{\Delta})$ restricted to $D_\delta' := \{(x, s, u) \in \mathcal{V}; u \in B_\delta'\}$ is the graph of (X_1, S_1), where

$$X_1(h) = F_x\big(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)\big),$$

and

$$S_1(h) = F_s\big(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)\big),$$

for $h \in B_\delta'$. We will need the following lemmas.

Lemma 2.2.4. *For all $u \in B_\delta$, $G'(u)$ is an isomorphism on $\mathbb{R}_u^p$. Moreover,*

$$[G'(u)]^{-1} = \Big(\sum_{m \geq 0} (-H(u))^m\Big).[\partial_u F_u(X(u), S(u), u)]^{-1},$$

where $H(u) := [\partial_u F_u(X(u), S(u), u)]^{-1}.[\partial_x F_u(X(u), S(u), u).X'(u) + \partial_s F_u(X(u), S(u), u).S'(u)]$.

Proof. $G(u) = F_u(X(u), S(u), u)$ gives by derivation

$$G'(u) = \partial_x F_u(X(u), S(u), u).X'(u) + \partial_s F_u(X(u), S(u), u).S'(u) + \partial_u F_u(X(u), S(u), u).$$

Recall that the linear map $\partial_u F_u(x, s, u)$ is invertible for all $(x, s, u) \in \mathcal{V}$, and satisfies:

$$\|[\partial_u F_u(x, s, u)]^{-1}\| < \overline{\lambda} < 1.$$

Then one can write
$$G'(u) = [\partial_u F_u(X(u), S(u), u)].[I + H(u)],$$
where $H(u) := [\partial_u F_u(X(u), S(u), u)]^{-1}.[\partial_x F_u(X(u), S(u), u).X'(u) + \partial_s F_u(X(u), S(u), u).S'(u)]$. Since $\partial_u F_u(X(u), S(u), u)$ is invertible, it is enough to prove that $I + H(u)$ is invertible too. It is the case if $\|H(u)\| < 1$ because it is an endomorphism of $\mathbb{R}^p_u$. So now we will prove that $\|H(u)\| < 1$. It is easy to see that, by definition of ε_ν (equation (2.6)), for all $u \in B_\delta$,
$$\|H(u)\| < 2\overline{\lambda}\nu\varepsilon_\nu = \frac{1-\overline{\lambda}}{6} < 1.$$
Therefore $I + H(u)$ is invertible on $\mathbb{R}^p_u$ and $[I + H(u)]^{-1} = \sum_{m\geq 0}(-H(u))^m$. This ends the proof of Lemma 2.2.4. $\square$

Lemma 2.2.5. *For all $u \in B_\delta$, $\|[G'(u)]^{-1}\| < 1$.*

Proof. This easily follows from the previous lemma. In fact,
$$\begin{aligned}
\|[G'(u)]^{-1}\| &\leq \| \sum_{m\geq 0} (-H(u))^m \|.\|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\
&\leq \tfrac{1}{1-\|H(u)\|}.\|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\
&< \tfrac{1}{1-2\overline{\lambda}\nu\varepsilon_\nu}.\|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\
&< \tfrac{6}{5+\overline{\lambda}}.\|[\partial_u F_u(X(u), S(u), u)]^{-1}\| < \tfrac{6\overline{\lambda}}{5+\overline{\lambda}} < 1.
\end{aligned}$$
$\square$

We will now prove that G is invertible.

Proposition 2.2.6. *There exists an open set B'_δ in $\mathbb{R}^p_u$ strictly containing B_δ, such that G is a homeomorphism from B_δ onto B'_δ.*

Proof. Without loss of generality, we can assume that ξ is defined on $\overline{B}_\delta$. We introduce an auxiliary map defined on $\overline{B}_\delta$,
$$\chi(u) = [\partial_u F_u(X(0), S(0), 0)]^{-1}.G(u) = [\partial_u F_u(P)]^{-1}.G(u).$$
We will first study the invertibility of χ, from which that of G easily follows. Let y be in a subset of $\mathbb{R}^p_u$ to be specified later on. We are looking for the conditions under which there exists a unique $x \in B_\delta$, such that $y = \chi(x)$. We let $\psi(x) := x - \chi(x) + y$, so that the point y has a unique preimage under χ if and only if ψ has a unique fixed point. To prove this last property, we will need the next lemma.

Lemma 2.2.7. *For all $u \in \overline{B}_\delta$, $\|I - \chi'(u)\| < \frac{1-\overline{\lambda}}{2}$.*

Proof. By derivating $\chi(u) = [\partial_u F_u(P)]^{-1}.G(u)$, one gets
$$\chi'(u) = [\partial_u F_u(P)]^{-1}.[\partial_u F_u(X(u), S(u), u)].[I + H(u)].$$

We set $\mathcal{W} := [\partial_u F_u(P)]^{-1}.[\partial_u F_u(X(u), S(u), u)]$ and $\mathcal{T} := \mathcal{W} - I$ so that $\mathcal{W} = \mathcal{T} + I$ and $\chi'(u) - I = \mathcal{W}.(I + H(u)) - I$. Recall that $\|H(u)\| < 2\overline{\lambda}\nu\varepsilon_\nu = \frac{1-\overline{\lambda}}{6}$ (see the proof of Lemma 2.2.4). Therefore,

$$\begin{aligned}
\|\mathcal{T}\| &= \|[\partial_u F_u(P)]^{-1}.[\partial_u F_u(X(u), S(u), u)] - I\| \\
&= \|[\partial_u F_u(P)]^{-1}.[\partial_u F_u(X(u), S(u), u) - \partial_u F_u(P)]\| \\
&\leq \|[\partial_u F_u(P)]^{-1}\|.\|[\partial_u F_u(X(u), S(u), u) - \partial_u F_u(X(0), S(0), 0)\| \\
&\leq \overline{\lambda}(2C_2\nu + C_2)\|u\|,
\end{aligned}$$

by the Mean Value Theorem. Writing $\chi'(u) - I = (\mathcal{T} + I).(I + H(u)) - I = H(u) + \mathcal{T}.(I + H(u))$ gives

$$\begin{aligned}
\|\chi'(u) - I\| &\leq \|H(u)\| + \|\mathcal{T}\|.(1 + \|H(u)\|) \\
&< 2\overline{\lambda}\nu\varepsilon_\nu + \overline{\lambda}(2C_2\nu + C_2)\|u\|(1 + 2\varepsilon_\nu\overline{\lambda}\nu) \\
&< \tfrac{1-\overline{\lambda}}{6} + C_2(2\nu + 1)^2\|u\|,
\end{aligned}$$

because $\overline{\lambda} < 1$ and $\overline{\lambda}\varepsilon_\nu < 1$ using equation (2.6).

Recall that $\|u\| < \frac{1-\overline{\lambda}}{3C_2(2\nu+1)^2}$ by equation (2.8), which yields

$$\begin{aligned}
\|\chi'(u) - I\| &< \tfrac{1-\overline{\lambda}}{6} + \tfrac{1-\overline{\lambda}}{3} \\
&< \tfrac{1-\overline{\lambda}}{2}.
\end{aligned}$$

This ends the proof of Lemma 2.2.7. □

• We now go back to proving the invertibility of χ. Let $\kappa := \frac{1-\overline{\lambda}}{2}$. Clearly $\kappa < 1$. The last lemma shows that $\psi = I_{B_\delta} - \chi + y$ is a contracting map. In order for it to have a unique fixed point, one needs to have $\psi(\overline{B}_\delta) \subset \overline{B}_\delta$. And this condition is satisfied if $\|y\| \leq \delta(1-\kappa)$. Therefore $\chi : \overline{B}_\delta \longrightarrow \operatorname{Im}\chi$ is bijective and satisfies $\overline{B}_{\delta(1-\kappa)} \subset \operatorname{Im}\chi$.

• The invertibility of G easily follows from that of χ. Recall that $\chi = [\partial_u F_u(P)]^{-1}.G$. Therefore, $G : \overline{B}_\delta \longrightarrow \operatorname{Im} G$ is an homeomorphism and satisfies $B'_\delta := \operatorname{Im} G \supset \overline{B}_{\frac{\delta(1-\kappa)}{\overline{\lambda}}}$.

Recall that $\kappa = \frac{1-\overline{\lambda}}{2}$ which gives $(1-\kappa) > \overline{\lambda}$ and thus B'_δ, which contains $B_{\frac{\delta(1-\kappa)}{\overline{\lambda}}}$, strictly contains B_δ. This ends the proof of the proposition. □

Therefore, the proof of the graph property in Proposition 2.2.3 for the case $n = 1$ is complete. Let $\widehat{\Delta}^1 = F(\widehat{\Delta}) \cap D_\delta = \operatorname{graph}\xi_1 = \{(X_1(u), S_1(u), u); u \in B_\delta\}$. The next proposition will not only end the proof of the case $n = 1$, but will also be a preliminary step to estimating $\lim\limits_{n\to\infty} \|\xi'_n\|$ in Section 2.2.5.

Proposition 2.2.8. *If we set* $\|\xi'_1\| := \sup_{u \in B_\delta}(\|\xi'_1(u)\|) = \max(\|X'_1\|, \|S'_1\|)$, *then* $\|\xi'_1\| < \nu$.

Proof. We recall that for $h \in B'_\delta$,

$$X_1(h) = F_x\left(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)\right),$$

and

$$S_1(h) = F_s\left(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)\right).$$

Since we are only interested in uniform norms over B_δ, we consider h to belong to B_δ from now on. We let $u := G^{-1}(h)$. Then $u \in G^{-1}(B_\delta) \subsetneq B_\delta$. We write

$$X_1(G(u)) = F_x(X(u), S(u), u),$$

and

$$S_1(G(u)) = F_s(X(u), S(u), u).$$

By derivating the two sides with respect to u and inverting $G'(u)$, one gets for all $u \in B_\delta$

$$\begin{aligned} X_1'(G(u)) &= \partial_x F_x(X(u), S(u), u).X'(u).[G'(u)]^{-1} + \partial_s F_x(X(u), S(u), u).S'(u).[G'(u)]^{-1} \\ &+ \partial_u F_x(X(u), S(u), u).[G'(u)]^{-1}, \end{aligned}$$

and

$$\begin{aligned} S_1'(G(u)) &= \partial_x F_s(X(u), S(u), u).X'(u).[G'(u)]^{-1} + \partial_s F_s(X(u), S(u), u).S'(u).[G'(u)]^{-1} \\ &+ \partial_u F_s(X(u), S(u), u).[G'(u)]^{-1}. \end{aligned}$$

Let us begin by studying $T := \|\partial_x F_x(X(u), S(u), u)\|.\|[G'(u)]^{-1}\|$. Using the estimates in Lemma 2.2.5, one gets

$$\begin{aligned} T &\leq \|\partial_x F_x(X(u), S(u), u)\|.\|[I + H(u)]^{-1}\|.\|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\ &< \overline{\lambda}.\|[I + H(u)]^{-1}\| < \tfrac{6\overline{\lambda}}{5+\overline{\lambda}} := \widetilde{\alpha}, \end{aligned}$$

where we can easily see that $0 < \overline{\lambda} < \widetilde{\alpha} < 1$. Recall that by Proposition 2.2.1, for all $u \in B_\delta$,

$$\max\left(\|\partial_s F_x(X(u), S(u), u)\|, \|\partial_x F_s(X(u), S(u), u)\|\right) < \frac{5 - 5\lambda}{2(11 + \lambda)},$$

which yields

$$\|\xi_1'\| < \left(\widetilde{\alpha} + \frac{5 - 5\lambda}{2(11 + \lambda)}\right)\|\xi'\| + \sup_{u \in B_\delta} \max\left(\|\partial_u F_x(X(u), S(u), u)\|, \|\partial_u F_s(X(u), S(u), u)\|\right).$$

On the one hand, $\widetilde{\alpha} = \frac{6\overline{\lambda}}{5+\overline{\lambda}} = \frac{6+6\lambda}{11+\lambda}$, and thus $\widetilde{\alpha} + \frac{5-5\lambda}{2(11+\lambda)} = \frac{1+\widetilde{\alpha}}{2} := \beta$, with $0 < \widetilde{\alpha} < \beta < 1$. On the other hand, using the particular form of F on the unstable manifold (Equations (2.2) and (2.3)), for $X \in \widetilde{W}^u(N)$, the derivative $DF(X)$ at X has the following form

$$DF(X) = \begin{pmatrix} \partial_x F_x(X) & \partial_s F_x(X) & 0 \\ 0 & \partial_s F_s(X) & 0 \\ \partial_x F_u(X) & \partial_s F_u(X) & \partial_u F_u(X) \end{pmatrix}. \tag{2.9}$$

Therefore, using this particular form and the Mean Value Theorem, one gets

$$\sup_{u \in B_\delta} \max\left(\|\partial_u F_x(X(u), S(u), u)\|, \|\partial_u F_s(X(u), S(u), u)\|\right) \leq C_2 \sup_{u \in B_\delta} \|S(u)\|,$$

and so

$$\|\xi_1'\| < \beta\|\xi'\| + C_2 \sup_{u \in B_\delta} \|S(u)\|. \tag{2.10}$$

Using item 1 of Proposition 2.2.1, and the fact that $\nu \geq 1$, one gets

$$\|\xi_1'\| < \left(\beta + \frac{5 - 5\lambda}{2(11 + \lambda)}\right)\nu = \nu.$$

This ends the proof of the lemma. □

Observe that the fact that ν is larger than 1 is crucial to show that $\|\xi_1'\| < \nu$ which explains our initial choice in the beginning of Section 2.2.4. Since all the computations in the previous lemmas are independent of n, the proof of the inductive step easily follows.

We then set $\widehat{\Delta}^n = \widetilde{\Delta}^n \cap D_\delta = \operatorname{graph} \xi_n = \{(X_n(u), S_n(u), u); u \in B_\delta\}$ for all $n \in \mathbb{N}$. This ends the proof of Proposition 2.2.3.

2.2.5 Linear straightening along the graphs

We will now see how tangent vectors along the graphs are straightened. We will use the estimates of the previous section to prove the following proposition.

Recall that $\widehat{\Delta}^n = \operatorname{graph} \xi_n = \operatorname{graph}(X_n, S_n) = \{(X_n(u), S_n(u), u); u \in B_\delta\}$ for all $n \in \mathbb{N}$.

Proposition 2.2.9. *For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $\|\xi_n'\| < \varepsilon$.*

Proof. Generalizing to all the iterates Inequality (2.10), since the estimates are uniform with respect to the order of the iteration, one gets

$$\|\xi_{n+1}'\| < \beta \|\xi_n'\| + C_2 \sup_{u \in B_\delta} \|S_n(u)\|. \tag{2.11}$$

By the Mean Value Theorem, one can prove by induction that $\sup_{u \in B_\delta} \|S_n(u)\| \leq \overline{\lambda}^n \sup_{u \in B_\delta} \|S(u)\|$. More precisely, for all $u \in B_\delta$ and for all $n \in \mathbb{N}^*$, there exists $Z = (Z_1, Z_2, Z_3) \in \widehat{\Delta}^{n-1}$ such that $S_n(u) = F_s(Z) = F_s(Z) - F_s(Z_1, 0, Z_3)$. Therefore $\|S_n(u)\| \leq \overline{\lambda} \|Z_2\| \leq \overline{\lambda} \sup_{u \in B_\delta} \|S_{n-1}(u)\|$, since $\widehat{\Delta}^{n-1} = \operatorname{graph}(X_{n-1}, S_{n-1})$, which proves our claim. Since $C_2 \sup_{u \in B_\delta} \|S(u)\| < 1$, then by Inequality (2.11),

$$\|\xi_{n+1}'\| < \beta \|\xi_n'\| + \overline{\lambda}^n.$$

The proof of the convergence follows the same lines as that of (b_n) in Section 2.2.2, since $\beta < 1$. $\square$

2.2.6 Nonlinear straightening and proof of Theorem 1

We can now end the proof of Theorem 1 by a simple application of the Mean Value Theorem. We get for $n \geq n_0$,

$$\begin{aligned}
\sup_{u \in B_\delta} d\big(\xi_n(u), (P_0^n, 0)\big) &\leq \sup_{u \in B_\delta} d\big(\xi_n(u), \xi_n(0)\big) + d\big(\xi_n(0), (P_0^n, 0)\big) \\
&< \varepsilon + \|S_n(0)\| \\
&< \varepsilon + \overline{\lambda}^n \|S(0)\|,
\end{aligned}$$

where we have used that $\|u\| < 1$. The convergence easily follows. This completes the proof of Theorem 1.

2.3 Proof of Theorem 2

We will now prove Theorem 2 which will be a consequence of Theorem 1. Let φ be the diffeomorphism given by Proposition B and $\mathcal{U}$ be as in Section 2.1.1. Let δ be given by Theorem 1. We set $\overline{U} := \varphi^{-1}(D_\delta)$. Let $m \in \mathbb{N}$ be fixed, then $\psi_{(m,\overline{U})} = f^m \circ \varphi^{-1}{}_{|D_\delta}$. Let $(\widehat{\Delta}^n)$

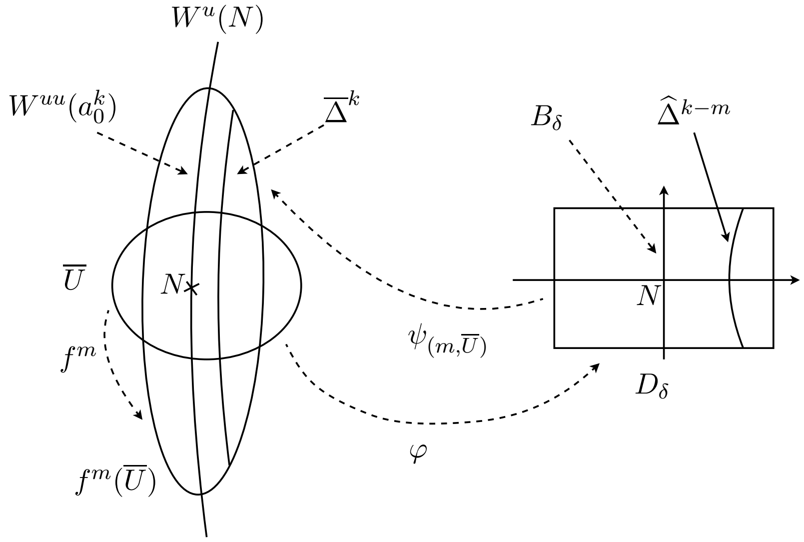

Figure 2.4: In $f^m(\overline{U})$

be as in Theorem 1. For all $k \geq m$, let $\overline{\Delta}^k := \psi_{(m,\overline{U})}(\widehat{\Delta}^{k-m})$. The $(m,\overline{U})$-graph property of $\overline{\Delta}^k$ is an immediate consequence of Theorem 1. As for the convergence, the C^0 part of the convergence is obvious by uniform continuity of $\psi_{(m,\overline{U})}$. It is now enough to prove the convergence of the second term of the $(C^1, m, \overline{U})$-distance. There exist two positive real numbers $\overline{C}$ and $\overline{\overline{C}}$, such that for all $u \in B_\delta$, for all $v_1 \in B_{\mathbb{R}^p_u}$, for all $n \in \mathbb{N}$, if we set $T := \|D\psi_{(m,\overline{U})}(\xi_n(u), u).(\xi_n{}'(u).v_1, v_1) - D\psi_{(m,\overline{U})}(X_n(0), 0, u).(0, v_1)\|$, then

$$\begin{aligned}
T \leq\ & \|D\psi_{(m,\overline{U})}(\xi_n(u), u).(\xi_n{}'(u).v_1, v_1) - D\psi_{(m,\overline{U})}(\xi_n(u), u).(0, v_1)\| \\
& +\|D\psi_{(m,\overline{U})}(\xi_n(u), u).(0, v_1) - D\psi_{(m,\overline{U})}(X_n(0), 0, u).(0, v_1)\| \\
\leq\ & \|D\psi_{(m,\overline{U})}(\xi_n(u), u)\|.\|\xi_n{}'(u)\| + \|D\psi_{(m,\overline{U})}(\xi_n(u), u) - D\psi_{(m,\overline{U})}(X_n(0), 0, u)\| \\
\leq\ & \overline{C}.\|\xi_n{}'(u)\| + \overline{\overline{C}} d\left((\xi_n(u), u), (X_n(0), 0, u)\right)
\end{aligned}$$

by the Mean Value Theorem. The convergence follows from Theorem 1. By setting $n := k - m$, the proof of Theorem 2 is now complete.

2.4 Application to diffusion

We will now use the basic λ-lemma to prove a diffusion result. We will prove the existence of a shadowing orbit for a finite family of invariant dynamically minimal sets, contained in a normally hyperbolic manifold, and having successive heteroclinic connections. We will see that the existence of Arnold's diffusion orbit easily follows from this application.

Note that, in Chapter 5, we will prove a slightly more general result (Proposition 5.5.4) which will be adapted to the setting there, but can easily be adapted to the setting here as well.

2.4.1 Shadowing orbits for a finite family of invariant minimal sets

In this section, we prove a corollary of the basic λ-lemma that gives the existence of a shadowing orbit for a transition chain. Let f, M and N be as in Section 1.4. If A is an invariant *dynamically*

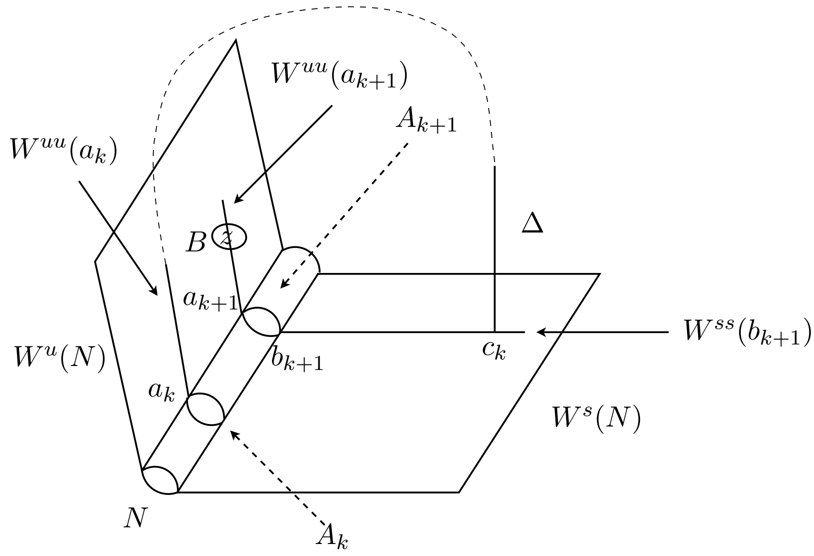

Figure 2.5: Heteroclinic connections

minimal set contained in N, that is, a set in which the orbit of each point is dense, we set

$$W^u(A) := \bigcup_{a \in A} W^{uu}(a).$$

Definition 2.4.1. **[Transition chain].** Let $n \in \mathbb{N}$, $(n > 1)$. If $(A_k)_{1 \le k \le n}$ is a finite family of invariant dynamically minimal sets contained in N, we say that (A_k) is a transition chain if, for all $k = 1, \ldots, n-1$, $W^u(A_k) \cap W^s(A_{k+1}) \neq \emptyset$.

Note that we do not require any regularity for the sets. In the Hamiltonian nearly integrable case, they can be general Aubry-Mather sets for instance.

We will only need the convergence in the C^0 topology stated in the basic λ-lemma to prove the following result. In Figure 2.5, we illustrate the assumptions of Corollary 2.4.2, in the particular case $n_0 = 2$ and $p = 1$. Of course, since the invariant manifolds are 3-dimensional, this is only a rough representation of the situation.

Corollary 2.4.2. *Let f, M and N be as in Section 1.4. Let $(A_k)_{1 \le k \le n}$ be a transition chain in N such that, for all $k = 1, \ldots, n-1$, there exist $a_k \in A_k$, $b_{k+1} \in A_{k+1}$ and $c_k \in W^{uu}(a_k) \cap W^{ss}(b_{k+1})$ such that $W^{uu}(a_k)$ and $W^s(N)$ transversely intersect at c_k. Then,*

$$W^u(A_n) \subset \overline{W^u(A_1)}.$$

Proof. We first prove that $W^u(A_{k+1}) \subset \overline{W^u(A_k)}$ for all $k = 1, \ldots, n-1$. Let z be in $W^u(A_{k+1})$ and let $\eta > 0$ be an arbitrary real number. Let B be the ball in M of radius η and centered at z. We will prove that $B \cap W^u(A_k) \neq \emptyset$.

Since B can be transported into $\mathcal{U}$ by means of the backwards iterates of f, then without loss of generality, we can restrict the problem to the straightening neighborhood $\mathcal{V} \subset V$ (see Proposition B and Proposition 2.2.1).

Since z is in $W^u(A_{k+1})$, there exists a unique $a_{k+1} \in A_{k+1}$ such that $z \in W^{uu}(a_{k+1})$. Let us set $\Delta := W^{uu}(a_k)$, then Δ is an immersed p-dimensional C^3-submanifold of M, transversely intersecting $W^s(N)$. We let $b^m_{k+1} := f^m(b_{k+1})$ and Δ^m be the connected component of $f^m(\Delta)\cap\mathcal{V}$ containing $c^m_k := f^m(c_k)$.

• By the basic λ-lemma, for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m \geq N_1$, Δ^m is ε-close to $W^{uu}(b^m_{k+1})$ (in the sense of Definition 2.1.2).

• Since A_{k+1} is invariant, the sequence $(b^m_{k+1})_{m\in\mathbb{N}}$ lies in A_{k+1}. Since this set is also minimal, we can extract a subsequence $(b^{m_j}_{k+1})_{j\in\mathbb{N}}$ such that $\lim\limits_{j\to\infty} b^{m_j}_{k+1} = a_{k+1}$. More precisely,

$$\forall\ \varepsilon > 0, \exists\ N_2 \in \mathbb{N};\ j \geq N_2 \Rightarrow d(b^{m_j}_{k+1}, a_{k+1}) < \varepsilon.$$

• The foliations being straightened, for j large enough, $W^{uu}(b^{m_j}_{k+1})$ is ε-close to $W^{uu}(a_{k+1})$.

• Recall that $\Delta \subset W^u(A_k)$ which is invariant. There exists then $N_0 \in \mathbb{N}$ such that $\Delta^{m_{N_0}}$ intersects B. Therefore, $W^u(A_{k+1}) \subset \overline{W^u(A_k)}$, for $k = 1, \ldots, n-1$.

By induction, $W^u(A_n) \subset \overline{W^u(A_1)}$. This ends the proof of the corollary. □

Comments. In particular, this corollary proves the existence of an orbit that wanders arbitrarily close to both A_1 and A_n. More precisely, given two arbitrary neighborhoods U_1 and U_n of A_1 and A_n respectively, there exists then an orbit intersecting both U_1 and U_n (take the orbit of any point in $U_n \cap W^u(A_1)$ for instance). In Chapter 5, we will see that we can actually prove the existence of an orbit that wanders arbitrarily close to each A_i (Proposition 5.5.4).

2.4.2 Particular case: Arnold's example

We will see in this section that Arnold's system ([Arn64]) satisfies all the assumptions of Corollary 2.4.2 and thus, one easily deduces the existence of drifting orbits. In Arnold's example, the stable manifold of a torus transversely intersects the unstable manifold of the next torus. These manifolds are Lagrangian and the Lagrangian/Lagrangian intersections will easily yield the isotropic/coisotropic intersections needed in Corollary 2.4.2 ($W^{uu}(a_k)$ and $W^s(N)$ transversely intersecting at c_k). We start with a reminder on Arnold's example and define the objects (F, M, N, the transition chain,...) needed to set up the context of Corollary 2.4.2.

The *autonomous* version of the Hamiltonian used by Arnold is defined on $\mathbb{T}^3 \times \mathbb{R}^3$ and is given by

$$H_{\varepsilon,\mu}(\theta, r) = \frac{1}{2}(r_1^2 + r_2^2) + r_3 + \varepsilon(\cos\theta_1 - 1) + \varepsilon\mu(\cos\theta_1 - 1)(\cos\theta_2 + \sin\theta_3),$$

where $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ and $0 < |\mu| << |\varepsilon| << 1$.

Theorem 2.4.3. [Arnold] Given $A < B$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0; \varepsilon_0[$ there exists μ_0 such that for all $\mu \in]0; \mu_0[$, the system $H_{\varepsilon,\mu}$ admits an orbit whose projection on the action space $\mathbb{R}^3$ intersects the open sets $r_2 < A$ and $r_2 > B$.

The Hamiltonian $H_{\varepsilon,\mu}$ is a perturbation of $H_{0,0} = \frac{1}{2}(r_1^2 + r_2^2) + r_3$, and the parameters ε and μ play asymmetric roles: ε preserves the integrability and creates hyperbolicity, and μ breaks down the integrability and causes instability. More precisely, when $\varepsilon = 0$, $\mathbb{T}^3 \times \mathbb{R}^3$ is foliated by invariant lagrangian tori, and when $\varepsilon > 0$ and $\mu = 0$, the system is equivalent to the uncoupled product of a pendulum ($H_p(\theta_1, r_1) = \frac{1}{2}r_1^2 + \varepsilon(\cos\theta_1 - 1)$) with the completely integrable system $H_r(\theta_2, \theta_3, r_2, r_3) = \frac{1}{2}r_2^2 + r_3$. The resonant surface given by the equation $r_1 = 0$, which is invariant and foliated by invariant tori when $\varepsilon = 0$, is destroyed. It gives rise to a one-parameter family of 2-dimensional invariant tori which are partially hyperbolic, whose union is the normally hyperbolic invariant manifold $N' := \{0,0\} \times \mathbb{T}^2 \times \mathbb{R}^2$. The invariant manifolds of N' are the product of those of the hyperbolic point $(\theta_1 = 0, r_1 = 0)$ with the annulus $\mathbb{T}^2 \times \mathbb{R}^2$. When $|\mu| > 0$, we lose the integrability and the invariant manifolds of the tori do not coincide anymore. The Poincaré-Melnikov integrals show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0; \varepsilon_0[$ there exists μ_0 such that for all $\mu \in]0; \mu_0[$, the invariant manifolds transversely intersect along a homoclinic orbit. Note that Arnold chose the last term of the perturbation in such a way that it vanishes on the invariant tori (because $\theta_1 = 0$), and thus the previous partially hyperbolic tori are preserved when $\mu > 0$, as well as the normally hyperbolic manifold.

It is possible to choose a section $\mathcal{S}$ (see [Mar96]) contained in an energy level $\mathfrak{H}$ and transverse (in $\mathfrak{H}$) to the Hamiltonian flow. The Poincaré map associated to $\mathcal{S}$ and defined in a neighborhood of $N := N' \cap \mathcal{S}$ (which is also normally hyperbolic) in $\mathcal{S}$ will play the role of F (this of course is immediate with the nonautonomous form of the system). Note that $\mathcal{S}$ can be chosen so that the invariant manifolds of N are the intersections of those of N' with $\mathcal{S}$.

Let ω be irrational and let T_ω be the torus in N given by the equation $r_2 = \omega$. It is invariant and minimal (because ω is irrational). Arnold proved the existence of a finite family $(T_{\omega_i})_{i \in I}$ of those tori that have in addition Lagrangian invariant manifolds with transverse heteroclinic connections: $W^u(T_{\omega_i}) \pitchfork W^s(T_{\omega_{i+1}})$.

To get Arnold's orbits, it suffices now to apply Corollary 2.4.2 to the family (T_{ω_i}), since this family is contained in a normally hyperbolic manifold. The Lagrangian/Lagrangian intersection implies the isotropic/coisotropic intersection needed in the corollary. More precisely, for all $i \in I$, let $c_i \in W^u(T_{\omega_i}) \pitchfork W^s(T_{\omega_{i+1}})$. We set a_i the point in T_{ω_i} such that $c_i \in W^{uu}(a_i)$. It is easy to see that $W^{uu}(a_i)$ transversely intersects $W^s(N)$ at c_i. One gets then a transition chain as in Corollary 2.4.2.

Comments. The existence of the drifting orbits is therefore a simple application of the λ-lemma while in [Arn64], the author used an obstruction argument for each torus. The obstruction argument is a geometrical way to say that $W^u(T_{\omega_{i+1}}) \subset \overline{W^u(T_{\omega_i})}$. This argument was not completely proved in Arnold's paper. The first proof was given in ([Mar96]) by means of a partially hyperbolic λ-lemma, which yields the straightening property for arcs lying in some suitable "vertical cone". The proof was improved later on in [FM00]. Our λ-lemma holds for normally hyperbolic manifolds instead of partially hyperbolic tori and does not require any "verticality" assumption for the transversal arcs, which enables us to significatively simplify the previous proofs for transition chains of tori contained in a normally hyperbolic manifold, and apply to more general situations as well.

Chapter 3

A “fibered" λ-lemma and an application to diffusion

In this chapter, we state and prove a “fibered" λ-lemma for normally hyperbolic controllable manifolds, which is a generalization of the basic λ-lemma to higher dimensional transverse submanifolds. We consider f, M and N as in Section 1.4. The basic λ-lemma is valid for submanifolds Δ of dimension p (the dimension of the unstable leaves) transverse to the stable manifold of N. We generalize this result to submanifolds of dimension $p+r$, with $0 \leq r \leq n_0$ (recall that $p+n_0$ is the dimension of the unstable manifold of N).

To prove the basic λ-lemma in Chapter 2, we used that Δ and $W^s(N)$ transversely intersect *at a point* and we introduced the *graph property* (Definition 2.1.1). The notion of C^1-convergence was based on the persistence under iteration of this property *over a fixed strip*. Here, because of dimensional reasons, we are not able to adapt the same method to the general context. Therefore, new techniques are needed. We will define a new notion of C^1-convergence by means of diffeomorphisms and tangent vectors (Theorem 3 below).

In Section 3.3, we apply the fibered λ-lemma to prove the transitivity of transversal heteroclinic connections for systems under additional assumptions.

3.1 A fibered λ-lemma for normally hyperbolic manifolds

Since we are only interested in the behavior of the manifolds (transversely intersecting $W^s(N)$) after a large number of iterations, we will restrict our study to the straightening neighborhood defined in Proposition B (Section 1.1.3), but it would be possible to give abstract definitions as well.

Let us start with fixing some general assumptions on the neighborhood in which the fibered λ-lemma (Theorem 3) will be stated. We keep the notation of Sections 2.1.1 and 2.2.1. Let $\mathcal{V}$ be as in Section 2.1.1 and Proposition 2.2.1. We set

$$C := \max(C_1, \sup_{Z\in\mathcal{V}} \|[\partial_x F_x(Z)]^{-1}\|). \tag{3.1}$$

We need additional assumptions in the neighborhood of N to state Theorem 3.

Proposition 3.1.1. *There exists a neighborhood $\widehat{V}$ of N in $\mathcal{V}$ of the form $N \times \mathcal{B}_s^p \times \mathcal{B}_u^p$ (where $\mathcal{B}_{s,u}^p$ are open balls in $\mathbb{R}_{s,u}^p$ centered at 0) such for all $(x,s,u) \in \widehat{V}$, $\|s\| < s_{\max}$, where*

$s_{\max} = \frac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2CC_2(1+\overline{\lambda})}$. *Moreover,*

$$\text{for all } (Z,Z)' \in \widehat{V}^2,\ \|\partial_s F_s(Z)\|.\|[\partial_x F_x(Z')]^{-1}\| < \overline{\lambda}.$$

Proof. The proof of the last estimate is immediate by uniform continuity. □

Note that one gets $-2\overline{\lambda}^2 + 3\overline{\lambda} - 1 = -(2\overline{\lambda} - 1)(\overline{\lambda} - 1) > 0$ by definition of $\overline{\lambda}$.

As we did in Chapter 2, for ϱ small enough, we set

$$B_\varrho := \{u \in B_u^p\,;\ \|u\| < \varrho\} \text{ and } D_\varrho := \{(x,s,u) \in \widehat{V}\,;\ u \in B_\varrho\}.$$

We now state an easy lemma that will give the existence of an extension of Γ to a suitably fibered submanifold Γ^0, which will enable us to state and prove Theorem 3.

Lemma 3.1.2. *Let Γ be a C^2-submanifold in $\widehat{V} \subset N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$ of dimension $p+r$, with $0 \le r \le n_0$, which transversely intersects $\widetilde{W}^s(N)$ along γ'. When $r > 0$, we suppose that γ' is a relatively compact submanifold and that there exist a submanifold $\widehat{\gamma}'$ of N and a C^1-map $T' : \widehat{\gamma}' \to \mathcal{B}_s^p$ such that $\gamma' = \{(x, T'(x), 0); x \in \widehat{\gamma}'\}$. Then, there exist $\varrho' > 0$ and a relatively compact submanifold $\gamma \supseteq \gamma'$ in $\widehat{V}$ such that:*

1. *there exist a submanifold $\widehat{\gamma} \supseteq \widehat{\gamma}'$ of N and a C^1-map $T : \widehat{\gamma} \to \mathcal{B}_s^p$ such that $T_{|\widehat{\gamma}'} = T'$ and $\gamma = \{(x, T(x), 0); x \in \widehat{\gamma}\}$,*
2. *there exists a C^2-extension Γ^0 of the connected component of $\Gamma \cap D_{\varrho'}$ containing γ' such that $\Gamma^0 \pitchfork \widetilde{W}^s(N)$ along γ and $\Gamma \cap D_{\varrho'} \subset \Gamma^0 \subset \widehat{\gamma} \times \mathbb{R}_s^p \times B_{\varrho'}$. In addition, there exists a C^2-map $\widehat{S}$ defined on $\widehat{\gamma} \times B_{\varrho'}$ with values in $\mathcal{B}_s^p$ such that Γ^0 is the image of the following map:*

$$\begin{array}{rccc} \psi^0 : & \widehat{\gamma} \times B_{\varrho'} & \longrightarrow & D_{\varrho'} \\ & (x,u) & \longmapsto & (x, \widehat{S}(x,u), u), \end{array}$$

that is, Γ^0 is the graph of $\widehat{S}$.

We prove Lemma 3.1.2 in Section 3.2.1. Let Γ^0 be as in Lemma 3.1.2. For all $n \in \mathbb{N}$, we denote by Γ^{n+1} the connected component of $F(\Gamma^n) \cap \widehat{V}$ that contains $\gamma^{n+1} := F^{n+1}(\gamma)$, and we set $\widehat{\gamma}^{n+1} := F_{|N}^{n+1}(\widehat{\gamma})$. In the sequel, we identify γ^0 with γ and $\widehat{\gamma}^0$ with $\widehat{\gamma}$. The fibered λ-lemma states the following.

Theorem 3. **[Fibered λ-lemma]**. *Let Γ be a C^2-submanifold in $\widehat{V} \subset N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$ of dimension $p+r$, with $0 \le r \le n_0$, which transversely intersects $\widetilde{W}^s(N)$ along γ'. When $r > 0$, we suppose that γ' is a relatively compact submanifold and that there exist a submanifold $\widehat{\gamma}'$ of N and a C^1-map $T' : \widehat{\gamma}' \to \mathcal{B}_s^p$ such that $\gamma' = \{(x, T'(x), 0); x \in \widehat{\gamma}'\}$. Then, there exist $\varrho > 0$ and two extended submanifolds $\Gamma^0 \supseteq \Gamma \cap D_\varrho$ and $\widehat{\gamma} \supseteq \widehat{\gamma}'$ such that:*

1. C^0-convergence. *For all $n \in \mathbb{N}$, there exists a C^2-diffeomorphism ψ^n from $\widehat{\gamma}^n \times B_\varrho$ onto $\Gamma^n \cap D_\varrho$. Moreover, if we keep the notation above and denote by Υ^n, for all $n \in \mathbb{N}$, the following diffeomorphism:*

$$\begin{array}{rccc} \Upsilon^n : & \widehat{\gamma}^n \times B_\varrho & \longrightarrow & \widetilde{W}^u(\widehat{\gamma}^n) \cap D_\varrho \\ & (x,u) & \longmapsto & (x,0,u) \end{array}$$

where we have set $\widetilde{W}^u(\widehat{\gamma}^n) := \bigcup_{x \in \widehat{\gamma}^n} \widetilde{W}^{uu}(x)$, we have the following convergence

$$\lim_{n\to\infty} d_{C^0}(\psi^n, \Upsilon^n) = 0,$$

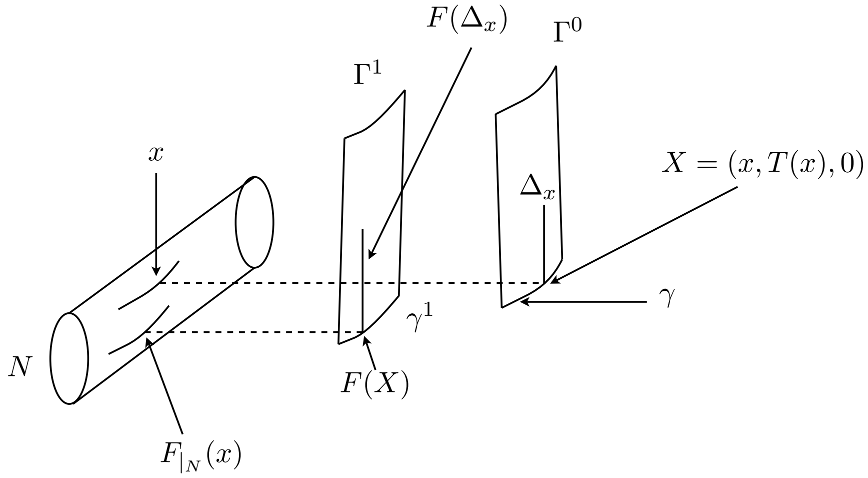

Figure 3.1: Iteration of Γ^0 and its intersection

where $d_{C^0}(\psi^n, \Upsilon^n) = \sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} d\big(\psi^n(x,u), \Upsilon^n(x,u)\big)$,

2. C^1**-convergence**. $\forall\,\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ *such that* $\forall\, n \geq n_0$, $\forall\, q \in \Gamma^n \cap D_\varrho$, *for all* unit vector $Z = (Z_x, Z_s, Z_u)$ *tangent to* $\Gamma^n \cap D_\varrho$ *at* q, $\|Z_s\| < \varepsilon$,

3. Fibration. *When* $r > 0$, Γ^0 *is fibered with p-dimensional submanifolds transverse to* $\widetilde{W}^s(N)$ *that are straightened in the sense of the basic* λ*-lemma.*

Note that ψ^0 in Theorem 3 will coincide with that defined in Lemma 3.1.2. We prove Theorem 3 in Section 3.2.3.

When F satisfies additional assumptions, we get a version of the fibered λ-lemma that gives "complete" C^1-convergence.

Corollary 3.1.3. *For all* $n \in \mathbb{N}$, *let* Γ^n, ψ^n *and* Υ^n *be as in Theorem 3. We suppose that* F *satisfies the additional assumption*

$$\forall\ (x, s, u) \in \widehat{V},\ F_x(x, s, u) = g(x),$$

where $g = F_{|N}$. *Then,*

$$\lim_{n\to\infty} d_{C^1}(\psi^n, \Upsilon^n) = 0,$$

where $d_{C^1}(\psi^n, \Upsilon^n) = d_{C^0}(\psi^n, \Upsilon^n) + \sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} \|D\psi^n(x,u) - D\Upsilon^n(x,u)\|$. *More precisely, there exists* $\varrho > 0$ *such that for all* $n \in \mathbb{N}$, *there exits a* C^2*-map* $\widehat{S}^n$ *defined on* $\widehat{\gamma}^n \times B_\varrho$ *with values in* $\mathcal{B}_s^p$ *such that* $\Gamma^n \cap D_\varrho$ *is the image of the following diffeomorphism:*

$$\begin{array}{cccc} \psi^n: & \widehat{\gamma}^n \times B_\varrho & \longrightarrow & D_\varrho \\ & (x,u) & \longmapsto & (x, \widehat{S}^n(x,u), u), \end{array}$$

that is, $\Gamma^n \cap D_\varrho$ *is the graph of* $\widehat{S}^n$ *and moreover*

$$\lim_{n\to\infty} \|\widehat{S}^n\|_{C^1(\widehat{\gamma}^n\times B_\varrho)} = 0.$$

This means that for n large enough, and in the strip D_ϱ, Γ^n is as close as desired in the C^1-topology to $\widetilde{W}^u(\widehat{\gamma}^n)$. We prove Corollary 3.1.3 in Section 3.2.4. We end this section with a few remarks.

Remark 3.1.4. 1. *Recall that in Chapter 2 we set* $\overline{\lambda} = \frac{1+\lambda}{2}$. *For the proof of the basic* λ*-lemma, we could have chosen any other value of* $\overline{\lambda}$ *in* $]\lambda; 1[$ *without altering the result. It is not the case here. To prove the fibered* λ*-lemma, we need* $\overline{\lambda}$ *to be larger than* $\frac{1}{2}$. *Therefore, the computations in this chapter can be adjusted to go with any value of* $\overline{\lambda}$ *in* $]\max(\frac{1}{2}, \lambda); 1[$.

2. *In Theorem 3, we supposed that* Γ *transversely intersects* $\widetilde{W}^s(N)$ *along* γ' *which is a graph over a submanifold in* N. *The assumption* $\gamma' \subset \widehat{V}$ *is no restriction to us since the persistence of the graph property in* $\widetilde{W}^s(N)$ *depends only on the straightening of the leaves as we will prove in Section 3.2.2.*

3. *In Section 3.3 and in Chapter 4, we will need a weaker version of Theorem 3. More precisely, we will apply the fibered* λ*-lemma to the unstable manifolds of suitable tori in a transition chain. We will need to "straighten" submanifolds* contained in the unstable manifolds. *Therefore, we will not need to extend* $\Gamma \cap D_{\varrho'}$ *to* Γ^0, *instead it will suffice to take* $\Gamma^0 \subset \Gamma$ *with the same properties. The existence of such inclusion is immediate for the same reasons.*

3.2 Proof of the fibered λ-lemma

This section is devoted to the proof of the fibered λ-lemma. We first start with the proof of Lemma 3.1.2. Then, we briefly study the dynamics on $\widetilde{W}^s(N)$. The biggest part of Section 3.2 is dedicated to the proof of Theorem 3. We end this section with the proof of Corollary 3.1.3. Note that since Γ transversely intersects $\widetilde{W}^s(N)$ along γ', then the dimension of γ' is r.

3.2.1 Proof of Lemma 3.1.2

Here, we prove Lemma 3.1.2. The extension of γ' to γ, of $\widehat{\gamma}'$ to $\widehat{\gamma}$ and of Γ in the vicinity of γ' to a submanifold in $\widehat{\gamma} \times \mathbb{R}^p_s \times \mathbb{R}^p_u$ is a consequence of the Whitney continuation theorem. We will not detail this technical (and elementary) construction here. Moreover, one needs to take a real number ρ small enough so that for all $x \in \widehat{\gamma}$, the submanifold $\{x\} \times \mathbb{R}^p_s \times \mathbb{R}^p_u$ is transverse to the extended manifold intersected with D_ρ in $\widehat{\gamma} \times \mathbb{R}^p_s \times \mathbb{R}^p_u$. The existence of ρ is a consequence of the transversality in $\widetilde{W}^s(N)$. More precisely, since γ is a graph over $\widehat{\gamma}$, then, for all $x \in \widehat{\gamma}$, $\{x\} \times \mathbb{R}^p_s \times \mathbb{R}^p_u$ is transverse to γ. Therefore, the result easily follows since transversality is an open property. We denote by Γ^0_ρ the extension of the connected component of $\Gamma \cap D_\rho$ containing γ'.

For $x \in \widehat{\gamma}$, we let $\Delta_x := \Gamma^0_\rho \cap (\{x\} \times \mathbb{R}^p_s \times \mathbb{R}^p_u)$ so that $\Gamma^0_\rho = \cup_{x \in \widehat{\gamma}} \Delta_x$. This is a disjoint union and it follows from above that each Δ_x is a C^2-submanifold of $N \times \mathbb{R}^p_s \times \mathbb{R}^p_u$ of dimension p and transverse to $\widetilde{W}^s(N)$. Applying the same reasoning as in Section 2.2.3 to each Δ_x, one gets that for all $x \in \widehat{\gamma}$, there exists $\alpha_x \in]0; \rho]$ and a C^2-map $\xi_x = (X_x, S_x) : B_{\alpha_x} \to N \times \mathcal{B}^p_s$ such that $\Delta_x \cap D_{\alpha_x} = \{(X_x(u), S_x(u), u); u \in B_{\alpha_x}\}$, that is, $\Delta_x \cap D_{\alpha_x}$ is the graph of ξ_x. The submanifold $\widehat{\gamma}$ being relatively compact, there exists $\varrho' > 0$ such that for all $x \in \widehat{\gamma}$,

$$\Delta_x \cap D_{\varrho'} = \text{graph}\,(\xi_x) = \{(X_x(u), S_x(u), u); u \in B_{\varrho'}\},$$

where we kept the notation ξ_x for $\xi_{x|_{B_{\varrho'}}}$. On the other hand, $\Delta_x \subset (\{x\} \times \mathbb{R}^p_s \times \mathbb{R}^p_u)$ and thus

$$\Delta_x \cap D_{\varrho'} = \mathrm{graph}(\xi_x) = \{(X_x(u), S_x(u), u); u \in B_{\varrho'}\} = \{(x, S_x(u), u); u \in B_{\varrho'}\}.$$

We set $\widehat{S}$ the map defined on $\widehat{\gamma} \times B_{\varrho'}$ such that, for all $(x,u) \in \widehat{\gamma} \times B_{\varrho'}$, $\widehat{S}(x,u) = S_x(u)$. We set $\Gamma^0 := \Gamma^0_\rho \cap D_{\varrho'}$. It follows then that Γ^0 is the graph of $\widehat{S}$, since $\Gamma^0 = \cup_{x\in\widehat{\gamma}} \Delta_x \cap D_{\varrho'}$. This ends the proof of Lemma 3.1.2.

3.2.2 Straightening inside $\widetilde{W}^s(N)$

Let $\widehat{\gamma}$ be given by Lemma 3.1.2. Recall that $g := F_{|N}$ and that for all $n \in \mathbb{N}$, we set $\widehat{\gamma}^n := g^n(\widehat{\gamma})$. The dynamics on $\widetilde{W}^s(N)$ are described in the following proposition.

Proposition 3.2.1. *For all $n \in \mathbb{N}$, there exists a C^1-map T_n defined on $\widehat{\gamma}^n$ such that γ^n is the graph of T_n, in the sense that $\gamma^n = \{(x, T_n(x), 0); x \in \widehat{\gamma}^n\}$. Moreover,*

$$\lim_{n\to\infty} \|T_n\|_{C^1(\widehat{\gamma}^n)} = 0.$$

Proof. We start with proving the existence of (T_n). Then we show the convergence in the C^0-topology. We end this proof with the C^0-convergence of the derivatives.

• The persistence of the graph property is an immediate consequence of the straightening of the stable manifold and the stable leaves. We proceed by induction. The case $n = 0$ is given by Lemma 3.1.2. We suppose now that for a fixed $n \in \mathbb{N}^*$, $\gamma^{n-1} = \{(x, T_{n-1}(x), 0); x \in \widehat{\gamma}^{n-1}\}$. Then

$$\begin{aligned} \gamma^n = \ & F(\gamma^{n-1}) \\ = \ & \left\{\left(F_x(x, T_{n-1}(x), 0), F_s(x, T_{n-1}(x), 0), F_u(x, T_{n-1}(x), 0)\right);\ x \in \widehat{\gamma}^{n-1}\right\} \\ = \ & \left\{\left(F_x(x, 0, 0), F_s(x, T_{n-1}(x), 0), 0\right);\ x \in \widehat{\gamma}^{n-1}\right\}, \end{aligned}$$

thanks to equations (2.1) and (2.3). Since g is a diffeomorphism on N, one gets

$$\begin{aligned} \gamma^n = \ & \left\{\left(y, F_s(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0), 0\right);\ y \in g(\widehat{\gamma}^{n-1})\right\} \\ = \ & \left\{\left(y, T_n(y), 0\right);\ y \in \widehat{\gamma}^n\right\}, \end{aligned}$$

where $T_n(y) = F_s(g^{-1}(y), T_{n-1}(g^{-1}(y), 0)$. This yields that γ^n is a graph of a C^1-map T_n over $\widehat{\gamma}^n$.

• We now prove the C^0-convergence. We show that $\|T_n\|_{C^0} < \overline{\lambda}\|T_{n-1}\|_{C^0}$ for all $n \in \mathbb{N}^*$. Let $y \in \widehat{\gamma}^n$, then if $x = g^{-1}(y)$, $T_n(y) = F_s(x, T_{n-1}(x), 0)$. Therefore,

$$\begin{aligned} \|T_n(y)\| = \ & \|F_s(x, T_{n-1}(x), 0)\| \\ = \ & \|F_s(x, T_{n-1}(x), 0) - F_s(x, 0, 0)\| \\ \leq \ & \overline{\lambda}\|T_{n-1}(x)\| \\ \leq \ & \overline{\lambda}\|T_{n-1}\|_{C^0}, \end{aligned}$$

using the Mean Value Theorem. Therefore, one gets $\|T_n\|_{C^0} < \overline{\lambda}\|T_{n-1}\|_{C^0}$ which yields

$$\|T_n\|_{C^0} < \overline{\lambda}^n \|T\|_{C^0}$$

and the convergence easily follows.

• We now check the convergence of the derivatives. Let y be in $\widehat{\gamma}^n$. By derivating the equality $T_n(y) = F_s(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0)$, one gets

$$\begin{aligned} T_n'(y) = \;& \partial_x F_s\left(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0\right).(g^{-1})'(y) \\ & + \partial_s F_s\left(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0\right).T_{n-1}'(g^{-1}(y)).(g^{-1})'(y). \end{aligned}$$

On the one hand, using the Mean Value Theorem,

$$\begin{aligned} \|\partial_x F_s(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0)\| &= \|\partial_x F_s(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0) - \partial_x F_s(g^{-1}(y), 0, 0)\| \\ &\leq C_2 \|T_{n-1}(g^{-1}(y))\| \\ &\leq C_2 \|T_{n-1}\|_{C^0} \\ &\leq C_2 \overline{\lambda}^{n-1} \|T\|_{C^0}. \end{aligned}$$

On the other hand, thanks to Proposition 3.1.1, $\|\partial_s F_s(g^{-1}(y), T_{n-1}(g^{-1}(y)), 0)\|.\|(g^{-1})'(y)\| < \overline{\lambda}$, for all $y \in \widehat{\gamma}^n$. Therefore, for all $y \in \widehat{\gamma}^n$, $\|T_n'(y)\| < CC_2\|T\|_{C^0}.\overline{\lambda}^{n-1} + \overline{\lambda}\|T_{n-1}'\|$, where C is given by equation (3.1). Proposition 3.1.1 also implies that $CC_2\|T\|_{C^0} < 1$ since $\|T\|_{C^0} < s_{\max}$, which yields

$$\|T_n'\| < \overline{\lambda}^{n-1} + \overline{\lambda}\|T_{n-1}'\|.$$

The proof of the convergence follows the same lines as that of the sequence (b_n) in the basic λ-lemma (Section 2.2.2). $\square$

3.2.3 Proof of Theorem 3

This section is devoted to the proof of Theorem 3. When $r = 0$, the proof is immediate because one recovers a weaker version of the basic λ-lemma (Theorem 1). In the sequel, we suppose that $r > 0$.

To prove Theorem 3, we will start with stating and proving an auxiliary proposition (Proposition 3.2.2 below). Theorem 3 will easily follow from the proof of Proposition 3.2.2. Let Γ^0, γ, $\widehat{\gamma}$ and ϱ' be as in Lemma 3.1.2. Recall that, for all $n \in \mathbb{N}$, we denote by Γ^{n+1} the connected component of $F(\Gamma^n) \cap \widehat{V}$ that contains $\gamma^{n+1} := F^{n+1}(\gamma)$, and that $\widehat{\gamma}^{n+1} := g^{n+1}(\widehat{\gamma})$, where $g = F_{|N}$.

3.2.3.1 Auxiliary proposition

In this section, we state Proposition 3.2.2 where we claim the existence of the sequence of parametrization diffeomorphisms (ψ^n). In addition, Proposition 3.2.2 will yield intermediate convergence results needed to prove the convergences stated in Theorem 3. We will prove Proposition 3.2.2 in Section 3.2.3.2.

Proposition 3.2.2. *There exists $\varrho > 0$ such that for all $n \in \mathbb{N}$, there exist two C^2-maps $\widehat{X}^n$ and $\widehat{S}^n$ defined on $\widehat{\gamma}^n \times B_\varrho$ with values in N and $\mathbb{R}^p_s$ respectively, such that*

i) $\forall\ (x, u) \in \widehat{\gamma}^n \times B_\varrho$, $\partial_x \widehat{X}^n(x, u)$ is injective on $T_x\widehat{\gamma}^n$,

ii) $\psi^n := (\widehat{X}^n, \widehat{S}^n, I)$ is a C^2-diffeomorphism from $\widehat{\gamma}^n \times B_\varrho$ onto $\Gamma^n \cap D_\varrho$.

Moreover,

iii) $\exists\ \nu \geq 1;\ \forall\ n \in \mathbb{N},\ \sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} \|\partial_u \widehat{X}^n(x,u)\| \leq \nu,$

iv) $\lim_{n\to\infty} \sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} \left\|\partial_u \widehat{S}^n(x,u)\right\| = 0,$

v) $\lim_{n\to\infty} \sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} \left\|\partial_x \widehat{S}^n(x,u).\left[\partial_x \widehat{X}^n(x,u)\right]^{-1}\right\| = 0.$

3.2.3.2 Proof of the auxiliary proposition (Proposition 3.2.2)

To prove Proposition 3.2.2, we use Theorem 1. We start with proving the existence of the parametrization maps (ψ^n) and the main idea is the following: we construct a fibration of Γ^0 with submanifolds of dimension p transverse to $\widetilde{W}^s(N)$ (mimicking our Δ in the basic λ-lemma). Then, we fiber Γ^n, for all $n \in \mathbb{N}$, with the images under F^n of the initial fibers. Before defining ϱ, we will introduce a real number $\sigma > 0$ such that in $D_\sigma := \{(x,s,u) \in \widehat{V}; \|u\| < \sigma\}$, all the fibers and all their iterates are graphs over $B_\sigma := \{u \in \mathbb{R}^p_u; \|u\| < \sigma\}$. Using these graphs, we introduce a parametrization of Γ^n, for all $n \in \mathbb{N}$, by a "vertical" parameter and a "horizontal" one. Theorem 1 will ensure the uniform convergence of the C^1-norms of the graph maps and thus the straightening in the "vertical" direction. Then we define $\varrho \in]0;\sigma]$ and (ψ^n). We prove that (ψ^n) is a sequence of diffeomorphisms and that in D_ϱ we have the convergence of item v).

3.2.3.2.1 Fibering Γ and its iterates. In this Section, we construct a fibration of Γ^n, for all $n \in \mathbb{N}$, with submanifolds of dimension p transverse to $\widetilde{W}^s(N)$. When $n = 0$, the fibration of Γ^0 was constructed in the proof of Lemma 3.1.2 in Section 3.2.1, where we defined ϱ' and wrote $\Gamma^0 = \cup_{x\in\widehat{\gamma}}\Delta_x \cap D_{\varrho'}$. We then consider the fibration of Γ^n, for all $n \in \mathbb{N}^*$, by the connected components of the images under F of the fibers of Γ^{n-1} intersected with $\widehat{V}$ containing the corresponding points in γ^n. These fibrations have the same properties as that of Γ^0 and will help us parameterize Γ^n, in order to highlight the graph property and thus the straightening in the "vertical" direction.

3.2.3.2.2 Definition of σ and proof of item iii). Recall that $\gamma = \{(x,T(x),0); x \in \widehat{\gamma}\}$. For all $x \in \widehat{\gamma}$, Δ_x passes through $(x,T(x),0)$. We index the fibers of Γ^0 with the variable x in $\widehat{\gamma}$, and the fibers of the iterates with $g^n(x)$. More precisely, for $\sigma > 0$ small enough, we set $\Delta^0_{x,\sigma} := \Delta_x \cap D_\sigma$. For all $n \in \mathbb{N}$, we denote by $\Delta^{n+1}_{g^{n+1}(x),\sigma}$ the connected component of $F(\Delta^n_{g^n(x),\sigma}) \cap D_\sigma$ containing $F^{n+1}(x,T(x),0)$. We have the following proposition that states that all the fibers of all the iterates are graphs *over a fixed domain.*

Proposition 3.2.3. *There exists a real number $\sigma > 0$ such that for all $x \in \widehat{\gamma}$, for all $n \in \mathbb{N}$, $\Delta^n_{g^n(x),\sigma}$ is the graph over B_σ of a C^2-map, that we will denote by $\xi^n_{g^n(x)}$. Moreover, there exists $\nu \geq 1$ such that $\forall\ n \in \mathbb{N}$,*

$$\sup_{(x,u)\in\widehat{\gamma}\times B_\sigma} \|(\xi^n_{g^n(x)})'(u)\| \leq \nu.$$

Proof. We keep the notation of Section 3.2.1, where we wrote

$$\Delta_x \cap D_{\varrho'} = \mathrm{graph}(\xi_x) = \{(X_x(u), S_x(u), u); u \in B_{\varrho'}\} = \{(x, S_x(u), u); u \in B_{\varrho'}\}.$$

Let us examine the other iterates. We will apply the same method as in Section 2.2.4 to prove the persistence of the graph property for $(\widehat{\Delta}^n)$. Since the fibers of Γ^0 depend on the variable x

in $\widehat{\gamma}$, we need to slightly change ν and ε_ν (given by Equation (2.6)) in order for these constants to be independent of x.

Let $\nu := \max(1, \sup\{\|\xi'_x(u)\|; x \in \widehat{\gamma}, u \in B_{\varrho'}\})$. Because of the particular form of ξ_x, one gets $\nu = \max(1, \sup\{\|S'_x(u)\|; x \in \widehat{\gamma}, u \in B_{\varrho'}\})$. Let $\varepsilon_\nu := \frac{1-\overline{\lambda}}{2\nu\overline{\lambda}}$. We introduce η as we did in Section 2.2.4. Let $\sigma := \min(1, \varrho', \eta, \frac{1-\overline{\lambda}}{3C_2(2\nu+1)^2})$ which will play the role of δ given by Theorem 1. The proof in Section 2.2.4 shows that $\forall\ x \in \widehat{\gamma}$, $\forall\ n \in \mathbb{N}$, $\Delta^n_{g^n(x),\sigma}$ is a graph over B_σ of $\xi^n_{g^n(x)}$ and $\sup\{\|(\xi^n_{g^n(x)})'(u)\|; x \in \widehat{\gamma}, u \in B_\sigma\} \leq \nu$. $\square$

Remark 3.2.4. *It is clear that the first set of fibers, that of* Γ^0*, has a particular form:* $X_x(u) = x$*, for all* $x \in \widehat{\gamma}$*, for all* $u \in B_\sigma$*. It is not the case for the set of fibers of* Γ^n *for* $n > 0$*, that is why we will stick to the first parametrization (*$\Delta_x \cap D_\sigma = \operatorname{graph} \xi_x = \{(X_x(u), S_x(u), u); u \in B_\sigma\}$*) whenever we want to generalize a certain statement to other iterates, and we will keep in mind that the first parametrization is a particular one.*

3.2.3.2.3 Parametrization in D_σ and proof of item iv). The persistence of the graph property will enable us to parameterize Γ^n, for all $n \in \mathbb{N}$. We will write

$$\Gamma^n \cap D_\sigma = \bigcup_{x \in \widehat{\gamma}^n} \operatorname{graph} \xi^n_x = \bigcup_{x \in \widehat{\gamma}^n} \operatorname{graph}(X^n_x, S^n_x) = \{(X^n_x(u), S^n_x(u), u); x \in \widehat{\gamma}^n, u \in B_\sigma\}.$$

So n will indicate the number of iterations, x will determine in a unique way the fiber, and u represents the “height" on the fiber. Note that this parametrization uses the preservation of the graph property for the fibers and for the sequence $(\widehat{\gamma}^n)$: x being in $\widehat{\gamma}^n$, there exists a unique point $(x, T_n(x), 0)$ in γ^n and a unique fiber $\Delta^n_{x,\sigma}$ passing through this point. The variable x will be called the “horizontal" parameter and the variable u the “vertical" parameter.

The next proposition states the straightening of the fibers in the “vertical" direction. Moreover, it will yield item iv) of Proposition 3.2.2 and item 3 of Theorem 3.

Proposition 3.2.5. *We have the following uniform convergence*

$$\lim_{n\to\infty} \sup\{\|(\xi^n_x)'(u)\|; x \in \widehat{\gamma}^n, u \in B_\sigma\} = 0.$$

Proof. The proof is deduced from that of Proposition 3.2.3. In the latter, ν and ε_ν are chosen to be uniform with respect to the variable x. One can follow the same lines as in Sections 2.2.4 and 2.2.5 to prove the convergence. $\square$

Item iv) of Proposition 3.2.2 follows from Proposition 3.2.5.

Remark 3.2.6. *The tangent space to* Γ *at any point of* γ *is “straightened". More precisely, for all* $x \in \gamma$*, one has the following direct sum:* $T_x\Gamma = T_x\gamma \oplus T_x\Delta_{x_0}$ *where* $x := (x_0, T(x_0), 0)$*. It is also the case for all the iterates:* $\forall\ n \in \mathbb{N}$*,* $\forall\ y \in \gamma^n$*,* $T_y\Gamma^n = T_y\gamma^n \oplus T_y\Delta^n_{y_0,\sigma}$*. Section 3.2.2 states that* $T_y\gamma^n$ *is straightened and Proposition 3.2.5 states that* $T_y\Delta^n_{y_0,\sigma}$ *is also straightened.*

For all $n \in \mathbb{N}$, we set $\widehat{X}^n$ and $\widehat{S}^n$ the maps defined on $\widehat{\gamma}^n \times B_\sigma$ such that for all $x \in \widehat{\gamma}^n$ and $u \in B_\sigma$,

$$\widehat{X}^n(x, u) := X^n_x(u) \text{ and } \widehat{S}^n(x, u) := S^n_x(u), \tag{3.2}$$

to stress out on the fact that they are functions of two variables. In the sequel, both notations will be used for different purposes. We will use the notation $X_x^n(u)$ when we want to highlight the fibered structure of Γ^n and we will use the notation $\widehat{X}^n(x,u)$ when we want to differentiate this map with respect to one (or two) of its variables.

3.2.3.2.4 Definition of ϱ and of the parametrization maps ψ^n. We will now define ϱ. First, let

$$A := \max\left(1, \sup\left\{\left\|\partial_x \widehat{S}(x,u)\right\| ; x \in \widehat{\gamma}, u \in B_\sigma\right\}\right) \tag{3.3}$$

and

$$\varepsilon_{A,\nu} := \frac{1-\overline{\lambda}}{2(1+\overline{\lambda})C\nu(1+A)}. \tag{3.4}$$

By uniform continuity and thanks to Equations (2.1) and (2.3), there exists $\eta_A > 0$ such that for all $(x,s,u) \in D_\sigma$, if $\|u\| < \eta_A$ then

$$\max\left(\|\partial_x F_u(x,s,u)\|, \|\partial_s F_u(x,s,u)\|, \|\partial_s F_x(x,s,u)\|\right) < \varepsilon_{A,\nu}. \tag{3.5}$$

We set

$$\varrho := \min(\sigma, \eta_A). \tag{3.6}$$

From now on, we will restrict our work to D_ϱ. We will keep the notation $\widehat{X}^n$, $\widehat{S}^n$, X_x^n and S_x^n for the restriction of these maps to $\widehat{\gamma}^n \times B_\varrho$.

We now define a sequence of maps using the parametrization of Γ^n. For $n \in \mathbb{N}$, let

$$\begin{array}{rccl} \psi^n : & \widehat{\gamma}^n \times B_\varrho & \longrightarrow & \Gamma^n \cap D_\varrho \\ & (x,u) & \longmapsto & (X_x^n(u), S_x^n(u), u) = (\widehat{X}^n(x,u), \widehat{S}^n(x,u), u). \end{array}$$

For all $n \in \mathbb{N}$, ψ^n is a C^2 bijection from $\widehat{\gamma}^n \times B_\varrho$ onto $\Gamma^n \cap D_\varrho$ by definition of the fibration and thanks to the graph property.. We will prove later on that it is a diffeormorphism.

3.2.3.2.5 Proof of item i). We devote Section 3.2.3.2.5 to prove Proposition 3.2.8 below and thus item i) of Proposition 3.2.2. We start with an easy lemma that will be needed in the proof of Proposition 3.2.8.

Lemma 3.2.7. *Let E be a normed vector space and let E_1 and E_2 be two subspaces of E equipped with the induced norm. Let $\mathcal{H} : E_1 \to E_2$ be a linear map such that $\|\mathcal{H}\| < 1$. We denote by I_{E_1} the identity map on E_1. Then $I_{E_1} - \mathcal{H}$ is injective.*

Proof. Let w_1 and w_2 be two arbitrary vectors in E_1. Then,

$$\begin{aligned} \|(I_{E_1} - \mathcal{H})(w_1) - (I_{E_1} - \mathcal{H})(w_2)\| &= \|(w_1 - w_2) - (\mathcal{H}(w_1) - \mathcal{H}(w_2))\| \\ &= \|(w_1 - w_2) - \mathcal{H}(w_1 - w_2)\| \\ &\geq \|w_1 - w_2\|(1 - \|\mathcal{H}\|), \end{aligned}$$

and the injectivity easily follows. □

The following Proposition will not only imply item i) of Proposition 3.2.2, but it will also yield Corollary 3.2.9 and will give the preliminary estimates to prove item v).

Proposition 3.2.8. *Let A be as in Equation (3.3) and let $\widehat{X}^n$ and $\widehat{S}^n$ be as in Equation (3.2). Then, for all $n \in \mathbb{N}$, for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$, $\partial_x \widehat{X}^n(x,u) : T_x\widehat{\gamma}^n \longrightarrow T_{\widehat{X}^n(x,u)}N$ is injective and satisfies*

$$\left\|\partial_x \widehat{S}^n(x,u).\left[\partial_x \widehat{X}^n(x,u)\right]^{-1}\right\| \leq A.$$

Proof. We will proceed by induction. For $n=0$, the statement is true since for all $(x,u) \in \widehat{\gamma} \times B_\varrho$, $\partial_x \widehat{X}(x,u)$ is the identity map on $T_x\widehat{\gamma}$ (Remark 3.2.4), and because of the definition of A (Equation (3.3)). Since the first iteration is special, we will prove that the statement is true for $n=1$ before the induction step.

• **<u>The case $n=1$</u>**
We will need to write $\partial_x \widehat{X}^1$ and $\partial_x \widehat{S}^1$ in terms of $\partial_x \widehat{X}$ and $\partial_x \widehat{S}$. Recall that we have the C^2-bijections

$$\begin{array}{rccl} \psi: & \widehat{\gamma} \times B_\varrho & \longrightarrow & \Gamma \cap D_\varrho \\ & (x,u) & \longmapsto & (X_x(u), S_x(u), u) = (\widehat{X}(x,u), \widehat{S}(x,u), u), \end{array}$$

and

$$\begin{array}{rccl} \psi^1: & \widehat{\gamma}^1 \times B_\varrho & \longrightarrow & \Gamma^1 \cap D_\varrho \\ & (y,t) & \longmapsto & (X^1_y(t), S^1_y(t), t) = (\widehat{X}^1(y,t), \widehat{S}^1(y,t), t). \end{array}$$

Let Y be in $\Gamma^1 \cap D_\varrho$. There exist then $y \in \widehat{\gamma}^1$ and $t \in B_\varrho$ such that $Y = (X^1_y(t), S^1_y(t), t)$. Moreover, $\Gamma^1 \cap D_\varrho \subset F(\Gamma) \cap D_\varrho \subset F(\Gamma \cap D_\varrho)$. There exist then $x \in \widehat{\gamma}$ and $u \in B_\varrho$ such that $Y = F(X_x(u), S_x(u), u)$, and $y = F_{|N}(x) = g(x)$ by definition of the fibers of Γ^1. This yields the following equalities:

$$\begin{aligned} X^1_{g(x)}(t) &= F_x(X_x(u), S_x(u), u), \\ S^1_{g(x)}(t) &= F_s(X_x(u), S_x(u), u), \\ t &= F_u(X_x(u), S_x(u), u). \end{aligned}$$

Therefore,

$$X^1_{g(x)}(F_u(X_x(u), S_x(u), u)) = F_x(X_x(u), S_x(u), u), \tag{3.7}$$

$$S^1_{g(x)}(F_u(X_x(u), S_x(u), u)) = F_s(X_x(u), S_x(u), u). \tag{3.8}$$

For the readability of our formulas, we set $J := (X_x(u), S_x(u), u) = \psi(x,u)$. By derivating Equations (3.7) and (3.8) with respect to x, and using the notation given by Equation (3.2), one gets

$$\begin{aligned} \partial_x \widehat{X}^1\big(g(x), F_u(J)\big).g'(x) + \partial_u \widehat{X}^1\big(g(x), F_u(J)\big).\big[\partial_x F_u(J).\partial_x \widehat{X}(x,u) + \partial_s F_u(J).\partial_x \widehat{S}(x,u)\big] = \\ \partial_x F_x(J).\partial_x \widehat{X}(x,u) + \partial_s F_x(J).\partial_x \widehat{S}(x,u), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \partial_x \widehat{S}^1\big(g(x), F_u(J)\big).g'(x) + \partial_u \widehat{S}^1\big(g(x), F_u(J)\big).\big[\partial_x F_u(J).\partial_x \widehat{X}(x,u) + \partial_s F_u(J).\partial_x \widehat{S}(x,u)\big] = \\ \partial_x F_s(J).\partial_x \widehat{X}(x,u) + \partial_s F_s(J).\partial_x \widehat{S}(x,u). \end{aligned} \tag{3.10}$$

If $(y,t) \in \widehat{\gamma}^1 \times B_\varrho$ is chosen, then $(x,u) \in \widehat{\gamma} \times B_\varrho$ is uniquely determined. To prove the injectivity of $\partial_x \widehat{X}^1(y,t)$, for all $(y,t) \in \widehat{\gamma}^1 \times B_\varrho$, since $g'(x)$ is invertible, it is enough to prove the injectivity of $\partial_x \widehat{X}^1(y,t).g'(x)$. Since $\partial_x F_x(J) = \partial_x F_x(X_x(u), S_x(u), u)$ is invertible for all $(x,u) \in \widehat{\gamma} \times B_\varrho$ (see Section 2.2.1), Equation (3.9) yields

$$\begin{aligned}\partial_x \widehat{X}^1(y,t).g'(x) = \; & \partial_x F_x(J).\Big[I_{T_x\widehat{\gamma}} - (\partial_x F_x(J))^{-1}.\partial_u \widehat{X}^1(y,t).\big[\partial_x F_u(J) + \partial_s F_u(J).\partial_x \widehat{S}(x,u). \\ & (\partial_x \widehat{X}(x,u))^{-1}\big] + (\partial_x F_x(J))^{-1}.\partial_s F_x(J).\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}\Big]\partial_x \widehat{X}(x,u).\end{aligned}$$

It is enough to prove the injectivity of $I_{T_x\widehat{\gamma}} - \mathcal{H}(x,u)$ where

$$\begin{aligned}\mathcal{H}(x,u) := \; & (\partial_x F_x(J))^{-1}.\partial_u \widehat{X}^1(y,t).[\partial_x F_u(J) + \partial_s F_u(J).\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}] \\ & + (\partial_x F_x(J))^{-1}.\partial_s F_x(J).\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}.\end{aligned}$$

To do so, we will use Lemma 3.2.7 so that it is enough to prove that $\|\mathcal{H}(x,u)\| < 1$, for all $(x,u) \in \widehat{\gamma} \times B_\varrho$. Since $(\partial_x \widehat{X}(x,u))^{-1} = I_{T_x\widehat{\gamma}}$ and $\|\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}\| = \|\partial_x \widehat{S}(x,u)\| \leq A$, one gets the following by using that for all $n \in \mathbb{N}$, $\sup\{\|(\xi_x^n)'(u)\|; x \in \widehat{\gamma}^n, u \in B_\sigma\} \leq \nu$ and Equation (3.6):

$$\begin{aligned}\|\mathcal{H}(x,u)\| & \leq C\nu(\varepsilon_{A,\nu} + A\varepsilon_{A,\nu}) + C\varepsilon_{A,\nu}A \\ & \leq \varepsilon_{A,\nu}\left(C\nu(1+A) + CA\right) \\ & \leq \varepsilon_{A,\nu}\left(C\nu(1+A) + C\nu(1+A)\right),\end{aligned}$$

because $\nu \geq 1$. This yields $\|\mathcal{H}(x,u)\| \leq 2\varepsilon_{A,\nu}C\nu(1+A) = \frac{1-\overline{\lambda}}{1+\overline{\lambda}} < 1$ for all $(x,u) \in \widehat{\gamma} \times B_\varrho$ by definition of $\varepsilon_{A,\nu}$ (Equation (3.4)). Thus $\|\mathcal{H}(x,u)\| < 1$ and $I - \mathcal{H}(x,u)$ is injective for all (x,u) in $\widehat{\gamma} \times B_\varrho$.

This way, the injectivity of $\partial_x \widehat{X}^1(y,t)$ is immediate for all $(y,t) \in \widehat{\gamma}^1 \times B_\varrho$. Moreover $\partial_x \widehat{X}^1(y,t)$ is a bijection from $T_y\widehat{\gamma}^1$ onto $\mathrm{Im}\,(\partial_x \widehat{X}^1(y,t))$ for all $(y,t) \in \widehat{\gamma}^1 \times B_\varrho$ and satisfies

$$\left[\partial_x \widehat{X}^1(y,t).g'(x)\right]^{-1} = \left(\partial_x \widehat{X}(x,u)\right)^{-1}.\sum_{m\geq 0}\left(-\mathcal{H}(x,u)\right)^m.[\partial_x F_x(J)]^{-1}$$

defined on $\mathrm{Im}\,(\partial_x \widehat{X}^1(y,t))$, where $y = g(x)$, $J = (X_x(u), S_x(u), u)$ and $t = F_u(J)$. We finish the proof of the case $n = 1$ by bounding $T_1 := \|\partial_x \widehat{S}^1(y,t).[\partial_x \widehat{X}^1(y,t)]^{-1}\|$. Using Equation (3.10), one gets

$$\begin{aligned}T_1 = \; & \big\|\partial_x \widehat{S}^1(y,t).g'(x).\big[\partial_x \widehat{X}^1(y,t).g'(x)\big]^{-1}\big\| \\ = \; & \big\|\big[-\partial_u \widehat{S}^1\big(g(x),(F_u(J)\big).[\partial_x F_u(J) + \partial_s F_u(J).\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}] + \partial_x F_s(J)+ \\ & \partial_s F_s(J).\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}\big].\sum_{m\geq 0}(-\mathcal{H}(x,u))^m.[\partial_x F_x(J)]^{-1}\big\|.\end{aligned}$$

Now recall that for all $(x,u) \in \widehat{\gamma} \times B_\varrho$, the following inequalities hold true

- $\|\partial_x \widehat{S}(x,u).(\partial_x \widehat{X}(x,u))^{-1}\| \leq A$,
- $\|\partial_u \widehat{S}^1\big(g(x), F_u(J)\big)\| \leq \nu$,
- $\|\partial_s F_s(J)\|.\|[\partial_x F_x(J)]^{-1}\| < \overline{\lambda}$,

- $\| \sum_{m\geq 0} (-\mathcal{H}(x,u))^m \| \leq \frac{1}{1-\|\mathcal{H}(x,u)\|} \leq \frac{1+\overline{\lambda}}{2\overline{\lambda}}$.

On the other hand, using the Mean Value Theorem, Proposition 3.1.1 and Equation (2.2) imply,

$$\begin{aligned}\|\partial_x F_s(J)\| &= \|\partial_x F_s(X_x(u), S_x(u), u) - \partial_x F_s(X_x(u), 0, u)\| \\ &\leq C_2 \|S_x(u)\| \\ &\leq C_2 . \tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2CC_2(1+\overline{\lambda})} \\ &\leq \tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2C(1+\overline{\lambda})},\end{aligned}$$

which yields

$$\begin{aligned}\left\|\partial_x \widehat{S}^1(y,t).[\partial_x \widehat{X}^1(y,t)]^{-1}\right\| &\leq \left(\nu\varepsilon_{A,\nu}(1+A) + \tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2C(1+\overline{\lambda})}\right).C.\tfrac{1+\overline{\lambda}}{2\overline{\lambda}} + \overline{\lambda}.A.\tfrac{1+\overline{\lambda}}{2\overline{\lambda}} \\ &\leq \left(\tfrac{1-\overline{\lambda}}{2(1+\overline{\lambda})} + \tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2(1+\overline{\lambda})}\right).\tfrac{1+\overline{\lambda}}{2\overline{\lambda}} + A.\tfrac{1+\overline{\lambda}}{2} \\ &\leq \tfrac{1-\overline{\lambda}}{2}.1 + \tfrac{1+\overline{\lambda}}{2}.A \\ &\leq \tfrac{1-\overline{\lambda}}{2}.A + \tfrac{1+\overline{\lambda}}{2}.A \\ &\leq A\end{aligned}$$

where we used that $A \geq 1$ and the definition of $\varepsilon_{A,\nu}$ (Equation (3.4)). This ends the proof of the statement for $n = 1$.

• **The inductive step**

We assume now that $\partial_x \widehat{X}^n(x,u)$ is injective for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$ and satisfies:

$$\left\|\partial_x \widehat{S}^n(x,u).\left[\partial_x \widehat{X}^n(x,u)\right]^{-1}\right\| \leq A.$$

We want to prove that the statements are true for $n+1$. As we did for the first iterate, we write $\partial_x \widehat{X}^{n+1}$ and $\partial_x \widehat{S}^{n+1}$ in terms of $\partial_x \widehat{X}^n$ and $\partial_x \widehat{S}^n$. For all $(y,t) \in \widehat{\gamma}^{n+1} \times B_\varrho$ there exists a unique $(x,u) \in \widehat{\gamma}^n \times B_\varrho$ such that $y = g(x)$ and $t = F_u(X_x^n(u), S_x^n(u), u)$. By setting $R := (X_x^n(u), S_x^n(u), u)$, the same calculations as those in the step $n = 1$ yield the following equalities

$$\begin{aligned}\partial_x \widehat{X}^{n+1}\big(g(x), (F_u(R)\big).g'(x) + \partial_u \widehat{X}^{n+1}\big(g(x), (F_u(R)\big).[\partial_x F_u(R).\partial_x \widehat{X}^n(x,u) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u)] \\ = \partial_x F_x(R).\partial_x \widehat{X}^n(x,u) + \partial_s F_x(R).\partial_x \widehat{S}^n(x,u),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\partial_x \widehat{S}^{n+1}\big(g(x), (F_u(R)\big).g'(x) + \partial_u \widehat{S}^{n+1}\big(g(x), (F_u(R)\big).[\partial_x F_u(R).\partial_x \widehat{X}^n(x,u) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u)] \\ = \partial_x F_s(R).\partial_x \widehat{X}^n(x,u) + \partial_s F_s(R).\partial_x \widehat{S}^n(x,u).\end{aligned} \tag{3.12}$$

By writing $I_{T_x\widehat{\gamma}^n} = \left[\partial_x \widehat{X}^n(x,u)\right]^{-1}.\partial_x \widehat{X}^n(x,u)$ and using Equation (3.11), one gets

$$\begin{aligned}\partial_x \widehat{X}^{n+1}(y,t).g'(x) = \partial_x F_x(R).\Big[I_{\mathcal{A}} - (\partial_x F_x(R))^{-1}.\partial_u \widehat{X}^{n+1}(y,t).\big[\partial_x F_u(R) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u). \\ (\partial_x \widehat{X}^n(x,u))^{-1}\big] + (\partial_x F_x(R))^{-1}.\partial_s F_x(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}\Big].\partial_x \widehat{X}^n(x,u),\end{aligned}$$

where $\mathcal{A} = \mathrm{Im}\,(\partial_x \widehat{X}^n(x,u))$.

Since $\partial_x \widehat{X}^n(x,u)$ and $\partial_x F_x(R)$ are injective for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$, it is enough to prove that $I_{\mathrm{Im}\,(\partial_x \widehat{X}^n(x,u))} - \mathcal{H}^n(x,u)$ is injective where

$$\begin{aligned}\mathcal{H}^n(x,u) := \quad & (\partial_x F_x(R))^{-1}.\partial_u \widehat{X}^{n+1}(y,t).[\partial_x F_u(R) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}] \\ & +(\partial_x F_x(R))^{-1}.\partial_s F_x(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}.\end{aligned}$$

The proof follows the same lines as that of $I_{T_x\widehat{\gamma}} - \mathcal{H}(x,u)$ (for $(x,u) \in \widehat{\gamma} \times B_\varrho$) because $\left\|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\right\| \leq A$, and all the other estimates are independent of n. The same computations give that $\|\mathcal{H}^n(x,u)\| \leq \frac{1-\overline{\lambda}}{1+\overline{\lambda}} < 1$, for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$.

We get then that $\partial_x \widehat{X}^{n+1}(y,t)$ is injective for all (y,t) in $\widehat{\gamma}^{n+1} \times B_\varrho$ (by applying Lemma 3.2.7) and thus bijective onto $\mathrm{Im}\,(\partial_x \widehat{X}^{n+1}(y,t))$. Moreover,

$$\left[\partial_x \widehat{X}^{n+1}(y,t).g'(x)\right]^{-1} = \left(\partial_x \widehat{X}^n(x,u)\right)^{-1}.\sum_{m\geq 0}(-\mathcal{H}^n(x,u))^m.[\partial_x F_x(R)]^{-1}.$$

We use the same method as before and set $T_2 := \partial_x \widehat{S}^{n+1}(y,t).[\partial_x \widehat{X}^{n+1}(y,t)]^{-1}$. Then, by using Equation (3.12),

$$\begin{aligned} T_2 = \quad & \partial_x \widehat{S}^{n+1}(y,t).g'(x).[\partial_x \widehat{X}^{n+1}(y,t).g'(x)]^{-1} \\ = \quad & \big[\partial_u \widehat{S}^{n+1}\big(g(x), F_u(R)\big).[\partial_x F_u(R) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}] \\ & +\partial_x F_s(R) + \partial_s F_s(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}\big].\sum_{m\geq 0}(-\mathcal{H}^n(x,u))^m.[\partial_x F_x(R)]^{-1},\end{aligned}$$

to prove that

$$\left\|\partial_x \widehat{S}^{n+1}(y,t).\left[\partial_x \widehat{X}^{n+1}(y,t)\right]^{-1}\right\| \leq A$$

because $\|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\| \leq A$ and all the other estimates are uniform with respect to n. This ends the proof of the induction and thus the proposition. □

3.2.3.2.6 Proof of item ii). We will now prove item ii) of Proposition 3.2.2. It will be a direct consequence of Proposition 3.2.8.

Corollary 3.2.9. *For all $n \in \mathbb{N}$, ψ^n is a C^2-diffeomorphism from $\widehat{\gamma}^n \times B_\varrho$ onto $\Gamma^n \cap D_\varrho$.*

Proof. Since ψ^n is a C^2-bijection onto $\Gamma^n \cap D_\varrho$, it is enough to prove that for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$, $D\psi^n(x,u)$ is injective. By differentiating $\begin{array}{rccc}\psi^n : & \widehat{\gamma}^n \times B_\varrho & \longrightarrow & \Gamma^n \cap D_\varrho \\ & (x,u) & \longmapsto & (\widehat{X}^n(x,u), \widehat{S}^n(x,u), u)\end{array}$ one gets, for all $n \in \mathbb{N}$, for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$,

$$\begin{array}{rccc} D\psi^n(x,u): & T_x\widehat{\gamma}^n \times \mathbb{R}^p_u & \longrightarrow & T_{\psi^n(x,u)}(\Gamma^n \cap D_\varrho) \\ & (v_1,v_2) & \longmapsto & \begin{pmatrix} \partial_x \widehat{X}^n(x,u).v_1 + \partial_u \widehat{X}^n(x,u).v_2 \\ \partial_x \widehat{S}^n(x,u).v_1 + \partial_u \widehat{S}^n(x,u).v_2 \\ v_2 \end{pmatrix}. \end{array}$$

Let $(v_1,v_2) \in T_x\widehat{\gamma}^n \times \mathbb{R}^p_u$ such that $D\psi^n(x,u).(v_1,v_2) = (0,0,0)$. Therefore, $v_2 = 0$ and $\partial_x \widehat{X}^n(x,u).v_1 = 0$, which yields $v_1 = 0$ since $\partial_x \widehat{X}^n(x,u)$ is injective thanks to Proposition 3.2.8. Therefore $D\psi^n(x,u)$ is injective for all $n \in \mathbb{N}$, for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$. This completes the proof of Corollary 3.2.9. □

3.2.3.2.7 Proof of item v), end of the proof of Proposition 3.2.2. We now prove item v) of Proposition 3.2.2.

Proposition 3.2.10. *For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho} \left\|\partial_x \widehat{S}^n(x,u).\left[\partial_x \widehat{X}^n(x,u)\right]^{-1}\right\| \leq \varepsilon.$$

Proof. At the end of the proof of Proposition 3.2.8, we showed that for all $n \in \mathbb{N}$, for all $(y,t) \in \widehat{\gamma}^{n+1} \times B_\varrho$, there exists a unique $(x,u) \in \widehat{\gamma}^n \times B_\varrho$ such that $y = g(x)$ and $t = F_u(R)$, where $R := (X_x^n(u), S_x^n(u), u)$. Moreover, on $\mathrm{Im}\,(\partial_x \widehat{X}^{n+1}(y,t))$

$$\begin{aligned} T_2 = \;& \partial_x \widehat{S}^{n+1}(y,t).[\partial_x \widehat{X}^{n+1}(y,t)]^{-1} \\ = \;& \partial_x \widehat{S}^{n+1}(y,t).g'(x).[\partial_x \widehat{X}^{n+1}(y,t).g'(x)]^{-1} \\ = \;& \big[-\partial_u \widehat{S}^{n+1}\big(g(x),F_u(R)\big).[\partial_x F_u(R) + \partial_s F_u(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}] \\ & +\partial_x F_s(R) + \partial_s F_s(R).\partial_x \widehat{S}^n(x,u).(\partial_x \widehat{X}^n(x,u))^{-1}\big].\sum_{m\geq 0}(-\mathcal{H}^n(x,u))^m.[\partial_x F_x(R)]^{-1}, \end{aligned}$$

We will improve the estimates in Proposition 3.2.8. Recall that the following inequalities hold true for all $n \in \mathbb{N}$, and for all $(x,u) \in \widehat{\gamma}^n \times B_\varrho$:

- $\max\big(\|\partial_x F_u(R)\|, \|\partial_s F_u(R)\|\big) < \varepsilon_{A,\nu}$, where $\varepsilon_{A,\nu} = \frac{1-\overline{\lambda}}{2(1+\overline{\lambda})C\nu(1+A)}$,
- $\|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\| \leq A$,
- $\|\sum_{m\geq 0}(-\mathcal{H}(x,u))^m\| \leq \frac{1+\overline{\lambda}}{2\overline{\lambda}}$,
- $\|[\partial_x F_x(R)]^{-1}\| \leq C$,
- $\|\partial_s F_s(R)\|.\|[\partial_x F_x(R)]^{-1}\| < \overline{\lambda}$.

On the other hand, using the Mean Value Theorem and Proposition 3.1.1

$$\begin{aligned} \|\partial_x F_s(R)\| & = \|\partial_x F_s(X_x^n(u), S_x^n(u), u) - \partial_x F_s(X_x^n(u), 0, u)\| \\ & \leq C_2 \|S_x^n(u)\| \\ & \leq C_2.\overline{\lambda}^n(\sup\{\|S_x(u)\|; (x,u) \in \widehat{\gamma} \times B_\varrho\}) \\ & \leq \overline{\lambda}^n.\tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2C(1+\overline{\lambda})}. \end{aligned}$$

Then, for all $(y,t) \in \widehat{\gamma}^{n+1} \times B_\varrho$

$$\begin{aligned} \left\|\partial_x \widehat{S}^{n+1}(y,t).\left[\partial_x \widehat{X}^{n+1}(y,t)\right]^{-1}\right\| & \leq \sup_{y,t}\left\|(\xi_y^{n+1})'(t)\right\|.\varepsilon_{A,\nu}.(1+A).C.\tfrac{1+\overline{\lambda}}{2\overline{\lambda}} + \\ & \overline{\lambda}^n\tfrac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2(1+\overline{\lambda})}.\tfrac{1+\overline{\lambda}}{2\overline{\lambda}} + \sup_{x,u}\left\|\partial_x \widehat{S}^n(x,u).\left[\partial_x \widehat{X}^n(x,u)\right]^{-1}\right\|\overline{\lambda}\tfrac{1+\overline{\lambda}}{2\overline{\lambda}}. \end{aligned}$$

Note that

$$\varepsilon_{A,\nu}.(1+A).C.\frac{1+\overline{\lambda}}{2\overline{\lambda}} \leq \frac{1-\overline{\lambda}}{4\overline{\lambda}\nu} < 1,$$

because $\nu \geq 1$ and $\overline{\lambda} = \frac{1+\lambda}{2} > \frac{1}{5}$ and that

$$\frac{-2\overline{\lambda}^2+3\overline{\lambda}-1}{2(1+\overline{\lambda})}.\frac{1+\overline{\lambda}}{2\overline{\lambda}}<1,$$

If we set $\widetilde{\lambda}:=\frac{1+\overline{\lambda}}{2}$, then

$$\sup_{y,t}\left\|\partial_x\widehat{S}^{n+1}(y,t).\left[\partial_x\widehat{X}^{n+1}(y,t)\right]^{-1}\right\|\leq\sup_{y,t}\left\|(\xi_y^{n+1})'(t)\right\|+\overline{\lambda}^n+\widetilde{\lambda}\sup_{x,u}\left\|\partial_x\widehat{S}^n(x,u).\left[\partial_x\widehat{X}^n(x,u)\right]^{-1}\right\|$$

which yields

$$\sup_{y,t}\left\|\partial_x\widehat{S}^{n+1}(y,t).\left[\partial_x\widehat{X}^{n+1}(y,t)\right]^{-1}\right\|\leq\sup_{y,t}\left\|(\xi_y^{n+1})'(t)\right\|+\widetilde{\lambda}^n+\widetilde{\lambda}\sup_{x,u}\left\|\partial_x\widehat{S}^n(x,u).\left[\partial_x\widehat{X}^n(x,u)\right]^{-1}\right\|$$

because $\overline{\lambda}<\widetilde{\lambda}$. The convergence follows the same lines as that of (c_n) in Section 2.2.2, because $\widetilde{\lambda}<1$ and the convergence of $(\sup_{y,t}\|(\xi_y^{n+1})'(t)\|)$ is given in Proposition 3.2.5. □

This ends the proof of Proposition 3.2.2.

3.2.3.3 Proof of Theorem 3

We now prove Theorem 3.

3.2.3.3.1 Proof of item 1. Let ϱ be as in Proposition 3.2.2. In Corollary 3.2.9, we proved that, for all $n\in\mathbb{N}$, ψ^n is a C^2-diffeomorphism from $\widehat{\gamma}^n\times B_\varrho$ onto $\Gamma^n\cap D_\varrho$. The C^0-convergence will follow from Proposition 3.2.5 (and the Mean Value Theorem) and Proposition 3.2.1. We will apply the same method as in Section 2.2.6. First recall that, for all $n\in\mathbb{N}$,

$$\begin{array}{rccc}\Upsilon^n: & \widehat{\gamma}^n\times B_\varrho & \longrightarrow & \widetilde{W}^u(\widehat{\gamma}^n)\cap D_\varrho\\ & (x,u) & \longmapsto & (x,0,u)\end{array}$$

and

$$\begin{array}{rccl}\psi^n: & \widehat{\gamma}^n\times B_\varrho & \longrightarrow & \Gamma^n\cap D_\varrho\\ & (x,u) & \longmapsto & (X_x^n(u),S_x^n(u),u)=(\xi_x^n(u),u)\end{array}$$

so that $d_{C^0}(\psi^n,\Upsilon^n)=\sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho}d\big(\xi_x^n(u),(x,0)\big)$.

We fix $n\in\mathbb{N}$. For all $x\in\widehat{\gamma}^n$,

$$\begin{aligned}\sup_{u\in B_\varrho}d\left(\xi_x^n(u),(x,0)\right)&\leq\sup_{u\in B_\varrho}d\left(\xi_x^n(u),\xi_x^n(0)\right)+d\left(\xi_x^n(0),(x,0)\right)\\&\leq\sup_{u\in B_\varrho}\|(\xi_x^n)'(u)\|+\|T_n\|\\&\leq\sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho}\|(\xi_x^n)'(u)\|+\|T_n\|,\end{aligned}$$

by the Mean Value Theorem and since $\varrho\leq1$, where (T_n) is given by Proposition 3.2.1. Therefore,

$$\sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho}d\left(\xi_x^n(u),(x,0)\right)\leq\sup_{(x,u)\in\widehat{\gamma}^n\times B_\varrho}\|(\xi_x^n)'(u)\|+\|T_n\|.$$

The convergence follows from Proposition 3.2.5 and Proposition 3.2.1.

3.2.3.3.2 Proof of item 2. Item 2 of Theorem 3 is an immediate consequence of Proposition 3.2.2. Thanks to the latter, we are able to rewrite the s-component of any unit tangent vector to $\Gamma^n \cap D_\varrho$ and prove the C^1-convergence by using the convergences in Proposition 3.2.2.

Corollary 3.2.11. *Let ϱ be as in Proposition 3.2.2. For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0$, $\forall\, q \in \Gamma^n \cap D_\varrho$, for all* unit vector $Z = (Z_x, Z_s, Z_u)$ *tangent to $\Gamma^n \cap D_\varrho$ at q, $\|Z_s\| < \varepsilon$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Let q be in $\Gamma^n \cap D_\varrho$ and let $Z = (Z_x, Z_s, Z_u)$ be a unit vector tangent to $\Gamma^n \cap D_\varrho$ at q. There exist then $x \in \widehat{\gamma}^n$ and $u \in B_\varrho$ such that $Z \in T_{\psi^n(x,u)}(\Gamma^n \cap D_\varrho)$, that is, $\operatorname{Im}(D\psi^n(x,u))$ since ψ^n is a diffeomorphism. Therefore, there exists a unique (v_1, v_2) in $T_x\widehat{\gamma}^n \times \mathbb{R}^p_u$ such that

$$
\begin{aligned}
Z_x &= \partial_x \widehat{X}^n(x,u).v_1 + \partial_u \widehat{X}^n(x,u).v_2,\\
Z_s &= \partial_x \widehat{S}^n(x,u).v_1 + \partial_u \widehat{S}^n(x,u).v_2,\\
Z_u &= v_2.
\end{aligned}
$$

Z being of norm 1, then, in particular,

$$\|v_2\| \leq 1 \tag{3.13}$$

and

$$\left\|\partial_x \widehat{X}^n(x,u).v_1 + \partial_u \widehat{X}^n(x,u).v_2\right\| \leq 1. \tag{3.14}$$

As for $\|Z_s\|$, one has the more refined estimates using Inequalities (3.13) and (3.14)

$$
\begin{aligned}
\|Z_s\| &= \|\partial_x \widehat{S}^n(x,u).v_1 + \partial_u \widehat{S}^n(x,u).v_2\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).v_1\| + \|\partial_u \widehat{S}^n(x,u).v_2\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).v_1\| + \|\partial_u \widehat{S}^n(x,u)\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}.\partial_x \widehat{X}^n(x,u).v_1\| + \|\partial_u \widehat{S}^n(x,u)\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\|.\|\partial_x \widehat{X}^n(x,u).v_1\| + \|\partial_u \widehat{S}^n(x,u)\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\|.(\|\partial_x \widehat{X}^n(x,u).v_1 + \partial_u \widehat{X}^n(x,u).v_2\| + \|\partial_u \widehat{X}^n(x,u).v_2\|)\\
&\quad + \|\partial_u \widehat{S}^n(x,u)\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\|.(1 + \|\partial_u \widehat{X}^n(x,u)\|.\|v_2\|) + \|\partial_u \widehat{S}^n(x,u)\|\\
&\leq \|\partial_x \widehat{S}^n(x,u).[\partial_x \widehat{X}^n(x,u)]^{-1}\|.(1 + \nu) + \|\partial_u \widehat{S}^n(x,u)\|,
\end{aligned}
$$

by item iii) of Proposition 3.2.2. The convergence follows from Proposition 3.2.2 items iv) and v). ☐

This ends the proof of Theorem 3 item 2.

3.2.3.3.3 Proof of item 3. Item 3 of Theorem 3 is an immediate consequence of Sections 3.2.3.2.1, 3.2.3.2.2 and 3.2.3.2.3. Proposition 3.2.5 yields the convergence of the fibers in the sense of the basic λ-lemma.

3.2.4 Proof of Corollary 3.1.3

We now prove Corollary 3.1.3. We keep the same notation as above.

Proof. In the particular case $F_x(x, s, u) = g(x)$ for all $(x, s, u) \in \widehat{V}$, it is easy to see that for all $n \in \mathbb{N}$, for all $(x, u) \in \widehat{\gamma}^n \times B_\varrho$,

$$X_x^n(u) = \widehat{X}^n(x, u) = x.$$

Therefore, for all $n \in \mathbb{N}$, ψ^n takes the following form:

$$\begin{array}{rccc} \psi^n : & \widehat{\gamma}^n \times B_\varrho & \longrightarrow & \Gamma^n \cap D_\varrho \\ & (x, u) & \longmapsto & \left(x, \widehat{S}^n(x, u), u\right). \end{array}$$

To prove the convergence of

$$d_{C^1}(\psi^n, \Upsilon^n) = d_{C^0}(\psi^n, \Upsilon^n) + \sup_{(x,u) \in \widehat{\gamma}^n \times B_\varrho} \|D\psi^n(x, u) - D\Upsilon^n(x, u)\|,$$

it is enough to prove the convergence of $\sup\limits_{(x,u) \in \widehat{\gamma}^n \times B_\varrho} \|D\widehat{S}^n(x, u)\|$, thanks to item 1 of Theorem 3 and the particular form of ψ^n.

The convergence of $\sup\limits_{(x,u) \in \widehat{\gamma}^n \times B_\varrho} \left\|\partial_u \widehat{S}^n(x, u)\right\|$ is given by Proposition 3.2.2 item iv), and that of $\sup\limits_{(x,u) \in \widehat{\gamma}^n \times B_\varrho} \left\|\partial_x \widehat{S}^n(x, u)\right\|$ follows from Proposition 3.2.2 item v) since $\left[\partial_x \widehat{X}^n(x, u)\right]^{-1} = I_{T_x\widehat{\gamma}^n}$ in this particular case. This ends the proof of Corollary 3.1.3. $\square$

3.3 Applications

Here we state and prove an application of the fibered λ-lemma. Given a finite family of invariant minimal tori with successive transversal heteroclinic connections, we give necessary conditions that ensure the existence of a transversal heteroclinic connection between the extremal tori of this chain.

3.3.1 Transitivity of transversal heteroclinic connections

We begin this section with defining the *strong torsion* property, a property that is stronger than the simple torsion (twist) in the sense that it requires the notion of C^1-convergence of graph maps. Then we state and prove a corollary of the fibered λ-lemma that gives the transitivity of transversal heteroclinic connections for systems having the strong torsion property and the normal form needed for Corollary 3.1.3.

Let us first introduce the notation for the next definition. In the sequel, we consider F, M and N as in Sections 1.4 and 2.1.1. Moreover, we suppose that $N \subset \mathbb{T}^n \times \mathbb{R}^n$ $(n \geq 1)$ endowed with its angle-action coordinate system (θ, r) and we denote by Π_θ the projection over the angle variable θ. Recall that $g := F_{|_N}$. Let T be an invariant torus in N and let b be a point in T.

For $m \in \mathbb{N}$ and $\alpha \in]0; \frac{1}{2}[$, we denote by D_m^α the strip in the θ-variable centered at $g^m(b)$ and of radius α. More precisely, we set

$$D_m^\alpha := \{(\theta, r) \in N; d(\theta, \Pi_\theta(g^m(b))) < \alpha\}.$$

Definition 3.3.1. **[The strong torsion property].** *Let F, g, M and N be as above. Let T be an invariant torus in N. We suppose that T is the graph of a C^1-map $R : \mathbb{T}^n \longrightarrow \mathbb{R}^n$ over the angle variable. We say that F has the strong torsion property over T if for any C^1-submanifold $\widehat{\gamma}$ of N transverse to T in N at a point b, one has the following property:*

$\exists\, m_0 \in \mathbb{N}$, $\exists\, \alpha \in]0; \frac{1}{2}[$ *such that* $\forall\, m \geq m_0$, *the connected component of $g^m(\widehat{\gamma}) \cap D_m^\alpha$ containing $g^m(b)$ is a graph over $\Pi_\theta(D_m^\alpha)$ of a C^1-map H_m that satisfies:*

$$\lim_{m\to\infty} \|H_m - R\|_{C^1\left(\Pi_\theta(D_m^\alpha)\right)} = 0.$$

Note that, for notational simplicity, we kept the notation R for the restriction of R to $\Pi_\theta(D_m^\alpha)$. In the rest of the chapter and in Chapter 4, if H_m is also defined on a domain containing $\Pi_\theta(D_m^\alpha)$, we also keep the same notation for the restricted map.

Roughly speaking, having the strong torsion property over T means that for any $\widehat{\gamma}$ transverse to T, there exists a strip in the angle variable with a fixed width, such that (in this strip) $g^m(\widehat{\gamma})$ is close to T in the C^1-topology for m large enough.

If F has the strong torsion property over each torus of a finite family of invariant minimal tori with transversal heteroclinic connections (and some additional assumptions), one can expect the transitivity of the transversal heteroclinic connections. More precisely, we have the following corollary resulting from the fibered λ-lemma (Corollary 3.1.3).

Corollary 3.3.2. *Let F, M, N and g be as above and let $\widehat{V}$ be as in Proposition 3.1.1. We suppose that for all $(x, s, u) \in \widehat{V}$, $F_x(x, s, u) = g(x)$. Moreover, we suppose that there exists a transition chain $(T_k)_{1\leq k\leq n}$ of tori in N (see Definition 2.4.1) such that for all $k = 1, \dots, n-1$, $W^u(T_k) \pitchfork W^s(T_{k+1})$. In addition, we suppose that for all $k = 2, \dots, n$, the torus T_k is the graph of a C^1-map R_k over the angle variable. We suppose that F has the strong torsion property over T_k, for all $k = 2, \dots, n$. Then, $W^u(T_1) \pitchfork W^s(T_n)$.*

Proof. For all $k = 1, \dots, n-1$, we fix $c_k \in W^u(T_k) \cap W^s(T_{k+1})$. Let a_k (respectively b_{k+1}) be the point in T_k (respectively T_{k+1}) such that $c_k \in W^{uu}(a_k)$ (respectively $c_k \in W^{ss}(b_{k+1})$). Let γ_k be the submanifold containing c_k such that $W^u(T_k)$ transversely intersects $W^s(N)$ along γ_k, for all $k = 1, \dots, n-1$. Since $W^u(T_k) \pitchfork W^s(T_{k+1})$ at c_k, it is easy to see that γ_k is transverse to $W^s(T_{k+1})$ in the neighborhood of c_k and thus “transverse" to the stable foliation. Therefore, it is a graph over its $\widehat{\gamma}_k := P^s(\gamma_k)$ in the neighborhood of c_k, where

$$\begin{array}{rccc} P^s: & W^s(N) & \longrightarrow & N \\ & x & \longmapsto & x^s \end{array}$$

such that $x \in W^{ss}(x^s)$. For all $m \in \mathbb{N}$, for all $k = 1, \dots, n-1$, we set $b_{k+1}^m := g^m(b_{k+1})$ and $\widehat{\gamma}_k^m := g^m(\widehat{\gamma}_k)$.

Note that for all $k = 1, \dots, n-1$, $\widehat{\gamma}_k$ is transverse to T_{k+1} in N by the transversality of γ_k and $W^s(T_{k+1})$ in $W^s(N)$.

To prove the transitivity of transversal heteroclinic connections, it is enough to investigate the case of three tori having successive heteroclinic connections, the result will follow by induction.

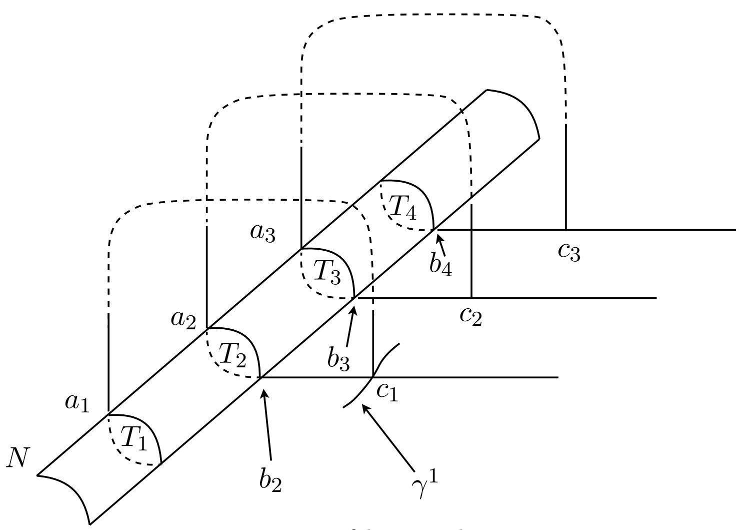

Figure 3.2: Transitivity of heteroclinic connections

More precisely, we suppose that $W^u(T_1) \pitchfork W^s(T_2)$ and $W^u(T_2) \pitchfork W^s(T_3)$. We will prove that $W^u(T_1) \pitchfork W^s(T_3)$. Let $\varepsilon > 0$ be fixed. Without any loss of generality, we can assume $\gamma_1 \subset \widehat{V}$ and that it is a graph over $\widehat{\gamma}_1$. If this is not the case, we iterate (and shrink if necessary) γ_1 and reset the counters.

- Let $\Gamma := W^u(T_1)$. The fibered λ-lemma applied to Γ implies that for q large enough, Γ^q is $\frac{\varepsilon}{2}$-close to $W^u(\widehat{\gamma}_1^q)$ in the C^1-topology (in the sense of Corollary 3.1.3).
- On the other hand, F has the strong torsion property over T_2, and $\widehat{\gamma}_1$ is transverse to T_2. Then $\exists\, \alpha \in]0; \frac{1}{2}[$ such that for all $\delta > 0$, for m large enough, $\widehat{\gamma}_1^m \cap D_m^\alpha$ is δ-close to $T_2 \cap D_m^\alpha$ in the C^1-topology (in the sense of Definition 3.3.1).
- By the C^1-regularity of the foliation of the unstable manifold, for m large enough, one has that $W^u(\widehat{\gamma}_1^m \cap D_m^\alpha)$ is $\frac{\varepsilon}{2}$-close to $W^u(T_2 \cap D_m^\alpha)$ in the C^1-topology.
- The sequence (b_2^m) lies in T_2. This latter being minimal, then for j large enough, $D_{m_j}^\alpha$ contains a_2. Therefore, for j large enough, $W^u(T_2 \cap D_{m_j}^\alpha)$ transversely intersects $W^s(T_3)$.

Then the result easily follows since transversality is an open property. □

Comments. • The transitivity of the transversal heteroclinic connections we proved in Corollary 3.3.2 provides another tool to prove the existence of the drifting orbits for systems satisfying the assumptions of the corollary.

• In order to prove the transitivity of transversal heteroclinic connections, we need the fibered λ-lemma that ensures the C^1-convergence of Γ, which requires a suitable normal form. If one is able to prove the C^1-convergence with an assumption weaker than the normal form, then the transitivity follows too. The next step will be to find a suitable fibered λ-lemma with weaker assumptions.

3.3.2 A system having the strong torsion property

We now investigate the case of a particular system useful for the next chapter and prove that this system has the strong torsion property. We consider the case where $N \subset \mathbb{T} \times \mathbb{R}$ and $F_{|N} = g$

is the following rotation $g : \begin{array}{ccc} N & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (\theta,r) & \longmapsto & (\theta+r,r). \end{array}$

Proposition 3.3.3. *Let F be as above. Let $r_0 \in \mathbb{R}$. Then F has the strong torsion property over $T := \mathbb{T}\times\{r_0\}$.*

Proof. Let $\widehat{\gamma}$ be a C^1-submanifold that transversely intersects T in N at a point $b := (\theta_0, r_0)$. Given a neighborhood I_r of r_0 in $\mathbb{R}$, we set

$$D_r := \{(\theta,r)\in \mathbb{T}\times\mathbb{R}\,;r\,\in I_r\}$$

For all $m \in \mathbb{N}$, we set $b^m := g^m(b)$ and $\widehat{\gamma}^m := g^m(\widehat{\gamma})$.

By the transversality of $\widehat{\gamma}$ and T in $\mathbb{T}\times\mathbb{R}$, following the same method as in Section 2.2.3, there exist a neighborhood I_r of r_0 in $\mathbb{R}$ and a C^1-map $k : I_r \longrightarrow \mathbb{T}$ such that

$$\widehat{\gamma}\cap D_r = \{(k(r),r); r\in I_r\}$$

and $k(r_0) = \theta_0$. Note that for all $m \in \mathbb{N}$,

$$g^m(\widehat{\gamma}\cap D_r) = \widehat{\gamma}^m \cap D_r = \{(k(r)+mr,r); r\in I_r\}.$$

Let $m_0 \in \mathbb{N}$ such that, for all $r \in I_r$, $k'(r) + m_0 > 0$, that is, such that $k_{m_0} := k + mId_{|I_r}$, when seen as a map with values in $\mathbb{R}$, is increasing on I_r. Up to shrinking I_r if necessary, we can suppose that $k_{m_0}(I_r)$ is centered at $\theta_0 + m_0r_0$ and of radius $\alpha \in]0; \frac{1}{2}[$. We set

$$I_\theta := k_{m_0}(I_r).$$

Therefore, k_{m_0} is a diffeomorphism from I_r onto $I_\theta \subset \mathbb{T}$. Let $h_{m_0} := (k_{m_0})^{-1}$. Then,

$$\widehat{\gamma}^{m_0}\cap D_r = \{(k_{m_0}(r),r); r\in I_r\} = \{(\theta, h_{m_0}(\theta)); \theta\in I_\theta\}\,. \tag{3.15}$$

Therefore, for all $m \geq m_0$, $\widehat{\gamma}^m \cap D_r = \{(\theta + (m-m_0)h_{m_0}(\theta), h_{m_0}(\theta)); \theta \in I_\theta\}$. Let

$$h_m := Id_{|I_\theta} + (m-m_0)h_{m_0}.$$

When seen as a map with values in $\mathbb{R}$, h_m is increasing on I_θ, for all $m \geq m_0$, since h_{m_0} is increasing, hence it is a diffeomorphism from I_θ onto its image $h_m(I_\theta)$. Moreover, it is easy to see that for all $m \geq m_0$, $h_m(I_\theta)$ strictly contains a segment I_m centered at θ_0+mr_0 and of radius $\alpha \in]0; \frac{1}{2}[$. Now, let us look at h_m as having its values in $\mathbb{T}$. It follows from above that for all $m \geq m_0$, if we set

$$D_m^\alpha := \{(\theta,r)\in N\,;\, d\,(\theta,\theta_0+mr_0) < \alpha\} = \{(\theta,r)\in N\,;\,\theta\,\in\, I_m\}\,,$$

then the connected component of $\widehat{\gamma}^m \cap D_m^\alpha$ containing b^m is the graph of

$$H_m := h_{m_0}\circ h_m^{-1}$$

(defined on I_m). Now let us investigate the C^1-convergence of $(H_m - r_0)$. Since h_{m_0} is increasing, there exist two real numbers C_1^+ and C_2^+ such that for all $\theta \in I_\theta$,

$$0 < C_1^+ \leq h'_{m_0}(\theta) \leq C_2^+. \tag{3.16}$$

Therefore, for all $m \geq m_0$,

$$\begin{aligned} \|H_m - r_0\|_{C^1(I^m)} &= \|H_m - r_0\|_{C^0} + \|H'_m\| \\ &= \sup_{\varphi \in I^m} \left\| h_{m_0} \circ h_m^{-1}(\varphi) - h_{m_0} \circ h_m^{-1}(\theta_0 + m r_0) \right\| + \left\| (h_{m_0} \circ h_m^{-1})' \right\| \\ &\leq (1+\alpha) \left\| (h_{m_0} \circ h_m^{-1})' \right\|, \end{aligned}$$

by the mean value theorem. On the other hand, $\|(h_{m_0} \circ h_m^{-1})'\| \leq \|h'_{m_0}\|.\|(h_m^{-1})'\| \leq \frac{C_2^+}{1+(m-m_0)C_1^+}$. The convergence easily follows. This ends the proof of Proposition 3.3.3.

□

Chapter 4

Windows and estimates of diffusion times

In this chapter, we investigate another application of the fibered λ-lemma. We construct correctly aligned windows along a transition chain and using Easton's shadowing method, we deduce the existence of diffusion orbits and we estimate the diffusion time.

We will work in a particular setting. We consider f, M and N as in Section 1.4. Moreover, we suppose that $N \subset \mathbb{T} \times \mathbb{R}$ and that $f_{|_N}$ is an integrable twist map. In addition, we suppose that f admits a specific normal form near N, which will enable us to use the version of the fibered λ-lemma that ensures the C^1-convergence of any manifold transversely intersecting $W^s(N)$ (Corollary 3.1.3). We also suppose that N contains a transition chain of Diophantine circles. We will deal with this specific context, because our goal is to set out a simple construction where we highlight the different quantities that are involved when computing diffusion times. More precisely, we prove that the time needed to drift along our transition chain splits into three characteristic parameters: the *ergodization* time associated with each circle of the chain, the *straightening* time given by the estimates in the fibered λ-lemma, and the *torsion* time on each circle.

Moreover, our construction is well chosen so that the estimates do not deteriorate with each iteration: the time needed to wander ε-close from one circle to the next is uniform with respect to the chain and depends on parameters originating from our assumptions. We think of our construction as a first step to compute diffusion times in a more complex context.

The chapter is organized as follows. In the first section, we describe the assumptions for our results and we state Theorem 4 that gives one proof for the existence of orbits drifting along our transition chain (another proof can be deduced from Corollary 2.4.2, up to a slight generalization like we will do in Proposition 5.5.4 to prove the existence of an orbit passing arbitrarily close to *each* torus of the chain). The method is based on the construction of correctly aligned windows. We give a brief reminder on this method in Section 4.2. In Section 4.3, we compute the ergodization times for the rotation on the Diophantine circles $\mathbb{T} \times \{r_i\}$. We prove our first result in Section 4.4. In Section 4.5, we state and prove our second result Corollary 4.5.1 which gives estimates of the time needed for the orbit (whose existence was proved in Theorem 4) to drift.

4.1 Assumptions and first result

Let us first describe the assumptions for the results of this chapter (Theorem 4 below and Corollary 4.5.1 in Section 4.5).

• *A(1)* We consider f, M and N as in Section 1.4 and F as in Section 2.1.1. We suppose that $N = \mathbb{T} \times I \subset \mathbb{T} \times \mathbb{R}$ and that it is equipped with the angle-action variables (θ, r). Moreover, we suppose that $F_{|_N} : (\theta, r) \longmapsto (\theta + r, r)$, for all $(\theta, r) \in N$, so that N is foliated with invariant circles $\mathbb{T} \times \{r\}$, $r \in I$.

• *A(2)* Let $\widehat{V}$ be as in Proposition 3.1.1. We suppose that for all $(x, s, u) \in \widehat{V}$, $F_x(x, s, u) = g(x)$, where $g = F_{|_N}$.

• *A(3)* We suppose that N contains a transition chain of Diophantine circles. More precisely, we suppose that there exists a finite family of invariant circles $(T_i)_{1\leq i\leq n}$ in N such that $T_i = \mathbb{T} \times \{r_i\}$ for all $i = 1, \ldots, n$, where r_i is (C_i, τ)-Diophantine with $C_i > 0$ and $\tau \geq 1$ (see Definition 1.2.3). Moreover, for all $i = 1, \ldots, n-1$, we suppose that the unstable manifold of T_i transversely intersects the stable manifold of T_{i+1}.

An example of a dynamical system that satisfies the assumptions *A(1)*, *A(2)* and *A(3)* can be found in [LM05]. One only needs to note that the Diophantine circles with exponent $\tau \geq 1$ are dense in $\mathbb{T} \times \mathbb{R}$.

We now state Theorem 4 which yields the existence of a diffusion orbit along the transition chain.

Theorem 4. *We consider a system that satisfies A(1), A(2) and A(3). Then, for all $\varepsilon > 0$, there exist an orbit $(x_i)_{1\leq i\leq n}$ and a sequence of positive integers $(k_i)_{1\leq i\leq n-1}$ such that:*

$$\begin{aligned} &d(x_i, T_i) < \varepsilon, \ \text{ for all } i = 1, \ldots, n, \\ &x_{i+1} = f^{k_i}(x_i), \ \text{ for all } i = 1, \ldots, n-1, \end{aligned}$$

where k_i splits into three characteristic parameters and will be made explicit in Corollary 4.5.1.

We will prove Theorem 4 in Section 4.4 by means of windows that properly lign up, as we will see in the next section. In Section 4.5, we will give explicit estimates of the diffusion time $\sum\limits_{i=1}^{n-1} k_i$.

4.2 Correctly aligned windows

To prove the results of this chapter, we will use Easton's method of correctly aligned windows. In this section, we give a brief reminder on this method. For a more elaborate study, we refer to [Eas81] and to [GR07] and [GR09] for a topological version of Easton's method. We first recall the definition of Easton's correctly aligned windows.

Definition 4.2.1. **[Windows.]** Let M be a d-dimensional manifold and let d_h and d_v be two positive integers such that $d_h + d_v = d$. A (d_h, d_v)-window R in M is the image of the rectangle $[-1, 1]^{d_h} \times [-1, 1]^{d_v}$ under a C^1-diffeomorphism B_R with values in M. The horizontals are the images of $[-1, 1]^{d_h}$ under $B_R(., y_v)$ for all $y_v \in [-1, 1]^{d_v}$, and the verticals are the images of $[-1, 1]^{d_v}$ under $B_R(y_h, .)$ for all $y_h \in [-1, 1]^{d_h}$. The exit set is $B_R\left(\partial[-1, 1]^{d_h} \times [-1, 1]^{d_v}\right)$.

Definition 4.2.2. **[Correctly aligned windows.]** Let M be a manifold and let R_1, R_2 be two (d_h, d_v)-windows in M. We say that R_1 is correctly aligned with R_2 if :
- each horizontal $B_{R_1}\left([-1,1]^{d_h}, y_v\right)$ of R_1 transversely intersects each vertical $B_{R_2}\left(y_h, [-1,1]^{d_v}\right)$ of R_2,
- the intersection of $B_{R_1}\left([-1,1]^{d_h}, y_v\right)$ and $B_{R_2}\left(y_h, [-1,1]^{d_v}\right)$ is a unique point a such that $a = B_{R_1}(x_h, y_v) = B_{R_2}(y_h, x_v)$ where $(x_h, x_v) \in\ ]-1,1[^d$.

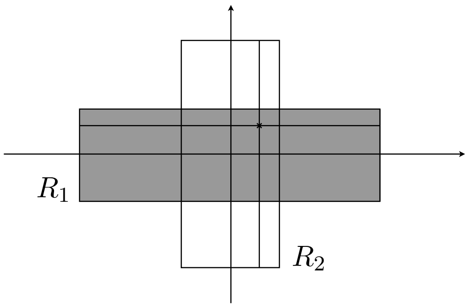

Figure 4.1: Correctly aligned windows

The next Theorem (see [Eas81]) states that, given a family of correctly aligned windows, there exists an orbit that crosses each window.

Theorem 4.2.3. [Easton.] Let $(R_i)_{1\le i\le n}$ be a sequence of (d_h, d_v)-windows in M and let $(F_i)_{1\le i\le n-1}$ be a sequence of diffeomorphisms on M such that $F_i\left(R_i\right)$ is correctly aligned with R_{i+1} for all $i = 1, \ldots, n-1$. Then, there exists a point z in R_1 such that for all $i = 1, \ldots, n-1$

$$F_i \circ \ldots \circ F_1(z) \in R_{i+1}.$$

In our construction, the maps F_i will represent different powers of the diffeomorphism f that is given by the assumptions of Theorem 4. The sum of these powers is the drifting time.

4.3 Ergodization times for the rotations of T_i

Here, we keep the notation of Section 1.2. In order to compute diffusion times in Section 4.5, we need to compute the ergodization time for the rotation F_{r_i} on each circle T_i. In Section 1.2, we stated results on ergodization times in a general context (dimension $n \ge 1$, nonresonant and resonant at a high level vectors). These results follow from the works of Dumas, Bourgain, Golse, Wennberg, Berti, Bolle, Biasco,...

In this section, we adapt these results to our context, that is, when $n = 1$ and (further in this section) when r_i is Diophantine of exponent $\tau \ge 1$. Moreover, we give a new (and simple) proof of Theorem 1.2.2 in the case $n = 1$ by means of Dirichlet's box principle, instead of the common continued fractions method.

Let $r \in \mathbb{R}$. Recall that $F_r : \mathbb{T} \to \mathbb{T}$ maps θ to $F_r(\theta) = \theta + r \bmod 1$ and that $q_r(\alpha)$ is the time, when it exists, needed for r to α-ergodize $\mathbb{T}$.

When $r \notin \mathbb{Q}$, the rotation is minimal and $q_r(\alpha)$ exists for any $0 < \alpha \le \frac{1}{2}$. When $r = \frac{s}{p} \in \mathbb{Q}^*$ $(p \in \mathbb{N}^*)$, it is easy to see that $q_{s/p}(\alpha)$ exists if and only if $\alpha \ge p^{-1}$, in which case $q_{s/p}(\alpha) \le p-1$.

From now on, we suppose that $r \notin \mathbb{Q}$. Recall that $|\,.\,|_{\mathbb{Z}}$ is the distance to $\mathbb{Z}$, that is,

$$|x|_{\mathbb{Z}} = \inf \{|x-s|\ ;\ s \in \mathbb{Z}\}$$

and that for $K > 1$, we set

$$\begin{aligned} \Psi_r(K) := & \max \left\{ |pr|_{\mathbb{Z}}^{-1}\ ; p \in \mathbb{Z},\ 1 \leq |p| \leq K \right\} \\ = & \max \left\{ |pr|_{\mathbb{Z}}^{-1}\ ; p \in \mathbb{Z},\ 1 \leq p \leq K \right\} \end{aligned}$$

which is always finite. Ψ_r is a non decreasing unbounded and piecewise constant function defined on $]1, +\infty[$.

Let us describe some additional properties satisfied by Ψ_r. By Dirichlet's box principle, for any $K > 1$, one can always find $(p, s) \in \mathbb{N}^* \times \mathbb{Z}$ such that

$$1 \leq p < K \quad \text{and} \quad |pr - s| \leq K^{-1}.$$

Therefore, for any $K > 1$,

$$\Psi_r(K) \geq K, \tag{4.1}$$

that is, the function Ψ_r grows *at least linearly* for any $r \notin \mathbb{Q}$.

In the Diophantine case, we can moreover characterize an upper bound for the growth. More precisely, let $c > 0$ and $\tau \geq 1$, and r be (c, τ)-Diophantine. Then $|p\alpha|_{\mathbb{Z}} \geq cp^{-\tau}$ for any $p \in \mathbb{N}^*$. This yields that, for any $K > 1$,

$$\Psi_r(K) \leq c^{-1} K^{\tau}, \tag{4.2}$$

that is, Ψ_r grows *at most polynomially* for r Diophantine.

The next lemma gives the ergodization time for any irrational r.

Lemma 4.3.1. *Let $r \in \mathbb{R} \setminus \mathbb{Q}$ and $\alpha \in]0; \frac{1}{2}]$. Then*

$$q_r(\alpha) \leq \Psi_r(2\alpha^{-1}).$$

Proof. It follows from Property (4.1) that $\Psi_r(2\alpha^{-1}) \geq 2\alpha^{-1} > 1$. Therefore, one can apply Dirichlet's box principle to $\Psi_r(2\alpha^{-1})$. More precisely, there exists $(p, s) \in \mathbb{N}^* \times \mathbb{Z}$ such that

$$|pr - s| \leq \left(\Psi_r(2\alpha^{-1})\right)^{-1} \tag{4.3}$$

and

$$1 \leq p < \Psi_r(2\alpha^{-1}). \tag{4.4}$$

Thanks to Inequality (4.4), it is enough to prove that $q_r(\alpha) \leq p - 1$. Inequality (4.3) and the definition of Ψ_r yield that $\Psi_r(p) \geq \Psi_r(2\alpha^{-1})$, which implies that $p \geq 2\alpha^{-1}$ (since Ψ_r is non decreasing) and thus that $\alpha/2 \geq p^{-1}$. Therefore,

$$q_{s/p}(\alpha/2) \leq p - 1. \tag{4.5}$$

Using Inequality (4.3) again and the fact that $\left(\Psi_r(2\alpha^{-1})\right)^{-1} \leq \alpha/2$ (which follows from Property (4.1)), it is easy to see that the distance between the two sets $\{\theta, F_r(\theta), \ldots, F_r^{p-1}(\theta)\}$ and $\{\theta, F_{s/p}(\theta), \ldots, F_{s/p}^{p-1}(\theta)\}$ for $\theta \in \mathbb{T}$, is at most $\alpha/2$. Since the latter set is $\alpha/2$-dense by Inequality (4.5), then the former set is α-dense. Therefore, $q_r(\alpha) \leq p - 1$. This ends the proof of Lemma 4.3.1. $\square$

In the Diophantine case, one recovers Corollary 1.2.3. More precisely, we have the following corollary.

Corollary 4.3.2. *Let* $\alpha \in]0; \frac{1}{2}]$ *and let* r *be* (c, τ)*-Diophantine, where* $c > 0$ *and* $\tau \geq 1$*. Then*

$$q_r(\alpha) \leq c^{-1} 2^\tau \alpha^{-\tau}.$$

Proof. The proof is immediate using Lemma 4.3.1 and Property (4.2). □

Let $(T_i)_{1 \leq i \leq n}$ be as in Assumption *A(3)*. Since for all $i = 1, \ldots, n$, r_i is (C_i, τ)-Diophantine ($C_i > 0$,$\tau \geq 1$), then, for all $\alpha \in]0; \frac{1}{2}]$,

$$q_{r_i}(\alpha) \leq C_i^{-1} 2^\tau \alpha^{-\tau}. \tag{4.6}$$

4.4 Proof of Theorem 4

In this section, we prove Theorem 4. We construct correctly aligned windows arbitrarily close to the transition chain. Theorem 4.2.3 yields the existence of an orbit which intersects these windows. Moreover, we give preliminary estimates of the diffusion time that will be computed in Section 4.5.

The mechanism consists of three stages due to the features of our system: assumption *A(1)* and Proposition 3.3.3 yield that F has the *stong torsion property* over each circle of the chain. The alignment of our windows in the central direction will follow from this property. Assumption *A(2)* and assumption *A(3)* set the context to apply the fibered λ-lemma (Corollary 3.1.3) in order to prove the *straightening* of our horizontals and thus align our windows in the stable direction. In addition, the control of the *ergodization* follows from assumption *A(3)*, which will enable us to carry our windows around the circles.

4.4.1 Set-up and notation

Let us start by fixing the notation. For all $i = 1, \ldots, n-1$, we fix c_i in $W^u(T_i) \cap W^s(T_{i+1})$. We call a_i the point in T_i and b_{i+1} the point in T_{i+1} such that $c_i \in W^{uu}(a_i) \cap W^{ss}(b_{i+1})$ for all $i = 1, \ldots, n-1$. We call γ_i the 1-dimensional (relatively compact) submanifold containing c_i such that $W^u(T_i)$ transversely intersects $W^s(N)$ along γ_i.
We introduce the following maps

$$\begin{array}{rccc} P^u: & W^u(N) & \longrightarrow & N \\ & x & \longmapsto & x^u \end{array}$$

such that $x \in W^{uu}(x^u)$ and

$$\begin{array}{rccc} P^s: & W^s(N) & \longrightarrow & N \\ & x & \longmapsto & x^s \end{array}$$

such that $x \in W^{ss}(x^s)$. For all $i = 1, \ldots, n-1$, let

$$\widehat{\gamma}_i := P^s(\gamma_i).$$

As we mentioned in the proof of Corollary 3.3.2, since $W^u(T_i)$ is transverse to $W^s(T_{i+1})$ for all $i = 1, \ldots, n-1$, then γ_i is transverse to $W^s(T_{i+1})$ in the neighborhood of c_i and thus "transverse" to the stable foliation. Therefore, it is a graph over $\widehat{\gamma}_i$ in the neighborhood of c_i in the

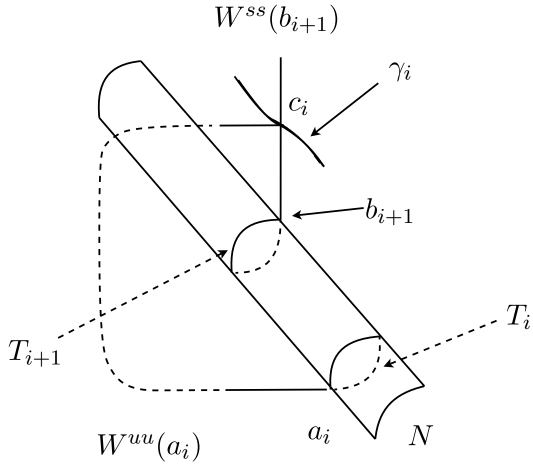

Figure 4.2: Transition chain

sense that it intersects each stable leaf of the points in $\widehat{\gamma}_i$ once at most. Therefore, without any loss of generality, we can suppose that γ_i is a graph over $\widehat{\gamma}_i$ (one only needs to take the restriction to the appropriate neighborhood and keep the notation) or equivalently, that the restriction of P^s to γ_i is a diffeomorphism.

• We give here a brief reminder on the notion of the scattering maps. We do not mean to give the most general definition, we rather limit ourselves to that needed for our context. For more details on scattering maps, we refer to [DDLLS06] and the references therein.

Definition 4.4.1. **[Scattering map.]** *Let N be a normally hyperbolic manifold, and let P^s and P^u be as above. Let γ be a submanifold in the homoclinic intersection $W^u(N) \cap W^s(N)$. We set $\widehat{\gamma} := P^s(\gamma)$. We suppose that ${P^s}_{|_\gamma} : \gamma \longmapsto \widehat{\gamma}$ is invertible. The scattering map $\mathcal{S}_\gamma$ associated with γ is the map defined on $\widehat{\gamma}$ with values in N such that $\mathcal{S}_\gamma = P^u \circ \left({P^s}_{|_\gamma}\right)^{-1}$.*

Note that the definition of the scattering map depends on the homoclinic manifold γ. In the sequel, we will drop the subscript γ from the notation $\mathcal{S}_\gamma$ since it will not lead to confusion depending on the context. It is then easy to see that for all $i = 1, \dots, n-1$, $\mathcal{S}(\widehat{\gamma}_i) \subset T_i$.

In order to define the "size" of the windows that we will construct in the following sections, we need first to introduce two parameters α and ϱ, parameters whose existence follows from the assumptions and that will be *uniform* with respect to the chain. The size of the windows in the *horizontal* directions will depend on α and ϱ (as we will see in Equations (4.9) and (4.10)). The existence of the parameter α will follow from the strong torsion property of F over the circles of the chain, and that of the parameter ϱ from the fibered λ-lemma applied to $(W^u(T_i))$.

• **Definition of** α**.** Note that for all $i = 1, \dots, n-1$, $\widehat{\gamma}_i$ is transverse to T_{i+1} in N by the transversality of γ_i and $W^s(T_{i+1})$ in $W^s(N)$. Therefore, one can apply the strong torsion property to $(\widehat{\gamma}_i)$. More precisely, thanks to Proposition 3.3.3, for all $i = 1, \dots, n-1$, there exist $j_i \in \mathbb{N}$ and $\alpha_i \in]0; \frac{1}{2}[$ such that $\forall\, m \geq j_i$, the connected component of $g^m(\widehat{\gamma}_i) \cap D_m^{\alpha_i}$ containing $g^m(b_{i+1})$ is a graph over $\Pi_\theta(D_m^{\alpha_i})$ of a C^1-map $h^{i,m}$ that satisfies:

$$\lim_{m\to\infty} \|h^{i,m} - r_{i+1}\|_{C^1\left(\Pi_\theta\left(D_m^{\alpha_i}\right)\right)} = 0.$$

Let

$$j := \max_{1\le i\le n-1} j_i \quad \text{and} \quad \alpha = \min_{1\le i\le n-1} \alpha_i. \tag{4.7}$$

Since we are looking for orbits wandering arbitrarily close to N, the windows will be defined in the coordinate system given by Proposition B (Section 1.1.3), where the stable and unstable manifolds and leaves are straightened. Let $\widehat{V}$ be as in Proposition 3.1.1 and let $\widehat{\mathcal{U}} := \varphi^{-1}\left(\widehat{V}\right)$. Without any loss of generality, we can suppose that the homoclinic points (c_i) are chosen so that $j = 0$ and for all $i = 1, \dots, n-1$,

$$\gamma_i \in \widehat{\mathcal{U}}.$$

For notational simplicity, we keep the same notation for (γ_i), $(\widehat{\gamma}_i)$, (a_i) and (b_i), and their identifications in $\widehat{V}$ by φ. For all $i = 1, \dots, n-1$ and for all $m \in \mathbb{N}$, we set $a_i^m := g^m(a_i)$. The same holds for the iterates of b_i and $\widehat{\gamma}_i$ by g, and the iterates of γ_i and c_i by F. In the sequel, we will identify f with F since we will be mainly interested in their powers.

• **Definition of ϱ.** Since for all $i = 1, \dots, n-1$, $W^u(T_i)$ transversely intersects $W^s(N)$ along γ_i, using the same reasoning as in Section 3.2.1 and by reducing γ_i (and resetting α) if necessary (and keeping the same notation), there exist a real number $\varrho_i' > 0$, a submanifold $\Gamma_i \subset W^u(T_i)$ containing γ_i and a C^2-map $\widehat{S}_i$ defined on $\widehat{\gamma}_i \times B_{\varrho_i'}$ with values in $\mathcal{B}_s^p$ such that Γ_i is the image of the following diffeomorphism:

$$\begin{array}{cccc} \psi^i: & \widehat{\gamma}_i \times B_{\varrho_i'} & \longrightarrow & D_{\varrho_i'} \\ & (x,u) & \longmapsto & (x, \widehat{S}_i(x,u), u), \end{array}$$

that is, Γ_i is the graph of $\widehat{S}_i$. Thanks to Assumption *A(2)*, we can apply the fibered λ-lemma (Corollary 3.1.3) to Γ_i for all $i = 1, \dots, n-1$, which yields that, for all $i = 1, \dots, n-1$, there exists $\varrho_i > 0$ such that, for all $m \in \mathbb{N}$, there exits a C^2-map $\widehat{S}^{i,m}$ defined on $\widehat{\gamma}_i^m \times B_{\varrho_i}$ with values in $\mathcal{B}_s^p$ such that $\Gamma_i^m \cap D_{\varrho_i}$ is the image of the following diffeomorphism:

$$\begin{array}{cccc} \psi^{i,m}: & \widehat{\gamma}_i^m \times B_{\varrho_i} & \longrightarrow & D_{\varrho_i} \\ & (x,u) & \longmapsto & (x, \widehat{S}^{i,m}(x,u), u), \end{array}$$

that is, $\Gamma_i^m \cap D_{\varrho_i}$ is the graph of $\widehat{S}^{i,m}$. Moreover,

$$\lim_{m\to\infty} \|\widehat{S}^{i,m}\|_{C^1(\widehat{\gamma}_i^m \times B_{\varrho_i})} = 0.$$

Note that we identified $\widehat{S}_i$ with $\widehat{S}^{i,0}$. We set

$$\varrho := \min_{1\le i\le n-1} \varrho_i. \tag{4.8}$$

4.4.2 The static/mobile windows

We will use Theorem 4.2.3 to prove Theorem 4. We will construct a sequence of windows $(R_i)_{1\le i\le n}$ (the *static* windows) such that R_i is arbitrarily close to T_i, for all $i = 1, \dots, n$. We will also construct another sequence of windows $(\widetilde{R}_i)_{1\le i\le n-1}$ (the *mobile* windows) such that for all $i = 1, \ldots, n-1$, R_i is correctly aligned with $\widetilde{R}_i$, and $f^{k_i}\left(\widetilde{R}_i\right)$ is correctly aligned with R_{i+1}, where k_i is a suitable integer that will be computed. We will use Theorem 4.2.3 to deduce the existence of an orbit that intersects each R_i and thus is arbitrarily close to each T_i, for $i = 1, \ldots, n$.

4.4.2.1 The static windows

We will now construct the sequence of *static* windows $(R_i)_{1\leq i\leq n}$. Let $\varepsilon > 0$ be fixed. The *static* windows $(R_i)_{1\leq i\leq n}$ will be defined in the coordinate system given by Proposition B (Section 1.1.3). Let $\widehat{V}$ be as in Proposition 3.1.1. The window R_i (which is in $\mathbb{T}\times\mathbb{R}\times\mathbb{R}_s^p\times\mathbb{R}_u^p$) will take the following form:

$$R_i = a_i^{-n_0} + [-\varepsilon_\theta, \varepsilon_\theta] \times \left[-\frac{\varepsilon_r}{2}, \frac{\varepsilon_r}{2}\right] \times \left[-\frac{\varepsilon_s}{2}, \frac{\varepsilon_s}{2}\right]^p \times [-\varepsilon_u, \varepsilon_u]^p,$$

where n_0 is an integer, ε_θ, ε_r, ε_s and ε_u are positive real numbers, all to be defined in the sequel. The horizontal directions will be (θ, u) and the vertical ones (r, s).

Let ε_r, ε_s and ε_u be positive real numbers small enough so that, for all $i = 1, \ldots, n$,

$$V_i := \mathbb{T} \times [r_i - \varepsilon_r, r_i + \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p \times [-\varepsilon_u, \varepsilon_u]^p \ \subset\ \widehat{V}$$

and

$$\forall\, z \,\in\, V_i,\ \ d(z, T_i) < \varepsilon.$$

• **Definition of ε_θ and ε_u.** Up to reducing ε_u if necessary, we suppose that there exists $\varepsilon_u' > \varepsilon_u$ such that

$$[-\varepsilon_u, \varepsilon_u]^p \subsetneq [-\varepsilon_u', \varepsilon_u']^p \subset B_{\frac{\varrho}{2}}, \tag{4.9}$$

where ϱ was given in Equation (4.8) and $B_{\frac{\varrho}{2}} = \{u \in \mathbb{R}_u^p;\ \|u\| < \frac{\varrho}{2}\}$. We choose $\varepsilon_\theta > 0$ such that

$$\varepsilon_\theta < \frac{\alpha}{4}, \tag{4.10}$$

where α was given in Equation (4.7).

• **Definition of n_0.** For all $i = 1, \ldots, n-1$, let

$$\mathcal{H}_{0,0}^i = \mathbb{T} \times \{r_i\} \times \{0\} \times \left[-\frac{\varepsilon_u}{2}, \frac{\varepsilon_u}{2}\right]^p,$$

which is in the straightened unstable manifold of T_i. For all $i = 1, \ldots, n-1$, let Γ_i be as in Section 4.4.1. Let n_0 be in $\mathbb{N}$ such that, for all $i = 1, \ldots, n-1$,

$$F^{-n_0}(\Gamma_i) \ \subset\ \mathcal{H}_{0,0}^i. \tag{4.11}$$

Such an integer exists by the definition of the global unstable manifold of T_i. Note that we identified $\widehat{\mathcal{U}}$ with $\widehat{V}$ and f with F outside $\widehat{\mathcal{U}}$ (see Section 4.4.1).

• **Definition of ε_r and ε_s.** We now introduce some notation useful to state the following lemma. For $\varepsilon_r > 0$ and $\varepsilon_s > 0$ small enough, and for $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$, we set, for all $i = 1, \ldots, n-1$,

$$\mathcal{H}_{x_r,x_s}^i = \mathbb{T} \times \{r_i + x_r\} \times \{x_s\} \times \left[-\frac{\varepsilon_u}{2}, \frac{\varepsilon_u}{2}\right]^p. \tag{4.12}$$

We will think of $\left\{\mathcal{H}^i_{x_r,x_s};\,(x_r,x_s)\in[-\varepsilon_r,\varepsilon_r]\times[-\varepsilon_s,\varepsilon_s]^p\right\}$ as a "thickening" of $\mathcal{H}^i_{0,0}$ in the directions r and s. The next lemma gives the existence of a "thickening" of Γ_i with similar submanifolds, on which we have *full control* of the strong torsion, and the straightening by the fibered λ-lemma.

Lemma 4.4.2. *Let n_0 be as in Equation (4.11). There exist $\varepsilon_r>0$ and $\varepsilon_s>0$ small enough such that, for all $i=1,\dots,n-1$, for all $(x_r,x_s)\in[-\varepsilon_r,\varepsilon_r]\times[-\varepsilon_s,\varepsilon_s]^p$, there exists a submanifold $\Gamma^i_{x_r,x_s}\subset F^{n_0}\left(\mathcal{H}^i_{x_r,x_s}\right)$ such that*

1. *$\Gamma^i_{x_r,x_s}$ transversely intersects $\widetilde{W}^s(N)$ along a relatively compact submanifold $\gamma^i_{x_r,x_s}$ such that $\gamma^i_{x_r,x_s}$ is a graph over $\widehat{\gamma}^i_{x_r,x_s}:=P^s(\gamma^i_{x_r,x_s})$,*
2. *the fibered λ-lemma can be applied to $\Gamma^i_{x_r,x_s}$ with $\varrho^i_{x_r,x_s}>\frac{\varrho}{2}$ and the correspondent C^2-maps $\widehat{S}^{i,m}_{x_r,x_s}$ such that*
$$\lim_{m\to\infty}\left\|\widehat{S}^{i,m}_{x_r,x_s}\right\|_{C^1\left(\left(\widehat{\gamma}^i_{x_r,x_s}\right)^m\times B_{\varrho^i_{x_r,x_s}}\right)}=0.$$
3. *$\widehat{\gamma}^i_{x_r,x_s}$ is transverse to T_{i+1} at a point $b^i_{x_r,x_s}$ such that $d(b^i_{x_r,x_s},b_{i+1})<\frac{\alpha}{8}$,*
4. *there exists $\alpha^i_{x_r,x_s}>\frac{\alpha}{2}$ such that the strong torsion property applies to $\widehat{\gamma}^i_{x_r,x_s}$ with $\alpha^i_{x_r,x_s}$ and the correspondent C^1 graph maps $\left(h^{i,m}_{x_r,x_s}\right)_{m\geq 0}$ and strips in the θ-variable $D^{\alpha^i_{x_r,x_s}}_{\substack{m\\x_r,x_s}}$ centered on $b^i_{x_r,x_s}$.*

where we identified $\Gamma^i_{0,0}$ with Γ_i, $\gamma^i_{0,0}$ with γ_i, $\widehat{\gamma}^i_{0,0}$ with $\widehat{\gamma}_i$, $\varrho^i_{0,0}$ with ϱ_i, $\widehat{S}^{i,m}_{0,0}$ with $\widehat{S}^{i,m}$, $b^i_{0,0}$ with b_{i+1}, $\alpha^i_{0,0}$ with α_i and $h^{i,m}_{0,0}$ with $h^{i,m}$.

Proof. The proof is immediate because transversality is an open property, and because of the definition of α (see the proof of Proposition 3.3.3) and the definition of ϱ (see Equation (3.6)). ☐

In the sequel, for notational simplicity, we will drop the subsubscript x_r,x_s from $D^{\frac{\alpha}{2}}_{\substack{m\\x_r,x_s}}$ since we will only use the latter in the context: $\operatorname{graph}(h^{i,m}_{x_r,x_s})\cap D^{\frac{\alpha}{2}}_{\substack{m\\x_r,x_s}}$, so whenever we write $\operatorname{graph}(h^{i,m}_{x_r,x_s})\cap D^{\frac{\alpha}{2}}_m$, we mean the intersection with the strip in the θ-variable of radius $\frac{\alpha}{2}$ centered at $(b^i_{x_r,x_s})^m$.

Let ε_u be as in Equation (4.9), ε_θ as in Equation (4.10), and ε_r and ε_s as in Lemma 4.4.2. For all $i=1,\dots,n$, we define the static window (with a slight abuse of notation) as follows
$$R_i=a_i^{-n_0}+[-\varepsilon_\theta,\varepsilon_\theta]\times\left[-\frac{\varepsilon_r}{2},\frac{\varepsilon_r}{2}\right]\times\left[-\frac{\varepsilon_s}{2},\frac{\varepsilon_s}{2}\right]^p\times[-\varepsilon_u,\varepsilon_u]^p,\tag{4.13}$$
where n_0 is given by Estimate (4.11). Namely, for all $i=1,\dots,n$, the window R_i is centered at $a_i^{-n_0}$. The horizontal directions are (θ,u) and the vertical ones are (r,s). The horizontals are
$$H^i_{x_r,x_s}=a_i^{-n_0}+[-\varepsilon_\theta,\varepsilon_\theta]\times\{x_r\}\times\{x_s\}\times[-\varepsilon_u,\varepsilon_u]^p$$
for $(x_r,x_s)\in\left[-\frac{\varepsilon_r}{2},\frac{\varepsilon_r}{2}\right]\times\left[-\frac{\varepsilon_s}{2},\frac{\varepsilon_s}{2}\right]^p$, which will also form a thickening of $H^i_{0,0}$ (which lies in $\widetilde{W}^u(T_i)$, that is, the straightened unstable manifold of T_i) in the directions r and s.

Note that for all $i=1,\dots,n$, the window R_i has the same "size", which is a feature of our construction: the size of the windows are uniform with respect to the chain and does not deteriorate with each iteration. The same will hold for the mobile windows.

4.4.2.2 The mobile windows

We now state and prove the main result of this section which yields the existence of the *mobile* windows and the correct alignements needed to prove Theorem 4. Moreover, the proof of the following proposition gives the preliminary estimates of the diffusion time.

Proposition 4.4.3. *Let $(R_i)_{1\le i\le n}$ be as in Equality (4.13). Then, for all $i = 1,\dots,n-1$, there exist a window $\widetilde{R}_i$ and an integer k_i such that*

$$\begin{aligned}&R_i \textit{ is correctly aligned with } \widetilde{R}_i \textit{ and}\\ &f^{k_i}\left(\widetilde{R}_i\right) \textit{ is correctly aligned with } R_{i+1}.\end{aligned}$$

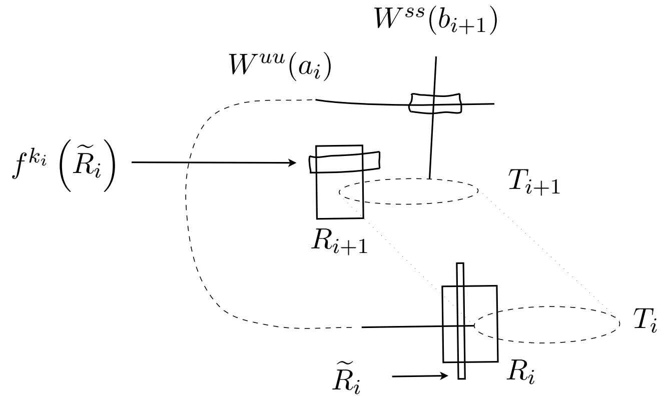

Figure 4.3: Static windows (R_i) and mobile windows $(\widetilde{R}_i)$

Proof. Let ε_u be as in Equation (4.9), ε_θ as in Equation (4.10), and ε_r and ε_s as in Lemma 4.4.2. Let n_0 be as in Estimate (4.11). Recall the definition of the following "auxiliary horizontals". For $i = 1,\dots,n-1$ and $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$,

$$\mathcal{H}^i_{x_r,x_s} = \mathbb{T} \times \{r_i + x_r\} \times \{x_s\} \times \left[-\frac{\varepsilon_u}{2}, \frac{\varepsilon_u}{2}\right]^p.$$

Fix $i = 1,\dots,n-1$. For all $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$, let $\Gamma^i_{x_r,x_s}$ be as in Lemma 4.4.2. The mobile window $\widetilde{R}_i$ will be constructed in such a way that each horizontal $\widetilde{H}^i_{x_r,x_s}$ of $\widetilde{R}_i$ will satisfy

$$f^{n_0}\left(\widetilde{H}^i_{x_r,x_s}\right) \subset \Gamma^i_{x_r,x_s}$$

(which yields in particular that $\widetilde{H}^i_{x_r,x_s}$ is contained in $\mathcal{H}^i_{x_r,x_s}$). Therefore, we start with investigating the behavior of $\Gamma^i_{x_r,x_s}$ under iteration.

The key-feature of our approach to prove the alignments is that both the fibered λ-lemma and the strong torsion give rise to *graph properties*, the first over the u-variable, and the second over the θ-variable. This will yield the transversality of the horizontals of R_i (resp. $f^{k_i}(\widetilde{R}_i)$) with the verticals of $\widetilde{R}_i$ (resp. R_{i+1}). The alignments will be due to three phenomena that arise when iterating $\cup_{x_r,x_s}\Gamma^i_{x_r,x_s}$.

1. Straightening. In order to align $\widetilde{R}_i$ with R_{i+1} under a certain iterate of f in the s-direction, we will need to iterate $\Gamma^i_{x_r,x_s}$ until the s-component of any point in the iterated manifold is of norm smaller than $\frac{\varepsilon_s}{2}$, which is the "size" of R_{i+1} in the s-direction. More precisely, we apply the fibered λ-lemma to $\Gamma^i_{x_r,x_s}$ which yields the existence, for all $i = 1,\dots,n-1$, of an integer m_i such that, for all $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$,

$$\forall\, m \geq m_i, \quad \left\| \widehat{S}^{i,m}_{x_r,x_s} \right\|_{C^0} < \frac{\varepsilon_s}{2}. \tag{4.14}$$

Later on, this will ensure the correct alignment of $f^{k_i}(\widetilde{R}_i)$ with R_{i+1} in the s-direction (see Figure (4.4) which roughly illustrates the case $p = 1$).

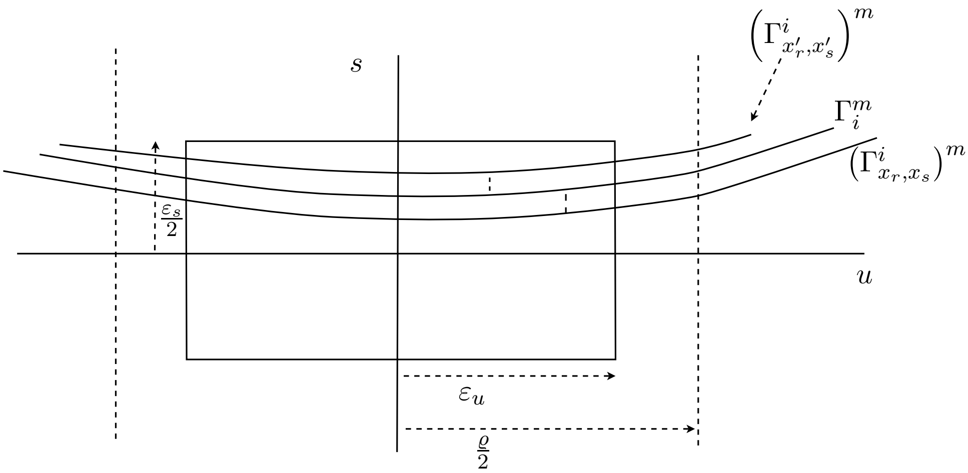

Figure 4.4: Alignment in the normal directions

2. Torsion. For all $i = 1,\dots,n-1$, for all $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$, let $\widehat{\gamma}^i_{x_r,x_s}$ be as in Lemma 4.4.2. The projections over N of the iterates of $\Gamma^i_{x_r,x_s}$ are the iterates of $\widehat{\gamma}^i_{x_r,x_s}$ (thanks to the normal form given by Assumption *A(2)*). Let us investigate the behavior of $\widehat{\gamma}^i_{x_r,x_s}$ under iteration. We will limit ourselves to the strips in the θ-variable of radius $\frac{\alpha}{2}$, over which the suitable connected components of the iterates of $\widehat{\gamma}^i_{x_r,x_s}$ are graphs.

Since F has the strong torsion property over T_{i+1} (see Proposition 3.3.3), and by Lemma 4.4.2, for all $i = 1,\dots,n-1$, there exists p_i such that for all $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$,

$$\forall\, p \geq p_i, \quad \left\| h^{i,p}_{x_r,x_s} - r_{i+1} \right\|_{C^0\left(\Pi_\theta\left(D_p^{\frac{\alpha}{2}}\right)\right)} < \frac{\varepsilon_r}{2} \quad \text{and}$$

$$\Pi_\theta \circ F^{-(p+n_0)}\left(\left(\Gamma^i_{x_r,x_s}\right)^p \cap \mathcal{D}^{\frac{\alpha}{2}}_{p,x_r,x_s}\right) \subsetneq \Pi_\theta(a_i^{-n_0}) + [-\varepsilon_\theta, \varepsilon_\theta], \tag{4.15}$$

where $\mathcal{D}^{\frac{\alpha}{2}}_{p,x_r,x_s} := \left\{X = (\theta, r, s, u) \in \widehat{V}\,;\, (\theta, r) \in \operatorname{graph}(h^{i,p}_{x_r,x_s} \cap D_p^{\frac{\alpha}{2}})\right\}$ and Π_θ is the projection over the θ-variable. The second inclusion implies in particular that, when $x_r = x_s = 0$, $\mathcal{S}\left(\operatorname{graph} h^{i,p} \cap D_p^{\frac{\alpha}{2}}\right) \subsetneq a_i^p + [-\varepsilon_\theta, \varepsilon_\theta] \times \{0\}$ where $\mathcal{S}$ is the scattering map (see Definition 4.4.1) and where we identified N with $N \times \{0\} \times \{0\}$. The first estimate will ensure, later on, the correct alignment in the central direction. The second estimate will ensure the alignment of R_i with $\widetilde{R}_i$ in the θ-direction.

3. Ergodization. Since for $i = 1,\dots,n$, r_i is (C_i, τ)-Diophantine ($C_i > 0$ and $\tau \geq 1$), then, the $\frac{\alpha}{8}$-ergodization time for the rotation on T_{i+1} exists for all $i = 1,\dots,n-1$ (see Section 4.3). More precisely, there exists $Q_i \in \mathbb{N}$ such that

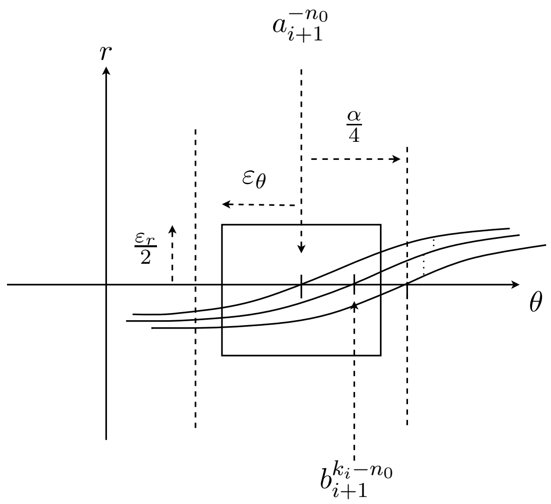

Figure 4.5: Alignment in the central directions

$$\bigcup_{0\leq k\leq Q_i} B\left(F_{r_{i+1}}^k\left(\Pi_\theta\left(b_{i+1}\right)\right), \frac{\alpha}{8}\right) \times \{r_{i+1}\} = T_{i+1}, \tag{4.16}$$

where we kept the notation of Definition 1.2.1. In particular, there exists $q_i \leq Q_i$ such that

$$d\left(b_{i+1}^{q_i}, a_{i+1}^{-n_0}\right) < \frac{\alpha}{8}. \tag{4.17}$$

These three phenomena happen simultaneously. More precisely, under iteration by F, (Γ_{x_r,x_s}^i) will be subject to straightening (in the sense of the fibered λ-lemma) and to torsion in the central direction while the intersection of its projection over N with T_{i+1} undergoes ergodization on T_{i+1} and approaches the center of R_{i+1}.

• **Definition of the diffusion time k_i.** We now define the diffusion time k_i for $i = 1, \dots, n-1$, which depends on the “slowest" phenomenon. We set

$$k_i = \begin{cases} n_0 + q_i, & \text{if } q_i \geq \max(m_i, p_i) \\ n_0 + \max(m_i, p_i) + q_i' & \text{if } q_i < \max(m_i, p_i), \end{cases} \tag{4.18}$$

where $q_i' \leq Q_i$ and such that

$$d\left(b_{i+1}^{\max(m_i,p_i)+q_i'}, a_{i+1}^{-n_0}\right) < \frac{\alpha}{8}. \tag{4.19}$$

• **Construction of $\widetilde{R}_i$.** First, for $i = 1, \dots, n-1$, let

$$\Lambda_i := \bigcup_{(x_r,x_s)\in[-\varepsilon_r,\varepsilon_r]\times[-\varepsilon_s,\varepsilon_s]^p} \left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}.$$

We are only interested in the parts of $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}$ which are graphs over the u-variable and the θ-variable, which is necessary to ensure the alignment. We set

$$D_\theta\left(a_{i+1}^{-n_0},\frac{\alpha}{4}\right) := \left\{(\theta,r,s,u)\in\widehat{V};\ (\theta,r)\in\bigcup_{x_r,x_s}\mathrm{graph}\,(h_{x_r,x_s}^{i,k_i-n_0}\cap D_{k_i-n_0}^{\frac{\alpha}{2}}),\ d\left(\theta,\Pi_\theta(a_{i+1}^{-n_0})\right)\le\frac{\alpha}{4}\right\}.$$

Intersecting Λ_i with $D_\theta\left(a_{i+1}^{-n_0},\frac{\alpha}{4}\right)$ will ensure that the projection over the central variable of each $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}$, when restricted to the projected strip, is a graph over a domain that contains $\Pi_\theta(a_{i+1}^{-n_0})+[-\varepsilon_\theta,\varepsilon_\theta]$ (which is the projection of R_{i+1} over the θ-variable), since $\varepsilon_\theta<\frac{\alpha}{4}$ (Inequality (4.10)). Let us now consider a convenient strip in the u-direction. Let ε_u' be as in Estimate (4.9) and set

$$D_{\varepsilon_u'} := \left\{(\theta,r,s,u)\ \in\ \widehat{V}\,;\ u\ \in\ [-\varepsilon_u',\varepsilon_u']^p\right\}.$$

Intersecting Λ_i with $D_{\varepsilon_u'}$ will ensure that each $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}$, when intersected with the strip, is a graph over the u-variable over a domain that contains $[-\varepsilon_u,\varepsilon_u]^p$ which is the projection of R_{i+1} over the u-variable. For $i=1,\dots,n-1$, let

$$\widetilde{\Lambda}_i := \Lambda_i\cap D_\theta\left(a_{i+1}^{-n_0},\frac{\alpha}{4}\right)\cap D_{\varepsilon_u'}.$$

We can think of $\widetilde{\Lambda}_i$ as a window with horizontals

$$\left\{\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}\cap D_\theta\left(a_{i+1}^{-n_0},\frac{\alpha}{4}\right)\cap D_{\varepsilon_u'};\ (x_r,x_s)\in[-\varepsilon_r,\varepsilon_r]\times[-\varepsilon_s,\varepsilon_s]^p\right\}$$

and verticals

$$\left\{\widetilde{\Lambda}_i\cap\left(a_{i+1}^{-n_0}+\{x_\theta\}\times\left[-\frac{\varepsilon_r}{2},\frac{\varepsilon_r}{2}\right]\times\left[-\frac{\varepsilon_s}{2},\frac{\varepsilon_s}{2}\right]^p\times\{x_u\}\right);(x_\theta,x_u)\ \in\ \left[-\frac{\alpha}{4},\frac{\alpha}{4}\right]\times[-\varepsilon_u',\varepsilon_u']^p\right\}$$

which means that we consider $\widetilde{\Lambda}_i$ as having the same vertical directions as R_{i+1}, that is, in the (r,s) direction. This is possible because of the graph properties over the (θ,u)-variables.

- **Construction of $\widetilde{R}_i$.** We set $\widetilde{R}_i := f^{-k_i}\left(\widetilde{\Lambda}_i\right)$. Note that $\widetilde{R}_i\subset\bigcup_{x_r,x_s}\mathcal{H}_{x_r,x_s}^i$ and that intersecting each $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}$ with $D_\theta\left(a_{i+1}^{-n_0},\frac{\alpha}{4}\right)$ and $D_{\varepsilon_u'}$, then applying f^{-k_i}, means that each $f^{-k_i}\left(\left(\Gamma_{x_r,x_s}^i\right)\right)^{k_i-n_0}\subset\mathcal{H}_{x_r,x_s}^i$ was intersected in the (θ,u)-direction. We just defined the exit set of $\widetilde{R}_i$ (see Definition 4.2.1). The horizontals and the verticals of $\widetilde{R}_i$ are the images under f^{-k_i} of those of $\widetilde{\Lambda}_i$. Let us now prove the correct alignments.

1. $f^{k_i}\left(\widetilde{R}_i\right)$ is correctly aligned with R_{i+1}. More precisely, since each horizontal of $f^{k_i}\left(\widetilde{R}_i\right)$ is, by construction, contained in a $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}$ and since the verticals of R_{i+1} are in the (r,s)-direction, then

- the transversality of the horizontals of $f^{k_i}\left(\widetilde{R}_i\right)$ and the verticals of R_{i+1} follows from the following:
  - $\left(\Gamma_{x_r,x_s}^i\right)^{k_i-n_0}\cap D_{\varepsilon_u'}$, by definition, is a graph over the u-variable (a property given by the fibered λ-lemma),

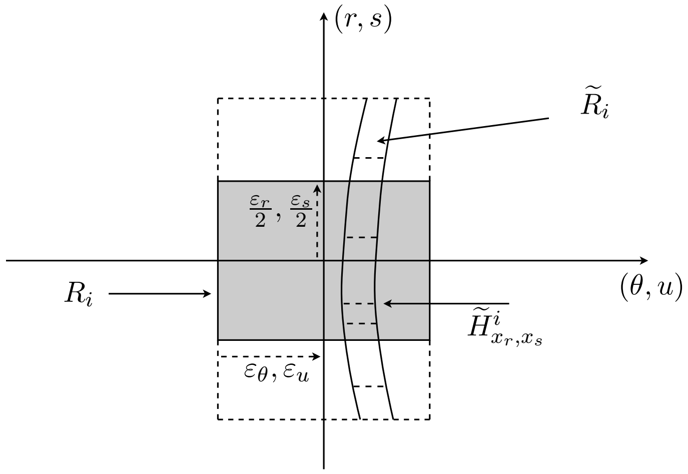

Figure 4.6: Alignment of R_i with $\widetilde{R}_i$

- the projection of $\left(\Gamma^i_{x_r,x_s}\right)^{k_i-n_0} \cap D_\theta\left(a^{-n_0}_{i+1}, \frac{\alpha}{4}\right)$ over the central direction is a graph over the θ-variable (a property given by the strong torsion),

- each intersection point lies inside $f^{k_i}\left(\widetilde{R}_i\right) \cap R_{i+1}$:
  - in the u-direction thanks to Inclusion (4.9),
  - in the s-direction thanks to the choice of m_i in Inequality (4.14),
  - in the central direction thanks to the first inequality in (4.15), Inequality (4.10), Inequalities (4.17) or (4.19) which yield that $d\left(b^{k_i-n_0}_{i+1}, a^{-n_0}_{i+1}\right) < \frac{\alpha}{8}$, and the fact that, for all $(x_r, x_s) \in [-\varepsilon_r, \varepsilon_r] \times [-\varepsilon_s, \varepsilon_s]^p$, $d(b^i_{x_r,x_s}, b_{i+1}) < \frac{\alpha}{8}$ (see Lemma 4.4.2) which yield, thanks to the form of F, that $d\left((b^i_{x_r,x_s})^{k_i-n_0}, b^{k_i-n_0}_{i+1}\right) < \frac{\alpha}{8}$.

2. R_i is correctly aligned with $\widetilde{R}_i$. More precisely,

- the horizontals of R_i are in the same directions as those of $\widetilde{R}_i$ which are transverse to the verticals of $\widetilde{R}_i$ by construction of the verticals and horizontals of $\widetilde{R}_i$ and since f^{-k_i} is a diffeomorphism,

- the intersection points are inside $R_i \cap \widetilde{R}_i$:
  - in the θ-direction because of Estimate (4.15),
  - in the rest of the directions because of the definitions of R_i (Estimate (4.13)) and $\mathcal{H}^i_{x_r,x_s}$ (Estimate (4.12)).

This ends the proof of Proposition 4.4.3. □

Theorem 4 follows from Proposition 4.4.3 and Theorem 4.2.3.

4.5 Diffusion Times

In the proof of Proposition 4.4.3, we described, for all $i = 1, \dots, n-1$, the diffusion time k_i. The next corollary gives an upper bound for the diffusion time.

Corollary 4.5.1. *We keep the assumptions of Theorem 4 and the notation of the proof of Proposition 4.4.3. The diffusion time of the orbit $(x_i)_{1\le i\le n}$ is*

$$\sum_{i=1}^{n-1} k_i \;\le\; \sum_{i=1}^{n-1} \left(n_0 + \max(m_i, p_i) + C_{i+1}^{-1} 16^\tau \alpha^{-\tau}\right),$$

where

- $\alpha \in \,]0; \frac{1}{2}[$ *and is given in Equation (4.7),*
- n_0 *is a unifom integer along the chain given by Equation (4.11) which translates the heteroclinic transition,*
- m_i *is the straightening time given by Equation (4.14),*
- p_i *is the torsion time given by Equation (4.15).*

Proof. The proof is immediate thanks to the estimates in the proof of Proposition 4.4.3. One only needs to note that the ergodization time Q_i given by Equation (4.16) is $Q_i = q_{r_{i+1}}(\frac{\alpha}{8})$ if we follow the notation of Section 4.5, that is, $Q_i \le C_{i+1}^{-1} 16^\tau \alpha^{-\tau}$ by Corollary 4.3.2. □

We end this chapter with some comments.

• Given an explicit example satisfying the assumptions of Theorem 4 (like the system in [LM05] for instance), one can compute the characteristic quantities in the diffusion time, since n_0 depends on the straightened neighborhood, m_i highly depends on the splitting of the heteroclinic connection ($W^u(T_i) \pitchfork W^s(T_{i+1})$) as can be seen in Chapters 2 and 3, and p_i depends on the splitting $\widehat{\gamma}_i \cap T_{i+1}$. If these quantities can be computed for the central horizontal Γ_i, and if one provides upper and lower bounds of the splitting in the neighborhood of the heteroclinic connections (like in [LM05]), one can give explicit estimates for m_i and p_i. The next step will be to estimate the diffusion time in this particular example, and in more general cases. This could not be accomplished here because of the lack of time.

• The dominant quantity in Equation (4.18) determines the diffusion time. More precisely, the torsion time p_i is linear in ε^{-1} as can be seen in Corollary 3.3.3 and the ergodization time is polynomial in α^{-1} by Corollary 4.3.2. As for the straightening time, the situation is more complicated because of the splitting. Since the straightening in the fibered λ-lemma follows from the *uniform* straightening of the leaves, we will illustrate the situation for the leaves for simplicity. By the estimates in Chapter 2 (keeping the same notation) the time needed to straighten Δ within ε is mainly the integer m such that

$$\beta^m \|\xi'\| + m\beta^{m-1} \le \varepsilon.$$

Therefore, roughly speaking, if the splitting is large ($\|\xi'\|$ small), then the time needed to straighten the horizontals is Logarithmic in ε^{-1} and therefore, the ergodization time controls the diffusion time. However, if the splitting is small, then the time needed to straighten the horizontal is large and thus dominates the ergodization.

Chapter 5

Asymptotically dense projected orbits

The aim of this chapter is to construct a simple class of *a priori* stable nearly integrable systems on $\mathbb{A}^3$, for which the dynamical behavior caused by a double resonance plays the central role and yields the existence of "asymptotically dense projected orbits", that is, orbits at fixed energy whose projection on the energy level passes within an arbitrarily small distance from each point of the projected energy level, when the size of the perturbation tends to 0.

5.1 Introduction and main result

Given an integer $m \geq 1$, we denote by $\mathbb{A}^m = \mathbb{T}^m \times \mathbb{R}^m$ the cotangent bundle of the torus $\mathbb{T}^m$, that we endow with its usual angle-action coordinates (θ, r) and its Liouville symplectic form $\Omega = \sum_{i=1}^m dr_i \wedge d\theta_i$. We denote by Π the projection $\mathbb{A}^m \to \mathbb{R}^m$ and by $\mathbf{d}$ the Hausdorff distance for compact subsets of $\mathbb{R}^m$. When H is a C^κ function on an open set of $\mathbb{A}^m$, $\kappa \geq 2$, we denote by X^H its Hamiltonian vector field and by Φ_t^H its local flow. Given a function H and an element a in its range, we write $H^{-1}(a)$ instead of $H^{-1}(\{a\})$, even if H is not a bijection.

Our systems will be defined on $\mathbb{A}^3$ and have the following form

$$H_n(\theta, r) = \tfrac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3) + f_n(\theta, r),$$

where $\|\cdot\|$ stands for the Euclidean norm, $U \in C^\kappa(\mathbb{T}^2)$ is a generic potential function and $f_n \in C^\kappa(\mathbb{A}^3)$ is an additional perturbation such that $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \leq \frac{1}{n}$. For the sake of simplicity, we limit ourselves to the case where κ is an integer ≥ 2, but the construction could easily be extended to the C^∞ or Gevrey cases as well.

The system H_n is a perturbation of the integrable system $h(r) = \frac{1}{2}\|r\|^2$. We will focus on the energy level $H_n^{-1}(\frac{1}{2})$ but any other positive energy level would have the same properties. The frequency map associated with h is

$$\omega(r) = r,$$

and the double resonance under concern is the set of actions r such that $\omega_2(r) = \omega_3(r) = 0$, that is, the line $r_2 = r_3 = 0$. This line intersects the unperturbed level $\mathbb{S} = h^{-1}(\frac{1}{2})$ (the unit sphere) at the points $D_\pm = (\pm 1, 0, 0)$, and the "principal part" of the averaged system at these points coincides and reads

$$\overline{H}_n(\theta_2, \theta_3, r_2, r_3) = \tfrac{1}{2}\big(r_2^2 + r_3^2\big) + \tfrac{1}{n}U(\theta_2, \theta_3)$$

(the full averaged systems also contain the average of f_n, but this will be insignificant thanks to a proper choice of this additional perturbation). This averaged system is of "classical form", the sum of a kinetic part and a potential part. The potential will be arbitrarily chosen in a residual subset of $C^\kappa(\mathbb{T}^2)$; in particular, the system $\overline{H}_n$ will be nonintegrable. This property is in contrast with the previous studies on double resonances, where the averaged system was usually assumed to be integrable or nearly integrable (see [Bes97]). However, this nonintegrability and the associated "chaotic behavior" are essential features of generic nearly integrable systems, as proved in recent studies on Arnold diffusion ([Che12, Marc, Mar12, Kal12]). On the contrary, the last term f_n of the perturbation will be a "very nongeneric" bump function, especially designed to easily create and control the so-called "splitting of separatrices", in the spirit of [Dou88, MS02].

The truncated system

$$\mathcal{H}_n(\theta, r) = \tfrac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3), \qquad (\theta, r) \in \mathbb{A}^3, \tag{5.1}$$

does not admit diffusion orbits. Indeed, it appears as the direct product of the one-degree-of-freedom Hamiltonian $\frac{1}{2}r_1^2$ with the previous averaged system, and the conservation of the energy in both factors prevents from any diffusion phenomenon. It is only when the perturbation f_n is added that the splitting of separatrices appears and makes the diffusion possible. The structure of our system is therefore in some sense analogous to that of Arnold's initial model for diffusion along a simple resonance ([Arn64]). But while in Arnold's model the diffusion phenomenon occurs only along a single resonance, in our model the diffusion takes place along a very large family of simple resonances, namely the great circles of $\mathbb{S}$ orthogonal to the vectors $k = (0, k_2, k_3)$, where k_2, k_3 are coprime integers. The previous double resonant points $D_\pm$ are the places where exchanges of resonances are made possible by the structure of the averaged systems in their neighborhood.

Let us now state our main result. For $2 \le \kappa < +\infty$, we endow the spaces $C^\kappa(\mathbb{T}^2)$ of C^κ functions on $\mathbb{T}^2$ with their usual C^κ norms

$$\|U\|_{C^\kappa(\mathbb{T}^2)} = \operatorname*{Max}_{|\alpha| \le \kappa} \operatorname*{Max}_{\theta \in \mathbb{T}^2} |\partial^\alpha U(\theta)|,$$

which make them Banach spaces. Throughout this chapter, the triples $x = (x_1, x_2, x_3)$ in $\mathbb{T}^3$ or $\mathbb{R}^3$ will also be denoted by

$$x = (x_1, \overline{x}), \qquad \overline{x} = (x_2, x_3).$$

We also introduce a formal definition for the notion of "approximative density". Given a metric space (E, d) and $\delta > 0$, we say that a subset S of E is δ-dense in a subset $F \subset E$ when F is contained in the union of the family of all open δ-balls centered at points of S. We will prove the following diffusion result.

Theorem 5. *Let $\kappa \ge 2$ be a fixed integer. Then there exists a residual subset $\mathscr{U}$ in $C^\kappa(\mathbb{T}^2)$ such that for each $U \in \mathscr{U}$, there is a sequence $(f_n)_{n\ge1}$ of C^κ functions on $\mathbb{A}^3$, with $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \le \frac{1}{n}$, such that for any $\delta > 0$, there exists n_0 such that for $n \ge n_0$, the system*

$$H_n(\theta, r) = \tfrac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\overline{\theta}) + f_n(\theta, r), \qquad (\theta, r) \in \mathbb{A}^3, \tag{5.2}$$

admits an orbit Γ_n with energy $\frac{1}{2}$ such that $\Pi(\Gamma_n)$ is δ–dense in $\Pi(H_n^{-1}(\frac{1}{2}))$.

Since we only aim at producing examples, the fact that $\mathscr{U}$ is nonempty would be enough. However the residual character of $\mathscr{U}$ makes it plausible –eventhough we do not try to prove this– that the wild behavior of orbits described in our examples is in fact generic for a priori stable perturbations of integrable systems. We could also work in the class of diffeomorphisms, in which case an analogous construction yields examples of nearly-integrable diffeomorphisms with a large class of orbits biasymptotic to infinity. However, we will limit here to the Hamiltonian case, which is indeed richer and sligthly more difficult, due to the additional geometrical difficulty induced by the preservation of energy. The chapter is organized as follows.

• Section 2 is devoted to the description of those properties of classical systems which we will need to construct our examples, namely the existence of suitable *chains of annuli*. Here we summarize [Marc]. However, our present construction is to a large extent independent of this latter work (apart from the necessary definitions), the concern of which is the genericity of the potential U for which the associated classical system possesses suitable chains of annuli.

• In Section 3, we deduce from the previous properties of classical systems the existence of *chains of cylinders* in our systems $\mathcal{H}_n$, and we prove that these chains project in the space of actions asymptotically close to a dense family of great circles in the unit sphere (the simple resonance lines). These cylinders are normally hyperbolic invariant manifolds diffeomorphic to $\mathbb{T}^2 \times [0,1]$ and admit a foliation by invariant tori diffeomorphic to $\mathbb{T}^2$.

• In Section 4, we construct the sequence (f_n), in such a way that each invariant torus in the previous family admits a homoclinic orbit along which its stable and unstable manifolds intersect transversely in a weak sense. This in particular yields the existence of heteroclinic connections between nearby enough tori contained in the same cylinders. Other transversality properties for heteroclinic orbits between tori in different cylinders of the chains are also proved.

• Finally in Section 5, we prove the existence of the diffusion orbits. The key lemma there is the basic λ–lemma proved in Chapter 2 which is especially designed for normally hyperbolic manifolds and which enables us to prove very easily the necessary shadowing results.

5.2 Classical systems

This section is devoted to the description of the generic hyperbolic properties of classical systems on the torus $\mathbb{T}^2$ which will be needed in the construction of our examples. Given a potential function $U \in C^\kappa(\mathbb{T}^2)$, we define here the associated classical system as the Hamiltonian on $\mathbb{A}^2$

$$C_U(x,y) = \tfrac{1}{2}\|y\|^2 + U(x), \tag{5.3}$$

where $x \in \mathbb{T}^2$ and $y \in \mathbb{R}^2$. We will always require the potential U to admit a single maximum at some x^0, which is nondegenerate in the sense that the Hessian of U is negative definite. This is of course true for a U in a residual subset $\mathscr{U}_0 \subset C^\kappa(\mathbb{T}^2)$. It is then easy to check that the lift of x^0 to the zero section of $\mathbb{A}^2$ is a hyperbolic fixed point for X^C.

1. We denote by $\pi : \mathbb{A}^2 \to \mathbb{T}^2$ the canonical projection and we fix $U \in \mathscr{U}_0$ together with the associated classical system $C := C_U$.

Definition 5.2.1. *Let $c \in H_1(\mathbb{T}^2, \mathbb{Z})$ and let $I \subset \mathbb{R}$ be an interval. An* annulus for X^C realizing c and defined over I *is a submanifold A, contained in $C^{-1}(I) \subset \mathbb{A}^2$, such that*

- *for each $e \in I$, $\mathsf{A} \cap C^{-1}(e)$ is the orbit of a periodic solution γ_e of X^C, which is hyperbolic in $C^{-1}(e)$, with orientable stable and unstable bundles, and such that the projection $\pi \circ \gamma_e$ on $\mathbb{T}^2$ realizes c,*
- *the frequency $\omega(e)$ of the solution γ_e is an increasing function of e and, in the case where $I =]\widehat{e}, e_m[$, $\omega(e) \to 0$ and $\omega'(e) \to +\infty$ when $e \to \widehat{e}$,*

- *there exists a covering $I = \cup_{1\le i\le i^*} I_i^*$ of I by open subintervals of I such that for $1 \le i \le i^*$ and for $e \in I_i^*$, the solution γ_e admits a homoclinic solution ω_e along which the stable and unstable manifolds of γ_e intersect transversely inside $C^{-1}(e)$.*

Since the solutions γ_e are hyperbolic and vary continuously with e (since A is assumed to be a submanifold), the annulus A is a $C^{\kappa-1}$ submanifold of $\mathbb{A}^2$, with boundary $\partial\mathsf{A} \sim \mathbb{T}\times\partial I$. It is clearly normally hyperbolic (the boundary causes no trouble is this simple setting), and its stable and unstable manifolds are the unions of those of the periodic solutions γ_e. Note that when I has a boundary point, the family γ_e can be continued over a slightly larger open interval, but it will be interesting to allow the intervals to be compact in our subsequent constructions.

It is not difficult to prove that an annulus A is a $C^{\kappa-1}$ *symplectic* submanifold of $\mathbb{A}^2$ and, using Moser's isotopy method, that there exists a symplectic embedding $\phi : \mathbb{T}\times I \to \mathbb{A}^2$ whose image is A and which satisfies

$$C \circ \phi(\varphi, e) = e,$$

where of course $\mathbb{T}\times I$ is equipped with its usual symplectic structure. Note that obviously $\phi(\mathbb{T}\times\{e\}) = \mathsf{A}\cap C^{-1}(e)$.

2. Due to the reversibility of C, the solutions of the vector field X^C occur in *opposite pairs* (pairs of symmetric solutions whose time parametrizations are exchanged by the symmetry $t \mapsto -t$). This is in particular the case for the solutions homoclinic to the hyperbolic fixed point O associated with the maximum x^0 of U. We set

$$\widehat{e} = \mathrm{Max}\ U = U(x^0).$$

Definition 5.2.2. *Let $c \in H_1(\mathbb{T}^2,\mathbb{Z}) \setminus \{0\}$. A* singular annulus for X^C *realizing* $\pm c$, *with* parameters $\widetilde{e} > \widehat{e}$ and $e^0 < \widehat{e}$, *is a C^1 invariant manifold $\mathsf{A}^\bullet$ with boundary , diffeomorphic to the sphere S^2 minus three disjoint open discs, such that, setting $I =]\widehat{e}, \widetilde{e}\,[$ and $I_0 =]e^0, \widehat{e}[$:*

- *$\mathsf{A}^\bullet \cap C^{-1}(\widehat{e})$ is the union of the hyperbolic fixed point O and a pair of opposite homoclinic orbits,*
- *$\mathsf{A}^\bullet \cap C^{-1}(I)$ admits two connected components $\mathsf{A}^\bullet_+$ and $\mathsf{A}^\bullet_-$, which are annuli defined over I and realizing c and $-c$ respectively,*
- *$\mathsf{A}^\bullet_0 = \mathsf{A}^\bullet \cap C^{-1}(I_0)$ is an annulus defined over I_0 and realizing the null class 0,*
- *$\mathsf{A}^\bullet$ admits a C^1 stable (resp. unstable) manifold, in which the union of the stable (resp. unstable) manifolds of $\mathsf{A}^\bullet_+$, $\mathsf{A}^\bullet_-$ and $\mathsf{A}^\bullet_0$ is dense.*
- *both homoclinic orbits admit homoclinic connections along which the stable and unstable manifolds of $\mathsf{A}^\bullet$ intersect transversely in $C^{-1}(\widehat{e})$.*

Note that a singular annulus $\mathsf{A}^\bullet$ is "almost everywhere $C^{\kappa-1}$", since the connected components of $\mathsf{A}^\bullet\cap C^{-1}(I)$ and $\mathsf{A}^\bullet\cap C^{-1}(I_0)$ are annuli, so $C^{\kappa-1}$ submanifolds of $\mathbb{A}^2$. Note also that $\mathsf{A}^\bullet$ is a center manifold for both homoclinic orbits in $\mathsf{A}^\bullet\cap C^{-1}(\widehat{e})$, with hyperbolic transverse spectrum. One can in fact prove that a singular annulus is slightly more regular than C^1 (depending on the Lyapunov exponents of the fixed point O), but this is useless here.

A singular annulus is depicted in Figure 5.1: it is essentially the part of the phase space of a simple pendulum limited by two essential invariant curves at the same energy, from which a neighborhood of the elliptic fixed point has been removed. More precisely, on the annulus $\mathbb{A}$ equipped with the coordinates (φ, I), for $\lambda > 0$ we define the pendulum Hamiltonian

$$P_{\widehat{e},\lambda}(\varphi, I) = \tfrac{1}{2}I^2 + \lambda(\cos 2\pi\varphi - 1) + \widehat{e}$$

and for $a < \widehat{e} < b$ we introduce the subset $\mathcal{A}^\bullet(a,b)$ defined by $a \leq P_\lambda(\varphi, I) \leq b$. So $\mathcal{A}^\bullet(a,b)$ is the zone bounded by the two invariant curves of equation $P_\lambda = b$, together with an invariant curve surrounding the elliptic point. We call $\mathcal{A}^\bullet(\widehat{e}, \lambda, a, b)$ the *standard singular annulus* with parameters $(\widehat{e}, \lambda, a, b)$. One easily proves that a singular annulus is C^1 symplectically diffeomorphic to some standard annulus, by a diffeomorphism $\phi^\bullet : \mathcal{A}^\bullet \to \mathsf{A}^\bullet$ such that $C_{|\mathsf{A}^\bullet} \circ \phi^\bullet = P_{\widehat{e},\lambda}$.

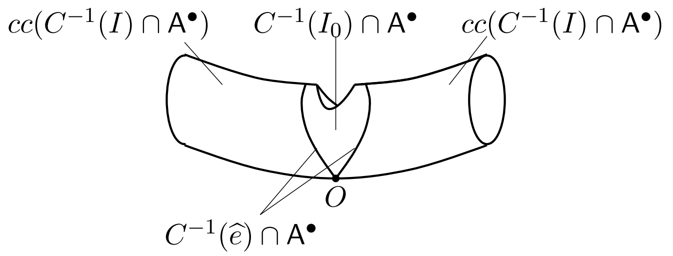

Figure 5.1: A singular annulus

Note finally that there exist embeddings $\phi_\pm : \mathbb{T}\times\,]\widehat{e}, \widetilde{e}] \to \mathsf{A}^\bullet_\pm$ and $\phi_0 : \mathbb{T}\times\,]e^0, \widehat{e}] \to \mathsf{A}^\bullet_0$ for the 3 subannuli of a singular annulus.

3. Let us now turn to the definition of chains of annuli for the classical system C. We say that a family $(I_i)_{1\leq i\leq m}$ of nontrivial closed subintervals of $]\widehat{e}, +\infty[$ is *ordered* when $\mathrm{Max}\, I_i = \mathrm{Min}\, I_{i+1}$ for $1 \leq i \leq m-1$.

Definition 5.2.3. *Let $c, c' \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$.*

- *A* chain of annuli realizing c *is a family $(\mathsf{A}_i)_{0\leq i\leq m}$ of annuli realizing c, defined over an ordered family (I_i) of closed subintervals of $]\widehat{e}, +\infty[$, with the additional property that for $0 \leq i \leq m-1$*

$$W^u(\mathsf{A}_i) \cap W^s(\mathsf{A}_{i+1}) \neq \emptyset, \qquad W^s(\mathsf{A}_i) \cap W^u(\mathsf{A}_{i+1}) \neq \emptyset,$$

  the intersection being transverse in $\mathbb{A}^2$.

- *A* generalized chain of annuli realizing c and c' *is the union of two chains $(\mathsf{A}_i)_{0\leq i\leq m}$ and $(\mathsf{A}'_i)_{0\leq i\leq m'}$ realizing c and c' respectively, together with a singular annulus $\mathsf{A}^\bullet$, such that*

$$W^u(\mathsf{A}_0) \cap W^s(\mathsf{A}^\bullet) \neq \emptyset, \qquad W^s(\mathsf{A}_0) \cap W^u(\mathsf{A}^\bullet) \neq \emptyset,$$
$$W^u(\mathsf{A}'_0) \cap W^s(\mathsf{A}^\bullet) \neq \emptyset, \qquad W^s(\mathsf{A}'_0) \cap W^u(\mathsf{A}^\bullet) \neq \emptyset,$$

  the intersections being transverse in $\mathbb{A}^2$.

Note that we do not specify the homology of the singular annulus $\mathsf{A}^\bullet$, this latter turns out to be fixed independently of the classes c and c' in our subsequent construction.

4. We now state a genericity result due to Marco which was proved in [Marc]. We say that a nonzero class $c \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$ is primitive (or is a primary class) when the equality $c = mc'$ with $m \in \mathbb{Z}$ implies $m = \pm 1$. We denote by $\mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$ the set of primitive homology classes. Here $\mathbf{d}$ is the Hausdorff distance for compact subsets of $\mathbb{R}^2$ and $\Pi : \mathbb{A}^2 \to \mathbb{R}^2$ is the canonical projection.

Theorem 5.2.4. [Marco.] *Let $2 \leq \kappa \leq +\infty$. Then there exists a residual subset $\mathscr{U} \subset \mathscr{U}_0$ in $C^\kappa(\mathbb{T}^2)$ such that for $U \in \mathscr{U}$, the associated classical system C_U defined in (5.3) satisfies the following properties.*

1. *For each $c \in \mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$ there exists a chain $\mathbf{A}(c) = (\mathsf{A}_0, \ldots, \mathsf{A}_m)$ of annuli realizing c, defined over ordered intervals $I_0, \ldots, I_m$, such that the first and last intervals are of the form*
$$I_0 =]\,\mathrm{Max}\, U, e_m] \quad \textit{and} \quad I_m = [e_P, +\infty[,$$
for suitable constants e_m and e_P.

2. *Let $c = (c_1, c_2)$ in the canonical identification of $H_1(\mathbb{T}^2, \mathbb{Z})$ with $\mathbb{Z}^2$ and for $e > 0$, set*
$$Y_c(e) = \frac{\sqrt{e}\, c}{\|c\|}$$
Setting $\Gamma_e = \mathsf{A}_m \cap C_U^{-1}(e)$ for $e \in [e_P, +\infty[$, then
$$\lim_{e \to +\infty} \mathbf{d}\big(\Pi(\Gamma_e), \{Y_c(e)\}\big) = 0$$

3. *Given two primitive classes $c \neq c'$, there exists $\sigma \in \{-1, +1\}$ such that the two chains $\mathbf{A}(c) = (\mathsf{A}_i)_{0 \leq i \leq m}$ and $\mathbf{A}(\sigma c') = (\mathsf{A}'_i)_{0 \leq i \leq m'}$ satisfy*
$$W^u(\mathsf{A}_0) \cap W^s(\mathsf{A}'_0) \neq \emptyset \quad \textit{and} \quad W^u(\mathsf{A}'_0) \cap W^s(\mathsf{A}_0) \neq \emptyset,$$
both heteroclinic intersections being transverse in $\mathbb{A}^2$.

4. *There exists a singular annulus $\mathsf{A}^\bullet$ which admits transverse heteroclinic connections with every first annulus in the previous chains.*

The existence of the "high energy annuli" A_m is proved by a simple argument due to Poincaré, on the creation of hyperbolic orbits near perturbations of resonant tori, so we call e_P the Poincaré energy for the class c. The other annuli are proved to exist by minimization arguments of Morse and Hedlund.

There exist in general several singular annuli with the previous intersection property, but one will be enough for our future needs. We say that a chain with I_0 and I_m as in the first item above is *biasymptotic to $\widehat{e} := \mathrm{Max}\, U$ and $+\infty$*. It may be useful to rephrase Theorem 5.2.4 in a more concise way.

Corollary 5.2.5. *For $U \in \mathscr{U}$ and for each pair of classes $c, c' \in \mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$, there exists a generalized chain of annuli, union of $(\mathsf{A}_i)_{0 \leq i \leq m}$, $(\mathsf{A}'_i)_{0 \leq i \leq m'}$ and $\mathsf{A}^\bullet$, such that $(\mathsf{A}_i)_{0 \leq i \leq m}$ and $(\mathsf{A}'_i)_{0 \leq i \leq m'}$ are biasymptotic to $\widehat{e}$ and $+\infty$ and realize c and c' respectively.*

In the y–plane, one therefore gets the following picture for the projection of generalized chains of annuli, along some lines of rational slope (which obviously correspond to homology classes when the energy tends to $+\infty$, by Theorem 5.2.4, see [Marc]).

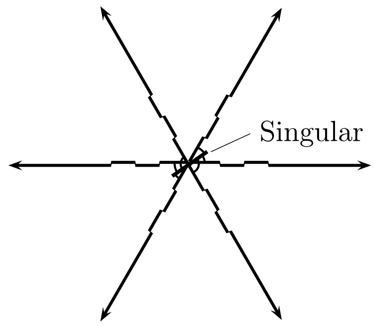

Figure 5.2: Projected generalized chains of annuli in a classical system

5.3 Chains of cylinders for $\mathcal{H}_n$

Here we call *cylinder* for a vector field defined on the cylinder $\mathbb{A}^3$ a normally hyperbolic invariant manifold with boundary, diffeomorphic to $\mathbb{T}^2 \times [0,1]$. In particular, the stable and unstable manifolds of a cylinder are well-defined and the definition of chains of cylinders can be properly stated. In this section, we prove the existence of a family of chains of cylinders for $\mathcal{H}_n$ in the energy level $\mathcal{H}_n^{-1}(\frac{1}{2})$, whose projection by Π forms an asymptotically dense subset of the unit sphere.

5.3.1 Cylinders and chains

1. Let us set out a first definition.

Definition 5.3.1. *Let X be a vector field on $\mathbb{A}^3$.*

- *We say that $\mathscr{C} \subset \mathbb{A}^3$ is a C^p* invariant cylinder with boundary *for X if $\mathscr{C}$ is a submanifold of $\mathbb{A}^3$, C^p–diffeomorphic to $\mathbb{T}^2 \times [0,1]$, such that X is everywhere tangent to $\mathscr{C}$ and is moreover tangent to $\partial\mathscr{C}$ at each point of $\partial\mathscr{C}$.*
- *Given an invariant cylinder with boundary $\mathscr{C}$, we say that it is* normally hyperbolic *when there exist a neighborhood N of $\mathscr{C}$ and a complete vector field $X_\circ$ on $\mathbb{A}^3$ such that $X \equiv X_\circ$ in N and such that $X_\circ$ admits a normally hyperbolic invariant submanifold $\mathscr{C}_\circ$, diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, which contains $\mathscr{C}$.*

Note first that $\mathscr{C}$ is invariant under the flow, thanks to the tangency hypothesis on the boundary. In particular, both connected components of $\partial\mathscr{C}$ are invariant 2-dimensional tori. In the following, when the context is clear, normally hyperbolic cylinders with boundary will be called *compact invariant cylinders* for short.

The main interest of the previous definition is that it is possible to properly define the stable and unstable manifolds of compact invariant cylinders. Indeed, one checks that the stable manifold $W^{ss}(x)$ of a point $x \in \mathscr{C}$ is well-defined and *independent of the choice of* $(X_\circ, \mathscr{C}_\circ)$ (recall that $W^{ss}(x)$ is the set of all initial conditions y such that $\mathrm{dist}(\Phi^{tX}(x), \Phi^{tX}(y))$ tends to 0 at an exponential rate e^{-ct}, where c dominates the contraction exponent on $\mathscr{C}_\circ$). The stable manifold of $\mathscr{C}$ is then well-defined, as the union of the stable manifolds $W^{ss}(x)$ for $x \in \mathscr{C}$, which turns out to have the same regularity as $\mathscr{C}$. The same remark obviously also holds for the unstable manifolds.

2. In addition to our previous invariant cylinders, it will be necessary to introduce slightly more complicated objects which we call *singular cylinders*. Recall that $\mathcal{A}^\bullet$ is the standard singular annulus defined in the previous section.

Definition 5.3.2. *Let X be a vector field on $\mathbb{A}^3$.*

- *We say that $\mathscr{C}^\bullet \subset \mathbb{A}^3$ is an* invariant singular cylinder *for X if $\mathscr{C}^\bullet$ is a C^1 submanifold with boundary of $\mathbb{A}^3$, diffeomorphic to $\mathbb{T} \times \mathcal{A}^\bullet$, such that X is everywhere tangent to $\mathscr{C}^\bullet$ (and is moreover tangent to $\partial\mathscr{C}^\bullet$ at each point of $\partial\mathscr{C}^\bullet$).*
- *Given an invariant singular cylinder $\mathscr{C}^\bullet$, we say that it is* normally hyperbolic *when there exist a neighborhood N of $\mathscr{C}^\bullet$ and a complete vector field $X_\circ$ on $\mathbb{A}^3$ such that $X \equiv X_\circ$ on N and which admits a normally hyperbolic invariant submanifold $\mathscr{C}_\circ$, diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, which contains $\mathscr{C}^\bullet$.*

As above, we simply say *compact singular cylinders* instead of normally hyperbolic compact invariant singular cylinders. Again, the stable and unstable manifolds of a point $x \in \mathscr{C}^\bullet$ are well-defined and independent of the choice of $(X_\circ, \mathscr{C}_\circ)$, and this is also the case for the stable and unstable manifolds of $\mathscr{C}^\bullet$.

3. Let H be a Hamiltonian on $\mathbb{A}^3$ and let $\mathbf{e}$ be a regular value of H.

Definition 5.3.3. *A* chain of cylinders *for H at energy $\mathbf{e}$ is a finite family $(\mathscr{C}_i)_{1\leq i\leq i^*}$ of compact invariant cylinders* or singular cylinders*, contained in $H^{-1}(\mathbf{e})$, such that $W^u(\mathscr{C}_i)$ intersects $W^s(\mathscr{C}_{i+1})$ for $1 \leq i \leq i^* - 1$.*

Note in particular that, for the sake of simplicity, we do not make any distinction between "regular" cylinders and singular cylinders in a chain. Note also that the definition here slitghly differs from that of chains of annuli above. In the following we will have to add suitable transversality conditions for the various homoclinic and heteroclinic intersections in a chain of cylinders, which could be stated in a general context but will be easier to make explicit in the case of our Hamiltonians H_n, this will be done in Section 5.4.

5.3.2 Cylinders for $\mathcal{H}_n$

1. We now go back to the truncated Hamiltonian $\mathcal{H}_n$ defined in (5.1). Let $k = (k_2, k_3) \in \mathbb{Z}^2$ be a primary integer vector and let $\mathbb{S}_k$ be the half great circle of the unit sphere $\mathbb{S}$ formed by the actions $r = (r_1, \overline{r}) = (r_1, r_2, r_3)$ such that

$$\overline{r} \cdot k = 0, \qquad (-r_3, r_2) \cdot k \geq 0, \qquad r \in \mathbb{S}.$$

The main result of this section is the following.

Proposition 5.3.4. *Let $U \in \mathscr{U}$ and set, for $(\theta, r) \in \mathbb{A}^3$*

$$\mathcal{H}_n(\theta, r) = \tfrac{1}{2}\|r\|^2 + \tfrac{1}{n}U(\theta_2, \theta_3).$$

Fix k as above and fix $\delta > 0$. Then:

- *there is $n_0(k) > 0$ such that for $n \geq n_0(k)$, there are regular cylinders $\mathscr{C}_{-m}, \ldots, \mathscr{C}_{-1}$, $\mathscr{C}_0$, $\mathscr{C}_1, \ldots, \mathscr{C}_m$, where the integer m depends on k, which satisfy*

$$\mathbf{d}\big(\cup_j \Pi(\mathscr{C}_j), \mathbb{S}_k\big) < \delta, \tag{5.4}$$

and such that both ordered families $\mathscr{C}_{-m}, \ldots, \mathscr{C}_{-1}$, $\mathscr{C}_0$, $\mathscr{C}_1, \ldots, \mathscr{C}_m$ and $\mathscr{C}_m, \ldots, \mathscr{C}_1$, $\mathscr{C}_0$, $\mathscr{C}_{-1}, \ldots, \mathscr{C}_{-m}$ are chains.

- *there exist two singular cylinders* $\mathscr{C}_-^\bullet$ *and* $\mathscr{C}_+^\bullet$, independent of k, *such that the extremal cylinders* $\mathscr{C}_{-m}$ *and* $\mathscr{C}_m$ *admit transverse heteroclinic connections with* $\mathscr{C}_-^\bullet$ *and* $\mathscr{C}_+^\bullet$ *respectively,*

- *each cylinder* $\mathscr{C}_j$ *admits a foliation by isotropic tori, such that the union of the subfamily of dynamically minimal tori is a dense subset of* $\mathscr{C}_j$*, and each singular cylinder* $\mathscr{C}_\pm^\bullet$ *admits a foliation by isotropic tori on an open and dense subset, such that the union of the subfamily of dynamically minimal tori is a dense subset of* $\mathscr{C}_\pm^\bullet$.

We will moreover prove that the cylinders $\mathscr{C}_j$ and $\mathscr{C}_{-j}$ are exchanged with one another by a natural symmetry.

Proof. We can assume without loss of generality that $\widehat{e} = \operatorname{Max} U = 0$. We first perform a standard rescaling to get rid of the parameter n, namely, setting

$$\sigma_n(\theta, r) = (\theta, nr), \tag{5.5}$$

one immediately checks the conjugacy relation

$$\Phi_{nt}^{\mathcal{H}_n} = \sigma_n^{-1} \circ \Phi_t^{\mathcal{H}} \circ \sigma_n, \tag{5.6}$$

where $\mathcal{H} := \mathcal{H}_1$, while σ_n sends the energy level $\mathcal{H}_n^{-1}(\frac{1}{2})$ onto the level $\mathcal{H}^{-1}(\frac{n}{2})$. We can therefore examine the behavior of the system $\mathcal{H}$ at high energy $\mathbf{e}$ and get our results by inverse rescaling. We will fix two coprime integers (k_2, k_3) and concentrate on the neighborhood of the half great circle $\sqrt{2\mathbf{e}}\,\mathbb{S}_k$ on the sphere of radius $\sqrt{2\mathbf{e}}$.

• We will apply Theorem 5.2.4 to $c \sim (-k_3, k_2)$. Reversing the ordering of the intervals for compatibility reasons, there is an ordered family $I_m, \ldots, I_0$, with $I_m =]0, b_m]$ and $I_0 = [e_P, +\infty[$ such that the system C_U admits a chain of annuli $\mathsf{A}_m, \ldots, \mathsf{A}_0$ realizing c and defined over $I_m, \ldots, I_0$. For $0 \le j \le m$, we denote by $\phi_j : \mathbb{T} \times I_j \to \mathbb{A}^2$ the embedding of A_j defined in the previous section and for $e \in I_j$, we denote by $\Gamma_j(e) = \phi_j(\mathbb{T} \times \{e\})$ the periodic orbit at energy e in A_j.

• Let us fix an energy $\mathbf{e} > e_P$. The level $\mathcal{H}^{-1}(\mathbf{e})$ contains the union

$$\bigcup_{0 \le e_1 \le \mathbf{e}} \big\{\theta_1 \in \mathbb{T},\ \tfrac{1}{2} r_1^2 = e_1\big\} \times C_U^{-1}(\mathbf{e} - e_1).$$

which will serve as a guide to construct embeddings for our cylinders.

• Consider first an annulus A_j with $1 \le j \le m$ and set $I_j = [a_j, b_j]$ for $1 \le j \le m-1$ and $I_m =]0, b_m]$, so that I_j is contained in $]0, e_P[$. We set

$$J_j = \Big[\sqrt{2(\mathbf{e} - b_j)}, \sqrt{2(\mathbf{e} - a_j)}\Big]$$

and introduce the map

$$F_j^+ : \mathbb{T}^2 \times J_j \longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2$$

defined componentwise by

$$F_j^+(\varphi_1, \varphi_2, r_1) = \Big((\varphi_1, r_1), \phi_j\big(\varphi_2, \mathbf{e} - \tfrac{1}{2} r_1^2\big)\Big).$$

One immediately checks that F_j^+ is an embedding. Let $\mathscr{C}_j \subset \mathcal{H}^{-1}(\mathbf{e})$ be its image. Then $\mathscr{C}_j$ admits a regular foliation whose leaves are the tori

$$\mathscr{T}_{r_1} = F_j^+(\mathbb{T}^2 \times \{r_1\}).$$

The torus $\mathscr{T}_{r_1}$ is the direct product of the circle $\mathbb{T} \times \{r_1\}$ in the first factor of the product $\mathbb{A} \times \mathbb{A}^2$ with the hyperbolic periodic orbit $\Gamma_j(\mathbf{e} - \frac{1}{2}r_1^2)$ in the second factor. For each $z = (z_1, z_2) \in \mathscr{C}_j$, there is a single hyperbolic orbit in the annulus A_j which contains z_2. This yields a decomposition of the tangent space $T_z\mathcal{H}^{-1}(\mathbf{e})$ as a sum $T_z\mathscr{C}_j \oplus E^+(z) \oplus E^-(z)$, where $E^\pm(z)$ are the stable and unstable directions of that orbit at the point z_2. All these considerations also make sense for any small enough hyperbolic continuation of A_j, which immediately proves that $\mathscr{C}_j$ is a compact invariant cylinder in the sense of Definition 5.3.1.

- One gets a parallel construction using the embedding

$$F_j^- : \mathbb{T}^2 \times \big(- J_j\big) \longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2$$

defined by

$$F_j^-(\varphi_1, \varphi_2, r_1) = F_j^+(\varphi_1, \varphi_2, -r_1).$$

whose image will be denoted by $\mathscr{C}_{-j}$ and is a compact invariant cylinder as well. Moreover, $\mathscr{C}_j$ and $\mathscr{C}_{-j}$ are obviously symmetric.

- As for A_0, we introduce the interval $J_0 = \big[- \sqrt{2(\mathbf{e} - e_P)}, \sqrt{2(\mathbf{e} - e_P)}\big]$ and the map

$$F^0 : \mathbb{T}^2 \times J_0 \longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2$$

defined by

$$F^0(\varphi_1, \varphi_2, r_1) = \Big((\varphi_1, r_1), \phi_j\big(\varphi_2, \mathbf{e} - \tfrac{1}{2}r_1^2\big)\Big).$$

One easily checks that F_0 is again an embedding and that its image $\mathscr{C}_0$ is a compact invariant cylinder (note that now $\mathscr{C}_0$ is a two sheeted ramified covering of the corresponding part of A_0). This completes the construction of the family $\mathscr{C}_{-m}, \ldots, \mathscr{C}_m$.

- Fix now an integer $j \in \{0, \ldots, m-1\}$ and let $e := e^j = e_{j+1}$ be the intersection point of the intervals I_j and I_{j+1}. So there exists a heteroclinic connection

$$\Omega_j^{j+1} \subset W^u(\Gamma_j(e)) \cap W^s(\Gamma_{j+1}(e))$$

between the extremal periodic orbits of A_j and A_{j+1}, which gives rise to a *manifold* of heteroclinic orbits between $\mathscr{C}_j$ and $\mathscr{C}_{j+1}$, namely $\big(\mathbb{T} \times \{\sqrt{2(\mathbf{e} - e)}\}\big) \times \Omega_j^{j+1}$, which is therefore diffeomorphic to $\mathbb{A}$. Again, a parallel construction using now the heteroclinic connection

$$\Omega_{j+1}^{j} \subset W^s(\Gamma_j(e)) \cap W^u(\Gamma_{j+1}(e))$$

proves the existence of an annulus of heteroclinic orbits $W_{-(j+1)}^{-j}$ between $\mathscr{C}_{-(j+1)}$ and $\mathscr{C}_{-j}$. This proves that the family $\mathscr{C}_{-m}, \ldots, \mathscr{C}_m$ is a chain of cylinders. The proof for the opposite ordering is similar.

- Finally, to exhibit the singular cylinders, one uses the embedding $\phi^\bullet : \mathcal{A}^\bullet \to \mathbb{A}^2$ whose image is the singular annulus $\mathsf{A}^\bullet$ of the system C_U depicted in Theorem 5.2.4. This enables one to introduce the two maps $F_\pm^\bullet : \mathbb{T} \times \mathcal{A}^\bullet \longrightarrow \mathcal{H}^{-1}(\mathbf{e})$ defined by

$$F_\pm^\bullet(\varphi_1, \varphi_2, r_2) = \Big(\Big(\varphi_1, \pm\sqrt{2(\mathbf{e} - C_U\big(\phi^\bullet(\varphi_2, r_2)\big)}\Big), \phi^\bullet(\varphi_2, r_2)\Big).$$

Again, one easily checks that these are embeddings. We set $\mathscr{C}_\pm^\bullet = F_\pm^\bullet(\mathbb{T}\times \mathsf{A}^\bullet)$. The existence of manifolds $W_\pm^\bullet$ of heteroclinic connections between $\mathscr{C}_{\pm m}$ and $\mathscr{C}_\pm^\bullet$ follows from exactly the same considerations as above.

• The cylinders $\mathscr{C}_j$, $1 \leq j \leq m$, are foliated by the invariant tori $\mathscr{T}_{r_1}$ for $r_1 \in J_j$. Let us prove that, when $\mathbf{e}$ is large enough, they are dynamically minimal for r_1 in a dense subset of J_j. First, observe that, by Definition 5.2.1, the frequency map $\omega_j = 1/T_j : I_j \to \mathbb{R}$ of the annulus A_j satisfies

$$\omega_j'(e) \geq \mu > 0$$

for $e \in I_j$ and $1 \leq j \leq m$. Fix the integer j. Reparametrizing by the first action r_1 yields a frequency $\Omega_2(r_1) := \omega_j(\mathbf{e} - \frac{1}{2}r_1^2)$ on the second factor, so that $\Omega_2(\sqrt{2\mathbf{e}}) = 0$ and

$$\Omega_2'(r_1) = -r_1\omega_2'(\mathbf{e} - \tfrac{1}{2}r_1^2) \leq -\mu\sqrt{2(\mathbf{e} - b_1)}.$$

Now the frequency on the first factor $\mathbb{A}$ is $\Omega_1(r_1) = r_1$, so $\Omega_1'(r_1) = 1$. Since $\mathbf{e} > 0$, this proves that the frequency curve $(\Omega_1, \Omega_2) \subset \mathbb{R}^2$ of $\mathscr{C}_j$ satisfies $\Omega_1 \geq 0$, $\Omega_2 \geq 0$ and is transverse to each vector line in $\mathbb{R}^2$, so that the ratio Ω_2/Ω_1 is irrational for r_1 in a dense subset of J_j. This proves that the corresponding torus $\mathscr{T}_{r_1}$ is dynamically minimal. Similar arguments show the same property for the cylinders $\mathscr{C}_j$, $-m \leq j \leq -1$, as well as for the singular cylinders $\mathscr{C}_\pm^\bullet$, using the embeddings $\phi_\pm^\bullet$ and $\phi_0^\bullet$ of their associated subannuli.

• It remains to examine the torsion properties of $\mathscr{C}_0$. One could use an argument similar to but a little more involved than the previous one, but one can also observe that up to a standard linear change of variables (see [Marc]), one can assume that $c = (1, 0)$. In the new variables, that we still denote by (θ, r), the kinetic part of the Hamiltonian $\mathcal{H}$ takes the form

$$T(r) = r_1^2 + Q(r_2, r_3),$$

and for $\mathbf{e}$ large enough X^T is the dominant term of $X^{\mathcal{H}}$ since X^U is bounded. Moreover, by the asymptotic property of the projection of the Poincaré annulus A_0 ($\sim \mathsf{A}_m$ in Theorem 5.2.4), its frequency map is a $O(1)$ pertubation of

$$\omega : e := \tfrac{1}{2}Q(r_2, r_3) \longmapsto \partial_{r_2}Q(r_2, r_3).$$

Therefore $\omega'(e) \to 0$ when $e \to \infty$. Since the frequency map on the first factor still has a derivative equal to 1, the same argument as above proves now the same torsion property for the part of $\mathscr{C}_0$ associated with the subannulus of A_0 defined over $[e^*, +\infty[$, when e^* is large enough (note that this requires $\mathbf{e} > e^*$ for being consistent). The case of the remaining part of $\mathscr{C}_0$, associated with the subannulus defined over $[e_P, e^*[$, is analogous to that of the cylinders $\mathscr{C}_j$ above, by compactness. As a consequence, $\mathscr{T}_{r_1}$ is dynamically minimal for r_1 in a dense subset of J_0.

• It only remains to prove (5.4), but this is an immediate consequence of Theorem 5.2.4, taking into account the reparametrization (5.5). This concludes the proof. □

We will say that the cylinder $\mathscr{C}_j$ exhibited in Proposition 5.3.4 is associated with the annulus A_j, for $1 \leq 0 \leq m$, and that $\mathscr{C}_\pm^\bullet$ is associated with $\mathsf{A}_\pm^\bullet$.

In the following we will apply the previous proposition to an increasing family of simple resonances $\cup_{1\leq \ell\leq n}\mathbb{S}_{k_\ell}$, and we need to exhibit a single chain of cylinders for $\mathcal{H}_n$ whose projection follows each line in this family. To this aim, we order the subset $\widehat{\mathbb{Z}}^2 \subset \mathbb{Z}^2$ formed by the vectors

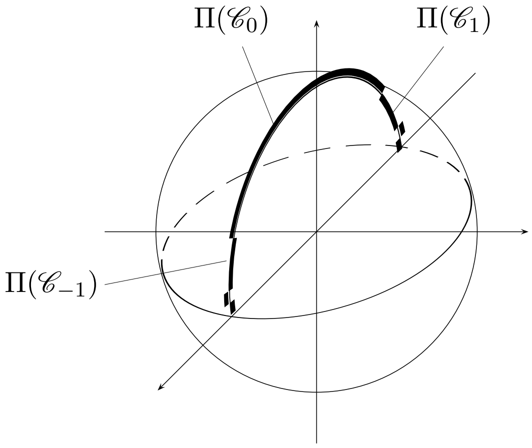

Figure 5.3: Projected cylinders

$k = (k_2, k_3)$ with coprime components, in such a way that the resulting sequence $(k_\ell)_{\ell \geq 1}$ satisfies $\|k_\ell\| \leq \|k_{\ell+1}\|$. For each $k \sim c \in \widehat{\mathbb{Z}}^2$, we denote by $\mathrm{Cyl}_k(\mathcal{H}_n)$ the set of cylinders associated with the annuli $\mathsf{A}_0(c), \ldots, \mathsf{A}_m(c)$ of Theorem 5.2.4, together with the singular cylinders $\mathscr{C}_\pm^\bullet$, so that $\#\mathrm{Cyl}_k(\mathcal{H}_n) = 2m + 3$. Finally we set

$$\mathrm{Cyl}(\mathcal{H}_n) = \bigcup_{1 \leq \ell \leq n} \mathrm{Cyl}_{k_\ell}(\mathcal{H}_n).$$

Recall that for each $k \in \widehat{\mathbb{Z}}^2$, the cylinders in $\mathrm{Cyl}_k(\mathcal{H}_n)$ form *two* chains, depending on the way they are ordered: a chain from $\mathscr{C}_-^\bullet$ to $\mathscr{C}_+^\bullet$ and a chain from $\mathscr{C}_+^\bullet$ to $\mathscr{C}_-^\bullet$. The first one will be denoted by $\mathrm{Chain}_k^+(\mathcal{H}_n)$ while the second one will be denoted by $\mathrm{Chain}_k^-(\mathcal{H}_n)$.

Definition 5.3.5. *Let $n \geq 1$ be fixed. We denote by* $\mathrm{Chain}(\mathcal{H}_n)$ *the chain formed by the concatenation of the chains* $\mathrm{Chain}_{k_\ell}^{(-1)^\ell}(\mathcal{H}_n)$*, $1 \leq \ell \leq n$.*

Finally, we denote by $\mathrm{Tori}(\mathcal{H}_n)$ the set of all isotropic tori of the form $\mathscr{T}_{r_1}$ contained in the cylinders and singular cylinders of $\mathrm{Cyl}(\mathcal{H}_n)$. The rest of the chapter is devoted to the construction of a perturbation which will create shadowing orbits along $\mathrm{Chain}(\mathcal{H}_n)$, passing close to a δ-dense family of dynamically minimal tori in $\mathrm{Tori}(\mathcal{H}_n)$.

5.4 Construction of the perturbation

This section is devoted to the construction of a perturbation $f_n \in C^\kappa(\mathbb{A}^3)$ such that the system $H_n = \mathcal{H}_n + f_n$ admits the same family of cylinders as $\mathcal{H}_n$, with additional splitting properties for their invariant manifolds.

5.4.1 The transversality conditions

We first set out some definitions for the splitting of separatrices. In this section f_n denotes a function in $C^\infty(\mathbb{A}^3)$ whose support is contained in the complement of the union of the cylinders of $\mathrm{Cyl}(\mathcal{H}_n)$. As a consequence, each $\mathscr{C} \in \mathrm{Cyl}(\mathcal{H}_n)$ is also invariant under the flow generated by $H_n = \mathcal{H}_n + f_n$, and contained in $H_n^{-1}(\frac{1}{2})$. We can therefore set

$$\mathrm{Cyl}(H_n) := \mathrm{Cyl}(\mathcal{H}_n), \qquad \mathrm{Chain}(H_n) := \mathrm{Chain}(\mathcal{H}_n), \qquad \mathrm{Tori}(H_n) := \mathrm{Tori}(\mathcal{H}_n).$$

Given a point x in a cylinder of $\mathrm{Cyl}(H_n)$, note that its stable and unstable manifolds $W^{s,u}(x)$ are well-defined, and that this is also the case for the stable and unstable manifolds of any $\mathscr{T} \in \mathrm{Tori}(H_n)$, that we denote by $W^{s,u}(\mathscr{T})$. Let us now introduce our conditions.

Definition 5.4.1. *Let $\mathscr{T} \in \mathscr{C}$ and $\mathscr{T}' \in \mathscr{C}'$ be two elements of* $\mathrm{Tori}(H_n)$ *(recall that $\mathscr{C}$ and $\mathscr{C}'$ can be singular cylinders). We say that the pair $(\mathscr{T}, \mathscr{T}')$ satisfies* Condition (W) *if there exists $a \in \mathscr{T}$ whose unstable manifold $W^{uu}(a)$ intersects $W^s(\mathscr{C}')$ transversely in $H_n^{-1}(\frac{1}{2})$ at some point of $W^s(\mathscr{T}')$.*

Definition 5.4.2. *Fix a cylinder $\mathscr{C} \in \mathrm{Cyl}(H_n)$ with associated embedding $F : \mathbb{T}^2 \times J \to H_n^{-1}(\frac{1}{2})$ ($\mathscr{C}$ regular) and for $r_1 \in J$, set $\mathscr{T}(r_1) = F(\mathbb{T}^2 \times \{r_1\})$. We say that $\mathscr{C}$ satisfies* Condition (T) *when there is $\rho > 0$ such that for each pair $(r_1, r_1') \in J^2$ with $|r_1 - r_1'| < \rho$, the pair $(\mathscr{T}(r_1), \mathscr{T}(r_1'))$ satisfies Condition (W).*

Given one of the two singular cylinders $\mathscr{C}^\bullet$ associated with the singular annulus $\mathsf{A}^\bullet$, we denote by $F_\pm^\bullet$ and $F_0^\bullet$ the embeddings associated with the corresponding embeddings $\phi_\pm^\bullet$ and $\phi_0^\bullet$ as in the proof of Proposition 5.3.4. Given an invariant circle S at energy e of the pendulum system on the standard singular annulus $\mathcal{A}^\bullet$ (with suitable parameters), we set $\mathscr{T}_\pm(e) = F_\pm^\bullet(\mathbb{T} \times S)$ when $e > \widehat{e}$ and $\mathscr{T}_0(e) = F_0^\bullet(\mathbb{T} \times S)$ when $e < \widehat{e}$.

Definition 5.4.3. *We say that $\mathscr{C}^\bullet$ satisfies* Condition (T) *when there is $\rho > 0$ such that:*

- *given $\sigma \in \{-1, 0, +1\}$, each pair $(\mathscr{T}_\sigma(e), \mathscr{T}_\sigma(e'))$ such that $|e - e'| < \rho$ satisfies Condition (W),*
- *the same condition holds true for each pair $(\mathscr{T}_\sigma(e), \mathscr{T}_{\sigma'}(e'))$ when one sign σ or σ' is zero and the other one is in $\{-1, 1\}$.*

As for chains, we have to add a transversality condition for tori contained in distinct cylinders.

Definition 5.4.4. *We say that a chain of cylinders $(\mathscr{C}_k)_{1 \le k \le k^*}$ satisfies* Condition (S) *when each cylinder $\mathscr{C}_k$ satifies Condition (T) and when moreover, for $1 \le k \le k^* - 1$, there are open subsets $O_k \subset \mathscr{C}_k$ and $O_{k+1} \subset \mathscr{C}_{k+1}$, union of elements of* $\mathrm{Tori}(H_n)$*, such that for each $\mathscr{T} \subset O_k$ and $\mathscr{T}' \in O_{k+1}$, the pair $(\mathscr{T}, \mathscr{T}')$ satisfies Condition (W).*

Note that Condition (S) is obviously open in the following sense.

Lemma 5.4.5. *Assume that* $\mathrm{Chain}(H_n)$ *satisfies Condition (S). Given a small enough function f in the C^2 topology, with support contained in the complement of the union of the cylinders of* $\mathrm{Cyl}(H_n)$*, then* $\mathrm{Chain}(H_n + f) := \mathrm{Chain}(H_n)$ *is a chain at energy $\frac{1}{2}$ for $H_n + f$ and satisfies Condition (S).*

Our purpose in this section is to prove the following result.

Proposition 5.4.6. *Fix $\kappa \ge 2$. Then for each $n \ge 1$, there exists a function $f_n \in C^\infty(\mathbb{A}^3)$, whose support is contained in the complement of the union of the cylinders of* $\mathrm{Cyl}(\mathcal{H}_n)$*, and which satisfies $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \le \frac{1}{n}$, such that* $\mathrm{Chain}(\mathcal{H}_n + f_n)$ *satisfies Condition (S).*

The rest of the section is devoted to the proof, which will requires two steps: we will first exhibit a perturbation which creates the heteroclinic connections for the pairs of tori contained in the same cylinder, and we will then use the previous openness property to add a second perturbation adapted to the heteroclinic conditions for extremal tori of the chain.

5.4.2 Flow boxes near homoclinic intersections of cylinders

In order to properly define the various pertubations, we first need "flow boxes" centered on suitable parts of the homoclinic manifolds of the cylinders of $\mathcal{H}_n$, and located "far from these cylinders". Given $U \in \mathscr{U}$, we fix an annulus A of the system C_U defined over I. We denote by $\Gamma(e)$ the periodic orbit $\mathsf{A} \cap C_U^{-1}(e)$ and we fix an open subinterval $I^* \subset I$ over which $W^{s,u}(\Gamma(e))$ intersect transversely along a homoclinic orbit $\Omega(e)$ (see Definition 5.2.1). Therefore, there exists a 3–dimensional section Σ in $\mathbb{A}^2$, transverse to X^{C_U}, such that the union $\cup_{e\in I^*}\Omega(e)$ intersects Σ along a $C^{\kappa-1}$ curve σ.

1. Since Σ is transverse to X^{C_U}, for $e \in I^*$ the intersection $\Sigma \cap C_U^{-1}(e)$ is symplectic. Reducing Σ if necessary, one easily proves the existence of a ball $B \subset \mathbb{R}^2$ centered at 0 and a $C^{\kappa-1}$ diffeomorphism $\chi_0 : I^* \times B \to \Sigma$, such that

- $C_U \circ \chi_0(e, s, u) = e$,
- the connected component of $W^u(\mathsf{A}) \cap \Sigma$ containing σ admits the equation $s = 0$,
- the connected component of $W^s(\mathsf{A}) \cap \Sigma$ containing σ admits the equation $u = 0$,
- for each $e \in I^*$, $\chi_0(e, \cdot)$ is symplectic for the usual structure on B and the induced structure on $\Sigma \cap C_U^{-1}(e)$.

2. For $\tau_0 > 0$ small enough, the Hamiltonian flow $\Phi^{C_U} :]-\tau_0, \tau_0[\times \Sigma \to \mathbb{A}^2$ is a diffeomorphism onto its image $\mathscr{O}$. This enables one to construct a symplectic diffeomorphism

$$\chi :]-\tau_0, \tau_0[\times I^* \times B \longrightarrow \mathscr{O}$$

by setting

$$\chi(\tau, e, s, u) = \Phi_\tau^{C_U}\big(\chi_0(e, s, u)\big).$$

One immediately checks that these coordinates are symplectic. By construction, the Hamiltonian C_U takes the simple expression

$$C_U \circ \chi(\tau, e, s, u) = e.$$

This in turn yields a symplectic diffeomorphism $\widehat{\chi} : \mathscr{D} \longrightarrow \mathbb{A} \times \mathscr{O} \subset \mathbb{A}^3$, where $\mathscr{D}$ is the subset of all $(\tau, \mathbf{e}, s, u, \theta_1, r_1) \in]-\tau_0, \tau_0[\times R \times B \times \mathbb{A}$ such that $\mathbf{e} - \frac{1}{2}r_1^2 \in I^*$ (note that now $\mathbf{e}$ stands for the *total* energy of the system), defined by

$$\widehat{\chi}(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \Phi_\tau^{\mathcal{H}}\Big((\theta_1, r_1), \chi\big(0, \mathbf{e} - \tfrac{1}{2}r_1^2, s, u\big)\Big),$$

which clearly satisfies

$$\mathcal{H} \circ \widehat{\chi}(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mathbf{e}.$$

3. The effect of the rescaling (5.5) is immediately computed in the previous straightening coordinates. Setting

$$\chi_n = \sigma_n^{-1} \circ \chi$$

one gets

$$\chi_n^{-1} \circ \Phi_t^{\mathcal{H}_n} \circ \chi_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = (\tau + \tfrac{1}{n}\,t, \mathbf{e}, s, u, \theta_1, r_1). \tag{5.7}$$

5.4.3 Perturbation and Condition (T) for cylinders

We now construct a first perturbation $f_n^{(1)}$ which produces heteroclinic connections between nearby elements of $\mathrm{Tori}(\mathcal{H}_n + f_n^{(1)})$ contained in the same cylinder and yields Condition (T) for each cylinder.

To begin with, let us fix a regular cylinder $\mathscr{C} \in \mathrm{Cyl}(\mathcal{H}_n)$, associated with some annulus A of C_U defined over an interval I, and let I^* be a subinterval of I as in the previous section. Let $F : \mathbb{T}^2 \times J \to \mathcal{H}_n^{-1}(\frac{1}{2})$ be the embedding of $\mathscr{C}$ introduced in Section 5.3.2, where J is the interval associated with I in the r_1–line. We denote by J^* the subset of J associated with I^* by the same process. As above, we set $\mathscr{T}_{r_1} = F(\mathbb{T}^2 \times \{r_1\})$.

Beginning with expressions in the straightening coordinates and with the assumptions and notation of the previous section, we set, for $(\tau, \mathbf{e}, s, u, \theta_1, r_1) \in \mathscr{D}$:

$$f \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mu\eta(\tau)\eta(\theta_1)\frac{\theta_1^2}{2}, \tag{5.8}$$

where $\mu > 0$ is a small enough constant and η is a bump function, all these data being suitably chosen in the following lemma.

Lemma 5.4.7. *With the notation of the previous section, given an arbitrary neighborhood of the homoclinic curve σ, given $n \geq 1$ and $\nu > 0$, there exist $\mu > 0$, $\eta \in C^\infty(\mathbb{R})$ and $\rho > 0$ such that the pair $(\mathscr{T}_{r_1}, \mathscr{T}_{r_1'})$ satisfy Condition (W) for $r_1, r_1' \in J^*$ with $|r_1 - r_1'| < \rho$, and such that moreover*

$$\|f\|_{C^\kappa(\mathbb{A}^3)} \leq \nu.$$

Proof. We have fixed the energy $\mathbf{e} = \frac{1}{2}$, so the coordinates $(\tau, s, u, \theta_1, r_1)$ form a chart in the neighborhood of Σ. In this chart, the vector field generated by $H_n \circ \widehat{\chi}_n$ reads:

$$\left|\begin{array}{l} \dot{\tau} = 1 \\ \dot{s} = 0 \\ \dot{u} = 0 \\ \dot{\theta}_1 = 0 \\ \dot{r}_1 = \mu\eta(\tau)\eta(\theta_1)\theta_1 \end{array}\right. \tag{5.9}$$

We require the function η to have its support localized in a small enough neighborhood of 0 in order for f_n to satisfy the condition on its support, and to take only nonnegative values. This way, the variation of the variable r_1 when passing through the support of the function f_n is easily computed:

$$\Delta r_1 = \mu\|\eta\|_1^2\theta_1,$$

where $\|\eta\|_1$ is the L^1 norm. The conclusion for the existence of weak transverse intersections then easily follows. Finally, the statement on the upper bound for the norm of f_n comes from the possibility of chosing $\mu > 0$ arbitrarily small, to control the growth on the various derivatives in the composition

$$f = (f \circ \widehat{\chi}_n) \circ \widehat{\chi}_n^{-1}$$

when $n \to \infty$. □

It remains to examine the case of the singular cylinders. In fact exactly the same considerations as above apply, since we assumed for the system C_U the existence of transverse homoclinic orbits for the two homoclinic orbits attached to the fixed point O (see Definition 5.2.2). This enables one to find a finite family of sections Σ and homoclinic curves enjoying the same properties as in the regular case. This yields the following intermediate result.

Corollary 5.4.8. *Given $n \geq 1$, there exists $f_n^{(1)} \in C^\infty(\mathbb{A}^3)$ such that each $\mathscr{C} \in \mathrm{Cyl}(\mathcal{H}_n + f_n^{(1)})$ satisfies Condition (T), with $\|f_n^{(1)}\|_{C^\kappa(\mathbb{A}^3)} \leq \frac{1}{2n}$.*

Proof. We apply the previous lemma inductively, after a preliminary ordering of all subintervals $(I_*(k))_{1\leq k\leq k^*}$ attached with all regular cylinders in $\mathrm{Cyl}(\mathcal{H}_n)$ and a choice of pairwise disjoint attached sections Σ and homoclinic curves σ (which is obviously possible thanks to the structure of the set of annuli of C_U). Using the possibility to choose the support of the function f in Lemma 5.4.7 inside an arbitrary neighborhood of σ, we can therefore obtain a finite family of perturbations f^k, $1 \leq k \leq k^*$, with pairwise disjoint supports, such that the sum $f_n^{(1)} = \sum_k f^k$ satisfies the two claims of our statement since its norm is just the supremum of the individual norms. □

5.4.4 Perturbation and Condition (S) for chains

So far we have constructed a perturbed Hamiltonian $\mathcal{H}_n + f_n^{(1)}$ such that each cylinder of the family $\mathrm{Cyl}(\mathcal{H}_n + f_n^{(1)})$ satisfies Condition (T). It remains now to add a new (and smaller) perturbation term to ensure that the pairs of tori located in consecutive cylinders of the associated chain satisfy Condition (W). We begin with a classical lemma on the existence of heteroclinic intersections for tori with the same homology.

Lemma 5.4.9. *Let $(\mathscr{C}_k)_{1\leq k\leq k^*} = \mathrm{Chain}(\mathcal{H}_n + f_n^{(1)})$ be the chain of cylinders for the Hamiltonian. Then for $1 \leq k \leq k^*-1$, there are tori $\mathscr{T}_k \subset \mathscr{C}_k$ and $\mathscr{T}_{k+1} \subset \mathscr{C}_{k+1}$ (of the family $\mathrm{Tori}(\mathcal{H}_n + f_n^{(1)})$) which admit a heteroclinic connection.*

Proof. Let us begin with the unperturbed situation generated by $\mathcal{H}_n$. Fix two consecutive (regular or singular) cylinders $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$, associated with annuli A_k and A_{k+1}. Then there exists an energy e for which the periodic orbits $\mathsf{A}_k \cap C_U^{-1}(e)$ and $\mathsf{A}_{k+1} \cap C_U^{-1}(e)$ admit a transverse heteroclinic orbit, and this situation persists in a neighborhood of e (using if necessary small hyperbolic continuations for the annuli). As a consequence, as above, there exists a transverse section $\Sigma \subset \mathcal{H}_n^{-1}(\frac{1}{2})$, endowed with symplectic coordinates (s, u, θ_1, r_1), such that $W^u(\mathscr{C}_k) \cap \Sigma$ and $W^s(\mathscr{C}_{k+1}) \cap \Sigma$ read $\{u = 0\}$ and $\{s = 0\}$. The subset $\{u = s = 0\}$ is the (local) intersection with Σ of a manifold of heteroclinic orbits between $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$. This manifold $\mathscr{A}$ is symplectic and diffeomorphic to $\mathbb{T} \times I$, where I is some (small) open interval. The invariant manifolds $W^u(\mathscr{T}_k(r_1^0))$ and $W^s(\mathscr{T}_{k+1}(r_1^0))$ intersect $\mathscr{A}$ along one and the same essential circle $\{r_1 = r_1^0\}$.

Now for n large enough the perturbed situation for $\mathcal{H}_n + f_n^{(1)}$ is only slightly distorted. One can still find a section Σ with coordinates (s, u, θ_1, r_1) in which $W^u(\mathscr{C}_k) \cap \Sigma$ and $W^s(\mathscr{C}_{k+1}) \cap \Sigma$ have the same equations and so intersect along the slightly perturbed annulus $\mathscr{A}'$ with equation $u = s = 0$ in the new coordinates (all this being deduced from the various transversality properties). Again, by transversality, $W^u(\mathscr{T}_k(r_1^0)) \cap \mathscr{A}$ and $W^s(\mathscr{T}_{k+1}(r_1^0)) \cap \mathscr{A}$ are embedded essential circles but they do not coincide any longer (in general).

However, it is easy to see that they still intersect each other, using the fact that the coordinates (θ_1, r_1) are exact symplectic on $\mathscr{A}$ together with the Lagrangian character of the invariant manifolds $W^u(\mathscr{T}_k(r_1^0))$ and $W^s(\mathscr{T}_{k+1}(r_1^0))$ (see [LMS03] for more details). Indeed, since the tori $\mathscr{T}_k(r_1^0)$ and $\mathscr{T}_{k+1}(r_1^0)$ are left unchanged when the perturbation is added, the intersections $C_k = W^u(\mathscr{T}_k(r_1^0)) \cap \mathscr{A}$ and $C_{k+1} = W^s(\mathscr{T}_{k+1}(r_1^0)) \cap \mathscr{A}$ have the same homology in $\mathscr{A}'$, meaning that the symplectic area between them vanishes. This comes from the fact that this assertion is trivially true in the unperturbed situation along with the Lagrangian character of $W^u(\mathscr{T}_k(r_1^0))$ and $W^s(\mathscr{T}_{k+1}(r_1^0))$. This proves our claim. ☐

Our next lemma will enable us to complete the proof of Proposition 5.4.6

Lemma 5.4.10. *For $n \geq n_0$ large enough, there exists a function $f_n \in C^\infty(\mathbb{A}^3)$ with support constained in the complement of $\cup_{1\leq k\leq k^*}\mathscr{C}_k$, with $\|f_n\|_{C^\kappa(\mathbb{A}^3)} \leq \frac{1}{n}$, such that the chain $(\mathscr{C}_k)_{1\leq k\leq k^*}$ for the system $H_n := \mathcal{H}_n + f_n$ satisfies Condition (S).*

Proof. The proof is similar and even simpler than that of Lemma 5.4.7. With the notation of Lemma 5.4.9, if the circles C_k and C_{k+1} intersect transversely in $\mathscr{A}$, there is obviously nothing to do. Now if they intersect tangentially, one constructs a flow-box as in Section 5.4.2 and again use a perturbation of the form

$$\ell_n \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mu\eta(\tau)\eta(\theta_1)\frac{\theta_1^2}{2}.$$

The support of ℓ_n can be chosen arbitrarilly small, and its norm is controlled by means of the constant μ. In particular, it can be chosen small enough to preserve the Condition (T) for all cylinders. One can therefore proceed by induction as above, using now the natural ordering of the heteroclinically connected pairs of tori inside consecutive cylinders of the chain. This proves the existence of a finite family of functions ℓ_n^j, with controlled supports and norms, such that $f_n = f_n^{(1)} + \sum_j \ell_n^j$ fulfills our claims. ☐

5.5 Diffusion orbits

We first recall the basic λ-lemma (Theorem 1 in Chapter 2) in a version adapted to our present setting and state an abstract shadowing result for chains of cylinders that is slightly more general than Corollary 2.4.2. We then apply this result to prove Theorem 5.

5.5.1 Shadowing orbits along chains of minimal sets

Theorem 1 requires the existence of the straightening neighborhood (Proposition B in Section 1.1.3) for the cylinders. In the case of general normally hyperbolic manifolds (like in the previous chapters) such results need abstract additional assumptions given by Section 1.4, but here we will take advantage of the very simple geometric structure of the problem.

1. Let us begin with a straightening result in the neighborhood of the annuli. Let $U \in \mathscr{U}$ be fixed.

Lemma 5.5.1. *Let A be an annulus defined over I for C_U. Then there exist a neighborhood $\mathscr{O}$ of A, an interval $\widehat{I}$ containing I and a symplectic diffeomorphism $\Psi : \mathbb{T} \times \widehat{I} \times B \to \mathscr{O}$, where $B = [-\alpha, \alpha]^2$ is a ball in $\mathbb{R}^2$, such that $\mathsf{A} = \Psi(\mathbb{T} \times I \times \{0\})$ and*

$$C_U \circ \Psi(\varphi, \rho, s, u) = C_U \circ \Psi(\varphi, \rho, 0, 0) + O_2(s, u).$$

In particular, the stable and unstable manifolds $W^{s,u}(\mathsf{A})$, together with the stable and unstable manifolds $W^{ss,uu}(x)$ for $x \in \mathsf{A}$, are straightened in these coordinates and read:

$$\Psi^{-1}\big(W^s(\mathsf{A})\big) = \{u = 0\}, \qquad \Psi^{-1}\big(W^u(\mathsf{A})\big) = \{s = 0\},$$

$$\Psi^{-1}\big(W^{ss}(x)\big) = \{(\varphi,\rho,s,0) \mid s \in [-\alpha,\alpha]\}, \quad \Psi^{-1}\big(W^{uu}(x)\big) = \{(\varphi,\rho,0,u) \mid u \in [-\alpha,\alpha]\},$$

where (φ,ρ) *is defined by* $\Psi(x) = (\varphi,\rho,0,0)$.

The proof is a simple application of the Moser isotopy lemma. One proves indeed the straightening result first and deduces the normal form from the structure of the Hamiltonian system in such a neighborhood. The previous lemma immediately yields the following straightening result in the neighborhood of the cylinders of $\mathrm{Cyl}(H_n)$.

Lemma 5.5.2. *Let* $\mathscr{C}$ *be a cylinder of the family* $\mathrm{Cyl}(H_n)$ *and let* A *be the associated annulus, defined over* I. *Let* $\mathscr{O}$ *and* Ψ *be defined as in the previous lemma. Then, up to shrinking* B *if necessary, the product diffeomorphism*

$$\widehat{\Psi} = \mathrm{Id}_{\mathbb{A}} \times \Psi \ : \ \mathbb{A} \times \mathbb{T} \times \widehat{I} \times B \longrightarrow \mathbb{A} \times \mathscr{O}$$

is symplectic and satisfies

$$H_n \circ \widehat{\Psi}(\theta_1, r_1, \varphi, \rho, s, u) = \tfrac{1}{2} r_1^2 + C_U \circ \Psi(\varphi,\rho,0,0) + O_2(s,u).$$

Proof. This is an immediate consequence of the fact that if B is small enough, the neighborhood $\mathbb{A} \times \mathscr{O}$ and the support of f_n are disjoint, so that $(H_n)_{|\mathscr{O}} = (\mathcal{H}_n)_{|\mathscr{O}}$. The claim then immediately follows from the previous lemma. $\square$

Note that $\mathscr{C}$ is then the set of all $\widehat{\Psi}(\theta_1, r_1, \varphi, \rho, 0, 0)$ such that

$$\tfrac{1}{2} r_1^2 + C_U \circ \Psi(\varphi,\rho,0,0) = \tfrac{1}{2}.$$

The basic λ-lemma proved in Chapter 2 was stated in the framework of symplectic diffeomorphisms and normally hyperbolic invariant submanifolds in a symplectic manifold. We therefore need to adapt it to the present context, since the cylinders $\mathscr{C}$ are not normally hyperbolic in $\mathbb{A}^3$, but rather in $H_n^{-1}(\frac{1}{2})$. The simplest way to overcome this (easy) problem is to apply the lemma to the full normally hyperbolic manifold $\mathscr{N} = \widehat{\Psi}(\mathbb{A} \times \mathbb{T} \times \widehat{I} \times \{0\})$ (with the notation of the previous lemma) and the symplectic diffeomorphism Φ^{H_n} (the time-one map). This is made possible by the previous straightening result (see Chapter 2 for a proof, the lack of compactness obviously causes no trouble here, due to the preservation of energy and the fact that $\mathscr{C}$ is relatively compact). We set $\Phi := \Phi^{H_n}$.

The λ-lemma. *Let* $\mathscr{C} \in \mathrm{Cyl}(H_n)$ *be a cylinder at energy* $\frac{1}{2}$ *for the Hamiltonian system* H_n *and let* $\mathscr{N}$ *be the normally hyperbolic manifold of* $\mathbb{A}^3$ *defined above. Let* Δ *be a* 1*–dimensional submanifold of* A^3 *which transversely intersects* $W^s(\mathscr{N})$ *at some point* a. *Then* $\Phi^n(\Delta)$ *converges to the unstable leaf* $W^{uu}\big(\Phi^n(\ell(a))\big)$ *in the* C^0 *compact open topology, where* $\ell(a)$ *is the unique element of* $\mathscr{C}$ *such that* a *belongs to the stable leaf* $W^{ss}(\ell(a))$.

Let us make clear the notion of convergence used here (see Chapter 2 for details). The simplest way to define it is to use Lemma 5.5.2. In the neighborhood $\mathbb{A} \times \mathscr{O}$ and relatively to the previous coordinates, if $x \sim (\theta_1, r_1, \varphi, \rho, 0, 0) \in \mathscr{C}$, the unstable leaf $W^{uu}(x)$ reads

$$W^{uu}(x) = \{(\theta_1, r_1, \varphi, \rho, 0, u) \mid u \in \ [-\alpha,\alpha]\}.$$

The first result in the basic λ-lemma (Theorem 1) is that for n large enough, the connected component Δ^n of $\Phi^n(a)$ in $\Phi^n(\Delta) \cap (\mathbb{A} \times \mathscr{O})$ is a *graph* over the unstable direction, that is, it admits the equation

$$\Delta^n = \Big\{\big(\theta_1^n(u), r_1^n(u), \varphi^n(u), \rho^n(u), s^n(u), u\big) \mid u \in \,]-\overline{u}, \overline{u}[\Big\}.$$

The convergence statement then just says that

$$\|(\theta_1^n(u), r_1^n(u), \varphi^n(u), \rho^n(u), s^n(u)) - (\theta_1^n(0), r_1^n(0), \varphi^n(0), \rho^n(0), 0)\| \to 0$$

uniformly in u when n tends to $+\infty$. Of course one then gets more global formulation by using the definion of $W^u(\mathscr{C})$ as the union of the images by Φ of the local unstable manifold. Note that this is only a C^0-convergence while a stronger C^1-convergence result was proved in Chapter 2. The same definitions apply to the following case.

Corollary 5.5.3. *Let $\mathscr{C} \in \mathrm{Cyl}(H_n)$. Let Δ be a 1–dimensional submanifold of $H_n^{-1}(\frac{1}{2})$ which transversely intersects $W^s(\mathscr{C})$ in $H_n^{-1}(\frac{1}{2})$ at some point a. Then $\Phi^n(\Delta)$ converges to the unstable leaf $W^{uu}\big(\Phi^n(\ell(a))\big)$ in the C^0 compact open topology.*

Proof. Observe that the fact that Δ intersects $W^s(\mathscr{C})$ transversely in $H_n^{-1}(\frac{1}{2})$ implies that Δ transversely intersects $W^s(\mathscr{N})$. Then apply the λ-lemma and use the invariance of energy. $\square$

2. We can now state our shadowing result, which is slightly more general than the analogous result in Corollary 2.4.2.

Proposition 5.5.4. [Shadowing lemma.] *Set $(\mathscr{C}^i)_{1\leq i\leq i^*} := \mathrm{Chain}(H_n)$ and for $1 \leq i \leq i^*$, let $(\mathscr{T}_j^i)_{1\leq j\leq j_i^*}$ be a family of dynamically minimal invariant tori contained in $\mathscr{C}^i$, such that*

- *for $1 \leq j \leq j_i^* - 1$, there exists $a_j^i \in \mathscr{T}_j^i$ such that $W^{uu}(a_j^i)$ intersects $W^s(\mathscr{C}^i)$ transversely in $H_n^{-1}(\frac{1}{2})$, at some point contained in $W^s(\mathscr{T}_{j+1}^i)$,*
- *for $1 \leq i \leq i^* - 1$, there exists $a_{j_i^*}^i \in \mathscr{T}_{j_i^*}^i$ such that $W^{uu}(a_{j_i^*}^i)$ intersects $W^s(\mathscr{C}^{i+1})$ transversely in $H_n^{-1}(\frac{1}{2})$, at some point contained in $W^s(\mathscr{T}_1^{i+1})$.*

Then, for each $\rho > 0$, there exists an orbit Γ at energy $\frac{1}{2}$ of H_n which intersects each ρ–neighborhood $\mathscr{V}_\rho(\mathscr{T}_j^i)$, for $1 \leq i \leq i^*$ and $1 \leq j \leq j_i^*$.

Proof. The argument is a refinement of that introduced in Corollary 2.4.2. Fix i and forget about the corresponding superscript. Fix a ball B_{j+1} centered at some point of $W^u(\mathscr{T}_{j+1})$ and contained in $\mathscr{V}_\rho(\mathscr{T}_{j+1})$. Then the previous λ-lemma and the minimality of the torus $\mathscr{T}_{j+1}$ immediately yield the existence of a positive integer p such that $\Phi^p(W^{uu}(a_j))$ intersects B_{j+1}, so that there is $z \in W^{uu}(\Phi^p(a_j)) \cap B_{j+1}$ (see the proof of Corollary 2.4.2 for more details). Then, for q large enough, $\Phi^{-q}(z) \in \mathscr{V}_\rho(\mathscr{T}_j)$. Therefore there exists a ball B_j centered at $\Phi^{-q}(z)$ and contained in $\mathscr{V}_\rho(\mathscr{T}_j)$ such that

$$\Phi^q(B_j) \subset B_{j+1}.$$

Therefore, we proved the existence of a ball B_j centered on $W_j^u(\mathscr{T}_j)$ and a positive q such that $\Phi^q(B_j) \subset B_{j+1}$.

One obviously has a similar result for the "extremal" tori $\mathscr{T}_{j_i^*}^i$ and $\mathscr{T}_1^{i+1}$.

Therefore, given a ball B^* centered on $W^u(\mathscr{T}_{j_i^*}^{i^*})$ and contained in $\mathscr{V}_\rho(\mathscr{T}_{j_i^*}^{i^*})$, an immediate induction proves the existence of an integer q^* and a ball B centered on $W^u(\mathscr{T}_1^1)$ and contained in $\mathscr{V}_\rho(\mathscr{T}_1^1)$, whose sequence of iterates intersects each $\mathscr{V}_\rho(\mathscr{T}_j^i)$ and which moreover satisfies $\Phi^{q^*}(B) \subset B^*$. This proves our claim. $\square$

5.5.2 Asymptotic density: proof of Theorem 5

It only remains now to gather the results of the previous sections and apply the shadowing lemma (Proposition 5.5.4) to the chain of cylinders $\mathrm{Chain}(H_n)$ and a suitable family of minimal tori inside. Fix $\delta > 0$. Given $n \geq 1$, we set

$$(\mathscr{C}^i)_{1\leq i\leq i^*(n)} := \mathrm{Chain}(H_n).$$

• There exists n_0 such that for $n \geq n_0$, the union of the lines $(\mathbb{S}_{k_\ell})_{1\leq \ell\leq n}$ is $\delta/4$–dense in $\mathbb{S}$.

• There exists $n_1 \geq n_0$ such that for $n \geq n_1$, the sphere $\mathbb{S}$ is $\delta/4$–dense in $H_n^{-1}(\frac{1}{2})$, and therefore the union of the lines $(\mathbb{S}_{k_\ell})_{1\leq \ell\leq n}$ is $\delta/2$–dense in $H_n^{-1}(\frac{1}{2})$.

• By construction, there exists $n_2 \geq n_1$ such that for $n \geq n_2$,

$$\mathbf{d}\Big(\bigcup_{1\leq i\leq i^*(n)} \Pi(\mathscr{C}^i), \bigcup_{1\leq \ell\leq n} \mathbb{S}_{k_\ell}\Big) \leq \delta/6.$$

• By density of the minimal tori in the cylinders (see Proposition 5.3.4), and since the chains satisfy Condition (S), for each $n \geq n_2$, one can exhibit a family of minimal tori $(\mathscr{T}_j^i)$ satisfying the assumptions of the shadowing lemma and such that the union $\cup_{i,j}\Pi(\mathscr{T}_j^i)$ is $\delta/6$–dense in $\cup_{1\leq i\leq i^*(n)}\Pi(\mathscr{C}^i)$.

• Proposition 5.5.4, applied with $\rho = \delta/6$, shows the existence of an orbit of H_n whose projection is $\delta/6$–dense in $\cup_{i,j}\Pi(\mathscr{T}_j^i)$ and therefore $\delta/2$–dense in $\bigcup_{1\leq \ell\leq n}\mathbb{S}_{k_\ell}$, so also δ–dense in $H_n^{-1}(\frac{1}{2})$. This concludes the proof of Theorem 5.

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