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Liana Heuberger

Deux points de vue sur les variétés de Fano : géométrie du diviseur anticanonique et classification des surfaces à singularités 1/3(1,1)

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dirigée par Andreas Höring

Soutenue le 23 juin 2016 devant le jury composé de :

M. Sebastien Boucksom	Université Paris VI	Examinateur
M. Frédéric Campana	Université Nancy I	Examinateur
M. Ivan Cheltsov	University of Edinburgh	Rapporteur
M. Olivier Debarre	Université Paris VII et ENS	Examinateur
M. Antoine Ducros	Université Paris VI	Examinateur
M. Andreas Höring	Université de Nice Sophia Antipolis	Directeur de thèse

Institut de mathématiques de Jussieu-Paris
Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Université Pierre et Marie Curie.
École doctorale de sciences
mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

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Deux points de vue sur les variétés de Fano : géométrie du diviseur anticanonique et classification des surfaces à singularités 1/3(1,1)

Le sujet principal de cette thèse est l'étude des variétés de Fano, qui sont des objets centraux de la classification des variétés algébriques.

La première question abordée concerne les variétés de Fano lisses de dimension quatre. On cherche a étudier les potentielles singularités d'un diviseur anticanonique de sorte qu'on puisse les écrire sous une forme locale explicite. En tant qu'étape intermédiaire, on démontre aussi que ces points sont au plus des singularités terminales, c'est-à-dire les singularités les plus proches du cas lisse du point de vue de la géométrie birationnelle. On montre ensuite que ce dernier résultat se généralise en dimension arbitraire en admettant une conjecture de non-annulation de Kawamata.

De façon complémentaire, on s'intéresse à des variétés de Fano de dimension plus petite, mais admettant des singularités. Il s'agit des surfaces de del Pezzo ayant des singularités de type $\frac{1}{3}(1,1)$. Ceci est l'exemple le plus simple de singularité rigide, c'est-à-dire qui reste inchangée à une déformation $\mathbb{Q}$-Gorenstein près. On classifie entièrement ces objets en trouvant 29 familles. On obtient ainsi un tableau contenant des modèles de ces surfaces, qui pour la plupart sont des intersections complètes dans des variétés toriques. Ce travail s'inscrit dans un contexte plus large, où les auteurs de [OP] calculent leur cohomologie quantique pour ensuite vérifier si les Conjectures A et B de [ACC$^+$16] sont valides dans ces cas.

Bien que ces problématiques soient très proches, elles font appel à deux points de vue opposées. L'étude des surfaces, abordée au Chapitre 7, commence par l'utilisation d'une variante du programme des modèles minimaux afin de les construire, et se conclut grâce à des modèles obtenus par des techniques informatiques. Le cas des variétés de Fano de dimension quatre utilise des techniques modernes des singularités des paires pour retrouver le résultat de terminalité. Ce qui donne lieu à une étude de cas purement locale, alors que les méthodes sus-citées dans le cas des surfaces sont globales.

Variétés de Fano et systèmes anticanoniques

La problématique principale provient du constat suivant : en dimension strictement inférieure à quatre, soit le système anticanonique $|-K_X|$ est globalement engendré, soit, par un résultat non trivial de Shokurov, un diviseur général $D \in |-K_X|$ est lisse. Par contre, en dimension quatre, il existe des exemples où un tel D est singulier en un point. Les travaux de Höring et Voisin [HV11] montrent qu'il s'agit du cas extrémal, i.e. D ne peut être qu'à singularités isolées. Le but du Chapitre 3 est de trouver une forme explicite des équations locales de D afin de décrire précisément ses singularités. En utilisant des méthodes des singularités des paires, on relie la géométrie d'une variété de Fano ambiante X avec celle des diviseurs anticanoniques et on obtient le premier résultat suivant :

Theorem 0.1. Soit X une variété de Fano de dimension quatre et $D \in |-K_X|$ un élément général. Alors D a au plus des singularités terminales.

On remarquera que la notion de singularité terminale, introduite dans le deuxième chapitre, est le bon équivalent du résultat de Shokurov en dimension trois, puisque si D est une surface, un point terminal est bien un point lisse. On peut ensuite améliorer ce résultat. En effet les singularités terminales des variétés de dimension trois, bien que classifiées, sont en nombre infini, ce qui n'est pas le cas pour les familles de déformations de variétés de Fano ambiantes. Il s'agit du point de départ des travaux présentés ensuite.

Une conséquence du Théorème 0.1 est le résultat principal du Chapitre 3, que l'on énonce ci-dessous.

Theorem 0.2. Soit X une variété de Fano de dimension quatre et $D \in |-K_X|$ un élément général du système anticanonique. Alors, dans une carte analytique locale, toute singularité de D peut s'écrire sous l'une des deux formes suivantes :

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \text{ ou } x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0.$$

Une direction naturelle dans laquelle poursuivre ces travaux est de s'intéresser aux multiples du système anticanonique, c'est-à-dire aux systèmes linéaires de la forme $|-mK_X|$, où $m \geq 1$. La conjecture de Fujita, démontrée dans les travaux de Reider (1988), Ein et Lazarsefeld (1995) et Kawamata (1997), dans le cas où X est de dimension au plus quatre, implique que le système $|-(n-1)K_X|$ est sans point de base pour tout $n \geq \dim X$. Il est donc naturel de se poser la question suivante :

Question : Dans le cas général où le système linéaire $|-(n-2)K_X|$ a éventuellement des points de base, peut-on déduire qu'il contient un élément lisse ?

Le résultat de Shokurov sus-cité montre que la réponse est affirmative si $n = 3$, car tous les diviseurs anticanoniques généraux sont lisses sur une variété de dimension trois. On trouve des résultats intermédiaires dans cette direction en dimension quatre en utilisant systématiquement des théorèmes d'extension à partir des centres log canoniques :

Proposition 0.3. Si C est un centre log canonique de la paire $(X, Y_1 + Y_2)$ qui est au plus de dimension 1, alors $\mathrm{Bs}\,|-2K_X| \cap C = \emptyset$.

Le Chapitre 5 est une étude du Théorème 0.1 en dimension arbitraire, en relation avec la conjecture de non-annulation de Kawamata. On montre d'abord que la terminalité est une conséquence de cette conjecture et on cherche ensuite à trouver le contexte le plus général pour que cela reste vrai.

Une autre approche consiste à trouver des exemples de variétés de Fano ayant le lieu de base $\mathrm{Bs}|-K_X|$ non-vide, puis à décrire sa géométrie. Ceci est problématique car la plupart des exemples connus de variétés de Fano de dimension quatre sont des variétés toriques, où tout fibré ample est globalement engendré.

Un bon point de départ sont les variétés décrites en tant qu'intersection complète $X_{d_1 \ldots d_c}$ dans des espaces projectifs à poids $\mathbb{P}(a_0, \ldots, a_n)$. On introduit les notions fondamentales nous permettant de vérifier des propriétés de $X_{d_1 \ldots d_c}$, notamment sa lissité et l'amplitude du diviseur anticanonique. On explicite ensuite une classification des variétés de codimension un qui sont de cette forme-là et on donne enfin deux exemples en codimension deux, un lisse et un singulier. En particulier, on trouve $X_{6,10} \in \mathbb{P}(1,1,1,2,2,5,5)$, dont le lieu de base $\mathrm{Bs}|-K_X|$ est formé de trois courbes rationnelles qui s'intersectent deux à deux. Ceci démontre, entre autres, que ce lieu de base n'est pas toujours uniréglé (ce qui n'était pas clair a priori).

Surfaces de del Pezzo avec des singularités 1/3(1,1)

Les singularités considérées, les points du type $\frac{1}{3}(1,1)$, sont des singularités quotient cycliques sur la surface de del Pezzo considérée, c'est-à-dire qu'elles peuvent être exprimées localement en tant que quotient de $\mathbb{C}^2$ par le groupe de l'unité d'ordre 3. Les déformations $\mathbb{Q}-$Gorenstein ont la propriété de ne pas changer le degré K^2 de la surface de départ. On bénéficie donc de deux outils pour aborder la classification : le programme des modèles minimaux et le calcul des invariants à partir d'adaptations de formules classiques (Riemann Roch, formule de Noether).

On utilise d'abord les invariants pour borner le nombre de singularités qui peuvent apparaître sur une surface S de del Pezzo (on trouve que le nombre maximal est 6). Ensuite, le programme de Mori pour ces surfaces nous permet de terminer la classification. En effet, les singularités n'admettent qu'un certain nombre de configurations possibles sur un rayon extrémal de S. Si on choisit de leur assigner un ordre de contraction, on arrive à effectuer un programme des modèles minimaux nous permettant de conclure rapidement.

Voici le résultat principal de cette section :

Theorem 0.4. Il existe exactement 29 familles de déformations $\mathbb{Q}$-Gorenstein des surfaces de del Pezzo ayant $k \geq 1$ points singuliers de type $\frac{1}{3}(1,1)$.

Le Chapitre 7 termine avec la présentation de deux tableaux contenant des invariants des surfaces, ainsi que des données combinatoires. Celles-ci permettant de reconstruire (dans la plupart des cas) des variétés toriques F telles que nos surfaces de del Pezzo $X \subset F$ soient des intersections complètes dans F. On explique comment faire ces constructions en toute généralité, ainsi que sur un exemple.

Mots-clés

Variétés de Fano, système anticanonique, centres log-canoniques, surfaces log-del Pezzo, variétés toriques, programme des modèles minimaux

Contents

Chapter 1

Introduction

1.1 Fano manifolds

Fano varieties constitute a fundamental part of the classification of algebraic varieties. Their study began with classical research of Italian algebraic geometers at the beginning of the XX-th century and has rapidly developed since the 1980s due to the introduction of the Minimal Model Program. Throughout this thesis, we will consider Fano manifolds over the complex numbers. By definition, these are projective manifolds with ample anticanonical class, which from a geometric point of view signifies that they are positively curved. These objects are rationally connected (by [Cam92] and [KMM87]), their Kodaira dimension is $-\infty$, and there exist only finitely many deformation types in each dimension (again by [KMM87]).

Fano manifolds have been completely classified up to dimension three. The only one-dimensional Fano variety over an algebraically closed field is the projective line. In dimension two there are ten such deformation families, termed del Pezzo surfaces: they are either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ in up to 8 points in general position. Fano threefolds have been classified into 105 deformation families by the work of Mori and Mukai [MM82] in the case where $\rho > 1$ and Iskovskikh [Isk77, Isk78] for $\rho = 1$. The following result is central to this classification:

Theorem 1.1. [Šok79] Let X be a smooth Fano threefold. Then the anticanonical system $|-K_X|$ is not empty and a general divisor $D \in |-K_X|$ is a smooth $K3$ surface.

In Chapter 3, we study a Fano fourfold X from the perspective of the singularities of the anticanonical divisor, a general element of the system $|-K_X|$. We sometimes refer to this object as a "general elephant", terminology due to Reid [Rei87]. This analysis is trivial in dimension two, as for all but one of the del Pezzo surfaces the base locus $\mathrm{Bs}\,|-K_X|$ is empty. For the remaining surface, isomorphic to $\mathrm{Bl}_{p_1 \ldots p_8}\mathbb{P}^2$, the anticanonical system has a single point as its base locus, however its general elephant is smooth at this point.

When considering the threefold case, naturally Theorem 1.1 leads to an immediate conclusion, though when examining $\mathrm{Bs}\,|-K_X|$ the discussion is not as straightforward as in the surface case. More precisely, a key observation in the proof of Theorem 1.1 is that if $\mathrm{Bs}|-K_X| \neq \emptyset$, then it is isomorphic to $\mathbb{P}^1$. It is perhaps interesting to remark that here a result on the geometry of the anticanonical system is crucial in completing the classification, while for surfaces the opposite is the case.

The methods used in proving Threorem 1.1 are not generalizable in higher dimensions since they rely on the geometry of $K3$ surfaces. Moreover, the statement does not hold in its current form if X is a fourfold, as Höring and Voisin [HV11] constructed an example where a general $D \in |-K_X|$

is not even $\mathbb{Q}$-factorial. For this reason, a suitable generalization is one in which we allow a class of singularities on D as long as points[1] of this type are smooth if D is a surface. This is exactly the case for terminal singularities. Let us finish by presenting the example:

Example 1.2. [HV11, Ex. 2.12] Let S be the blow-up of $\mathbb{P}^2$ in eight points in general position. As previously mentioned, S is a del Pezzo surface whose anticanonical system has exactly one base point, denoted by p. Set $X = S \times S$ and $S_i := p_i^{-1}(p)$ where p_i is the projection on the i-th factor. Then X is a smooth Fano fourfold and

$$\mathrm{Bs}\,|-K_X| = S_1 \cup S_2.$$

Let $D \in |-K_X|$ be a general element, then $\mathrm{Bs}\,|-K_X| \subset D$, so the surfaces S_1 and S_2 are Weil divisors on D. If they were $\mathbb{Q}$-Cartier, their intersection $S_1 \cup S_2$ would have dimension at least one, yet we have $S_1 \cup S_2 = (p,p)$. We have thus constructed a variety D that is not only singular, but not even $\mathbb{Q}-$factorial.

1.2 Previous results

We present classification results in arbitrary dimension, provided that the manifold X is of high Fano index.

Definition 1.3. The Fano index of a variety X is the number

$$i(X) := \sup\{t \in \mathbb{Q} \mid -K_X \equiv tH, \text{ for some ample Cartier divisor } H\}.$$

If X has klt singularities, then $\mathrm{Pic}(X)$ is torsion-free (by [IP99, Prop.2.1.2]) and the H in the definition is uniquely determined. We call it the *fundamental divisor* of X.

It is known that $i(X) \leq \mathrm{dim}X + 1$, and if $i(X) \geq \mathrm{dim}X$ then by the Kobayashi-Ochiai criterion X is either a hyperquadric or a projective space. Smooth Fanos of index $\mathrm{dim}X - 1$ are called del Pezzo manifolds and have been classified by Fujita, while terminal Fano manifolds of index larger than $\mathrm{dim}X - 2$ were classified by Campana-Flenner [CF93] and Sano [San96].

Finally, the works of Mukai have classified smooth Fano n-folds X of index $n-2$ (called Mukai manifolds), provided that the linear system $|H|$ contains a smooth divisor. In the article [Mel99], Mella proved a stronger result: aside from two threefold cases, a Mukai variety with at worst klt singularities always contains good divisors (i.e. the fundamental divisor of X has at worst the same singularities as X). Since the exceptional cases are both of singular varieties, this implies that Mukai's initial classification holds for all Fano varieties of index $n-2$.

Though no complete classification exists in this case, smooth and mildly singular Fano varieties of index $n-3$ have been studied by Floris in [Flo13]. One of the fundamental results of this article is the following:

Theorem 1.4. [Flo13, Thm.1.2] Let X be a smooth Fano variety of dimension $n \geq 4$ and index $n-3$, with H fundamental divisor.

1. If the dimension of X is $n = 4, 5$, then $h^0(X, H) \geq n-2$.

2. If $n = 6, 7$ and the tangent bundle T_X is H-semistable, then $h^0(X, H) \geq n-2$.

3. If $n \geq 8$, then $h^0(X, H) \geq n-2$.

[1] a posteriori the singularities will be isolated.

We will extensively use part 1 of this statement in Chapter 3, where the existence of two independent sections of $|-K_X|$ is crucial to the proof of Proposition 3.6, as well as to eliminate some of the cases leading to the result in Theorem 3.8. We display a brief computation in 5.10 to show that $h^0(X, -K_X) \geq 2$ in arbitrary index if X is a fourfold.

The starting point for the description of the general elephant in Chapter 3 was a result (which we fully state as Proposition 5.2 in Chapter 5) of Kawamata in [Kaw00], generalized by Floris as follows:

Theorem 1.5. [Flo13, Thm.4.1] Let X be a Fano variety of dimension $n \geq 4$ with at most Gorenstein canonical singularities and index $n-3$, with H fundamental divisor. Suppose that $h^0(X, H) \neq 0$ and let $Y \in |H|$ be a general element. Then Y has at most canonical singularities.

1.3 Main results

Following the discussion in 1.1, here is the first result in Chapter 3:

Theorem 1.6. Let X be a four-dimensional Fano manifold and let $D \in |-K_X|$ be a general divisor. Then D has at most terminal singularities.

Theorem 1.6 is indeed the natural generalization of Theorem 1.1 in dimension four, and it relies on and improves parts of 1.4 and 1.5 in the smooth case.

Although terminal Gorenstein singularities of threefolds are a well understood class, the statement of Theorem 1.6 may be refined to a more specific result: while there only exists a finite number of deformation families of Fano varieties in dimension four, there is a priori an infinite amount of possible types of singularities on their anticanonical divisors.

Theorem 1.7. Let X be a four-dimensional Fano manifold and let $D \in |-K_X|$ be a general divisor. Then the singularities of D are locally analytically given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \text{or} \quad x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0.$$

In particular, this result is consistent with the case in Example 1.2, which is a singularity of the first type. We do not yet know of any examples of the second type of singularity on a general elephant.

Throughout the proof of Theorem 1.7 we combine information obtained from the geometry of the ambient space X together with the fact that the threefolds belong to the same linear system inside it. We begin the analysis of the isolated terminal points by dividing the discussion into two cases: depending on the geometry of all general elements in $|-K_X|$, the anticanonical system either has fixed or moving singularities. Specifically, either a point $x \in \mathrm{Bs}|-K_X|$ is a singularity of all the general elephants or there exists a subvariety $V \subset \mathrm{Bs}|-K_X|$ of strictly positive dimension along which these singularities move. We further separate each of these cases according to the rank of the degree two part of a local expression of D and obtain the result in Theorem 1.7. It is perhaps important to remark that the two situations do not generate identical outcomes: if $|-K_X|$ has moving singularities, then they are necessarily of rank four, while if the singularities are fixed we are additionally left with one case in rank three.

The fundamental tool in the proofs of both Theorem 1.6 and Theorem 1.7 is the following inequality:

$$a_i + 1 \geq 2r_i \tag{1.1}$$

where, given a resolution $\mu : X' \to X$ of $\mathrm{Bs}|-K_X|$, we denote by a_i the discrepancies of each exceptional divisor E_i with respect to $(X, 0)$ and by r_i the coefficients of E_i appearing in μ^*D (cf.

Notation 3.2). We prove this result in Proposition 3.6 with techniques using singularities of pairs and multiplier ideals. The inequality straightforwardly implies the terminality result for all cases except $a_i = r_i = 1$, which we show does not in fact occur. It also allows us to systematically eliminate most of the cases leading to the statement of Theorem 1.7, providing the necessary liaison between the geometry of X and that of D. The general strategy of this proof is to explicitly build a sequence of blow-ups, starting from a center in X containing either the fixed singularities or the subvariety V, that ultimately contradicts condition (1.1).

The motivation behind Chapter 4 is a consequence of Fujita's conjecture, which is proven up to dimension four:

Theorem 1.8. If X is a smooth Fano variety and $\dim(X) \leq 4$, then the linear system $|-(n-1)K_X|$ is base-point free for all $n \geq \dim X$.

In our case, this translates to $|-3K_X|$ being base-point-free. Having just proven that this is not the case for the anticanonical system, it is natural to ask whether $|-2K_X|$ contains a smooth element.

Using $h^0(X, -K_X) \geq 2$, we obtain intermediate results on this topic:

Proposition 1.9. Let X be a smooth Fano fourfold and Y_1, Y_2 two general anticanonical divisors. If C is a log-canonical center of the pair $(X, Y_1 + Y_2)$ that is at most of dimension one, then $\operatorname{Bs}|-2K_X| \cap C = \emptyset$, for two general elephants Y_1 and Y_2.

The next step would be to extend this result to the case where the center is a surface. If this is a smooth subvariety S, it is a surface of general type. We carry out an extensive case study on whether $|-2K_X|_S|$ has a component of multiplicity higher than one in Proposition 4.4. The fact that S is a minimal surface of general type certainly simplifies the analysis, as much progress has already been made in this direction (for example in [Wen95, Xia85] in the case where $K_S^2 \leq 4$).

We then prove the following conjecture in the case where S is not singular and irreducible:

Conjecture 1.10. Let X be a Fano fourfold. If $\operatorname{Bs}|-K_X| \leq 1$, there exists a smooth element $D \in |-2K_X|$.

If S is smooth, the proof is elementary by using [BHPVdV04, VII,Thm.7.4], whose statement summarizes numerous results on minimal surfaces of general type. If S is reducible, we show this through an analysis of the interactions between log canonical centers of the pair $(X, Y_1 + Y_2)$ and the base loci of both $|-K_X|$ and $|-2K_X|$. We discuss how to adapt these methods if S is singular and irreducible.

In Chapter 5 we examine the necessary conditions for obtaining Theorem 1.6 in arbitrary dimension. What allows us to prove the pivotal Proposition 3.6 and ultimately arrive at condition (1.1) in Chapter 3 is the fact that Kawamata's effective nonvanishing Conjecture holds up to dimension two:

Conjecture 1.11. (Kawamata nonvanishing) Let D be a numerically effective Cartier divisor on a normal projective variety X. If there exists an effective $\mathbb{R}$-divisor B such that the pair (X, B) is Kawamata log terminal and such that the $\mathbb{R}$-Cartier divisor $D - (K_X + B)$ is big and nef, then the bundle $\mathcal{O}_X(D)$ has a non-zero global section.

This result was in turn motivated by Shokurov's famous non-vanishing theorem:

Theorem 1.12. Let X be a nonsingular projective variety, and let D be a nef Cartier divisor. Let G be a $\mathbb{Q}-$divisor such that $\lceil G \rceil$ is effective. Assume that the $\mathbb{Q}-$divisor $aD + G - K_X$ is ample for some $a \in \mathbb{Q}$, $a > 0$, and the support of the fractional part $\{G\}$ has only normal crossings, that is, all its irreducible components are nonsingular and intersect transversally. Then

$$H^0(X, \mathcal{O}_X(mD + \lceil G \rceil)) \neq 0$$

for any sufficiently large integer $m \gg 0$.

We carry out an analysis on whether weaker versions of Conjecture 1.11, for instance in the case where X is a Fano variety and D a general elephant, imply the fact that D has at most terminal singularities.

Chapter 6 is dedicated to finding examples of Fano fourfolds having non-empty base locus $\mathrm{Bs}|-K_X|$ and then to describe the geometry of this locus. This is problematic, as most of the Fano fourfolds classified so far are toric and therefore ample line bundles on them are globally generated.

We look for suitable examples among complete intersections $X_{d_1,\ldots,d_c}$ in the weighted projective space $\mathbb{P}(a_0 \ldots a_n)$, since properties like smoothness or ampleness of $|-K_X|$ are easy to verify. We show that there are precisely 10 families of smooth Fano hypersurfaces in weighted projective space that are not linear cones. Out of these, the only nontrivial $\mathrm{Bs}|-K_X|$ is in the case of $X_{10} \subset \mathbb{P}(1^4, 2, 5)$ and in particular it is of dimension zero. We also analyze examples of codimension two complete intersections, among which $X_{6,10} \subset \mathbb{P}(1^3, 2^2, 5^2)$ yields a base locus of $|-K_X|$ consisting of three rational curves. More work in even higher codimensions are likely to produce further examples, though the combinatorics involved become increasingly complex.

In Chapter 7 we diverge from the study of smooth varieties and classify del Pezzo surfaces admitting quotient singularities of type $\frac{1}{3}(1,1)$. This is part of a joint work with Alessio Corti [CH]. The motivation behind this classification and subsequent construction of models of these surfaces as complete intersections in toric varieties is to provide enough data in order to compute their quantum orbifold cohomology (this is done in [OP]). We choose the simplest type of cyclic quotient singularity that is rigid under $\mathbb{Q}$-Gorenstein(qG) deformation, that is to say points of type $\frac{1}{3}(1,1)$. This type of deformation ensures that the canonical class is well-behaved in families, in particular K^2 and $h^0(-K)$ are locally constant in such a family (crucially, this is not true in the case of the Gorenstein index).

We restrict our attention to qG-rigid singularities for the following reason: a surface cyclic quotient singularity $(x \in X) \cong \frac{1}{n}(1,q)$ has a unique qG-deformation component. Furthermore, the general surface of the miniversal family has a unique singularity of class R, the *R-content* of $(x \in X)$, cf. [ACC+16] and [AK, Definition 2.4] and the discussion following it.

It is known [ACC+16, Lemma 6] that, if X is a del Pezzo surface with cyclic quotient singularities $x_i \in X$, the natural transformation of qG-deformation functors:

$$\mathrm{Def}^{\mathrm{qG}} X \to \prod \mathrm{Def}^{\mathrm{qG}}(X, x_i)$$

is smooth: a choice for each i of a local qG-deformation of the singularity (X, x_i) can always be globalized to a qG-deformation of X. In other words X can be qG-deformed to a surface that has only the residues of the (X, x_i) as singularities.

Our point of view here is that, when we classify del Pezzo surfaces, and study mirror symmetry for them, it is natural to classify first the locally qG-rigid ones, for these are the generic surfaces that we are most likely to encounter, and study their qG-degenerations as a second step. In particular, the singularity $\frac{1}{3}(1,1)$ is qG-rigid and a singularity has R-content $\frac{1}{3}(1,1)$ if and only if it is of the form $\frac{1}{3(3m+1)}\big(1, 2(3m+1) - 1\big)$.

Here is the main result of this chapter:

Theorem 1.13. A del Pezzo surface with $k \geq 1\ \frac{1}{3}(1,1)$ points is one of the following, all constructed in Tables 7.2 and 7.3 and in the statement and proof of theorem 7.21:

(1) A surface of the family $X_{1,\,d}$ is the blow-up of $25/3 - d \leq 8$ nonsingular points on $\mathbb{P}(1,1,3)$. If $d < 16/3$, then it is also the blow-up of a surface of the family $B_{1,\,16/3}$ in $1 \leq 16/3 - d \leq 5$ nonsingular points;

(2) A surface of the family $X_{2,\,d}$ is the blow-up of $17/3 - d \leq 5$ nonsingular points on $X_{2,\,17/3}$. If $d < 8/3$, then it is also the blow-up of a surface of the family $B_{2,\,8/3}$ in $1 \leq 8/3 - d \leq 2$ nonsingular points;

(3) A surface of the family $X_{3,\,d}$ is the blow-up of $5 - d \leq 4$ nonsingular points on $X_{3,\,5}$;

(4) A surface of the family $X_{4,\,d}$ is the blow-up of $7/3 - d \leq 2$ nonsingular points on $X_{4,\,7/3}$;

(5) A surface of the family $X_{5,\,2/3}$ is the blow-up of a nonsingular point on $X_{5,\,5/3}$;

(6) $X_{6,\,1}$ is the blow-up of a nonsingular point on $X_{6,\,2}$.

The notations in Theorem 1.13 are as follows: the symbol $X_{k,d}$ signifies the family of surfaces X with k singular points, degree $K_X^2 = d$ and Fano index 1. The families $B_{1,\,16/3}$, $B_{2,\,8/3}$ and $\mathbb{P}(1,1,3) = S_{1,25/3}$ follow the same notation yet are distinguished as their Fano index is $f > 1$.

We start by constructing birational models for the surfaces: for a given k, they are organized in *cascades* - terminology due to [RS03] - they can be constructed from one another through blow-ups of smooth points. The difficulty resides in determining the surfaces that are at the bottom of the cascades, the ones of highest possible degree K^2 for a fixed k. This is done through a detailed combinatorial study of the configuration of negative curves on their minimal resolutions.

Once we have obtained a complete list, we need to find good model constructions. In the case of most surfaces, this is equivalent to describing it as a complete intersection of type $L_1 \ldots L_c$ in a toric variety F such that the line bundles L_i are nef on F and $-K_F - \sum_{i=1}^{c} L_i$ is ample. The models we find are by no means unique, however they provide the data required to compute the quantum cohomology of the surfaces. Tables 7.2 and 7.3 plot many invariants of these surfaces, as well as display the model constructions. In Sections 7.5 and 7.6 we explain how to read the tables, as well as how to verify that the models coincide with the surfaces we classified.

1.4 Outlook

An interesting and concrete problem would be the construction of more smooth Fano fourfolds with singular anticanonical divisors. Essentially, Example 1.2 is the only one that is known, and, as mentioned before, even finding Fano varieties with nontrivial base locus turns out to be problematic. These examples have to be built by hand since computer databases, though vast, principally contain information on toric varieties. Finding such a divisor with a singularity of rank three would be particularly interesting. Completing the classification in Chapter 6 of weighted complete intersections of codimension two would be a first step. A much more ambitious project would be to classify all smooth Fano fourfolds for which the anticanonical divisor is not smooth.

A second, more abstract question would be whether variants of the terminality result are true in arbitrary dimension, for example for the fundamental divisor on Fano varieties of index $n - 3$.

Some of the methods in Chapter 5 may apply in this case, as by Theorem 1.4 it has at least n-2 global sections.

Concerning the log del Pezzo surfaces, the motivation for doing the classification was eventually proving Conjectures A and B in [ACC+16]. Though Conjecture B is out of reach with currently available methods (if one tried to calculate quantum cohomology for surfaces with slightly more complicated rigid cyclic quotient singularities, the computations would be incredibly complicated), the same is not true for Conjecture A. It states that there exists a one-to-one correspondence between mutation equivalence classes of Fano polygons and $\mathbb{Q}$-Gorenstein deformation classes of locally $\mathbb{Q}$-rigid del Pezzo surfaces with cyclic quotient singularities. Basically, a (class of a) polygon P goes to a (any) generic $\mathbb{Q}-$Gorenstein deformation of the toric surface X_P. In [ACC+16], the authors have already proved that this is a surjective map, the difficulty resides in proving injectivity.

Chapter 2

Preliminaries

The principal results of this thesis concern Fano varieties, we therefore briefly resume some of their elementary, but fundamental features which significantly facilitate this study. We then present the technical results that use singularities of pairs, which will extensively be used throughout Chapters 3, 4 and 5. All definitions in this chapter follow [KM98], [IP99] and [Laz04].

2.1 Fano varieties

We will be working over the field of complex numbers. Here is a basic result of the definition of a Fano variety:

Theorem 2.1. [IP99, Prop.2.1.2] Let X be a smooth Fano variety. We have the following:

(a) $H^i(X, \mathcal{O}_X) = 0,\ \forall\, i > 0$.

(b) $\mathrm{Pic}(X)$ is a finitely generated torsion-free $\mathbb{Z}$-module and $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$.

(c) numerical equivalence on Cartier divisors on X coincides with linear equivalence.

(d) X has Kodaira dimension $-\infty$.

We say a few words about the proof. By the Kodaira vanishing theorem we immediately obtain (a). We use the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

that induces the long exact sequence in cohomology

$$\ldots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \ldots$$

which, together with (a) and the fact that $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$, proves the second part of (b). We also have that if $D \equiv 0$ then by the Riemann-Roch formula $h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X) = 1$. Since X is projective, this implies $D = 0$ and that $\mathrm{Pic}(X)$ is torsion-free.

The concepts of rationality and ruledness are key in the classification of algebraic surfaces. Even though del Pezzo surfaces are rational (over any algebraically closed field), this is not true for three dimensional Fano manifolds. A natural replacement for rationality and ruledness in higher dimension are rational connectedness and uniruledness, respectively.

Theorem 2.2 ([KMM87, Cam92]). A Fano manifold over a field of characteristic zero is rationally connected.

The same authors have shown in [KMM92, Cam91] that rationally connected varieties are simply connected, therefore the same holds for a Fano manifold.

Theorem 2.3. [Pet11, Thm.5.5] Let X be a projective manifold with $-K_X$ semi-ample. Then T_X is generically nef.

Naturally this remains true in the case of Fano varieties, and so does the following corollary which we use in the Riemann Roch computation in Chapter 5:

Proposition 2.4. [Pet11, Cor.5.6] Let X be an n-dimensional projective manifold with $-K_X$ nef. Then

$$c_2(X) \cdot H_1 \cdot \ldots \cdot H_n - 2 \geq 0$$

for all ample line bundles H_j on X.

2.2 Singularities of pairs

Definition 2.5. Let X be an irreducible normal variety. A $\mathbb{Q}$-Weil divisor (or $\mathbb{Q}$-divisor) on X is a formal linear combination $D = \sum d_i D_i$, where $d_i \in \mathbb{Q}$ and D_i are distinct prime divisors. Such a divisor is said to be $\mathbb{Q}$-Cartier if mD is Cartier for some $0 \neq m \in \mathbb{Z}$. X is called $\mathbb{Q}$-factorial if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

Definition 2.6. A pair (X, Δ) is the data of a normal variety X and a $\mathbb{Q}-$Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Definition 2.7. Let (X, Δ) be a pair and write $\Delta = \sum d_i \Delta_i$, where Δ_i are distinct prime divisors. Let $\mu : Y \to X$ be a birational morphism, Y normal. We then write

$$K_Y \equiv \mu^*(K_X + \Delta) + \sum a(E_i, X, \Delta) E_i,$$

where $E_i \subset Y$ are distinct prime divisors and $a(E_i, X, \Delta) \in \mathbb{R}$. Further, we adopt the convention that a nonexceptional divisor E appears in the sum if and only if $E = \mu_*^{-1} \Delta_i$ for some i and then its coefficient $a(E, X, \Delta) = -d_i$.

We call the $a(E_i, X, \Delta)$ the discrepancies of the pair (X, Δ).

A divisor $E \subset Y$ is exceptional over X if there exists a birational morphism $\mu : Y \to X$ such that E is μ-exceptional.

The aim of this definition is to construct an invariant from the data provided by the pair (that is, independent of both μ and Y). Note that the $a(E_i, X, \Delta)$ themselves are not an invariant of (X, Δ), since we can always blow up along smooth centers on a given Y in order to create new exceptional divisors. However, continuing this process arbitrarily will only result in additional positive coefficients, therefore it becomes apparent that the discrepancies containing information on the singularities of (X, Δ) are the smallest ones. It is then natural to introduce the following definition:

Definition 2.8. We set

$$\mathrm{discrep}(X, \Delta) = \inf\{a(E_i, X, \Delta) \mid E \text{ is an exceptional prime divisor over } X\} \text{ and}$$

$$\mathrm{totaldiscrep}(X, \Delta) = \inf\{a(E_i, X, \Delta) \mid E \text{ is a divisor over } X\}$$

A pair (X, Δ) is said to be

- terminal if $\mathrm{discrep}(X, \Delta) > 0$,
- canonical if $\mathrm{discrep}(X, \Delta) \geq 0$,
- klt (kawamata log terminal) if $\mathrm{discrep}(X, \Delta) > -1$ and $\lfloor \Delta \rfloor \leq 0$,
- plt (purely log terminal) if $\mathrm{discrep}(X, \Delta) > -1$,
- lc (log canonical) if $\mathrm{discrep}(X, \Delta) \geq -1$.

These types of singularities arise naturally in birational geometry: assuming $\Delta = 0$, terminal singularities are the mildest singularities for which the Minimal Model Program is stated (starting with dimension 3, since in dimension 2 these are smooth points), canonical singularities arise on canonical models of surfaces of general type. Kawamata log terminal singularities are the most general case for which certain (non-)vanishing theorems are true (we devote Chapter 5 to studying a result of this type). Finally, log canonical singularities are the largest class for which the notion of discrepancy makes sense:

Remark 2.9. [KM98, Cor.2.31] If E is a divisor over X whose coefficient $a(E, X, \Delta)$ is smaller than -1, then one can produce arbitrarily small coefficients by blowing up more centers. More precisely, either $\mathrm{discrep}(X, \Delta) = -\infty$ or $-1 \leq \mathrm{totaldiscrep}(X, \Delta) \leq \mathrm{discrep}(X, \Delta) \leq 1$.

Proof. We sketch the first part: take E to be a divisor over X such that $a(E, X, \Delta) = 1 - c$ with $c > 0$. Take a birational morphism $\mu : Y \to X$ such that $\mathrm{center}_Y E$ is a divisor on Y and let $K_Y + \Delta_Y = f^*(K_X + \Delta)$.

Let Z_0 be a codimension 2 locus contained in E but not in any other exceptional divisor of f or in $f - 1_* \Delta$. We can assume Y is smooth at the generic point of Z_0. Let $g_1 = Y_1 = \mathrm{Bl}_{Z_0} Y \to Y$ with exceptional divisor $E_1 \subset Y_1$. Then

$$a(E_1, X, \Delta) = a(E_1, Y, \Delta_Y) = -c.$$

Let $Z_1 \subset Y_2$ be the intersection of E_1 with he strict transform of E. Let $g_2 : Y_2 = \mathrm{Bl}_{Z_1} Y_1 \to Y_1$ with exceptional divisor $E_2 \subset Y_2$. Then

$$a(E_2, X, \Delta) = a(E_2, Y, \Delta_Y) = -2c.$$

By repeating this process we obtain divisors with discrepancies $-3c$, $-4c$ etc., which proves the remark.

□

Singularities that are klt, canonical or terminal are all rational, that is for some resolution $\mu : Y \to X$ we have $R^i \mu_* \mathcal{O}_Y = 0$ for all $i > 0$. This does not hold however in the case of log canonical singularities, as there are counterexamples even in dimension two: the cone over a smooth cubic curve is log canonical and does not have rational singularities. The exceptional locus of its minimal resolution is an elliptic curve, which leads to a nontrivial $R^1 \mu_* \mathcal{O}_Y$.

Finally, we present the generalization of log-canonical to the case where X is not normal:

Definition 2.10 ([Kaw00]). Let X be a reduced equi-dimensional algebraic scheme and Δ an effective $\mathbb{Q} - divisor$ on X. The pair (X, Δ) is said to be *slc* (semi-log canonical) if the following conditions are satisfied:

(a) X satisfies the Serre condition S_2, and has only normal crossing singularities in codimension 1.

(b) The singular locus of X does not contain any irreducible component of Δ.

(c) $K_X + \Delta$ is a $\mathbb{Q}-$Cartier divisor.

(d) For any birational morphism $\mu : Y \to X$ from a normal variety, if we write $K_Y = \mu^*(K_X + \Delta) + \Delta_Y$, then all coefficients in Δ_Y are at least -1.

We now discuss log canonical centers and thresholds.

Definition 2.11. Let(X, Δ) be a log canonical pair. A place for (X, Δ) is a prime divisor on some birational model $\mu : Y \to X$ of X such that $a(E, X, B) = -1$. The closure of $\mu(E)$ in X is called a log canonical center and is denoted as $\mathrm{Center}_X(E)$. We use the notation $CLC(X, \Delta)$ when referring to the set of all centers, while the alternative $\mathbb{LCS}(X, \Delta)$ is also known to appear in literature.

Remark 2.12. If the pair (X, Δ) is log canonical,

$$\{x \,|\, (X, \Delta) \text{ is not klt near x}\} = \cup_E \mathrm{Center}_X(E),$$

where the union runs over the places.

Centers appear in three fundamental instances throughout the thesis: in the proof of Proposition 3.6 when showing an anticanonical divisor on a fourfold is terminal, in a similar result in Chapter 5 while sketching out a generalization of this proof and in Chapter 4, where we mostly study the interactions of such objects. An important result in this direction is the following:

Proposition 2.13. If (X, Δ) is a klt pair and D an effective $\mathbb{Q}$-Cartier $\mathbb{Q}-$divisor such that $(X, \Delta + D)$ is log canonical. Then $CLC(X, D)$ is a finite set and if $W_1, W_2 \in CLC(X, D)$, all the irreducible components of $W_1 \cap W_2$ are in $CLC(X, D)$.

This motivates the following:

Definition 2.14. We refer to a center as minimal when it is minimal with respect to the inclusion. This concept can be either global, i.e. a minimal lc center of the pair (X, Δ), or local, i.e. the minimal lc center of (X, Δ) at a point $x \in X$. The latter is of course unique.

Definition 2.15. Let (X, Δ) be a klt pair, D an effective $\mathbb{Q}$-Cartier divisor. The log canonical threshold of D for (X, Δ) is

$$\mathrm{lct}((X, \Delta), D) = \sup\{t \,|\, (X, \Delta + tD) \text{ is lc}\}.$$

Note that this is the same as $\sup\{t \,|\, (X, \Delta + tD) \text{ is klt}\}$.

Remark 2.16.
- A pair is properly log canonical (plc) if it is lc and not klt, therefore if $c = \mathrm{lct}((X, \Delta), D)$, the pair $(X, \Delta + tD)$ is plc.
- The lct is a rational number and the supremum appearing in the definition is actually a maximum.

The notion of log canonical center is precisely the object of the following fundamental theorems, which are instrumental for the result in Chapter 3:

Theorem 2.17. [Fuj11, Thm.2.2] Let (X, B) be a projective log canonical pair. Let D be a Cartier divisor on X such that $D - (K_X + B)$ is ample. Let C be an lc center of (X, B) with a reduced scheme structure. Then

$$H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(D)) = 0$$

for every $i > 0$, where $\mathcal{I}_C$ is the defining ideal sheaf of C. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C(D))$$

is surjective.

This section extension theorem is a generalization of the Nadel-Shokurov vanishing theorem, which is formulated in the case of a smooth variety and is, in turn, a restatement of the classical vanishing theorem of Kawamata-Viehweg in terms of multiplier ideals. However, the methods used to prove Fujino's result are hard to reach using previous techniques, and in particular its novelty resides in the non-minimality of the center C. The statement is a significant technical tool when working with the minimal model program for log canonical pairs.

We continue with a result by Fujino and Gongyo, which is a stronger version of Kawamata's subadjunction theorem:

Theorem 2.18 (Subadjunction theorem)**.** [FG12, Thm.1.2] Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is Kawamata log terminal. In particular, W is normal and has only rational singularities.

Theorem 2.19. (Kawamata nonvanishing) Let D be a numerically effective Cartier divisor on a normal projective variety X of dimension at most three. If there exists an effective $\mathbb{Q}$-divisor B such that the pair (X, B) is Kawamata log terminal and such that the $\mathbb{Q}$-Cartier divisor $D - (K_X + B)$ is big and nef, then the bundle $\mathcal{O}_X(D)$ has a non-zero global section.

Chapter 3

Fano fourfolds

3.1 Terminality

For the convenience of the reader, we recall the known results on anticanonical divisors on Fano fourfolds that we have discussed in the Introduction:

Proposition 3.1. [HV11, Thm.1.7][Kaw00, Thm 5.2] Let X be a four-dimensional Fano manifold and $D \in |-K_X|$ be a general divisor. We have the following:

1) $h^0(X, -K_X) \geq 2$.

2) D is irreducible. In particular, this implies that $\mathrm{Bs}|-K_X|$ is at most a surface.

3) D has at most isolated canonical singularities.

Notation 3.2. Let X be a projective manifold. Given a resolution $\mu : X' \to X$ of the base locus of $|-K_X|$ such that $E = \sum_{i=1}^{m} E_i$ is its exceptional locus and $D \in |-K_X|$ is a general element, we then write:

- $|\mu^* D| = |D'| + \sum_{i=1}^{m} r_i E_i$, where D' is the strict transform of D and r_i are positive integers for $i \in \{1 \ldots m\}$,

- $K_{X'} = \mu^* K_X + \sum_{i=1}^{m} a_i E_i$, where $a_i > 0$, for all $i \in \{1 \ldots m\}$.

These coefficients will be extensively used throughout our proofs.

Definition 3.3. [Laz04, Def.1.1.8] The base ideal of a linear system $|L|$, denoted by

$$\mathfrak{b}(|L|) = \mathfrak{b}(X, |L|) \subseteq \mathcal{O}_X$$

is the image of the map $L \otimes_{\mathbb{C}} L^* \to \mathcal{O}_X$ determined by the evaluation map of $|L|$. The base locus

$$\mathrm{Bs}\,|L| \subseteq X$$

of $|L|$ is the closed subset of X cut out by the base ideal $\mathfrak{b}(|L|)$. When we wish to emphasize the scheme structure on $\mathrm{Bs}(|L|)$ determined by $\mathfrak{b}(|L|)$ we refer to $\mathrm{Bs}(|L|)$ as the base scheme of $|L|$.

Remark 3.4. By Bertini's Theorem, if $|L|$ is a linear system on a smooth variety X and $D_1, D_2 \in |L|$ are two general elements, then the divisor $D_1 + D_2$ is SNC outside $\mathrm{Bs}|L|$.

Definition 3.5. [Laz04, Def. 9.2.10] Let $|L|$ be a non-empty linear series on a smooth complex variety X and let $\mu : X' \to X$ be a log resolution of $|L|$, with

$$\mu^*|L| = |W| + F,$$

where $F + \mathrm{exc}(\mu)$ is a divisor with SNC support and $W \subseteq H^0(X', \mathcal{O}_{X'}(\mu^* L - F))$. Given a rational number $c > 0$, the *multiplier ideal* $\mathcal{J}(c \cdot |L|)$ corresponding to c and $|L|$ is

$$\mathcal{J}(c \cdot |L|) = \mathcal{J}(X, c \cdot |L|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [c \cdot F]).$$

Proposition 3.6. Let X be a four-dimensional Fano manifold. Then for every $c < 2$ we have that

$$\mathcal{J}(c| - K_X|) = \mathcal{O}_X.$$

In terms of the coefficients in Notation 3.2, this is equivalent to

$$\forall\, i \in \{1 \dots m\} \; : \; a_i + 1 \geq 2r_i. \tag{3.1}$$

Proof. Arguing by contradiction, we suppose there exists a rational number $c < 2$ such that $\mathcal{J}(c| - K_X|) \subsetneq \mathcal{O}_X$. By [Laz04, Prop.9.2.26], this is equivalent to the fact that the pair $\left(X, c\,\frac{D_1+D_2}{2}\right)$ is not klt for two general divisors D_1, $D_2 \in | - K_X|$. Take $c_0 < c$ to be the log canonical threshold of $\left(X, \frac{D_1+D_2}{2}\right)$, thus producing a properly log canonical pair $\left(X, c_0\frac{D_1+D_2}{2}\right)$ which admits a minimal log canonical center, denoted in what follows by C.

As by Remark 3.4 the divisor $D_1 + D_2$ has simple normal crossings outside the base locus of $| - K_X|$, the identity map is a log resolution of the pair $\left(X \setminus \mathrm{Bs}| - K_X|, c_0\frac{D_1+D_2}{2}\right)$. Since $\frac{c_0}{2} < 1$ we deduce that this pair is klt, which shows that the log canonical center C must be included in the base locus. As the dimension of $\mathrm{Bs}| - K_X|$ is at most two by Proposition 3.1, then C is also at most a surface.

Since $c_0 < 2$, we can apply Theorem 2.17 to the pair $\left(X, c_0\frac{D_1+D_2}{2}\right)$ together with the anticanonical divisor. As the divisor $-K_X - (K_X + c_0\frac{D_1+D_2}{2}) \sim (2 - c_0)(-K_X)$ is ample, we have obtained a surjective map

$$H^0(X, \mathcal{O}_X(-K_X)) \twoheadrightarrow H^0(C, \mathcal{O}_C(-K_X)).$$

This map is the zero map since C is contained in $\mathrm{Bs}| - K_X|$, and in order to obtain a contradiction we show that the target is nontrivial.

Indeed, using the minimality of C and the Theorem 2.18, there exists an effective $\mathbb{Q}$-divisor $B \subset C$ such that

$$\left(K_X + c_0\frac{D_1 + D_2}{2}\right)|_C \sim_{\mathbb{Q}} K_C + B$$

and the pair (C, B) is klt. This provides the ingredients to apply Kawamata's Nonvanishing Theorem (Theorem 2.19) to the pair (C, B) and the divisor $-K_X|_C$. As the divisor

$$-K_X|_C - (K_C + B) \sim_{\mathbb{Q}} -K_X|_C - \left(K_X + c_0\frac{D_1 + D_2}{2}\right)|_C \sim_{\mathbb{Q}} (2 - c_0)(-K_X|_C)$$

is ample, it follows that $H^0(C, \mathcal{O}_C(-K_X)) \neq \emptyset$. $\square$

This statement immediately implies our first result:

Theorem 3.7. Let X be a four-dimensional Fano manifold and let $D \in |-K_X|$ be a general divisor. Then D has at most terminal singularities.

Proof. Let μ be a resolution as in Notation 3.2. The adjunction formula for a general elephant gives us:

$$K_{D'} = (\mu|_{D'})^* K_D + \sum_{i=1}^{m} (a_i - r_i)(E_i \cap D') \tag{3.2}$$

which means that the discrepancy of $(D, 0)$ is at least $\inf_i \{a_i - r_i \mid E_i \text{ is } \mu\text{-exceptional}\}$. As by Proposition 3.1,3 we already know that this is non-negative, the aim of what follows is to show that the discrepancy of this pair is non-zero. Note that since we have considered a log resolution, the intersection $E_i \cap D'$ is reduced for all $i \in \{1 \dots m\}$.

We further argue that the inequality in condition (3.1) is sufficient in order to obtain terminality. Indeed, the only case in which this doesn't imply $a_i - r_i > 0$ is if both a_i and r_i are equal to one.

Since the coefficients do not depend on the choice of the resolution, we can assume that we have been working with one in which all blow-ups were made along smooth centers.

Claim: We can only obtain $a_i = 1$ for a certain $i \in \{1 \dots m\}$ if $\operatorname{codim}_X \mu(E_i) = 2$.

We start with a simple example, then prove the claim in all its generality. Suppose that after a series of two blow-ups along smooth centers

$$X' = X_2 \xrightarrow{\mu_2} X_1 \xrightarrow{\mu_1} X$$

we have constructed a resolution as in Notation 3.2 such that $a_2 = 1$. We can also suppose $a_1 > 1$, since we may assume we obtained the desired coefficient only after the last blow-up. We show that the center corresponding to E_2 has codimension two.

Suppose the center corresponding to μ_i is of codimension m_i, $i = 1, 2$. We have the following:

$$\begin{aligned} K_{X_1} &= \mu_1^* K_X + (m_1 - 1)E_1 \\ K_{X_2} &= \mu_2^* K_{X_1} + (m_2 - 1)E_2, \end{aligned}$$

By combining these two relations we get:

$$\begin{aligned} K_{X_2} &= \mu_2^*(\mu_1^* K_{X_1} + (m_1 - 1)E_1) + (m_2 - 1)E_2 \\ &= (\mu_1 \circ \mu_2)^* K_X + (m_1 - 1)\mu_2^* E_1 + (m_2 - 1)E_2 \\ &= (\mu_1 \circ \mu_2)^* K_X + (m_1 - 1)(E_1 + \nu_1 E_2) + (m_2 - 1)E_2 \\ &= (\mu_1 \circ \mu_2)^* K_X + (m_1 - 1)E_1 + ((m_1 - 1)\nu_1 + m_2 - 1)E_2, \end{aligned}$$

where $\nu_1 > 0$ if and only if $\mu_2(E_2) \subset E_1$. Our hypothesis is precisely that $(m_1 - 1)\nu_1 + m_2 - 1 = 1$ and since $\nu_1 \geq 0$ and $m_1 - 1 = a_1 > 1$, the only chance for this to occur is if $\nu_1 = 0$ and $m_2 = 2$.

We move on to the general case. Similarly, without loss of generality, we can assume we obtained this coefficient by doing the very last blow-up of the resolution, denoted by μ_m:

$$X' = X_m \xrightarrow{\mu_m} X_{m-1} \xrightarrow{\psi} X$$

where $\mu = \psi \circ \mu_m$. We have that

$$a_m = \lambda + \sum_{i=1}^{m-1} a_i \nu_i$$

where $\lambda = \operatorname{codim}_{X_{m-1}} \mu_m(E_m) - 1 \geq 1$ and $\nu_i > 0$ if and only if $\mu_m(E_m) \subset E_i$. Having $a_m = 1$ implies $\lambda = 1$ and $\nu_i = 0\ \forall\, i$, which proves the claim since the former condition shows that $\mu_m(E_m)$ is exactly of codimension two in X_{m-1}, while the latter signifies that ψ does not contract $\mu_m(E_m)$.

The fact that $\operatorname{codim}_X \psi(E_i) = 2$ implies that $\psi(E_i)$ is a divisor on D. Therefore the intersection $E_i \cap D'$ is not $\mu|_{D'}$-exceptional, this divisor does not contribute to the discrepancy of the pair $(D, 0)$ as computed in (3.2). Together with condition (3.1) this proves that the discrepancy can never be zero, therefore a general elephant D has at most terminal singularities. $\square$

3.2 Separating strict transforms

The main result of this section is the following:

Theorem 3.8. Let X be a four-dimensional Fano manifold and let $D \in |-K_X|$ be a general divisor. Then the singularities of D are locally analytically given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \text{or} \quad x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0.$$

We provide the set-up for the discussion describing the local equations of these singular points.

Notation 3.9. Let $|L|$ be a linear system on a projective manifold X and let D be an effective prime divisor on X. We denote by $|L|_D$ the linear system on D given by the image of the restriction morphism:

$$H^0(X, \mathcal{O}_X(L)) \to H^0(D, \mathcal{O}_D(L)).$$

We obtain a linear system on D that will not only be determined by the intrinsic properties of D, but which fundamentally depends on the behavior of $|L|$ on X. An immediate consequence of this is:

$$\mathrm{Bs}|L|_D = \mathrm{Bs}|L| \cap D. \tag{3.3}$$

Notation 3.10. Given a linear system $|L|$ on a projective variety X and a point $x \in \mathrm{Bs}|L|$, denote by $|L|_x$ the following closed subset of $|L|$:

$$|L|_x := \{D \in |L| \mid x \in D_{sing}\},$$

where by D_{sing} we denote the singularities of D.

Lemma 3.11 (Tangency lemma)**.** Let X be a projective manifold and let $|L|$ be a linear system on X. Let $x \in X$ be a point in $\mathrm{Bs}|L|$ such that $\operatorname{codim}_{|L|}|L|_x = 1$. Then the tangent spaces at x of all divisors in $|L| \setminus |L|_x$ coincide.

Proof. Let D_1, $D_2 \in |L| \setminus |L|_x$ be two divisors and denote by $P := \langle D_1, D_2 \rangle$ the pencil that they generate.

As $|L|$ is a projective space, any intersection between a codimension one subset and a line is non-empty, thus there exists a divisor $D \in |L|_x \cap P$. If f_i are local equations of D_i around x, there exist two scalars $\lambda, \eta \in \mathbb{C}$ such that D is given by:

$$f = \lambda f_1 + \eta f_2.$$

We differentiate and obtain

$$\nabla f = \lambda \nabla f_1 + \eta \nabla f_2,$$

and as D is singular at x, the left hand side vanishes at this point. On the other hand, both $\nabla f_1(x)$ and $\nabla f_2(x)$ are nonzero, thus they must be proportional. This is the same as saying that the tangent spaces of D_1 and D_2 coincide at the point x. $\square$

Remark 3.12. Throughout this chapter, we use the lemma above in two particular cases, for a divisor $D \in |L|$, where $|L|$ is a linear system on an irreducible projective variety X (a posteriori obtained by a sequence of blow-ups starting from the initial Fano fourfold):

- there exists a curve $C \subset X$ such that for all $D \in |L|$ there exists a point $x \in D_{sing} \cap C$ and the union of these points is dense in C.

- there exists a surface $S \subset X$ such that for all $D \in |L|$ there exists a curve $C \subset D_{sing} \cap S$ and the union of all such curves is dense in S.

We show that the first case satisfies the hypotheses of Lemma 3.11. The second case is similar.

Let $|L|^0$ be the Zariski open set in $|L|$ such that for all $D \in |L|^0$ we have $D_{sing} \subset \mathrm{Bs}|L|$. Denote by $\mathcal{U} = \{(D, x) \mid D \in |L|^0, \, x \in D_{sing}\}$ the universal family over $|L|^0$ and take p_1 and p_2 to be the projections on the first and second factor respectively.

Since every $D \in |L|^0$ has an isolated singularity we have that p_1 is a finite morphism and the fiber of p_2 over a point $x \in C \cap D_{sing}$ is $|L|_x$. As p_2 is dominant, we obtain that

$$\dim |L| = \dim \mathcal{U} = \dim |L|_x + \dim C = \dim |L|_x + 1.$$

Remark 3.13. Let X be a smooth projective fourfold, D and D' two effective divisors and C a curve such that that $T_D|_{C_{gen}} = T_{D'}|_{C_{gen}}$. If we blow up X along C, the strict transforms of D and D' will intersect along a surface.

Proof. Let $\mu_1 : X_1 \to X$ be the blow-up of X along C with exceptional divisor E_1 and denote by D_1 and D'_1 the strict transforms of D and D' respectively. Then the surfaces

$$Q := D_1|_{E_1} \simeq \mathbb{P}(\mathcal{N}^*_{C/D}) \text{ and } Q' := D'_1|_{E_2} \simeq \mathbb{P}(\mathcal{N}^*_{C/D'}),$$

coincide as the normal sheaves are the same by the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{T}_{C_{gen}} & \to & \mathcal{T}_D|_{C_{gen}} & \to & \mathcal{N}_{C/D} & \to & 0 \\ & & \| & & \| & & & & \\ 0 & \to & \mathcal{T}_{C_{gen}} & \to & \mathcal{T}_{D'}|_{C_{gen}} & \to & \mathcal{N}_{C/D'} & \to & 0, \end{array}$$

which proves the remark. □

Notation 3.14. In what follows, if

$$X \xleftarrow{\mu_1} X_1 \xleftarrow{\mu_2} \ldots \xleftarrow{\mu_i} X_i \xleftarrow{\mu_{i+1}} \ldots$$

is a sequence of blow-ups, we set $|L_0| = |-K_X|$ and we recursively define $|L_i|$ as the linear system on X_i which is the proper transform of the system $|L_{i-1}|$. The fact that $|-K_X|$ has no fixed components implies that the same holds for all $|L_i|$. It is perhaps important to stress that despite the notation (which we use in the scope of not confusing divisors and linear systems), the linear systems $|L_i|$ are not complete. The index i is a good way to keep track of the level that we are on: as before, exceptional divisors of μ_i are denoted by E_i, while members of $|L_i|$ are denoted by D_i, D'_i, $\widetilde{D}_i$ etc.

3.2.1 Log pairs with mobile boundaries

Throughout this study we sometimes need to focus not on individual properties of a general elephant, but to examine the entire anticanonical system. This is especially present when discussing the tangency lemma, but also when working with the linear systems involved in the above notation. There is a notion of mobile log pair which takes this precise fact into consideration. Such a pair $(X, \mathcal{M})$ consists in a variety X and a formal finite $\mathbb{Q}$-linear combination of linear systems $\mathcal{M} = \sum \mathcal{M}_{\rangle}$ on X such that each $\mathcal{M}_i$ has no fixed component and each coefficient a_i is nonnegative. The notions of discrepancies, log terminality and log canonicity can be defined for such pair as for usual log pairs. In particular, for a birational morphism $\mu : X' \to X$, the pullback $\mu^*(\mathcal{M}_i)$ may obtain fixed components from the exceptional divisors of μ. However, the proper transform $\mu^{-1}(\mathcal{M}_i)$ has no fixed component. One can work with such pairs as with usual log pairs, by replacing each linear system with its general member or with the $\mathbb{Q}-$divisor $\dfrac{M_j^1 + M_j^2 + \ldots + M_j^N}{N}$ for a sufficiently large N. A more detailed account of this concept can be found in [CS16, §2.2] and [CP07, §.3.3], and also in the works of Alexeev [Ale94] and Kollàr [Kol97].

We begin to examine the local picture around a singular point of a general elephant. In order to obtain the equations in Theorem 3.8, we construct a sequence of blow-ups contradicting condition (3.1) until we are only left with the possibilities in the statement. The choice of the first blow-up depends on the nature of the singular point relative to the entire linear system $|-K_X|$. Essentially, there are two possible cases: fixed and moving singularities.

3.2.2 Fixed singularities

Let X be a four-dimensional Fano manifold and suppose that there exists a point $x \in X$ such that for all $D \in |-K_X|$ we have $x \in D_{sing}$. The point x is called a fixed singularity of the linear system $|-K_X|$.

Terminal Gorenstein singularities have been classified (refer to [Mor85a], [Rei87] and [Kol91]) into different classes of compound du Val singularities. They are isolated points defined by an equation of the form:

$$g(x_1, x_2, x_3) + x_4 h(x_1, x_2, x_3, x_4) = 0,$$

where h is arbitrary and g is the expression of an A-D-E surface singularity in the coordinates x_1, x_2 and x_3. However, as some of these cases overlap (for example a compound A_n singularity can also be compound D_{n+2}), we choose a different way of organizing the local equations for x. We nonetheless retain the fact that all these points are of multiplicity two on D.

We order the singularities of a general elephant according to the rank of the hessian of a local equation, using the Morse Lemma for holomorphic functions:

Lemma 3.15. [AGZV12, Thm.11.1] There exists a neighborhood of a critical point where the rank of the second differential is equal to k, in which a holomorphic function in n variables can locally analytically be written as:

$$f(x_1 \ldots x_n) = x_1^2 + \ldots + x_k^2 + g(x_{k+1}, \ldots, x_n),$$

where the second differential of g at zero is equal to zero, that is g is at least of degree three in the variables $x_{k+1}, \ldots, x_n$.

Here is the main result of this section:

Theorem 3.16. Let X be a four-dimensional Fano manifold and suppose that there exists a point $x \in X$ such that for all $D \in |-K_X|$ we have $x \in D_{sing}$. Then around this point each general elephant is defined by an equation of one of the following two forms:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \text{ or } x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0.$$

Throughout the proof we repeatedly use the following lemma:

Lemma 3.17. Under the assumptions of Theorem 3.16, let $\mu_1 : X_1 \to X$ be the blow-up of X at the point x and E_1 its exceptional divisor. Let $|L_1|$ be as defined in Notation 3.14. Then the intersection $\mathrm{Bs}|L_1| \cap E_1$ is at most a curve.

Proof. Suppose that $\dim(\mathrm{Bs}|L_1| \cap E_1) = 3$. This implies that E_1 is a fixed component of each D_1, a contradiction since a strict transform doesn't contain the exceptional divisor.

If $\mathrm{Bs}|L_1| \cap E_1$ contains a surface S, let $\mu_2 : X_2 \to X_1$ be its blow-up and E_2 the unique exceptional divisor mapping onto S. As a general elephant D has multiplicity two at x, we have that S is either a plane or a quadric cone. Therefore, the blow-up of X_1 along S will either be smooth or singular along the $\mathbb{P}^1$ above the vertex of the cone, respectively (in particular, the exceptional locus of μ_2 is irreducible). We compute the discrepancies a_i and the coefficients r_i, for $i = 1, 2$, as introduced in Notation 3.2.

The dimensions of the centers give us that:

$$\begin{aligned} K_{X_1} &= \mu_1^* K_X + 3E_1 \\ K_{X_2} &= \mu_2^* K_{X_1} + E_2, \end{aligned}$$

where F consists of other exceptional divisors not mapping onto S. As E_1 is smooth along S, we have that $\mu_2^* E_1 = E_1' + E_2$, thus obtaining $a_1 = 3$ and $a_2 = 4$.

The computations of the r_i depend on the multiplicity of x on D, which we denote by $m_1 \geq 2$ since D is singular at x. Set $m_2 \geq 1$ to be the multiplicity of D_1 along S, then the coefficients are:

$$\mu_1^* D = D_1 + m_1 E_1$$

$$\mu_2^* \mu_1^* D = D_2 + m_1 E_1' + (m_1 + m_2) E_2,$$

where $D_2 \subset X_2$ is the strict transform of D_1. Thus $r_1 = m_1$ and $r_2 = m_1 + m_2$. Clearly

$$2r_2 = 2(m_1 + m_2) \geq 2m_2 + 4 \geq 6$$

is strictly larger than $a_2 + 1 = 5$, hence by condition (3.1) we obtain a contradiction. $\square$

Definition 3.18. Let D be a divisor on a smooth variety X and take a point $x \in D$. Let $\mu : X_1 \to X$ be a blow-up at a point $x \in X$, let D_1 be the strict transform of D and E the exceptional divisor. The tangent cone of D at x is the union of the tangent lines to D at x, which is clearly isomorphic to $D_1 \cap E$. Since D has terminal Gorenstein singularities, the tangent cone is necessarily a (possibly reducible or non-reduced) quadric in $\mathbb{P}^2$.

From the point of view of the local equations, if we assume that x is the origin and that D is given by the ideal I, the tangent cone of D at x is the variety whose ideal is $\mathrm{in}(I)$, the initial ideal associated to I (i.e. the ideal generated by the lowest-degree homogenous components of all polynomials in I).

As we apply the Morse Lemma, the second interpretation shows that the tangent cone is always given by the quadratic terms of a local equation of the general elephant. Combining this with the more geometric third interpretation will allow us to derive information on the singularities of blow-ups.

Proof of Theorem 3.16. Theorem 3.7 states that D is terminal, and by the classification of terminal singularities (refer to [Mor85a], [Rei87] and [Kol91]) we obtain that the singularity is a point of multiplicity two on D. We analyze each of the cases in the Morse Lemma. Namely, we are in one of four situations corresponding to the rank of the hessian of a local expression of D:

Rank one: By Lemma 3.15, a general member of $D \in |-K_X|$ is locally given by:

$$x_1^2 + g(x_2, x_3, x_4) = 0,$$

where the degree of g is at least equal to three. The tangent cone $D_1 \cap E_1$ is singular along the entire surface $[0 : x_2 : x_3 : x_4]$. By Bertini's Theorem all the singularities of general divisors in the linear system $|L_1|_{E_1}$ are contained in $\mathrm{Bs}|L_1|_{E_1}$. Since $\mathrm{Bs}|L_1|_{E_1} = \mathrm{Bs}|L_1| \cap E_1$ by (3.3), this contradicts Lemma 3.17.

Rank two: In this case, the equation of a general $D \in |-K_X|$ is the following:

$$x_1^2 + x_2^2 + g(x_3, x_4) = 0,$$

where again g is of degree three or higher. The tangent cone is a union of two different planes in $E_1 \simeq \mathbb{P}^3$ which intersect along the line $l_{D_1} = [0 : 0 : x_3 : x_4]$ for a general $D_1 \in |L_1|$. Note that there is no immediate contradiction, since by Lemma 3.17 the intersection $\mathrm{Bs}|L_1| \cap E_1$ can be a curve C. By Bertini's Theorem the curve C contains l_{D_1} as an irreducible component. Since there are only finitely many irreducible components of C and an infinite numbers of strict transforms, the tangent cones are in fact singular along the same component C_1, that is the line $l_{D_1} = C_1$ is independent of the choice of D. At this point, the first coefficients are $a_1 = 3$ and $r_1 = 2$ and condition (3.1) is satisfied.

First suppose that a general $D_1 \in |L_1|$ is singular along C_1 and denote its multiplicity by $m \geq 2$. By blowing up X_1 along C_1 we obtain the coefficients $a_2 = 3 + 2 = 5$ and $r_2 = m + 2$, which contradict condition (3.1) since $a_2 + 1 = 6 < 8 \leq 2(m + 2) = 2r_2$.

Thus both D_1 and E_1 are smooth at the generic point of C_1 and their intersection is a surface S which is singular along C_1. This is precisely the previously mentioned tangent cone. As S is singular and both jacobian matrices $J_{E_1}|_{C_1}$ and $J_{D_1}|_{C_1}$ are of maximal rank, we obtain that

$$T_{E_1}|_{C_1} = T_{D_1}|_{C_1}, \ \forall\ D_1 \in |L_1| \text{ general.}$$

Since D_1 is tangent to E_1 along C_1, all general members $D_1 \in |L_1|$ are therefore tangent along C_1.

Let $\mu_2 : X_2 \to X_1$ be the blow up of X_1 along C_1, let E_2 be its exceptional divisor and set $\mu = \mu_1 \circ \mu_2$. We obtain the following coefficients:

$$K_{X_2} = \mu^* K_X + 3E_1' + 5E_2$$
$$\mu^* D = D_2 + 2E_1' + 3E_2,$$

where D_2 and E_1' are the strict transforms of D_1 and E_1 respectively. Take two distinct general divisors $D_1', D_1'' \in |L_1|$. As they are tangent at the generic point of C_1, its blow-up μ_2 will not separate their strict transforms D_2' and D_2''. Indeed, by Remark 3.13 we obtain that the two strict transforms intersect along a surface $S \subset X_2$. Similarly, using that $T_{E_1}|_{C_1} = T_{D_1'}|_{C_1}$, we have that S is also contained in E_1'. If $\mu_3 : X_3 \to X_2$ is the blow-up of X_2 along S, from the following computations:

$$\mu_3^* E_1 = E_1' + E_3,$$
$$\mu_3^* E_2 = E_2' + E_3,$$
$$\mu_3^* D_2 = D_3 + E_3,$$

we obtain $a_3 = 9$ and $r_3 = 6$, where D_3 is the strict transform of D_2 through μ_3. This again contradicts condition (3.1).

Rank three: By Lemma 3.15, in this case the polynomial g only depends on the variable x_4 and modulo a change of coordinates the equation of a general anticanonical member $D \in |-K_X|$ is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^k = 0,$$

where $k \geq 3$. The strict transform D_1 of such a divisor is either smooth or has a unique singularity of multiplicity two at the point $[0 : 0 : 0 : 1]$. The rank of its hessian remains at least equal to three, while the compound term becomes of degree $k-2$. As before, the singularities of D_1 are contained in the curve $C := \mathrm{Bs}|L_1| \cap E_1$.

We show that in fact every general D_1 is smooth, allowing us to conclude that $k = 3$. This is essentially done by contradiction and using the same sequence of blow-ups as in rank two, only this time we need to be more precise in order to obtain the tangency condition.

Step 1: Assume $k > 3$ and thus each general $D_1 \in |L_1|$ is singular. Then these singularities are contained in the same irreducible component of C. As in the previous case, this is immediate since we have a finite number of irreducible components and an infinite number of divisors. Denote this component by C_1.

In what follows, let $|L_1|^0$ be the Zariski open set in $|L_1|$ such that for all $D_1 \in |L_1|^0$ we have $D_{1,sing} \subset \mathrm{Bs}|L_1|$. Consider

$$\mathcal{U} := \{(D_1, x_1) \mid D_1 \in |L_1|^0,\ x_1 \in D_{1,sing} \cap C_1\} \subset |L_1|^0 \times X_1$$

to be the universal family over $|L_1|^0$ and denote by p_1 and p_2 the projections on its two components.

Step 2: Two general members of $|L_1|^0$ are not singular at the same point of the curve C_1. Choose the first general member $D_1' \in |L_1|^0$ and let $x_1 \in C_1$ be its singular point. The set

$$F_1 := \left\{D_1 \in |L_1|^0 |\ x_1 \in D_{1,sing}\right\}$$

is a fiber of p_2, so it is closed. First observe that this set cannot be dense. Otherwise, every general element $D_1 \in |L_1|^0$ would be singular at c_1, say of multiplicity $m \geq 2$. Then by blowing up X_1 at c_1 one would obtain a contradiction to condition (3.1), since $a_2 = 6$ and $r_2 = 2 + m \geq 4$. Hence F_1 is closed, but not dense, and by generality we can choose D_1'' in its complement, this way making sure that its singular point does not coincide with c_1.

Step 3: Having fixed D_1' and D_1'' as above, let $P := \langle D_1', D_1'' \rangle$ to be the pencil that they generate. Then the singular loci of the members of P cover the entire curve C_1. The set

$$\mathcal{T} = \{(D_1, x_1) \mid D_1 \in P,\ x_1 \in D_{1,sing} \cap E_1\}$$

is a closed subset of $\mathcal{U}$, and by Step 1 the second projection $p_2 : \mathcal{U} \to X_1$ maps it onto a closed subset of C_1. Using Step 2 and the continuity of p_2, we conclude that its image must be all of C_1.

Step 4: We obtain a contradiction and conclude that $k = 3$.

By Remark 3.12 we obtain that $T_{D_1'}|_{C_{1gen}} = T_{D_1''}|_{C_{1gen}}$. Exactly as in the rank two case, we blow up X_1 along C_1 and by Remark 3.13 we deduce that the strict transforms D_2' and D_2'' intersect along a surface S. However, contrary to the rank two case, here the strict transform of E_1 does not contain S since the tangent cone is smooth at the generic point of C_1.

By doing the same blow-up of X_2 along S and using that $S \not\subset E_1$ we again arrive at a contradiction of condition (3.1) as the coefficients are $a_3 = 6$ and $r_3 = 4$.

Rank four: This case does occur, and it is precisely the one illustrated in the example 1.2 mentioned in the introduction. Together with the only possible case in rank three, this proves the theorem. □

Note that throughout the proof, despite having started with a fixed singularity of $|-K_X|$, we have come across singularities of elements in $|L_1|$ "moving" along a curve C_1. We now see what happens if that had already been the case for $|-K_X|$.

3.2.3 Moving singularities

As usual, X is a four-dimensional Fano manifold. Proposition 3.1, 3 states that all general elephants have isolated singularities. Consider the set

$$V := \{x \in D_{sing} \cap \mathrm{Bs}|-K_X| \,|\, D \in |-K_X| \text{ general}\}.$$

If it contains a component of strictly positive dimension, we say that $|-K_X|$ has moving singularities along the component in question. As all our computations are local and the base locus of $|-K_X|$ is at most a surface, we analyze the two possible dimensions separately.

Before we start, we state a result of Kollàr that we use in the classification. For this we need to introduce the following definition:

Definition 3.19. [Kol97, Def. 4.3] Let $0 \in H \subset X$ (where X is smooth at 0) be a hypersurface singularity. In local coordinates $H = (g = 0)$ and let g_2 denote the quadratic part of g. We say that H has singularities of type cA if either H is smooth or g_2 has rank at least 2 as a quadratic form.

Theorem 3.20. [Kol97, Thm. 4.4] Let X be a smooth variety over a field of characteristic zero and $|B|$ a linear system of Cartier divisors. Assume that for every $x \in X$ there is a $B(x) \in |B|$ such that $B(x)$ is smooth at x (or $x \notin B(x)$). Then a general member $B^g \in |B|$ has only type cA singularities.

The curve case

We are in the most elementary situation of a moving singularity: in the base locus of $|-K_X|$ there exists a curve, which we denote by C_0, such that for all general $D \in |-K_X|$ there exists a point $x \in D_{sing} \cap C_0$. Suppose that the set of such points is dense in C_0. Here is the central result of this section:

Theorem 3.21. Let X be a four-dimensional Fano manifold and using the terminology above suppose that $|-K_X|$ has moving singularities along a curve C_0. Then in a neighborhood of each movable singular point in C_0 the general elephant is locally defined by an equation of the form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

Just as we have done previously, a good approach is to start by blowing up C_0, the difference being that instead of focusing on a single point and a specific divisor, we must consider the entire linear system. This discussion fits into the more general context of log pairs with mobile boundaries mentioned in 3.2.1.

Global geometric context: Lemma 3.11 and Remark 3.12 imply that $T_D|_{C_{0_{gen}}}$ is independent of $D \in |-K_X|$. This means that by blowing up X along C_0 the strict transforms of general elephants will have a surface in common, as shown in Remark 3.13. Denote this blow-up by $\mu_1 : X_1 \to X$, the exceptional divisor by E_1 and the common surface by S_1. By Bertini's Theorem, if a general divisor D is singular at a point $x \in C_0$, then

$$D_{1,sing} \cap \mathbb{P}^2 \subset D_{1,sing} \cap \mathbb{P}^2 \cap S_1,$$

where D_1 is the strict transform of D and $\mathbb{P}^2$ is the fiber of the natural projection $E_1 \to C_0$ mapping onto x. In particular, this discussion allows for a precise description of the tangent cone of D at x: we know that $D_1 \cap E_1$ is either a double plane or a quadric cone, and it is the latter because S_1 (which is invariant with respect to the choice of $D \in |L|$) cannot coincide with the $\mathbb{P}^2$ that maps onto x.

Note that after this step the coefficients in Notation 3.2 are $a_1 = 2$ and $r_1 = 1$.

Local coordinates: The purpose of this discussion is to find a way to check if the strict transform of a general elephant is singular, and then to analyze its singularities.

For simplicity, start by choosing coordinates on an open set U such that C_0 is given by:

$$C_0 \,:\, \{x_2 = x_3 = x_4 = 0\}$$

and single out two general elephants: one denoted by D, which is singular at the origin and the second, denoted by $\widetilde{D}$, which is smooth inside U. By restricting U we may assume that the origin is the only singular point of D and, up to a coordinate change, that the local equation of $\widetilde{D}$ is precisely

$$x_3 = 0.$$

This choice of coordinates, convenient for obtaining a straightforward expression of μ_1, comes at the expense of the precise form of D given by the Morse Lemma in Section 3.2.2.

Denote by $\mu_1 : U_1 \to U$ the blow-up of U along C_0. We choose the chart on U_1 given by

$$\left\{ (x_1, x_2, x_3, x_4)(z_2, z_3) \;\middle|\; \begin{array}{l} x_3 = x_2 z_2 \\ x_4 = x_2 z_3 \end{array} \right\}$$

and the following local coordinates on it:

$$(u_1, u_2, u_3, u_4) \longrightarrow (u_1, u_4, u_2 u_4, u_3 u_4)(u_2, u_3).$$

Note that in this chart μ_1 is given by

$$(u_1, u_2, u_3, u_4) \to (u_1, u_4, u_2 u_4, u_3 u_4),$$

the exceptional divisor E_1 has the equation $u_4 = 0$, the projective plane above the origin has the equations $u_1 = u_4 = 0$ and $\widetilde{D}_1$ is given by $u_2 = 0$. In what follows we will build different sequences of blow-ups starting from μ_1, recall that we use Notation 3.14 in the cases of both D and $\widetilde{D}$ throughout.

We now combine the two perspectives. We know that the surface $D_1 \cap E_1$ has two irreducible components: S_1 and the $\mathbb{P}^2$ which is situated above the origin in this coordinate charts. By intersecting them, we obtain a curve, denoted by C, which will contain the singularities of D_1. Since the singular point of $\widetilde{D}$ is outside of U we have that $S_1 = \widetilde{D}_1 \cap E_1$. From the previous paragraph we obtain that the local expressions of S_1 and C are

$$S_1 = \{u_4 = u_2 = 0\} \text{ and } C = \{u_1 = u_4 = u_2 = 0\}.$$

If f is the local equation of D, write $f = q + h$ where q is the quadratic part of f and h contains higher order terms. Denote by $M_D = (m_{ij})_{1 \le i,j \le 4}$ the matrix of q viewed as a quadratic form. This is a symmetric matrix such that $2M_D$ is the hessian of f at the origin.

Since $C_0 \subset D$, we have that f contains no pure monomials in the variable x_1. Moreover, as the tangent spaces of D and $\widetilde{D}$ coincide at every point of $C_{0,gen}$, the jacobian matrix of f is proportional to the vector (0 0 1 0) at each point $x \in C_{0,gen}$. We deduce that f also does not contain monomials of type $x_1^k x_2$ or $x_1^k x_4$ and in particular M_D is of the following form:

$$M_D = \begin{pmatrix} 0 & 0 & m_{13} & 0 \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ 0 & m_{24} & m_{34} & m_{44} \end{pmatrix}.$$

This matrix will be the main object of study in our case-by-case analysis.

Proof of Theorem 3.21. The argument is somewhat different in the two following situations:

Case 1: $m_{13} \neq 0$.

Denote by f_1 the local equation of D_1 which was obtained from f. More precisely, if we take an arbitrary monomial in f and trace it throughout this first blow-up we get :

$$f \ni \mathfrak{m} = x_1^{d_1} x_2^{d_2} x_3^{d_3} x_4^{d_4} \longrightarrow \overline{\mathfrak{m}} = u_1^{d_1} u_2^{d_3} u_3^{d_4} u_4^{d_2+d_3+d_4-1} \in f_1.$$

When restricting the jacobian matrix J_{D_1} to C, the only remaining terms are the following :

$$J_{D_1}|_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_{44}u_3^2 + 2m_{24}u_3 + m_{22} \end{pmatrix},$$

thus the singular points of D_1 are given by:

$$\begin{cases} u_1 = u_2 = u_4 = 0 \\ m_{44}u_3^2 + 2m_{24}u_3 + m_{22} = 0 \end{cases} \tag{3.4}$$

We are now ready to examine the cases corresponding to different ranks of M_D.

Rank one: This case doesn't occur because by Theorem 3.20 each moving singularity is of type cA, meaning that the degree two part of f is at least of rank two as a quadratic form.

Rank two: First we show that D_1 is singular at C_{gen}, i.e. that all the coefficients of the degree two equation in (3.4) are zero. Denote by M_{ij} the 3×3 minors in M_D obtained by eliminating row i and column j. Since M_D has rank two, all of the M_{ij} are zero. Note that

$$M_{44} = m_{13}^2 m_{22}, \quad M_{24} = m_{13}^2 m_{24} \text{ and } M_{22} = m_{13}^2 m_{44},$$

and since $m_{13} \neq 0$ the result is immediate.

By Lemma 3.11 and Remark 3.12, this implies that all elements in $D_1 \in |L_1|$ are tangent along S_1. If we blow-up U_1 along this surface, the strict transforms of these divisors will have a surface in common, which is in its generic point defined by $\mathbb{P}(\mathcal{N}^*_{S_1/D_1})$. Denote it by S_2. We construct another blow-up in order to obtain the following sequence :

$$U \xleftarrow{\mu_1} U_1 \xleftarrow{\mu_2} U_2 \xleftarrow{\mu_3} U_3,$$

where μ_2 and μ_3 are the blow-ups of U_1 along S_1 and of U_2 along S_2 respectively. By computing the coefficients introduced in Notation 3.2 :

$$\begin{cases} a_2 = a_1 + 1 = 2 + 1 = 3 \\ r_2 = r_1 + 1 = 1 + 1 = 2 \end{cases} \text{and} \begin{cases} a_3 = a_2 + 1 = 4 \\ r_3 = r_2 + 1 = 3 \end{cases}$$

we obtain an immediate contradiction to condition (3.1).

Rank three: All of the coefficients of the equation $m_{44}u_3^2 + 2m_{24}u_3 + m_{22} = 0$ being equal to zero would imply $\mathrm{rank}(M_D) = 2$. As

$$det(M_D) = m_{13}^2 \times (m_{22}m_{44} - m_{24}^2) = 0,$$

we may assume, up to choosing a different chart on U_1, that $m_{44} \neq 0$. Since $m_{22}m_{44} - m_{24}^2 = 0$, we have a degree two equation with the double root $u_3 = -\dfrac{m_{24}}{m_{44}}$, namely

$$D_{1,sing} = \left(0,\, 0,\, -\frac{m_{24}}{m_{44}},\, 0\right).$$

Thus if M_D is of rank three, the strict transform D_1 has exactly one singular point, the same being true for all general elephants. As $\widetilde{D}_1$ is smooth at $D_{1,sing}$, the singular point does not coincide for two general strict transforms. This means that in this case the linear system of strict transforms $|L_1|$ has singularities that are moving inside of S_1.

In order to show that these singularities will not cover the entire surface it suffices to prove that $\mathrm{Bs}|L_1|$ has a reduced structure at S_1 (see discussion in Section 3.2.3 for details). This becomes apparent when we look at the local equations: since as before m_{44}, m_{24} and m_{22} cannot be all at once equal to zero, the equation of $D_1 \cap \widetilde{D}_1 = D_1|_{\{u_2=0\}}$ contains at least one of the monomials u_3u_4, $u_3^2u_4$ or u_4, thus this intersection only contains $S_1 = \{u_2 = u_4 = 0\}$ with multiplicity one.

The singularities of general divisors in $|L_1|$ must then move along a curve $C_1 \subset S_1$. By Remark 3.12, all elements of $|L_1|$ must be tangent along C_1. We want to explicitly construct the blow-up of U_1 along C_1 and at the same time keep track of the singular points of D_1.

In order to do this, we need to control the rank of the singularity of D_1 so that we understand whether the initial situation was improved by doing the first blow-up. We do a coordinate change that brings the singular point onto the origin:

$$(u_1,\, u_2,\, u_3,\, u_4) \mapsto (u_1,\, u_2,\, u_3 + \frac{m_{24}}{m_{44}},\, u_4).$$

If f_1 a local equation of D_1 after this transformation, let $f_1 = q_1 + h_1$, where q_1 is the quadratic part of f_1. Denote by $M_{D_1} = (p_{ij})_{1\leq i,j,\leq 4}$ the matrix of q_1 as a quadratic form.

We claim that $p_{12} = m_{13}$ and M_{D_1} is of the form:

$$M_{D_1} = \begin{pmatrix} 0 & p_{12} & 0 & p_{14} \\ p_{12} & 0 & 0 & p_{24} \\ 0 & 0 & 0 & 0 \\ p_{14} & p_{24} & 0 & p_{44} \end{pmatrix}. \tag{3.5}$$

Indeed, take an arbitrary monomial of f and trace it throughout the first blow-up and the coordinate change:

$$\mathfrak{m} = x_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4} \longrightarrow \overline{\mathfrak{m}} = u_1^{d_1}u_2^{d_3}u_3^{d_4}u_4^{d_2+d_3+d_4-1} \longrightarrow$$
$$\longrightarrow \mathfrak{m}_1 = u_1^{d_1}u_2^{d_3}\left(u_3 - \frac{m_{24}}{m_{44}}\right)^{d_4} u_4^{d_2+d_3+d_4-1}$$

Note that $\mathfrak{m}_1$ splits into d_4+1 monomials of degrees $d_1+d_2+2d_3+d_4+k-1$, where $k \in \{0 \ldots d_4\}$.

Step 1: $p_{22} = p_{23} = p_{33} = 0$. A monomial in $\mathfrak{m}_1$ contributing to M_{D_1} will be of degree two and it follows from the expression above that there will be none of type u_2^2, u_2u_3 or u_3^2. Indeed, for all of these monomials we have $d_3 + k = 2$, automatically increasing the total degree to at least three.

Step 2: $p_{13} = 0$. The monomial u_1u_3 can only be obtained if $d_1 = 1$ and $k = 1$, while all other powers are zero. This implies $d_4 = 1$ and $d_3 = d_2 = 0$, thus the coefficient of u_1u_3 is $2m_{14} = 0$.

Step 3: $p_{12} = m_{13}$. As in the previous step, we obtain that the coefficient of u_1u_2 is $2m_{13}$.

Step 4: $p_{34} = 0$. The monomial u_3u_4 can be obtained from either x_2x_4 or x_4^2 in f, corresponding to $k = 1$, $d_4 = 2$ and $k = 1$, $d_2 = d_4 = 1$ respectively. Its coefficient $2p_{34}$ will be

$$m_{44} \times 2\left(-\frac{m_{24}}{m_{44}}\right) + 2m_{24} = 0.$$

Step 5: $p_{11} = 0$. The contributions to p_{11} come from monomials of the form $x_1^2x_2$ and $x_1^2x_4$. Because of the tangency condition relating the jacobian of D with that of $\widetilde{D}$, the coefficients in f of these two monomials are zero.

This proves the claim. We have thus obtained a much simpler matrix M_{D_1} after the first blow-up, though its rank can still be equal to three.

We now construct the second blow-up. Since we cannot change coordinates while maintaining the format of M_{D_1}, we will consider an arbitrary curve in S_1 passing through the origin and denote it by C_1. This curve is smooth at the origin because of the general choice of D. Locally it is given by:

$$\begin{cases} u_2 = u_4 = 0 \\ g(u_1, u_3) = 0 \end{cases},$$

where g is an arbitrary holomorphic function such that $g(0,0) = 0$.

Denote by $\mu_2 : U_2 \to U_1$ the blow-up of U_1 along C_1. One of the charts on U_2 is given by

$$\left\{ (u_1, u_2, u_3, u_4)(t_1, t_2) \;\middle|\; \begin{cases} u_2 = t_1 g(u_1, u_3) \\ u_4 = t_2 g(u_1, u_3) \end{cases} \right\}$$

and we choose the following local coordinates on it: $v_1 := t_1$, $v_2 := t_2$, $v_3 := u_1$ and $v_4 := u_3$. The chart becomes:

$$(v_1, v_2, v_3, v_4) \longrightarrow (v_4, v_1 g(v_3, v_4), v_3, v_2 g(v_3, v_4))(v_1, v_2).$$

In this chart the exceptional divisor E_2 is given by $g(v_3, v_4) = 0$ and the projective plane above the origin has the equations $v_3 = v_4 = 0$. The strict transform of E_1, denoted by E_1', is given by $v_2 = 0$.

Denote by D_2 the strict transform of D_1 through μ_2. We will show that D_2 is singular at the origin and we will compute the rank of M_{D_2} at this point. Indeed, consider an arbitrary monomial in f_1, denoted by $\mathfrak{m}_1 = u_1^{d_1}u_2^{d_2}u_3^{d_3}u_4^{d_4}$. Note that since $S_1 \subset D_1$ we have $d_2 + d_4 > 0$. As before, its contribution to the local equation of D_2 is

$$\mathfrak{m}_1 = u_1^{d_1}u_2^{d_2}u_3^{d_3}u_4^{d_4} \longrightarrow \mathfrak{m}_2 = v_1^{d_2}v_2^{d_4}v_3^{d_3}v_4^{d_1}g(v_3, v_4)^{d_2+d_4-1}.$$

The partial derivatives of $\mathfrak{m}_2$ vanish at the origin: consider for example $\frac{\partial \mathfrak{m}}{\partial v_3}$, which is a sum of two terms. The only situations in which it would not vanish at the origin are if (for the first term) $d_3 - 1 = d_2 = d_4 = d_1 = d_2 + d_4 - 1 = 0$ or (for the second term) $d_2 = d_4 = d_3 = d_1 = d_2 + d_4 - 2 = 0$, and neither can occur.

If the origin is not an isolated singular point, we are done. Indeed, if D_2 is singular along an entire curve, by Remark 3.12 all elements in $|L_2|$ are tangent along a surface denoted by S_2. We construct the following blow-up sequence:

$$U \xleftarrow{\mu_1} U_1 \xleftarrow{\mu_2} U_2 \xleftarrow{\mu_3} U_3 \xleftarrow{\mu_4} U_4,$$

where $U_3 = \mathrm{Bl}_{S_2}U_2$ and $U_4 = \mathrm{Bl}_{S_3}U_3$, where $S_3 = \mathbb{P}(\mathcal{N}^*_{S_2/D_2})$ is the surface that the strict transforms of elements in $|L_2|$ have in common. The coefficients are:

$$\left\{ \begin{array}{l} a_2 = a_1 + 2 = 4 \\ r_2 = r_1 + 1 = 2 \end{array} \right. , \left\{ \begin{array}{l} a_3 = a_2 + 1 = 5 \\ r_3 = r_2 + 1 = 3 \end{array} \right. \text{ and } \left\{ \begin{array}{l} a_4 = a_3 + 1 = 6 \\ r_4 = r_3 + 1 = 4 \end{array} \right. ,$$

a contradiction to condition (3.1).

Assume now that the origin is an isolated singular point of D_2. Then it is neither a fixed singularity for the system $|L_2|$ nor is it a moving singularity along a surface. Indeed, a short computation shows that $\widetilde{D}_2$ is smooth at the origin and Proposition 3.22 allows us to eliminate the surface case if $\mathrm{Bs}|L_2|$ has a reduced structure at $S_{2,gen}$. If this is not the case, all elements in $|L_2|$ are tangent along S_2 and we repeat the sequence of blow-ups above in order to derive a contradiction.

We now proceed to determining the rank of this singularity. If a monomial in $\mathfrak{m}_2$ is of degree two then $d_2 + d_4 = 1$ and $d_1 + d_3 = 1$, in particular it comes from certain degree two monomials in f_1. Using that M_{D_1} is of the form in (3.5), we obtain that M_{D_2} has the following form:

$$M_{D_2} = \begin{pmatrix} 0 & 0 & 0 & p_{12} \\ 0 & 0 & 0 & p_{14} \\ 0 & 0 & 0 & 0 \\ p_{12} & p_{14} & 0 & 0 \end{pmatrix}.$$

This is a rank two matrix, therefore we can apply the same strategy as in the case where M_D was of rank two : for the most part of the discussion, we know what we obtain geometrically without having to perform any local computations. The process consists in doing three additional blow-ups : the first along a curve and each of the other two along a suitably chosen surface. The coefficients will not be exactly the same since we have already blown up two subvarieties, we will however obtain the same type of contradiction.

In the end we will have constructed a sequence of five blow-ups :

$$U \xleftarrow{\mu_1} U_1 \xleftarrow{\mu_2} U_2 \xleftarrow{\mu_3} U_3 \xleftarrow{\mu_4} U_4 \xleftarrow{\mu_5} U_5,$$

where as before $U_1 = Bl_{C_0}U$ and $U_2 = Bl_{C_1}U_1$. The morphisms μ_3, μ_4 and μ_5 will be the blow-ups along a curve C_2 and two surfaces denoted by S_3 and S_4 respectively.

The choices of the centers are straightforward, we proceed exactly as in the previous rank two case:

- $C_2 \subseteq U_2$ is the curve along which the singularities of general members of $|L_2|$ move,
- $S_3 \subseteq U_3$ is the surface along which general members of $|L_3|$ are tangent,
- $S_4 \subseteq U_4$ is the surface in $\mathrm{Bs}|L_4|$ that exists because of this tangency, given by $\mathbb{P}(\mathcal{N}^*_{S_3/D_3})$.

The only detail we need to additionally keep track of is the interaction between the exceptional loci.

We briefly come back to the local picture in order to eventually describe μ_4. The first observation is that after the change of coordinates, the origin belongs to both divisors E_2 and E_1', the strict transform of E_1 through μ_2. As the sequence of blow-ups does not depend on the coordinate choice, we have just used a local computation to show that the singular point of the strict transform of a general $D \in |-K_X|$ through $\mu_1 \circ \mu_2$ belongs to $E_1' \cap E_2$. The same must be then true for the curve formed precisely by these singular points, that is to say C_2. This implies that both E_2 and E_1' will contribute to a_3 and r_3, the coefficients of E_3 in Notation 3.2.

We claim that the only exceptional divisor that S_3 belongs to is E_3. Indeed, a short computation shows that

$$T_{E_2}|_{C_{2,gen}} \neq T_{\widetilde{D}_2}|_{C_{2,gen}} \text{ and } T_{E'_1}|_{C_{2,gen}} \neq T_{\widetilde{D}_2}|_{C_{2,gen}}$$

where $\widetilde{D}_2$ is the strict transform of $\widetilde{D}$ through $\mu_1 \circ \mu_2$. As $T_{\widetilde{D}_2}|_{C_{2,gen}} = T_{D_2}|_{C_{2,gen}}$ by Remark 3.12, this proves that the divisor E_4 will be disjoint from both strict transforms of E_1 and E_2, but will have a surface in common with the strict transform of E_3. At this stage, the coefficients are:

$$\begin{cases} a_2 = a_1 + 2 = 4 \\ r_2 = r_1 + 1 = 2 \end{cases}, \begin{cases} a_3 = a_1 + a_2 + 2 = 8 \\ r_3 = r_1 + r_2 + 1 = 4 \end{cases} \text{ and } \begin{cases} a_4 = a_3 + 1 = 9 \\ r_4 = r_3 + 1 = 5 \end{cases},$$

which doesn't yet allow us to conclude. The fifth blow-up is of the surface $S_4 \subset E_4$ defined above, which may or may not also be included in E_3. The two cases both lead us to a contradiction to condition (3.1):

$$\begin{cases} a_5 = a_4 + a_3 + 1 = 18 \\ r_5 = r_4 + r_3 + 1 = 10 \end{cases} \text{ or } \begin{cases} a_5 = a_4 + 1 = 10 \\ r_5 = r_4 + 1 = 6 \end{cases}.$$

Case 2: $m_{13} = 0$.

This is a degeneration of the previous situation and as such will be easier to exclude. Geometrically, the condition says that the restriction $D_1|_{E_1}$ is not reduced along the $\mathbb{P}^2$ above the origin. As before, we have that

$$S_1 = \{u_4 = u_2 = 0\} \text{ and } C = \{u_1 = u_4 = u_2 = 0\}.$$

The problem here is that D_1 may be singular outside C: if we restrict the jacobian of D_1 just to the $\mathbb{P}^2$ above the origin, we obtain

$$J_{D_1}|_{\mathbb{P}^2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ m_{22} + 2m_{23}u_2 + m_{33}u_2^2 + 2m_{34}u_2u_3 + 2m_{24}u_3 + m_{44}u_3^2 \end{pmatrix}$$ [1]

which is included in C iff either $m_{23} \neq 0$ while $m_{22} = m_{33} = m_{34} = m_{24} = m_{44} = 0$ or $m_{33} \neq 0$ and $m_{22} = m_{23} = m_{34} = m_{24} = m_{44} = 0$, both cases leading to D_1 being singular along the entire curve C. The latter is impossible since it would mean that the rank of M_D is one and [Kol97, Thm. 4.4] implies it should be at least equal to two. The former also leads to a contradiction: we have that M_D is of rank two and D_1 is singular along a curve. We perform the same blow-ups as in the rank two case and obtain the coefficients $a_3 = 4$ and $r_3 = 3$ which contradict condition (3.1).

Rank four: This is the case in the statement of Theorem 3.21. Following the same strategy of the proof so far, the strict transform of a general elephant can either contain one or two singular points and despite the lack of examples we have yet to find a reason for this situation not to occur. □

[1] this is not the case if $m_{13} \neq 0$, as the first row of $J_{D_1}|_{\mathbb{P}^2}$ is $2m_{13}u_2$.

The surface case

Now that we have discussed the situation where the singularities of general elements in $|-K_X|$ move along a curve C, we claim that this is the maximal-dimensional case that we need to consider.

The argument holds in the following more general case: let $|L|$ be a linear system without fixed components on a fourfold X. As before denote by $|L|^0$ the Zariski open set in $|L|$ such that for all $D \in |L|^0$ we have that $D_{sing} \subset \mathrm{Bs}|L|$.

We show that the set

$$W = \{x \in D_{sing} | D \in |L|^0\}$$

cannot contain a surface: we prove that if $\mathrm{Bs}|L|$ contains a reduced surface S, then in fact the singular points of all general $D \in |L|^0$ belong to a curve included in S.

We are only concerned with the smooth points $x \in S$, since the singular ones already belong to a subset of the desired codimension. Fix a general element $D \in |L|^0$ that is singular at x. The divisor $S \subset D$ is not Cartier at x, as otherwise all points in $D_{sing} \cap S$ would be singular points of S. Since every $M \in |L|_D$ is Cartier at x we see that there exists another component $R \subset M$ with $x \in R$. We can then decompose $|L|_D$ as follows:

$$|L|_D = S + |R_D|, \tag{3.6}$$

such that $x \in \mathrm{Bs}|R_D|$. Note that the linear system $|R_D|$ may have fixed components, as $\mathrm{Bs}|L|$ possibly contains other surfaces aside from S.

We show that $\mathrm{Bs}|R_D|$ is independent of the initial choice of D. Indeed, fix $D' \in |L|$ to be another general element and through the same process construct $|R_{D'}|$. Then $D \cap D'$ is a subscheme of D having S as an irreducible component. Denote by $T := \mathrm{Supp}(D \cap D' \setminus S)$. We have that $||L|_D|_T = ||L|_{D'}|_T = |L|_T$ since the following diagram of restrictions is commutative:

$$\begin{tikzcd}
 & H^0(D, \mathcal{O}_D(L)) \arrow[dr] & \\
H^0(X, \mathcal{O}_X(L)) \arrow[ur] \arrow[dr] & & H^0(T, \mathcal{O}_T(L)) \\
 & H^0(D', \mathcal{O}_{D'}(L)) \arrow[ur] &
\end{tikzcd}$$

By restricting (3.6) and the analogous decomposition of $|L|_{D'}$ with respect to $|R_{D'}|$ to T we obtain that

$$|L|_T = S_T + |R_D|_T = S_T + |R_{D'}|_T.$$

So $|R_D|_T = |R_{D'}|_T$, which means $\mathrm{Bs}|R_D| = \mathrm{Bs}|R_{D'}|$ since both base loci are included in T.

Proposition 3.22. Let $|L|$ be a linear system without fixed components on a projective four-dimensional manifold X. If $S \subset \mathrm{Bs}|L|$ is a reduced surface and all general elements in $|L|$ are smooth in codimension one, then the set $W = \{x \in D_{sing} \,|\, D \in |L|^0\} \cap S$ is at most of dimension one.

Proof. Since every $x \in W$ is either a singular point of S or belongs to $\mathrm{Bs}|R_D| \cap S$ by the previous discussion, it is enough to show that $S \cap \mathrm{Bs}|R_D| \subsetneq S$. Indeed, suppose $S \subset \mathrm{Bs}|R_D|$, then we can write

$$|L|_D = 2S + |R'_D|,$$

meaning that for every $D' \in |L|$ we have that the threefolds D and D' are tangent along S. This is equivalent to $\mathrm{Bs}|L|$ having a non-reduced structure at S_{gen}, which contradicts the hypothesis. $\square$

Proof of Theorem 3.8. We now check that the hypotheses of Proposition 3.22 are true for a four-dimensional Fano manifold X and the anti-canonical system $|-K_X|$. Suppose for a contradiction that $\mathrm{Bs}|-K_X|$ (viewed as a scheme cf. Definition 3.3) has a non-reduced structure along an irreducible surface S. By [Kaw00, Prop 4.2], if we consider two general elephants D and D', they give rise to an lc pair $(D, D \cap D')$. But S is a component of $D \cap D'$ of at least multiplicity two, which gives rise to a discrepancy equal to -2, a contradiction.

We then apply Proposition 3.22 and conclude that the anticanonical system $|-K_X|$ either has fixed singularities or singularities moving along a curve. By Theorem 3.16 and Theorem 3.21, we obtain that locally analytically these points are of the form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \text{ or } x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0.$$

□

Chapter 4

The bi-anticanonical system

The advances in Fujita's Conjecture, particularly the works of Reider, Ein and Lazarsfeld and Kawamata show that for a Fano variety X of dimension at most four we have that $|-(n-1)K_X|$ is base-point free for all $n \geq \dim X$. It is then natural to ask whether a general $D \in |-2K_X|$ is smooth, this being the middle ground between the mildly singular general elephants and the base-point freeness of $|-3K_X|$.

Let Y_1 and Y_2 be two general elephants. As discussed in Proposition 3.6, the pair (X, Y_1+Y_2) is properly log canonical, it therefore admits a log-canonical center. The strategy is to determine the existence of smooth members of $|-2K_X|$ by extending sections from such an object.

If $S = Y_1 \cap Y_2$ and let C be a reduced (but not necessarily minimal) log-canonical center included in S. This exists since S itself is a union of log-canonical centers. Using Theorem 2.17 for the pair (X, Y_1+Y_2) and the divisor $-2K_X$, as $-2K_X - (K_X+Y_1+Y_2)$ is ample, we have that the map

$$H^0(X, \mathcal{O}_X(-2K_X)) \to H^0(C, \mathcal{O}_C(-2K_X|_C)) \tag{4.1}$$

is surjective. We want to find smooth elements of $|-2K_X|_C|$, which we will then extend to the entire variety X.

We analyze a number of particular cases. The first one is that any center of dimension one of the pair (X, Y_1+Y_2) is nodal. In the works of Kawamata [Kaw97] and in a more general context Ambro [Amb] it is shown that any finite union of log canonical centers is seminormal. The situation of the nodal curve is a particular case of this situation. We expect that the same strategy of the proof of Proposition smoothcenter works in the case of Gorenstein curves. In the general context one would have to analyze the base-point freeness of the pull-back of $|-2K_X|_C|$ on the normalization of C.

Proposition 4.1. Suppose C is a center of the pair (X, Y_1+Y_2) of at most dimension one, and that if it is a curve, it is at most nodal. Then a general $D \in |-2K_X|$ is smooth along C.

Remark 4.2. We formulated this proposition in terms of smoothness as it is the subject of the chapter, however we prove something stronger, namely that for every center C as in the statement we have that $C \cap \mathrm{Bs}\,|-2K_X| = \emptyset$. We will use this particular statement when discussing Conjecture 4.8.

Proof. If $\dim C = 0$, then $C \notin \mathrm{Bs}\,|-2K_X|$. This is clear since all log canonical centers are irreducible, therefore C is a point and we have $H^0(C, \mathcal{O}_C(-2K_X|_C)) \simeq \mathbb{C}$. Then the surjective map (4.1) is non-zero, which would be impossible if C were included in the base locus.

If C is a curve, the strategy is to use an equivalent of the Subadjunction Theorem 2.18 to acquire information about $H^0(C, \mathcal{O}_C(-2K_X|_C))$.

We claim that $C \cap \mathrm{Bs}|-2K_X| = \emptyset$. First note that a singular point $p \in C_{sing}$ is a log canonical center, and we have determined such a point misses $\mathrm{Bs}|-2K_X|$.

The curve C may be singular (in particular Theorem 2.18 does not directly apply), but as we have assumed it is at most nodal, therefore Gorenstein. This makes our job much easier.

We show that there exists an effective $\mathbb{Q}$-divisor $\Delta \subset C \setminus C_{sing}$ such that

$$K_C + \Delta \sim_{\mathbb{Q}} (K_X + Y_1 + Y_2)|_C \sim_{\mathbb{Q}} -K_X|_C. \tag{4.2}$$

Indeed, let $\nu : \widetilde{C} \to C$ be the normalization of C. The weak version of the Subadjunction Theorem [Kol13, Thm-Defn.4.45] (proof in [BHN15, Lm.3.1]) states that there exists an effective divisor $\Delta_{\widetilde{C}}$ on $\widetilde{C}$ such that

$$K_{\widetilde{C}} + \Delta_{\widetilde{C}} \sim_{\mathbb{Q}} \nu^*(K_X + Y_1 + Y_2)|_C \tag{4.3}$$

and further if a point $p \in C$ is an lc center, then we have a set-theoretical inclusion $\nu^{-1}(p) \in \Delta_{\widetilde{C}}$. As C is nodal, the preimage of each point $p \in C_{sing}$ consists of exactly two points p_1 and p_2, and since p is an lc center we have that $p_1,\, p_2 \in \Delta_{\widetilde{C}}$. We then write

$$K_{\widetilde{C}} \quad = \quad \nu^* K_C - \textstyle\sum_{p \in C_{sing}} (p_1 + p_2)$$

and by adding $\Delta_{\widetilde{C}}$ to both sides we obtain

$$K_{\widetilde{C}} + \Delta_{\widetilde{C}} \quad \sim_{\mathbb{Q}} \quad \nu^* K_C + \Delta',$$

where Δ' is an effective $\mathbb{Q}$-divisor. Using (4.3), we conclude that there exists $\Delta = \nu_*(\Delta') \subset C \setminus C_{sing}$ such that (4.2) holds.

We show that $\deg(-2K_X|_C) \geq 2p_a(C)$, where $p_a(C)$ is the arithmetic genus of C, allowing us to conclude that $\mathrm{Bs}|-2K_X|_C| = \emptyset$. It suffices to show that

$$\deg(-2K_X|_C - K_C) \geq 2 \quad \Leftrightarrow \quad \deg(-K_X|_C + \Delta) \geq 2.$$

As C is nodal, we still have that $\deg(K_C) = 2p_a(C) - 2$ by [Har77, Ex.1.9] and therefore cannot be odd. As $\deg -K_X|_C \geq 1$, we obtain that $\deg(-2K_X|_C - K_C)$ is both larger than one and even, hence it is at least equal to two.

To complete the proof, we use [Har86, Prop.1.5]. This article enables us to apply many basic facts about smooth curves to the nodal case, such as Serre duality or the criterion for base point freeness of a linear system. As

$$\begin{aligned} \deg(K_C - (-2K_X|_C)) \quad &\leq \quad 2g - 2 - 2g = -2 \text{ and} \\ \deg(K_C - (2K_X|_C - p)) \quad &\leq \quad 2g - 2 - 2g + 1 = -1 \end{aligned}$$

we obtain

$$\begin{aligned} h^0(C, K_C - (-2K_X|_C)) \quad &= \quad h^1(C, -2K_X|_C) = 0 \\ h^0(C, K_C - (-2K_X|_C)) \quad &= \quad h^1(C, -2K_X - p) = 0 \end{aligned}$$

for every point $p \in C$. Using the Riemann Roch formula, we see that

$$\dim|-2K_X - p| \quad = \quad \dim|-2K_X| - 1,$$

therefore $|-2K_X|_C|$ is base-point free, proving the claim. As all its sections extend from C to X by (4.1), a general $D \in |-2K_X|$ will be smooth along C. $\square$

We move to analyzing the case when the center is a surface S. We first show that this surface cannot be contained in $\mathrm{Bs}|-2K_X|$.

Remark 4.3. If X is a Fano fourfold, then $\dim \mathrm{Bs}|-2K_X| \leq 1$.

Proof. Suppose by contradiction that S_0 is an irreducible surface contained in the base locus of $|-2K_X|$. As $\mathrm{Bs}|-2K_X| \subseteq \mathrm{Bs}|-K_X|$, we have that $S_0 \subseteq Y_1 \cap Y_2 = S$, where Y_1 and Y_2 are as above. As they are ample, S is connected. We are in one of two cases: either $S_0 = S$ is a minimal log canonical center of the pair $(X, Y_1 + Y_2)$, or there exists a center $C \subset S_0$ such that $\dim C \leq 1$. Either way, the map (4.1) is zero. In the first case, since S is minimal, it is klt, and we can apply the effective Kawamata Nonvanishing theorem 2.19 to the pair $(S, 0)$ and the divisor $2K_S$. We obtain a first contradiction. If we are in the second case, using Riemann-Roch [Har86, Thm.1.3] for the curve C and $-2K_X|_C$ we conclude that the target of (4.1) is nontrivial, which is absurd since we have just deduced that it is a surjective zero map. We conclude that $Bs|-2K_X|$ is at most a curve. $\square$

We now study the second particular case, in which the center is a surface. We further suppose that $S = Y_1 \cap Y_2$ is smooth, then it is a surface of general type. What is more $K_X|_S = K_S$ is ample, therefore S is minimal. The aim of what follows is to analyze the bicanonical system $|2K_S|$. It is already known by [BHPVdV04, VII,Thm.7.4] that if $p_g(S) \geq 1$ then this system is base point free, therefore we are able to successfully extend the smooth sections to X. We now restrict to the case where $p_g(S) = 0$. A first step is to examine whether it is possible for the fixed part of $|2K_S|$ to contain a smooth irreducible component of high multiplicity.

Proposition 4.4. Let S be a smooth surface with K_S ample and $p_g(S) = 0$. Then there exists a reduced divisor $D \in |2K_S|$, with the exception of the following situations in which $K_S^2 = 4$, $|2K_S| = V + |M|$, where $|M|$ is the moving part of $|2K_S|$ and the V constitute its fixed part, with the following numerical data:

- $V = 2C_0 + C_1$, where C_i are irreducible curves, such that:

$$\begin{cases} C_0 \text{ is a genus 1 curve such that } C_0^2 = -1, \\ C_1 \text{ is a genus 0 curve such that } C_1^2 = -3, \\ M^2 = K_S \cdot M = 5 \text{ and } K_S \cdot C_0 = K_S \cdot C_1 = 1. \end{cases}$$

- $V = 2C_0 + C_1 + C_2$, where C_i are irreducible curves, such that:

$$\begin{cases} C_0 \text{ and } C_1 \text{ are genus 1 curves such that } C_0^2 = C_1^2 = -1, \\ C_2 \text{ is a genus 0 curve such that } C_2^2 = -3, \\ M^2 = K_S \cdot M = 4 \text{ and } K_S \cdot C_0 = K_S \cdot C_1 = K_S \cdot C_2 = 1. \end{cases}$$

- $V = 2C_0 + C_1 + C_2$, where C_i are irreducible curves, such that:

$$\begin{cases} C_0 \text{ is a genus 1 curve such that } C_0^2 = -1, \\ C_1 \text{ and } C_2 \text{ are genus 0 curves such that } C_1^2 = C_2^2 = -3, \\ M^2 = K_S \cdot M = 4 \text{ and } K_S \cdot C_0 = K_S \cdot C_1 = K_S \cdot C_2 = 1. \end{cases}$$

This very last case is divided into two subcases, depending on the intersection of each component of the fixed locus with the mobile part of the linear system. We give a detailed account of this during the proof.

We first state three results that are relevant for the proof:

Remark 4.5. The fixed part of $|2K_S|$ contains no smooth elliptic curve C such that $K_S - C$ is nef and big.

Proof. We proceed by contradiction. Suppose C exists, then from the following exact sequence:

$$0 \to \mathcal{O}_S(2K_S - C) \to \mathcal{O}_S(2K_S) \to \mathcal{O}_C(2K_S) \to 0$$

we get the corresponding sequence in cohomology

$$H^0(S, \mathcal{O}_S(-2K_S)) \to H^0(C, \mathcal{O}_C(2K_S)) \to H^1(S, \mathcal{O}_S(K_S + K_S - C)).$$

Since $K_S - C$ is nef and big, by the Kodaira vanishing theorem we obtain that the first arrow is surjective. If C were a component of the fixed locus of $2K_S$, this would be the zero map, however $H^0(C, \mathcal{O}_C(2K_S)) = \mathbb{C}^2$ since C is an elliptic curve and $K_S \cdot C > 0$. This proves the statement. $\square$

Another statement concerns the case in degree two:

Theorem 4.6. [Xia85, Thm3] Let S be a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 2$. Then the bicanonical map Φ_{2K} is not associated to a genus two fibration, therefore it is a surjection on $\mathbb{P}^2$.

Lastly, we are able to eliminate cases for which $K_S^2 = 3, 4$ thanks to the work of Weng in [Wen95], in which he computes most of the intersection numbers for the fixed and moving components of $|2K_S|$.

Proposition 4.7. [Wen95, Cor.] Let S be a smooth minimal surface of general type with $p_g(S) = 0$ and $|2K_S| = V + |M|$, the decomposition of the bicanonical system into the fixed part V and the moving part $|M|$. We assume that $V \neq 0$.

(1) Case $K_S^2 = 3$. We only have the following possibilities:

   (a) $K_SM = 4$, $K_SV = 2$, $M^2 = 4$, $MV = 4$, $V^2 = 0$;
   (b) $K_SM = 5$, $K_SV = 1$, $M^2 = 5$, $MV = 5$, $V^2 = -3$;
   (c) $K_SM = 5$, $K_SV = 1$, $M^2 = 7$, $MV = 3$, $V^2 = -1$;

(2) Case $K_S^2 = 4$. We only have the following possibilities:

   (a) $K_SM = 4$, $K_SV = 4$, $M^2 = 4$, $MV = 4$, $V^2 = 4$;
   (b) $K_SM = 5$, $K_SV = 3$, $M^2 = 5$, $MV = 5$, $V^2 = 1$;
   (c) $K_SM = 6$, $K_SV = 2$, $M^2 = 4$, $MV = 8$, $V^2 = -4$;
   (d) $K_SM = 6$, $K_SV = 2$, $M^2 = 6$, $MV = 6$, $V^2 = -2$;
   (e) $K_SM = 6$, $K_SV = 2$, $M^2 = 8$, $MV = 4$, $V^2 = 0$;
   (f) $K_SM = 7$, $K_SV = 1$, $M^2 = 9$, $MV = 5$, $V^2 = -3$;
   (g) $K_SM = 7$, $K_SV = 1$, $M^2 = 11$, $MV = 3$, $V^2 = -1$.

In particular, the number of irreducible components of the fixed part of $|2K_S|$ is at most 2 (resp. 4) when $K_S^2 = 3$ (resp. 4).

Proof of Proposition 4.4. The proof is split in various cases according to the number of irreducible components contained in the fixed locus of $|2K_S|$. The approach is essentially the same: we have a main formula that describes $|2K_S|$, which we intersect in turn with the canonical class, with each component of the fixed locus and with the mobile part of $|2K_S|$. We then deduce information on the self-intersection of these divisors using the Hodge index theorem and use positivity properties to derive contradictions. In some cases, not all these steps are necessary.

The first two cases are the easiest. To fix notation, suppose that

$$|2K_S| = V + |M| \tag{4.4}$$

is the decomposition into the fixed, respectively mobile part of the bicanonical system of S.

If $K_S^2 = 1$, suppose we can write $V = mC + B$, where $m \geq 2$ and C is irreducible. We intersect both members of (4.4) with K_S and obtain

$$2 = mCK_S + BK_S + MK_S.$$

As all terms are positive, this can only happen if $m = 2$ and $B = M = 0,\ K_SC = 1$. Because S is a smooth dimension two complete intersection (of two smooth ample divisors), its Picard group is torsion-free by the Lefschetz Theorem. This means $C \in |K_S|$, contradicting the fact that $p_g(S) = 0$.

If $K_S^2 = 2$, Theorem 4.6 concludes that $p_2(S) = 3$. We write $V = mC + B$ just as in the previous case, this time we obtain that

$$4 = mCK_S + BK_S + MK_S. \tag{4.5}$$

Again, we cannot have $B = M = 0$, since this would imply $m \in \{2, 4\}$, therefore $p_g(S) \neq 0$. We are left with the following possibilities:

- ◁ If $K_SM = 2$ then $B = 0$, $m = 2$ and $CK_S = 1$. The Hodge index theorem gives that $4 = (K_SM)^2 \geq K_S^2M^2 = 2M^2$, as well as $1 = (CK_S)^2 \geq 2C^2$, therefore $M^2 \in \{0, 1, 2\}$ and C^2 is negative by adjunction. The adjunction theorem also implies that C^2 and CK_S are congruent modulo 2, therefore $C^2 \in \{-1, -3\}$. If we intersect (4.4) with M, we obtain that $4 = 2CM + M^2$, therefore $M^2 \in \{0, 2\}$ and $CM \in \{1, 2\}$. If we intersect (4.4) with C, we get $2 = 2C^2 + CM$, which is impossible.
- ◁ If $K_SM = 1$, then $M^2 \leq 0$ by the Hodge index theorem since $1 = (KM)^2 \geq K^2M^2 = 2M^2$. As $|M|$ does not have base components, we must have $M^2 = 0$. There are two possibilities given by (4.5): either $(m, CK_S, BK_S) = (2, 1, 1)$ or $(m, CK_S, BK_S) = (3, 1, 0)$.
  - In the former case, as $BK_S = 1$ we have that B is irreducible and further B^2 is odd. If we intersect (4.4) respectively with B and M, we obtain:

    $$2 = 2CM + BM$$
    $$2 = 2CB + B^2 + BM,$$

    which contradict each other with respect to the parity of BM.
  - The latter case implies that $B = 0$. We intersect (4.4) with M and obtain $2 = 3CM$, impossible.
- ◁ The case $M = 0$ is impossible since $p_2(S) \geq 2$.

We repeatedly use Proposition 4.7 to prove our claim in the cases $K_S^2 = 3$ and $K_S^2 = 4$.

If $K_S^2 = 3$, suppose there exists a non-reduced curve in $\mathrm{Bs}|2K_S|$. By Proposition 4.7, the number of irreducible components in the fixed locus of $2K_S$ is at most two, thus $V = 2C$, where C is an irreducible curve. Since $2K_S = 2C + M$ and $K_SM \geq 4$ we obtain that $K_SC = 1$ and $K_SM = 4$. But this corresponds to case $(1.a)$ in Proposition 4.7, which also implies $V^2 = 0$, a contradiction since $\deg K_C$ is always odd.

If $K_S^2 = 4$, we have further possibilities. We denote by C_0 the non-reduced curve in the fixed locus, which this time may contain up to four irreducible components. The condition $K_SM \geq 4$ is still valid in this case.

Suppose C_0 is the only curve in $\mathrm{Bs}|2K_S|$. We have the following cases, ranging from the highest possible multiplicity of C_0 to the lowest:

◁ $2K_S = 4C_0 + M$. Then $K_SM = 4$ and $K_SC_0 = 1$. We are thus in case $(2.a)$ of Proposition 4.7, implying that $(4C_0)^2 = 4$, a contradiction.

◁ $2K_S = 3C_0 + M$. By the same type of considerations, we are either in case $(2.a)$ or $(2.b)$ in Proposition 4.7, implying that $(3C_0)^2$ is either equal to 1 or 4, both leading to a contradiction.

◁ $2K_S = 2C_0 + M$. Firstly, $K_SM = 5$ cannot happen for parity reasons. If $K_SM = 4$ and $K_SC_0 = 2$, then by case $(2.a)$ we have that $C_0^2 = 1$. But as $\deg K_{C_0}$ is always even, the adjunction theorem gives us a contradiction.

If $K_SM = 6$ and $K_SC_0 = 1$, we have that $(2C_0)^2 \in \{-4, -2, 0\}$, so $C_0^2 \in \{-1,\ 0\}$. Again by the adjunction theorem we have that C_0^2 must be even, therefore it is equal to -1 and C_0 is an elliptic curve. In order to apply Remark 4.5 to the curve C_0, one must check that $K_S - C_0 \equiv \frac{1}{2}M$ is nef and big, which is clear as $M^2 = 8 > 0$ and $|M|$ is mobile. Therefore C_0 is not in the base locus of $|2K_S|$, a contradiction.

Suppose the fixed locus V contains an additional curve C_1 (which can appear with higher multiplicity). Here is a list of possibilities:

◁ $2K_S = 3C_0 + C_1 + M$. By intersecting this relation with K_S we obtain:

$$8 = 3K_SC_0 + K_SC_1 + K_SM.$$

This forces $K_SC_0 = K_SC_1 = 1$ and $K_SM = 4$, the corresponding case in Proposition 4.7 being $(2.a)$, which implies $V^2 = 4$. Now $(C_1)^2$ has to be odd, and by the Hodge index theorem we know that $1 = (K_SC_1) \geq K_S^2C_1^2 = 4C_1^2$. The adjunction theorem states that $C^2 = K_C - K_SC \geq -2 - 1 = -3$, therefore $C_1^2 \in \{-1, -3\}$. As

$$V^2 = 3(3C_0^2 + 2C_0C_1) + C_0^2 = 4,$$

clearly none of these two cases work.

◁ $2K_S = 2C_0 + 2C_1 + M$. Intersection with the canonical class gives

$$8 = 2K_SC_0 + 2K_SC_1 + K_SM.$$

Again we must have $K_SC_0 = K_SC_1 = 1$ (thus C_0^2 and C_1^2 must be odd) and $K_SM = 4$, which by Proposition 4.7 also results in $V^2 = 4(C_0 + C_1)^2 = 4$. The relation $C_0^2 + C_1^2 + 2C_0C_1 = 1$ is impossible for parity reasons.

◁ $2K_S = 2C_0 + C_1 + M$. The relation

$$8 = 2K_SC_0 + K_SC_1 + K_SM$$

leaves room for two possibilities.

If $K_SM = 4$, then $K_SC_0 = 1$ and $K_SC_1 = 2$ (thus C_0^2 is odd and C_1^2 must be even). Applying the Hodge index theorem as before we obtain $C_0^2 \in \{-1,\ -3\}$ and $C_1^2 \in \{0,\ -2\}$. Proposition 4.7 gives that $V^2 = 4$, so

$$4C_0^2 + 4C_0C_1 + C_1^2 = 4,$$

therefore $C_1^2 = 0$ and $C_0C_1 \in \{2, 4\}$. Since

$$4 = 2K_SC_1 = 2C_0C_1 + MC_1,$$

we get that $C_0C_1 = 2$ and $C_0^2 = -1$, so once again C_0 is an elliptic curve. In oder to apply Remark 4.5, we need to verify that $K_S - C_0 \equiv \frac{1}{2}(C_1 + M)$ is nef (the bigness is clear as $M^2 > 0$). This numerical case corresponds to $(2.a)$ in Proposition 4.7, therefore we additionally know that $M^2 = 4$ and $VM = 2C_0M = 4$.

Since $(K_S - C_0)C_1 = 0$ is positive and $|M|$ is mobile, we indeed have that $\frac{1}{2}(C_0 + M)$ is nef and thus C_0 is not in the base locus of $|2K_S|$, a contradiction.

If $K_SM = 5$, we come across the case in the statement of our proposition, which we cannot exclude using the current methods. Let us show that the intersection numbers match.

We are in case $(2.b)$ of the proposition while also having $K_SC_0 = K_SC_1 = 1$. Firstly, the adjunction theorem shows that C_i^2 is odd, and by the index theorem we obtain $C_i^2 \in \{-1, -3\}$ for $i = 0, 1$. We also know that $V^2 = 2$, i.e. $4C_0^2 + 4C_0C_1 + C_1^2 = 1$. Congruence modulo 4 finds $C_1^2 = -3$, and $C_0^2 + C_0C_1 = 1$ means $C_0C_1 \in \{2, 4\}$.

By intersecting the formula for $2K_S$ with C_1 we get $2 = 2C_0C_1 - 3 + MC_1 \Leftrightarrow 2C_0C_1 + MC_1 = 5$. Since $MC_1 \geq 0$, we get $C_0C_1 = 2$ and $MC_1 = 1$. Then $C_0^2 = -1$ and as $MV = 5$, we have $2MC_0 + MC_1 = 5$ so $MC_0 = 2$.

Finally, suppose the fixed locus V contains an additional curve C_2 (which can also appear with higher multiplicity). The only case is:

$\lhd$ $2K_S = 2C_0 + C_1 + C_2 + M$. As

$$8 = 2K_SC_0 + K_SC_1 + K_SC_2 + K_SM$$

and $K_SM \geq 4$ we must have $K_SM = 4$ and $K_SC_0 = K_SC_1 = K_SC_2 = 1$. As before, $C_i^2 \in \{-1, -3\}$ for $i = 0, 1, 2$ by the adjunction theorem and the index theorem. This is the case $(2.a)$ in Proposition 4.7, i.e. we also know that $V^2 = 4$ and $MV = 4$:

$$4C_0^2 + C_1^2 + C_2^2 + 4C_0C_1 + 4C_0C_2 + 2C_1C_2 = 4 \tag{4.6}$$

$$2C_0M + C_1M + C_2M = 4. \tag{4.7}$$

By intersecting the formula for $2K_S$ with C_0, C_1 and C_2 we also have the following three relations:

$$2 = 2C_0^2 + C_0C_1 + C_0C_2 + C_0M \tag{4.8}$$

$$2 = 2C_0C_1 + C_1^2 + C_1C_2 + C_1M \tag{4.9}$$

$$2 = 2C_0C_2 + C_1C_2 + C_2^2 + C_2M \tag{4.10}$$

As M is nef and $C_0M \leq 2$ by (4.7), we have that $2 - 2C_0^2 \geq C_0C_1 + C_0C_2 \geq -2C_0^2$.

We split this study into three cases: $C_1^2 \neq C_2^2$, $C_1^2 = C_2^2 = -1$ and $C_1^2 = C_2^2 = -3$.

– If $C_1^2 \neq C_2^2$ then $C_1^2 + C_2^2 = -4$. By (4.6) we deduce that C_1C_2 is even. Again because of parity, (4.9) and (4.10) give that both C_1M and C_2M are odd.

  If they are equal, by (4.7) we immediately get $C_1M = C_2M = C_0M = 1$. By (4.8) we have that $C_0C_1 + C_0C_2$ is odd, and (4.6) implies that $\frac{1}{2}C_1C_2$ is even, i.e. C_1C_2 is divisible by 4. As at least one of C_0C_1 or C_0C_2 is strictly positive, by (4.9) and (4.10) we must have $C_1C_2 = 0$. It is now clear that

$$(C_0^2, C_1^2, C_2^2, C_0C_1, C_0C_2, C_1C_2) = (-1, -1, -3, 1, 2, 0),$$

up to a permutation of C_1 and C_2.

If on the other hand $C_1M \neq C_2M$, as they are both odd we obtain $C_0M = 0$ by (4.7). Without loss of generality we can assume $C_1M = 1$ and $C_2M = 3$. This time we have that $C_0C_1 + C_0C_2$ is even, therefore by (4.6) we get that C_1C_2 is not divisible by 4. As it cannot be larger than 6 because of (4.9) and (4.10), we obtain $C_1C_2 = 2$. We find the following:

$$(C_0^2, C_1^2, C_2^2, C_0C_1, C_0C_2, C_1C_2) = (-1, -1, -3, 0, 0, 2),$$

which contradicts (4.8).

– If $C_1^2 = C_2^2 = -1$, then by (4.6) we have that C_1C_2 must be odd. We claim that C_0C_1 and C_0C_2 can not both be 0. Indeed, by (4.8) that would imply that $C_0M \geq 4$, which contradicts (4.7). We may assume without loss of generality that $C_0C_1 \geq 1$ and by (4.9) we have

$$3 = 2 - C_1^2 = 2C_0C_1 + C_1C_2 + C_1M \geq 2 + 1 = 3,$$

therefore in order to verify the equality we must take $C_1M = 0$ and $C_1C_2 = C_0C_1 = 1$. As $C_0M \leq 2$, we again deduce from (4.8) that $C_0C_2 \geq 1$. Then the inequality above holds if we replace C_1 by C_2, we therefore obtain $C_2M = 0$ and $C_0C_2 = 1$. By (4.7) we get that $C_0M = 2$ and finally

$$(C_0^2, C_1^2, C_2^2, C_0C_1, C_0C_2, C_1C_2) = (-1, -1, -1, 1, 1, 1).$$

This case is however eliminated using Remark 4.5. Indeed, we only need to check that $K_S - C_0 \equiv \frac{1}{2}(C_1 + C_2 + M)$ is nef, as the bigness is clear since $M^2 = 4$ and $C_1^2 = C_2^2 = -1$. Since C_1 and C_2 have the same numerical data, we only verify this for C_1. As $C_1^2 + C_1C_2 + C_1M = -1 + 1 + 0 = 0$, the divisor is nef and we apply Remark 4.5 to obtain a contradiction.

– If $C_1^2 = C_2^2 = -3$, C_1C_2 is again odd by (4.6). The relations (4.9) and (4.10) become:

$$5 = 2C_0C_1 + C_1C_2 + C_1M \tag{4.11}$$
$$5 = 2C_0C_2 + C_1C_2 + C_2M. \tag{4.12}$$

Since every term of the sums on the right hand side is non-negative, this means that C_0C_1 and C_0C_2 are at most equal to two. The same holds for C_0M because of (4.7). As by (4.8) we have $2 - 2C_0^2 = C_0C_1 + C_0C_2 + C_0M \leq 6$ and $C_0^2 \in \{-1, -3\}$, we deduce $C_0^2 = -1$. The relation (4.8) becomes

$$4 = C_0C_1 + C_0C_2 + C_0M. \tag{4.13}$$

Substituting what we know so far in (4.6) we obtain

$$4(C_0C_1 + C_0C_2) + 2C_1C_2 = 14. \tag{4.14}$$

As $C_0M \leq 2$, by (4.13) we have $C_0C_1 + C_0C_2 \in \{2, 3, 4\}$, the last possibility being excluded by (4.14). We deduce that $C_0M \in \{1, 2\}$ and by substituting these cases in the relations above we obtain the following possibilities:

$$(C_0^2, C_1^2, C_2^2, C_0C_1, C_0C_2, C_1C_2) = (-1, -3, -3, 1, 2, 1) \text{ if } C_0M = 1 \text{ and}$$
$$(C_0^2, C_1^2, C_2^2, C_0C_1, C_0C_2, C_1C_2) = (-1, -3, -3, 1, 1, 3) \text{ for } C_0M = 1.$$

In both instances, Remark 4.5 cannot be applied.

□

We finish this chapter by proposing a more geometric conjecture:

Conjecture 4.8. Let X be a Fano fourfold. If $\mathrm{Bs}\,|-K_X| \leq 1$, there exists a smooth element $D \in |-2K_X|$.

Here is the motivation for this result: as before, we denote $Y_1 \cap Y_2$ by S. If S is smooth and irreducible, then it is a surface of general type and the hypothesis implies that $h^0(S, -K_X|_S) = h^0(S, K_S) \neq 0$. By previous results on minimal smooth surfaces of general type [BHPVdV04, VII,Thm.7.4] we know that if $p_g(S) \neq 0$ the bicanonical system $|2K_S|$ is base point free. Using (4.1) we deduce that $D \in |-2K_X|$ is smooth along S. But $\mathrm{Bs}\,|-2K_X| \subset \mathrm{Bs}\,|-K_X| \subset S$, therefore we have obtained the conclusion in this case. The situation where S is reducible is more complicated, however information about the interactions of $\mathrm{Bs}\,|-2K_X|$ and the components of S can be derived via Remark 4.2.

The case where S is singular is naturally more challenging. We show that Conjecture 4.8 holds if S is reducible and briefly discuss the remaining case.

Proof of the reducible case. Suppose $S = Y_1 \cap Y_2$ is reducible, and write $S = \sum_{i=1}^{d} S_i$, where $d > 1$ and S_i are reduced, distinct and irreducible, but not necessarily smooth. We know that $|-K_X|_{Y_i}|$ has no fixed component, and further that $Y_1|_{Y_2}$ is connected and smooth outside $\mathrm{Bs}\,|-K_X|$. For each irreducible component $S_i \subset Y_1 \cap Y_2$ there exists S_j such that $\emptyset \neq S_i \cap S_j$. As $S_i \cap S_j$ is a union of centers of dimension at most one of $(X, Y_1 + Y_2)$, Remark 4.2 implies that $\mathrm{Bs}\,|-2K_X| \cap S_i \cap S_j = \emptyset$.

If $\dim(\mathrm{Bs}\,|-K_X|) = 0$ or if $|-K_X|$ is base-point free, then $\mathrm{Bs}\,|-2K_X| \subseteq \mathrm{Bs}\,|-K_X| \subseteq S_i \cap S_j$, therefore we must have $\mathrm{Bs}\,|-2K_X| = \emptyset$ thus a general $D \in |-2K_X|$ is smooth.

If $S_i \cap S_j \subseteq \mathrm{Bs}\,|-K_X|$, we want to show, as in the smooth case, that $|-2K_X|_{S_i}|$ is base-point free. A short analysis of the morphism associated to $|-K_X|_{Y_1}|$ shows that $[S_i] = [S_j] \in \mathrm{Cl}(Y_1)$ for all $i, j = 1 \ldots d$.

Indeed, denote by $W \subseteq \mathbb{P}(|-K_X|_{Y_1}|)$ the image of Y_1 through $\varphi_{|-K_X|_{Y_1}|}$ and by $\mu : \widetilde{Y_1} \to Y_1$ a desingularization of Y_1. If $\varphi : \widetilde{Y_1} \to W$ is the composition of the two maps, we are in one of two cases. Firstly, if $\dim W \geq 2$ then a divisor $D \in |-K_X|_{Y_1}|$ is integral. In fact, we have that

$$\mu^*|-K_X|_{Y_1}| = E + \varphi^*|H|,$$

where E is the exceptional locus and H is ample on W, therefore $\varphi^*(H)$ is connected and irreducible as $\widetilde{Y_1}$ is smooth. If $\dim W = 1$, as points on W are numerically equivalent, the same is true about the fibers.

Therefore $dS_i \sim -K_X|_{Y_1}$ and by the adjunction formula we have:

$$K_{S_i} \quad = \quad (K_{Y_1} + S_i)_{S_i} \quad = \quad S_i|_{S_i} \quad = \quad \frac{1}{d}\,(-K_X|_{S_i}).$$

For any lc center $Z \subsetneq S_i$ of $(X, Y_1 + Y_2)$, by Remark 4.2 we have that $\mathrm{Bs}\,|-2K_X| \cap Z = \emptyset$. This is equivalent to the fact that

$$\mathrm{Bs}\,|-2K_X| \cap S_i \subseteq \{x \in S_i \,|\, S_i \text{ is the minimal lc center of } (X, Y_1 + Y_2) \text{ at } x\} =: B.$$

We are only interested in the points $x \in B$ that are in the smooth locus of Y_1. Indeed, if $x \in Y_{1,sing}$, it can be either a fixed singularity of multiplicity two, or a moving singularity (see Section 3.2.3

in Chapter 3 for the definition). In both cases we obtain log canonical centers of dimension at most one (we recall that moving singularities can at most describe a curve) which contain x, thus contradicts Remark 4.2.

As S_i is klt at the points in $B \setminus Y_{1,sing}$ (being a local minimal center), our goal is to show that the linear system $|-K_X|_{S_i}|$ is base point free at the klt locus of S_i. We prove this by adapting the ideas in [Kaw00, Thm.3.1] to our context.

Take $D \in |-K_X|_{S_i}|$ and by the short exact sequence

$$0 \to \mathcal{O}_{S_i}(D) \to \mathcal{O}_{S_i}(2D) \to \mathcal{O}_D(2D) \to 0.$$

We know that D is an ample Cartier divisor such that $h^0(S_i, D) \neq 0$, and that S_i is slc by [Kaw00, Prop.4.2]. In particular, this means we can apply the version for slc varieties of Kodaira's vanishing theorem [Fuj14, Thm.1.8]: as $D - K_{S_i} = \left(1 - \frac{1}{d}\right) D$ is ample, we obtain that $H^1(S_i, D) = 0$. Note that this does not hold in the case where S is not reducible. Therefore it is enough to prove the freeness of $|\mathcal{O}_D(2D)|$ in order to conclude.

We now apply the argument of Kawamata in [Kaw00, $Prop.$4.2]. Let $\mathfrak{m}$ be an ideal sheaf of $\mathcal{O}_D$ of colength one (a priori this is the ideal corresponding to a point that can be singular). We show that $H^1(D, \mathfrak{m}(2D)) = 0$, which by the Serre duality is equivalent to showing $\operatorname{Hom}(\mathfrak{m}, \omega_D(-2D)) = 0$, where $\omega_D \simeq \mathcal{O}_D(K_{S_i} + D)$. It's easy to compute its degree: $\deg \omega_D = (K_{S_i} + D)D = \left(1 + \frac{1}{d}\right) D^2$, therefore $\deg \omega_D(-2D) = \left(\frac{1}{d} - 1\right) D^2 < 0$ since $d \geq 2$. We conclude since $\chi(D, \mathfrak{m}) = \chi(D, \mathcal{O}_D) - 1$ by the definition of $\mathfrak{m}$.

□

If S is irreducible and singular, we use hypothesis to single out another anticanonical divisor Y_3. Let $C = Y_1 \cap Y_2 \cap Y_3 \subset \operatorname{Bs}|-K_X|$, which is a (possibly singular) curve. Following the discussion above, we are interested in points $x \in \operatorname{Bs}|-2K_X| \cap S_{norm}$ such that $x \in Y_{i,smooth}$. While so far we have not been able to conclude in this case, there are several possible ways to approach this problem:

1. If x is a smooth point of S, but not of C, we can deduce that x is an lc center of $(X, Y_1+Y_2+Y_3)$. Indeed, the fact that $C = Y_1 \cap Y_2 \cap Y_3$ is not smooth at x guarantees the existence of a two-dimensional vector space $V = T_{Y_1,x} \cap T_{Y_2,x}$. If we blow up X with center x, the strict transforms of Y_1 and Y_2 will meet along a line l. After a second blow-up, this time along l, we obtain that E_2 is a log canonical place. The computations are similar to the ones in Chapter 3 and we do not reproduce them in their entirety, however for verification purposes we mention that $a_1 = 3, r_1 = 1$ and $a_2 = 5, r_2 = 2$. The pair $(X, Y_1 + Y_2 + Y_3)$ is not necessarily lc, we could however use its log-canonical threshold to obtain further information. Similarly, if x is a singular point of S one could attempt finding a smaller-dimensional lc center of the pair $(X, Y_1 + Y_2)$ in order to contradict the minimality of S.

2. Another approach is to apply generalized results on Gorenstein curves to the case of C (see [CFHR99, Thm.3.3]), which require adapting properties from the smooth case (such as 2-connectedness of the bicanonical divisor on a minimal surface of general type) in order to deduce base-point freeness of $|-2K_X|$ on C, therefore at x.

Chapter 5

Effective Nonvanishing

In this chapter, we further our study of the anticanonical system on a Fano variety by looking at higher dimensional cases. We show how the Kawamata Effective Nonvanishing conjecture implies the terminality result in Chapter 3 and, complementarily, we discuss the minimal necessary hypotheses for this argument to work. We begin by stating the conjecture in its generality:

Conjecture 5.1. (Kawamata nonvanishing) Let D be a numerically effective Cartier divisor on a normal projective variety X. If there exists an effective $\mathbb{Q}$-divisor B such that the pair (X, B) is Kawamata log terminal and such that the $\mathbb{Q}$-Cartier divisor $D - (K_X + B)$ is big and nef, then the bundle $\mathcal{O}_X(D)$ has a non-zero global section.

This implies the following results:

Proposition 5.2. Let X be a normal projective variety with at most canonical Gorenstein singularities, and let D be a Cartier divisor. Assume that Conjecture 5.1 holds, that $K_X \equiv 0$ and D is ample. Let $Y \in |D|$ be a general member. Then the pair (X, Y) is log-canonical. In particular, Y is slc.

If X is of dimension three, this is shown in [Kaw00, Prop 4.2] without assuming Conjecture 5.1. The main theorems we use throughout the proofs are stated in the preliminaries.

Proof. Remark that $|D|$ is not empty precisely because we apply the Nonvanishing Conjecture to D and the pair $(X, 0)$.

Suppose the pair is not lc and let $c < 1$ be the log-canonical threshold such that (X, cY) is properly lc. Let W be a minimal center of the pair (X, cY). Then by the Subadjunction formula for lc centers (Theorem 2.18), there exists an effective $\mathbb{Q}$-divisor B' on W such that $(K_X + cY)|_W \sim_{\mathbb{Q}} K_W + B'$ and (W, B') is klt.

As the divisor $D - (K_X + cY) = (1 - c)Y$ is ample, we may apply Theorem 2.17 to the pair (X, cY) and the minimal center W, thus obtaining that the restriction map

$$H^0(X, D) \to H^0(W, D|_W) \tag{5.1}$$

is surjective. We once again apply Conjecture 5.1, only this time to $D|_W$ and the pair (W, B'). Indeed, we have that

$$D|_W - (K_W + B') = D|_W - (K_X + cY)|_W = (1 - c)Y|_W,$$

and since this is ample we get that $H^0(W, D|_W) \neq 0$. Since the map (5.1) is surjective, this implies that W is not contained in the base locus of $|D|$, contradicting the fact that by Bertini's theorem

$Y_{sing} \subset \mathrm{Bs}|D|$ and that W is a minimal lc center for the pair (X, cY). Indeed, as $Y_{X \setminus \mathrm{Bs}|D|}$ is SNC, the identity map is a resolution of the pair $(X \setminus \mathrm{Bs}|D|, cY)$. Since $c < 1$, the pair is klt, therefore the non-klt locus is contained inside $\mathrm{Bs}|D|$, a contradiction. □

In order to show the next result, we need the following two statements, the first of which is also known as inversion of adjunction:

Theorem 5.3. [Kol97, Thm.7.5] Let X be normal and $S \subset X$ an irreducible Cartier divisor. Let B be an effective $\mathbb{Q} - divisor$ and assume that $K_X + S + B$ is $\mathbb{Q} - Cartier$. Then

- $(X, S + B)$ is plt near $S \Leftrightarrow (S, B|_S)$ is klt
- $(X, S + B)$ is lc near $S \Leftrightarrow (S, B|_S)$ is lc.

Lemma 5.4. [Flo13, Lemma 2.8] Let X be a normal variety and Δ a divisor such that (X, Δ) is klt. Let H be an ample Cartier divisor on X and $Y \in |H|$ a general element. Suppose that $(X, \Delta + Y)$ is not plt and let c be the log canonical threshold. Then the union of all centers of log canonicity of $(X, \Delta + cY)$ is contained in the base locus of $|H|$.

Proposition 5.5. Suppose that Conjecture 5.1 holds. Let X be a smooth Fano manifold. Then $H^0(X, -K_X) \neq 0$. Moreover, if D is a general member of the linear system $|-K_X|$, we have the following:

(a) D is integral and has at most canonical singularities

(b) D is smooth in codimension 2

(c) D has at most terminal singularities.

Proof. The first part of the statement immediately follows by applying Conjecture 5.1 for $D = -K_X$.

(a) We first prove that D has at most canonical singularities by following the proof of [Flo13, Prop. 4.1]. In this article, the result is stated in the case of Fano varieties of index $n - 3$, however the strategy holds in the general case.

Suppose that the statement is false. Then the pair $(D, 0)$ is not klt and by Theorem 5.3 this impiles that (X, D) is not plt. Let c be the log canonical threshold of (X, D). By Lemma 5.4, the pair (X, cY) is plt in the complement of the base locus of $|-K_X|$. Since (X, cD) is properly lc there exists a minimal center W.

Using Theorem 2.18, there exists an effective $\mathbb{Q}$-divisor B' on W such that $(K_X + cD)|_W \sim_{\mathbb{Q}} K_W + B'$ and (W, B') is klt. As $Y \in |-K_X|$ and $c < 1$, we obtain that the divisor $K_W + B'$ is anti-ample.

As the divisor $-K_X - (K_X + cD) = (2 - c)D$ is ample, we may apply Theorem 2.17 to the pair (X, cD) and the minimal center W, thus obtaining that the restriction map

$$H^0(X, -K_X) \to H^0(W, -K_X|_W) \tag{5.2}$$

is surjective. By applying Conjecture 5.1, only this time to $-K_X|_W$ and the pair (W, B'), and as the divisor $-K_X|_W - (K_W + B')$ is ample (being the sum of two ample divisors) we get that $H^0(W, -K_X|_W) \neq 0$. Since the map (5.2) is surjective, this implies that W is not contained in the base locus of $|-K_X|$, which is false by Lemma 5.4.

Assuming that D is not irreducible, let D_1 and D_2 be two of its components. Since $-K_X$ is ample, their intersection will be nonempty and as D_1 and D_2 are lc centers, so will $D_1 \cap D_2$. This contradicts the fact that the pair (X,D) is plt.

(b) This argument follows the proof of [HV11, Thm.1.6]. Consider the restriction sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-K_X) \to \mathcal{O}_D(-K_X) \to 0.$$

Since $h^1(X, \mathcal{O}_X) = 0$, we have a surjection

$$H^0(X, \mathcal{O}_X(-K_X)) \to H^0(D, \mathcal{O}_D(-K_X)).$$

By Conjecture 5.1, the right side is nonzero. Further, a general element of the linear system $|-K_X|_D|$ is obtained by intersecting D with another element $D' \in |-K_X|$. By Proposition 5.2 we know that for

$$S := D \cap D' \subset |-K_X|_D|$$

general, the pair (D, S) is log-canonical. In particular, S is reduced and so the singular locus of S has at least codimension 3 in X. By Bertini's Theorem we know that both the singular loci of the divisors D and D' are contained in the base locus of $|-K_X|$. In particular, they are contained in S and more precisely since S is a complete intersection cut out by D and D' we have

$$D_{sing} \subset S_{sing} \text{ and } D'_{sing} \subset S_{sing}.$$

Moreover, by inversion of adjunction (Theorem 5.3), the pair $(X, D + D')$ is log-canonical near the divisor D.

We will now argue by contradiction and suppose that a general element in $D \in |-K_X|$ is singular along a subvariety of codimension 2 in D. Take a general element Z in the pencil $\langle D, D' \rangle \subset |-K_X|$ spanned by D and D'. Then we have

$$Z \cap D = D \cap D' = S,$$

so as above we see that $Z_{sing} \subset S_{sing}$.

Since for a general member of the pencil $Z \in \langle D, D' \rangle$ we have

$$\mathrm{codim}_X Z_{sing} = 3 \text{ and } \mathrm{codim}_X D_{sing} \geq 3,$$

there exists a subvariety $V \subset S_{sing}$ such that $\mathrm{codim}_X V = 3$ and every general Z is singular along V. By upper semicontinuity of the multiplicity, this shows that both D and D' are singular along V.

Let $\mu : X' \to X$ be the blow-up of X along V. Since X is smooth along V, we have:

$$K_{X'} = \mu^* K_X + 2E,$$

where E is the exceptional divisor. Moreover, since D and D' are singular along V we have:

$$\mu^* D = \tilde{D} + aE \text{ and } \mu^* D' = \tilde{D}' + bE,$$

where a and b are at least two. Thus the pair $(X, D + D')$ is not log-canonical, a contradiction.

(c) If we assume Conjecture 5.1, the proof of Proposition 3.6 can be carried out identically in the arbitrary dimensional case. In fact, the only instance where the dimension hypothesis is used is when bounding the dimension of the log-canonical center in order for the Nonvanishing Theorem to apply. We will do the details in the more general Proposition 5.9.

We are therefore allowed to use condition (3.1) in order to prove that the singularities are terminal in arbitrary dimension by following the steps of the proof of Theorem 3.7. □

We further want to determine exactly how loose the hypotheses can be, given that in practice we do not need to apply the full generality provided by the Nonvanishing Conjecture in order to obtain terminality. In fact, if this conjecture is proven in two particular cases (for Fano and Calabi Yau varieties) as well as in dimension $n-2$, one can deduce that $D \in |-K_X|$ is terminal when X is a Fano manifold of dimension n. We list the complete statements for future reference:

Conjecture 5.6 (Effective nonvanishing for Fano varieties)**.** If D is an anticanonical divisor on an n-dimensional smooth projective Fano variety X, then the bundle $\mathcal{O}_X(D)$ has a non-zero global section.

Conjecture 5.7 (Effective nonvanishing for Calabi-Yau varieties)**.** Let D be an ample Cartier divisor on a normal projective Calabi-Yau variety Y with at most Gorenstein canonical singularities. Then the bundle $\mathcal{O}_X(D)$ has a non-zero global section.

Remark 5.8. Our principal tool in proving terminality is Proposition 3.6, which requires two anticanonical sections, it is natural to demand that $h^0(X, -K_X) \geq 2$. This condition automatically follows if we suppose that weaker versions of Conjecture 5.1 hold: the existence of an anticanonical divisor $D \in |-K_X|$ is the consequence of Conjecture 5.6 and by applying Conjecture 5.7 to the pair $(D, 0)$ and divisor $-K_X|_D$ we obtain that $h^0(D, -K_X|_D) \neq 0$. As X is Fano and all higher degree cohomology of its structure sheaf vanishes, we have the following exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, -K_X) \to H^0(D, -K_X|_D) \to 0.$$

We are therefore able to lift the section of $-K_X|_D$ from D to X. Moreover, since $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$, the condition $h^0(D, -K_X|_D) \geq 1$ implies exactly that $h^0(X, -K_X) \geq 2$.

Proposition 5.9. Let X be a smooth Fano manifold of dimension n. Suppose Conjectures 5.6 and 5.7 hold, as well as Conjecture 5.1 for varieties of dimension at most $n-2$. If a general $D \in |-K_X|$ is normal, then D has at most terminal singularities.

Proof. We follow the steps of Proposition 3.6. By Remark 5.8 we know that $h^0(X, -K_X) \geq 2$. Let D_1 and D_2 be two independent general divisors in $|-K_X|$. We first remark that if $H^0(X, -K_X) \neq 0$ we can identically repeat part of the proof of (a) in Proposition 5.5 in order to show that general elephants are irreducible in arbitrary dimension: any intersection of two irreducible components leads to a log canonical center of the plt pair (X, D), where $D \in |-K_X|$, a contradiction.

Step 1: We show that the pair $\left(X, c\frac{D_1+D_2}{2}\right)$ is klt for all $c < 2$.

Suppose by contradiction that this is false, then the log-canonical threshold of the pair $\left(X, \frac{D_1+D_2}{2}\right)$ is $c_0 < 2$. Denote by C a minimal log-canonical center of $\left(X, c_0\frac{D_1+D_2}{2}\right)$. In Proposition 3.6 we introduce this object in order to extend sections of $-K_X|_C$ to X and obtain a contradiction. The arbitrary dimensional equivalent of this will use the nonvanishing hypothesis of the statement, therefore the first condition to verify is $\dim C \leq n-2$.

Indeed, as Remark 3.4 holds in arbitrary dimension, we again have that the identity map is a log resolution of $\left(X \setminus \mathrm{Bs}|-K_X|, c_0\frac{D_1+D_2}{2}\right)$, therefore the pair is klt, showing that $C \subset \mathrm{Bs}|-K_X|$. Since D_1 and D_2 are irreducible, their intersection will be of codimension two, therefore

$$\mathrm{dim} C \leq \mathrm{dim} \mathrm{Bs}|-K_X| \leq n-2.$$

The hypotheses for applying Theorem 2.17 are independent of dimensions, thus we proceed identically as in Proposition 3.6 in order to obtain the surjective map

$$H^0(X, \mathcal{O}_X(-K_X)) \twoheadrightarrow H^0(C, \mathcal{O}_C(-K_X)).$$

As this is the zero map since $C \subset \mathrm{Bs}|-K_X|$, it remains to verify that the target is non-trivial, which we prove by applying the nonvanishing conjecture to C. Since C is a minimal center, it is normal, only has rational singularities, and moreover it is of dimension at most $n-2$ by the previous discussion. We need to produce a pair on C, which is generally done by applying Theorem 2.18. This implies the existence of an effective divisor $B \subset C$ such that the pair (C, B) is klt and

$$\left(K_X + c_0\frac{D_1 + D_2}{2}\right)|_C \sim_{\mathbb{Q}} K_C + B.$$

This numerical data allows us to verify the positivity condition in the statement of Conjecture 5.1 applied to the pair (C, B) and the divisor $-K_X|_C$, and as in Proposition 3.6, we obtain a contradiction because $H^0(C, \mathcal{O}_C(-K_X)) \neq 0$.

Let us now translate this into conditions on discrepancies. Set $\mu : X' \to X$ to be a resolution, let $E = \sum_{i=1}^{m} E_i$ be its exceptional locus and recall (cf. Notation 3.2) that a_i and r_i are the following coefficients:

- $|\mu^* D| = |D'| + \sum_{i=1}^{m} r_i E_i$, where D' is the strict transform of D and $r_i > 0$ for $i \in \{1 \dots m\}$,
- $K_{X'} = \mu^* K_X + \sum_{i=1}^{m} a_i E_i$, where $a_i > 0$, for all $i \in \{1 \dots m\}$.

The adjunction formula for a general elephant gives that:

$$K_{D'} = (\mu|_{D'})^* K_D + \sum_{i=1}^{m} (a_i - r_i)(E_i \cap D') \tag{5.3}$$

which means that the discrepancy of $(D, 0)$ is $\inf_i \{a_i - r_i \mid E_i \text{ is } \mu|_D\text{-exceptional}\}$. As the pair $\left(X, c\frac{D_1+D_2}{2}\right)$ is klt for all $c < 2$, we have that $a_i - cr_i > -1$, $\forall\, i \in \{1 \dots m\}$ and therefore

$$a_i + 1 \geq 2r_i,\ \forall\, i.$$

In particular, this means that D has at most canonical singularities, and the aim of what follows is to show that the discrepancy of the pair $(D, 0)$ is non-zero. The only case in which the above inequality doesn't imply $a_i - r_i > 0$ is if both a_i and r_i are equal to one. Note that since we have considered a log resolution, the intersection $E_i \cap D'$ is reduced for all $i \in \{1 \dots m\}$.

Step 2: We show that the case $a_i = r_i = 1$ does not come into play when computing the discrepancies on a general $D \in |-K_X|$, which completes the proof.

Since the coefficients do not depend on the choice of the resolution, we can assume that we have been working with one in which all blow-ups were made along smooth centers.

Claim: We can only obtain $a_i = 1$ for a given $i \in \{1 \dots m\}$ if $\mathrm{codim}_X \mu(E_i) = 2$.

Without loss of generality, we can assume we obtained this coefficient by doing the very last blow-up of the resolution, denoted by We have that

$$a_m = \lambda + \sum_{i=1}^{m-1} a_i \nu_i$$

where $\lambda = \mathrm{codim}_{X_{m-1}} \mu_m(E_m) - 1 \geq 1$ and $\nu_i > 0$ if and only if $\mu_m(E_m) \subset E_i$. Having $a_m = 1$ implies $\lambda = 1$ and $\nu_i = 0\ \forall\, i \in \{1 \dots m-1\}$. This proves the claim since the former condition shows that $\mu_m(E_m)$ is exactly of codimension two in X_{m-1}, while the latter signifies that ψ does not contract $\mu_m(E_m)$.

As $\operatorname{codim}_X \psi(E_i) = 2$ implies that the intersection $E_i \cap D'$ is not $\mu|_{D'}$-exceptional, this divisor does not contribute to the discrepancy of the pair $(D, 0)$ as computed in (5.3). Together with the above inequality, this proves that the discrepancy can never be zero, therefore a general elephant D has at most terminal singularities. $\square$

Note 5.10. In lower dimension, the number of sections in $H^0(X, -K_X)$ can be computed by hand. For instance, let us show that $h^0(X, -K_X) \geq 2$ if X is a Fano fourfold.

We use the Riemann-Roch formula to compute $\chi(X, -tK_X)$, which is the same as $h^0(X, -tK_X)$ by the Kodaira vanishing theorem. Using Serre duality we obtain an additional symmetry property: for every integer t we have that $\chi(X, -tK_X) = (-1)^4 \chi(X, K_X + tK_X) = \chi(X, (1+t)K_X)$.

In general, the Riemann-Roch formula for an n-dimensional projective variety X with at most Gorenstein terminal singularities and a Cartier divisor D is the following:

$$\chi(X, tD) = \frac{D^n}{n!} t^n + \frac{-K_X \cdot D^{n-1}}{2(n-1)!} t^{n-1} + \frac{(K_X^2 + c_2(X))D^{n-2}}{12(n-2)!} t^{n-2} + p(t) + \chi(X, \mathcal{O}_X),$$

where $p(t)$ is a polynomial of degree $n-3$ with no constant term.

As $\chi(X, \mathcal{O}_X) = 1$ for any Fano variety and in our case $\dim X = 4$, this becomes:

$$h^0(X, -tK_X) = \frac{(-K_X)^4}{4!} t^4 + \frac{(-K_X)^4}{2 \cdot 3!} t^3 + \frac{(-K_X)^4 + c_2(X)(-K_X)^2}{4!} t^2 + st + 1.$$

Using the symmetry property above for $t = 1$, we easily determine that $s = \frac{c_2(X) \cdot (-K_X)^2}{4!}$.

By Proposition 2.4 we have that $c_2 \cdot (-K_X) \cdot (-K_X) \geq 0$, therefore

$$h^0(X, -K_X) = (-K_X)^4 \left(\frac{2}{4!} + \frac{1}{2 \cdot 3!} \right) + \frac{2c_2(X) \cdot (-K_X)^2}{4!} + 1 \geq (-K_X)^4 \cdot \frac{1}{6} + 1 > 1.$$

Since $h^0(X, -K_X)$ is an integer, we obtain the conclusion.

Chapter 6

Examples in Weighted Projective Spaces

6.1 Introduction

We start with the definition of our main object, the weighted projective space or WPS. All notations follow [IF00]. Our goal is to build Fano varieties that can be described as complete intersections in these spaces, and to study the base loci of their anticanonical systems. The reason for doing this is that singularities of weighted projective spaces have a specific description as cyclic quotient singularities. Subvarieties of these spaces having certain properties, such as quasismoothness and well-formedness, only pick up the singularities of the ambient space, making them easier to analyze.

Definition 6.1 (WPS as a scheme, [IF00, Def.5.1])**.** Let $Q = (a_0, \dots, a_n)$ be an $n+1$-tuple of positive integers, which we refer to as the set of weights. Denote by $|Q|$ the integer $\sum a_i$ and by $S(Q)$ the polynomial ring $\mathbb{C}[T_0 \dots T_n]$ graded by $deg(T_i) = a_i$. The weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is the scheme defined by

$$\mathbb{P}(a_0, \dots, a_n) = \text{Proj S(Q)}.$$

Note 6.2. Let $x_0, \dots, x_n$ be the affine coordinates on $\mathbb{A}^{n+1}$ and let the group $\mathbb{C}^*$ act via:

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n).$$

Then $\mathbb{P}(a_0, \dots, a_n)$ is the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{C}^*$. Under this group action, $x_0, \dots x_n$ are the homogeneous coordinates on $\mathbb{P}(a_0, \dots a_n)$, which is an n-dimensional projective variety.

We now move to studying the singularities of these objects.

Definition 6.3 (Cyclic quotient singularity, [IF00, Def.5.13])**.** Let $r > 0$ and $a_1, \dots, a_n$ be integers and let $x_1, \dots, x_n$ be coordinates on $\mathbb{A}^n$. Suppose that μ_r acts on $\mathbb{A}^n$ via:

$$x_i \mapsto \varepsilon^{a_i} x_i \text{ for all i},$$

where ε is a fixed primitive rth root of unity. Given a normal variety X, we say that singularity $Q \in X$ is a quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$ if (X, Q) is isomorphic to an analytic neighborhood of $(\mathbb{A}^n, 0)/\mu_r$.

Notation 6.4 ([IF00, Not.5.14]). Write $P_i \in \mathbb{P}(a_0, \dots, a_n)$ for the point $[0 : \dots : 0 : 1 : 0 : \dots : 0]$, where the 1 is in the ith position. We call P_i a vertex, the 1-dimensional toric stratum P_iP_j an edge etc. The fundamental simplex (that is, the union of all the coordinate hyperplanes $P_0 \dots \hat{P}_i \dots P_n$, where we use hats to denote vertices that are omitted) will be denoted by Δ.

Remark 6.5 (The singular locus of $\mathbb{P}(a_0, \dots, a_n)$, [IF00, 5.15]).
Define $h_{i_1,i_2,\dots,i_k} = hcf(a_{i_1}, a_{i_2}, \dots, a_{i_k})$. The vertex P_j is a singularity of type $\frac{1}{a_j}(a_0, \dots, \widehat{a_j}, \dots a_n)$, which is not necessarily isolated. Each generic point P of the edge $P_{i_1}P_{i_2}$ has an analytic neighborhood $P \in U$ which is analytically isomorphic to $(0, Q) \in \mathbb{A}^1 \times Y$, where $Q \in Y$ is a singularity of type $\frac{1}{h_{i_1,i_2}}(a_0 \dots \widehat{a_{i_1}} \dots \widehat{a_{i_2}} \dots a_n)$. Similar results hold for higher dimensional toric strata, however singularities may only occur on the fundamental simplex Δ.

Proof. We only deal with the situation of vertices and edges, the general case is an immediate consequence of this discussion.

The vertex case is almost tautological. Suppose for simplicity that we want to study the type of singularity we have at the vertex $P_n \in \mathbb{P}$. Now P_n is the origin of the chart

$$U_n = \{[x_0 : \dots : x_n] \in \mathbb{P}(a_0 \dots a_n) |\ x_n = 1\},$$

so in order to finish the proof we show that $U_n = \mathbb{A}^n/\mu_{a_n}$. Recall that $\mathbb{P}(a_0, \dots, a_n)$ is described by the action

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n),$$

thus when setting the last coordinate equal to one we still have to quotient by the residual action of μ_{a_n} on the first n coordinates of each point.

Now consider $P = [0 : \dots 0 : p_{n-1} : p_n]$ to be a general point of the edge $P_{n-1}P_n$. Then P is the origin of the chart

$$U := \{[x_0 : \dots : x_n] \in \mathbb{P}(a_0, \dots, a_n) |\ x_n = 1,\ x_{n-1} = \frac{p_{n-1}}{p_n^{a_n/a_{n-1}}}\}.$$

Next we need to determine an action of a certain cyclic group μ_r such that $U \simeq \mathbb{A}^{n-1}/\mu_r$. Since we built U by intersecting U_n with another chart, we can work with the residual action of μ_{a_n} introduced in the vertex case. As before, if ε is a primitive a_n-th root of unity, U_n is the quotient of $\mathbb{A}^n$ by the action

$$\varepsilon \cdot (x_0 \dots x_{n-1}, 1) = (\varepsilon^{a_0} x_0, \dots, \varepsilon^{a_{n-1}} x_{n-1}, 1).$$

If we fix the $(n-1)$-th coordinate equal to $\frac{p_{n-1}}{p_n^{a_n/a_{n-1}}}$, we again recover an action on the first $n-2$ coordinates, namely that of $\varepsilon \in \mu_{a_n}$ such that $\varepsilon^{a_{n-1}} = 1$. As $\mu_{a_{n-1}} \cap \mu_{a_n} = \mu_{hcf(a_{n-1},a_n)} = \mu_{h_{n-1,n}}$, we have obtained that P is a singularity of type $\frac{1}{h_{n-1,n}}(a_0, \dots, a_{n-2})$. $\square$

What follows is a list of useful properties of the Fano complete intersections that we construct. Varieties with these properties can only inherit the singularities of the ambient space, which we can later easily manipulate. All definitions have equivalents for general subvarieties $X \in \mathbb{P}$ (see [IF00, §6]), however for us X is always a weighted complete intersection, which in many cases allows for a definition that is easier to check.

Definition 6.6 (Weighted complete intersection, [IF00, Def.6.4]). Let I be a homogeneous ideal of the graded ring S and define X_I to be

$$X_I = Proj(S/I) \subset \mathbb{P}.$$

Suppose furthermore that I is generated by a regular sequence $\{f_i\}$ of homogeneous elements of S. Then $X_I \subset \mathbb{P}$ is called a weighted complete intersection of multidegree $\{d_i = \deg f_i\}$. In this case, we denote by $X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n)$ a general element of the family of all weighted complete intersections of multidegree $\{d_i\}$.

The generality of such a $X_{d_1,\dots,d_c}$ is fundamental throughout the chapter, in particular for the following definition:

Definition 6.7 (Linear cone, [IF00, Def.6.5])**.** Using the notation in Definition 6.6, if there exist $i \in \{1 \dots c\}$ and $j \in \{0 \dots n\}$ such that $d_i = a_j$, we say that $X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n)$ is a linear cone. It is clear that in this case the weighted complete intersection is isomorphic to an intersection of lower codimension, that is $X_{d_1,\dots,\widehat{d_i},\dots d_c} \subset \mathbb{P}(a_0,\dots,\widehat{a_j},\dots a_n)$, or possibly a weighted projective space.

Definition 6.8 ([IF00, Def.6.1])**.** Let X be a closed subvariety of a weighted projective space $\mathbb{P}$ and $p : \mathbb{A} \setminus \{0\} \to \mathbb{P}$ the canonical projection. The punctured affine cone C_X^* over X is given by $C_X^* = p^{-1}(X)$ and the affine cone C_X over X is the completion of C_X^* in $\mathbb{A}^{n+1}$.

Notice that $\mathbb{C}^*$ acts on C_X^* to give $X = C_X^*/\mathbb{C}^*$.

Definition 6.9 ([IF00, Def.6.3])**.** Let X be a closed subvariety in $\mathbb{P}$. We say that it is quasismooth of dimension m if its affine cone C_X is smooth of dimension $m+1$ outside its vertex $\underline{0}$.

All our weighted complete intersections will be either of codimension one or two. We discuss the notion of quasismoothness in these two cases.

Remark 6.10. If $X \subset \mathbb{P}$ is quasismooth, its singularities are due to the $\mathbb{C}^*$-action and hence are cyclic quotient singularities. We will apply this property to the general elephant, since the ambient Fano varieties that we construct will be smooth.

Definition 6.11 (Quasi-smoothness for hypersurfaces, [IF00, Thm.8.1])**.** The general hypersurface X_d in $\mathbb{P}(a_0,\dots,a_n)$ of degree d, where $n \geq 1$ is quasismooth if and only if at least one of the following holds:

1. there exists a variable x_i for some $i \in \{1 \dots n\}$ of weight d (therefore X_d is a linear cone),
2. for each $k \in \{1 \dots n\}$ and every nonempty subset $I = \{i_0,\dots,i_{k-1}\}$ of $\{0,1,\dots,n\}$, we have:
   (a) there exists a monomial $x_I^M = x_{i_0}^{m_0} \dots x_{i_{k-1}}^{m_{k-1}}$ of degree d, or
   (b) for $\mu = 1 \dots k$, there exist monomials $x_I^M x_{e_\mu} = x_{i_0}^{m_0} \dots x_{i_{k-1}}^{m_{k-1}} x_{e_\mu}$ of degree d, where $\{e_\mu\}$ are k distinct elements.

Definition 6.12 (Quasi-smoothness in codimension two, [IF00, Def.8.7])**.** The general codimension 2 weighted complete intersection X_{d_1,d_2} in $\mathbb{P}(a_0,\dots,a_n)$ of multidegree $\{d_1,d_2\}$, where $n \geq 2$ is quasismooth if and only if at least one of the following holds:

1. there exists a variable x_i for some $i \in \{1 \dots n\}$ of weight either d_1 or d_2, in which case X_{d_1,d_2} is a linear cone, therefore we refer to Definition 6.11 to discuss its quasismoothness
2. for each $k \in \{1 \dots n\}$ and every nonempty subset $I = \{i_0,\dots,i_{k-1}\}$ of $\{0,1,\dots,n\}$, we have either:

(a) there exists a monomial $x_I^{M_1}$ of degree d_1 and there exists a monomial $x_I M_2$ of degree d_2

(b) there exists a monomial $x_I^{M_1}$ of degree d_1 and for $\mu = 1 \dots k$, there exist monomials $x_I^{M_2} x_{e_\mu}$ of degree d_2

(c) there exists a monomial x_i^M of degree d_2 and for $\mu = 1 \dots k-1$ there exist monomials $x_I^{M_\mu} x_\mu$ of degree d_1, where $\{e_\mu\}$ are $k-1$ distinct elements

(d) for $\mu = 1 \dots k$ there exist monomials $x_I^{M_\mu^1} x_\mu^1$ of degree d_1 and $x_I^{M_\mu^2} x_\mu^2$ of degree d_2, such that $\{e_\mu^1\}$ are k distinct elements, $\{e_\mu^2\}$ are k distinct elements and $\{e_\mu^1, e_\mu^2\}$ contains at least $k+1$ distinct elements.

We now introduce a necessary notion for applying the adjunction formula for a weighted complete intersection.

Definition 6.13 (Well formed WPS, [IF00, Def.5.22])**.** The weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is well formed if

$$\mathrm{hcf}(a_0, \dots, \widehat{a_i}, \dots, a_n) = 1 \text{ for every i} \in \{0 \dots n\}.$$

Even if a weighted projective space is not well formed, one can easily find a well formed weighted projective space isomorphic to it. Indeed, if $a_0, \dots, a_n$ is the set of weights and $q = \mathrm{hcf}(a_1, \dots, a_n)$ then by [IF00, Lem.5.7] we have $\mathbb{P}(a_0, \dots, a_n) = \mathrm{Proj}\ \mathrm{S}(a_0, \dots, a_n) \simeq \mathrm{Proj}\ \mathrm{S}(a_0, a_1/q \dots, a_n/q)$. It is therefore sufficient to only consider well formed weighted projective spaces in this thesis.

Definition 6.14 (Well-formedness of WCI, [IF00, Def.6.12])**.** The weighted complete intersection

$$X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0, \dots, a_n)$$

is well formed if and only if

1. $\mathbb{P}(a_0, \dots, a_n)$ is well formed
2. for all $\mu \in \{1, \dots, c\}$, the highest common factor of any $(n-1-c+\mu)$ of the $\{a_i\}_{i \in \{0 \dots n\}}$ must divide at least μ of the $\{d_j\}_{j \in \{1 \dots c\}}$.

Theorem 6.15 (The Adjunction Formula, [IF00, Def.6.1])**.** If a weighted complete intersection $X_{d_1,\dots,d_c}$ in $\mathbb{P}(a_0, \dots, a_n)$ is well formed and quasismooth, then $\omega_X \simeq \mathcal{O}_X(\sum d_i - \sum a_j)$. We define the amplitude to be this difference of sums, usually denoting it by α.

In particular, this formula shows that for a Fano weighted complete intersection we have $\sum a_j - \sum d_i > 0$.

Example 6.16 ([IF00, Ex.6.15])**.** Here is an illustrative example showing that well-formedness is mandatory to applying the adjunction formula: consider the surface $S_9 \subset \mathbb{P}(1, 2, 2, 3)$. This surface is both quasismooth and nonsingular, and if it is well formed then $\alpha = 1$, therefore $K_S^2 = \frac{3}{4}$. But the degree of a nonsingular surface is always an integer, therefore we have obtained a contradiction.

However, the following result shows that well-formedness only needs to be checked for dimensions one and two:

Theorem 6.17 ([IF00, Def.6.17])**.** Let $X = X_{d_1,\dots,d_c}$ in $\mathbb{P}(a_0, \dots, a_n)$ be a quasismooth weighted complete intersection of dimension greater than 2. Then:

1. either X is well formed

2. or X is the intersection of a linear cone with other hypersurfaces (that is, $a_i = d_\lambda$ for some i and λ).

Remark 6.18. In case 2, the weighted complete intersection is isomorphic to either a complete intersection of lower codimension, that is, $X_{d_1,\dots,\widehat{d_\lambda},\dots d_c}$ in $\mathbb{P}(a_0,\dots,\widehat{a_i},\dots,a_n)$, or possibly a weighted projective space.

Finally, we have a combinatorial relation between the degrees of the complete intersection and the weights of the ambient projective space. We now have the necessary tools to produce relevant examples of nontrivial base loci of anticanonical systems on Fano fourfolds.

Lemma 6.19 ([IF00, Lm.18.14])**.** Let $X_{d_1\dots d_c}$ in $\mathbb{P}(a_0,\dots,a_n)$ be a quasismooth weighted complete intersection, but not an intersection of a linear cone with other hypersurfaces. Suppose also that $d_1,\dots,d_c$ and $a_0,\dots,a_n$ are in increasing order. Then:

(i) $d_c > a_n,\ d_{c-1} > a_{n-1},\dots,$ and $d_1 > a_{n-c+1}$

(ii) if $d_{c-1} < a_n$, then $a_n | d_c$.

Proof. (i) Let f_i be local equations for X_{d_i}, $i \in \{1\dots c\}$. Suppose by contradiction that $d_c > a_n,\dots,d_{c-k+1} > a_{n-k+1}$ and $d_{c-k} < a_{n-k}$ for some $k \in \{0\dots c-1\}$. Then $d_i < a_{n-k}$ for all $i \le c-k$, i.e. the polynomials $f_1,\dots,f_{n-k}$ do not involve the variables $x_{n-k},\dots,x_n$.

If Π is the coordinate $(k+1)$-plane in $\mathbb{A}^{n+1}$ given by

$$x_0 = \ldots = x_{n-k-1} = 0,$$

then $f_1,\dots,f_{n-k}$ are identically zero on Π. Let $Z = (f_{c-k+1} = \ldots = f_c = 0) \cap \Pi$. We have that $\dim Z \ge 1$ and so $Z \setminus \underline{0} \ne \emptyset$. Let $Q \in Z \setminus \underline{0}$, then $\partial f_i/\partial x_j$ are zero at Q for all $i \le c-k$ and for all $j \in \{0\dots n\}$. Therefore

$$\operatorname{rank}\begin{pmatrix} \partial f_1/\partial x_0(Q) & \dots & \partial f_1/\partial x_n(Q) \\ \vdots & & \vdots \\ \partial f_c/\partial x_0(Q) & \dots & \partial f_c/\partial x_n(Q) \end{pmatrix} \le k-c.$$

Thus $Q \in C_X^*$ is singular and so X is not quasismooth.

Case (ii) is treated likewise. □

We also need to compute the cohomology of these weighted complete intersections. We start with that of the ambient projective space:

Lemma 6.20 ([Dol82, §1.4])**.**

- $H^i(\mathbb{P},\mathcal{O}_{\mathbb{P}}(k)) = 0$ for $i \ne 0,n,\ k \in \mathbb{Z}$.
- $H^n(\mathbb{P},\mathcal{O}_{\mathbb{P}}(k)) = S_{-n-|Q|}$, where S is the polynomial ring with variables of weights Q.

Lemma 6.21 ([Dol82, §.3.4.3])**.** Let $X = X_{d_1,\dots,d_c} = V(f_1,\dots,f_c) \subset \mathbb{P}(a_0,\dots,a_n)$ be a well-formed quasismooth weighted complete intersection.

Let A be the graded ring $S(a_0,\dots,a_n)/(f_1,\dots,f_c)$ and A_n be the n-th graded part of A. Then

$$H^i(X,\mathcal{O}_X(n)) \simeq \begin{cases} A_n & \text{if } i=0 \\ 0 & \text{if } i = 1,\dots,\dim X - 1 \\ A_{-n-\alpha} & \text{if } i = \dim X \end{cases}$$

for all $n \in \mathbb{Z}$.

In particular, Lemma 6.20 shows the following: if $X = X_{d_1,\dots,d_c} \subset \mathbb{P}$ is a Fano quasismooth weighted complete intersection that is not a linear cone, consider the elementary short exact sequence:

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0.$$

If we twist this with $\mathcal{O}_{\mathbb{P}}(\sum a_i - \sum d_i)$ and use the previous cohomology results as well as the Adjunction Formula 6.15, we obtain that $H^0(X, \mathcal{O}_X(-K_X)) \simeq H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\sum a_i - \sum d_i))$ provided that $\sum a_i - 2\sum d_i < 0$. This will hold for all cases in which we analyze the base locus of $|-K_X|$ throughout the chapter.

6.2 The hypersurface case

We start by determining all possibilities for a smooth Fano fourfold X to be a hypersurface in a weighted projective space, that is we find the degree of the fourfold and the weights of the ambient space. We then study the base loci of $|-K_X|$.

First of all, in this discussions we will not consider the case of linear cones. If a degree d hypersurface $X \subset \mathbb{P}$ is a linear cone, then automatically X is isomorphic to a weighted projective space. As we want X to be a smooth Fano fourfold, we deduce that $X \simeq \mathbb{P}^4$, the only non-singular weighted projective space of dimension four. But the anticanonical system of this space has an empty base locus, thus providing no interesting new examples.

Theorem 6.22. There are exactly 10 families of smooth Fano 4-fold weighted hypersurfaces that are not linear cones:

$$\begin{gathered} X_2,\ X_3,\ X_4,\ X_5 \in \mathbb{P}(1,1,1,1,1,1) \\ X_4,\ X_6 \in \mathbb{P}(1,1,1,1,1,2) \\ X_6 \in \mathbb{P}(1,1,1,1,1,3) \\ X_8 \in \mathbb{P}(1,1,1,1,1,4) \\ X_6 \in \mathbb{P}(1,1,1,1,2,3) \\ X_{10} \in \mathbb{P}(1,1,1,1,2,5). \end{gathered}$$

Proof of Theorem 6.22. Let X be a degree d hypersurface in $\mathbb{P}(a_0, \dots, a_5)$. By the Adjunction Formula [Theorem 6.15], a Fano variety X must verify the following condition:

$$\sum a_i - d > 0. \tag{6.1}$$

We will use the following

Remark 6.23. We deduce some general smoothness conditions for complete intersections:

1. As X is smooth, it must avoid the singular locus of $\mathbb{P}(a_0, \dots, a_5)$ and in particular the equation of X should not be verified by the coordinates of the vertices introduced in Remark 6.5. If we denote by f the defining function of X, this implies that for each $i \in \{0 \dots 5\}$ for which $a_i > 1$, f contains a monomial of the type x_i^m. As X is a hypersurface of degree d, we get that $a_i | d$ for all $i = 0 \dots 5$. The proof is identical in the general n-dimensional case.

2. By Remark 6.5, any common factor of at least two weights will produce a singular simplex Δ in $\mathbb{P}(a_0, \dots, a_5)$. As X_d is Cartier since $\mathrm{lcm}(\mathrm{a}_0 \dots \mathrm{a}_5)|d$ and the linear system $|X_d|$ is ample, one of its general elements will always intersect such a simplex (even in the case where Δ is a line), thus inducing a singularity on X. We impose the condition that all weights a_i are pairwise coprime in order to obtain a smooth hypersurface.

For the following, we assume a_i are in increasing order. Using Remark 6.23, for all $i = 0 \dots 5$ we have that $a_i | d$ and all a_i are pairwise coprime. By condition (6.1) and Lemma 6.19, we deduce that

$$6a_5 > d \geq 2a_5.$$

So $d = \lambda a_5$, where $\lambda \in \{2, 3, 4, 5\}$ and condition (6.1) transforms into:

$$(a_0 + \dots + a_4) - (\lambda - 1)a_5 > 0. \tag{6.2}$$

As the $\{a_i\}$ are pairwise coprime then $\prod_{i=0}^{5} a_i | d$ and so $\prod_{i=0}^{4} a_i | \lambda$. We have the following cases:

(i) $\lambda = 5$. Then either $(a_0, \dots, a_4) = (1, 1, 1, 1, 1)$ and by condition (6.2) we get $a_5 = 1$, or $(a_0, \dots, a_4) = (1, 1, 1, 1, 5)$ and $a_5 \leq 2$, which is impossible by the initial convention on the order of the a_i.

(ii) $\lambda = 4$. The only possibilities are:

   (a) $(a_0, \dots, a_4) = (1, 1, 1, 1, 1)$ and $a_5 = 1$

   (b) $(a_0, \dots, a_4) = (1, 1, 1, 1, 2)$ and $a_5 < 2$

   (c) $(a_0, \dots, a_4) = (1, 1, 1, 1, 4)$ and $a_5 < 3$,

   and the last two also contradict the convention on the increasing order on the a_i.

(iii) $\lambda = 3$. Either $(a_0, \dots, a_4) = (1, 1, 1, 1, 1)$ and $a_5 \in \{1, 2\}$ or $(a_0, \dots, a_4) = (1, 1, 1, 1, 3)$ and $a_5 \leq 3$ (which contradicts the coprime condition).

(iv) $\lambda = 2$. Either $(a_0, \dots, a_4) = (1, 1, 1, 1, 1)$ and $a_5 \in \{1, 2, 3, 4\}$ or $(a_0, \dots, a_4) = (1, 1, 1, 1, 2)$ and $a_5 \in \{3, 5\}$.

We are left with exactly the hypersurfaces appearing in the statement. These are automatically quasi-smooth since from the very beginning we required that $a_i | d$. □

We consider each case separately and find the base loci (if nonempty) of their anticanonical systems.

1. $X_{10} \in \mathbb{P}(1, 1, 1, 1, 2, 5)$ is a Fano hypersurface with $-K_X = \mathcal{O}_X(1)$, thus the restriction $Y|_X$, where $Y \in |-K_X|$ is a general element, is isomorphic to a degree 10 hypersurface in $\mathbb{P}(1, 1, 1, 2, 5)$. By further intersecting with elements in the same linear system, we obtain that the base locus is zero-dimensional and of degree 10, given by an equation of the type $x_5^2 + x_4^5 = 0$ in $\mathbb{P}(2, 5)$.

2. $X_6 \in \mathbb{P}(1, 1, 1, 1, 2, 3)$ is a Fano hypersurface with $-K_X = \mathcal{O}_X(3)$, and by applying the same method as before we obtain that $Y|_X$ is isomorphic to a degree 6 hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$. However, the linear system of such hypersurfaces is very ample, since by [BR86, Lemma 4B.6] it is enough to check that the algebra $\oplus \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k \cdot 6H))$ is generated by $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6H))$.

3. $X_8 \in \mathbb{P}(1, 1, 1, 1, 1, 4)$ is a Fano hypersurface and $\mathcal{O}_X(-K_X) = \mathcal{O}_X(1)$. However, here the anticanonical system will have an empty base locus, as by Lemma 6.21 we have $h^0(X, \mathcal{O}_X(1)) = \dim(\mathbb{C}[x_0 \dots x_5]/(f_8))_1 = 5$, where f_8 is a local equation of X_8 and $\mathbb{C}[x_0 \dots x_5]$ is the graded polynomial ring corresponding to $\mathbb{P}(1, 1, 1, 1, 1, 4)$. This means that $|-K_X|$ has empty base locus.

4. $X_6 \in \mathbb{P}(1,1,1,1,1,3)$ has $-K_X = \mathcal{O}_X(2)$ and it is clear that the base locus of $\mathcal{O}_{\mathbb{P}}(2)$ is the point $[0 : \ldots : 0 : 1]$, which doesn't belong to X_6.

5. $X_6 \in \mathbb{P}(1,1,1,1,1,2)$ has $-K_X = \mathcal{O}_X(1)$ and the base locus of $\mathcal{O}_{\mathbb{P}}(1)$ is $[0 : \ldots : 0 : 1] \notin X_6$.

6. $X_4 \in \mathbb{P}(1,1,1,1,1,2)$ has $-K_X = \mathcal{O}_X(3)$, which is very ample by [BR86, Lemma 4B.6].

6.3 Complete intersection of two hypersurfaces

An immediate corollary of Definition 6.12 is the following:

Corollary 6.24 ([IF00, Cor.8.8])**.** Suppose X_{d_1,d_2} in $\mathbb{P}$ is quasismooth and is not the intersection of a linear cone with another hypersurface. We have the following:

(i) Every variable x_i occurs in at least one of the defining equations.

(ii) All but at most one variable are in both equations.

(iii) If x_i does not appear in one defining equation, then there is a monomial x_i^m occurring in the other equation.

6.3.1 The examples

Example 6.25. $X_{6,10} \subset \mathbb{P}(1,1,1,2,2,5,5)$.

This complete intersection is a Fano variety with $-K_X = \mathcal{O}_X(1)$. Though quasismooth, it will not be smooth since X_6 contains the singular line $\{x_0 = \ldots = x_4 = 0\}$, while X_{10} is very ample on $\mathbb{P}$. As sections of $\mathcal{O}_X(1)$ only involve the first three variables, the base locus of $|-K_X|$ is isomorphic to the curve $C_{6,10} \subset \mathbb{P}(2,2,5,5)$. In this projective space, the linear system of degree ten hypersurfaces $|C_{10}|$ is free, while in degree six all hypersurfaces contain $\mathbb{P}(5,5)$ (since any degree six equation cannot involve variables of weight five). This is precisely the base locus of $|C_6|$, since it includes the intersection of $C' : x_0^3 = 0$ and $C'' : x_1^3 = 0$, implying that it includes the entire base locus.

By Bertini's Theorem we have that $C_{6,sing} \subset \mathbb{P}(5,5)$, therefore $C_{6,10,sing} \subset X_{10} \cap \mathbb{P}(5,5)$, which is a finite number of points satisfying the equation

$$x_2^2 + x_3^2 + x_2x_3 = 0$$

As this base locus is not quasismooth, we cannot apply the adjunction theorem in order to find its canonical class. A general element in $|C_6|$ has the equation

$$x_0^3 + x_1^3 + x_0^2x_1 + x_0x_1^2 = 0 \Leftrightarrow (x_0 + x_1)(x_0 + ix_1)(x_0 - ix_1) = 0,$$

thus each member has three components that are part of $|C_2|$. Each of these in turn will give a quasismooth intersection with C_{10}, and its canonical class will be $\mathcal{O}_{\mathbb{P}(2,2,5,5)}(-2)$, which means we have obtained three rational curves.

Example 6.26. $X_{6,6} \subset \mathbb{P}(1,1,1,2,2,3,3)$.

This is a Fano manifold with $-K_X = \mathcal{O}_X(1)$, and similarly to the previous example it is easy to deduce that $Bs|-K_X| \simeq X_{6,6} \subset \mathbb{P}(2,2,3,3)$. On the other hand, here the base locus is smooth, as it is easy to show that the two singular lines in $\mathbb{P}(2,2,3,3)$ aren't cut at the same point by both general surfaces of degree six. This allows the use of the adjunction formula Theorem 6.15 and we deduce that $K_{Bs|-K_X|} = \mathcal{O}_{\mathbb{P}}(2)$.

Remark 6.27. If we impose the condition that both X_{d_1} and X_{d_2} be Cartier divisors on $\mathbb{P}(a_0, \ldots, a_6)$ such that X_{d_1,d_2} is not a linear cone, one needs to verify the following two relations:

$$d_1 + d_2 \leq a_0 + \ldots a_6$$
$$\operatorname{lcm}(a_0, \ldots, a_6) \,|\, \operatorname{hcf}(d_1, d_2),$$

where the first is the necessary for X_{d_1,d_2} to be Fano and the second is the Cartier hypothesis, since $\operatorname{Pic}(\mathbb{P}(a_0, \ldots, a_n)) = \mathbb{Z}$ is generated by $[\mathcal{O}_{\mathbb{P}}(\operatorname{lcm}(a_0, \ldots, a_6))]$ by [BR86, Thm7.1]. A straightforward computation using Lemma 6.19 shows that the only occurring case of a smooth X_{d_1,d_2} is that of Example 6.26.

Chapter 7

Del Pezzo surfaces with $1/3(1,1)$ singularities

7.1 Introduction

We classify non-smooth del Pezzo surfaces with $\frac{1}{3}(1,1)$ points in precisely 29 qG-deformation families. We structure the classification into six unprojection cascades, determine their biregular invariants and their directed MMP together with a distinguished configuration of negative curves on the minimal resolution. This overlaps with work of Fujita and Yasutake [FY].

The classification is summarised in Table 7.2 and Table 7.3, which also plot invariants and provide good model constructions of surfaces in all families. With the exception of two surfaces, the constructions are degeneration loci in simplicial toric varieties. In Section 7.5 we display the construction models and explain the computations needed to verify that they coincide with the surfaces we classified in Theorem 7.6 and Corollary 7.8. We refer to [CH] for details on the cases that are not complete intersections in toric varieties.

This work is part of a program to understand mirror symmetry for orbifold del Pezzo surfaces [ACC+16, KNP, OP, Tve, Pri, CKP] and it is aimed specifically at giving evidence for the conjectures made in [ACC+16].

7.1.1 The results

The classification and its cascade structure

Definition 7.1. A $\frac{1}{n}(a,b)$ *point* is a surface cyclic quotient singularity $\mathbb{C}^2/\mu_n$ where μ_n acts linearly on $\mathbb{C}^2$ with weights $a,\, b \in \left(\frac{1}{n}\mathbb{Z}\right)/\mathbb{Z}$. We always assume no stabilisers in codimension 0, 1, that is, $\operatorname{hcf}(a,n) = \operatorname{hcf}(b,n) = 1$.

A *del Pezzo surface* is a surface X with cyclic quotient singularities and $-K_X$ ample.

The *Fano index* of X is the largest positive integer $f > 0$ such that $-K_X = fA$ in the Class group $\operatorname{Cl} X$.

Remark 7.2. We view a del Pezzo surface X with quotient singularities as a variety. Such a surface is in a natural way a smooth orbifold (or DM stack), but we mostly ignore this structure. Thus for us $\operatorname{Cl} X$ is the Class group of Weil divisors on X modulo linear equivalence. In particular, although K_X is a Cartier divisor on the orbifold, we think of it as a $\mathbb{Q}$-Cartier divisor on the underlying variety (the coarse moduli space of the orbifold) and then to say that it is ample is to say that a positive integer multiple is Cartier and ample.

See [ACC+16] for a discussion of qG-deformations of del Pezzo surfaces with cyclic quotient singularities. In particular, it is explained there that the singularity $\frac{1}{3}(1,1)$ is qG-rigid and the degree $d = K^2$ is locally constant in qG-families.

We classify qG-deformation families of del Pezzo surfaces with $k \geq 1$ $\frac{1}{3}(1,1)$ points. What is more, [CH, §7] shows that with the exception of $X_{4,\,1/3}$, $X_{5,\,2/3}$ and $X_{6,\,1}$ (all of which have $h^0(X, -K_X) = 0$), all other families admit a qG-degeneration to a toric surface.

The following two theorems are the main results of this chapter:

Theorem 7.3. There are precisely 3 qG-deformation families of del Pezzo surfaces with $k \geq 1$ $\frac{1}{3}(1,1)$ points and Fano index $f \geq 2$. They are:

(1) $S_{1,\,25/3} = \mathbb{P}(1,1,3)$ with $k = 1$, $K^2 = \frac{25}{3}$ and $f = 5$;

(2) $B_{1,\,16/3}$: the family of weighted hypersurfaces $X_4 \subset \mathbb{P}(1,1,1,3)$ with $k = 1$, $K^2 = \frac{16}{3}$ and $f = 2$;

(3) $B_{2,\,8/3}$: the family of weighted hypersurfaces $X_6 \subset \mathbb{P}(1,1,3,3)$ with $k = 2$, $K^2 = \frac{8}{3}$ and $f = 2$.

Theorem 7.4. There are precisely 26 qG-deformation families of del Pezzo surfaces with $k \geq 1$ $\frac{1}{3}(1,1)$ points and Fano index $f = 1$. The numerical invariants of these surfaces are shown in Table 7.3 in Section 7.5. In that table $X_{k,\,d}$ denotes the unique family with k $\frac{1}{3}(1,1)$ points, $K^2 = d$ and $f = 1$. The table also gives a good model construction of a surface X in all families.

Next we discuss the finer structure of the classification.

Definition 7.5. A negative curve on X is a compact curve $C \subset X$ with negative self-intersection number $C^2 < 0$. We say that a projective curve $C \subset X$ is a $(-m)$-curve if $C^2 = -m$. Note that in general $m \in \mathbb{Q}$. Let $P_1, \ldots, P_k \in X$ be the singular points and denote by $X^0 = X^{\text{nonsing}} = X \setminus \{P_1, \ldots P_k\}$ the nonsingular locus of X. A (-1)-curve $C \subset X^0$ is called a *floating* (-1)-curve.

Theorem 7.3 and Theorem 7.4 are a straightforward logical consequence of the minimal model program and the following, which is proved in Section 7.4:

Theorem 7.6. Let X be a del Pezzo surface with $k \geq 1$ $\frac{1}{3}(1,1)$ points. If X has no floating (-1)-curves, then X is one of the following surfaces, all constructed in Table 7.2 and 7.3 and in the statement and proof of Theorem 7.21:

(1) $k = 1$ and either X is a surface of the family of weighted hypersurfaces $B_{1,\,16/3} = X_4 \subset \mathbb{P}(1,1,1,3)$, or $X = S_{1,\,25/3} = \mathbb{P}(1,1,3)$. Here $B_{1,\,16/3}$ and $S_{1,\,25/3}$ are surfaces of anticanonical degrees $\frac{16}{3}$ and $\frac{25}{3}$ respectively, both containing exactly one singular point of type $\frac{1}{3}(1,1)$;

(2) $k = 2$ and either $X = X_{2,\,17/3}$, or X is a surface of the family of weighted hypersurfaces $B_{2,\,8/3} = X_6 \subset \mathbb{P}(1,1,3,3)$, where $B_{2,\,8/3}$ is a surface of anticanonical degree $\frac{8}{3}$ containing two singular points of type $\frac{1}{3}(1,1)$;

(3) $k = 3$ and $X = X_{3,\,5}$;

(4) $k = 4$ and $X = X_{4,\,7/3}$;

(5) $k = 5$ and $X = X_{5,\,5/3}$;

(6) $k = 6$ and $X = X_{6,\,2}$.

Remark 7.7. With the exception of families $B_{1,\,16/3}$, $B_{2,\,8/3}$, all the surfaces in Theorem 7.6 are qG-rigid: in other words, they are the only isomorphism class of surfaces in that family.

Theorem 7.21 of Section 7.4 is a more precise version of Theorem 7.6 just stated. In particular, the statement of Theorem 7.21 in Section 7.4 has images showing the directed MMP for X that provide a birational construction of X, and pictures of a distinguished configuration of negative curves in the minimal resolution $f\colon Y \to X$.

In all cases, we could have pushed the analysis to the point where we could have made a list of all negative curves on Y and X, and computed generators of the nef cones $\operatorname{Nef} Y$, $\operatorname{Nef} X$. We did not pursue this as we don't have a compelling reason to do so.

Surfaces with a given k are all obtained by a *cascade*—the terminology is due to [RS03]—of blow-ups of smooth points starting with one of the surfaces in Theorem 7.6. In our case, a cascade signifies a sequence of del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities in which two elements differ by a blow-up of a smooth point or, conversely, by a contraction of a floating $(-1)-$curve. Each sequence we consider is of maximal length: the surfaces at the bottom of the cascade are "minimal" in this sense (all of the remaining extremal rays pass through the singular points), while those at the top will have the lowest possible anticanonical degree while preserving the ampleness of $-K_X$.

Corollary 7.8. (1) A surface of the family $X_{1,\,d}$ is the blow-up of $25/3 - d \le 8$ nonsingular points on $\mathbb{P}(1,1,3)$. If $d < 16/3$, then it is also the blow-up of a surface of the family $B_{1,\,16/3}$ in $1 \le 16/3 - d \le 5$ nonsingular points;

(2) A surface of the family $X_{2,\,d}$ is the blow-up of $17/3 - d \le 5$ nonsingular points on $X_{2,\,17/3}$. If $d < 8/3$, then it is also the blow-up of a surface of the family $B_{2,\,8/3}$ in $1 \le 8/3 - d \le 2$ nonsingular points;

(3) A surface of the family $X_{3,\,d}$ is the blow-up of $5 - d \le 4$ nonsingular points on $X_{3,\,5}$;

(4) A surface of the family $X_{4,\,d}$ is the blow-up of $7/3 - d \le 2$ nonsingular points on $X_{4,\,7/3}$;

(5) A surface of the family $X_{5,\,2/3}$ is the blow-up of a nonsingular point on $X_{5,\,5/3}$;

(6) $X_{6,\,1}$ is the blow-up of a nonsingular point on $X_{6,\,2}$.

□

Remark 7.9. In the cases $k = 1$ and $k = 2$, Corollary 7.8 is not an immediate consequence of Theorem 7.6. Indeed, given a surface X, it is clear that a sequence of contractions of floating (-1)-curves leads to one of the surfaces listed in Theorem 7.6. We need to check, in addition, that:

(1) If $X \to B_{1,\,16/3}$ is the blow-up of a nonsingular point, there is an alternative sequence of 4 blow-downs of floating (-1)-curves starting from X and ending in $\mathbb{P}(1,1,3)$;

(2) If $X \to B_{2,\,8/3}$ is the blow-up of a nonsingular point, then there is an alternative sequence of 4 blow-downs of floating (-1)-curves starting from X and ending in the surface $X_{2,\,17/3}$.

These facts are easy to verify from the explicit birational constructions given in Theorem 7.21.

7.2 Invariants

Here X is a surface of one of the 29 families of Del Pezzo surfaces with $k \ge 1$ $\frac{1}{3}(1,1)$ points, and $X^0 = X^{\text{nonsing}} = X \setminus \operatorname{Sing} X$ is the nonsingular locus of X. The main scope of the study of the following invariants is to obtain a bound for the number of possible $\frac{1}{3}(1,1)$ singularities on X. This will be the starting point in our classification. We are interested in:

(i) k, the number of singular points of X;

(ii) $K^2 = K_X^2$, the anticanonical *degree* of X. It is obvious that $K_X^2 > 0$ and $K_X^2 \equiv \frac{k}{3} \pmod{\mathbb{Z}}$;

(iii) $h^0(X, -K_X)$, an integer ≥ 0 and, more generally, $h^0(X, -nK_X)$ for all integers $n \geq 0$;

(iv) $r = \rho(Y) = \rho(X) + k$, the Picard rank of the minimal resolution $f\colon Y \to X$;

(v) $n = e(X^0) = \widehat{c}_2(X) - k/3 = 2 + \rho(X) - k$, where e is the (homological) topological Euler number and $\widehat{c}_2(X) = c_2(\widehat{T}_X)$ is the orbifold second Chern class of X;

(vi) σ, the *defect* of X, defined as follows: let $L = H^2(Y;\mathbb{Z})$, viewed as a unimodular lattice by means of the intersection form, let $N = \langle -3\rangle^{\perp k} \subset L$ be the sublattice spanned by the (-3)-curves, and let $\overline{N} = \{\mathbf{v} \in L \mid \exists d \in \mathbb{Z} \text{ with } d\mathbf{v} \in N\}$ be the saturation of N in L, then, for some integer $\sigma > 0$, $\overline{N}/N \cong \mathbb{F}_3^\sigma$. Indeed, note that $\overline{N}/N \subset N^*/N$ where $N^* = \operatorname{Hom}(N,\mathbb{Z})$ and $N \subset N^*$ the natural inclusion given by the quadratic form. Note that N^*/N is 3-torsion and isomorphic to $(\mathbb{Z}/3\mathbb{Z})^k$, so $\overline{N}/N$ is also 3-torsion. Equivalently, $\sigma = k - \operatorname{rk}\operatorname{Im}[N\otimes\mathbb{F}_3 \to L\otimes\mathbb{F}_3]$. We prove in Lemma 7.14 below that $\mathbb{F}_3^\sigma \cong H_1(X^0;\mathbb{Z})$;

Since in this chapter we focus the birational constructions and the classification, these are the only invariants we need to consider. In the paper [CH] we take this further in order to find the complete intersection models: for each of these families, we compute the number of moduli, the Fano index and the fundamental group $\pi_1(X^0)$. This information is plotted in the Tables in Section 7.5.

Remark 7.10. The Riemann-Roch [Rei87, § 3] and Noether formula state:

$$h^0(X, -K_X) = 1 + K_X^2 - \frac{k}{3} \quad \text{and} \quad K_X^2 = 12 - n - \frac{5k}{3}$$

so one can compute $h^0(X, -K_X)$, n and r from k and K_X^2 (vanishing implies that $h^0(X, -nK_X) = \chi(X, -nK_X)$ for $n \geq 0$).

It is easy from these data to compute the Poincaré series $P_X(t) = \sum_{n\geq 0} t^n h^0(X, -nK_X)$:

$$P_X(t) = \frac{1 + \left(K_X^2 - 1 - \frac{k}{3}\right)t + \left(K_X^2 + \frac{2k}{3}\right)t^2 + \left(K_X^2 - 1 - \frac{k}{3}\right)t^3 + t^4}{(1-t)^2(1-t^3)}$$

Remark 7.11.

- If X admits a toric qG-degeneration, then $n = e(X^0) \geq 0$. Indeed, in this case n is the number of T-cones of the Fano polygon corresponding to the toric degenerate surface, see [AK].

- If X admits a toric qG-degeneration, then $h^0(X, -K_X) \geq 1$. Indeed, by the Riemann–Roch formula, $h^0(X, -K_X)$ is constant on a qG family and if X_0 is toric then $H^0(X_0, -K_{X_0}) \neq (0)$ since it contains at least the toric boundary divisor of X_0.

- It follows [K$^+$92, Chapter 10] from the generic semi-positivity of $\widehat{T}_X$ [KM99, 1.8 Corollary] that $\widehat{c}_2(X) \geq 0$.

The main result of this section is Proposition 7.15 where we derive an almost exact table of invariants of del Pezzo surfaces with $\frac{1}{3}(1,1)$ points from elementary lattice theory and elementary covering space theory. These methods are surprisingly effective in producing an almost exact table of invariants and we hope that they can be useful in other problems of classification of orbifold del Pezzo surfaces. We use the result in Section 7.4 to cut down on the cases we need to consider in the proof of theorems 7.6 and 7.21. We start with a study of the defect invariant.

Lemma 7.12. Using the notation introduced in Section 7.2, $k - r/2 \leq \sigma \leq k/2$.

Proof. $\overline{N}/N \subset N^*/N$ is totally isotropic where N^*/N is endowed with the discriminant quadratic form, hence $\sigma = \dim_{\mathbb{F}_3} \overline{N}/N \leq \frac{1}{2} \dim_{\mathbb{F}_3} N^*/N = \frac{k}{2}$. Also, $\mathrm{Im}[N \otimes \mathbb{F}_3 \to L \otimes \mathbb{F}_3]$ is totally isotropic, hence it has dimension $\leq r/2$, thus the kernel has dimension $\geq k - r/2$. $\square$

Remark 7.13. In fact one can do better, but we won't need to do so here. For example, if $k = 2$, then the discriminant bilinear form $A(x, y) = x^2 + y^2$ has no isotropic vector, hence $\sigma = 0$ in this case.

Lemma 7.14. $H_1(X^0; \mathbb{Z}) \cong \mathbb{F}_3^\sigma$.

Proof. Denote by $E = \cup_{i=1}^k E_i \subset Y$ the exceptional divisor of the minimal resolution morphism $Y \to X$, and note that of course $X^0 = Y \setminus E$. Because $Y \setminus E$ is smooth, the Poincaré homomorphism $H_c^i(Y \setminus E; \mathbb{Z}) \to H_{4-i}(Y; \mathbb{Z})$ is an isomorphism. The long exact sequence for compactly supported cohomology fits into a commutative diagram:

$$\begin{tikzcd}
H^2(Y;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(E;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_c^3(Y \setminus E;\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^3(Y;\mathbb{Z}) = (0) \arrow[d, equal] \\
L \arrow[r] & N^\star \arrow[r] & \mathbb{F}_3^\sigma \arrow[r] & (0)
\end{tikzcd}$$

$\square$

We proceed to making our first significant restriction on the number of cases, bounding the number of possible singularities k on X and correlating it to the degree of the canonical class $K^2 = K_X^2$:

Proposition 7.15. $k \leq 6$ and moreover:

(1) If $k = 1$ then $K^2 \equiv 1/3 \pmod{\mathbb{Z}}$ and $1/3 \leq K^2 \leq 25/3$;

(2) If $k = 2$ then $K^2 \equiv 2/3 \pmod{\mathbb{Z}}$ and $2/3 \leq K^2 \leq 20/3$;

(3) If $k = 3$ then $K^2 \equiv 0 \pmod{\mathbb{Z}}$ and $1 \leq K^2 \leq 5$;

(4) If $k = 4$ then $K^2 \equiv 1/3 \pmod{\mathbb{Z}}$ and $1/3 \leq K^2 \leq 10/3$;

(5) If $k = 5$ then $K^2 \equiv 2/3 \pmod{\mathbb{Z}}$ and $2/3 \leq K^2 \leq 8/3$;

(6) If $k = 6$ then $K^2 \equiv 0 \pmod{\mathbb{Z}}$ and $1 \leq K^2 \leq 2$.

Remark 7.16. It follows from the proof of Theorems 7.6 and 7.21 that the possibilities $k = 2, K^2 = 20/3$; $k = 4, K^2 = 10/3$; $k = 5, K^2 = 8/3$ do not actually occur.

Proof of Proposition 7.15. By [KM99, 1.8 Corollary] the orbi-tangent bundle is generically semi-positive, and then by [K+92, Chapter 10] $\widehat{c}_2 = n + k/3 \geq 0$. It follows from this that $K^2 = 12 - n - \frac{5k}{3} \leq 12 - \frac{4k}{3}$. By using $K^2 \equiv k/3 \pmod{\mathbb{Z}}$ and:

$$0 < K^2 \leq 12 - \frac{4k}{3} \quad \text{and} \quad h^0(X, -K) = 1 + K^2 - k/3 \geq 0$$

we immediately conclude that $k \leq 7$. Indeed, if $k = 8$ the first set of inequalities forces $K^2 = 2/3$ but this would imply $h^0(X, -K) = 1 + K^2 - 8/3 = 1 + 2/3 - 8/3 = -1 < 0$, a contradiction.

Similarly, when $k = 7$ the first inequality gives $1/3 \leq K^2 \leq 7/3$, but $K^2 = 1/3$ does not occur because it would imply $h^0(X, -K) = 1 + 1/3 - 7/3 = -1$, again a contradiction. With a bit more work we can exclude a few more cases: if $k = 1$, then $K^2 = 31/3$ implies $r = 0$, $K^2 = 28/3$ implies $r = 1$ and both are impossible because $r = k + \rho(X) > k$. Similarly, $k = 2$ and $K^2 = 26/3$ implies $r = 2$, impossible; $k = 3$ and $K^2 = 8$ implies $r = 3$, also impossible. Thus $k \leq 7$ and we are left with the following possibilities:

(1) If $k = 1$ then $K^2 \equiv 1/3 \pmod{\mathbb{Z}}$ and $1/3 \leq K^2 \leq 25/3$;

(2) If $k = 2$ then $K^2 \equiv 2/3 \pmod{\mathbb{Z}}$ and $2/3 \leq K^2 \leq 23/3$;

(3) If $k = 3$ then $K^2 \equiv 0 \pmod{\mathbb{Z}}$ and $1 \leq K^2 \leq 7$;

(4) If $k = 4$ then $K^2 \equiv 1/3 \pmod{\mathbb{Z}}$ and $1/3 \leq K^2 \leq 19/3$;

(5) If $k = 5$ then $K^2 \equiv 2/3 \pmod{\mathbb{Z}}$ and $2/3 \leq K^2 \leq 14/3$;

(6) If $k = 6$ then $K^2 \equiv 0 \pmod{\mathbb{Z}}$ and $1 \leq K^2 \leq 4$;

(7) If $k = 7$ then $K^2 \equiv 1/3 \pmod{\mathbb{Z}}$ and $4/3 \leq K^2 \leq 7/3$.

We are still quite some way from proving what we need. We exclude the remaining possibilities by studying the invariant σ. The key observation is that, by Lemma 7.12, we have that $\sigma \geq k - r/2$ so, for example, if $k = 2$ and $K^2 = 23/3$, we must have $r = 3$ and then $\sigma > 0$. It is easy to see that this case does not occur: by Lemma 7.14 $H_1(X^0; \mathbb{Z}) \cong \mathbb{F}_3^\sigma$, so by covering space theory there is a 3-to-1 covering $Y \to X$, étale above X^0, from a surface Y, necessarily a del Pezzo surface, with $1/3(1, 1)$ points and degree $K_Y^2 = 3 \times \frac{23}{3} = 23$ and we already know that such a surface does not exist.

As another example, $k = 7$, $K^2 = 4/3$ implies $\sigma \geq 2$ so there is a 9-to-1 cover $Y \to X$ from a del Pezzo surface Y with $1/3(1, 1)$ points and $K_Y^2 = 12$ and we know that such a surface does not exist.

In Table 7.1 we summarise the cases where we can definitely conclude $\sigma > 0$. All but two are excluded at once by the same method (the other two cases actually occur) and the result follows.

k	K^2	r	σ	Occurs
2	23/3	3	> 0	No
3	6	5	> 0	No
3	7	4	> 0	No
4	13/3	7	> 0	No
4	16/3	6	> 0	No
4	19/3	5	> 1	No
5	8/3	9	> 0	No
5	11/3	8	> 0	No
5	14/3	7	> 1	No
6	1	11	> 0	Yes
6	2	10	> 0	Yes
6	3	9	> 1	No
7	4/3	11	> 1	No
7	7/3	10	> 1	No

Table 7.1: Necessarily defective possibilities

All other possibilities are excluded by the same method, except $k = 5$, $K^2 = 8/3$, $\sigma \geq 1$: this possibility is not excluded at this point, and it is not excluded by the statement of Proposition 7.15. Table 7.1 states that it does not occur, but this fact will only follow from the proof of theorems 7.6 and 7.21 in Section 7.4.

□

7.3 MMP

In the proof of Theorem 7.6 in Section 7.4 we systematically use the following elementary result, which we state without proof. Analogous statements for surfaces with canonical singularities can be found in [Mor85b, Fur86].

Theorem 7.17. Let X be a projective surface having $k \times \frac{1}{3}(1,1)$, $n_2 \times A_2$, and $n_1 \times A_1$ singularities. Assume that $k + 2n_2 + n_1 \leq 6$.

Let $f\colon X \to X_1$ be an extremal contraction. Then exactly one of the following holds:

(E) $f\colon (X, E) \to (X_1, P)$ is a divisorial contraction. Denote by $Y \to X$ and $Y_1 \to X_1$ the minimal resolutions, and $E' \subset Y$ the proper transform of the exceptional curve. Then $E' \subset Y$ is a (-1)-curve meeting transversely at most one exceptional curve of $Y \to X$ above each singularity, and one of the following holds:

(E.1) E is contained in the nonsingular locus. Then E is a (-1)-curve and we call it a *floating* (-1)-curve;

(E.2) (A1 contraction) E contains one A_1-singularity, $P \in X_1$ is a nonsingular point;

(E.3) (A2 contraction) E contains one A_2-singularity, $P \in X_1$ is a nonsingular point;

(E.4) E contains one $\frac{1}{3}(1,1)$-singularity, $P \in X_1$ is a A_1-point;

(E.5) E contains one $\frac{1}{3}(1,1)$-singularity and one A_1 singularity, $P \in X_1$ is a nonsingular point;

(E.6) E contains two $\frac{1}{3}(1,1)$-singularities, $P \in X_1$ is an A_2-point.

(C) $X_1 = \mathbb{P}^1$, that is, f is generically a conic bundle. Denote by $F \subset X$ a special fibre of f, and by $Y \to X$ and $Y_1 \to X_1$ the minimal resolutions and $F' \subset Y$ the proper transform of F. Then F' is a (-1)-curve and one of the following holds:

(C.1) F contains two A_1-singularities, and F' meets each of the (-2)-curves transversely;

(C.2) F contains one $\frac{1}{3}(1,1)$-singularity and one A_2 singularity, and F' meets the (-3)-curve and one of the (-2)-curves transversely.

(D) $X_1 = \{\mathrm{pt}\}$ is a point, that is, X is a del Pezzo surface of Picard rank one, and X is one of the following surfaces:

(D.1) $\mathbb{P}^2$;

(D.2) $\mathbb{P}(1,1,2)$ (this surface has exactly one A_1 singular point);

(D.3) $\mathbb{P}(1,2,3)$ (this surface has exactly one A_1 and one A_2 singularities);

(D.4) $\mathbb{P}^2/\mu_3$ where μ_3 acts with weights $1, \omega, \omega^2$. This surface has exactly $3 \times A_2$ singularities;[1]

(D.5) $\mathbb{P}(1,1,3)$.

Remark 7.18. Consider the class of projective surfaces X be having $k \times \frac{1}{3}(1,1)$, $n_2 \times A_2$, and $n_1 \times A_1$ singularities and $k + 2n_2 + n_1 \leq 6$. It follows from the previous statement that a MMP starting from a surface in the class only involves surfaces in the class.

The directed minimal model program In the proof of theorems 7.6 and 7.21 in the following section, we run the MMP starting with a del Pezzo surface with $\frac{1}{3}(1,1)$ points.

In all cases, we perform extremal contractions in the order that they are listed in Theorem 7.17 above: that is, we first contract all the floating (-1)-curves, then we contract a ray of type (E.2) if available, or else one of type (E.3), etc.

We call this the *directed MMP*.

Lemma 7.19. Let X be a del Pezzo surface with $k \geq 1$ $\frac{1}{3}(1,1)$ singular points. Assume that X contains no floating (-1)-curves. Denote by

$$X = X_0 \xrightarrow{\varphi_0} \dots \longrightarrow X_{i-1} \xrightarrow{\varphi_{i-1}} X_i \longrightarrow \dots$$

the contractions and surfaces occurring in a MMP for X (not necessarily directed).

(1) All surfaces X_i are del Pezzo surfaces.

(2) Denote by $f_i : Y_i \to X_i$ the minimal resolution of X_i and let $C \subset Y_i$ be a (reduced and irreducible) curve with negative self-intersection $C^2 = -m$. Then:

(2.1) if C is f_i-exceptional, then $m = 2$ or 3,

(2.2) if C is not f_i-exceptional, then $m = 1$ and C intersects at least one f_i-exceptional curve. In particular, none of the surfaces X_i contain a floating (-1)-curve.

Proof of Lemma 7.19. We prove the statement by induction on i. We first show that X_i is a del Pezzo surface. Suppose X_{i-1} is del Pezzo and let $E \subset X_{i-1}$ be the effective divisor such that $K_{X_{i-1}} = \varphi_{i-1}^\star K_{X_i} + aE$, $a > 0$. Let $\Gamma \subset X_i$ be a curve. Denoting by $\Gamma' \subset X_{i-1}$ the proper transform, we have that:

$$K_{X_i} \cdot \Gamma = K_{X_i} \cdot \varphi_{i-1\star}\Gamma' = \varphi_{i-1}^\star(K_{X_i}) \cdot \Gamma' = (K_{X_{i-1}} - aE) \cdot \Gamma' < 0.$$

As $K_{X_i}^2 > K_{X_{i-1}}^2$, by the Nakai-Moishezon criterion we conclude that $-K_{X_i}$ is ample.

Assuming that (2.1) holds for X_{i-1}, then it also holds for X_i, by the structure of divisorial contractions listed in Theorem 7.17.

Let now C be a $(-m)$-curve on Y_i that is not contracted by f_i, then since $-K_{Y_i} + f_i^\star K_{X_i} \geq 0$ we have that:

$$-K_{Y_i} \cdot C = f_i^\star(-K_{X_i}) \cdot C + (-K_{Y_i} + f_i^\star K_{X_i}) \cdot C \geq f_i^\star(-K_{X_i}) \cdot C = -K_{X_i} \cdot f_{i\star}C > 0.$$

Then $K_C = (K_{Y_i} + C)|_C < 0$, therefore C is a rational curve and $K_{Y_i} \cdot C = m - 2 < 0$ implies $m = 1$, that is, $C \subset Y$ is a (-1)-curve, and the image $C_i = f_i(C) \subset X_i$ is a floating (-1)-curve. Now C_i does not contain the image of the φ_{i-1}-exceptional curve: otherwise, the proper transform $C' \subset Y_{i-1}$

[1] X is the toric surface obtained by blowing up 3 vertices on the hexagon of lines of a degree 6 nonsingular del Pezzo surface. Explicitly, it is a cubic surface given by $xyz = t^3$.

would be a curve of negative self-intersection ${C'}^2 < -1$ not contracted by $f_{i-1}\colon Y_{i-1} \to X_{i-1}$, contradicting (2.1) for X_{i-1}. Thus, C_i is the image of a floating (-1)-curve in X_{i-1} and then in fact, by descending induction on i, C_i is the image of a floating (-1)-curve on X, a contradiction to the main assumption that these do not appear. This shows (2.2). $\square$

Remark 7.20. In Section 7.4 we use the following type of argument very frequently. Suppose that

$$X = X_0 \xrightarrow{\varphi_0} \ldots \longrightarrow X_i \xrightarrow{\varphi_i} X_{i+1} \xrightarrow{\varphi_{i+1}} X_{i+2} \ldots$$

is the sequence of contractions and surfaces occurring in a directed MMP for X. If φ_i is of type (E.6), then φ_{i+1} is not of type (E.3). Indeed denote by $f_i : Y_i \to X_i$ the minimal resolution. If φ_{i+1} is of type (E.3), the proper transform $C \subset Y_{i+1}$ of the curve contracted by φ_{i+1} is a (-1)-curve, and its proper transform on Y_i is a (-1)-curve that shows that a contraction of type (E.3) or (E.4) was available on X_i in the first place, and this is a contradiction.

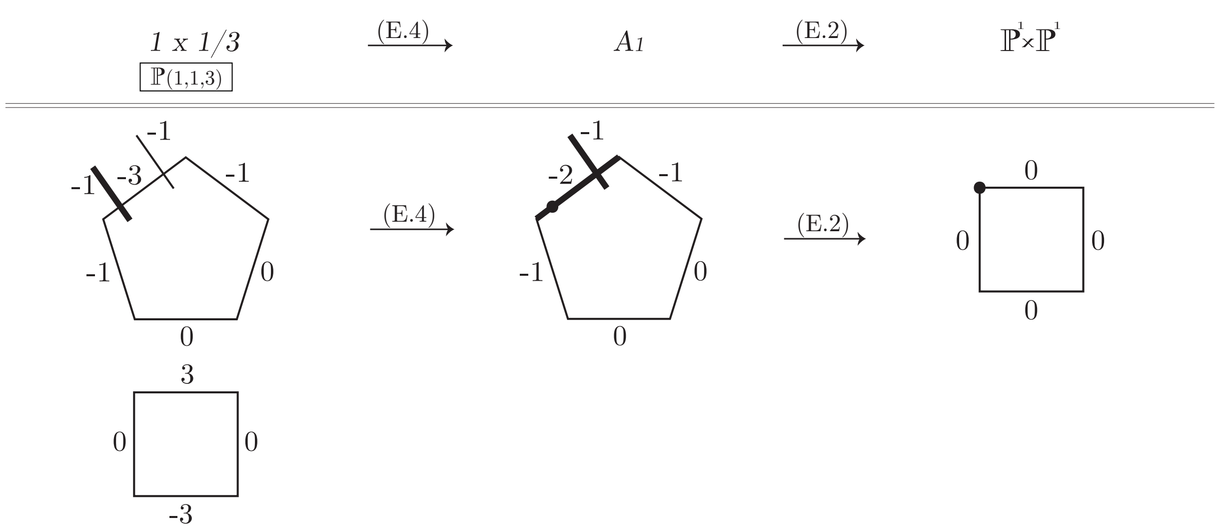
7.4 Trees

This is the main section of this chapter. We prove Theorem 7.21, from which Theorem 7.6 of the introduction immediately follows. The proof uses Proposition 7.15. Before we begin constructing the surfaces, we take apart the first case and explain how to read the contractions on this example.

Theorem 7.21. Let X be a del Pezzo surface with $k \geq 1$ $\frac{1}{3}(1,1)$ singular points. If X has no floating (-1)-curves, then it is one of the following surfaces. The images show the sequence of contractions and surfaces of the directed MMP for X providing a birational construction of it, followed by—and separated by a double horizontal rule—a picture of the minimal resolutions of the surfaces of the MMP showing a configuration of curves on them:

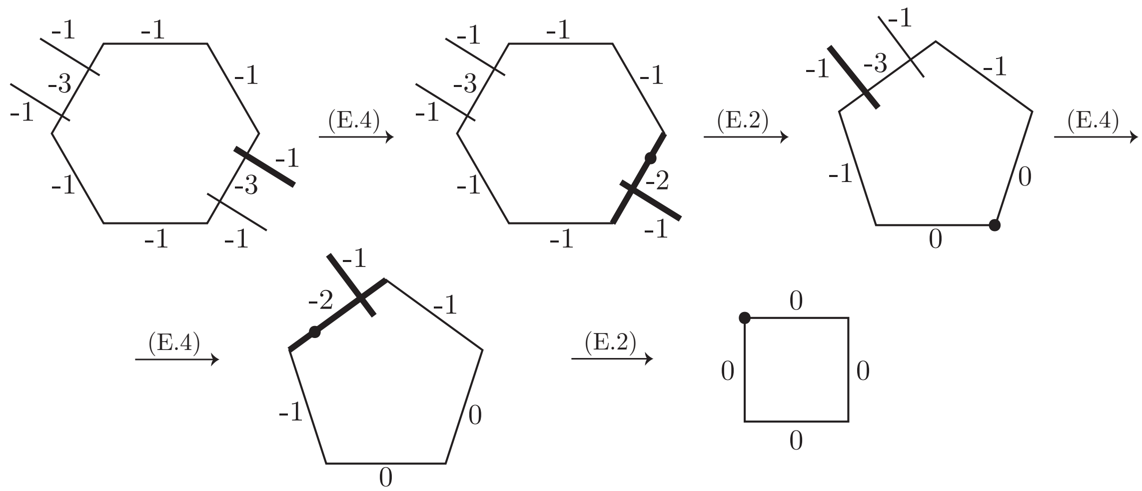
- In the images showing the sequence of contractions we record the singularities on each intermediate surface. For example, "$2 \times 1/3 + A_2$" signifies a surface with two $\frac{1}{3}(1,1)$ singularities and one A_2 singularity.
- In the pictures showing the minimal resolutions the contracted curves are in bold and their images are denoted by a bold point.

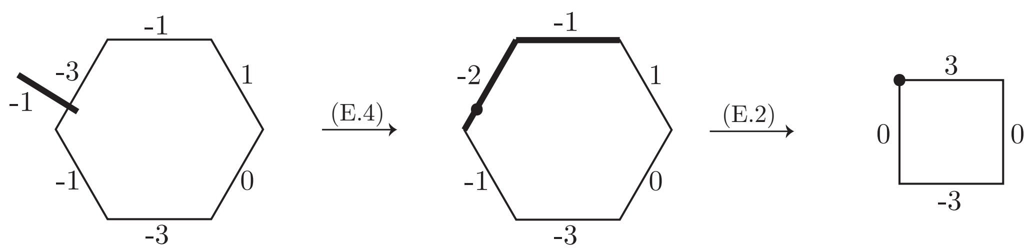
(1) $k = 1$ and either $X = B_{1,16/3}$ (the first case pictured), or $X = \mathbb{P}(1,1,3)$ (the second case pictured):



(2) $k = 2$ and either $X = B_{2,\,8/3}$ (the first case pictured), or $X_{2,\,17/3}$ (the second case):

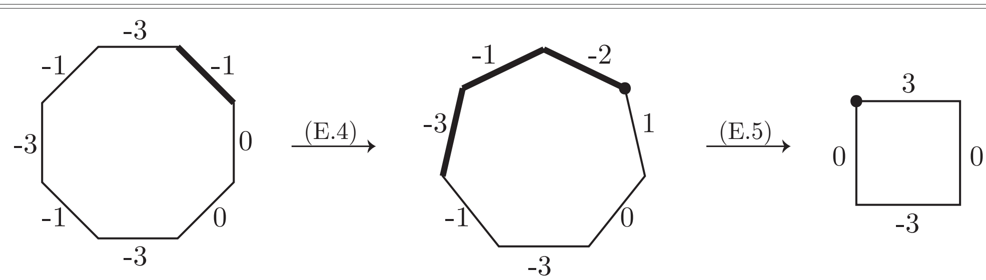
$$2 \text{ x } 1/3 \xrightarrow{\text{(E.4)}} 1/3 + A_1 \xrightarrow{\text{(E.2)}} \underset{\boxed{\mathbb{P}(1,1,3)}}{1 \text{ x } 1/3} \xrightarrow{\text{(E.4)}} A_1 \xrightarrow{\text{(E.2)}} \mathbb{P}^1\times\mathbb{P}^1$$





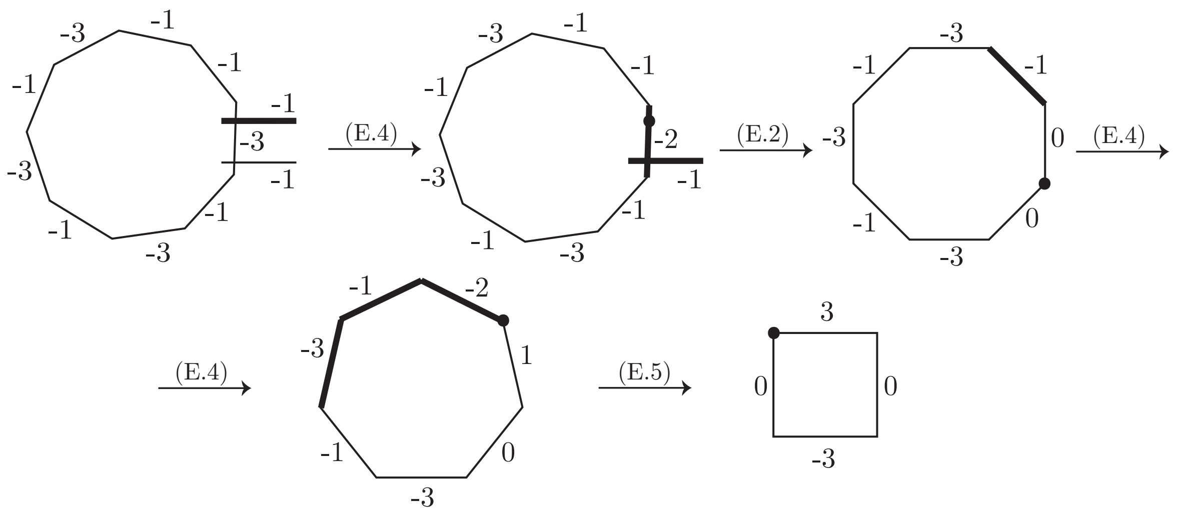
(3) $k = 3$ and $X = X_{3,\,5}$:

$$3 \text{ x } 1/3 \xrightarrow{\text{(E.4)}} 2 \text{ x } 1/3 + A_1 \xrightarrow{\text{(E.5)}} \mathbb{P}(1,1,3)$$



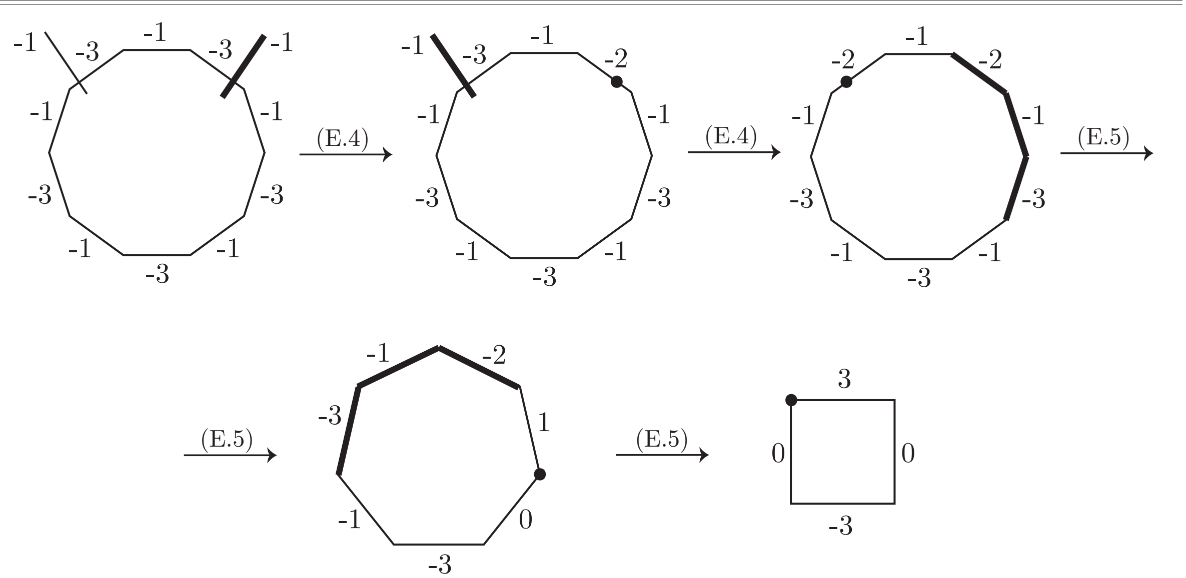
(4) $k = 4$ and $X = X_{4,\,7/3}$:

$$4 \text{ x } 1/3 \xrightarrow{\text{(E.4)}} 3 \text{ x } 1/3 + A_1 \xrightarrow{\text{(E.2)}} 3 \text{ x } 1/3 \xrightarrow{\text{(E.4)}} 2 \text{ x } 1/3 + A_1 \xrightarrow{\text{(E.5)}} \mathbb{P}(1,1,3)$$

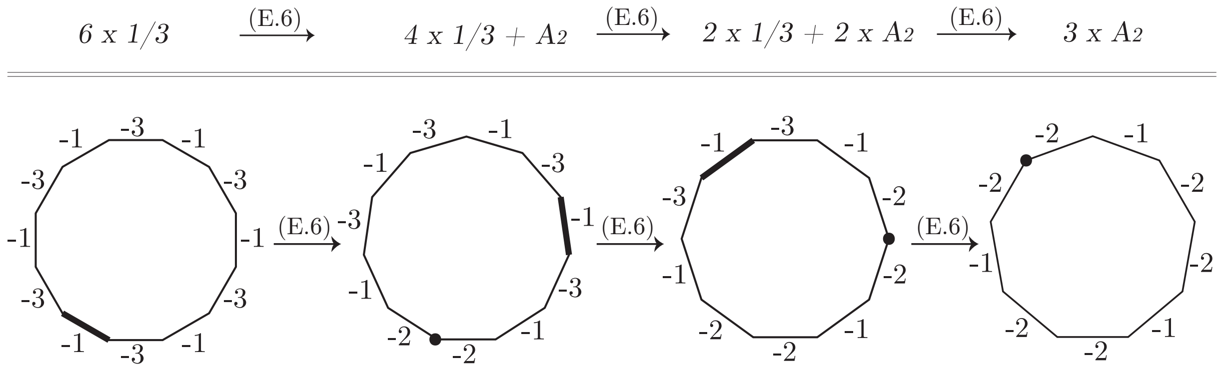


(5) $k = 5$ and $X = X_{5,\,5/3}$:

$$5 \text{ x } 1/3 \xrightarrow{\text{(E.4)}} 4 \text{ x } 1/3 + A_1 \xrightarrow{\text{(E.4)}} 3 \text{ x } 1/3 + 2 \text{ x } A_1 \xrightarrow{\text{(E.5)}}$$
$$\xrightarrow{\text{(E.5)}} 2 \text{ x } 1/3 + A_1 \xrightarrow{\text{(E.5)}} \mathbb{P}(1,1,3)$$

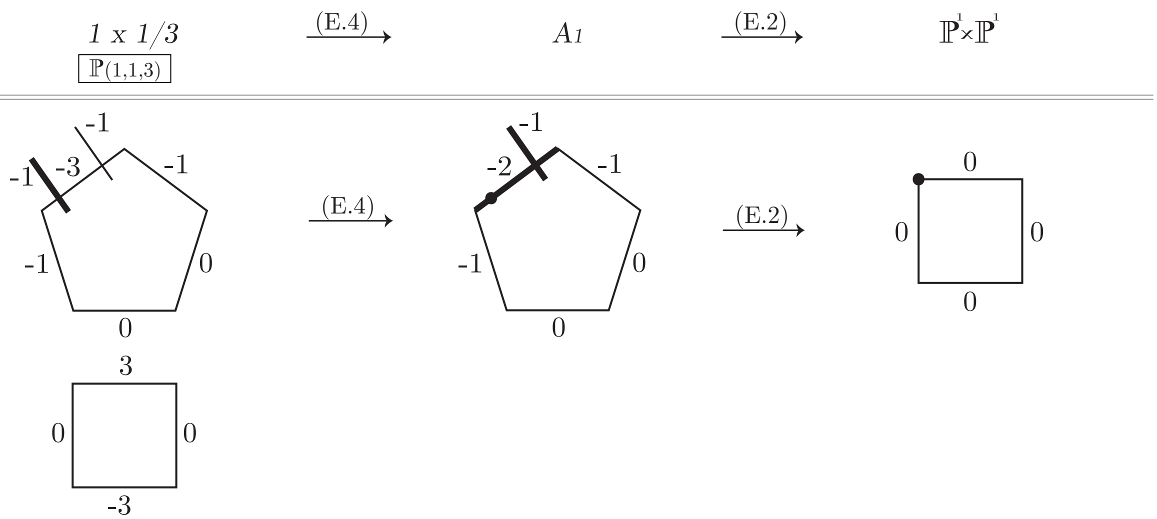


(6) $k = 6$ and $X = X_{6,\,2}$:



We go back to the case where $k = 1$ in order to explain the image:

For $k = 1$ we obtain either $X = B_{1,\,16/3}$ (the first case pictured), or $X = \mathbb{P}(1, 1, 3)$ (the second case pictured):



We start with one $\frac{1}{3}(1, 1)$ singularity on our surface X. We are either on $\mathbb{P}(1, 1, 3)$ whose Picard number equals one, or we have a sequence of two contractions available: a contraction of type (E.4), followed by one of type (E.2). This is the first row of the picture, depicting the effects of applying the directed MMP to X. Underneath the horizontal rule, we describe what happens to the minimal resolution Y of X during this process. If $X = \mathbb{P}(1, 1, 3)$ this is trivial and we have only to draw its minimal resolution, the Hirzebruch surface $\mathbb{F}_3$.

However, if we are in the other case, denote the contraction morphisms by $X \to X_1 \to X_2 = \mathbb{P}^1 \times \mathbb{P}^1$ and $Y \to Y_1 \to Y_2 = \mathbb{P}^1 \times \mathbb{P}^1$ respectively. Y contains a pentagon of curves various self-intersections, notably the $(-3)-$curve is the exceptional curve above the $\frac{1}{3}(1, 1)$ singularity. Out of the four (-1)-curves that intersect it, the one in bold is that which corresponds to the extremal ray of the (E.4) contraction on X. It will naturally induce a contraction on Y and the image of this curve is depicted as a bold point. The intersection numbers on the new surface change accordingly. The curves that are to be contracted next are also in bold. Note that when considering (E.2), though with $X_1 \to X_2$ only one curve is contracted, $Y_1 \to Y_2$ is a contraction of two curves: by starting with the bold (-1)-curve, the (-2)-curve becomes of self-intersection (-1) and is automatically also contracted. We then obtain the representation of $\mathbb{P}^1 \times \mathbb{P}^1$ with the usual two rulings.

Proof. In all cases we run the directed MMP for X. In other words, at each step we choose rays exactly in the order that they are listed in Theorem 7.17. The figures in the statement show the sequence of contractions as they occur in the directed minimal model program.

We begin the proof by drawing a tree representing the directed MMPs that can potentially occur (Figure 7.1 is an example). For each branch, corresponding to a sequence of contractions, we construct a configuration of curves on the minimal resolution Y. In many cases, the configuration of curves shows that at some stage in the MMP there was the option of performing a contraction higher up in the list of Theorem 7.17: that is, the MMP represented by that branch is not directed and hence it does not actually occur. At the end we are left with the directed MMPs that actually take place.

Here we only treat in detail the cases $k = 4$ and $k = 6$; the other cases are very similar and can be done by the same methods. Figures 7.6 to 7.9 at the end of the proof list the remaining trees for all k.

The $k = 4$ case We first argue that the sequence of extremal contractions of the directed MMP must be one of those shown on Figure 7.1.

In the argument that follows we denote by

$$X = X_0 \xrightarrow{\varphi_0} \ldots \longrightarrow X_{i-1} \xrightarrow{\varphi_{i-1}} X_i \longrightarrow \ldots$$

the sequence of contractions and surfaces occurring in a directed MMP for X. Also we denote by $f_i\colon Y_i \to X_i$ the minimal resolutions. By Theorem 7.17, φ_0 is either of type (E.6) or (E.4) and we claim that (E.6) does not occur.

Suppose for a contradiction that φ_0 is an (E.6) contraction. By Theorem 7.17, φ_1 is of type (E.3), (E.4) or (E.6): indeed, φ_1 can not be a conic bundle because X_1 has an odd number of singular points and from the classification of fibres every special fibre has two singularities on it, and it is clear from the classification that X_1 is not a del Pezzo surface with $\rho = 1$.

By Remark 7.20, (E.3) can not follow (E.6). If φ_1 is of type (E.4), then this contraction would have been already available on X_0, a contradiction. Finally, if φ_1 is of type (E.6), X_2 has $2 \times A_2$ singularities and then, by Theorem 7.17, φ_2 is of type (E.3): just as before, none of the $\rho = 1$ del Pezzo surfaces have $2 \times A_2$ singularities, and from the classification of fibres φ_2 can not be a conic bundle. But again (E.3) can not follow (E.6).

All of this shows that φ_0 is of type (E.4), therefore X_1 has $3 \times \frac{1}{3}(1,1) + A_1$ singularities. Thus φ_1 can be of type (E.2), (E.4), (E.5) or (E.6), and we claim that the last two do not occur.

Suppose for a contradiction that φ_1 is of type (E.5). X_2 is a del Pezzo surface with $2 \times \frac{1}{3}(1,1)$ singularities. By the case $k = 2$ of the theorem, which we assume to have already proved, φ_2 is of type (E.4) and this contraction was available on X_1, a contradiction.

If φ_1 is of type (E.6) the surface X_2 has $A_1 + A_2 + \frac{1}{3}(1,1)$ singularities. The contraction φ_2 can not be of type (E.2), (E.4) or (E.5) because otherwise the same contraction would have been available on X_1. It can not be of type (E.3) either because by Remark 7.20 (E.3) can not follow (E.6). By Theorem 7.17 these were the only possibilities thus this case does not occur.

If φ_1 is of type (E.2) then X_2 is a del Pezzo surface with $k = 3$ and the tree continues as in the $k = 3$ case, which we assume already known.

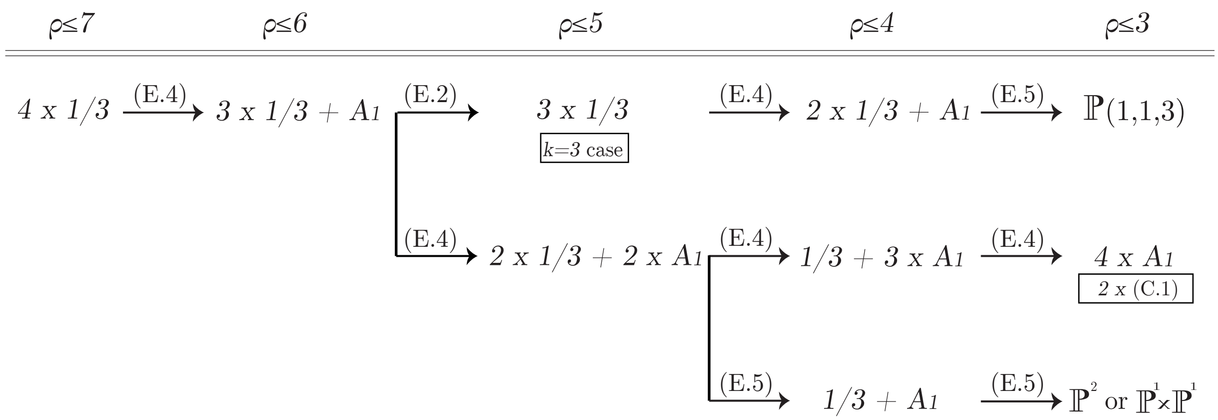
If φ_1 is of type (E.4) then X_2 has $2 \times A_1 + 2 \times \frac{1}{3}(1,1)$ singularities.

The next contraction φ_2 is not of type (E.2) because it would have been available earlier.

If φ_2 were of type (E.6) then X_3 would have $A_2 + 2A_1$ singularities. The next contraction φ_3 is not of type (E.2) because it would have been available earlier; it is not of type (E.3) because by Remark 7.20 (E.3) does not follow (E.6); it is not of fibering type because X_3 has an odd number of singularities; and X_3 is not a del Pezzo surface with $\rho = 1$ by the classification of Theorem 7.17.

Thus, φ_2 is not of type (E.2) or (E.6) and it can be only of type (E.4) or (E.5), which can be shown to lead respectively to the two remaining possibilities in Figure 7.1.

Figure 7.1: $k = 4$ tree of possibilities

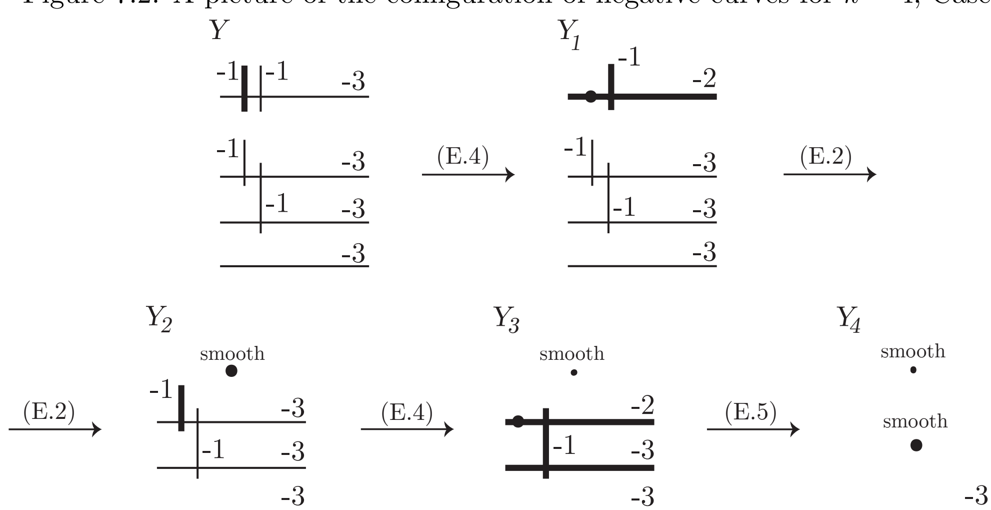


We now explore the branches of this tree one at a time and show that only one actually occurs.

Case 1 $(E.4) + (E.2) + (E.4) + (E.5)$

If this sequence of contractions occurs, then Y must contain the configuration of curves depicted in Figure 7.2 below. The figure shows the effect of the contractions of the MMP on the minimal resolutions: the contracted curves are in bold, as are the points onto which they map.

Figure 7.2: A picture of the configuration of negative curves for $k = 4$, Case 1

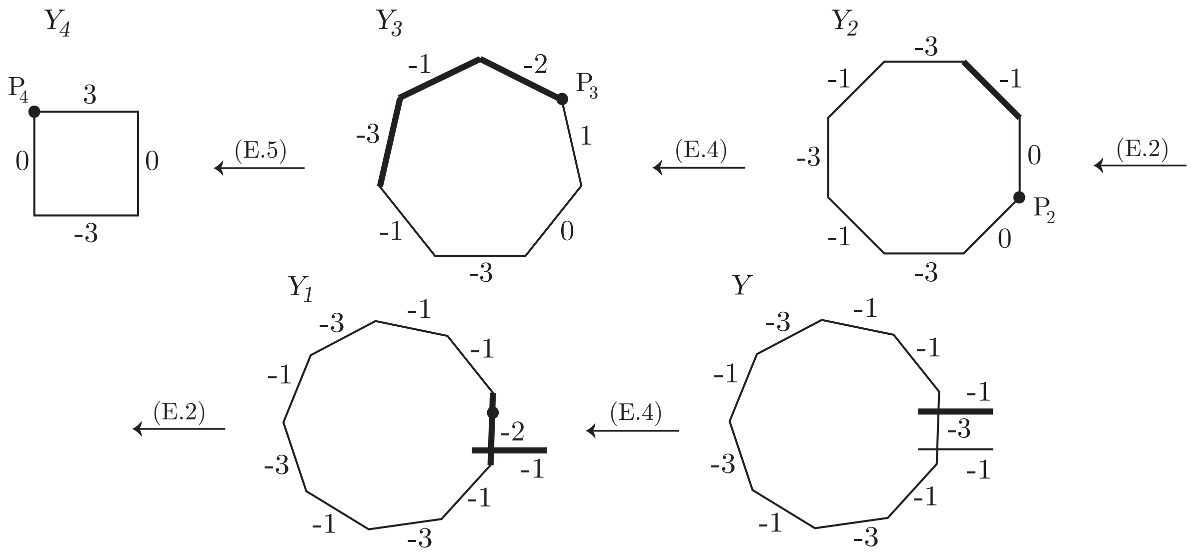


Looking more closely at how Y is built from $Y_4 = \mathbb{F}_3$ by a sequence of blow-ups, we argue that Y must have more negative curves, shown in Figure 7.3 below.

We use the following result:

Remark 7.22. Let X be a smooth del Pezzo surface and $C \subset X$ an irreducible rational curve with positive self intersection. Then C moves in a free linear system. Indeed, the map $H^0(X, \mathcal{O}_X(C)) \to H^0(C, \mathcal{O}_C(C))$ is surjective because $-K_X$ is ample. Since $C^2 > 0$ we have that $\mathcal{O}_C(C)$ is base point free and the conclusion immediately follows from the vanishing of $H^1(X, \mathcal{O}_X)$.

Figure 7.3: A better picture of the configuration of negative curves for $k = 4$, Case 1

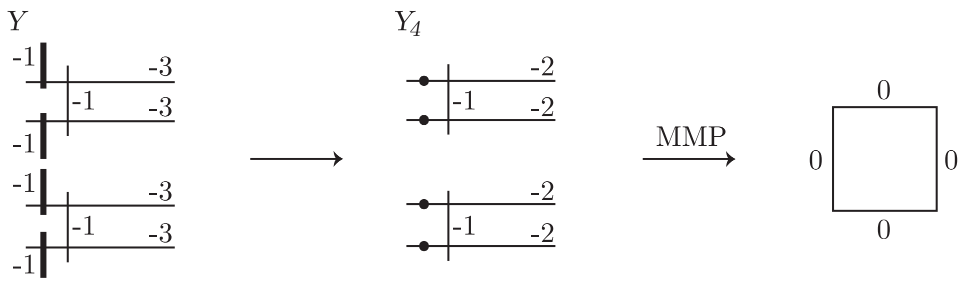


Indeed, the point $P_4 \in Y_4$ that is the image of the exceptional curve in Y_3 does not lie on the (-3)-curve and then by Remark 7.22 we can choose a configuration of curves as shown in the Figure displaying P_4 as the intersection of a fibre and a curve of self-intersection $+3$. At the next step we need to blow up a point $P_3 \in Y_3$ on the (-2)-curve and not contained in any other negative curve. By Remark 7.22 again, we can "move" the curve with self-intersection 1 until it contains P_3 as in the figure. At the next step again we need to blow up a nonsingular point $P_2 \in Y_2$ not lying on any negative curve, and we use Remark 7.22 to "move" the two curves with self-intersection 0 until they both contain P_2.

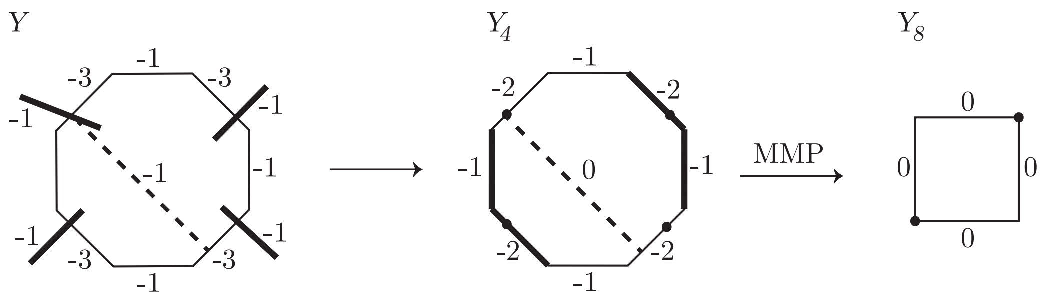
We are left with the minimal resolution of the surface $X_{4,7/3}$.

Case 2 $(E.4) + (E.4) + (E.4) + (E.4)$

We contract one (-1)-curve intersecting each (-3)-curve on Y and end up with a surface fibering over $\mathbb{P}^1$, denoted by Y_4, corresponding to an extremal contraction $X_4 \to \mathbb{P}^1$ having two singular fibers of type (C.1). As Y_4 is a nonsingular surface, we next run the classical Minimal Model Program for Y_4 relative to the existing fibration $Y_4 \to \mathbb{P}^1$, which ends in a Segre surface $\mathbb{F}_k$, and we claim that we can assume that $k = 0$:



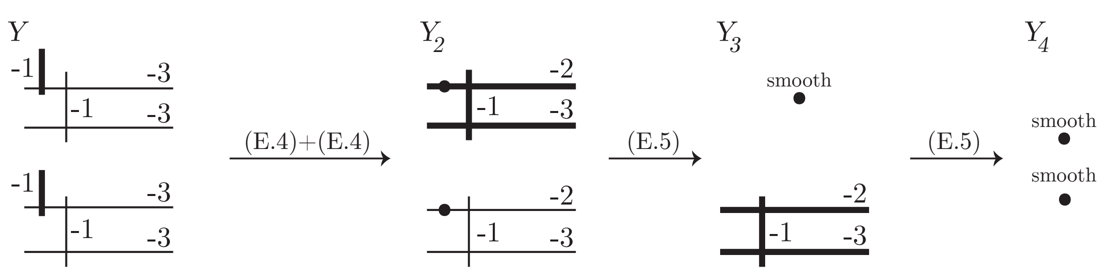
Indeed, since all negative curves left on Y_4 have self-intersection ≥ -2, the same is true about $\mathbb{F}_k$. We are left with $k \in \{0, 1, 2\}$. If $k = 2$, this leads to a (-2)-curve on Y not contracted by the morphism to X, contradicting lemma 7.19. If on the other hand $k = 1$ then by choosing the last contraction differently we would have landed on $\mathbb{F}_0$.



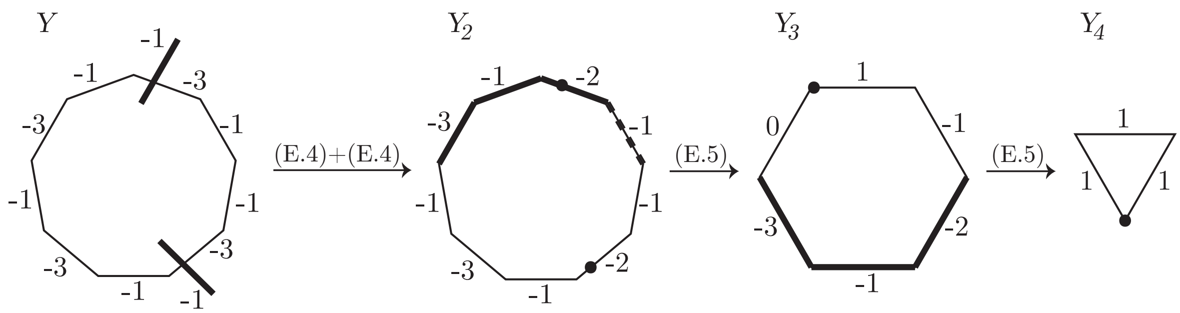
The horizontal ruling of $\mathbb{F}_0$ transforms to a free linear system of (0)-curves intersecting two of the opposing (-2)-curves in the special fibers on Y_4, depicted as a dashed line in the figure. As both of these (-2)-curves contain a point that is to be blown up in the process of building Y, we denote by the dotted line the (0)-curve passing through the point $P_4 \in Y_4$ that is the image of the exceptional curve in Y_3 corresponding to the last (E.4)-type contraction (as well as its strict transform on Y). Exactly before this contraction is performed, in Y_3 (and, ultimately, also in Y), the dotted line is a (-1)-curve intersecting only the (-1)-curve that is to be contracted and the (-2)-curve on the other special fiber. This means that an (E.2)-type contraction was available on X_3, which contradicts the fact that a directed MMP was used in obtaining X_4. Therefore this case does not occur.

Case 3 $(E.4) + (E.4) + (E.5) + (E.5)$

In this case the minimal resolutions must contain the following configurations of curves:

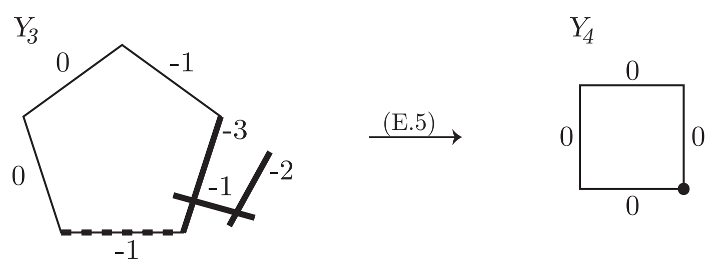


Since the resulting surface X_4 is smooth and rationally connected and, by Lemma 7.19, it contains no negative curves, it is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of the projective plane, looking more closely at how Y is built from $Y_4 = \mathbb{P}^2$ by a sequence of blow-ups, we see that Y must have the negative curves shown in the following figure:



Note that after having done the first two steps of the MMP, Y_2 contains two (-1)-curves showing that two (E.2)-type contractions were available on X_2, which proves that the MMP is not directed and that this case does not occur.

If $Y_4 = \mathbb{P}^1 \times \mathbb{P}^1$ we reach a contradiction without having to study the minimal resolution of X. Indeed, the following figure shows the contraction $Y_3 \to Y_4$:



From the picture it is clear that an (E.4)-type contraction was available on X_3, a contradiction. This case also does not occur, and the only surface with four $\frac{1}{3}(1,1)$ points is the one described in Case 1.

The $k = 6$ case We argue that the sequence of extremal contractions of the directed MMP is one of the two shown in Figure 7.4.

We use the same notations as in the $k = 4$ case, i.e we denote by

$$X = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \ldots \longrightarrow X_{i-1} \xrightarrow{\varphi_{i-1}} X_i \longrightarrow \ldots$$

the sequence of contractions and surfaces occurring in a directed MMP for X and by $f_i \colon Y_i \to X_i$ the minimal resolutions. Proposition 7.15, which we use repeatedly in the course of the proof, implies that $\rho(X) \in \{1 \ldots 5\}$. By Theorem 7.17, φ_0 is either of type (E.4) or (E.6) and we claim that (E.4) does not occur.

Suppose for a contradiction that φ_0 is an (E.4) contraction, then X_1 has $5 \times \frac{1}{3}(1,1) + A_1$ singularities. Theorem 7.17 implies that φ_1 is either of type (E.2), (E.4), (E.5) or (E.6): indeed, φ_1 can not be a conic bundle because X_1 contains singularities of type $\frac{1}{3}(1,1)$ but none of type A_2, and it is clear from the classification that X_1 is not a del Pezzo surface with $\rho = 1$.

If φ_1 is of type (E.2), then X_2 is a del Pezzo surface with $k = 5$ singularities. As we assume the case $k = 5$ is known, this implies that $X_2 = X_{5,5/3}$, a contradiction since $\rho(X_2) \leq 3$ while $\rho(X_{5,5/3}) = 5$.

If φ_1 is a type (E.4) contraction, the surface X_2 has $4 \times \frac{1}{3}(1,1) + 2 \times A_1$ singularities and its Picard number is at most three. From this surface, only contractions of type (E.2), (E.4) and (E.6) are possible: once again φ_2 cannot be of fibering type because of the presence of $\frac{1}{3}(1,1)$ singularities and the absence of those of type A_2, and by Theorem 7.17 it is not a del Pezzo surface of $\rho = 1$.

We show that none of these possibilities occur. Indeed, if φ_2 is of type (E.2), the del Pezzo surface X_3 has $\rho \leq 2$ and $4 \times \frac{1}{3}(1,1) + A_1$ singularities. The following contraction φ_3 cannot be a fibration because the number of singularities is even, and X_3 cannot be of Picard rank one by the classification in Theorem 7.17. Since a second (E.2) contraction should have already been performed as φ_1, φ_3 can only be of type (E.4), (E.5) or (E.6). These lead to surfaces of $\rho = 1$ and singularities of type $3 \times \frac{1}{3}(1,1) + 2 \times A_1$, $3 \times \frac{1}{3}(1,1)$ and $2 \times \frac{1}{3}(1,1) + A_1 + A_2$ respectively, none of which appear in the list in Theorem 7.17.

The same reasoning holds if φ_2 is of type (E.4). In this case, the del Pezzo surface X_3 has $\rho \leq 2$ and $3 \times \frac{1}{3}(1,1) + 3 \times A_1$ singularities. By the classification in Theorem 7.17, X_3 is clearly not a conic bundle over $\mathbb{P}^1$, nor does it have $\rho = 1$. The next contraction φ_3 can be either of type (E.2), (E.4), (E.5) or (E.6) and leads to a surface X_4 of Picard rank one. The first three cases result in at least two points of type $\frac{1}{3}(1,1)$ on X_4, while the contraction of type (E.6) means that X_4 has $\frac{1}{3}(1,1) + A_2 + 3 \times A_1$ singularities. None of these correspond to one of the $\rho = 1$ surfaces in Theorem 7.17, since the only surface of this type containing a $\frac{1}{3}(1,1)$ point is $\mathbb{P}(1,1,3)$.

Finally, if φ_2 is of type (E.6), then X_3 has $2 \times \frac{1}{3} + 2 \times A_1 + A_2$ singularities. Theorem 7.17 implies that the only possible contractions on X_3 are of type (E.3) and (E.6). Indeed, the odd number of singularities and the presence of two $\frac{1}{3}(1,1)$ points allow us to respectively eliminate the possibilities of φ_3 being of fibering type and X_3 having $\rho = 1$. A contraction of type (E.2) would have been available earlier in the directed MMP since by performing φ_2 no new singularities of type A_1 were created. The same is true for contractions of type (E.4) and (E.5) which, if available, should have been done prior to the one of type (E.6). As before, both contractions would lead to a non-existent del Pezzo surface of $\rho = 1$: if φ_3 is of type (E.3), then X_4 has $2 \times \frac{1}{3} + 2 \times A_1$ singularities and if φ_3 is of type (E.6) we obtain $2 \times A_1 + 2 \times A_2$ singularities, a contradiction to Theorem 7.17.

If φ_1 is of type (E.5), the del Pezzo surface X_2 has exactly four $\frac{1}{3}(1,1)$ points. From our discussion in the case $k = 4$ we obtain that $X_2 = X_{4,7/3}$ and since $\rho(X_2) \leq 2$ and $\rho(X_{4,7/3}) = 5$, this is a contradiction.

If φ_1 is of type (E.6), X_2 has $3 \times \frac{1}{3} + A_1 + A_2$ singularities. This surface is not of Picard rank one, and by theorem 7.17 it is not a conic fibration, as it has an odd number of singularities and too many $\frac{1}{3}(1,1)$ points. Since by remark 7.20 a contraction of type (E.3) cannot follow one of type (E.6) and considering the order of the contractions in the directed MMP, the only possibility left is that φ_2 is of type (E.6). X_3 now has singularities of type $\frac{1}{3} + A_1 + 2 \times A_2$ and no further divisorial contractions are available, again because we cannot follow with one of type (E.3) and all other possiblities would have been performed earlier. The singularities on X_3 do not however correspond to any of the surfaces of $\rho = 1$ in Theorem 7.17, nor can they be paired on singular fibers of a conic fibration, for instance because there is only one point of type A_1. We have thus exhausted all the possibilities for φ_1.

All of this shows that φ_0 can not be of type (E.4), thus it is a contraction of type (E.6) and all the divisorial contractions that follow must be of the same type. The surface X_1 can not be of $\rho = 1$ or of fibering type since it has too many $\frac{1}{3}(1,1)$ points and an odd number of singularities, thus φ_1 is also an (E.6) contraction. As depicted in Figure 7.4, by Theorem 7.17 we have two possiblities: either X_2 is a conic bundle with two special fibres of type (C.2), or one last divisorial contraction is available, leading to the $\rho = 1$ surface (D.4).

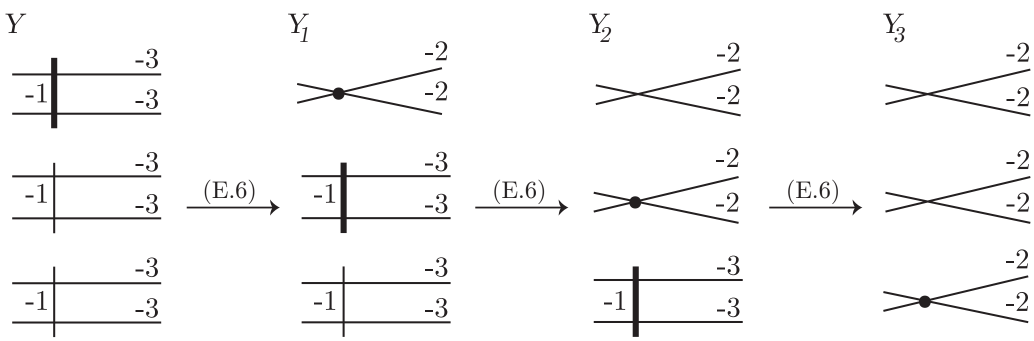
Figure 7.4: $k = 6$ tree of possibilities

$\rho \leq 5$		$\rho \leq 4$		$\rho \leq 3$		$\rho \leq 2$
6 x 1/3	$\xrightarrow{\text{(E.6)}}$	4 x 1/3 + A_2	$\xrightarrow{\text{(E.6)}}$	2 x 1/3 + 2 x A_2 $\boxed{\text{2 x (C.2)}}$	$\xrightarrow{\text{(E.6)}}$	3 x A_2

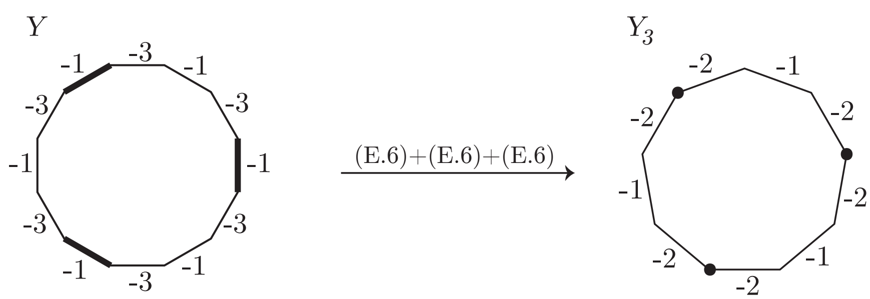
Both instances occur and, as we will see, they lead to the same surface X. This makes sense since at the very beginning of the MMP there are a total of six contractions of the same type available and at each step we choose one at the expense of two others. Depending on their configuration we stop after either two or three contractions, thus obtaining two end products of the directed MMP for the same X.

Case 1 $(E.6) + (E.6) + (E.6)$

The curve configuration for this case is:



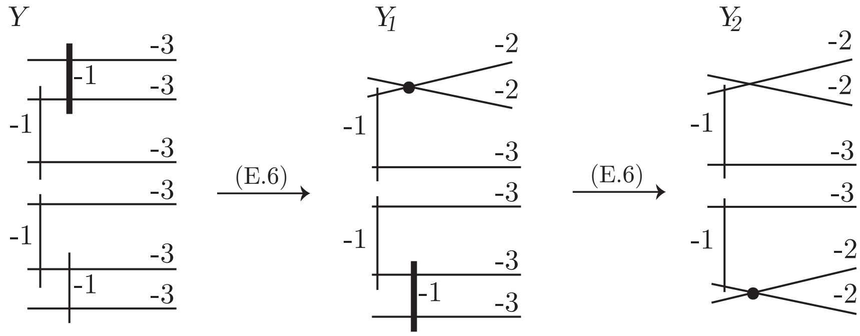
Looking more closely at how Y is built from Y_3—the surface of Picard rank one having three singular points of type A_2 described in Theorem 7.17 —by a sequence of blow-ups, we see that Y must have the negative curves shown in the following figure:



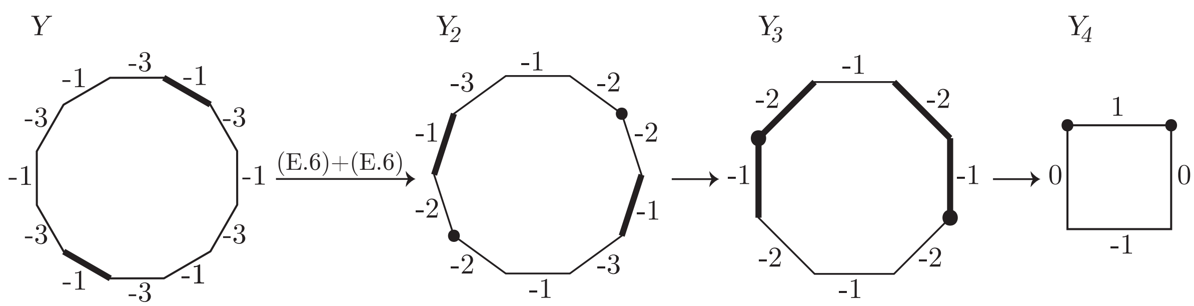
Case 2 $(E.6) + (E.6)$

This sequence ends with a conic fibration having two singular fibers. On the minimal resolutions, the contractions are the following:

Figure 7.5: Configuration of curves for $k = 6$, Case 2



We again proceed to run the nonsingular minimal model program for the surface Y_2 relative to the current fibration over $\mathbb{P}^1$. As before, we eventually reach a surface $\mathbb{F}_k$, where $k \in \{0, 1, 2\}$, and we choose the sequence of contractions such that k is maximal. Figure 7.5 shows that all (-3)-curves on Y either already existed on the special fibers of Y_2 or they come from blowing up points inside these fibers. Thus if $k = 2$, the (-2)-section remains as such even on Y, which is impossible according to lemma 7.19. By maximality, $k = 1$, and then the minimal resolutions must have the negative curves shown in the following figure:



Cases 1 and 2 are two different directed MMPs starting from the del Pezzo surface $X_{6,2}$.

We further give the trees of possibilities for the four remaining cases. When constructing the tree for a surface with k_0 singularities, we are allowed to use the final working cases for the surfaces with $k < k_0$ points of type $\frac{1}{3}(1, 1)$. This not only shortens the process, but also allows us to further exclude certain sequences of contractions. Indeed, suppose that in the k_0 tree a branch leads to a del Pezzo surface with singularity content $k \times \frac{1}{3}(1, 1)$, where $k < k_0$. If the sequence in the statement of Theorem 7.21 for k doesn't correlate with the directed MMP obtained thus far in the k_0 tree, then the entire branch can be removed.

Figure 7.6: $k = 1$ Tree

$\rho \leq 9$		$\rho \leq 8$		$\rho \leq 7$
1 x 1/3 $\boxed{\mathbb{P}(1,1,3)}$	$\xrightarrow{(E.4)}$	A_1	$\xrightarrow{(E.2)}$	$\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$

Figure 7.7: $k = 2$ Tree

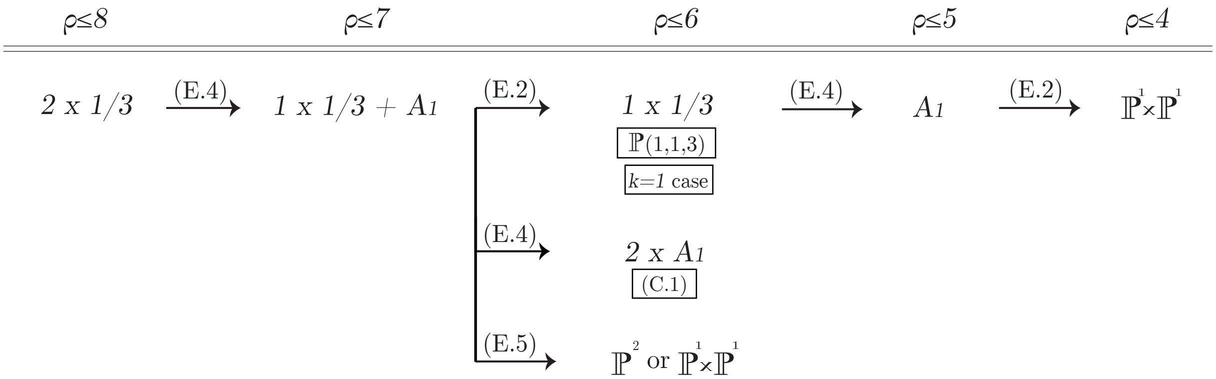
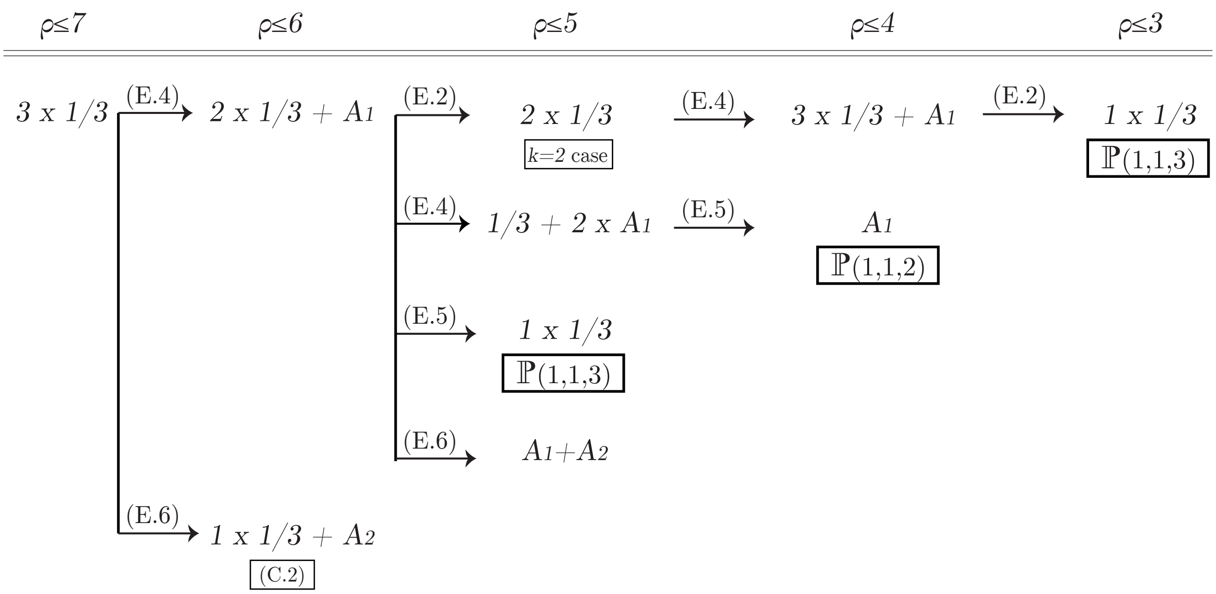


Figure 7.8: $k = 3$ Tree



Finally, using the techniques presented so far in this section will lead to a systematic elimination of the branches so that we are left with exactly the surfaces in the statement.

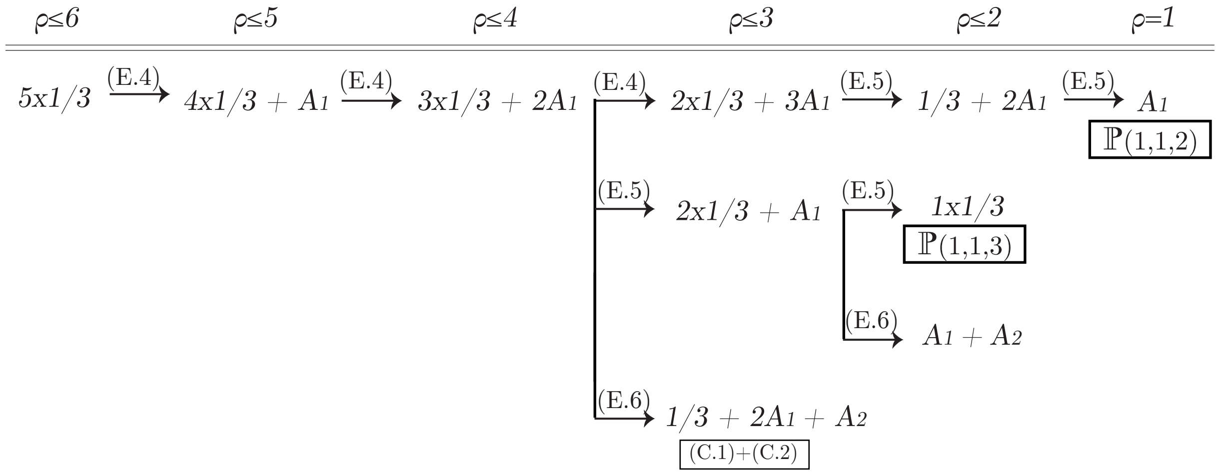
□

7.5 Tables

Tables 7.2 and 7.3 summarise the classification and provide constructions for a general surface in each family. We explain how to read the tables. We focus on Table 7.3 since Table 7.2 is a straightforward illustration of Theorem 7.3. We then show, in the particular case of $X_{1,10/3}$, that the toric data leads to the surface which we found using the classification.

The symbol $X_{k,d}$ in the first column of Table 7.3 signifies the family of surfaces X with k singular points, degree $K_X^2 = d$ and $f = 1$: the next three columns display the invariants $h^0(X, -K_X)$, the rank $r = \operatorname{rk} H^2(Y; \mathbb{Z}) = \rho(Y) = k + \rho(X)$ where Y is the minimal resolution of X, and the dimension

Figure 7.9: $k = 5$ Tree



of the family.

In all cases except $X_{1,7/3}$, $X_{5,5/3}$, $X_{5,2/3}$, $X_{6,2}$ and $X_{6,1}$, the next column shows a well-formed simplicial toric variety F and line bundles $L_1, \ldots, L_c$ on F such that a general complete intersection of type $(L_1, \ldots, L_c)$ on F is a quasi-smooth and well-formed surface of the family $X_{k,d}$. The last column computes the cone $\operatorname{Nef} F$. This information is necessary to verify that the following conditions hold:

(a) the $L_i \in \operatorname{Nef} F$ and

(b) $-K_F - \Lambda \in \operatorname{Amp} F$, where $\Lambda = \sum_{i=1}^{c} L_i$.

Since, by the adjunction formula, $-K_X = -(K_F + \Lambda)_{|X}$ is ample, the constructions make it manifest that X is a Fano variety. We then verify that a general complete intersection of type $(L_1, \ldots, L_c)$ on F has $k\ \frac{1}{3}(1,1)$ singularities, and compute k and the anticanonical degree $d = K_X^2$.

We explain in more detail how to read the information in the last two columns, and reference [CH] for details on the five remaining cases.

The typical entry

All cases except $X_{1,7/3}$, $X_{5,5/3}$, $X_{5,2/3}$, $X_{6,2}$ and $X_{6,1}$ are typical. In a typical case, the table gives the weight matrix of an action of $\mathbb{C}^{\times l}$ on $\mathbb{C}^m$ such that $F = \mathbb{C}^m // (\mathbb{C}^{\times l})$ and, to the right of this and separated by a vertical line, a sequence of column vectors representing the line bundles L_i.

For example, the entry for $X_{4,4/3}$ shows that an example of a surface X with $k = 4$ singularities and $K^2 = \frac{4}{3}$ can be constructed as a complete intersection of two general sections of the line bundles $L_1 = (2,4)$ and $L_2 = (4,2)$ in the Fano simplicial toric variety F given by weight matrix:

x_0	x_1	x_2	x_3	x_4	x_5
1	2	2	1	1	0
0	1	1	2	2	1

and $\operatorname{Nef} F = \langle (2,1), (1,2) \rangle$ (the notation is explained fully in Section 7.6 below). Here $\Lambda = L_1 + L_2 \sim (6,6)$, $-(K_F + \Lambda) \sim (1,1)$ and $-K_F \sim (7,7)$ are all ample.

Remark 7.23. Now that we have established what we are looking for, let us discuss the motivation of obtaining this precise type of model. In [ACC+16] mirror symmetry for a locally qG-rigid del Pezzo surface is stated in terms of a qG-degeneration to a toric surface: thus it is crucial for us to determine which families admit such a degeneration. The mirror symmetry conjecture B of [ACC+16] computes the quantum orbifold cohomology of a locally qG-rigid surface X from data attached to the toric qG-degeneration. In order to compute the quantum orbifold cohomology of a surface X by the known technology of abelian/nonabelian correspondence and quantum Lefschetz [CFKS08, CG07, Tse10], and thus give evidence for conjecture B of [ACC+16], we need a model of X as a complete intersection in a rep quotient variety (cf. [CH, Def.10]). In this context, we need conditions (a) and (b) to control the asymptotics of certain I-functions, and this motivates our constructions here. Conditions (a) and (b) are of course also natural from a purely classification-theoretic perspective. Paper [OP] computes (part of) the quantum orbifold cohomology of our surfaces.

Table 7.2: del Pezzo surfaces with $1/3(1,1)$ and $f > 1$

Name	$h^0(X, -K_X)$	r	No. moduli	Model Construction	f
$S_{1,\,25/3}$	9	2	-8	$\mathbb{P}(1,1,3)$	5
$B_{1,\,16/3}$	6	5	-2	$X_4 \subset \mathbb{P}(1,1,1,3)$	2
$B_{2,\,8/3}$	3	8	2	$X_6 \subset \mathbb{P}(1,1,3,3)$	2

Table 7.3: del Pezzo surfaces with $1/3(1,1)$ and $f = 1$

Name	$h^0(X, -K_X)$	r	No. moduli	Weights and Line bundles	Nef F
$X_{1,\,22/3}$	8	3	-6	$\begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array}$
$X_{1,\,19/3}$	7	4	-4	$\begin{array}{ccccc} 1 & 3 & 3 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{array}$	$\begin{array}{ccc} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{array}$
$X_{1,\,16/3}$	6	5	-2	$\begin{array}{ccccc\|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}$	$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$
$X_{1,\,13/3}$	5	6	0	$\begin{array}{ccccc\|c} 1 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}$	$\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}$
$X_{1,\,10/3}$	4	7	2	$\begin{array}{cccccc\|cc} 1 & 1 & 2 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 \end{array}$	$\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array}$
$X_{1,\,7/3}$	3	8	4	$F = \mathrm{wGr}(2,5)$ and $\mathcal{O}_F(2)^{\oplus 4}$ where $w = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right)$	1
$X_{1,\,4/3}$	2	9	6	$\begin{array}{ccccc\|cc} 1 & 1 & 2 & 2 & 3 & 4 & 4 \end{array}$	1

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Name	$h^0(X, -K_X)$	r	No. moduli	Weights and Line bundles	Nef F
$X_{1,\,1/3}$	1	10	8	$\begin{array}{cccc\|c} 1 & 2 & 3 & 5 & 10 \end{array}$	1
$X_{2,\,17/3}$	6	5	-4	$\begin{array}{ccccc\|c} 1 & 1 & 2 & 3 & 0 & 4 \\ -1 & 0 & 1 & 3 & 1 & 2 \end{array}$	$\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}$
$X_{2,\,14/3}$	5	6	-2	$\begin{array}{ccccc\|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 & 1 & 4 \end{array}$	$\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}$
$X_{2,\,11/3}$	4	7	0	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 4 & 4 \end{array}$	$\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 4 & 4 & 4 \end{array}$
$X_{2,\,8/3}$	3	8	2	$\begin{array}{ccccc\|c} 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}$	$\begin{array}{cc} 1 & 1 \\ 1 & 3 \end{array}$
$X_{2,\,5/3}$	2	9	4	$\begin{array}{ccccc\|c} 1 & 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 3 & 3 & 1 & 6 \end{array}$	$\begin{array}{cc} 2 & 1 \\ 3 & 3 \end{array}$
$X_{2,\,2/3}$	1	10	6	$\begin{array}{ccccc\|cc} 1 & 2 & 2 & 3 & 3 & 4 & 6 \end{array}$	1
$X_{3,\,5}$	5	6	-4	$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{array}$	$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}$
$X_{3,\,4}$	4	7	-2	$\begin{array}{cccccc} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \end{array}$	$\begin{array}{ccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 & 4 \end{array}$
$X_{3,\,3}$	3	8	0	$\begin{array}{ccccc\|c} 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}$	$\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}$
$X_{3,\,2}$	2	9	2	$\begin{array}{ccccc\|c} 1 & 3 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}$	$\begin{array}{cc} 2 & 0 \\ 1 & 1 \end{array}$
$X_{3,\,1}$	1	10	4	$\begin{array}{cccccc\|c} 1 & 0 & 0 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 2 & 4 \end{array}$	$\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}$
$X_{4,\,7/3}$	2	9	0	$\begin{array}{ccccc\|c} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 3 & 3 & 2 & 1 & 6 \end{array}$	$\begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array}$
$X_{4,\,4/3}$	1	10	2	$\begin{array}{cccccccc\|cc} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 2 \end{array}$	$\begin{array}{cccc} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}$
$X_{4,\,1/3}$	0	11	4	$\begin{array}{ccccc\|cc} 2 & 2 & 3 & 3 & 3 & 6 & 6 \end{array}$	1

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Name	$h^0(X, -K_X)$	r	No. moduli	Weights and Line bundles	Nef F
$X_{5,\,5/3}$	1	10	0	F and $D(s)$ where $s\colon E \otimes L \to E^\vee$ as in [CH, § 3.3]	
$X_{5,\,2/3}$	0	11	2	F and $L^{\oplus 2}$ as in [CH, § 3.4]	
$X_{6,\,2}$	1	10	-2	F/μ_3 (see [CH, § 2.2.5]) where F has weights $\begin{array}{cccccc} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}$	$\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}$
$X_{6,\,1}$	0	11	0	$\mathbb{P}^3/\mu_3$ and $\mathcal{O}(3)$ where μ_3 acts with weights $\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	

7.6 Computations

In what follows, we recall how to build a toric variety from the data in the table and how to compute its singularities.

From a GIT quotient to a fan Consider a rank r lattice $\mathbb{L}^* \cong \mathbb{Z}^r$ and denote by $\mathbb{G}$ the torus with character group $\mathbb{L}^*$. Consider now $\mathbb{Z}^{*\,m}$, denote by x_i the standard basis elements, let $D\colon \mathbb{Z}^{*\,m} \to \mathbb{L}^*$ be a group homomorphism such that the $D_i = D(x_i)$ span a strictly convex cone $\mathcal{C} \subset \mathbb{L}^*_{\mathbb{R}}$. D dualises to a group homomorphism $\mathbb{G} \to \mathbb{C}^{\times\,m}$ and hence $\mathbb{G}$ acts on $\mathbb{C}^m$.

Definition-Remark 7.24. It is easy to see [Ahm] that:

(1) Choose a basis of $\mathbb{L}^* \cong \mathbb{Z}^r$ and identify D with a $r \times m$ matrix, which we call the *weight matrix*. $\mathbb{G}$ acts faithfully if and only if the rows of D span a saturated sublattice of $\mathbb{Z}^r$, if and only if the hcf of all the $r \times r$ minors of D is 1. A matrix satisfying this condition is called *standard*.

(2) $\mathbb{G}$ acts faithfully on the divisor $D_i = (x_i = 0) \subset \mathbb{C}^m$ if and only if the matrix $D_{\widehat{i}} = (D_1 \ldots, \widehat{D}_i, \ldots, D_m)$ obtained from D by removing the i-th column, is standard.

Definition 7.25. The homomorphism $D\colon \mathbb{Z}^{*\,m} \to \mathbb{L}^*$ is *well-formed* if both the weight matrix D and the $D_{\widehat{i}}$ for all $i = 1, \ldots, m$ are standard.

Remark 7.26. We can take GIT quotients for any D; however, if D is the divisor homomorphism of some toric variety X_Σ, then D is well-formed. The aim of the considerations that follow is precisely to state that the converse is also true.

Given an element $\omega \in \mathbb{L}^*_{\mathbb{R}}$, which we refer to as a stability condition, we can form the GIT quotient

$$X_\omega := \mathbb{C}^m /\!/_\omega \mathbb{G},$$

which we detail below. There is a wall-and-chamber decomposition of $\mathcal{C} \subset \mathbb{L}^*_{\mathbb{R}}$, called the secondary fan, and if stability conditions ω_1, ω_2 lie in the same chamber then the GIT quotients X_{ω_1}, X_{ω_2} coincide. More precisely, the walls of the decomposition are the cones of the form $\langle D_{i_1}, \ldots, D_{i_k} \rangle \subset \mathbb{L}^*_{\mathbb{R}}$ that have codimension one. The chambers are the connected components of the complement of the union of all the walls; these are r-dimensional open cones in $\mathcal{C}$. By construction, a chamber is the intersection of the interiors of the simplicial r-dimensional cones $\langle D_{i_1}, \ldots, D_{i_r} \rangle \subset \mathbb{L}^*_{\mathbb{R}}$ that contain it. Choose now a chamber, and pick a stability condition ω in it. Given such an ω, the irrelevant ideal $I_\omega \subset \mathbb{C}[x_1, \ldots, x_m]$ is

$$I_\omega = \Big(x_{i_1} \cdots x_{i_r} \mid \omega \in \langle D_{i_1}, \ldots, D_{i_r} \rangle \Big)$$

the unstable locus is $Z_\omega = V(I_\omega)$; and the GIT quotient is the bona fide quotient

$$X_\omega = (\mathbb{C}^m \setminus Z_\omega)/\mathbb{G}\,.$$

Note that I_ω, Z_ω and the quotient X_ω depend only on the chamber that ω sits in and not on ω itself. Given such an ω we can also form a simplicial fan Σ where

$$\sigma \in \Sigma \quad \text{if and only if} \quad \omega \in \langle D_i \mid i \notin \sigma \rangle$$

and Σ also depends only on the chamber that ω sits in.

Charts on GIT quotients We explain how to set up an explicit atlas of charts on $X_\omega = [(\mathbb{C}^m \setminus Z_\omega)/\mathbb{G}]$, which we use repeatedly in the calculations needed to validate the entries of Table 7.3. Fix a well-formed $D \colon \mathbb{Z}^{*m} \to \mathbb{L}^*$, choose a basis of $\mathbb{L}^*$, identify D with an integral $r \times m$ matrix. We have that $\mathbb{C}^m \setminus Z_\omega$ is a union of $\mathbb{G}$-invariant open subsets:

$$\mathbb{C}^m \setminus Z_\omega = \bigcup_{\{(i_1,\ldots,i_r) \mid \omega \in \langle D_{i_1},\ldots,D_{i_r} \rangle\}} U_{i_1,\ldots,i_r} \quad \text{where} \quad U_{i_1,\ldots,i_r} = \{x_{i_1} \neq 0, \ldots, x_{i_r} \neq 0\} \subset \mathbb{C}^m$$

Let now $V_{i_1,\ldots,i_r} = \{x_{i_1} = \cdots = x_{i_r} = 1\} \subset \mathbb{C}^m$, then $[U_{i_1,\ldots,i_r}/\mathbb{G}] = [V_{i_1,\ldots,i_r}/\mu\!\!\!/]$ where $\mu\!\!\!/$ is the finite subgroup of $\mathbb{G}$ that fixes $V_{i_1,\ldots,i_r}$. Concretely, $\mu\!\!\!/$ is the finite group with character group A, the cokernel of the homomorphism:

$$D_{i_1,\ldots,i_r} = (D_{i_1}, \ldots, D_{i_r}) \colon \mathbb{Z}^{*r} \to \mathbb{L}^*$$

Complete intersections in toric varieties Consider a well-formed $D \colon \mathbb{Z}^{*m} \to \mathbb{L}^*$ as above. Fix a chamber of the secondary fan, a stability condition in it, and let $F = X_\Sigma$ be the corresponding simplicial toric variety. (the models constructed here are such that F is always Fano, and we assume that $\omega = D_1 + \cdots + D_m$ is the anticanonical divisor of F. This assumption however is irrelevant for the present discussion.) We consider complete intersections $X \subset F$ of general elements of linear systems $|L_1|, \ldots, |L_c|$ where $L_i \in \mathbb{L}^*$ are $\mathbb{G}$-linearised line bundles, that is, line bundles on X_Σ.[2] The space of sections $H^0(F, L_i)$ is the vector subspace of $\mathbb{C}[x_1, \ldots, x_m]$ with basis consisting of monomials $x^v \in \mathbb{C}[x_1, \ldots, x_m]$ where $v \in \mathbb{Z}^{*m}$ has homogeneity type L_i, that is $D(v) = L_i$. Let $f_i \in H^0(F, L_i)$, then $V(f_1, \ldots, f_c)$ is stable under the action of $\mathbb{G}$, and we consider the subvariety $X = (V(f_1, \ldots, f_c) \setminus Z_\omega)/\mathbb{G} \subset F$:

[2] More precisely, line bundles on the canonical DM stack of X_Σ.

Definition 7.27. 1. $X \subset F$ is *quasi-smooth* if either: $V(f_1, \ldots, f_c) \subset Z_\omega$, or

$$V(f_1, \ldots, f_c) \setminus Z_\omega \subset \mathbb{C}^m \setminus Z_\omega$$

is a smooth subvariety of codimension c;

2. Suppose that $X \subset F$ is quasi-smooth. We say that X is *well-formed* if the following holds: For all toric strata $S \subset F$ with nontrivial stabilizer, $S \subset X$ implies $\operatorname{codim}_X S \geq 2$.

7.6.1 A sample computation

In the column labelled "Weights and Line bundles," all lines of Table 7.3, except those corresponding to families $X_{1,7/3}$, $X_{5,5/3}$, $X_{5,2/3}$, $X_{6,2}$ and $X_{6,1}$, list a well-formed weight matrix

$$D\colon \mathbb{Z}^{*\,m} \to \mathbb{L}^* = \mathbb{Z}^r$$

for constructing a simplicial toric variety F and, to the right of it and separated by a vertical line, a list of column vectors $L_i \in \mathbb{Z}^r$, representing line bundles on F such that X is a complete intersection of general members of the $|L_i|$. The last column is a list of column vectors in $\mathbb{Z}^r$, the generators of Nef F, which is the chamber of the secondary fan that contains the stability conditions that give F as GIT quotient. In all cases it is immediate to verify that the $L_i \in \operatorname{Nef} F$ and that $-K_F - \Lambda \in \operatorname{Amp} F$ where $\Lambda = \sum L_i$. In particular it follows from this that X is a Fano variety.

Family $X_{1,10/3}$

As stated in Corollary 7.8, a surface X in this family is either: (i) The blow-up of $\mathbb{P}(1,1,3)$ at $d = 5$ general points; or (equivalently) (ii) The blow-up of $B_{1,16/3}$ at $d = 2$ general points.

According to the table, a surface in this family can be constructed as a codimension 2 complete intersection of type $L_1 = (2,2)$, $L_2 = (2,2)$ in the (manifestly well-formed) simplicial toric variety F with weight matrix:

x_0	x_1	x_2	x_3	x_4	x_5
1	1	2	1	0	0
0	0	1	2	1	1

and Nef $F = \langle L + 2M, 2L + M\rangle$, where $L = (1,0)$ and $M = (0,1)$ are the standard basis vectors of $\mathbb{L}^*$. Note that both L_1, L_2 are ample, and $-(K_F + L_1 + L_2) \sim L + M$ is ample.

First we examine all the charts of F and verify that X is a quasi-smooth well-formed complete intersection with $1 \times \frac{1}{3}(1,1)$ singularities. Finally we calculate $K_X^2 = 10/3$.

The chamber is $\langle x_2, x_3\rangle$, so the irrelevant ideal for the given stability condition is

$$\operatorname{Irr} = (x_0, x_1, x_2)(x_3, x_4, x_5),$$

and the charts are the U_{ij} with $i \leq 2$, $3 \leq j$.

Let us first look at the chart $U_{03} = \{x_0 \neq 0, x_3 \neq 0\}$. Considering $V_{03} = \{x_0 = x_3 = 1\} \subset \mathbb{C}^6$ it is immediate that:

$$U_{03} = \frac{1}{2}(0,1,1,1)_{x_1,x_2,x_4,x_5}$$

the quotient of $V_{03} \cong \mathbb{C}^4$ with coordinates x_1, x_2, x_4, x_5 by the action of μ_2 with weights $(0,1,1,1)$. We see that the x_1-axis C is a curve toric stratum of the 4-fold F with stabilizer μ_2 at the generic point. We claim that $C \cap X = \emptyset$. Indeed $C = \{x_2 = x_4 = x_5 = 0\}$ is the toric variety with weight matrix:

$$\begin{array}{ccc} x_0 & x_1 & x_3 \\ \hline 1 & 1 & 1 \\ 0 & 0 & 2 \end{array}$$

Note, however, that this matrix is not well-formed. Applying the algorithm in [Ahm], we see that C, together with the line bundles $L_{1|C}$, $L_{2|C}$, is the toric variety with well-formed weight matrix

$$\begin{array}{ccc} x_0 & x_1 & x_3 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

and line bundles $L_{1|C} = L_{2|C} = (1,1)$, which is manifestly the same as $\mathbb{P}^1$ with $L_{1|C} = L_{2|C} = \mathcal{O}(1)$. It is clear that the two restriction maps $H^0(F, L_i) = \langle x_0x_3, x_1x_3 \rangle \to H^0(C, L_{i|C})$ are surjective and thus two general members of L_1 and L_2 do not intersect anywhere on C.

The chart U_{13} is very similar; and the charts U_{04}, U_{05}, U_{14}, U_{15} are smooth and it is immediate that none of the strata passing through those charts are contained in the base locus of $|L_i|$; thus, we only need to look at U_{23}.

Considering $V_{23} = \{x_2 = x_3 = 1\} \subset \mathbb{C}^6$ it is easy to see that:

$$U_{23} = \frac{1}{3}(1,1,1,1)_{x_0,x_1,x_4,x_5}$$

the quotient of $V_{23} \cong \mathbb{C}^4$ with coordinates x_0, x_1, x_4, x_5 by the action of μ_3 with weights $(1,1,1,1)$. Denote by $f_i \in H^0(F, L_i)$ general members: the monomials $x_0x_3, x_1x_3, x_2x_4, x_3x_4$ all appear in f_i with nonzero coefficient, thus the surface X must contain the origin of this chart, it is quasi-smooth there, and it has a singularity $1/3(1,1)$ there. This completes the verification that X is well-formed and has $1 \times 1/3(1,1)$ singularities.

We now compute the degree of X. The Chow ring of F is generated by $L = (1,0)$ and $M = (0,1)$ with the relations $L^2(2L+M) = 0$, $(L+2M)M^2 = 0$ (corresponding to the components (x_0, x_1, x_2), (x_3, x_4, x_5) of Irr), and, for example, $L^2M^2 = 1/3$ obtained by looking at the chart U_{23}. From this information, we get that $L^3M = -(1/2)L^2M^2 = -1/6$ and $L^4 = -(1/2)L^3M = 1/12$ and similarly $M^4 = 1/2$, $M^3L = -1/6$ and then it is easy to compute:

$$K_X^2 = L_1L_2(-K_F - L_1 - L_2)^2 = (2L+2M)^2(L+M)^2 = 4(L+M)^4 = 4\Big(\frac{1}{12} - \frac{4}{6} + \frac{6}{3} - \frac{4}{6} + \frac{1}{12}\Big) = \frac{10}{3}.$$

Bibliography

[ACC+16] Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten. Mirror symmetry and the classification of orbifold del Pezzo surfaces. *Proc. Amer. Math. Soc.*, 144(2):513–527, 2016.

[AGZV12] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1.* Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition.

[Ahm] Hamid Ahmadinezhad. On pliability of del Pezzo fibrations and Cox rings. `arXiv:1304.4357 [math.AG]`.

[AK] Mohammad Akhtar and Alexander Kasprzyk. Singularity content. `arXiv:1401.5458 [math.AG]`.

[Ale94] Valery Alexeev. General elephants of $\mathbf{Q}$-Fano 3-folds. *Compositio Math.*, 91(1):91–116, 1994.

[Amb] Florin Ambro. The locus of log canonical singularities. `arXiv:math/9806067 [math.AG]`.

[BHN15] Mauro C. Beltrametti, Andreas Höring, and Carla Novelli. Fano varieties with small non-klt locus. *Int. Math. Res. Not. IMRN*, (11):3094–3120, 2015.

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[BR86] Mauro Beltrametti and Lorenzo Robbiano. Introduction to the theory of weighted projective spaces. *Exposition. Math.*, 4(2):111–162, 1986.

[Cam91] F. Campana. On twistor spaces of the class C. *J. Differential Geom.*, 33(2):541–549, 1991.

[Cam92] F. Campana. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4)*, 25(5):539–545, 1992.

[CF93] F. Campana and H. Flenner. Projective threefolds containing a smooth rational surface with ample normal bundle. *J. Reine Angew. Math.*, 440:77–98, 1993.

[CFHR99] Fabrizio Catanese, Marco Franciosi, Klaus Hulek, and Miles Reid. Embeddings of curves and surfaces. *Nagoya Math. J.*, 154:185–220, 1999.

[CFKS08] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Claude Sabbah. The abelian/nonabelian correspondence and Frobenius manifolds. *Invent. Math.*, 171(2):301–343, 2008.

[CG07] Tom Coates and Alexander Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.

[CH] Alessio Corti and Liana Heuberger. Del Pezzo surfaces with $\frac{1}{3}(1,1)$ points. arxiv:1505.02092 [math.AG].

[CKP] Tom Coates, Alexander Kasprzyk, and Thomas Prince. Laurent inversion. arXiv:1505.01855 [math.AG].

[CP07] Ivan Cheltsov and Jihun Park. Two remarks on sextic double solids. *J. Number Theory*, 122(1):1–12, 2007.

[CS16] Ivan Cheltsov and Constantin Shramov. *Cremona groups and the icosahedron*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[FG12] Osamu Fujino and Yoshinori Gongyo. On canonical bundle formulas and subadjunctions. *Michigan Math. J.*, 61(2):255–264, 2012.

[Flo13] Enrica Floris. Fundamental divisors on Fano varieties of index $n-3$. *Geom. Dedicata*, 162:1–7, 2013.

[Fuj11] Osamu Fujino. Fundamental theorems for the log minimal model program. *Publ. Res. Inst. Math. Sci.*, 47(3):727–789, 2011.

[Fuj14] Osamu Fujino. Fundamental theorems for semi log canonical pairs. *Algebr. Geom.*, 1(2):194–228, 2014.

[Fur86] Mikio Furushima. Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbf{C}^3$. *Nagoya Math. J.*, 104:1–28, 1986.

[FY] Kento Fujita and Kazunori Yasutake. Classification of log del Pezzo surfaces of index three. arXiv:1401.1283 [math.AG].

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har86] Robin Hartshorne. Generalized divisors on Gorenstein curves and a theorem of Noether. *J. Math. Kyoto Univ.*, 26(3):375–386, 1986.

[HV11] Andreas Höring and Claire Voisin. Anticanonical divisors and curve classes on Fano manifolds. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1371–1393, 2011.

[IF00] A. R. Iano-Fletcher. Working with weighted complete intersections. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 101–173. Cambridge Univ. Press, Cambridge, 2000.

[IP99] V.A Iskovskikh and Yuri Prokhorov. *Algebraic geometry. V*, volume 47 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1999. Fano varieties, A translation of ıt Algebraic geometry. 5 (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation edited by A. N. Parshin and I. R. Shafarevich.

[Isk77] V. A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(3):516–562, 717, 1977.

[Isk78] V. A. Iskovskih. Fano threefolds. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(3):506–549, 1978.

[K$^+$92] János Kollár et al. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Kaw97] Yujiro Kawamata. On Fujita's freeness conjecture for 3-folds and 4-folds. *Math. Ann.*, 308(3):491–505, 1997.

[Kaw00] Yujiro Kawamata. On effective non-vanishing and base-point-freeness. *Asian J. Math.*, 4(1):173–181, 2000. Kodaira's issue.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[KM99] Seán Keel and James M^{c}Kernan. Rational curves on quasi-projective surfaces. *Mem. Amer. Math. Soc.*, 140(669):viii+153, 1999.

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[KNP] Alexander M. Kasprzyk, Benjamin Nill, and Thomas Prince. Minimality and mutation-equivalence of polygons. `arXiv:1501.05335 [math.AG]`.

[Kol91] János Kollár. Flips, flops, minimal models, etc. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 113–199. Lehigh Univ., Bethlehem, PA, 1991.

[Kol97] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Mel99] Massimiliano Mella. Existence of good divisors on Mukai varieties. *J. Algebraic Geom.*, 8(2):197–206, 1999.

[MM82] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82.

[Mor85a] Shigefumi Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.

[Mor85b] David R. Morrison. The birational geometry of surfaces with rational double points. *Math. Ann.*, 271(3):415–438, 1985.

[OP] Alessandro Oneto and Andrea Petracci. On the quantum periods of del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities. `arXiv:1507.08589 [math.AG]`.

[Pet11] Thomas Peternell. Generically nef vector bundles and geometric applications. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 345–368. Springer, Heidelberg, 2011.

[Pri] Thomas Prince. Smoothing Toric Fano Surfaces Using the Gross–Siebert Algorithm. `arXiv:1504.05969 [math.AG]`.

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[RS03] Miles Reid and Kaori Suzuki. Cascades of projections from log del Pezzo surfaces. In *Number theory and algebraic geometry*, volume 303 of *London Math. Soc. Lecture Note Ser.*, pages 227–249. Cambridge Univ. Press, Cambridge, 2003.

[San96] Takeshi Sano. Classification of non-Gorenstein $\mathbf{Q}$-Fano d-folds of Fano index greater than $d-2$. *Nagoya Math. J.*, 142:133–143, 1996.

[Šok79] V. V. Šokurov. Smoothness of a general anticanonical divisor on a Fano variety. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):430–441, 1979.

[Tse10] Hsian-Hua Tseng. Orbifold quantum Riemann-Roch, Lefschetz and Serre. *Geom. Topol.*, 14(1):1–81, 2010.

[Tve] Ketil Tveiten. Period integrals and mutation. `arXiv:1501.05095 [math.AG]`.

[Wen95] Lin Weng. A result on bicanonical maps of surfaces of general type. *Osaka J. Math.*, 32(2):467–473, 1995.

[Xia85] Gang Xiao. Finitude de l'application bicanonique des surfaces de type général. *Bull. Soc. Math. France*, 113(1):23–51, 1985.