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Catégories amassées supérieures et frises tropicales

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To my dearest parents and
to Zhao Zhonghua

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Résumé

Résumé

Cette thèse est consacrée aux objets m-amas basculants dans les catégories m-amassées généralisées et aux frises tropicales associées aux diagrammes de Dynkin. La catégorie amassée généralisée qui provient d'une algèbre différentielle graduée 3-Calabi-Yau convenable a été introduite par C. Amiot. Elle est Hom-finie, 2-Calabi-Yau et admet un objet amas-basculant canonique. Dans cette thèse, nous étendons ces résultats au cas où l'algèbre différentielle graduée initiale est $(m+2)$-Calabi-Yau pour un entier positif arbitraire m. Nous montrons que la catégorie m-amassée généralisée associée est Hom-finie, $(m+1)$-Calabi-Yau et admet un objet m-amas basculant canonique. Dans cette catégorie triangulée, nous obtenons une classe d'objets m-amas basculants grâce aux mutations d'objets pré-basculants et aux équivalences dérivées. Pour les catégories m-amassées généralisées qui proviennent des algèbres différentielles graduées fortement $(m+2)$-Calabi-Yau, nous prouvons que chaque P-objet amas-basculant presque complet admet exactement $m+1$ compléments avec la propriété de péridicité. Finalement, inspiré par le travail de Ringel sur les fonctions amas-additives sur des carquois à translation stables, nous introduisons les frises tropicales sur des catégories 2-Calabi-Yau munies d'objet amas-basculant. Nous montrons que chaque frise tropicale sur la catégorie amassée d'un carquois de Dynkin est d'une forme spéciale et donnons une preuve d'une conjecture de Ringel sur la forme des fonctions amas-additives.

Mots-clefs

Catégories m-amassées généralisées, Objets m-amas basculants, Compléments, Frises tropicales.

On generalized higher cluster categories and tropical friezes

Abstract

This thesis is concerned with higher cluster tilting objects in generalized higher cluster categories and tropical friezes associated with Dynkin diagrams. The generalized cluster category arising from a suitable 3-Calabi-Yau differential graded algebra was introduced by C. Amiot. It is Hom-finite, 2-Calabi-Yau and admits a canonical cluster-tilting object. In this thesis, we extend these results to the case where the initial differential graded algebra is $(m + 2)$-Calabi-Yau for an arbitrary positive integer m. We show that its associated generalized m-cluster category is Hom-finite, $(m + 1)$-Calabi-Yau and admits a canonical m-cluster tilting object. In this triangulated category, we obtain a class of m-cluster tilting objects by taking advantage of silting mutation and derived equivalence. For generalized m-cluster categories arising from strongly $(m+2)$-Calabi-Yau differential graded algebras, we prove that each almost complete m-cluster tilting P-object admits exactly $m+1$ complements with periodicity property. Finally, inspired by Ringel's work on cluster-additive functions on stable translation quivers, we introduce tropical friezes on 2-Calabi-Yau categories with cluster-tilting object. We show that any tropical frieze on the cluster category of a Dynkin quiver is of a special form and give a proof of a conjecture of Ringel on the form of cluster-additive functions.

Keywords

Generalized m-cluster categories, m-cluster tilting objects, Complements, Tropical friezes.

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Chapitre 1

Preliminaries

1.1 Triangulated categories

In this section, we recall some basic definitions and properties of triangulated categories, and recall the facts on t-structure which we will use in Chapters 3 and 4.

Our main references for this section are [11], [44], [62] and [77].

1.1.1 Foundations

Let $\mathcal{T}$ be an additive category endowed with an automorphism Σ, which is usually called the *suspension functor*. The inverse of Σ is denoted by Σ^{-1}. A *sextuple* (X, Y, Z, u, v, w) is given by three objects $X, Y, Z \in \mathcal{T}$ and three morphisms $u : X \to Y, v : Y \to Z, w : Z \to \Sigma X$. A more customary notation of sextuples is

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

A *morphism of sextuples* from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a tuple (f, g, h) of morphisms such that the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'. \end{tikzcd}$$

Moreover, if f, g and h are isomorphisms in $\mathcal{T}$, then (f, g, h) is called an *isomorphism of sextuples.*

Definition 1.1.1. An additive category $\mathcal{T}$ with suspension functor Σ is called a *triangulated category* if it is endowed with a class $\mathcal{U}$ of sextuples (called *triangles*) which satisfies the following axioms (TR1) to (TR4) :

(TR1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $u : X \to Y$ in $\mathcal{T}$ can be embedded into a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$. For every object X of $\mathcal{T}$, the sextuple $X \xrightarrow{id_X} X \to 0 \to \Sigma X$ is a triangle.

(TR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle, then $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is a triangle.

(TR3) Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w'), and morphisms f and g satisfying $u'f = gu$, there exists a morphism (f, g, h) of triangles.

(TR4) Let

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{j} \Sigma X$$

$$Y \xrightarrow{v} Z \xrightarrow{i'} X' \xrightarrow{j'} \Sigma Y$$

$$X \xrightarrow{vu} Z \xrightarrow{i''} Y' \xrightarrow{j''} \Sigma X$$

be three triangles. There exist two morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that the following diagram commutes :

$$\begin{tikzcd}
 & \Sigma^{-1}X' \arrow[r, equal] \arrow[d, "-\Sigma^{-1}j'"] & \Sigma^{-1}X' \arrow[d] & \\
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "j"] \arrow[d, "f"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "i''"] \arrow[d, "i'"] & Y' \arrow[r, "j''"] \arrow[d, "g"] & \Sigma X \arrow[d, "\Sigma u"] \\
 & X' \arrow[r, equal] & X' \arrow[r, "j'"] & \Sigma Y
\end{tikzcd}$$

where the two middle rows and the two middle columns are triangles.

Remark 1.1.2. A different way of displaying the axiom (TR4) is given by an octahedron. Therefore, axiom (TR4) is also called the *octahedral axiom*.

Let $(\mathcal{T}, \Sigma, \mathcal{U})$ and $(\mathcal{T}', \Sigma', \mathcal{U}')$ be two triangulated categories. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ is called a *triangle functor* or an *exact functor* if there exists an invertible natural transformation $\alpha : F\Sigma \to \Sigma' F$ such that (FX, FY, FZ, Fu, Fv, Fw) is a triangle in $\mathcal{U}'$ whenever (X, Y, Z, u, v, w) is a triangle in $\mathcal{U}$.

Proposition 1.1.3 ([77])**.** *Let $\mathcal{T}$ be a triangulated category. Let (X, Y, Z, u, v, w) be a triangle and M an object of $\mathcal{T}$. Then*

a) $vu = wv = 0$.

b) The following long exact sequences are exact :

$$\cdots \to \mathcal{T}(M, \Sigma^i X) \to \mathcal{T}(M, \Sigma^i Y) \to \mathcal{T}(M, \Sigma^i Z) \to \mathcal{T}(M, \Sigma^{i+1} X) \to \dots$$

$$\dots \to \mathcal{T}(\Sigma^{i+1} X, M) \to \mathcal{T}(\Sigma^i Z, M) \to \mathcal{T}(\Sigma^i Y, M) \to \mathcal{T}(\Sigma^i X, M) \to \dots$$

c) Let (f, g, h) be a morphism of triangles. If two of the three morphisms are isomorphisms, then so is the third.

Proposition 1.1.4 ([44])**.** *Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be two triangles in a triangulated category $\mathcal{T}$. Let $g : Y \to Y'$ be a morphism. Then the following are equivalent :*

a) $v'gu = 0$.

b) There exists a morphism (f, g, h) from the first triangle to the second.

1.1.2 t-structure

Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is given by two strictly (*i. e.* stable under isomorphisms) full subcategories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ which satisfy the following three conditions :

a) for $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$, we have that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$,

b) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$,

c) for any object $X \in \mathcal{T}$, there exists a triangle $X' \to X \to X'' \to \Sigma X'$ such that $X' \in \mathcal{T}^{\leq 0}$ and $X'' \in \mathcal{T}^{\geq 1}$,

where $\mathcal{T}^{\leq n}$ denotes $\Sigma^{-n}(\mathcal{T}^{\leq 0})$ and $\mathcal{T}^{\geq n}$ denotes $\Sigma^{-n}(\mathcal{T}^{\geq 0})$ for any $n \in \mathbb{Z}$.

Denote by $\mathcal{H}$ the full subcategory $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ of $\mathcal{T}$. It is called the *heart of the t-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. The heart $\mathcal{H}$ of a t-structure is an abelian category [11].

Now we recall the work on aisles of Keller-Vossieck, who gave an alternative description of t-structures. In Chapters 3 and 4, we use this work to obtain the existence of a canonical t-structure as in Section 2 of [2].

A strictly full subcategory $\mathcal{A}$ of $\mathcal{T}$ is called an *aisle* if it is stable under shifts Σ^l ($l \in \mathbb{N}$) and extensions, and the inclusion $\mathcal{A} \to \mathcal{T}$ admits a right adjoint.

For a full subcategory $\mathcal{U}$ of $\mathcal{T}$ we denote by $\mathcal{U}^{\perp}$ (resp. ${}^{\perp}\mathcal{U}$) the full subcategory consisting of the objects $Y \in \mathcal{T}$ such that $\mathrm{Hom}(X, Y) = 0$ (resp. $\mathrm{Hom}(Y, X) = 0$) for all $X \in \mathcal{U}$.

Proposition 1.1.5 ([62])**.** *A strictly full subcategory $\mathcal{A}$ is an aisle if and only if $(\mathcal{A}, (\Sigma\mathcal{A})^{\perp})$ is a t-structure.*

1.2 Derived categories and derived functors

An important class of triangulated categories is the one of the derived categories of differential graded algebras. Besides, we also use the theory of derived functors between derived categories.

Our main references for this section are [50] and [52].

1.2.1 Derived categories

Let k be a commutative ring.

Definition 1.2.1. A *differential graded k-algebra* (for simplicity, dg k-algebra) is a graded k-algebra $A = \oplus_{n \in \mathbb{Z}} A^n$ equipped with a k-linear homogeneous map $d_A : A \to A$ of degree 1 such that $d_A^2 = 0$ and the graded Leibniz rule $d_A(ab) = d_A(a)b + (-1)^n a d_A(b)$ holds, where $a \in A^n$ and $b \in A$. The map d_A is called a *differential* on A.

An ordinary k-algebra can be viewed as a dg k-algebra concentrated in degree 0 whose differential is trivial. A graded k-algebra can be viewed as a dg k-algebra with the zero differential.

Definition 1.2.2. A (right) *differential graded A-module* (for simplicity, dg A-module) is a (right) graded A-module $M = \oplus_{n \in \mathbb{Z}} M^n$ equipped with a k-linear homogeneous map $d_M : M \to M$ of degree 1 such that $d_M^2 = 0$ and the graded Leibniz rule $d_M(ma) = d_M(m)a + (-1)^n m d_A(a)$ holds for all $m \in M^n$ and $a \in A$. The map d_M is called a *differential* on M.

The *homology* of a dg algebra A is defined on each degree by

$$H^n A = \mathrm{Ker} d_A^n / \mathrm{Im} d_A^{n-1}.$$

In a similar way, the *homology* of a dg A-module M is defined on each degree by

$$H^n M = \mathrm{Ker} d_M^n / \mathrm{Im} d_M^{n-1}.$$

Let M and N be two dg A-modules. The dg k-module $\mathrm{Hom}_A^{\bullet}(M, N)$ is defined as follows :

a) for each integer n, the nth-component $\mathrm{Hom}_A^n(M,N)$ of $\mathrm{Hom}_A^\bullet(M,N)$ is the subset of $\prod_{j\in\mathbb{Z}} \mathrm{Hom}_k(M^j, N^{j+n})$ whose elements $f=(f_j)_j$ satisfy that

$$f_j(m)a = f_{j+l}(ma), \quad m \in M^j, a \in A^l;$$

b) the differential of $\mathrm{Hom}_A^\bullet(M,N)$ is defined by

$$d^n(f) = d_N \circ f - (-1)^n f \circ d_M,$$

where f is in $\mathrm{Hom}_A^n(M,N)$.

The kernel of d^0 (denoted by $Z^0\mathrm{Hom}_A^\bullet(M,N)$) consists of the elements f in the zeroth component $\mathrm{Hom}_A^0(M,N)$ which commute with the differentials, that is, $d_N \circ f = f \circ d_M$. The zeroth homology $H^0\mathrm{Hom}_A^\bullet(M,N)$ is just the quotient of $Z^0\mathrm{Hom}_A^\bullet(M,N)$ by the homotopy relation :

$$f \sim g \Longleftrightarrow \exists\, s \in \mathrm{Hom}_A^{-1}(M,N) \,\text{such that}\, f-g = d_N \circ s + s \circ d_M.$$

The *category of dg A-modules* $\mathcal{C}(A)$ is the category whose objects are dg A-modules and morphism spaces are given by

$$\mathrm{Hom}_{\mathcal{C}(A)}(M,N) = Z^0\mathrm{Hom}_A^\bullet(M,N)$$

for all dg A-modules M and N. The category $\mathcal{C}(A)$ is an abelian category. The *homotopy category* $\mathcal{H}(A)$ of A has the same objects as $\mathcal{C}(A)$, and its morphism spaces are given by

$$\mathrm{Hom}_{\mathcal{H}(A)}(M,N) = H^0\mathrm{Hom}_A^\bullet(M,N)$$

for all dg A-modules M and N. The category $\mathcal{H}(A)$ is a triangulated category. A *quasi-isomorphism* from M to N is a morphism in $\mathcal{H}(A)$ which induces an isomorphism $H^iM \simeq H^iN$ for each i.

Let $\mathcal{N}$ be the full subcategory of $\mathcal{H}(A)$ whose objects are those N such that there is a triangle

$$X \xrightarrow{s} Y \to N \to \Sigma X$$

with s a quasi-isomorphism. Then $\mathcal{N}$ is a thick subcategory (definition see 1.3.1) of $\mathcal{H}(A)$.

Definition 1.2.3. The *derived category* $\mathcal{D}(A)$ of A is defined as the triangulated quotient category (definition see section 1.3)

$$\mathcal{D}(A) := \mathcal{H}(A)/\mathcal{N}.$$

The derived category $\mathcal{D}(A)$ is triangulated, has arbitrary coproducts and products.

1.2.2 Derived functors

Let k be a commutative ring. Let A and B be two dg k-algebras.

Definition 1.2.4. A *differential graded left A right B bimodule* (for simplicity, dg A-B-bimodule) is a graded A-B-bimodule $M = \oplus_{n\in\mathbb{Z}} M^n$ equipped with a differential $d_M : M \to M$ such that M is a left dg A-module and a right dg B-module.

Let L be a right dg A-module and N a right dg B-module. Then $L \otimes_A M$ admits a natural right B-module structure and $\mathrm{Hom}^\bullet_B(M, N)$ admits a natural right A-module structure. The dg A-B-module M gives rise to an adjoint pair $(- \otimes_A M, \mathrm{Hom}^\bullet_B(M, -))$ between the categories of dg modules :

$$\begin{tikzcd} \mathcal{C}(A) \arrow[r, shift left, "- \otimes_A M"] & \mathcal{C}(B) \arrow[l, shift left, "\mathrm{Hom}^\bullet_B(M,-)"] \end{tikzcd}$$

This adjoint pair induces an adjoint pair between the homotopy categories :

$$\begin{tikzcd} \mathcal{H}(A) \arrow[r, shift left, "- \otimes_A M"] & \mathcal{H}(B) \arrow[l, shift left, "\mathrm{Hom}^\bullet_B(M,-)"] \end{tikzcd}$$

However, in general these two functors are not well-defined triangle functors between the derived categories.

Definition 1.2.5. a) A dg A-module P is *cofibrant* if

$$\mathrm{Hom}_{\mathcal{C}(A)}(P, L) \xrightarrow{s_*} \mathrm{Hom}_{\mathcal{C}(A)}(P, N)$$

is surjective for each quasi-isomorphism $s : L \to N$ which is surjective in each component.

b) A dg A-module I is *fibrant* if

$$\mathrm{Hom}_{\mathcal{C}(A)}(N, I) \xrightarrow{i^*} \mathrm{Hom}_{\mathcal{C}(A)}(L, I)$$

is surjective for each quasi-isomorphism $i : L \to N$ which is injective in each component.

A dg A-module is cofibrant if and only if it is a direct summand of a dg A-module P which admits a filtration

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \ldots \subset F_p \subset F_{p+1} \subset \ldots \subset P, \quad p \in \mathbb{N}$$

in $\mathcal{C}(A)$ such that

a) P is the union of the F_p, $p \in \mathbb{N}$;
b) as graded A-modules, for each p, F_p is a direct summand of F_{p+1} ;
c) for each p, the subquotient F_{p+1}/F_p is isomorphic in $\mathcal{C}(A)$ to a direct summand of a direct sum of modules of the form $\Sigma^n A$, $n \in \mathbb{Z}$.

Proposition 1.2.6 ([50]). *The canonical triangle functor $\pi : \mathcal{H}(A) \to \mathcal{D}(A)$ admits a left adjoint $\mathbf{p}$ and a right adjoint $\mathbf{i}$ such that for each object X of $\mathcal{D}(A)$,*

a) the object $\mathbf{p}X$ is cofibrant and the object $\mathbf{i}X$ is fibrant, and
b) there exist quasi-isomorphisms $\mathbf{p}X \to X$ and $X \to \mathbf{i}X$.

We call $\mathbf{p}X$ a *cofibrant resolution* of X and $\mathbf{i}X$ a *fibrant resolution* of X.

Now we have the following diagram of triangle functors :

$$\begin{tikzcd} \mathcal{H}(A) \arrow[r, shift left, "- \otimes_A M"] \arrow[d, shift left, "\pi_A"] & \mathcal{H}(B) \arrow[l, shift left, "\mathrm{Hom}^\bullet_B(M,-)"] \arrow[d, shift right, "\pi_B"'] \\ \mathcal{D}(A) \arrow[u, shift left, "\mathbf{p}"] & \mathcal{D}(B) \arrow[u, shift right, "\mathbf{i}"'] \end{tikzcd}$$

Definition 1.2.7. Let M be a dg A-B-bimodule. The *left derived functor* $-\overset{L}{\otimes}_A M : \mathcal{D}(A) \to \mathcal{D}(B)$ is defined as the composition $\pi_B \circ (-\otimes_A M) \circ \mathbf{p}$. The *right derived functor* $\mathrm{RHom}_B(M,-) : \mathcal{D}(B) \to \mathcal{D}(A)$ is defined as the composition $\pi_A \circ \mathrm{Hom}^\bullet_B(M,-) \circ \mathbf{i}$.

There is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(B)}(L \overset{L}{\otimes}_A M, N) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L, \mathrm{RHom}_B(M,N))$$

for each dg A-module L and dg B-module N. The left derived functor $-\overset{L}{\otimes}_A M$ and the right derived functor $\mathrm{RHom}_B(M,-)$ form an adjoint pair.

1.3 Triangulated quotients

In this thesis, we study triangulated quotients of subcategories (namely, perfect derived categories) of the derived categories of suitable dg algebras. Our main references for this section are [2] and [77].

Let $\mathcal{T}$ be a triangulated category.

Definition 1.3.1. An additive subcategory $\mathcal{N}$ of $\mathcal{T}$ is a *thick* subcategory if $\mathcal{N}$ is a full triangulated subcategory (*i. e.* Σ is an automorphism of $\mathcal{N}$ and $\mathcal{N}$ is closed under extensions) of $\mathcal{T}$ and satisfies that : for all triangles

$$X \xrightarrow{f} Y \to N \to \Sigma X$$

where $N \in \mathcal{N}$ and f factors through an object of $\mathcal{N}$, the objects X and Y belong to $\mathcal{N}$.

Theorem 1.3.2 (J. Rickard)**.** *Let $\mathcal{N}$ be a full triangulated subcategory of $\mathcal{T}$. Then $\mathcal{N}$ is thick if and only if $\mathcal{N}$ is closed under taking direct summands.*

Given a thick subcategory $\mathcal{N}$ of $\mathcal{T}$, the *triangulated quotient* (denoted as $\mathcal{T}/\mathcal{N}$) is the category constructed as follows :

- The objects of $\mathcal{T}/\mathcal{N}$ are the objects of $\mathcal{T}$.
- The morphisms in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X,Y)$ are the equivalence classes $s^{-1}f$ of diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

where s and f are morphisms in $\mathcal{T}$, and s is contained in a triangle

$$Z \xrightarrow{s} X \to N \to \Sigma Z$$

with N an object of $\mathcal{N}$, while the equivalence relation is given by :

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

are equivalent if there exist another such diagram

$$\begin{tikzcd} & Z'' \arrow[dl, "s''"'] \arrow[dr, "f''"] & \\ X & & Y \end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & Z'' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u, "t"'] \arrow[d, "t'"] & Y \\ & Z' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Let $s^{-1}f$ be in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X,Y)$ and $t^{-1}g$ in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(Y,Z)$. Suppose that f is in $\mathrm{Hom}_{\mathcal{T}}(X',Y)$ and t is in $\mathrm{Hom}_{\mathcal{T}}(Y',Y)$, and the morphism t is contained in a triangle

$$Y' \xrightarrow{t} Y \xrightarrow{q} N \to \Sigma Y'$$

with $N \in \mathcal{N}$. The morphism $qf \in \mathrm{Hom}_{\mathcal{T}}(X',N)$ can be embedded into a triangle

$$W \to X' \xrightarrow{qf} N \to \Sigma W.$$

The commutative diagram

$$\begin{tikzcd} X' \arrow[r, "qf"] \arrow[d, "f"'] & N \arrow[d, equal] \\ Y \arrow[r, "q"] & N \end{tikzcd}$$

can be completed to the following commutative diagram

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d, "h"'] & X' \arrow[r, "qf"] \arrow[d, "f"'] & N \arrow[r] \arrow[d, equal] & \Sigma W \arrow[d, "\Sigma h"'] \\ Y' \arrow[r, "t"'] & Y \arrow[r, "q"'] & N \arrow[r] & \Sigma Y'. \end{tikzcd}$$

Then there is a new diagram

$$\begin{tikzcd} & & W \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

with $fr = th$. The octahedral axiom (TR4) ensures that $(sr)^{-1}(gh)$ lies in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X,Z)$. The *composition* of $s^{-1}f$ and $t^{-1}g$ is defined as the morphism $(sr)^{-1}(gh)$. It is well-defined.

For each morphism $s \in \mathrm{Hom}_{\mathcal{T}}(X,Y)$ which is contained in a triangle

$$X \xrightarrow{s} Y \to N \to \Sigma X$$

with $N \in \mathcal{N}$, the morphism $(id_X)^{-1}s$ is an isomorphism in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Y)$ whose inverse is $s^{-1}(id_X)$.

The canonical functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ sends each object to itself and sends each morphism $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ to the morphism $(id_X)^{-1}f \in \mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Y)$. The image of the objects in $\mathcal{N}$ under Q are zero objects in $\mathcal{T}/\mathcal{N}$. The functor Q induces a triangulated structure on $\mathcal{T}/\mathcal{N}$.

Proposition 1.3.3 ([77]). *For any triangle functor $F : \mathcal{T} \to \mathcal{T}'$ which sends the objects of a thick subcategory $\mathcal{N}$ of $\mathcal{T}$ to zero objects of $\mathcal{T}'$, there exists a unique triangle functor $F' : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$ such that $F' \circ Q = F$:*

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{T}' \\ & \mathcal{T}/\mathcal{N} \arrow[ur, dotted, "F'"'] & \end{tikzcd}$$

Let k be a field and $\mathcal{T}$ a k-linear triangulated category. Assume that there is an automorphism ν on $\mathcal{T}$ such that $\nu(\mathcal{N}) \subset \mathcal{N}$ and a non-degenerate bilinear form

$$\beta_{N,X} : \mathcal{T}(N, X) \times \mathcal{T}(X, \nu(N)) \longrightarrow k$$

which is bifunctorial both in $N \in \mathcal{N}$ and in $X \in \mathcal{T}$. In Section 1 of [2], Amiot constructed a related form on the triangulated quotient :

$$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X, Y) \times \mathcal{T}/\mathcal{N}(Y, \Sigma^{-1}\nu(X)) \longrightarrow k$$

The form β' is well-defined, bilinear and bifunctorial.

Definition 1.3.4. Let X and Y be two objects in $\mathcal{T}$.

a) A morphism $p : N \to X$ is called a *local $\mathcal{N}$-cover of X relative to Y* if N is in $\mathcal{N}$ and p induces an exact sequence

$$0 \to \mathcal{T}(X, Y) \xrightarrow{p^*} \mathcal{T}(N, Y).$$

b) A morphism $i : X \to N$ is called a *local $\mathcal{N}$-envelope of X relative to Y* if N is in $\mathcal{N}$ and i induces an exact sequence

$$0 \to \mathcal{T}(Y, X) \xrightarrow{i_*} \mathcal{T}(Y, N).$$

Theorem 1.3.5 ([2]). *Let X and Y be two objects of $\mathcal{T}$. If there exists a local $\mathcal{N}$-cover of X relative to Y and a local $\mathcal{N}$-envelope of νX relative to Y, then the bilinear form $\beta'_{X,Y}$ is non-degenerate.*

1.4 2-Calabi-Yau categories

Cluster algebras were invented by S. Fomin and A. Zelevinsky [34] in order to develop a combinatorial approach to the total positivity in algebraic groups [66] and the canonical bases in quantum groups [49] [65]. The categorification of cluster algebras has attracted a lot of attention. In this section, we list three classes of triangulated categories whose most important property is to be Calabi-Yau of dimension 2. Each of them gives rise to some additive categorification of cluster algebras.

Our main references for this section are [2], [17], [20], [37] and [70].

1.4.1 Cluster categories

Let k be an algebraically closed field. Let Q be a connected finite acyclic quiver with vertex set $\{1, \ldots, n\}$. We denote the finite-dimensional derived category of the module category of finite-dimensional right kQ-modules by $\mathcal{D}_{fd}(\mathrm{mod}kQ)$, the Auslander-Reiten translation in $\mathcal{D}_{fd}(\mathrm{mod}kQ)$ by τ and the suspension functor by Σ.

Definition 1.4.1. The *cluster category* $\mathcal{C}_Q$ of Q is the orbit category $\mathcal{D}_{fd}(\mathrm{mod}kQ)/\tau^{-1}\Sigma$:
- The objects are the objects of $\mathcal{D}_{fd}(\mathrm{mod}kQ)$.
- For any $X, Y \in \mathcal{C}_Q$, the morphism space is

$$\mathrm{Hom}_{\mathcal{C}_Q}(X, Y) = \coprod_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_{fd}(\mathrm{mod}kQ)}((\tau^{-1}\Sigma)^n X, Y).$$

Proposition 1.4.2 ([53])**.** *The cluster category* $\mathcal{C}_Q$ *is triangulated and the canonical functor* $\pi : \mathcal{D}_{fd}(\mathrm{mod}kQ) \to \mathcal{C}_Q$ *is a triangle functor.*

Several important properties of the cluster category $\mathcal{C}_Q$ were proved in [17] :

a) It is a Krull-Schmidt category.

b) It is 2-Calabi-Yau, that is, there are bifunctorial isomorphisms

$$D\mathrm{Hom}_{\mathcal{C}_Q}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}_Q}(Y, \Sigma^2 X), \quad X, Y \in \mathcal{C}_Q.$$

c) The set $\mathrm{ind}(kQ) \coprod \{\Sigma P_i | i \in Q_0\}$ is a complete set of representatives for indecomposable objects of $\mathcal{C}_Q$, where ind(kQ) is a complete set of representatives for indecomposable right kQ-modules and P_i are indecomposable projective right kQ-modules.

An object T of $\mathcal{C}_Q$ is called a *cluster-tilting object* if T is rigid, *i. e.* $\mathrm{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$ and if for each object L satisfying $\mathrm{Ext}^1_{\mathcal{C}_Q}(T, L) = 0$, we have that L belongs to the subcategory $\mathrm{add}T$ of direct summands of finite direct sums of copies of T. The image of kQ in $\mathcal{C}_Q$ is a cluster-tilting object. Each rigid indecomposable object of $\mathcal{C}_Q$ is contained in a cluster-tilting object. Let T be a basic cluster-tilting object of $\mathcal{C}_Q$ with $T = T_1 \oplus \ldots \oplus T_n$ a decomposition of T into indecomposables. The quiver Q_T of the endomorphism algebra of T is defined by :
- the vertex set is $\{1, \ldots, n\}$;
- the number of arrows from i to j equals the dimension of the vector space $\mathrm{irr}_T(T_i, T_j)$ (by definition, this is $\mathrm{rad}_T(T_i, T_j)/\mathrm{rad}^2_T(T_i, T_j)$, where $\mathrm{rad}_T(T_i, T_j)$ denotes the vector space of non isomorphisms from T_i to T_j).

Proposition 1.4.3 ([20])**.** *The quiver* Q_T *does not have loops or 2-cycles.*

A basic rigid object $\overline{T}$ in $\mathcal{C}_Q$ is called an *almost complete cluster-tilting object* if there exists an indecomposable object M (not in $\mathrm{add}T$) such that $\overline{T} \oplus M$ is a cluster-tilting object. The object M is called a *complement* of $\overline{T}$.

Theorem 1.4.4 ([17])**.** *Each almost complete cluster-tilting object in* $\mathcal{C}_Q$ *admits exactly two complements.*

More precisely, given one complement M to an almost complete cluster-tilting object $\overline{T}$, the other can be constructed using approximation theory. Indeed, there is a triangle

$$M^* \to B \xrightarrow{f} M \to \Sigma M^*$$

in $\mathcal{C}_Q$, where f is a minimal right (add$\overline{T}$)-approximation of M in $\mathcal{C}_Q$ and M^* is the other complement of $\overline{T}$. Dually, there is another triangle

$$M \xrightarrow{g} B' \to M^* \to \Sigma M$$

in $\mathcal{C}_Q$ with g a minimal left (add$\overline{T}$)-approximation of M in $\mathcal{C}_Q$. It was shown also in [17] that two indecomposable objects M and M^* form such an exchange pair if and only if

$$\mathrm{dimExt}^1_{\mathcal{C}_Q}(M, M^*) = 1 = \mathrm{dimExt}^1_{\mathcal{C}_Q}(M^*, M).$$

Let T^* denote $\overline{T} \oplus M^*$. It is called the *mutation* of T at M.

Proposition 1.4.5 ([17]). *Any two basic cluster-tilting objects are linked by a finite sequence of mutations.*

Let Q be a finite quiver without loops or 2-cycles and i a vertex of Q. The *mutation* of Q at i is the quiver $\mu_i(Q)$ obtained from Q as follows :

i) for each subquiver $j \xrightarrow{b} i \xrightarrow{a} l$, add a new arrow $j \xrightarrow{[ab]} l$;
ii) reverse all arrows α incident with i, denote the new ones as α^* ;
iii) remove all the arrows in a maximal set of pairwise disjoint 2-cycles.

Theorem 1.4.6 ([20]). *The quiver Q_{T^*} of the endomorphism algebra of T^* over $\mathcal{C}_Q$ is the mutation of Q_T at its vertex corresponding to M.*

1.4.2 Preprojective algebras

Let k be an algebraically closed field and Q a connected finite acyclic quiver. Its *double quiver* $\overline{Q}$ is obtained from Q by adding an arrow a^* in the opposite direction for each arrow a of Q. The *preprojective algebra* Λ_Q is the quotient of the path algebra $k\overline{Q}$ by the ideal generated by the element $\sum_{a\in Q_1}(aa^* - a^*a)$. It is a selfinjective algebra. Let modΛ_Q denote the category of finite-dimensional right Λ_Q-modules and nilΛ_Q the subcategory of modΛ_Q consisting of the objects which admit composition series given by the simple modules associated with the vertices. The algebra Λ_Q is finite-dimensional if and only if Q is of Dynkin type A, D or E if and only if modΛ_Q = nilΛ_Q.

Definitions 1.4.7. An object T in nilΛ_Q is

- *rigid* if $\mathrm{Ext}^1_{\Lambda_Q}(T, T) = 0$;
- *maximal rigid* if T is rigid and X lies in add$T$ whenever $T \oplus X$ is rigid ;
- *cluster-tilting* if T is rigid and X lies in add$T$ whenever $\mathrm{Ext}^1_{\Lambda_Q}(T, X) = 0$.

An easy fact is that each indecomposable projective-injective Λ_Q-module is a direct summand of any maximal rigid Λ_Q-module.

From now on, assume that Q is a Dynkin quiver. Let $T = T_1 \oplus \ldots \oplus T_r$ be a basic maximal rigid Λ_Q-module with T_i indecomposable for all i. Without loss of generality, assume that $T_{r-n+1}, \ldots, T_r$ are projective-injective. The quiver Q_T of the endomorphism algebra of T over Λ_Q is defined as in subsection 1.4.1. The *ice quiver* Q^0_T of the endomorphism algebra of T over Λ_Q is the subquiver of Q_T such that there are no arrows between any vertices $i, j \in \{r - n + 1, \ldots, r\}$. The vertices $r - n + 1, \ldots, r$ are often called *frozen vertices*. The mutation of an ice quiver is defined as the mutation of a quiver but only mutation with respect to non frozen vertices are allowed and no arrows are drawn between the frozen vertices.

The following results all come from the work of Geiss, Leclerc and Schröer.

Theorem 1.4.8 ([37]). *Let Q be a Dynkin quiver and T a Λ_Q-module. Then T is maximal rigid if and only if T is cluster-tilting. Moreover, the quiver Q_T of the endomorphism algebra of T over Λ_Q does not have loops or 2-cycles.*

Theorem 1.4.9 ([37]). *Let T be a basic maximal rigid object in* $\mathrm{mod}\Lambda_Q$ *with Q a Dynkin quiver. Suppose that $T = \overline{T} \oplus X$ with X indecomposable non projective-injective and that $f : X \to T'$ is a minimal left* ($\mathrm{add}\overline{T}$)*-approximation of X. Then f is a monomorphism and there is a short exact sequence*

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0.$$

Moreover, the object Y is indecomposable and not isomorphic to X such that $\overline{T} \oplus Y$ is a new maximal rigid object.

Let $\mu_X(T)$ denote $\overline{T} \oplus Y$. We call it the *mutation* of T at X.

Theorem 1.4.10 ([37]). *Keep the above notation. Then*

$$\mathrm{dimExt}^1_{\Lambda_Q}(X, Y) = 1 = \mathrm{dimExt}^1_{\Lambda_Q}(Y, X),$$

$$\textit{and} \quad Q^0_{\mu_X(T)} = \mu_X(Q^0_T),$$

where $Q^0_{\mu_X(T)}$ is the ice quiver of the endomorphism algebra of $\mu_X(T)$ and $\mu_X(Q^0_T)$ is the mutation of the ice quiver Q^0_T at the vertex corresponding to X.

1.4.3 Generalized cluster categories

Let k be a field. Let Q be a finite quiver and kQ its path algebra. Let $[kQ, kQ]$ denote the subspace of kQ generated by all commutators $[a, b] = ab - ba$. The quotient $kQ/[kQ, kQ]$ admits a basis formed by the cycles of Q. For each arrow a of Q, the *cyclic derivative with respect to a* is the unique linear map

$$\partial_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path p to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path p as a concatenation of paths u, a, v. A *potential* on Q is an element W of $kQ/[kQ, kQ]$ which is a linear combination of cycles of length ≥ 1 in Q.

Definitions 1.4.11. Let Q be a finite quiver and W a potential on Q. Let $\widetilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (they all have degree 0),
- the arrows $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- the loops t_i of degree -2 associated with each vertex i of Q.

a) The *Ginzburg dg algebra* $\Gamma(Q, W)$ is the dg k-algebra whose underlying graded algebra is the graded path algebra $k\widetilde{Q}$ and whose endowed differential is the unique linear endomorphism homogeneous of degree 1 such that on the generators
   - $d(a) = 0$ for each arrow a of Q,
   - $d(a^*) = \partial_a W$ for each arrow a of Q,
   - $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q, where e_i is the idempotent associated with i and the sum runs over the set of arrows of Q.

b) The *Jacobian algebra* $J(Q, W)$ is the zeroth homology of the Ginzburg dg algebra $\Gamma(Q, W)$ defined as

$$J(Q, W) = kQ/\langle \partial_a W | a \in Q_1 \rangle.$$

Let Γ denote $\Gamma(Q,W)$. Let $\mathcal{D}(\Gamma)$ be the derived category of Γ, $\mathrm{per}\Gamma$ the *perfect derived category* of Γ, *i.e.* the smallest triangulated subcategory of $\mathcal{D}(\Gamma)$ which is the closure under shifts, extensions and passage to direct summands of the free right Γ-module Γ_Γ, $\mathcal{D}_{fd}(\Gamma)$ the finite-dimensional derived category of Γ consisting of objects of $\mathcal{D}(\Gamma)$ with finite-dimensional total homology.

Lemma 1.4.12 ([54]). *The finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$ is contained in the perfect derived category* $\mathrm{per}\Gamma$.

The *generalized cluster category* of (Q,W) is defined to be the triangulated quotient

$$\mathcal{C}_{(Q,W)} = \mathrm{per}\Gamma/\mathcal{D}_{fd}(\Gamma).$$

The quiver with potential (Q,W) is called *Jacobi-finite* if the Jacobian algebra $J(Q,W)$ is finite-dimensional.

Theorem 1.4.13 ([2]). *Let (Q,W) be a Jacobi-finite quiver with potential. Then the generalized cluster category $\mathcal{C}_{(Q,W)}$ is Hom-finite and 2-Calabi-Yau. Moreover, the image of the free module Γ in $\mathcal{C}_{(Q,W)}$ is a cluster-tilting object and its endomorphism algebra is isomorphic to the Jacobian algebra $J(Q,W)$.*

When (Q,W) is not Jacobi-finite, the generalized cluster category $\mathcal{C}_{(Q,W)}$ has infinite-dimensional morphism spaces and is not 2-Calabi-Yau. In [70], Plamondon gives an approach to study the Jacobi-infinite case.

Let T be an object of $\mathcal{C} = \mathcal{C}_{(Q,W)}$. Let $\mathrm{pr}_{\mathcal{C}}T$ denote the full subcategory of $\mathcal{C}$ whose objects are those X such that there exists a triangle

$$T_1 \to T_0 \to X \to \Sigma T_1$$

with T_0, T_1 in $\mathrm{add}T$. We refer to [70] for the definition of the mutation of suitable objects in $\mathcal{C}$.

Proposition 1.4.14 ([70]). *The category $\mathrm{pr}_{\mathcal{C}}\Gamma$ is a Krull-Schmidt category and depends only on the mutation class of the object Γ in $\mathcal{C}$.*

Definition 1.4.15. The subcategory $\mathcal{D}$ of $\mathcal{C}_{(Q,W)}$ is the full subcategory of $\mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma \cap \mathrm{pr}_{\mathcal{C}}\Gamma$ whose objects are those X such that $\mathrm{Ext}^1_{\mathcal{C}}(\Gamma, X)$ is finite-dimensional.

These subcategories still have the good properties which hold in the Jacobi-finite case.

Proposition 1.4.16 ([70]). *Let X be an object in $\mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma \cup \mathrm{pr}_{\mathcal{C}}\Gamma$ and Y an object of $\mathrm{pr}_{\mathcal{C}}\Gamma$. Then there exists a canonical bifunctorial bilinear form*

$$\overline{\beta}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 X) \to k$$

which is non-degenerate.

Chapter 2

Summary of results

In this chapter, we give a short summary of the main results in this thesis.

This thesis is concerned with

- higher cluster tilting objects in generalized higher cluster categories, and
- tropical friezes associated with Dynkin diagrams.

The thesis is organized as follows. In Chapter 3, we show that the generalized m-cluster category arising from a suitable $(m+2)$-Calabi-Yau dg algebra is Hom-finite, $(m+1)$-Calabi-Yau and admits a canonical m-cluster tilting object. In Chapter 4, we study the complements of an almost complete m-cluster tilting object in a generalized m-cluster category. In Chapter 5, we turn to a relatively independent subject, namely, tropical friezes associated with Dynkin diagrams. We prove that each such frieze is obtained by composing a linear form with the index with respect to a cluster-tilting object.

The results of Chapter 3 were published in [40]. These of Chapters 4 and 5 are contained in the preprints [41] and [42], which were submitted for publication.

2.1 Existence of m-cluster tilting objects

Let k be a field and A a (pseudo-compact) dg k-algebra. Denote by $\mathcal{D}(A)$ the derived category of A, $\mathrm{per}A$ the perfect derived category of A, $\mathcal{D}_{fd}(A)$ the finite-dimensional derived category of A, A^e the dg algebra $A^{op} \otimes_k A$. Let m be a positive integer. Suppose that A has the following four properties:

a) A is (topologically) homologically smooth;

b) the p-th homology H^pA vanishes for each positive integer p;

c) the 0-th homology H^0A is finite-dimensional;

d) A is $(m+2)$-Calabi-Yau as a bimodule.

Thanks to the property b), the derived category $\mathcal{D}(A)$ carries a standard t-structure $(\mathcal{D}(A)^{\leq 0}, \mathcal{D}(A)^{\geq 0})$. The property a), namely, (topological) homological smoothness implies (Lemma 4.1, [54]) that $\mathcal{D}_{fd}(A)$ is Hom-finite and is contained in $\mathrm{per}A$. The properties a), b) and c) together imply the following proposition:

Proposition (3.2.5)**.** *The category* $\mathrm{per}A$ *is Hom-finite.*

The property d) implies (Lemma 4.1, [54]) that for all objects L of $\mathcal{D}(A)$ and M of $\mathcal{D}_{fd}(A)$, there is a canonical isomorphism

$$D\mathrm{Hom}_{\mathcal{D}(A)}(M,L) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L,\Sigma^{m+2}M),$$

where D is the duality functor $\mathrm{Hom}_k(-,k)$. Let $\mathcal{C}_A$ denote the triangulated quotient $\mathrm{per}A/\mathcal{D}_{fd}(A)$ and $\pi : \mathrm{per}A \to \mathcal{C}_A$ the canonical projection functor. The category $\mathcal{C}_A$ is called the *generalized m-cluster category* of A. We have the following proposition:

Proposition (3.2.7)**.** *The category $\mathcal{C}_A$ is $(m+1)$-Calabi-Yau.*

Let $\mathcal{F}$ be the full subcategory $\mathcal{D}(A)^{\leq 0} \cap {}^{\perp}\mathcal{D}(A)^{\leq -m-1} \cap \mathrm{per}A$ of $\mathrm{per}A$. We call $\mathcal{F}$ the *fundamental domain.* For each object X of $\mathcal{F}$, we can construct m triangles as in Lemma 3.2.8, using which we can deduce the following proposition:

Proposition (3.2.15)**.** *The projection functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}_A$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}_A$.*

Thanks to this proposition, we can show that the image πA is an *m-cluster tilting object* in $\mathcal{C}_A$, that is, the spaces $\mathrm{Hom}_{\mathcal{C}_A}(\pi A, \Sigma^r L) = 0$ vanish for all integers $1 \leq r \leq m$ if and only if L belongs to $\mathrm{add}\pi A$ the full subcategory of $\mathcal{C}_A$ consisting of the direct summands of finite direct sums of copies of πA.

In conclusion, we generalize Amiot's work (Theorem 2.1 for $m = 1$ case, [2]) to the following theorem:

Theorem (3.2.2)**.** *Let A be a (pseudo-compact) dg k-algebra with the four properties stated at the beginning of this section. Then*

1) The category $\mathcal{C}_A$ is Hom*-finite and $(m+1)$-Calabi-Yau;*

2) The object πA is an m-cluster tilting object in $\mathcal{C}_A$;

3) The endomorphism algebra of πA over $\mathcal{C}_A$ is isomorphic to $H^0 A$.

2.2 Two classes of generalized m-cluster categories

Let (Q,W) be a graded quiver with superpotential in the sense of [56], where the author defined its Ginzburg dg category $\Gamma_n(Q,W)$. It is proved also in [56] that $\Gamma_n(Q,W)$ is homologically smooth and n-Calabi-Yau. For simplicity, we write $\Gamma^{(n)}$ for $\Gamma_n(Q,W)$. Let N_Q denote the minimal degree of the arrows in Q. Suppose that the arrows of Q are concentrated in nonpositive degrees and that m is a positive integer satisfying $m \geq -N_Q$. As a direct application of Theorem 3.2.2, we get the following theorem:

Theorem (3.3.3)**.** *Suppose that the zeroth homology of the Ginzburg dg category $\Gamma^{(m+2)}$ is finite-dimensional. Then the* generalized m-cluster category

$$\mathcal{C}_{(Q,W)} = \mathrm{per}\Gamma^{(m+2)}/\mathcal{D}_{fd}(\Gamma^{(m+2)})$$

associated to (Q,W) *is* Hom*-finite and $(m+1)$-Calabi-Yau. Moreover, the image of the free module $\Gamma^{(m+2)}$ in $\mathcal{C}_{(Q,W)}$ is an m-cluster tilting object whose endomorphism algebra is isomorphic to the zeroth homology of $\Gamma^{(m+2)}$.*

Let Q be an acyclic quiver and W the zero potential. Then its generalized m-cluster category $\mathcal{C}_{(Q,0)}$ recovers the (classical) m-cluster category $\mathcal{C}_Q^{(m)} = \mathcal{D}_{fd}(\mathrm{mod}kQ)/\tau^{-1}\Sigma^m$, which was first mentioned in [53].

Corollary (3.3.4)**.** *Let k be an algebraically closed field and m a positive integer. Suppose that Q is an acyclic quiver. Then the generalized m-cluster category $\mathcal{C}_{(Q,0)}$ is triangle equivalent to the (classical) m-cluster category $\mathcal{C}_Q^{(m)}$.*

Now we turn to another class of generalized m-cluster categories which arise from finite-dimensional algebras A of global dimension $\leq m+1$. Let B be the trivial extension $A \oplus \Sigma^{-m-2}DA$. Then $\mathrm{per}B$ is contained in $\mathcal{D}_{fd}(B)$. Denote by $p : B \to A$ the canonical projection and $p_* : \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(B)$ the induced triangulated functor. Let $\langle A \rangle_B$ be the thick subcategory of $\mathcal{D}_{fd}(B)$ generated by the image of p_*. We call the triangulated hull $\mathcal{C}_A = \langle A \rangle_B/\mathrm{per}B$ the *m-cluster category* of A.

We give a complete proof for the following well-known lemma.

Lemma (3.4.6)**.** *Let A be a dg k-algebra. Then for all dg A-modules L, M, the objects* $\mathrm{RHom}_A(L, M)$ *and* $\mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(L, M))$ *are isomorphic in the derived category of dg A-A-bimodules.*

We specialize L to DA and M to A in the above lemma and we deduce that the derived $(m+2)$-preprojective algebra $\Pi_{m+2}(A)$ defined in [56] is quasi-isomorphic to the tensor algebra $T_A(\Sigma^{m+1}\mathrm{RHom}_A(DA, A))$. As an application of Theorem 3.2.2, if $Tor^A_{m+1}(-, DA)$ is nilpotent, the generalized m-cluster category $\mathcal{C} = \mathrm{per}\Pi_{m+2}(A)/\mathcal{D}_{fd}(\Pi_{m+2}(A))$ is Hom-finite, $(m+1)$-Calabi-Yau and the image of $\Pi_{m+2}(A)$ in $\mathcal{C}$ is an m-cluster tilting object.

Then we construct a triangle equivalence between the m-cluster category $\mathcal{C}_A$ and the above generalized m-cluster category $\mathcal{C}$. As a consequence, we have the following theorem:

Theorem (3.4.9)**.** *Let A be a finite-dimensional k-algebra of global dimension $\leq m+1$. If the functor* $\mathrm{Tor}^A_{m+1}(-, DA)$ *is nilpotent, then the m-cluster category $\mathcal{C}_A$ of A is Hom-finite,$(m+1)$-Calabi-Yau and the image of A_B is an m-cluster tilting object in $\mathcal{C}_A$.*

2.3 Complements of almost complete m-cluster tilting P-objects

Let k be an algebraically closed field of characteristic zero. Let A be a (pseudo-compact) dg k-algebra which satisfies the four properties at the beginning of Section 2.1. Then the category $\mathrm{per}A$ is k-linear Hom-finite and has split idempotents. It follows that $\mathrm{per}A$ is a Krull-Schmidt triangulated category. Denote by $\mathcal{C}_A$ the generalized m-cluster category.

Definition (4.2.5)**.** An object $X \in \mathrm{per}A$ is *silting* (resp. *tilting*) if $\mathrm{per}A = \mathrm{thick}X$ the smallest thick subcategory of $\mathrm{per}A$ containing X, and the spaces $\mathrm{Hom}_{\mathcal{D}(A)}(X, \Sigma^i X)$ are zero for all integers $i > 0$ (resp. $i \neq 0$).

Theorem (4.3.3)**.** *The image of any silting object under the projection functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ is an m-cluster tilting object in $\mathcal{C}_A$.*

The dg algebra A itself is a silting object. Notice that under the assumptions we made on A, tilting objects do not exist. Assume that H^0A is a basic algebra. Let e be a primitive idempotent of H^0A. We call eA a *P-indecomposable.* Let M be $(1-e)A$. In Section4.2, we inductively construct the right mutations RA_t and the left mutations LA_t with respect to the dg module M for all positive integers t, where $RA_0 = LA_0 = P$.

Theorem (4.2.7, [1])**.** *For each nonnegative integer t, the objects $M \oplus RA_t$ and $M \oplus LA_t$ are silting objects in* per$A$. *Moreover, any basic silting object containing M as a direct summand is either of the form $M \oplus RA_t$ or of the form $M \oplus LA_t$.*

Using the standard t-structure on $\mathcal{D} = \mathcal{D}(A)$, we obtain that RA_t belongs to $\mathcal{D}^{\leq t} \cap {}^{\perp}\mathcal{D}^{\leq -1} \cap \mathrm{per}A$ and LA_t belongs to $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -t-1} \cap \mathrm{per}A$. Thus, the objects LA_t $(0 \leq t \leq m)$ lie in the fundamental domain $\mathcal{F}$.

Definition (4.3.4)**.** An object X in $\mathcal{C}_A$ is called an *almost complete m-cluster tilting object* if there exists some indecomposable object X' in $\mathcal{C}_A\backslash$ (addX) such that $X \oplus X'$ is an m-cluster tilting object. Here X' is called a *complement* of X. In particular, we call $\pi(M)$ an *almost complete m-cluster tilting P-object.*

Theorem (4.3.6)**.** *The almost complete m-cluster tilting P-object $\pi(M)$ has at least $m+1$ complements in $\mathcal{C}_A$.*

Let l be a finite-dimensional separable k-algebra. We use the same notation $PCAlgc(l)$ as in [76] to denote the category of pseudo-compact augmented dg l-algebras whose augmentation ideal equals their radical. We mainly consider the strongly $(m+2)$-Calabi-Yau (see [76]) case in Chapter 4.

Theorem (4.4.7, [76])**.** *Let A be a strongly $(m+2)$-Calabi-Yau dg algebra with components concentrated in degrees ≤ 0. Suppose that A lies in $PCAlgc(l)$ for some finite-dimensional separable commutative k-algebra l. Then A is quasi-isomorphic to some good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$.*

We study the truncations of minimal cofibrant resolutions of simple modules of strongly $(m+2)$-Calabi-Yau algebras (or good completed deformed preprojective dg algebras) in Proposition 4.4.10. Then we prove the following theorems:

Theorem (4.4.11)**.** *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ and i a vertex of Q. Assume that there are no loops of Q at vertex i and $H^0\Pi$ is finite-dimensional. Then the image of RA_t is isomorphic to the image of LA_{m+1-t} in the generalized m-cluster category $\mathcal{C}_\Pi$ for each integer $0 \leq t \leq m+1$.*

Theorem (4.5.2)**.** *Under the assumptions of Theorem 4.4.11, for each positive integer t,*

1) the image of RA_t is isomorphic to the image of $RA_{t(\mathrm{mod}\, m+1)}$ in $\mathcal{C}_\Pi$,

2) the image of LA_t is isomorphic to the image of $LA_{t(\mathrm{mod}\, m+1)}$ in $\mathcal{C}_\Pi$.

For the category $\mathcal{C}_\Pi$, we use the higher AR theory of [48] to give a more explicit criterion than the general Theorem 5.8 of [48] for determining the number of complements of an almost complete m-cluster tilting P-object. The associated AR $(m+3)$-angle is constructed in the proof of the following theorem:

Theorem (4.6.3)**.** *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ and i a vertex of Q. Assume that the zeroth homology $H^0\Pi$ is finite-dimensional and there are no loops of Q at vertex i. Then the almost complete m-cluster tilting P-object $\Pi/e_i\Pi$ has exactly $m+1$ complements in the generalized m-cluster category $\mathcal{C}_\Pi$.*

2.4 Liftable almost complete m-cluster tilting objects

In this section, we summarize some results concerning the complements of liftable almost complete m-cluster tilting objects. Let k be an algebraically closed field of characteristic zero. Let A be a (pseudo-compact) dg k-algebra which satisfies the four properties at the beginning of Section 2.1. Let $\mathcal{C}_A$ denote its generalized m-cluster category.

Definition (4.3.4)**.** An almost complete m-cluster tilting object Y is said to be *liftable* if there exists a basic silting object Z in per$A$ such the $\pi(Z/Z')$ is isomorphic to Y for some indecomposable direct summand Z' of Z.

The following two propositions state that if the initial dg algebra A is 3-Calabi-Yau or A is the completed Ginzburg dg algebra $\widehat{\Gamma}_{m+2}(Q,0)$ of an acyclic quiver Q, then all almost complete m-cluster tilting object in $\mathcal{C}_A$ are liftable.

Proposition (4.3.5)**.** *Let A be a 3-Calabi-Yau dg algebra satisfying the assumptions at the beginning of Section 2.1. Then any $(1-)$cluster tilting object in $\mathcal{C}_A$ is induced by a silting object in $\mathcal{F}$ under the canonical projection π.*

Proposition (4.8.6)**.** *Let Q be an acyclic quiver and B its path algebra. Let Γ be the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,0)$ and $\mathcal{C}_\Gamma$ the generalized m-cluster category. Then any m-cluster tilting object in $\mathcal{C}_\Gamma$ is induced by a silting object in $\mathcal{F}$ under the canonical projection $\pi : \mathrm{per}\Gamma \to \mathcal{C}_\Gamma$.*

Using our method, we obtain the following theorem (which can also be deduced from [48]).

Theorem (4.3.7)**.** *Each liftable almost complete m-cluster tilting object has at least $m+1$ complements in $\mathcal{C}_A$.*

Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ whose zeroth homology $H^0\Pi$ is finite-dimensional. Let Z be a basic silting object in perΠ which is minimal perfect and cofibrant. Denote by E the dg algebra $\tau_{\leq 0}(\mathrm{Hom}^{\bullet}_{\Pi}(Z,Z))$. We study the properties of E. The dg algebra E is strongly $(m+2)$-Calabi-Yau and lies in $PCAlgc(l)$, where $l = \prod_{|Q_0|} k$. We give a theoretical criterion for an liftable almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ to admit exactly $m+1$ complements.

Theorem (4.7.4)**.** *Keep the above notation. Then*

1) *the dg algebra E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$, where the quiver Q' has the same number of vertices as Q and $H^0\Pi'$ is finite-dimensional;*

2) *let Y be a liftable almost complete m-cluster tilting object of the form $\pi(Z/Z')$ in $\mathcal{C}_\Pi$ for some indecomposable direct summand Z' of Z. If we further assume that there are no loops at the vertex j of Q', where $e_j\Pi' \overset{L}{\otimes}_{\Pi'} Z = Z'$, then Y has exactly $m+1$ complements in $\mathcal{C}_\Pi$.*

It is not easy to check the 'non-loop' assumption on Q' in the second statement of the above theorem. This leads us to consider a class of dg algebras which satisfy m-rigidity.

Definition (4.7.5)**.** Let r be a positive integer. An algebra $A \in PCAlgc(l)$ is said to be *r-rigid* if

$$HH_0(A) \simeq l, \quad \text{and} \quad HH_p(A) = 0 \ (1 \leq p \leq r-1),$$

where $HH_*(A)$ is the pseudo-compact version of the Hochschild homology of A.

The completed Ginzburg dg algebras $\widehat{\Gamma}_{m+2}(Q,0)$ of acyclic quivers Q are m-rigid. The definition of 1-rigidity coincides with the definition of rigidity in [31]. In this case, the quiver Q' never contains loops.

However, we do not obtain too much progress on determining the number of complements of liftable almost complete m-cluster tilting objects even under the m-rigidity condition. A conjecture is as follows:

Conjecture (4.7.10). *Let $\Pi = \widehat{\Pi}(Q, m+2, W)$ be an m-rigid good completed deformed preprojective dg algebra whose zeroth homology $H^0\Pi$ is finite-dimensional. Then any liftable almost complete m-cluster tilting object has exactly $m+1$ complements in $\mathcal{C}_\Pi$.*

2.5 Tropical friezes associated with Dynkin diagrams

Inspired by a conjecture of Ringel on cluster-additive functions on stable translation quivers and by the tropicalized version of Coxeter-Conway's frieze patterns of integers, we introduce tropical friezes on 2-Calabi-Yau categories $\mathcal{C}$ with cluster-tilting object in Chapter 5.

Definition (5.2.2). A *tropical frieze* on $\mathcal{C}$ with values in $\mathbb{Z}$ is a map

$$f : obj(\mathcal{C}) \to \mathbb{Z}$$

such that

d1) $f(X) = f(Y)$ if X and Y are isomorphic,

d2) $f(X \oplus Y) = f(X) + f(Y)$ for all objects X and Y,

d3) for all objects L and M such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(L, M) = 1$, the equality

$$f(L) + f(M) = \max\{f(E), f(E')\}$$

holds, where E and E' are the middle terms of the non-split triangles

$$L \to E \to M \to \Sigma L \quad \text{and} \quad M \to E' \to L \to \Sigma M$$

with end terms L and M.

If we specialize the 2-Calabi-Yau category $\mathcal{C}$ to the cluster category $\mathcal{C}_Q$ of a Dynkin quiver, we obtain the following proposition:

Proposition (5.3.4). *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q and $T = T_1 \oplus \ldots \oplus T_n$ a basic cluster-tilting object of $\mathcal{C}_Q$. Then the map*

$$\Phi_T : \{\textit{tropical friezes on } \mathcal{C}_Q\} \longrightarrow \mathbb{Z}^n$$

given by $\Phi_T(f) = (f(T_1), \ldots, f(T_n))$ is a bijection.

Let X be an object of a 2-Calabi-Yau category $\mathcal{C}$ and T a basic cluster-tilting object of $\mathcal{C}$. The *index* of X with respect to T is defined by $\mathrm{ind}_T(X) = [T_0^X] - [T_1^X]$, where T_0^X and T_1^X belong to $\mathrm{add}T$ such that there exists a triangle

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X.$$

Let $T = T_1 \oplus \ldots \oplus T_n$ be a decomposition of T into indecomposables. We denote the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ by B, the indecomposable right projective B-module $\mathcal{C}(T, T_i)$ by P_i, the simple top of P_i by S_i.

Let $K_0^{sp}(\mathrm{mod}B)$ denote the split Grothendieck group of the abelian category $\mathrm{mod}B$. Define a bilinear form

$$\langle\, ,\, \rangle : K_0^{sp}(\mathrm{mod}B) \times K_0^{sp}(\mathrm{mod}B) \to \mathbb{Z}$$

by setting

$$\langle X, Y \rangle = \dim \mathrm{Hom}_B(X, Y) - \dim \mathrm{Ext}^1_B(X, Y)$$

for all finite-dimensional B-modules X and Y. In particular, if X is a projective B-module, then

$$\langle X, Y \rangle = \dim \mathrm{Hom}_B(X, Y),$$

in this case, the linear form $\langle X, ? \rangle$ on $K_0^{sp}(\mathrm{mod}B)$ induces a well-defined form

$$\langle X, ? \rangle : K_0(\mathrm{mod}B) \to \mathbb{Z},$$

where $K_0(\mathrm{mod}B)$ is the Grothendieck group of $\mathrm{mod}B$. Define an antisymmetric bilinear form on $K_0^{sp}(\mathrm{mod}B)$ by setting

$$\langle X, Y \rangle_a = \langle X, Y \rangle - \langle Y, X \rangle$$

for all finite-dimensional B-modules X and Y. In [69] Palu has proved that the antisymmetric bilinear form $\langle\, ,\, \rangle_a$ descends to the Grothendieck group $K_0(\mathrm{mod}B)$.

Let F denote the functor $\mathcal{C}(T, ?)$. Let m be an element in $K_0(\mathrm{mod}B)$. The function $f_{T,m} : obj(\mathcal{C}) \to \mathbb{Z}$ which sends an object X to the integer $\langle F(\mathrm{ind}_T(X)), m \rangle$, is a well-defined function.

Theorem (5.3.1)**.** *Assume that $\langle S_i, m \rangle_a \geq 0$ for each simple B-module S_i $(1 \leq i \leq n)$. Then the function $f_{T,m}$ is a tropical frieze on $\mathcal{C}$.*

The main theorem of Chapter 5 is as follows:

Theorem (5.5.1)**.** *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q. Then all tropical friezes on $\mathcal{C}_Q$ are of the form $f_{T,m}$, where T is a cluster-tilting object of $\mathcal{C}_Q$ and m an element in the Grothendieck group $K_0(\mathrm{modEnd}_{\mathcal{C}_Q}(T))$.*

As an application of Theorem 5.5.1, we show the following sign-coherence property:

Theorem (5.5.7)**.** *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q and f a tropical frieze on $\mathcal{C}_Q$. Then there exists a cluster-tilting object T such that*

$$f(T_i) \geq 0 \quad (\textit{resp. } f(T_i) \leq 0)$$

for all indecomposable direct summands T_i of T.

Using similar techniques, in section 5.6, we give a proof of a conjecture of Ringel (Section 6 of [74]) on cluster-additive functions on stable translation quivers.

Chapter 3

Generalized m-cluster categories

We prove the existence of an m-cluster tilting object in a generalized m-cluster category which is $(m+1)$-Calabi-Yau and Hom-finite, arising from an $(m+2)$-Calabi-Yau dg algebra. This is a generalization of the result for the $m = 1$ case in Amiot's Ph. D. thesis. Our results apply in particular to higher cluster categories associated with Ginzburg dg categories coming from suitable graded quivers with superpotential, and higher cluster categories associated with suitable finite-dimensional algebras of finite global dimension.

3.1 Introduction

In recent years, the categorification of cluster algebras has attracted a lot of attention. Notice that there are two quite different notions of categorification: additive categorification, studied in many articles, and monoidal categorification as introduced in [46]. One important class of categories arising in additive categorification is that of the cluster categories associated with finite-dimensional hereditary algebras. These were introduced in [17] (for quivers of type A in [24]), and investigated in many subsequent articles, e.g. [18] [21] [23] [25] [26] ... , cf. [73] for a survey. The cluster category $\mathcal{C}_Q$ associated with the path algebra of a finite acyclic quiver Q is constructed as the orbit category of the finite-dimensional derived category $\mathcal{D}_{fd}(\mathrm{mod}kQ)$ under the action of the autoequivalence $\tau^{-1}\Sigma$, where Σ is the suspension functor and τ the Auslander-Reiten translation. This category is Hom-finite, triangulated and 2-Calabi-Yau. Analogously, for a positive integer m, the m-cluster category $\mathcal{C}_Q^{(m)}$ is constructed as the orbit category of $\mathcal{D}_{fd}(\mathrm{mod}kQ)$ under the action of the autoequivalence $\tau^{-1}\Sigma^m$. This higher cluster category is Hom-finite, triangulated, and $(m+1)$-Calabi-Yau. It was first mentioned in [53], and has been studied in more detail in several articles [3] [59] [60] [75] ... . Many results about cluster categories can be generalized to m-cluster categories. In particular, combinatorial descriptions of higher cluster categories of type A_n and D_n are studied in [8] [9], the existence of exchange triangles in m-cluster categories was shown in [48], both [78] and [79] proved that there are exactly $m+1$ non isomorphic complements to an almost complete tilting object, and so on.

C. Amiot [2] generalized the construction of the cluster categories to finite-dimensional algebras A of global dimension ≤ 2. In order to show that there is a triangle equivalence between $\mathcal{C}_A$, constructed as a triangulated hull [53], and the quotient category $\mathrm{per}\Pi_3(A)/\mathcal{D}_{fd}\Pi_3(A)$, where $\Pi_3(A)$ is the 3-derived preprojective algebra [56] of A, she first studied the category $\mathcal{C}_A = \mathrm{per}A/\mathcal{D}_{fd}(A)$ associated with a dg algebra A with the following four properties:

1) A is homologically smooth;

2) A has vanishing homology in positive degrees;

3) A has finite-dimensional homology in degree 0 and

4) A is 3-Calabi-Yau as a bimodule.

She proved that the category $\mathcal{C}_A$ is Hom-finite and 2-Calabi-Yau. Moreover, the image of the free dg module A is a cluster tilting object in $\mathcal{C}_A$ whose endomorphism algebra is the zeroth homology of A. She applied these results in particular to the Ginzburg dg algebras $\Gamma = \Gamma(Q, W)$ associated [38] with Jacobi-finite quivers with potential (Q, W), and then introduced generalized cluster categories $\mathcal{C}_{(Q,W)} = \mathrm{per}\Gamma/\mathcal{D}_{fd}\Gamma$, which specialize to the cluster categories $\mathcal{C}_Q$ in the case where Q is acyclic and W is the zero potential.

The motivation of this article is to investigate the existence of cluster tilting objects in generalized higher cluster categories. We change the above fourth property of the dg algebra A to:

4') A is $(m+2)$-Calabi-Yau as a bimodule.

Similarly as in [2], using the inherited t-structure on per$A$ we prove in Section 2 that the quotient category $\mathcal{C}_A$ =per$A/\mathcal{D}_{fd}(A)$ is Hom-finite and $(m+1)$-Calabi-Yau. We call it the generalized m-cluster category. The image of the free dg module A is an m-cluster tilting object in $\mathcal{C}_A$ whose endomorphism algebra is the zeroth homology of A.

We apply these main results in Section 3 to higher cluster categories $\mathcal{C}_{(Q,W)}$ associated with Ginzburg dg categories [56] arising from suitable graded quivers with superpotential (Q, W). In order for the Ginzburg dg categories to satisfy the four properties, we assume that their zeroth homologies are finite-dimensional, that the graded quivers are concentrated in nonpositive degrees, and that the degrees of the arrows of Q are greater than or equal to $-m$. This generalized higher cluster category $\mathcal{C}_{(Q,W)}$ specializes to the higher cluster category $\mathcal{C}_Q^{(m)}$ when Q is an acyclic ordinary quiver and W is the zero superpotential.

In the last section, we work with finite-dimensional algebras A of global dimension $\leq n$. If the functor $\mathrm{Tor}_n^A(-, DA)$ is nilpotent, then the $(n-1)$-cluster category $\mathcal{C}_A$ defined as in Section 4 of A is Hom-finite, n-Calabi-Yau and the image of A is an $(n-1)$-cluster tilting object in $\mathcal{C}_A$. This section is a straightforward generalization of Section 4 in [2], so we only list the main steps of the proof.

3.2 Existence of higher cluster tilting objects

Let k be a field and A a differential graded (dg) k-algebra. We write per$A$ for the *perfect derived category* of A, i.e. the smallest triangulated subcategory of the derived category $\mathcal{D}(A)$ containing A and stable under passage to direct summands. We denote by $\mathcal{D}_{fd}(A)$ the finite-dimensional derived category of A whose objects are those of $\mathcal{D}(A)$ with finite-dimensional total homology, and denote by A^e the dg algebra $A^{op} \otimes_k A$. Usually, we write [1] in this chapter for the suspension functors Σ in triangulated categories. Let D denote the duality functor $\mathrm{Hom}_k(-, k)$.

Lemma 3.2.1 ([54], Lemma 4.1)**.** *Suppose that A is homologically smooth. Define*

$$\Omega = R\mathrm{Hom}_{A^e}(A, A^e)$$

and view it as an object in $\mathcal{D}(A^e)$. *Then for all objects* L *of* $\mathcal{D}(A)$ *and* M *of* $\mathcal{D}_{fd}(A)$, *we have a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}(A)}(M, L) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L \overset{L}{\otimes}_A \Omega, M).$$

If we have an isomorphism $\Omega \simeq A[-d]$ *in* $\mathcal{D}(A^e)$ *for some positive integer* d, *then* $\mathcal{D}_{fd}(A)$ *is* d*-Calabi-Yau, i.e. we have*

$$D\mathrm{Hom}_{\mathcal{D}(A)}(M, L) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L, M[d]).$$

From the proof given in [54] of Lemma 3.2.1, we can see that $\mathcal{D}_{fd}(A)$ is Hom-finite and is a thick triangulated subcategory in $\mathrm{per}A$. We denote by π the canonical projection functor from $\mathrm{per}A$ to $\mathcal{C}_A = \mathrm{per}A/\mathcal{D}_{fd}(A)$.

Let $m \geq 1$ be a positive integer. Suppose that A has the following properties ($\star$):

a) A is homologically smooth, *i.e.* A belongs to $\mathrm{per}(A^e)$ when considered as a bimodule over itself;

b) the p-th homology H^pA vanishes for each positive integer p;

c) the 0-th homology H^0A is finite-dimensional;

d) A is $(m+2)$-Calabi-Yau as a bimodule, *i.e.* there is an isomorphism in $\mathcal{D}(A^e)$

$$R\mathrm{Hom}_{A^e}(A, A^e) \simeq A[-m-2].$$

The main generalized result is the following theorem:

Theorem 3.2.2. *Let* A *be a dg* k*-algebra with the four properties* ($\star$). *Then*

1) the category $\mathcal{C}_A = \mathrm{per}A/\mathcal{D}_{fd}(A)$ *is* Hom*-finite and* $(m+1)$*-Calabi-Yau;*

2) the object $T = \pi A$ *is an* m-cluster tilting object *in* $\mathcal{C}_A$, *i.e. we have*

$$\mathrm{Hom}_{\mathcal{C}_A}(T, T[r]) = 0,\ r = 1, \ldots, m,$$

and for each object L *in* $\mathcal{C}_A$, *if* $\mathrm{Hom}_{\mathcal{C}_A}(T, L[r])$ *vanishes for each* $r = 1, \ldots, m$, *then* L *belongs to* $\mathrm{add}T$ *the full subcategory of* $\mathcal{C}_A$ *consisting of direct summands of finite direct sums of copies of* πA;

3) the endomorphism algebra of T *over* $\mathcal{C}_A$ *is isomorphic to* H^0A.

We call $\mathcal{C}_A$ the *generalized* m*-cluster category* associated with A.

From now on, we simply denote $\mathcal{D}(A)$ by $\mathcal{D}$, and denote $\mathcal{C}_A$ by $\mathcal{C}$.

Let $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) be the full subcategory of $\mathcal{D}$ whose objects are the dg modules X such that H^pX vanishes for all $p > 0$ (resp. $p < 0$). Similar as in [2], the proof of Theorem 3.2.2 also depends on the existence of a canonical t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathrm{per}A$. For a complex of k-modules X, we denote by $\tau_{\leq 0}X$ the subcomplex with $(\tau_{\leq 0}X)^i = X^i$ for $i < 0$, $(\tau_{\leq 0}X)^0 = \mathrm{Ker}d^0$ and zero otherwise. Set $\tau_{\geq 1}X = X/\tau_{\leq 0}X$. By the assumptions on A, the canonical inclusion $\tau_{\leq 0}A \rightarrow A$ is a quasi-isomorphism of dg algebras. Thus, we can assume that A^p is zero for all $p > 0$.

Proposition 3.2.3 ([2]). *Let* $\mathcal{H}$ *be the heart of the t-structure, i.e.* $\mathcal{H}$ *is the intersection* $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. *Then*

1) the functor H^0 induces an equivalence from $\mathcal{H}$ onto the category $\mathrm{Mod}H^0A$ *of right H^0A-modules;*

2) for all X and Y in $\mathcal{H}$, we have an isomorphism

$$\mathrm{Ext}^1_{H^0A}(X,Y) \simeq \mathrm{Hom}_{\mathcal{D}}(X, Y[1]).$$

Lemma 3.2.4. *For each integer n, the space H^nA is finite-dimensional.*

Proof. By our assumptions, the space H^mA is zero for every positive integer m and H^0A is finite-dimensional. We use induction on n to show that

a) for all $M \in \mathrm{mod}H^0A$, the space $\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite-dimensional for $p \geq n$ and

b) the homology $H^{-n}A$ is finite-dimensional.

It is easy to check a) and b) for $n = 0$. Assume that a) and b) hold for some $n \geq 0$.

Let $p \geq n+1$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, M[p])$ to the triangle

$$(H^{-n}A)[n-1] \longrightarrow \tau_{\leq -n-1}A \longrightarrow \tau_{\leq -n}A \longrightarrow (H^{-n}A)[n],$$

we can get the long exact sequence

$$\ldots \to (\tau_{\leq -n}A, M[p]) \to (\tau_{\leq -n-1}A, M[p]) \to ((H^{-n}A)[n-1], M[p]) \to \ldots,$$

where we write $(,)$ for $\mathrm{Hom}_{\mathcal{D}}(,)$. By part a) of the induction hypothesis, the space $\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -n}A, M[p])$ is finite-dimensional, and by part b) of the induction hypothesis, the space $H^{-n}A$ is finite-dimensional. Moreover, the homological smoothness of A implies that $\mathcal{D}_{fd}(A)$ is Hom-finite, so the space $\mathrm{Hom}_{\mathcal{D}}((H^{-n}A)[n-1], M[p])$ is finite-dimensional. This implies a) for the '$n+1$' case.

Now we show b) for the '$n+1$' case. Apply the functor $\mathrm{Hom}_{\mathcal{D}}(-, M[n+1])$ to the triangle

$$(H^{-n-1}A)[n] \longrightarrow \tau_{\leq -n-2}A \longrightarrow \tau_{\leq -n-1}A \longrightarrow (H^{-n-1}A)[n+1].$$

Since the object $\tau_{\leq -n-2}A$ is in $\mathcal{D}^{\leq -n-2}$, there holds an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(H^{-n-1}A, M) \simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -n-1}A, M[n+1]),$$

whose right-hand side is finite-dimensional. Let M be the duality DH^0A. Then the following isomorphism

$$DH^{-n-1}A \simeq \mathrm{Hom}_{H^0A}(H^{-n-1}A, DH^0A)$$

implies that the space $H^{-n-1}A$ is finite-dimensional. This finishes the proof. $\square$

The subcategory of $(\mathrm{per}A)^{op} \times \mathrm{per}A$ whose objects are the pairs (X, Y) such that, the space $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is finite-dimensional, is stable under extensions and passage to direct factors. By Lemma 3.2.4, the space $H^nA(\simeq \mathrm{Hom}_{\mathcal{D}}(A, A[n]))$ is finite-dimensional. As a result, the following proposition holds.

Proposition 3.2.5. *The category* $\mathrm{per}A$ *is* Hom-*finite.*

Lemma 3.2.6 ([2]). *For each X in* $\mathrm{per}A$*, there exist integers N and M such that X belongs to $\mathcal{D}^{\leq N} \cap {}^{\perp}\mathcal{D}^{\leq M}$. Moreover, the t-structure on $\mathcal{D}$ canonically restricts to* $\mathrm{per}A$.

An obvious remark here is that the first statement in Lemma 3.2.6 has the following equivalent saying: *there exists a positive integer* N_0 *such that* X *belongs to* $\mathcal{D}^{\leq n} \cap {}^{\perp}\mathcal{D}^{\leq -n}$ *for any* $n \geq N_0$.

Proposition 3.2.7. *The category* $\mathcal{C}$ *is* $(m+1)$*-Calabi-Yau.*

Proof. Let $\mathcal{T}$ denote the category per$A$. Let $\mathcal{N}$ denote $\mathcal{D}_{fd}(A)$, which is a thick subcategory of $\mathcal{T}$. Because of the Calabi-Yau property, that is,

$$D\mathrm{Hom}_{\mathcal{D}}(N, X) \simeq \mathrm{Hom}_{\mathcal{D}}(X, N[m+2]) \quad \text{for each } N \in \mathcal{D}_{fd}(A) \text{ and } X \in \mathcal{D},$$

there is a bifunctorial non-degenerate bilinear form :

$$\beta_{N,X} : \mathrm{Hom}_{\mathcal{D}}(N, X) \times \mathrm{Hom}_{\mathcal{D}}(X, N[m+2]) \longrightarrow k$$

Therefore, by Section 1 in [2], there exists a bifunctorial form :

$$\beta'_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, X[m+1]) \longrightarrow k \quad \text{for } X, Y \in \mathcal{C}.$$

By Lemma 3.2.6, the object X belongs to ${}^{\perp}\mathcal{D}^{\leq r}$ for some integer r. Thus, we obtain an injection

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X, \tau_{>r}Y),$$

and the object $\tau_{>r}Y$ is in $\mathcal{D}_{fd}(A)$. Since per$A$ is Hom-finite by Proposition 3.2.5, still using Section 1 in [2], we can get that $\beta'_{X,Y}$ is non-degenerate. Therefore, we have

$$D\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X[m+1]) \quad \text{for } X, Y \in \mathcal{C}.$$

Thus, the category $\mathcal{C}$ is $(m+1)$-Calabi-Yau. $\square$

Let $\mathcal{F}$ be the full subcategory $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -m-1} \cap \mathrm{per}A$ of per$A$.

Lemma 3.2.8. *For each object* X *of* $\mathcal{F}$*, there exist* m *triangles (which are not unique in general)*

$$P_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow P_1[1],$$
$$P_2 \longrightarrow Q_1 \longrightarrow P_1 \longrightarrow P_2[1],$$
$$\cdots \quad \cdots$$
$$P_m \longrightarrow Q_{m-1} \longrightarrow P_{m-1} \longrightarrow P_m[1],$$

where $Q_0, Q_1, \ldots, Q_{m-1}$ *and* P_m *are in* add$A$.

Proof. For each object X in per$A$, the following isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(A, X) \simeq H^0X \simeq \mathrm{Hom}_{H^0A}(H^0A, H^0X)$$

hold. Therefore, we can find a morphism $Q_0 \longrightarrow X$ with Q_0 a free dg A-module, which induces an epimorphism $H^0Q_0 \twoheadrightarrow H^0X$. Take X in $\mathcal{F}$ and form a triangle

$$P_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow P_1[1].$$

Step 1. *The object* P_1 *is in* $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -m} \cap \mathrm{per}A$.

Since the objects Q_0 and X are in $\mathcal{D}^{\leq 0}$, P_1 is in $\mathcal{D}^{\leq 1}$. Moreover, we have a long exact sequence

$$\ldots \rightarrow H^0Q_0 \twoheadrightarrow H^0X \rightarrow H^1P_1 \rightarrow H^1Q_0 = 0.$$

It follows that $H^1 P_1 = 0$. Thus, the object P_1 belongs to $\mathcal{D}^{\leq 0}$.

Let Y be in $\mathcal{D}^{\leq -m}$. Consider the long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Q_0, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_1, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X[-1], Y) \longrightarrow \ldots.$$

Since X belongs to ${}^{\perp}\mathcal{D}^{\leq -m-1}$ and Y is in $\mathcal{D}^{\leq -m}$, the space $\mathrm{Hom}_{\mathcal{D}}(X[-1], Y)$ vanishes. The object Q_0 is free and $H^0 Y$ is zero, so the space $\mathrm{Hom}_{\mathcal{D}}(Q_0, Y)$ also vanishes. Thus P_1 belongs to ${}^{\perp}\mathcal{D}^{\leq -m}$.

Moreover, since $\mathrm{per}A$ is closed under extensions in $\mathcal{D}$, the object P_1 belongs to $\mathrm{per}A$. Thus, the object P_1 belongs to $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -m} \cap \mathrm{per}A$. Similarly as above, we can find a morphism $Q_1 \longrightarrow P_1$ with Q_1 a free dg A-module, which induces an epimorphism $H^0 Q_1 \twoheadrightarrow H^0 P_1$. Then we form a triangle

$$P_2 \longrightarrow Q_1 \longrightarrow P_1 \longrightarrow P_2[1].$$

Step 2. *For $1 \leq r \leq m$, the object P_t is in $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq t-m-1} \cap \mathrm{per}A$.*

By the same argument as in step 1, we obtain that the object P_2 is in $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq 1-m} \cap \mathrm{per}A$.

In this way, we inductively construct m triangles

$$P_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow P_1[1],$$

$$P_2 \longrightarrow Q_1 \longrightarrow P_1 \longrightarrow P_2[1],$$

$$\ldots \quad \ldots$$

$$P_m \longrightarrow Q_{m-1} \longrightarrow P_{m-1} \longrightarrow P_m[1],$$

where $Q_0, Q_1, \ldots, Q_{m-1}$ are free dg A-modules and P_t belongs to $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq t-m-1} \cap \mathrm{per}A$, for each $1 \leq t \leq m$.

The following two steps are quite similar to the proof of Lemma 2.10 in [2]. However, for the convenience of the reader, we give a complete proof.

Step 3. *$H^0 P_m$ is a projective $H^0 A$-module.*

Since P_m belongs to $\mathcal{D}^{\leq 0}$, there exists a triangle

$$\tau_{\leq -1} P_m \longrightarrow P_m \longrightarrow H^0 P_m \longrightarrow (\tau_{\leq -1} P_m)[1].$$

Take an object M in the heart $\mathcal{H}$, and consider the long exact sequence

$$\ldots \longrightarrow ((\tau_{\leq -1} P_m)[1], M[1]) \longrightarrow (H^0 P_m, M[1]) \longrightarrow (P_m, M[1]) \longrightarrow \ldots,$$

where we write $(\,,)$ for $\mathrm{Hom}_{\mathcal{D}}(\,,)$. The space $\mathrm{Hom}_{\mathcal{D}}((\tau_{\leq -1} P_m)[1], M[1])$ vanishes because $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq -2}, \mathcal{D}^{\geq -1})$ is zero. Since P_m belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$, the space $\mathrm{Hom}_{\mathcal{D}}(P_m, M[1])$ also vanishes. As a result, the space

$$\mathrm{Ext}^1_{\mathcal{H}}(H^0 P_m, M) \simeq \mathrm{Hom}_{\mathcal{D}}(H^0 P_m, M[1])$$

is zero. Thus, $H^0 P_m$ is a projective $H^0 A$-module.

Step 4. *P_m is isomorphic to an object in* $\mathrm{add}A$.

From step 3, we deduce that it is possible to find an object P in $\mathrm{add}A$ and a morphism $P \longrightarrow P_m$ such that $H^0 P$ and $H^0 P_m$ are isomorphic. Then we form a new triangle

$$E \longrightarrow P \longrightarrow P_m \longrightarrow E[1].$$

Since P and P_m are in $\mathcal{D}^{\leq 0}$, the object E is in $\mathcal{D}^{\leq 1}$. Moreover, there is a long exact sequence

$$\ldots \longrightarrow H^0 E \longrightarrow H^0 P \simeq H^0 P_m \longrightarrow H^1 E \longrightarrow H^1 P = 0.$$

So E is in $\mathcal{D}^{\leq 0}$. Since P_m belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$, the space $\mathrm{Hom}_{\mathcal{D}}(P_m, E[1])$ vanishes. Therefore, the object P is isomorphic to the direct sum of P_m and E. Then we have an isomorphism

$$H^0 P \simeq H^0 P_m \oplus H^0 E.$$

We obtain that $H^0 E$ is zero. As a consequence, there is no nonzero morphism from P to E, since P is a free A-module. Therefore, E is the zero object and P_m is isomorphic to P which is an object in $\mathrm{add}A$. □

Let X be an object of $\mathcal{F}$. By Lemma 3.2.8, there are m triangles related to the object X. Denote by ν the Nakayama functor on mod $H^0 A$. Clearly, $\nu H^0 P_m$ and $\nu H^0 Q_{m-1}$ are injective $H^0 A$-modules. Let M be the kernel of the morphism $\nu H^0 P_m \longrightarrow \nu H^0 Q_{m-1}$. It lies in the heart $\mathcal{H}$. Let $N = X[1]$.

Lemma 3.2.9. *1) There are isomorphisms of functors:*

$$\mathrm{Hom}_{\mathcal{D}}(-, X[2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_1[3])|_{\mathcal{H}} \simeq \ldots$$

$$\ldots \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{H}}(-, M).$$

2) There is a monomorphism of functors:

$$\mathrm{Ext}^1_{\mathcal{H}}(-, M) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+2])|_{\mathcal{H}}.$$

Proof. Let L be in $\mathcal{H}$. Let us prove part 1).

Step 1. *There is an isomorphism of functors:*

$$\mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{H}}(-, M).$$

Applying $\mathrm{Hom}_{\mathcal{D}}(L, -)$ to the m-th triangle

$$P_m \longrightarrow Q_{m-1} \longrightarrow P_{m-1} \longrightarrow P_m[1],$$

we obtain a long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, Q_{m-1}[m+1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_{m-1}[m+1]) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_m[m+2]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, Q_{m-1}[m+2]) \longrightarrow \ldots.$$

Since L belongs to $\mathcal{D}_{fd}(A)$, by the Calabi-Yau property one can easily see the following isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(L, Q_{m-1}[m+1]) \simeq D\mathrm{Hom}_{\mathcal{D}}(Q_{m-1}, L[1]).$$

The space vanishes since the object Q_{m-1} is a free dg A-module and $H^1 L$ is zero. Consider the triangle

$$\tau_{\leq -1} P_m \longrightarrow P_m \longrightarrow H^0 P_m \longrightarrow (\tau_{\leq -1} P_m)[1].$$

We can get a long exact sequence

$$\ldots \longrightarrow ((\tau_{\leq -1} P_m)[1], L) \longrightarrow (H^0 P_m, L) \longrightarrow (P_m, L) \longrightarrow (\tau_{\leq -1} P_m, L) \longrightarrow \ldots,$$

where we write $(,)$ for $\mathrm{Hom}_{\mathcal{D}}(,)$. Since the space $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq -1-i}, \mathcal{D}^{\geq 0})$ is zero, the space $\mathrm{Hom}_{\mathcal{D}}((\tau_{\leq -1}P_m)[i], L)$ vanishes for $i = 0, 1$. Thus, we have

$$\mathrm{Hom}_{\mathcal{D}}(P_m, L) \simeq \mathrm{Hom}_{\mathcal{D}}(H^0P_m, L) \simeq \mathrm{Hom}_{\mathcal{H}}(H^0P_m, L).$$

Combining with the Calabi-Yau property, we get the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(L, P_m[m+2]) &\simeq D\mathrm{Hom}_{\mathcal{D}}(P_m, L)\\ &\simeq D\mathrm{Hom}_{\mathcal{H}}(H^0P_m, L) \simeq \mathrm{Hom}_{\mathcal{H}}(L, \nu H^0P_m).\end{aligned}$$

Similarly, we can see that

$$\mathrm{Hom}_{\mathcal{D}}(L, Q_{m-1}[m+2]) \simeq \mathrm{Hom}_{\mathcal{H}}(L, \nu H^0Q_{m-1}).$$

Therefore, the functor $\mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}}$ is isomorphic to the functor $\mathrm{Hom}_{\mathcal{H}}(-, M)$, which is the kernel of the morphism

$$\mathrm{Hom}_{\mathcal{H}}(-, \nu H^0P_m) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(-, \nu H^0Q_{m-1}).$$

Step 2. *There are isomorphisms of functors:*

$$\mathrm{Hom}_{\mathcal{D}}(-, X[2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_1[3])|_{\mathcal{H}} \simeq \ldots \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}}.$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(L, -)$ to the $(m-1)$-th triangle

$$P_{m-1} \longrightarrow Q_{m-2} \longrightarrow P_{m-2} \xrightarrow{h_{m-2}} P_{m-1}[1],$$

we obtain a long exact sequence

$$\begin{aligned}\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, Q_{m-2}[m]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_{m-2}[m]) \longrightarrow\\ \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_{m-1}[m+1]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, Q_{m-2}[m+1]) \longrightarrow \ldots.\end{aligned}$$

Since Q_{m-2} is a free A-module and L is in $\mathcal{H}$, the space $\mathrm{Hom}_{\mathcal{D}}(Q_{m-2}, L[r])$ vanishes for each positive integer r. As a result, by the Calabi-Yau property, the following two isomorphisms hold

$$\mathrm{Hom}_{\mathcal{D}}(L, Q_{m-2}[m]) \simeq D\mathrm{Hom}_{\mathcal{D}}(Q_{m-2}, L[2]) = 0,$$

$$\mathrm{Hom}_{\mathcal{D}}(L, Q_{m-2}[m+1]) \simeq D\mathrm{Hom}_{\mathcal{D}}(Q_{m-2}, L[1]) = 0.$$

Therefore, we have

$$\mathrm{Hom}_{\mathcal{D}}(-, P_{m-2}[m])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}},$$

where the isomorphism is induced by the left multiplication by $h_{m-2}[m]$.

We inductively work with each triangle and get a corresponding isomorphism induced by the left multiplication by $h_{m-r}[m-r+2]$,

$$\mathrm{Hom}_{\mathcal{D}}(-, P_{m-r}[m-r+2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_{m-r+1}[m-r+3])|_{\mathcal{H}}, \quad 2 \leq r \leq m-1,$$

while the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(-, X[2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_1[3])|_{\mathcal{H}}$$

is induced by the left multiplication by $h_0[2]$. Therefore, the first assertion in this lemma holds.

Let us prove part 2).

Consider the following long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_m[m+2]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, Q_{m-1}[m+2]) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_{m-1}[m+2]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(L, P_m[m+3]) \longrightarrow \ldots.$$

By the Calabi-Yau property, the space $\mathrm{Hom}_{\mathcal{D}}(L, P_m[m+3])$ is isomorphic to the zero space $D\mathrm{Hom}_{\mathcal{D}}(P_m[1], L)$.

Hence, the functor $\mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+2])|_{\mathcal{H}}$ is isomorphic to the cokernel of the morphism

$$\mathrm{Hom}_{\mathcal{H}}(-, \nu H^0 P_m) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(-, \nu H^0 Q_{m-1}).$$

As an H^0A-module, M admits an injective resolution of the following form

$$0 \longrightarrow \nu H^0 P_m \longrightarrow \nu H^0 Q_{m-1} \longrightarrow I \longrightarrow \ldots,$$

where I is an injective H^0A-module. Then $\mathrm{Ext}^1_{\mathcal{H}}(-, M)$ is the first homology of the following complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{H}}(-, \nu H^0 P_m) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(-, \nu H^0 Q_{m-1}) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(-, I) \longrightarrow \ldots.$$

Therefore, we get a monomorphism of functors

$$\mathrm{Ext}^1_{\mathcal{H}}(-, M) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+2])|_{\mathcal{H}}.$$

□

Following Step 1 in the proof of Lemma 3.2.9, there is an isomorphism of functors:

$$\mathrm{Hom}_{\mathcal{D}}(-, P_{m-1}[m+1])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{H}}(-, M).$$

We denote it by φ_1, and when φ_1 is applied to an object V in $\mathcal{H}$, we denote the isomorphism by $\varphi_{1,V}$. Let ρ be the preimage of the identity map on M under the isomorphism

$$\varphi_{1,M} : \mathrm{Hom}_{\mathcal{D}}(M, P_{m-1}[m+1]) \simeq \mathrm{Hom}_{\mathcal{H}}(M, M).$$

Now we can form a triangle

$$P_{m-1}[m] \longrightarrow Y' \longrightarrow M \xrightarrow{\rho} P_{m-1}[m+1].$$

Lemma 3.2.10. *The object Y' is in $\mathcal{F}$.*

Proof. Since M belongs to $\mathcal{H}$ and $P_{m-1}[m+1]$ belongs to $\mathrm{per}A$, it follows that Y' is also in $\mathrm{per}A$. Moreover, Y' is in $\mathcal{D}^{\leq 0}$, since the objects M and P_{m-1} are in $\mathcal{D}^{\leq 0}$. Let Z be an object in $\mathcal{D}^{\leq -m-1}$. Then there is a long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(M, Z) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y', Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(M[-1], Z) \longrightarrow \ldots.$$

Since Z belongs to $\mathcal{D}^{\leq -m-1}$, we have the following triangle

$$\tau_{\leq -m-2} Z \longrightarrow Z \longrightarrow (H^{-m-1} Z)[m+1] \longrightarrow (\tau_{\leq -m-2} Z)[1].$$

By the Calabi-Yau property, the space

$$\mathrm{Hom}_{\mathcal{D}}(M[-1], (\tau_{\leq -m-2}Z)[i]) \simeq D\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -m-2}Z, M[m+1-i])$$

is zero for $i = 0, 1$. As a result, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(M[-1], Z) &\simeq \mathrm{Hom}_{\mathcal{D}}(M[-1], (H^{-m-1}Z)[m+1]) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-1}Z, M).\end{aligned}$$

From Step 2 in the proof of Lemma 3.2.8, we know that the object P_{m-1} is in ${}^{\perp}\mathcal{D}^{\leq -2}$. So the m-th shift $P_{m-1}[m]$ is in ${}^{\perp}\mathcal{D}^{\leq -m-2}$. Combining with the Calabi-Yau property, the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m], Z) &\simeq \mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m], (H^{-m-1}Z)[m+1]) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-1}Z, P_{m-1}[m+1])\end{aligned}$$

hold. Now by Lemma 3.2.9, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m], Z) \simeq \mathrm{Hom}_{\mathcal{D}}(M[-1], Z).$$

Consider the following commutative diagram

$$\begin{tikzcd}
(P', Z') \arrow[r] \arrow[d, "a"] & (P', Z) \arrow[r] \arrow[d, "b"] & (P', (H^{-m-1}Z)[m+1]) \arrow[r] \arrow[d, "c"] & (P', Z'[1]) \arrow[d, "d"] \\
(M, Z') \arrow[r] & (M, Z) \arrow[r] & (M, (H^{-m-1}Z)[m+1]) \arrow[r] & (M, Z'[1]),
\end{tikzcd}$$

where we write $(,)$ for $\mathrm{Hom}_{\mathcal{D}}(,)$, P' for $P_{m-1}[m+1]$, and Z' for $\tau_{\leq -m-2}Z$. Since the object $P_{m-1}[m+1]$ is in ${}^{\perp}\mathcal{D}^{\leq -m-3}$, we have that the space $\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], (\tau_{\leq -m-2}Z)[1])$ vanishes, and then the rightmost morphism d is a zero map. By the Calabi-Yau property and Proposition 3.2.3, one can easily get the following isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], (H^{-m-1}Z)[m+1]) \simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-1}Z, P_{m-1}[m+2]),$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(M, (H^{-m-1}Z)[m+1]) &\simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-1}Z, M[1]) \\ &\simeq D\mathrm{Ext}^1_{\mathcal{H}}(H^{-m-1}Z, M).\end{aligned}$$

Then by Lemma 3.2.9, the morphism c is surjective. Consider the triangle

$$\tau_{\leq -m-3}Z \longrightarrow \tau_{\leq -m-2}Z \longrightarrow (H^{-m-2}Z)[m+2] \longrightarrow (\tau_{\leq -m-3}Z)[1].$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, M[m+2])$ to this triangle and by the Calabi-Yau property, we can obtain isomorphisms as follows:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(M, \tau_{\leq -m-2}Z) &\simeq D\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -m-2}Z, M[m+2]) \\ \simeq D\mathrm{Hom}_{\mathcal{D}}((H^{-m-2}Z)[m+2], M[m+2]) &\simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-2}Z, M).\end{aligned}$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], -)$ to the same triangle and by the Calabi-Yau property, we can get isomorphisms as follows:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], \tau_{\leq -m-2}Z) &\simeq \mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m+1], (H^{-m-2}Z)[m+2]) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}}(H^{-m-2}Z, P_{m-1}[m+1]).\end{aligned}$$

Therefore, following Lemma 3.2.9, the leftmost morphism a is an isomorphism. Then by Five-Lemma, the morphism b is surjective. From the long exact sequence at the beginning of the proof, we can see that the space $\mathrm{Hom}_{\mathcal{D}}(Y', Z)$ vanishes for any $Z \in \mathcal{D}^{\leq -m-1}$. Hence, the object Y' is in $\mathcal{F}$. $\square$

Let φ_r $(2 \leq r \leq m-1)$ denote the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(-, P_{m-r}[m-r+2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_{m-r+1}[m-r+3])|_{\mathcal{H}}$$

in Lemma 3.2.8, and let φ_m denote the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(-, X[2])|_{\mathcal{H}} \simeq \mathrm{Hom}_{\mathcal{D}}(-, P_1[3])|_{\mathcal{H}}.$$

We write f for the composition $h_{m-2}[m] \ldots h_0[2]$, and θ for the composition $\varphi_1 \ldots \varphi_m$. Let ε be the preimage of the identity map on M under the isomorphism

$$\theta_M : \mathrm{Hom}_{\mathcal{D}}(M, X[2]) \simeq \mathrm{Hom}_{\mathcal{H}}(M, M).$$

As a result, we have that

$$\theta_M(\varepsilon) = id_M = \varphi_{1M}(\rho).$$

Thus, the following equalities hold,

$$f\varepsilon = \varphi_{2,M} \ldots \varphi_{m,M}(\varepsilon) = \rho.$$

Now we form a new triangle

$$N \longrightarrow Y \longrightarrow M \xrightarrow{\varepsilon} N[1].$$

Lemma 3.2.11. *The object Y is in $\mathcal{F}$ and $\tau_{\leq -1} Y$ is isomorphic to N.*

Proof. Since M belongs to $\mathcal{H}$ and N belongs to $\mathcal{D}^{\leq 0} \cap \mathrm{per} A$, the object Y is also in $\mathcal{D}^{\leq 0} \cap \mathrm{per} A$. Our aim is to show that Y is in ${}^{\perp}\mathcal{D}^{\leq -m-1}$. Let Z be an object in $\mathcal{D}^{\leq -m-1}$. There is a long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(N[1], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(M, Z) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(N, Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(M[-1], Z) \longrightarrow \ldots.$$

Since N is in ${}^{\perp}\mathcal{D}^{\leq -m-2}$ and the functors $\mathrm{Hom}_{\mathcal{D}}(-, N[1])|_{\mathcal{H}}$ and $\mathrm{Hom}_{\mathcal{H}}(-, M)$ are isomorphic, by the same argument as in Lemma 3.2.10, we can obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(N, Z) \simeq \mathrm{Hom}_{\mathcal{D}}(M[-1], Z).$$

Since ρ is the composition $f\varepsilon$, there exists a morphism $g : Y \longrightarrow Y'$ such that the following diagram is commutative

$$\begin{tikzcd}
N \arrow[r] \arrow[d, "{f[-1]}"] & Y \arrow[r] \arrow[d, dotted, "g"] & M \arrow[r, "\varepsilon"] \arrow[d, equal] & N[1] \arrow[d, "f"] \\
P_{m-1}[m] \arrow[r] & Y' \arrow[r] & M \arrow[r, "\rho"] & P_{m-1}[m+1].
\end{tikzcd}$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Z)$ to this diagram, then we get the following commutative diagram

$$\begin{tikzcd}
(M, Z) \arrow[r] \arrow[d, equal] & (Y', Z) \arrow[r] \arrow[d, "{\mathrm{Hom}_{\mathcal{D}}(g,-)|_Z}"] & (P_{m-1}[m], Z) \arrow[r] \arrow[d, "{\mathrm{Hom}_{\mathcal{D}}(f[-1],-)|_Z}"] & (M[-1], Z) \arrow[d, equal] \\
(M, Z) \arrow[r] & (Y, Z) \arrow[r] & (N, Z) \arrow[r] & (M[-1], Z),
\end{tikzcd}$$

where we write $(,)$ for $\mathrm{Hom}_{\mathcal{D}}(,)$. The morphism

$$\mathrm{Hom}_{\mathcal{D}}(f[-1], -)|_Z : \mathrm{Hom}_{\mathcal{D}}(P_{m-1}[m], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(N, Z)$$

is an isomorphism. We can see this as follows:
applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Z)$ to triangles

$$P_1[1] \longrightarrow Q_0[1] \longrightarrow N \stackrel{h_0[1]}{\longrightarrow} P_1[2], \quad \text{and}$$

$$P_r[r] \longrightarrow Q_{r-1}[r] \longrightarrow P_{r-1}[r] \stackrel{h_{r-1}[r]}{\longrightarrow} P_r[r+1], \quad 2 \leq r \leq m-1,$$

we can get long exact sequences (here we denote N by $P_0[1]$, i.e. X by P_0)

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Q_{r-1}[r+1], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_r[r+1], Z) \longrightarrow$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{r-1}[r], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Q_{r-1}[r], Z) \longrightarrow \ldots, \quad 1 \leq r \leq m-1.$$

The objects $Z[-r-i]$ $(i = 0, 1)$ are in $\mathcal{D}^{\leq r+i-m-1}$ $(\subset \mathcal{D}^{\leq -1})$. Since Q_{r-1} is a free A-module, the space $\mathrm{Hom}_{\mathcal{D}}(Q_{r-1}[r+i], Z)$ vanishes for $i = 0, 1$. Thus, the morphism

$$\mathrm{Hom}_{\mathcal{D}}(h_{r-1}[r], -)|_Z : \mathrm{Hom}_{\mathcal{D}}(P_r[r+1], Z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{r-1}[r], Z)$$

is an isomorphism for each $1 \leq r \leq m-1$. As a consequence, the functor $\mathrm{Hom}_{\mathcal{D}}(f[-1], -)|_Z$ is an isomorphism. By Five-Lemma, we can obtain that $\mathrm{Hom}_{\mathcal{D}}(g, -)|_Z$ is an epimorphism. From Lemma 3.2.10, we know that the object Y' is in $\mathcal{F}$, and the space $\mathrm{Hom}_{\mathcal{D}}(Y', Z)$ vanishes. It follows that the space $\mathrm{Hom}_{\mathcal{D}}(Y, Z)$ is also zero, hence Y is in $\mathcal{F}$.

Since N is in $\mathcal{F}[1]$, the spaces $H^0 N$ and $H^1 N$ are zero. Thus, the object $H^0 Y$ is isomorphic to M. Moreover, the space $\mathrm{Hom}_{\mathcal{D}}(N, H^0 Y)$ is zero. Hence, we can obtain a commutative diagram of triangles

$$\begin{tikzcd}
\tau_{\leq -1} Y \arrow[r] & Y \arrow[r, "p_Y"] & H^0 Y \arrow[r] & (\tau_{\leq -1} Y)[1] \\
N \arrow[r] \arrow[u, dotted, "\delta_2"'] & Y \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u, dotted, "\delta_1"'] & N[1], \arrow[u, dotted]
\end{tikzcd}$$

where $\delta_1 : M \longrightarrow H^0 Y$ is an epimorphism between isomorphic terms. Therefore, δ_1 is an isomorphism. Thus, $\tau_{\leq -1} Y$ is isomorphic to N. $\square$

Lemma 3.2.12. *The image of the functor $\tau_{\leq -i}$ restricted to $\mathcal{F}$ is in $\mathcal{F}[i]$ and the functor $\tau_{\leq -i} : \mathcal{F} \longrightarrow \mathcal{F}[i]$ is fully faithful for each positive integer i.*

Proof. Let X be an object in $\mathcal{F}$. Then $\tau_{\leq -i} X$ is in $\mathcal{D}^{\leq -i}$, and there is a triangle in $\mathcal{D}$

$$\tau_{\leq -i} X \longrightarrow X \longrightarrow \tau_{> -i} X \longrightarrow (\tau_{\leq -i} X)[1].$$

Following Lemma 3.2.6, the object $\tau_{\leq -i} X$ belongs to $\mathcal{D}^{\leq -i} \cap \mathrm{per} A$. Let Y be an object in $\mathcal{D}^{\leq -m-i-1}$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ to this triangle, then we can get a long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -i} X, Y) \to \mathrm{Hom}_{\mathcal{D}}((\tau_{> -i} X)[-1], Y) \to \ldots.$$

The space $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ vanishes because X is in ${}^{\perp}\mathcal{D}^{\leq -m-1}$ and i is a positive integer. Since $\tau_{> -i} X$ is in $\mathcal{D}_{fd}(A)$, by the Calabi-Yau property, we have the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}((\tau_{> -i} X)[-1], Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, (\tau_{> -i} X)[m+1]) = 0.$$

Hence the space $\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -i}X, Y)$ is also zero. It follows that $\tau_{\leq -i}X$ belongs to $\mathcal{F}[i]$.

Let X, Y be two objects in $\mathcal{F}$ and $f : \tau_{\leq -i}X \longrightarrow \tau_{\leq -i}Y$ a morphism. Consider the following diagram

$$\begin{tikzcd}
(\tau_{>-i}X)[-1] \arrow[r] & \tau_{\leq -i}X \arrow[r, "s^i_X"] \arrow[d, "f"] & X \arrow[r] \arrow[d, dotted] & \tau_{>-i}X \\
(\tau_{>-i}Y)[-1] \arrow[r] & \tau_{\leq -i}Y \arrow[r, "s^i_Y"] & Y \arrow[r] & \tau_{>-i}Y.
\end{tikzcd}$$

For $j = 0, 1$, by the Calabi-Yau property, the isomorphism holds

$$\mathrm{Hom}_{\mathcal{D}}((\tau_{>-i}X)[-j], Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, (\tau_{>-i}X)[m+2-j]) = 0,$$

since Y is an object in ${}^{\perp}\mathcal{D}^{\leq -m-1}$.

Since the space $\mathrm{Hom}_{\mathcal{D}}((\tau_{>-i}X)[-1], Y)$ vanishes, the composition $s^i_Y f$ factors through s^i_X. Thus, the functor $\tau_{\leq -i}$ is full.

Let $g : X \longrightarrow Y$ be a morphism in $\mathcal{F}$ satisfying $\tau_{\leq -i}g$ is zero. Then it induces the following commutative diagram

$$\begin{tikzcd}
\tau_{\leq -i}X \arrow[r, "s^i_X"] \arrow[d, "\tau_{\leq -i}g"] & X \arrow[r, "p^i_X"] \arrow[d, "g"] & \tau_{>-i}X \arrow[r] \arrow[dl, dotted, "g_1"] & (\tau_{\leq -i}X)[1] \\
\tau_{\leq -i}Y \arrow[r, "s^i_Y"] & Y \arrow[r, "p^i_Y"] & \tau_{>-i}Y \arrow[r] & (\tau_{\leq -i}X)[1]
\end{tikzcd}$$

such that the morphism gs^i_X is zero. So the morphism g factors through p^i_X. That is, there exists a morphism $g_1 : \tau_{>-i}X \longrightarrow Y$ such that $g = g_1 p^i_X$. The morphism g_1 is zero, since the space $\mathrm{Hom}_{\mathcal{D}}(\tau_{>-i}X, Y)$ vanishes. Thus, the morphism g is zero. It follows that the functor $\tau_{\leq -i}$ is faithful. Now this lemma holds. $\square$

Together by Lemma 3.2.11 and Lemma 3.2.12, we know that the functor $\tau_{\leq -1} : \mathcal{F} \longrightarrow \mathcal{F}[1]$ is an equivalence.

By the same arguments as Step 1 and Step 2 in the proof of Proposition 2.9 in [2], we can get the following two lemmas. However, for the convenience of our later Proposition 3.2.15, we would like to write down the proof of the second lemma, which presents a procedure of constructing the needed object.

Lemma 3.2.13. *The functor π (restricted to $\mathcal{F}$): $\mathcal{F} \longrightarrow \mathcal{C}$ is fully faithful.*

Lemma 3.2.14. *For any object X in* per$A$*, there exists an integer r and an object Z in $\mathcal{F}[-r]$ such that πX and πZ are isomorphic objects in the category $\mathcal{C}$.*

Proof. Let X be an object in per$A$. By Lemma 3.2.6, there exists an integer r such that X is in $\mathcal{D}^{\leq m+1-r} \cap {}^{\perp}\mathcal{D}^{\leq r-m-1}$. Consider the triangle

$$\tau_{\leq r}X \longrightarrow X \longrightarrow \tau_{>r}X \longrightarrow (\tau_{\leq r}X)[1].$$

Let Y be an object in $\mathcal{D}^{\leq r-m-1}$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Y)$, we can get a long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq r}X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}((\tau_{>r}X)[-1], Y) \to \ldots.$$

Clearly, the space $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ is zero. By the Calabi-Yau property, we have the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}((\tau_{>r}X)[-1], Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, (\tau_{>r}X)[m+1]) = 0.$$

Therefore, the object $\tau_{\leq r}X$ is in ${}^{\perp}\mathcal{D}^{\leq r-m-1}$. Thus, we have that $\tau_{\leq r}X$ is in $\mathcal{F}[-r]$. Let Z denote $\tau_{\leq r}X$. Since $\tau_{>r}X$ is in $\mathcal{D}_{fd}(A)$, the objects πX and πZ are isomorphic in $\mathcal{C}$. □

Proposition 3.2.15. *The projection functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}$.*

Proof. We only need to show that π restricted to $\mathcal{F}$ is dense. Let X be an object in $\mathrm{per}A$. Then there exists an integer r such that, the object X is in $\mathcal{D}^{\leq m+1-r} \cap {}^{\perp}\mathcal{D}^{\leq r-m-1}$, the object $\tau_{\leq r}X$ is in $\mathcal{F}[-r]$, and πX is isomorphic to $\pi(\tau_{\leq r}X)$ in $\mathcal{C}$. Now we do induction on the number r. From the remark right after Lemma 3.2.6, we can suppose that $r \leq 0$.

If $r = 0$, the object $\tau_{\leq 0}X$ is in $\mathcal{F}$, and $\pi(\tau_{\leq 0}X)$ is isomorphic to the image $\pi X$of X in $\mathcal{C}$.

Suppose when $r = r_0 \leq 0$, one can find an object Y in $\mathcal{F}$ such that πY is isomorphic to πX in $\mathcal{C}$.

Consider the case $r = r_0 - 1$. Then $\tau_{\leq r_0-1}X$ is in $\mathcal{F}[1 - r_0]$. Set $Z = (\tau_{\leq r_0-1}X)[-1]$. Thus, the object Z is in $\mathcal{F}[-r_0]$. By hypothesis, there exists an object Y in $\mathcal{F}$ such that πY is isomorphic to πZ in $\mathcal{C}$. Therefore, we have following isomorphisms in $\mathcal{C}$

$$\pi Y \simeq \pi Z = \pi((\tau_{\leq r_0-1}X)[-1]) \simeq (\pi(\tau_{\leq r_0-1}X))[-1] \simeq (\pi X)[-1].$$

Since $Y[1]$ is in $\mathcal{F}[1]$ and $\tau_{\leq -1} : \mathcal{F} \longrightarrow \mathcal{F}[1]$ is an equivalence, there exists an object N in $\mathcal{F}$ such that $\tau_{\leq -1}N$ is isomorphic to $Y[1]$. As a consequence, the following isomorphisms hold in $\mathcal{C}$

$$\pi N \simeq \pi(\tau_{\leq -1}N) \simeq \pi(Y[1]) \simeq (\pi Y)[1] \simeq \pi X.$$

Hence we can deduce that for each object T in $\mathcal{C}$, there exists an object T' in $\mathcal{F}$ such that $\pi T'$ is isomorphic to T in $\mathcal{C}$. □

We call $\mathcal{F}$ the *fundamental domain.*

Proof of the main Theorem 3.2.2.

Proof. Proposition 3.2.5 and Proposition 3.2.7 have shown that the category $\mathcal{C}$ is Hom-finite and $(m + 1)$-Calabi-Yau, respectively.

Now we only need to show that the object πA is an m-cluster tilting object whose endomorphism algebra is isomorphic to the zeroth homology H^0A of A.

Since A is in the subcategory $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -1}$, its shift $A[i]$ is in $\mathcal{D}^{\leq -i} \cap {}^{\perp}\mathcal{D}^{\leq -i-1}$. Thus, the objects $A[i]\,(1 \leq i \leq m)$ are in the fundamental domain $\mathcal{F}$. Following Proposition 3.2.15, the functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}$ induces an equivalence between $\mathcal{F}$ and $\mathcal{C}$, so we have that

$$\mathrm{Hom}_{\mathcal{C}}(\pi A, \pi(A[i])) \simeq \mathrm{Hom}_{\mathcal{F}}(A, A[i]) = \mathrm{Hom}_{\mathcal{D}}(A, A[i])$$

$$\simeq H^iA = \begin{cases} H^0A, & i = 0; \\ 0, & 1 \leq i \leq m. \end{cases}$$

Therefore, the endomorphism algebra of πA is isomorphic to the zeroth homology H^0A of A, and

$$\mathrm{Hom}_{\mathcal{C}}(\pi A, (\pi A)[r]) = 0,\ r = 1, \ldots, m.$$

Let X be an object in $\mathcal{F}$. According to Lemma 3.2.8, there exist m triangles where $Q_0, Q_1, \ldots, Q_{m-1}$ are free A-modules and P_m is in add$A$.

Now we will show the following isomorphisms

$$\mathrm{Ext}^1_{\mathcal{D}}(P_{m-1}, Y) \simeq \mathrm{Ext}^2_{\mathcal{D}}(P_{m-2}, Y) \simeq \ldots \simeq \mathrm{Ext}^m_{\mathcal{D}}(X, Y), \quad Y \in \mathcal{D}_{\leq 0}. \qquad (1)$$

Applying $\mathrm{Hom}_{\mathcal{D}}(-, Y[j])$ to the triangle (here we write P_0 instead of X)

$$P_{m-j+1} \longrightarrow Q_{m-j} \longrightarrow P_{m-j} \longrightarrow P_{m-j+1}[1], \quad j = 2, \ldots m,$$

we can get a long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Q_{m-j}[1], Y[j]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{m-j+1}[1], Y[j]) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{D}}(P_{m-j}, Y[j]) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Q_{m-j}, Y[j]) \longrightarrow \ldots.$$

Since Q_{m-j} are free A-modules, the spaces $\mathrm{Hom}_{\mathcal{D}}(Q_{m-j}[i], Y[j])$ are zero for $i = 0, 1$. Therefore, we have the following isomorphisms

$$\mathrm{Ext}^j_{\mathcal{D}}(P_{m-j}, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(P_{m-j}, Y[j]) \simeq \mathrm{Hom}_{\mathcal{D}}(P_{m-j+1}[1], Y[j])$$
$$\simeq \mathrm{Ext}^{j-1}_{\mathcal{D}}(P_{m-j+1}, Y), \quad j = 2, \ldots m.$$

It follows that (1) is true.

Next applying $\mathrm{Hom}_{\mathcal{D}}(-, Y[j])$ to the triangle

$$P_{m-j} \longrightarrow Q_{m-j-1} \longrightarrow P_{m-j-1} \longrightarrow P_{m-j}[1], \quad j = 2, \ldots m-1,$$

similarly we can obtain the following isomorphisms

$$\mathrm{Ext}^1_{\mathcal{D}}(P_{m-2}, Y) \simeq \mathrm{Ext}^2_{\mathcal{D}}(P_{m-3}, Y) \simeq \ldots \simeq \mathrm{Ext}^{m-1}_{\mathcal{D}}(X, Y), \quad Y \in \mathcal{D}_{\leq 0}.$$

Thus, we can get a list of isomorphisms

$$\mathrm{Ext}^1_{\mathcal{D}}(P_{m-i}, Y) \simeq \mathrm{Ext}^{m+1-i}_{\mathcal{D}}(X, Y), \quad 1 \leq i \leq m, \, Y \in \mathcal{D}_{\leq 0}.$$

Suppose that Z is an object in $\mathcal{C}$ such that the space $\mathrm{Hom}_{\mathcal{C}}(Z, (\pi A)[i])$ vanishes for each $1 \leq i \leq m$. Since the functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}$ induces an equivalence between $\mathcal{F}$ and $\mathcal{C}$, there exists an object X in $\mathcal{F}$ such that πX is isomorphic to Z in $\mathcal{C}$. Therefore, we have the following isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(Z, (\pi A)[i]) \simeq \mathrm{Hom}_{\mathcal{C}}(\pi X, (\pi A)[i]) \simeq \mathrm{Hom}_{\mathcal{D}}(X, A[i])$$
$$\simeq \mathrm{Ext}^i_{\mathcal{D}}(X, A), \quad 1 \leq i \leq m.$$

Hence, we have

$$\mathrm{Ext}^1_{\mathcal{D}}(P_{m-i}, A) \simeq \mathrm{Ext}^{m+1-i}_{\mathcal{D}}(X, A) = 0, \quad 1 \leq i \leq m.$$

As a consequence, the triangle

$$P_m \longrightarrow Q_{m-1} \longrightarrow P_{m-1} \longrightarrow P_m[1]$$

splits, then the object P_{m-1} is in add$A$. Next the triangle

$$P_{m-1} \longrightarrow Q_{m-2} \longrightarrow P_{m-2} \longrightarrow P_{m-1}[1]$$

also splits, then the object P_{m-2} is also in add$A$. By iterated arguments, we can get that P_i $(1 \leq i \leq m)$ and X are all in add$A$. Thus, the object Z, which is isomorphic to πX in $\mathcal{C}$, is in the subcategory addπA. Hence, the object πA is an m-cluster tilting object in the category $\mathcal{C}$. $\square$

3.3 The cluster categories of Ginzburg dg categories

In [38], V. Ginzburg defined the Ginzburg dg algebra $\Gamma(Q,W)$ associated with a quiver with potential (Q,W), where the arrows of the quiver Q are concentrated in degree 0. Generally, let Q be a graded k-quiver such that the set Q_0 of objects is finite and $Q(x,y)$ is a finite-dimensional k-module for all objects x and y. Let $\mathcal{R}$ be the discrete k-category on the set Q_0 of vertices. Denote by $\mathcal{A}$ the tensor category $T_{\mathcal{R}}(Q)$. Let $Q^\vee$ be the dual of the $\mathcal{R}$-bimodule Q over $\mathcal{R}^e$ endowed with the canonical involution (see Section 3.1 in [56]). Fixing an integer n and a superpotential W in the cyclic homology $HC_{n-3}(\mathcal{A})$, *i.e.* a linear combination of cycles of degree $3-n$ considered up to cyclic permutation 'with signs', the Ginzburg dg category $\Gamma_n(Q,W)$ is defined in [56] as the tensor category over $\mathcal{R}$ of the bimodule

$$\widetilde{Q} = Q \oplus Q^\vee[n-2] \oplus \mathcal{R}[n-1]$$

endowed with the unique differential which

a) vanishes on Q;

b) takes the element a^* of $Q^\vee[n-2]$ to the cyclic derivative $\partial_a W$ for each arrow a in Q_1, where the map ∂_a takes a path p to the sum $(-1)^{deg(a)}\sum_{p=uav} \pm vu$ (here $deg(a)$ denotes the degree of an arrow a and the sign $\pm$ is computed by Koszul sign rule);

c) takes the element t_x of $\mathcal{R}[n-1]$ to $(-1)^n\,\mathrm{id}_x(\sum_{v\in Q_1}[v,v^*])\,\mathrm{id}_x$ for each object x in Q_0, where [,] denotes the supercommutator.

Remark 3.3.1. The $\mathcal{R}$-bimodule (or graded quiver) $\widetilde{Q}$ has an intuitional expression (as the graded quivers in the ordinary Ginzburg dg algebras) as follows

- the same vertices as Q,
- the arrows are
  - i) the arrows of Q of the same degree,
  - ii) an arrow $a^* : j \to i$ of degree '$2-n-deg(a)$' for each arrow $a : i \to j$ of Q,
  - iii) a loop $t_x : x \to x$ of degree $1-n$ for each vertex x of Q.

Theorem 3.3.2 ([56])**.** *The Ginzburg dg category $\Gamma_n(Q,W)$ is homologically smooth and n-Calabi-Yau.*

For simplicity, set $\Gamma^{(n)}$ as the Ginzburg dg category $\Gamma_n(Q,W)$ associated with a graded quiver with superpotential (Q,W). Moreover, we assume that the arrows of Q are concentrated in nonpositive degrees. We denote the minimal degree by N_Q.

Theorem 3.3.3. *Let m be a positive integer satisfying $m \geq -N_Q$. Suppose that the zeroth homology of the Ginzburg dg category $\Gamma^{(m+2)}$ is finite-dimensional. Then the* generalized m-cluster category

$$\mathcal{C}_{(Q,W)} = \mathrm{per}\Gamma^{(m+2)}/\mathcal{D}_{fd}(\Gamma^{(m+2)})$$

associated with (Q,W) *is* Hom*-finite and $(m+1)$-Calabi-Yau. Moreover, the image of the free module $\Gamma^{(m+2)}$ in $\mathcal{C}_{(Q,W)}$ is an m-cluster tilting object whose endomorphism algebra is isomorphic to the zeroth homology of $\Gamma^{(m+2)}$.*

Proof. Since the nonpositive integer $N_Q \geq -m$, the elements of $Q^\vee[m]$ are concentrated in nonpositive degrees. Then the Ginzburg dg category $\Gamma^{(m+2)}$ has its homology concentrated in nonpositive degrees. We have that the p-th homology $H^p\Gamma^{(m+2)}$ is zero for each integer $p > 0$. By assumption, the space $H^0\Gamma^{(m+2)}$ is finite-dimensional. Combining with Theorem 3.3.2, the dg algebra $\Gamma^{(m+2)}$ satisfies the four properties $(\star)$. We apply the main Theorem 3.2.2 in particular to $\Gamma^{(m+2)}$. Then the result clearly holds. $\square$

The following corollary considers acyclic quivers with zero superpotential. In this case, the generalized m-cluster category $\mathcal{C}_{(Q,0)}$ recovers the (classical) m-cluster category $\mathcal{C}_Q^{(m)}$.

Corollary 3.3.4. *Let k be an algebraically closed field and m a positive integer. Suppose that Q is an acyclic ordinary quiver. Then the generalized m-cluster category $\mathcal{C}_{(Q,0)}$ is triangle equivalent to the orbit category $\mathcal{C}_Q^{(m)}$ of the finite-dimensional derived category $\mathcal{D}_{fd}(\mathrm{mod}kQ)$ under the action of the automorphism $\tau^{-1}\Sigma^m (= \nu^{-1}\Sigma^{m+1})$, where Σ (resp. ν) is the suspension functor (resp. Serre functor) and τ is the Auslander-Reiten translation.*

Proof. Since Q is an acyclic ordinary quiver, the degrees of the arrows of $\widetilde{Q}$ concentrate in $0, -m, -m-1$ and the homology $H^{-i}\Gamma^{(m+2)}$ vanishes for each $1 \leq i \leq m-1$.

Since W is zero (in fact, if $m \geq 2$, the only superpotential is the zero one, otherwise, the degrees of the homogeneous summands of superpotentials are $1 - m (\leq -1)$, while the degrees of the arrows are zero), the zeroth homology of $\Gamma^{(m+2)}$ is the finite-dimensional path algebra kQ.

Following Theorem 3.3.3, the generalized m-cluster category $\mathcal{C}_{(Q,0)}$ is $(m+1)$-Calabi-Yau, and the image of $\Gamma^{(m+2)}$ (denoted by T) is an m-cluster tilting object whose endomorphism algebra is isomorphic to the finite-dimensional hereditary algebra kQ.

Moreover, from the proof of the main Theorem 3.2.2, we know that the objects $\Sigma^i\Gamma^{(m+2)} (0 \leq i \leq m)$ are in the fundamental domain $\mathcal{F}$. Therefore, the following isomorphisms hold

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(T, \Sigma^{-i}T) &\simeq \mathrm{Hom}_{\mathcal{C}}(\Sigma^i T, T) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^i\Gamma^{(m+2)}, \Gamma^{(m+2)}) \\ &\simeq H^{-i}\Gamma^{(m+2)} = 0, \quad \text{for each } 1 \leq i \leq m-1,\end{aligned}$$

where $\mathcal{C}$ denotes the generalized m-cluster category $\mathcal{C}_{(Q,0)}$. Hence, following Theorem 4.2 in [60], there is a triangle equivalence from $\mathcal{C}_{(Q,0)}$ to $\mathcal{C}_Q^{(m)}$. $\square$

Example 3.3.5. Suppose m is 2. Let us consider the graded quiver Q

$$\begin{tikzcd} & 2 \arrow[dr, "c"] & \\ 1 \arrow[ur, "a"] & & 3, \arrow[ll, "b"] \end{tikzcd}$$

where $deg(a) = -1, deg(b) = deg(c) = 0$, with superpotential $W = abc$.

The Ginzburg dg category $\Gamma^{(4)} = \Gamma_4(Q, W)$ is the tensor category whose underlying graded quiver is $\widetilde{Q}$

$$\begin{tikzcd} & 2 \arrow[loop above, "t_2"] \arrow[dl, shift left, "a^*"] \arrow[dr, shift left, "c"] & \\ 1 \arrow[loop left, "t_1"] \arrow[ur, shift left, "a"] \arrow[rr, shift left, "b^*"] & & 3 \arrow[loop right, "t_3"] \arrow[ul, shift left, "c^*"] \arrow[ll, shift left, "b"] \end{tikzcd}$$

where $deg(a^*) = -1$, $deg(b^*) = deg(c^*) = -2$ and $deg(t_i) = -3$ for $1 \leq i \leq 3$. Its differential takes the following values on the arrows of $\widetilde{Q}$:

$$d(a^*) = -bc, \quad d(b^*) = ca, \quad d(c^*) = ab,$$
$$d(t_1) = bb^* + a^*a, \quad d(t_2) = aa^* - c^*c, \quad d(t_3) = cc^* - b^*b.$$

The zeroth homology $H^0\Gamma^{(4)}$ equals to the path algebra with relation $kQ^{(0)}/(bc)$, whose k-basis is $\{e_1, e_2, e_3, b, c\}$. Therefore, the dimension of $H^0\Gamma^{(4)}$ is 5.

Following Theorem 3.3.3, the image of $\Gamma^{(4)}$ in the generalized 2-cluster category $\mathcal{C}_{(Q,W)}$ is a 2-cluster tilting object, whose endomorphism algebra is given by the following quiver with the relation

$$2 \xrightarrow{c} 3 \xrightarrow{b} 1, \qquad bc = 0.$$

In the following, we will show that the generalized 2-cluster category $\mathcal{C}_{(Q,W)}$ and the orbit category $\mathcal{C}^{(2)}_{A_3}$ are triangle equivalent.

Let Q' be the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] \\ 1 \arrow[r, "\beta"] & 3, \end{tikzcd}$$

with $deg(\alpha) = deg(\beta) = 0$. We denote the indecomposable module e_ikQ' by P'_i, and its corresponding simple module by S'_i for $1 \leq i \leq 3$. Notice that we consider right modules so that, for example, the support of P'_1 is $\{1, 2\}$. Let T be the almost complete tilting module $P'_2 \oplus P'_3$. Its two complements are P'_1 and S'_3. We write $\overline{T}$ as the direct sum $S'_3 \oplus P'_2 \oplus P'_3$. Then we have the derived equivalence

$$\mathcal{D}\mathrm{End}(\overline{T}) \simeq \mathcal{D}(\mathrm{mod}kQ').$$

Following Proposition 4.2 in [56], the derived 4-preprojective dg algebras $\Pi_4(\mathrm{End}(\overline{T}), 0)$ and $\Pi_4(kQ', 0)$ are Morita equivalent. Moreover, by Theorem 6.3 in [56], the derived 4-preprojective dg algebra $\Pi_4(kQ', 0)$ is quasi-isomorphic to the Ginzburg dg category $\Gamma_4(Q', 0)$.

The underlying graded quiver of the algebra $\mathrm{End}(\overline{T})$ is

$$Q'' \qquad \begin{tikzcd} & 2 \arrow[d, "\gamma"] \\ 1 & 3, \arrow[l, "\delta"] \end{tikzcd}$$

with the relation $\delta\gamma = 0$, where $deg(\gamma) = deg(\delta) = 0$. Thus, the algebra $\mathrm{End}(\overline{T})$ is quasi-isomorphic to the path algebra of the following graded quiver Q'''

$$\begin{tikzcd} & 2 \arrow[dl, "\eta"'] \arrow[dr, "\gamma"] & \\ 1 & & 3, \arrow[ll, "\delta"] \end{tikzcd}$$

with the differential $d(\eta) = -\delta\gamma$, where $deg(\eta) = -1$. Following Proposition 6.6 in [56], the derived 4-preprojective dg algebra $\Pi_4(\mathrm{End}(\overline{T}), 0)$ is quasi-isomorphic to the tensor category $T_{\mathcal{R}}(\widetilde{Q'''})$, endowed with the unique differential such that

$$d(\eta) = \partial_{\eta^*}W' = -\delta\gamma, \quad d(\delta^*) = \partial_\delta W' = \gamma\eta^*, \quad d(\gamma^*) = \partial_\gamma W' = \eta^*\delta,$$
$$d(t_1) = \delta\delta^* + \eta\eta^*, \quad d(t_2) = \eta^*\eta - \gamma^*\gamma, \quad d(t_3) = \gamma\gamma^* - \delta^*\delta,$$

where $W' = \eta^*\delta\gamma$, and $\widetilde{Q'''} = Q''' \oplus (Q''')^\vee[2] \oplus \mathcal{R}[3]$.

It is easy to check that the tensor category $T_{\mathcal{R}}(\widetilde{Q'''})$ endowed with the differential equals the Ginzburg dg category $\Gamma_4(\mathcal{Q}, W')$, where $\mathcal{Q}$ is the graded quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\gamma"] & \\ 1 \arrow[ur, "\eta^*"] & & 3, \arrow[ll, "\delta"] \end{tikzcd}$$

obtained from Q''' by replacing η by η^*, and W' is still the superpotential $\eta^*\delta\gamma$. Obviously, the graded quivers $\mathcal{Q}$ and Q are isomorphic, while the superpotentials W' and W correspond to each other. Hence, the derived 4-preprojective dg algebra $\Pi_4(\mathrm{End}(\overline{T}), 0)$ is quasi-isomorphic to the Ginzburg dg category $\Gamma_4(Q, W)$.

As a consequence, the Ginzburg dg categories $\Gamma_4(Q, W)$ and $\Gamma_4(Q', 0)$ are Morita equivalent. Therefore, the generalized 2-cluster categories $\mathcal{C}_{(Q,W)}$ and $\mathcal{C}_{(Q',0)}$ are triangle equivalent. By Corollary 3.3.4, we can conclude that the generalized 2-cluster category $\mathcal{C}_{(Q,W)}$ and the orbit category $\mathcal{C}_{A_3}^{(2)}$ are triangle equivalent.

3.4 For algebras of finite global dimension

Let A be a finite-dimensional k-algebra of finite global dimension. Let n be a positive integer. The finite-dimensional derived category $\mathcal{D}_{fd}(A)$ admits a right Serre functor

$$\nu_A = - \overset{L}{\otimes}_A DA.$$

Unfortunately, the orbit category $\mathcal{O}_A$ of $\mathcal{D}_{fd}(A)$ under the autoequivalence $\nu_A[-n]$ is not triangulated in general, then we take advantage of the triangulated hull of $\mathcal{O}_A$ which was constructed in [53]. Let B be the trivial extension $A \oplus DA[-n-1]$ with A in degree 0 and DA in degree $n+1$. The dg B-bimodule DB is isomorphic to $B[n+1]$, and the perfect derived category $\mathrm{per}B$ is contained in $\mathcal{D}_{fd}(B)$ under this construction. It is not hard to check that for each object X in $\mathrm{per}B$ and Y in $\mathcal{D}_{fd}(B)$, there is a functorial isomorphism

$$D\mathrm{Hom}_{\mathcal{D}(B)}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(Y, X[n+1]).$$

Denote by $p : B \to A$ the canonical projection and $p_* : \mathcal{D}_{fd}(A) \to \mathcal{D}_{fd}(B)$ the induced triangulated functor. Let $\langle A \rangle_B$ be the thick subcategory of $\mathcal{D}_{fd}(B)$ generated by the image of p_*. We call the triangulated hull

$$\mathcal{C}_A = \langle A \rangle_B / \mathrm{per} B$$

of $\mathcal{O}_A$ the $(n-1)$-*cluster category* of A. Here we would like to point out that the authors of [68] make use of the d-Calabi-Yau generalized cluster categories, whereas they do not give any explicit proof for their construction. In general, this category $\mathcal{C}_A$ has infinite-dimensional morphism spaces. In [2] C. Amiot dealt with the case $n \leq 2$. By $\Pi_3 A$ we denote the derived 3-preprojective algebra of A as introduced in [56].

Theorem 3.4.1 ([2])**.** *Let A be a finite-dimensional k-algebra of global dimension ≤ 2. If the functor $Tor_2^A(-, DA)$ is nilpotent, then the cluster category $\mathcal{C}_A$ is Hom-finite, 2-Calabi-Yau and the object A is a cluster tilting object. Moreover, there exists a triangle equivalence from $\mathcal{C}_A$ to the generalized cluster category $\mathcal{C} = \mathrm{per}\Pi_3 A / \mathcal{D}_{fd}(\Pi_3 A)$ sending the object A to the image of the derived 3-preprojective algebra $\Pi_3 A$ in $\mathcal{C}$.*

In this section, we will investigate the generalization of the above theorem to the case that A is a finite-dimensional k-algebra of global dimension $\leq n$ (instead of ≤ 2). Since the generalization is straightforward, we only list the main steps here, and leave the proofs to the interested reader.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{T}$.

Definition 3.4.2 ([2])**.** Let X and Y be objects in $\mathcal{T}$. A morphism $p : N \to X$ is called a *local $\mathcal{N}$-cover of X relative to Y* if N is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(X, Y) \xrightarrow{p^*} \mathcal{T}(N, Y).$$

A morphism $i : X \to N$ is called a *local $\mathcal{N}$-envelope of X relative to Y* if N is in $\mathcal{N}$ and if it induces an exact sequence:

$$0 \longrightarrow \mathcal{T}(Y, X) \xrightarrow{i_*} \mathcal{T}(Y, N).$$

We can read the following lemma from the proof of Theorem 4.2 in [2].

Lemma 3.4.3 ([2])**.** *Let X and Y be objects of $\mathcal{D}_{fd}(B)$ such that the space $\mathrm{Hom}_{\mathcal{D}(B)}(X, Y)$ is finite-dimensional. Then there exists a local $\mathrm{per}B$-cover of X relative to Y.*

Under the assumption of the above lemma, both $\mathrm{Hom}_{\mathcal{D}(B)}(N, X)$ and $\mathrm{Hom}_{\mathcal{D}(B)}(X, N)$ are finite-dimensional for N in $\mathrm{per}B$ and X in $\mathcal{D}_{fd}(B)$. Therefore, there exists a local $\mathrm{per}B$-envelope of $X[n+1]$ relative to Y. Hence the bilinear form

$$\beta'_{X,Y} : \mathrm{Hom}_{\mathcal{C}_A}(X, Y) \times \mathrm{Hom}_{\mathcal{C}_A}(Y, X[n]) \longrightarrow k, \quad X, Y \in \mathcal{C}_A$$

constructed in the first section of [2] is non-degenerate. Therefore, if $\mathcal{C}_A$ is Hom-finite, then it is n-Calabi-Yau as a triangulated category.

We denote by $\mathcal{D}$ the derived category $\mathcal{D}(A)$ of the algebra A. Let us recall the following important properties of the Serre functor ν_A:

- $\nu_A(\mathcal{D}^{\geq 0}) \subset \mathcal{D}^{\geq -n}$;
- $\mathrm{Hom}_{\mathcal{D}}(U, V)$ vanishes for all $U \in \mathcal{D}^{\geq 0}$ and $V \in \mathcal{D}^{\leq -n-1}$;

- ν_A admits an inverse
$$\nu_A^{-1} = - \overset{L}{\otimes}_A \mathrm{RHom}_A(DA, A),$$
where the homology of the complex $\mathrm{RHom}_A(DA, A)$ is given by
$$\begin{aligned} H^i\mathrm{RHom}_A(DA, A) &\simeq \mathrm{Hom}_{\mathcal{D}}(DA, A[i]) \\ &\simeq \begin{cases} \mathrm{Hom}_{\mathcal{D}}(DA, A), & i = 0, \\ \mathrm{Ext}^i_A(DA, A), & i = 1, \ldots, n, \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$
- $\nu_A^{-1}(\mathcal{D}^{\leq 0}) \subset \mathcal{D}^{\leq n}$.

Using these properties we obtain the following generalization of Proposition 4.7 of [2].

Proposition 3.4.4. *Let A be a finite-dimensional k-algebra of global dimension $\leq n$ and X the A-A-bimodule $\mathrm{Ext}^n_A(DA, A)$. Then the endomorphism algebra $\widetilde{A} = \mathrm{End}_{\mathcal{C}_A}(A)$ is isomorphic to the tensor algebra T_AX of X over A. As a consequence, if the category $\mathcal{C}_A$ is Hom-finite, then the functor $-\otimes_A \mathrm{Ext}^n_A(DA, A)$ is nilpotent.*

In fact, the converse statement of the consequence in Proposition 3.4.4 is also true. Taking advantage of the above properties of the Serre functor ν_A, we also have the following variant of Proposition 4.9 of [2].

Proposition 3.4.5. *Let A be a finite-dimensional k-algebra of global dimension $\leq n$. The following properties are equivalent:*

1) the category $\mathcal{C}_A$ is Hom-finite,

2) the functor $-\otimes_A \mathrm{Ext}^n_A(DA, A)$ is nilpotent,

3) the functor $Tor^A_n(-, DA)$ is nilpotent.

Now we give a complete proof for the following well-known lemma.

Lemma 3.4.6. *Let A be a dg k-algebra. Then for all dg A-modules L, M, the objects $\mathrm{RHom}_A(L, M)$ and $\mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(L, M))$ are isomorphic in the derived category of dg A-A-bimodules.*

Proof. Let N be an A-A-bimodule. We construct two maps Φ and Ψ as follows
$$\begin{aligned} \Phi : \mathrm{Hom}_A(L \otimes_A N, M) &\longrightarrow \mathrm{Hom}_{A^e}(N, \mathrm{Hom}_k(L, M)) \\ f &\longmapsto (\Phi(f)(n) : l \mapsto (-1)^{|l||n|} f(l \otimes n)), \\ \Psi : \mathrm{Hom}_{A^e}(N, \mathrm{Hom}_k(L, M)) &\longrightarrow \mathrm{Hom}_A(L \otimes_A N, M) \\ g &\longmapsto \Psi(g)(l \otimes n) = (-1)^{|l||n|} g(n)(l). \end{aligned}$$
It is not hard to check that Φ and Ψ are A-A-bihomomorphisms homogeneous of degree 0 and satisfy
$$\Phi\Psi = \mathbf{1}, \quad \Psi\Phi = \mathbf{1}.$$
Moreover, the morphisms Φ and Ψ commute with the differentials. Thus, they induce inverse isomorphisms
$$\mathrm{Hom}_{\mathcal{C}(A)}(L \otimes_A N, M) \simeq \mathrm{Hom}_{\mathcal{C}(A^e)}(N, \mathrm{Hom}_k(L, M)),$$

where $\mathcal{C}(E)$ denotes the category of dg E-modules for a dg algebra E. The morphisms Φ and Ψ also induce inverse isomorphisms

$$\mathrm{Hom}_{\mathcal{H}(A)}(L \otimes_A N, M) \simeq \mathrm{Hom}_{\mathcal{H}(A^e)}(N, \mathrm{Hom}_k(L, M)),$$

where $\mathcal{H}(E)$ denotes the category up to homotopy of dg E-modules for a dg algebra E. If we specialize N to A, then we have

$$\mathrm{Hom}_{\mathcal{H}(A)}(\mathbf{p}L, \mathbf{i}M) \simeq \mathrm{Hom}_{\mathcal{H}(A^e)}(A, \mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M)),$$

where $\mathbf{p}L$ is a cofibrant resolution of L, and $\mathbf{i}M$ is a fibrant resolution of M.

Now we show that the complex $\mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M)$ is a fibrant resolution of $\mathrm{Hom}_k(L, M)$ in $\mathcal{C}(A^e)$. Let $\iota : U \to V$ be a quasi-isomorphism in $\mathcal{C}(A^e)$ which is injective in each component. We have the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(A^e)}(U, \mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M))) &\simeq \mathrm{Hom}_{\mathcal{C}(A)}(\mathbf{p}L \otimes_A U, \mathbf{i}M),\\ \mathrm{Hom}_{\mathcal{C}(A^e)}(V, \mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M))) &\simeq \mathrm{Hom}_{\mathcal{C}(A)}(\mathbf{p}L \otimes_A V, \mathbf{i}M).\end{aligned}$$

Since $\mathbf{p}L$ is cofibrant, the morphism $\mathbf{p}L \otimes \iota :\ \mathbf{p}L \otimes U \to \mathbf{p}L \otimes V$ is a quasi-isomorphism in $\mathcal{C}(A)$ which is injective in each component. Since $\mathbf{i}M$ is fibrant, it follows that the morphism

$$\mathrm{Hom}_{\mathcal{C}(A)}(\mathbf{p}L \otimes_A V, \mathbf{i}M) \longrightarrow \mathrm{Hom}_{\mathcal{C}(A)}(\mathbf{p}L \otimes_A U, \mathbf{i}M)$$

is surjective. Thus, the complex $\mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M)$ is fibrant. Therefore, we have the following isomorphisms in the derived category of dg A-A-bimodules

$$\begin{aligned}\mathrm{RHom}_A(L, M) \simeq \mathrm{Hom}_{\mathcal{H}(A)}(\mathbf{p}L, \mathbf{i}M) &\simeq \mathrm{Hom}_{\mathcal{H}(A^e)}(A, \mathrm{Hom}_k(\mathbf{p}L, \mathbf{i}M))\\ &\simeq \mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(L, M)).\end{aligned}$$

□

Lemma 3.4.7. *Assume that A is a proper (i.e. $\dim_k H^*A < \infty$) dg algebra. Then the objects $\mathrm{RHom}_A(DA, A)$ and $\mathrm{RHom}_{A^e}(A, A^e)$ are isomorphic in the derived category of dg A-A-bimodules.*

Proof. If we particularly choose L as DA and M as A in Lemma 3.4.6, then we have the isomorphism in the derived category of dg A-A-bimodules

$$\mathrm{RHom}_A(DA, A) \simeq \mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(DA, A)).$$

Since A is proper, the object A^e $(= A^{op} \otimes_k A)$ is quasi-isomorphic to $A^{op} \otimes_k D(DA)$ and DA is perfect over k. Therefore, we have the quasi-isomorphisms

$$A^e \simeq A^{op} \otimes_k D(DA) \simeq \mathrm{Hom}_k(DA, A).$$

As a result, we have the following isomorphisms in the derived category of dg A-A-bimodules

$$\mathrm{RHom}_{A^e}(A, A^e) \simeq \mathrm{RHom}_{A^e}(A, \mathrm{Hom}_k(DA, A)) \simeq \mathrm{RHom}_A(DA, A).$$

□

Let Θ be a cofibrant resolution of the dg A-bimodule $\mathrm{RHom}_A(DA, A)$. Therefore, following [56], the derived $(n+1)$-preprojective algebra is defined as

$$\Pi_{n+1}(A) = T_A(\Theta[n]).$$

It is homologically smooth and $(n+1)$-Calabi-Yau as a bimodule. Moreover, the complex $\mathrm{RHom}_A(DA, A)[n]$ has its homology concentrated in nonpositive degrees $-n, \ldots, -1, 0$, and

$$H^0(\Theta[n]) \simeq H^0(\mathrm{RHom}_A(DA, A)[n]) \simeq H^n(\mathrm{RHom_A(DA, A)}) \simeq \mathrm{Ext}^n_A(DA, A).$$

Thus, the homology of the dg algebra $\Pi_{n+1}(A)$ vanishes in positive degrees, and we have the following isomorphisms

$$H^0(\Pi_{n+1}(A)) \simeq T_A(H^0(\Theta[n])) \simeq T_A(\mathrm{Ext}^n_A(DA, A)) \simeq \widetilde{A}.$$

In order for the derived $(n+1)$-preprojective algebra $\Pi_{n+1}(A)$ to satisfy the four properties in Section 2, we would like to have that $H^0(\Pi_{n+1}(A))$ is finite-dimensional.

Corollary 3.4.8. *Let A be a finite-dimensional k-algebra of global dimension $\leq n$. If the functor $\mathrm{Tor}^A_n(-, DA)$ is nilpotent, then the generalized $(n-1)$-cluster category*

$$\mathcal{C} = \mathrm{per}\Pi_{n+1}(A)/\mathcal{D}_{fd}(\Pi_{n+1}(A))$$

is Hom-finite, n-Calabi-Yau and the image of the free dg module $\Pi_{n+1}(A)$ is an $(n-1)$-cluster tilting object in $\mathcal{C}$.

Proof. If the functor $\mathrm{Tor}^A_n(-, DA)$ is nilpotent, then the functor $-\otimes_A \mathrm{Ext}^n_A(DA, A)$ is nilpotent by Proposition 3.4.5. Thus, the zeroth homology of $\Pi_{n+1}(A)$ is finite-dimensional. Now we apply Theorem 3.2.2 in particular to the derived $(n+1)$-preprojective algebra $\Pi_{n+1}(A)$, then this corollary holds. □

Theorem 3.4.9. *Let A be a finite-dimensional k-algebra of global dimension $\leq n$. If the functor $\mathrm{Tor}^A_n(-, DA)$ is nilpotent, then the $(n-1)$-cluster category $\mathcal{C}_A$ of A is Hom-finite, n-Calabi-Yau and the image of A_B is an $(n-1)$-cluster tilting object in $\mathcal{C}_A$.*

Proof. Similarly as [2], we will construct a triangle equivalence between the $(n-1)$-cluster category $\mathcal{C}_A$ of A and the generalized $(n-1)$-cluster category $\mathcal{C}$ of $\Pi_{n+1}(A)$. Then the statement will follow from Corollary 3.4.8. □

Recall that $\langle A\rangle_B$ denotes the thick subcategory generated by A_B in the derived category $\mathcal{D}_{fd}(B)$. First we will construct a triangle equivalence from $\langle A\rangle_B$ to $\mathrm{per}\Pi_{n+1}(A)$. Consider the functor $\mathrm{RHom}_B(A_B, -)$. By Section 8 in [50], it induces a triangle equivalence between $\langle A\rangle_B$ and $\mathrm{per}C$, where C is the dg algebra $\mathrm{RHom}_B(A_B, A_B)$. The following lemma is an easy extension of Lemma 4.13 of [2].

Lemma 3.4.10. *The dg algebras $\Pi_{n+1}(A)$ and $\mathrm{RHom}_B(A_B, A_B)$ are isomorphic objects in the homotopy category of dg algebras.*

As a result, the functor $\mathrm{RHom}_B(A_B, -)$ induces a triangle equivalence between $\langle A\rangle_B$ and $\mathrm{per}\Pi_{n+1}(A)$, which sends the object A_B to the free module $\Pi_{n+1}(A)$ and sends the free B-module B to the object $A_{\Pi_{n+1}(A)}$. So the functor also induces an equivalence between the category $\mathrm{per}B$ and the thick subcategory $\langle A\rangle_{\Pi_{n+1}(A)}$ of $\mathcal{D}(\Pi_{n+1}(A))$ generated by A. Moreover, as in Lemma 4.15 of [2], we still have that the category $\langle A\rangle_{\Pi_{n+1}(A)}$ is the finite-dimensional derived category $\mathcal{D}_{fd}(\Pi_{n+1}(A))$. Hence, the categories $\mathcal{C}_A$ and $\mathcal{C}$ are triangle equivalent and Theorem 3.4.9 holds for an arbitrary positive integer n.

Chapter 4

Complements of almost complete m-cluster tilting objects

We study higher cluster tilting objects in generalized higher cluster categories arising from dg algebras of higher Calabi-Yau dimension. Taking advantage of silting mutations of Aihara-Iyama, we obtain a class of m-cluster tilting objects in generalized m-cluster categories. For generalized m-cluster categories arising from strongly $(m+2)$-Calabi-Yau dg algebras, by using truncations of minimal cofibrant resolutions of simple modules, we prove that each almost complete m-cluster tilting P-object has exactly $m+1$ complements with periodicity property. This leads us to the conjecture that each liftable almost complete m-cluster tilting object has exactly $m+1$ complements in generalized m-cluster categories arising from m-rigid good completed deformed preprojective dg algebras.

4.1 Introduction

Cluster categories associated with acyclic quivers were introduced in [17], where the authors gave an additive categorification of the finite type cluster algebras introduced by Fomin and Zelevinsky [34] [35]. The cluster category of an acyclic quiver Q is defined as the orbit category of the finite-dimensional derived category of the category of finite-dimensional representations of Q under the action of $\tau^{-1}\Sigma$, where τ is the AR-translation and Σ the suspension functor. If we replace the autoequivalence $\tau^{-1}\Sigma$ with $\tau^{-1}\Sigma^m$ for an integer $m \geq 2$, we obtain the m-cluster category, which was first mentioned and proved to be triangulated in [53], cf. also [75]. In the cluster category, the exchange relations of the corresponding cluster algebra are modeled by exchange triangles. It was shown in [17] that every almost complete cluster tilting object admits exactly two complements. In the higher cluster category, exchange triangles are replaced by AR-angles, whose existence (in the more general set up of Krull-Schmidt Hom-finite triangulated categories with Serre functors) was shown in [48]. Both [78] and [79] proved that each almost complete m-cluster tilting object has exactly $m+1$ complements in an m-cluster category. In this paper, we study the analogous statements for almost complete m-cluster tilting objects in certain $(m+1)$-Calabi-Yau triangulated categories.

Amiot [2] constructed generalized cluster categories using 3-Calabi-Yau dg algebras which satisfy some suitable assumptions. A special class is formed by the generalized cluster categories associated with Ginzburg algebras [38] coming from suitable quivers with potentials. If the quiver is acyclic, the generalized cluster category is triangle equivalent to the classical cluster category. Amiot's results were extended by the author to generalized

m-cluster categories in [40] by changing the Calabi-Yau dimension from 3 to $m + 2$ for an arbitrary positive integer m. As one of the applications, she particularly considered generalized higher cluster categories associated with Ginzburg dg categories [56] coming from suitable graded quivers with superpotentials.

In the representation theory of algebras, mutation plays an important role. Here we recall several kinds of mutation. Cluster algebras associated with finite quivers without loops or 2-cycles are defined using mutation of quivers. As an extension of quiver mutation, the mutation of quivers with potentials was introduced in [31]. Moreover, the mutation of decorated representations of quivers with potentials, which can be viewed as a generalization of the BGP construction, was also studied in [31]. Tilting modules over finite-dimensional algebras are very nice objects, although some of their direct summands can not be mutated. In the cluster category associated with an acyclic quiver, mutation of cluster tilting objects is always possible [17]. It is determined by exchange triangles and corresponds to mutation of clusters in the corresponding cluster algebra via a certain cluster character [25]. In the derived categories of finite-dimensional hereditary algebras, a mutation operation was given in [22] on silting objects, which were first studied in [62]. Silting mutation of silting objects in triangulated categories, which is always possible, was investigated recently by Aihara and Iyama in [1].

The aim of this paper is to study higher cluster tilting objects in generalized higher cluster categories arising from dg algebras of higher Calabi-Yau dimension. Under certain assumptions on the dg algebras (Assumptions 4.2.1), tilting objects do not exist in the derived categories (Remark 4.2.6). Thus, we consider silting objects, e.g., the dg algebras themselves. The author was motivated by the construction of tilting complexes in Section 4 of [47].

This article is organized as follows: In Section 2, we list our assumptions on dg algebras and use the standard t-structure to situate the silting objects which are iteratively obtained from P-indecomposables with respect to the fundamental domain. In Section 3, using silting objects we construct higher cluster tilting objects in generalized higher cluster categories. We show that in such a category each liftable almost complete m-cluster tilting object has at least $m+1$ complements. In Section 4, we specialize to strongly higher Calabi-Yau dg algebras. By studying minimal cofibrant resolutions of simple modules of good completed deformed preprojective dg algebras, we obtain isomorphisms in generalized higher cluster categories between images of some left mutations and images of some right mutations of the same P-indecomposable. Using this, we derive the periodicity property of the images of iterated silting mutations of P-indecomposables in Section 5, where we also construct $(m + 1)$-Calabi-Yau triangulated categories containing infinitely many indecomposable m-cluster tilting objects. We obtain an explicit description of the terms of Iyama-Yoshino's AR angles in this situation, and we deduce that each almost complete m-cluster tilting P-object in the generalized m-cluster category associated with a suitable completed deformed preprojective dg algebra has exactly $m+1$ complements in Section 6. We show that the truncated dg subalgebra at degree zero of the dg endomorphism algebra of a silting object in the derived category of a good completed deformed preprojective dg algebra is also strongly Calabi-Yau in Section 7. Then we conjecture a class (namely m-rigid) of good completed deformed preprojective dg algebras such that each liftable almost complete m-cluster tilting object should have exactly $m+1$ complements in the associated generalized m-cluster category. In Section 8, we give a long exact sequence to show the relations between extension spaces in generalized higher cluster categories and extension spaces in derived categories. This sequence generalizes the short exact sequence obtained by Amiot [2] in the 2-Calabi-Yau case. At the end, we show that any almost complete

m-cluster tilting object in $\mathcal{C}_\Pi$ is liftable if Π is the completed deformed preprojective dg algebra arising from an acyclic quiver.

Notation

For a collection $\mathcal{X}$ of objects in an additive category $\mathcal{T}$, we denote by add$\mathcal{X}$ the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{X}$ and is closed under finite direct sums, summands and isomorphisms. Let k be an algebraically closed field of characteristic zero.

4.2 Silting objects in derived categories

Let A be a differential graded (for simplicity, write 'dg') k-algebra. We write per$A$ for the *perfect derived category* of A, i.e. the smallest triangulated subcategory of the derived category $\mathcal{D}(A)$ containing A and stable under passage to direct summands. We denote by $\mathcal{D}_{fd}(A)$ the *finite-dimensional derived category* of A whose objects are those of $\mathcal{D}(A)$ with finite-dimensional total homology.

A dg k-algebra A is *pseudo-compact* if it is endowed with a complete separated topology which is generated by two-sided dg ideals of finite codimension. A (pseudo-compact) dg algebra A is *(topologically) homologically smooth* if A lies in perA^e, where A^e is the (completed) tensor product of A^{op} and A over k. For example, suppose that A is of the form $(\widehat{kQ}, d)$, where $\widehat{kQ}$ is the completed path algebra of a finite graded quiver Q with respect to the two-sided ideal $\mathfrak{m}$ of $\widehat{kQ}$ generated by the arrows of Q, and the differential d takes each arrow of Q to an element of $\mathfrak{m}$; it was stated in [63] that A is pseudo-compact and topologically homologically smooth.

Assumptions 4.2.1. Let m be a positive integer. Suppose that A is a (pseudo-compact) dg k-algebra and has the following four additional properties:

a) A is (topologically) homologically smooth;

b) the pth homology H^pA vanishes for each positive integer p;

c) the zeroth homology H^0A is finite-dimensional;

d) A is $(m+2)$-Calabi-Yau as a bimodule, *i.e.*, there is an isomorphism in $\mathcal{D}(A^e)$

$$\mathrm{RHom}_{A^e}(A, A^e) \simeq \Sigma^{-m-2}A.$$

Theorem 4.2.2 ([56]). *(Completed) Ginzburg dg categories $\Gamma_{m+2}(Q, W)$ associated with graded quivers with superpotentials (Q, W) are (topologically) homologically smooth and $(m+2)$-Calabi-Yau.*

Lemma 4.2.3 ([54]). *Suppose that A is (topologically) homologically smooth. Then the category $\mathcal{D}_{fd}(A)$ is contained in* per$A$. *If moreover A is $(m+2)$-Calabi-Yau for some positive integer m, then for all objects L of $\mathcal{D}(A)$ and M of $\mathcal{D}_{fd}(A)$, we have a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}(A)}(M, L) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(L, \Sigma^{m+2}M).$$

Throughout this paper, we always consider the dg algebras satisfying Assumptions 4.2.1.

Proposition 4.2.4 ([40]). *Under Assumptions 4.2.1, the triangulated category* per$A$ *is* Hom-*finite.*

Let $(\mathcal{D}A)^c$ denote the full subcategory of $\mathcal{D}(A)$ consisting of compact objects. Since each idempotent in $\mathcal{D}(A)$ is split and $(\mathcal{D}A)^c$ is closed under direct summands, each idempotent in $(\mathcal{D}A)^c$ is also split. Therefore, the category per$A$ which is equal to $(\mathcal{D}A)^c$ by [52] is a k-linear Hom-finite category with split idempotents. It follows that per$A$ is a Krull-Schmidt triangulated category.

Definitions 4.2.5. Let A be a dg algebra satisfying Assumptions 4.2.1.

a) An object $X \in \mathrm{per}A$ is *silting* (resp. *tilting*) if $\mathrm{per}A = \mathrm{thick}X$ the smallest thick subcategory of per$A$ containing X, and the spaces $\mathrm{Hom}_{\mathcal{D}(A)}(X, \Sigma^i X)$ are zero for all integers $i > 0$ (resp. $i \neq 0$).

b) An object $Y \in \mathrm{per}A$ is *almost complete silting* if there is some indecomposable object Y' in $(\mathrm{per}A)\backslash(\mathrm{add}Y)$ such that $Y \oplus Y'$ is a silting object. Here Y' is called a *complement* of Y.

Clearly the dg algebra A itself is a silting object since $\mathrm{Hom}_{\mathcal{D}(A)}(A, \Sigma^i A)$ is isomorphic to $H^i A$ which is zero for each positive integer.

Remark 4.2.6. Under Assumptions 4.2.1, tilting objects do not exist in per$A$. Otherwise, let T be a tilting object in per$A$. By definition, the object T generates per$A$. Then for any object M in $\mathcal{D}(A)$, it belongs to the subcategory $\mathcal{D}_{fd}(A)$ if and only if $\sum_{p\in\mathbb{Z}} \mathrm{dimHom}_{\mathcal{D}(A)}(T, \Sigma^p M)$ is finite. Since the space $\mathrm{Hom}_{\mathcal{D}(A)}(T,T)$ is finite-dimensional by Proposition 4.2.4 and the space $\mathrm{Hom}_{\mathcal{D}(A)}(T, \Sigma^p T)$ vanishes for any nonzero integer p, the object T belongs to $\mathcal{D}_{fd}(A)$. Note that $\mathcal{D}_{fd}(A)$ is $(m+2)$-Calabi-Yau as a triangulated category by Lemma 4.2.3. Thus, we have the following isomorphism

$$(0 =)\mathrm{Hom}_{\mathcal{D}(A)}(T, \Sigma^{m+2} T) \simeq D\mathrm{Hom}_{\mathcal{D}(A)}(T,T)(\neq 0).$$

Here we obtain a contradiction. Therefore, tilting objects do not exist.

Assume that $H^0 A$ is a basic algebra. Let e be a primitive idempotent element of $H^0 A$. We denote by P the indecomposable direct summand eA (in the derived category $\mathcal{D}(A)$) of A and call it a P-*indecomposable*. We denote by M the dg module $(1-e)A$. It follows from Proposition 4.2.4 that the subcategory add$M$ is functorially finite [7] in add$A$. Let us write RA_0 for P (later we will also write LA_0 for P).

By induction on $t \geq 1$, we define RA_t as follows: take a minimal right (add$M$)-approximation $f^{(t)} : A^{(t)} \to RA_{t-1}$ of RA_{t-1} in $\mathcal{D}(A)$ and form a triangle in $\mathcal{D}(A)$

$$RA_t \xrightarrow{\alpha^{(t)}} A^{(t)} \xrightarrow{f^{(t)}} RA_{t-1} \longrightarrow \Sigma RA_t.$$

Dually, for each integer $t \geq 1$, we take a minimal left (add$M$)-approximation $g^{(t)} : LA_{t-1} \to B^{(t)}$ of LA_{t-1} in $\mathcal{D}(A)$, and form a triangle in $\mathcal{D}(A)$

$$LA_{t-1} \xrightarrow{g^{(t)}} B^{(t)} \xrightarrow{\beta^{(t)}} LA_t \longrightarrow \Sigma LA_{t-1}.$$

The object RA_t is called the *right mutation* of RA_{t-1} (with respect to M), and LA_t is called the *left mutation* of LA_{t-1} (with respect to M).

Theorem 4.2.7 ([1]). *For each nonnegative integer t, the objects $M \oplus RA_t$ and $M \oplus LA_t$ are silting objects in* $\mathrm{per}A$. *Moreover, any basic silting object containing M as a direct summand is either of the form $M \oplus RA_t$ or of the form $M \oplus LA_t$.*

From the construction and the above theorem, we know that the morphisms $\alpha^{(t)}$ (resp. $\beta^{(t)}$) are minimal left (resp. minimal right) $(\mathrm{add}M)$-approximations in $\mathcal{D}(A)$ and that the objects RA_t and LA_t are indecomposable objects in $\mathcal{D}(A)$ which do not belong to $\mathrm{add}M$.

We simply denote $\mathcal{D}(A)$ by $\mathcal{D}$. Let $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$) be the full subcategory of $\mathcal{D}$ whose objects are the dg modules X such that H^pX vanishes for each positive (resp. nonpositive) integer p. For a complex X of k-modules, we denote by $\tau_{\leq 0}X$ the subcomplex with $(\tau_{\leq 0}X)^0 = \mathrm{Ker}d^0$, and $(\tau_{\leq 0}X)^i = X^i$ for negative integers i, otherwise zero. Set $\tau_{\geq 1}X = X/\tau_{\leq 0}X$.

Proposition 4.2.8. *For each integer $t \geq 0$, the object RA_t belongs to the subcategory $\mathcal{D}^{\leq t} \cap {}^{\perp}\mathcal{D}^{\leq -1} \cap \mathrm{per}A$, and the object LA_t belongs to the subcategory $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -t-1} \cap \mathrm{per}A$.*

Proof. We consider the triangles appearing in the constructions of RA_t, and similarly for LA_t.

The object $RA_0 (= P)$ belongs to $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -1} \cap \mathrm{per}A$ since the dg algebra A has its homology concentrated in nonpositive degrees. The object RA_t is an extension of $A^{(t)}$ by $\Sigma^{-1}RA_{t-1}$, which both belong to the subcategory $\mathcal{D}^{\leq t} \cap \mathrm{per}A$. Thus, the object RA_t belongs to $\mathcal{D}^{\leq t} \cap \mathrm{per}A$. We do induction on t to show that RA_t belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$. Let Y be an object in $\mathcal{D}^{\leq -1}$. By applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ to the triangle

$$RA_t \longrightarrow A^{(t)} \longrightarrow RA_{t-1} \longrightarrow \Sigma RA_t,$$

we obtain the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(A^{(t)}, Y) \to \mathrm{Hom}_{\mathcal{D}}(RA_t, Y) \to \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-1}RA_{t-1}, Y) \to \ldots.$$

Since ΣY belongs to $\mathcal{D}^{\leq -2}$, by hypothesis, the space $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-1}RA_{t-1}, Y)$ is zero. Thus, the object RA_t belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$. □

Assume that $\{e_1, \cdots, e_n\}$ is a collection of primitive idempotent elements of H^0A. We denote by S_i the simple module corresponding to e_iA. For any object X in $\mathrm{per}A$, we define the support of X as follows:

Definition 4.2.9. The *support* of X is defined as the set

$$\mathit{supp}\,(X) := \{j \in \mathbb{Z} \mid \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^j S_i) \neq 0 \,\text{for some simple module}\, S_i\}.$$

Proposition 4.2.10. *For any nonnegative integer t, we have the following inclusions:*

1) $\{-t\} \subseteq \mathit{supp}\,(RA_t) \subseteq [-t, 0]$,

2) $\{t\} \subseteq \mathit{supp}\,(LA_t) \subseteq [0, t]$.

Proof. We only show the first statement, since the second one can be deduced in a similar way.

By Proposition 4.2.8, the object RA_t belongs to $\mathcal{D}^{\leq t} \cap {}^{\perp}\mathcal{D}^{\leq -1} \cap \mathrm{per}A$. Therefore, the space $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^r S_i)$ vanishes for each integer $r \geq 1$ since $\Sigma^r S_i$ lies in $\mathcal{D}^{\leq -1}$, and the space $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{r'} S_i)$ vanishes for each integer $r' \leq -t-1$ since $\Sigma^{r'} S_i$ lies in $\mathcal{D}^{\geq t+1}$. Thus, we have the inclusion $\mathit{supp}\,(RA_t) \subseteq [-t, 0]$.

Let S_P be the simple module corresponding to the P-indecomposable P from which RA_t and LA_t are obtained by mutation. We will show that $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-t}S_P)$ is nonzero. Clearly, the space $\mathrm{Hom}_{\mathcal{D}}(P, S_P)$ is nonzero. We do induction on the integer t. Assume that $\mathrm{Hom}_{\mathcal{D}}(RA_{t-1}, \Sigma^{1-t}S_P)$ is nonzero. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, \Sigma^{1-t}S_P)$ to the triangle

$$RA_t \to A^{(t)} \to RA_{t-1} \to \Sigma RA_t,$$

where $A^{(t)}$ belongs to $(\mathrm{add}A)\backslash(\mathrm{add}P)$, we get the long exact sequence

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma A^{(t)}, \Sigma^{1-t}S_P) \to \mathrm{Hom}_{\mathcal{D}}(\Sigma RA_t, \Sigma^{1-t}S_P) \to$$

$$\mathrm{Hom}_{\mathcal{D}}(RA_{t-1}, \Sigma^{1-t}S_P) \to \mathrm{Hom}_{\mathcal{D}}(A^{(t)}, \Sigma^{1-t}S_P),$$

where both the leftmost term and the rightmost term are zero. Therefore, we obtain that $\mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-t}S_P)$ is nonzero. This completes the proof. $\square$

Now we deduce the following corollary, which can also be deduced from Theorem 2.43 in [1].

Corollary 4.2.11. *1) For any two nonnegative integers $r \neq t$, the object RA_r is not isomorphic to RA_t, and the object LA_r is not isomorphic to LA_t.*

2) For any two positive integers r and t, the objects RA_r and LA_t are not isomorphic.

Proof. Assume that $r > t \geq 0$. Following Proposition 4.2.10, we have that

$$\mathrm{Hom}_{\mathcal{D}}(RA_r, \Sigma^{-r}S_P) \neq 0, \quad \text{while} \quad \mathrm{Hom}_{\mathcal{D}}(RA_t, \Sigma^{-r}S_P) = 0.$$

Thus, the objects RA_r and RA_t are not isomorphic. Similarly for LA_r and LA_t. Also in a similar way, we can obtain the second statement. $\square$

Combining Theorem 4.2.7 with Proposition 4.2.10, we can deduce the following corollary, which is analogous to Corollary 4.2 of [47]:

Corollary 4.2.12. *For any positive integer l, up to isomorphism, the object M admits exactly $2l-1$ complements whose supports are contained in $[1-l, l-1]$. These give rise to basic silting objects and they are the indecomposable objects RA_t and LA_t for $0 \leq t < l$.*

4.3 From silting objects to m-cluster tilting objects

Let $\mathcal{F}$ be the full subcategory $\mathcal{D}^{\leq 0} \cap {}^{\perp}\mathcal{D}^{\leq -m-1} \cap \mathrm{per}A$ of $\mathcal{D}$. It is called the *fundamental domain* in [40]. Following Lemma 4.2.3, the category $\mathcal{D}_{fd}(A)$ is a thick subcategory of $\mathrm{per}A$. The triangulated quotient category $\mathcal{C}_A = \mathrm{per}A/\mathcal{D}_{fd}(A)$ is called the *generalized m-cluster category* [40]. We denote by π the canonical projection functor from $\mathrm{per}A$ to $\mathcal{C}_A$.

Proposition 4.3.1 ([40])**.** *Under Assumptions 4.2.1, the projection functor $\pi : \mathrm{per}A \longrightarrow \mathcal{C}_A$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}_A$.*

Theorem 4.3.2 ([40] Theorem 2.2, [63] Theorem 7.21)**.** *If A satisfies Assumptions 4.2.1, then*

1) the generalized m-cluster category $\mathcal{C}_A$ is Hom*-finite and $(m+1)$-Calabi-Yau;*

*2) the object $T = \pi(A)$ is an m-*cluster tilting object *in $\mathcal{C}_A$, i.e.,*

$$\mathrm{add}T = \{L \in \mathcal{C}_A | \,\mathrm{Hom}_{\mathcal{C}_A}(T, \Sigma^r L) = 0,\, r = 1, \ldots, m\}.$$

Theorem 4.3.3. *The image of any silting object under the projection functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ is an m-cluster tilting object in $\mathcal{C}_A$.*

Proof. Assume that Z is an arbitrary silting object in $\mathrm{per}A$. Without loss of generality, we can assume that Z is a cofibrant dg A-module [50]. We denote by Γ the dg endomorphism algebra $\mathrm{Hom}^{\bullet}_A(Z, Z)$. Since the spaces $\mathrm{Hom}_{\mathcal{D}}(Z, \Sigma^i Z)$ are zero for all positive integers i, the dg algebra Γ has its homology concentrated in nonpositive degrees. The zeroth homology of Γ is isomorphic to the space $\mathrm{Hom}_{\mathcal{D}}(Z, Z)$ which is finite-dimensional by Proposition 4.2.4.

Since Z is a compact generator of $\mathcal{D}$, the left derived functor $F = - \overset{L}{\otimes}_{\Gamma} Z$ is a Morita equivalence [50] from $\mathcal{D}(\Gamma)$ to $\mathcal{D}$ which sends Γ to Z. Therefore, the dg algebra Γ is also (topologically) homologically smooth and $(m+2)$-Calabi-Yau. Thus, the generalized m-cluster category $\mathcal{C}_{\Gamma}$ is well-defined. The equivalence F also induces a triangle equivalence from $\mathcal{C}_{\Gamma}$ to $\mathcal{C}_A$ which sends $\pi(\Gamma)$ to $\pi(Z)$. By Theorem 4.3.2, the image $\pi(\Gamma)$ is an m-cluster tilting object in $\mathcal{C}_{\Gamma}$. Hence, the image of Z is an m-cluster tilting object in $\mathcal{C}_A$. $\square$

We use the same notation LA_t, RA_t and M as in subsection 4.2. A direct corollary of Theorem 4.3.3 is that for each nonnegative integer t, the images of $LA_t \oplus M$ and $RA_t \oplus M$ in the generalized m-cluster category $\mathcal{C}_A$ are m-cluster tilting objects.

Definitions 4.3.4. Let A be a dg algebra satisfying Assumptions 4.2.1 and $\mathcal{C}_A$ its generalized m-cluster category.

a) An object X in $\mathcal{C}_A$ is called an *almost complete m-cluster tilting object* if there exists some indecomposable object X' in $\mathcal{C}_A \backslash$ (addX) such that $X \oplus X'$ is an m-cluster tilting object. Here X' is called a *complement* of X. In particular, we call $\pi(M)$ an *almost complete m-cluster tilting P-object.*

b) An almost complete m-cluster tilting object Y is said to be *liftable* if there exists a basic silting object Z in $\mathrm{per}A$ such the $\pi(Z/Z')$ is isomorphic to Y for some indecomposable direct summand Z' of Z.

Here the functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ and the dg A-module M are the same as before.

Proposition 4.3.5. *Let A be a 3-Calabi-Yau dg algebra satisfying Assumptions 4.2.1. Then any $(1-)$cluster tilting object in $\mathcal{C}_A$ is induced by a silting object in $\mathcal{F}$ under the canonical projection π.*

Proof. Let T be a cluster tilting object in $\mathcal{C}_A$. By Proposition 4.3.1, we know that there exists an object Z in the fundamental domain $\mathcal{F}$ such that $\pi(Z) = T$.

First we will claim that Z is a partial silting object, that is, the spaces $\mathrm{Hom}_{\mathcal{D}}(Z, \Sigma^i Z)$ are zero for all positive integers i. Since Z belongs to $\mathcal{F}$, clearly these spaces vanish for all integers $i \geq 2$. Consider the case $i = 1$. The following short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{D}}(X, Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X, Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y, X) \to 0$$

was shown to exist in [2] for any objects X, Y in $\mathcal{F}$. We specialize both X and Y to the object Z. The middle term in the short exact sequence is zero since T is a cluster tilting object. Thus, the object Z is partial silting.

Second we will show that Z generates per$A$. Consider the following triangle

$$A \xrightarrow{f} Z_0 \to Y \to \Sigma A$$

in $\mathcal{D}$, where f is a minimal left (add$Z$)-approximation in $\mathcal{D}$. It is easy to see that Y also belongs to $\mathcal{F}$. Therefore, the above triangle can be viewed as a triangle in $\mathcal{C}_A$ with f a minimal left (add$Z$)-approximation in $\mathcal{C}_A$. Applying the functor $\mathrm{Hom}_{\mathcal{C}_A}(-, Z)$ to the triangle, we get the exact sequence

$$\mathrm{Hom}_{\mathcal{C}_A}(Z_0, Z) \to \mathrm{Hom}_{\mathcal{C}_A}(A, Z) \to \mathrm{Hom}_{\mathcal{C}_A}(\Sigma^{-1}Y, Z) \to \mathrm{Hom}_{\mathcal{C}_A}(\Sigma^{-1}Z_0, Z).$$

Therefore, the space $\mathrm{Hom}_{\mathcal{C}_A}(Y, \Sigma Z)$ becomes zero. As a consequence, Y belongs to add$Z$ in $\mathcal{C}_A$. Since both Y and Z are in $\mathcal{F}$, the object Y also belongs to add$Z$ in $\mathcal{D}$. Therefore, the dg algebra A belongs to the subcategory thick$Z$ of per$A$. It follows that Z generates per$A$. □

Theorem 4.3.6. *The almost complete m-cluster tilting P-object $\pi(M)$ has at least $m+1$ complements in $\mathcal{C}_A$.*

Proof. By Proposition 4.2.8 and Corollary 4.2.11, the pairwise non isomorphic indecomposable objects LA_t $(0 \leq t \leq m)$ belong to the fundamental domain $\mathcal{F}$. Then by Proposition 4.3.1, the $m+1$ objects $\pi(P)$, $\pi(LA_1), \ldots, \pi(LA_m)$ are indecomposable and pairwise non isomorphic in $\mathcal{C}_A$. It follows that $\pi(M)$ has at least $m+1$ complements in $\mathcal{C}_A$. □

Let us generalize the above theorem:

Theorem 4.3.7. *Each liftable almost complete m-cluster tilting object has at least $m+1$ complements in $\mathcal{C}_A$.*

Proof. Let Y be a liftable almost complete m-cluster tilting object. By definition there exists a basic silting object Z (assume that Z is cofibrant) in per$A$ such that $\pi(Z/Z')$ is isomorphic to Y for some indecomposable direct summand Z' of Z. Let Γ be the dg endomorphism algebra $\mathrm{Hom}^{\bullet}_A(Z, Z)$. Then $H^0\Gamma$ is a basic algebra.

Similarly as in the proof of Theorem 4.3.3, the dg algebra Γ satisfies Assumptions 4.2.1, and the left derived functor $F := - \overset{L}{\otimes}_{\Gamma} Z$ induces a triangle equivalence from $\mathcal{C}_\Gamma$ to $\mathcal{C}_A$ which sends $\pi(\Gamma)$ to $\pi(Z)$. Let Γ' be the object $\mathrm{Hom}^{\bullet}_A(Z, Z/Z')$ in perΓ. Then $\pi(\Gamma')$ is the almost complete m-cluster tilting P-object in $\mathcal{C}_\Gamma$ which corresponds to Y under the functor F. It follows from Theorem 4.3.6 that $\pi(\Gamma')$ has at least $m+1$ complements in $\mathcal{C}_\Gamma$. So does the liftable almost complete m-cluster tilting object Y in $\mathcal{C}_A$. □

Remark 4.3.8. Let $\mathcal{T}$ be a Krull-Schmidt Hom-finite triangulated category with a Serre functor. In fact, following [48], one can get that any almost complete m-cluster tilting object Y in $\mathcal{T}$ has at least $m+1$ complements. Note that the notation in [40] and [48] has some differences with each other, for example, m-cluster tilting objects in [40] correspond to $(m+1)$-cluster tilting subcategories (or objects) in [48]. Here we use the same notation as [40]. Set $\mathcal{Y} = \mathrm{add}Y$, $\mathcal{Z} = \cap_{i=1}^{m} {}^{\perp}(\Sigma^i\mathcal{Y})$ and $\mathcal{U} = \mathcal{Z}/\mathcal{Y}$. Let X be an m-cluster tilting object in $\mathcal{T}$ which contains Y as a direct summand. Set $\mathcal{X} = \mathrm{add}X$. Then by Theorem 4.9 in [48], the subcategory $\mathcal{L} := \mathcal{X}/\mathcal{Y}$ is m-cluster tilting in the triangulated category $\mathcal{U}$. The subcategories $\mathcal{L}$, $\mathcal{L}\langle 1\rangle$, …, $\mathcal{L}\langle m\rangle$ are distinct m-cluster tilting subcategories of $\mathcal{U}$, where $\langle 1\rangle$ is the shift functor in the triangulated category $\mathcal{U}$. Also by the same theorem, the one-one correspondence implies that the number of m-cluster tilting objects of $\mathcal{T}$ containing Y as a direct summand is at least $m+1$.

4.4 Minimal cofibrant resolutions of simple modules for strongly $(m+2)$-Calabi-Yau case

The well-known Connes long exact sequence (SBI-sequence) for cyclic homology [64] associated with a dg algebra A is as follows

$$\ldots \to HH_{m+3}(A) \xrightarrow{I} HC_{m+3}(A) \xrightarrow{S} HC_{m+1}(A) \xrightarrow{B} HH_{m+2}(A) \xrightarrow{I} \ldots,$$

where $HH_*(A)$ denotes the Hochschild homology of A and $HC_*(A)$ denotes the cyclic homology.

Let M and N be two dg A-modules with M in $\mathrm{per}A^e$. Then in $\mathcal{D}(k)$ we have the isomorphism

$$\mathrm{RHom}_{A^e}(\mathrm{RHom}_{A^e}(M, A^e), N) \simeq M \overset{L}{\otimes}_{A^e} N.$$

An element $\xi = \sum_{i=1}^{s} \xi_{1i} \otimes \xi_{2i} \in H^r(M \overset{L}{\otimes}_{A^e} N)$ is *non-degenerate* if the corresponding map

$$\xi^+ : \mathrm{RHom}_{A^e}(M, A^e) \to \Sigma^r N$$

given by $\xi^+(\phi) = \sum_{i=1}^{s}(-1)^{|\phi||\xi|}\phi(\xi_{1i})_2 \xi_{2i} \phi(\xi_{1i})_1$ is an isomorphism. Throughout this chapter, we write $|\cdot|$ to denote the degrees.

Let l be a finite-dimensional separable k-algebra. We fix a trace $Tr : l \to k$ and let $\sigma' \otimes \sigma''$ be the corresponding Casimir element (i.e., $\sigma' \otimes \sigma'' = \sum \sigma'_i \otimes \sigma''_i$ and $Tr(\sigma'_i \sigma''_j) = \delta_{ij}$). An *augmented dg l-algebra* is a dg algebra A equipped with dg k-algebra homomorphisms $l \xrightarrow{\varsigma} A \xrightarrow{\epsilon} l$ such that $\epsilon\varsigma$ is the identity. Following [76] we write $PCAlgc(l)$ for the category of pseudo-compact augmented dg l-algebras satisfying $\mathrm{Ker}(\epsilon) = \mathrm{Coker}(\varsigma) = \mathrm{rad}A$. When forgetting the grading, $\mathrm{rad}A$ is just the Jacobson radical of the underlying ungraded algebra $A^u := \prod_r A^r$ of the dg algebra $A = (A^r)_r$.

The SBI-sequence can be extended to the case that $A \in PCAlgc(l)$, where $HH_*(A)(= H_*(A \overset{L}{\otimes}_{A^e} A))$ is computed by the pseudo-compact Hochschild complex. For more details, see Section 8 and Appendix B in [76].

Definition 4.4.1 ([76]). An algebra $A \in PCAlgc(l)$ is *strongly $(m+2)$-Calabi-Yau* if A is topologically homologically smooth and $HC_{m+1}(A)$ contains an element η such that $B\eta$ is non-degenerate in $HH_{m+2}(A)$.

Theorem 4.4.2 ([76]). *Let $A \in PCAlgc(l)$. Assume that $A = (A^r)_{r \le 0}$ is concentrated in nonpositive degrees. Then A is strongly $(m+2)$-Calabi-Yau if and only if there is a quasi-isomorphism $(\widehat{T_l V}, d) \to A$ as augmented dg l-algebras with V having the following properties*

a) $d(V) \cap V = 0$;

b) $V = V_c \oplus lz$ with z an l-central element of degree $-m-1$, V_c finite-dimensional and concentrated in degrees $[-m, 0]$;

c) $dz = \sigma' \eta \sigma''$ with $\eta \in V_c \otimes_{l^e} V_c$ non-degenerate and antisymmetric under the map $F : v_1 \otimes v_2 \to (-1)^{|v_1||v_2|} v_2 \otimes v_1$ for any v_1, v_2 in V_c.

We would like to present the explicit construction of Ginzburg dg categories in the following straightforward proposition.

Proposition 4.4.3. *The completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,W)$ associated with a finite graded quiver Q concentrated in degrees $[-m,0]$ and with a reduced superpotential W being a linear combination of paths of Q of degree $1-m$ and of length at least 3, is strongly $(m+2)$-Calabi-Yau.*

Proof. We check that $\widehat{\Gamma}_{m+2}(Q,W)$ satisfies the assumptions and the conditions in Theorem 4.4.2.

Let l be the separable k-algebra $\prod_{i\in Q_0} ke_i$. Let $\overline{Q}^G$ be the double quiver obtained from Q by adjoining an opposite arrow a^* of degree $-m-|a|$ for each arrow $a\in Q_1$. Let $\widetilde{Q}^G$ be obtained from $\overline{Q}^G$ by adjoining a loop t_i of degree $-m-1$ for each vertex i. Then the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,W)$ is the completed path category $\widehat{T_l(\widetilde{Q}^G)}$ with the following differential

$$\begin{gathered} d(a)=0, \quad a\in Q_1;\\ d(t_i)=e_i(\textstyle\sum_{a\in Q_1}[a,a^*])e_i, \quad i\in Q_0;\\ d(a^*)=(-1)^{|a|}\frac{\partial W}{\partial a}=(-1)^{|a|}\textstyle\sum_{p=uav}(-1)^{(|a|+|v|)|u|}vu, \quad a\in Q_1; \end{gathered}$$

where the sum in the third formula runs over all homogeneous summands $p=uav$ of W.

Thus, the components of $\widehat{\Gamma}_{m+2}(Q,W)$ are concentrated in nonpositive degrees and $\widehat{\Gamma}_{m+2}(Q,W)$ $(=l\oplus\prod_{s\geq 1}(\widetilde{Q}^G)^{\otimes_l s})$ lies in $PCAlgc(l)$.

The differential above which is induced by the reduced superpotential W satisfies that $d(\widetilde{Q}^G)\cap\widetilde{Q}^G=0$. Set $z=\sum_{i\in Q_0}t_i$. Then z is an l-central element of degree $-m-1$. Clearly, $\widetilde{Q}^G=\overline{Q}^G\oplus lz$, the double quiver $\overline{Q}^G$ is finite and concentrated in degrees $[-m,0]$, and the element $d(z)=\sum_{a\in Q_1}(aa^*-(-1)^{|a||a^*|}a^*a)$ is antisymmetric under the flip F.

The last step is to show that $\eta:=\sum_{a\in Q_1}[a,a^*]$ is non-degenerate, that is, the corresponding map

$$\eta^+ : \mathrm{Hom}_{l^e}(\overline{Q}^G,l^e)\longrightarrow\overline{Q}^G, \quad \phi\to(-1)^{|\phi||\eta|}\phi(\eta_1)_2\eta_2\phi(\eta_1)_1$$

is an isomorphism. Define morphisms $\phi_\gamma(\gamma\in\overline{Q}^G):\overline{Q}^G\to l^e$ as follows

$$\phi_\gamma(\alpha)=\delta_{\alpha\gamma}e_{t(\alpha)}\otimes e_{s(\alpha)}.$$

Then $\{\phi_\gamma|\gamma\in\overline{Q}^G\}$ is a basis of the space $\mathrm{Hom}_{l^e}(\overline{Q}^G,l^e)$. Applying the map η^+, we obtain the images $\eta^+(\phi_a)=(-1)^{m|a|}a^*$ and $\eta^+(\phi_{a^*})=(-1)^{1+|a^*|^2}a$ for arrows $a\in Q_1$. Thus, $\{\eta^+(\phi_\gamma)|\gamma\in\overline{Q}^G\}$ is a basis of $\overline{Q}^G$. Therefore, the element η is non-degenerate. □

Now we write down the explicit construction of deformed preprojective dg algebras as described in [76]. Let Q be a finite graded quiver and L the subset of Q_1 consisting of all loops a of odd degree such that $|a|=-m/2$. Let $\overline{Q}^V$ be the double quiver obtained from Q by adjoining an opposite arrow a^* of degree $-m-|a|$ for each $a\in Q_1\setminus L$ and putting $a^*=a$ without adjoining an extra arrow for each $a\in L$. Let N be the Lie algebra $k\overline{Q}^V/[k\overline{Q}^V,k\overline{Q}^V]$ endowed with the necklace bracket $\{-,-\}$ (cf. [13], [39]). Let W be a superpotential which is a linear combination of homogeneous elements of degree $1-m$ in N and satisfies $\{W,W\}=0$ (in order to make the differential well-defined). Let $\widetilde{Q}^V$ be obtained from $\overline{Q}^V$ by adjoining a loop t_i of degree $-m-1$ for each vertex i. Then the *deformed preprojective dg algebra* $\Pi(Q,m+2,W)$ is the dg algebra $(k\widetilde{Q}^V,d)$ with the differential

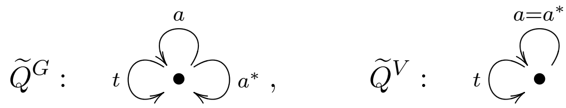
$$\begin{gathered} da = \{W, a\} = (-1)^{(|a|+1)|a^*|} \frac{\partial W}{\partial a^*} = (-1)^{(|a|+1)|a^*|} \sum_{p=ua^*v} (-1)^{(|a^*|+|v|)|u|} vu; \\ da^* = \{W, a^*\} = (-1)^{|a|+1} \frac{\partial W}{\partial a} = (-1)^{|a|+1} \sum_{p=uav} (-1)^{(|a|+|v|)|u|} vu; \\ dt_i = e_i (\textstyle\sum_{a \in Q_1} [a, a^*]) e_i; \end{gathered}$$

where $a \in Q_1$ and $i \in Q_0$. Later we will denote the homogeneous elements $rvu\,(r \in k)$ appearing in $d\alpha\,(\alpha \in \overline{Q}^V)$ by $y(\alpha, v, u)$.

Remark 4.4.4. As in Proposition 4.4.3, we see that the completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ associated with a finite graded quiver Q concentrated in degrees $[-m, 0]$ and with a reduced superpotential W being a linear combination of paths of $\overline{Q}^V$ of length at least 3, is also strongly $(m+2)$-Calabi-Yau.

Suppose that -1 is a square in the field k and denote by $\sqrt{-1}$ a chosen square root. Then the class of deformed preprojective dg algebras is strictly greater than the class of Ginzburg dg categories. Suppose that Q does not contain *special loops* (i.e., loops of odd degree which is equal to $-m/2$). Then we can easily see that $\Gamma_{m+2}(Q, W) = \Pi(Q, m+2, -W)$. Otherwise, let Q^0 be the subquiver of Q obtained by removing the special loops. For each special loop a in Q_1, we add a pair of loops a' and a'' to Q^0 which are also special at the same vertex of Q^0. Denote the new quiver by Q'. Let W' be the superpotential obtained from W by replacing each special loop by the corresponding element $a' + a''\sqrt{-1}$. Now we define a map $\iota : \Gamma_{m+2}(Q, W) \to \Pi(Q', m+2, -W')$, it sends each special loop a of Q_1 to the element $a' + a''\sqrt{-1}$ and its dual a^* to the element $a' - a''\sqrt{-1}$ in $\Pi(Q', m+2, -W')$, and it keeps the other arrows of $\widetilde{Q}^G$. Then it is not hard to check that ι is a dg algebra isomorphism. It follows that Ginzburg dg categories are deformed preprojective dg algebras. For the strictness, see the following example.

Example 4.4.5. Suppose that m is 2. Let Q be the quiver consisting of only one vertex '$\bullet$' and one loop a of degree -1. Then the Ginzburg dg category $\Gamma_4(Q, 0)$ and the deformed preprojective dg algebra $\Pi(Q, 4, 0)$ respectively have the the following underlying graded quivers



where $|a| = |a^*| = -1$ and $|t| = -3$. The differential takes the following values

$$d(a) = 0 = d(a^*), \quad d_{\Gamma_4(Q,0)}(t) = aa^* + a^*a, \quad d_{\Pi(Q,4,0)}(t) = 2a^2.$$

Then $\dim H^{-1}(\Gamma_4(Q, 0)) = 2$ while $\dim H^{-1}(\Pi(Q, 4, 0)) = 1$. Hence, these two dg algebras are not quasi-isomorphic. Moreover, it is obvious that the dg algebra $\Pi(Q, 4, 0)$ can not be realized as a Ginzburg dg category.

Lemma 4.4.6. *Let $\Pi = \widehat{\Pi}(Q, m+2, W)$ be a completed deformed preprojective dg algebra. Let x (resp. y) denote the minimal (resp. maximal) degree of the arrows of $\overline{Q}^V$. Then there exist a canonical completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$ isomorphic to Π as a dg algebra, where the quiver Q' is concentrated in degrees $[-m/2, y]$.*

Proof. We can construct directly a quiver Q' and a superpotential W'.

We claim first that $x + y = -m$. Let x_1 (resp. y_1) denote the minimal (resp. maximal) degree of the arrows of Q. Then $\overline{Q}^V \setminus Q$ is concentrated in degrees $[-m - y_1, -m - x_1]$. If $x_1 \leq -m - y_1$, then $x = x_1$ and $y_1 \leq -m - x_1$. Hence, $x + y = x_1 + (-m - x_1) = -m$. Similarly for the case '$-m - y_1 \leq x_1$'.

Let Q^0 be the subquiver of Q which has the same vertices as Q and whose arrows are those of Q with degree belonging to $[-m/2, y]$ $(= [(x+y)/2, y])$. In this case $|a^*| = -m - |a| \in [-m-y, -m/2] = [x, -m/2]$. For each arrow b of Q whose dual b^* has degree in $(-m/2, y]$, we add a corresponding arrow b' to Q^0 with the same degree as b^*. Denote the new quiver by Q'. Therefore, the quiver Q' has arrow set

$$\{a \in Q_1 \,|\, |a| \in [-m/2, y]\} \cup \{b' \,|\, |b'| = |b^*|, b \in Q_1 \text{ and } |b^*| \in (-m/2, y]\}.$$

We define a map $\iota : \widetilde{Q}^V \to \widetilde{Q'}^V$ by setting

$$\iota(a) = a,\ \iota(a^*) = a^*; \quad \iota(t_i) = t_i; \quad \iota(b) = (-1)^{|b||b^*|+1} b'^*,\ \iota(b^*) = b'.$$

Let W' be the superpotential obtained from W by replacing each arrow α in W by $\iota(\alpha)$. Then it is not hard to check that the map ι can be extended to a dg algebra isomorphism from Π to Π'. $\square$

In particular, if Q is concentrated in degrees $[-m, 0]$, then by the above lemma, the new quiver Q' is concentrated in degrees $[-m/2, 0]$. If the following two conditions

V1) Q a finite graded quiver concentrated in degrees $[-m/2, 0]$,

V2) W a reduced superpotential being a linear combination of paths of $\overline{Q}^V$ of degree $1-m$ and of length ≥ 3,

hold, then we will say that the completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m{+}2, W)$ is *good*.

Theorem 4.4.7 ([76]). *Let A be a strongly $(m{+}2)$-Calabi-Yau dg algebra with components concentrated in degrees ≤ 0. Suppose that A lies in $PCAlgc(l)$ for some finite-dimensional separable commutative k-algebra l. Then A is quasi-isomorphic to some completed deformed preprojective dg algebra.*

We consider the strongly $(m+2)$-Calabi-Yau case in this section, by Theorem 4.4.7, it suffices to consider good completed deformed preprojective dg algebras $\Pi = \widehat{\Pi}(Q, m+2, W)$. The simple Π-module S_i (attached to a vertex i of Q) belongs to the finite-dimensional derived category $\mathcal{D}_{fd}(\Pi)$, hence it also belongs to perΠ. We will give a precise description of the objects RA_t and LA_t obtained from iterated mutations of a P-indecomposable $e_i\Pi$, where e_i is the primitive idempotent element associated with a vertex i of Q.

Definition 4.4.8 ([70]). Let $A = (\widehat{kQ}, d)$ be a dg algebra, where Q is a finite graded quiver and d is a differential sending each arrow to a (possibly infinite) linear combination of paths of length ≥ 1. A dg A-module M is *minimal perfect* if

a) its underlying graded module is of the form $\oplus_{j=1}^N R_j$, where R_j is a finite direct sum of shifted copies of direct summands of A, and

b) its differential is of the form $d_{int} + \delta$, where d_{int} is the direct sum of the differentials of these R_j $(1 \leq j \leq N)$, and δ, as a degree 1 map from $\oplus_{j=1}^N R_j$ to itself, is a strictly upper triangular matrix whose entries are in the ideal $\mathfrak{m}$ of A generated by the arrows of Q.

Lemma 4.4.9 ([70]). *Let M be a dg $A(= (\widehat{kQ}, d))$-module such that M lies in* per$A$. *Then M is quasi-isomorphic to a minimal perfect dg A-module.*

In the second part of this section, we illustrate how to obtain minimal perfect dg modules which are quasi-isomorphic to simple Π-modules from cofibrant resolutions [63]. If a cofibrant resolution $\mathbf{p}X$ of a dg module X is minimal perfect, then we say $\mathbf{p}X$ a *minimal cofibrant resolution* of X.

Let i be a vertex of Q and $P_i = e_i\Pi$. Consider the short exact sequence in the category $\mathcal{C}(\Pi)$ of dg modules

$$0 \to \mathrm{Ker}(p) \xrightarrow{\iota} P_i \xrightarrow{p} S_i \to 0,$$

where in the category $\mathrm{Grmod}(\Pi)$ of graded modules $\mathrm{Ker}(p)$ is the direct sum of $\rho P_{s(\rho)}$ over all arrows $\rho \in \widetilde{Q}_1^V$ with $t(\rho) = i$. Here $\rho P_{s(\rho)}$ denotes the image in P_i of the map $P_{s(\rho)} \to P_i$ given by the left multiplication by ρ. The simple module S_i is quasi-isomorphic to $cone(\mathrm{Ker}(p) \xrightarrow{\iota} P_i)$, *i.e.*, the dg module

$$X = (\underline{X} = P_i \oplus \Sigma X_0' \oplus \ldots \oplus \Sigma X_{m+1}', d_X = \begin{pmatrix} d_{P_i} & \iota \\ 0 & -d_{\mathrm{Ker}(p)} \end{pmatrix}),$$

where for each integer $0 \leq j \leq m+1$, the object X_j' is the direct sum of $\rho P_{s(\rho)}$ ranging over all arrows $\rho \in \widetilde{Q}_1^V$ with $t(\rho) = i$ and $|\rho| = -j$. By Section 2.14 in [63], the dg module X is a cofibrant resolution of the simple module S_i.

Now let $P_j' \, (0 \leq j \leq m+1)$ be the direct sum of $P_{s(\rho)}$ where ρ ranges over all arrows in $\widetilde{Q}_1^V$ satisfying $t(\rho) = i$ and $|\rho| = -j$. Clearly, $P_{m+1}' = P_i$. We require that the ordering of direct summands $P_{s(\rho)}$ in P_j' is the same as the ordering of direct summands $\rho P_{s(\rho)}$ in X_j' for each integer $0 \leq j \leq m+1$. Let Y be an object whose underlying graded module is $\underline{Y} = P_i \oplus \Sigma P_0' \oplus \Sigma^2 P_1' \oplus \ldots \oplus \Sigma^{m+2} P_{m+1}'$. We endow $\underline{Y}$ with a degree 1 graded endomorphism $d_{int} + \delta_Y$, where d_{int} is the same notation as in Definition 4.4.8. The columns of δ_Y have the following two types: $(\alpha, 0, \ldots, -y_{red}(\alpha, v, u), \ldots, 0)^t$, and $(t_i, \ldots, -a^*, \ldots, (-1)^{|b||b^*|} b, \ldots, 0)^t$ for the last column. Here α is an arrow in $\overline{Q}^V$, while a is an arrow in Q and b is an arrow in $\overline{Q}^V \setminus Q$. Here $y_{red}(\alpha, v, u)$ is obtained from the path $y(\alpha, v, u) = \beta_s \ldots \beta_1$ (this notation is defined just before Remark 4.4.4) by removing the factor β_s. The ordering of the elements in each column is determined by the ordering of Y.

Let $f : Y \to X$ be a map constructed as the diagonal matrix whose elements are all arrows in $\widetilde{Q}_1^V$ with target at i, together with e_i as the first element. Moreover, we require that the ordering of these arrows is determined by Y (hence also by X), that is, the components of f are of the form

$$f_\rho : \Sigma^{|\rho|+1} P_{s(\rho)} \longrightarrow \Sigma \rho P_{s(\rho)}, \qquad u \mapsto \rho u.$$

It is not hard to check the identity $f(d_{int} + \delta_Y) = d_X f$. Hence, the morphism f is an isomorphism in $\mathcal{C}(\Pi)$, and the map $d_{int} + \delta_Y$ makes the object Y to be a dg module which is minimal perfect. Therefore, the dg module Y is a minimal cofibrant resolution of the simple module S_i.

In the third part of this section, we show that when there are no loops of Q at vertex i, the truncations of the minimal cofibrant resolution Y of the simple module S_i produce RA_t and $LA_t \, (0 \leq t \leq m+1)$ obtained from the P-indecomposable P_i by iterated mutations. If we write M for the dg module Π/P_i, then the dg modules $P_j' \, (0 \leq j \leq m)$ appearing in Y lie in $\mathrm{add}M$. Let $\varepsilon_{\leq t} Y$ be the submodule of Y with the inherited differential whose underlying graded module is the direct sum of those summands of Y with copies of shift $\leq t$. Let $\varepsilon_{\geq t+1} Y$ be the quotient module $Y/(\varepsilon_{\leq t} Y)$. Notice that $\varepsilon_{\leq t} Y$ is a truncation of Y for the canonical weight structure on $\mathrm{per}\Pi$, cf. Bondarko [15], Keller-Nicolás [58].

Proposition 4.4.10. *Let* Π *be a good completed deformed preprojective dg algebra* $\widehat{\Pi}(Q, m+2, W)$ *and* i *a vertex of* Q. *Assume that there are no loops of* Q *at vertex* i. *Then the following two isomorphisms*

$$\Sigma^{-t}\varepsilon_{\leq t}Y \simeq RA_t \quad \textit{and} \quad \Sigma^{-t-1}\varepsilon_{\geq t+1}Y \simeq LA_{m+1-t}$$

hold in the derived category $\mathcal{D} := \mathcal{D}(\Pi)$ *for each integer* $0 \leq t \leq m+1$.

Proof. We only consider the first isomorphism. Then the second one can be obtained dually. For arrows of $\overline{Q}^V$ of degree $-j$ ending at vertex i, we write α_j; for the symbols $-y_{red}(\alpha, v, u)$ of degree $-j$, we simply write $-y^j_{red}$, and for morphisms f of degree $-j$, we write f_j, where $0 \leq j \leq m$. Moreover, we use the notation $[x]$ to denote a matrix whose entries x have the same 'type' (in some obvious sense).

Clearly, when $t = 0$, we have that $\varepsilon_{\leq 0}Y = P_i = RA_0$.

When $t = 1$, we have the following isomorphisms

$$\Sigma^{-1}\varepsilon_{\leq 1}Y \simeq (\Sigma^{-1}P_i \oplus P'_0, \begin{pmatrix} d_{\Sigma^{-1}P_i} & -[\alpha_0] \\ 0 & d_{P'_0} \end{pmatrix}) \simeq \Sigma^{-1}cone(P'_0 \xrightarrow{h^{(1)}} P_i),$$

where each component of $h^{(1)}(= [\alpha_0])$ is the left multiplication by some α_0. Since W is reduced, the left multiplication by α_0 is nonzero in the space $\mathrm{Hom}_{\mathcal{D}}(P'_0, P_i)$. Moreover, only the trivial paths e_i have zero degree, and there are no loops of $\overline{Q}^V$ of degree zero at vertex i. It follows that $h^{(1)}$ is a minimal right $(\mathrm{add}M)$-approximation of P_i. Then $\Sigma^{-1}\varepsilon_{\leq 1}Y$ and RA_1 are isomorphic in $\mathcal{D}$.

In general, assume that $\Sigma^{-t}\varepsilon_{\leq t}Y \simeq RA_t\,(1 \leq t \leq m)$. We will show that

$$\Sigma^{-t-1}\varepsilon_{\leq t+1}Y \simeq RA_{t+1}.$$

First we have the following isomorphism

$$\Sigma^{-t-1}\varepsilon_{\leq t+1}Y \simeq (\Sigma^{-t-1}P_i \oplus \Sigma^{-t}P'_0 \oplus \ldots \oplus P'_t,$$

$$\begin{pmatrix} d_{\Sigma^{-t-1}P_i} & (-1)^{t+1}[\alpha_0] & \ldots & (-1)^{t+1}[\alpha_{t-1}] & (-1)^{t+1}[\alpha_t] \\ 0 & d_{\Sigma^{-t}P'_0} & \ldots & (-1)^t[y^{t-2}_{red}] & (-1)^t[y^{t-1}_{red}] \\ & \ldots & & \ldots & \\ 0 & 0 & \ldots & d_{\Sigma^{-1}P'_{t-1}} & (-1)^t[y^0_{red}] \\ 0 & 0 & \ldots & 0 & d_{P'_t} \end{pmatrix})$$

$$\simeq \Sigma^{-1}cone(P'_t \xrightarrow{h^{(t+1)}} RA_t),$$

where $h^{(t+1)} = ((-1)^t[\alpha_t], (-1)^{t-1}[y^{t-1}_{red}], \ldots, (-1)^{t-1}[y^0_{red}])^t$. Each column of $h^{(t+1)}$ is a nonzero morphism in $\mathrm{Hom}_{\mathcal{D}}(P'_t, RA_t)$, since the superpotential W is reduced. Otherwise, the arrow α_t will be a linear combination of paths of length ≥ 2. It follows that $h^{(t+1)}$ is right minimal. Let L be an arbitrary indecomposable object in $\mathrm{add}A$ and $f = (f_t, [f_{t-1}], \ldots, [f_1], [f_0])^t$ an arbitrary morphism in $\mathrm{Hom}_{\mathcal{D}}(L, RA_t)$. Then the vanishing of $d(f)$ implies that $d(f_t) = -[\alpha_0][f_{t-1}] - \ldots - [\alpha_{t-2}][f_1] - [\alpha_{t-1}][f_0]$. Since there are no loops of $\overline{Q}^V$ of degree $-t$ at vertex i, the map f_t which is homogeneous of degree $-t$ is a linear combination of the following forms:

(i) $f_t = \alpha_t g_0$, where $|g_0| = 0$. In this case, the differential

$$d(f_t) = d(\alpha_t g_0) = d(\alpha_t)g_0 = [\alpha_0][y^{t-1}_{red}]g_0 + \ldots + [\alpha_{t-1}][y^0_{red}]g_0,$$

which implies that $[f_r]$ is equal to $-[y^r_{red}]g_0\,(0 \le r \le t-1)$. Then the equalities

$$f = \begin{pmatrix} f_t \\ [f_{t-1}] \\ \dots \\ [f_1] \\ [f_0] \end{pmatrix} = \begin{pmatrix} \alpha_t g_0 \\ -[y^{t-1}_{red}]g_0 \\ \dots \\ -[y^1_{red}]g_0 \\ -[y^0_{red}]g_0 \end{pmatrix} = \begin{pmatrix} (-1)^t\alpha_t \\ (-1)^{t-1}[y^{t-1}_{red}] \\ \dots \\ (-1)^{t-1}[y^1_{red}] \\ (-1)^{t-1}[y^0_{red}] \end{pmatrix} (-1)^t g_0.$$

hold. Thus, the morphism f factors through $h^{(t+1)}$.

(ii) $f_t = \alpha_r g_{t-r}$, where $|g_{t-r}| = r-t\,(0 \le r \le t-1)$. In these cases, the differentials

$$d(f_t) = d(\alpha_r)g_{t-r} + (-1)^r\alpha_r d(g_{t-r}) =$$
$$[\alpha_0][y^{r-1}_{red}]g_{t-r} + \ldots + [\alpha_{r-1}][y^0_{red}]g_{t-r} + (-1)^r\alpha_r d(g_{t-r}),$$

which implies that

$$[f_{t-1}] = -[y^{r-1}_{red}]g_{t-r}\,,\ \dots\,,\ [f_{t-r}] = -[y^0_{red}]g_{t-r} \quad \text{and}$$
$$[f_{t-r-1}] = (-1)^{r+1}d(g_{t-r}).$$

Then we have that

$$\begin{pmatrix} f_t \\ [f_{t-1}] \\ \dots \\ [f_1] \\ [f_0] \end{pmatrix} = \begin{pmatrix} \alpha_r g_{t-r} \\ -[y^{r-1}_{red}]g_{t-r} \\ \dots \\ -[y^0_{red}]g_{t-r} \\ (-1)^{r+1}d(g_{t-r}) \\ 0 \\ \dots \\ 0 \end{pmatrix} = d_{RA_t}\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ (-1)^t g_{t-r} \\ 0 \\ \dots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ (-1)^t g_{t-r} \\ 0 \\ \dots \\ 0 \end{pmatrix} d_L$$

is a zero element in $\mathrm{Hom}_{\mathcal{D}}(L, RA_t)$. Therefore, the morphism $h^{(t+1)}$ is a minimal right $(\mathrm{add}A)$-approximation of $RA_t\,(1 \le t \le m)$. Hence, the isomorphism $\Sigma^{-t-1}\varepsilon_{\le t+1}Y \simeq RA_{t+1}$ holds. □

We further assume that the zeroth homology $H^0\Pi$ is finite-dimensional. Then the dg algebra Π satisfies Assumptions 4.2.1 and moreover it is strongly $(m+2)$-Calabi-Yau.

The simple module S_i is zero in the generalized m-cluster category $\mathcal{C}_\Pi = \mathrm{per}\Pi/\mathcal{D}_{fd}(\Pi)$, so its corresponding minimal cofibrant resolution Y also becomes zero in $\mathcal{C}_\Pi$. Taking truncations of Y, we obtain $m+2$ triangles in $\mathcal{C}_\Pi$

$$\pi(\varepsilon_{\le t}Y) \longrightarrow 0 \longrightarrow \pi(\varepsilon_{\ge t+1}Y) \longrightarrow \Sigma\pi(\varepsilon_{\le t}Y), \qquad 0 \le t \le m+1,$$

where $\pi : \mathrm{per}\Pi \to \mathcal{C}_\Pi$ is the canonical projection functor. Therefore, the following theorem holds:

Theorem 4.4.11. *Under the assumptions in Proposition 4.4.10 and the assumption that $H^0\Pi$ is finite-dimensional, the image of RA_t is isomorphic to the image of LA_{m+1-t} in the generalized m-cluster category $\mathcal{C}_\Pi$ for each integer $0 \le t \le m+1$.*

Proof. The following isomorphisms

$$\pi(RA_t) \simeq \pi(\Sigma^{-t}\varepsilon_{\le t}Y) \simeq \pi(\Sigma^{-t-1}\varepsilon_{\ge t+1}Y) \simeq \pi(LA_{m+1-t})$$

are true in $\mathcal{C}_\Pi$ for all integers $0 \le t \le m+1$. □

In the presence of loops, the objects RA_t and LA_r do not always satisfy the relations in Theorem 4.4.11. See the following example.

Example 4.4.12. Suppose that m is 2. Let Q be the quiver whose vertex set Q_0 has only one vertex '$\bullet$' and whose arrow set Q_1 has two loops α and β of degree -1. Then the completed deformed preprojective dg algebra $\Pi = \widehat{\Pi}(Q, 4, 0)$ has the underlying graded quiver as follows

$$\widetilde{Q}^V : \qquad \begin{tikzcd} \bullet \arrow[loop left, "t"] \arrow[loop above, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

with $|\alpha| = |\beta| = -1$ and $|t| = -3$. The differential takes the following values

$$d(\alpha) = 0 = d(\beta), \quad d(t) = 2\alpha^2 + 2\beta^2.$$

The algebra Π is an indecomposable object in the derived category $\mathcal{D}(\Pi)$. Let $P = \Pi$. Then we have the equality $\Pi = P \oplus M$, where $M = 0$. Then LA_r is isomorphic to $\Sigma^r P$ and RA_r is isomorphic to $\Sigma^{-r} P$ for all $r \geq 0$.

The zeroth homology $H^0\Pi$ is one-dimensional and generated by the trivial path $e_\bullet$. Let $\mathcal{C}_\Pi$ be the generalized 2-cluster category. We claim that the image of RA_1 in $\mathcal{C}_\Pi$ is not isomorphic to the image of LA_2. Otherwise, assume that $\pi(RA_1)$ is isomorphic to $\pi(LA_2)$. Then the following isomorphisms hold

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_2), \Sigma\pi(LA_2)) &\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_2), \Sigma\pi(RA_1)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^2 P, \Sigma\pi(\Sigma^{-1}P)) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^2 P, P) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(\Sigma^2 P, P) \simeq H^{-2}\Pi. \end{aligned}$$

The left end term of these isomorphisms vanishes since $\pi(LA_2)$ is a 2-cluster tilting object, while the right end term is a 3-dimensional space whose basis is $\{\alpha^2, \alpha\beta, \beta\alpha\}$. Therefore, we obtain a contradiction.

4.5 Periodicity property

Lemma 4.5.1. *Let A be a dg algebra satisfying Assumptions 4.2.1. Let x and y be two integers satisfying $x \leq y + m + 1$. Suppose that the object X lies in $\mathcal{D}^{\leq x} \cap \mathrm{per}A$ and the object Y lies in ${}^{\perp}\mathcal{D}^{\leq y} \cap \mathrm{per}A$. Then the quotient functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi(X), \pi(Y)).$$

Proof. This proof is quite similar to the proof of Lemma 2.9 given in [70].

First, we show the injectivity.

Assume that $f : X \to Y$ is a morphism in $\mathcal{D}$ whose image in $\mathcal{C}_A$ is zero. It follows that f factors through some N in $\mathcal{D}_{fd}(A)$. Let $f = hg$. Consider the following diagram

$$\begin{tikzcd} & X \arrow[rr, "f"] \arrow[dr, "g"] \arrow[dl, dotted] & & Y & & \\ \tau_{\leq x}N \arrow[rr] & & N \arrow[ur, "h"'] \arrow[rr] & & \tau_{\geq x+1}N \arrow[r] & \Sigma(\tau_{\leq x}N). \end{tikzcd}$$

We have that g factors through $\tau_{\leq x}N$ because $X \in \mathcal{D}^{\leq x}$ and the space $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq x}, \tau_{\geq x+1}N)$ vanishes.

Now since $\tau_{\leq x}N$ is still in $\mathcal{D}_{fd}(A)$, by the Calabi-Yau property, the following isomorphism

$$D\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq x}N, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2}(\tau_{\leq x}N))$$

holds. Since $\Sigma^{m+2}(\tau_{\leq x}N)$ belongs to $\mathcal{D}^{\leq x-m-2}(\subseteq \mathcal{D}^{\leq y-1})$, the right hand side of the above isomorphism is zero. Therefore, the morphism f is zero in the derived category $\mathcal{D}$.

Second, we show the surjectivity.

Consider an arbitrary fraction $s^{-1}f$ in $\mathcal{C}_A$

$$\begin{tikzcd} X \arrow[dr, "f"] & & Y \arrow[dl, "s"'] \\ & U \arrow[dl, "r"] & \\ N & & \end{tikzcd}$$

where the cone N of s is in $\mathcal{D}_{fd}(A)$. Now look at the following diagram

$$\begin{tikzcd} X \arrow[r, dotted, "w"] \arrow[dr, "f"'] & Y \arrow[d, "s"] \arrow[dr, "v"] & & \\ & U \arrow[r, dotted, "g"'] \arrow[d, "r"] & Z \arrow[d] & \\ \tau_{\leq x}N \arrow[r] & N \arrow[r, "\pi_{x+1}"] \arrow[d, "u"] & \tau_{\geq x+1}N \arrow[r] \arrow[dl, dotted, "h"] & \Sigma(\tau_{\leq x}N) \\ & \Sigma Y. & & \end{tikzcd}$$

By the Calabi-Yau property, the space $\mathrm{Hom}_{\mathcal{D}}(\tau_{\leq x}N, \Sigma Y)$ is isomorphic to the space $D\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+1}(\tau_{\leq x}N))$, which is zero since $x-m-1 \leq y$. Thus, there exists a morphism h such that $u = h \circ \pi_{x+1}$. Now we embed h into a triangle in $\mathcal{D}$ as follows

$$Y \xrightarrow{v} Z \longrightarrow \tau_{\geq x+1}N \xrightarrow{h} \Sigma Y.$$

It follows that the morphism v factors through s by some morphism g. Then we can get a new fraction

$$\begin{tikzcd} X \arrow[dr, "g\circ f"'] & & Y \arrow[dl, "v"] \\ & Z & \end{tikzcd}$$

where the cone of v is $\tau_{\geq x+1}N(\in \mathcal{D}_{fd}(A))$. This fraction is equal to the one we start with because

$$v^{-1}(g \circ f) = (g \circ s)^{-1}(g \circ f) \sim s^{-1}f.$$

Moreover, since the space $\mathrm{Hom}_{\mathcal{D}}(X, \tau_{\geq x+1}N)$ vanishes, there exists a morphism $w : X \to Y$ such that $g \circ f = v \circ w$. Therefore, the fraction above is exactly the image of w in $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ under the quotient functor π. $\square$

Note that in the assumptions of the above lemma, we do not necessarily suppose that the objects X and Y lie in some shifts of the fundamental domain.

A special case of Lemma 4.5.1 is that, if X lies in $\mathcal{D}^{\leq m} \cap \mathrm{per}A$, then the quotient functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(X, RA_t) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi(X), \pi(RA_t))$$

for any nonnegative integer t, where RA_t belongs to ${}^{\perp}\mathcal{D}^{\leq -1}$.

Theorem 4.5.2. *Under the assumptions of Theorem 4.4.11, for each positive integer t,*

1) *the image of RA_t is isomorphic to the image of $RA_{t(\mathrm{mod}\, m+1)}$ in $\mathcal{C}_\Pi$,*
2) *the image of LA_t is isomorphic to the image of $LA_{t(\mathrm{mod}\, m+1)}$ in $\mathcal{C}_\Pi$.*

Proof. We only show the first statement. Then the second one can be obtained similarly.

Following Theorem 4.4.11, the image of RA_{m+1} in $\mathcal{C}_\Pi$ is isomorphic to P, which is RA_0 by definition. Let us denote '$t\,(\mathrm{mod}\, m+1)$' by $\bar{t}$. We prove the statement by induction.

Assume that the image of RA_t is isomorphic to the image of $RA_{\bar{t}}$ in $\mathcal{C}_\Pi$. Consider the following two triangles in $\mathcal{D}(\Pi)$

$$RA_{t+1} \longrightarrow A^{(t+1)} \overset{f^{(t+1)}}{\longrightarrow} RA_t \longrightarrow \Sigma RA_{t+1},$$

$$RA_{\overline{t+1}} \longrightarrow A^{(\overline{t+1})} \overset{f^{(\overline{t+1})}}{\longrightarrow} RA_{\bar{t}} \longrightarrow \Sigma RA_{\overline{t+1}},$$

and also consider their images in $\mathcal{C}_\Pi$. By Lemma 4.5.1, the isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\Pi)}(L, RA_t) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(L, \pi(RA_t))$$

holds for any object $L \in \mathrm{add}M$ and any nonnegative integer t. Hence, the images $\pi(f^{(t+1)})$ and $\pi(f^{(\overline{t+1})})$ are minimal right $(\mathrm{add}M)$-approximations of $\pi(RA_t)$ and $\pi(RA_{\bar{t}})$ in $\mathcal{C}_\Pi$, respectively. By hypothesis, $\pi(RA_t)$ is isomorphic to $\pi(RA_{\bar{t}})$. Therefore, the objects $A^{(t+1)}$ and $A^{(\overline{t+1})}$ are isomorphic, and $\pi(RA_{t+1})$ is isomorphic to $\pi(RA_{\overline{t+1}})$ in $\mathcal{C}_\Pi$. This completes the statement. $\square$

Remark 4.5.3. Section 10 in [48] gave a class of $(2n+1)$-Calabi-Yau (only for even integers $2n$, not for all integers $m \geq 2$) triangulated categories (arising from certain Cohen-Macaulay rings) which contain infinitely many indecomposable $2n$-cluster tilting objects.

In the following, for every integer $m \geq 2$, we construct an $(m+1)$-Calabi-Yau triangulated category which contains infinitely many indecomposable m-cluster tilting objects.

When $m = 2$, we use the same quiver Q as in Example 4.4.12.

When $m > 2$, let Q be the quiver consisting of one vertex $\bullet$ and one loop α of degree -1.

Let $\Pi = \widehat{\Pi}(Q, m+2, 0)$ be the associated completed deformed preprojective dg algebra. Clearly, Π is an indecomposable object in the derived category $\mathcal{D}(\Pi)$, the zeroth homology $H^0\Pi$ is one-dimensional and the path α^s is a nonzero element in the homology $H^{-s}\Pi$ ($s \in \mathbb{N}^*$). Let $\mathcal{C}_\Pi$ be the generalized m-cluster category and $\pi : \mathrm{per}\Pi \to \mathcal{C}_\Pi$ the canonical projection functor. Set $P = \Pi$. Then $\Pi = P \oplus 0$. For each integer $t \geq 0$, the object LA_t is isomorphic to $\Sigma^t P$ and the object RA_t is isomorphic to $\Sigma^{-t} P$. Now we claim that

1) For any two integers $r > t \geq 0$, the object $\pi(RA_r)$ is not isomorphic to $\pi(RA_t)$ in $\mathcal{C}_\Pi$, and the object $\pi(LA_r)$ is not isomorphic to $\pi(LA_t)$ in $\mathcal{C}_\Pi$.

2) For any two integers $r_1, r_2 > 0$, the objects $\pi(RA_{r_1})$ and $\pi(LA_{r_2})$ are not isomorphic in $\mathcal{C}_\Pi$.

Otherwise, similarly as in Example 4.4.12, the following contradictions will appear

$$\begin{aligned}
(0=)\,\mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(RA_t), \Sigma\pi(RA_t)) &= \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(RA_t), \Sigma\pi(RA_r))\\
&\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^{-t}P, \Sigma^{1-r}P) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{t-r+1}P)\\
&\simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{t-r+1}P) \simeq H^{t-r+1}\Pi\,(\neq 0);\\
(0=)\,\mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_r), \Sigma\pi(LA_r)) &= \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_r), \Sigma\pi(LA_t))\\
&\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^{r}P, \Sigma^{t+1}P) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{t-r+1}P)\\
&\simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{t-r+1}P) \simeq H^{t-r+1}\Pi\,(\neq 0);\\
(0=)\,\mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_{r_2}), \Sigma\pi(LA_{r_2})) &= \mathrm{Hom}_{\mathcal{C}_\Pi}(\pi(LA_{r_2}), \Sigma\pi(RA_{r_1}))\\
&\simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(\Sigma^{r_2}P, \Sigma^{1-r_1}P) \simeq \mathrm{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{1-r_1-r_2}P)\\
&\simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{1-r_1-r_2}P) \simeq H^{1-r_1-r_2}\Pi\,(\neq 0);
\end{aligned}$$

where the left end terms become zero, the right end terms are nonzero since $t-r+1 \leq 0$ and $1-r_1-r_2 < 0$, and the isomorphism

$$\mathrm{Hom}_{\mathcal{C}_\Pi}(P, \Sigma^{-s}P) \simeq \mathrm{Hom}_{\mathcal{D}(\Pi)}(P, \Sigma^{-s}P)$$

holds for any $s \in \mathbb{N}$ following Lemma 4.5.1.

Therefore, the $(m+1)$-Calabi-Yau triangulated category $\mathcal{C}_\Pi$ contains infinitely many m-cluster tilting objects, and in the presence of loops, the objects $\pi(RA_t)$ and $\pi(LA_r)$ do not satisfy the relations in Theorem 4.4.11 and Theorem 4.5.2.

4.6 AR $(m+3)$-angles related to P-indecomposables

Let $\mathcal{T}$ be an additive Krull-Schmidt category. We denote by $J_{\mathcal{T}}$ the *Jacobson radical* [6] of $\mathcal{T}$. Let $f \in \mathcal{T}(X,Y)$ be a morphism. Then f is called (in [48]) a *sink map* of $Y \in \mathcal{T}$ if f is right minimal, $f \in J_{\mathcal{T}}$, and

$$\mathcal{T}(-,X) \xrightarrow{f\cdot} J_{\mathcal{T}}(-,Y) \longrightarrow 0$$

is exact as functors on $\mathcal{T}$. The definition of *source maps* is given dually.

Let n be a positive integer. Given n triangles in a triangulated category,

$$X_{i+1} \overset{b_{i+1}}{\to} B_i \overset{a_i}{\to} X_i \to \Sigma X_{i+1}, \quad 0 \leq i < n,$$

the complex

$$X_n \overset{b_n}{\to} B_{n-1} \xrightarrow{b_{n-1}a_{n-1}} B_{n-2} \to \ldots \to B_1 \xrightarrow{b_1a_1} B_0 \overset{a_0}{\to} X_0$$

is called (in [48]) an $(n+2)$-*angle*.

Definition 4.6.1 ([48]). Let H be an m-cluster tilting object in a Krull-Schmidt triangulated category. We call an $(m+3)$-angle with X_0, X_{m+1} and all $B_i\,(0 \leq i \leq m)$ in add$H$ an *AR* $(m+3)$-*angle* if the following conditions are satisfied

a) a_0 is a sink map of X_0 in add$H$ and b_{m+1} is a source map of X_{m+1} in add$H$, and

b) a_i (resp. b_i) is a minimal right (resp. left) (add$H$)-approximation of X_i for each integer $1 \leq i \leq m$.

Remark 4.6.2. An AR $(m+3)$-angle with right term X_0 (resp. left term X_{m+1}) depends only on X_0 (resp. X_{m+1}) and is unique up to isomorphism as a complex.

We will use the higher AR theory to show the following theorem, which gives a more explicit criterion than the general Theorem 5.8 in [48] for the category $\mathcal{C}_\Pi$.

Theorem 4.6.3. *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ and i a vertex of Q. Assume that the zeroth homology $H^0\Pi$ is finite-dimensional and there are no loops of Q at vertex i. Then the almost complete m-cluster tilting P-object $\Pi/e_i\Pi$ has exactly $m+1$ complements in the generalized m-cluster category $\mathcal{C}_\Pi$.*

Proof. Set $RA_0 = P_i = e_i\Pi$ and $M = \Pi/e_i\Pi$. Section 4 gives us a construction of iterated mutations RA_t of P_i in the derived category $\mathcal{D}(\Pi)$, that is, the morphism $h^{(1)} : P'_0 \to P_i$ is a minimal right (add$M$)-approximation of P_i, and morphisms $h^{(t+1)} : P'_t \to RA_t$ $(1 \leq t \leq m)$ are minimal right (add$A$)-approximations of RA_t with P'_t in add$M$. Let $\mathcal{A}$ (resp. $\mathcal{M}$) denote the subcategory add$\pi(\Pi)$ (resp. add$\pi(M)$) in the generalized m-cluster category $\mathcal{C}_\Pi$.

Step 1. Since P'_0, P_i and M are in the fundamental domain, the morphism $h^{(1)}$ can be viewed as a minimal right $\mathcal{M}$-approximation in $\mathcal{C}_\Pi$, that is, the sequence

$$\mathcal{A}(-, P'_0)|_{\mathcal{M}} \xrightarrow{h^{(1)}_*} \mathcal{A}(-, P_i)|_{\mathcal{M}} = J_{\mathcal{A}}(-, P_i)|_{\mathcal{M}} \to 0,$$

is exact as functors on $\mathcal{M}$. Since there are no loops of $\overline{Q}^V$ of degree zero at vertex i, the Jacobson radical of $\mathrm{End}_{\mathcal{A}}(P_i)$ $(\simeq \mathrm{End}_{\mathcal{D}(\Pi)}(P_i))$ consists of combinations of cyclic paths $p = a_1 \dots a_r$ $(r \geq 2)$ of $\overline{Q}^V$ of degree zero. The path p factors though $e_{s(a_1)}\Pi$ and factors through $h^{(1)}$. Therefore, we have an exact sequence

$$\mathcal{A}(P_i, P'_0) \xrightarrow{h^{(1)}_*} \mathrm{rad}\,\mathrm{End}_{\mathcal{A}}(P_i) \longrightarrow 0.$$

Thus, the morphism $h^{(1)}$ is a sink map in the subcategory $\mathcal{A}$.

Step 2. The morphisms $h^{(t+1)}$ $(1 \leq t \leq m)$ are minimal right (add$A$)-approximations of RA_t with P'_t in add$M$. Since the objects $RA_t(1 \leq t \leq m)$ and P'_t lie in the shift $\Sigma^{-m}\mathcal{F}$ of the fundamental domain by Proposition 4.2.8, the images of $h^{(t+1)}$ are minimal right $\mathcal{A}$-approximations in $\mathcal{C}_\Pi$.

Step 3. Consider the morphisms $\alpha^{(t)}$ in the triangles of constructing RA_t in $\mathcal{D}(\Pi)$

$$\Sigma^{-1}RA_{t-1} \longrightarrow RA_t \xrightarrow{\alpha^{(t)}} P'_{t-1} \xrightarrow{h^{(t)}} RA_{t-1}, \quad 1 \leq t \leq m.$$

We already know that the maps $\alpha^{(t)}$ are minimal left (add$M$)-approximations in $\mathcal{D}(\Pi)$. Now applying the functor $\mathrm{Hom}_{\mathcal{D}(\Pi)}(-, P_i)$ to the above triangles, we obtain long exact sequences

$$\dots \to (P'_{t-1}, P_i) \xrightarrow{\alpha^{(t)*}} (RA_t, P_i) \longrightarrow (\Sigma^{-1}RA_{t-1}, P_i) \to \dots,$$

where $(,)$ denotes $\mathrm{Hom}_{\mathcal{D}(\Pi)}(,)$. The terms $\mathrm{Hom}_{\mathcal{D}(\Pi)}(\Sigma^{-1}RA_{t-1}, P_i)$ are zero since all RA_{t-1} lie in ${}^{\perp}\mathcal{D}(\Pi)^{\leq -1}$. Hence, the morphisms $\alpha^{(t)}$ are minimal left (add$A$)-approximations in $\mathcal{D}(\Pi)$. Since the objects $RA_t(1 \leq t \leq m)$ and P_t' lie in the shift $\Sigma^{-m}\mathcal{F}$, the images of $\alpha^{(t)}$ are minimal left $\mathcal{A}$-approximations in $\mathcal{C}_\Pi$.

Step 4. Consider the following two triangles in $\mathcal{D}(\Pi)$

$$RA_{m+1} \overset{\alpha^{(m+1)}}{\longrightarrow} P_m' \overset{h^{(m+1)}}{\longrightarrow} RA_m \longrightarrow \Sigma RA_{m+1},$$

$$P_i \overset{g^{(1)}}{\longrightarrow} P_m' \overset{\beta^{(1)}}{\longrightarrow} LA_1 \longrightarrow \Sigma P_i.$$

Since the objects P_i, P_m' and LA_1 are in the fundamental domain $\mathcal{F}$, the second triangle can also be viewed as a triangle in $\mathcal{C}_\Pi$ and the morphism $\beta^{(1)}$ is a minimal right $\mathcal{M}$-approximation of LA_1. Note that the objects RA_m and P_m' belong to $\Sigma^{-m}\mathcal{F}$. Hence, the image of the first triangle

$$\pi(RA_{m+1}) \overset{\pi(\alpha^{(m+1)})}{\longrightarrow} P_m' \overset{\pi(h^{(m+1)})}{\longrightarrow} \pi(RA_m) \longrightarrow \Sigma\pi(RA_{m+1})$$

is a triangle in $\mathcal{C}_\Pi$ with $\pi(h^{(m+1)})$ a minimal right $\mathcal{M}$-approximation of $\pi(RA_m)$. By Theorem 4.4.11, the image of RA_m is isomorphic to the image of LA_1 in $\mathcal{C}_\Pi$. Thus, the images of these two triangles in $\mathcal{C}_\Pi$ are isomorphic. We can also check that $g^{(1)}$ is a source map in $\mathcal{A}$ as Step 1. Therefore, the image $\pi(\alpha^{(m+1)})$ is also a source map in $\mathcal{A}$ with $\pi(RA_{m+1})$ isomorphic to P_i in $\mathcal{C}_\Pi$.

Step 5. Now we form the following $(m+3)$-angle in $\mathcal{C}_\Pi$

$$P_i = \pi(RA_{m+1}) \xrightarrow{\varphi_{m+1}} P_m' \xrightarrow{\varphi_m} P_{m-1}' \longrightarrow \cdots \longrightarrow P_1' \xrightarrow{\varphi_1} P_0' \xrightarrow{\varphi_0} P_i,$$

where φ_0 is equal to $\pi(h^{(1)})$, the morphism φ_t $(1 \leq t \leq m)$ is the composition $\pi(\alpha^{(t)})\pi(h^{(t+1)})$, and φ_{m+1} is equal to $\pi(\alpha^{(m+1)})$. From the above four steps, we know that φ_0 is a sink map in $\mathcal{A}$, and φ_{m+1} is a source map in $\mathcal{A}$. As a consequence, this $(m+3)$-angle is the AR $(m+3)$-angle determined by P_i. Since the indecomposable object P_i does not belong to $\mathrm{add}(\oplus_{t=0}^m P_t')$, following Theorem 5.8 in [48], the almost complete m-cluster tilting P-object $\Pi/e_i\Pi$ has exactly $m+1$ complements $e_i\Pi$, $\pi(RA_1), \ldots, \pi(RA_m)$ in $\mathcal{C}_\Pi$. The proof is completed. □

4.7 Liftable almost complete m-cluster tilting objects for strongly $(m+2)$-Calabi-Yau case

Keep the assumptions as in Theorem 4.6.3. Let $\Pi = \widehat{\Pi}(Q, m+2, W)$. Let Y be a liftable almost complete m-cluster tilting object in the generalized m-cluster category $\mathcal{C}_\Pi$. Assume that Z is a basic cofibrant silting object in $\mathrm{per}\Pi$ such that $\pi(Z/Z')$ is isomorphic to Y, where $\pi : \mathrm{per}\Pi \to \mathcal{C}_\Pi$ is the canonical projection and Z' is an indecomposable direct summand of Z. Let A be the dg endomorphism algebra $\mathcal{H}om^\bullet_\Pi(Z, Z)$ and F the left derived functor $-\overset{L}{\otimes}_A Z$. From the proof of Theorem 4.3.3, we know that F is a Morita equivalence from $\mathcal{D}(A)$ to $\mathcal{D}(\Pi)$ and A satisfies Assumptions 4.2.1. We denote the truncated dg subalgebra $\tau_{\leq 0}A$ by E. Since A has its homology concentrated in nonpositive degrees, the canonical inclusion $E \hookrightarrow A$ is a quasi-isomorphism. Then the left derived functor $-\overset{L}{\otimes}_E A$ is a Morita equivalence from $\mathcal{D}(E)$ to $\mathcal{D}(A)$.

Theorem 4.7.1 ([51]). *Let l be a commutative ring. Let B and B' be two dg l-algebras and X a dg B-B'-bimodule which is cofibrant over B. Assume that B and B' are flat as dg l-modules and*

$$- \overset{L}{\otimes}_{B'} X : \mathcal{D}(B') \to \mathcal{D}(B)$$

is an equivalence. Then the dg algebras B and B' have isomorphic cyclic homology and isomorphic Hochschild homology.

A corollary of Theorem 4.7.1 is that B' is strongly $(m+2)$-Calabi-Yau if and only if so is B.

The object Z is canonically an k-module, and the dg algebras A and E are k-algebras. Thus, the derived equivalent dg algebras Π, A and E are flat as dg k-modules. Following Remark 4.4.4 and Theorem 4.7.1, the dg algebras A and E are also strongly $(m+2)$-Calabi-Yau.

We will show that the dg algebra E satisfies the assumption in Theorem 4.4.7, that is E lies in $PCAlgc(l')$ for some finite-dimensional separable commutative k-algebra l'. In fact, $l' = \prod_{|Z|} k$, where $|Z|$ is the number of indecomposable direct summands of Z in perΠ. Furthermore, from the following lemma, we can deduce that $l' = l$.

Lemma 4.7.2. *Suppose that B is a dg algebra with positive homologies being zero. Then all basic cofibrant silting objects have the same number of indecomposable direct summands in* per$B$.

Proof. The triangulated category per$B$ contains an additive subcategory $\mathcal{B} := \mathrm{add}B$. Since the dg algebra B has its homology concentrated in nonpositive degrees, it follows that

$$\mathrm{Hom}_{\mathrm{per}B}(\mathcal{B}, \Sigma^p \mathcal{B}) = 0, \quad p > 0.$$

Since the category per$B$, which consists of the compact objects in $\mathcal{D}(B)$, and the category add$B$ are both idempotent split, by Proposition 5.3.3 of [15], the isomorphism

$$K_0(\mathrm{per}B) \simeq K_0(\mathrm{add}B)$$

holds, where $K_0(-)$ denotes the Grothendieck group.

Let Z be any basic cofibrant silting object in per$B$ and B' its dg endomorphism algebra $\mathrm{Hom}^\bullet_B(Z, Z)$. Then B' has its homology concentrated in nonpositive degrees and perB' is triangle equivalent to per$B$. Therefore, we have $K_0(\mathrm{per}B') \simeq K_0(\mathrm{add}B')$ and $K_0(\mathrm{per}B') \simeq K_0(\mathrm{per}B)$. As a consequence, the following isomorphisms hold

$$K_0(\mathrm{add}B) \simeq K_0(\mathrm{add}B') \simeq K_0(\mathrm{add}Z).$$

Thus, any basic cofibrant silting object in per$B$ has the same number of indecomposable direct summands as that of the dg algebra B itself. □

When forgetting the grading, the dg algebra E becomes to be $E^u := Z^0 A \oplus (\prod_{r<0} A^r)$, where $Z^0 A (= \mathrm{Hom}_{\mathcal{C}(\Pi)}(Z, Z))$ consists of the zeroth cycles of A. For any $x \in \prod_{r<0} A^r$, the element $1 + x$ clearly has an inverse element. It follows that $\prod_{r<0} A^r$ is contained in $\mathrm{rad}(E^u)$. We have the following canonical short exact sequence

$$0 \to B^0 A \to Z^0 A \xrightarrow{p} H^0 A \to 0,$$

where $B^0 A$ is the two-sided ideal of the algebra $Z^0 A$ consisting of the zeroth boundaries of A.

Following from Lemma 4.4.9, without loss of generality, we can assume that the basic silting object Z is a minimal perfect dg Π-module.

Lemma 4.7.3. *Keep the above notation and suppose that Z is a minimal perfect dg Π-module. Then B^0A lies in the radical of Z^0A.*

Proof. Let f be an element in B^0A. Then f is of the form $d_Z h + hd_Z$ for some degree -1 morphism $h : Z \to Z$. Since Z is minimal perfect, the entries of f lie in the ideal $\mathfrak{m}$ generated by the arrows of $\widetilde{Q}^V$. Then for any morphism $g : Z \to Z$, the morphism $1_Z - gf$ admits an inverse $1_Z + gf + (gf)^2 + \ldots$. Similarly for the morphism $1_Z - fg$. It follows that f lies in the radical of the algebra Z^0A. This completes the proof. $\square$

The epimorphism p in the above short exact sequence induces an epimorphism

$$\overline{p} : Z^0A/\mathrm{rad}(Z^0A) \to H^0A/\mathrm{rad}(H^0A).$$

Since B^0A lies in the radical of Z^0A, the epimorphism $\overline{p}$ is an isomorphism. Therefore, the following isomorphisms

$$E^u/\mathrm{rad}(E^u) \simeq Z^0A/\mathrm{rad}(Z^0A) \simeq H^0A/\mathrm{rad}(H^0A)$$

are true. Note that $\mathrm{per}\Pi$ is Krull-Schmidt and Hom-finite. Since the algebra $E_i := \mathrm{End}_{\mathrm{per}\Pi}(Z_i)$ is local and k is algebraically closed, the quotient $E_i/\mathrm{rad}(E_i)$ is isomorphic to k. Then we have that

$$\begin{aligned} H^0A/\mathrm{rad}(H^0A) &\simeq \mathrm{End}_{\mathrm{per}\Pi}(Z)/\mathrm{rad}(\mathrm{End}_{\mathrm{per}\Pi}(Z)) \\ &\simeq \prod\nolimits_{|Z|} E_i/\mathrm{rad}(E_i) \simeq \prod\nolimits_{|Z|} k\,(=l). \end{aligned}$$

Hence, the dg algebra E lies in $PCAlgc(l)$. Therefore, E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\widehat{\Pi}(Q', m+2, W')$ (denoted by Π'). Moreover, $H^0\Pi'$ is equal to H^0A which is finite-dimensional.

The following diagram

$$\begin{tikzcd} \mathrm{per}\Pi' \arrow[r, "-\overset{L}{\otimes}_{\Pi'}E"] \arrow[d] & \mathrm{per}E \arrow[r, "-\overset{L}{\otimes}_{E}A"] \arrow[d] & \mathrm{per}A \arrow[r, "-\overset{L}{\otimes}_{A}Z"] \arrow[d] & \mathrm{per}\Pi \arrow[d] \\ \mathcal{C}_{\Pi'} \arrow[r] & \mathcal{C}_E \arrow[r] & \mathcal{C}_A \arrow[r] & \mathcal{C}_\Pi \end{tikzcd}$$

is commutative, where each functor in the rows is an equivalence and the functor in each column is the canonical projection. The preimage of Z in $\mathrm{per}\Pi'$, under the equivalence F given by the composition of the functors in the top row, is Π'. Let $\Pi'_0 = e_j\Pi'$ be the P-indecomposable dg Π'-module such that $F(\Pi'_0) = Z'$ in $\mathrm{per}\Pi$, where j is a vertex of Q'. Assume that there are no loops of Q' at vertex j. It follows from Theorem 4.6.3 that the almost complete m-cluster tilting P-object Π'/Π'_0 has exactly $m+1$ complements in $\mathcal{C}_{\Pi'}$. Note that the image of Π'/Π'_0 in $\mathcal{C}_\Pi$, under the equivalence given by the composition of the functors in the bottom row, is Y. Therefore, the liftable almost complete m-cluster tilting object Y has exactly $m+1$ complements in $\mathcal{C}_\Pi$.

As a conclusion, we write down the following theorem.

Theorem 4.7.4. *Let Π be a good completed deformed preprojective dg algebra $\widehat{\Pi}(Q, m+2, W)$ whose zeroth homology $H^0\Pi$ is finite-dimensional. Let Z be a basic silting object in* $\mathrm{per}\Pi$ *which is minimal perfect and cofibrant. Denote by E the dg algebra $\tau_{\leq 0}(\mathrm{Hom}^\bullet_\Pi(Z, Z))$. Then*

1) E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$, where the quiver Q' has the same number of vertices as Q and $H^0\Pi'$ is finite-dimensional;

2) let Y be a liftable almost complete m-cluster tilting object of the form $\pi(Z/Z')$ in $\mathcal{C}_\Pi$ for some indecomposable direct summand Z' of Z. If we further assume that there are no loops at the vertex j of Q', where $e_j\Pi' \overset{L}{\otimes}_{\Pi'} Z = Z'$, then Y has exactly $m+1$ complements in $\mathcal{C}_\Pi$.

Here we would like to point out a special case of the above theorem, namely $m = 1$ and $Z = LA_1^{(k)} \oplus \Pi/e_k\Pi$ with respect to some vertex k of Q. Let $(Q^\star, W^\star)$ denote the (reduced) mutation $\mu_k(Q, W)$ defined in [31] of the quiver with potential (Q, W) at vertex k. Let A be the dg endomorphism algebra $\mathrm{Hom}^\bullet_\Pi(Z, Z)$ and $\Pi^\star$ the good completed deformed preprojective dg algebra $\widehat{\Pi}(Q^\star, m+2, W^\star)$. By [63], there is a canonical morphism from $\Pi^\star$ to A. Define three functors as follows:

$$F = -\overset{L}{\otimes}_{\Pi^\star} Z, \quad F_1 = -\overset{L}{\otimes}_{\Pi^\star} A, \quad F_2 = -\overset{L}{\otimes}_A Z.$$

Clearly, we have that $F = F_2F_1$ and F_2 is a quasi-inverse equivalence. It was shown in [63] that F is a quasi-inverse equivalence. The following isomorphisms

$$H^n(\Pi^\star) \simeq \mathrm{Hom}_{\mathcal{D}(\Pi^\star)}(\Pi^\star, \Sigma^n\Pi^\star) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(A, \Sigma^n A) \simeq H^n A$$

become true, which implies that $\Pi^\star$ and A are quasi-isomorphic. Therefore, the quiver with potential (Q', W') appearing in Theorem 4.7.4 1) for this special case can be chosen as $\mu_k(Q, W)$.

As the end part of this section, we state a 'reasonable' conjecture about the non-loop assumption in the above theorem for completed deformed preprojective dg algebras.

Definition 4.7.5. Let r be a positive integer. An algebra $A \in PCAlgc(l)$ is said to be *r-rigid* if

$$HH_0(A) \simeq l, \quad \text{and} \quad HH_p(A) = 0 \ (1 \le p \le r-1),$$

where $HH_*(A)$ is the pseudo-compact version of the Hochschild homology of the dg algebra A.

Remark 4.7.6. For completed Ginzburg algebras associated with quivers with potentials, our definition of 1-rigidity coincides with the definition of rigidity in [31]. Proposition 8.1 in [31] states that any rigid reduced quiver with potential is 2-acyclic. Then no loops will be produced following their mutation rule. Although we do not know whether the quiver Q' related to such a silting object as in Theorem 4.7.4 can be obtained from mutation of quivers with potentials, we can still obtain that the quiver Q' always does not contain loops in the condition of 1-rigidity (see Corollary 4.7.9).

Proposition 4.7.7. *The completed deformed preprojective dg algebras $\Pi = \widehat{\Pi}(Q, m+2, 0)$ associated with acyclic quivers Q are m-rigid.*

Proof. It is clear that the zeroth component Π^0 of Π is just the finite-dimensional path algebra kQ (denoted by B) and the $(-p)$th component of Π is zero for $1 \le p \le m-1$. Thus, the Hochschild homology of Π is given by

$$HH_0(\Pi) = B/[B, B] = \textstyle\prod_{|Q_0|} k,$$

$$HH_p(\Pi) = HH_p(B) = \mathrm{Ker}(\partial_p^0)/\mathrm{Im}(\partial_{p+1}^0) \quad (1 \le p \le m-1),$$

where $\partial_p^0 : B^{\otimes_k(p+1)} \to B^{\otimes_k p}$ is the pth row differential of the uppermost row in the Hochschild complex $X := \Pi \overset{L}{\otimes}_{\Pi^e} \Pi$.

Since the path algebra kQ is of finite dimension and of finite global dimension and k is algebraically closed, we have $HH_p(B) = 0$ for all integers $p > 0$, cf. Proposition 2.5 of [51]. It follows that the dg algebra $\widehat{\Pi}(Q, m+2, 0)$ is m-rigid. □

Proposition 4.7.8. *Let $\Pi = \widehat{\Pi}(Q, m+2, W)$ be a good completed deformed preprojective dg algebra and p a fixed integer in the segment $[0, m]$. Suppose the p-th Hochschild homology of Π satisfies the isomorphism*

$$HH_p(\Pi) \simeq \begin{cases} \prod_{|Q_0|} k & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

Then $\overline{Q}^V$ does not contain loops with zero differential and of degree $-p$.

Proof. Let a be a loop of $\overline{Q}^V$ at some vertex i with zero differential and of degree $-p$. The element a lies in the rightmost column of the Hochschild complex X of Π. By assumption the differential $d(a)$ is zero, so a is an element in $HH_p(\Pi)$. Now we claim that a is a nonzero element in $HH_p(\Pi)$.

First, the superpotential W is a linear combination of paths of length at least 3, so $d(\widetilde{Q}_1^V) \subseteq \mathfrak{m}^2$, where $\mathfrak{m}$ is the two-sided ideal of Π generated by the arrows of $\widetilde{Q}^V$. Second, it is obvious that the relation $\mathrm{Im}\partial_1 \cap \{\text{loops of } \widetilde{Q}^V\} = \emptyset$ holds. Therefore, the loop a can not be written in the form $\sum d(\gamma) + \sum \partial_1(u \otimes v)$ for paths $\gamma \in e_i \mathfrak{m} e_i$ and u, v paths of $\widetilde{Q}^V$, which means that a is a nonzero element in $HH_p(\Pi)$.

Note that the trivial paths associated with the vertices of Q are nonzero elements in $HH_0(\Pi)$. Hence, we get a contradiction to the isomorphism in the assumption. As a result, the quiver $\overline{Q}^V$ does not contain loops with zero differential and of degree $-p$. □

Corollary 4.7.9. *Keep the notation as in Theorem 4.7.4 and let $m = 1$. Suppose that Π is 1-rigid. Then the new quiver Q' does not contain loops.*

Proof. It follows from statement 1) in Theorem 4.7.4 that E is quasi-isomorphic to some good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', 3, W')$. Then following Theorem 4.7.1 and the analysis before Theorem 4.7.4, we can obtain that the dg algebras Π and Π' have isomorphic Hochschild homology. Therefore, the new dg algebra Π' is also 1-rigid. Note that every arrow of Q' has zero degree and thus has zero differential. Hence, by Proposition 4.7.8 the quiver Q' does not contain loops. □

Conjecture 4.7.10. *Let $\Pi = \widehat{\Pi}(Q, m+2, W)$ be an m-rigid good completed deformed preprojective dg algebra whose zeroth homology $H^0\Pi$ is finite-dimensional. Then any liftable almost complete m-cluster tilting object has exactly $m+1$ complements in $\mathcal{C}_\Pi$.*

By the same procedure as in the proof of Corollary 4.7.9, we know that the good completed deformed preprojective dg algebra $\Pi' = \widehat{\Pi}(Q', m+2, W')$ in Theorem 4.7.4 is also m-rigid, and the new quiver Q' does not contain loops of degree zero. It seems that we would like to get a stronger result than Proposition 4.7.8, that is, m-rigidity implies that $\overline{Q'}^V$ does not contain loops (not only loops with zero differential). If this is true, then it follows from statement 2) in Theorem 4.7.4 that any liftable almost complete m-cluster tilting object has exactly $m+1$ complements in $\mathcal{C}_\Pi$.

If Conjecture 4.7.10 holds, then the m-rigidity property shown in Proposition 4.7.7 of the dg algebra $\Pi = \widehat{\Pi}(Q, m+2, 0)$ with Q an acyclic quiver implies that any liftable almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ has exactly $m+1$ complements. Later Proposition 4.8.6 shows that any almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ is liftable in the 'acyclic quiver' case. Thus, on one hand, if Conjecture 4.7.10 holds, we can deduce a common result both in [78] and [79], namely, any almost complete m-cluster tilting object in the classical m-cluster category $\mathcal{C}_Q^{(m)}$ has exactly $m+1$ complements. On the other hand, it follows from this common result for the classical m-cluster category $\mathcal{C}_Q^{(m)}$, which is triangle equivalent to the corresponding generalized m-cluster category $\mathcal{C}_\Pi$, that any almost complete m-cluster tilting object in $\mathcal{C}_\Pi$ should have exactly $m+1$ complements.

4.8 A long exact sequence and the acyclic case

Let A be a dg algebra satisfying Assumptions 4.2.1. In the first part of this section, we give a long exact sequence to see the relations between extension spaces in generalized m-cluster categories $\mathcal{C}_A$ and extension spaces in derived categories $\mathcal{D}(=\mathcal{D}(A))$. If the extension spaces between two objects of $\mathcal{C}_A$ are zero, in some cases, we can deduce that the extension spaces between these two objects are also zero in the derived category $\mathcal{D}$.

Proposition 4.8.1. *Suppose that X and Y are two objects in the fundamental domain $\mathcal{F}$. Then there is a long exact sequence*

$$\begin{gathered}0 \to \mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^m_{\mathcal{D}}(Y,X)\\ \to \mathrm{Ext}^2_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^2_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^{m-1}_{\mathcal{D}}(Y,X)\\ \to \quad \cdots \qquad \cdots \quad \to\\ \mathrm{Ext}^m_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^m_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y,X) \to 0.\end{gathered}$$

Proof. We have the canonical triangle

$$\tau_{\leq -m}X \to X \to \tau_{\geq 1-m}X \to \Sigma(\tau_{\leq -m}X),$$

which yields the long exact sequence

$$\cdots \to (\Sigma^{-t}(\tau_{\geq 1-m}X),Y) \to (\Sigma^{-t}X,Y) \to (\Sigma^{-t}(\tau_{\leq -m}X),Y) \to \cdots, \quad t\in\mathbb{Z},$$

where $(\,,\,)$ denotes $\mathrm{Hom}_{\mathcal{D}}(\,,\,)$.

Step 1. *The isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m}X),Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+2-t}X)$$

holds when $t \leq m+1$.

By the Calabi-Yau property, there holds the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m}X),Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+2-t}(\tau_{\geq 1-m}X)), \quad t\in\mathbb{Z}. \qquad (8.1).$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(Y,-)$ to the triangle which we start with, we obtain the exact sequence

$$\begin{gathered}\mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+2-t}(\tau_{\leq -m}X)) \to \mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+2-t}X) \to\\ \mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+2-t}(\tau_{\geq 1-m}X)) \to \mathrm{Hom}_{\mathcal{D}}(Y,\Sigma^{m+3-t}(\tau_{\leq -m}X)).\end{gathered}$$

When $t \leq m+1$, we have that $(-m)-(m+2-t) \leq -m-1$. Then the objects $\Sigma^{m+2-t}(\tau_{\leq -m}X)$ and $\Sigma^{m+3-t}(\tau_{\leq -m}X)$ belong to $\mathcal{D}^{\leq -m-1}$. Note that Y is in ${}^{\perp}\mathcal{D}^{\leq -m-1}$. Therefore, the following isomorphism holds

$$\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}(\tau_{\geq 1-m}X)) \simeq \mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}X). \qquad (8.2).$$

As a consequence, when $t \leq m+1$, together by (8.1) and (8.2), we have the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m}X), Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}X).$$

Moreover, if $t \leq 1$, the object $\Sigma^{m+2-t}X$ belongs to $\mathcal{D}^{\leq -m-1}$, so the space $\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-t}X)$ vanishes, and so does the space $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\geq 1-m}X), Y)$.

Step 2. *When $t \leq m$, we have the following isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m}X), Y) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)).$$

Consider the triangles

$$\tau_{\leq s-1}X \to \tau_{\leq s}X \to \Sigma^{-s}(H^sX) \to \Sigma(\tau_{\leq s-1}X), \qquad s \in \mathbb{Z}.$$

Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ to these triangles, we can obtain the following long exact sequences

$$\cdots \to (\Sigma^{-s-t}(H^sX), Y) \to (\Sigma^{-t}(\tau_{\leq s}X), Y) \to$$
$$(\Sigma^{-t}(\tau_{\leq s-1}X), Y) \to (\Sigma^{-s-t-1}(H^sX), Y) \to \cdots,$$

where $(,)$ denotes $\mathrm{Hom}_{\mathcal{D}}(,)$. Using the Calabi-Yau property, we have that

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t}(H^sX), Y) \simeq D\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma^{m+2-s-t}(H^sX)), \qquad t \in \mathbb{Z}.$$

When $t \leq -s$, the inequality $m+2-s-t-1 \geq m+1$ holds. So the two objects $\Sigma^{m+2-s-t}(H^sX)$ and $\Sigma^{m+2-s-t-1}(H^sX)$ belong to $\mathcal{D}^{\leq -m-1}$. Therefore, we obtain that the spaces $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t}(H^sX), Y)$ and $\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-s-t-1}(H^sX), Y)$ are zero, and the following isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq s}X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq s-1}X), Y)$$

holds. As a consequence, we can get the following isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -t}X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -t-1}X), Y) \simeq \cdots$$
$$\simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m}X), Y), \ t \leq m. \qquad (8.3).$$

Since the functor $\pi : \mathrm{per}A \to \mathcal{C}_A$ induces an equivalence from $\Sigma^t\mathcal{F}$ to $\mathcal{C}$ (Proposition 4.3.1 applies to shifted t-structure), the following bijections

$$\mathrm{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi(\tau_{\leq -t}X), \pi(\Sigma^tY)) \simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq -t}X, \Sigma^tY) \quad (8.4)$$

hold. Hence, when $t \leq m$, together by (8.3) and (8.4), we have the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\Sigma^{-t}(\tau_{\leq -m}X), Y) \simeq \mathrm{Hom}_{\mathcal{C}_A}(\pi X, \Sigma^t(\pi Y)).$$

Therefore, the long exact sequence at the beginning becomes

$$0 = \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-1}(\tau_{\geq 1-m}X), Y) \to \mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^m_{\mathcal{D}}(Y,X)$$

$$\to \mathrm{Ext}^2_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^2_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^{m-1}_{\mathcal{D}}(Y,X)$$

$$\to \quad \cdots \qquad \cdots \quad \to$$

$$\mathrm{Ext}^m_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^m_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y,X) \to \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-m-1}X, Y) = 0.$$

This concludes the proof. $\square$

Remarks 4.8.2. 1) When $m = 1$, the long exact sequence in Proposition 4.8.1 becomes the following short exact sequence (already appearing in the proof of Proposition 4.3.5)

$$0 \to \mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_A}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}}(Y,X) \to 0 \qquad (8.5),$$

which was presented in [2] for the Hom-finite 2-Calabi-Yau case, and also was presented in [70] for the Jacobi-infinite 2-Calabi-Yau case.

2) If T is an object in the fundamental domain $\mathcal{F}$ satisfying

$$\mathrm{Ext}^i_{\mathcal{D}}(T,T) = 0, \quad i = 1, \ldots, m,$$

then the long exact sequence in Proposition 4.8.1 implies that the spaces $\mathrm{Ext}^i_{\mathcal{C}_A}(T,T)$ also vanish for integers $1 \leq i \leq m$.

Suppose that X and Y are two objects in the fundamental domain. It is clear that $\mathrm{Ext}^i_{\mathcal{D}}(X,Y)$ vanishes when $i > m$, since X belongs to $\mathcal{F}$ and $\Sigma^i Y$ lies in $\mathcal{D}^{\leq -m-1}$. Now we assume that the spaces $\mathrm{Ext}^i_{\mathcal{C}_A}(X,Y)$ are zero for integers $1 \leq i \leq m$. What about the extension spaces $\mathrm{Ext}^i_{\mathcal{D}}(X,Y)$ in the derived category? Do they always vanish?

When $m = 1$, the short exact sequence (8.5) implies that the space $\mathrm{Ext}^1_{\mathcal{D}}(X,Y)$ vanishes.

When $m > 1$, we will give the answer for completed Ginzburg dg categories (the same as completed deformed preprojective dg algebras in this case) arising from acyclic quivers.

Proposition 4.8.3. *Let Q be an acyclic quiver. Let Γ be the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,0)$ and $\mathcal{C}_\Gamma$ the generalized m-cluster category. Suppose that X and Y are two objects in the fundamental domain $\mathcal{F}$ which satisfy*

$$\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(X,Y) = 0, \quad i = 1, \ldots, m.$$

Then the extension spaces $\mathrm{Ext}^i_{\mathcal{D}(\Gamma)}(X,Y)$ vanish for all positive integers i.

Proof. Let B be the path algebra kQ and Ω the inverse dualizing complex $\mathrm{RHom}_{B^e}(B, B^e)$. Set $\Theta = \Sigma^{m+1}\Omega$. Then the $(m+2)$-Calabi-Yau completion [56] of B is the tensor dg category

$$\Pi_{m+2}(B) = T_B(\Theta) = B \oplus \Theta \oplus (\Theta \otimes_B \Theta) \oplus \ldots.$$

Theorem 6.3 in [56] shows that $\Pi_{m+2}(B)$ is quasi-isomorphic to the completed Ginzburg dg category Γ. Thus, we can write Γ as

$$\Gamma = B \oplus \Theta \oplus (\Theta \overset{L}{\otimes}_B \Theta) \oplus \ldots = \oplus_{p \geq 0} \Theta^{\overset{L}{\otimes}_B p}.$$

Let X', Y' be two objects in $\mathcal{D}_{fd}(B)$. The following isomorphisms hold

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{D}(\Gamma)}(X' \overset{L}{\otimes}_B \Gamma, Y' \overset{L}{\otimes}_B \Gamma) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B \Gamma|_B)\\ \simeq\ &\mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B (\oplus_{p\geq 0}\Theta^{\overset{L}{\otimes}_B p})) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(X', \oplus_{p\geq 0}(Y' \overset{L}{\otimes}_B \Theta^{\overset{L}{\otimes}_B p}))\\ \simeq\ &\oplus_{p\geq 0}\mathrm{Hom}_{\mathcal{D}(B)}(X', Y' \overset{L}{\otimes}_B \Theta^{\overset{L}{\otimes}_B p}).\end{aligned}$$

By the proof in [54] of Lemma 4.2.3, the category $\mathcal{D}_{fd}(B)$ admits a Serre functor S whose inverse is $- \overset{L}{\otimes}_B \Omega$. Therefore, the functor $- \overset{L}{\otimes}_B \Theta$ is equal to the functor $S^{-1}\Sigma^{m+1}(\simeq \tau^{-1}\Sigma^m)$, where τ is the Auslander-Reiten translation. As a consequence, we have that

$$\mathrm{Hom}_{\mathcal{D}(\Gamma)}(X' \overset{L}{\otimes}_B \Gamma, Y' \overset{L}{\otimes}_B \Gamma) \simeq \oplus_{p\geq 0}\mathrm{Hom}_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y').$$

Let $\mathcal{C}_Q^{(m)}$ be the classical m-cluster category $\mathcal{D}_{fd}(B)/(\tau^{-1}\Sigma^m)^{\mathbb{Z}}$. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{D}_{fd}(B) \arrow[r, "-\overset{L}{\otimes}_B\Gamma"] \arrow[d, "\pi_B"'] & \mathrm{per}\Gamma \arrow[d, "\pi_\Gamma"] \\ \mathcal{C}_Q^{(m)} \arrow[r, "-\overset{L}{\otimes}_B\Gamma", "\simeq"'] & \mathcal{C}_\Gamma. \end{tikzcd}$$

Under the equivalence, let $X = X' \overset{L}{\otimes}_B \Gamma$ and $Y = Y' \overset{L}{\otimes}_B \Gamma$, so the vanishing of spaces $\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(X,Y)$ implies that $\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(X',Y')$ also vanish for integers $1 \leq i \leq m$. Note that

$$\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(X',Y') \simeq \oplus_{p\in\mathbb{Z}}\mathrm{Ext}^i_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y').$$

Hence, we obtain that

$$\mathrm{Ext}^i_{\mathcal{D}(\Gamma)}(X,Y) \simeq \oplus_{p\geq 0}\mathrm{Ext}^i_{\mathcal{D}_{fd}(B)}(X', (\tau^{-1}\Sigma^m)^p Y') = 0, \qquad 1 \leq i \leq m.$$

□

Let Q be an ordinary acyclic quiver and B the path algebra kQ. Let Γ be its completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,0)$. Let T be an m-cluster tilting object in $\mathcal{C}_Q^{(m)}$. Then T is induced from an object T' (that is, $T = \pi(T')$) in the fundamental domain

$$\mathcal{S}_m := \mathcal{S}_m^0 \vee \Sigma^m B, \quad \text{where } \mathcal{S}_m^0 := \mathrm{mod}B \vee \Sigma(\mathrm{mod}B) \ldots \vee \Sigma^{m-1}(\mathrm{mod}B).$$

Lemma 4.8.4 ([22])**.** *The object T' is a partial silting object, that is,*

$$\mathrm{Hom}_{\mathcal{D}_{fd}(B)}(T', \Sigma^i T') = 0, \qquad i > 0;$$

and T' is maximal with this property.

An object in $\mathcal{D}_{fd}(B)$ which satisfies the 'maximal partial silting' property as in Lemma 4.8.4 is called a 'silting' object in [22]. Next we will show that our definition for silting object in per$B$ coincides with their definition.

Lemma 4.8.5. *Let U be a basic partial silting object in $\mathcal{D}_{fd}(B)$. Then U is maximal partial silting if and only if U generates* per$B$*.*

Proof. On one hand, assume that U is a basic partial silting object and generates per$B$. By Lemma 4.7.2, the object U has the same number of indecomposable direct summands as that of the dg algebra B itself. That is, U is a basic partial silting object with $|Q_0|$ indecomposable direct summands. Following from Lemma 2.2 in [22], we obtain that U is a maximal partial silting object.

On the other hand, assume that U is a maximal partial silting object in $\mathcal{D}_{fd}(B)$. We decompose U into a direct sum $\Sigma^{k_1}U_1 \oplus \ldots \oplus \Sigma^{k_r}U_r$ such that each U_i lies in mod$B$ and $k_1 < \ldots < k_r$. Set $U' = \oplus_{i=1}^{r} U_i$. It follows from Lemma 2.2 in [22] that the object U' can be ordered to a complete exceptional sequence. Let $C(U')$ be the smallest full subcategory of mod$B$ which contains U' and is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms. By Lemma 3 in [14], the subcategory $C(U')$ is equal to mod$B$. As a consequence, the object U generates $\mathcal{D}_{fd}(B)$ which is equal to per$B$. $\square$

Since B is finite-dimensional and hereditary, the subcategory $\mathcal{S}_m^0$ is contained in ${}^{\perp}\mathcal{D}(B)^{\leq -m-1}$. The isomorphism

$$\mathrm{Hom}_{\mathcal{D}(B)}(\Sigma^m B, M) \simeq H^{-m}M \quad (M \in \mathcal{D}(B))$$

implies that $\Sigma^m B$ is in ${}^{\perp}\mathcal{D}(B)^{\leq -m-1}$. So $\mathcal{S}_m$ is contained in $\mathcal{D}(B)^{\leq 0} \cap^{\perp} \mathcal{D}(B)^{\leq -m-1} \cap \mathcal{D}_{fd}(B)$.

Set $Z = T' \overset{L}{\otimes}_B \Gamma$. For any object N in $\mathcal{D}(\Gamma)$, we have the following canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\Gamma)}(T' \overset{L}{\otimes}_B \Gamma, N) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(T', \mathrm{RHom}_{\Gamma}(\Gamma, N)).$$

When N lies in $\mathcal{D}(\Gamma)^{\leq -m-1}$, the right hand side of the above isomorphism becomes zero. Thus, the object Z is in the fundamental domain of $\mathcal{D}(\Gamma)$. The spaces $\mathrm{Ext}^i_{\mathcal{C}_Q^{(m)}}(T,T)$ vanish for integers $1 \leq i \leq m$, following the proof of Proposition 4.8.3, the space $\mathrm{Hom}_{\mathcal{D}(\Gamma)}(Z, \Sigma^i Z)$ is zero for each positive integer i. In addition, Lemma 4.8.4 and Lemma 4.8.5 together imply that T' generates $\mathcal{D}_{fd}(B)$. Hence, the object Z generates perΓ. So Z is a basic silting object whose image in $\mathcal{C}_\Gamma$ is $T \overset{L}{\otimes}_B \Gamma$.

Now we conclude the above analysis to get the following proposition.

Proposition 4.8.6. *Let Q be an acyclic quiver and B its path algebra. Let Γ be the completed Ginzburg dg category $\widehat{\Gamma}_{m+2}(Q,0)$ and $\mathcal{C}_\Gamma$ the generalized m-cluster category. Then any m-cluster tilting object in $\mathcal{C}_\Gamma$ is induced by a silting object in $\mathcal{F}$ under the canonical projection $\pi : \mathrm{per}\Gamma \to \mathcal{C}_\Gamma$.*

Proof. Let $\overline{T}$ be an m-cluster tilting object in $\mathcal{C}_\Gamma$. Then $\overline{T}$ can be written as $T \overset{L}{\otimes}_B \Gamma$ for some m-cluster tilting object T in $\mathcal{C}_Q^{(m)}$, where T is induced by some silting object T' in $\mathcal{S}_m$. The object $T' \overset{L}{\otimes}_B \Gamma$ (denoted by Z) is a silting object in the fundamental domain $\mathcal{F}$ whose image under the canonical projection $\pi : \mathrm{per}\Gamma \to \mathcal{C}_\Gamma$ is equal to $\overline{T}$. This completes the proof. $\square$

Chapter 5

Tropical friezes associated with Dynkin diagrams

Tropical friezes are the tropical analogues of Coxeter-Conway's frieze patterns. In this note, we study them using triangulated categories. A tropical frieze on a 2-Calabi-Yau triangulated category $\mathcal{C}$ is a function satisfying a certain addition formula. We show that when $\mathcal{C}$ is the cluster category of a Dynkin quiver, the tropical friezes on $\mathcal{C}$ are in bijection with the n-tuples in $\mathbb{Z}^n$, any tropical frieze f on $\mathcal{C}$ is of a special form, and there exists a cluster-tilting object such that f simultaneously takes non-negative values or non-positive values on all its indecomposable direct summands. Using similar techniques, we give a proof of a conjecture of Ringel for cluster-additive functions on stable translation quivers.

5.1 Introduction

Cluster algebras introduced by S. Fomin and A. Zelevinsky [34], are subrings of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of rational functions in n indeterminates endowed with a distinguished set of generators called cluster variables, which are constructed recursively via an operation called mutation. A cluster algebra is said to be of finite type if the number of cluster variables is finite. The classification of finite type cluster algebras was achieved [35] in terms of Dynkin diagrams.

Motivated by close relations between tilting theory of finite-dimensional hereditary algebras and the combinatorics of mutation in cluster algebras, the cluster category $\mathcal{C}_Q$ of a finite acyclic quiver Q was introduced in [24] for type A_n and in [17] for the general case. The cluster category provides a natural model for the combinatorics of its corresponding cluster algebra. It is triangulated [53], Krull-Schmidt and 2-Calabi-Yau [17] in the sense that there are bifunctorial isomorphisms

$$\mathrm{Ext}^1(X, Y) \simeq D\mathrm{Ext}^1(Y, X), \quad X, Y \in \mathcal{C}_Q.$$

There are also many other 2-Calabi-Yau triangulated categories, for example, the stable module categories of preprojective algebras of Dynkin type studied by Geiss-Leclerc-Schröer in their series of papers, the generalized cluster categories of Jacobi-finite quivers with potential [31] and of finite-dimensional algebras of global dimension ≤ 2, which were investigated in [2] by C. Amiot.

Starting from a 2-Calabi-Yau Hom-finite triangulated category $\mathcal{C}$ with a cluster-tilting object T, Palu [69] introduced the notion of a cluster character χ from $\mathcal{C}$ to a commutative ring which satisfies the multiplication formula

$$\chi(L)\chi(M) = \chi(E) + \chi(E')$$

for all objects L and M such that $\mathrm{Ext}^1_{\mathcal{C}}(L, M)$ is one-dimensional, where E and E' are the middle terms of the non-split triangles with end terms L and M. He explicitly constructed cluster characters from cluster-tilting objects.

In this article, we introduce tropical friezes f on $\mathcal{C}$ mainly by replacing the above multiplication formula with an addition formula

$$f(L) + f(M) = \max\{f(E), f(E')\}.$$

Our inspiration comes from the definition of cluster-additive functions [74] on stable translation quivers and from the tropicalized version of Coxeter-Conway's frieze patterns. To the best of our knowledge, such tropical frieze patterns first appeared implicitly in Fock-Goncharov's preprint [32] and explicitly in Section 4 of J. Propp's preprint [72].

The paper is organized as follows.

In Section 2, after recalling some facts on frieze patterns and stating the assumptions on the categories $\mathcal{C}$ we consider (namely, 2-Calabi-Yau categories with cluster-tilting object), we introduce the notion of tropical friezes. Then we study their first properties and some links to cluster characters, using which we give an example and a counter-example of tropical friezes.

In Section 3, taking advantage of the indices [59] of objects of $\mathcal{C}$, for each cluster-tilting object T and each element m in the Grothendieck group $K_0(\mathrm{modEnd}_{\mathcal{C}}(T))$, we define a function $f_{T,m}$ on $\mathcal{C}$. A criterion for $f_{T,m}$ to be a tropical frieze is given in Theorem 5.3.1, which is also a necessary condition when $\mathcal{C}$ is the cluster category $\mathcal{C}_Q$ of a Dynkin quiver Q. We also show that the tropical friezes on $\mathcal{C}_Q$ with Q Dynkin are in bijection with the n-tuples in $\mathbb{Z}^n$ by composing Palu's cluster character with a morphism of semifields. Then we investigate the cluster-hammock functions introduced by Ringel [74], which always give rise to tropical friezes while their sums do not.

Section 4 just consists of simple illustrations for the cases A_1 and A_2, in order to give the reader an intuitive impression.

In Section 5, for a cluster-tilting object T and a tropical frieze f on $\mathcal{C}$, we define an element $g(T)$ in the Grothendieck group $K_0(\mathrm{add}T)$, which transforms in the same way as the index with respect to cluster-tilting objects. The main result (Theorem 5.5.1) states that each tropical frieze on $\mathcal{C}_Q$ with Q a Dynkin quiver is of the form $f_{T,m}$. A different approach of this fact is given in Section 5 of [33]. As an application, we show that for any tropical frieze f on $\mathcal{C}_Q$, there exists a cluster-tilting object T' (resp. T'') such that f simultaneously takes non-negative (resp. non-positive) values on all its indecomposable direct summands.

Section 6 gives a proof of a conjecture of Ringel which concerns the universal form of cluster-additive functions f on the stable translation quiver $\mathbb{Z}\Delta$ with Δ a simply laced Dynkin diagram, namely, f is a non-negative linear combination of pairwise 'compatible' (in the sense of Ringel) cluster-hammock functions.

5.2 First properties of tropical friezes

In this section, we recall Coxeter-Conway's frieze patterns at the beginning, then inspired by a tropicalized version of Coxeter-Conway's frieze patterns of integers, we introduce tropical friezes on 2-Calabi-Yau triangulated categories. Apart from studying their first properties, we also investigate some links between tropical friezes and cluster characters.

5.2.1 Frieze patterns

In early 1970s, Coxeter and Conway studied frieze patterns and triangulated polygons in [27, 28, 29]. A frieze pattern $\mathcal{F}_n$ of order n consists of $n-1$ infinite rows of positive numbers, whose first and last rows are filled with 1. Besides, the essential point is the *unimodular rule*, that is, for every four adjacent numbers in $\mathcal{F}_n$ forming a diamond shape

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

the relation $ad = bc + 1$ is satisfied. For example, the following diagram is a frieze pattern of order 6:

$$\begin{array}{cccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \ \ 1 \ \cdots \\
& \cdots & 2 & & 2 & & 2 & & 1 & & 4 & & 1 \ \ 2 \ \ 2 & \cdots \\
& & \cdots & 3 & & 3 & & 1 & & 3 & & 3 & 1 \ \ 3 \ \ 3 & \cdots \\
& & & \cdots & 4 & & 1 & & 2 & & 2 & & 2 \ \ 1 \ \ 4 \ \ 1 & \cdots \\
& & & & \cdots & 1 & & 1 & & 1 & & 1 & 1 \ \ 1 \ \ 1 \ \ 1 & \cdots
\end{array}$$

A notable property of $\mathcal{F}_n$ is its periodicity with period a divisor of n. More precisely, it is invariant under a glide reflection σ which is $[\frac{n}{2}]$ times horizontal translation composed with a horizontal reflection.

A frieze pattern $\mathcal{F}_n$ is determined by the elements in one of its diagonals (say $b_1 = 1$, $b_2, \ldots, b_{n-2}, b_{n-1} = 1$), and it consists of integers if and only if b_s divides $b_{s-1} + b_{s+1}$ for $s = 2, \ldots, n-2$. Let $a_0 = b_2$ and $a_1, a_2, \ldots$ be the numbers lying to the right of a_0 in the second row. Then we have

$$a_s = \frac{b_s + b_{s+2}}{b_{s+1}}, \qquad 1 \leq s \leq n-3.$$

A frieze pattern $\mathcal{F}_n$ can also be derived from $a_0, \ldots, a_{n-4}$, since a_{n-3} satisfies the linear equation

$$\begin{vmatrix} a_0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & a_1 & 1 & \ldots & 0 & 0 \\ 0 & 1 & a_2 & \ldots & 0 & 0 \\ & \ldots & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & a_{n-3} \end{vmatrix} = b_{n-1} = 1$$

and $\mathcal{F}_n$ is symmetrical by the glide reflection σ. Moreover, $\mathcal{F}_n$ consists of integers if and only if $a_0, \ldots, a_{n-4}, a_{n-3}$ are integers.

Let $\mathcal{P}_n$ be a regular n-gon with vertices $0, \ldots, n-1$. A triangulation T of $\mathcal{P}_n$ is a maximal set of non-crossing diagonals of $\mathcal{P}_n$, whose cardinality is always equal to $n-3$. Such a pair $(\mathcal{P}_n, T)$ is called a triangulated n-gon. Let a_r denote the number of triangles at vertex r with respect to some triangulation T. Then

$$\ldots \quad a_0 \quad a_1 \ldots a_{n-1} \quad a_0 \quad \ldots$$

is the second row of a frieze pattern of integers. Furthermore, the frieze patterns of positive integers of order n are in bijection with triangulated n-polygons.

Associated with an acyclic quiver Q, the authors observed in [23] a generalized version of Coxeter-Conway's frieze patterns. The elements of the generalized frieze pattern $\mathcal{F}_Q$ associated with Q are cluster variables in the cluster algebra $\mathcal{A}_Q$. Moreover, the sequences in $\mathcal{F}_Q$ satisfy linear recurrence relations if and only if Q is of Dynkin or affine type (see [35, 5, 61]). Of course, there are more connections between frieze patterns and cluster algebras (see for instance [36, 4, 10]).

The *tropical semifield* $(\mathbb{Z}, \odot, \oplus)$ is the set $\mathbb{Z}$ of integers with multiplication and addition given by

$$a \odot b = a + b, \qquad a \oplus b = \max\{a, b\}.$$

Notice that the unit in the tropical semifield with respect to the given multiplication is the number 0.

If we view the unimodular rule as an equation in the tropical semifield, then it becomes

$$a + d = \max\{b + c, 0\},$$

which is deduced from

$$a \odot d = a + d \quad \text{and} \quad (b \odot c) \oplus 1 = \max\{b + c, 0\}.$$

Example 5.2.1. One can easily check that for every adjacent numbers a, b, c, d forming a diamond shape with a left and d right in the following diagram

$$\begin{array}{cccccccccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \cdots & & & \\
& \cdots & 2 & & 1 & & 1 & & -1 & & 4 & & -2 & & 2 & & 1 & \cdots & & \\
& & \cdots & 3 & & 2 & & -2 & & 3 & & 2 & & -2 & & 3 & & 2 & \cdots & \\
& & & \cdots & 4 & & -2 & & 2 & & 1 & & 1 & & -1 & & 4 & & -2 & \cdots \\
& & & & \cdots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \; \cdots
\end{array}$$

the relation $a + d = \max\{b + c, 0\}$ is satisfied. Notice that if we omit the first and last rows which are filled with 0, nothing will change. We call such a diagram a *tropicalized frieze pattern* of order 6. This diagram is also periodic with period a divisor of 6, it is also invariant under the same glide reflection σ (as frieze patterns). In fact, this is a general phenomenon: every tropicalized frieze pattern of order n is periodic. We will explain this fact right after Proposition 5.3.4.

In the following, we will study tropical friezes on 2-Calabi-Yau triangulated categories, especially on the cluster categories associated with Dynkin diagrams. As we will see after Proposition 5.3.4, this generalizes the above tropicalization of frieze patterns of integers.

5.2.2 Definitions and first properties

Let k be an algebraically closed field. Let $\mathcal{C}$ be a k-linear triangulated category with suspension functor Σ where all idempotents split. We further assume that the category $\mathcal{C}$

a) is Hom-finite, *i.e.* the morphism space $\mathcal{C}(X, Y)$ is finite-dimensional for any two objects X, Y in $\mathcal{C}$ (which implies that $\mathcal{C}$ is Krull-Schmidt);

b) is 2-Calabi-Yau, *i.e.* there exist bifunctorial isomorphisms

$$D\mathcal{C}(X, Y) \simeq \mathcal{C}(Y, \Sigma^2 X), \quad X, Y \in \mathcal{C},$$

where D denotes the duality functor $\operatorname{Hom}_k(?, k)$;

c) admits a cluster-tilting object T, *i.e.*

i) T is rigid (that is, $\mathcal{C}(T, \Sigma T) = 0$), and T is basic (that is, its indecomposable direct summands are pairwise non-isomorphic),

ii) for each object X of $\mathcal{C}$, if $\mathcal{C}(T, \Sigma X)$ vanishes, then X belongs to the subcategory $\mathrm{add}T$ of direct summands of finite direct sums of copies of T.

If a category $\mathcal{C}$ satisfies all these assumptions, we say that $\mathcal{C}$ is a *2-Calabi-Yau category with cluster-tilting object*. A typical class of such categories is the class of cluster categories [17] of connected finite acyclic quivers. Throughout this article, our category $\mathcal{C}$ is always a 2-Calabi-Yau category with cluster-tilting object.

Definition 5.2.2. A *tropical frieze* on $\mathcal{C}$ with values in the integer ring $\mathbb{Z}$ is a map

$$f : obj(\mathcal{C}) \to \mathbb{Z}$$

such that

d1) $f(X) = f(Y)$ if X and Y are isomorphic,

d2) $f(X \oplus Y) = f(X) + f(Y)$ for all objects X and Y,

d3) for all objects L and M such that $\mathrm{dimExt}^1_{\mathcal{C}}(L, M) = 1$, the equality

$$f(L) + f(M) = \max\{f(E), f(E')\}$$

holds, where E and E' are the middle terms of the non-split triangles

$$L \to E \to M \to \Sigma L \quad \text{and} \quad M \to E' \to L \to \Sigma M$$

with end terms L and M.

Let f and g be two tropical friezes on the same category $\mathcal{C}$. The sum $f + g$ clearly satisfies items d1) and d2). For item d3), we have that

$$\begin{aligned}(f+g)(L) + (f+g)(M) &= (f(L) + f(M)) + (g(L) + g(M)) \\ &= \max\{f(E), f(E')\} + \max\{g(E), g(E')\}.\end{aligned}$$

Then $f + g$ is a tropical frieze if and only if for all pairs (E, E') as in item d3) the equality

$$\max\{f(E), f(E')\} + \max\{g(E), g(E')\} = \max\{(f+g)(E), (f+g)(E')\}$$

holds. Notice that for two integers a, b, the number

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}.$$

Thus, the sum $f + g$ is a tropical frieze if and only if for all pairs (E, E') as in item d3) the equality

$$|f(E) - f(E')| + |g(E) - g(E')| = |(f(E) - f(E')) + (g(E) - g(E'))|$$

holds, if and only if the inequality

$$(f(E) - f(E'))(g(E) - g(E')) \geq 0$$

holds. If two tropical friezes satisfy such a property, then we say that they are *compatible*.

Now we state a simple property of tropical friezes.

Proposition 5.2.3. *Let $f_1, \ldots, f_n$ be tropical friezes on the same category $\mathcal{C}$. Then the sum $\sum_i f_i$ is a tropical frieze if and only if the functions f_i are pairwise compatible.*

Proof. This statement is a trivial generalization of the above analysis:

the sum $\sum_i f_i$ is a tropical frieze if and only if for all pairs (E, E') as in item d3) the equality

$$\sum_i |f_i(E) - f_i(E')| = |\sum_i (f_i(E) - f_i(E'))|$$

holds, if and only if $f_i(E) - f_i(E')$ are simultaneously non-negative or simultaneously non-positive for all integers $1 \leq i \leq n$, if and only if the tropical friezes f_i are pairwise compatible. □

Similarly, one can obtain that the difference $f - g$ is a tropical frieze if and only if for all pairs (E, E') as in item d3) the equality

$$|f(E) - f(E')| - |g(E) - g(E')| = |(f(E) - f(E')) - (g(E) - g(E'))|$$

holds, if and only of the inequalities

$$|f(E) - f(E')| \geq |g(E) - g(E')| \quad \text{and} \quad (f(E) - f(E'))(g(E) - g(E')) \geq 0$$

hold. If two tropical friezes satisfy such a property, then we say that they are *strongly compatible.*

Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q. For any indecomposable object X of $\mathcal{C}_Q$, the space $\mathrm{Hom}_{\mathcal{C}_Q}(X, X)$ is one-dimensional, so we have that $\dim\mathrm{Ext}^1_{\mathcal{C}_Q}(\Sigma X, X) = 1$. The associated non-split triangles are of the following form

$$\Sigma X \to E \xrightarrow{g} X \to \Sigma^2 X \quad \text{and} \quad X \to 0 \xrightarrow{g'} \Sigma X \to \Sigma X, \qquad (*)$$

where g' denotes the zero morphism in $\mathcal{C}_Q$ from the object 0 to the object ΣX.

The following proposition is quite similar to the statements for cluster-additive functions on stable translation quivers given in Section 1 of [74].

Proposition 5.2.4. *Let Q be a Dynkin quiver. Then any tropical frieze on $\mathcal{C}_Q$ which takes only non-positive values or only non-negative values is the zero function.*

Proof. Let f be a non-zero tropical frieze on $\mathcal{C}_Q$ with non-positive values and X an indecomposable object such that $f(X) < 0$. From the non-split triangles $(*)$ above, we have that

$$f(\Sigma X) = \max\{f(E), 0\} - f(X) \geq 0 - f(X) > 0,$$

which is a contradiction. Therefore, any tropical frieze with only non-positive values is the zero function.

Let f be a tropical frieze on $\mathcal{C}_Q$ with non-negative values. We lift f in the natural way to a function f' which is $(\tau^{-1}\Sigma)$-invariant on the bounded derived category $\mathcal{D}_Q$ of the category $\mathrm{mod}kQ$. Here τ is the Auslander-Reiten translation on $\mathcal{D}_Q$. Denote by ϕ the canonical equivalence [43] from the mesh category of the translation quiver $\mathbb{Z}Q$ to the full subcategory $\mathrm{ind}(\mathcal{D}_Q)$ of indecomposables of $\mathcal{D}_Q$. We define a function f'' on $\mathbb{Z}Q$ by setting $f'' = f'\phi$. Let z be any vertex of $\mathbb{Z}Q$. In $\mathcal{D}_Q$ we have the Auslander-Reiten triangle [44] as follows

$$\phi(\tau z) \to \bigoplus_{y \to z} \phi(y) \to \phi(z) \to \Sigma\phi(\tau z),$$

where '$y \to z$' in the middle term are arrows in $\mathbb{Z}Q$. Its image (still use the same notation) in $\mathcal{C}_Q$ is a non-split triangle. The other non-split triangle with end terms $\phi(z)$ and $\phi(\tau z)$ in $\mathcal{C}_Q$ is

$$\phi(z) \to 0 \to \phi(\tau z) \xrightarrow{\simeq} \Sigma\phi(z).$$

Hence, we can deduce that

$$\begin{aligned} f''(\tau z) + f''(z) &= f'(\phi(\tau z)) + f'(\phi(z)) = \max\{\sum_{y \to z} f'(\phi(y)), 0\} \\ &= \sum_{y \to z} f'(\phi(y)) = \sum_{y \to z} f''(y). \end{aligned}$$

As a consequence, the function f'' is an additive function on $\mathbb{Z}Q$ with non-negative values, which implies that f'' is the zero function [45]. Therefore, the function f is the zero function on $\mathcal{C}_Q$. □

5.2.3 Cluster characters and tropical friezes

In this subsection, we will see some links between cluster characters and tropical friezes.

Let d2′) denote the item obtained from item d2) in Definition 5.2.2 in which the equality becomes $f(X \oplus Y) = f(X)f(Y)$, and d3′) the item obtained from item d3) in Definition 5.2.2 in which the equality becomes $f(L)f(M) = f(E) + f(E')$. A map $\chi : obj(\mathcal{C}) \to A$, where A is a commutative ring, is called a *cluster character* in [69] if it satisfies items d1), d2′) and d3′).

Remark 5.2.5. Let χ be a cluster character mapping from $\mathcal{C}$ to the tropical semifield $(\mathbb{Z}, \odot, \oplus)$. Then we obtain the following equalities

$$\chi(X \oplus Y) = \chi(X) \odot \chi(Y) = \chi(X) + \chi(Y),$$

$$\chi(L) + \chi(M) = \chi(L) \odot \chi(M) = \chi(E) \oplus \chi(E') = \max\{\chi(E), \chi(E')\}.$$

As a result, the map χ is a tropical frieze mapping to the integer ring $\mathbb{Z}$.

Let Q be a connected finite acyclic quiver with vertex set $\{1, \ldots, n\}$ and $\mathcal{C}_Q$ its associated cluster category. It was proved, in [26] for Dynkin quivers and in [25] for acyclic quivers, that the Caldero-Chapoton map

$$CC : obj(\mathcal{C}_Q) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

defined in [23] is a cluster character.

Example 5.2.6. Let X be an object of $\mathcal{C}_Q$. Then the image $CC(X)$ can be written uniquely as

$$CC(X) = \frac{h(x_1, \ldots, x_n)}{\prod_{i=1}^{n} x_i^{d_i(X)}},$$

where the polynomial $h(x_1, \ldots, x_n)$ is not divisible by any x_i, $1 \le i \le n$. Look at the function

$$d_i : obj(\mathcal{C}_Q) \to \mathbb{Z}$$

with $d_i(X)$ given as in the above expression for each object X of $\mathcal{C}_Q$.

We use elementary properties of polynomials. From the equality

$$CC(X \oplus Y) = CC(X)CC(Y)$$

in item d2′), one clearly sees that $d_i(X \oplus Y) = d_i(X) + d_i(Y)$. It is also not hard to calculate the denominators of the two hand sides of the equality

$$CC(L)CC(M) = CC(E) + CC(E')$$

in item d3′), which gives us the equality $d_i(L) + d_i(M) = \max\{d_i(E), d_i(E')\}$. Therefore, each function d_i is a tropical frieze on $\mathcal{C}_Q$.

However, the sum $d_i + d_j$ is not always a tropical frieze on $\mathcal{C}_Q$. We choose the linear orientation of A_3. The Auslander-Reiten quiver of the cluster category $\mathcal{C}_{\vec{A}_3}$ is

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & P_3 \arrow[dr] & & \Sigma P_1 \arrow[dr] & & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & \Sigma P_2 \arrow[dr] & & \\
P_1 \arrow[ur] & & S_2 \arrow[ur] & & S_3 \arrow[ur] & & \Sigma P_3 &
\end{tikzcd}$$

where P_i (resp. I_i, S_i) is the right projective (resp. injective, simple) $k\vec{A}_3$-module associated with vertex i. By definition $CC(\Sigma P_2) = x_2$ and one can calculate that

$$CC(P_1) = \frac{1+x_2}{x_1},\ CC(S_3) = \frac{1+x_2}{x_3},\ CC(P_3) = \frac{x_1 + x_1x_2 + x_3 + x_2x_3}{x_1x_2x_3}.$$

The space $\mathrm{Ext}^1_{\mathcal{C}_{\vec{A}_3}}(\Sigma P_2, P_3)$ is 1-dimensional and the non-split triangles are

$$\Sigma P_2 \to P_1 \to P_3 \to I_2 \quad \text{and} \quad P_3 \to S_3 \to \Sigma P_2 \to \Sigma P_3.$$

Consider the function $d_1 + d_3$. We have that

$$(d_1 + d_3)(\Sigma P_2) + (d_1 + d_3)(P_3) = (0 + 0) + (1 + 1) = 2$$
$$\max\{(d_1 + d_3)(P_1), (d_1 + d_3)(S_3)\} = \max\{1 + 0, 0 + 1\} = 1.$$

Thus, the sum $d_1 + d_3$ is not a tropical frieze. In another way, since

$$(d_1(P_1) - d_1(S_3))(d_3(P_1) - d_3(S_3)) = (1 - 0)(0 - 1) = -1 < 0,$$

the tropical friezes d_1 and d_3 are not compatible. As a consequence, the difference $d_1 - d_3$ is not a tropical frieze on $\mathcal{C}_{\vec{A}_3}$ either.

Let T be a cluster-tilting object of $\mathcal{C}$ and T_1 an indecomposable direct summand of T. Iyama and Yoshino proved in [48] that, up to isomorphism, there is a unique indecomposable object T_1^* not isomorphic to T_1 such that the object $\mu_1(T)$ obtained from T by replacing the indecomposable direct summand T_1 with T_1^* is cluster-tilting. We call $\mu_1(T)$ the *mutation* of T at T_1. There are non-split triangles (namely, *exchange triangles*), unique up to isomorphism,

$$T_1^* \to E \xrightarrow{f} T_1 \to \Sigma T_1^* \quad \text{and} \quad T_1 \xrightarrow{g} E' \to T_1^* \to \Sigma T_1$$

such that f is a minimal right $\mathrm{add}(T/T_1)$-approximation and g a minimal left $\mathrm{add}(T/T_1)$-approximation.

Lemma 5.2.7 ([55]). *The quiver of the endomorphism algebra of T does not have a loop at the vertex corresponding to T_1 if and only if we have* $\dim \mathrm{Ext}^1_{\mathcal{C}}(T_1, T_1^*) = 1$ *if and only if* $\mathrm{Ext}^1_{\mathcal{C}}(T_1, T_1^*)$ *is a simple module over* $\mathcal{C}(T_1, T_1)$*. In this case, in the exchange triangles, we have*

$$E = \bigoplus_{i \to 1} T_i \quad \textit{and} \quad E' = \bigoplus_{1 \to j} T_j.$$

We say that the mutation of T at T_1 is *simple* if the conditions of the above lemma hold. A category $\mathcal{C}$ is said to be *cluster-transitive* if any two basic cluster-tilting objects of $\mathcal{C}$ can be obtained from each other by a finite sequence of simple mutations.

The following property of tropical friezes on a cluster-transitive category $\mathcal{C}$ is quite similar to that [69] of cluster characters on $\mathcal{C}$.

Proposition 5.2.8. *Let $\mathcal{C}$ be a cluster-transitive category and $T = T_1 \oplus \ldots \oplus T_n$ a basic cluster-tilting object of $\mathcal{C}$ with T_i indecomposable. Suppose that f and g are two tropical friezes on $\mathcal{C}$ such that $f(T_i) = g(T_i)$, $1 \leq i \leq n$. Then f and g coincide on all subcategories* $\mathrm{add}T'$*, where T' is any cluster-tilting object of $\mathcal{C}$.*

Proof. By assumption we know that f and g coincide on all indecomposable direct summands of T. We will prove this proposition by recursion on the minimal number of mutations linking a basic cluster-tilting object to T.

Now let $T' = T_1' \oplus \ldots \oplus T_n'$ be a basic cluster-tilting object with T_i' indecomposable satisfying that $f(T_i') = g(T_i')$ for all integers $1 \leq i \leq n$. Assume that $T'' = \mu_1(T') = T_1^{''} \oplus T_2' \oplus \ldots \oplus T_n'$ is the mutation of T' in direction 1. Then we have the non-split triangles

$$T_1^{''} \to E \to T_1' \to \Sigma T_1^{''} \quad \text{and} \quad T_1' \to E' \to T_1^{''} \to \Sigma T_1'$$

with middle terms E and E' both belonging to $\mathrm{add}(T'/T_1')$. Hence, the following equlities

$$f(T_1^{''}) = \max\{f(E), f(E')\} - f(T_1') = \max\{g(E), g(E')\} - g(T_1') = g(T_1^{''})$$

hold. This completes the proof. $\square$

Let $\mathcal{C}_Q$ be the cluster category of a connected finite acyclic quiver Q. It was shown in [17] that $\mathcal{C}_Q$ is cluster-transitive and any rigid indecomposable object of $\mathcal{C}_Q$ is a direct summand of a cluster-tilting object. If f and g are two tropical friezes on $\mathcal{C}_Q$ which coincide on all indecomposable direct summands of some cluster-tilting object, by Proposition 5.2.8, they coincide on all rigid objects. In particular, when Q is Dynkin, the two tropical friezes f and g are equal.

5.3 Tropical friezes from indices

5.3.1 Reminder on indices

Let X be an object of $\mathcal{C}$ and T a cluster-tilting object of $\mathcal{C}$. Following [59], we have triangles

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \quad \text{and} \quad X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1 \to \Sigma X,$$

where T_1^X, T_0^X, T_X^0 and T_X^1 belong to $\mathrm{add}T$. Recall that the index and coindex of X with respect to T are defined to be the classes in the split Grothendieck group $K_0(\mathrm{add}T)$ of the additive category $\mathrm{add}T$ as follows

$$\mathrm{ind}_T(X) = [T_0^X] - [T_1^X] \quad \text{and} \quad \mathrm{coind}_T(X) = [T_X^0] - [T_X^1],$$

which do not depend on the choices of the above triangles.

Assume that T is the direct sum of n pairwise non-isomorphic indecomposable objects $T_1, \ldots, T_n$. Let B be the endomorphism algebra of T over $\mathcal{C}$. We denote the indecomposable right projective B-module $\mathcal{C}(T, T_i)$ by P_i and denote its simple top by S_i.

Let $K_0^{sp}(\mathrm{mod}B)$ denote the split Grothendieck group of the abelian category $\mathrm{mod}B$ of finite-dimensional right B-modules, that is, the quotient of the free abelian group on the set of isomorphism classes of finite-dimensional right B-modules, modulo the subgroup generated by all elements

$$[X \oplus Y] - [X] - [Y].$$

Define a bilinear form

$$\langle\, , \,\rangle : K_0^{sp}(\mathrm{mod}B) \times K_0^{sp}(\mathrm{mod}B) \to \mathbb{Z}$$

by setting

$$\langle X, Y\rangle = \dim \mathrm{Hom}_B(X, Y) - \dim \mathrm{Ext}^1_B(X, Y)$$

for all finite-dimensional B-modules X and Y. In particular, if X is a projective B-module, then

$$\langle X, Y\rangle = \dim \mathrm{Hom}_B(X, Y),$$

in this case, the linear form $\langle X, ?\rangle$ on $K_0^{sp}(\mathrm{mod}B)$ induces a well-defined form

$$\langle X, ?\rangle : K_0(\mathrm{mod}B) \to \mathbb{Z},$$

where $K_0(\mathrm{mod}B)$ is the Grothendieck group of $\mathrm{mod}B$. Define an antisymmetric bilinear form on $K_0^{sp}(\mathrm{mod}B)$ by setting

$$\langle X, Y\rangle_a = \langle X, Y\rangle - \langle Y, X\rangle$$

for all finite-dimensional B-modules X and Y. In [69] Palu has proved that the antisymmetric bilinear form $\langle\, , \,\rangle_a$ descends to the Grothendieck group $K_0(\mathrm{mod}B)$.

Let F denote the functor $\mathcal{C}(T, ?)$. It was shown in [59] that F induces an equivalence of categories

$$\mathcal{C}/\mathrm{add}(\Sigma T) \xrightarrow{\simeq} \mathrm{mod}B.$$

Moreover, this functor F sends the objects in $\mathrm{add}T$ to finite-dimensional projective B-modules.

Let m be a class in $K_0(\mathrm{mod}B)$. We define a function $f_{T,m}$ from $\mathcal{C}$ to $\mathbb{Z}$ as

$$f_{T,m}(X) = \langle F(\mathrm{ind}_T(X)), m\rangle, \quad X \in \mathcal{C}.$$

When it does not cause confusion, we simply write $\mathrm{ind}(X)$ instead of $\mathrm{ind}_T(X)$.

5.3.2 Tropical friezes

In this subsection, we will give a sufficient condition for the function $f_{T,m}$ to be a tropical frieze on $\mathcal{C}$. Moreover, when $\mathcal{C} = \mathcal{C}_Q$ the cluster category of a Dynkin quiver Q, we will see that this sufficient condition is also a necessary condition.

Theorem 5.3.1. *Assume that $\langle S_i, m\rangle_a \geq 0$ for each simple B-module $S_i\,(1 \leq i \leq n)$. Then the function $f_{T,m}$ is a tropical frieze.*

Proof. The function $f_{T,m}$ clearly satisfies the terms d1) and d2) in Definition 5.2.2. Now Let L and M be objects of $\mathcal{C}$ such that $\mathrm{dimExt}^1_{\mathcal{C}}(L, M) = 1$. Let

$$L \xrightarrow{h} E \xrightarrow{g} M \to \Sigma L \quad \text{and} \quad M \xrightarrow{h'} E' \xrightarrow{g'} L \to \Sigma M$$

be the associated non-split triangles.

First, let $C \in \mathcal{C}$ be any lift of $\mathrm{Coker}(Fg)$. We know from [69] that

$$\begin{aligned} \mathrm{ind}(E) &= \mathrm{ind}(L) + \mathrm{ind}(M) - \mathrm{ind}(C) - \mathrm{ind}(\Sigma^{-1}C) \quad \text{and} \\ \langle FC, m\rangle_a &= \langle F(\mathrm{ind}(C)), m\rangle + \langle F(\mathrm{ind}(\Sigma^{-1}C)), m\rangle. \end{aligned}$$

By assumption $\langle S_i, m\rangle_a \geq 0$ for each simple B-module $S_i\,(1 \leq i \leq n)$. So we have that $\langle FC, m\rangle_a \geq 0$. Thus,

$$\begin{aligned} \langle F(\mathrm{ind}(E)), m\rangle &= \langle F(\mathrm{ind}(L)), m\rangle + \langle F(\mathrm{ind}(M)), m\rangle - \langle FC, m\rangle_a \\ &\leq \langle F(\mathrm{ind}(L)), m\rangle + \langle F(\mathrm{ind}(M)), m\rangle. \end{aligned}$$

Similarly, we obtain another inequality

$$\langle F(\mathrm{ind}(E')), m\rangle \leq \langle F(\mathrm{ind}(L)), m\rangle + \langle F(\mathrm{ind}(M)), m\rangle.$$

It follows that

$$\max\{f_{T,m}(E), f_{T,m}(E')\} \leq f_{T,m}(L) + f_{T,m}(M).$$

Second, we consider the identity maps $id_M : M \to M$ and $id_L : L \to L$. Thanks to the dichotomy phenomenon shown in [69], exactly one of the conditions $FM = (Fg)(FE)$ and $FL = (Fg')(FE')$ is true. Assume that the first condition holds, then Fg is an epimorphism and FC vanishes. Therefore, we have that

$$\begin{aligned} f_{T,m}(E) = \langle F(\mathrm{ind}(E)), m\rangle &= \langle F(\mathrm{ind}(L)), m\rangle + \langle F(\mathrm{ind}(M)), m\rangle \\ &= f_{T,m}(L) + f_{T,m}(M). \end{aligned}$$

As a consequence, the equality

$$f_{T,m}(L) + f_{T,m}(M) = \max\{f_{T,m}(E), f_{T,m}(E')\}$$

holds and $f_{T,m}$ is a tropical frieze. □

5.3.3 Another proof

For $L \in \mathcal{C}$ and $e \in \mathbb{N}^n$, we denote by $Gr_e(\mathrm{Ext}^1_{\mathcal{C}}(T, L))$ the quiver Grassmannian of B-submodules of the B-module $\mathrm{Ext}^1_{\mathcal{C}}(T, L)$ whose dimension vector is e and we denote by $\chi(Gr_e(\mathrm{Ext}^1_{\mathcal{C}}(T, L)))$ its Euler-Poincaré characteristic for étale cohomology with proper support.

For $1 \leq i \leq n$, we define the integer $g_i(L)$ to be the multiplicity of $[T_i]$ in the index $\mathrm{ind}(L)$ and define the element X'_L of the field $\mathbb{Q}(x_1, \ldots, x_n)$ by

$$X'_L = \prod_{i=1}^{n} x_i^{g_i(L)} \sum_e \chi(Gr_e(\mathrm{Ext}^1_{\mathcal{C}}(T, L))) \prod_{i=1}^{n} x_i^{\langle S_i, e\rangle_a},$$

where the sum ranges over all tuples $e \in \mathbb{N}^n$. This is a vastly generalized form of the CC map. It was proved in [69] that the function $X'_?$ is a cluster character from $\mathcal{C}$ to $\mathbb{Q}(x_1, \ldots, x_n)$. If we define functions d_i on $\mathcal{C}$ as in Example 5.2.6 by replacing CC map with $X'_?$, then each function d_i is also a tropical frieze.

We will use the tropical semifield $(\mathbb{Z}, \odot, \oplus)$ to give another proof of Theorem 5.3.1 for $\mathcal{C} = \mathcal{C}_Q$ where Q is a Dynkin quiver with n vertices. Let T be a cluster-tilting object of $\mathcal{C}$ and B its endomorphism algebra. Notice that any indecomposable object of $\mathcal{C}$ is a direct summand of some cluster-tilting object which is obtained from T by a finite sequence of mutations. Since $X'_{T_i} = x_i$ and $X'_?$ is a cluster character, the image X'_L lies in the universal semifield $\mathbb{Q}_{sf}(x_1, \ldots, x_n)$ (Section 2.1 in [12]). For an element $m \in K_0(\mathrm{mod}B)$, we define the map

$$\varphi_m : \mathbb{Q}_{sf}(x_1, \ldots, x_n) \longrightarrow (\mathbb{Z}, \odot, \oplus)$$

as the unique homomorphism between semifields which takes $x_i = X'_{T_i}$ to the integer $\langle F(\mathrm{ind}(T_i)), m\rangle$. Then the composition $\varphi_m X'_?$ is a cluster character from $\mathcal{C}$ to $(\mathbb{Z}, \odot, \oplus)$ and thus a tropical frieze from $\mathcal{C}$ to the integer ring $\mathbb{Z}$ by Remark 5.2.5. When $\mathcal{C} = \mathcal{C}_Q$ with Q a Dynkin quiver, Nakajima [67] showed that $\chi(Gr_e(\mathrm{Ext}^1_{\mathcal{C}}(T, L)))$ is a non-negative integer. Now we write down explicitly the function

$$\begin{aligned}
\varphi_m X'_L &= \max_e \{\sum_{i=1}^n (g_i(L) + \langle S_i, e\rangle_a)\langle F(\mathrm{ind}(T_i)), m\rangle\} \\
&= \max_e \{\langle F(\mathrm{ind}(L)), m\rangle + \sum_{i=1}^n \langle S_i, e\rangle_a \langle F(\mathrm{ind}(T_i)), m\rangle\} \\
&= \max_e \{\langle F(\mathrm{ind}(L)), m\rangle - \sum_{i=1}^n \langle (g_i(\Sigma^{-1}Y) + g_i(Y))\langle F(\mathrm{ind}(T_i)), m\rangle\} \\
&= \max_e \{\langle F(\mathrm{ind}(L)), m\rangle - \langle F(\mathrm{ind}(\Sigma^{-1}Y) + \mathrm{ind}(Y)), m\rangle\} \\
&= \max_e \{\langle F(\mathrm{ind}(L)), m\rangle - \langle FY, m\rangle_a\} \\
&= \max_e \{\langle F(\mathrm{ind}(L)), m\rangle - \sum_{i=1}^n e_i \langle S_i, m\rangle_a\}
\end{aligned}$$

where e ranges over all elements in $K_0(\mathrm{mod}B)$ such that $\chi(Gr_e(\mathrm{Ext}^1_{\mathcal{C}}(T, L)))$ is non zero and Y is an object of $\mathcal{C}$ satisfying $FY = e = (e_i)_i \in K_0(\mathrm{mod}B)$. If $\langle S_i, m\rangle_a \geq 0$ for each simple B-module S_i, then we have that

$$\varphi_m X'_L = \langle F(\mathrm{ind}(L)), m\rangle = f_{T,m}(L).$$

Thus, the function $f_{T,m}$ is equal to $\varphi_m X'_?$ and is a tropical frieze.

Remark 5.3.2. Let $\mathcal{C}_Q$ be the cluster category associated with a Dynkin quiver Q. Let T be a cluster-tilting object of $\mathcal{C}_Q$ and B its endomorphism algebra. Let F be the functor $\mathrm{Hom}_{\mathcal{C}_Q}(T, ?)$. In fact, the sufficient condition for a function $f_{T,m}$ to be a tropical frieze in Theorem 5.3.1 is also a necessary condition in this situation.

For any indecomposable object X of $\mathcal{C}_Q$, look at the second triangle associated with X in $(*)$ before Proposition 5.2.4, whose image under F is

$$FX \to 0 \xrightarrow{Fg'} F(\Sigma X) \xrightarrow{\simeq} F(\Sigma X).$$

We have that $Fg' = 0$ and $\mathrm{Coker}(Fg') = F(\Sigma X)$. If X does not belong to $\mathrm{add}T$, then $\mathrm{Coker}(Fg')$ is not zero which implies that $\mathrm{Coker}(Fg)$ vanishes by the dichotomy phenomenon. Let m be a class in $K_0(\mathrm{mod}B)$. From the proof of Theorem 5.3.1, we know that

$$f_{T,m}(E) = f_{T,m}(\Sigma X) + f_{T,m}(X) = \langle F(\Sigma X), m\rangle_a.$$

Assume that $f_{T,m}$ is a tropical frieze. Then it follows that

$$f_{T,m}(\Sigma X) + f_{T,m}(X) = \max\{f_{T,m}(E), 0\} \geq 0.$$

Thus, for every indecomposable object $X \notin \mathrm{add}T$, the value $\langle F(\Sigma X), m\rangle_a$ is non-negative, particularly when $F(\Sigma X)$ is a simple B-module S_i.

Example 5.3.3. Let Q be an acyclic quiver and j a sink of Q (that is, no arrows of Q starting at j). Let T be the image of kQ in $\mathcal{C}_Q$ under the canonical inclusion and B its endomorphism algebra $\mathrm{End}_{\mathcal{C}_Q}(T)$. For each simple B-module S_i, we have that

$$\begin{aligned}\langle S_i, S_j\rangle_a &= -\mathrm{dim}\mathrm{Ext}^1_B(S_i, S_j) + \mathrm{dim}\mathrm{Ext}^1_B(S_j, S_i) = \mathrm{dim}\mathrm{Ext}^1_B(S_j, S_i)\\ &= \text{the number of arrows from } i \text{ to } j \text{ in } Q\,(\geq 0).\end{aligned}$$

As an application of Theorem 5.3.1, the function f_{T,S_j} is a tropical frieze.

Similarly, if j is a source of an acyclic quiver Q (that is, no arrows of Q ending at j), then $f_{T,-S_j}$ is a tropical frieze.

Using a similar method as in the second proof of Theorem 5.3.1, it is not hard to get the following proposition:

Proposition 5.3.4. *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q and $T = T_1 \oplus \ldots \oplus T_n$ a basic cluster-tilting object of $\mathcal{C}_Q$ with T_i indecomposable. Then the map*

$$\Phi_T : \{\textit{tropical friezes on } \mathcal{C}_Q\} \longrightarrow \mathbb{Z}^n$$

given by $\Phi_T(f) = (f(T_1), \ldots, f(T_n))$ is a bijection.

Proof. For any fixed n-tuple $\underline{a} = (a_1, \ldots, a_n)$ in $\mathbb{Z}^n$, there is a unique homomorphism of semifields

$$\phi_{\underline{a}} : \mathbb{Q}_{sf}(x_1, \ldots, x_n) \longrightarrow (\mathbb{Z}, \odot, \oplus)$$

such that $\phi_{\underline{a}}(x_i) = a_i$. We denote the composition $\phi_{\underline{a}} X'_?$ by $f_{\underline{a}}$. Then $f_{\underline{a}}$ is a tropical frieze on $\mathcal{C}_Q$ satisfying $f_{\underline{a}}(T_i) = a_i$. Therefore, the map Φ_T is a surjection. The injectivity follows from Proposition 5.2.8. Hence, the map Φ_T is bijective. □

Now we give an explanation of the periodicity phenomenon which is stated at the end of subsection 2.1. Let $\mathcal{F}_n^t$ be a tropicalized frieze pattern of order $n(> 3)$. Let Q be a quiver of type A_{n-3}. Then $\mathcal{F}_n^t$ gives a function (denoted by f) on the Auslander-Reiten quiver Γ of $\mathcal{D}_Q$. Each subquiver (y or z may not appear)

$$\begin{tikzcd} & y \arrow[dr] & \\ \tau x \arrow[ur] \arrow[dr] & & x \\ & z \arrow[ur] & \end{tikzcd}$$

in Γ induces an Auslander-Reiten triangle in $\mathcal{D}_Q$

$$\tau x \to y \oplus z \to x \to \Sigma\tau x.$$

Since $\mathcal{F}_n^t$ is a tropicalized frieze pattern, the function f satisfies that

$$f(\tau x) + f(x) = \max\{f(y) + f(z), 0\}.$$

Let $\mathcal{S}$ be any slice in Γ. Set $T = \bigoplus_{y\in\mathcal{S}} y$. Then the image of T is a basic cluster-tilting object of $\mathcal{C}_Q$. By Proposition 5.3.4, there exists a unique tropical frieze $g : \mathcal{C}_Q \to \mathbb{Z}$ such that $g(y) = f(y)$ for all $y \in \mathcal{S}$. We extend g in a natural way to a $(\tau^{-1}\Sigma)$-invariant function on $\mathcal{D}_Q$ (still denote as g). Then g also satisfies the above equation as f. Therefore, the two functions f and g are equal. Moreover, for each integer i, we have that

$$f((\tau^{-1}\Sigma)^i x) = g((\tau^{-1}\Sigma)^i x) = g(x) = f(x) \quad \text{and}$$

$$f((\tau^{-n})^i x) = g((\tau^{-n})^i x) = g((\tau^{-2}\tau^{2-n})^i x) = g((\tau^{-1}\Sigma)^{2i} x) = g(x).$$

In conclusion, $\mathcal{F}_n^t$ is periodic with period a divisor of n, and it is invariant under the glide reflection σ.

5.3.4 Cluster-hammock functions and tropical friezes

In this subsection, we will see that the cluster-hammock functions defined by Ringel [74] always give rise to tropical friezes, while their sums do not, even for pairwise 'compatible' (in the sense of Ringel) cluster-hammock functions.

Let $\Gamma = \mathbb{Z}Q$ be the translation quiver of a Dynkin quiver Q. For any vertex x of Γ, Ringel [74] defined the *cluster-hammock function* $h_x : \Gamma_0 \to \mathbb{Z}$ by the following properties

a) $h_x(x) = -1$;

b) $h_x(y) = 0$ for $y \neq x \in \mathcal{S}$, where $\mathcal{S}$ is any slice containing x;

c) $h_x(z) + h_x(\tau z) = \sum_{y \to z} \max\{h_x(y), 0\}$ for all $z \in \Gamma_0$.

As shown in [74], the cluster-hammock function h_x is $(\tau^{-1}\Sigma)$-invariant and takes the value -1 on the $(\tau^{-1}\Sigma)$-orbit of x while it takes non-negative values on the other vertices. Thus, h_x naturally induces a well-defined function on $\mathrm{ind}(\mathcal{C}_Q)$, which we still denote as h_x on $\mathrm{ind}(\mathcal{C}_Q)$. We extend h_x to a function defined on $\mathcal{C}_Q$ by requiring that $h_x(X \oplus Y) = h_x(X) + h_x(Y)$ for all objects X, Y of $\mathcal{C}_Q$. Let $\mathcal{S}'_x$ be the slice in $\mathbb{Z}Q$ with x its unique sink and $\mathcal{S}''_x$ the slice in $\mathbb{Z}Q$ with x its unique source.

Let Z be an indecomposable object of $\mathcal{C}_Q$. If there is an arrow from x to Z in the Auslander-Reiten quiver of $\mathcal{C}_Q$, then Z and τZ both lie in the $(\tau^{-1}\Sigma)$-orbit of the convex hull of $\mathcal{S}'_x$ and $\mathcal{S}''_x$. Thus, both $h_x(Z)$ and $h_x(\tau Z)$ are zero, which implies that all $h_x(y)$ appearing in the right hand side of item c) are non-positive. Hence, we have that

$$h_x(Z) + h_x(\tau Z) = \sum_{y \to Z} \max\{h_x(y), 0\} = 0 = \max\{\sum_{y \to Z} h_x(y), 0\},$$

where '$y \to Z$' are arrows in Γ. If there is no arrow from x to Z in the Auslander-Reiten quiver of $\mathcal{C}_Q$, then we have the following equalities

$$\begin{aligned} h_x(Z) + h_x(\tau Z) &= \sum_{y \to Z} \max\{h_x(y), 0\} \\ &= \sum_{y \to Z} h_x(y) = \max\{\sum_{y \to Z} h_x(y), 0\}, \end{aligned}$$

where '$y \to Z$' are arrows in Γ. Therefore, for all non-split triangles as the triangles $(*)$ before Proposition 5.2.4, the function h_x satisfies item d3) in Definition 5.2.2. Besides, by Proposition 5.3.4, there is a unique tropical frieze $g : \mathcal{C}_Q \to \mathbb{Z}$ such that $g(Y) = h_x(Y)$ for all indecomposables Y which come from the same slice containing x. Thus, we have that $h_x = g$ and h_x is a tropical frieze on $\mathcal{C}_Q$.

Let $\mathcal{S}_x$ be any slice in $\mathbb{Z}$Q with x a source. Set $T = \oplus_{Y \in \mathcal{S}_x} Y$. It is a basic cluster-tilting object of $\mathcal{C}_Q$. Let B be the endomorphism algebra of T and S_x the simple B-module corresponding to x. Clearly $\mathcal{S}_x$ is the quiver of B. Set $m = -S_x$. Then $f_{T,-S_x}$ is a tropical frieze and takes the same values as h_x on all indecomposable direct summands of T. As a result, the function h_x is equal to $f_{T,-S_x}$.

However, the sum $\sum_x h_x$ of cluster-hammock functions with all x coming from the same slice $\mathcal{S}$ in $\mathbb{Z}Q$ is not always a tropical frieze, which is quite different to the Corollary in Section 6 of [74]. Here we also use the same counter-example on $\mathcal{C}_{\vec{A}_3}$ as in Subsection 5.2.3. We already know that the functions d_1 and $h_{\Sigma P_1}$ are tropical friezes. Let $T = \Sigma P_1 \oplus \Sigma P_2 \oplus \Sigma P_3$. Then d_1 and $h_{\Sigma P_1}$ coincide on all ΣP_i $(1 \leq i \leq 3)$. Thus, $h_{\Sigma P_1}$ is equal to d_1. Similarly, the tropical frieze $h_{\Sigma P_3}$ is equal to d_3. But the sum $h_{\Sigma P_1} + h_{\Sigma P_3} = d_1 + d_3$ is not a tropical frieze.

5.4 Simple illustrations for the cases A_1 and A_2

Let us first look at the cluster category $\mathcal{C} = \mathcal{C}_Q$ of the quiver Q of type A_1. Let X and ΣX be the two indecomposable objects of $\mathcal{C}_Q$. Assume f is a tropical frieze on $\mathcal{C}_Q$. Then we have that

$$f(X) + f(\Sigma X) = 0.$$

Set $T = X$ and $m = f(X)S_X$, where S_X is the unique simple $(\mathrm{End}_{\mathcal{C}_Q}(X))$-module. Since $\langle S_X, m\rangle_a$ is zero, by Theorem 5.3.1 the function $f_{T,m}$ is a tropical frieze. The following equalities

$$f_{T,m}(X) = \langle F(\mathrm{ind}(X)), f(X)S_X\rangle = f(X) \quad \text{and}$$
$$f_{T,m}(\Sigma X) = \langle F(\mathrm{ind}(\Sigma X)), f(X)S_X\rangle = -f(X) = f(\Sigma X)$$

clearly hold. Therefore, the tropical frieze f is equal to $f_{T,m}$.

Now let us look at the cluster category $\mathcal{C} = \mathcal{C}_Q$ of a quiver Q of type A_2. Assume that f is a non-zero tropical frieze on $\mathcal{C}_Q$. Following Proposition 5.2.4, we know that there exist an indecomposable object X such that $f(X) < 0$. Let Y and Y' be the two non-isomorphic indecomposables such that $X \oplus Y$ and $X \oplus Y'$ are cluster-tilting objects of $\mathcal{C}_Q$. Then we have that

$$f(Y) + f(Y') = \max\{f(X), 0\} = 0.$$

Therefore, there must exist a cluster-tilting object $T = T_1 \oplus T_2$ with T_i indecomposable such that

$$f(T_1) \geq 0 \quad \text{and} \quad f(T_2) < 0.$$

Let Q_T be the quiver of the endomorphism algebra $B = \mathrm{End}_{\mathcal{C}_Q}(T)$. The quiver Q_T is also of type A_2. Let P_i be the indecomposable projective B-module and S_i its corresponding simple top, $i = 1, 2$.

If S_1 attaches to the sink in Q_T, set $m = f(T_1)S_1 + f(T_2)S_2$, then

$$\langle S_1, m\rangle_a = -f(T_2)\mathrm{dimExt}^1_B(S_1, S_2) = -f(T_2) > 0 \quad \text{and}$$
$$\langle S_2, m\rangle_a = f(T_1)\mathrm{dimExt}^1_B(S_1, S_2) = f(T_1) \geq 0,$$

which implies that $f_{T,m}$ is a tropical frieze by Theorem 5.3.1. Moreover, the tropical friezes f and $f_{T,m}$ coincide on T_i. Therefore, the tropical frieze f is equal to $f_{T,m}$.

If S_1 attaches to the source in Q_T, set $T' = \mu_2\mu_1(T) = T_1' \oplus T_2'$, where T_1' and T_2' come from the following non-split triangles in $\mathcal{C}_Q$

$$T_1 \to T_2 \to T_1' \to \Sigma T_1, \qquad T_1' \to 0 \to T_1 \to \Sigma T_1';$$

$$T_2 \to T_1' \to T_2' \to \Sigma T_2, \qquad T_2' \to 0 \to T_2 \to \Sigma T_2'.$$

We can calculate that

$$f(T_1') = -f(T_1) \leq 0 \quad \text{and} \quad f(T_2') = -f(T_2) > 0.$$

Notice that the quiver $Q_{T'}$ of the endomorphism algebra $B' = \mathrm{End}_{\mathcal{C}_Q}(T')$ is $T_1' \to T_2'$. Let S_i' be the simple B'-module corresponding to T_i'. Now we go back to the above case. Set $m' = f(T_1')S_1' + f(T_2')S_2'$. Then we have that $f_{T',m'}$ is a tropical frieze and takes the same values as f on T_i'. Thus, the tropical frieze f is equal to $f_{T',m'}$.

In fact, such a phenomenon for the cases A_1 and A_2 is a common phenomenon for the Dynkin case, which we will state in Theorem 5.5.1 in the next section.

Let $f_{T,m}$ be a tropical frieze on $\mathcal{C}_Q$ with Q a quiver of type A_2. Suppose that the quiver Q_T of the endomorphism algebra $B = \mathrm{End}_{\mathcal{C}_Q}(T)$ is $(T_1 \to T_2)$ and $m = m_1S_1 + m_2S_2$. From Remark 5.3.2 we know that $\langle S_i, m\rangle_a \geq 0$ for $i = 1, 2$, that is,

$$\langle S_1, m\rangle_a = m_2 \mathrm{dimExt}^1_B(S_2, S_1) = m_2 \geq 0, \quad \text{and}$$
$$\langle S_2, m\rangle_a = -m_1 \mathrm{dimExt}^1_B(S_2, S_1) = -m_1 \geq 0.$$

Notice that $f_{T,m}(T_i) = \langle FT_i, m\rangle = m_i$ for $i = 1, 2$. Set $T' = \mu_1(T) = T_1' \oplus T_2$ and $T'' = \mu_2(T) = T_1 \oplus T_2''$. Then the following expressions hold

$$f_{T,m}(T_1') = \max\{f_{T,m}(T_2), 0\} - f_{T,m}(T_1) \geq -f_{T,m}(T_1) \geq 0,$$
$$f_{T,m}(T_2'') = \max\{f_{T,m}(T_1), 0\} - f_{T,m}(T_2) = -f_{T,m}(T_2) \leq 0.$$

Therefore, in the A_2 case, there exist cluster-tilting objects T' and T'' such that $f_{T,m}$ takes non-negative values on direct summands of T' and non-positive values on direct summands of T''.

5.5 The main theorem (Dynkin case)

As a generalization of the phenomenon illustrated in Section 5.4, the aim of this section is to show the following theorem:

Theorem 5.5.1. *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q. Then each tropical frieze on $\mathcal{C}_Q$ is of the form $f_{T,m}$, where T is a cluster-tilting object of $\mathcal{C}_Q$ and m an element in the Grothendieck group $K_0(\mathrm{modEnd}_{\mathcal{C}_Q}(T))$.*

We will prove this theorem in Subsections 5.5.1 and 5.5.2. First, we need to introduce some notation:

Let $\mathcal{C}$ be a 2-Calabi-Yau category with cluster-tilting object. Let f be a tropical frieze on the category $\mathcal{C}$ and $T = T_1 \oplus \ldots \oplus T_n$ a basic cluster-tilting object of $\mathcal{C}$. Suppose that the quiver Q of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ does not have loops nor 2-cycles. Let b_{ij} denote the number of arrows $i \to j$ minus the number of arrows $j \to i$ in Q (notice that at least one of these two numbers is zero). For each integer $1 \leq i \leq n$, let $g_i(T)$ be the integer

$$g_i(T) = \sum_r [b_{ri}]_+ f(T_r) - \sum_s [b_{is}]_+ f(T_s),$$

where $[b_{kl}]_+ = \max\{b_{kl}, 0\}$ is equal to the number of arrows $k \to l$ in Q. Denote by $g(T)$ the class $\sum_{i=1}^n g_i(T)[T_i]$ in the Grothendieck group $K_0(\mathrm{add}T)$.

5.5.1 Transformations of the class $g(T)$ under mutations

Since the quiver Q does not have loops, for each T_k, there is a unique indecomposable object T'_k such that the space $\mathrm{Ext}^1_{\mathcal{C}}(T'_k, T_k)$ is one-dimensional and the non split triangles are given [57] by

$$T'_k \to E \to T_k \to \Sigma T'_k \quad \text{and} \quad T_k \to E' \to T'_k \to \Sigma T_k,$$

where

$$E = \bigoplus_r [b_{rk}]_+ T_r \quad \text{and} \quad E' = \bigoplus_s [b_{ks}]_+ T_s.$$

Let $T' = \mu_k(T) = T'_k \oplus (\bigoplus_{i \neq k} T_i)$. Define linear transformations ϕ_+ and ϕ_- from $K_0(\mathrm{add}T)$ to $K_0(\mathrm{add}T')$ as in [30] by

$$\begin{aligned}
\phi_+(T_i) &= \phi_-(T_i) = [T_i] \qquad \text{for } i \neq k, \text{ and} \\
\phi_+(T_k) &= [E] - [T'_k] = -[T'_k] + \sum_r [b_{rk}]_+ [T_r] \\
\phi_-(T_k) &= [E'] - [T'_k] = -[T'_k] + \sum_s [b_{ks}]_+ [T_s].
\end{aligned}$$

It was shown in [30] that if X is a rigid object of $\mathcal{C}$, then the index of X with respect to cluster-tilting objects transforms as follows:

$$\mathrm{ind}_{T'}(X) = \begin{cases} \phi_+(\mathrm{ind}_T(X)) & \text{if } [\mathrm{ind}_T(X) : T_k] \geq 0, \\ \phi_-(\mathrm{ind}_T(X)) & \text{if } [\mathrm{ind}_T(X) : T_k] \leq 0, \end{cases}$$

where $[\mathrm{ind}_T(X) : T_k]$ denotes the coefficient of T_k in the decomposition of $\mathrm{ind}_T(X)$ in the category $K_0(\mathrm{add}T)$.

Proposition 5.5.2. *Suppose that the quivers Q and Q' of the endomorphism algebras $\mathrm{End}_{\mathcal{C}}(T)$ and $\mathrm{End}_{\mathcal{C}}(T')$ do not have loops nor 2-cycles. Then the element $g(T)$ transforms in the same way as above, i.e.*

$$g(T') = \begin{cases} \phi_+(g(T)) & \text{if } g_k(T) \geq 0, \\ \phi_-(g(T)) & \text{if } g_k(T) \leq 0. \end{cases}$$

Proof. We first assume that $g_k(T) \geq 0$, that is,

$$f(E) = \sum_r [b_{rk}]_+ f(T_r) \geq \sum_s [b_{ks}]_+ f(T_s) = f(E').$$

Since f is a tropical frieze, we have that $f(T_k) + f(T'_k) = f(E) = \sum_r [b_{rk}]_+ f(T_r)$. We compute $\phi_+(g(T))$:

$$\begin{aligned}
\phi_+(g(T)) = \phi_+(\sum_{i=1}^n g_i(T)[T_i]) &= \sum_{i \neq k} g_i(T)[T_i] + g_k(T)\phi_+(T_k) \\
&= \sum_{i \neq k} g_i(T)[T_i] - g_k(T)[T'_k] + \sum_r g_k(T)[b_{rk}]_+[T_r] \\
&= \sum_{i \neq k} (g_i(T) + [b_{ik}]_+ g_k(T))[T_i] - g_k(T)[T'_k].
\end{aligned}$$

By assumption, the quivers Q and Q' do not have loops nor 2-cycles. Following [16], we know that $Q' = \mu_k(Q)$ is the mutation of the quiver Q at vertex k. Let b'_{ij} be the number of arrows $i \to j$ minus the number of arrows $j \to i$ in Q'. Then it is known from [34] that

$$b'_{ij} = \begin{cases} b_{ji} & \text{if}\, i = k \,\text{or}\, j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

It is obvious that

$$\begin{aligned} g_k(T') &= \sum_r [b'_{rk}]_+ f(T_r) \ - \sum_s [b'_{ks}]_+ f(T_s) \\ &= \sum_r [b_{kr}]_+ f(T_r) \ - \sum_s [b_{sk}]_+ f(T_s) = -g_k(T). \end{aligned}$$

For vertices $i \neq k$, we distinguish three cases to compute $g_i(T')$.
If $b_{ik} = b_{ki} = 0$, then $b'_{ij} = b_{ij}$ and $b'_{ji} = b_{ji}$ for all vertices j. In this case, we have that

$$\begin{aligned} g_i(T') &= \sum_r [b'_{ri}]_+ f(T_r) - \sum_s [b'_{is}]_+ f(T_s) \\ &= \sum_r [b_{ri}]_+ f(T_r) - \sum_s [b_{is}]_+ f(T_s) = g_i(T). \end{aligned}$$

If $b_{ik} > 0$, then

$$\begin{aligned} g_i(T') = \sum_r [b'_{ri}]_+ f(T_r) - \sum_s [b'_{is}]_+ f(T_s) &= (\sum_r [b_{ri}]_+ f(T_r) + b_{ik} f(T'_k)) \\ &\quad - (\sum_s [b_{is}]_+ f(T_s) - b_{ik} f(T_k) + \sum_{s'} b_{ik} [b_{ks'}]_+ f(T_{s'})) \\ = g_i(T) + b_{ik}(f(T'_k) + f(T_k) - \sum_{s'} [b_{ks'}]_+ f(T_{s'})) & \\ = g_i(T) + b_{ik}(\sum_r [b_{rk}]_+ f(T_r) - \sum_s [b_{ks}]_+ f(T_s)) & \\ = g_i(T) + b_{ik} g_k(T). & \end{aligned}$$

If $b_{ik} < 0$, then $b_{ki} = -b_{ik} > 0$, and

$$\begin{aligned} g_i(T') &= \sum_r [b'_{ri}]_+ f(T_r) - \sum_s [b'_{is}]_+ f(T_s) \\ &= (\sum_r [b_{ri}]_+ f(T_r) - b_{ki} f(T_k) + \sum_{r'} [b_{r'k}]_+ b_{ki} f(T_{r'})) \\ &\qquad - (\sum_s [b_{is}]_+ f(T_s) + b_{ki} f(T'_k)) \\ &= g_i(T) - b_{ki}(f(T_k) + f(T'_k) - \sum_{r'} [b_{r'k}]_+ f(T_{r'})) \\ &= g_i(T) - b_{ki}(\sum_r [b_{rk}]_+ f(T_r) - \sum_{r'} [b_{r'k}]_+ f(T_{r'})) = g_i(T). \end{aligned}$$

Therefore, we obtain that $g(T') = \phi_+(g(T))$ when $g_k(T) \geq 0$. In a similar way we can also obtain that $g(T') = \phi_-(g(T))$ when $g_k(T) \leq 0$. □

5.5.2 Proof of the main theorem

Let T_0 and T_1 be two objects in add$T$ which do not have a direct summand in common. Let η be a morphism in $\mathcal{C}(T_1, T_0)$. Denote by $C(\eta)$ the cone of η. Then we have the following triangle in $\mathcal{C}$

$$T_1 \xrightarrow{\eta} T_0 \to C(\eta) \to \Sigma T_1. \qquad (**)$$

The algebraic group $\mathrm{Aut}(T_0) \times \mathrm{Aut}(T_1)$ acts on $\mathcal{C}(T_1, T_0)$ via

$$(g_0, g_1)\eta' = g_0 \eta' g_1^{-1}.$$

Let $\mathcal{O}_\eta$ denote the orbit of η in the space $\mathcal{X} := \mathcal{C}(T_1, T_0)$ under the above action of $\mathrm{Aut}(T_0) \times \mathrm{Aut}(T_1)$.

It is not hard to obtain the following lemma. For the convenience of the reader we include a proof.

Lemma 5.5.3. *Let η and η' be two morphisms in $\mathcal{X}$. Then $\mathcal{O}_\eta = \mathcal{O}_{\eta'}$ if and only if $C(\eta) \simeq C(\eta')$.*

Proof. First we assume that $\mathcal{O}_\eta = \mathcal{O}_{\eta'}$. Then there exists an element $(g_0, g_1) \in \mathrm{Aut}(T_0) \times \mathrm{Aut}(T_1)$ such that $\eta' = g_0 \eta g_1^{-1}$. The commutative square $g_0\eta = \eta' g_1$ can be completed to a commutative diagram of triangles as follows

$$\begin{tikzcd}
T_1 \arrow[r, "\eta"] \arrow[d, "g_1"] & T_0 \arrow[r, "\iota"] \arrow[d, "g_0"] & C(\eta) \arrow[r, "p"] \arrow[d, "h"] & \Sigma T_1 \arrow[d, "\Sigma g_1"] \\
T_1 \arrow[r, "\eta'"] & T_0 \arrow[r, "\iota'"] & C(\eta') \arrow[r, "p'"] & \Sigma T_1.
\end{tikzcd}$$

Here the morphism h is an isomorphism from $C(\eta)$ to $C(\eta')$.

Second we assume that $C(\eta) \simeq C(\eta')$. Let h be an isomorphism from $C(\eta)$ to $C(\eta')$ and h^{-1} its inverse. Since the space $\mathcal{C}(T_0, \Sigma T_1)$ vanishes, we have that (keeping the notation as in the above commutative diagram)

$$p'h\iota = 0 \quad \text{and} \quad ph^{-1}\iota' = 0.$$

Thus, there exist two morphisms g_0 and g_0' in $\mathcal{C}(T_0, T_0)$ such that

$$\iota' g_0 = h\iota \quad \text{and} \quad \iota g_0' = h^{-1}\iota'.$$

As a consequence, the equalities

$$\iota g_0' g_0 = h^{-1}\iota' g_0 = h^{-1}h\iota = \iota \quad \text{and} \quad \iota' g_0 g_0' = h\iota g_0' = hh^{-1}\iota' = \iota'$$

hold. Thus, we have that $g_0 g_0' = 1 = g_0' g_0$. The morphism g_0 is an element in $\mathrm{Aut}(T_0)$. The commutative square $h\iota = \iota' g_0$ can be completed to a commutative diagram of triangles as above. Thus, there exists an element $g_1 \in \mathrm{Aut}(T_1)$ such that $g_0\eta = \eta' g_1$. Therefore, the two orbits $\mathcal{O}_\eta$ and $\mathcal{O}_{\eta'}$ are the same. $\square$

Lemma 5.5.4. *Keep the above notation. We have the equality*

$$\mathrm{codim}_{\mathcal{X}} \mathcal{O}_\eta = 1/2\, \mathrm{dim} \mathrm{Ext}^1_{\mathcal{C}}(C(\eta), C(\eta)).$$

Proof. Let F be the functor $\mathcal{C}(T, ?)$ and B the endomorphism algebra FT. We denote the space $\mathrm{Hom}_B(FT_1, FT_0)$ by $F\mathcal{X}$. Since F induces a category equivalence from $\mathcal{C}/\mathrm{add}(\Sigma T)$ to $\mathrm{mod}B$, we have that

$$\mathrm{codim}_{\mathcal{X}}\mathcal{O}_\eta = \mathrm{codim}_{F\mathcal{X}}\mathcal{O}_{F\eta}.$$

The algebra B is a finite-dimensional algebra, both FT_1 and FT_0 are finitely generated B-modules. As in [71], we view $F\eta$ as a complex in $K^b(\mathrm{proj}B)$ and define the space $E(F\eta)$ as

$$E(F\eta) = \mathrm{Hom}_{K^b(\mathrm{proj}B)}(\Sigma^{-1}F\eta, F\eta).$$

Following Lemma 2.16 in [71], we have the equality

$$\mathrm{codim}_{F\mathcal{X}}\mathcal{O}_{F\eta} = \mathrm{dim}E(F\eta).$$

The exact sequence

$$FT_1 \xrightarrow{F\eta} FT_0 \to F(C(\eta)) \to 0$$

is a minimal projective presentation of $F(C(\eta))$. Still following from [71], the equality

$$\mathrm{dim}E(F\eta) = \mathrm{dimHom}_B(F(C(\eta)), \tau F(C(\eta)))$$

holds, where τ is the Auslander-Reiten translation. Moreover, by Section 3.5 in [59], we have that $F(\Sigma C(\eta)) \simeq \tau F(C(\eta))$.

For two objects X and Y of $\mathcal{C}$, let $(\Sigma T)(X, Y)$ be the subspace of $\mathcal{C}(X, Y)$ consisting of morphisms from X to Y factoring through an object in $\mathrm{add}(\Sigma T)$, let $\mathcal{C}/_{(\Sigma T)}(X, Y)$ denote the space $\mathcal{C}(X, Y)/(\Sigma T)(X, Y)$. Lemma 3.3 in [69] shows that there is a bifunctorial isomorphism

$$\mathcal{C}/_{(\Sigma T)}(X, \Sigma Y) \simeq D(\Sigma T)(Y, \Sigma X).$$

If we choose $Y = X$, then we can deduce that

$$\dim \mathcal{C}/_{(\Sigma T)}(X, \Sigma X) = \dim(\Sigma T)(X, \Sigma X) = 1/2 \dim \mathcal{C}(X, \Sigma X).$$

Notice that the equivalence F gives the following equality

$$\mathrm{dimHom}_B(F(C(\eta)), \tau F(C(\eta))) = \dim \mathcal{C}/_{(\Sigma T)}(C(\eta), \Sigma C(\eta)).$$

Finally, if we combine all the equalities about dimensions together, then we can obtain that

$$\mathrm{codim}_{\mathcal{X}}\mathcal{O}_\eta = 1/2\, \mathrm{dimExt}^1_{\mathcal{C}}(C(\eta), C(\eta)).$$

□

If we do not assume that T_0 and T_1 do not have a common direct summand, then the equality in Lemma 5.5.4 becomes

$$\mathrm{codim}_{\mathcal{X}}\mathcal{O}_\eta \geq 1/2\, \mathrm{dimExt}^1_{\mathcal{C}}(C(\eta), C(\eta)).$$

This is because the third equality in the proof becomes

$$\mathrm{dim}E(F\eta) \geq \mathrm{dimHom}_B(F(C(\eta)), \tau F(C(\eta)))$$

for arbitrary projective presentations.

Lemma 5.5.5. *Suppose that $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects. Then the set $\{[C(\eta)] | \eta \in \mathcal{C}(T_1, T_0)\}$ is finite, where $[C(\eta)]$ denotes the isomorphism class of $C(\eta)$ in $\mathcal{C}$.*

Proof. We use the same exact sequence

$$FT_1 \overset{F\eta}{\to} FT_0 \to F(C(\eta)) \to 0$$

as in the proof of Lemma 5.5.4, which is a projective presentation of $F(C(\eta))$. By assumption $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects. So the number of isomorphism classes of indecomposable B-modules is also finite. Notice that the dimension of $F(C(\eta))$ is bounded by the dimension of FT_0. Hence, the set of $\{[F(C(\eta))] | \eta \in \mathcal{C}(T_1, T_0)\}$ is finite, where $[F(C(\eta))]$ denotes the isomorphism class of $F(C(\eta))$ in mod$B$.

Now we decompose $C(\eta)$ as $X_\eta \oplus \Sigma T_\eta$, where X_η does not contain a direct summand in add(ΣT). We have that $F(C(\eta)) = F(X_\eta)$. Since $\mathcal{C}(T_0, \Sigma T_\eta)$ vanishes, we can rewrite the triangle $(**)$ before Lemma 5.5.3 as

$$T_1 \xrightarrow{\begin{pmatrix} \eta \\ 0 \end{pmatrix}} T_0 \oplus 0 \xrightarrow{\begin{pmatrix} \iota_\eta & 0 \\ 0 & 0 \end{pmatrix}} X_\eta \oplus \Sigma T_\eta \to \Sigma T_1,$$

which is the direct sum of the following two triangles

$$\Sigma^{-1} C(\iota_\eta) \to T_0 \overset{\iota_\eta}{\to} X_\eta \to C(\iota_\eta), \qquad \text{and}$$

$$T_\eta \to 0 \to \Sigma T_\eta \to \Sigma T_\eta.$$

Here $C(\iota_\eta)$ denotes the cone of the morphism ι_η. Therefore, the object T_η is a direct summand of T_1, and there are only finitely many choices. In conclusion, there are only finitely many isomorphism classes of $C(\eta)$ when η runs over the space $\mathcal{C}(T_1, T_0)$. $\square$

Under the assumption that $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects, by combining Lemma 5.5.3 and Lemma 5.5.5 we can obtain that there are only finitely many orbits $\mathcal{O}_\eta$ in the affine space $\mathcal{X}$. Therefore, there must exist some morphism η such that

$$\mathrm{codim}_{\mathcal{X}} \mathcal{O}_\eta = 0,$$

which implies that $C(\eta)$ is a rigid object by Lemma 5.5.4. We say a morphism η *generic* if its cone $C(\eta)$ is rigid. We deduce the following proposition

Proposition 5.5.6. *Suppose that $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable objects. Then there exists a generic morphism $\eta \in \mathcal{C}(T_1, T_0)$ with the cone $C(\eta)$ rigid.*

Now we are ready to prove our main theorem.

Proof of Theorem 5.5.1. Let $T = T_1 \oplus \ldots \oplus T_n$ be any basic cluster-tilting object in $\mathcal{C}_Q$. Keeping the notation at the beginning of this section, we define two objects

$$L = \bigoplus_{g_i(T) < 0} T_i^{-g_i(T)} \qquad \text{and} \qquad R = \bigoplus_{g_i(T) > 0} T_i^{g_i(T)}.$$

By Proposition 5.5.6, there exists a morphism $\eta \in \mathrm{Hom}_{\mathcal{C}_Q}(L, R)$ such that the cone $C(\eta)$ is rigid. The triangle

$$L \xrightarrow{\eta} R \to C(\eta) \to \Sigma L$$

implies that the index

$$\mathrm{ind}_T(C(\eta)) = [R] - [L] = g(T).$$

Since $C(\eta)$ is rigid, there exists a cluster-tilting object T' of $\mathcal{C}_Q$ such that $C(\eta) \in \mathrm{add}T'$. The triangle

$$\Sigma^{-1}C(\eta) \to 0 \to C(\eta) \to C(\eta)$$

gives us that

$$\mathrm{ind}_{\Sigma^{-1}T'}(C(\eta)) \in \mathbb{Z}^n_{\leq 0}.$$

Set $T'' = \Sigma^{-1}T'$. It was shown in [20] that the quiver of the endomorphism algebra of a cluster-tilting object of $\mathcal{C}_Q$ does not have loops nor 2-cycles. Therefore, it follows from Proposition 5.5.2 that

$$g(T'') = \mathrm{ind}_{T''}(C(\eta)) \in \mathbb{Z}^n_{\leq 0},$$

that is, $g_i(T'') \leq 0$.

Let B'' denote the endomorphism algebra $\mathrm{End}_{\mathcal{C}_Q}(T'')$ and Q'' its associated quiver. Let S_i'' be the simple top of the indecomposable projective B''-module $P_i'' = \mathrm{Hom}_{\mathcal{C}_Q}(T'', T_i'')$. Set

$$m'' = \sum_{j=1}^{n} f(T_j'')S_j'' \; (\in K_0(\mathrm{mod}B'')).$$

Then for each simple B''-module S_i'', we have that

$$\begin{aligned}
\langle S_i'', m''\rangle_a &= \sum_{j=1}^{n} f(T_j'')\langle S_i'', S_j''\rangle_a \\
&= \sum_{j=1}^{n} f(T_j'')(-\mathrm{dimExt}^1_{B''}(S_i'', S_j'')) + \sum_{j=1}^{n} f(T_j'')\mathrm{dimExt}^1_{B''}(S_j'', S_i'') \\
&= -\sum_{j=1}^{n} [b_{ji}'']_+ f(T_j'') + \sum_{j=1}^{n} [b_{ij}'']_+ f(T_j'') = -g_i(T'') \geq 0,
\end{aligned}$$

where b_{kl}'' denotes the number of arrows $k \to l$ minus the number of arrows $l \to k$ in Q''. Therefore, by Theorem 5.3.1 the function $f_{T'',m''}$ is a tropical frieze. Since we have

$$f_{T'',m''}(T_i'') = \langle P_i'', m''\rangle = \langle P_i'', f(T_i'')S_i''\rangle = f(T_i''),$$

the tropical friezes f and $f_{T'',m''}$ coincide on all T_i''. Now it follows from Proposition 5.2.8 that f is equal to $f_{T'',m''}$.

5.5.3 Sign-coherence property

For any tropical frieze f on $\mathcal{C}_Q$ with Q a Dynkin quiver, we will see in this subsection the existence of cluster-tilting objects whose indecomposable direct summands have sign-coherent values under f.

Theorem 5.5.7. *Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q and f a tropical frieze on $\mathcal{C}_Q$. Then there exists a cluster-tilting object T such that*

$$f(T_i) \geq 0 \quad (\textit{resp. } f(T_i) \leq 0)$$

for all indecomposable direct summands T_i of T.

Proof. Since f is a tropical frieze on $\mathcal{C}_Q$, it follows from Theorem 5.5.1 that f is equal to some $f_{T,m}$ with T a cluster-tilting object and m an element in $K_0(\mathrm{mod}\mathrm{End}_{\mathcal{C}_Q}(T))$. We divide the proof into three steps.

Step 1. For any cluster-tilting object S of $\mathcal{C}_Q$, we define its associated positive cone as

$$C(S) = \{\mathrm{ind}_T(U) \,|\, U \in \mathrm{add}S\} \ (\subset K_0(\mathrm{add}T)).$$

Each element $X \in K_0(\mathrm{add}T)$ can be written uniquely as

$$X = [T_0] - [T_1],$$

where T_0, $T_1 \in \mathrm{add}T$ without common indecomposable direct summands. By Proposition 5.5.6, there exists some morphism $\eta \in \mathrm{Hom}_{\mathcal{C}_Q}(T_1, T_0)$ such that the cone $C(\eta)$ is rigid. Moreover, we have that

$$\mathrm{ind}_T(C(\eta)) = [T_0] - [T_1] = X.$$

Since $C(\eta)$ is rigid, it belongs to $\mathrm{add}S$ for some cluster-tilting object S of $\mathcal{C}_Q$, which implies that the element X belongs to the positive cone $C(S)$. As a consequence, we can obtain that

$$K_0(\mathrm{add}T) = \bigcup_S C(S),$$

where S ranges over all (finitely many) cluster-tilting objects of $\mathcal{C}_Q$.

Step 2. Let $T_1, \ldots, T_n$ be the pairwise non-isomorphic indecomposable direct summands of T. Suppose that $m = \sum_{i=1}^n m_i S_i$ with S_i the simple $\mathrm{End}_{\mathcal{C}_Q}(T)$-module corresponding to T_i. Let F be the functor $\mathrm{Hom}_{\mathcal{C}_Q}(T, ?)$. Set

$$H_m^{\geq 0} = \{X \in K_0(\mathrm{add}T) \,|\, \langle FX, m\rangle \geq 0\}.$$

It is clear that

$$\langle \mathrm{sgn}(m_i) FT_i, m\rangle = |m_i| \geq 0,$$

where

$$\mathrm{sgn}(m_i) = \begin{cases} 1 & \text{if}\, m_i \geq 0, \\ -1 & \text{if}\, m_i < 0. \end{cases}$$

Let H be the hyperquadrant of $K_0(\mathrm{add}T)$ consisting of the non-negative linear combinations of the $\mathrm{sgn}(m_i)[T_i]$, $1 \leq i \leq n$. Then we have that

$$H \subset H_m^{\geq 0}.$$

Step 3. It was shown in Section 2.4 of [30] that each positive cone $C(S)$ is contained in a hyperquadrant of $K_0(\mathrm{add}T)$ with respect to the given basis $[T_i]$, $1 \leq i \leq n$. Thus, each hyperquadrant of $K_0(\mathrm{add}T)$ is a union of positive cones. Let T' be a cluster-tilting object satisfying

$$C(T') \subset H \subset H_m^{\geq 0}.$$

We obtain that

$$f(T_i') = f_{T,m}(T_i') = \langle F(\mathrm{ind}_T(T_i')), m\rangle \geq 0$$

for all indecomposable direct summands T_i' of T'.

Similarly, there exists some cluster-tilting object T'' such that $f(T_i'') \leq 0$ for all indecomposable direct summands T_i'' of T''. $\square$

5.5.4 Another approach to the main theorem

Let $\mathcal{C}_Q$ be the cluster category of a Dynkin quiver Q. In this subsection, we will see another approach to Theorem 5.5.1 by using the work of V. Fock and A. Goncharov [33]. For simplicity, we write $\mathbb{Z}_{tr}$ for the tropical semifield $(\mathbb{Z}, \odot, \oplus)$.

Let $\mathcal{A}_{Q^{op}}(\mathbb{Z}_{tr})$ and $\mathcal{X}_{Q^{op}}(\mathbb{Z}_{tr})$ be the set of tropical $\mathbb{Z}$-points of $\mathcal{A}$-variety and $\mathcal{X}$-variety [33] associated with the opposite quiver Q^{op}, respectively. For a vertex k of Q, the mutation $\mu_k : \mathcal{A}_{Q^{op}}(\mathbb{Z}_{tr}) \to \mathcal{A}_{\mu_k(Q^{op})}(\mathbb{Z}_{tr})$ is given by the tropicalization of formula (14) in [33]:

$$A_k + (\mu_k A)_k = \max\{\sum_j [b_{jk}]_+ A_j, \sum_j [b_{kj}]_+ A_j\},$$

where $[b_{rs}]_+$ is the number of arrows from r to s in Q (or from s to r in Q^{op}). Let T be the image of kQ in $\mathcal{C}_Q$. Then for each tropical $\mathbb{Z}$-point A in $\mathcal{A}_{Q^{op}}(\mathbb{Z}_{tr})$, there is a unique tropical frieze h on $\mathcal{C}_Q$ such that $h(T_j) = A_j$ for each $1 \leq j \leq n$. Moreover, this correspondence commutes with mutation. Besides, we know from [71] that the isomorphism $\mathcal{X}_{Q^{op}}(\mathbb{Z}_{tr}) \simeq K_0(\mathrm{add}T)$ commutes with mutation. Given a seed $\underline{i}$, in [33] V. Fock and A. Goncharov considered the function $P_{\underline{i}} = \sum_{i=1}^n a_i x_i$ on $\mathcal{A}(\mathbb{Z}_{tr}) \times \mathcal{X}(\mathbb{Z}_{tr})$. Now we can transform the function $P_{\underline{i}}$ in our case as

$$P_S = \sum_{i=1}^n h(S_i)[\mathrm{ind}_S(Y) : S_i]$$

where S is the cluster-tilting object of $\mathcal{C}_Q$ corresponding to the seed $\underline{i}$, the elements a_i correspond to $h(S_i)$ and x_i correspond to $[\mathrm{ind}_S(Y) : S_i]$ for some object Y of $\mathcal{C}_Q$.

Let f be a tropical frieze on $\mathcal{C}_Q$. Let L and R be the same objects as in the proof of Theorem 5.5.1. Assume X is an object of $\mathcal{C}_Q$ with

$$\mathrm{ind}_T(X) = [R] - [L] \, (= g(T)).$$

For example, the cone $C(\eta)$ as in the proof of Theorem 5.5.1. For the pair $N = (f, \mathrm{ind}_T(X))$ in $\mathcal{A}(\mathbb{Z}_{tr}) \times \mathcal{X}(\mathbb{Z}_{tr})$, by Theorem 5.2 in [33], there exists a cluster-tilting object T' such that all coordinates $[\mathrm{ind}_{T'}(X) : T'_i]$ are non-negative. It follows that there exists some rigid object $X_0 \in \mathrm{add}T'$ with the same index as X. Set $T'' = \Sigma^{-1}T'$, as in the proof of Theorem 5.5.1, we can also obtain that

$$g(T'') = \mathrm{ind}_{T''}(X) = \mathrm{ind}_{T''}(X_0) \in \mathbb{Z}^n_{\leq 0}.$$

This gives another approach to the main theorem.

Moreover, our definition for positive cones in Step 1 in the proof of Theorem 5.5.7 coincides with Fock-Goncharov's. From the equality

$$K_0(\mathrm{add}T) = \bigcup_S C(S),$$

where S ranges over all (finitely many) cluster-tilting objects of $\mathcal{C}_Q$, we can also obtain that a finite type cluster $\mathcal{X}$-variety is of definite type (see Corollary 5.5 and Conjecture 5.7 in [33]).

5.6 Proof of a conjecture of Ringel

Definition 5.6.1 (Ringel [74]). Let $\Gamma = \mathbb{Z}\Delta$ with Δ one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ and Γ_0 the vertex set of Γ. A function $f : \Gamma_0 \to \mathbb{Z}$ is said to be *cluster-additive* on Γ if

$$f(z) + f(\tau z) = \sum\nolimits_{y \to z} \max\{f(y), 0\}, \qquad \text{for all } z \in \Gamma_0,$$

where the sum runs over all arrows $y \to z$ ending at z in Γ.

The following theorem confirms a conjecture by Ringel [74].

Theorem 5.6.2. *Each cluster-additive function on $\Gamma = \mathbb{Z}\Delta$ with Δ one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ is a non-negative linear combination of cluster-hammock functions (and therefore of the form*

$$\sum\nolimits_{x \in \mathcal{U}} n_x h_x$$

for a tilting set $\mathcal{U}$ and integers $n_x \in \mathbb{N}_0$, for all $x \in \mathcal{U}$).

Proof. Let Q be an orientation of the Dynkin diagram Δ. Then Γ can be viewed as the Auslander-Reiten quiver of the bounded derived category $\mathcal{D}_Q$ of the category modkQ. Let I_i be the i-th indecomposable right injective kQ-module. Define a dimension vector $\underline{d} = (d_i)_{i \in Q_0}$

$$d_i = \begin{cases} f(I_i) & \text{if } f(I_i) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let rep$(Q, \underline{d})$ be the affine variety of representations of the opposite quiver Q^{op} with dimension vector $\underline{d}$. Choose a right kQ-module M whose associated point in rep$(Q, \underline{d})$ is generic, so that M is rigid.

Define an object T of the cluster category $\mathcal{C}_Q$ as $M \oplus (\bigoplus_{f(I_i)<0} (\Sigma P_i)^{-f(I_i)})$. For each i satisfying $f(I_i) < 0$, we have the following isomorphisms

$$\mathrm{Ext}^1_{\mathcal{C}_Q}(\Sigma P_i, M) \simeq \mathrm{Hom}_{\mathcal{C}_Q}(P_i, M) \simeq \mathrm{Hom}_{kQ}(P_i, M),$$

where the second isomorphism follows from Proposition 1.7 (d) in [17]. Notice that the space $\mathrm{Hom}_{kQ}(P_i, M)$ vanishes since M does not contain S_i as a composition factor. Thus, the object T is rigid.

Let $M = M_1^{a_1} \oplus \ldots \oplus M_r^{a_r}$ be a decomposition of M with M_j $(1 \le j \le r)$ indecomposable and pairwise non-isomorphic. Let $\mathcal{T}$ be the set

$$\{M_j | 1 \le j \le r\} \cup \{\Sigma P_i | i \in Q_0 \text{ such that } f(I_i) < 0\}.$$

Then $\mathcal{T}$ is a partial tilting set [74]. Denote by $\Sigma\mathcal{T}$ the set $\{\Sigma Y | Y \in \mathcal{T}\}(= \{\Sigma M_j | 1 \le j \le r\} \cup \{I_i | i \in Q_0 \text{ such that } f(I_i) < 0\})$. Let T^+ be a basic cluster-tilting object of $\mathcal{C}_Q$ which contains every element in $\mathcal{T}$ as a direct summand. For an indecomposable object X, we use the notation $[N : X]$ to denote the multiplicity of X appearing as a direct summand in $\mathcal{C}_Q$ of an object N.

Define a new function f' as $\sum_{X \in \Sigma\mathcal{T}} [\Sigma T : X] h_X$, where h_X is the cluster-hammock function on Γ associated with X. Then f' is a cluster-additive function by the Corollary in Section 6 of [74]. Notice that

$$[\Sigma T : I_i] = [T : \Sigma P_i] = -f(I_i) \quad \text{and} \quad [\Sigma T : \Sigma M_j] = a_j \, (1 \le j \le r).$$

Now we rewrite f' as

$$\sum_{I_i \in \Sigma\mathcal{T}} [\Sigma T : I_i] h_{I_i} + \sum_{\Sigma M_j \in \Sigma\mathcal{T}} [\Sigma T : \Sigma M_j] h_{\Sigma M_j} = \sum_{I_i \in \Sigma\mathcal{T}} (-f(I_i)) h_{I_i} + \sum_{j=1}^{r} a_j h_{\Sigma M_j}.$$

In the following we will show that f and f' coincide on all indecomposable injective kQ-modules. Notice that for any pair $X \neq X'$ in a partial tilting set, the value $h_X(X')$ is zero (Section 5, [74]).

Step 1. Look at the indecomposable injective kQ-modules I_l satisfying $f(I_l) < 0$.

It is easy to see that

$$f'(I_l) = -f(I_l) h_{I_l}(I_l) = f(I_l).$$

Step 2. Look at the indecomposable injective kQ-modules I_l satisfying $f(I_l) = 0$.

We have the following isomorphisms

$$\mathrm{Ext}^1_{\mathcal{C}_Q}(T, \Sigma^{-1} I_l) \simeq \mathrm{Hom}_{\mathcal{C}_Q}(T, I_l) \simeq \mathrm{Hom}_{\mathcal{C}_Q}(T, \Sigma^2 P_l) \simeq D\mathrm{Hom}_{\mathcal{C}_Q}(P_l, T)$$

$$\simeq D\mathrm{Hom}_{kQ}(P_l, M) \oplus D\mathrm{Ext}^1_{\mathcal{C}_Q}(P_l, \bigoplus_{f(I_i)<0} (-f(I_i)) P_i) = 0.$$

Hence, the set $\Sigma\mathcal{T} \cup \{I_l | f(I_l) = 0\}$ is also a partial tilting set, which implies that

$$h_X(I_l) = 0, \quad X \in \Sigma\mathcal{T}.$$

As a result, we obtain that

$$f'(I_l) = 0 = f(I_l).$$

Step 3. Look at the indecomposable injective kQ-modules I_l satisfying $f(I_l) > 0$.

We compute the dimension of $\mathrm{Hom}_{\mathcal{C}_Q}(T, I_l)$. As in step 2, we obtain the following isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_Q}(T, I_l) \simeq D\mathrm{Hom}_{kQ}(P_l, M) \simeq \mathrm{Hom}_{kQ}(M, I_l).$$

It follows that

$$\mathrm{dimHom}_{\mathcal{C}_Q}(T, I_l) = \mathrm{dimHom}_{\mathcal{C}_Q}(M, I_l) = d_l = f(I_l).$$

Let B denote the endomorphism algebra $\mathrm{End}_{\mathcal{C}_Q}(T^+)$ and S_{M_j} the simple B-module which corresponds to the indecomposable projective B-module $\mathrm{Hom}_{\mathcal{C}_Q}(T^+, M_j)$. For each object M_j, since I_l does not lie in $\mathrm{add}(\Sigma T^+)$, we have that

$$\mathrm{dimHom}_{\mathcal{C}_Q}(M_j, I_l) = \mathrm{dimHom}_B(\mathrm{Hom}_{\mathcal{C}_Q}(T^+, M_j), \mathrm{Hom}_{\mathcal{C}_Q}(T^+, I_l))$$

$$\begin{aligned} &= \text{the multiplicity of } S_{M_j} \text{ as a composition factor of } \mathrm{Hom}_{\mathcal{C}_Q}(T^+, I_l) \\ &= h_{\Sigma M_j}(I_l), \end{aligned}$$

where the last equality appears in the end of the proof of the Lemma in Section 10 of [74]. Since $h_{I_i}(I_l) = 0$ for all $I_i \in \Sigma\mathcal{T}$, the following equalities

$$f(I_l) = \mathrm{dimHom}_{\mathcal{C}_Q}(M, I_l) = \sum_{j=1}^{r} a_j \mathrm{dimHom}_{\mathcal{C}_Q}(M_j, I_l)$$

$$= \sum_{j=1}^{r} a_j h_{\Sigma M_j}(I_l) = f'(I_l)$$

hold.

Therefore, the cluster-additive functions f and f' coincide on all indecomposable injective kQ-modules, which implies that f is equal to f'. Set $\mathcal{U} = \{\Sigma Z \mid Z$ is an indecomposable direct summand of $T^+\}$, which is a tilting set. Then we obtain that

$$\begin{aligned} f = f' &= \sum\nolimits_{X \in \Sigma \mathcal{T}} [\Sigma T : X] h_X \\ &= \sum\nolimits_{X \in \Sigma \mathcal{T}} [\Sigma T : X] h_X + \sum\nolimits_{X' \in \mathcal{U} \setminus \Sigma \mathcal{T}} 0 \cdot h'_X \\ &= \sum\nolimits_{x \in \mathcal{U}} n_x h_x, \end{aligned}$$

where

$$n_x = \begin{cases} [\Sigma T : x] & \text{if } x \in \Sigma \mathcal{T}, \\ 0 & \text{if } x \in \mathcal{U} \setminus \Sigma \mathcal{T}. \end{cases}$$

This completes the proof. □

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