A higher Albanese map for smooth projective complex threefolds based on a construction by M. Green

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Chapter 1

General Introduction

1.1 Algebraic Cycles and Chow Groups

We begin by introducing the basic object of our study.

Let X be a smooth projective complex manifold (and later on also Y). An *algebraic k-cycle* is a finite formal sum $\sum_i n_i[V_i],\ n_i \in \mathbb{Z}$, where the V_i are k-dimensional reduced and irreducible subvarieties of X. The group $\mathcal{Z}_k(X)$ of algebraic k-cycles is the free abelian group on such subvarieties.

A k-cycle Z is *rationally equivalent to zero* if there exist a finite number of irreducible $k+1$-dimensional subvarieties $W_1, ..., W_r$ of X and functions $f_i \in K(W_i)^*$ for $i = 1, ..., r$ such that

$$Z \in \sum_{i=1}^{r} [div(f_i)].$$

Here $K(W_i)$ denotes the function field of W_i.

The quotient group $CH_k(X) = \mathcal{Z}_k(X)/\mathcal{Z}_k(X)_{rat}$ is called the *k-th Chow group of X*. We will often write $CH^p(X) = CH_{n-p}(X)$ for the group of codimension p cycles of X modulo rational equivalence.

The Chow groups satisfy many properties (W. Fulton's book on intersection theory ([Ful]) is the classic reference for this):

1) Proper pushforward: Let $f : X \longrightarrow Y$ be a proper morphism. If $A \in \mathcal{Z}_k(X)_{rat}$, then $f_*A \in \mathcal{Z}_k(Y)_{rat}$, so that f induces a homomorphism

$$f_* : CH_k(X) \longrightarrow CH_k(Y).$$

If $\alpha \in CH_k(X)$ and $g : Y \longrightarrow Z$ is another proper morphism, then

$$(g \circ f)_*\alpha = g_*(f_*\alpha).$$

2) Flat pullback: Let $f : X \longrightarrow Y$ be a flat morphism. If $B \in \mathcal{Z}^p(Y)_{rat}$, then $f^*B \in \mathcal{Z}^p(X)_{rat}$, so that f induces a homomorphism

$$f^* : CH^p(Y) \longrightarrow CH^p(X).$$

If $\gamma \in CH^p(Z)$ and $g : Y \longrightarrow Z$ is another flat morphism, then

$$(g \circ f)^*\gamma = f_*(g_*\gamma).$$

3) Intersection product: Let $\alpha \in CH^p(X)$, $\beta \in CH^q(X)$. Then there is an intersection product

$$CH^p(X) \otimes CH^q(X) \longrightarrow CH^{p+q}(X),\ \alpha \otimes \beta \mapsto \alpha \cdot \beta.$$

4) Cycle class: Let $A \in \mathcal{Z}^p(X) = \mathcal{Z}_{n-p}(X)$. A has a fundamental class $cl(A) \in H_{2n-2p}(X, \mathbb{Z})$, which by Poincaré duality is isomorphic to $H^{2p}(X, \mathbb{Z})$. If $A \in \mathcal{Z}^p(X)_{rat}$, then $cl(A) = 0$, so that we have an induced map

$$cl : CH^p(X) \longrightarrow H^{2p}(X, \mathbb{Z}).$$

The cycles in the kernel of this map define the subgroup $CH^p(X)_{hom}$. Such cycles are called *homologically equivalent to zero*. Of course this gives a much coarser equivalence relation than rational equivalence. Since $cl(\alpha)$ for any $\alpha \in CH^p(X)$ is mapped to $H^{p,p}(X)$ under the natural map $H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$, the class of an algebraic cycle is a so-called *Hodge class*. The cycle class map is compatible with proper pushforward and flat pullback. It is also compatible with the intersection product: Let $\beta \in CH^q(X)$. Then

$$cl(\alpha \cdot \beta) = cl(\alpha) \cup cl(\beta) \in H^{2p+2q}(X, \mathbb{Z}).$$

5) Correspondences: A correspondence between two smooth projective varieties X and Y is a cycle $\Gamma \in CH^k(X \times Y)$. It induces a map

$$\Gamma_* : CH^p(X) \longrightarrow CH^{p+k-\dim X}(Y)$$

by defining

$$\Gamma_*(\alpha) = pr_{Y*}\big(pr_X^*(\alpha) \cdot \Gamma\big),$$

and a map

$$\Gamma^* : CH^q(Y) \longrightarrow CH^{q+k-\dim Y}(X)$$

by defining

$$\Gamma^*(\beta) = pr_{X*}\big(pr_Y^*(\beta) \cdot \Gamma\big).$$

Let $\Gamma' \in CH^l(Y \times Z)$ be another correspondence. Then we have the *composition* of correspondences

$$\Gamma' \circ \Gamma := pr_{XZ*}\big(pr_{XY}^*\Gamma \cdot pr_{YZ}^*\Gamma'\big) \in CH^{k+l-\dim Y}(X \times Z).$$

If $\Gamma = \Gamma_f$ is the graph of a morphism $f : X \to Y$, then

$$\Gamma' \circ \Gamma_f = (f \times id_Z)^*\Gamma',$$

and if $\Gamma' = \Gamma_g$ is the graph of a morphism $g : Y \to Z$, then

$$\Gamma_g \circ \Gamma = (id_X \times g)_*\Gamma.$$

The class map is compatible with the action of a correspondence: For $\Gamma \in CH^k(X \times Y)$, we have $cl(\Gamma) \in H^{2k}(X \times Y, \mathbb{Z})$. It induces a map in cohomology

$$cl(\Gamma)_* : H^{2p}(X,\mathbb{Z}) \to H^{2(p+k-\dim X)}(Y,\mathbb{Z}),\ cl(\alpha) \mapsto pr_{Y*}\big(pr_X^* cl(\alpha) \cup cl(\Gamma)\big).$$

Then we have

$$cl(\Gamma)_*\big(cl(\alpha)\big) = cl(\Gamma_*(\alpha)).$$

1.2 Divisors on curves, the Jacobian of a curve, Abel's Theorem, and Jacobi's Inversion Theorem

In the case of 0-cycles on smooth complete complex curves (i.e. compact Riemann surfaces), the structure of 0-cycles modulo rational equivalence, or equivalently, divisors modulo linear equivalence, has been understood perfectly well since the middle of the 19th century.

For a smooth curve C of genus g, the space of holomorphic differentials $H^{1,0}(C) = H^0(C, \Omega_C)$ is a complex vector space of dimension g. Let $\omega_1, ..., \omega_g$ be a basis. Let $\gamma_1, ..., \gamma_{2g}$ be a basis for the free $\mathbb{Z}$-module $H_1(C,\mathbb{Z})$ satisfying the conditions

$$\gamma_i \cdot \gamma_{g+i} = 1, \gamma_{g+i} \cdot \gamma_i = -1,\ \text{for } i = 1, ..., g,$$

and

$$\gamma_i \cdot \gamma_j = 0,\ \text{otherwise}.$$

Define the $2j$ *period vectors*

$$\pi_j = {}^t\Big(\int_{\gamma_j} \omega_1, ..., \int_{\gamma_j} \omega_g\Big) \in \mathbb{C}^g.$$

It is well known that these period vectors are linearly independent over $\mathbb{R}$, so that they span a lattice Λ in $\mathbb{C}^g$. The Jacobian variety $J(C)$ of the curve C is then defined to be the quotient $\mathbb{C}^g/\Lambda$; it is a g-dimensional complex torus. More precisely, it is a polarized abelian variety, since the intersection pairing on $H_1(C,\mathbb{Z})$ leads, via the natural identification $H_2(J(C),\mathbb{Z}) = \bigwedge^2(H_1(C,\mathbb{Z}))$, to the polarizing class

$$[\omega_C] : H_2(J(C),\mathbb{Z}) \longrightarrow \mathbb{Z},\ \gamma_i \wedge \gamma_j \mapsto \gamma_i \cdot \gamma_j.$$

Now fix a point p on C. Then for any divisor $D = n_1 d_1 + ... + n_r d_r$ on C, we can define the *Abel-Jacobi map*

$$\mu_C : Div(C) \longrightarrow J(C),\ D \mapsto {}^t\Big(\sum_{i=1}^r n_i \int_p^{d_i} \omega_1, ..., \sum_{i=1}^r n_i \int_p^{d_i} \omega_g\Big).$$

If the divisor D is of degree zero, i.e. $D = d_1 + ... + d_r - e_1 + ...e_r$, where some or all of the points d_i may coincide, just as may the e_i, then we have a natural map

$$\mu_C : Div^0(C) \longrightarrow J(C),\ D \mapsto {}^t\Big(\sum_{i=1}^r \int_{e_i}^{d_i} \omega_1, ..., \sum_{i=1}^r \int_{e_i}^{d_i} \omega_g\Big)$$

without recourse to the choice of a point p on C.

The classic result for curves - the Abel-Jacobi theorem - now gives a complete description of divisors modulo linear equivalence. Let us recall that two divisors D and E on a curve C are *linearly equivalent*, written $D \sim E$, if their difference is a principal divisor, i.e. $D - E = (f)$, where $f \in K^*(C)$ is a meromorphic function on C and the brackets mean taking its divisor.

The Abel-Jacobi Theorem. *Let C be a smooth curve, and let $D, E \in Div^0(C)$ be divisors of degree zero. Then we have:*
(i) $\mu_C(D) = \mu_C(E)$ on $J(C)$ if and only if $D \sim E$.
(ii) The map μ_C is surjective.

The first statement is known as Abel's theorem and the second one is usually called the Jacobi inversion theorem. We remark that the Abel-Jacobi map

$$\mu_C : C^{(g)} \longrightarrow J(C)$$

is surjective and generically injective, where $C^{(g)}$ denotes the g-th symmetric product of the curve C, which is also a smooth, g-dimensional complex variety. Also, for $g \geq 1$, the map $\mu_C : C^{(1)} = C \longrightarrow J(C)$ is injective, since two distinct points on such a curve are never linearly equivalent.

For a long time there was hope that the situation would be similar for algebraic cycles on higher dimensional varieties. For divisors this is true, and the results just described pretty much carry over to the general situation. However, as soon as the codimension of the cycle on the variety rises, things become much, much more complicated. In fact, the simplest case to consider next, that of 0-cycles on surfaces, is sufficient to show how radically the structure of the cycle group modulo rational equivalence changes - in fact, it is still unknown today and precisely what M. Green tried to describe with his higher Abel-Jacobi map. The key result is due to D. Mumford ([Mum]) and we give an account of it in the following section.

1.3 A result from the late '60s by Mumford over $\mathbb{C}$

D. Mumford's article "Rational equivalence of 0-cycles on surfaces" could very well have been published in his "Pathologies in Algebraic Geometry" series. In it he takes up work by F. Severi on 0-cycles. Mumford considers the three following statements:

1) $\exists n$ such that $\forall$0-cycles A with $\deg(A) \geq n, A \sim_{rat} A',\ A'$ effective.

2) $\exists n$ such that the natural map $S^{(n)} \times S^{(n)} \longrightarrow CH_0(S), (A, B) \mapsto [A - B]$, is surjective.

3) $\exists n$ such that $\forall m, A \in S^{(n)}, \exists$ a subvariety $W : A \in W \subset S^{(n)}$ of codimension $\leq n$ consisting of points rationally equivalent to A.

One often says that $CH_0(S)$ is *representable* if it satisfies condition (2) (Roitman's theorem in the next paragraph will explain this terminology), and that it is *finite-dimensional* if it satisfies condition (3). The three conditions are equivalent (for (2) $\Leftrightarrow$ (3) see for example [Voi2], Proposition 10.4).

Now Mumford proved that for projective complex surfaces, as long as they have geometric genus $p_g > 0$, these three conditions no longer hold:

Mumford's theorem for 0-cycles on surfaces. *Let S be a projective complex surface such that $p_g(S) = dimH^0(S, K_S) > 0$. Then $CH_0(S)$ is not finite-dimensional.*

Mumford calls the technique used to arrive at this result *induced differentials*. By pulling back a 2-form ω on the surface S to the form $pr_1^*\omega + ... + pr_n^*\omega$

on $S^n = S \times ... \times S$ via the n projection maps, and then pushing it forward via the quotient map

$$\pi : S^n \longrightarrow S^n/\Sigma_n = S^{(n)},$$

one obtains a 2-form $\omega^{(n)}$ that is still non-degenerate on an open subset of $S^{(n)}$ ($S^{(n)}$ is no longer necessarily smooth, as in the case of curves.) If one is now given a morphism

$$f : W \longrightarrow S^{(n)},$$

where W is smooth, it is possible to define an induced differential 2-form $\omega_f^{(n)}$ on W. In case W is a smooth subset of $S^{(n)}$,

$$i : W \hookrightarrow S^{(n)},$$

then this induced differential form $\omega_f^{(n)}$ is just the restriction of $\omega^{(n)}$ to W, i.e. the pull-back via the inclusion map i.

The main effort is to prove the following

Theorem 1.3.1. *([Mum]) Let $\eta \in H^0(\Omega^2_{S^{(n)}})$ be a 2-form on $S^{(n)}$. Then for all $f : W \longrightarrow S^{(n)}$ such that all the 0-cycles $f(w)$, $w \in W$, are rationally equivalent, it follows that $\eta_f = 0$.*

But since $\eta = \omega^{(n)}$ is non-degenerate on an open subset of $S^{(n)}$, it can only vanish on a subvariety of dimension at most n. This means that any subvariety W parametrizing rationally equivalent 0-cycles can be at most n-dimensional. So if one lets n grow, this contradicts the assertion that $CH_0(S)$ is finite-dimensional.

Nowadays Mumford's result is often shown using a technique by S. Bloch and V. Srinivas (see [BlSri]) called "the decomposition of the diagonal", which considerably simplifies the proof.

The reason for which we give this description of Mumford's induced differentials, or *Mumford's pullback of a differential form* in the special case just considered, is that they are still a fundamental tool in the study of algebraic cycles. The higher Abel-Jacobi invariants we construct in this work can also be used to pull back differential forms on a variety X we consider to a variety parametrizing 0-cycles on X. We call this *Green's pullback of a differential form.* It is the main result of chapter 4 that these two pullbacks coincide.

In the next paragraph we discuss some generalizations of Mumford's result by Roitman, who in particular managed to describe exactly what $CH_0(X)$ of a variety X (of any dimension) looks like when it *is* finite-dimensional.

1.4 Contributions by Roitman

In this paragraph we assume throughout that X is a smooth projective complex variety, of any dimension. To study 0-cycles A. Roitman, a student of Yu. I. Manin, introduced some other equivalence relations between them (in addition to rational and homological equivalence). His first result was this:

Roitman's theorem. *([Roi1]) If $CH_0(X)_{hom}$ is representable, then the Albanese map*

$$alb_X : CH_0(X)_{hom} \longrightarrow \mathrm{Alb}(X)$$

is an isomorphism.

This justifies the term *representable* by the fact that the Albanese variety $\mathrm{Alb}(X)$ is an algebraic group (an abelian variety in this case, just like the Jacobian of a curve) and by the representation theory attached to it. For an extension of this result, see the article "Roitman's theorem for singular projective varieties" by J. Biswas and V. Srinivas ([BiSri]).

In particular, if in this case $q(X) = \frac{1}{2} dim H^1(X, \mathbb{C}) = 0$, then $\mathrm{Alb}(X) = 0$ and it follows that $CH_0(X) \cong \mathbb{Z}$, which means that the class of a 0-cycle is completely described by its degree.

A similar necessary and sufficient condition for $CH_0(X)_{hom}$ to be representable says that all the information about it is then already contained on a smooth curve C lying on X:

Proposition 1.4.1. *$CH_0(X)_{hom}$ is representable if and only if there is a smooth curve $C = Y_1 \cap ... \cap Y_{n-1}$ which is a complete intersection of ample hypersurfaces $Y_j \subset X$ such that the map*

$$i_* : CH_0(C)_{hom} = J(C) \longrightarrow CH_0(X)_{hom}$$

is surjective. Here $i : C \hookrightarrow X$ is the inclusion map.

So in retrospect it is possible to add these two conditions to the three given by Mumford above - they are all equivalent.

In a later paper Roitman managed to describe the *torsion points* of $CH_0(X)_{hom}$, whether finite-dimensional or not.

Roitman's theorem on the torsion of $CH_0(X)_{hom}$. *([Roi2]) The Albanese map*

$$alb_X : CH_0(X)_{hom} \longrightarrow \mathrm{Alb}(X)$$

induces an isomorphism on the torsion points.

S. Bloch (see [Bl2]) independently arrived at a weaker form of this theorem. Next, we discuss some of his results on 0-cycles.

1.5 Work by Bloch on 0-cycles

S. Bloch makes the conjecture of the counterpart of Mumford's theorem in [Bl3]:

Bloch's conjecture for 0-cycles on surfaces.

1.6 The Bloch-Beilinson Conjectures

According to the Bloch-Beilinson conjectures for algebraic cycles on a smooth projective variety X, there should exist a filtration $F^\bullet CH^p(X)$ of the Chow group $CH^p(X)$ of codimension p cycles modulo rational equivalence which satisfies certain properties. In particular, in the case of 0-cycles on complex varieties these properties are:

1) $F^{n+1}CH_0(X) = 0$, where $n = dimX$.

2) The filtration F^i is stable under correspondences,
i.e. for $\Gamma \in CH^m(X \times Y)$, m=dimY, the induced map

$$\Gamma_* : CH_0(X)_{\mathbb{Q}} \to CH_0(Y)_{\mathbb{Q}}$$

satisfies

$$\Gamma_*(F^iCH_0(X)_{\mathbb{Q}}) \subset F^iCH_0(Y)_{\mathbb{Q}}.$$

3) The graded pieces of the filtration are governed by the global holomorphic forms and vice versa, i.e. the map

$$Gr^i_F\Gamma_* : Gr^i_F CH_0(X)_{\mathbb{Q}} \to Gr^i_F CH_0(Y)_{\mathbb{Q}}$$

is zero if and only if the map

$$[\Gamma]^* : H^0(Y, \Omega^i_Y) \to H^0(X, \Omega^i_X)$$

is zero.

The third point is consistent with the fact that both the Chow group of 0-cycles of a connected variety X and its spaces of holomorphic forms are birational invariants. It follows from 3) that $F^1CH_0(X) = CH_0(X)_{hom}$ (and we may think of this subgroup as the kernel of the degree map: $CH_0(X) \to H^{2n}(X,\mathbb{Z}) \cong \mathbb{Z}$), and also that $F^2CH_0(X) = CH_0(X)_{alb}$.

For example, S.Bloch has defined in [Bl1] a filtration for 0-cycles on abelian varieties of arbitrary dimension. The filtration satisfies

$$F^1CH_0(A) = CH_0(A)_{hom},$$

the subgroup of 0-cycles homologically equivalent to zero,

$$F^2CH_0(A) = CH_0(A)_{alb},$$

the subgroup of cycles lying in the kernel of the Albanese map $F^1CH_0(A) \to Alb(A)$, and

$$F^nCH_0(A) \neq 0 \text{ (when the ground field is } \mathbb{C}\text{), but } F^{n+1}CH_0(A) = 0,$$

where $n = dimX$.

However, Bloch relies on the Pontrjagin product for 0-cycles, which in turn is defined using the group structure of the abelian variety, and so it is impossible to generalize his filtration to arbitrary varieties. See also the article [Beau] by Beauville for further results about 0-cycles on abelian varieties.

1.7 M. Green's construction of a higher Abel-Jacobi map for 0-cycles on smooth complex surfaces

For surfaces, Green defined in [G] a map ψ_2^2 from $F^2CH_0(S) := ker(alb_S)$ to a higher Jacobian J_2^2 and conjectured that it was an isomorphism.

Had this been true, the construction would have in particular given a positive answer to Bloch's conjecture for surfaces that if there are no global holomorphic 2-forms on S, then $F^1CH_0(S) \cong Alb(S)$, since the Jacobian J_2^2 is built from the transcendental part of $H^2(S, \mathbb{Z})$ (see Conjecture 2.4. and Theorem 2.5. in [G]). A filtration on $CH_0(S)$ would have been obtained by setting $F^3CH_0(S) = ker(\psi_2^2) = 0$.

1.8 C. Voisin's counterexample, but also two positive results about Green's map

But soon after C.Voisin published a counterexample in [V], and it isn't at all clear how to modify the construction to avoid this. However, in the same paper, she shows that Green's higher Abel-Jacobi invariant has the merit of governing Mumford's pullback of holomorphic 2-forms on the surface S to a variety parametrizing 0-cycles on S. Since this technique, developed by Severi and Mumford (see [Mum]), remains one of the most important in the study of algebraic cycles (for a recent generalization see [Voi]), it seems useful to prove an analogous result for threefolds.

We first extend the definition of ψ_2^2 to 0-cycles on threefolds, which is straightforward. This gives us a map which we call ψ_2^3. We then construct a higher Abel-Jacobi map ψ_3^3 defined on the kernel of ψ_2^3. We plan to show in a subsequent work that ψ_2^3 resp. ψ_3^3 govern the pullback of holomorphic 2-forms resp. 3-forms for families of 0-cycles on X.

1.9 A short excursion to the case of char $k = p$

In the case where the characteristic of the ground field is $p > 0$, G. Welters has very recently constructed a map Δ_2^2 for 0-cycles on surfaces which involves the Brauer group. This map is defined on the kernel of a map Δ_2^1, which is a lifting of the Albanese map. References for this section are [Mil] or [Tam], and we follow more or less verbatim [Wel].

Let z be a 0-cycle of degree 0 of the surface X, and let $Z \subset X$ be a 0-dimensional closed subset containing the support $|z|$ of Z. Let $\overline{Z} \subset \overline{X}$ be the 0-dimensional closed subset obtained as the inverse image of Z in $\overline{X} = X \otimes_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k.

There is an exact sequence of discrete G_k-modules (G_k is the absolute Galois group of k)

$$0 \longrightarrow \frac{H^0(\overline{Z}, \mathbb{G}_m)}{H^0(\overline{X}, \mathbb{G}_m)} \longrightarrow Pic(\overline{X}, \overline{Z}) \longrightarrow Pic(\overline{X}) \longrightarrow 0 \tag{1.1}$$

($\mathbb{G}_m$ is the sheaf associated to the multiplicative group scheme

$$Spec(\mathbb{Z}[t, t^{-1}] \times_{Spec\mathbb{Z}} X).$$

The group $Pic(\overline{X}, \overline{Z})$ consists of isomorphism classes of couples $(\overline{\mathcal{L}}, \overline{\theta})$, with $\overline{\mathcal{L}}$ an invertible sheaf on $\overline{X}$ and

$$\overline{\theta} : \mathcal{O}_{\overline{Z}} \longrightarrow \overline{\mathcal{L}} \otimes \mathcal{O}_{\overline{Z}}$$

an isomorphism. Now the norm map for $Z \longrightarrow Spec(k)$ induces a morphism of G_k-modules, also denoted by z,

$$\frac{H^0(\overline{Z}, \mathbb{G}_m)}{H^0(\overline{X}, \mathbb{G}_m)} \xrightarrow{z} \overline{k}^*, \tag{1.2}$$

and by pushing out (1.1) by (1.2) one obtains a 1-extension representing $\Delta_2^1([z])$:

$$0 \longrightarrow \overline{k}^* \longrightarrow \overline{E}_z \longrightarrow Pic(\overline{X}) \longrightarrow 0. \tag{1.3}$$

For the construction of the map Δ_2^2, Welters considers a 0-cycle z of degree 0, such that $\Delta_2^1([z]) = 0$. There exists a morphism of G_k-modules $\tilde{z}$,

$$Pic(\overline{X}, \overline{Z}) \xrightarrow{\tilde{z}} \overline{k}^*, \tag{1.4}$$

restricting to the map z on $\frac{H^0(\overline{Z},\mathbb{G}_m)}{H^0(\overline{X},\mathbb{G}_m)}$. He then constructs for all 0-dimensional closed subsets $Z \subset X$ explicit 2-extensions of discrete G_k-modules

$$0 \longrightarrow Pic(\overline{X}, \overline{Z}) \longrightarrow \prod_{\overline{C} \subset \overline{X}} Pic(\overline{C}, \overline{Z} \cap \overline{C}) \longrightarrow \widetilde{Br}(\overline{X}, \overline{Z}) \longrightarrow Br(\overline{X}) \longrightarrow 0 \tag{1.5}$$

defining a canonical element $\alpha_Z \in Ext^2_{G_k}\big(Br(\overline{X}), Pic(\overline{X}, \overline{Z})\big)$. This leads to the

Definition 1.9.1. ([Wel], Definition 3. 4)*: Let z be a 0-cycle of degree 0 on the surface X, such that $\Delta_2^1([z]) = 0$. Let $Z \subset X$ be any 0-dimensional closed subset containing the support $|z|$ of z. Then $\Delta_2^2([z]) \in Ext^2_{G_k}(Br(\overline{X}), \overline{k}^*)_{\mathbb{Q}}$ is the image of α_Z under the morphism (1.4), $\Delta_2^2([z]) = \tilde{z}(\alpha_Z)$.*

In the penultimate section of his paper, Welters shows (Theorem 6.8) that in case the surface considered is smooth, projective and geometrically irreducible, this 2-extension class is described as the composition of two 1-extensions, "one of them reflecting the relation of the 0-cycle with its curve environment, and the second one relating the curve to the surface" - just as in M. Green's construction of the map ψ_2^2.

Finally, Welters compares his map Δ_2^2 to the higher l-adic Abel-Jacobi map

$$d_2^2 : CH_0(X)_{alb} \otimes \mathbb{Q} \longrightarrow H^2_{cont}\big(G_k, H^2(\overline{X}, \mathbb{Q}_l(2))\big)$$

defined by Raskind (see [Ras], [Jan1]). This leads to the comparison Theorem 5.1, which in turn is used to restate Theorem 6.8 in a different form (see Corollary 6.15).

Chapter 2

Construction of ψ_2^3 - Green's higher A-J map

2.1 Introduction

In this paragraph we construct a higher Abel-Jacobi map on $F^2CH_0(X) := ker(alb_X)$, which is immediately derived from Green's map for surfaces.

2.2 Construction of ψ_2^3

Let X be a smooth projective complex threefold. Let Z_0 be a 0-cycle of degree 0 in the kernel of the Albanese map:

$$alb_X(Z_0) = 0 \in \mathrm{Alb}(X).$$

Choose a smooth ample surface $S \subset X$ containing $supp(Z_0)$: by the Lefschetz hyperplane theorem $H^1(X,\mathbb{Z}) \cong H^1(S,\mathbb{Z})$, so $\mathrm{Alb}(X) \cong \mathrm{Alb}(S)$ and $alb_X(Z_0) = 0 \Rightarrow alb_S(Z_0) = 0$.

Let $i : C \to S$ be a smooth, but not necessarily connected curve, which maps generically one to one onto its image $i(C)$ on S, and Z a 0-cycle supported on C such that $i_*(Z) = Z_0$.

In general, we will consider all smooth surfaces $j : S \to X$ and smooth curves $i : C \to S$ carrying a 0-cycle Z such that $(j \circ i)_*(Z) = Z_0$ and $alb_S(i_*(Z)) = 0$.

Given such S, C and Z we can apply Green's construction of the higher Abel-Jacobi map ψ_2^2 for a 0-cycle on a surface (see [G] or [V] for the description). He obtains two extension classes

$$e_{S,C} \in H^2(S,\mathbb{Z})^*_{tr} \otimes_{\mathbb{Z}} H^1(C,\mathbb{Z})_{new} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$$

and

$$f_{C,Z} \in H^1(C,\mathbb{Z})^*_{new} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Here

$$H^1(C,\mathbb{Z})_{new} := \frac{H^1(C,\mathbb{Z})}{H^1(S,\mathbb{Z})} \text{ and } H^2(S,\mathbb{Z})_{tr} := NS(S)^{\perp}.$$

Notice that, using Poincaré duality on $H^2(S,\mathbb{Z})$, we have

$$H^2(S,\mathbb{Z})^*_{tr} = \frac{H^2(S,\mathbb{Z})}{NS(S)}.$$

Green takes the tensor product of these two extension classes and contracts via the map $H^1(C,\mathbb{Z})^*_{new} \otimes H^1(C,\mathbb{Z})_{new} \to \mathbb{Z}$ to obtain an element

$$e_{S,C} \cdot f_{C,Z} \in H^2(S,\mathbb{Z})^*_{tr} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Finally, in order to make the invariant $e_{S,C} \cdot f_{C,Z}$ independent of the chosen lifting of Z_0, he quotients by the subgroup $U_2^2(S) \subset H^2(S,\mathbb{Z})^*_{tr} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$ defined as the group generated by those $e_{S,C} \cdot f_{C,Z}$ for which $i_*(Z) = 0$ as a 0-cycle of S.

Let now

$$j_* : H^2(S,\mathbb{Z}) \to H^4(X,\mathbb{Z})$$

be the Gysin morphism induced by $j : S \to X$ and also call

$$j_* : H^2(S,\mathbb{Z})^*_{tr} \to H^4(X,\mathbb{Z})/j_*NS(S)$$

the induced map on the quotient.

Definition 2.2.1. $H^4(X,\mathbb{Z})^*_{tr} = H^4(X,\mathbb{Z})/ < j_*NS(S) >, \forall j : S \to X, S$ *is a smooth surface.*

We have the projection

$$\pi : \frac{H^4(X,\mathbb{Z})}{j_*NS(S)} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \to H^4(X,\mathbb{Z})^*_{tr} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Definition 2.2.2. *Let $U_2^3(X)$ be the group generated by all $j_*(e_{S,C} \cdot f_{C,Z})$ for which $(j \circ i)_*(Z) = 0$ as a 0-cycle of X.*

Let

$$J_2^3(X) = \frac{H^4(X,\mathbb{Z})^*_{tr} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}}{U_2^3(X)}.$$

Lemma 2.2.3. *The projection of $j_*(e_{S,C} \cdot f_{C,Z})$ in $J_2^3(X)$ is independent of the surface S chosen in the construction.*

Proof. Suppose $Z_1 \subset C_1 \xrightarrow{i_1} S_1 \xrightarrow{j_1} X$ and $Z_2 \subset C_2 \xrightarrow{i_2} S_2 \xrightarrow{j_2} X$ are both 0-cycles mapping to Z_0 on X. Then

$$Z_1 - Z_2 \subset C_1 \sqcup C_2 \xrightarrow{i'} S_1 \sqcup S_2 \xrightarrow{j'} X$$

obviously satisfies $(j' \circ i')_*(Z_1 - Z_2) = 0$, and we have

$$j_{1*}(e_{S_1,C_1} \cdot f_{C_1,Z_1}) - j_{2*}(e_{S_2,C_2} \cdot f_{C_2,Z_2}) = j'_*(e_{S_1 \sqcup S_2, C_1 \sqcup C_2} \cdot f_{C_1 \sqcup C_2, Z_1 - Z_2}) \in U_2^3(X).$$

□

So this, being independent of the choices made, defines a map

$$\begin{array}{rcl} \psi_2^3 : ker(alb_X) & \to & J_2^3(X), \\ Z_0 & \mapsto & [j_*(e_{S,C} \cdot f_{C,Z})], \end{array}$$

which factors through rational equivalence. For if a cycle Z_0 is rationally equivalent to 0, there is a smooth curve $C \xrightarrow{i} X$ and a cycle Z rationally equivalent to 0 in C, such that $i_*Z = Z_0$. Then since $f_{C,Z} = 0$, $i'(Z)$ is Albanese equivalent to 0 in S, and furthermore we have $j_*(e_{S,C} \cdot f_{C,Z}) = 0$ for any $C \xrightarrow{i'} S \xrightarrow{j} X$, with S a smooth surface.

Chapter 3

Construction of ψ_3^3

3.1 Introduction

In this paragraph we construct a higher Abel-Jacobi map defined on $ker(\psi_2^3)$.

3.2 Construction of ψ_3^3

Let Z_0 be a 0-cycle in the kernel of the map $\psi_2^3 : ker(alb_X) \to J_2^3(X)$. For an adequate choice $Z \to C \to S$ we construct invariants

$$d_{X,S} \in J^3(X)_{AJ} \otimes H^2(S,\mathbb{Z})^*_{new},$$

$$e_{S,C} \cdot f_{C,Z} \in H^2(S,\mathbb{Z})^*_{tr,new} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z},$$

which we contract to an element lying in $J^3(X)_{AJ} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}^{\otimes 2}$. This element we then project onto a certain quotient $J_3^3(X)$ to obtain our third Abel-Jacobi-invariant of the 0-cycle Z.

The invariant $e_{S,C} \cdot f_{C,Z}$ is essentially the one from Green's construction for 0-cycles on surfaces. We show how to obtain $d_{X,S}$. Let $\Gamma_j \subset S \times X$ be the graph of the morphism $j : S \to X$. It has its cohomology class $[\Gamma_j]$ in $H^6(S \times X, \mathbb{Z})$.

Let $\Delta_X \subset X \times X$ be the diagonal in $X \times X$, and let $[\Delta_3] = id_{H^3(X,\mathbb{Z})}$ be the component of $[\Delta_X]$ lying in $H^3(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^3(X,\mathbb{Z})$ in the Künneth decomposition of $H^6(X \times X, \mathbb{Z})$. Note that $[\Delta_3]$ is a Hodge class (i.e. it maps to $H^{3,3}(X)$), but that it is not necessarily an analytic (algebraic) cycle class.

Let $\Delta_S \in S \times S$ be the diagonal in $S \times S$, and let $[\Delta_2] = id_{H^2(S,\mathbb{Z})}$ be the component of $[\Delta_S]$ lying in $H^2(S,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})$ in the Künneth decompostion of $H^4(S \times S, \mathbb{Z})$. Note that $[\Delta_2]$ is not only a Hodge class, but by a result of Murre ([M]) also analytic (algebraic).

For what is to follow, we recall that for any complex analytic manifold X there is a short exact sequence

$$0 \to J^{2p-1}(X) \to H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) \to Hdg^{2p}(X) \to 0, \tag{3.1}$$

where

$$J^{2p-1}(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p H^{2p-1}(X) + H^{2p-1}(X, \mathbb{Z})}$$

is the p-th Griffiths intermediate Jacobian of X, $H_{\mathcal{D}}$ denotes Deligne cohomology, which is defined as the hypercohomology of the complex of sheaves on X

$$\mathbb{Z}(p)_{\mathcal{D}} : 0 \to \mathbb{Z}(p) \to \mathcal{O}_X \to \Omega_X^1 \to ... \to \Omega_X^{p-1} \to 0,$$

and $Hdg^{2p}(X)$ are the Hodge classes

$$\{\eta \in H^{2p}(X, \mathbb{Z}) : \alpha_* \eta \in H^{p,p}(X)\},$$

with $\alpha_* : H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$ the natural map (see [EV] and [GMV] for descriptions).

We define for two classes S in $X \times Y$ and T in $Y \times Z$ (which may be cohomology classes in $H^*(X \times Y)$, Deligne cohomology classes in $H_{\mathcal{D}}^*(X \times Y)$, or cycle classes in $CH^p(X \times Y)$ resp. $...(Y \times Z)$) the *composed correspondence* $S \circ T := p_{13*}\{p_{12}^* S \cdot p_{23}^* T\}$ on $X \times Z$. Here "$\cdot$" denotes cup-product, the product in Deligne-cohomology and intersection product of cycles, respectively, and the p_{ij} are the various projection maps from $X \times Y \times Z$ (see chapter 16 in [F]). In the case of Deligne cohomology, we will denote the composition by "$\circ_{\mathcal{D}}$".

Now, since Γ_j and Δ_2 are both algebraic cycles, we can compose them to obtain $\Gamma_j \circ \Delta_2 \subset S \times X$, with $[\Gamma_j \circ \Delta_2] \in H^6(S \times X, \mathbb{Z})$.

Lemma 3.2.1. *$[\Delta_3] \circ [\Gamma_j \circ \Delta_2] = 0$ on $S \times X$.*

Proof. At first, our projection maps refer to $S \times S \times X$. For example, p_{12} is the map $S \times S \times X \to S \times S$. By definition

$$[\Delta_2] \in H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \subset H^4(S \times S, \mathbb{Z}),$$

and it follows that

$$p_{12}^*[\Delta_2] \in H^2(S, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \otimes H^0(X, \mathbb{Z}) \subset H^6(S \times S \times X, \mathbb{Z}).$$

We have

$$[\Gamma_j \circ \Delta_2] \in H^2(S, \mathbb{Z}) \otimes H^4(X, \mathbb{Z}) \subset H^6(S \times X, \mathbb{Z}).$$

Next we consider the projection maps from $S \times X \times X$. Again, by definition

$$[\Delta_3] \in H^3(X,\mathbb{Z}) \otimes H^3(X,\mathbb{Z}) \subset H^6(X \times X,\mathbb{Z}),$$

and it follows that

$$p_{23}^*[\Delta_3] \in H^0(S,\mathbb{Z}) \otimes H^3(X,\mathbb{Z}) \otimes H^3(X,\mathbb{Z}) \subset H^6(S \times X \times X,\mathbb{Z}).$$

Similarly,

$$p_{12}^*[\Gamma_j \circ \Delta_2] \in H^2(S,\mathbb{Z}) \otimes H^4(X,\mathbb{Z}) \otimes H^0(X,\mathbb{Z}) \subset H^6(S \times X \times X,\mathbb{Z}).$$

So finally, for the reason of cohomological degree, we see that

$$p_{23}^*[\Delta_3] \cup p_{12}^*[\Gamma_j \circ \Delta_2] = 0,$$

which implies that

$$[\Delta_3] \circ [\Gamma_j \circ \Delta_2] = 0.$$

□

For the cycle $\Gamma_j \circ \Delta_2$ we have the Deligne cycle class map

$$\begin{aligned} cl_{\mathcal{D}} : CH^3(S \times X) &\rightarrow H^6_{\mathcal{D}}(S \times X, \mathbb{Z}(3)) \\ \Gamma_j \circ \Delta_2 &\mapsto [\Gamma_j \circ \Delta_2]_{\mathcal{D}}, \end{aligned}$$

which is compatible with the usual cycle class map cl, i.e. $[\Gamma_j \circ \Delta_2]_{\mathcal{D}}$ goes to $[\Gamma_j \circ \Delta_2]$ in the short exact sequence

$$0 \rightarrow J^5(S \times X) \rightarrow H^6_{\mathcal{D}}(S \times X, \mathbb{Z}(3)) \rightarrow Hdg^6(S \times X) \rightarrow 0. \tag{3.2}$$

Since we don't know whether $[\Delta_3] \in Hdg^6(X \times X)$ is algebraic, we can't use the Deligne cycle class map, but we can still lift it to a class

$$[\Delta_3]_{\mathcal{D}} \in H^6_{\mathcal{D}}(X \times X, \mathbb{Z}(3))$$

via the sequence

$$0 \rightarrow J^5(X \times X) \rightarrow H^6_{\mathcal{D}}(X \times X, \mathbb{Z}(3)) \rightarrow Hdg^6(X \times X) \rightarrow 0. \tag{3.3}$$

Then it is clear that

$$[\Delta_3]_{\mathcal{D}} \circ_{\mathcal{D}} [\Gamma_j \circ \Delta_2]_{\mathcal{D}} =\in H^6_{\mathcal{D}}(S \times X, \mathbb{Z}(3)),$$

will map to

$$[\Delta_3] \circ [\Gamma_j \circ \Delta_2] \in H^6_{\mathcal{D}}(S \times X, \mathbb{Z}(3)),$$

which we have just proven to be zero. Hence $[\Delta_3]_{\mathcal{D}} \circ_{\mathcal{D}} [\Gamma_j \circ \Delta_2]_{\mathcal{D}}$ is in fact an element $\delta \in J^5(S \times X) = J(H^5(S \times X))$.

For what is to come we introduce a convenient form of notation for the Jacobian of a (pure) Hodge structure of odd weight:

Definition 3.2.2. *Let $(V_{\mathbb{Z}}, F^\bullet)$ be a pure Hodge structure of weight $2k-1$, i.e. $V_{\mathbb{Z}}$ is a finitely generated free abelian $\mathbb{Z}$-module and there is a decomposition*

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=2k-1} V^{p,q},$$

where $V^{p,q} := F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$ is defined via the Hodge filtration $F^\bullet$. Then we put

$$J(V) := \frac{V_{\mathbb{C}}}{F^k V_{\mathbb{C}} + V_{\mathbb{Z}}}.$$

Now using the Künneth decomposition

$$H^5(S \times X) = \sum_{p=0}^{5} H^p(S) \otimes H^{5-p}(X),$$

we obtain a natural projection from $J(H^5(S \times X))$ to

$$J(H^2(S) \otimes H^3(X)) \cong H^3(X, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z},$$

and a further projection to

$$J(\frac{H^2(S)}{j^* H^2(X)} \otimes \frac{H^3(X)}{j_* H^1(S)}) \cong \frac{H^3(X, \mathbb{Z})}{j_* H^1(S, \mathbb{Z})} \otimes \frac{H^2(S, \mathbb{Z})}{j^* H^2(X, \mathbb{Z})} \otimes \mathbb{R}/\mathbb{Z}$$

(see Lemma 1.2. in [G] for these isomorphisms). The projection δ' of δ in this quotient can be seen as an element of $\frac{J^3(X)}{\mathrm{Pic}^0(S)} \otimes_{\mathbb{Z}} \frac{H^2(S,\mathbb{Z})}{j^* H^2(X,\mathbb{Z})}$.

Lemma 3.2.3. *δ' is independent of the lifting $[\Delta_3]_{\mathcal{D}}$ of $[\Delta_3]$.*

Proof. Let us suppose that we had chosen a different lifting $[\Delta_3]_{\mathcal{D}} + \nu$, $\nu \in J^5(X \times X)$. We work with the projection maps from $S \times S \times X \times X$. We have

$$p_{34}^*([\Delta_3]_{\mathcal{D}} + \nu) \cdot_{\mathcal{D}} p_{23}^*[\Gamma_j]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{12}^*[\Delta_2]_{\mathcal{D}} =$$

$$= (p_{34}^*[\Delta_3]_{\mathcal{D}} + p_{34}^*\nu) \cdot_{\mathcal{D}} p_{23}^*[\Gamma_j]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{12}^*[\Delta_2]_{\mathcal{D}} =$$

$$= p_{34}^*[\Delta_3]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{23}^*[\Gamma_j]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{12}^*[\Delta_2]_{\mathcal{D}} + p_{34}^*\nu \cdot p_{23}^*[\Gamma_j] \cdot p_{12}^*[\Delta_2]$$

in $H^{12}_{\mathcal{D}}(S \times S \times X \times X, \mathbb{Z}(6))$, since the intermediate Jacobian is an ideal of square zero (see Proposition 7.10 in [E-V]), and where $\cdot$ denotes the action of a cohomology class on the intermediate Jacobian. Now via the Künneth formula we have the decomposition

$$\nu \in J^5(X \times X) = J(H^5(X \times X)) = \bigoplus_{i=0}^{5} J(H^i(X) \otimes H^{5-i}(X)).$$

The class $[\Gamma_j]$ acts on $J(H^5(X \times X))$ as $(j^* \times id_X)$ (see Proposition 16.1.1 in [F]), and the class $[\Delta_2]$ kills all the components other than $J(H^2(S) \otimes H^3(X))$. But anything in the image of $j^* : H^2(X) \to H^2(S)$ goes to zero in the quotient defined above, so we are done.

□

Remark 3.2.4. *It is a result of Lieberman [L] that for abelian varieties the Künneth components of the diagonal are all algebraic, so in particular the* $[\Delta_3]$ *used here. This is a consequence of the fact that numerical and homological equivalence coincide for cycles on abelian varieties.*

Lemma 3.2.5. δ' *is independent of the choice made for the representative* $\Delta_2 \in CH^2(S \times S)$ *of* $[\Delta_2]$*.*

Proof. The proof is similar to the previous one. Murre stresses in his paper [Mur] that the choice of Δ_2 as a cycle class is not canonical (see section 5, "A remark about the uniqueness of the motives"). What is clear, however, is that such a representative will be sent to $[\Delta_2] \in H^4(S \times S, \mathbb{Z})$, and so again any two representatives will at most differ by an element $\mu \in J^3(S \times S)$ when mapped into the Deligne cohomology group $H^4_{\mathcal{D}}(S \times S, \mathbb{Z}(2))$.

The difference to the previous proof is that μ will be pushed forward via the graph of j, not pulled back, but this kind of element will be taken care of by the quotient $\frac{H^3(X)}{j_* H^1(S)}$. We obtain

$$p_{34}^*[\Delta_3]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{23}^*[\Gamma_j]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{12}^*([\Delta_2]_{\mathcal{D}} + \mu) =$$

$$= p_{34}^*[\Delta_3]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{23}^*[\Gamma_j]_{\mathcal{D}} \cdot_{\mathcal{D}} p_{12}^*[\Delta_2]_{\mathcal{D}} + p_{34}^*[\Delta_3] \cdot p_{23}^*[\Gamma_j] \cdot p_{12}^*\mu$$

in $H^{12}_{\mathcal{D}}(S \times S \times X \times X, \mathbb{Z}(6))$. We have the decomposition

$$\mu \in J(H^3(S \times S)) = \bigoplus_{i=0}^{3} J(H^i(S) \otimes H^{3-i}(S)).$$

Now $[\Gamma_j]$ acts on $J^3(S \times S)$ as $(id_S \times j_*)$, and this time $[\Delta_3]$ only leaves $J(H^2(S) \otimes H^3(X))$, which completes the proof.

□

We have to work a little bit more before obtaining our final invariant $d_{X,S}$.

Definition 3.2.6. *Let*

$$H^2(S,\mathbb{Z})_{new} = ker(j_* : H^2(S,\mathbb{Z}) \to H^4(X,\mathbb{Z})).$$

Then we have

$$H^2(S,\mathbb{Z})^*_{new} = \frac{H^2(S,\mathbb{Z})}{j^*H^2(X,\mathbb{Z})}.$$

In order to contract $d_{X,S}$ and $e_{S,C} \cdot f_{C,Z}$ to $d_{X,S} \cdot e_{S,C} \cdot f_{C,Z}$, we quotient $J^3(X)$ by the image of the Abel-Jacobi map

$$\phi_X : \mathcal{Z}^2(X)_{hom} \to J^3(X),$$

where $\mathcal{Z}^2(X)_{hom}$ denotes the group of codimension two algebraic cycles of X which are homologically equivalent to zero.

Definition 3.2.7. $J^3(X)_{AJ} := \frac{J^3(X)}{im(\phi_X)}$.

Note that $im(\phi_X) \supseteq j_*H^1(S,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} = j_* \operatorname{Pic}^0(S)$.

Definition 3.2.8. *We call $d_{X,S}$ the projection of δ' in*

$$J^3(X)_{AJ} \otimes H^2(S,\mathbb{Z})^*_{new}.$$

We now specify our choice of C and S: since by assumption Z_0 lies in the kernel of the map ψ_2^3, we know that there exists a surface S, a curve C and a lifting Z of Z_0 such that the invariant $e_{S,C} \cdot f_{C,Z} \in H^2(S,\mathbb{Z})^*_{tr} \otimes \mathbb{R}/\mathbb{Z}^{\otimes 2}$ belongs to $H^2(S,\mathbb{Z})^*_{tr,new} \otimes \mathbb{R}/\mathbb{Z}^{\otimes 2}$, where

$$H^2(S,\mathbb{Z})^*_{tr,new} := ker(j_* : H^2(S,\mathbb{Z})^*_{tr} = \frac{H^2(S,\mathbb{Z})}{NS(S)} \to \frac{H^4(X,\mathbb{Z})}{j_*NS(S)}).$$

Lift $e_{S,C} \cdot f_{C,Z}$ to an element in $H^2(S,\mathbb{Z})_{new} \otimes \mathbb{R}/\mathbb{Z}^{\otimes 2}$. This lifting will depend upon the choice of an element

$$\alpha \in ker(j_* : NS(S) \to H^4(X,\mathbb{Z})).$$

In order to have a well-defined contraction between $d_{X,S}$ and $e_{S,C} \cdot f_{C,Z}$ via the pairing

$$H^2(S,\mathbb{Z})^*_{new} \otimes H^2(S,\mathbb{Z})_{new} \to \mathbb{Z},$$

we must prove the following:

Lemma 3.2.9.

$$d_{X,S} \cdot e_{S,C} \cdot f_{C,Z} \in J^3(X)_{AJ} \otimes \mathbb{R}/\mathbb{Z}^{\otimes 2}$$

is independent of the element α chosen in the lifting.

Proof. We can view

$$\delta' \in \frac{J^3(X)}{j_* \operatorname{Pic}^0(S)} \otimes \frac{H^2(S,\mathbb{Z})}{j^* H^2(X,\mathbb{Z})}$$

as an element

$$\delta' \in Hom(H^2(S,\mathbb{Z}) \cap ker j_*, \frac{J^3(X)}{j_* \operatorname{Pic}^0(S)}).$$

Lemma 3.2.10. *The restriction of δ' (the coboundary map in the below diagram) to $NS(S) \cap ker j_*$ is equal to the composition of maps*

$$\phi_X \circ j_* : NS(S) \cap ker j_* \to \mathcal{Z}(X)^2_{hom}/j_* \operatorname{Pic}^0(S) \to J^3(X)/j_* \operatorname{Pic}^0(S)$$

Proof of lemma 3.2.10. Consider the action of $\delta = [\Delta_3]_{\mathcal{D}} \circ_{\mathcal{D}} [\Gamma_j]_{\mathcal{D}} \circ_{\mathcal{D}} [\Delta_2]_{\mathcal{D}}$ on a class $\alpha \in NS(S) \cap ker j_* \subseteq NS(S) = Hdg^2(S)$, which we can lift to an element $\tilde{\alpha} \in H^2_{\mathcal{D}}(S, \mathbb{Z}(1)) \cong \operatorname{Pic}(S)$:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}^0(S) \arrow[r] \arrow[d] & H^2_{\mathcal{D}}(S,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\circ_{\mathcal{D}}[\Delta_2]_{\mathcal{D}}"] & Hdg^2(S) \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \operatorname{Pic}^0(S) \arrow[r] \arrow[d, "j_*"] & H^2_{\mathcal{D}}(S,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\circ_{\mathcal{D}}[\Gamma_j]_{\mathcal{D}}=j_*"] & Hdg^2(S) \arrow[r] \arrow[d, "j_*"] & 0 \\
0 \arrow[r] & J^3(X) \arrow[r] \arrow[d, "id"] & H^4_{\mathcal{D}}(X,\mathbb{Z}(2)) \arrow[r] \arrow[d, "\circ_{\mathcal{D}}[\Delta_3]_{\mathcal{D}}"] & Hdg^4(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^3(X) \arrow[r] & H^4_{\mathcal{D}}(X,\mathbb{Z}(2)) \arrow[r] & Hdg^4(X) \arrow[r] & 0.
\end{tikzcd}$$

Since $j_*(\alpha) = 0 \in Hdg^4(X)$, $j_*(\tilde{\alpha})$ gives an element of $J^3(X)$, which is precisely the Abel-Jacobi image $\phi_X(j_*(\alpha))$ (see Proposition 1 in [EZ], for example), and which is well-defined up to an element in $j_* \operatorname{Pic}^0(S)$.

□

But since we have quotiented by the image of the Abel-Jacobi map ϕ_X in the definition of $d_{X,S}$, we see that the choice of α makes no difference in our construction, and so we are done with the proof of Lemma 3.2.9.

□

Now we can define the higher Jacobian $J_3^3(X)$ and the corresponding map ψ_3^3 for 0-cycles on X:

Definition 3.2.11. *Let $U_3^3(X)$ be the group generated by all $d_{X,S} \cdot e_{S,C} \cdot f_{C,Z}$ for which $(j \circ i)_*(Z) = 0$ as a 0-cycle of X,*

$$J_3^3(X) = \frac{J^3(X)_{AJ} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}^{\otimes 2}}{U_3^3(X)},$$

and define the map

$$\begin{array}{rcl} \psi_3^3 : ker(\psi_2^3) & \to & J_3^3(X), \\ Z_0 & \mapsto & [d_{X,S} \cdot e_{S,C} \cdot f_{C,Z}]. \end{array}$$

What remains to be shown in order to have a well-defined map is this:

Lemma 3.2.12. *$\psi_3^3(Z_0)$ is independent of the choice of the surface S.*

Proof. The proof is like the one of lemma 2.3: suppose again that $Z_1 \to C_1 \to S_1$ and $Z_2 \to C_2 \to S_2$ are both 0-cycles mapping to Z_0 on X.

Then we immediately have for their difference

$$d_{X,S_1} \cdot e_{S_1,C_1} \cdot f_{C_1,Z_1} - d_{X,S_2} \cdot e_{S_2,C_2} \cdot f_{C_2,Z_2} =$$

$$d_{X,S_1 \sqcup S_2} \cdot e_{S_1 \sqcup S_2, C_1 \sqcup C_2} \cdot f_{C_1 \sqcup C_2, Z_1 - Z_2} \in U_3^3(X).$$

□

Finally we mention that - for exactly the same reason as ψ_2^3 in part 2 - the map ψ_3^3 factors through rational equivalence, and so indeed defines a map from $ker(\psi_2^3) \subset CH_0(X)$ to $J_3^3(X)$.

Chapter 4

A formula for the pullback of holomorphic 3-forms

4.1 Introduction

In this section we prove that Mumford's pullback of a holomorphic 3-form on X to a variety parametrizing 0-cycles on X can be computed with our three Abel-Jacobi invariants.

4.2 A formula for the pullback of holomorphic 3-forms

4.2.1 Statement of the Theorem

We let our curves and surfaces vary in families parametrized by the same open complex ball B. Consider the following situation: Let $\mathcal{C}$ and $\mathcal{S}$ be smooth complex varieties with proper submersive holomorphic maps π and ρ to B of relative dimensions 1 and 2, respectively. Let $i : \mathcal{C} \to \mathcal{S}$ and $j : \mathcal{S} \to X \times B$ be analytic morphisms making both squares in the diagram below commute:

$$\begin{array}{ccccccc}
\mathcal{C} & \xrightarrow{i} & \mathcal{S} & \xrightarrow{j} & B \times X & \xrightarrow{pr_2} & X \\
\pi \downarrow & & \rho \downarrow & & pr_1 \downarrow & & \\
B & = & B & = & B & & .
\end{array}$$

Let $s_1, ..., s_N$ be holomorphic sections of $\pi : \mathcal{C} \to B$, and let $m_1, ..., m_N$ be integers such that the 0-cycle

$$Z_b := \sum_{i=0}^{N} m_i s_i(b)$$

satisfies the following conditions for all $b \in B$:
1) it is of degree 0 on each component of the curve C_b,
2) $i_{b*}Z_b$ is Abel-Jacobi equivalent to 0 in S_b, and
3)
$$j_{b*}(e_b \cdot f_b) = 0 \;\; in \;\; H^4(X, \mathbb{Z})/j_{b*}NS(S_b) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2},$$
where $e_b = e_{S_b,C_b}$, $f_b = f_{C_b,Z_b}$, and
$$e_b \cdot f_b \in H^2(S_b, \mathbb{Z})^*_{tr} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$$
is Green's contraction introduced in the previous section.

We would like to differentiate the invariants $(e_b)_{b\in B}$ with respect to the parameters. However, since the groups $H^2(S_b, \mathbb{Z})_{tr}$ do not form a local system (the Neron-Severi group may jump), we can't do that.

Instead, let $e'_b = e'_{S_b,C_b}$ be the extension class of the short exact sequence of mixed Hodge structures
$$0 \longrightarrow \frac{H^1(C_b)}{i_b^* H^1(S_b)} \longrightarrow H^2(S_b, C_b) \longrightarrow H^2(S_b)_{C_b} \to 0, \tag{4.1}$$
where
$$H^2(S_b)_{C_b} := ker\big(i_b^* : H^2(S_b) \to H^2(C_b)\big).$$
Identifying the intermediate Jacobian in which e'_b lies to its underlying real torus, we have
$$\begin{aligned} e'_b &\in \frac{H^1(C_b, \mathbb{Z})}{i_b^* H^1(S_b, \mathbb{Z})} \otimes (H^2(S_b, \mathbb{Z})_{C_b})^* \otimes \mathbb{R}/\mathbb{Z} \\ &= \frac{H^1(C_b, \mathbb{Z})}{i_b^* H^1(S_b, \mathbb{Z})} \otimes \frac{H^2(S_b, \mathbb{Z})}{< C_{b,r} >} \otimes \mathbb{R}/\mathbb{Z}, \end{aligned}$$
where the $C_{b,r}$ are the components of the curve C_b. We know (cf. Appendix A) that the previous invariant e_b is obtained by projecting e'_b via the quotient map
$$\frac{H^2(S_b, \mathbb{Z})}{< C_{b,r} >} \longrightarrow \frac{H^2(S_b, \mathbb{Z})}{< NS(S_b) >} = H^2(S, \mathbb{Z})^*_{tr}.$$

Now since the first two quotients in the last expression above are locally constant, and B is simply connected, we have natural identifications
$$\frac{H^1(C_b, \mathbb{Z})}{i_b^* H^1(S_b, \mathbb{Z})} \cong \frac{H^1(C_0, \mathbb{Z})}{i_b^* H^1(S_0, \mathbb{Z})} \;\; \text{and} \;\; \frac{H^2(S_b, \mathbb{Z})}{< C_{b,r} >} \cong \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >}.$$

Next, the Abel-Jacobi invariant
$$f_b = alb\, Z_b \in J(C_b) \cong H^1(C_b, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

belongs to

$$H^1(C_b, \mathbb{Z})_{new} \otimes \mathbb{R}/\mathbb{Z}$$

by our second assumption, where

$$H^1(C_b, \mathbb{Z})_{new} := ker\big(i_{b*} : H^1(C_b, \mathbb{Z}) \to H^3(S_b, \mathbb{Z})\big).$$

The groups $H^1(C_b, \mathbb{Z})_{new}$ form a local system, and as above we identify canonically

$$H^1(C_b, \mathbb{Z})_{new} \cong H^1(C_0, \mathbb{Z})_{new}.$$

So now we can view $(e'_b)_{b\in B}$ and $(f_b)_{b\in B}$ as maps

$$e' : B \longrightarrow \frac{H^1(C_0, \mathbb{Z})}{i_b^* H^1(S_0, \mathbb{Z})} \otimes \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >} \otimes \mathbb{R}/\mathbb{Z},\ b \mapsto e'_b,$$

and

$$f : B \longrightarrow H^1(C_0, \mathbb{Z})_{new} \otimes \mathbb{R}/\mathbb{Z},\ b \mapsto f_b,$$

which are differentiable - in fact they are even real-analytic, since we know by Griffiths' results that the Abel-Jacobi invariants vary holomorphically in the family. Differentiating these two functions gives maps

$$de' : T_B^{\mathbb{R}} \longrightarrow \frac{H^1(C_0, \mathbb{Z})}{i_0^* H^1(S_0, \mathbb{Z})} \otimes_{\mathbb{Z}} \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >} \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$df : T_B^{\mathbb{R}} \longrightarrow H^1(C_0, \mathbb{Z})_{new} \otimes_{\mathbb{Z}} \mathbb{R},$$

and by adjunction real differential forms

$$de' \in \frac{H^1(C_0, \mathbb{Z})}{i_0^* H^1(S_0, \mathbb{Z})} \otimes_{\mathbb{Z}} \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >} \otimes_{\mathbb{Z}} \Omega_B^{\mathbb{R}}$$

and

$$df \in H^1(C_0, \mathbb{Z})_{new} \otimes_{\mathbb{Z}} \Omega_B^{\mathbb{R}}.$$

Remark 4.2.1. *Instead of trivializing the local systems and differentiating the functions $b \mapsto e'_b$, $b \mapsto f_b$, we could have introduced the Gauss-Manin connections ∇ on the local systems considered, and applied it to our sections e'_b and f_b. The two points of view are equivalent and will be adopted in the sequel. The advantage of the second point of view is that one does not need to distinguish one point* 0.

Finally, contracting via the natural pairing between $H^1(C_0, \mathbb{Z})_{new}$ and

$$\frac{H^1(C_0, \mathbb{Z})}{i_0^* H^1(S_0, \mathbb{Z})} = H^1(C_0, \mathbb{Z})^*_{new},$$

and using wedge-product for 1-forms, we obtain a real 2-form

$$de' \wedge df \in \bigwedge^2 \Omega_B^{\mathbb{R}} \otimes_{\mathbb{Z}} \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >}. \tag{4.2}$$

Now we do the following : Firstly, using the duality

$$\frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >} \cong (H^2(S_b, \mathbb{Z})_{C_b})^*,$$

this 2-form can be viewed as an element

$$de' \wedge df \in Hom_{\mathbb{Z}}(H^2(S_b, \mathbb{Z})_{C_b}, \bigwedge^2 \Omega_B^{\mathbb{R}}),$$

which gives, by $\mathbb{C}$-linear extension, an element

$$de' \wedge df_{\mathbb{C}} \in Hom_{\mathbb{C}}\big(H^2(S_b, \mathbb{C})_{C_b}, \bigwedge^2 \Omega_B^{\mathbb{C}}\big).$$

Secondly, it follows from our third hypothesis that the 2-form (4.2) actually takes values in

$$\bigwedge^2 \Omega_B^{\mathbb{R}} \otimes_{\mathbb{Z}} ker\Big(j_{0*} : \frac{H^2(S_0, \mathbb{Z})}{< C_{0,r} >} \longrightarrow \frac{H^4(X, \mathbb{Z})}{j_{0*}(NS(S_0))}\Big),$$

so that our homomorphism $de' \wedge df_{\mathbb{C}}$ in fact vanishes on

$$j_0^*\Big(H^2(X, \mathbb{C}) \cap \big(j_{0*}(NS(S_0))^{\perp}\big)\Big).$$

In particular, it will vanish on $j_0^* H^{2,0}(X)$, which will be crucial for a pairing to come later, since we will be dealing with a complex 2-form of type $(2,0)$ on the surface S_0 which is only defined up to the pull-back of a $(2,0)$-form on X.

We will similarly need to differentiate the third Abel-Jacobi invariant: Recall that

$$\delta_b' \in \frac{H^2(S_b, \mathbb{Z})}{j_b^* H^2(X, \mathbb{Z})} \otimes (H^3(X, \mathbb{Z})_{S_b})^* \otimes \mathbb{R}/\mathbb{Z},$$

where

$$H^3(X, \mathbb{Z})_{S_b} := ker\big(j_b^* : H^3(X, \mathbb{Z}) \longrightarrow H^3(S_b, \mathbb{Z})\big).$$

The groups $\frac{H^2(S_b,\mathbb{Z})}{j_b^*H^2(X,\mathbb{Z})}$ and $H^3(X,\mathbb{Z})_{S_b}$ form a local system, hence we have natural identifications

$$\frac{H^2(S_b,\mathbb{Z})}{j_b^*H^2(X,\mathbb{Z})} \cong \frac{H^2(S_0,\mathbb{Z})}{j_0^*H^2(X,\mathbb{Z})} \quad \text{and} \quad H^3(X,\mathbb{Z})_{S_b} \cong H^3(X,\mathbb{Z})_{S_0}.$$

This allows us to view δ' as a map

$$B \longrightarrow \frac{H^2(S_0,\mathbb{Z})}{j_0^*H^2(X,\mathbb{Z})} \otimes (H^3(X,\mathbb{Z})_{S_0})^* \otimes \mathbb{R}/\mathbb{Z},\ b \mapsto \delta'_b,$$

which again by Griffiths' results is differentiable and even real-analytic. Its differential is a map

$$d\delta' : T_B^{\mathbb{R}} \longrightarrow \frac{H^2(S_0,\mathbb{Z})}{j_0^*H^2(X,\mathbb{Z})} \otimes_{\mathbb{Z}} (H^3(X,\mathbb{Z})_{S_0})^* \otimes_{\mathbb{Z}} \mathbb{R},$$

or a real 1-form

$$d\delta' \in \frac{H^2(S_0,\mathbb{Z})}{j_0^*H^2(X,\mathbb{Z})} \otimes_{\mathbb{Z}} (H^3(X,\mathbb{Z})_{S_0})^* \otimes_{\mathbb{Z}} \Omega_B^{\mathbb{R}}.$$

This 1-form in turn can be viewed as an element

$$d\delta' \in Hom_{\mathbb{Z}}\Big(H^3(X,\mathbb{Z})_{S_0}, \frac{H^2(S_0,\mathbb{Z})}{j_0^*H^2(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \Omega_B^{\mathbb{R}}\Big),$$

and again, by $\mathbb{C}$-linear extension, as

$$d\delta'_{\mathbb{C}} \in Hom_{\mathbb{C}}\Big(H^3(X,\mathbb{C})_{S_0}, \frac{H^2(S_0,\mathbb{C})}{j_0^*H^2(X,\mathbb{C})} \otimes_{\mathbb{C}} \Omega_B^{\mathbb{C}}\Big).$$

The main result of this section concerns pull-backs to B of global holomorphic 3-forms on X. Before we can state it, we need a lemma. Note that simply by reason of type, a $(3,0)$-form ω on X will vanish when pulled back to the surface S_b, so that its class $[\omega]$ belongs to $H^3(X,\mathbb{C})_{S_b}$. Hence we can apply our homomorphism $d\delta'_{\mathbb{C}}$ to it, and the following statement makes sense:

Lemma 4.2.2. *Let ω be a form of type $(3,0)$ on X. Then at $0 \in B$,*

$$d\delta'_{\mathbb{C}}\big([\omega]\big) \in \frac{H^{2,0}(S_0)}{j_0^*H^{2,0}(X)} \otimes \Omega_{B,0}^{1,0} \subset \frac{H^2(S_0,\mathbb{C})}{j_0^*H^2(X,\mathbb{C})} \otimes_{\mathbb{C}} \Omega_{B,0}^{\mathbb{C}}.$$

This lemma is an immediate consequence of Proposition 4.2.6, which gives an explicit description of the map $d\delta'_{\mathbb{C}}$ and will be proved later in this section. Admitting it, and using the remarks made after the definition of

$$de' \wedge df_{\mathbb{C}} \in Hom_{\mathbb{C}}\Big(H^2(S_0, \mathbb{C})_{C_0}, \bigwedge^2 \Omega_B^{\mathbb{C}}\Big),$$

it follows that

$$de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big) \in \Omega^3_{B,0,\mathbb{C}}$$

is well defined. Here we use contraction and wedge products of forms. Notice that, as mentioned in Remark 4.2.1, Lemma 4.2.2 holds true at any point (replacing 0 by $b \in B$ everywhere), so that the construction in fact works over B and thus provides a 3-form on B.

Definition 4.2.3. *We call*

$$d\delta' \wedge de' \wedge df\big([\omega]\big) := de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big)$$

Green's pull-back *of the holomorphic* 3*-form* ω *on* X *to* B*.*

Next, following Mumford [Mum], another way to obtain a 3-form on B from one on X is this: Define the maps

$$\theta_i = pr_2 \circ j \circ i \circ s_i : B \longrightarrow X,\ i = 1, ..., N,$$

and recall that $m_1, ...m_N$ are the coefficients of the 0-cycles

$$Z_b = \sum_{i=0}^{N} m_i s_i(b), b \in B,$$

on the curves C_b.

Definition 4.2.4. *We call*

$$M^*(\omega) := \sum_{i=0}^{N} m_i \theta_i^*(\omega)$$

Mumford's pull-back *of the holomorphic* 3*-form* ω *to the variety* B *parametrizing* 0*-cycles on* X*.*

Now we can finally state our main result of this section:

Theorem 4.2.5. *Let* ω *be a holomorphic* 3*-form on* X*. Then* $d\delta' \wedge de' \wedge df\big([\omega]\big)$ *is holomorphic and equal (up to sign) to* $M^*(\omega)$*, i.e. Green's and Mumford's pull-back of* ω *to* B *are the same (up to sign).*

4.2.2 The Proof of Theorem 4.2.5

For the proof of Theorem 4.2.5, we shall use the following alternative description of the Abel-Jacobi invariants e' and δ'; we refer to Appendix A for more details.

Let $\phi : X \to Y$ be a morphism of smooth projective varieties. There is a mixed Hodge structure on the relative cohomology groups

$$H^*(Y, X, \mathbb{Z}),$$

which fit in the long exact sequence

$$\ldots \to H^*(Y, X) \to H^*(Y) \to H^*(X) \to H^{*+1}(Y, X) \ldots$$

This sequence splits into short exact sequences of mixed Hodge structures

$$0 \to \frac{H^{*-1}(X)}{\phi^* H^{*-1}(Y)} \to H^*(Y, X) \to H^*(Y)_X \to 0,$$

where as usual

$$H^*(Y)_X := ker\big(H^*(Y) \to H^*(X)\big).$$

It is known by results by Carlson's [Car1, Car2] that this exact sequence of mixed Hodge structures is described by its extension class

$$e_{Y,X} \in J\Big(Hom\,(H^*(Y)_X, \frac{H^{*-1}(X)}{\phi^* H^{*-1}(Y)})\Big).$$

It turns out that these extension classes are exactly the Abel-Jacobi invariants associated to the cycle $\Gamma_\phi = graph(\phi)$ of $X \times Y$. From now on we will use this fact freely. For details and a proof of this fact, see Appendix B.

Step 1) The action of $d\delta'_{\mathbb{C}}$

Let ω be a holomorphic 3-form on X, let Ω^k_B and $\Omega^k_{\mathcal{S}}$ denote the sheaves of holomorphic k-forms on B and $\mathcal{S}$, respectively. Define $\mathcal{H}^2_{\mathcal{S}} = \mathbf{R}^2\rho_*\mathbb{C} \otimes \mathcal{O}_B$, and let $\mathcal{H}^2_X$ be the trivial bundle over B with fiber $H^2(X, \mathbb{C})$. Consider the following sequence of sheaves on $\mathcal{S}$, defining K:

$$0 \longrightarrow K \longrightarrow \Omega^3_{\mathcal{S}} \xrightarrow{p} \Omega^2_{\mathcal{S}/B} \otimes \rho^*\Omega^1_B \longrightarrow 0. \tag{4.3}$$

We project the section $j^*\omega$ of $\Omega^3_{\mathcal{S}}$ to the section $p(j^*\omega)$ of $\Omega^2_{\mathcal{S}/B} \otimes \rho^*\Omega^1_B$, further via ρ_* to a section still denoted by $p(j^*\omega)$ of

$$\rho_*\Omega^2_{\mathcal{S}/B} \otimes \rho_*\rho^*\Omega^1_B = \mathcal{H}^{2,0}_{\mathcal{S}} \otimes \Omega^1_B \subseteq \mathcal{H}^2_{\mathcal{S}} \otimes \Omega^1_B,$$

and finally via the projection

$$\mathcal{H}_{\mathcal{S}}^{2,0} \longrightarrow \frac{\mathcal{H}_{\mathcal{S}}^{2,0}}{j^*\mathcal{H}_X^{2,0}}$$

to a section

$$\alpha(\omega) \in \frac{\mathcal{H}_{\mathcal{S}}^{2,0}}{j^*\mathcal{H}_X^{2,0}} \otimes \Omega_B^1.$$

Here $\mathcal{H}_{\mathcal{S}}^{2,0} \subset \mathcal{H}_{\mathcal{S}}^2$ and $\mathcal{H}_X^{2,0} \subset \mathcal{H}_X^2$ are the Hodge bundles with fibers $H^{2,0}(S_b) \cong H^0(\Omega^2_{S_b}) \subset H^2(S_b, \mathbb{C})$ and $H^{2,0}(X) \subset H^2(X, \mathbb{C})$, respectively.

Proposition 4.2.6. *Let ω be a holomorphic 3-form on X. Then for any $b \in B$ we have the equality*

$$d\delta'_{\mathbb{C}}\big([\omega]\big)_b = \alpha(\omega)_b$$

via the inclusion

$$\frac{H^{2,0}(S_b,\mathbb{C})}{j_b^* H^{2,0}(X,\mathbb{C})} \otimes \Omega^1_{B,b} \subseteq \frac{H^2(S_b,\mathbb{C})}{j_b^* H^2(X,\mathbb{C})} \otimes \Omega^{\mathbb{C}}_{B,b} \cong \frac{H^2(S_0,\mathbb{C})}{j_0^* H^2(X,\mathbb{C})} \otimes \Omega^{\mathbb{C}}_{B,b}.$$

Proof. By Carlson's description of δ'_{X,S_b} as an extension class (see appendix A), it is the class of $\sigma_F - \sigma_{\mathbb{Z}}$ in the quotient

$$\frac{Hom_{\mathbb{C}}\big(H^3(X,\mathbb{C})_{S_b}, H^2(S_b,\mathbb{C})/j_b^* H^2(X,\mathbb{C})\big)}{F^0 Hom_{\mathbb{C}}(.,.) \oplus Hom_{\mathbb{Z}}(.,.)}, \tag{4.4}$$

where $F^\bullet$ is the natural Hodge filtration on the homomorphism group, and $Hom_{\mathbb{Z}}$ is the integral structure on the homomorphism group. Here

$$\sigma_F : H^3(X,\mathbb{C})_{S_b} \longrightarrow H^3(X, S_b, \mathbb{C})$$

and

$$\sigma_{\mathbb{Z}} : H^3(X,\mathbb{Z})_{S_b} \longrightarrow H^3(X, S_b, \mathbb{Z})$$

are splittings of the sequence (4.5)

$$0 \longrightarrow \frac{H^2(S_b)}{j_b^* H^2(X)} \longrightarrow H^3(X, S_b) \longrightarrow H^3(X)_{S_b} \longrightarrow 0, \tag{4.5}$$

which preserve the Hodge filtration and the integral structure, respectively.

The identification of the Jacobian (4.4) with its underlying real torus

$$\frac{Hom_{\mathbb{R}}\big(H^3(X,\mathbb{R})_{S_b}, H^2(S_b,\mathbb{R})/j_b^* H^2(X,\mathbb{R})\big)}{Hom_{\mathbb{Z}}\big(H^3(X,\mathbb{Z})_{S_b}, H^2(S_b,\mathbb{Z})/j^* H^2(X,\mathbb{Z})\big)}$$

comes from the identification

$$\frac{Hom_{\mathbb{C}}\big(H^3(X,\mathbb{C})_{S_b}, H^2(S_b,\mathbb{C})/j_b^*H^2(X,\mathbb{C})\big)}{F^0 Hom_{\mathbb{C}}\big(H^3(X,\mathbb{C})_{S_b}, H^2(S_b,\mathbb{C})/j_b^*H^2(X,\mathbb{C})\big)}$$

$$\cong Hom_{\mathbb{R}}\big(H^3(X,\mathbb{R})_{S_b}, H^2(S_b,\mathbb{R})/j_b^*H^2(X,\mathbb{R})\big).$$

This isomorphism translates into the fact that there is one and only one real splitting that, when complexified, also preserves the Hodge filtration. We call this splitting $\sigma_{\mathbb{R},F}$.

Hence we conclude that the Abel-Jacobi invariant we want to differentiate, which is the Abel-Jacobi invariant δ'_{X,S_b} seen as an element of the real torus

$$Hom_{\mathbb{Z}}\big(H^3(X,\mathbb{Z})_{S_b}, H^2(S_b,\mathbb{Z})/j^*H^2(X,\mathbb{Z})\big) \otimes \mathbb{R}/\mathbb{Z}, \tag{4.6}$$

is the class of

$$\sigma_{\mathbb{R},F} - \sigma_{\mathbb{Z}} \in Hom_{\mathbb{R}}\big(H^3(X,\mathbb{R})_{S_b}, H^2(S_b,\mathbb{R})/H^2(X,\mathbb{R})\big)$$

in the quotient

$$\frac{Hom_{\mathbb{R}}\big(H^3(X,\mathbb{R})_{S_b}, H^2(S_b,\mathbb{R})/j_b^*H^2(X,\mathbb{R})\big)}{Hom_{\mathbb{Z}}\big(H^3(X,\mathbb{Z})_{S_b}, H^2(S_b,\mathbb{Z})/j_b^*H^2(X,\mathbb{Z})\big)}.$$

Now, starting from the cohomology class $[\omega] \in H^{3,0}(X)$, we have a class $\sigma_{\mathbb{R},F}\big([\omega]\big) \in F^3H^3(X,S_b)$; on the other hand, since $j_b^*\omega = 0$ on S_b for all b, the $(3,0)$ form ω also determines a class in $H^3(X,S_b)$, which is easily seen to vary holomorphically with b, so that we get a section $\tilde{\omega}$ of the bundle $\mathcal{H}^{3,0}_{X,\mathcal{S}}$, the holomorphic vector bundle over B with fiber $H^{3,0}(X,S_b) \subset H^3(X,S_b,\mathbb{C})$. We have

Lemma 4.2.7. *For any $b \in B$, the equality $\sigma_{\mathbb{R},F,b}\big([\omega]\big) = \tilde{\omega}_b$ holds.*

Proof of Lemma 4.2.7. We have

$$F^3\frac{H^2(S_b)}{j_b^*H^2(X)} = 0.$$

Since the class $[\omega]$ belongs to $F^3H^3(X)$, any Hodge lifting of $[\omega]$ belongs to $F^3H^3(X,S_b)$. Since

$$F^3H^3(X,S_b) \cap \frac{H^2(S_b)}{j_b^*H^2(X)} = F^3\frac{H^2(S_b)}{j_b^*H^2(X)} = 0,$$

it follows that a Hodge lifting of $[\omega]$ is unique. Hence $\tilde{\omega}$ and $\sigma_{\mathbb{R},F,b}$ must coincide. $\square$

Recall (see Remark 4.2.1) that $d\delta'_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of

$$\nabla\delta'_{X,S_b},$$

where ∇ is the Gauss-Manin connection applied to the family of tori (4.6).

On the other hand, we have just shown that

$$\delta'_{X,S_b} = \sigma_{F,\mathbb{R}} - \sigma_{\mathbb{Z}}$$

modulo

$$Hom_{\mathbb{Z}}\big(H^3(X,\mathbb{Z})_{S_b}, H^2(S_b,\mathbb{Z})/j_b^*H^2(X,\mathbb{Z})\big).$$

Since $\sigma_{\mathbb{Z}}$ is integral, hence flat, we get

$$\nabla\delta'_{X,S_b} = \nabla\sigma_{F,\mathbb{R}}.$$

Next, since $[\omega]$ gives a flat section of the bundle $\mathcal{H}^3_{\mathcal{S}}$ with fiber $H^3(X)_{S_b}$, we find that

$$\nabla\sigma_{F,\mathbb{R}}([\omega]) = \nabla^{X,S}(\sigma_{F,\mathbb{R}}([\omega])).$$

Here, on the right hand side, the Gauss-Manin connection acts on the bundle $\mathcal{H}^3_{X,S}$ of complex relative cohomology, with fiber $H^3(X,S_b)$ at $b \in B$, but the resulting form lies in

$$\Omega^{\mathbb{C}}_B \otimes (\mathcal{H}^2_{\mathcal{S}}/\mathcal{H}^2_X) \subset \Omega^{\mathbb{C}}_B \otimes \mathcal{H}^3_{X,\mathcal{S}},$$

because the projection of $\sigma_{F,\mathbb{R}}([\omega])$ in $\mathcal{H}^3_{\mathcal{S}}$ is flat.

By Lemma 4.2.7, we have $\sigma_{F,\mathbb{R}}([\omega]) = \tilde{\omega}$, and

$$\begin{aligned} d\delta'_{\mathbb{C}}([\omega]) &= \nabla\delta'_{X,S_b}([\omega]) = \\ = \nabla\sigma_{F,\mathbb{R}}([\omega]) &= \nabla^{X,\mathcal{S}}(\sigma_{F,\mathbb{R}}([\omega])) = \nabla^{X,\mathcal{S}}(\tilde{\omega}). \end{aligned} \tag{4.7}$$

The proof of the proposition will be complete once we have shown that

$$\nabla^{X,\mathcal{S}}(\tilde{\omega}) = \alpha(\omega). \tag{4.8}$$

This follows from a general fact concerning the Gauss-Manin connection acting on the relative cohomology of a family (cf. Appendix C and [?]).

Let $\mathcal{X} \subset \mathcal{Y}$ be an immersion of two families of compact differentiable manifolds parametrized by a basis B. Let α be a closed differentiable k-form on $\mathcal{Y}$ which vanishes on each fiber X_b. Then α also provides a natural section $\tilde{\alpha}$ of the bundle $\mathcal{H}^k_{\mathcal{Y},\mathcal{X}}$ of relative cohomology. The projection of $\tilde{\alpha}$ in $\mathcal{H}^k_{\mathcal{Y}}$ is flat, so that $\nabla\tilde{\alpha}$ belongs to

$$\frac{\mathcal{H}^{k-1}_{\mathcal{X}}}{\mathcal{H}^{k-1}_{\mathcal{Y}}} \otimes \Omega^{\mathbb{C}}_B.$$

On the other hand, since α vanishes on X_b, the form α projects into a section of $A^{k-1}(X_b) \otimes \Omega_{B,b}$ for each $b \in B$, which is closed. Hence we get an element

$$p(\alpha_b) \in H^{k-1}(X_b) \otimes \Omega_{B,b}.$$

Lemma 4.2.8. *The projection of $p(\alpha_b)$ in $\big(H^{k-1}(X_b)/H^{k-1}(Y_b)\big) \otimes \Omega_{B,b}$ coincides with $\nabla\tilde{\alpha}$ at any point $b \in B$.*

This lemma shows equality (4.8) and completes the proof of Proposition 4.2.6. □

Step 2) The action of $de'_{\mathbb{C}}$

Recall from the beginning that we have the map

$$de' \wedge df_{\mathbb{C}} \in Hom_{\mathbb{C}}\Big(H^2(S_0,\mathbb{C})_{C_0}, \bigwedge^2 \Omega_B^{\mathbb{C}}\Big).$$

By equations (4.7) and (4.8) we have

$$d\delta'_{\mathbb{C}}\big([\omega]\big) = \nabla^{X,\mathcal{S}}(\tilde{\omega}) = \alpha(\omega) \in \frac{\mathcal{H}^{2,0}_{\mathcal{S}}}{j^*\mathcal{H}^{2,0}_{X}} \otimes \Omega^1_B,$$

so that we can calculate $de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big)$ as $de' \wedge df_{\mathbb{C}}\big(\alpha(\omega)\big)$. Now, since $\alpha(\omega)$ is the projection of $p(j^*\omega) \in \mathcal{H}^{2,0}_{\mathcal{S}} \otimes \Omega_B$ in the quotient $(\mathcal{H}^{2,0}_{\mathcal{S}}/\mathcal{H}^{2,0}_{X}) \otimes \Omega_B$, we have as well

$$de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big) = de' \wedge df_{\mathbb{C}}(p(j^*\omega)). \tag{4.9}$$

We begin the computation again by writing

$$de' \in \frac{H^1(C_0,\mathbb{Z})}{i_0^*H^1(S_0,\mathbb{Z})} \otimes_{\mathbb{Z}} \frac{H^2(S_0,\mathbb{Z})}{< C_{0,r} >} \otimes_{\mathbb{Z}} \Omega_B^{\mathbb{R}}$$

as a homomorphism (after $\mathbb{C}$-linear extension)

$$de'_{\mathbb{C}} \in Hom_{\mathbb{C}}\Big(H^2(S_0,\mathbb{C})_{C_0}, \frac{H^1(C_0,\mathbb{C})}{i_0^*H^1(S_0,\mathbb{C})} \otimes_{\mathbb{C}} \Omega_B^{\mathbb{C}}\Big).$$

This map in turn defines an extended homomorphism

$$de'_{\mathbb{C},1} \in Hom_{\mathbb{C}}\Big(H^2(S_0,\mathbb{C})_{C_0} \otimes_{\mathbb{C}} \Omega_B^{\mathbb{C}}, \frac{H^1(C_0,\mathbb{C})}{i_0^*H^1(S_0,\mathbb{C})} \otimes_{\mathbb{C}} \bigwedge^2 \Omega_B^{\mathbb{C}}\Big)$$

by the rule

$$de'_{\mathbb{C},1}(\alpha \otimes \eta) = de'_{\mathbb{C}}(\alpha) \wedge \eta. \tag{4.10}$$

Similarly, we have

$$df \in \big(ker\, i_{0*} : H^1(C_0, \mathbb{C}) \to H^3(S_0, \mathbb{C})\big) \otimes \Omega_B^{\mathbb{C}},$$

which we view as a homomorphism

$$df_{\mathbb{C}} \in Hom\big(\frac{H^1(C_0, \mathbb{C})}{i_0^* H^1(S_0, \mathbb{C})}, \Omega_B^{\mathbb{C}}\big),$$

which can be extended to

$$df_{\mathbb{C},2} \in Hom\big(\frac{H^1(C_0, \mathbb{C})}{i_0^* H^1(S_0, \mathbb{C})} \otimes \bigwedge^2 \Omega_B^{\mathbb{C}}, \bigwedge^3 \Omega_B^{\mathbb{C}}\big)$$

by the rule

$$df_{\mathbb{C},2}(\alpha \otimes \eta) = df_{\mathbb{C}}(\alpha) \wedge \eta.$$

It is clear from these definitions that

$$de' \wedge df_{\mathbb{C}}\big(p(j^*\omega)\big) = -df_{\mathbb{C},2}\big(de'_{\mathbb{C},1}(p(j^*\omega)\big). \tag{4.11}$$

(Notice that we are not allowed to write this equality with $p(j^*\omega)$ replaced by $\alpha(\omega)$, since $de'_{\mathbb{C},1}$ does not act on $\alpha(\omega)$.)

So first we have to compute $de'_{\mathbb{C},1}\big(p(j^*\omega)\big)$. On $\mathcal{C}$ we have a natural surjective morphism

$$q : \Omega^3_{\mathcal{C}} \longrightarrow \Omega_{\mathcal{C}/B} \otimes \pi^*\Omega^2_B.$$

By pulling back via $j \circ i$, the holomorphic 3-form ω on X gives a section $(j \circ i)^*\omega$ of $\Omega^3_{\mathcal{C}}$; we project it via q to the section $q((j \circ i)^*\omega)$ of $\Omega^1_{\mathcal{C}/B} \otimes \pi^*\Omega^2_B$, further via π_* to a section still denoted by $q((j \circ i)^*\omega)$ of

$$\pi_*\Omega^1_{\mathcal{C}/B} \otimes \pi_*\pi^*\Omega^2_B = \mathcal{H}^{1,0}_{\mathcal{C}} \otimes \Omega^2_B \subseteq \mathcal{H}^1_{\mathcal{C}} \otimes \Omega^2_B,$$

and then to a section

$$\beta(\omega) \in \frac{\mathcal{H}^{1,0}_{\mathcal{C}}}{i^*\mathcal{H}^{1,0}_{\mathcal{S}}} \otimes \Omega^2_B.$$

Here $\mathcal{H}^{1,0}_{\mathcal{C}}$ and $\mathcal{H}^{1,0}_{\mathcal{S}}$ denote the Hodge bundles with fibers $H^{1,0}(C_b)$ and $H^{1,0}(S_b)$, respectively.

Proposition 4.2.9. *Let ω be a holomorphic 3-form on X. Then for any $b \in B$ we have the equality*

$$de'_{\mathbb{C},1}\big(p(j^*\omega)\big)_b = \beta(\omega)_b$$

via the inclusion

$$\frac{H^{1,0}(C_b,\mathbb{C})}{i_b^* H^{1,0}(S_b,\mathbb{C})} \otimes \Omega^2_{B,b} \subseteq \frac{H^1(C_b,\mathbb{C})}{i_b^* H^1(S_b,\mathbb{C})} \otimes \Omega^{2,\mathbb{C}}_{B,b} \cong \frac{H^1(C_0,\mathbb{C})}{i_0^* H^1(S_b,\mathbb{C})} \otimes \Omega^{2,\mathbb{C}}_{B,b}.$$

Proof. Carlson's description of e'_{S_b,C_b} as an extension class says it is the class of $\sigma'_F - \sigma'_{\mathbb{Z}}$ in the quotient

$$\frac{Hom_{\mathbb{C}}\big(H^2(S_b,\mathbb{C})_{C_b}, H^1(C_b,\mathbb{C})/i_b^*H^1(S_b,\mathbb{C})\big)}{F^0 Hom_{\mathbb{C}}(.,.) + Hom_{\mathbb{Z}}(.,.)},$$

where σ'_F and $\sigma'_{\mathbb{Z}}$ are Hodge and integral splittings of the sequence

$$0 \longrightarrow H^1(C_b)/j_b^*H^1(S_b) \longrightarrow H^2(S_b,C_b) \longrightarrow H^2(S_b)_{C_b} \longrightarrow 0 \qquad (4.12)$$

of mixed Hodge structures.

Just as in the previous step we obtain that e'_{S_b,C_b} is the class of $\sigma'_{\mathbb{R},F} - \sigma'_{\mathbb{Z}}$ in the quotient

$$\frac{Hom_{\mathbb{Z}}\big(H^2(S_b,\mathbb{Z})_{C_b}, H^1(C_b,\mathbb{R})/i_b^*H^1(S_b,\mathbb{R})\big)}{Hom_{\mathbb{Z}}\big(H^2(S_b,\mathbb{Z})_{C_b}, H^1(C_b,\mathbb{Z})/i_b^*H^1(S_b,\mathbb{Z})\big)}$$

$$\cong Hom_{\mathbb{Z}}\big(H^2(S_b,\mathbb{Z})_{C_b}, H^1(C_b,\mathbb{Z})/i_b^*H^1(S_b,\mathbb{Z})\big) \otimes \mathbb{R}/\mathbb{Z},$$

where $\sigma'_{\mathbb{R},F}$ is the unique real splitting of (4.12) that also preserves the Hodge filtration.

Now we have by the above

$$de' = \nabla(\sigma'_{F,\mathbb{R}} - \sigma'_{\mathbb{Z}}) = \nabla\sigma'_{F,\mathbb{R}}, \qquad (4.13)$$

since $\sigma'_{\mathbb{Z}}$ is an integral, hence flat section of the bundle with fiber

$$Hom\big(H^2(S_b)_{C_b}, H^1(C_b)/i_b^*H^1(S_b)\big).$$

In the sequel we shall denote by

$$\mathcal{H}^2_{\mathcal{C}} \subset \mathcal{H}^2_{\mathcal{S}}$$

the flat subbundle with fiber $H^2(S_b)_{C_b}$ and by

$$\mathcal{H}^2_{\mathcal{S},\mathcal{C}}$$

the flat bundle with fiber $H^2(S_b, C_b)$. The Gauss-Manin connection acts in a compatible way on all of these bundles.

Since $p(\omega)$ is a section of $\mathcal{H}^2_{\mathcal{C}} \otimes \Omega_B$ which is closed with respect to the Gauss-Manin connection, it follows from Leibniz's rule, (4.10), and the formula (4.13) that

$$de'_{\mathbb{C},1}(p(j^*\omega)) = (\nabla\sigma'_{F,\mathbb{R}})(p(j^*\omega)) = \nabla\big(\sigma'_{F,\mathbb{R}}(p(\omega))\big), \tag{4.14}$$

where ∇ on the right hand side acts on the bundle $\mathcal{H}^2_{\mathcal{S},\mathcal{C}}$ of relative cohomology.

Now, the argument in the proof of the Lemma 4.2.7 gives as well

Lemma 4.2.10. *If $\alpha \in H^{2,0}(S_b)$, the class $\sigma'_{F,\mathbb{R}}(\alpha) \in H^2(S_b, C_b, \mathbb{C})$ is the class of the closed form α, which vanishes on C_b.*

It follows from this lemma that

$$\sigma'_{F,\mathbb{R}}(p(j^*\omega)) \in \mathcal{H}^2_{\mathcal{S},\mathcal{C}} \otimes \Omega^{\mathbb{C}}_B$$

is equal to $\widetilde{p(\omega)}$, where the tilde denotes the lifting $H^{2,0}(S_b) \longrightarrow H^2(S_b, C_b)$ described in the statement of the lemma. Combining (4.14) and the lemma above, we get

$$de'_{\mathbb{C},1}(p(j^*\omega)) = \nabla(\widetilde{p(j^*\omega)}), \tag{4.15}$$

where ∇ on the right hand side acts on $\mathcal{H}^2_{\mathcal{S},\mathcal{C}} \otimes \Omega^{\mathbb{C}}_B$, but the resulting differential lies in

$$\frac{\mathcal{H}^1_{\mathcal{C}}}{i^*\mathcal{H}^1_{\mathcal{S}}} \otimes \bigwedge^2 \Omega^{\mathbb{C}}_B,$$

because the projection of $\widetilde{p(j^*\omega)}$ in $\mathcal{H}^2_{\mathcal{C}} \otimes \Omega^{\mathbb{C}}_B$ is ∇-closed.

The proof of Proposition 4.2.9 is then concluded by the following

Lemma 4.2.11. *We have the equality*

$$\nabla(\widetilde{p(j^*\omega)}) = \beta(\omega) \tag{4.16}$$

in $\frac{\mathcal{H}^1_{\mathcal{C}}}{i^*\mathcal{H}^1_{\mathcal{S}}} \otimes \bigwedge^2 \Omega^{\mathbb{C}}_B$.

This lemma follows from a slight generalization of Lemma 4.2.8, which concerned the Gauss-Manin connection acting on the relative cohomology of a family of pairs. Remember that $p(j^*\omega) \in \mathcal{H}^2_{\mathcal{S}} \otimes \Omega_B$ is obtained from the 3-form $j^*\omega$ on $\mathcal{S}$, observing that $j^*\omega$ is closed and in $L^1 A^3_{\mathcal{S}}$ - here $L^\bullet A^\bullet_{\mathcal{S}}$ denotes the Leray filtration on the sheaf of differential forms on $\mathcal{S}$ (cf. [?]).

Now the form $i^*(j^*\omega)$ on $\mathcal{C}$ belongs to $L^2A^3_{\mathcal{C}}$, so this provides on the one hand the lifting

$$\widetilde{p(j^*\omega)}$$

of $p(j^*\omega)$ to $\mathcal{H}^2_{S,C} \otimes \Omega_B$, and on the other hand the projection

$$q(i^*(j^*\omega)) \in \mathcal{H}^1_{\mathcal{C}} \otimes \Omega^2_B.$$

The equality (4.16) is then essentially the same statement as the equality given in Lemma 4.2.8, just for the level L^2 of the Leray filtration. This completes the proof of Proposition 4.2.9. □

Step 3) The action of $df_{\mathbb{C}}$

In this third and last step we finally calculate

$$de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big),$$

that is Green's pull-back of ω. We already noted that

$$de' \wedge df_{\mathbb{C}}\big(d\delta'_{\mathbb{C}}\big([\omega]\big)\big) = -df_{\mathbb{C},2}(de'_{\mathbb{C},1}(p(j^*\omega)).$$

Since by Proposition 4.2.9 we know that

$$de'_{\mathbb{C},1}(p(j^*\omega)) = \beta(\omega) \in \frac{\mathcal{H}^{1,0}_{\mathcal{C}}}{i^*\mathcal{H}^{1,0}_{\mathcal{S}}} \otimes \Omega^2_B,$$

we do this by computing $df_{\mathbb{C},2}\big(\beta(\omega)\big)$. Recall that $\beta(\omega)$ is the projection of $q(i^*(j^*\omega)) \in \Omega_{\mathcal{C}/B} \otimes \bigwedge^2 \Omega_B$ in

$$\frac{\mathcal{H}^{1,0}_{\mathcal{C}}}{i^*\mathcal{H}^{1,0}_{\mathcal{S}}} \otimes \Omega^2_B.$$

It follows that we have in fact

$$df_{\mathbb{C},2}(\beta(\omega)) = df_{\mathbb{C},2}(q(\omega)).$$

Theorem 4.2.5 then follows from this last result:

Proposition 4.2.12. *Let ω' be a form of type $(3,0)$ on $\mathcal{C}$, and $q(\omega')$ the section of $\mathcal{H}^{1,0}_{\mathcal{C}} \otimes \Omega^2_B$ associated to it. Then we have*

$$df_{\mathbb{C},2}\big(q(\omega)\big) = \sum_{i=0}^{N} m_i s_i^*(\omega') \in \Omega^3_B.$$

This proposition is proved exactly as in [**?**], Lemma 8, where it is stated for a 2-form on $\mathcal{C}$.

Chapter 5

On the image of the map ψ_3^3

5.1 Introduction

In this section we use Theorem 4.2.5 comparing Mumford's and Green's pull-backs to prove the following result:

Theorem 5.1.1. *Let X be a 3-fold with $h^{1,0}(X) = b_{4,tr}(X) = 0$ (this implies that the maps alb_X and ψ_2^3 are identically zero). Then if*

$$h^{3,0}(X) := dim H^0(\Omega_X^3) > 0,$$

the map $\psi_3^3 : ker(\psi_2^3) \longrightarrow J_3^3(X)$ has, modulo torsion, positive-dimensional image.

Some preliminary remarks are in order here.

Remark 5.1.2. *Note that the second and third conditions are fulfilled, for example, by 3-dimensional complete intersections of ample hypersurfaces in projective space because of Lefschetz's hyperplane theorem.*

Remark 5.1.3. *The result above is obtained by arguing that if ψ_3^3 is 0, then we can find a diagram as in the beginning of the previous section, with $(j \circ i)_* Z_b = x_b - y_b$, $\delta_b' \cdot e_b \cdot f_b = 0$ for any b, and the map $b \mapsto (x_b, y_b) \in X \times X$ is submersive. One can then apply Theorem 4.2.5 together with Proposition 5.2.2 below to conclude that $H^{3,0}(X) = 0$.*

More generally, Mumford's technique of pulling back a holomorphic form on the symmetric product $X^{(k)} \times X^{(k)}$ of the variety X, induced by a holomorphic form on X, to a variety parametrizing 0-cycles, in order to bound the dimension of the fibers of a map defined on this symmetric product (see [Mum]), can be combined with the reasoning above to prove that if $H^{3,0}(X) \neq 0$, the image of ψ_3^3 is actually infinite dimensional.

Remark 5.1.4. *In the same spirit, the assumption that* $h^{1,0} = 0$ *is not necessary here. Indeed, the set*

$$\mathcal{Z}_{k,alb} \subset X^{(k)} \times X^{(k)}$$

consisting of couples (Z, Z') *such that* $alb_X(Z - Z') = 0$ *is a Zariski closed algebraic subset of codimension* $\leq g := dim \mathrm{Alb} X$ *in* $X^{(k)} \times X^{(k)}$*. Hence we can apply the argument alluded to in the previous remark to* $\mathcal{Z}_{k,alb}$ *for* $k \to \infty$ *to conclude that* $Im\,\psi_3^3$ *is infinite dimensional if* $h^{3,0}(X)$ *is non zero, even if* $g \neq 0$.

5.2 On the image of the map ψ_3^3

We work with the diagram from the beginning of the last section, in which the 0-cycles, curves and surfaces are parametrized by a complex ball B. The first step in the proof of the theorem is the following result:

Proposition 5.2.1. *Let* $V \subset B$ *be a smooth, real-analytic subset, such that for all* $b \in V$*, the Green contraction*

$$\delta'_b \cdot e_b \cdot f_b \in J^3(X)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

is torsion. Then for any holomorphic 3*-form* ω *on* X*, the* 3*-form* $d\delta' \wedge de' \wedge df([\omega])$ *defined in the previous section vanishes on* V.

Here $\delta'_b = \delta'_{X,S_b}$, $e_b = e_{S_b,C_b}$, and $f_b = f_{C_b,Z_b}$.

Proof. To make the calculation a bit simpler, we will assume that the surface $S = S_0$ is regular, i.e. that $H^1(S, \mathbb{Z}) = 0$. Let $C = C_0$ and let

$$\{\alpha_i, \beta_i\},\ i = 1, ..., g$$

be a symplectic basis for $H^1(C, \mathbb{Z})$. Let

$$\{\gamma_j\},\ j = 1, ..., b_2(S) \text{ and } \{\gamma_j^*\},\ j = 1, ..., b_2(S)_{new}$$

be respectively a basis of $H^2(S, \mathbb{Z})$ such that $\{\gamma_j\},\ j = 1, ..., b_2(S)_{new}$, induces a basis of

$$H^2(S, \mathbb{Z})_{new} = ker\big(j_* : H^2(S, \mathbb{Z}) \to H^4(X, \mathbb{Z})\big),$$

and the dual basis of $H^2(S, \mathbb{Z})^*_{new} = \frac{H^2(S,\mathbb{Z})}{j^* H^2(X,\mathbb{Z})}$. Hence by definition we have $< \gamma_i, \gamma_j^* >= 0$ for $i \neq j$, $i \leq b_2(S)_{new}$ and $< \gamma_j, \gamma_j^* >= 1$, where $< .,. >$ is the duality between $H^2(S, \mathbb{Z})_{new}$ and $\frac{H^2(S,\mathbb{Z})}{j^* H^2(X,\mathbb{Z})}$. Finally, let

$$\{\delta_n\},\ n = 1, ..., b_3(X)$$

be a basis of

$$H^3(X,\mathbb{Z})^*_S = \frac{H^3(X,\mathbb{Z})}{j_*H^1(S,\mathbb{Z})},$$

which is equal to $H^3(X,\mathbb{Z})$ by our assumption that S is regular. Then we can write explicitly

$$f_b = \sum_{i=1}^{g} \phi_i(b) \otimes \alpha_i + \sum_{i=1}^{g} \psi_i(b) \otimes \beta_i \in \mathbb{R}/\mathbb{Z} \otimes H^1(C,\mathbb{Z}),$$

$$e'_b = \sum_{j=1}^{b_2(S)} \Big\{ \sum_{i=1}^{g} \rho_{ij}(b) \otimes \gamma_j \otimes \alpha_i + \sum_{i=1}^{g} \chi_{ij}(b) \otimes \gamma_j \otimes \beta_i \Big\},$$

modulo

$$\mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} NS(S) \otimes H^1(C,\mathbb{Z}),$$

and

$$\delta'_b = \sum_{n=1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} \eta_{nj}(b) \otimes \delta_n \otimes \gamma_j^*,$$

where $\phi_i, \psi_i, \rho_{ij}, \chi_{ij}$ and $\eta_{nj} : B \longrightarrow \mathbb{R}/\mathbb{Z}$ are differentiable functions.

By assumption, Green's contraction

$$e'_b \cdot f_b$$

belongs to

$$ker\Big(j_* : \frac{H^2(S,\mathbb{Z})}{NS(S)} \to \frac{H^4(X,\mathbb{Z})}{j_*NS(S)} \Big),$$

which is isomorphic to $H^2(S,\mathbb{Z})_{new}/(NS(S) \cap ker\, j_*)$. This means that

$$\sum_{j=1}^{b_2(S)} \Big(\sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big) \otimes \gamma_j$$

belongs to $(\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes H^2(S,\mathbb{Z})_{new}$ modulo $(\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes NS(S)$. Let us write

$$\sum_{j=1}^{b_2(S)} \Big(\sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big) \otimes \gamma_j = a + b,$$

with

$$a \in (\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes H^2(S,\mathbb{Z})_{new},$$
$$b \in (\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes NS(S).$$

Then we have that

$$a = \sum_{j=1}^{b_2(S)_{new}} \Big(\sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big) \otimes \gamma_j$$

modulo $(\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes (NS(S) \cap ker\, j_*)$. Furthermore, by definition of our contraction $\delta'_b \cdot e_b \cdot f_b$, it is equal (modulo $(\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes Im\mathrm{AJ}_X$) to the contraction $\delta'_b \cdot a$ obtained using the pairing between $H^2(S,\mathbb{Z})_{new}$ and $\frac{H^2(S,\mathbb{Z})}{j^* H^2(X,\mathbb{Z})}$. Hence $\delta'_b \cdot e_b \cdot f_b =$

$$= \sum_{n=1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} \eta_{nj}(b) \otimes \Big\{ \sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big\} \otimes \delta_n \quad (5.1)$$

modulo $(\mathbb{R}/\mathbb{Z})^{\otimes 2} \otimes Im\mathrm{AJ}_X$.

We have to show that if the expression (5.1) vanishes modulo torsion in $J^3(X)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$ for any point b of V, then for any $\omega \in H^{3,0}(X)$, the 3-form $d\delta \wedge de \wedge df([\omega])$ defined on B vanishes on V.

First of all, we note that this assumption means that for any $v \in V$ there exists a non-zero integer $m \in \mathbb{N}$ such that $m\delta'_b \cdot e'_b \cdot f_b = 0$. The formula for $\delta'_b \cdot e'_b \cdot f_b$ now shows that the locus $V_m \subset V$ where $m\delta'_b \cdot e'_b \cdot f_b = 0$ is a countable union of real analytic subsets of V. The assumption says that V is the union over m of the V_m's and by Baire's theorem it follows that V must be equal to some V_m. So we may assume that

$$m\delta'_b \cdot e'_b \cdot f_b = 0$$

on V.

The image of the Abel-Jacobi map

The image of the Abel-Jacobi map $Im\mathrm{AJ}_X$ is a subgroup of $J^3(X)$; it is an extension of an Abelian subvariety $J^3(X)_{alg}$ of $J^3(X)$, its connected component of 0, by a countable group Griff(X). $J^3(X)_{alg}$ corresponds to a certain real subtorus

$$H^3(X,\mathbb{Z})_{alg} \otimes \mathbb{R}/\mathbb{Z} \subset H^3(X,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} = J^3(X),$$

where the sublattice

$$H^3(X,\mathbb{Z})_{alg} \subset H^3(X,\mathbb{Z})$$

has the property that

$$H^3(X,\mathbb{Z})_{alg} \otimes \mathbb{C} \subset H^3(X,\mathbb{C})$$

is contained in $H^{2,1}(X) \oplus H^{1,2}(X)$, and thus is perpendicular to $H^{3,0}(X)$ with respect to Poincaré duality.

A special case

Assume first that the stronger condition

$$m\delta'_b \cdot e'_b \cdot f_b = 0 \text{ in } (J^3(X)/J^3(X)_{alg}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \tag{5.2}$$

is satisfied along V. We may suppose that the basis δ_n has been chosen so that the

$$\delta_k,\ k \leq b_3(X)_{alg} := rk\, H^3(X,\mathbb{Z})_{alg}$$

induce a basis of $H^3(X,\mathbb{Z})_{alg}$. Then from (5.1) we conclude that

$$m \cdot \sum_{n=b_3(X)_{alg}+1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} \eta_{nj}(b) \otimes \Big\{ \sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big\} \otimes \delta_n \tag{5.3}$$

vanishes in $\frac{H^3(X,\mathbb{Z})}{H^3(X,\mathbb{Z})_{alg}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3}$. Now we observe that since $[\omega]$ is perpendicular to δ_n, $n \leq b_3(X)_{alg}$, we have

$$d\delta'_{\mathbb{C}}([\omega]) = \sum_{n=b_3(X)_{alg}+1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} d\eta_{nj}(b) < \delta_n, [\omega] > \otimes \gamma_j^* \tag{5.4}$$

in $H^2(S,\mathbb{C})_{new} \otimes \Omega_B^{\mathbb{C}}$. It follows that the coefficients η_{nj} with $n \leq b_3(X)_{alg}$ won't play any role in the computation of $d\delta \wedge de \wedge df([\omega])$.

We then conclude exactly as in [**?**]. Along V, the coefficients of (5.3) are tensor products over $\mathbb{Z}$ of differentiable functions on V with values in $\mathbb{R}/\mathbb{Z}$. The vanishing of the expression (5.3) provides linear relations with constant (rational) coefficients among the elements of each set of coefficients

$$\eta_{nj},$$

$$\rho_{ij},\ \chi_{ij},$$

$$\phi_i,\ \psi_i.$$

Differentiating these relations provides linear relations with constant (rational) coefficients among the elements of each set of differentials on V

$$d\eta_{nj},$$

$$d\rho_{ij},\ d\chi_{ij},$$

$$d\phi_i,\ d\psi_i.$$

It is an easy but also messy computation to deduce from these relations the vanishing of the restriction to V of the 3-form $d\delta \wedge de \wedge df([\omega])$.

The general case

We now consider the general case where

$$m\delta'_b \cdot e_b \cdot f_b \in Im\mathrm{AJ}_X \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2},$$

in contrast to (5.2). With the same notations as above, this means that the term (5.3) vanishes in the quotient of $\frac{H^3(X,\mathbb{Z})}{H^3(X,\mathbb{Z})_{alg}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3}$ by $\mathrm{Griff}(X) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$, where the group

$$\mathrm{Griff}(X) \subset \frac{H^3(X,\mathbb{Z})}{H^3(X,\mathbb{Z})_{alg}} \otimes \mathbb{R}/\mathbb{Z} \cong \frac{J^3(X)}{J^3(X)_{alg}}$$

is countable. This implies the existence of finitely many $c_{n,k} \in \mathbb{R}/\mathbb{Z}$ such that we obtain the following equality in $\frac{H^3(X,\mathbb{Z})}{H^3(X,\mathbb{Z})_{alg}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3}$:

$$m \cdot \sum_{n=b_3(X)_{alg}+1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} \eta_{nj}(b) \otimes \Big\{ \sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big\} \otimes \delta_n =$$
$$= \sum_{n=b_3(X)_{alg}+1}^{b_3(X)} \sum_k c_{n,k} \otimes \tau_{n,k} \otimes \delta_n \quad (5.5)$$

where $\tau_{n,k} \in \mathcal{C}^\infty(V, \mathbb{R}/\mathbb{Z})^{\otimes 2}$.

It is clear that the right hand side vanishes under the triple derivative

$$d^3 : \mathcal{C}^\infty(V, \mathbb{R}/\mathbb{Z})^{\otimes 3} \longrightarrow \bigwedge^3 \Omega_V^{\mathbb{R}},$$
$$f \otimes g \otimes h \mapsto df \wedge dg \wedge dh.$$

But if we go back to the construction of $d\delta' \wedge de' \wedge df([\omega])$, we see that it is obtained by an adequate contraction using various Poincaré dualities of

$$d^3(\delta' \otimes e' \otimes f \otimes [\omega]).$$

It follows that the equation (5.5) has the same implication on $d\delta' \wedge de' \wedge df([\omega])$ as the equation (5.3)

$$m \cdot \sum_{n=b_3(X)_{alg}+1}^{b_3(X)} \sum_{j=1}^{b_2(S)_{new}} \eta_{nj}(b) \otimes \Big\{ \sum_{i=1}^{g} \rho_{ij}(b) \otimes \psi_i(b) - \sum_{i=1}^{g} \chi_{ij}(b) \otimes \phi_i(b) \Big\} \otimes \delta_n = 0$$

considered before. So we also conclude in this case that $d\delta' \wedge de' \wedge df([\omega])$ vanishes on V.

□

Putting together Theorem 4.2.5 and Proposition 5.2.1 we obtain the following result:

Proposition 5.2.2. *Let $V \subset B$ be a smooth, real-analytic subset, such that for all $b \in V$, the product $\delta'_b \cdot e_b \cdot f_b$ is torsion in $J^3(X)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$. Then Mumford's pull-back $M^*(\omega)$ of the holomorphic 3-form ω on X vanishes on V.*

The other half of the proof of Theorem 5.1.1 is:

Proposition 5.2.3. *Assume that $h^{1,0}(X) = b_{4,tr}(X) = 0$, and that the map ψ_3^3 vanishes modulo torsion in $J_3^3(X)$. Then there exist data*

$$\begin{array}{ccccc} \mathcal{C} & \rightarrow & \mathcal{S} & \rightarrow & B \times X \\ \pi \downarrow & & \downarrow & & \downarrow \\ B & = & B & = & B \end{array}$$

together with sections s_i of π, and integers m_i, defining a family of zero-cycles $Z_b = \sum_i m_i s_i(b)$ homologous to zero on C_b, which satisfy the properties:

a) There exists a map $\Psi = (\Psi_1, \Psi_2) : B \longrightarrow X \times X$ such that

$$(pr_2 \circ j \circ i)_* Z_b = \Psi_1(b) - \Psi_2(b)$$

as a zero-cycle of X, for any $b \in B$.

b) There is a smooth locally closed real analytic subset $V \subset B$ such that, for all $b \in V$,

$$\delta'_b \cdot e_b \cdot f_b = 0 \in \big(J^3(X)_{AJ} \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^{\otimes 2}\big)/T,$$

where T is defined as the torsion subgroup of the group $J^3(X)_{AJ} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$, and

c) $\Psi_{|V}$ is a submersion.

Proof. Since $H^{1,0}(X) = 0 = H^{2,0}(X)$, the maps alb_X and ψ_2^3 are zero. So ψ_3^3 is defined on the cycles $x - y$, $x, y \in X$. The assumption that $\psi_3^3 = 0$ modulo torsion implies that for any $x, y \in X$ there exist a curve C, a surface S, together with morphisms

$$i : C \longrightarrow S,\ j : S \longrightarrow X$$

and a cycle Z on C homologous to 0, such that

1. The cycle $j_*(i_*Z)$ is equal to $x - y$ as a zero cycle on X.

2. The cycle i_*Z is albanese equivalent to 0 on S.

3. For some non zero integer m we have
$$m\delta'_{X,S} \cdot e_{S,C} \cdot f_{C,Z} = 0$$
in $J^3(X)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$.

There are countably many algebraic varieties B_m parametrizing the data above, except for condition 3. For each such variety B_m we have a morphism
$$(\Psi_1^m, \Psi_2^m) : B_m \longrightarrow X \times X$$
given by property 1.

And for each of these varieties the equations provided by condition 3 are satisfied on a countable union of locally closed real analytic subsets $V_{n,m}$. Our assumption is that $X \times X$ is filled in by the countable union of the images $(\Psi_1^m, \Psi_2^m)(V_{m,n})$. It follows then from Sard's and Baire's theorems that some (Ψ_1^m, Ψ_2^m) must be submersive at some point of some $V_{m,n}$. $\square$

This proposition, together with Proposition 5.2.2, now implies the theorem as follows:

If $\psi_3^3 = 0$ modulo torsion, we are in the position of applying Proposition 5.2.3. On the one hand, by property a) we have that for any holomorphic 3-form ω on X, Mumford's pull-back $M^*(\omega) = \Psi_1^*(\omega) - \Psi_2^*(\omega)$. On the other hand, by property b) and by Proposition 5.2.2, we know that $M^*\omega$ vanishes on V.

Finally, by property c) the map $\Psi_{|V} : V \longrightarrow X \times X$ is submersive, so the vanishing of
$$\Psi_1^*(\omega) - \Psi_2^*(\omega)_{|V} = \Psi_{|V}^*\big(pr_1^X(\omega) - pr_2^X(\omega)\big)$$
implies that $pr_1^X(\omega) - pr_2^X(\omega)$ is zero on an open set of $X \times X$, hence that ω is zero, since it is holomorphic.

In sum, we have shown that, under the assumptions $h^{1,0}(X) = b_{4,tr}(X) = 0$, if $\psi_3^3 = 0$ modulo torsion, then there are no holomorphic 3-forms on X, which is just Theorem 5.1.1.

Chapter 6

The case of the product of a surface with a curve

6.1 Introduction

In this section we will consider the special case where our 3-fold X is the product of a smooth surface S and a smooth curve C, and establish a connection between the Abel-Jacobi map alb_C for C, the map ψ_2^2 for S and the map ψ_3^3 for X. Concretely, we will define the map k indicated below and prove that the following diagram commutes:

$$\begin{array}{ccc} CH_0(C)_0 \otimes CH_0(S)_{alb} & \rightarrow & ker(\psi_2^3) \\ alb_C \otimes \psi_2^2 \downarrow & & \downarrow \\ J(C) \otimes J_2^2(S) & \xrightarrow{k} & J_3^3(C \times S). \end{array}$$

6.2 The case of the product of two curves

Before we do this, let us consider first the case of a surface which is the product of two curves, $S = C \times D$, and compare the two Abel-Jacobi maps alb_C and alb_D for C and D with the map ψ_2^2 for S.

There is a product "$\times$" for 0-cycles which factors through rational equivalence (see [F], 1.10):

$$CH_0(C) \otimes CH_0(D) \longrightarrow CH_0(S),$$

$$\sum_i n_i[c_i] \times \sum_j m_j[d_j] \mapsto \sum_{i,j} n_i m_j [c_i \times d_j].$$

Obviously the product of two cycles of degree zero is again of degree zero, but the point is that it is even in the kernel of the Albanese map for S. It suffices to show this in the case where our 0-cycle Z is of the form $Z_C \times Z_D$, where $[Z_D] \in CH_0(D)_0$ is simply of the form $[p-q]$. For this we note that we have two inclusion maps

$$i_p : C \to C \times p \subset S$$

and

$$i_q : C \to C \times q \subset S,$$

which induce Gysin maps in cohomology

$$i_{p*}, i_{q*} : H^1(C, \mathbb{Z})^* \to H^1(S, \mathbb{Z})^*.$$

These morphisms are morphisms of Hodge structures which in turn give morphisms of the corresponding Jacobians

$$i_{p*}, i_{q*} : J(C) \to \mathrm{Alb}(S).$$

But these two maps are identical because they are induced by topological maps i_{p*} and i_{q*}, and i_p and i_q are homotopic. So we have

$$alb_S([Z_C] \times [p-q]) = i_{p*}(alb_C[Z_C]) - i_{q*}(alb_C[Z_C]) = 0 \in \mathrm{Alb}(S).$$

The general case follows immediately by linearity.

Next, after we define the map l indicated below, we can ask whether the following diagram, where ψ_2^2 is Green's higher Abel-Jacobi map, commutes:

$$\begin{array}{ccc} CH_0(C)_0 \otimes CH_0(D)_0 & \to & ker(alb_S) \\ alb_C \otimes alb_D \downarrow & & \downarrow \psi_2^2 \\ J(C) \otimes J(D) & \xrightarrow{l} & J_2^2(S) \end{array}.$$

From the Künneth formula we have the inclusion

$$i : H^1(C, \mathbb{Z}) \otimes H^1(D, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{Z}),$$

and the map l will be based upon this inclusion. As previously we have the isomorphism

$$\alpha : H^{1,0}(C)^* = \frac{H^1(C, \mathbb{C})}{F^1 H^1(C)} \cong H^1(C, \mathbb{R}),$$

which allows us to write $J(C) = \frac{H^{1,0}(C)^*}{H^1(C,\mathbb{Z})}$ in the form of a real torus

$$\frac{H^1(C, \mathbb{R})}{H^1(C, \mathbb{Z})} \cong H^1(C, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}.$$

Doing the same for $J(D)$, we obtain a map

$$H^1(C,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}\otimes H^1(D,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}\hookrightarrow H^2(S,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}\otimes\mathbb{R}/\mathbb{Z},$$

and by projecting successively to $H^2(S,\mathbb{Z})^*_{tr}\otimes\mathbb{R}/\mathbb{Z}\otimes\mathbb{R}/\mathbb{Z}$, using the projection

$$H^2(S,\mathbb{Z})\longrightarrow H^2(S,\mathbb{Z})/NS(S)=H^2(S,\mathbb{Z})^*_{tr},$$

and then to

$$\frac{H^2(S,\mathbb{Z})^*_{tr}\otimes\mathbb{R}/\mathbb{Z}\otimes\mathbb{R}/\mathbb{Z}}{U^2_2(S)}=J^2_2(S),$$

we arrive at our desired map l.

In order to prove the commutativity of the diagram above, we assume again that $Z=Z_C\times Z_D$, with

$$Z_D=p-q\in\mathcal{Z}_0(D)_0.$$

Let

$$[\Gamma_{i_p}-\Gamma_{i_q}]\in CH^2(C\times S)=CH^2(C\times C\times D)$$

be the correspondence given by the difference of the graphs of the two inclusion maps

$$i_p: C\longrightarrow C\times p\subset S$$

and

$$i_q: C\longrightarrow C\times q\subset S.$$

We can view our 0-cycle Z as the cycle supported on $(C\times p)\cup(C\times q)$, which is equal to Z_C on $C\times p$ and to $-Z_C$ on $C\times q$.

It follows that in order to compute Green's higher invariant of Z we have to make the Green contraction of the Abel-Jacobi invariant of Z_C in C with the Abel-Jacobi invariant

$$e_{S,C\times p}-e_{S,C\times q}\in H^2(S,\mathbb{Z})^*_{tr}\otimes H^1(C,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}.$$

But this is exactly the projection of (the real version of)

$$AJ(\Gamma_{i_p}-\Gamma_{i_q})\in J^3(C\times S)$$

in the quotient above.

We first compute the last Abel-Jacobi invariant. Let Δ_C be the diagonal in $C\times C$; then it is clear that

$$\Gamma_{i_p}-\Gamma_{i_q}=p^*_{12}(\Delta_C)\cdot p^*_3(p-q).$$

It follows that the Abel-Jacobi image of $\Gamma_{i_p}-\Gamma_{i_q}$ in $J^3(C\times S)=J(H^3(C\times S))$ is given by

$$AJ\big(\Gamma_{i_p}-\Gamma_{i_q}\big)=AJ\big(p_{12}^*(\Delta_C)\cdot p_3^*(p-q)\big)=p_{12}^*[\Delta_C]\cdot p_3^*\big(alb_D(p-q)\big),$$

where $[\Delta_C]$ is the cycle class of the diagonal in $H^2(C\times C,\mathbb{Z})$, and " $\cdot$ " denotes the action by cup-product of the Hodge class $[\Delta_C]$ on the Jacobian

$$p_3^*J(D)\subset J^1(C\times C\times D)$$

with value in $J^3(C\times C\times D)$. Since $H^3(C\times S)$ contains the Künneth component $H^1(C)\otimes H^2(S)$, we can project the intermediate Jacobian via

$$pr: J(H^3(C\times S))\longrightarrow J(H^1(C)\otimes H^2(S));$$

comparing the Künneth types, we find that the following diagram commutes:

$$\begin{array}{llcc} p_{12}^*[\Delta_C] & : p_3^*J(D) & \rightarrow & J(H^3(C\times S)) \\ & \quad\downarrow & & pr\downarrow \\ p_{12}^*[\Delta_C]_{(1,1)} & : p_3^*J(D) & \rightarrow & J(H^1(C)\otimes H^2(S)) \end{array}$$

where $[\Delta_C]_{(1,1)}$ is the Künneth component of type $(1,1)$ of $[\Delta_C]$, and the horizontal maps are induced by cup-product. Hence we conclude that

$$pr\big(AJ\big(\Gamma_{i_p}-\Gamma_{i_q}\big)\big)=p_{12}^*\big([\Delta_C]_{(1,1)}\big)\cdot p_3^*\big(alb_D(p-q)\big). \tag{6.4}$$

The element

$$e_{S,C\times p}-e_{S,C\times q}\in J(H^1(C)\otimes H^2(S)^*_{tr})$$

is just the projection of $pr\big(AJ\big(\Gamma_{i_p}-\Gamma_{i_q}\big)\big)$ via the quotient map $H^2(S,\mathbb{Z})\longrightarrow H^2(S,\mathbb{Z})^*_{tr}$.

Next recall that in order to construct Green's contraction

$$f_{C,Z_C}\cdot(e_{S,C\times p}-e_{S,C\times q})\in H^2(S,\mathbb{Z})^*_{tr}\otimes(\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

we need to write our Jacobians as real tori.

We have the identifications

$$J(D)\cong H^1(D,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z},$$

$$J(H^1(C)\otimes H^2(S))\cong H^1(C,\mathbb{Z})\otimes H^2(S,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}.$$

Since the cup-product with $p_{12}^*[\Delta_C]_{(1,1)}$ is a morphism of Hodge structures

$$H^1(D,\mathbb{Z})\longrightarrow H^1(C,\mathbb{Z})\otimes H^2(S,\mathbb{Z}),$$

we have a commutative diagram

$$\begin{array}{ccc} J(D) & \cong & H^1(D,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z} \\ [\Delta_C]_{(1,1)}\downarrow & & [\Delta_C]_{(1,1)}\downarrow \\ J(H^1(C)\otimes H^2(S)) & \cong & H^1(C,\mathbb{Z})\otimes H^2(S,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z} \end{array}.$$

Hence we have proved the following

Lemma 6.2.1. *The real version of* $e_{S,C\times p}-e_{S,C\times q}$ *is the projection of*

$$[\Delta_C]_{(1,1)}\cdot f_{D,p-q}\in H^1(C,\mathbb{Z})\otimes H^1(C,\mathbb{Z})\otimes H^1(D,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z}$$

in $H^1(C,\mathbb{Z})\otimes H^2(S,\mathbb{Z})^*_{tr}\otimes\mathbb{R}/\mathbb{Z}$*.*

Here the $\cdot$ product identifies to the tensor product, and $f_{D,p-q}$ is the real version of the Abel-Jacobi invariant of $p-q$ in D.

It remains now only to perform the Green contraction of f_{C,Z_C} and $e_{S,C\times p}-e_{S,C\times q}$. For this we apply the following tautological lemma

Lemma 6.2.2. *Let* $\alpha\in H^1(C,\mathbb{Z})$*. Then its contraction with* $[\Delta_C]_{(1,1)}\in H^1(C,\mathbb{Z})\otimes H^1(C,\mathbb{Z})$ *is equal to* α*.*

Proof. The contraction is obtained by identifying $H^1(C,\mathbb{Z})$ to $H^1(C,\mathbb{Z})^*$ by Poincaré duality. Under this identification, the class $[\Delta_C]_{(1,1)}$ is sent to the endomorphism

$$\alpha\mapsto pr_{2*}(pr_1^*\alpha\cup[\Delta_C])$$

of $H^1(C,\mathbb{Z})$, and this endomorphism is the identity since Δ_C is the diagonal. □

Using Lemma 6.2.1, we now apply this to the coefficients of

$$f_{C,Z_C}\otimes f_{D,p-q}\in H^1(C,\mathbb{Z})\otimes H^1(D,\mathbb{Z})\otimes(\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

to conclude that the Green contraction

$$f_{C,Z_C}\cdot(e_{S,C\times p}-e_{S,C\times q})$$

is the projection of $f_{C,Z_C}\otimes f_{D,p-q}$ in $H^2(S,\mathbb{Z})^*_{tr}\otimes(\mathbb{R}/\mathbb{Z})^{\otimes 2}$. Hence we have proved:

Proposition 6.2.3. *Let the surface* S *be the product of two curves* C *and* D*. Then the diagram*

$$\begin{array}{ccc} CH_0(C)_0\otimes CH_0(D)_0 & \to & ker(alb_S) \\ alb_C\otimes alb_D\downarrow & & \downarrow\psi_2^2 \\ J(C)\otimes J(D) & \xrightarrow{l} & J_2^2(S) \end{array}$$

commutes.

6.3 The case of the product of a surface with a curve

We now turn to the case of zero-cycles on a threefold X, which is the product of a curve C and a surface S. We begin by showing

Lemma 6.3.1. *The map induced by the product* $\times$

$$CH_0(C)_0 \otimes CH_0(S)_{alb} \longrightarrow CH_0(X)_{alb}$$

has its image in $ker(\psi_2^3)$.

Proof. We will again start with the case when $Z = Z_C \times Z_S$, with $Z_C = p - q$, p, $q \in C$, and $Z_S \in CH_0(S)_{alb}$ arbitrary. Then we can choose the surface $(p \times S) \cup (q \times S) \subset X$ to compute $\psi_2^3(Z)$, since it contains the support of $Z = (p - q) \times Z_S$. Now by construction the map ψ_2^3 is given by pushing forward $\psi_2^2(Z_S)$ on $p \times S$ and $-\psi_2^2(Z_S)$ on $q \times S$ via the inclusion map, and this is equivalent to pushing forward $\psi_2^2(Z_S)$ via the Gysin morphism

$$j_{p*} - j_{q*} : H^2(S) \longrightarrow H^4(X).$$

But the two push-forward maps in cohomology j_{p*} and j_{q*} are identical, since they are induced by the homotopic maps j_p and j_q, and it follows that

$$(p - q) \times Z_S \in ker(\psi_2^3).$$

Again, the case of an arbitrary cycle $Z \in CH_0(C)_0 \otimes CH_0(S)_{alb}$ follows by linearity from this special case. $\square$

We now construct the map

$$k : J(C) \otimes J_2^2(S) \longrightarrow J_3^3(S \times C),$$

which just as in the previous case will be based on the Künneth inclusion

$$i : H^1(C, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \hookrightarrow H^3(C \times S, \mathbb{Z}).$$

By definition,

$$J_2^2(S) = \frac{H^2(S, \mathbb{Z})^*_{tr} \otimes \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z}}{U_2^2(S)},$$

where $H^2(S, \mathbb{Z})^*_{tr} = \frac{H^2(S,\mathbb{Z})}{NS(S)}$, and as before we write $J(C) = H^1(C, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$.

Lemma 6.3.2. *The map*

$$i : H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \otimes H^2(S,\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \hookrightarrow H^3(C \times S,\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3}$$

followed by the projections

$$\pi_1 : H^3(C \times S,\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3} \longrightarrow J^3(C \times S)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

and

$$\pi_2 : J^3(C \times S)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \longrightarrow \frac{J^3(C \times S)_{AJ} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}}{U_3^3(C \times S)} = J_3^3(C \times S)$$

factors through $J(C) \otimes J_2^2(S)$.

(We recall that π_1 is the composition of the identifications

$$\begin{aligned} H^3(C \times S,\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3} &= H^3(C \times S,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \\ &\cong J^3(C \times S) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \end{aligned}$$

and of the projection in $J^3(C \times S)_{AJ}$, that is modulo the image of the Abel-Jacobi map of $C \times S$.)

We will denote by

$$k : J(C) \otimes J_2^2(S) \longrightarrow J_3^3(C \times S)$$

the induced map on the higher Jacobians.

Proof. To begin, we show that for an element $\eta \in NS(S) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$ and an element $\gamma \in J(C) = H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$, $i(\gamma \otimes \eta)$ belongs to $Im(AJ) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$ and thus vanishes when projected via π_1. In order to see this, note that the class η, being a Hodge class, induces a morphism of Hodge structures

$$p_2^*\eta \cup : H^1(C,\mathbb{Z}) \longrightarrow H^3(C \times S,\mathbb{Z}),$$

and the corresponding morphism of Jacobians

$$p_2^*\eta \cup : J(C) \longrightarrow J^3(C \times S)$$

is the Abel-Jacobi map associated to the family of 1-cycles $Z_C \times \tilde{\eta}$ of $C \times S$, where Z_C is a 0-cycle on C of degree zero, and $\tilde{\eta}$ is a 1-cycle on S of class η. Hence the image of $p_2^*\eta \cup$ (which identifies to the tensor product map via the Künneth decomposition), is contained in the image of the Abel-Jacobi map of $C \times S$. It follows that for any $\eta = \sum_i \eta_i \otimes \alpha_i$, $\eta_i \in NS(S)$, $\alpha_i \in (\mathbb{R}/\mathbb{Z})^{\otimes 2}$, and any $\gamma \in J(C)$, $i(\gamma \otimes \eta)$ belongs to $Im\,AJ \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$, which proves our claim.

The second thing to show is that this composed map factors through $J(C) \otimes U_2^2(S)$, but it can be seen that an element of $J(C) \otimes U_2^2(S)$ maps to an element of $U_3^3(X)$ via $\pi_1 \circ i$ and hence vanishes when projected via π_2. Indeed, these elements are of the form $alb_C(p-q) \otimes \alpha$, where α is obtained by Green's contraction of some $f_{\tilde{C},\tilde{Z}} \cdot e_{\tilde{C},S}$, where $\tilde{Z}$ is a zero cycle of $\tilde{C}$ and $\psi : \tilde{C} \to S$ is a morphism such that $\psi_* \tilde{Z} = 0$ as a cycle on S. We will show below that $alb_C(p-q) \otimes \alpha$ is the Green contraction of α with the Abel-Jacobi invariant of the cycle $\Gamma_{i_p} - \Gamma_{i_q}$ of $S \times C \times S$. Hence, by definition of ψ_3^3, we find that $alb_C(p-q) \otimes \alpha$ belongs to $U_3^3(C \times S)$.

$\square$

Now we can discuss the commutativity of the diagram of the beginning of this section.

Proposition 6.3.3. *The diagram*

$$\begin{array}{ccc} CH_0(C)_0 \otimes CH_0(S)_{alb} & \rightarrow & ker(\psi_2^3) \\ alb_C \otimes \psi_2^2 \downarrow & & \downarrow \\ J(C) \otimes J_2^2(S) & \xrightarrow{k} & J_3^3(C \times S) \end{array} . \tag{6.5}$$

commutes.

Proof. Just as in the proof of Proposition 6.2.3, this will result from the following description of the Abel-Jacobi invariant $\delta'_{X,p\times S} - \delta'_{X,q\times S}$.

Let $\Gamma_{j_p} - \Gamma_{j_q}$ be the difference of the graphs of the two inclusion maps

$$j_p : S \longrightarrow p \times S \subset X$$

and

$$j_q : S \longrightarrow q \times S \subset X.$$

Then by definition $\delta'_{X,p\times S} - \delta'_{X,q\times S}$ is obtained by projecting

$$AJ(\Gamma_{j_p} - \Gamma_{j_q}) \in J^5(S \times X)$$

in the adequate quotient.

By definition of ψ_3^3, since we can see $(p-q) \times Z_S$ as the cycle Z_S on $p \times S$ and $-Z_S$ on $q \times Z_S$, we find that $\psi_3^3((p-q) \times Z_S)$ is obtained as the projection in $J_3^3(X)$ of the Green contraction of any lifting

$$\widetilde{\psi_2^2(Z_S)} \in H^2(S, \mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

of $\psi_2^2(Z_S)$ and of the real version of $\delta'_{X,p\times S} - \delta'_{X,q\times S}$. Since the latter is obtained as a projection of the (real version of) the Abel-Jacobi invariant of $\Gamma_{j_p} - \Gamma_{j_q}$, we start by computing

$$AJ(\Gamma_{j_p} - \Gamma_{j_q}) \in J^5(S \times X).$$

Let $\Delta_S \subset S \times S$ be the diagonal; we see immediately that

$$\Gamma_{j_p} - \Gamma_{j_q} = p_{13}^*(\Delta_S) \cdot p_2^*(p-q) \tag{6.6}$$

as a cycle of $S \times C \times S$.

Now by (6.6) we have the following formula for its Abel-Jacobi image in $J^5(S \times X) = J(H^5(S \times X))$:

$$AJ\big(\Gamma_{j_p} - \Gamma_{j_q}\big) = p_{13}^*\big([\Delta_S]\big) \cdot p_2^*\big(alb_C(p-q)\big). \tag{6.7}$$

We project this invariant via

$$pr : J(H^5(S \times X)) \longrightarrow J(H^2(S) \otimes H^3(X));$$

from (6.7), by analysing Künneth types, it follows that

$$pr\big(AJ\big(\Gamma_{j_p} - \Gamma_{j_q}\big)\big) = p_{13}^*\big([\Delta_S]_{(2,2)}\big) \cdot p_2^*\big(alb_C(p-q)\big), \tag{6.8}$$

where $[\Delta_S]_{(2,2)}$ is the $(2,2)$ Künneth component of Δ_S. Now since $[\Delta_S]_{(2,2)}$ is a Hodge class, the identifications

$$J(C) \cong H^1(C, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z},$$

$$J^5(S \times C \times S) \cong H^5(S \times C \times S, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$$

are compatible with the cup-product with $p_{13}^*[\Delta_S]_{(2,2)}$ (which via the Künneth decomposition is given by the tensor product with $[\Delta_S]_{(2,2)}$.) Hence we have proved

Lemma 6.3.4. *The invariant $AJ(\Gamma_{j_p} - \Gamma_{j_q})$, viewed as an element of*

$$H^2(S, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}) \otimes H^2(S, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z},$$

is equal to

$$p_{13}^*[\Delta_S]_{(2,2)} \cup p_2^* f_{C,p-q}.$$

(Here the cup-product identifies to a tensor product, and $f_{C,p-q} \in H^1(C, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$ is the real version of $alb_C(p-q)$.)

The conclusion of the proof is identical to the case of a product of two curves. In order to compute $\psi_3^3(Z)$, we have to make the Green contraction of $\widetilde{\psi_2^2(Z_S)}$ and $AJ(\Gamma_{j_p} - \Gamma_{j_q})$. We now use the following lemma which is proved just as Lemma 6.2.2:

Lemma 6.3.5. *The map*

$$H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathbb{Z}),$$

which is given by contraction with the class $[\Delta_S]_{(2,2)} \in H^2(S,\mathbb{Z}) \otimes H^2(S,\mathbb{Z})$ *using Poincaré duality, is equal to the identity.*

It follows from this lemma that Green's contraction of $\widetilde{\psi_2^2(Z_S)}$ with

$$p_{13}^*[\Delta_S]_{(2,2)} \cup p_2^* f_{p-q,C}$$

is equal to

$$f_{C,p-q} \otimes \widetilde{\psi_2^2(Z_S)} \in H^1(C,\mathbb{Z}) \otimes H^2(S,\mathbb{Z}) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 3}.$$

This proves the commutativity of the diagram (6.5). $\square$

To conclude this section, we point out the following fact:

Remark 6.3.6. *One knows that the map* ψ_2^2 *is not injective (see [V3], section 2: "The non-injectivity of* ψ_2^2*"). The description of the map* ψ_3^3 *just given in this special case shows that it is not injective, if we admit that the following statement is true:*

Let Z_S *be a zero-cycle on a surface* S *which is not rationally equivalent to* 0*. Then there exist a curve* C *and a zero-cycle* Z_C *of degree* 0 *on* C *such that* $Z_S \times Z_C$ *is not rationally equivalent to* 0 *on* $C \times S$*.*

Appendix A

Carlson vs. Griffiths

In this appendix we want to show why the two ways of obtaining the extension class $e_{X,S}$ are actually the same, as we claim in section 4. We found a proof for this kind of statement in the case of a 0-cycle on a curve in [Car1] (Proposition 9), but for the higher-dimensional analogue we have not been able to find a reference.

The first way, which we used for constructing the map ψ_3^3, is morally given by intersecting the graph Γ_j of the morphism $j : S \longrightarrow X$ with cycles Δ_2 and Δ_3 representing the $(2,2)$- and $(3,3)$-Künneth-components of the diagonals in $S \times S$ and $X \times X$, respectively, to get a 4-cycle of $S \times X$ which is homologically trivial. This would then be mapped to Griffiths' intermediate Jacobian. However, only morally, since we don't know if there is such a cycle Δ_3, i.e. we don't know if the $(3,3)$-Künneth-component is algebraic. So all we can do is lift it to a Deligne cohomology class $[\Delta_3]_{\mathcal{D}}$, intersect it with $[\Gamma_j \circ \Delta_2]_{\mathcal{D}}$ in Deligne cohomology, and via the short exact sequence

$$0 \to J^5(S \times X) \to H^6_{\mathcal{D}}(S \times X, \mathbb{Z}(3)) \to Hdg^6(S \times X) \to 0$$

obtain an element in Griffiths intermediate Jacobian. Finally, this element is mapped via the Künneth formula to a quotient of the Jacobian variety

$$J\Big(\frac{H^2(S)}{j^*H^2(X)} \otimes \frac{H^3(X)}{j_*H^1(S)}\Big).$$

The second way, which we used for the calculations in section 4, comes from the mixed Hodge structures derived from a part of the long exact sequence in relative cohomology induced by the morphism $j : S \longrightarrow X$.

A *mixed Hodge structure* (MHS) H is a triple $(H_{\mathbb{Z}}, F^{\bullet}, W_{\bullet})$ consisting of
- a lattice $H_{\mathbb{Z}} \subset H = H_{\mathbb{Z}} \otimes \mathbb{R}$
- a finite increasing (weight) filtration $W_{\bullet}$ on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$

- and a finite decreasing (Hodge) filtration $F^\bullet$ on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$,
such that each graded piece $Gr_i^W = W_i/W_{i-1}$ of the weight filtration is a pure Hodge structure.

Consider the short exact sequence

$$0 \longrightarrow \frac{H^2(S)}{j^*H^2(X)} \xrightarrow{i} H^3(X,S) \xrightarrow{p} ker\big(j^* : H^3(X) \to H^3(S)\big) \longrightarrow 0,$$

where we intentionally leave out the coefficients. The middle component is endowed with a MHS, and the cokernel and kernel on the left and right are even pure Hodge structures. As a sequence of torsion-free $\mathbb{Z}$-modules it splits over the integers, i.e. there exists a section

$$\sigma_{\mathbb{Z}} : ker(j^*) \longrightarrow H^3(X,S,\mathbb{Z})$$

such that $p \circ \sigma_{\mathbb{Z}} = id$.

Of course the sequence also splits over $\mathbb{C}$, but there exists a section

$$\sigma_F : ker(j^*) \longrightarrow H^3(X,S,\mathbb{C})$$

that also respects the Hodge filtration, i.e. such that

$$\sigma_F(F^i ker(j^*)) \subseteq F^i Gr_3^W H^3(X,S,\mathbb{C}),$$

for i=0,...,3. The difference of two such sections defines a map

$$\sigma_F - \sigma_{\mathbb{Z}} : ker(j^*) \longrightarrow coker(j^*),$$

and in the quotient

$$Ext^1_{MHS}\big(ker(j^*), coker(j^*)\big) := \frac{Hom_{\mathbb{C}}\big(ker(j^*), coker(j^*)\big)}{F^0 Hom_{\mathbb{C}}(.,.) + Hom_{\mathbb{Z}}(.,.)}$$

it will be independent of the choices made for $\sigma_{\mathbb{Z}}$ and σ_F. The notation means that elements of $F^0 Hom_{\mathbb{C}}(.,.)$ are homomorphisms σ_F such that

$$\sigma_F\big(F^i ker(j^*)\big) \subseteq F^i\big(coker(j^*)\big)$$

for all i = 0,...,3, and elements of $Hom_{\mathbb{Z}}(.,.)$ are homomorphisms $\sigma_{\mathbb{Z}}$ such that

$$\sigma_{\mathbb{Z}}\big(ker(j^*)\big)_{\mathbb{Z}} \subseteq \big(coker(j^*)\big).$$

Now that we have described the two methods, we want to show that they lead to the same result. Consider the dual sequence of ( ):

$$0 \longleftarrow ker\big(j_* : H_2(S,\mathbb{Z}) \to H_2(X,\mathbb{Z})\big) \longleftarrow H_3(X-S,\mathbb{Z}) \longleftarrow \frac{H_3(X,\mathbb{Z})}{j_*H_3(S,\mathbb{Z})} \longleftarrow 0.$$

We can interpret elements of $H_3(X-S,\mathbb{Z})$ as topological 3-cycles on X whose boundaries lie in S. Let $\gamma_1,...,\gamma_n$ be a $\mathbb{Z}$-basis for $ker(j_*)$, and $\gamma^1,...,\gamma^n$ the dual basis for $coker(j^*)$. Let $\Gamma_1,...,\Gamma_n$ be 3-chains on X such that $\partial\Gamma_i=\gamma_i$; Γ_i represents a class in $H_3(X-S,\mathbb{Z})$. We have the following formula by Carlson ([Car1], 2.d):

The homomorphisms representing the extension given above are all of the form

$$\omega \mapsto \sum_i \gamma^i \int_{\Gamma_i} \Omega,$$

where $\Omega=\sigma_F(\omega)$.

Appendix B

Deligne cohomology and cycle class map

B.1 Deligne cohomology

In this section we want to give the definition of the Deligne cohomology groups and state some of their properties.

Definition B.1.1. *Let X be a complex variety, $p \geq 1$ an integer and denote $\mathbb{Z}(p) = (2\pi i)^p \cdot \mathbb{Z}$. We define the* **Deligne complex** $\mathbb{Z}_{\mathcal{D}}(p)$ *to be the complex*

$$0 \longrightarrow \mathbb{Z}(p) \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow ... \longrightarrow \Omega_X^{p-1} \longrightarrow 0 \ ,$$

where $\mathbb{Z}(p)$ is in degree zero and Ω^q in degree $q+1$. The **Deligne cohomology groups** $H^q_{\mathcal{D}}(X, \mathbb{Z}(p))$ *are the hypercohomology groups $\mathbb{H}^q(X, \mathbb{Z}_{\mathcal{D}}(p))$.*

For example, if $p = 0$, then the complex $\mathbb{Z}_{\mathcal{D}}(0)$ is just $\mathbb{Z}$, and we have $H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) = H^q(X, \mathbb{Z})$ for all q. For $p = 1$ we obtain a quasi-isomorphism between the complexes $\mathbb{Z}_{\mathcal{D}}(1)$ and $\mathcal{O}_X^*$ (put in degree 1) from the exponential sheaf sequence. Hence $H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^0(X, \mathcal{O}^*)$ and $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X, \mathcal{O}^*)$, which is just the Picard group.

In case our variety is kähler the following result holds:

Proposition B.1.2. *Let X be a compact kähler variety. Then the Deligne cohomology groups fit into the long exact sequence*

$$... \longrightarrow H^{q-1}(X, \mathbb{Z}) \longrightarrow \frac{H^{q-1}(X, \mathbb{C})}{F^p H^{q-1}(X)} \longrightarrow H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow H^q(X, \mathbb{Z}) \longrightarrow ...$$

Proof. Let $\Omega_X^{\leq p-1}$ denote the complex

$$0 \longrightarrow 0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow ... \longrightarrow \Omega_X^{p-1} \longrightarrow 0 \ ,$$

where again Ω^q_X is in degree $q+1$. This gives rise to a short exact sequence of complexes

$$0 \longrightarrow \Omega_X^{\leq p-1} \longrightarrow \mathbb{Z}_{\mathcal{D}}(p) \longrightarrow \mathbb{Z}(p) \longrightarrow 0 \ ,$$

which in its turn induces a long exact sequence

$$... \longrightarrow \mathbb{H}^{q-1}(X, \mathbb{Z}(p)) \longrightarrow \mathbb{H}^{q-1}(X, \Omega_X^{\leq p-1}) \longrightarrow H^q_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow \mathbb{H}^q(X, \mathbb{Z}(p)) \longrightarrow ...$$

in hypercohomology. Obviously $\mathbb{H}^{q-1}(X, \mathbb{Z}(p)) \cong H^{q-1}(X, \mathbb{Z})$, so it remains to show that

$$\mathbb{H}^{q-1}(X, \Omega_X^{\leq p-1}) = \frac{H^{q-1}(X, \mathbb{C})}{F^p H^{q-1}(X)}.$$

For this consider the short exact sequence of complexes

$$0 \longrightarrow \Omega_X^{\geq p} \longrightarrow \Omega_X^* \longrightarrow \Omega_X^{\leq p-1} \longrightarrow 0.$$

The proposition now follows from the facts that for the hypercohomology of the truncated holomorphic de Rham complex we have $\mathbb{H}^q(X, \Omega_X^{\geq p}) = F^p H^q(X)$ (see [V1], Proposition 7.3), $\mathbb{H}^q(X, \Omega_X^*) = H^q(X, \mathbb{C})$, and that the map $\mathbb{H}^q(X, \Omega_X^*) \longrightarrow \mathbb{H}^q(X, \Omega_X^{\leq p-1})$ is surjective. □

The case where $q = 2p$ is of special interest. First we define the subgroup of Hodge classes in $H^{2p}(X, \mathbb{Z})$:

Definition B.1.3. *Let $Hdg^{2p}(X, \mathbb{Z}) \subseteq H^{2p}(X, \mathbb{Z})$ denote the subgroup of those classes whose image in $H^{2p}(X, \mathbb{C})$ is of type (p,p).*

Clearly $Hdg^{2p}(X, \mathbb{Z}) = ker\big(H^{2p}(X, \mathbb{Z}) \longrightarrow \frac{H^{2p}(X, \mathbb{C})}{F^p H^{2p}(X)}\big)$, and we obtain from the proposition in this case the

Corollary B.1.4. *The Deligne cohomology group $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ is an extension of the group of Hodge classes $Hdg^{2p}(X, \mathbb{Z})$ by Griffiths' intermediate Jacobian $J^{2p-1}(X)$:*

$$0 \longrightarrow J^{2p-1}(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \longrightarrow Hdg^{2p}(X, \mathbb{Z}) \longrightarrow 0.$$

B.2 The Deligne cycle class map

Here we describe some of the properties of the Deligne cycle class map.

Proposition B.2.1. a) *The Deligne cycle class map factors through rational equivalence: Let $U, V \in \mathcal{Z}^p(X)$ be two codimension p cycles on X that are rationally equivalent. Then $cl_{\mathcal{D}}(U) = cl_{\mathcal{D}}(V)$, i.e. it descends to a map*

$$cl_{\mathcal{D}} : CH^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

b) *It is compatible with the product structure on* $H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$*: Let* $[U] \in CH^p(X)$ *and* $[V] \in CH^q(X)$*. Then*

$$cl_{\mathcal{D}}([U]\cdot[V]) = cl_{\mathcal{D}}([U]) \cdot_{\mathcal{D}} cl_{\mathcal{D}}([V]),$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
CH^p(X) \times CH^q(X) \arrow[r, "\cdot"] \arrow[d, "cl_{\mathcal{D}} \times cl_{\mathcal{D}}"] & CH^{p+q}(X) \arrow[d, "cl_{\mathcal{D}}"] \\
H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \times H^{2q}_{\mathcal{D}}(X,\mathbb{Z}(q)) \arrow[r, "\cdot_{\mathcal{D}}"] & H^{2(p+q)}_{\mathcal{D}}(X,\mathbb{Z}(p+q))
\end{tikzcd}$$

c) *It commutes with flat pull-back: Let* $f : X \longrightarrow Y$ *be a flat morphism and* $[U] \in CH^p(X)$*. Then*

$$cl_{\mathcal{D}}(f^*([U]) = f^*(cl_{\mathcal{D}}([U])),$$

i.e.

$$\begin{tikzcd}
CH^p(Y) \arrow[r, "f^*"] \arrow[d, "cl_{\mathcal{D}}"] & CH^p(X) \arrow[d, "cl_{\mathcal{D}}"] \\
H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p)) \arrow[r, "f^*"] & H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))
\end{tikzcd}$$

commutes.

Proof. Propositions 7.6, 7.4 and 7.5 in [EV], Théorème 9.20 in [V2] □

As we have seen, there is a generalization of the short exact sequence for line-bundles (or divisors)

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow NS(X) \longrightarrow 0,$$

which makes use of Deligne cohomology:

$$0 \longrightarrow J^{2p-1}(X) \longrightarrow H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \longrightarrow Hdg^{2p}(X) \longrightarrow 0.$$

We reproduce here an illustrative diagram by Murre from [GMV] that brings out the analogy between the two sequences above:

$$\begin{tikzcd}
 & & CH^p(X)_{hom} \arrow[d] & & \\
 & & CH^p(X) \arrow[d, "cl_{\mathcal{D}}"] & & \\
0 \arrow[r] & J^{2p-1}(X) \arrow[r] & H^{2p}_{\mathcal{D}}(Y,\mathbb{Z}(p)) \arrow[r] & Hdg^{2p}(X) \arrow[r] & 0
\end{tikzcd}$$

Appendix C

The Gauss-Manin connection on relative cohomology

If $X \subset Y$ is a differentiable submanifold of a manifold Y, the relative cohomology groups

$$H^k(Y, X, \mathbb{R})$$

can be computed in de Rham cohomology as the cohomology of the complex $A^{\cdot}_{Y,X}$ of differential forms on Y vanishing on X. The exact sequence of relative cohomology

$$H^{k-1}(X, \mathbb{R}) \xrightarrow{i} H^k(Y, X, \mathbb{R}) \to H^k(Y, \mathbb{R}) \to H^k(X, \mathbb{R})$$

is associated to the short exact sequence of complexes

$$0 \to A^\bullet_{Y,X} \to A^\bullet_Y \to A^\bullet_X \to 0.$$

It follows from this and the general definition of the coboundary map in the long exact sequence associated to a short exact sequence of complexes, that the map i above is explicitly computed as follows: starting from a closed $k-1$-form α on X with class $[\alpha] \in H^{k-1}(X, \mathbb{R})$, let $\tilde{\alpha}$ be a $k-1$-form on Y extending α. Then the k-form $d\alpha$ on Y vanishes along X and is closed. Its class $[d\alpha] \in H^k(Y, X)$ is equal to $i([\alpha])$. Suppose now that we have a diagram

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathcal{Y} \\ \downarrow \pi & & \downarrow \rho \\ B & = & B \end{array}$$

where π and ρ are differentiable submersive proper maps between differentiable manifolds. Locally on B we may assume that

$$\mathcal{X} \cong X \times B, \mathcal{Y} \cong Y \times B \tag{C.1}$$

and π and ρ are the second projection, where X and Y are the fibers of π and ρ, respectively, over $0 \in B$. Since B is locally contractible, it follows that we have three sets of local systems on B, namely

$$H_X^{\cdot} := R^{\cdot}\pi_*\mathbb{R},\; H_Y^{\cdot} := R^{\cdot}\rho_*\mathbb{R}$$

and the local system of relative cohomology $H_{Y,X}^{\cdot}$ which fits into the long exact sequence

$$H_X^{k-1} \to H_{Y,X}^k \to H_Y^k \to H_X^k \to \dots$$

We want to compute the Gauss-Manin connection $\nabla^{Y,X}$ on the associated bundle

$$\mathcal{H}_{Y,X}^k = H_{Y,X}^k \otimes \mathcal{C}^\infty{}_B.$$

Assume we are given a family $(\alpha_b)_{b\in B}$ of closed differential forms on Y, vanishing on B. Via the isomorphisms (C.1), $b \mapsto [\alpha_b]$ gives a section $[\alpha]$ of $\mathcal{H}_{Y,X}^k$. Then

$$\nabla^{Y,X}([\alpha_b]) \in \mathcal{H}_{Y,X}^k \otimes \Omega_B$$

is equal to the map

$$u \mapsto [d_u\alpha],\, T_B \to H^k(Y, X),$$

where $d_u\alpha$ is the derivative with respect to u of the family of forms α_b, and $[\,.\,]$ means the cohomology class of the considered closed form.

We suppose now that ω is a closed k-form on $\mathcal{Y}$ such that $\omega_b := \omega_{|Y_b}$ vanishes on X_b for each $b \in B$. The family of closed forms ω_b vanishing on X_b provides a section

$$\tilde{\omega} \in \mathcal{H}_{Y,X}^k.$$

We note that since $\tilde{\omega}$ lifts the section

$$b \mapsto [\omega_b] = [\omega]_{|Y_b}$$

of $\mathcal{H}_Y^k$, which is obviously flat, the derivative $\nabla^{Y,X}\tilde{\omega}$ belongs to

$$i(\mathcal{H}_X^{k-1}) \otimes \Omega_B \subset \mathcal{H}_{Y,X}^k \otimes \Omega_B.$$

Next, consider the restriction of ω to $\mathcal{X}$. Since it vanishes on the fibers X_b, it admits a projection $p(\omega)_b$ in $A^{k-1}(X_b) \otimes \pi^*\Omega_{B,b}$ for each $b \in B$. This form with coefficients in $\Omega_{B,b}$ is easily seen to be closed on X_b, thus providing a class

$$\alpha_b(\omega) \in H^{k-1}(X_b) \otimes \Omega_{B,b}.$$

We have the following

Lemma C.0.2. *One has the equality at the point $b \in B$:*

$$(\nabla^{Y,X}\tilde{\omega})_b = i(\alpha_b(\omega)).$$

Proof. We may assume that B is one dimensional, say $B \cong \mathbb{R}$, and we use the isomorphisms (C.1). Let us write $\omega = \omega_1 + dt \wedge \omega_2$, where ω_1 and ω_2 are k and $k-1$-forms on Y varying with $t\ in \mathbb{R}$. Then the form $\omega_{1|Y_b}$ is closed, vanishes on X_b, and its class is equal to $[\omega_b] \in H^k(Y, X)$. Hence we have

$$(\nabla^{Y,X}\tilde{\omega})_b = [\frac{d}{dt}\omega_1]_{t=b} \otimes dt.$$

Next we observe that

$$p(\omega)_b = dt \otimes \omega_{2|X\times b}.$$

Hence the class $\alpha_b(\omega)$ is equal to $dt \otimes [\omega_{2|X\times b}]$. So our claim is that

$$i([\omega_{2|X\times b}]) = [\frac{d}{dt}\omega_1]_{t=b}$$

in $H^k(Y, X)$. But since ω is closed, we have $d\omega_1 - dt \wedge d\omega_2 = 0$, hence

$$(\frac{d}{dt}\omega_1)_{t=b} = (d\omega_2)_{|Y\times b}.$$

So it follows from the construction of i described above that

$$[\frac{d}{dt}\omega_1]_{Y\times b} = i([\omega_{2|X\times b}]),$$

which is our claim. $\square$

Appendix D

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