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Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles.

Perturbations of the metric in Seiberg-Witten equations.

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Contents

Introduction

Cohomologie du schéma de Hilbert ponctuel d'une surface à valeurs dans certaines représentations d'un fibré tautologique

Cette partie du travail est consacrée à la cohomologie du schéma de Hilbert ponctuel sur une surface algébrique projective lisse X à valeurs dans certaines représentations d'un fibré vectoriel tautologique $E^{[n]}$, associé à un fibré en droites E sur X. En particulier, on traitera en détail le cas de la cohomologie $H^*(X^{[n]}, S^2E^{[n]})$ de la puissance symétrique $S^2E^{[n]}$, et de la cohomologie $H^*(X^{[n]}, \Lambda^2E^{[n]})$ de la puissance extérieure $\Lambda^2E^{[n]}$.

Motivations. Ce travail est motivé par les problèmes rencontrés dans les phénomènes et la conjecture de *dualité étrange*, que nous allons expliquer. La dualité étrange est une relation de dualité entre des espaces de sections de fibrés déterminants sur des différents espaces de modules de faisceaux semi-stables sur une variété algébrique lisse. Les premiers exemples des phénomènes de dualité étrange ont été découverts sur des courbes par Beauville [3] et Beauville-Narasimhan-Ramanan [6]. Soit X une courbe projective lisse de genre $g \geq 2$, et $\mathcal{SU}(n)$ l'espace de modules de fibrés semi-stables de rang n et de degré 0 sur X. Soit $\mathcal{D}$ le générateur ample de $\mathrm{Pic}(\mathcal{SU}(n))$, qui est un groupe abélien libre de rang 1, par le théorème de Drezet-Narasimhan [30]: $\mathcal{D}$ s'appelle le fibré déterminant sur $\mathcal{SU}(n)$. Si Θ représente le diviseur théta dans la Jacobienne $J^{g-1}(X)$, on a une dualité étrange:

$$H^0(J^{g-1}(X), \mathcal{O}(n\Theta))^* \simeq H^0(\mathcal{SU}(n), \mathcal{D})$$

entre l'espace de sections du fibré déterminant $\mathcal{D}$ sur $\mathcal{SU}(n)$ et l'espace de fonctions théta de niveau n sur $J^{g-1}(X)$. Le calcul de la dimension de $H^0(\mathcal{SU}(n), \mathcal{D})$ et, plus généralement, de $H^0(\mathcal{SU}(n), \mathcal{D}^{\otimes^k})$ a été effectué par Beauville and Laszlo [5] en utilisant la formule de Verlinde, conjecturée par Verlinde [116], et démontrée par plusieurs auteurs, parmis lesquels Tsuchiya-Ueno-Yamada [113], Beauville [4], Faltings [37], Thaddeus [110], Jeffrey-Kirwan [71].

Sur une surface algébrique projective lisse simplement connexe X, Le Potier a proposé la conjecture suivante, dite de dualité étrange. L'algèbre de Grothendieck $K_{\mathrm{top}}(X)$ des fibrés topologiques sur X est isomorphe, comme groupe abélien, à $\mathbb{Z} \times H^2(X, \mathbb{Z}) \times \mathbb{Z}$. En d'autres mots, une classe en $K_{\mathrm{top}}(X)$ est identifiée par son rang, sa première classe de Chern c_1 et sa caractéristique d'Euler-Poincaré χ. La caractéristique d'Euler-Poincaré définit une forme quadratique entière sur $K_{\mathrm{top}}(X)$, en posant:

$$\langle u, v \rangle = \chi(u \cdot v) \qquad \text{if } u, v \in K_{\mathrm{top}}(X) \ .$$

Prenons ensuite deux classes $u, v \in K_{\mathrm{top}}(X)$, orthogonales pour $\langle \cdot, \cdot \rangle$ et considérons l'espace de modules de faisceaux semi-stables M_u, M_v avec classes de Grothendieck fixées u et v, respectivement. Soit $\mathcal{D}_{u,v}$ et $\mathcal{D}_{v,u}$ les deux fibrés déterminants (voir [86], [68]) sur M_u et M_v associés aux classes v et u, respectivement. Sous certaines hypothèses techniques, Le Potier trouve une section canonique $\sigma_{v,u} \in H^0(M_u \times M_v, \mathcal{D}_{u,v} \boxtimes \mathcal{D}_{v,u})$, qui permet de définir le morphisme de dualité étrange:

$$D_{v,u} : H^0(M_u, \mathcal{D}_{u,v})^* \longrightarrow H^0(M_v, \mathcal{D}_{v,u}) \ .$$

Le Potier a conjecturé, sous certaines hypothèses, que $D_{v,u}$ est un isomorphisme, si M_v n'est pas vide.

Danila [21] a étudié la conjecture de la dualité étrange de Le Potier sur $\mathbb{P}_2$. En particulier, elle a résolu la conjecture pour les classes de Grothendieck $c = (2, 0, n)$, $u = (0, 1, 0)$, $n \leq 19$. Soit $M_n := M_{(2,0,n)}$. La stratégie consiste à utiliser les espaces de modules de systèmes cohérents (voir [66], [84], [85]) S_α afin d'établir une relation — pour $l(l-1) \leq n \leq (l+1)(l+2)$ —entre l'espace de sections $H^0(M_n, \mathcal{D}_{n,u})$ et l'espace de sections $H^0(U, S^l\mathcal{R} \otimes \mathcal{D}_u)$ d'un faisceau cohérent $\mathcal{R}$ sur un ouvert U du schéma de Hilbert $X^{[n+l^2]}$, où $\mathcal{R}$ est localement libre (voir [22]). Danila construit une résolution localement libre $K^\bullet \longrightarrow \mathcal{R}$, dont les termes K^i dépendent de la puissance symétrique $S^r(\mathcal{O}(k)^{[n+l^2]})$ du fibré tautologique $\mathcal{O}(k)^{[n+l^2]}$ associé au fibré de droites $\mathcal{O}(k)$ sur $\mathbb{P}_2$. Pour calculer la suite spectrale d'hypercohomologie, il faut calculer des groupes de cohomologie du genre:

$$H^q(X^{[m]}, S^k(\mathcal{O}(k)^{[m]}) \otimes \mathcal{D}_u) ,$$

$\mathcal{D}_u$ étant le déterminant de Donaldson sur $X^{[n]}$ associé à la classe u. Ce sont précisément les obstacles d'ordre technique dans ces calculs qui limitent les résultats de Danila à $n \leq 19$. La connaisssance de ces groupes de cohomologie entraînerait la preuve complète de la conjecture pour le plan projectif, au moins pour $c = (2, 0, n)$, $u = (0, 1, 0)$. Danila a démontré dans [23] et [24] des formules générales pour la cohomologie de fibrés tautologiques sur le schéma de Hilbert $H^*(X^{[n]}, L^{[n]})$ associés à un fibré en droites L sur X, et de la puissance symétrique double $H^*(X^{[n]}, S^2L^{[n]})$, pour $n \leq 3$.

Dans ce travail, on généralise ses résultats pour $S^2L^{[n]}$ pour tout n, et on obtient également des formules générales pour la cohomologie de la puissance extérieure double $H^*(X^{[n]}, \Lambda^2L^{[n]})$ pour tout n. Il s'avère que ces derniers groupes sont également impliqués dans la vérification de la conjecture de la dualité étrange sur le plan projective pour les espaces de modules M_2 et $M_{(1,d,m)}$ (cf. [83]).

La méthode: correspondance de McKay. La méthode qu'on utilise est différente de celle de Danila et elle provient de développements récents dans la correspondance de McKay. Le début de la correspondance de McKay date de bien avant McKay et, en effet, a commencé avec Klein vers 1870 et avec Coxeter et Du Val vers 1930. Quand on quotiente $\mathbb{C}^2$ par un sous-groupe fini G de $SL(2, \mathbb{C})$ et on prend la résolution minimale Y de $\mathbb{C}^2/G$, alors Y est crépant et le lieu exceptionnel consiste en un ensemble de courbes, dont le graphe dual est un diagramme de Dynkin du type A_n, D_n, E_6, E_7, E_8.

McKay (voir [89], [90], [44]) a observé que les diagrammes de Dynkin issus de résolutions de singularités kleiniennes sont reliés aux représentations de G. Si ρ est une représentation de G, et ρ_i sont de représentations irréductibles, le *graphe de McKay* est le diagramme de Dynkin associé à la matrice $a_{ij} - 2\mathrm{id}$, où a_{ij} est défini comme

$$\rho \otimes \rho_i = \sum_j a_{ij}\rho_j ;$$

le graphe de McKay ainsi défini est alors le diagramme de Dynkin donné par le graphe dual (étendu) du lieu exceptionnel quand G opère sur $\mathbb{C}^2$ via la représentation ρ. De plus, McKay a suggeré qu'il y a une bijection entre les composantes du lieu exceptionnel et les représentations irréductibles de G. Dans [53], Gonzalez-Springer and Verdier ont démontré une version de cet énoncé en K-théorie, en montrant un isomorphisme:

$$K_G(\mathbb{C}^2) \xrightarrow{\simeq} K(Y)$$

entre le groupe de K-théorie G-équivariante de $\mathbb{C}^2$ et le groupe de K-théorie de Y, qui rend précise la correspondance entre l'ensemble des représentations irréductibles de G et l'ensemble des composantes irréductibles du lieu exceptionnel.

La généralisation du théorème de Gonzalez-Springer-Verdier qui nous intéresse a été publiée en 2001, avec le résultat suivant de Bridgeland-King-Reid [16]. Soit M une variété quasi-projective lisse sur $\mathbb{C}$ et G un groupe fini d'automorphismes de M tel que ω_M est localement trivial comme G-faisceau. Soit $Y = \mathrm{Hilb}^G(M)$ le G-schéma de Hilbert (selon Nakamura) des G-orbites. Alors, si la dimension de $Y \times_{M/G} Y$ n'est pas trop grande, on obtient l'équivalence:

$$\Phi : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M) ,$$

entre la catégorie dérivée des faisceaux cohérents sur Y et la catégorie dérivée des faisceaux G-équivariants sur M, où Φ est la transformée de Fourier-Mukai ayant comme noyau la famille universelle $\mathcal{Z} \subset Y \times M$.

L'étape suivante a été menée par Haiman ([61], [62], [60]), qui a démontré que l'action du groupe symétrique $\mathfrak{S}_n$ sur le produit X^n d'une surface X satisfait l'hypothèse du théorème BKR, et que $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ peut être identifié avec le schéma de Hilbert $X^{[n]}$. Par conséquent, on obtient l'équivalence:

$$\mathbf{\Phi} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

obtenue par la transformée de Fourier-Mukai de noyau B^n, le schéma de Hilbert isospectral. Ainsi, les calculs de cohomologie sur le schéma de Hilbert peuvent être obtenus plus simplement par des calculs de cohomologie $\mathfrak{S}_n$-équivariante sur le produit X^n.

Résultats. Notre première question a été de trouver l'image d'un faisceau tautologique $F^{[n]}$ associé à un faisceau cohérent F sur la surface X pour l'équivalence BKRH $\mathbf{\Phi}$. On rappelle que le faisceau tautologique $F^{[n]}$ est le faisceau défini par le foncteur de Fourier-Mukai:

$$F^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) = \mathbf{R}p_{X^{[n]}\,*}(\mathcal{O}_\Xi \otimes^L p_X^* F) \ ,$$

où Ξ est la famille universelle sur le schéma de Hilbert. Dans le diagramme commutatif

$$\begin{tikzcd} B^n \arrow[r, "p"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^n X \end{tikzcd}$$

B^n est le schéma de Hilbert isospectral, S^nX est la variété symétrique et μ est le morphisme de Hilbert-Chow. L'équivalence de Bridgeland-King-Reid-Haiman

$$\mathbf{\Phi} = \mathbf{R}p_* \circ q^*$$

calculée sur $F^{[n]}$ devient alors simplement la composition des foncteurs de Fourier-Mukai:

$$\mathbf{\Phi}(F^{[n]}) = \mathbf{\Phi} \circ \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) = \Phi^{\mathbf{R}f_*\mathcal{O}_Z}_{X \to X^n}(F) \ ,$$

que l'on sait être un troisième foncteur de Fourier-Mukai de noyau $\mathbf{R}f_*\mathcal{O}_Z$, où Z est la famille universelle sur le schéma de Hilbert isospectral, et f le morphisme: $f \colon B^n \times X \longrightarrow X^n \times X$. On démontre que l'image directe dérivée $\mathbf{R}f_*\mathcal{O}_Z$ est quasi isomorphe au faisceau structural $\mathcal{O}_D$ de l'union schématique $D = \cup_{i=1}^n \Delta_{i,n+1}$ des diagonales $\Delta_{i,n+1}$ de $X^n \times X$. Afin de pouvoir calculer la cohomologie $\mathfrak{S}_n$-équivariante de $\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X \to X^n}(F)$, il faut extraire des informations effectivement utiles à partir du noyau $\mathcal{O}_D$. Cette étape a été achevée en démontrant que le faisceau $\mathcal{O}_D$ admet une résolution $\mathcal{K}^\bullet$ de type Čech en termes des diagonales $\Delta_{i,n+1}$ et de leurs intersections. Ceci permet de construire un complexe simple $\mathcal{C}^\bullet_F$ sur X^n (qui n'est autre que la projection sur X^n de la résolution de Čech $\mathcal{K}^\bullet$ tensorisée par p_X^*F), qui peut être identifié, dans la catégorie dérivée équivariante $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$, avec l'image voulue $\mathbf{\Phi}(F^{[n]})$. On a alors obtenu le premier résultat nouveau de ce travail:

Theorem 0.1. *Soient X une surface quasi-projective lisse et F un faisceau cohérent sur X. L'image du faisceau tautologique $F^{[n]}$ sur le schéma de Hilbert $X^{[n]}$ par l'équivalence de Bridgeland-King-Reid-Haiman $\mathbf{\Phi}$ est isomorphe dans la catégorie dérivée $\mathbf{D}^b_{\mathfrak{S}_n}(X)$ au complexe $\mathcal{C}^\bullet_F$:*

$$\mathbf{\Phi}(F^{[n]}) \simeq \mathcal{C}^\bullet_F \ .$$

Le comportement du complexe $\mathcal{C}^\bullet_F$ sous l'action de $\mathfrak{S}_n$ est très simple, et permet aisément le calcul de la cohomologie $\mathfrak{S}_n$-équivariante de X^n à valeurs dans $\mathcal{C}^\bullet_F$. Elle est isomorphe à la cohomologie du schéma de Hilbert $X^{[n]}$ à valeurs dans $F^{[n]}$. On obtient la généralisation suivante du résultat de Danila:

Theorem 0.2. *Soient X une surface algébrique, F un faisceau cohérent et A un fibré en droites sur X. Soit $\mathcal{D}_A$ le déterminant de Donaldson sur $X^{[n]}$ relatif à A. Alors*

$$H^*(X^{[n]}, F^{[n]} \otimes \mathcal{D}_A) \simeq H^*(X, F \otimes A) \otimes S^{n-2} H^*(X, A)\ .$$

On s'est proposé ensuite d'exploiter notre connaissance de l'image $\mathbf{\Phi}(F^{[n]})$ d'un faisceau tautologique $F^{[n]}$ en termes du complexe $\mathcal{C}^\bullet_F$ afin de comprendre l'image de la puissance tensorielle d'un fibré tautologique $E^{[n]}$ associé à un fibré en droites E sur X et sa cohomologie $\mathfrak{S}_n$-équivariante. La stratégie décrite ci-dessus marche partiellement aussi dans ce cas. La seule différence est qu'ici le noyau de la composition des foncteurs de Fourier-Mukai qui interviennent n'est pas du tout trivial. Ceci conduit à utiliser un résultat profond de Haiman sur polygraphes ([61], [62]). L'image du produit tensoriel de faisceaux tautologiques est alors donnée par:

$$\mathbf{\Phi}({E^{[n]}}^{\otimes^k}) \simeq \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(E^{\boxtimes^k})$$

où $D(n,k)$ est le polygraphe de Haiman dans $X^n \times X^k$. Il généralise pour $k \geq 1$ le schéma D décrit précédemment. Le polygraphe $D(n,k)$ est, en général, un schéma beaucoup plus compliqué que D et son faisceau structural n'admet pas, à notre connaissance, une bonne résolution analogue à celle de D. Il n'est donc pas possible de trouver un complexe simple permettant d'interpréter l'image $\mathbf{\Phi}({E^{[n]}}^{\otimes^k})$. Néanmoins nous avons démontré que le mapping-cone du morphisme naturel:

$$\underbrace{\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E}_{k\text{-times}} \longrightarrow \mathbf{\Phi}({E^{[n]}}^{\otimes^k})$$

est acyclique en degré supérieur à zéro, c'est-à-dire, les images directes supérieures s'annulent: $R^q p_* q^*({E^{[n]}}^{\otimes^k}) = 0$ si $q > 0$, et en degré 0 le morphisme:

$$p_*(q^* E^{[n]}) \otimes \ldots \otimes p_*(q^* E^{[n]}) \longrightarrow p_* q^*(E^{[n]} \otimes \ldots \otimes E^{[n]})$$

est surjectif et son noyau est le sous-faisceau de torsion. Ce résultat nous permet d'identifier l'image $p_* q^*(E^{[n]} \otimes \ldots \otimes E^{[n]})$ au terme $E^{0,0}_\infty$ de la suite spectrale hyperdérivée associée à $\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E$. Le calcul de ce terme est en général techniquement difficile, mais il n'est pas vraiment nécessaire pour comprendre la cohomologie $\mathfrak{S}_n$-équivariante de l'image $\mathbf{\Phi}({E^{[n]}}^{\otimes^k})$. Tout ce dont on a besoin est de connaître les invariants $\mathbf{\Phi}({E^{[n]}}^{\otimes^k})^{\mathfrak{S}_n}$, qui peuvent être identifiés avec l'image directe par le morphisme de Hilbert-Chow $\mu_*({E^{[n]}}^{\otimes^k})$. Vu que l'image directe $\mathfrak{S}_n$-invariante $\pi_*^{\mathfrak{S}_n}$ sur la variété symétrique $S^n X$ est un foncteur exact, ceci équivaut à connaître les invariants $(E^{0,0}_\infty)^{\mathfrak{S}_n}$, ou le terme $\mathcal{E}^{0,0}_\infty$de la suite spectrale des invariants:

$$\mathcal{E}^{p,q}_1 \simeq (E^{p,q}_1)^{\mathfrak{S}_n}\ .$$

Cette nouvelle suite spectrale de faisceaux sur $S^n X$ est beaucoup plus simple que l'originale et permet le calcul explicite de l'image directe de la puissance tensorielle double $E^{[n]} \otimes E^{[n]}$ par le morphisme de Hilbert-Chow μ dans le cas où E est un fibré en droites X. On a démontré la généralisation suivante de la formule de Danila-Brion ([23]):

Theorem 0.3. *Soient X une surface quasi-projective lisse, E un fibré en droites sur X. Alors, l'image directe dérivée $\mathbf{R}\mu_*(E^{[n]} \otimes E^{[n]})$ de la puissance tensorielle double d'un fibré vectoriel tautologique $E^{[n]}$ par le morphisme de Hilbert-Chow μ est quasi-isomorphe au complexe à deux termes:*

$$0 \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^0_E)^{\mathfrak{S}_n} \xrightarrow{d} (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^{\mathfrak{S}_n} \longrightarrow 0\ ,$$

acyclique en degré supérieur à zéro, où le morphisme d est donné par $d = \mathrm{id} \otimes d^0_{\mathcal{C}^\bullet_E}$.

Il est maintenant simple d'en tirer des conclusions sur la cohomologie équivariante de $\mathbf{\Phi}(E^{[n]} \otimes E^{[n]})$, ou, en d'autres mots, de la cohomologie $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$. En décomposant la puissance tensorielle en composantes symétrique et extérieure, on obtient le résultat:

Theorem 0.4. *Soient X une surface quasi-projective lisse, E un fibré en droites sur X. La cohomologie du schéma de Hilbert $X^{[n]}$ à valeurs dans la puissance extérieure $\Lambda^2 E^{[n]}$ d'un fibré tautologique $E^{[n]}$ associé au fibré en droites E sur X, est donnée par l'isomorphisme de modules gradués:*

$$H^*(X^{[n]}, \Lambda^2 E^{[n]}) \simeq \Lambda^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X) \ .$$

La cohomologie du schéma de Hilbert $X^{[n]}$ à valeurs dans la puissance symétrique $S^2 E^{[n]}$ est donnée par l'isomorphisme de modules gradués:

$$H^*(X^{[n]}, S^2 E^{[n]}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus S^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X)$$

où $\mathcal{J}$ est l'idéal dans $S^{n-1} H^*(X, \mathcal{O}_X)$ des classes de cohomologie qui s'annulent dans le schéma $\{a\} \times S^{n-2} X$, avec a un point fixé dans X.

Les deux énoncés du théorème peuvent être réunis dans la formule:

$$H^*(X^{[n]}, E^{[n]^{\otimes^2}}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus H^*(X, E)^{\otimes^2} \otimes S^{n-2} H^*(X, \mathcal{O}_X) \ .$$

Perturbations de la métrique dans les équations de Seiberg-Witten

A la fin des années 1980, Donaldson [28], [29] construit les premiers invariants différentiels pour des variétés différentielles de dimension 4 compactes simplement connexes. Ces types d'invariants permettent de faire la distinction entre variétés qui sont homéomorphes mais pas difféomorphes. Par exemple, il est possible de démontrer que la quintique lisse dans $\mathbb{P}^3_{\mathbb{C}}$ et la variété $9\mathbb{P}^2_{\mathbb{C}} \sharp 44 \overline{\mathbb{P}^2_{\mathbb{C}}}$ sont homéomorphes mais pas difféomorphes. Les invariants de Donaldson sont des invariants polynomiaux:

$$q_d : H_2(M, \mathbb{Z}) \times \cdots \times H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Q}$$

construits à partir de la $SU(2)$-théorie de jauge des instantons, ou connexions anti-autoduales. En d'autres mots, étant donné un fibré vectoriel hermitien E de rang 2, de déterminant trivial, considérons l'espace $\mathcal{S}$ de $SU(2)$-connexions A satisfaisantes à la condition:

$$F_A^+ = 0 \ . \tag{1}$$

L'espace de modules des instantons M_E est le quotient:

$$M_E = \mathcal{S}/\mathcal{G}$$

où $\mathcal{G}$ est le groupe d'automorphismes de E. Il est toujours possible de donner sur M_E une structure d'espace analytique complexe, mais il n'y a pas de raisons pour que M_E soit lisse. Afin d'assurer que M_E est une variété lisse, il faut avoir une action libre et il faut démontrer que le morphisme $A \longrightarrow F_A^+$ est transverse à 0, de sorte que $\mathcal{S}$ est une sous-variété (banachique) lisse de l'espace affine (de Banach) $\mathcal{A}$ des connexions $SU(2)$. Le premier point s'obtient simplement en considérant l'action d'un groupe réduit $\bar{\mathcal{G}} = \mathcal{G}/C(G)$ et en éliminant les connexions réductibles par modification de la métrique. Le deuxième est non triviale et constitue une étape fondamentale dans la construction d'espaces de modules des instantons. Le problème a été résolu par Freed et Uhlenbeck [46] qui ont considéré des perturbations des équations (1) de la forme:

$$F_A^{+,g} = 0 \tag{2}$$

où la métrique g sur la variété M est vue comme un paramètre additionnel. Les deux auteurs ont démontré que l'application perturbée $(A,g) \longrightarrow F_A^{+,g}$ est transverse à 0; par conséquent, l'espace des solutions $\mathcal{S}$ peut être équipé d'une structure de variété lisse (de Banach). Une application standard du théorème de Sard-Smale entraîne alors que, pour une métrique générique $\mathcal{C}^k$ h dans $\mathrm{Met}(M)$, l'espace de modules des instantons M_E^h, relative à la métrique h, est une variété lisse. Ce fait fondamental, combiné avec la preuve (difficile) de l'existence d'une compactification (faite par Donaldson [29], Uhlenbeck [114], [115]), permet la construction des invariants polynomiaux de Donaldson.

En octobre 1994, Seiberg et Witten (voir [104], [105], [119]) construisent un autre type d'invariants différentiels, numériques, basés sur une théorie de jauge de type $U(1)$ beaucoup plus simple, qui peut être interprétée du point de vue de la théorie quantique de champs comme une théorie "duale" de celle de Donaldson. Sur la base de considérations profondes de physique théorique, Witten a prévu que les invariants de Seiberg-Witten seraient capables de saisir la richesse et subtilité des invariants de Donaldson; de plus, il a précisément conjecturé que les polynômes de Donaldson pourraient être exprimés en termes d'invariants de Seiberg-Witten. La conjecture de Witten est en train d'être démontrée, sur la base d'une idée de Pidstrigach et Tyurin, suite au travail long et technique de Okonek, Teleman [99], [108], [109] et, surtout, de Feehan-Leness [38], [39], [40], [42], [41]. Les invariants de Seiberg-Witten sont construits à partir des équations de Seiberg-Witten: une fois fixée une structure $Spin^c$ sur une variété riemannienne compacte orientable (M,g) de dimension 4, de fibré de spineurs $W \simeq W_+ \oplus W_-$ et de fibré en droites hermitien fondamental $L \simeq \det W_+$, les équations sont:

$$D_A\psi = 0 \tag{3a}$$
$$F_A^+ = [\psi^* \otimes \psi]_0 \tag{3b}$$

où A est une connexion unitaire sur L, ψ est un spineur positif $\psi \in \Gamma(W_+)$, et $[\psi^* \otimes \psi]_0$ est la partie de trace nulle dans $i\mathfrak{su}(W_+) \simeq i\Lambda_+^2 T^*M$ de l'opérateur $\psi^* \otimes \psi \in \mathfrak{u}(W_+)$. Le groupe de jauge est ici $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$ et il opère sur les solutions via $(A,\psi) \longmapsto ((g^2)^*A, g\psi)$. Le groupe opère librement sur les solutions des équations (3a), (3b) de la forme (A,ψ), avec $\psi \neq 0$, qui sont dites des monopoles irréductibles. L'espace des modules des monopoles de Seiberg-Witten est le quotient:

$$\mathcal{M}^{SW} = \mathcal{S}/\mathcal{G}$$

où $\mathcal{S}$ est l'espace de solutions des équations de Seiberg-Witten. Afin d'assurer que l'espace de modules est lisse, il faut garantir que l'action $\mathcal{G}$ est libre (ce qui peut être fait comme pour les instantons, en changeant la métrique afin d'éliminer les monopoles réductibles) et que l'espace de solutions $\mathcal{S}$ est une sous-variété (banachique) de l'espace de configurations $\mathcal{A}_L \times \Gamma(W_+)$, obtenue comme image réciproque de 0 par un morphisme transverse à la section nulle. Le deuxième problème est classiquement résolu par une perturbation des équations du type:

$$D_A\psi = 0 \tag{4a}$$
$$F_A^+ = [\psi^* \otimes \psi]_0 + \eta \tag{4b}$$

où η est une 2-forme autoduale imaginaire $\eta \in iA_+^2(M)$. On peut ainsi obtenir la transversalité désirée et la lissité des espaces de modules $\mathcal{M}_\eta^{SW}$ de Seiberg-Witten pour une 2-forme générique $\eta \in iA_+^2(M)$. Bien que cette perturbation soit très simple, elle ne semble pas la plus naturelle, ni la plus géométrique; comme on l'a vu précédemment dans la théorie de Donaldson, la transversalité des équations est obtenue par la perturbation de la seule métrique, procédure qui permet en même temps de se débarasser des connexions réductibles. La perturbation de la métrique dans la théorie de Donaldson a une signification géométrique plus profonde; par contre, la 2-forme η manque de toute interprétation géométrique ou physique. En plus, le comportement des équations de Seiberg-Witten sous des variations de la métrique est intéressant en soi: peu de choses sont connues sur cette question. La seule référence dans la littérature sur les perturbations de la métrique dans les équations de Seiberg-Witten est un article de Eichhorn et Friedrich [31], où les deux auteurs prétendent avoir démontré un résultat de transversalité pour des métriques génériques, mais une lecture attentive de leur démonstration révèle plusieurs fautes qui ne peuvent pas être facilement corrigées.

On s'est proposé alors de clarifier la question. La première difficulté qu'on a rencontrée est la variation de l'opérateur de Dirac correspondant à une variation de la métrique: la question a déjà été étudiée par Bourguignon et Gauduchon ([12], [11]). Les deux auteurs construisent des isomorphismes entre les fibrés de spineurs associés à des métriques différentes, afin de comparer les opérateurs de Dirac qui opèrent dans les espaces de spineurs correspondants. On a décidé d'aborder ce problème d'une autre façon, qu'on va expliquer. Se donner une structure *Spin*^{c} sur une variété riemannienne compacte (M, g) de dimension 4 est équivalent à se donner une représentation spinorielle (W, ρ), c'est-à-dire, les données d'un fibré hermitien W sur M et un morphisme de fibrés:

$$\rho : TM \longrightarrow \mathrm{End}(W)$$

tel que $\rho(x)^* = -\rho(x)$, $\rho(x)^2 = -g(x, x)$, pour tout $x \in TM$ (cf. [81], [38], [40])). On fixe dorénavant un fibré de spineurs W sur la variété riemannienne (M, g). A un changement de la métrique $g_t = \varphi_t^* g$, $\varphi_t \in \mathrm{Aut}(TM)$ on associe la multiplication de Clifford suivant le diagramme:

$$\begin{tikzcd}
(TM, g) \arrow[r, "\rho"] & \mathrm{End}(W) \\
(TM, g_t) \arrow[u, "\varphi_t"] \arrow[ur, "\rho_t"] &
\end{tikzcd}$$

Le couple (W, ρ_t), donné par le même fibré de spineur W, avec la nouvelle multiplication de Clifford ρ_t, devient une nouvelle structure *Spin*^{c} pour la nouvelle variété riemannienne (M, g_t). Il est évident que, de cette façon, il est *inévitable* de changer la multiplication de Clifford quand on change la métrique. On se demande alors ce que signifie perturber *uniquement* la métrique, dès qu'on est obligé de changer la multiplication de Clifford chaque fois qu'on veut changer la métrique. Afin de répondre à cette question, on est induit à étudier les relations entre les multiplications de Clifford (ou structures *Spin*^{c}, une fois que le fibré de spineur est fixé) et les métriques. Si on fixe le fibré de spineurs W et on prend l'ensemble des couples compatibles (g, ρ):

$$\Xi = \{(g, \rho) \mid g \in \mathrm{Met}(M)\,,\ \rho : TM \longrightarrow \mathrm{End}(W)\ , \rho(u)^* = \rho(-u)\,,\ \rho(u)^2 = -g(u, u)\}$$

alors $\Xi \longrightarrow \mathrm{Met}(M)$ est une fibration $\mathcal{C}^\infty(M, PU(W))$ sur l'espace de la métrique, sur laquelle opère $\mathrm{Aut}(TM)$. Dans ce cadre, le concept de la perturbation de la seule métrique correspond, en un sens faible, à choisir des variations de la structure *Spin*^{c} transversales à la distribution verticale: en d'autre mots, cela nécessite la notion d'une connexion sur cette fibration. Il y a maintenant une connexion naturelle, la distribution horizontale en un point (g, ρ) étant donnée par l'espace tangent à l'image de la section $\sigma(g, \rho)$:

$$\begin{aligned} \mathrm{Sym}^+(TM, g) &\xrightarrow{\ \sigma(g,\rho)\ } \Xi \\ \varphi &\longmapsto (\varphi^* g, \rho \circ \varphi) \end{aligned}$$

transverse à la fibre Ξ_g, où $\mathrm{Sym}^+(TM, g)$ désigne les automorphismes symétriques positifs du fibré tangent. Cette connexion clarifie le concept de perturbation de la seule métrique dans un sens plus fort. On définit les équations de Seiberg-Witten et, par conséquent, un espace de modules $\mathcal{M}$ paramétrisés par Ξ, dont la fibre au point (g, ρ) est l'espace de modules de Seiberg-Witten standard $\mathcal{M}^{SW}_{g,\rho}$ associé à la structure *Spin*^{c} donnée par le couple (g, ρ). On démontre que le groupe d'automorphismes unitaires du fibré de spineurs opère sur la fibration Ξ (verticalement) et sur les solutions des équations de Seiberg-Witten; dans le cas où M est simplement connexe cette action est transitive sur les fibres: par conséquent, deux espaces de modules de Seiberg-Witten pour deux multiplications de Clifford différentes sur la même métrique sont isomorphes:

$$\mathcal{M}^{SW}_{(g,\rho)} \simeq \mathcal{M}^{SW}_{(g,\rho')} .$$

On utilise des variations de la structure *Spin*^{c} tangentes à la distribution horizontale naturelle pour calculer la variation des équations de Seiberg-Witten. En particulier, la variation de l'opérateur de Dirac qu'on obtient par ce moyen est la même obtenue par Bourguignon et Gauduchon. On calcule la différentielle $D\mathbb{F}$ de la fonctionnelle de Seiberg-Witten perturbée (en termes de variations de la connexion unitaire A, du spineur φ et de la métrique g) et son adjoint (formel) $D\mathbb{F}^*$, et on étudie les équations du noyau $D\mathbb{F}^* u = 0$. Démontrer que le noyau est nul en un point (A, ψ, g, ρ) implique la transversalité des équations de Seiberg-Witten pour la métrique générique au voisinage de g. Dans le cas général, les équations sont compliquées et on n'a toujours pas de réponse.

Quand M est une surface complexe de Kähler avec fibré en droites canonique K_M, les équations de Seiberg-Witten ont une interprétation en termes de couples holomorphes (∂_A, α), où ∂_A est une semiconnexion holomorphe sur un fibré en droites N tel que $K_M^* \otimes N^{\otimes^2} \simeq L$, et α est une section holomorphe de (N, ∂_A). Ce fait permet une grande simplification des équations de Seiberg-Witten et, par conséquent, de notre question. On interprète tous les objets précédents dans le contexte de la géométrie complexe et on utilise la décomposition des endomorphismes symétriques en hermitiens et anti-hermitiens; les équations du noyau deviennent alors extrêmement plus simples. On obtient que les équations de Seiberg-Witten sont transverses pour une métrique hermitienne générique suffisamment proche de la métrique de Kähler g. On a précisément démontré:

Theorem 0.5. *Soit (M, g, J) une surface de Kähler. Soit N un fibré en droites hermitien sur M tel que $2\deg(N) - \deg(K) < 0$. Considérons la structure Spin^c canonique sur M tordue par le fibré en droites hermitien N. Pour une métrique générique h dans un voisinage ouvert de g dans* $\mathrm{Met}(M)$ *l'espace de modules de Seiberg-Witten $\mathcal{M}^{SW}_h$ est lisse. En effet, l'énoncé est vrai pour une métrique hermitienne générique h dans un voisinage ouvert de g.*

On trouve un contre-exemple qui montre qu'il faut obligatoirement sortir de la classe des métriques de Kähler afin d'obtenir la transversalité.

Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles

Introduction

This part of our work deals with the cohomology of the Hilbert scheme of points over a smooth algebraic projective surface X with values in some representations of a tautological vector bundle $E^{[n]}$, associated to a line bundle E on X. In particular, we will treat in detail the case of the cohomology $H^*(X^{[n]}, S^2E^{[n]})$ of the symmetric power $S^2E^{[n]}$, and the cohomology $H^*(X^{[n]}, \Lambda^2E^{[n]})$ of the exterior power $\Lambda^2E^{[n]}$.

Motivations. The motivations of this work lie in *strange duality* phenomenons and conjecture, which we will now explain. Strange duality is a duality relation between spaces of sections of determinant line bundles on different moduli spaces of semistable sheaves on a smooth algebraic variety. The first examples of strange duality phenomenons were discovered on curves by Beauville [3] and Beauville-Narasimhan-Ramanan [6]. Let X be a smooth projective curve of genus $g \geq 2$, and $\mathcal{SU}(n)$ the moduli space of semistable vector bundles of rank n over X with degree 0. Let $\mathcal{D}$ be the ample generator of $\mathrm{Pic}(\mathcal{SU}(n))$, which is free abelian of rank 1, by Drezet-Narasimhan theorem [30]: $\mathcal{D}$ is called the determinant line bundle over $\mathcal{SU}(n)$. If Θ denotes the theta divisor on the Jacobian $J^{g-1}(X)$, we have a strange duality:

$$H^0(J^{g-1}(X), \mathcal{O}(n\Theta))^* \simeq H^0(\mathcal{SU}(n), \mathcal{D})$$

between the space of sections of the determinant line bundle $\mathcal{D}$ on $\mathcal{SU}(n)$ and the space of theta functions of level n on $J^{g-1}(X)$. The computation of the dimension of $H^0(\mathcal{SU}(n), \mathcal{D})$ and, more generally, of $H^0(\mathcal{SU}(n), \mathcal{D}^{\otimes^k})$ has been performed by Beauville and Laszlo [5], by means of Verlinde formula, conjectured by Verlinde [116], and proved by several authors, among others Tsuchiya-Ueno-Yamada [113], Beauville [4], Faltings [37], Thaddeus [110], Jeffrey-Kirwan [71].

On a smooth simply connected algebraic projective surface X Le Potier proposed the following strange duality conjecture. Suppose X is simply connected. Then the Grothendieck algebra $K_{\mathrm{top}}(X)$ of topological vector bundles on X is isomorphic, as an abelian group, to $\mathbb{Z} \times H^2(X, \mathbb{Z}) \times \mathbb{Z}$. In other words, a class in $K_{\mathrm{top}}(X)$ is identified by its rank, its first Chern classe c_1 and its Euler-Poincaré characteristic χ. The Euler-Poincaré characteristic defines an integral quadratic form on $K_{\mathrm{top}}(X)$, setting:

$$\langle u, v \rangle = \chi(u \cdot v) \qquad \text{if } u, v \in K_{\mathrm{top}}(X) \ .$$

Take now two classes $u, v \in K_{\mathrm{top}}(X)$, orthogonal for $\langle \cdot, \cdot \rangle$ and form the moduli spaces of semistable sheaves M_u, M_v with fixed Grothendieck classes u and v, respectively. Let $\mathcal{D}_{u,v}$ and $\mathcal{D}_{v,u}$ be the two determinant line bundles (see [86], [68]) on M_u and M_v associated to the classes v and u respectively. Under some technical hypothesis Le Potier finds a canonical section $\sigma_{v,u} \in H^0(M_u \times M_v, \mathcal{D}_{u,v} \boxtimes \mathcal{D}_{v,u})$ which allows to define the strange duality morphism:

$$D_{v,u} : H^0(M_u, \mathcal{D}_{u,v})^* \longrightarrow H^0(M_v, \mathcal{D}_{v,u}) \ .$$

Le Potier conjectured under some hypothesis that $D_{v,u}$ is an isomorphism, if M_v is not empty.

Danila [21] addressed Le Potier strange duality conjecture on $\mathbb{P}_2$. In particular, she solved affirmatively the conjecture for Grothendieck classes $c = (2, 0, n)$, $u = (0, 1, 0)$, $n \leq 19$. Let $M_n := M_{(2,0,n)}$. The strategy followed is to use moduli spaces of coherent systems (see [66], [84], [85]) S_α to relate — for $l(l-1) \leq n \leq (l+1)(l+2)$ — the space of sections $H^0(M_n, \mathcal{D}_{n,u})$ to the space of sections $H^0(U, S^l\mathcal{R} \otimes \mathcal{D}_u)$ of a coherent sheaf $\mathcal{R}$ on an open set U of the Hilbert scheme $X^{[n+l^2]}$, where $\mathcal{R}$ is locally free (see [22]). Danila resolves $\mathcal{R}$ with a locally free resolution $K^\bullet \longrightarrow \mathcal{R}$, whose terms K^i depend on the symmeric power $S^r(\mathcal{O}(k)^{[n+l^2]})$ of the tautological bundle $\mathcal{O}(k)^{[n+l^2]}$ associated to the line bundle $\mathcal{O}(k)$ on $\mathbb{P}_2$. To compute the hypercohomology spectral sequence, it is then necessary to handle and compute cohomology groups of the kind:

$$H^q(X^{[m]}, S^k(\mathcal{O}(k)^{[m]}) \otimes \mathcal{D}_u) \ ,$$

$\mathcal{D}_u$ being the Donaldson determinant on $X^{[m]}$ associated to the class u. It is precisely the technical difficulties in these computations that limited Danila's results on the conjecture to $n \leq 19$. The comprehensive knowledge of these cohomology groups would lead to a complete proof of the conjecture for the projective plane, at least for $c = (2, 0, n)$, $u = (0, 1, 0)$. Danila proved in [23] and [24] general formulas for the cohomology of tautological bundles on the Hilbert scheme $H^*(X^{[n]}, L^{[n]})$ associated to a line bundle on X, and of the double symmetric power $H^*(X^{[n]}, S^2 L^{[n]})$, for $n \leq 3$.

In our work we generalize her results for $S^2 L^{[n]}$ for all n, and we give also general formulas for the cohomology of the double exterior power $H^*(X^{[n]}, \Lambda^2 L^{[n]})$ for all n. It turns out that the latter groups are involved as well in the verification of strange duality conjecture on the projective plane for M_2 and $M_{(1,d,m)}$. (cf. [83])

The method: McKay correspondence. The method we use is quite different from Danila's and was provided by recent developments in McKay correspondence. The beginning of McKay correspondence dates back long before McKay and actually started with Klein around 1870 and with Coxeter and Du Val around 1930. When we quotient $\mathbb{C}^2$ by a finite subgroup G of $SL(2, \mathbb{C})$, and we take a minimal resolution Y of $\mathbb{C}^2/G$, then Y is crepant and the exceptional locus consists of a bunch of curves, whose dual graph is a Dynkin diagram of the kind A_n, D_n, E_6, E_7, E_8.

McKay (see [89], [90], [44]) made the observation that the Dynkin diagrams arising from resolutions of kleinian singularities are in connection with the representations of G. If ρ is a representation of G, and ρ_i are the irreducible representations, the *McKay graph* is the Dynkin diagram associated to the matrix $a_{ij} - 2\mathrm{id}$, where a_{ij} is defined as

$$\rho \otimes \rho_i = \sum_j a_{ij} \rho_j \ ;$$

it turns out that the McKay graph just defined is exactly the Dynkin diagram given by the (extended) dual graph of the exceptional locus when G acts on $\mathbb{C}^2$ via the representation ρ. Moreover, McKay suggested that there is a one-to-one correspondence between the components of the exceptional locus and the irreducible representations of G. In [53], Gonzalez-Springer and Verdier proved a K-theoretic version of this statement, showing an isomorphism:

$$K_G(\mathbb{C}^2) \xrightarrow{\simeq} K(Y)$$

between the K-theory of Y and the G-equivariant K-theory of $\mathbb{C}^2$, making precise the correspondence between irreducible representations of G and irreducible components of the exceptional locus.

The generalization of Gonzalez-Springer-Verdier theorem we are interested in, came out in 2001, with the following Bridgeland-King-Reid result [16]. Let M be a smooth quasi-projective variety over $\mathbb{C}$ and G a finite group of automorphisms of M such that ω_M is locally trivial as G-sheaf. Let $Y = \mathrm{Hilb}^G(M)$ the G-Hilbert scheme (according to Nakamura) of G-orbits. Then, under some smallness hypothesis on $Y \times_{M/G} Y$, we have an equivalence of derived categories:

$$\Phi : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M)$$

between the derived category of coherent sheaves on Y and the derived category of G-equivariant sheaves on M, where Φ is the Fourier-Mukai transform with kernel the universal family $\mathcal{Z} \subset Y \times M$.

The next step was made by Haiman ([61], [62], [60]), who proved that the action of the symmetric group $\mathfrak{S}_n$ on the product X^n of a surface X satisfies the hypothesis of BKR-theorem, and that $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ can be identified with the Hilbert scheme $X^{[n]}$. As a consequence, we have an equivalence:

$$\mathbf{\Phi} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

obtained by the Fourier-Mukai transform of kernel B^n, the isospectral Hilbert scheme. Consequently, cohomology computations on the Hilbert scheme can be obtained as simpler $\mathfrak{S}_n$-equivariant (hyper)cohomology computations on the product X^n.

Results. Our first concern was finding the image of a tautological sheaf $F^{[n]}$ associated to a coherent sheaf F on the surface X for the BKRH equivalence $\mathbf{\Phi}$. We recall that the tautological sheaf $F^{[n]}$ is the sheaf defined by means of the Fourier-Mukai functor:

$$F^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) = \mathbf{R}p_{X^{[n]}*}(\mathcal{O}_\Xi \otimes^L_{\mathcal{O}_{X^{[n]}\times X}} p_X^* F)$$

where Ξ is the universal family on the Hilbert scheme. In the commutative diagram

$$\begin{tikzcd} B^n \arrow[r, "p"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^n X \end{tikzcd}$$

B^n is the isospectral Hilbert scheme, S^nX is the symmetric variety, μ is the Hilbert-Chow morphism. The Bridgeland-King-Reid-Haiman equivalence

$$\mathbf{\Phi} := \mathbf{R}p_* \circ q^*$$

computed on $F^{[n]}$ is then simply the composition of Fourier-Mukai functors:

$$\mathbf{\Phi}(F^{[n]}) = \mathbf{\Phi} \circ \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) = \Phi^{\mathbf{R}f_*\mathcal{O}_Z}_{X \to X^n}(F) \ ,$$

which we know being a third Fourier-Mukai functor of kernel $\mathbf{R}f_*\mathcal{O}_Z$, where Z is the universal family on the isospectral Hilbert scheme, and f is the morphism: $f \colon B^n \times X \longrightarrow X^n \times X$. We proved that the derived direct image $\mathbf{R}f_*\mathcal{O}_Z$ is quasi isomorphic to the structural sheaf $\mathcal{O}_D$ of the scheme-theoretic union $D = \cup_{i=1}^n \Delta_{i,n+1}$ of diagonals $\Delta_{i,n+1}$ in $X^n \times X$. In order to be able to compute the $\mathfrak{S}_n$-equivariant cohomology of $\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X \to X^n}(F)$, we needed to extract some effectively useful information from the kernel $\mathcal{O}_D$. This task was achieved by showing that the sheaf $\mathcal{O}_D$ affords a Čech-type resolution $\mathcal{K}^\bullet$ in terms of the diagonals $\Delta_{i,n+1}$ and their intersections. As a consequence, we succeeded in defining a simple complex $\mathcal{C}^\bullet_F$ on X^n (which is nothing but the projection onto X^n of the Čech resolution $\mathcal{K}^\bullet$ twisted by p_X^*F), which could be identified, in the equivariant derived category $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$, with the searched image $\mathbf{\Phi}(F^{[n]})$. We got the first new result of this work:

Theorem 0.6. *Let X a smooth quasi-projective surface and F a coherent sheaf on X. The image of the tautological sheaf $F^{[n]}$ on the Hilbert scheme $X^{[n]}$ for the Bridgeland-King-Reid-Haiman equivalence $\mathbf{\Phi}$ is isomorphic in $\mathbf{D}^b_{\mathfrak{S}_n}(X)$ to the complex $\mathcal{C}^\bullet_F$:*

$$\mathbf{\Phi}(F^{[n]}) \simeq \mathcal{C}^\bullet_F \ .$$

The behaviour of the complex $\mathcal{C}^\bullet_F$ under the action of $\mathfrak{S}_n$ is very simple, and allows, without any effort, to compute the $\mathfrak{S}_n$-equivariant hypercohomology of $\mathcal{C}^\bullet_F$ on X^n, which is isomorphic to the cohomology of $F^{[n]}$ on the Hilbert scheme. We got the following generalization of a Danila-Brion result (cf. [23]):

Theorem 0.7. *Let X be a smooth algebraic surface, F a coherent sheaf and A a line bundle on X. Let $\mathcal{D}_A$ the Donaldson determinant on $X^{[n]}$ relative to A. Then*

$$H^*(X^{[n]}, F^{[n]} \otimes \mathcal{D}_A) \simeq H^*(X, F \otimes A) \otimes S^{n-2} H^*(X, A) \ .$$

The second task we proposed ourselves was to exploit our knowledge of the image $\mathbf{\Phi}(F^{[n]})$ of a tautological sheaf $F^{[n]}$ in terms of the complex $\mathcal{C}^\bullet_F$ to understand the image of a tensor power of a tautological vector bundle $E^{[n]}$ associated to a line bundle E on X and its $\mathfrak{S}_n$-equivariant hypercohomology. The strategy decribed above partially works in this case as well. The only difference here is that the kernel of the resulting composition of Fourier-Mukai functors is quite nontrivial and we have to make use of a deep result by Haiman on polygraphs ([61], [62]). The image of the tensor product of tautological sheaves is then given by:

$$\mathbf{\Phi}(E^{[n]^{\otimes^k}}) \simeq \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(E^{\boxtimes^k})$$

where $D(n,k)$ is Haiman's polygraph in $X^n \times X^k$ and generalizes for $k \geq 1$ the scheme D described above. The polygraph $D(n,k)$ is, in general, a far more complicated scheme than D and its structural sheaf does not admit, to our knowledge, any nice resolution in the way D has. Consequently, we could not find any "simple" complex quasi-isomorphic to the image $\mathbf{\Phi}(E^{[n]^{\otimes^k}})$. Nonetheless, we could prove that the mapping cone of the natural morphism:

$$\underbrace{\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E}_{k\text{-times}} \longrightarrow \mathbf{\Phi}(E^{[n]^{\otimes^k}})$$

is acyclic in degree higher than zero, that is, the higher direct images vanish: $R^j p_* q^*(E^{[n]^{\otimes^k}}) = 0$ if $j > 0$ and in degree 0 the morphism:

$$\underbrace{p_*(q^* E^{[n]}) \otimes \ldots \otimes p_*(q^* E^{[n]})}_{k\text{-times}} \longrightarrow p_* q^*(\underbrace{E^{[n]} \otimes \ldots \otimes E^{[n]}}_{k\text{-times}})$$

is surjective and its kernel is the torsion subsheaf. This result allows us to identify the image $p_* q^*(E^{[n]} \otimes \ldots \otimes E^{[n]})$ with the term $E^{0,0}_\infty$ of the hyperderived spectral sequence associated to $\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E$. The computation of this term can be technically difficult in general, but it is not really necessary to understand the $\mathfrak{S}_n$-equivariant hypercohomology of the image $\mathbf{\Phi}(E^{[n]^{\otimes^k}})$. All what is needed is the knowledge of the invariants $\mathbf{\Phi}(E^{[n]^{\otimes^k}})^{\mathfrak{S}_n}$, which can be identified with the image of the Hilbert-Chow morphism: $\mu_*(E^{[n]^{\otimes^k}})$. Since the $\mathfrak{S}_n$-invariant push-forward $\pi^{\mathfrak{S}_n}_*$ on the symmetric variety is an exact functor, this amounts to knowing the invariants $(E^{0,0}_\infty)^{\mathfrak{S}_n}$, or the $\mathcal{E}^{0,0}_\infty$ term of the spectral sequence of invariants:

$$\mathcal{E}^{p,q}_1 \simeq (E^{p,q}_1)^{\mathfrak{S}_n} \ .$$

It turns out that this new spectral sequence of sheaves on $S^n X$ is much simpler than the original one and it allows to explicitely compute the direct image of the double tensor power $E^{[n]} \otimes E^{[n]}$ for the Hilbert-Chow morphism μ in the case E is a line bundle on X. We proved the following generalization of Danila-Brion formula ([23]):

Theorem 0.8. *Let X a smooth quasi-projective surface, E a line bundle on X. Then the derived direct image $\mathbf{R}\mu_*(E^{[n]} \otimes E^{[n]})$ of the double tensor power of a tautological vector bundle $E^{[n]}$ for the Hilbert-Chow morphism μ is quasi-isomorphic to the two-terms complex:*

$$0 \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^0_E)^{\mathfrak{S}_n} \xrightarrow{d} (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^{\mathfrak{S}_n} \longrightarrow 0 \ ,$$

acyclic in degree higher than zero, where the morphism d is given by $d = \mathrm{id} \otimes d^0_{\mathcal{C}_E^\bullet}$.

It is simple now to draw consequences about equivariant cohomology of $\mathbf{\Phi}(E^{[n]} \otimes E^{[n]})$, or, in other words, of the cohomology $H^*(X^{[n]}, E^{[n]} \otimes E^{[n]})$. Splitting the tensor power into symmetric and exterior components, we get the following :

Theorem 0.9. *Let X be a smooth quasi-projective surface, E a line bundle on X. Then the cohomology of the exterior power $\Lambda^2 E^{[n]}$ of a tautological vector bundle $E^{[n]}$ on the Hilbert scheme $X^{[n]}$ associated to the line bundle E on X, is given by the isomorphism of graded modules:*

$$H^*(X^{[n]}, \Lambda^2 E^{[n]}) \simeq \Lambda^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X) \ .$$

The cohomology of the symmetric power $S^2E^{[n]}$ is given by the following isomorphism of graded modules:

$$H^*(X^{[n]}, S^2E^{[n]}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus S^2H^*(X,E) \otimes S^{n-2}H^*(X, \mathcal{O}_X)$$

where $\mathcal{J}$ is the ideal of the classes in $S^{n-1}H^*(X, \mathcal{O}_X)$ vanishing on the scheme $\{a\} \times S^{n-2}X$, with a a fixed point in X.

The two statements of the theorem can be gathered in the formula:

$$H^*(X^{[n]}, E^{[n]^{\otimes^2}}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus H^*(X,E)^{\otimes^2} \otimes S^{n-2}H^*(X, \mathcal{O}_X) \ .$$

1 Preliminaries and notations

In this part we are primarily concerned with schemes and varieties over $k = \mathbb{C}$.

1.1 Hilbert schemes of points on a surface

1.1.1 Hilbert schemes of points

Let X a quasi-projective variety over the field k. Consider the functor:

$$\underline{\mathrm{Hilb}}^n_X : \mathrm{Sch}_k \longrightarrow \mathrm{Sets}$$

from the category of noetherian schemes over k to the category of sets, defined by:

$$\underline{\mathrm{Hilb}}^n_X(T) = \{Z \subseteq X \times T \mid Z \text{ closed subscheme, } Z \text{ flat and finite over } T \text{ of relative degree } n\ \} \ .$$

Grothendieck proved the following fundamental theorem:

Theorem 1.1. *The functor $\underline{\mathrm{Hilb}}^n_X$ is representable by a quasi-projective variety $X^{[n]}$. If X is projective, then $X^{[n]}$ is projective. $X^{[n]}$ is called the Hilbert scheme of n points on the variety X.*

Actually Grothendieck proved in [55] a much more general version of the previous statement. Since $X^{[n]}$ represents the functor $\underline{\mathrm{Hilb}}^n_X$ we have, for any noetherian k-scheme T:

$$\mathrm{Mor}_{\mathrm{Sch}_k}(T, X^{[n]}) \simeq \underline{\mathrm{Hilb}}^n_X(T) \ .$$

Setting $T = X^{[n]}$ and taking the identity in the last bijection we get a universal family Ξ of subschemes $\Xi \subset X \times X^{[n]}$, flat and finite over $X^{[n]}$ of relative degree n such that any other family of subschemes of X of length n parametrized by a scheme T is the pull-back of Ξ by a unique morphism: $T \longrightarrow X$.

There is another construction, close to the Hilbert scheme of points on a variety, which parametrizes points on X as well: the symmetric variety S^nX.

Definition 1.2. Let X a quasi-projective variety. The symmetric variety S^nX, for $n \in \mathbb{N}$, $n \geq 1$, is the quotient:

$$X^n/\mathfrak{S}_n$$

of the n-product of the variety X by the symmetric group $\mathfrak{S}_n$.

The symmetric variety S^nX actually parametrizes 0-dimensional effective cycles on X of degree n. A point on S^nX can always be written as the formal sum $\sum_i n_i x_i$, where $x_i \in X$, $n_i \in \mathbb{N}$, $\sum_i n_i = n$. The general relation between the symmetric variety and the Hilbert scheme of points is given by the Hilbert-Chow morphism (see [95], [94], [59], [56]):

Theorem 1.3. *There exists a morphism:*

$$X^{[n]}_{red} \xrightarrow{\mu} S^n X$$

defined by

$$\mu(\xi) = \sum_{x \in X} \mathrm{length}(\xi_x)\, x \ .$$

1.1.2 Hilbert scheme of points on a surface

In general even if the variety X is nonsingular, the Hilbert scheme $X^{[n]}$ can be very singular if $n \geq 3$. Let X a nonsingular variety. The symmetric variety $S^n X$ is normal and has only rational singularities, because a quotient of a smooth variety by a finite group (see [17], [13]). Since it is Gorenstein, this is equivalent of having canonical singularities ([80].) The following important result gives the non-singularity of the Hilbert scheme for a quasi-projective nonsingular surface (see [43], [95]).

Theorem 1.4. *Let X a quasi-projective nonsingular surface. Then*

1. *The Hilbert scheme $X^{[n]}$ of n points on X is nonsingular.*
2. *The Hilbert-Chow morphism:*
$$\mu : X^{[n]} \longrightarrow S^n X$$
is a resolution of singularities.

The fact that $S^n X$ has rational singularities implies that the higher direct images of structural sheaf of the Hilbert-Chow morphism μ vanish; furthermore, since $\mu_* \mathcal{O}_{X^{[n]}} \simeq \mathcal{O}_{S^n X}$, because μ is birational and $S^n X$ is normal, we have:

$$\mathbf{R}\mu_* \mathcal{O}_{X^{[n]}} = \mathcal{O}_{S^n X} \ .$$

We are now interested to smallness properties of the Hilbert-Chow morphism. The dimension of the fibers of μ is given by the following proposition:

Proposition 1.5. *The fiber of the Hilbert-Chow morphism over a point $\sum_{x \in X} n_x x \in S^n X$ is irreducible of dimension $\sum_{x \in X} (n_x - 1)$.*

This proposition is a consequence of results by Hartshorne [63] and Fogarty [43] on the dimension of $\mathrm{Hilb}^n(\mathbb{C}\{x, y\})$, by Briançon [15] on the irreducibility of $\mathrm{Hilb}^n(\mathbb{C}\{x, y\})$. These results were proved and generalized in a different and more geometric way by Ellingsrud and Stromme [35] and later by Ellingsrud and Lehn [34].

Definition 1.6. Let $X \xrightarrow{f} Y$ a proper surjective map of algebraic varieties. Let $Y^d_f := \{y \in Y \mid \dim f^{-1}(y) = d\}$. Then f is semismall if $\mathrm{codim}_Y\, Y^d_f \geq 2d$, for all $d > 0$, and small if $\mathrm{codim}_Y\, Y^d_f > 2d$, for all $d > 0$.

One sees immediately that a proper surjective semismall map is generically finite and hence $\dim X = \dim Y$.

Remark 1.7. Let X a smooth quasi-projective surface. Let $\nu = \nu_1 \geq \cdots \geq \nu_k$ a partition of n. We define the stratum $S^n_\nu X$ of $S^n X$ as:

$$S^n_\nu X := \left\{ \sum_{i=1}^{k} \nu_i x_i \mid x_i \neq x_j \text{ for } i \neq j \right\} \ .$$

Then $\mathrm{codim}_{S^n X}\, S^n_\nu X = 2(n-k)$ and $S^n X$ stratifies into: $S^n X = \bigcup_\nu S^n_\nu X$. Since $\dim \mu^{-1}(\sum_{i=1}^k \nu_i x_i) = \sum_{i=1}^k \nu_i - 1 = n - k$ we get that:

$$(S^n X)^m_\mu = \bigcup_{\substack{\nu \\ \mathrm{length}(\nu) = k}} S^n_\nu X \ .$$

An immediate consequence of the previous remark is that:

Corollary 1.8. *The Hilbert-Chow morphism* $\mu : X^{[n]} \longrightarrow S^n X$ *is semismall.*

1.1.3 Nested Hilbert schemes.

We will now describe briefly incidence varieties, or nested Hilbert schemes. See [18], [19], [111], [112], [87], [36], [23].

Proposition 1.9 (Cheah-Tikhomirov). *Let X a quasi-projective surface. Let n, $m \in \mathbb{N} \setminus \{0\}$, $n > m$. Let F the functor:*

$$F : \ Sch/k \longrightarrow \ Sets$$

associating to a scheme S the set of couples (ζ, ξ) of subschemes $\zeta \subseteq S \times X$, $\xi \subseteq S \times X$, both flat over S of relative length n and m respectively, and such that $\xi \subseteq \zeta$. The functor F is representable by a scheme $X^{[n,m]}$, called the nested Hilbert scheme or incidence scheme.

$X^{[n,m]}$ is naturally a closed subscheme of $X^{[n]} \times X^{[m]}$ and hence equipped with two projections:

$$X^{[n]} \xleftarrow{p_1} X^{[n,m]} \xrightarrow{p_2} X^{[m]} .$$

The nested Hilbert scheme $X^{[n,m]}$ parametrizes two flat families $\zeta_n \subseteq X^{[n,m]} \times X$, $\xi_m \subseteq X^{[n,m]} \times X$ of length n and m, respectively, defined as:

$$\zeta_n := (p_1 \times \mathrm{id})^{-1}(\Xi_n) \quad ; \quad \xi_m := (p_2 \times \mathrm{id})^{-1}(\Xi_m) .$$

with $\xi_m \subseteq \zeta_n$. Their structural sheaves fit in the exact sequence:

$$0 \longrightarrow \mathcal{I}_{n,m} \longrightarrow \mathcal{O}_{\zeta_n} \longrightarrow \mathcal{O}_{\xi_m} \longrightarrow 0 . \tag{5}$$

The sheaf $\mathcal{I}_{n,m}$ is a coherent sheaf on $X^{[n,m]} \times X$, flat over $X^{[n,m]}$, fiberwise zero dimensional of relative length $n - m$. Therefore it induces a morphism into the symmetric variety:

$$\rho : X^{[n,m]} \longrightarrow S^{n-m} X , \tag{6}$$

defined as $\rho(\eta) = \sum_{x \in X} \mathrm{length}((\mathcal{I}_{n,m})_{\eta,x}) x$.

The interesting case for us is when $n = m + 1$. Cheah [19] and Tikhomirov [111], [112] prove that this is the only case where the incidence scheme is smooth.

Theorem 1.10. *The incidence scheme $X^{[n+1,n]}$ is smooth irreducible variety.*

Moreover, in this case:

Proposition 1.11. *Let $\Xi_n \subseteq X^{[n]} \times X$ be the universal family over $X^{[n]}$ and $\mathcal{I}_{\Xi_n}$ its ideal sheaf. Then the incidence variety $X^{[n+1.n]}$ is isomorphic to the projectivization $\mathbb{P}(\mathcal{I}_{\Xi_n})$ and to the blow up $\mathrm{Bl}_{\Xi_n}(X^{[n]} \times X)$ of the product $X^{[n]} \times X$ along the universal family Ξ_n:*

$$\mathrm{Bl}_{\Xi_n}(X^{[n]} \times X) \simeq \mathbb{P}(\mathcal{I}_{\Xi_n}) \simeq X^{[n+1,n]} .$$

These are isomorphisms of schemes over $X^{[n]} \times X$.

In this case the ideal sheaf $\mathcal{I}_{n+1,n}$ on $X^{[n+1,n]}$ gives rise to a third flat family over $X^{[n+1,n]}$ via the morphism (6), of relative length 1, which will be called η. It turns out that in this case the ideal sheaf $\mathcal{I}_{n+1,n} \simeq \mathcal{O}_\eta(-E)$, (cf. [23]) where E is the exceptional divisor over $\mathrm{Bl}_{\Xi_n}(X^{[n]} \times X)$, viewed on $\eta \subseteq X^{[n+1,n]} \times X$ via the identification given by the first projection.

The exact sequence (5) becomes (in the identification $X^{[n+1,n]} \simeq \mathrm{Bl}_{\Xi_n}(X^{[n]} \times X)$):

$$0 \longrightarrow \mathcal{O}_\eta(-E) \longrightarrow \mathcal{O}_{\zeta_{n+1}} \longrightarrow \mathcal{O}_{\xi_n} \longrightarrow 0 \tag{7}$$

where E denote the exceptional divisor. Danila [23] proves that we have another exact sequence:

$$0 \longrightarrow \mathcal{O}_{\zeta_{n+1}} \longrightarrow \mathcal{O}_{\xi_n} \oplus \mathcal{O}_\eta \longrightarrow \mathcal{O}_\eta\Big|_E \longrightarrow 0 \tag{8}$$

Finally if $g : X^{[n+1,n]} \longrightarrow X^{[n]} \times X$ is the blow-up projection, we have (cf. [23])

$$\mathbf{R}g_*\mathcal{O}_{X^{[n+1,n]}} \simeq \mathcal{O}_{X^{[n]}\times X} \quad ; \quad \mathbf{R}g_*\mathcal{O}_E \simeq \mathcal{O}_\Xi \ . \tag{9}$$

1.2 Equivariant derived categories and Fourier-Mukai functors

1.2.1 Equivariant sheaves

In this section we will explain briefly some basic facts about equivariant sheaves. We will follow [16], [7]. Let X a variety and G a finite group acting (on the left) on X. Let $\mathrm{Coh}(X)$ the category of coherent sheaves on X.

Definition 1.12. A G-linearization of a coherent sheaf F is a collection of isomorphisms: $\lambda_g^F : F \longrightarrow g^*F$ for all $g \in G$ such that $g^*(\lambda_h^F) \circ \lambda_g = \lambda_{hg}$, and $\lambda_1 = \mathrm{id}_F$, for all $g, h \in G$. We also say that F is a coherent G-sheaf.

Let now E, F two coherent G-sheaves with given G-linearizations $\{\lambda_g^E\}_{g\in G}$, $\{\lambda_g^F\}_{g\in G}$. The group G acts (on the right) on the vector space $\mathrm{Hom}_X(E, F)$ in the following way: for $\theta \in \mathrm{Hom}_X(E, F)$

$$\theta g := \lambda_{g^{-1}}^F \circ g^*\theta \circ \lambda_g^E \ .$$

The category of G-equivariant sheaves $\mathrm{Coh}_G(X)$ is the category whose objects are G-linearized sheaves and whose morphisms (between two objects) $\mathrm{GHom}_X((E, \{\lambda_g^E\}_g), (F, \{\lambda_g^F\}_g))$ are the G-invariant morphisms $\mathrm{Hom}_X(E, F)^G$. In the same way we can define the category $\mathrm{QCoh}_G(X)$ of G-equivariant quasi-coherent sheaves, or more generally G-equivariant sheaves $\mathrm{Sh}_G(X)$ on X.

For two G-sheaves E and F on a variety X, the representation $\mathrm{Hom}_X(E, F)$ decomposes into irreducible representations ρ_i, $i = 0, \dots m$, where we indicated with ρ_0 the trivial representation:

$$\mathrm{Hom}_X(E, F) \simeq \oplus_{i=0}^m \mathrm{GHom}_X(E \otimes \rho_i, F) \otimes \rho_i \ .$$

If G acts trivially on X every G-sheaf decomposes as a direct sum over the irreducible representations:

$$E \simeq \oplus_{i=0}^m E_i \otimes \rho_i \ , \tag{10}$$

where E_i are simply coherent sheaves on X. In particular, we can define the functor of fixed points

$$[-]^G : \mathrm{Coh}_G(X) \longrightarrow \mathrm{Coh}(X)$$

which associates to a G-sheaf E the sheaf E_0, appearing in the direct sum above. Since the group is finite and we are taking fixed points of $\mathcal{O}_X$-modules, the functor $[-]^G$ is exact. If λ_g^F is the linearization of a coherent sheaf F on X, directly from the decomposition into irreducible representations, and from the definition of the functor $[-]^G$, we get:

$$\left[\lambda_g^F\right]^G = \mathrm{id}_{[F]^G} \ . \tag{11}$$

Indeed it comes directly from (10) the restriction of λ_g^F to $[F]^G$ is the identity.

Now let X, Y two varieties acted upon by finite groups G and H respectively. Suppose there exist a morphism of groups $\varphi : G \longrightarrow H$. If $f : X \longrightarrow Y$ is an equivariant morphism, then f^* commutes with the actions and then it defines a functor $f^* : \mathrm{Coh}_H(Y) \longrightarrow \mathrm{Coh}_G(X)$. On the other hand, we need the surjectivity of the morphism $\varphi : G \longrightarrow H$ to define a good equivariant push-forward. Let K the kernel of the epimorphism $\varphi : G \twoheadrightarrow H$. If F is a G-sheaf on X, then f_*F is naturally G-linearized via φ, where G acts trivially on Y. Then (see [7]) we can define a good H-linearization of the fixed points sheaf $(f_*F)^K$. Consequently, the functor

$$f_*^K := [-]^K \circ f_* : \mathrm{Coh}_G(X) \longrightarrow \mathrm{Coh}_H(Y)$$

is a well defined equivariant push forward. It turns out that f^* is the left adjoint of f_*^K. An important case occurs when Y is the quotient $Y = X/G$, $H = \{1\}$ and $f : X \longrightarrow X/G$ is the quotient map. Then $K = G$ and the functor f_*^G is the exactly the composition

$$f_*^G : [-]^G \circ f_* \; .$$

Since the morphism f is finite and G is a finite group, f_*^G is an exact functor. Si $F \in \mathrm{Coh}_G(X)$, we will denote also $F^G := f_*^G F$.

1.2.2 Equivariant derived categories

Definition 1.13. Let X a variety, G a finite group acting on X. The equivariant derived category $\mathbf{D}_G(X)$ is defined as the derived category of the abelian category $\mathrm{Coh}_G(X)$ of G-equivariant coherent sheaves on X:

$$\mathbf{D}_G(X) := \mathbf{D}(\mathrm{Coh}_G(X)) \; .$$

Remark 1.14. It is well known that the category of coherent sheaves on an algebraic variety does not have enough injectives in general. One can pass by this difficulty and define and compute derived functors by seeing the equivariant derived category as the full subcategory of the derived category of $\mathrm{QCoh}_G(X)$ consisting of complexes with coherent cohomology.

Remark 1.15. If X is a smooth variety, we will work with the bounded equivariant derived category $\mathbf{D}_G^b(X) := \mathbf{D}^b(\mathrm{Coh}_G(X))$, since all geometric derived functors take their values there, (thanks to the syzygy theorem). Again we can see the bounded equivariant derived category as the full subcategory of the (unbounded) derived category $\mathbf{D}^b(\mathrm{QCoh}_G(X))$ of quasi-coherent G-sheaves, consisting of complexes with bounded and coherent cohomology.

The functors $\mathrm{GExt}_X^i(-,-)$ are defined as the derived functors of the functor $\mathrm{GHom}_X(-,-)$, and coincides with the G-invariant part of $\mathrm{Ext}_X^i(-,-)$ by the universal property of the derived functor. In other words, taken $E, F \in \mathbf{D}_G(X)$, then $\mathrm{GExt}_X^i(E,F) = \mathrm{Hom}_X^i(E,F)^G = \mathrm{Hom}_X(E, F[i])^G$.

1.2.3 Equivariant Fourier-Mukai functors

Let X and Y two varieties equipped with the actions of two finite groups G and H. Then $G \times H$ acts on the product $X \times Y$ via the diagonal action. The projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are equivariant with respect to the projections $G \times H \longrightarrow G$ and $G \times H \longrightarrow H$. As a consequence of general facts seen before, they define functors:

$$\pi_X^* : \mathrm{Coh}_G(X) \longrightarrow \mathrm{Coh}_{G\times H}(X \times Y)$$
$$\pi_{Y*}^{\;G} : \mathrm{Coh}_{G\times H}(X \times Y) \longrightarrow \mathrm{Coh}_H(Y)$$

which can be derived, defining equivariant pull-back and push-forwards:

$$\pi_X^* = \mathbf{L}\pi_X^* : \mathbf{D}_G(X) \longrightarrow \mathbf{D}_{G\times H}(X \times Y)$$
$$\mathbf{R}\pi_{Y}{}_*^G : \mathbf{D}_{G\times H}(X \times Y) \longrightarrow \mathbf{D}_H(Y)\ .$$

The bifunctor $- \otimes -$ pass as well on the G-equivariant level, hence, deriving it, we get a bifunctor:

$$- \otimes^L - : \mathbf{D}_{G\times H}(X \times Y) \times \mathbf{D}_{G\times H}(X \times Y) \longrightarrow \mathbf{D}_{G\times H}(X \times Y)\ .$$

The choice of a kernel $P \in \mathbf{D}_{G\times H}(X \times Y)$ defines a functor :

$$- \otimes^L P : \mathbf{D}_{G\times H}(X \times Y) \longrightarrow \mathbf{D}_{G\times H}(X \times Y)\ .$$

The composition of these functors defines the equivariant Fourier-Mukai functor with kernel P:

$$\Phi^P_{X\to Y} := \mathbf{R}\pi_{Y}{}_*^G(\pi_X^*(-) \otimes^L P) : \mathbf{D}_G(X) \longrightarrow \mathbf{D}_H(Y)\ .$$

Let us consider the diagram:

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, "u \times v"] & Y \arrow[dr, "v"] & \\
X \arrow[dr, "u"] & X/G \times Y/H \arrow[r, "\pi_{Y/H}"] \arrow[d, "\pi_{X/G}"] & Y/H \\
& X/G &
\end{tikzcd}$$

Taking the invariants of P by $G \times H$, we get a kernel $P^{G\times H} \in \mathbf{D}(X/G \times Y/H)$ and consequently an associated Fourier-Mukai functor:

$$\Phi^{P^{G\times H}}_{X/G\to Y/H} : \mathbf{D}(X/G) \longrightarrow \mathbf{D}(Y/H)\ .$$

This new functor is linked with the previous by the relation:

Proposition 1.16.
$$\Phi^{P^{G\times H}}_{X/G\to Y/H} = v_*^H \circ \Phi^P_{X\to Y} \circ \mathbf{L}u^*\ .$$

We first prove the following lemma:

Lemma 1.17. *Let R a k-algebra,* $\mathrm{char}(k) = 0$ *and M an $R[G]$-module. Let N a R-module, (that is, G acts trivially on N). Then*
$$(M \otimes_R^L N)^G = M^G \otimes_R^L N\ .$$

Proof. Resolve M with a projective resolution $P^\bullet \longrightarrow M$. Applying the fixed points functor we get a resolution: $(P^G)^\bullet \longrightarrow M^G$ of M^G. Now the $(P^i)^G$ are projective elements, because they are direct factors of the P^i, which are themselves projective, and a direct factor of a projective is projective. Hence $(P^G)^\bullet \longrightarrow M^G$ is a projective resolution of M^G. Now the tensor product $- \otimes N$ commutes with the fixed points functor, since G does not act on N: $(-)^G \otimes N \simeq (- \otimes N)^G$. Deriving, since the fixed points functor takes projectives to projectives, we are done.

Proof of the proposition 1.16. If $F \in \mathbf{D}(X/G)$ we have:

$$\begin{aligned}
v_*^H \circ \Phi_{X\to Y}^P \circ \mathbf{L}u^* F &= v_*^H \circ \mathbf{R}\pi_{Y*}^{\ G}(\pi_X^*(\mathbf{L}u^*F) \otimes^L_{\mathcal{O}_{X\times Y}} P) \\
&= v_*^H \circ \mathbf{R}\pi_{Y*}^{\ G}\Big(\mathbf{L}(u\times v)^*\pi_{X/G}^*(F) \otimes^L_{\mathcal{O}_{X\times Y}} P\Big) \\
&= \mathbf{R}\pi_{Y/H*}(u\times v)_*^{G\times H}\Big(\mathbf{L}(u\times v)^*\pi_{X/G}^*(F) \otimes^L_{\mathcal{O}_{X\times Y}} P\Big) \\
&= \mathbf{R}\pi_{Y/H*}\Big[\pi_{X/G}^*(F) \otimes^L_{\mathcal{O}_{X/G\times Y/G}} (u\times v)_*P\Big]^{G\times H} \\
&= \mathbf{R}\pi_{Y/H*}\Big[\pi_{X/G}^*(F) \otimes^L_{\mathcal{O}_{X/G\times Y/G}} (u\times v)_*^{G\times H}P\Big] \\
&= \Phi_{X/G\to Y/G}^{P^{G\times H}}(F)
\end{aligned}$$

where we used projection formula in the fourth equality and lemma 1.17 in the fifth equality. □

1.2.4 Equivariant cohomology

Let G a finite group and R a (commutative) ring. Let $\mathrm{Mod}_{R[G]}$ the category of $R[G]$-modules, or G-modules over R. Then the *group cohomology* $H^i(G,-)$ is the i-th right derived functor $R^i[-]^G$ of the fixed points functor $[-]^G : \mathrm{Mod}_{R[G]} \longrightarrow \mathrm{Mod}_R$.

When G is a finite group and R is a k-algebra with $\mathrm{char}(k) = 0$ the existence of the Reynolds operator ensures that the fixed points functor is exact: therefore, in this case, $H^i(G,M) = 0$, for $i > 0$ and $M \in \mathrm{Mod}_{R[G]}$.

Definition 1.18. (cf. [55], [7]) Let X an algebraic variety, G a finite group acting on X. The equivariant cohomology $H^i_G(X,-)$ (with values in a G-sheaf $\mathcal{F} \in \mathrm{Coh}_G(X)$) is the i-th right derived functor of the functor of invariant sections Γ_X^G:

$$H^i_G(X,-) = R^i\Gamma^G_X \ .$$

As a consequence the equivariant cohomology can be computed as the limit of the spectral sequence:

$${}'E_2^{p,q} = H^p(G, H^q(X,\mathcal{F})) \implies H_G^{p+q}(X,\mathcal{F}) \ .$$

Take now the quotient $X \xrightarrow{f} Y = X/G$; the functor of invariant sections is then also: $\Gamma_X^G = \Gamma_Y^G \circ f_* = \Gamma_Y \circ f_*^G$. Therefore we have a second spectral sequence:

$${}''E_2^{p,q} = H^p(Y, R^q f_*^G\mathcal{F}) \implies H_G^{p+q}(X,\mathcal{F}) \ .$$

Since the group G is finite, and $\mathrm{char}(k) = 0$, the two spectral sequence degenerate:

$$H^i_G(X,\mathcal{F}) \simeq H^i(X,\mathcal{F})^G \simeq H^i(Y,\mathcal{F}^G) \ , \tag{12}$$

that is, the equivariant cohomology reduces to the *invariant cohomology*, or the *cohomology of the invariants*. Since we will be interested in actions of finite groups on varieties over $\mathbb{C}$, this will always be the case.

Equivariant cohomology has the property of recovering the cohomology of the quotient:

Proposition 1.19. *Let G a finite group. Let $p : X \longrightarrow Y$ a proper G-equivariant morphism of algebraic varieties. Let $q : X \longrightarrow X/G$ and $\pi : Y \longrightarrow Y/G$ the quotients of X and Y by G. Let $\mu : X/G \longrightarrow Y/G$ the map induced by p on the quotient level. Let Φ the functor:*

$$\Phi : \mathbf{R}p_* \circ \mathbf{L}q^* : \mathbf{D}(X/G) \longrightarrow \mathbf{D}_G(Y) \ .$$

Then:

1. $\mathbf{R}\mu_* \simeq \pi_*^G \circ \Phi$*;*

2. the hypercohomology of a complex $\mathcal{F}^\bullet \in \mathbf{D}(X/G)$ *is the* G*-equivariant hypercohomology of* $\Phi(\mathcal{F}^\bullet)$*:*

$$\mathbb{H}(X/G, \mathcal{F}^\bullet) \simeq \mathbb{H}_G(Y, \Phi(\mathcal{F}^\bullet)) \ .$$

Proof.

$$\begin{aligned} \pi_*^G \circ \Phi(\mathcal{F}^\bullet) &= \pi_*^G \circ \mathbf{R}p_* \circ \mathbf{L}q^*(\mathcal{F}^\bullet) \\ &\simeq [\mathbf{R}\mu_* \circ q_*(\mathbf{L}q^*(\mathcal{F}^\bullet))]^G \\ &\simeq \mathbf{R}\mu_* q_*^G(\mathbf{L}q^*(\mathcal{F}^\bullet)) \end{aligned}$$

because μ is G-invariant, then

$$\begin{aligned} \pi_*^G \circ \Phi(\mathcal{F}^\bullet) &\simeq \mathbf{R}\mu_*(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_{X/G}} q_*^G \mathcal{O}_X) \\ &\simeq \mathbf{R}\mu_*(\mathcal{F}^\bullet) \end{aligned}$$

by lemma 1.17, the projection formula and the fact that $q_*^G \mathcal{O}_X \simeq \mathcal{O}_{X/G}$. The second statement is now an easy consequence:

$$\begin{aligned} \mathbb{H}_G(Y, \Phi(\mathcal{F}^\bullet)) &= \mathbf{R}\Gamma_Y^G \circ \Phi(\mathcal{F}^\bullet) = \mathbf{R}\Gamma_{Y/G} \circ \pi_*^G \circ \Phi(\mathcal{F}^\bullet) \\ &\simeq \mathbf{R}\Gamma_{Y/G} \circ \mathbf{R}\mu_*(\mathcal{F}^\bullet) \\ &\simeq \mathbf{R}\Gamma_{X/G}(\mathcal{F}^\bullet) = \mathbb{H}(X/G, \mathcal{F}^\bullet) \ . \end{aligned}$$

Remark 1.20. The Bridgeland-King-Reid situation (theorem 1.23) is a particular case of this picture.

1.3 The G-orbit Hilbert scheme

We will briefly describe the G-orbit Hilbert scheme as explained in Nakamura [96] and Reid [102]. See also [70] and [69].

Let G a finite group and M a smooth quasi-projective variety on which the group G acts. Let $M^{[n]}$ the Hilbert scheme of n points on M and S^nM the symmetric variety. Let $n = |G|$. The group acts naturally on S^nM and $M^{[n]}$ in such a way that the Hilbert-Chow morphism μ:

$$\mu : M^{[n]} \longrightarrow S^nM$$

is G equivariant; therefore we have a well defined surjective map between the fixed points sets:

$$(M^{[n]})^G \xrightarrow{\mu^G} (S^nM)^G \ .$$

The quotient variety M/G can be embedded in $(S^nM)^G$ with the reduced structure:

$$\begin{aligned} j : M/G &\hookrightarrow (S^nM)^G \\ [x] &\xrightarrow{j} \sum_{g \in G} gx \end{aligned}$$

Definition 1.21. The G-orbit Hilbert scheme $\mathrm{Hilb}^G(M)$ is the irreducible component of $(M^{[n]})^G$ dominating $j(M)$, that is containing smooth orbits. We will refer to $\mathrm{Hilb}^G(M)$ also as the Nakamura G-Hilbert scheme.

As a consequence we get a G-Hilbert-Chow morphism:

$$\tau : \mathrm{Hilb}^G(M) \longrightarrow M/G \; . \tag{13}$$

A G-cluster is a closed subscheme Z of M of length $|G|$ such that $H^0(\mathcal{O}_Z) \simeq \mathbb{C}[G]$ as representations of G. The G-Hilbert scheme $\mathrm{Hilb}^G(M)$ is a fine moduli space for G-clusters on M. We will denote with $\mathcal{Z}$ the universal subscheme: $\mathcal{Z} \subseteq \mathrm{Hilb}^G(M) \times M$. The family $\mathcal{Z} \longrightarrow \mathrm{Hilb}^G(M)$ is also called the universal family of G-clusters on M.

Remark 1.22. In [16] the G-Hilbert scheme $\mathrm{GHilb}(M)$ is defined set theoretically as the set of G-invariant subschemes of M of length $|G|$ such that $H^0(\mathcal{O}_Z) \simeq \mathbb{C}[G]$, and it is the scheme representing the functor $\underline{\mathrm{GHilb}(M)}$ which takes a scheme S and associates the set $\{$ G-invariant subschemes Z of $M \times S$, finite over S, such that $H^0(\mathcal{O}_{Z_s}) \simeq \mathbb{C}[G]$ for all $s \in S$ $\}$. Ito and Nakamura [70] proved that $\mathrm{Hilb}^G(M) \subseteq \mathrm{GHilb}(M)$, but the converse is not at all obvious and probably false: $\mathrm{GHilb}(M)$ is not known to be irreducible or even connected and may even be not equidimensional in general. In [16] the authors define Y as the irreducible component of $\mathrm{GHilb}(M)$ containing free orbits. As a consequence $Y = \mathrm{Hilb}^G(M)$. $\mathrm{Hilb}^G(M)$ is also said to be a "dynamic" definition, $\mathrm{GHilb}(M)$ an "algebraic" one. See [20].

1.4 The BKR construction

Let M a smooth quasiprojective variety and G a finite group acting on M with the property that the canonical sheaf ω_M is locally trivial as a G-sheaf. By the Drezet-Kempf-Narasimhan lemma [30] it descends to the quotient M/G, hence M/G is Gorenstein. Let $Y = \mathrm{Hilb}^G(M)$. In [16] the authors build a Fourier-Mukai functor by means of the universal family $\mathcal{Z} \subseteq Y \times M$. In the diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "q"'] & M \arrow[d, "\pi"] \\ Y \arrow[r, "\mu"] & M/G \end{tikzcd}$$

π and q are finite of degree $|G|$, q is flat, and p and μ are birational. Now G acts on M and $\mathcal{O}_{\mathcal{Z}}$ can be seen as as a $\{1\} \times G$-equivariant sheaf on $Y \times M$. Therefore we can define an equivariant Fourier-Mukai functor:

$$\Phi^{\mathcal{O}_{\mathcal{Z}}}_{Y \to M} : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M) \; .$$

The main result proved by Bridgeland, King and Reid in [16] is the following theorem.

Theorem 1.23. *Let M a smooth quasi projective variety of dimension n, G a finite subgroup of $\mathrm{Aut}(M)$ such that the canonical line bundle ω_M is locally trivial as a G-sheaf. Let $Y = \mathrm{Hilb}^G(M)$ and $\mathcal{Z} \subseteq Y \times M$ the universal closed subscheme. Suppose that*

$$\dim Y \times_{M/G} Y \leq n+1 \; .$$

Then Y is a crepant resolution of M/G and the Fourier-Mukai functor

$$\Phi^{\mathcal{O}_{\mathcal{Z}}}_{Y \to M} : \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b_G(M)$$

is an equivalence of categories.

1.5 The isospectral Hilbert scheme

In this section we will present the construction of the isospectral Hilbert scheme as defined by Haiman [61], and a brief description of its properties. Haiman proves everything for the affine plane $\mathbb{A}^2_{\mathbb{C}}$, but all works for a general quasi-projective surface; we sketch here how to extend some of his proofs. Let X a smooth quasi-projective variety, $n \in \mathbb{N}$, $n \geq 1$, $X^{[n]}$ the Hilbert scheme of n points over X and $S^n X$ the symmetric variety.

Definition 1.24. The isospectral Hilbert scheme B^n is the reduced fiber product:

$$\begin{tikzcd} B^n \arrow[r, "p_n"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^n X \end{tikzcd}$$

that is $B^n := \left(X^{[n]} \times_{S^n X} X^n\right)_{\mathrm{red}}$.

Remark 1.25. In the definition above it is necessary to take the reduced scheme underlying the fiber product, since the simple fiber product $X^{[n]} \times_{S^n X} X^n$ is *never* reduced, if $n \geq 2$.

A first easy property of the isospectral Hilbert scheme is:

Proposition 1.26. *Let X a smooth quasi-projective surface. The isospectral Hilbert scheme B^n over X^n is irreducible of dimension $2n$.*

The following simple lemma allows to extend several of Haiman's results to an arbitrary quasi-projective surface.

Lemma 1.27. *Let X a quasi-projective variety. Then each point in X^n has an affine open neighbourhood of the form U^n, where U is an affine open set in X.*

Proof. It suffices to prove that given a smooth quasi-projective variety X and n points $x_1, \ldots, x_n$, there exists an affine open set U such that all $x_i \in U$. To prove this, embed X in a projective space $\mathbb{P}^N$ and take its projective closure $Y = \bar{X}$. Then $Z = Y \setminus X$ is a closed subset of the (possibly singular) projective variety Y. For large l, there exists sections $s_i \in H^0(Y, \mathcal{I}_Z(l)) \subseteq H^0(Y, \mathcal{O}_Y(l))$, vanishing on Z but nonzero on x_i. If l is large, the subspace $H_i \subseteq H^0(Y, \mathcal{I}_Z(l))$ consisting of sections of $\mathcal{I}_Z(l)$ vanishing on x_i form a hyperplane in $H^0(Y, \mathcal{I}_Z(l))$ for all i. Consider now a section $u \in H^0(Y, \mathcal{I}_Z(l)) \setminus \cup_{i=1}^n H_i$. The affine open set U defined by $u \neq 0$ in Y is contained in X and contains all the points x_i. Consequently, it is the wanted affine open set.

□

The first important fact Haiman proves is that B^n can be obtained as the blow-up of X^n along the union of pairwise diagonals in X^n:

Theorem 1.28. *The isospectral Hilbert scheme B^n can be identified with the blow up of X^n along the scheme-theoretic union of all its pairwise diagonals.*

Proof. The case of the affine plane is proved in Haiman [61]. For an affine surface the proof goes exactly as in the case of $\mathbb{A}^2_{\mathbb{C}}$. Passing to a quasi-projective variety X is now simple, owing to the preceding lemma. Consider the isospectral Hilbert scheme B^n and P a point on X^n. Let U^n be the affine open set containing P, found in the preceding lemma. The isospectral Hilbert scheme B^n_U associated to the affine surface U is now the blow-up $B^n_U \longrightarrow U^n$ of the pairwise diagonals in U^n. Since B^n_U can be identified with the inverse image of U^n for the projection $B^n \longrightarrow X^n$ and the statement is local on X^n, we are done.

□

The analogous of the nested Hilbert scheme for the isospectral Hilbert scheme is the *isospectral nested Hilbert scheme* which we will now introduce. Let $X^{[n+1,n]} \longrightarrow X^{[n]}$ the nested Hilbert scheme.

Definition 1.29. The nested isospectral Hilbert scheme $B^{n+1,n}$ is the reduced fiber product:

$$\begin{tikzcd} B^{n+1,n} \arrow[r, "u"] \arrow[d, "s"] & B^n \arrow[d, "q"] \\ X^{[n+1,n]} \arrow[r, "\tilde{u}"] & X^{[n]} \end{tikzcd}$$

that is, $B^{n+1,n} := \left(B^{n+1,n} \times_{X^{[n]}} B^n\right)_{\text{red}}$.

Analogously to the usual nested Hilbert scheme, there are two projections:

$$B^n \xleftarrow{u} B^{n+1,n} \xrightarrow{v} B^{n+1} .$$

Haiman uses the nested Hilbert scheme $X^{[n+1,n]}$ and its isospectral analougue $B^{n+1,n}$ as fundamental tools to prove one of his main theorems:

Theorem 1.30. *The isospectral Hilbert scheme is normal, Cohen-Macauley and Gorenstein.*

Sketch of the proof. Again, the case of the affine plane has been proved in [61]. The case of an affine surface goes exactly in the same way. To obtain the result for a quasi-projective surface X, take a point $Q \in B^n$, and its image P in X^n. Let U^n be the affine open set containing P found in the lemma 1.27. Then the isospectral Hilbert scheme B^n_U associated to the affine surface U has the wanted properties, contains Q and can be identified with an open set of B^n.

□

Remark 1.31. One of the technical step in the proof of the preceding theorem is proving that the morphism: $B^{n+1,n} \xrightarrow{v} B^{n+1}$ satisfies:

$$\mathbf{R}v_*\mathcal{O}_{B^{n+1,n}} \simeq \mathcal{O}_{B^{n+1}} . \tag{14}$$

As usual, Haiman proves this for the isospectral Hilbert scheme associated to the affine plane, and his proof works without any change for a smooth affine surface. By lemma 1.27 it is valid on an arbitrary smooth quasi-projective surface.

Remark 1.32. If $X \xrightarrow{f} Y$ is a Cohen-Macauley scheme, with f finite and surjective over the smooth variety Y, then X is flat over Y ([32], exercise 18.17). On the other hand if X is flat and finite over the Cohen-Macauley scheme Y, then X is Cohen-Macauley. In this case X is Gorenstein if and only if f has Gorenstein fibers. (cf. Bourbaki, [10], chapter 10, n.7, §2 and §3).

Remark 1.33. The preceding remark applies in particular to the isospectral Hilbert scheme B^n, since by theorem 1.30 it is Cohen-Macauley and $q : B^n \longrightarrow X^{[n]}$ is finite and surjective with $X^{[n]}$ smooth. Hence the Cohen-Macauley property of B^n is equivalent to the flatness of the morphism q. Therefore in the diagram:

$$\begin{tikzcd} B^n \arrow[r, "p_n"] \arrow[d, "q"] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^n X \end{tikzcd}$$

q is flat of degree $n!$, even if π is not.

Remark 1.34. The subscheme $Z \subseteq B^n \times X$, defined as the pull-back $(q \times \mathrm{id})^{-1}(\Xi)$ where Ξ is the universal family over $X^{[n]}$, is called the universal subscheme for the isospectral Hilbert scheme. We will also call it the isospectral universal family. Since Ξ is flat and finite over $X^{[n]}$, Z is flat and finite over B^n, hence Cohen-Macauley. It is reduced, because generically reduced. While Ξ is irreducible, Z has n irreducible components $Z_i = (p_n \times \mathrm{id})^{-1}(D_i)$, where D_i is the diagonal $D_i = \Delta_{i,n+1} \subseteq X^n \times X$. It is clear that $Z_i \simeq B^n$, hence Z_i are normal, Cohen-Macauley and Gorenstein.

Remark 1.35. In the diagram

$$\begin{tikzcd} B^{n+1,n} \arrow[r, "u"] \arrow[d, "s"'] & B^n \arrow[d, "q"] \\ X^{[n+1,n]} \arrow[r, "\tilde{u}"] & X^{[n]} \end{tikzcd}$$

the nested isospectral Hilbert scheme $B^{n+1,n}$ coincides with the fiber product $X^{[n+1,n]} \times_{X^{[n]}} B^n$ and the above diagram is cartesian. Indeed, since q is flat, the fiber product $X^{[n+1,n]} \times_{X^{[n]}} B^n$ is flat and finite of degree $n!$ over the nested Hilbert scheme $X^{[n+1,n]}$ which is smooth. Hence by remark 1.32 it is Cohen-Macauley. Since it is generically reduced (where s is unramified), it has to be reduced everywhere. Therefore it coincides with $B^{n+1,n}$.

Remark 1.36. Consider now the diagram:

$$\begin{tikzcd} B^{n+1,n} \arrow[r, "(u \times t)"] \arrow[d, "s"'] & B^n \times X \arrow[d, "q \times \mathrm{id}"] \\ X^{[n+1,n]} \arrow[r, "(\tilde{u} \times \tilde{t})"] & X^{[n]} \times X \end{tikzcd}$$

It is a flat base change. Since $X^{[n+1,n]}$ could be considered as the projectivization $\mathbb{P}(\mathcal{I}_\Xi)$, Ξ the universal family over $X^{[n]}$, by base change it is immediate to see that the isospectral nested Hilbert scheme $B^{n+1,n}$ can be seen as the projectivization:

$$B^{n+1,n} \simeq \mathbb{P}(\mathcal{I}_Z) ,$$

with Z the isospectral universal family. It can furthermore be seen, with the same kind of arguments made in [23], that

$$B^{n+1,n} \simeq \mathrm{Bl}_Z(B^n \times X) .$$

Always by flat base change applied to (9) we can prove:

$$\mathbf{R}(u \times t)_* \mathcal{O}_{B^{n+1,n}} \simeq \mathcal{O}_{B^n \times X} \qquad \mathbf{R}(u \times t)_* \mathcal{O}_{\mathcal{E}} \simeq \mathcal{O}_Z \tag{15}$$

where $\mathcal{E}$ is the exceptional divisor in $B^{n+1,n} \simeq \mathrm{Bl}_Z(B^n \times X)$.

Remark 1.37. We have just seen that the isospectral nested Hilbert scheme $B^{n+1,n}$ is obtained by the nested Hilbert scheme $X^{[n+1,n]}$ by a flat base change. Pulling back the exact sequences (7) and (8) on $B^{n+1,n} \times X$ via $s \times \mathrm{id}_X$ we get the two sequences:

$$0 \longrightarrow \mathcal{O}_{\eta_B}(-\mathcal{E}) \longrightarrow \mathcal{O}_{\zeta_B} \longrightarrow \mathcal{O}_{\xi_B} \longrightarrow 0 \tag{16}$$

and

$$0 \longrightarrow \mathcal{O}_{\zeta_B} \longrightarrow \mathcal{O}_{\xi_B} \oplus \mathcal{O}_{\eta_B} \longrightarrow \mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} \longrightarrow 0 \tag{17}$$

where we denoted with ζ_B, ξ_B, η_B the flat families over $B^{n+1,n} \times X$ of relative length $n+1$, n, 1, obtained by the pull back via $s \times \mathrm{id}_X$ of the families ζ_{n+1}, ξ_n, η on $X^{[n+1,n]} \times X$.

1.6 The Hilbert scheme $X^{[n]}$ as the $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$-scheme.

In this section we will explain briefly why the product X^n of a smooth quasi-projective surface X with the action of the symmetric group $\mathfrak{S}_n$ satisfies the hypothesis of the BKR theorem. In particular we will sketch why the scheme $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ can be identified with the Hilbert scheme $X^{[n]}$ of n points on X. We will follow [61] and [60].

Theorem 1.38. *The Hilbert scheme of n points over a smooth quasi-projective surface is isomorphic to the scheme* $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ *over the symmetric variety* S^nX.

Sketch of the proof. Let $\mathcal{Z} \subseteq \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \times X^n$ the universal family over $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$. It is flat over $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ of degree $n!$ and it has a natural $\mathfrak{S}_n$-action, where $\mathfrak{S}_n$ acts on the second factor. If we make $\mathfrak{S}_{n-1}$ act on X^n via the inclusion $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n$, $\mathcal{Z}$ becomes equipped with an $\mathfrak{S}_{n-1}$-action. Consider the quotient $\mathcal{Z}/\mathfrak{S}_{n-1}$: it can be identified with a subscheme $\mathcal{Z}/\mathfrak{S}_{n-1} \subseteq \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \times S^{n-1}X \times X$, flat over $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ of degree n. Consider now the embedding morphism:

$$
\begin{aligned}
i : \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \times X &\hookrightarrow \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \times S^{n-1}X \times X \\
(\xi, x) &\longmapsto (\xi, \tau(\xi) - x, x)
\end{aligned}
$$

The pullback of $\mathcal{Z}/\mathfrak{S}_{n-1}$ for this embedding can then be seen as a subscheme

$$
\mathcal{Y} := i^{-1}(\mathcal{Z}/\mathfrak{S}_{n-1}) \subseteq \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \times X
$$

flat over $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ of degree n. Actually the two families $\mathcal{Y}$ and $\mathcal{Z}/\mathfrak{S}_{n-1}$ are isomorphic over $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$. By definition of the Hilbert scheme $X^{[n]}$ the family $\mathcal{Y}$ defines a morphism:

$$
\phi : \mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \longrightarrow X^{[n]} \; .
$$

On the other hand, since the isospectral Hilbert scheme B^n is flat of degree $n!$ over $X^{[n]}$, it can be considered a family $B^n \subseteq X^{[n]} \times X^n$ of subschemes of length $n!$ of X^n, flat over $X^{[n]}$. The universal property of $(X^n)^{[n!]}$ then gives rise to a map:

$$
\psi : X^{[n]} \longrightarrow (X^n)^{[n!]}
$$

whose image is clearly contained in $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n) \subseteq (X^n)^{[n!]}$. They are clearly inverse one the other on the generic locus, hence everywhere. In the identification ϕ the universal families $\mathcal{Z}$ and B^n are identified.

□

Actually Haiman proves in [60] that this theorem is equivalent to the Cohen-Macauley property of B^n.

At this point, to prove that the $\mathfrak{S}_n$-action on X^n satisfies the hypothesis of theorem 1.23 we have to prove that ω_{X^n} is locally trivial as $\mathfrak{S}_n$-sheaf — which is easy, since the stabilizer of a point $x \in X^n$ acts as a subgroup of $SL(T_xX^n)$, and hence trivially on ω_{X^n} — and the smallness condition

$$
\dim(X^{[n]} \times_{S^nX} X^{[n]}) \leq 2n + 1 \; ;
$$

but this is a direct consequence of the fact that the Hilbert-Chow morphism:

$$
\mu : X^{[n]} \longrightarrow S^nX
$$

is a semismall resolution (see [27]):

Proposition 1.39. *Let* $f_1 : X_1 \longrightarrow Y$ *and* $f_2 : X_2 \longrightarrow Y$ *two proper surjective semismall maps, with* $m = \dim X_i = \dim Y$, $i = 1, 2$. *Then* $dim_Y X_1 \times_Y X_2 = m$.

As a consequence, we get the following remarkable particular case of 1.23:

Theorem 1.40 (Haiman)**.** *Let X a smooth quasi-projective surface, $X^{[n]}$ the Hilbert scheme of n-points on X, B^n the isospectral Hilbert scheme. Let $q : B^n \longrightarrow X^{[n]}$ and $p : B^n \longrightarrow X^n$ the projections on the Hilbert scheme and on the product variety, respectively. The Fourier-Mukai functor:*

$$\Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^n} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n) \tag{18}$$

defined by:

$$\Phi : \mathbf{R}p_* \circ q^*$$

is an equivalence.

We will refer to the previous equivalence as the Bridgeland-King-Reid-Haiman (BKRH)-equivalence. and we will indicate it with $\mathbf{\Phi}$.

2 The Čech complex for closed subschemes

The aim of this chapter is to prove, under some reasonable transversality hypothesis, the exactness of a Čech-like complex for a finite scheme theoretic union $Z = \cup_{i=1}^n Z_i$ of closed subschemes Z_i of a Cohen-Macauley scheme.

Let X a noetherian scheme and Z_i, $i = 1, \dots, n$ closed subschemes, defined by ideal sheaves I_{Z_i}. Let $Z = \cup_{i=1}^n Z_i$ the scheme-theoretic union, which we recall being defined by the ideal sheaf $I_Z = \cap_{i=1}^n I_{Z_i}$. If J is a subset of $\{1, \dots, n\}$, we indicate with Z_J the partial intersection $Z_J = \cap_{j \in J} Z_j$. We define the Čech complex $\check{\mathcal{C}}^\bullet$ as follows:

$$\check{\mathcal{C}}^\bullet : 0 \longrightarrow \mathcal{O}_Z \xrightarrow{\imath} \oplus_{i=1}^n \mathcal{O}_{Z_i} \xrightarrow{\partial^0} \oplus_{|J|=2} \mathcal{O}_{Z_J} \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} \mathcal{O}_{Z_{\{1,\dots,n\}}} \longrightarrow 0 \tag{19}$$

where the differential is defined by:

$$(\partial^p f)_J = \sum_{i \in J} \varepsilon_{i,J} f_{J \setminus \{i\}} \mid_{Z_J}$$

and $\varepsilon_{i,J}$ is the sign $\varepsilon_{i,J} = (-1)^{\sharp\{l \in J, l < i\}}$.

Remark 2.1. This complex is not exact in general. Let us take, for example, $X \simeq \mathbb{C}^2 \simeq \mathrm{Spec}(\mathbb{C}[x,y])$, l the x-axis, r the y-axis, s the diagonal of the first quadrant and $Z = l \cup r \cup s$. Let P the origin. The complex $\check{\mathcal{C}}^\bullet$ becomes:

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_l \oplus \mathcal{O}_r \oplus \mathcal{O}_s \longrightarrow \mathcal{O}_P \oplus \mathcal{O}_P \oplus \mathcal{O}_P \longrightarrow \mathcal{O}_P \longrightarrow 0 \, . \tag{20}$$

The first two differentials are given by $\imath(f) = (f|_l, f|_r, f|_s)$ and $\partial^0(\alpha, \beta, \gamma) = (\beta(0) - \alpha(0), \gamma(0) - \alpha(0), \gamma(0) - \beta(0))$. Now $\partial^0(\alpha, \beta, \gamma) = 0$ if and only if $\alpha(0) = \beta(0) = \gamma(0)$, that is, if and only if the three functions α, β, γ coincide at the origin. On the other hand, the image of $\imath$ is the restriction to the three lines of a function defined on Z. Therefore, in the identification $s = \mathrm{Spec}(\mathbb{C}[x,y]/(x-y)) \simeq \mathbb{C}[t]$ via $\mathbb{C} \ni t \longrightarrow (t,t) \in \mathbb{C}^2$, we have, for a function $f \in \mathcal{O}_Z$:

$$\frac{d}{dt} f(t,t)(0) = \frac{\partial}{\partial x} f(x,0)(0) + \frac{\partial}{\partial y} f(0,y)(0) \, . \tag{21}$$

This means that if $(\alpha, \beta, \gamma) \in \mathcal{O}_l \oplus \mathcal{O}_r \oplus \mathcal{O}_s$ are in the image of $\imath$, then they have to satisfy a nontrivial relation between their derivatives at the origin P, apart from coinciding in P. Since the condition (21) gives a one-dimensional restriction on $\ker \partial^0$, we have:

$$H^1(\check{\mathcal{C}}^\bullet) \simeq \mathbb{C} \, .$$

The question of the exactness of the complex $\check{\mathcal{C}}^\bullet$ in degree 1 is in relation with the seminormality of the local rings $\mathcal{O}_{Z,x}$. See Dayton and Roberts ([26], [25]). Orecchia ([101]) put in relation seminormality of the union Z with the transversality of the components Z_i.

We will prove, under transversality conditions, the exactness of the full complex $\check{\mathcal{C}}^\bullet$, and of a more general complex, of which the Čech complex is a particular case. The techical tool is the following proposition. Let $I \subseteq \{1, \ldots, n\}$. We will indicate with I' the complementary $I^{'} = \{1, \ldots, n\} \setminus I$ of I in $\{1, \ldots, n\}$. If $\mathcal{F}_i$, $i = 1, \ldots, n$ are sheaves on a scheme X and $I \subseteq \{1, \ldots, n\}$, we will indicate with $\mathcal{F}_I$ the tensor product: $\mathcal{F}_I = \otimes_{j \in I} \mathcal{F}_{j \in I}$.

Proposition 2.2. *Let X a noetherian scheme. Let $\mathcal{M}_i$, $i = 1, \ldots, n$ coherent sheaves on X. Consider, for all i, the exact sequences of sheaves:*

$$0 \longrightarrow \mathcal{N}_i \longrightarrow E_i \longrightarrow \mathcal{M}_i \longrightarrow 0$$

with E_i locally free. Let $\mathcal{K}_i^\bullet$ the complex (in degree 0 and 1):

$$\mathcal{K}_i{}^\bullet := E_i \longrightarrow \mathcal{M}_i \longrightarrow 0 \,.$$

If $\mathrm{Tor}_k(\mathcal{M}_{i_1}, \ldots, \mathcal{M}_{i_h}) = 0$ for $k > 0$, and for $0 \leq i_1 < \cdots < i_h \leq n$, $1 \leq h \leq n$, then the complex $\mathcal{K}^\bullet := \mathcal{K}_1{}^\bullet \otimes \ldots \otimes \mathcal{K}_n{}^\bullet$:

$$\begin{aligned} 0 \longrightarrow \otimes_{i=1}^n E_i \longrightarrow \oplus_{i=1}^n \mathcal{M}_i \otimes E_{\{i\}'} \longrightarrow & \\ \longrightarrow \oplus_{|I|=2} \mathcal{M}_I \otimes E_{I'} \longrightarrow \ldots \longrightarrow \otimes_{i=1}^n \mathcal{M}_i \longrightarrow 0 & \end{aligned}$$

is a right resolution of $\otimes_{i=1}^n \mathcal{N}_i$. In particular if $E_i = E$ for all i, the complex:

$$\begin{aligned} 0 \longrightarrow \otimes_{i=1}^n \mathcal{N}_i \longrightarrow E^{\otimes^n} \longrightarrow \oplus_{i=1}^n \mathcal{M}_i \otimes E^{\otimes^{n-1}} \longrightarrow & \\ \longrightarrow \oplus_{|I|=2} \mathcal{M}_I \otimes E^{\otimes^{n-2}} \longrightarrow \ldots \longrightarrow \otimes_{i=1}^n \mathcal{M}_i \longrightarrow 0 & \end{aligned} \tag{22}$$

is exact.

Proof. For all $i = 1, \ldots, n$ the complex of cochains

$$\mathcal{K}_i{}^\bullet := 0 \longrightarrow E_i \longrightarrow \mathcal{M}_i \longrightarrow 0$$

is clearly a right resolution of the sheaves $\mathcal{N}_i$. As a consequence the p-cohomology of the complex $\mathcal{K}_1{}^\bullet \otimes^L \ldots \otimes^L \mathcal{K}_n{}^\bullet$ in the bounded derived category $\mathbf{D}^b(X)$ is

$$H^p(\mathcal{K}_1{}^\bullet \otimes^L \ldots \otimes^L \mathcal{K}_n{}^\bullet) \simeq \mathrm{Tor}_{-p}(\mathcal{N}_1, \ldots, \mathcal{N}_n) \,.$$

To compute this cohomology group we can use the hypertor spectral sequence:

$${}^{'}E_1^{p,q} = \bigoplus_{i_1 + \cdots + i_n = p} \mathrm{Tor}_{-q}(\mathcal{K}_1^{i_1}, \ldots, \mathcal{K}_n^{i_n}) \implies \mathrm{Tor}_{-p-q}(\mathcal{N}_1, \ldots, \mathcal{N}_n) \,. \tag{23}$$

Now $\mathrm{Tor}_{-h-k}(\mathcal{K}_1^{i_1}, \ldots, \mathcal{K}_n^{i_n})$ is in turn the limit of the spectral sequence:

$${}^{''}E_2^{h,k} = \mathrm{Tor}_{-h}(\mathcal{K}_1^{i_1}, \mathrm{Tor}_{-k}(\mathcal{K}_2^{i_2}, \ldots, \mathcal{K}_n^{i_n}))$$

and since $E_i = \mathcal{K}_i^0$ is acyclic (because locally free), the term ${}^{'}E_1^{p,q}$ reduces to a sum:

$${}^{'}E_1^{p,q} = \bigoplus_{\substack{i_1 + \cdots + i_h = p \\ i_j \neq 0}} \mathrm{Tor}_{-q}(\mathcal{M}_{i_1}, \ldots, \mathcal{M}_{i_h})$$

which is zero by hypothesis, if $p \neq 0$. We remark that the complex ${}'E_1^{\bullet,0}$ is exactly $\mathcal{K}_1{}^{\bullet} \otimes \ldots \otimes \mathcal{K}_n{}^{\bullet}$. The spectral sequence then degenerates at level ${}'E_2$ and hence , if $p \neq 0$,

$$\begin{aligned} {}'E_2^{p,0} \simeq {}' E_\infty^{p,0} \simeq {}' E^p &\simeq H^p(\mathcal{K}_1{}^{\bullet} \otimes^L \ldots \otimes^L \mathcal{K}_n{}^{\bullet}) \\ &\simeq \operatorname{Tor}_{-p}(\mathcal{N}_1, \ldots, \mathcal{N}_n) \ . \end{aligned}$$

Now $\operatorname{Tor}_{-p}(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ is necessarily zero if $p > 0$, because $\mathcal{N}_i$ are sheaves. As a consequence, the only nonzero term in level 2 is

$$\begin{aligned} {}''E_2^{0,0} &\simeq H^0(\mathcal{K}_1^{\bullet} \otimes \ldots \otimes \mathcal{K}_n^{\bullet}) \\ &\simeq \mathcal{N}_1 \otimes \ldots \otimes \mathcal{N}_n \ . \end{aligned}$$

On the other hand,

$${}'E_2^{p,0} \simeq H^p({}'E_1^{\bullet,0}) \simeq H^p(\mathcal{K}_1^{\bullet} \otimes \ldots \otimes \mathcal{K}_n^{\bullet}) = 0 \ ,$$

if $p > 0$. As a consequence,

$$\mathcal{K}^{\bullet} \simeq \mathcal{K}_1^{\bullet} \otimes \ldots \otimes \mathcal{K}_n^{\bullet} \simeq \mathcal{K}_1^{\bullet} \otimes^L \ldots \otimes^L \mathcal{K}_n^{\bullet}$$

is a resolution of $\otimes_{i=1}^n \mathcal{N}_i$, since $H^0(\mathcal{K}^{\bullet}) \simeq \otimes_{i=1}^n \mathcal{N}_i$.

□

Remark 2.3. In the case where $\mathcal{M}_i$ are structural sheaves $\mathcal{O}_{Z_i}$ of closed subschemes Z_i of X, and E_i is chosen to be $\mathcal{O}_X$ for all i, the complex (22) is exactly the Čech complex (19).

We are now going to look for a simple criterion that allows us to decide if the hypothesis of the proposition are satisfied. Since every statement is in fact of local nature, we can set the discussion in the context of commutative algebra of local noetherian rings. The fundamental tool we will be using is the following result by Peskine and Szpiro (cf. [74]).

Lemma 2.4 (Peskine-Szpiro, Kempf-Laksov). *Let $(A, \mathfrak{m})$ a Cohen-Macauley noetherian local ring and $I \subseteq A$ an ideal. Let*

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \ldots \longrightarrow K^{n-1} \longrightarrow K^n \longrightarrow 0$$

be a complex of free modules. Suppose that

$$\operatorname{Supp}(K^{\bullet}) := \bigcup_{i=1}^n \operatorname{Supp} H^i(K^{\bullet}) \subseteq V(I) \ .$$

Then $H^i(K^{\bullet}) = 0$ for all $i < \operatorname{ht}(I)$.

□

Let now $(A, \mathfrak{m})$ be a noetherian local ring and $k := A/\mathfrak{m}$ its residue field. If M is a finite module over A with finite projective dimension $\operatorname{projdim} M$, the Auslander-Buchsbaum formula (see [88]) states that :

$$\operatorname{projdim} M + \operatorname{depth} M = \operatorname{depth} A \ . \tag{24}$$

If now $(A, \mathfrak{m})$ is a noetherian regular local ring, and M is a Cohen-Macauley module over A, Auslander-Buchsbaum formula implies that the length of a minimal free resolution of M equals its codimension.

Proposition 2.5. *Let $(A, \mathfrak{m})$ a noetherian regular local ring, $M_1, \ldots, M_k$ finite Cohen-Macauley modules over A of finite projective dimension. Let*

$$c(M_1, \ldots, M_k) = \left(\sum_{i=1}^{k} \operatorname{codim} M_i \right) - \operatorname{codim} M_1 \otimes \ldots \otimes M_k \ .$$

Then

$$\operatorname{Tor}_i(M_1, \ldots, M_k) = 0$$

for $i \gneq c(M_1, \ldots, M_k)$.

Proof. The proposition is an easy consequence of Peskine-Szpiro lemma and the existence, for the modules M_i, of minimal free resolutions of length equalling the codimensions $\operatorname{codim} M_i$. For every module M_i let's take its minimal free resolution $R_i^\bullet \longrightarrow M_i \longrightarrow 0$, written:

$$0 \longrightarrow R_i^0 \longrightarrow R_i^1 \longrightarrow \ldots \longrightarrow R_i^{\operatorname{codim} M_i} \longrightarrow M_i \longrightarrow 0 \ .$$

We can then compute $\operatorname{Tor}_i(M_1, \ldots, M_k)$ as the cohomology of the total complex: $R^\bullet := {R_1}^\bullet \otimes \ldots \otimes R_k^\bullet$. Now $R^\bullet$ is a finite complex of free modules of length $l = \sum_{i=1}^{k} \operatorname{codim} M_i$ and, for all i, $\operatorname{Tor}_i(M_1, \ldots, M_k) = H^{l-i}(R^\bullet)$ is supported in $\operatorname{Supp}(M_1 \otimes \ldots \otimes M_k) = V(\operatorname{Ann}(M_1 \otimes \ldots \otimes M_k))$. Therefore by Peskine and Szpiro lemma, $H^{l-i}(R^\bullet) = 0$ for $l - i < \operatorname{ht}(\operatorname{Ann}(M_1 \otimes \ldots \otimes M_k))$, that is if $i > l - \operatorname{ht}(\operatorname{Ann}(M_1 \otimes \ldots \otimes M_k))$. Now for a noetherian regular local ring, $\operatorname{ht}(I) = \operatorname{codim} V(I) = \dim A - \dim A/I$, and this implies the result.

□

Let now $(A, \mathfrak{m})$ a local ring, and $M_1, \ldots, M_k$ finite modules on A. We call the excess of dimension of $M_1 \otimes \ldots \otimes M_k$ the positive integer:

$$c(M_1, \ldots, M_k) = \left(\sum_{i=1}^{k} \operatorname{codim} M_i \right) - \operatorname{codim}(M_1 \otimes \ldots \otimes M_k)$$

For brevity's sake, if $M_1, \ldots M_n$ are modules over A and $H \subseteq \{1, \ldots, n\}$, $H = \{i_1, \ldots, i_h\}$, we will indicate the integer of proposition (2.5) with $c(M_H) := c(M_{i_1}, \ldots, M_{i_h})$.

Lemma 2.6. *Let $(A, \mathfrak{m})$ a regular local ring, $M_1, \ldots, M_k$ nonzero finite modules over A. Then*

1. $\operatorname{codim}(M_1 \otimes \ldots \otimes M_k) \leq \sum_{i=1}^{n} \operatorname{codim} M_i$
2. *For all $H \subseteq \{1, \ldots, n\}$ we have $0 \leq c(M_H) \leq c(M_1, \ldots, M_k)$.*

Proof. We embed $X = \operatorname{Spec} A$ in the product $X^n = \operatorname{Spec} A^{\otimes^n}$ via the diagonal immersion $X \xhookrightarrow{i} X^n$. We have: $\operatorname{codim}_{X^n}(M_1 \boxtimes \cdots \boxtimes M_k) = \sum_{i=1}^{n} \operatorname{codim}_X M_i$. Now $M_1 \otimes \ldots \otimes M_k = i^*(M_1 \boxtimes \cdots \boxtimes M_k)$. As a consequence

$$\operatorname{Supp}(M_1 \otimes \ldots \otimes M_k) = i^{-1}(\operatorname{Supp}(M_1 \boxtimes \cdots \boxtimes M_k)) \simeq \Delta \cap \operatorname{Supp}(M_1 \boxtimes \cdots \boxtimes M_k) \ .$$

Now, since X^n is a smooth scheme, we can estimate the dimension of the intersection $\Delta \cap \operatorname{Supp}(M_1 \boxtimes \cdots \boxtimes M_k)$ in the following way:

$$\dim \Delta \cap \operatorname{Supp}(M_1 \boxtimes \cdots \boxtimes M_k) \geq \dim \Delta + \dim \operatorname{Supp}(M_1 \boxtimes \cdots \boxtimes M_k) - \dim X^n$$

which implies $\operatorname{codim}_X(M_1 \otimes \ldots \otimes M_k) \leq \operatorname{codim}_{X^n}(M_1 \boxtimes \cdots \boxtimes M_k) = \sum_{i=1}^{k} \operatorname{codim}_X M_i$.

The second statement comes easily from the first, remarking that, if for all $H \subseteq \{1,\ldots,k\}$

$$\begin{aligned}
c(M_1,\ldots,M_k) &= \left(\sum_{i=1}^{n} \operatorname{codim} M_i\right) - c(M_H \otimes M_{H'}) \\
&\geq \left(\sum_{i=1}^{n} \operatorname{codim} M_i\right) - \operatorname{codim}(M_H) - \operatorname{codim}(M_{H'}) \\
&\geq c(M_H) + c(M_{H'}) \geq c(M_H) \ .
\end{aligned}$$

$\square$

Combining lemma 2.6 with proposition 2.5 we get the simple sufficient condition we needed for the vanishing of the Tor-s in the hypothesis of propositon 2.2. We now give the useful result in sheaf-theoretic terms in the following statement.

Theorem 2.7. *Let X a smooth variety and $\mathcal{M}_i$, $i = 1,\ldots,n$ Cohen-Macauley coherent sheaves on X. Consider the exact sequences:*

$$0 \longrightarrow \mathcal{N}_i \longrightarrow E_i \longrightarrow \mathcal{M}_i \longrightarrow 0$$

with E_i locally free. Therefore the complex:

$$\begin{aligned}
0 \longrightarrow \otimes_{i=1}^{n} \mathcal{N}_i \longrightarrow \otimes_{i=1}^{n} E_i \longrightarrow \oplus_{i=1}^{n} \mathcal{M}_i \otimes E_{\{i\}'} \longrightarrow& \\
\longrightarrow \oplus_{|I|=2} \mathcal{M}_I \otimes E_{I'} \longrightarrow \ldots \longrightarrow \otimes_{i=1}^{n} \mathcal{M}_i \longrightarrow 0&
\end{aligned}$$

is exact at point $x \in X$ such that:

$$\operatorname{codim}_X(\mathcal{M}_{H_x,x}) = \sum_{j \in H_x} \operatorname{codim}_X \mathcal{M}_j \ .$$

where $H_x = \{j, 1 \leq j \leq n \mid \mathcal{M}_{j,x} \neq 0\}$.

Example 2.8. Let X a smooth variety. Consider the product: $X^{n+1} \simeq X^n \times X$. Let $p_i : X^{n+1} \longrightarrow X$ the projection on the i-th factor and Δ the diagonal in X^2. Let

$$D_i := (p_i \times p_{n+1})^*(\Delta) \ .$$

Then $D_i \simeq X^n$, hence smooth. Moreover the intersection of all D_i: $\cap_{i=1}^{n} D_i \simeq \Delta_{1,\ldots,n+1}$, is the small diagonal in X^{n+1}, hence $\operatorname{codim}_{X^{n+1}} \cap_{i=1}^{n} D_i = \sum_{i=1}^{n} \operatorname{codim}_{X^{n+1}} D_i$. Theorem 2.7 then applies to the exact sequences

$$0 \longrightarrow \mathcal{I}_{D_i} \longrightarrow \mathcal{O}_{X^n \times X} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0 \ .$$

As a consequence of the theorem and of remark 2.3 the Čech complex:

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \longrightarrow \bigoplus_{|I|=2} \mathcal{O}_{D_I} \longrightarrow \ldots \longrightarrow \mathcal{O}_{D_{\{1,\ldots,n\}}} \longrightarrow 0$$

is exact and provides a right resolution of the sheaf $\mathcal{O}_D$.

Example 2.9. Let X a smooth quasi-projective surface, $n \in \mathbb{N}$, $n \geq 2$. Let B^n the isospectral Hilbert scheme and $Z \subseteq B^n \times X$ the isospectral universal family. As in the preceding example, let D the scheme theoretic union $D = \cup_{i=1}^{n} D_i$ of the diagonals $\Delta_{i,n+1}$. As we saw in remark 1.34, Z is the union of its irreducible components $Z_i = f^{-1}(D_i)$, where $f = (p_n \times \mathrm{id})$. While, by the previous example, the diagonals D_i are transverse in $X^n \times X$, the components Z_i are not transverse in $B^n \times X$: indeed the condition on the codimension does not hold any more: while $\operatorname{codim}_{B^n \times X} Z_i = 2$, we have:

$$\operatorname{codim}(Z_{i_1} \cap \cdots \cap Z_{i_h}) = h + 1 \ .$$

for $1 \leq i_1 < \cdots < i_h \leq n$. As a consequence, $\operatorname{Tor}_1(Z_{i_1}, \ldots, Z_{i_h}) \neq 0$ which implies that

$$\mathcal{I}_{Z_{i_1}} \cap \cdots \cap \mathcal{I}_{Z_{i_h}} \supsetneq \mathcal{I}_{Z_{i_1}} \cdot \cdots \cdot \mathcal{I}_{Z_{i_h}} \ .$$

Now, by transversality, since $\operatorname{Tor}_i(\mathcal{O}_{D_1}, \ldots, \mathcal{O}_{D_n}) = 0$,

$$\mathcal{I}_D = \mathcal{I}_{D_1} \cap \cdots \cap \mathcal{I}_{D_n} = \mathcal{I}_{D_1} \cdot \cdots \cdot \mathcal{I}_{D_n} \ .$$

Therefore

$$\begin{aligned} \mathcal{I}_{f^{-1}(D)} &= f^{-1}(\mathcal{I}_{D_1} \cdot \cdots \cdot \mathcal{I}_{D_n}) \\ &= \mathcal{I}_{f^{-1}(D_1)} \cdot \cdots \cdot \mathcal{I}_{f^{-1}(D_n)} \\ &= \mathcal{I}_{Z_1} \cdot \cdots \cdot \mathcal{I}_{Z_n} \subsetneq \mathcal{I}_Z \ . \end{aligned}$$

As a consequence the scheme $f^{-1}(D)$ is non reduced and $f^{-1}(D)_{\mathrm{red}} \simeq Z$. It is clear that $f^*\mathcal{O}_D \simeq \mathcal{O}_{\tilde{Z}}$. We will refer to $f^{-1}(D)$ as $\tilde{Z}$.

3 The image of a tautological vector bundle for the BKR equivalence

In this chapter we are going to compute the image of the tautological sheaf $F^{[n]}$ on the Hilbert scheme $X^{[n]}$, associated to a coherent sheaf F on the surface, for the Bridgeland-King-Reid-Haiman equivalence (18). More precisely we will find a $\mathfrak{S}_n$-equivariant complex $\mathcal{C}^\bullet_F$ in $\mathbf{D}^b_{\mathfrak{S}n}(X^n)$ such that $\mathbf{\Phi}(F^{[n]}) \simeq \mathcal{C}^\bullet_F$. This result will allow us to compute the cohomology $H^*(X^{[n]}, F^{[n]})$ of the tautological sheaf $F^{[n]}$ on the Hilbert scheme, thus giving another proof of results by Danila-Brion (cf. [23]) and Ellingsrud-Goettsche-Lehn (cf. [33]).

3.1 Preliminary results

We begin by proving a preliminary result on the vanishing of the higher direct images $R^i p_* \mathcal{O}_{B^n}$ of the structural sheaf $\mathcal{O}_{B^n}$ of the isospectral Hilbert scheme for the blow-up projection:

$$p : B^n \longrightarrow X^n \ .$$

We first recall Yoneda's Lemma (cf. [117], [72]):

Lemma 3.1 (Yoneda)**.** *Let $\mathcal{C}$ a category. Let $\mathcal{C}^\vee$ the category of controvariant functors from $\mathcal{C}$ to the category* Sets *of sets:*

$$\mathcal{C}^\vee := \operatorname{Funct}(\mathcal{C}^{op}, \mathrm{Sets}) \ .$$

Then the Yoneda functor:

$$\begin{aligned} h : \mathcal{C} &\longrightarrow \mathcal{C}^\vee \\ X &\longmapsto \operatorname{Hom}_{\mathcal{C}}(-, X) \end{aligned}$$

is fully faithful.

Remark 3.2. By applying Yoneda Lemma to $\mathcal{C}^{op}$ we have a fully faithful functor:

$$\begin{aligned} h' : \mathcal{C}^{op} &\longrightarrow (\mathcal{C}^{op})^{\vee} \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}^{op}}(-, X) = \mathrm{Hom}_{\mathcal{C}}(X, -) \end{aligned}$$

Now $X \simeq Y$ in $\mathcal{C}$ if and only if $X \simeq Y$ in $\mathcal{C}^{op}$. Therefore $X \simeq Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X) \simeq \mathrm{Hom}_{\mathcal{C}}(-, Y)$ if and only if $\mathrm{Hom}_{\mathcal{C}}(X, -) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, -)$.

Proposition 3.3. *Let $p : B^n \longrightarrow X^n$ the blow-up of the union of the pairwise diagonals in X^n. Then:*

$$\mathbf{R}p_*\mathcal{O}_{B^n} \simeq \mathcal{O}_{X^n} \ .$$

Proof. We recall that the BKRH functor:

$$\mathbf{\Phi} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

is an equivalence. Therefore, for all $F^\bullet$, $G^\bullet \in \mathbf{D}^b(X^{[n]})$

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathbf{\Phi}(F^\bullet), \mathbf{\Phi}(G^\bullet)) \simeq \mathrm{Hom}_{\mathbf{D}^b(X^{[n]})}(F^\bullet, G^\bullet) \ .$$

Therefore, for all $G^\bullet \in \mathbf{D}^b(X^{[n]})$:

$$\begin{aligned} \mathrm{Hom}^i_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathbf{\Phi}(\mathcal{O}_{X^{[n]}}), \mathbf{\Phi}(G^\bullet)) &\simeq \mathrm{Hom}^i_{\mathbf{D}^b(X^{[n]})}(\mathcal{O}_{X^{[n]}}, G^\bullet) \\ &\simeq \mathrm{Ext}^i_{X^{[n]}}(\mathcal{O}_{X^{[n]}}, G^\bullet) \\ &\simeq \mathbb{H}^i(X^{[n]}, G^\bullet) \ . \end{aligned}$$

Now, by proposition 1.19

$$\mathbb{H}^i(X^{[n]}, G^\bullet) \simeq \mathbb{H}^i(X^n, \mathbf{\Phi}(G^\bullet))^{\mathfrak{S}_n}$$

and the last term is

$$\mathbb{H}^i(X^n, \mathbf{\Phi}(G^\bullet))^{\mathfrak{S}_n} \simeq \mathfrak{S}_n\mathrm{Ext}^i_{X^n}(\mathcal{O}_{X^n}, \mathbf{\Phi}(G^\bullet)) \simeq \mathrm{Hom}^i_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathcal{O}_{X^n}, \mathbf{\Phi}(G^\bullet)) \ .$$

Therefore for all $G^\bullet \in \mathbf{D}^b(X^n)$:

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathbf{\Phi}(\mathcal{O}_{X^{[n]}}), \mathbf{\Phi}(G^\bullet)) \simeq \mathrm{Hom}_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathcal{O}_{X^n}, \mathbf{\Phi}(G^\bullet))$$

and since every object in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ can be written as $\mathbf{\Phi}(G^\bullet)$ for some $G^\bullet \in \mathbf{D}^b(X^{[n]})$, because $\mathbf{\Phi}$ is an equivalence, the following functors are isomorphic:

$$\mathrm{Hom}_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathbf{\Phi}(\mathcal{O}_{X^{[n]}}), -) \simeq \mathrm{Hom}_{\mathbf{D}^b_{\mathfrak{S}_n}(X^n)}(\mathcal{O}_{X^n}, -) \ .$$

By Yoneda Lemma we obtain:

$$\mathbf{\Phi}(\mathcal{O}_{X^{[n]}}) \simeq \mathcal{O}_{X^n} \ ,$$

which is exactly:

$$\mathbf{R}p_* \circ q^*\mathcal{O}_{X^{[n]}} \simeq \mathbf{R}p_*\mathcal{O}_{B^n} \simeq \mathcal{O}_{X^n} \ .$$

□

We now come to the definition of tautological sheaf. Consider the diagram:

$$\begin{tikzcd} X^{[n]} \times X \arrow[r, "p_X"] \arrow[d, "p_{X^{[n]}}"'] & X \\ X^{[n]} \end{tikzcd}$$

Definition 3.4. Let X a smooth algebraic surface. Let F a coherent sheaf on X. The tautological sheaf $F^{[n]}$ on $X^{[n]}$ associated to the coherent sheaf F on X is the element:

$$F^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) = (p_{X^{[n]}})_*(\mathcal{O}_\Xi \otimes^L_{\mathcal{O}_{X^{[n]}\times X}} p_X^* F) \in \mathbf{D}^b(X^{[n]}) \ .$$

Remark 3.5. Let F be a coherent sheaf on the surface X. The tautological sheaf $F^{[n]}$ is a sheaf. It suffices to prove that

$$\mathrm{Tor}_i^{\mathcal{O}_{X^{[n]}\times X}}(\mathcal{O}_\Xi, p_X^* F) = 0 \quad \text{for all } i > 0. \tag{25}$$

For any coherent sheaf F, the sheaves $\mathcal{O}_\Xi$ and $p_X^* F$ are transversely supported, that is,

$$\mathrm{codim}_{X^{[n]}\times X}(\mathcal{O}_\Xi \otimes p_X^* F) = \mathrm{codim}_{X^{[n]}\times X}(\mathcal{O}_\Xi) + \mathrm{codim}_{X^{[n]}\times X}(p_X^* F) \ .$$

If F is now Cohen-Macaulay, they are transverse by proposition 2.5, therefore (25) follows. Hence (25) is true for any 0-dimensional coherent sheaf, because Cohen-Macaulay. If F is of dimension 1, it can be written as the extension:

$$0 \longrightarrow F_0 \longrightarrow F \longrightarrow F_1 \longrightarrow 0$$

with F_0 of dimension 0 (hence Cohen-Macaulay) and F_1 of dimension 1 without immersed points (hence, again, Cohen-Macaulay); the long exact Tor-sequence gives the transversality. If F is of dimension 2, we can write the exact sequence:

$$0 \longrightarrow T \longrightarrow F \longrightarrow G \longrightarrow 0$$

where T is the torsion subsheaf (of dimension ≤ 1) and G is torsion-free. Therefore $\mathcal{O}_\Xi$ and T are transverse. To see that $\mathcal{O}_\Xi$ and G are transverse, write the short exact sequence:

$$0 \longrightarrow G \longrightarrow G^{**} \longrightarrow Q \longrightarrow 0 \ .$$

Now the bidual G^{**} is locally free and Q is of dimension 0. The long exact Tor-sequence gives again the wanted transversality.

Remark 3.6. If E is a vector bundle on X of rank r, then $\mathcal{O}_\Xi \otimes p_X^* E$ is flat over Ξ. Since $p_{X^{[n]}}\big|_\Xi : \Xi \longrightarrow X^{[n]}$ is flat and finite of degree n over $X^{[n]}$,

$$E^{[n]} := (p_{X^{[n]}})_*(\mathcal{O}_\Xi \otimes p_X^* E) = (p_{X^{[n]}}\big|_\Xi)_*(p_X\big|_\Xi^* E)$$

is a vector bundle of rank nr over $X^{[n]}$.

Remark 3.7. The definition we have given is compatible with the definition of the functor

$$-^{[n]} : K(X) \longrightarrow K(X^{[n]})$$

given, for exemple, in [87]. Actually the functor:

$$\Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X^{[n]})$$

induces the functor $-^{[n]}$ in K-theory.

Let X a smooth algebraic surface, $X^{[n]}$ the Hilbert scheme of points over X, Ξ the universal family on $X^{[n]}$. We recall that $B^n \xrightarrow{p} X^n$ is the isospectral Hilbert scheme and Z is the universal family on B^n.

Proposition 3.8. *The fiber product $\Xi \times_{X^{[n]}} B^n$ is reduced and isomorphic to the isospectral universal family $Z \subseteq B^n \times X$.*

Proof. The reduced scheme $(\Xi \times_{X^{[n]}} B^n)_{\mathrm{red}}$ underlying the fiber product coincides with Z, since Z and $(\Xi \times_{X^{[n]}} B^n)_{\mathrm{red}}$ are two reduced schemes supported on the same points. Consider the flat base change:

$$\begin{tikzcd}
\Xi \times_{X^{[n]}} B^n \arrow[r] \arrow[d] & B^n \arrow[d, "q"] \\
\Xi \arrow[r, "p_{X^{[n]}}"] & X^{[n]}
\end{tikzcd}$$

$\Xi \times_{X^{[n]}} B^n$ is flat and finite of degree n over B^n (because Ξ is flat and finite of degree n over $X^{[n]}$) and of degree $n!$ over Ξ (because such is B^n over $X^{[n]}$). Therefore, by remark 1.32, it is Cohen-Macaulay. Since $\Xi \times_{X^{[n]}} B^n$ is generically reduced and cannot have immersed components (because Cohen-Macaulay), it is reduced. Therefore

$$\Xi \times_{X^{[n]}} B^n \simeq Z \ .$$

□

Definition 3.9. Let $X \longrightarrow Y$ a morphism of schemes, F a coherent sheaf on Y. We say that f is transverse to F if

$$L^i f^* F = \mathrm{Tor}_{-i}^{\mathcal{O}_Y}(F, \mathcal{O}_X) = 0 \quad \text{for } i < 0 \ .$$

Lemma 3.10. *Let $f : X \longrightarrow Y$ a morphism of schemes. Let F a coherent sheaf on Y transverse to f. If $\mathbf{R} f_* \mathcal{O}_X \simeq \mathcal{O}_Y$, then*

$$\mathbf{R} f_*(F \otimes_{\mathcal{O}_Y} \mathcal{O}_X) \simeq F \ .$$

Proof. The proof is almost immediate, once seen that $F \otimes_{\mathcal{O}_Y} \mathcal{O}_X \simeq f^* F$. Now, since f is transverse to F,

$$f^* F \simeq \mathbf{L} f^* F \ .$$

Therefore

$$\mathbf{R} f_*(F \otimes_{\mathcal{O}_Y} \mathcal{O}_X) \simeq \mathbf{R} f_*(\mathbf{L} f^* F) \simeq F \otimes^L_{\mathcal{O}_Y} \mathbf{R} f_*(\mathcal{O}_X)$$

by projection formula. The hypothesis $\mathbf{R} f_*(\mathcal{O}_X) \simeq \mathcal{O}_Y$ allows to conclude.

□

Proposition 3.11. *Consider the morphism:*

$$f := (p \times \mathrm{id}) \colon B^n \times X \longrightarrow X^n \times X \ .$$

Then the structural sheaf $\mathcal{O}_D$ of the scheme $D = \cup_{i=1}^n D_i$ is transverse to f.

Proof. We recall (cf. example 2.8) that the complex

$$\mathcal{K}^\bullet \colon 0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \longrightarrow \bigoplus_{|I|=2} \mathcal{O}_{D_I} \longrightarrow \ldots \longrightarrow \mathcal{O}_{D_{\{1,\ldots,n\}}} \longrightarrow 0$$

is a right resolution of the sheaf $\mathcal{O}_D$. In other words:

$$\mathcal{O}_D \simeq \mathcal{K}^\bullet \ .$$

We now compute $\mathbf{L} f^* \mathcal{O}_D \simeq \mathbf{L} f^*(\mathcal{K}^\bullet)$. The hyperderived spectral sequence:

$$E_1^{p,q} = L^q f^*(\mathcal{K}^p) = \mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{K}^p)$$

converges to

$$E_1^{p,q} \implies L^{p+q} f^*(\mathcal{K}^\bullet) .$$

Now

$$\mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{K}^p) \simeq \bigoplus_{|I|=p+1} \mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{O}_{D_I})$$

and D_I is the smooth intersection of $|I|$ transverse diagonals of the kind D_j. Hence $\mathrm{codim}_{X^n \times X} D_I = 2(p+1)$ and we can resolve $\mathcal{O}_{D_I}$ with a locally free resolution $R^\bullet$ of length $2(p+1)$. Therefore

$$\mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{O}_{D_I}) = H^q(R^\bullet \otimes_{\mathcal{O}_{X^n \times X}} \mathcal{O}_{B^n \times X}) .$$

Since $R^\bullet \otimes_{\mathcal{O}_{X^n \times X}} \mathcal{O}_{B^n \times X}$ is now a complex of locally free sheaves on $B^n \times X^n$ of length $2(p+1)$, whose cohomology is supported in $Z_I = f^{-1}(\mathcal{O}_{D_I})$, hence in codimension $p+2$ by example 2.9, we deduce, by Peskine-Szpiro lemma 2.4 that

$$\mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{O}_{D_I}) = 0 \quad \text{if } -q > 2(p+2) - (p+2) = p .$$

Therefore $E_1^{p,q} = \mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_{B^n \times X}, \mathcal{K}^p) = 0$ if $p+q<0$. This implies

$$L^{p+q} f^*(\mathcal{K}^\bullet) = L^{p+q} f^*(\mathcal{O}_D) = 0 \quad \text{if } p+q<0$$

Now, since $\mathcal{O}_D$ is a sheaf, we always have $L^i f^* \mathcal{O}_D = \mathrm{Tor}_{-i}^{\mathcal{O}_{X^n \times X}}(\mathcal{O}_D, \mathcal{O}_{B^n \times X}) = 0$ for $i > 0$. Therefore

$$f^* \mathcal{O}_D \simeq \mathbf{L} f^* \mathcal{O}_D .$$

□

Corollary 3.12. *Let $\tilde{Z} \subseteq B^n \times X$ the pullback $f^{-1}(D)$ of the subscheme D of $X^n \times X$. Therefore*

$$\mathbf{R} f_* \mathcal{O}_{\tilde{Z}} \simeq \mathcal{O}_D$$

Proof. By example 2.9 we know that

$$\mathcal{O}_{\tilde{Z}} = f^* \mathcal{O}_D .$$

Therefore:

$$\mathbf{R} f_* \mathcal{O}_{\tilde{Z}} \simeq \mathbf{R} f_* (f^* \mathcal{O}_D) .$$

Since, by the previous proposition, the sheaf $\mathcal{O}_D$ is transverse to f, we can conclude by lemma 3.10 and proposition 3.3.

□

Proposition 3.13. *Let M a smooth algebraic variety and Y a smooth subvariety. Let Y_1 and Y_2 two smooth subvarieties of Y, transverse in Y, such that the intersection $Y_1 \cap Y_2$ is smooth. Then there is a canonical isomorphism:*

$$\mathrm{Tor}_i(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \simeq \Lambda^i N^*_{Y/M}\Big|_{Y_1 \cap Y_2} .$$

Proof. Let us prove first the case $Y = Y_1 = Y_2$. We want to prove:

$$\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^i N^*_Y .$$

To obtain the isomorphism, we can write locally Y as the scheme of zeros of a section s of a vector bundle F, transverse to the zero section. The Koszul complex $K^\bullet(s)$ associated to s is a resolution of $\mathcal{O}_Y$. Therefore

$$\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y) \simeq H^{-i}(K^\bullet(s) \otimes \mathcal{O}_Y) \simeq \Lambda^i F^*\Big|_Y .$$

Now it is immediate to see that the exact sequence:

$$0 \longrightarrow TY \longrightarrow TM\Big|_Y \xrightarrow{\mathrm{d}s} F\Big|_Y \longrightarrow 0$$

allows us to identify the restriction $F\big|_Y$ of the bundle F to the normal bundle $N_{Y/M}$ of Y in M. Hence we get the isomorphism $\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \Lambda^i N^*_{Y/M}$ on the considered open affine neighbourhood. To see that these local isomorphisms glue together, let now s' a section of another vector bundle $F^{'}$, transverse to the zero section. We can find on an open subset where the two vector bundles are defined an isomorphism of vector bundles: $\varphi : F \longrightarrow F^{'}$ such that $\varphi s = s'$. The morphism φ induces a morphism between the Koszul complexes $K^\bullet(s)$ and $K^\bullet(s')$ and a commutative diagram of isomorphisms:

$$\begin{tikzcd}
\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y) \arrow[r] \arrow[d] & \Lambda^i F^{'*}\Big|_Y \arrow[d] \arrow[dl, "\varphi\big|_Y"'] \\
\Lambda^i F^*\Big|_Y \arrow[r] & \Lambda^i N^*_{Y/M}
\end{tikzcd}$$

This shows that the identification of $\mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y)$ with $\Lambda^i N^*_{Y/M}$ is canonical.

We come now to the case one of Y_1 and Y_2 is a proper subvarieties of Y. Let $r = \operatorname{codim}_M Y$, $r + s_i = \operatorname{codim}_M Y_i$, $i = 1, 2$. The question is again local. Suppose that Y is defined by the scheme of zeros of $u : M \longrightarrow \mathbb{C}^r$, and Y_i by the scheme of zeros of: $(u, s_i) : M \longrightarrow \mathbb{C}^{r+s_i}$ and all these sections are transverse to the zero section. Suppose also that the section (u, s_1, s_2), whose scheme of zeros is $Y_1 \cap Y_2$, is again transverse to the zero section. Then:

$$\mathrm{Tor}_i(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \simeq H^{-i}(K^\bullet(u, s_1) \otimes K^\bullet(u, s_2)) \ .$$

The Koszul complex $K^\bullet(u, s_i)$ is the tensor product: $K^\bullet(u) \otimes K^\bullet(s_i)$. By transversality the complex $K^\bullet(s_1, s_2)$ is a resolution of $\mathcal{O}_{Y_1 \cap Y_2}$ in Y. Therefore:

$$\mathrm{Tor}_i(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \simeq \mathrm{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Y)\Big|_{Y_1 \cap Y_2} \simeq \Lambda^i N^*_{Y/M}\Big|_{Y_1 \cap Y_2}$$

□

We will prove now that

$$\mathbf{R}f_*\mathcal{O}_Z \simeq \mathcal{O}_D \ . \tag{26}$$

It will turn out that this is the techical key result in order to compute the image of a tautological vector bundle for the BKRH equivalence. We will prove (26) by induction on n; hence we start with the simpler case $n = 2$.

3.2 The case $n = 2$

We will now study in details the case $n = 2$. We recall that we indicated with $\tilde{Z}$ the pull back of the scheme $D = \cup_{i=1}^n D_i = \cup_{i=1}^n \Delta_{i,n+1}$ in $X^n \times X$ for the map $(p \times \mathrm{id})$. We saw that $\tilde{Z}$ is not reduced and $\tilde{Z}_{\mathrm{red}} \simeq Z$, the universal family for the isospectral Hilbert scheme. The morphism $p : B^2 \longrightarrow X^2$ the blow-up of the diagonal $\Delta \subseteq X^2$: B^2 is then smooth. Let E the exceptional divisor $E = p^{-1}(\Delta)$. Consider $D_1 = \Delta_{1,3}$ and $D_2 = \Delta_{2,3}$ in $X^2 \times X$ and $Z_1 = f^{-1}(D_1)$ and $Z_2 = f^{-1}(D_2)$ in $B^2 \times X$. We already know by example 2.9 that Z_1 and Z_2 are not transverse:

$$3 = \operatorname{codim}_{B^2 \times X}(Z_1 \cap Z_2) < 2 + 2 = \operatorname{codim}_{B^2 \times X}(Z_1) + \operatorname{codim}_{B^2 \times X}(Z_2) \ ;$$

hence we expect

$$\mathrm{Tor}_1^{\mathcal{O}_{B^2 \times X}}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) \neq 0 \ .$$

Lemma 3.14. *The ideal $\mathcal{I}_Z$ of $Z \subseteq B^2 \times X$ in $\tilde{Z}$ is isomorphic to $\mathcal{O}_E(E)$; we have the exact sequence on $B^2 \times X$:*

$$0 \longrightarrow \mathcal{O}_E(E) \longrightarrow \mathcal{O}_{\tilde{Z}} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ . \tag{27}$$

Proof. If J_1 and J_2 are two ideals of a commutative ring A, we have:

$$\mathrm{Tor}_1(A/J_1, A/J_2) \simeq J_1 \cap J_2/J_1J_2\ .$$

Now we know that $\mathcal{I}_{\tilde{Z}} \simeq \mathcal{I}_{Z_1}\mathcal{I}_{Z_2}$, while $\mathcal{I}_Z = \mathcal{I}_{Z_1} \cap \mathcal{I}_{Z_2}$. It's then clear that the ideal of Z in $\tilde{Z}$ identifies with $\mathrm{Tor}_1(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})$. We will now see that it is isomorphic to $\mathcal{O}_E(E)$. By proposition 3.13, since $B^2 \times X$ and Z_1 and Z_2 are smooth, we have:

$$\mathrm{Tor}_1(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) \simeq N^*_{Y/B^2\times X}\Big|_{Z_1\cap Z_2}$$

where Y is a smooth subvariety of $B^2 \times X$ containing Z_1 and Z_2 as transverse subvarieties. To work out what this term is we can suppose that the base X^3 is affine: $X \simeq \mathbb{C}^2$. In the product $X^2 \times X \simeq (\mathbb{C}^2)^2 \times \mathbb{C}^2$ we can now imagine that the diagonal is the second factor; as a consequence we can take coordinates $(x, y, \alpha, \beta, z, w)$ in $(\mathbb{C}^2)^2 \times \mathbb{C}^2$ such that $\mathcal{I}_\Delta = (x, y)$, $\mathcal{I}_{\Delta_{13}} = (z, w)$, $\mathcal{I}_{\Delta_{23}} = (x-z, y-w)$. As a consequence,

$$\mathrm{Bl}_\Delta(X^2) \times X \simeq \mathrm{Bl}_0(\mathbb{C}^2) \times \mathbb{C}^2 \times \mathbb{C}^2 \simeq H \times \mathbb{C}^2 \times \mathbb{C}^2\ ,$$

where $H \simeq \mathrm{Bl}_0(\mathbb{C}^2)$ is the total space of the Hopf line bundle on $\mathbb{P}^1$: $H \simeq \mathcal{O}_{\mathbb{P}_1}(-1)$. Here the exceptional divisor E on $\mathrm{Bl}_\Delta(X^2)$ is $\mathbb{P}_1 \times \mathbb{C}^2$. Let $\lambda \in H^0(\mathcal{O}(E))$ and u, v homogeneous coordinates in $H^0(\mathcal{O}(-E))$: we can set $x = \lambda u$, $y = \lambda v$. Then

$$\mathcal{I}_{Z_1} = (z, w)\ , \quad \mathcal{I}_{Z_2} = (\lambda u - z, \lambda v - w)\ .$$

We can easily see that Z_1 and Z_2 are transversely immersed in the smooth variety Y, defined by the ideal is $\mathcal{I}_Y = (zv - wu)$. Y is a divisor in $H \times \mathbb{C}^2 \times \mathbb{C}^2$ and corresponds exactly to the pull-back of another copy of the Hopf bundle $H^{'} \subseteq \mathbb{P}_1 \times \mathbb{C}^2$ via the projection:

$$\begin{tikzcd} H \times \mathbb{C}^2 \times \mathbb{C}^2 \arrow[r, "\pi_H \times \mathrm{pr}_3"] & \mathbb{P}_1 \times \mathbb{C}^2 \\ Y \simeq \pi^* H^{'} \arrow[r] \arrow[u, hook] & H^{'} \arrow[u, hook] \end{tikzcd}$$

where $\pi_H : H \times \mathbb{C}^2 \longrightarrow \mathbb{P}_1$ is the projection on $\mathbb{P}_1$. Now the normal bundle $N_{Y/H\times\mathbb{C}^2\times\mathbb{C}^2}$ is naturally identified with the pull-back π^*Q of the quotient:

$$0 \longrightarrow H^{'} \longrightarrow \mathcal{O}^2_{\mathbb{P}_1} \longrightarrow Q \longrightarrow 0$$

of $\mathcal{O}^2_{\mathbb{P}_1}$ by the Hopf bundle $H^{'}$. Since line bundles on the projective space are classified by their degree, (cf. [54]) we necessarily have $Q \simeq (H^{'})^* \simeq \mathcal{O}_{\mathbb{P}_1}(1)$. Therefore:

$$N^*_{Y/H\times\mathbb{C}^2\times\mathbb{C}^2}\Big|_{Z_1\cap Z_2} \simeq (\pi^*Q)^*\Big|_E \simeq \pi^*H^{'}\Big|_E \simeq \mathcal{O}_E(E)\ .$$

because the intersection $Z_1 \simeq Z_2$ is isomorphic to E via the projection onto B^2.

□

Corollary 3.15. *Let* $p\colon B^2 \longrightarrow X^2$ *the blow-up of the diagonal and* $D_i = \Delta_{i,3} \subseteq X^2 \times X$, $i = 1, 2$, $\tilde{Z} = (p \times \mathrm{id})^{-1}(D)$ *and* $Z = \tilde{Z}_{\mathrm{red}}$ *the isospectral universal family. Then*

$$\mathbf{R}(p \times \mathrm{id})_* \mathcal{O}_Z \simeq \mathcal{O}_D \ .$$

Proof. It is immediate from the exact sequence (27) and the well known fact that for a smooth blow-up $f\colon \mathrm{Bl}_Y(M) \longrightarrow M$ of a smooth subvariety Y in a smooth variety M we have (cf. [2]):

$$\mathbf{R}f_* \mathcal{O}_E(kE) = 0 \quad \text{if } 0 < k < \operatorname{codim}_M Y \ .$$

□

3.3 The general case

We now pass to the general case

$$f\colon B^n \times X \longrightarrow X^n \times X \ .$$

We first prove that $f_* \mathcal{O}_Z \simeq \mathcal{O}_D$ and, in a second time, that the higher direct images of the sheaf $\mathcal{O}_Z$ vanish. We will recall a fundamental result of local cohomology (see [64], [58]). Let X a locally noetherian scheme, Y a closed subscheme, $U = X \setminus Y$ and $j\colon U \hookrightarrow X$ the open immersion of U in X. We recall that, for a coherent sheaf F on X, the sheaf $\mathcal{H}^0_Y(F)$ is defined by the exact sequence:

$$0 \longrightarrow \mathcal{H}^0_Y(F) \longrightarrow F \longrightarrow j_* j^* F \ .$$

The i-th right derived functor of $\mathcal{H}^0_Y$ is denoted with $\mathcal{H}^i_Y$ and it is called the i-th sheaf of local cohomology with support in Y.

Theorem 3.16. *Let* X *a locally noetherian scheme,* Y *a closed subscheme. Let* F *a coherent sheaf on* X*. The following statements are equivalent:*

1. *For all* $\mathfrak{p} \in Y$, $\operatorname{depth} F_{\mathfrak{p}} \geq k$.
2. $\mathcal{H}^i_Y(F) = 0$ *for all* $i < k$.

If F is Cohen-Macauley we can rephrase the first condition by requiring:

$$1'. \quad \textit{For all } \mathfrak{p} \in Y,\ \dim_{\mathcal{O}_{X,\mathfrak{p}}} F_{\mathfrak{p}} \geq k \quad .$$

Therefore, for a Cohen-Macauley coherent sheaf F on X,

$$\dim_{\mathcal{O}_{X,\mathfrak{p}}} F_{\mathfrak{p}} \geq 2 \quad \text{for all } \mathfrak{p} \in Y \quad \Longleftrightarrow \quad F \simeq j_* j^* F \tag{28}$$

where $j\colon U \longrightarrow X$ is the open immersion of the complementary of Y.

Proposition 3.17. *Let* $f\colon B^n \times X \longrightarrow X^n \times X$ *and* Z *the isospectral universal family. Then*

$$f_* \mathcal{O}_Z \simeq \mathcal{O}_D \ .$$

Proof. $B^n \times X$ is a Cohen-Macauley normal variety, and $Z \subseteq B^n \times X$ is a Cohen-Macauley subvariety of codimension 2. Let now $\tilde{Y} \subseteq X^n$ be the scheme-theoretic union of all diagonals of length 3: $\tilde{Y} = \cup_{|I|=3} \Delta_I$, and let Y be its pull back on $X^n \times X$. It's clear that $\tilde{Y}$ and Y are of codimension 4 in X^n and $X^n \times X$. Moreover $Y \cap D$ is of codimension 4 in D. We will indicate $X^n_* := X^n \setminus \tilde{Y}$. Then $X^n \times X \setminus Y = X^n_* \times X$. We denote $\tilde{W} := p^{-1}(\tilde{Y})$, $B^n_* := p^{-1}(X^n_*) = B^n \setminus \tilde{W}$. Let $W = f^{-1}(Y)$. We remark that $\tilde{W}$ and W are of codimension 2 in B^n and $B^n \times X$ respectively, and $W \cap Z$ is of codimension 2 in Z. Now, since $\mathcal{O}_Z$ is Cohen-Macauley and for all $\mathfrak{p} \in W \cap Z$ the dimension

$$\dim_{B^n \times X} \mathcal{O}_{Z,\mathfrak{p}} \geq 2$$

we have by (28) and the facts on local cohomology that

$$\mathcal{O}_Z \simeq j_{W*} j_W{}^* \mathcal{O}_Z$$

where $j_W : B^n_* \times X \hookrightarrow B^n \times X$ is the open immersion of the complementary of W. Over $B^n_* \times X$ we have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{Z_i} \longrightarrow \bigoplus_{i<j} \mathcal{O}_{Z_i \cap Z_j} \longrightarrow 0$$

and the sheaves $\mathcal{O}_{Z_i \cap Z_j}$ are isomorphic (via the projection on B^n_*) to $\mathcal{O}_{E_{ij}}$, where E_{ij} are the irreducible components of the exceptional divisor E on B^n_*. Since B^n_* is a smooth blow-up of pairwise *disjoint* diagonals in X^n_*, over $B^n_* \times X$ we can treat the situation exactly as in the case $n = 2$: therefore the statement of the proposition is true on $X^n_* \times X$. Consider now the fiber product

$$\begin{tikzcd}
B^n \times X \arrow[r, "f"] & X^n \times X \\
B^n_* \times X \arrow[u, hook, "j_W"] \arrow[r, "f|_{B^n_* \times X}"'] & X^n_* \times X \arrow[u, hook, "j_Y"']
\end{tikzcd}$$

We have:

$$f_* \mathcal{O}_Z \simeq f_* j_{W*} j_W^* \mathcal{O}_Z \simeq j_{Y*} (f|_{B^n_* \times X})_* j_W^* \mathcal{O}_Z = j_{Y*} j_Y^* \mathcal{O}_D$$

It suffices to show that $j_{Y*} j_Y^* \mathcal{O}_D \simeq \mathcal{O}_D$. Applying the functor $j_{Y*} j_Y^*$ to the first three terms of the Čech complex, and recalling that $j_{Y*} j_Y^*$ is left exact we get the following morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_D \arrow[r] \arrow[d] & \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \arrow[r] \arrow[d] & \bigoplus_{|I|=2} \mathcal{O}_{D_I} \arrow[d] \\
0 \arrow[r] & j_{Y*} j_Y^* \mathcal{O}_D \arrow[r] & \bigoplus_{i=1}^{n} j_{Y*} j_Y^* \mathcal{O}_{D_i} \arrow[r] & \bigoplus_{|I|=2} j_{Y*} j_Y^* \mathcal{O}_{D_I}
\end{tikzcd}$$

Now $\mathcal{O}_{D_i}$ and $\mathcal{O}_{D_I}$ are Cohen-Macauley coherent sheaves, because structural sheaves of smooth subvarieties and, since $\operatorname{codim}_{D_i} D_i \cap Y = 4$ and $\operatorname{codim}_{D_I} D_I \cap Y = 2$ for $|I| = 2$, we have, by (28)

$$\mathcal{H}^i_Y(\mathcal{O}_{D_i}) = 0 = \mathcal{H}^i_Y(\mathcal{O}_{D_I}) \quad \text{if } i < 2 \;,$$

which means

$$\mathcal{O}_{D_i} \simeq j_{Y*} j_Y^* \mathcal{O}_{D_i} \quad \text{and} \quad \mathcal{O}_{D_I} \simeq j_{Y*} j_Y^* \mathcal{O}_{D_I} \;.$$

Since the last two vertical arrows in the previous diagram are isomorphisms, we get that the canonical morphism:

$$\mathcal{O}_D \longrightarrow j_{Y*} j_Y^* \mathcal{O}_D$$

is an isomorphism.

□

Before proving the next proposition, the technical heart of this chapter, we recall some facts on base change formulas.

Proposition 3.18. *Consider the fiber product:*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "u"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where X,Y,X', Y' *are noetherian* k*-schemes and* f *and* f' *are proper morphism. Let* $\mathcal{F}$ *a coherent* $\mathcal{O}_X$*-module, and* $\mathcal{G}$ *a coherent* $\mathcal{O}_{Y'}$*-module. Then:*

$$\mathbf{R}f'_*\big(\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G}\big) \simeq \mathbf{R}f_*\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G} \tag{29}$$

where $\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G}$ *is naturally seen as an element of* $\mathbf{D}(X')$*.*

Proof. It is a particular case of EGA III. Étude cohomologique des faisceaux cohérents, Seconde Partie, Proposition 6.9.8. [57].

Remark 3.19.

- Suppose Y' is flat over Y, then X' is flat over X. If $\mathcal{G} = \mathcal{O}_{Y'}$, we get the known formula for base flat change ([65]):

$$\mathbf{R}f'_*(u^*\mathcal{F}) \simeq v^*\mathbf{R}f_*\mathcal{F} \ .$$

- If X is flat over Y and $\mathcal{G} = \mathcal{O}_{Y'}$, we get:

$$\mathbf{R}f'_*(\mathbf{L}u^*\mathcal{F}) \simeq \mathbf{L}v^*(\mathbf{R}f_*\mathcal{F}) \ .$$

- If $Y = Y'$ and $X = X'$ formula (29) becomes the common projection formula:

$$\mathbf{R}f_*(\mathbf{L}f^*\mathcal{G} \otimes^L \mathcal{F}) \simeq \mathcal{G} \otimes^L \mathbf{R}f_*\mathcal{F} \ .$$

Notation 3.20. We explain here the slight abuse of notation we will be making for all the proof of next proposition. In the hypothesis of proposition 3.18, we will consider the sheaf $\mathcal{G}$ as a *sheaf on* Y and we will denote with $\mathbf{L}f^*\mathcal{G}$ the element $\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} \mathcal{G}$. It is canonically a complex of sheaves on X', but we will consider it *as a complex of sheaves on* X. If v is flat, the complex $\mathbf{L}f^*\mathcal{G}$ coincide with the complex $\mathbf{L}(f')^*\mathcal{G}$ (on X' and on X); if $\mathcal{G}$ is flat over Y, then $\mathbf{L}f^*\mathcal{G}$ is isomorphic to $(f')^*\mathcal{G}$. Moreover, if $\mathcal{H}$ is a sheaf on X', we will consider it as a sheaf on X and we will denote with $\mathbf{R}f_*\mathcal{H}$ the element $\mathbf{R}f'_*\mathcal{H}$, seen as a complex of sheaves on Y. In these notations base change formula (29) becomes simply a projection formula:

$$\mathbf{R}f_*(\mathbf{L}f^*\mathcal{G} \otimes^L_{\mathcal{O}_X} \mathcal{F}) \simeq \mathcal{G} \otimes^L_{\mathcal{O}_Y} \mathbf{R}f_*\mathcal{F}$$

and if v is flat the complex denoted with $\mathbf{L}f^*\mathcal{G}$ coincides exactly with $\mathbf{L}(f')^*\mathcal{G}$.

Proposition 3.21. *Let* $p : B^n \longrightarrow X^n$ *the blow-up of the pairwise diagonals. Let* $f := (p \times \mathrm{id}) : B^n \times X \longrightarrow X^n \times X$*. Then*

$$\mathbf{R}f_*\mathcal{O}_Z \simeq \mathcal{O}_D \ .$$

Proof. In proposition 3.17 we proved that $f_*\mathcal{O}_Z \simeq \mathcal{O}_D$. It remains to prove: $R^if_*\mathcal{O}_Z = 0$ for all $i > 0$. We prove the proposition by induction on n. The case $n = 2$ was previously proven. Suppose the proposition is true for $n \geq 2$. Consider the flat families ζ_B, ξ_B, η_B on $B^{n+1,n} \times X$, defined in remark 1.37 as the pull-back on $B^{n+1,n} \times X$ of the correspondent families on $X^{[n+1,n]} \times X$. We will consider the sheaves $\mathcal{O}_{\zeta_B}$, $\mathcal{O}_{\xi_B}$, $\mathcal{O}_{\eta_B}$ *as flat sheaves over* $B^{n+1,n}$, following the abuse of notation previously explained. More precisely, consider the morphisms:

$$B^{n+1,n} \xrightarrow{v} B^{n+1} \quad ; \quad B^{n+1,n} \xrightarrow{u} B^{n} \quad ; \quad B^{n+1,n} \xrightarrow{t} X$$

Then the three families above can be defined as the pull-back of the isospectral universal families:

$$\begin{aligned}
\mathcal{O}_{\zeta_B} &= v^*\mathcal{O}_{Z_{n+1}} = \mathcal{O}_{B^{n+1,n}} \otimes_{\mathcal{O}_{B^{n+1}}} \mathcal{O}_{Z_{n+1}} \\
\mathcal{O}_{\xi_B} &= u^*\mathcal{O}_{Z_{n}} = \mathcal{O}_{B^{n+1,n}} \otimes_{\mathcal{O}_{B^{n}}} \mathcal{O}_{Z_{n}} \\
\mathcal{O}_{\eta_B} &= t^*\mathcal{O}_{\Delta} = \mathcal{O}_{B^{n+1,n}} \otimes_{\mathcal{O}_{X}} \mathcal{O}_{\Delta}
\end{aligned}$$

where again $\mathcal{O}_{Z_{n+1}}$, $\mathcal{O}_{Z_n}$, $\mathcal{O}_{\Delta}$ are regarded as flat sheaves over B^{n+1}, B^n, X respectively. Consider the exact sequence (17):

$$0 \longrightarrow \mathcal{O}_{\zeta_B} \longrightarrow \mathcal{O}_{\xi_B} \oplus \mathcal{O}_{\eta_B} \longrightarrow \mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} \longrightarrow 0 \tag{30}$$

where the sheaf $\mathcal{O}_{\eta_B}\big|_{\mathcal{E}}$ is exactly the sheaf: $\mathcal{O}_{\eta_B} \otimes_{\mathcal{O}_{B^{n,n+1}}} \mathcal{O}_{\mathcal{E}}$. The principle of the proof is simple: consider the diagram:

$$\begin{tikzcd}
 & B^{n+1,n} \arrow[rr, "v"] \arrow[d, "u\times t"] \arrow[drr, "\varphi"] \arrow[ddl, bend right, "u"'] \arrow[dddr, "t"] & & B^{n+1} \arrow[d, "p_{n+1}"] & \\
 & B^n \times X \arrow[rr, "f"] \arrow[dl, "g"] \arrow[ddr, "h"] & & X^n \times X \arrow[dl, "\tilde{g}"] \arrow[ddr, "\tilde{h}"] & \\
B^n \arrow[rr, "p_n"] & & X^n & & \\
 & & X \arrow[rr, "\mathrm{id}"] & & X
\end{tikzcd}$$

We apply the functor

$$\begin{aligned}
\mathbf{R}\varphi_* &\simeq \mathbf{R}f_* \circ \mathbf{R}(u\times t)_* \\
&= \mathbf{R}{p_{n+1}}_* \circ \mathbf{R}v_* \,.
\end{aligned}$$

to the exact sequence (30). We get a distinguished triangle:

$$\mathbf{R}\varphi_*\mathcal{O}_{\zeta_B} \longrightarrow \mathbf{R}\varphi_*\mathcal{O}_{\xi_B} \oplus \mathbf{R}\varphi_*\mathcal{O}_{\eta_B} \longrightarrow \mathbf{R}\varphi_*\mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} \longrightarrow \mathbf{R}\varphi_*\mathcal{O}_{\zeta_B}[1] \tag{31}$$

We now compute the two central terms by going down $B^{n+1,n} \xrightarrow{\varphi} X^n \times X$ clockwise, and the extremal ones by going counterclockwise.

$$\mathbf{R}\varphi_*\mathcal{O}_{\xi_B} = \mathbf{R}\varphi_*(u^*\mathcal{O}_{Z_n}) = \mathbf{R}\varphi_*\big((u\times t)^* g^*\mathcal{O}_{Z_n}\big) \,.$$

We remark that, since $\mathcal{O}_{Z_n}$ is flat over B^n, and g is a flat morphism, then $g^*\mathcal{O}_{Z_n}$ is flat over $B^n \times X$. Then

$$(u\times t)^* g^*\mathcal{O}_{Z_n} \simeq \mathbf{L}(u\times t)^* g^*\mathcal{O}_{Z_n} \,.$$

We can apply the projection formula to obtain:

$$\begin{aligned}
\mathbf{R}\varphi_* \circ \mathbf{L}(u\times t)^* g^*\mathcal{O}_{Z_n} &\simeq \mathbf{R}f_* \circ \mathbf{R}(u\times t)_* \left(\mathbf{L}(u\times t)^* g^*\mathcal{O}_{Z_n}\right) \\
&\simeq \mathbf{R}f_* \left(g^*\mathcal{O}_{Z_n} \otimes^{L}_{\mathcal{O}_{B^n\times X}} \mathbf{R}(u\times t)_*\mathcal{O}_{B^{n+1,n}}\right) \\
&\simeq \mathbf{R}f_* g^*\mathcal{O}_{Z_n}
\end{aligned}$$

because $\mathbf{R}(u \times t)_* \mathcal{O}_{B^{n+1,n}} \simeq \mathcal{O}_{B^n \times X}$ by (15). Now g is a flat morphism: by flat base change we get easily:

$$\mathbf{R}\varphi_* \mathcal{O}_{\xi_B} \simeq \mathbf{R}f_* g^* \mathcal{O}_{Z_n} \simeq \tilde{g}^* \mathbf{R}p_{n*} \mathcal{O}_{Z_n} \simeq \tilde{g}^* \mathcal{O}_D$$

by induction hypothesis. The sheaf $\tilde{g}^* \mathcal{O}_D$, without any abuse of notation, is isomorphic to ${\mathrm{pr}_{13}}^* \mathcal{O}_D$ as a structural sheaf of a subscheme of $X^n \times X \times X$, where $\mathrm{pr}_{13} : X^n \times X \times X \longrightarrow X^n \times X$ is the projection onto the first and third factor.

Let us now compute $\mathbf{R}\varphi_* \mathcal{O}_{\eta_B}$. The computation is analogous to the previous one:

$$\mathbf{R}\varphi_* \mathcal{O}_{\eta_B} = \mathbf{R}f_* \circ \mathbf{R}(u \times t)_* \left((u \times t)^* h^* \mathcal{O}_\Delta \right) \ .$$

Remembering that $h^* \mathcal{O}_\Delta$ is flat over $B^n \times X$ (because $\mathcal{O}_\Delta$ is flat over X and h is flat) and applying the projection formula, we get:

$$\begin{aligned} \mathbf{R}\varphi_* \mathcal{O}_{\eta_B} &\simeq \mathbf{R}f_* \left(h^* \mathcal{O}_\Delta \otimes^L \mathbf{R}(u \times t)_* \mathcal{O}_{B^{n+1,n}} \right) \\ &\simeq \mathbf{R}f_* h^* \mathcal{O}_\Delta \end{aligned}$$

again because $\mathbf{R}(u \times t)_* \mathcal{O}_{B^{n+1,n}} \simeq \mathcal{O}_{B^n \times X}$ by (15). We have:

$$h^* \mathcal{O}_\Delta \simeq f^* \tilde{h}^* \mathcal{O}_\Delta$$

with $\tilde{h}^* \mathcal{O}_\Delta$ again flat over $X^n \times X$. Therefore, projection formula yields:

$$\mathbf{R}\varphi_* \mathcal{O}_{\eta_B} \simeq \mathbf{R}f_* h^* \mathcal{O}_\Delta \simeq \mathbf{R}f_* (f^* \tilde{h}^* \mathcal{O}_\Delta) \simeq \tilde{h}^* \mathcal{O}_\Delta \otimes^L \mathbf{R}f_* \mathcal{O}_{B^n \times X} \simeq \tilde{h}^* \mathcal{O}_\Delta \ ,$$

because $\mathbf{R}f_* \mathcal{O}_{B^n \times X} \simeq \mathcal{O}_{X^n \times X}$ by proposition 3.3. The $\tilde{h}^* \mathcal{O}_\Delta$ sheaf can be seen, as structural sheaf of a subscheme of $X^n \times X \times X$, without any abuse of notation, as $\mathrm{pr}_{23}^* \mathcal{O}_\Delta$, where $\mathrm{pr}_{23} : X^n \times X \times X \longrightarrow X \times X$ is the projection onto the second and the third factor and $\mathcal{O}_\Delta$ is seen as a subsheaf of $\mathcal{O}_{X \times X}$; in other words, $\tilde{h}^* \mathcal{O}_\Delta \simeq \mathcal{O}_{X^n} \boxtimes \mathcal{O}_\Delta$ over $X^n \times X \times X$.

Let us compute the final term $\mathbf{R}\varphi_* \mathcal{O}_{\eta_B}\Big|_{\mathcal{E}}$. We can see the sheaf $\mathcal{O}_{\eta_B}\Big|_{\mathcal{E}}$ as

$$\mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} \simeq \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{B^{n+1,n}}} t^* \mathcal{O}_\Delta \simeq \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{O}_\Delta \simeq \mathcal{O}_{\mathcal{E}} \otimes^L_{\mathcal{O}_X} \mathcal{O}_\Delta \ .$$

since $\mathcal{O}_\Delta$ is flat over X. Therefore:

$$\begin{aligned} \mathbf{R}\varphi_* \mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} &\simeq \mathbf{R}f_* \mathbf{R}(u \times t)_* \left(\mathbf{L}(u \times t)^* h^* \mathcal{O}_\Delta \otimes^L_{\mathcal{O}_{B^{n+1,n}}} \mathcal{O}_{\mathcal{E}} \right) \\ &\simeq \mathbf{R}f_* (h^* \mathcal{O}_\Delta \otimes^L_{\mathcal{O}_{B^n \times X}} \mathbf{R}(u \times t)_* \mathcal{O}_{\mathcal{E}}) \end{aligned}$$

by projection formula. Now $\mathbf{R}(u \times t)_* \mathcal{O}_{\mathcal{E}} \simeq \mathcal{O}_{Z_n}$, by (15). Therefore:

$$\begin{aligned} \mathbf{R}\varphi_* \mathcal{O}_{\eta_B}\Big|_{\mathcal{E}} &\simeq \mathbf{R}f_* (h^* \mathcal{O}_\Delta \otimes^L_{\mathcal{O}_{B^n \times X}} \mathcal{O}_{Z_n}) \\ &\simeq \mathbf{R}f_* (f^* \tilde{h}^* \mathcal{O}_\Delta \otimes^L_{\mathcal{O}_{B^n \times X}} \mathcal{O}_{Z_n}) \\ &\simeq \tilde{h}^* \mathcal{O}_\Delta \otimes_{\mathcal{O}_{X^n \times X}} \mathbf{R}f_* \mathcal{O}_{Z_n} \\ &\simeq \tilde{h}^* \mathcal{O}_\Delta \otimes_{\mathcal{O}_{X^n \times X}} \mathcal{O}_D \simeq \tilde{h}^* \mathcal{O}_\Delta\Big|_D \end{aligned}$$

where we used again projection formula and the induction hypothesis. The distinguished triangle (31) becomes:

$$\mathbf{R}\varphi_* \mathcal{O}_{\zeta_B} \longrightarrow \tilde{g}^* \mathcal{O}_D \oplus \tilde{h}^* \mathcal{O}_\Delta \longrightarrow \tilde{h}^* \mathcal{O}_\Delta\Big|_D \longrightarrow \mathbf{R}\varphi_* \mathcal{O}_{\zeta_B}[1] \ .$$

Now, counterclockwise:

$$\mathbf{R}\varphi_*\mathcal{O}_{\zeta_B} \simeq \mathbf{R}p_{n+1*}\mathbf{R}v_*(v^*\mathcal{O}_{Z_{n+1}})$$

and since $\mathcal{O}_{Z_{n+1}}$ is flat over B^{n+1}, projection formula gives:

$$\mathbf{R}\varphi_*\mathcal{O}_{\zeta_B} \simeq \mathbf{R}p_{n+1*}(\mathcal{O}_{Z_{n+1}} \otimes^L_{\mathcal{O}_{B^{n+1}}} \mathbf{R}v_*\mathcal{O}_{B^{n+1,n}}) \simeq \mathbf{R}p_{n+1*}\mathcal{O}_{Z_{n+1}}$$

because $\mathbf{R}v_*\mathcal{O}_{B^{n+1,n}} \simeq \mathcal{O}_{B^{n+1}}$ by (14). Therefore, the distinguished triangle (31) becomes:

$$\mathbf{R}p_{n+1*}\mathcal{O}_{Z_{n+1}} \longrightarrow \tilde{g}^*\mathcal{O}_D \oplus \tilde{h}^*\mathcal{O}_\Delta \longrightarrow \tilde{h}^*\mathcal{O}_\Delta\Big|_D \longrightarrow \mathbf{R}p_{n+1*}\mathcal{O}_{Z_{n+1}}[1]\ .$$

Now, since the second arrow is surjective, we get immediately from the long exact cohomology sequence that:

$$R^i p_{n+1*}\mathcal{O}_{Z_{n+1}} \simeq 0\ .$$

In the abuse of notations we explained before, this is equivalent to $R^i f_*\mathcal{O}_{Z_{n+1}} = 0$ for all $i > 0$, where $f : B^{n+1} \times X \longrightarrow X^{n+1} \times X$.

□

We have just proved the main technical point. We are now ready to compute the image $\mathbf{\Phi}(F^{[n]})$ of a tautological sheaf for the Bridgeland-King-Reid-Haiman equivalence:

$$\mathbf{\Phi} = \Phi^{\mathcal{O}_{B^n}}_{X^{[n]}\to X^n} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)\ .$$

We will need the following fact on the behaviour of Fourier-Mukai functors under composition (see [93] [67]).

Proposition 3.22. *Let X, Y, Z three algebraic varieties and $P^\bullet \in \mathbf{D}(X \times Y)$, $Q^\bullet \in \mathbf{D}(Y \times Z)$ given kernels. Consider the Fourier-Mukai functors of kernels $P^\bullet$ and $Q^\bullet$ respectively:*

$$\Phi^{P^\bullet}_{X\to Y} \quad \text{and} \quad \Phi^{Q^\bullet}_{Y\to Z}\ .$$

Then their composition is the Fourier-Mukai functor:

$$\Phi^{Q^\bullet}_{Y\to Z} \circ \Phi^{P^\bullet}_{X\to Y} \simeq \Phi^{P^\bullet * Q^\bullet}_{X\to Z}$$

where $P^\bullet * Q^\bullet$ is the kernel:

$$P^\bullet * Q^\bullet := \mathbf{R}\pi_{X,Z*}(\pi^*_{X,Y}P^\bullet \otimes^L_{\mathcal{O}_{X\times Y\times Z}} \pi^*_{Y,Z}Q^\bullet) \simeq \mathbf{R}\pi_{X,Z*}(P^\bullet \otimes^L_{\mathcal{O}_Y} Q^\bullet)$$

where $\pi_{X,Y}$, $\pi_{Y,Z}$, $\pi_{X,Z}$ are the projections:

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "\pi_{X,Y}"'] \arrow[d, "\pi_{X,Z}"] \arrow[dr, "\pi_{Y,Z}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

□

The complex $\mathcal{C}^\bullet_F$. We first introduce a complex of equivariant sheaves $\mathcal{C}^\bullet_F$ in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ and some notations. Let $\{1,\dots,n\}$ the set of the first positive n natural numbers. For every subset $J \subseteq \{1,\dots,n\}$, $|J| = p$, $p \geq 1$, we will denote with $p_J : X^n \longrightarrow X^J$ the projection onto the factors $X^{j_1} \times \cdots \times X^{j_p}$, if $J = \{j_1,\dots,j_p\}$. If $|J| \geq 2$, we will indicate with Δ_J the pull-back of the small diagonal in X^J via the projection p_J: $\Delta_J = p_J^{-1}(\Delta_{j_1,\dots,j_p})$. We set the convention $\Delta_J := X^n$ if $|J| = 1$, and $\Delta := \Delta_{\{1,\dots,n\}}$. Let now F a coherent sheaf on X. We will denote with F_J the sheaf $p_J^*(j_{J*}F)$, where j_J is the diagonal immersion of X into the small diagonal of X^J. We now begin building the complex $\mathcal{C}^\bullet_F$: Let

$$\mathcal{C}^p_F := \bigoplus_{|J|=p+1} F_J \ .$$

The differentials

$$\partial^p_F : \mathcal{C}^p_F \longrightarrow \mathcal{C}^{p+1}_F$$

are defined by:

$$\partial^p_F(x) := \sum_{i \in J} \varepsilon_{i,J} x_{J\setminus\{i\}}|_{\Delta_J}$$

where x is a local section of F_J and where $\varepsilon_{i,J}$ is the sign:

$$\varepsilon_{i,J} := (-1)^{\sharp\{b \in J | b < i\}} \ .$$

We will now endow the complex $(\mathcal{C}^\bullet_F, \partial^\bullet_F)$ with a $\mathfrak{S}_n$-linearization, in such a way that it becomes a complex of $\mathfrak{S}_n$-equivariant sheaves and it can be seen in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$. Let $\sigma \in \mathfrak{S}_n$ and $\sigma_* : X^n \longrightarrow X^n$ the permutation of the factors given by:

$$\sigma_*(x_1,\dots,x_n) \longrightarrow (x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)}) \ .$$

We have the following straightforward

Lemma 3.23.

$$\sigma_*(\Delta_J) \simeq \Delta_{\sigma(J)} \ .$$

□

Therefore we can give $\mathcal{C}^\bullet_F$ a natural $\mathfrak{S}_n$-linearization:

$$(\sigma.x)_J := \varepsilon_{\sigma,J}\sigma_* x_{\sigma^{-1}(J)}$$

where $\varepsilon_{\sigma,J}$ is the signature of the only permutation $\tau \in \mathfrak{S}_n$ such that $\sigma^{-1}\tau$ is strictly increasing. The sign $\varepsilon_{\sigma,J}$ is necessary to make the differential $\partial^p_F : \mathcal{C}^p_F \longrightarrow \mathcal{C}^{p+1}_F$ $\mathfrak{S}_n$-equivariant for this action. We have $\mathcal{C}^0_F \simeq \oplus_{i=1}^n F_i$ where $F_i = p_i^*(F)$, with $p_i : X^n \longrightarrow X$ the projection on the i-th factor. When $F = \mathcal{O}_X$, then $\mathcal{C}^0 \simeq \mathcal{O}_{X^n} \otimes_{\mathbb{C}} R$, where R is the natural representation of the symmetric group $\mathfrak{S}_n$: identifying R with $\mathbb{C}^n$, the group $\mathfrak{S}_n$ acts permutating the basis vectors. Moreover $\mathcal{C}^{n-1}_F \simeq F_\Delta \otimes \varepsilon_n$, where ε_n is the alternating representation of $\mathfrak{S}_n$ of dimension 1.

Example 3.24. For $n = 3$ the complex $\mathcal{C}^\bullet_F$ is given by:

$$0 \longrightarrow \bigoplus_{i=1}^n F_i \xrightarrow{\partial^0_F} F_{\Delta_{12}} \oplus F_{\Delta_{13}} \oplus_{\Delta_{23}} \xrightarrow{\partial^1_F} F_\Delta \otimes \varepsilon \longrightarrow 0$$

where:

$$\partial^0_F(x)_{ij} = x_j \mid_{\Delta_{ij}} - x_i \mid_{\Delta_{ij}}$$

and

$$\partial^1_F(x) = x_{23} \mid_\Delta - x_{13} \mid_\Delta + x_{12} \mid_\Delta \ .$$

We are now ready to prove the main theorem.

Theorem 3.25. *Let X a smooth algebraic surface and F a coherent sheaf on X. Let $F^{[n]}$ the tautological sheaf on the Hilbert scheme $X^{[n]}$ associated to F. Let*

$$\mathbf{\Phi} = \Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^n} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$$

the Bridgeland-King-Reid-Haiman equivalence. Then the image of the tautological sheaf $F^{[n]}$ for the equivalence $\mathbf{\Phi}$ is isomorphic in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ to the complex $(\mathcal{C}^\bullet_F, \partial^\bullet_F)$:

$$\mathbf{\Phi}(F^{[n]}) \simeq \mathcal{C}^\bullet_F \ .$$

Proof. The proof is now an easy consequence of propositions 3.21, 3.22, 3.8 and of example 2.8. The tautological sheaf $F^{[n]}$ has been defined as the image of the sheaf F by the Fourier-Mukai functor:

$$F^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) \ .$$

Therefore the searched image $\mathbf{\Phi}(F^{[n]})$ is exactly the image F for the composition of functors:

$$\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^n} \circ \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) \ .$$

This composition is, by proposition 3.22, the Fourier-Mukai functor

$$\Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^n} \circ \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}} \simeq \Phi^{\mathcal{O}_{B^n} * \mathcal{O}_\Xi}_{X \to X^n} \ .$$

Consider now the kernel:

$$\mathcal{O}_{B^n} * \mathcal{O}_\Xi := \mathbf{R}\pi_{X^n \times X *}(\pi^*_{X^{[n]} \times X^n}\mathcal{O}_{B^n} \otimes^L_{\mathcal{O}_{X^{[n]} \times X^n \times X}} \pi^*_{X^{[n]} \times X}\mathcal{O}_\Xi) \ .$$

The sheaves $\pi^*_{X^{[n]} \times X^n}\mathcal{O}_{B^n}$ and $\pi^*_{X^{[n]} \times X}\mathcal{O}_\Xi$ are transverse, because transversely supported and Cohen-Macauley, and their (derived) tensor product:

$$\pi^*_{X^{[n]} \times X^n}\mathcal{O}_{B^n} \otimes^L_{\mathcal{O}_{X^{[n]} \times X^n \times X}} \pi^*_{X^{[n]} \times X}\mathcal{O}_\Xi \simeq \pi^*_{X^{[n]} \times X^n}\mathcal{O}_{B^n} \otimes_{\mathcal{O}_{X^{[n]} \times X^n \times X}} \pi^*_{X^{[n]} \times X}\mathcal{O}_\Xi \simeq \mathcal{O}_{B^n} \otimes_{\mathcal{O}_{X^{[n]}}} \mathcal{O}_\Xi$$

is isomorphic to the structural sheaf of the fiber product $B^n \times_{X^{[n]}} \Xi$, which is, in turn, isomorphic to the isospectral universal family $Z \subseteq B^n \times X$ by proposition 3.8. Therefore the kernel reduces to:

$$\mathcal{O}_{B^n} * \mathcal{O}_\Xi \simeq \mathbf{R}\pi_{X^n \times X *}(\mathcal{O}_Z) \ .$$

Now the diagram:

$$\begin{tikzcd}
X^{[n]} \times X^n \times X \arrow[r, "\pi_{X^n \times X}"] & X^n \times X \\
B^n \times X \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. It means that

$$\mathbf{R}\pi_{X^n \times X *}(\mathcal{O}_Z) \simeq \mathbf{R}f_*\mathcal{O}_Z \simeq \mathcal{O}_D$$

by proposition 3.21. Therefore the BKRH transform of $F^{[n]}$ is simply:

$$\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X \to X^n}(F) \ .$$

Let us now study the last term. By definition it is

$$\Phi^{\mathcal{O}_D}_{X \to X^n}(F) \simeq \mathbf{R}\pi_{X^n *}(\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n \times X}} \pi^*_X F) \ .$$

We will now prove that:
$$\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{O}_D \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F \ .$$
By example 2.8 we can right resolve $\mathcal{O}_D$ by the complex $\mathcal{K}^\bullet$:
$$\mathcal{O}_D \simeq \mathcal{K}^\bullet \ .$$
Therefore
$$\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{K}^\bullet \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \ .$$
The hypertor spectral sequence associated to the last term is
$$E_1^{p,q} = \mathrm{Tor}_{-q}(\mathcal{K}^p, \pi_X^* F) \implies \mathrm{Tor}_{-p-q}(\mathcal{K}^\bullet, \pi_X^* F) = H^{p+q}(\mathcal{K}^\bullet \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \ .$$
Now
$$E_1^{p,q} = \bigoplus_{|I|=p+1} \mathrm{Tor}_{-q}^{\mathcal{O}_{X^n\times X}}(\mathcal{O}_{D_I}, \pi_X^* F) \ :$$
but
$$\mathrm{Tor}_{-q}^{\mathcal{O}_{X^n\times X}}(\mathcal{O}_{D_I}, \pi_X^* F) = H^q(\mathcal{O}_{D_I} \otimes^L_{\mathcal{O}_{X^n\times X}} (\mathcal{O}_{X^n\times X} \otimes^L_{\mathcal{O}_X} F)) = H^q(\mathcal{O}_{D_I} \otimes^L_{\mathcal{O}_X} F) = 0 \quad \text{if } q<0 \ ,$$
because D_I is flat over X. Therefore the E_1 level of the spectral sequence is reduced to the complex $E_1^{\bullet,0} \simeq \mathcal{K}^\bullet \otimes \pi_X^* F$: in other words the spectral sequence degenerates at level E_2: $E_2^{p,q} \simeq E_\infty^{p,q}$. Since $\mathcal{O}_D$ and $\pi_X^* F$ are sheaves we know *a priori* that
$$\mathrm{Tor}_{-q}(\mathcal{O}_D, \pi_X^* F) \simeq E_\infty^{q,0} \simeq E_2^{q,0} = 0 \quad \text{if } q>0 \ \ .$$
We get:
$$\mathcal{K}^\bullet \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{K}^\bullet \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F$$
and the last term is acyclic in degree >0. This yields in particular:
$$\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{K}^\bullet \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{K}^\bullet \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F \simeq \mathcal{O}_D \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F \ .$$
Therefore:
$$\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X\to X^n}(F) \simeq \mathbf{R}\pi_{X^n *}(\mathcal{O}_D \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \ .$$
Now
$$\mathbf{R}\pi_{X^n *}(\mathcal{O}_D \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \simeq \pi_{X^n *}(\mathcal{O}_D \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \simeq \pi_*(\mathcal{K}^\bullet \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F)$$
because
$$\pi_{X^n}\big|_D : D \longrightarrow X^n$$
is a finite morphism. The term in degree 0 is isomorphic to:
$$\pi_{X^n *}(\mathcal{K}^0 \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \simeq \bigoplus_{i=1}^n \pi_{X^n *}(\mathcal{O}_{D_i} \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \ .$$
Now $D_i \xrightarrow{\pi_{X^n}} X^n$ is an isomorphism and the diagram
$$\begin{tikzcd} D_i \arrow[r, "\pi_X"] \arrow[d, "\pi_{X^n}"'] & X \\ X^n \arrow[ur, "p_i"'] & \end{tikzcd}$$

commutes, hence:

$$\pi_{X^n*}(\mathcal{O}_{D_i} \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F) \simeq p_i^* F \simeq F_i \; .$$

For the other terms of the kind $\pi_{X^n*}(\mathcal{O}_{D_I} \otimes_{\mathcal{O}_{X^n\times X}} \pi_X^* F)$ it is sufficient to remark that:

$$\pi_{X^n} : D_I \longrightarrow \Delta_I$$

is an isomorphism. Therefore

$$\mathbf{\Phi}(F^{[n]}) \simeq \pi_{X^n*}(\mathcal{O}_D \otimes \pi_X^* F) \simeq \pi_{X^n*}(\mathcal{K}^\bullet \otimes \pi_X^* F) \simeq \mathcal{C}_F^\bullet \; .$$

□

The following result gives a generalization of Danila-Brion formula (cf. [23]).

Theorem 3.26. *Let X a smooth algebraic surface, F a coherent sheaf on X. Let $X^{[n]}$ the Hilbert scheme of n points on X, S^nX the symmetric variety and $\mu : X^{[n]} \longrightarrow S^nX$ the Hilbert-Chow morphism. Then*

$$\mathbf{R}\mu_*(F^{[n]}) \simeq \pi_*^{\mathfrak{S}_n}\left(\bigoplus_{i=1}^n F_i\right) \; .$$

Proof. By proposition 1.19, applied to the diagram:

$$\begin{tikzcd} B^n \arrow[r, "p"] \arrow[d, "q"'] & X^n \arrow[d, "\pi"] \\ X^{[n]} \arrow[r, "\mu"] & S^nX \end{tikzcd}$$

we have:

$$\mathbf{R}\mu_* \simeq \pi_*^{\mathfrak{S}_n} \circ \mathbf{\Phi} \; .$$

Therefore

$$\begin{aligned} \mathbf{R}\mu_*(F^{[n]}) &\simeq \pi_*^{\mathfrak{S}_n} \circ \mathbf{\Phi} \circ \Phi^{\mathcal{O}_\Xi}_{X\to X^{[n]}}(F) \\ &\simeq \pi_*^{\mathfrak{S}_n} \circ \Phi^{\mathcal{O}_D}_{X\to X^n}(F) \; . \end{aligned}$$

Now proposition 1.16, applied to the diagram:

$$\begin{tikzcd} X^n \times X \arrow[r, "\pi_X"] \arrow[d, "\pi_{X^n}"'] \arrow[dr, "\pi \times id"] & X \arrow[dr, "id"] & \\ X^n \arrow[dr, "\pi"'] & S^nX \times X \arrow[r] \arrow[d, "\pi_{S^nX}"] & X \\ & S^nX & \end{tikzcd}$$

easily yields:

$$\begin{aligned} \pi_*^{\mathfrak{S}_n} \circ \Phi^{\mathcal{O}_D}_{X\to X^n}(F) &\simeq \Phi^{\mathcal{O}_D^{\mathfrak{S}_n}}_{X\to S^nX}(F) \\ &\simeq \mathbf{R}\pi_{S^nX*}\left[\mathcal{O}_D^{\mathfrak{S}_n} \otimes^L_{\mathcal{O}_{S^nX\times X}} \pi_X^*(F)\right] \; . \end{aligned}$$

Now, by definition of the $\mathfrak{S}_n$-action on $\mathcal{O}_D$, it is clear that

$$\mathcal{O}_D^{\mathfrak{S}_n} \simeq \left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \right)^{\mathfrak{S}_n} .$$

Consequently,

$$\begin{aligned} \mathcal{O}_D^{\mathfrak{S}_n} \otimes^L_{\mathcal{O}_{S^nX\times X}} \pi_X^*(F) &\simeq \left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \right)^{\mathfrak{S}_n} \otimes^L_{\mathcal{O}_{S^nX\times X}} \pi_X^*(F) \\ &\simeq \left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^*(F) \right)^{\mathfrak{S}_n} \end{aligned}$$

where we applied lemma 1.17. Finally:

$$\begin{aligned} \mathbf{R}\pi_{S^nX*}\left[\mathcal{O}_D^{\mathfrak{S}_n} \otimes^L_{\mathcal{O}_{S^nX\times X}} \pi_X^*(F)\right] &\simeq \mathbf{R}\pi_{S^nX*}\left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \right)^{\mathfrak{S}_n} \\ &\simeq \mathbf{R}\pi_{S^nX*}\mathbf{R}(\pi\times \mathrm{id})_*^{\mathfrak{S}_n} \left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \right) \\ &\simeq \pi_*^{\mathfrak{S}_n}\mathbf{R}\pi_{X^n*} \left(\bigoplus_{i=1}^n \mathcal{O}_{D_i} \otimes^L_{\mathcal{O}_{X^n\times X}} \pi_X^* F \right) \\ &\simeq \pi_*^{\mathfrak{S}_n} \left(\bigoplus_{i=1}^n F_i \right) \end{aligned}$$

where the third isomorphism is obtained using the commutativity of the diagram:

$$\begin{tikzcd} X^n \times X \arrow[r, "\pi_{X^n}"] \arrow[d, "\pi\times \mathrm{id}"'] & X^n \arrow[d, "\pi"] \\ S^nX \times X \arrow[r, "\pi_{S^nX}"'] & S^nX \end{tikzcd}$$

and the fact that $\pi_{S^nX} \circ [-]^{\mathfrak{S}_n} \simeq [-]^{\mathfrak{S}_n} \circ \pi_{S^nX}$, because π_{S^nX} is $\mathfrak{S}_n$-invariant.

□

3.4 Applications

Thanks to the results obtained till now, we can show some applications: among others we reobtain and generalize some results of Danila [23].

We begin by introducing the *Donaldson line bundle* $\mathcal{D}_A$ on the Hilbert scheme, associated to a line bundle A on the surface X. Consider the bundle $A^{\boxtimes n} := A \boxtimes \cdots \boxtimes A$ on the product X^n. By Drezet-Kempf-Narasimhan lemma (cf. [30]), it descends to a line bundle on the quotient $X^n/\mathfrak{S}_n \simeq S^nX$.

Definition 3.27. We call Donaldson line bundle $\mathcal{D}_A$ on the Hilbert scheme $X^{[n]}$ the line bundle

$$\mathcal{D}_A := \mu^*(A^{\boxtimes n}/\mathfrak{S}_n) ,$$

pull back by the Hilbert-Chow morphism of the quotient by the symmetric group $\mathfrak{S}_n$ of the n-th exterior tensor product $A^{\boxtimes n}$ on X^n.

Theorem 3.28. *Let X a smooth algebraic surface, F a coherent sheaf and A a line bundle on X. Let $\mathcal{D}_A$ the Donaldson line bundle on the Hilbert scheme $X^{[n]}$. Then:*

$$H^*(X^{[n]}, F^{[n]} \otimes_{\mathcal{O}_{X^{[n]}}} \mathcal{D}_A) \simeq H^*(X, F \otimes A) \otimes S^{n-1} H^*(X, A) \ .$$

Proof.

$$\begin{aligned}
\mathbf{R}\Gamma_{X^{[n]}} \left(F^{[n]} \otimes_{\mathcal{O}_{X^{[n]}}} \mathcal{D}_A \right) &\simeq \mathbf{R}\Gamma_{S^nX} \circ \mathbf{R}\mu_* \left(F^{[n]} \otimes_{\mathcal{O}_{X^{[n]}}} \mu^*(A^{\boxtimes^n}/\mathfrak{S}_n) \right) \\
&\simeq \mathbf{R}\Gamma_{S^nX} \left(\left(\bigoplus_{i=1}^n F_i \right)^{\mathfrak{S}_n} \otimes_{\mathcal{O}_{S^nX}} A^{\boxtimes^n}/\mathfrak{S}_n \right) \\
&\simeq \mathbf{R}\Gamma_{S^nX} \circ \pi_*^{\mathfrak{S}_n} \left(\bigoplus_{i=1}^n F_i \otimes_{\mathcal{O}_{X^n}} \pi^* \left(A^{\boxtimes^n}/\mathfrak{S}_n \right) \right) \\
&\simeq \mathbf{R}\Gamma_{X^n}^{\mathfrak{S}_n} \left(\bigoplus_{i=1}^n F_i \otimes_{\mathcal{O}_{X^n}} A^{\boxtimes^n} \right) \\
&\simeq \mathbf{R}\Gamma_{X^n}^{\mathfrak{S}_n} \left(\bigoplus_{i=1}^n (F_i \otimes A) \boxtimes A^{\boxtimes^{n-1}} \right) \ .
\end{aligned}$$

Therefore, by Künneth formula, and taking the invariants:

$$\begin{aligned}
H^*(X^{[n]}, F^{[n]} \otimes_{\mathcal{O}_{X^{[n]}}} \mathcal{D}_A) &\simeq \left[\bigoplus_{i=1}^n H^*(X, F \otimes A) \otimes H^*(X, A)^{\otimes^{n-1}} \right]^{\mathfrak{S}_n} \\
&\simeq H^*(X, F \otimes A) \otimes S^{n-1} H^*(X, A) \ .
\end{aligned}$$

□

Remark 3.29. This proof follows closely Danila [23]. The key point is in any case theorem 3.26, which is obtained here more directly and in a more general form. Another proof is possible using theorem 3.25: it is only necessary to remark that:

$$q^*\mathcal{D}_{\mathcal{A}} \simeq p^*(A^{\boxtimes^n})$$

and then:

$$\mathbf{\Phi}(F^{[n]} \otimes \mathcal{D}_A) \simeq \mathcal{C}_F^\bullet \otimes A^{\boxtimes^n} \ .$$

Now, the computation of the $\mathfrak{S}_n$-equivariant hypercohomology of the complex $\mathcal{C}_F{}^\bullet \otimes A^{\boxtimes^n}$ gives the result. The only non-trivial part is to show that $\mathcal{C}_F^p \otimes A^{\boxtimes^n}$ has no $\mathfrak{S}_n$-invariants for $p > 0$.

Example 3.30. If A is trivial, then $\mathcal{D}_A \simeq \mathcal{O}_{X^{[n]}}$ and we get the cohomology of the tautological sheaf:

$$H^*(X^{[n]}, F^{[n]}) \simeq H^*(X, F) \otimes S^{n-1} H^*(X, \mathcal{O}_X) \ .$$

In the case X is affine we get:

$$H^*(X^{[n]}, F^{[n]}) \simeq H^0(X, F) \otimes S^{n-1} H^0(X, \mathcal{O}_X)$$

which implies the vanishing of the higher cohomology groups:

$$H^i(X^{[n]}, F^{[n]}) \simeq 0 \quad \text{for } i > 0$$

whereas, in the case X is projective, with $p_g = q = 0$ the formula becomes simply:

$$H^*(X^{[n]}, F^{[n]}) \simeq H^*(X, F) \ .$$

We want now to compute the cohomology of $F^{[n]} \otimes \det A^{[n]}$ with A a line bundle on X. We will prove that the cohomology $H^*(X^{[n]}, F^{[n]} \otimes \det A^{[n]})$ identifies to the anti-invariant hypercohomology of the complex $\mathcal{C}^\bullet_F \otimes A^{\boxtimes^n}$. Let R the natural representation of $\mathfrak{S}_n$, that is, $\mathbb{C}^n$ where $\mathfrak{S}_n$ acts permutating the vectors e_i of the canonical basis.

Lemma 3.31. *Let F a coherent sheaf on X and A a vector bundle on X.*

$$H^*(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) \simeq H^*_{\mathfrak{S}_n}(B^n, q^*(F^{[n]}) \otimes \det R) \ .$$

Proof. Let E the exceptional divisor in B^n. On $B^n \setminus p^{-1}(\cup_{|I|\geq 3} \Delta_I)$ we have an exact sequence of $\mathfrak{S}_n$-sheaves:

$$0 \longrightarrow \mathcal{O}_B^{[n]} \longrightarrow \mathcal{O}_B \otimes R \longrightarrow \mathcal{O}_E \longrightarrow 0 \ .$$

Taking the determinant yields a morphism of $\mathfrak{S}_n$-line bundles:

$$\det \mathcal{O}_B^{[n]} \longrightarrow \mathcal{O}_B \otimes \det R$$

whose scheme of zeros is the divisor E. We remark that $p_n^{-1}(\cup_{|I|\geq 3} \Delta_I)$ is a closed subscheme of codimension 2 and B^n is a normal variety. As a consequence the $\mathfrak{S}_n$-invariant sections of $\det \mathcal{O}_B^{[n]}$ on the $\mathfrak{S}_n$-invariant affine open sets V of B^n are the $\mathfrak{S}_n$-invariant sections of $\mathcal{O}_B \otimes R$ vanishing on E. The $\mathfrak{S}_n$-invariant sections of $\mathcal{O}_B \otimes \det R$ are the alternating regular functions:

$$g^*(f) = \varepsilon_g f$$

for all $g \in \mathfrak{S}_n$, As a consequence such functions necessarily vanish on E. Therefore:

$$q_*^{\mathfrak{S}_n}(\det \mathcal{O}_B^{[n]}) = q_*^{\mathfrak{S}_n}(\mathcal{O}_B \otimes \det R) \ .$$

By projection formula we get:

$$F^{[n]} \otimes \det \mathcal{O}_X^{[n]} = q_*^{\mathfrak{S}_n}(q^*(F^{[n]}) \otimes \det R) \ .$$

Taking the cohomology of the two members on the Hilbert scheme $X^{[n]}$ we get the result.

□

Theorem 3.32. *Let A a line bundle on X. We have:*

$$H^*(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) \simeq H^*_{\mathfrak{S}_n}(X^n, \mathcal{C}^\bullet_F \otimes A^{\boxtimes^n} \otimes \det R) \ .$$

Proof. We know that $\det A^{[n]} \simeq \det \mathcal{O}_X^{[n]} \otimes \mathcal{D}_A$ (see [83]). By remark 3.29, we know that

$$\mathbf{\Phi}(F^{[n]} \otimes \mathcal{D}_A) \simeq \mathcal{C}^\bullet_F \otimes A^{\boxtimes^n} \ .$$

Therefore

$$\begin{aligned} H^*(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) &\simeq H^*_{\mathfrak{S}_n}(B^n, q^*(F^{[n]} \otimes \mathcal{D}_A) \otimes \det R) \\ &\simeq H^*_{\mathfrak{S}_n}(X^n, \mathcal{C}^\bullet_F \otimes A^{\boxtimes^n} \otimes \det R) \ . \end{aligned}$$

Corollary 3.33. *The cohomology $H^*(X^{[n]}, F^{[n]} \otimes \det A^{[n]})$ is the limit of a spectral sequence $E_1^{p,q}$ given by*

$$E_1^{p,q} \simeq (H^*(X, F \otimes A^{\otimes^{p+1}}) \otimes \Lambda^{n-p-1} H^*(X, A))^q$$

for $0 \leq p \leq n-1$, and by $E_1^{p,q} = 0$ otherwise.

Proof. Consider the hypercohomology spectral sequence

$$E_1^{p,q} = H^q_{\mathfrak{S}_n}(X^n, \mathcal{C}_F^p \otimes A^{\boxtimes^n} \otimes \det R)\ .$$

To compute these terms one reduces to the computation of invariants of $H^*(\Delta_J, F_J \otimes A^{\boxtimes^n} \otimes \det R)$ for the action of the stabilizer of the diagonal Δ_J which is $\mathfrak{S}(J) \times \mathfrak{S}(\bar{J})$, $\bar{J}$ the complementary of J. We get an isomorphisms of vector spaces:

$$H^*_{\mathfrak{S}_n}(X^n, \mathcal{C}_F^p \otimes A^{\boxtimes^n} \otimes \det R) \simeq H^*(X, F \otimes A^{\otimes^{p+1}}) \otimes \Lambda^{n-p-1} H^*(X, A)$$

and this yields the result.

□

We suppose that X is a smooth projective surface. We denote with $H^+(A)$ and $H^-(A)$ the even and the odd part of the cohomology of X with values in A.

Corollary 3.34. *Let X be a smooth projective surface. The Euler-Poincaré characteristic of $F^{[n]} \otimes \det A^{\boxtimes^n}$ is given by:*

$$\chi(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) = \sum_{\substack{p \geq 1 \\ p+q \leq n}} (-1)^{p+q-1} \chi(F \otimes A^{\otimes^p}) \dim S^q H^-(A) \otimes \Lambda^{n-p-q} H^+(A)\ .$$

In particular, if $H^i(A) = 0$ if $i > 0$, then:

$$\chi(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) = \sum_{p \geq 1} (-1)^{p-1} \chi(F \otimes A^{\otimes^p}) \otimes \dim \Lambda^{n-p} H^0(A)\ .$$

Proof. The Euler-Poincaré characteristic $\chi(X^{[n]}, F^{[n]} \otimes \det A^{[n]})$is given by:

$$\begin{aligned} \chi(X^{[n]}, F^{[n]} \otimes \det A^{[n]}) &= \sum_{p,q} (-1)^{p+q} \dim E_1^{p,q} \\ &= \sum_{p \geq 1} (-1)^{p+q-1} \chi(F \otimes A^{\otimes^p}) \dim(\Lambda^{n-p} H^*(A))^q\ . \end{aligned}$$

The graduated exterior algebra $\Lambda H^*(A)$ identifies to graduated tensor product algebra:

$$\Lambda H^*(A) = \Lambda H^+(A) \otimes S H^-(A)\ .$$

As a consequence:

$$\sum_q (-1)^q (\Lambda^{n-p} H^*(A))^q = \sum_{p+l \leq n} (-1)^l \dim S^l H^-(A) \otimes \Lambda^{n-p-l} H^+(A)\ .$$

which gives the statement of the corollary.

□

Remark 3.35. The dimensions $\dim \Lambda^l H^+(A)$ and $\dim S^l H^-(A)$ can be computed thanks to the formulas:

$$\sum_l t^l (-1)^l \dim H^+(A) = (1+t)^{\dim H^+(A)}$$

$$\sum_l t^l \dim S^l H^-(A) = \frac{1}{(1-t)^{\dim H^-(A)}}\ .$$

Example 3.36. Let $X = \mathbb{P}_2$, and let $F = \mathcal{O}_{\mathbb{P}_2}(-1)$, $A = \mathcal{O}_{\mathbb{P}_2}(3)$. We take $n = 5$. Therefore:

$$\begin{aligned} \chi(\mathbb{P}_2^{[5]}, F^{[5]} \otimes \det A^{[5]}) &= \sum_{p=1}^{5} (-1)^{p-1} \dim H^0(\mathcal{O}_{\mathbb{P}_2}(3p-1)) \otimes \Lambda^{5-p} H^0(\mathcal{O}_{\mathbb{P}_2}(3)) \\ &= \sum_{p=1}^{5} (-1)^{p-1} \binom{3p-1}{2} \binom{10}{5-p} \\ &= 105\ . \end{aligned}$$

4 Cohomology of representations of a tautological vector bundle on the Hilbert scheme

4.1 Tensor powers of a tautological sheaf

Let X a smooth quasi-projective surface, $X^{[n]}$ the Hilbert scheme of n points on X, Ξ the universal family on $X^{[n]}$. Consider the diagram:

$$\begin{tikzcd}
\overbrace{\Xi \times_{X^{[n]}} \dots \times_{X^{[n]}} \Xi}^{k\text{-times}} \arrow[r, hook, "j"] \arrow[d, "w"'] \arrow[dr, phantom, "\square"] & \Xi^k \arrow[r, "p^k_X"] \arrow[d, "p^k_{X^{[n]}}"] & X^k \\
X^{[n]} \arrow[r, hook, "i"] & (X^{[n]})^k &
\end{tikzcd}$$

where the square is cartesian, and i and j denote the diagonal immersions. We remark that $p^k_{X^{[n]}}$ and w are flat and finite of degree n^k. Consider now vector bundles $E_1, \dots, E_k$ on X and their exterior tensor product $E_1 \boxtimes \cdots \boxtimes E_k$ on X^k. It's clear that

$$i^* {p^k_{X^{[n]}}}_* (p^k_X)^* (E_1 \boxtimes \cdots \boxtimes E_k) = E_1^{[n]} \otimes \ldots \otimes E_k^{[n]} = w_* j^* (p^k_X)^* (E_1 \boxtimes \cdots \boxtimes E_k)$$

by flat base change. $\Xi(n,k)$ is the k-th fiber product:

$$\Xi(n,k) := \underbrace{\Xi \times_{X^{[n]}} \dots \times_{X^{[n]}} \Xi}_{k\text{-times}} .$$

Remark that it embeds naturally in

$$\Xi(n,k) \hookrightarrow X^{[n]} \times X^k .$$

Therefore we can express the tensor product of k-tautological vector bundles $E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}$ as the Fourier-Mukai functor:

$$E_1^{[n]} \otimes \ldots \otimes E_k^{[n]} = \Phi^{\mathcal{O}_{\Xi(n,k)}}_{X^k \to X^{[n]}} (E_1 \boxtimes \cdots \boxtimes E_k) . \tag{32}$$

4.2 Haiman's result

In the same fashion we did in chapter 3, we want to use the BKRH correspondence $\mathbf{\Phi}$ to carry over X^n the tensor product of tautological bundles on the Hilbert scheme and then to compute equivariant cohomology there. The fundamental technical point is again the computation of the kernel of the resulting Fourier-Mukai functor. Haiman found this kernel for affine surfaces. It is very simple, using GAGA's principle [106] to extend this result for arbitrary smooth surfaces, thus making Haiman's result extremely useful in our context. We first fix some notations and definitions.

Definition 4.1. Let X a smooth quasi-projective variety. Let $D \subseteq X^n \times X$ the scheme-theoretic union of pairwise diagonals $D_i = \cup_{i=1}^n \Delta_{i,n}$. The polygraph $D(n,k) \subseteq X^n \times X^k$ is the reduced k-th fiber product

$$D(n,k) := \left(\underbrace{D \times_{X^n} \cdots \times_{X^n} D}_{k\text{-times}} \right)_{\text{red}} .$$

We now explain the term polygraph. Let f a function

$$f : \{1, \dots, k\} \longrightarrow \{1, \dots, n\} .$$

Consider the map

$$\begin{aligned} \pi_f : X^n &\longrightarrow X^k \\ (x_1, \dots, x_n) &\longmapsto (x_{f(1)}, \dots, x_{f(k)}) \end{aligned}$$

Let E_f its graph $E_f \subseteq X^n \times X^k$. Then $D(n,k)$ is the scheme-theoretic union:

$$D(n,k) = \bigcup_f E_f \ ,$$

where f ranges between all maps from $\{1, \dots, k\}$ to $\{1, \dots, n\}$ (cf. [61], [62]).

Consider now the isospectral universal family Z on B^n. By analogy with the prevous definitions we set:

$$Z(n,k) : \underbrace{Z \times_{B^n} \cdots \times_{B^n} Z}_{k\text{-times}} .$$

$Z(n,k)$ is flat and finite over B^n of degree n^k. Since Z is isomorphic to the fiber product of Ξ and B^n over the Hilbert scheme, it is immediate to see that:

$$Z(n,k) = \Xi(n,k) \times_{X^{[n]}} B^n$$

which implies that $Z(n,k)$ is Cohen-Macauly and hence reduced, since generically reduced. $Z(n,k)$ is naturally a subscheme of $B^n \times X^k$, and clearly isomorphic to the pull back:

$$Z(n,k) = (q \times \mathrm{id})^{-1}(\Xi(n,k)) \ ,$$

where $(q \times \mathrm{id}) : B^n \times X^k \longrightarrow X^{[n]} \times X^k$. We now give an easy application of GAGA principle, which we will use in the next theorem.

Lemma 4.2. *Let $f : X \longrightarrow Y$ a projective morphism of complex algebraic varieties and $f_{\mathrm{an}} : X_{\mathrm{an}} \longrightarrow S_{\mathrm{an}}$ the associated morphism of complex analytic spaces. Then:*

$$\mathbf{R}f_{\mathrm{an}*}(\mathcal{O}_{X_{\mathrm{an}}}) \simeq \mathbf{R}f_*(\mathcal{O}_X) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathrm{an}}} \ .$$

Proof. Since the statement is local, we can replace $\mathcal{O}_X$ by an algebraic coherent sheaf F on $\mathbb{P}^r_S := S \times \mathbb{P}^r$. Furthermore, after resolving F by locally free sheaves, we can suppose $F \simeq \mathcal{O}_{\mathbb{P}^r_S}(i) \simeq \mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}^r}(i)$ and S affine. The statement then becomes:

$$H^*(Y_{\mathrm{an}}, F_{\mathrm{an}}) \simeq H^*(Y, F) \otimes_{\mathcal{O}(S)} \mathcal{O}(S_{\mathrm{an}}) \ .$$

Since we supposed $F \simeq \mathcal{O}_S \boxtimes \mathcal{O}_{\mathbb{P}^r}(i)$, Künneth formula gives:

$$\begin{aligned} H^*(Y, F) &\simeq \mathcal{O}(S) \otimes_{\mathbb{C}} H^*(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) \\ H^*(Y_{\mathrm{an}}, F_{\mathrm{an}}) &\simeq H^*({\mathbb{P}^r}_{\mathrm{an}}, \mathcal{O}_{{\mathbb{P}^r}_{\mathrm{an}}}(i)) \otimes_{\mathbb{C}} \mathcal{O}(S_{\mathrm{an}}) \ . \end{aligned}$$

Therefore it suffices to verify:

$$H^*({\mathbb{P}^r}_{\mathrm{an}}, \mathcal{O}_{{\mathbb{P}^r}_{\mathrm{an}}}(i)) \simeq H^*(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) \ ,$$

but this comes directly from GAGA principle ([106]).

□

Theorem 4.3 (Haiman)**.** *Consider the map:*

$$f : (p \times \mathrm{id}) : B^n \times X^k \longrightarrow X^n \times X^k \quad .$$

Then the derived direct image $\mathbf{R}f_*\mathcal{O}_{Z(n,k)}$ of the structural sheaf of $Z(n,k)$ is the structural sheaf of the polygraph $D(n,k)$:

$$\mathbf{R}f_*\mathcal{O}_{Z(n,k)} \simeq \mathcal{O}_{D(n,k)} \ .$$

Proof. The case of the affine plane $X = \mathbb{A}^2_{\mathbb{C}}$ has been proved Haiman ([62]). To prove it for an arbitrary smooth quasi-projective variety we use the preceding lemma, applied to the morphism:

$$f : Z(n,k) \longrightarrow D(n,k)$$

and the fact that, by GAGA principle, for all complex algebraic variety S, the morphism of ringed spaces

$$(S_{\mathrm{an}}, \mathcal{O}_{S_{\mathrm{an}}}) \longrightarrow (S, \mathcal{O}_S)$$

is faithfully flat. This means that:

$$\mathbf{R}f_*\mathcal{O}_{Z(n,k)} \simeq \mathcal{O}_{D(n,k)} \iff \mathbf{R}f_*\mathcal{O}_{Z(n,k)_{\mathrm{an}}} \simeq \mathcal{O}_{D(n,k)_{\mathrm{an}}} .$$

Since the facts are local in nature, it suffices to prove the statement on a small analytic open subset V of $D(n,k)$. We can always choose it of the form:

$$V \simeq \prod_{j=1}^{s} D_{U_j}(n_j, k_j)$$

with U_j small analytic open set of $\mathbb{C}^2$, n_j, k_j positive natural numbers such that $\sum_j n_j = n$ and $\sum_j k_j = k$ and $D_{U_j}(n_j,k_j)$ the analytic polygraph relative to U_j. Over V, the analytic space $Z(n,k)_{\mathrm{an}}$ is now of the form

$$Z(n_j,k_j)_{\mathrm{an}}\Big|_V \simeq \prod_j Z_{U_j}(n_j,k_j)$$

and the map f is now the product map. Since the U_j are now analytic open sets of $\mathbb{C}^2$, and since the result is true for analytic open sets of $\mathbb{C}^2$, because it is true algebraically for $\mathbb{C}^2$, we are done.

□

We now want to find the image of a tensor product of tautological vector bundles by the BKRH equivalence. Consider the image of a tensor product of tautological vector bundles:

$$\mathbf{\Phi}(E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}) .$$

By (32) it is:

$$\begin{aligned} \mathbf{\Phi}(E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}) &\simeq \Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^k} \Phi^{\mathcal{O}_{\Xi(n,k)}}_{X^k \to X^{[n]}}(E_1 \boxtimes \cdots \boxtimes E_n) \\ &\simeq \Phi^{\mathcal{O}_{B^n} * \mathcal{O}_{\Xi(n,k)}}_{X^k \to X^n}(E_1 \boxtimes \cdots \boxtimes E_n) \end{aligned}$$

where the kernel $\mathcal{O}_{B^n} * \mathcal{O}_{\Xi(n,k)}$ is

$$\mathcal{O}_{B^n} * \mathcal{O}_{\Xi(n,k)} \simeq \mathbf{R}f_*(\mathcal{O}_{B^n} \otimes^L_{\mathcal{O}_{X^{[n]}}} \mathcal{O}_{\Xi(n,k)}) .$$

Since $\Xi(n,k)$ is flat over $X^{[n]}$, the latter reduces to:

$$\mathcal{O}_{B^n} * \mathcal{O}_{\Xi(n,k)} \simeq \mathbf{R}f_*(\mathcal{O}_{B^n} \otimes_{\mathcal{O}_{X^{[n]}}} \mathcal{O}_{\Xi(n,k)}) \simeq \mathbf{R}f_*(\mathcal{O}_{B^n \times_{X^{[n]}} \Xi(n,k)}) .$$

Now, using proposition 3.8,

$$B^n \times_{X^{[n]}} \Xi(n,k) \simeq Z(n,k)$$

and Haiman's theorem then yields

$$\mathcal{O}_{B^n} * \mathcal{O}_{\Xi(n,k)} \simeq \mathbf{R}f_*(\mathcal{O}_{Z(n,k)}) \simeq \mathcal{O}_{D(n,k)} .$$

Therefore

$$\mathbf{\Phi}(E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}) \simeq \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(E_1 \boxtimes \cdots \boxtimes E_n) \ . \tag{33}$$

The consequence of Haiman's theorem is the following. Recall that the complex $\mathcal{C}^\bullet_F$ defined in the previous chapter is quasi-isomorphic to $\mathbf{\Phi}(F^{[n]})$, when F is a coherent sheaf on X. We will denote with $F_B^{[n]}$ the pull-back on the isospectral Hilbert scheme B^n of a tautological sheaf $F^{[n]}$ on the Hilbert scheme $X^{[n]}$:

$$F_B^{[n]} := q^* F^{[n]} \ .$$

Theorem 4.4. *Let E_i, $i = 1, \ldots, k$ vector bundles on X. The mapping cone of the morphism:*

$$\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k} \longrightarrow \mathbf{\Phi}(E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}) \tag{34}$$

is acyclic in degree higher than zero. This means that:

$$R^q p_*(E_{1,B}^{[n]} \otimes \ldots E_{k,B}^{[n]}) = 0 \qquad \text{for } q > 0$$

and the morphism:

$$p_*(E_{1,B}^{[n]}) \otimes \ldots \otimes p_*(E_{k,B}^{[n]}) \longrightarrow p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]})$$

is surjective, its kernel being the torsion subsheaf.

Proof. We know by the previous arguments that the searched image of the BKRH equivalence is

$$\mathbf{\Phi}(E_1^{[n]} \otimes \ldots \otimes E_k^{[n]}) \simeq \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(E_1 \boxtimes \cdots \boxtimes E_n) \ .$$

We now work out the information of this Fourier-Mukai functor. Consider the diagram:

$$\begin{tikzcd} D(n,k) \arrow[r, hook, "j"] \arrow[d, "\tilde{w}"'] \arrow[dr, phantom, "\square"] & D^k \arrow[r, "p_X^k"] \arrow[d, "p^k_{X^n}"] & X^k \\ X^n \arrow[r, hook, "i"] & (X^n)^k & \end{tikzcd}$$

It is clear that

$$\Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n} \simeq \tilde{w}_* \circ L(p_X^k \circ j)^* \simeq \tilde{w}_* \circ Lj^* \circ L(p_X^k)^* \ .$$

Now the square in the previous diagram is cartesian, but $p^k_{X^n}$ is *not flat.* Therefore we cannot apply any flat base change theorem. In any case an easy application of base change formula for an arbitrary fiber product (3.18) yields a morphism:

$$\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k} \longrightarrow \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(E_1 \boxtimes \cdots \boxtimes E_n) \simeq \mathbf{R}p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]}) \ .$$

Now

$$H^q(\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k}) = \mathrm{Tor}_{-q}(p_*(E_{1,B}^{[n]}), \ldots, p_*(E_{k,B}^{[n]})) = 0 \qquad \text{if } q > 0 \ .$$

Moreover, since E_i are vector bundles on X and $E_{i,B}^{[n]}$ are consequently vector bundles on B^n,

$$R^q p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]}) = 0 \qquad \text{for } q > 0 \ ,$$

since the higher direct images coincide with

$$R^q p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]}) = R^q \tilde{w}_*[(p_X^k \circ j)^*(E_1 \boxtimes \cdots \boxtimes E_k)]$$

which is zero for $q > 0$, since the morphism $\tilde{w}$ is finite. Moreover in degree 0 we have the epimorphism:

$$p_*(E_{1,B}^{[n]}) \otimes \ldots \otimes p_*(E_{k,B}^{[n]}) \longrightarrow p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]}) \longrightarrow 0 \ .$$

Since B^n is integral, the term on the right is torsion free. As a consequence, the torsion subsheaf of $p_*(E_{1,B}^{[n]}) \otimes \ldots \otimes p_*(E_{k,B}^{[n]})$ is in the the kernel. Now the epimorphism above is an isomorphism out of the big diagonal of X^n; consequently, the kernel is torsion, hence it is exactly the torsion subsheaf.

$\square$

Corollary 4.5. *The term $p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]})$ can be identified with the term $E_\infty^{0,0}$ of the hyperderived spectral sequence associated to $\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k}$*

Proof. The hyperderived spectral sequence:

$$E_1^{p,q} := \bigoplus_{i_1+\cdots+i_k=p} \mathrm{Tor}_{-q}(\mathcal{C}^{i_1}_{E_1}, \ldots, \mathcal{C}^{i_k}_{E_k})$$

abuts to the hypercohomology

$$H^{p+q}(\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k}) \simeq \mathrm{Tor}_{-p-q}(p_*(E_{1,B}^{[n]}), \ldots, p_*(E_{k,B}^{[n]})) \ .$$

The term $E_1^{0,0} \simeq \mathcal{C}^0_{E_1} \otimes \ldots \otimes \mathcal{C}^0_{E_k}$ is torsion free, because a vector bundle on X^n, since each $\mathcal{C}^0_{E_i}$ is a vector bundle. Hence $E_\infty^{0,0}$ is torsion-free, because subsheaf of $E_1^{0,0}$. Furthermore the kernel of the epimorphism $H^0(\mathcal{C}^\bullet_{E_1} \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_{E_k}) \simeq p_*(E_{1,B}^{[n]}) \otimes \ldots \otimes p_*(E_{k,B}^{[n]}) \twoheadrightarrow E_\infty^{0,0}$ is torsion, because its support is contained in the union of supports of $E_1^{p,-p}$ for $p > 0$, hence in the big diagonal of X^n. Therefore the kernel is exactly the torsion subsheaf, and $E_\infty^{0,0}$ can be identified with $p_*(E_{1,B}^{[n]} \otimes \ldots \otimes E_{k,B}^{[n]})$.

$\square$

4.3 Action of $\mathfrak{S}_k$ on a tensor power of a complex. Derived action.

Consider the category of quasi-coherent sheaves over a scheme. The aim of this section is to describe how the group $\mathfrak{S}_k$ acts on the k-th tensor power $K^\bullet \otimes \ldots \otimes K^\bullet$ of a complex $K^\bullet$, and to extend this action to the derived tensor power $K^\bullet \otimes^L \ldots \otimes^L K^\bullet$. It is clear that this action is fully understood once it is understood on transpositions. For the moment we limit our study to a double tensor power $(C^\bullet \otimes C^\bullet, d_{C^\bullet \otimes C^\bullet})$ of a complex $C^\bullet$. We remind that the complex $C^\bullet \otimes C^\bullet$ is defined by

$$(C^\bullet \otimes C^\bullet)^n = \oplus_{p+q=n} C^p \otimes C^q$$
$$d^n_{C^\bullet \otimes C^\bullet} = \oplus_{p+q=n} \left[d^p_{C^\bullet} \otimes \mathrm{id}_{C^q} + (-1)^q \mathrm{id}_{C^p} \otimes d^q_{C^\bullet} \right] \ .$$

To define an action of $\mathfrak{S}_2$ on $C^\bullet \otimes C^\bullet$ it is necessary not simply to exchange $C^p \otimes C^q$ with $C^q \otimes C^p$, but to introduce a sign, in order to balance the effect of the sign $(-1)^p$ in the definition of the differential. The right involution (which is a *map of complexes*) is defined by:

$$i(u \otimes v) := (-1)^{pq} v \otimes u \qquad \text{if } u \in C^p,\, v \in C^q \ .$$

Suppose now that the complex $C^\bullet$ is right bounded. To extend this action to the derived tensor power $C^\bullet \otimes^L C^\bullet$ it suffices to replace the complex $C^\bullet$ by a $\otimes$-acyclic or projective resolution $R^\bullet$ of the complex $C^\bullet$ and to take the involution just defined on $R^\bullet \otimes R^\bullet$.

Definition 4.6. We denote with $S^2_L C^\bullet$ the subcomplex of invariants of the complex $R^\bullet \otimes R^\bullet$ for the group $\mathfrak{S}_2 = \langle i \rangle$ and we will call it the derived symmetric power of the complex $C^\bullet$. Analogously, we will denote $\Lambda^2_L C^\bullet$ and we will call it the derived exterior power of the complex $C^\bullet$, the subcomplex of anti-invariants of the complex $R^\bullet \otimes R^\bullet$. Their classes of isomorphism in $\mathbf{D}^-(\mathcal{A})$ do not depend on the choice of the resolution $R^\bullet$.

We want now to understand how this involution acts on the hyperderived spectral sequence associated to $C^\bullet \otimes^L C^\bullet$:

$$''E_1^{p,q} = \bigoplus_{h+k=n} \mathrm{Tor}_{-q}(C^h, C^k) \ .$$

To compute $\mathrm{Tor}_{-q}(C^h, C^k)$ we have to consider $\otimes$-acyclic resolutions of C^i:

$$R^{i,\bullet} \longrightarrow C^i \longrightarrow 0\ .$$

The involution will then act on the factors

$$\mathrm{Tor}_{-q}(C^h, C^k) \longrightarrow \mathrm{Tor}_{-q}(C^k, C^h)$$

as the induced application on the cohomology of the complex: $R^{h,\bullet} \otimes R^{k,\bullet}$ by the map

$$u \otimes v \longmapsto (-1)^{(h+i)(k+j)} \qquad \text{for } u \in R^{h,i},\ v \in R^{k,j}\ .$$

More generally we can consider the k-th derived tensor power $C^\bullet \otimes^L \dots \otimes^L C^\bullet$. If $R^\bullet$ is an $\otimes$-acyclic resolution of $C^\bullet$, the group $\mathfrak{S}_k$ acts on the tensor product

$$C^\bullet \otimes^L \dots \otimes^L C^\bullet \simeq R^\bullet \otimes \dots \otimes R^\bullet$$

by permutation of the factors, where the action of a transposition on two consecutive factors is exactly the one described above. This action does not depend on the choice of the resolution $R^\bullet$. As shown above for the case of two factors, we have a $\mathfrak{S}_k$-action on the hyperderived spectral sequence associated to $C^\bullet \otimes^L \dots \otimes^L C^\bullet$.

Definition 4.7. We define the derived symmetric power $S^k_L C^\bullet$ as the subcomplex of invariants of the complex $C^\bullet \otimes^L \dots \otimes^L C^\bullet$ by the action of the group $\mathfrak{S}_k$. Analogously, we define the derived exterior power $\Lambda^k_L C^\bullet$ as the complex of anti-invariants of the complex $C^\bullet \otimes^L \dots \otimes^L C^\bullet$ by the action of $\mathfrak{S}_k$.

Suppose now that X is a variety with the action of a finite group G and that $C^\bullet$ is a complex of G-equivariant sheaves on X, $C^\bullet \in \mathbf{D}^-_G(X)$. Then the $\mathfrak{S}_k$-action on $C^\bullet \otimes^L \dots \otimes^L C^\bullet$ commutes with the diagonal action of G on $C^\bullet \otimes^L \dots \otimes^L C^\bullet$, defined as the diagonal action on the complex $R^\bullet \otimes \dots \otimes R^\bullet$, where $R^\bullet$ is a locally free resolution of $C^\bullet$. We then a well defined $\mathfrak{S}_k \times G$ action on the derived tensor power $C^\bullet \otimes^L \dots \otimes^L C^\bullet$.

Remark 4.8. Let $C^\bullet$ a complex of coherent sheaves on a variety X. Then the k-th tensor power $C^\bullet \otimes \dots \otimes C^\bullet$ is naturally a complex of $\mathfrak{S}_k$-equivariant sheaves on X, where $\mathfrak{S}_k$ acts trivially on the variety X. Analogously, if $R^\bullet$ is a locally free resolution of $C^\bullet$, the k-th tensor power $R^\bullet \otimes \dots \otimes R^\bullet$ of the complex $R^\bullet$ is a complex of $\mathfrak{S}_k$-equivariant sheaves on X. Therefore we can see the symmetric derived power S^k_L as the composition of the derived tensor power and the $\mathfrak{S}_k$-fixed points functor:

$$S^k_L : \mathbf{D}^-(X) \xrightarrow{(-)^{(\otimes^L)^k}} \mathbf{D}^-_{\mathfrak{S}_k}(X) \xrightarrow{[-]^{\mathfrak{S}_k}} \mathbf{D}^-(X)\ .$$

Analogously the derived exterior power Λ^k_L is the composition of the derived tensor power with the $\mathfrak{S}_k$-anti-invariants functor, or the composition:

$$\Lambda^k_L : \mathbf{D}^-(X) \xrightarrow{(-)^{(\otimes^L)^k}} \mathbf{D}^-_{\mathfrak{S}_k}(X) \xrightarrow{-\otimes\varepsilon} \mathbf{D}^-_{\mathfrak{S}_k}(X) \xrightarrow{[-]^{\mathfrak{S}_k}} \mathbf{D}^-(X)\ ,$$

where ε is the alternating representation of $\mathfrak{S}_k$.

In the hypothesis of theorem 4.4, with $E_i = E$, for all $i = 1, \dots, k$ we have the corollary:

Corollary 4.9. *Consider the image $\mathbf{\Phi}(S^k E^{[n]})$ of the symmetric power of a tautological bundle $E^{[n]}$, by the BKRH equivalence. The mapping cone of the morphism:*

$$S^k_L \mathcal{C}^\bullet_E \longrightarrow \mathbf{\Phi}(S^k E^{[n]})$$

is acyclic in degree > 0*. As a consequence the higher direct images* $R^q p_*(S^k E_B^{[n]})$ *vanish for* $q > 0$ *and in degree* 0 *we have the epimorphism:*

$$S^k p_*(E_B^{[n]}) \twoheadrightarrow p_*(S^k E_B^{[n]}) \ ,$$

whose kernel is the torsion subsheaf. Therefore the image $\mathbf{\Phi}(S^k E^{[n]})$ *can be identified with the* $\mathfrak{S}_k$*-invariants* $(E_\infty^{0,0})^{\mathfrak{S}_k}$ *of the term* $E_\infty^{0,0}$ *of the hyperderived spectral sequence associated to* $\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet$*.*

Proof. The corollary is an immediate consequence of taking the $\mathfrak{S}_k$-invariants in the morphism (34), and the clear fact that the $\mathfrak{S}_k$-action on the tensor power commutes with the pull-back by q and push forward by p.

□

In the same way, taking $\mathfrak{S}_k$-anti-invariants in (34) we get

Corollary 4.10. *Consider the image* $\mathbf{\Phi}(\Lambda^k E^{[n]})$ *of the exterior power of a tautological bundle* $E^{[n]}$*, by the BKRH equivalence. The mapping cone of the morphism:*

$$\Lambda^k_L \mathcal{C}_E^\bullet \longrightarrow \Phi(\Lambda^k E^{[n]})$$

is acyclic in degree > 0*. As a consequence the higher direct images* $R^q p_*(\Lambda^2 E_B^{[n]})$ *vanish for* $q > 0$ *and in degree* 0 *we have the epimorphism:*

$$\Lambda^2 p_*(E_B^{[n]}) \twoheadrightarrow p_*(\Lambda^2 E_B^{[n]}) \ ,$$

whose kernel is the torsion subsheaf. Therefore the image $\Phi(\Lambda^k E^{[n]})$ *can be identified with the* $\mathfrak{S}_k$*-anti-invariants* $(E_\infty^{0,0} \otimes \varepsilon)^{\mathfrak{S}_k}$ *of the term* $E_\infty^{0,0}$ *of the hyperderived spectral sequence associated to* $\mathcal{C}_E^\bullet \otimes^L \ldots \otimes^L \mathcal{C}_E^\bullet$*.*

4.4 Derived Schur functors

We are now going to define general Schur functors of a complex $C^\bullet$ of locally free sheaves and its derived version. Let V_ν the irreducible representation of the group $\mathfrak{S}_k$ associated to the partition $\nu : \nu_1 \geq \nu_2 \geq \cdots \geq \nu_l$ of k (see Fulton-Harris [50]). We can obtain it as the left ideal of the algebra $\mathbb{C}[\mathfrak{S}_k]$ generated by the Young symmetrizer c_ν of ν. The Young symmetrizer can be computed from the Young tableau of ν: let P_ν the subgroup of $\mathfrak{S}_k$ that fixes the columns and Q_ν the subgroup fixing the rows. If

$$a_\nu = \sum_{g \in P_\nu} g$$

and

$$b_\nu = \sum_{g \in Q_\nu} \varepsilon_g g \ ,$$

where ε_g is the signature of the permutation g, then the Young symmetrizer is the element:

$$c_\nu = a_\nu b_\nu \ .$$

For all locally free $\mathfrak{S}_k$-sheaf E on an algebraic variety M, there is a decomposition of E in a direct sum of locally free subsheaves:

$$E \simeq \bigoplus_\nu V_\nu \otimes \mathcal{H}om_{\mathfrak{S}_k}(V_\nu \otimes \mathcal{O}_M, E) \ .$$

Now for every locally free sheaf W on M, the k-th tensor power $W^{\otimes^k}$ is naturally equipped with a $\mathfrak{S}_k$-action. We denote with $S^\nu W$ the Schur functor

$$S^\nu W := \mathcal{H}om_{\mathfrak{S}_k}(V_\nu \otimes \mathcal{O}_M, W^{\otimes^k})$$

associated with the partition ν of k. The symmetric power is the Schur functor associated to the partition k of k, and the exterior power is the one associated to the partition $(1, \ldots, 1)$.

Consider now a complex of coherent sheaves $C^\bullet$ on M and $R^\bullet$ a locally free resolution of $C^\bullet$. Let us form the k-th tensor power $K^\bullet$ of $R^\bullet$: $K^\bullet = R^\bullet \otimes \ldots \otimes R^\bullet$. We can decompose the complex $K^\bullet$ as follows:

$$K^\bullet = \bigoplus_\nu V_\nu \otimes \mathcal{H}om^\bullet_{\mathfrak{S}_k}(V_\nu \otimes \mathcal{O}_M, K^\bullet)\ .$$

We denote with $S^\nu_L C^\bullet \in \mathbf{D}^-(M)$ and we call it the derived Schur functor of the complex $C^\bullet$ the complex $\mathcal{H}om^\bullet_{\mathfrak{S}_k}(V_\nu \otimes \mathcal{O}_M, K^\bullet)$. Its isomorphism class does not depend on the choice of the resolution $R^\bullet$.

As we did for symmetric and exterior powers, we deduce from theorem 4.4:

Corollary 4.11. *Let ν a partition of n. Consider the image $\mathbf{\Phi}(S^\nu E^{[n]})$ of the Schur functor associated to the partition ν of a tautological bundle $E^{[n]}$, by the BKRH equivalence. The mapping cone of the morphism:*

$$S^\nu_L \mathcal{C}^\bullet_E \longrightarrow \mathbf{\Phi}(S^\nu E^{[n]})$$

is acyclic in degree > 0. As a consequence the higher direct images $R^q p_*(S^\nu E^{[n]}_B)$ vanish if $q > 0$ and in degree 0 we have the epimorphism:

$$S^\nu p_*(E^{[n]}_B) \twoheadrightarrow p_*(S^\nu E^{[n]}_B)$$

whose kernel is the torsion subsheaf. Therefore the image $\mathbf{\Phi}(S^\nu E^{[n]})$ can be identified with the sheaf

$$S^\nu E^{0,0}_\infty = \mathcal{H}om_{\mathfrak{S}_k}(V_\nu \otimes \mathcal{O}_{X^n}, E^{0,0}_\infty)\ ,$$

in terms of $E^{0,0}_\infty$ of the hyperderived spectral sequence associated to $\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E$.

4.5 The image of $E^{[n]} \otimes E^{[n]}$ by the Hilbert-Chow morphism

In the last section we identified (corollary 4.5) the image of a tensor product of a tautological bundle $E^{[n]}$ for the BKRH equivalence with the term $E^{0,0}_\infty$ of the hyperderived spectral sequence $E^{p,q}_r$ associated to the complex $\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E$. Working out the spectral sequence in all generality is hard, due to evident technical difficulties. Nonetheless, the knowledge of this image, although of great interest, is not really necessary for applications to computation of equivariant cohomology; all what we really need is the knowledge of the $\mathfrak{S}_n$-*invariants* of the image $\mathbf{\Phi}((E^{[n]})^{\otimes^k})$. We recall that if F is a coherent $\mathfrak{S}_n$-sheaf on the product X^n, we indicate with $F^{\mathfrak{S}_n}$ the $\mathfrak{S}_n$-invariant push forward $F^{\mathfrak{S}_n} := \pi_*^{\mathfrak{S}_n} F$ for the projection $\pi : X^n \longrightarrow S^n X$ onto the symmetric variety. By proposition 1.19 knowing the invariants $\mathbf{\Phi}(E^{[n]^{\otimes^k}})^{\mathfrak{S}_n}$ amounts to knowing the derived direct image $\mathbf{R}\mu_*((E^{[n]})^{\otimes^k})$ by the Hilbert-Chow morphism. It follows that $\mu_*((E^{[n]})^{\otimes^k})$ can be identified with the $\mathfrak{S}_n$-invariants $(E^{0,0}_\infty)^{\mathfrak{S}_n}$ of the term $E^{0,0}_\infty$ of the hyperderived spectral sequence associated to $\mathcal{C}^\bullet_E \otimes^L \ldots \otimes^L \mathcal{C}^\bullet_E$, or equivalently, since the functor $[-]^{\mathfrak{S}_n}$ is exact, with the term $\mathcal{E}^{0,0}_\infty$ of the spectral sequence

$$\mathcal{E}^{p,q}_1 = (E^{p,q}_1)^{\mathfrak{S}_n}$$

of invariants of the hyperderived spectral sequence $E^{p,q}_1$. This new spectral sequence of coherent sheaves (over $S^n X$) turns out to be much simpler than the original one and effectively useful (at least for $k = 2$) to compute the image $\mathbf{R}\mu_*(E^{[n]} \otimes E^{[n]})$. The tensor power $(E^{[n]})^{\otimes^2}$ splits into symmetric and exterior components:

$$E^{[n]} \otimes E^{[n]} \simeq S^2 E^{[n]} \oplus \Lambda^2 E^{[n]}$$

and we have seen that taking the symmetric (or exterior) power corresponds to taking invariants (or anti-invariants) by $\mathfrak{S}_2$ on $E^{[n]} \otimes E^{[n]}$. The $\mathfrak{S}_2$-action on the tensor power commutes with the geometric diagonal action of $\mathfrak{S}_n$, so we can further simplify the picture by looking at the spectral sequences:

$$'\mathcal{E}^{p,q}_1 = (E^{p,q}_1)^{\mathfrak{S}_2 \times \mathfrak{S}_n} \quad ; \quad ''\mathcal{E}^{p,q}_1 = (E^{p,q}_1 \otimes \varepsilon_2)^{\mathfrak{S}_2 \times \mathfrak{S}_n} \tag{35}$$

where ε_2 indicates the alternating representation of $\mathfrak{S}_2$. The terms ${}'\mathcal{E}_\infty^{0,0}$ and ${}''\mathcal{E}_\infty^{0,0}$ of these spectral sequences are quasi-isomorphic to the images $\mathbf{\Phi}(S^2 E^{[n]})$ and $\mathbf{\Phi}(\Lambda^2 E^{[n]})$, respectively.

The aim of this section is to prove:

$$\mathbf{R}\mu_*(E^{[n]} \otimes E^{[n]}) \simeq (\mathcal{C}_E^\bullet \otimes \mathcal{C}_E^\bullet)^{\mathfrak{S}_n}$$

or equivalently the two:

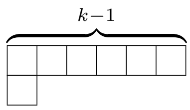
$$\mathbf{R}\mu_*(S^2 E^{[n]}) \simeq (S^2 \mathcal{C}_E^\bullet)^{\mathfrak{S}_n} \tag{36}$$

$$\mathbf{R}\mu_*(\Lambda^2 E^{[n]}) \simeq (\Lambda^2 \mathcal{C}_E^\bullet)^{\mathfrak{S}_n} \ . \tag{37}$$

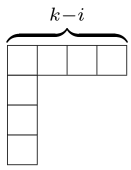
when E is a line bundle on X. We will always suppose from now on that $E^{[n]}$ is the tautological vector bundle associated to the line bundle E on X.

4.6 Preliminary results

We briefly review some basic facts about representations of $\mathfrak{S}_n$. Our main reference is Fulton-Harris [50]. We recall that ρ_k denotes the standard representation of $\mathfrak{S}_k$: its Young diagram is the hook:



Moreover, for all $0 \leq i \leq k-1$, $\Lambda^i \rho_k$ is the irreducible representation associated to the Young diagram:



In particular, $\Lambda^{k-1}\rho_k \simeq \varepsilon$, the alternating representation of $\mathfrak{S}_k$. Finally we recall, by Frobenius formula, that the characters of all irreducible representations of $\mathfrak{S}_k$ are rational. Let $R = \mathbb{C}^k$ the natural representation of $\mathfrak{S}_k$: $\mathfrak{S}_k$ acts on R by permutation of the canonical basis $e_1, \dots, e_k$. We recall, that the natural representation R splits as $R \simeq \rho_k \oplus 1$.

Lemma 4.12. *Let Y a smooth subvariety of codimension r of a smooth variety X. Let $N_{Y/X}$ the normal bundle of Y in X. Let ρ_k the standard representation of $\mathfrak{S}_k$. We have an isomorphism:*

$$\mathrm{Tor}_q(\underbrace{\mathcal{O}_Y, \dots, \mathcal{O}_Y}_{k\text{-}times}) \simeq \Lambda^q(N^*_{Y/X} \otimes \rho_k)$$

as $\mathfrak{S}_k$-representations.

Proof. We first verify the statement locally. Suppose Y is the scheme of zeros of a section s of a vector bundle F of rank r, transverse to the zero section. Consider the Koszul resolution $K^\bullet := K^\bullet(F, s)$ of the structural sheaf $\mathcal{O}_Y$. Let $R \simeq \mathbb{C}^k$ the natural representation of $\mathfrak{S}_k$ and e_i the vectors of the canonical basis. Let $\sigma = \sum_{i=1}^k e_i$ the canonical element, which is invariant for the $\mathfrak{S}_k$-action. The Koszul complex $K^\bullet(F \otimes R, s \otimes \sigma)$ is $\mathfrak{S}_k$-isomorphic to the tensor product $K^\bullet \otimes \ldots \otimes K^\bullet$ and consequently its $(-q)$-cohomology is $\mathfrak{S}_k$-isomorphic to $\mathrm{Tor}_q(\mathcal{O}_Y, \dots, \mathcal{O}_Y)$. On the other hand, consider the Koszul complex $K^\bullet(F \otimes \rho_k, 0)$; we have the isomorphism of $\mathfrak{S}_k$-representations:

$$K^\bullet(F \otimes R, s \otimes \sigma) \xrightarrow{\simeq} K^\bullet(F, s) \otimes K^\bullet(F \otimes \rho_k, 0)$$

where $\mathfrak{S}_k$ acts trivially on $K^\bullet(F,s)$. Consequently we obtain an isomorphism of $\mathfrak{S}_k$-representations:

$$\mathrm{Tor}_q(\mathcal{O}_Y,\dots,\mathcal{O}_Y) \simeq \mathcal{O}_Y \otimes \Lambda^q(F^* \otimes \rho_k) \simeq \Lambda^q(N^*_{Y/X} \otimes \rho_k)$$

because of the identification $F\big|_Y \simeq N_{Y/X}$. In the same fashion of proposition 3.13, we verify that this isomorphism does not depend on the choice of the vector bundle F and on the section s. Hence, the local isomorphisms glue together and allow to define the above isomorphism globally.

□

We already know that if Y_1 and Y_2 are two smooth subvarieties of a smooth variety X, contained transversally in a smooth subvariety Y of X we have:

$$\mathrm{Tor}_i(\mathcal{O}_{Y_1},\mathcal{O}_{Y_2}) \simeq \Lambda^i N^*_{Y/X}\Big|_{Y_1\cap Y_2}.$$

Lemma 4.13. *Let $\tau : \mathrm{Tor}_i(\mathcal{O}_{Y_1},\mathcal{O}_{Y_2}) \longrightarrow \mathrm{Tor}_i(\mathcal{O}_{Y_2},\mathcal{O}_{Y_1})$ the canonical transposition. Then the diagram:*

$$\begin{tikzcd} \mathrm{Tor}_i(\mathcal{O}_{Y_1},\mathcal{O}_{Y_2}) \arrow[r, "\tau"] \arrow[d] & \mathrm{Tor}_i(\mathcal{O}_{Y_2},\mathcal{O}_{Y_1}) \arrow[d] \\ \Lambda^i N^*_{Y/X}\big|_{Y_1\cap Y_2} \arrow[r, "(-1)^i"] & \Lambda^i N^*_{Y/X}\big|_{Y_1\cap Y_2} \end{tikzcd}$$

commutes.

Proof. In the notations of lemma 3.13, $K^\bullet(u,s_i)$ is the Koszul complex giving the resolution of $\mathcal{O}_{Y_i}$. We have the commutative diagram of complexes:

$$\begin{tikzcd} K^\bullet(u,s_1)\otimes K^\bullet(u,s_2) \arrow[r] \arrow[d] & K^\bullet(u)\otimes K^\bullet(u)\otimes K^\bullet(s_1)\otimes K^\bullet(s_2) \arrow[d] \\ K^\bullet(u,s_2)\otimes K^\bullet(u,s_1) \arrow[r] & K^\bullet(u)\otimes K^\bullet(u)\otimes K^\bullet(s_2)\otimes K^\bullet(s_1) \end{tikzcd}$$

where the first vertical arrow is the transposition of factors $K^\bullet(u,s_i)$ and the second vertical arrow is the transposition of factors $(1,2)$ and $(3,4)$. The lemma follows.

□

Lemma 4.14. *1. The representation $\Lambda^i(\rho_k \oplus \rho_k) \simeq \Lambda^i(\mathbb{C}^2 \otimes \rho_k)$, $0 \le i \le 2k-2$, has fixed points if and only if i is even. In this case the subspace of fixed points is one dimensional.*

2. Let u, v a basis of $\mathbb{C}^2$. Consider the $\mathfrak{S}_k$-invariant bivector:

$$\omega = \sum_{i=1}^{k} ue_i \wedge ve_i \in \Lambda^2(\mathbb{C}^2 \otimes R)$$

and let $\omega^l \in \Lambda^{2l}(\mathbb{C}^2 \otimes R)$ its l-th exterior power. Consider the projection: $\pi : R \longrightarrow \rho_k$. If $1 \le l \le k-1$ the image ω^R_l of $\omega^l/(l+1)!$ in $\Lambda^{2l}(\mathbb{C}^2 \otimes \rho_k)$ is nonzero.

3. Let $i \in \{1, \dots k\}$, and $G_i = \mathrm{Stab}_{\mathfrak{S}_k}\{\mathrm{i}\}$. Set $R_i = R/\langle e_i\rangle \simeq \mathbb{C}^{k-1}$. The projection

$$\varphi_i : \mathbb{C}^2 \otimes \rho_k \longrightarrow \mathbb{C}^2 \otimes \rho_{k-1}$$

induced from the projection from $R \longrightarrow R_i$, is G_i-equivariant and for $1 \le l \le k-2$ the image of ω_l^R for the projection $\Lambda^{2l}\varphi_i$ is exactly $\omega_l^{R_i}$.

Proof. 1. Given a finite group G, consider the following hermitian inner product on the space of central functions:

$$\langle u, v\rangle = \frac{1}{|G|}\sum_{g\in G} u(g)\bar{v}(g) \ .$$

Given two representations ρ and τ of the group G, we have

$$\dim \mathrm{Hom}(\tau, \rho) = \langle \chi_\tau, \chi_\rho\rangle \ .$$

In particular the dimension of the subspace of fixed points of a representation ρ is given by:

$$\dim \rho^G = \frac{1}{|G|}\sum_{g\in G} \chi_\rho(g) \ .$$

Since

$$\Lambda^i(\rho_k \oplus \rho_k) \simeq \bigoplus_{j=1}^{i} \Lambda^j\rho_k \otimes \Lambda^{i-j}\rho_k$$

and since the characters of the irreducible representations of $\mathfrak{S}_k$ are rational:

$$\begin{aligned}
\dim(\Lambda^i(\rho_k \oplus \rho_k))^{\mathfrak{S}_k} &= (\chi_{\Lambda^i(\rho_k\oplus\rho_k)}, \chi_1)\\
&= \sum_{j=0}^{i} (\chi_{\Lambda^j\rho_k\otimes\Lambda^{i-j}\rho_k}, \chi_1)\\
&= \sum_{j=0}^{i} (\chi_{\Lambda^j\rho_k}\cdot\chi_{\Lambda^{i-j}\rho_k}, \chi_1)\\
&= \sum_{j=0}^{i} (\chi_{\Lambda^j\rho_k}, \chi_{\Lambda^{i-j}\rho_k})
\end{aligned}$$

Now if $j \ne i-j$, $\Lambda^j\rho_k$ et $\Lambda^{i-j}\rho_k$ are two different irreducible representations of $\mathfrak{S}_k$, and then $(\chi_{\Lambda^j\rho_k}, \chi_{\Lambda^{i-j}\rho_k}) = 0$. The only possibility is then for $i-j=i$, which yields $i=2j$ and $(\chi_{\Lambda^i(\rho_k\oplus\rho_k)}, \chi_1) = 1$.

2. We can restrict ourselves to the case $l = k-1$. We have:

$$\begin{aligned}
\frac{\omega^{k-1}}{k!} &= \frac{1}{k!}\left(\sum_{i=1}^{k} ue_i \wedge ve_i\right)^{k-1}\\
&= \frac{1}{k!}(k-1)!\sum_{i=1}^{k} \widehat{ue_i \wedge ve_i}
\end{aligned}$$

where $\widehat{ue_i \wedge ve_i}$ indicates

$$\widehat{ue_i \wedge ve_i} = ue_1 \wedge ve_1 \wedge \ldots \wedge ue_{i-1} \wedge ve_{i-1} \wedge ue_{i+1} \wedge ve_{i+1} \wedge \ldots \wedge ue_k \wedge ve_k \ .$$

We now remark that the projection $\Lambda^{2k-2}(\mathrm{id}\otimes\pi)(\widehat{ue_i \wedge ve_i})$ is nonzero and always the same for every i: therefore the projection of $\omega^{k-1}/k!$ on $\Lambda^{2k-2}(\mathbb{C}^2\otimes\rho_k)$ is $\Lambda^{2k-2}(\mathrm{id}\otimes\pi)(\widehat{ue_1 \wedge ve_1})$, that is, a volume element in $\Lambda^{2k-2}(\mathbb{C}^2\otimes\rho_k)$.

3. This statement is evident once remarked that the commutative diagram:

$$\begin{tikzcd} \Lambda^{2l}(\mathbb{C}\otimes R) \arrow[r, "\Lambda^{2l}\pi_i"] \arrow[d, "\Lambda^{2l}\pi^R"'] & \Lambda^{2l}(\mathbb{C}\otimes R_i) \arrow[d, "\Lambda^{2l}\pi^{R_i}"'] \\ \Lambda^{2l}(\mathbb{C}\otimes \rho_k) \arrow[r, "\Lambda^{2l}\varphi_i"] & \Lambda^{2l}(\mathbb{C}\otimes \rho_{k-1}) \end{tikzcd}$$

is G_i-equivariant for $l \leq k-2$, since the diagram:

$$\begin{tikzcd} \mathbb{C}^2\otimes R \arrow[r, "\pi_i"] \arrow[d, "\pi^R"'] & R_i \arrow[d, "\pi^{R_i}"'] \\ \mathbb{C}\otimes \rho_k \arrow[r, "\varphi_i"] & \mathbb{C}\otimes \rho_k \end{tikzcd}$$

is G-equivariant.

□

We recall now a lemma by Danila ([23]). We recall the context. Let G a group acting on a variety Y, and $\varphi : G \longrightarrow \mathfrak{S}_k$ an epimorphism of groups. Let $F_1, \ldots, F_k$ a collection of coherent sheaves on Y such that the G action is compatible via φ with the permutation of the factors: this means that we have canonical isomorphisms:

$$h_g : F_{\varphi(g^{-1})(i)} \xrightarrow{\simeq} g^*F_i \ .$$

In this way the direct sum $\oplus_{i=1}^k F_i$ becomes a G-sheaf on Y. Let now $G_i = \mathrm{Stab}_G\{i\}$: for all $g \in G_i$ then

$$F_i \simeq g^*F_i \ .$$

The facts just listed induce corresponding facts on the spaces of sections $M_i = H^0(F_i)$ and $M = \oplus_{i=1}^k M_i$. We inherit morphisms:

$$\lambda_g : M_i \longrightarrow M_{\varphi(g)(i)}$$

setting $\lambda_g(s) := h_g s \circ g^{-1}$ In particular M becomes a left-G-representation.

Lemma 4.15. *The projection* $\mathrm{pr}_i : M \longrightarrow M_i$ *is* G_i*-equivariant and induces an isomorphism:*

$$M^G \longrightarrow M_i^{G_i} \ .$$

The lemmas 4.14 and 4.15 are the fundamental tools we will use to reduce the spectral sequences (35).

Remark 4.16. Suppose that the morphism $\varphi : G \longrightarrow \mathfrak{S}_k$ is not surjective. This is equivalent to saying that the G action on $\{1, \ldots, k\}$ is not transitive. Taking the orbits $I_1, \ldots, I_l$ for the G-action on $\{1, \ldots, k\}$ we can always reconduct us to a transitive action for which we can apply Danila's lemma 4.15 to G-homogeneous modules $M_{I_j} = \oplus_{i \in I_j} M_i$.

4.7 The $\mathfrak{S}_n$-equivariant spectral sequence

In this section we will proceed to work out the spectral sequences (35) ${}'\mathcal{E}_1^{p,q}$ and ${}''\mathcal{E}_1^{p,q}$. We begin with recalling some notations from the preceding chapter and pointing out some basic but important facts.

Let J a subset of $\{1, \ldots, n\}$. We denote with $\bar{J}$ the complementary of J in $\{1, \ldots, n\}$. We indicate with $p_J : X^n \longrightarrow X^J$ the projections onto factors in J, Δ_J the pull-back by p_J of the small diagonal in

X^J. We denote, for brevity's sake, the normal bundle N_{Δ_J} to the diagonal Δ_J simply with N_J. We will denote once for all the group $\mathfrak{S}_n$ with G, and the stabilizer in G of the subset J with G_J. Suppose $|J| = m$, $1 < m \leq n$. Then the normal bundle N_J is isomorphic as a sheaf on X^n to: $N_J \simeq N_{\Delta/X^m} \boxtimes \mathcal{O}_{X^{n-m}}$.

The stabilizer G_J acts on N_J as $\mathfrak{S}(J) \times \mathfrak{S}(\bar{J})$, but $\sigma(\bar{J})$ acts trivially on the normal bundle: the G_J-action on N_J is then isomorphic over a point $x \in \Delta_J$, to the representation $\rho_m \oplus \rho_m$ of $\mathfrak{S}_m$. Therefore the normal bundle N_J is locally isomorphic to the bundle:

$$N_J \simeq \mathcal{O}(\rho_m \oplus \rho_m) \boxtimes \mathcal{O}_{X^{n-m}} \ .$$

The general term of the spectral sequence $E_1^{p,q}$ becomes, due to the previouses lemmas:

$$\begin{aligned} E_1^{p,q} &\simeq \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}_E^i, \mathcal{C}_E^j) \simeq \bigoplus_{\substack{i+j=p \\ |I|=i+1 \\ |J|=j+1}} \mathrm{Tor}_{-q}(E_I, E_J) \\ &\simeq \bigoplus_{\substack{i+j=p \\ |I|=i+1 \\ |J|=j+1}} E_{I\cup J}^{\otimes^2} \otimes \Lambda^q N^*_{I\cap J}\Big|_{\Delta_{I\cup J}} \end{aligned}$$

Remark 4.17. If $I \cap J = \emptyset$ the diagonal Δ_J and Δ_I are transverse and

$$\mathrm{Tor}_{-q}(E_I, E_J) = 0 \quad \text{if } q < 0 \ .$$

If $I \cap J \neq \emptyset$ the intersection $\Delta_I \cap \Delta_J$ is the diagonal $\Delta_{I\cup J}$. Therefore we know that the vanishing of all the $\mathrm{Tor}_{-q}(E_I, E_J)$, $q < 0$ is equivalent to the transversality of the intersection of Δ_I and Δ_J. This happens if and only if

$$\operatorname{codim} \Delta_I + \operatorname{codim} \Delta_J = \operatorname{codim} \Delta_{I\cup J}$$

which reads

$$2(|I| - 1) + 2(|J| - 1) = 2(|I \cup J| - 1) \ .$$

Hence $|I| + |J| = |I \cup J| - 1$ which is equivalent to $|I \cap J| = 1$. The same thing can be obtained remarking that if $|I \cap J| = 1$, then, by definition, the normal bundle $N_{I\cap J}$ is the normal bundle of X^n to X^n, hence zero. If $|I \cap J| > 1$, $N_{I\cap J} \neq 0$.

Remark 4.18. Even if we have just seen that the general term $E_1^{p,q}$ of the hyperderived spectral sequence can be expressed in terms of the q-exterior power of conormal bundles of diagonals, this *does not mean* that the G-action on $E_1^{p,q}$ is the one induced by the G-action on $\Lambda^q N^*_{I\cap J}\big|_{I\cup J}$. The induced action explained above takes into account only the "geometric" action on the conormal bundle, while the G-representation of $\mathrm{Tor}_{-q}(\mathcal{C}_E^i, \mathcal{C}_E^j)$ have to take into account as well the G-action on the sheaves $\mathcal{C}_E^i$, $\mathcal{C}_E^j$ considered when we defined the G-action on the complex $\mathcal{C}_E^\bullet$ in the previous chapter (page 38). In this way only the spectral sequence $E_1^{p,q}$ is G-equivariant, and it makes sense to speak about invariants. The right G-action on $E_1^{p,q}$ in then induced by the "geometric" action on the conormal bundles of the diagonals *twisted by the signs* we get by the G-action on the sheaves $\mathcal{O}_{\Delta_I}$, as explained in section 3.3.

We now come to a closer look of the spectral sequences (35). We will denote once for all the group $\mathfrak{S}_2$ with H. We recall that the group H acts by transposition of the factors, while the group G acts in the way described above. The two actions commute: therefore we can define an action of the direct product group $G \times H$, which we will denote P. The spectral sequences (35) are defined as the invariants:

$$'\mathcal{E}_1^{p,q} = (E_1^{p,q})^{G\times H} \simeq \left(\bigoplus_{\substack{i+j=p\\|I|=i+1\\|J|=j+1}} \mathrm{Tor}_{-q}(E_I, E_J) \right)^{G\times H}$$

$$''\mathcal{E}_1^{p,q} = (E_1^{p,q} \otimes \varepsilon_H)^{G\times H} \simeq \left(\bigoplus_{\substack{i+j=p\\|I|=i+1\\|J|=j+1}} \mathrm{Tor}_{-q}(E_I, E_J) \otimes \varepsilon_H \right)^{G\times H}$$

Remark 4.19. When we want to use Danila's lemma, the first thing we remark is that the direct sums above are indexed by the disjoint union

$$\mathcal{P}^p := \coprod_{i+j=p} \mathcal{P}_{i+1} \times \mathcal{P}_{j+1}$$

where $\mathcal{P}_r$ is the set of parts of $\{1, \ldots, n\}$ with r elements. The group G acts transitively on $\mathcal{P}_{i+1}$, for $i = 1, \ldots, n$, and acts by diagonal action on $\mathcal{P}_{i+1} \times \mathcal{P}_{j+1}$; however, this last action is not transitive: the orbits are characterized by couples $(I, J) \in \mathcal{P}_{i+1} \times \mathcal{P}_{j+1}$ such that $|I \cap J| = r$. We therefore have $\min\{i+1, j+1\}+1$ orbits $\mathcal{O}_r$ of G in $\mathcal{P}_{i+1}\times\mathcal{P}_{j+1}$. We deduce an action of G on all $\mathcal{P}^p$. The group H acts on $\mathcal{P}^p$ as well by permutations of factors. The G and H actions on $\mathcal{P}^p$ commute, yielding an P-action on $\mathcal{P}^p$: the subsets $\mathcal{P}_{i+1} \times \mathcal{P}_{j+1} \coprod \mathcal{P}_{j+1} \times \mathcal{P}_{i+1}$ are invariant by the P-action, with orbits $H\mathcal{O}_r$. Let $(I, J) \in \mathcal{P}_{i+1}\times\mathcal{P}_{j+1}$. Let $G_{I,J}$ the stabilizer of the couple (I, J) for the G-action and $P_{I,J}$ the one for the P-action. It is clear that, if $|I| \neq |J|$, $P_{I,J} \simeq G_{I,J}$. If $|I| = |J|$ then $G_{I,J}$ is a normal subgroup of index 2 of $P_{I,J}$ and the quotient $P_{I,J}/G_{I,J}$ is isomorphic (not canonically) to a subgroup of $P_{I,J}$ generated by an element $\sigma_{I,J} \times \tau$ where τ is the transposition of factors and $\sigma_{I,J}$ is an involution of $\{1, \ldots, n\}$ fixing all elements in $I \cap J$ and such that $\sigma(I \setminus (I \cap J)) = J \setminus (I \cap J)$. In other words:

$$P_{I,J} \simeq \langle G_{I,J}, \sigma_{I,J} \times \tau\rangle \ .$$

When I=J, then we can choose the identity $\mathrm{id}_{\mathfrak{S}_m}$ as $\sigma_{I,J}$. Therefore:

$$P_{I,I} \simeq \langle G_{I,I}, \tau\rangle$$

with $G_{I,I} \simeq \mathfrak{S}(I) \times \mathfrak{S}(\bar{I})$.

At this point we are ready to apply Danila's lemma. We only have to split the direct sum:

$$E_1^{p,q} \simeq \bigoplus_{\substack{i+j=p\\|I|=i+1\\|J|=j+1}} \mathrm{Tor}_{-q}(E_I, E_J)$$

into homogeneous components, indexed by the P orbits in $\mathcal{P}^p$. The sums:

$$W_{r,-q}^{i,j} = \bigoplus_{\substack{|I|=i+1\\|J|=j+1}} \mathrm{Tor}_{-q}(E_I, E_J)$$

are always G-homogeneous. Hence, if $i < j$

$$W_{r,-q}^{i,j} \oplus W_{r,-q}^{j,i}$$

are always P-homogeneous and if $i = j$

$$W_{r,-q}^{i,i}$$

is always P-homogeneous. Therefore $E_1^{p.q}$ splits into homogeneous components in this way:

$$E_1^{p,q} \simeq \bigoplus_{r=0}^{\min\{i,j\}+2} \bigoplus_{\substack{i+j=p \\ i<j}} (W_{r,-q}^{i,j} \oplus W_{r,-q}^{j,i}) \qquad \text{if } p \text{ is odd}$$

and

$$E_1^{2p,q} \simeq \bigoplus_{r=0}^{\min\{i,j\}+2} \left(\bigoplus_{\substack{i+j=2p \\ i<j}} (W_{r,-q}^{i,j} \oplus W_{r,-q}^{i,j}) \bigoplus W_{r,-q}^{p,p} \right) \qquad \text{otherwise .}$$

In this context Danila's lemma reads:

Lemma 4.20. *Let $(i,j,r) \in \mathbb{N}^3$. Let $I, J \in \mathcal{P}_{i+1} \times \mathcal{P}_{j+1}$ and $|I \cap J| = r$ (that is $(I,J) \in \mathcal{O}_r$). Then:*

1. *If $i < j$, we have an isomorphism:*

$$\Gamma(X^n, W_{r,-q}^{i,j} \oplus W_{r,-q}^{i,j})^P \xrightarrow{\simeq} \Gamma(X^n, \mathrm{Tor}_{-q}(E_I, E_J))^{G_{I,J}} \ .$$

2. *There is an isomorphism:*

$$\Gamma(X^n, W_{r,-q}^{i,i})^P \xrightarrow{\simeq} \Gamma(X^n, \mathrm{Tor}_{-q}(E_I, E_J))^{P_{I,J}} \ .$$

We give some more notations and a final lemma, which will simplify the following discussion.

Notation 4.21. Let X a smooth quasi-projective surface. Let I, J multi-indexes, $I, J \subseteq \{1, \dots, n\}$, with $|J| = k$. Let $G_{\{1,\dots,k\}}$ the stabilizer of $\{1, \dots k\}$ in $\{1, \dots, n\}$. We will indicate with $\mathcal{A}_{k,q}$ the sheaf of invariants

$$\mathcal{A}_{k,q} := \left(\Lambda^q N^*_{\{1,\dots,k\}} \right)^{G_{\{1,\dots,k\}}}$$

considered as a sheaf over $\Delta_{X^k} \times S^{n-k}X$. It is clear that $\operatorname{rk} N_{\{1,\dots,k\}} = 2k-2$ hence $\mathcal{A}_{k,q} = 0$ for $q > 2k-2$. By lemma 4.14 it is zero if q is odd and it is a line bundle if q is even. Moreover, if U is an affine open subset of X, we will indicate with Δ_{U^I} the small diagonal in U^I; it is an open set of Δ_I. The G-invariant sections of $\oplus_{|J|=k} \Lambda^q N_J^*$ over U^n:

$$A_{k,q} := H^0(U^n, \oplus_{|J|=k} \Lambda^q N_J^*)^G$$

are isomorphic, by Danila's lemma, to the sections of $\mathcal{A}_{k,q}$ over the quotient $U^n \cap \Delta_{\{1,\dots k\}} / G_{\{1,\dots,k\}} \simeq \Delta_{U^k} \times S^{n-k}U$ (which is an open set of the diagonal: $\Delta_{X^k} \times S^{n-k}X$). Consider the embedding, for $0 < l < n-k$:

$$\begin{array}{ccc} \Delta_{\{1,\dots,k+l\}} \times S^{n-k-l}X & \overset{i}{\hookrightarrow} & \Delta_{\{1,\dots,k\}} \times S^{n-k}X \\ (\underbrace{(x,\dots,x)}_{(k+l)\text{-times}}, [y]) & \longmapsto & (\underbrace{(x,\dots,x)}_{k\text{-times}}, [lx+y]) \end{array}$$

We will denote with $\mathcal{A}_{k,q}\Big|_{\Delta_{\{1,\dots,k+l\}}}$ the pull-back $i^*\mathcal{A}_{k,q}$. It is identified with $i_* i * \mathcal{A}_{k,q}$. It is clear that

$$\mathcal{A}_{k,q}\Big|_{\Delta_{\{1,\dots,k+l\}}} \simeq \left(\Lambda^q N^*_{\{1,\dots,k\}}\Big|_{\Delta_{\{1,\dots,k+l\}} \times S^{n-k-l}} \right)^{\mathfrak{S}_k}$$

where $\Lambda^q N^*_{\{1,\dots,k\}}\Big|_{\Delta_{\{1,\dots,k+l\}} \times S^{n-k-l}X}$ is the quotient:

$$\left(\Lambda^q N^*_{\{1,\dots,k\}}\Big|_{\Delta_{\{1,\dots,k+l\}} \times X^{n-k-l}} \right)^{\mathfrak{S}(k+l+1,\dots,n)} .$$

Now for $k \geq 1$, consider the G-equivariant map:

$$\operatorname{Tor}_q(\mathcal{C}^{k-1}, \mathcal{C}^k) \longrightarrow \operatorname{Tor}_q(\mathcal{C}^k, \mathcal{C}^k)$$

induced by the map:

$$\mathcal{C}^{k-1} \longrightarrow \mathcal{C}^k . \tag{38}$$

It induces a G-equivariant morphism:

$$\bigoplus_{|I|=k+1} \bigoplus_{\substack{|J|=k \\ J \subseteq I}} \Lambda^q N_J^* \Big|_{\Delta_I} \longrightarrow \bigoplus_{|I|=k+1} \Lambda^q N_I^* \ ;$$

taking the component for $I = \{1, \ldots, k+1\}$ we get the $G_{\{1,\ldots,k+1\}}$-equivariant morphism:

$$\bigoplus_{\substack{|J|=k \\ J \subseteq \{1,\ldots,k+1\}}} \Lambda^q N_J^*|_{\Delta_{\{1,\ldots,k+1\}}} \xrightarrow{\beta} \Lambda^q N_{\{1,\ldots,k+1\}}^* \ . \tag{39}$$

Lemma 4.22. *The $G_{\{1,\ldots,k+1\}}$-equivariant morphism*

$$\bigoplus_{\substack{|J|=k \\ J \subseteq \{1,\ldots,k+1\}}} \Lambda^q N_J^*|_{\Delta_{\{1,\ldots,k+1\}}} \xrightarrow{\beta} \Lambda^q N_{\{1,\ldots,k+1\}}^* \tag{40}$$

induces isomorphisms:

$$\mathcal{A}_{k,q}\Big|_{\Delta_{\{1,\ldots,k+1\}}} \xrightarrow{\alpha} \mathcal{A}_{k+1,q} \tag{41}$$

$$H^0(\Delta_{U^{k+1}} \times S^{n-k-1}U, \Lambda^q N_{\{1,\ldots,k\}}^*)^{\mathfrak{S}_k} \xrightarrow{\simeq} A_{k+1,q} \ . \tag{42}$$

Proof. Taking the $G_{\{1,\ldots,k+1\}}$-invariants of the morphism (39) gives exactly the morphism α in (41). Now the stabilizer is isomorphic to $G_{\{1,\ldots,k+1\}} \simeq \mathfrak{S}_{k+1} \times \mathfrak{S}(k+2,\ldots,n)$. The $\mathfrak{S}_{k+1}$ and the $\mathfrak{S}(k+2,\ldots,n)$-actions commute, and the latter acts trivially on the fibers of the vector bundles over the diagonal $\Delta_{\{1,\ldots,k+1\}}$. Therefore we can reduce the question to the $\mathfrak{S}_{k+1}$-invariant map of vector bundles over $\Delta_{X^{k+1}} \times S^{n-k-1}X$:

$$\bigoplus_{\substack{|J|=k \\ J \subseteq \{1,\ldots,k+1\}}} \Lambda^q N_J^* \Big|_{\Delta_{X^{k+1}} \times S^{n-k-1}X} \xrightarrow{\beta} \Lambda^q N_{\{1,\ldots,k+1\}}^* \ . \tag{43}$$

It suffices then to prove that the morphism β (43) induces a nowhere zero morphism α between the line bundles of $\mathfrak{S}_{k+1}$-invariants:

$$\mathcal{A}_{k,q}|_{\Delta_{\{1,\ldots,k+1\}}} \xrightarrow{\alpha} \mathcal{A}_{k+1,q}$$

over the diagonal $\Delta_{X^{k+1}} \times S^{n-k-1}X$. It suffices to prove the same property for the dual morphism:

$$\Lambda^q N_{\{1,\ldots,k+1\}} \longrightarrow \bigoplus_{J \subseteq \{1,\ldots,k+1\}} \Lambda^q N_J \Big|_{\Delta_{X^{k+1}} \times S^{n-k-1}X} . \tag{44}$$

This reduces to the following easy consequence of lemma 4.14. Consider the $\mathfrak{S}_{k+1}$-equivariant diagram for $q = 2l$, where $R \simeq \mathbb{C}^{k+1}$ is the natural representation of $\mathfrak{S}_{k+1}$:

$$\begin{tikzcd}
\Lambda^{2l}(\mathbb{C} \otimes R) \arrow[r, "\Lambda^{2l}\tilde{\pi}_i"] \arrow[d, "\Lambda^{2l}\pi^R"'] \arrow[dr, "\Lambda^{2l}\psi"] & \oplus_{i=1}^{k+1} \Lambda^{2l}(\mathbb{C} \otimes R_i) \arrow[d, "\Lambda^{2l}\pi^{R_i}"] \\
\Lambda^{2l}(\mathbb{C} \otimes \rho_{k+1}) \arrow[r, "\Lambda^{2l}\tilde{\varphi}_i"] & \oplus_{i=1}^{k+1} \Lambda^{2l}(\mathbb{C} \otimes \rho_k(i))
\end{tikzcd}$$

where $\tilde{\pi}_i$ and φ_i are the obviuos projections *up to a sign*, in order to take into account the signs in the definition of the map (38). The property we want to prove for the map (44) is equivalent to prove that the morphism $\Lambda^{2l}\psi$ above induces a nonzero morphism between the vector spaces of invariants:

$$\Lambda^{2l}(\mathbb{C}\otimes R)^{\mathfrak{S}_{k+1}} \xrightarrow{(\Lambda^{2l}\psi)^{\mathfrak{S}_{k+1}}} \left[\oplus_{i=1}^{k+1}\Lambda^{2l}(\mathbb{C}\otimes\rho_k(i))\right]^{\mathfrak{S}_{k+1}} .$$

Let us take the $\mathfrak{S}_{k+1}$-invariant element $\omega^l \in \Lambda^{2l}(\mathbb{C}\otimes R)$ considered in lemma 4.14. We know that the images $\omega_{i,l} = \mathrm{pr}_i \circ \Lambda^{2l}\psi(\omega^l)$ are nonzero. Therefore $\Lambda^{2l}\psi(\omega^l) = (\omega_{i,l})_i$ is a nonzero element; since ω^l is $\mathfrak{S}_{k+1}$-invariant and $\Lambda^{2l}\psi$ is $\mathfrak{S}_{k+1}$-equivariant, the element $\Lambda^{2l}\psi(\omega^l)$ is necessarily $\mathfrak{S}_{k+1}$-invariant. Since we proved that it is nonzero, we are done.

The second statement follows from the first by taking the $G_{\{1,\dots k+1\}}$-invariant sections on U^n and by recalling Danila's lemma.

□

We will denote with $A_{k,q}\big|_{\Delta_{\{1,\dots,k+1\}}}$ the space of sections:

$$A_{k,q}\Bigg|_{\Delta_{\{1,\dots,k+1\}}} \simeq H^0(\Delta_{U^{k+1}} \times S^{n-k-1}U, \mathcal{A}_{k,q}\Bigg|_{\Delta_{\{1,\dots,k+1\}}}) ;$$

in this notation the second statement of the lemma then becomes:

$$A_{k,q}\Bigg|_{\Delta_{\{1,\dots,k+1\}}} \xrightarrow{\simeq} A_{k+1,q} .$$

4.8 Examples

Before attacking the equivariant spectral sequences (35) in all generality, we will work it out by hand for the cases $n = 2, 3, 4, 6$. It will then be evident the pattern of the general case. In these examples we will work with a trivial line bundle $E \simeq \mathcal{O}_X$: the introduction of a nontrivial line bundle E presents no difficulties at all. Let us begin with the simpler case, $n = 2$.

4.8.1 The case $n = 2$

The symmetric power. Let us begin with the spectral sequence ${}'\mathcal{E}_1^{p,q}$, the equivariant spectral sequence associated to $(\mathcal{C}^\bullet \otimes^L \mathcal{C}^\bullet)^{H\times G}$. We recall that for $n = 2$ the complex $\mathcal{C}^\bullet$ is simply

$$0 \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow 0 ,$$

where $\mathcal{C}^0 \simeq \mathcal{O}_{X^2} \otimes \mathbb{C}^2$, $\mathcal{C}^1 \simeq \mathcal{O}_\Delta$, where Δ is the diagonal of X^2. For $q = 0$ we clearly have the complex:

$${}'\mathcal{E}_1^{0,\bullet} \simeq (S^2\mathcal{C}^\bullet)^G .$$

For negative q the only possibly nonzero terms are:

$${}'\mathcal{E}_1^{2,q} \simeq \mathrm{Tor}_{-q}(\mathcal{C}^1, \mathcal{C}^1)^{G\times H} ,$$

since

$${}'E_1^{1,q} \simeq \mathrm{Tor}_{-q}(\mathcal{C}^0, \mathcal{C}^1) \oplus \mathrm{Tor}_{-q}(\mathcal{C}^1, \mathcal{C}^0) \simeq 0$$

since $\mathcal{C}^0$ is a locally free sheaf. Now $\mathrm{Tor}_1(\mathcal{C}^1, \mathcal{C}^1)$ is invariant by transposition of factors (since the transposition acts as $(-1)^{-1} = 1$ by section 4.3). Therefore the P-invariants $\mathrm{Tor}_1(\mathcal{C}^1, \mathcal{C}^1)^P$ are isomorphic to the G-invariants:

$$\mathrm{Tor}_1(\mathcal{C}^1, \mathcal{C}^1)^P \simeq \mathrm{Tor}_1(\mathcal{C}^1, \mathcal{C}^1)^G .$$

Now the group G acts pointwisely on $\mathrm{Tor}_1(\mathcal{C}^1, \mathcal{C}^1) \simeq \mathrm{Tor}_1(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq N^*_\Delta$ twisting the goemetric representation $\rho_2 \oplus \rho_2 = \varepsilon \oplus \varepsilon$ by the signs coming from its action on the two factors $\mathcal{O}_\Delta$: the G-sheaf $\mathcal{O}_\Delta$is then better written as $\mathcal{O}_\Delta \otimes \varepsilon$. The right G-action on $\mathrm{Tor}_q(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is then given by the representation:

$$\mathrm{Tor}_q(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq \Lambda^q N^*_\Delta \otimes \varepsilon^{\otimes^2} \simeq \Lambda^q(\varepsilon \oplus \varepsilon)$$

which does not have any invariants. Therefore ${}'\mathcal{E}_2^{p,-1} = 0$ for all p. At level ${}'E_1^{p,-2}$ the only nonzero term is $E_1^{2,-2} \simeq \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1)$ which has no H-invariants. Therefore the spectral sequence ${}'\mathcal{E}_1^{p,q}$ reduces to the complex ${}'\mathcal{E}_1^{\bullet,0} \simeq (S^2\mathcal{C}^\bullet)^G$, which has cohomology only in degree 0, by corollary 4.9. We now know by the same theorem and by the degeneration of the spectral sequence that

$${}'\mathcal{E}_\infty^{0,0} \simeq {}'\mathcal{E}_2^{0,0} \simeq H^0({}'\mathcal{E}_1^{\bullet,0}) \simeq \mu_*(S^2 E^{[2]}) \ .$$

As a consequence, we have proved:

$$\mathbf{R}\mu_*(S^2 E^{[2]}) \simeq (S^2 \mathcal{C}^\bullet_E)^G \ .$$

The exterior power. The case of exterior power is even simpler, since the only nonzero term in $(E_1^{p,q} \otimes \varepsilon_H)^H$, apart from the complex $(E_1^{\bullet,0} \otimes \varepsilon_H)^H \simeq \Lambda^2\mathcal{C}^\bullet$, is the term

$$(E_1^{2,-2} \otimes \varepsilon_H)^H \simeq \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1) \simeq \Lambda^2 N^*_\Delta$$

which is on the diagonal and is torsion; therefore, by corollary 4.10, we have again:

$$\mathbf{R}\mu_*(\Lambda^2 E^{[2]}) \simeq (\Lambda^2 \mathcal{C}^\bullet_E)^G \ .$$

4.8.2 The case $n = 3$.

The symmeric power The complex $\mathcal{C}^\bullet$ is in this case:

$$0 \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2 \longrightarrow 0 \ .$$

The complex ${}'\mathcal{E}_1^{\bullet,0}$ is, as we know,

$${}'\mathcal{E}_1^{\bullet,0} \simeq (S^2\mathcal{C}^\bullet)^G \ .$$

At level ${}'E_1^{p,-1} = (E_1^{p,-1})^H$ we have the (shifted) complex:

$$0 \longrightarrow \bigoplus_{i<j} N^*_{\Delta_{ij}} \longrightarrow \bigoplus_{i<j} N^*_{\Delta_{ij}}\Big|_\Delta \longrightarrow 0 \ ;$$

when we take the G-invariants the same reasonment of the case $n = 2$ proves that everything vanishes, because of lemma 4.14. We skip for a moment the level ${}'\mathcal{E}_1^{p,-2}$. At level ${}'E_1^{p,-3}$ the only nonzero term is

$${}'E_1^{4,-3} \simeq \mathrm{Tor}_3(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \simeq \Lambda^3 N^*_\Delta \ .$$

The G-action on $\Lambda^3 N^*_\Delta$ is isomorphic to the representation:

$$\Lambda^3(\rho_3 \oplus \rho_3) \otimes \varepsilon^{\otimes^2} \simeq \Lambda^3(\rho_3 \oplus \rho_3)$$

and has consequently no G- invariants by lemma 4.14. At level ${}'\mathcal{E}_1^{p,-4}$ we have the only term $\Lambda^4 N^*_\Delta$ which has invariants, but they are by no way harmful, since ${}'\mathcal{E}_1^{4,-4}$ is a torsion sheaf on the diagonal, and by theorem 4.4, we are interested to the limit ${}'\mathcal{E}_\infty^0$ modulo torsion. The only non trivial part here occurs at level -2. The complex ${}'E_1^{\bullet,-2}$ reduces to the complex

$$\mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^2) \longrightarrow \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2)$$

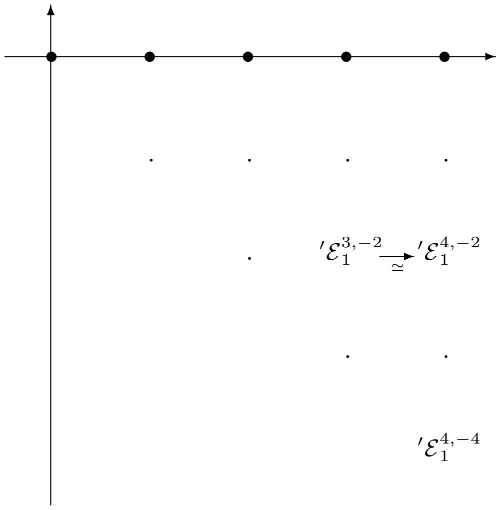
since $\mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1)$ is anti-invariant for the H-action. The complex reads:

$$0 \longrightarrow \bigoplus_{i<j} \Lambda^2 N^*_{\Delta_{ij}}\Big|_{\Delta} \xrightarrow{d_1^{3,-2}} \Lambda^2 N^*_{\Delta} \longrightarrow 0 \ .$$

By lemma 4.22 this morphism induces an isomorphism on the line bundles of G-invariants over Δ:

$$'\mathcal{E}_1^{2,-2} \simeq \mathcal{A}_{2,2}\Big|_{\Delta} \xrightarrow{(d_1^{3,-2})^G} \mathcal{A}_{3,2} \simeq {'\mathcal{E}_1^{3,-2}} \ .$$

The spectral sequence $'\mathcal{E}_1^{p,q}$ looks then like:



As a consequence $'\mathcal{E}_2^{3,-2} = {'\mathcal{E}_2^{4,-2}} = 0$. This clearly implies:

$$\mathbf{R}\mu_*(S^2 E^{[3]}) \simeq (S^2 \mathcal{C}^\bullet_E)^G \ .$$

The exterior power. The case of exterior powers presents less difficulties. The 0-level of the sequence $''E_1^{p.q}$ is always isomorphic to the searched exterior power complex:

$$''E_1^{\bullet,0} \simeq \Lambda^2 \mathcal{C}^\bullet$$

and the odd negative levels $''E_1^{p,q}$ have no G-invariants because of lemma 4.14. We remain with levels $''E_1^{p,-2}$ and $''E_1^{p,-4}$. The last one is actually 0, since the only nonzero term in $E_1^{\bullet,-4}$ is $E_1^{4,-4}$ and

$$''E_1^{4,-4} \simeq (\mathrm{Tor}_4(\mathcal{C}^2, \mathcal{C}^2) \otimes \varepsilon_H)^H \simeq 0$$

because $\mathrm{Tor}_4(\mathcal{C}^2, \mathcal{C}^2)$ is H-invariant. The $''E_1^{\bullet,-2}$ complex becomes:

$$''E_1^{2,-2} \xrightarrow{d_1^{2,-2}} {''E_1^{3,-2}}$$

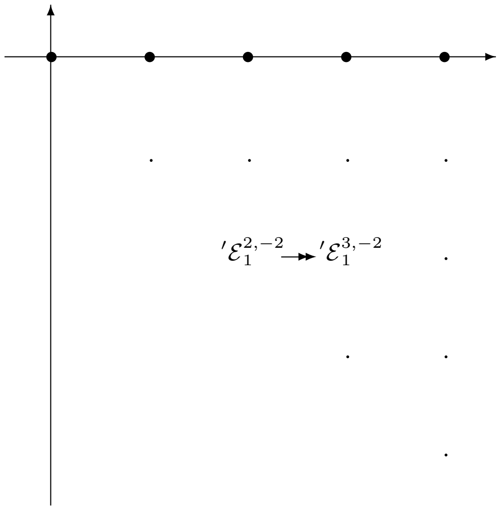
because

$$''E_1^{4,-2} \simeq (\mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2) \otimes \varepsilon_H)^H = 0 \ ,$$

since $\mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2)$ is G-invariant. The morphism $d_1^{2,-2}$ is the restriction:

$$\bigoplus_{i<j} \Lambda^2 N^*_{\Delta_{ij}} \longrightarrow \bigoplus_{i<j} \Lambda^2 N^*_{\Delta_{ij}}\Big|_{\Delta} \ ,$$

hence surjective. As a consequence the morphism induced on the G-invariants is surjective and hence $''\mathcal{E}_2^{3,-2} = 0$. The spectral sequence $''\mathcal{E}_1^{p,q}$ looks like:



On the $''\mathcal{E}_2^{p,q}$-level, only $''\mathcal{E}_2^{2,-2}$ in nonzero, if $q \neq 0$. This term contributes to the torsion of $\Lambda^2\mu_*(E^{[3]})$ and does not interest us. The $''\mathcal{E}_\infty^{0,0}$-term is rightly isomorphic to $\mu_*(\Lambda^2 E^{[3]})$ by corollary 4.10, and since the complex $''\mathcal{E}_1^{\bullet,0}$ is acycilic in degree greater than zero, we get:

$$\mathbf{R}\mu_*(\Lambda^2 E^{[3]}) \simeq (\Lambda^2 \mathcal{C}_E^\bullet)^G \ .$$

4.8.3 The case $n = 4$

Before starting we proceed to a further simplification. By lemma 1.27 the open sets S^4U, with U affine open subset in X, cover the symmetric variety S^4X. Since every consideration we will made is of local nature, it is then sufficient to prove (36) on every affine open set of the form S^4U. It is now equivalent to reason about the sheaves of invariants over S^4U and the modules of invariant sections over U^4. The quotient $U^4/G \simeq S^4U$ is naturally an open set of S^4X. We will indicate with $U^{[4]}$ the open set $\mu^{-1}(S^4U)$ of the Hilbert scheme. We will then consider the two spectral sequences:

$$'E_1^{p,q} = \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j)^H \tag{45}$$

$$'\mathcal{E}_1^{p,q} = \bigoplus_{i+j=p} H^0(U^4, \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j))^{G\times H} \tag{46}$$

which are the hyperderived spectral sequences of the complexes:

$$S^2_L\mathcal{C}^\bullet \quad ; \quad \Gamma_G(U^4, S^2_L\mathcal{C}^\bullet) \ .$$

We remark that the latter is acyclic in degree > 0. We recall that the complex $\mathcal{C}^\bullet$ is in this case:

$$0 \longrightarrow \mathcal{O}_X \otimes \mathbb{C}^4 \longrightarrow \bigoplus_{i<j} \mathcal{O}_{\Delta_{ij}} \longrightarrow \bigoplus_{i<j<k} \mathcal{O}_{\Delta_{ijk}} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \ .$$

The complexes $'\mathcal{E}_1^{\bullet,q}$. Since, by lemma 4.14, the complexes $'E_1^{p,q}$ have no G-invariants for q odd, we will consider only the case q even.

1. $q = 0$. By definition, the complex $'\mathcal{E}_1^{\bullet,0}$ is exactly the complex $\Gamma_G(U^4, S^2\mathcal{C}^\bullet)$.

2. $q = -2$. The modules ${}'\mathcal{E}_1^{p,-2}$ of $P = G \times H$-invariant sections are:

$$
\begin{aligned}
{}'\mathcal{E}_1^{3,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^2))^G \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{123}}))^{G_{\{12\}\times\{123\}}} \\
&\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{123}}))^{\mathfrak{S}_2}
\end{aligned}
$$

Now $\mathfrak{S}_2$ acts over each point of $\mathrm{Tor}_2(\Delta_{12}, \mathcal{O}_{\Delta_{123}}) \simeq \Lambda^2 N_{12}^*\Big|_{\Delta_{123}}$ with the representation:

$$
\Lambda^2(\varepsilon \oplus \varepsilon) \otimes \varepsilon^{\otimes^2} \simeq \Lambda^2(\varepsilon \oplus \varepsilon) \simeq 1
$$

hence trivially. Therefore

$$
{}'\mathcal{E}_1^{3,-2} \simeq H^0(\Delta_{U^3} \times U, \Lambda^2 N_{12}^*\Big|_{\Delta_{123}}) \simeq A_{2,2}\Big|_{\Delta_{123}} .
$$

$$
{}'\mathcal{E}_1^{4,-2} \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2))^P \oplus H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^3))^G .
$$

Now

$$
H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^3))^G \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta}))^{G_{\{12\}\times\{1234\}}}
$$

The stabilizer is isomorphic to

$$
G_{\{12\}\times\{1234\}} \simeq \mathfrak{S}(1,2) \times \mathfrak{S}(3,4)
$$

and acts on $\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta}) \simeq \Lambda^2 N_{12}^*\big|_{\Delta}$ with the representation:

$$
\Lambda^2(\varepsilon_2 \otimes \varepsilon_2) \otimes \varepsilon_2 \otimes \varepsilon_4 \simeq \varepsilon_2 \otimes \varepsilon_4 .
$$

As a consequence $\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta})$ has no $G_{\{12\}\times\{1234\}}$-invariants. Since $\mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2)$ is completely H-invariant, we have:

$$
\begin{aligned}
{}'\mathcal{E}_1^{4,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2))^P \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2))^G \\
&\simeq H^0(U^4, \bigoplus_{|I|=3} \Lambda^* N_I^*)^G \oplus H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{\{123\}}}, \mathcal{O}_{\Delta_{\{124\}}}))^{G_{\{123\}\times\{124\}}} .
\end{aligned}
$$

Now

$$
\begin{aligned}
H^0(U^4, \bigoplus_{|I|=3} \Lambda^* N_I^*)^G &\simeq H^0(U^4, \Lambda^2 N_{\{123\}}^*)^{\mathfrak{S}_3} \\
&\simeq H^0(\Delta_{U^3} \times U, (\Lambda^2 N_{\{123\}}^*)^{\mathfrak{S}_3})
\end{aligned}
$$

and by lemma 4.14 we can consider $(\Lambda^2 N_{\{123\}}^*)^{\mathfrak{S}_3}$ as a sublinebundle $\mathcal{A}_{3,2}$ of $(\Lambda^2 N_{\{123\}}^*)$ over the diagonal Δ_{123}. For the second term $H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{\{123\}}}, \mathcal{O}_{\Delta_{\{124\}}}))^{G_{\{123\}\times\{124\}}}$ we remark that the stabilizer $G_{\{123\}\times\{124\}}$ is isomoprhic to $\mathfrak{S}_{12}$ which acts on the fibers of

$$
\mathrm{Tor}_2(\mathcal{O}_{\Delta_{\{123\}}}, \mathcal{O}_{\Delta_{\{124\}}}) \simeq \Lambda^2 N_{\{12\}}^*\Big|_{\Delta}
$$

over each point of the small diagonal with the representation:

$$
\Lambda^2(\varepsilon_2 \oplus \varepsilon_2) \otimes \varepsilon_2^{\otimes^2} \simeq 1 ,
$$

hence trivially. As a consequence

$$
{}'\mathcal{E}_1^{4,-2} \simeq H^0(\Delta_{U^3} \times U, \mathcal{A}_{3,2}) \oplus H^0(\Delta_{U^4}, \mathcal{A}_{2,2}\Big|_{\Delta})
$$

since the invariants $(\Lambda^2 N^*_{\{12\}}\Big|_{\Delta})^{\mathfrak{S}_{12}}$ are exactly the restriction of the invariants $(\Lambda^2 N^*_{\{12\}})^{\mathfrak{S}_{12}} \simeq \mathcal{A}_{2,2}$ to the small diagonal.

The last nonzero term of the complex ${}'\mathcal{E}_1^{\bullet,-2}$ is :

$$\begin{aligned} {}'\mathcal{E}_1^{5,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^3))^G \\ &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta}))^{G_{\{123\}}} \end{aligned}$$

Now the stabilizer $G_{\{123\}} \simeq \mathfrak{S}_3$ acts on the fibers of $\mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta}) \simeq \Lambda^2 N^*_{\{123\}}\Big|_{\Delta}$ on each point of the small diagonal with the representation:

$$\Lambda^2(\rho_3 \oplus \rho_3) \otimes \varepsilon_3^{\otimes^2} \simeq \Lambda^2(\rho_3 \oplus \rho_3) \; .$$

Therefore the invariants $(\Lambda^2 N^*_{\{123\}}\Big|_{\Delta})^{\mathfrak{S}_3}$ can be interpreted as the restriction of the line bundle $\mathcal{A}_{3,2}$ restricted to the small diagonal. Therefore

$${}'\mathcal{E}_1^{5,-2} \simeq H^0(\Delta_{U^4}, \mathcal{A}_{3,2}\Big|_{\Delta}) \; .$$

The complex

$$0 \longrightarrow {}'\mathcal{E}_1^{3,-2} \longrightarrow {}'\mathcal{E}_1^{4,-2} \longrightarrow {}'\mathcal{E}_1^{5,-2} \longrightarrow 0$$

is clearly the complex of sections over U of the complex

$$0 \longrightarrow \mathcal{A}_{2,2}\Big|_{\Delta_{123}} \longrightarrow \mathcal{A}_{3,2} \oplus \mathcal{A}_{2,2}\Big|_{\Delta} \longrightarrow \mathcal{A}_{3,2}\Big|_{\Delta} \longrightarrow 0 \; . \tag{47}$$

The differentials are induced by the differentials of the complex ${}'E_1^{\bullet,-2}$: in particular the first component of the first map is obtained by taking the invariants of the map:

$$\bigoplus_{\substack{|J|=2,\\ J\subseteq\{123\}}} \Lambda^2 N^*_J\Big|_{\Delta_{\{123\}}} \longrightarrow \Lambda^2 N^*_{\{123\}} \tag{48}$$

which is in turn induced by the map :

$$\mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^2) \longrightarrow \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2) \; . \tag{49}$$

We are exactly in the situation the lemma 4.22: taking the $\mathfrak{S}_3$-invariants of (48) we obtain the isomorphism:

$$\alpha_2 : \mathcal{A}_{2,2}\Big|_{\Delta_{123}} \longrightarrow \mathcal{A}_{3,2} \; .$$

It is clear from the definition of the map (49) that the second component of the map:

$$\mathcal{A}_{2,2}\Big|_{\Delta_{123}} \longrightarrow \mathcal{A}_{3,2} \oplus \mathcal{A}_{2,2}\Big|_{\Delta}$$

is given by the restriction ν to the small diagonal. Let us consider now the second arrow in 47: it is obtained by considering the $\mathfrak{S}_3$-invariants of the morphism:

$$\bigoplus_{i=1}^{4} \mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta_{\{1234\}\setminus\{i\}}}) \longrightarrow \mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta}) \; .$$

The component on $\mathcal{A}_{3,2}$ is obtained setting $i=4$ and by definition of the diffrential $\partial^2 : \mathcal{C}^2 \longrightarrow \mathcal{C}^3$ this is exactly minus the restriction. The component on $\mathcal{A}_{2,2}\big|_{\Delta}$ is clearly the restriction of α_2 to the small diagonal: hence the complex:

$$0 \longrightarrow \mathcal{A}_{2,2}\Big|_{\Delta_{123}} \xrightarrow{(\alpha_2,\nu)} \mathcal{A}_{3,2} \oplus \mathcal{A}_{2,2}\Big|_{\Delta} \xrightarrow{(-\nu,\alpha_2|_{\Delta})} \mathcal{A}_{3,2}\Big|_{\Delta} \longrightarrow 0$$

is exact and ${}'\mathcal{E}_1^{\bullet,-2}$ is also exact.

3. $q=-4$.

$$\begin{aligned} {}'\mathcal{E}_1^{4,-4} &\simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{C}^2,\mathcal{C}^2))^P \oplus H^0(U^4,\mathrm{Tor}_4(\mathcal{C}^1,\mathcal{C}^3))^G \\ &\simeq H^0(U^4,\mathrm{Tor}_4(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta_{123}}))^{\mathfrak{S}_{123}} \oplus H^0(U^4,\mathrm{Tor}_4(\mathcal{O}_{\Delta_{12}},\mathcal{O}_{\Delta}))^{G_{\{12\}\times\{1234\}}} \end{aligned}$$

Now $\mathfrak{S}_3$ acts on the fibers of $\mathrm{Tor}_4(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta_{123}}) \simeq \Lambda^4 N^*_{\{123\}}$ over every point of the diagonal Δ_{123} with the representation:

$$\Lambda^4(\rho_3 \oplus \rho_3) \otimes \varepsilon_3^{\otimes^2} \simeq \Lambda^4(\rho_3\oplus\rho_3)$$

hence by lemma 4.14

$$\mathrm{Tor}_4(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta_{123}})^{\mathfrak{S}_3} \simeq \mathcal{A}_{3,4}\ .$$

As for the term: $\mathrm{Tor}_4(\mathcal{O}_{\Delta_{12}},\mathcal{O}_{\Delta})^{G_{\{12\}\times\{1234\}}}$ the stabilizer $G_{\{12\}\times\{1234\}} \simeq \mathfrak{S}(1,2)\times\mathfrak{S}(3,4)$ acts on the fibers of $\mathrm{Tor}_4(\mathcal{O}_{\Delta_{12}},\mathcal{O}_{\Delta}) \simeq \Lambda^4 N^*_{12}\Big|_{\Delta}$ over each point of the small diagonal with the representation:

$$\Lambda^4(\varepsilon_2\oplus\varepsilon_2)\otimes\varepsilon_2\otimes\varepsilon_4 \simeq \varepsilon_2\otimes\varepsilon_4$$

and therefore has no invariants. Therefore

$${}'\mathcal{E}_1^{4,-4} \simeq H^0(\Delta_{U^3}\times U, \mathcal{A}_{3,4})\ .$$

$$\begin{aligned} {}'\mathcal{E}_1^{5,-4} &\simeq H^0(U^4,\mathrm{Tor}_4(\mathcal{C}^2,\mathcal{C}^3))^P \\ &\simeq H^0(U^4,\mathrm{Tor}_4(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta}))^{\mathfrak{S}_3} \end{aligned}$$

which is clearly the space of sections over Δ_{U^4} of the restriction of the line bundle of invariants $\mathcal{A}_{3,4}$ to the small diagonal:

$${}'\mathcal{E}_1^{5,-4} \simeq H^0(\Delta_{U^4}, \mathcal{A}_{3,4}\Big|_{\Delta})\ .$$

Now $\mathrm{Tor}_4(\mathcal{C}^3,\mathcal{C}^3)$ is completely H-anti-invariant: therefore

$${}'\mathcal{E}_1^{6,-4} \simeq 0\ .$$

The map :

$${}'\mathcal{E}_1^{4,-4} \longrightarrow {}'\mathcal{E}_1^{5,-4} \longrightarrow$$

is induced by the restriction:

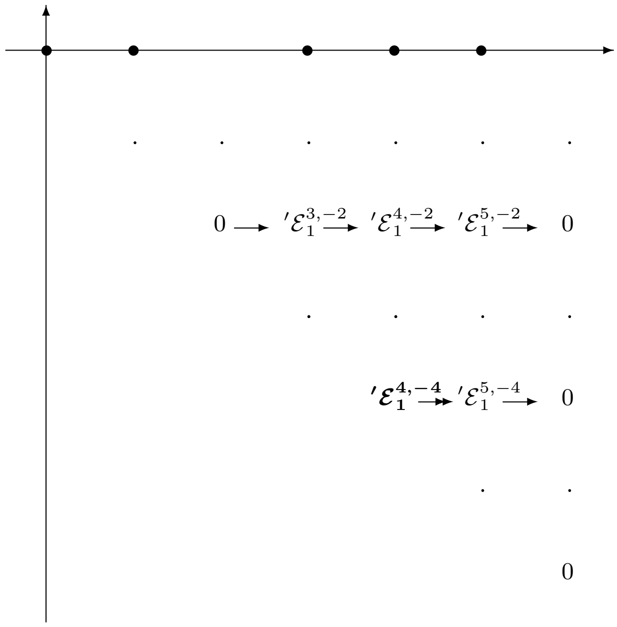
$$\mathcal{A}_{3,4} \longrightarrow \mathcal{A}_{3,4}\Big|_{\Delta} \longrightarrow 0$$

and hence it is surjective.

4. $q=-6$

$${}'\mathcal{E}_1^{6,-6} \simeq 0$$

since $\mathrm{Tor}_6(\mathcal{C}^3,\mathcal{C}^3)$ is completely H-anti-invariant. The spectral sequence ${}'\mathcal{E}_1^{p,q}$ looks like:



where we indicated with a bold character the degree where the complex ${}'\mathcal{E}_1^{4,-4}$ is not exact. Therefore if $q \neq 0$, ${}'\mathcal{E}_2^{p,q} = 0$ except for $p = q = -4$. Since we know, by corollary 4.9, that the complex $S^2_L\mathcal{C}^\bullet$ is acyclic in degree > 0, it follows that ${}'\mathcal{E}_2^{p,0} = 0$, if $p > 0$. As a consequence:

$$\Gamma(S^4U, \mu_*(S^2E^{[4]})) \simeq \Gamma(S^4U, (S^2\mathcal{C}^\bullet_E)^G) \; .$$

Since this is true for an arbitrary affine open set of S^4X of the form S^4U, with U affine open subset in X, and since such affine open subsets S^4U cover S^4X by lemma 1.27, we get, globally on S^4X:

$$\mathbf{R}\mu_*(S^2E^{[4]}) \simeq (S^2\mathcal{C}^\bullet_E)^G \; .$$

The exterior power. As we did in the symmetric case we consider the spectral sequences:

$$''E_1^{p,q} \simeq \Big(\bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j) \otimes \varepsilon_H\Big)^H$$

and

$$''\mathcal{E}_1^{p,q} \simeq \bigoplus_{i+j=p} H^0(U^4, \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j) \otimes \varepsilon_H)^{H \times G} \; ,$$

where U is a affine open subset in X. The complex $''\mathcal{E}_1^{\bullet,0}$ is as usual isomorphic to the complex $\Gamma_G(U^4, \Lambda^2\mathcal{C}^\bullet)$. By lemma 4.14 all the complexes $''\mathcal{E}_1^{p,q}$ with q odd vanish: as a consequence we can consider only the terms $''\mathcal{E}_1^{p,q}$ with q even.

The complex $''\mathcal{E}_1^{\bullet,-2}$.

$$\begin{aligned} ''\mathcal{E}_1^{2,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1) \otimes_H)^P \\ &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1))^G \end{aligned}$$

since $\mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^1)$ is completely H-anti-invariant. Therefore:

$$\begin{aligned} ''\mathcal{E}_1^{2,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{12}}))^{G_{\{12\}}} \\ &\simeq H^0(U^2 \times S^2U, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{12}}))^{\mathfrak{S}_2} \end{aligned}$$

and this is isomorphic to the space of sections of the sheaf of invariants $(\Lambda^2 N_{12}^*)^{\mathfrak{S}_2}$ over the diagonal $\Delta_{U^2} \times S^2U$. By lemma 4.14 it coincides with the line bundle $\det N_{12}^* \simeq \mathcal{A}_{2,2}$. Then :

$$''\mathcal{E}_1^{2,-2} \simeq H^0(\Delta_{U^2} \times S^2U, \mathcal{A}_{2,2}) \simeq A_{2,2} \ .$$

Next:

$$\begin{aligned} ''\mathcal{E}_1^{3,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^2))^G \\ &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{123}}))^{\mathfrak{S}_2} \ . \end{aligned}$$

Since $\mathfrak{S}_2$ acts on the fibres of $\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{123}})) \simeq \Lambda^2 N_{12}^*\big|_{\Delta_{123}}$ over each point of the diagonal Δ_{123} with the representation:

$$\Lambda^2(\varepsilon_2 \oplus \varepsilon_2) \otimes \varepsilon_2^{\otimes^2} \simeq 1$$

and since $\Lambda^2 N_{12}^*\big|_{\Delta_{123}}$ is isomorphic to $\mathcal{A}_{2,2}\big|_{\Delta_{123}}$ we get:

$$''\mathcal{E}_1^{3,-2} \simeq H^0(\Delta_{U^3} \times U, \mathcal{A}_{2,2}\Big|_{\Delta_{123}}) \simeq A_{2,2}\Big|_{\Delta_{123}} \ .$$

The term :

$$''\mathcal{E}_1^{4,-2} \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2) \otimes \varepsilon_H))^P \simeq 0$$

since $\mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^2)$ is completely H-invariant. Now

$$\begin{aligned} ''\mathcal{E}_1^{5,-2} &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^3))^G \\ &\simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta}))^{\mathfrak{S}_3} \end{aligned}$$

and , since $\mathfrak{S}_3$ acts on the fibers of $\mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta})) \simeq \Lambda^2 N_{123}^*\big|_{\Delta}$ over the points of the diagonal with the representation

$$\Lambda^2(\rho_3 \oplus \rho_3) \otimes \varepsilon_3^{\otimes^2} \simeq \Lambda^2(\rho_3 \oplus \rho_3) \ ,$$

by lemmas 4.14 and 4.22 we get that:

$$''\mathcal{E}_1^{5,-2} \simeq H^0(\Delta_{U^4}, (\Lambda^2 N_{123}^*\Big|_{\Delta})^{\mathfrak{S}_3}) \simeq H^0(\Delta_{U^4}, \mathcal{A}_{3,2}\Big|_{\Delta}) \simeq A_{3,2}\Big|_{\Delta} \ .$$

Finally:

$$''\mathcal{E}_1^{6,-2} \simeq H^0(U^4, \mathrm{Tor}_2(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}) \otimes \varepsilon_H)^P \simeq H^0(\Delta_{U^4}, \mathrm{Tor}_2(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}))^G$$

because $\mathrm{Tor}_2(\mathcal{C}^3, \mathcal{C}^3)$ is completely H-invariant. Then :

$$''\mathcal{E}_1^{6,-2} \simeq H^0(\Delta_{U^4}, \det N_{\Delta}^*)^{\mathfrak{S}_4} \simeq H^0(\Delta_{U^4}, \mathcal{A}_{4,2}) \simeq A_{4,2} \ .$$

The complex $''\mathcal{E}_1^{\bullet,-2}$ is then given (up to a shift) by the complex of sections of the complex:

$$0 \longrightarrow \mathcal{A}_{2,2} \longrightarrow \mathcal{A}_{2,2}\Big|_{\Delta_{123}} \longrightarrow 0 \longrightarrow \mathcal{A}_{3,2}\Big|_{\Delta} \longrightarrow \mathcal{A}_{4,2} \longrightarrow 0 \ .$$

The first map is induced by the map:

$$\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{12}}) \longrightarrow \mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}}, \mathcal{O}_{\Delta_{123}})$$

which is precisely the restriction map, hence surjective. The last map:

$$\mathcal{A}_{3,2}\Big|_{\Delta} \longrightarrow \mathcal{A}_{4,2}$$

is obtained by taking the $\mathfrak{S}_3$-invariants of

$$\Lambda^2 N^*_{123}\Big|_{\Delta} \longrightarrow \Lambda^2 N^*_{\Delta} \ .$$

By applying lemma 4.22 we find that it is the isomorphism α_3. Consequently, the complex:

$$0 \longrightarrow {}''\mathcal{E}_1^{2,2} \longrightarrow {}''\mathcal{E}_1^{3,2} \longrightarrow 0 \longrightarrow {}''\mathcal{E}_1^{5,2} \longrightarrow {}''\mathcal{E}^{6,2} \longrightarrow 0$$

is exact in degree different from 2.

The complex ${}''\mathcal{E}_1^{\bullet,-4}$.

$$''\mathcal{E}_1^{4,-4} \simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{C}^2, \mathcal{C}^2) \otimes \varepsilon_H)^P \simeq 0$$

since $\mathrm{Tor}_4(\mathcal{C}^2, \mathcal{C}^2)$ is completely H-invariant.

$$\begin{aligned} ''\mathcal{E}_1^{5,-4} &\simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{C}^2, \mathcal{C}^3))^G \\ &\simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta}))^{\mathfrak{S}_3} \simeq H^0(\Delta_{U^4}, \big(\Lambda^4 N^*_{123}\Big|_{\Delta}\big)^{\mathfrak{S}_3}) \\ &\simeq H^0(\Delta_{U^4}, \mathcal{A}_{3,2}\Big|_{\Delta}) \simeq A_{3,2}\Big|_{\Delta} \ . \end{aligned}$$

Finally the term:

$$\begin{aligned} ''\mathcal{E}_1^{6,-4} &\simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{C}^3, \mathcal{C}^3) \otimes \varepsilon_H)^P \simeq H^0(U^4, \mathrm{Tor}_4(\mathcal{C}^3, \mathcal{C}^3) \otimes \varepsilon_H)^G \\ &\simeq H^0(\Delta_{U^4}, \Lambda^4 N^*_{\Delta})^{\mathfrak{S}_4} \\ &\simeq H^0(\Delta_{U^4}, \mathcal{A}_{4,4}) \simeq A_{4,4} \end{aligned}$$

The map:

$$''\mathcal{E}_1^{5,4} \longrightarrow {}''\mathcal{E}_1^{6,4} \tag{50}$$

is induced by the map:

$$\mathcal{A}_{3,4}\Big|_{\Delta} \longrightarrow \mathcal{A}_{4,4} \tag{51}$$

which is in turn obtained by taking the $\mathfrak{S}_3$-invariants of:

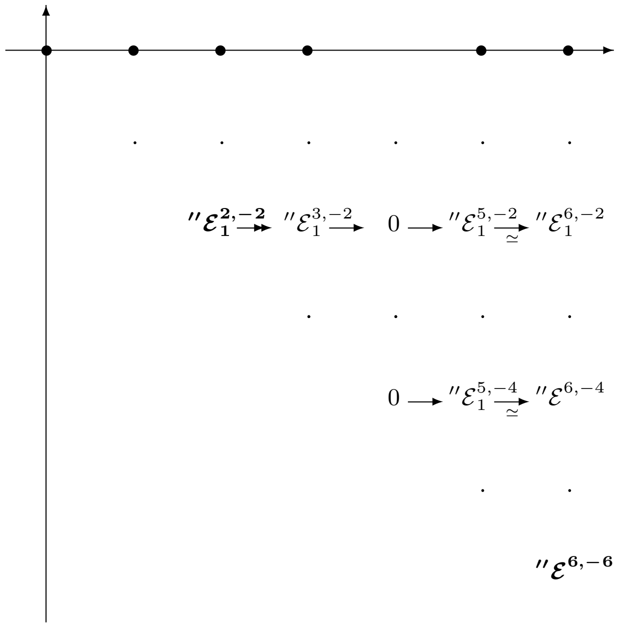
$$\Lambda^4 N^*_{123}\Big|_{\Delta} \longrightarrow \Lambda^4 N^*_{\Delta} \ .$$

Lemma 4.22 shows that (51) is the isomorphism α_3. Hence (50) is an isomorphism.

The complex ${}''\mathcal{E}_1^{\bullet,-6}$. It consists only of the term:

$$''\mathcal{E}_1^{6,-6} \simeq H^0(U^4, \det N^*_{\Delta})^{\mathfrak{S}_4} \simeq H^0(\Delta_{U^4}, \mathcal{A}_{4,6}) \simeq A_{4,6} \ .$$

Therefore the spectral sequence ${}''\mathcal{E}_1^{p,q}$ looks like:



where we indicated with a bold symbol the degrees where the complexes are not exact. Therefore the terms $''\mathcal{E}_2^{p,q}$, for $q \neq 0$ are nonzero if and only if $p+q=0$, $p=-2$, $p=-4$. Since, moreover, $''\mathcal{E}_2^{p,0}=0$ if $p>0$, we get by corollary 4.10:

$$\mathbf{R}\mu_*(\Lambda^2 E^{[4]}) \simeq (\Lambda^2 \mathcal{C}_E^\bullet)^G$$

on every affine open set S^4U of the symmetric variety. Since such affine open sets cover S^nX by lemma 1.27, we get the equality globally on S^nX.

Although all the difficulties of the general case are already present for $n=4$, the pattern of the general spectral sequence is very clearly expressed from $n=6$. Therefore we will skecth the situation for $n=6$ before giving the proof for the general case.

4.8.4 The case $n=6$

The symmetric power. As we did in the case $n=4$, we will place ourselves first over a G-invariant affine open set U^6 of X^6, with U affine open set in X, and consider the spectral sequence:

$$'\mathcal{E}_1^{p,q} = H^0(U^6, \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j))^P \ .$$

Here the complex $\mathcal{C}^\bullet$ has length 5. By lemma 4.14 it is sufficient to consider only the case q even. As usual, the complex $'\mathcal{E}_1^{\bullet,0}$ is isomorphic to G-invariants of the double symmetric power of the complex $\mathcal{C}^\bullet$:

$$'\mathcal{E}_1^{\bullet,0} \simeq (S^2\mathcal{C}^\bullet)^G \ .$$

The complex ${}'\mathcal{E}_1^{\bullet,-2}$. The computation of ${}'\mathcal{E}^{p,-2}-1$ is analogous to the one made for $n=4$, if $p\leq 4$. Therefore:

$$
\begin{aligned}
{}'\mathcal{E}_1^{2,-2} &\simeq 0\\
{}'\mathcal{E}_1^{3,-2} &\simeq A_{2,2}\Big|_{\Delta_{123}}\\
{}'\mathcal{E}_1^{4,-2} &\simeq A_{3,2}\oplus A_{2,2}\Big|_{\Delta_{\{1234\}}}
\end{aligned}
$$

For $p\geq 5$:

$$
\begin{aligned}
{}'\mathcal{E}_1^{5,-2} &\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^2,\mathcal{C}^3))^G\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^1,\mathcal{C}^4))^G\\
&\simeq H^0(\Delta_{U^4}\times S^2U,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta_{1234}}))^{\mathfrak{S}_3}\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}},\mathcal{O}_{\Delta_{1,\dots,5}}))^{G_{\{12\}\times\{12345\}}}\\
&\simeq A_{3,2}\Big|_{\Delta_{1234}}
\end{aligned}
$$

$$
\begin{aligned}
{}'\mathcal{E}_1^{6,-2} &\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^1,\mathcal{C}^5))^G\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^2,\mathcal{C}^4)^G\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^3,\mathcal{C}^3))^P\\
&\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12}},\mathcal{O}_{\Delta}))^{G_{\{12\}\times\{1\dots6\}}}\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta_{1\dots5}}))^{G_{\{123\}\times\{1\dots5\}}}\simeq 0
\end{aligned}
$$

$$
\begin{aligned}
{}'\mathcal{E}_1^{7,-2} &\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^3,\mathcal{C}^4))^G\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^2,\mathcal{C}^5))^G\\
&\simeq H^0(\Delta_{U^5}\times U,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{1\dots4}},\mathcal{O}_{\Delta_{1\dots5}}))^{\mathfrak{S}_4}\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}},\mathcal{O}_{\Delta}))^{G_{\{123\}\times\{1\dots6\}}}\\
&\simeq H^0(\Delta_{U^5}\times U,\Lambda^2N^*_{1\dots4}\Big|_{\Delta_{1\dots5}})^{\mathfrak{S}_4}\simeq A_{4,2}\Big|_{\Delta_{1\dots5}}
\end{aligned}
$$

$$
\begin{aligned}
{}'\mathcal{E}_1^{8,-2} &\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^4,\mathcal{C}^4))^P\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^3,\mathcal{C}^5))^G\\
&\simeq H^0(\Delta_{U^5}\times U,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{1\dots5}},\mathcal{O}_{\Delta_{1\dots5}}))^{\mathfrak{S}_5}\bigoplus_{i=2}^{4}\bigoplus_{\substack{|K|=|K'|=5\\|K\cap K'|=i}}H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_K},\mathcal{O}_{\Delta_{K'}}))^P\bigoplus\\
&\qquad\bigoplus H^0(U^6,\mathrm{Tor}_2\mathcal{O}_{\Delta_{1\dots4}},\mathcal{O}_{\Delta}))^{G_{\{1234\}\times\{1\dots6\}}}\\
&\simeq H^0(\Delta_{U^5}\times U,\Lambda^2N^*_{\Delta_1\dots5})^{\mathfrak{S}_5}\oplus H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{12345}},\mathcal{O}_{\Delta_{12346}}))^{P_{\{12345\}\times\{12346\}}}\\
&\simeq A_{5,2}\oplus H^0(\Delta_{U^6},\Lambda^2N^*_{1\dots4}\Big|_{\Delta})^{\mathfrak{S}_4}\\
&\simeq A_{5,2}\oplus A_{4,2}\Big|_{\Delta}
\end{aligned}
$$

$$
\begin{aligned}
{}'\mathcal{E}_1^{9,-2} &\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^4,\mathcal{C}^5))^G\\
&\simeq H^0(\Delta_{U^6},\mathrm{Tor}_2(\mathcal{O}_{\Delta_{1\dots5}},\mathcal{O}_{\Delta}))^{\mathfrak{S}_5}\\
&\simeq A_{5,2}\Big|_{\Delta}
\end{aligned}
$$

$$
{}'\mathcal{E}_1^{10,-2} \simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^5,\mathcal{C}^5))^P\simeq 0\,.
$$

In this computation we have used that:

1. for l odd, $\mathrm{Tor}_2(\mathcal{C}^l,\mathcal{C}^l)^P=0$, since $\mathrm{Tor}_2((\mathcal{C}^l,\mathcal{C}^l)$ is H-anti-invariant.
2. $\mathrm{Tor}_2(\mathcal{C}^i,\mathcal{C}^{i+l})^G=0$ if $l\geq 2$, because

$$
H^0(U^6,\mathrm{Tor}_2(\mathcal{C}^i,\mathcal{C}^{i+l}))^G\simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_{1\dots i+1}},\mathcal{O}_{\Delta_{1\dots i+l+1}}))^{G_{\{1\dots i+1\}\times\{i+2\dots i+l+1\}}}
$$

and the stabilizer

$$G_{\{1\dots i+1\}\times\{i+2\dots i+l+1\}} \simeq \mathfrak{S}(1,\dots i+1)\times\mathfrak{S}(i+2,i+l+1)\times\mathfrak{S}(i+l+2,n)$$

acts on the space of section like $\mathfrak{S}(1,\dots i+1)\times\mathfrak{S}(i+2,i+l+1)$, because the last factor acts trivially. The last group acts on the

$$\mathrm{Tor}_2(\mathcal{O}_{\Delta_{1\dots i+1}},\mathcal{O}_{\Delta_{1\dots i+l+1}}) \simeq \Lambda^2 N^*_{1\dots i+1}\Big|_{\Delta_{1\dots i+l+1}}$$

with the representation:

$$\Lambda^2(\rho_{i+1}\oplus\rho_{i+1})\otimes\varepsilon_{i+1}\otimes\varepsilon_{i+l+1}$$

and therefore has no invariants if $l\geq 2$.

3. For $h\geq k+2$,
$$H^0(U^6,\bigoplus_{\substack{|K|=|K'|=h\\|K\cap K'|=k}}\mathrm{Tor}_2(\mathcal{O}_{\Delta_K},\mathcal{O}_{\Delta_{K'}}))^P=0\,.$$

   This is because

$$H^0(U^6,\bigoplus_{\substack{|K|=|K'|=h\\|K\cap K'|=k}}\mathrm{Tor}_2(\mathcal{O}_{\Delta_K},\mathcal{O}_{\Delta_{K'}}))^P \simeq H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_I},\mathcal{O}_{\Delta_J}))^{P_{I,J}}$$

   where I and J are two chosen multi-indexes such that $|I|=|J|=h$, $|I\cap J|=k$ and there are transpositions in $\mathfrak{S}(I\setminus(I\cap J))\subseteq G_{I,J}$ which acts trivially on $\mathcal{O}_{\Delta_J}$, but with a sign on $\mathcal{O}_{\Delta_I}$ preventing P-invariants. This vanishing cannot happen if $h-k=1$, hence in this case:

$$H^0(U^6,\mathrm{Tor}_2(\mathcal{O}_{\Delta_I},\mathcal{O}_{\Delta_J}))^{P_{I,J}} \simeq H^0(\Delta_{U^{k+1}}\times S^{n-k-1}U,\Lambda^2 N_{1\dots,k}\Big|_{1\dots k+1})^{\mathfrak{S}_k}\simeq A_{k,2}\Big|_{\Delta_{1\dots k+1}}\,.$$

   The complex ${}'\mathcal{E}_1^{\bullet,-2}$ is then isomorphic (up to a shift) to the complex:

$$0\longrightarrow A_{2,2}\Big|_{\Delta_{123}}\xrightarrow{(\alpha_2,\nu)}A_{3,2}\oplus A_{2,2}\Big|_{\Delta_{1\dots4}}\xrightarrow{(\nu,\alpha_2|_{\Delta_{1\dots4}})}A_{3,2}\Big|_{\Delta_{1\dots4}}\longrightarrow 0\longrightarrow$$
$$\longrightarrow A_{4,2}\Big|_{\Delta_{1\dots5}}\xrightarrow{(\alpha_4,\nu)}A_{5,2}\oplus A_{4,2}\Big|_{\Delta}\xrightarrow{(\nu,\alpha_4|_{\Delta})}A_{5,2}\Big|_{\Delta}\longrightarrow 0$$

   which is exact.

Reasoning in this way we can find that:

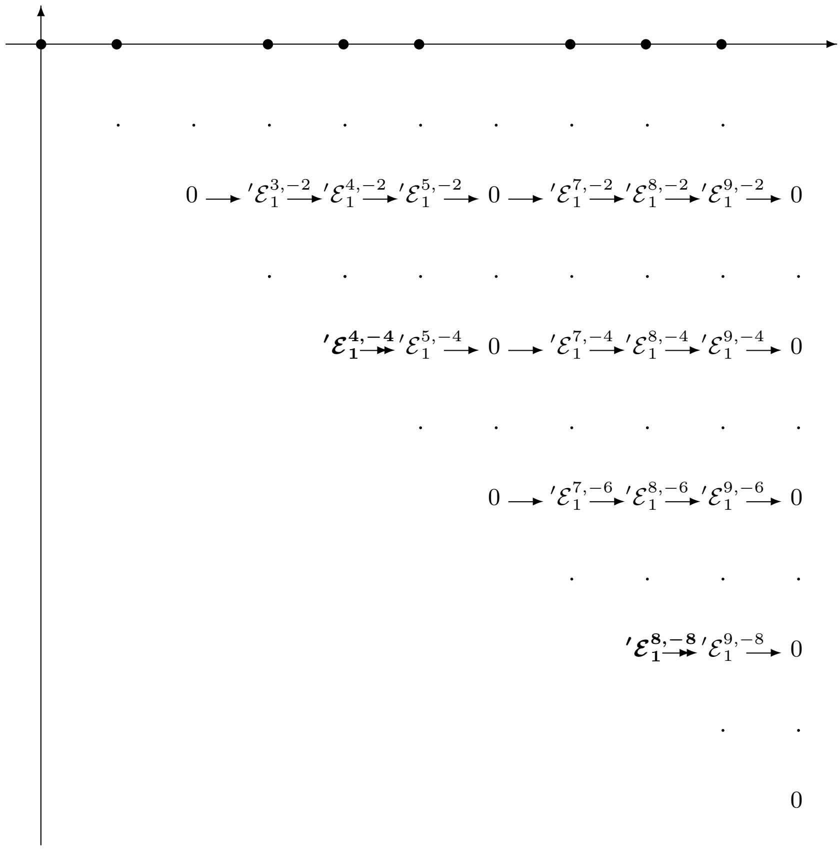
$${}'\mathcal{E}_1^{\bullet,-4}\simeq 0\longrightarrow A_{3,4}\twoheadrightarrow A_{3,4}\Big|_{\Delta_{1\dots4}}\longrightarrow 0\longrightarrow$$
$$\longrightarrow A_{4,4}\Big|_{\Delta_{1\dots5}}\xrightarrow{(\alpha_4,\nu)}A_{5,4}\oplus A_{4,4}\Big|_{\Delta}\xrightarrow{(\nu,\alpha_4|_{\Delta})}A_{5,4}\Big|_{\Delta}\longrightarrow 0$$

$${}'\mathcal{E}_1^{\bullet,-6}\simeq 0\longrightarrow A_{4,6}\Big|_{\Delta_{1\dots5}}\xrightarrow{(\alpha_4,\nu)}A_{5,6}\oplus A_{4,4}\Big|_{\Delta}\xrightarrow{(\nu,\alpha_4|_{\Delta})}A_{5,6}\Big|_{\Delta}\longrightarrow 0$$

$${}'\mathcal{E}_1^{\bullet,-8}\simeq 0\longrightarrow A_{5,8}\twoheadrightarrow A_{5,8}\Big|_{\Delta}\longrightarrow 0$$

$${}'\mathcal{E}_1^{\bullet,-10}\simeq 0$$

These complexes are exact out of the diagonal. The spectral sequence ${}'\mathcal{E}_1^{p,q}$ looks like:



where we indicated with a bold character the degrees at which the complexes are not exact. As a consequence, by what we just said and by corollary 4.9, we get ${}'\mathcal{E}_2^{p,q} = 0$ except

- ${}'\mathcal{E}_2^{0,0} \simeq {}'\mathcal{E}_\infty^{0,0}$
- ${}'\mathcal{E}_2^{4,-4}$ and ${}'\mathcal{E}_2^{8,-8}$, which are on the diagonal.

Therefore:

$$\mathbf{R}\mu_*(S^2 E^{[6]}) \simeq (S^2 \mathcal{C}_E^\bullet)^G \ ,$$

since it is true on every affine open set of the form $S^n U$, and these open sets cover $S^n X$.

The exterior power. As usual, let U^6 a G-invariant open affine subset of X^6, with U affine open set in X. We will use here the spectral sequence:

$$''\mathcal{E}_1^{p,q} \simeq H^0(U^6, \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i, \mathcal{C}^j) \otimes \varepsilon_H)^P \ .$$

We have $''\mathcal{E}_1^{\bullet,0} \simeq (\Lambda^2 \mathcal{C}^\bullet)^G$. We now sketch the computation of $''\mathcal{E}_1^{\bullet,-2}$. For the terms $''\mathcal{E}_1^{p,-2}$, $p \leq 4$, the computation is the same as that done for $n = 4$. We then begin with

$$
\begin{aligned}
''\mathcal{E}_1^{5,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^4))^G \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^3))^G \\
&\simeq 0 \oplus H^0(\Delta_{U^4} \times S^2U, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{123}}, \mathcal{O}_{\Delta_{1\ldots4}}))^{\mathfrak{S}_3} \simeq A_{3,2}\Big|_{\Delta_{1\ldots4}} \\
''\mathcal{E}_1^{6,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^3, \mathcal{C}^3) \otimes \varepsilon_H)^P \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^4))^G \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^1, \mathcal{C}^5))^G \\
&\simeq H^0(\Delta_{U^4} \times S^2U, \Lambda^2 N^*_{1\ldots4})^{\mathfrak{S}_4} \bigoplus_{i=1}^{3} H^0(U^6, \bigoplus_{\substack{|K|=|K'|=4 \\ |K\cap K'|=i}} \mathrm{Tor}_2(\mathcal{O}_{\Delta_K}, \mathcal{O}_{\Delta_{K'}}))^P \\
&\simeq A_{4,2} \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{O}_{\Delta_{1234}}, \mathcal{O}_{\Delta_{1235}}))^{P_{\{1234\}\times\{1235\}}} \\
&\simeq A_{4,2} \oplus H^0(\Delta_{U^5} \times U, \Lambda^2 N^*_{123}\Big|_{\Delta_{1\ldots5}})^{\mathfrak{S}_3} \\
&\simeq A_{4,2} \oplus A_{3,2}\Big|_{\Delta_{1\ldots5}} \\
''\mathcal{E}_1^{7,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^2, \mathcal{C}^5))^G \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^3, \mathcal{C}^4))^G \\
&\simeq H^0(\Delta_{U^5} \times U, \Lambda^2 N^*_{1\ldots4}\Big|_{\Delta_{1\ldots5}})^{\mathfrak{S}_4} \simeq A_{4,2}\Big|_{\Delta_{1\ldots5}} \\
''\mathcal{E}_1^{8,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^4, \mathcal{C}^4) \otimes \varepsilon_H)^P \oplus H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^3, \mathcal{C}^5))^G \simeq 0 \\
''\mathcal{E}_1^{9,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^4, \mathcal{C}^5))^G \simeq H^0(\Delta_{U^6}, \Lambda^2 N^*_{1\ldots5}\Big|_{\Delta})^{\mathfrak{S}_5} \simeq A_{5,2}\Big|_{\Delta} \\
''\mathcal{E}_1^{10,-2} &\simeq H^0(U^6, \mathrm{Tor}_2(\mathcal{C}^5, \mathcal{C}^5))^P \simeq H^0(\Delta_{U^6}, \Lambda^2 N^*_{1\ldots6})^{\mathfrak{S}_6} \simeq A_{6,-2}\,.
\end{aligned}
$$

Therefore the complex $''\mathcal{E}_1^{\bullet,-2}$ is isomorphic to:

$$
A_{2,2} \xrightarrow{\nu} A_{2,2}\Big|_{\Delta_{123}} \longrightarrow 0 \longrightarrow A_{3,2}\Big|_{\Delta_{1\ldots4}} \xrightarrow{(\alpha_3,\nu)} A_{4,2} \oplus A_{3,2}\Big|_{\Delta_{1\ldots5}} \xrightarrow{(-\nu,\alpha_3|_{\Delta_{1\ldots5}})} A_{4,2}\Big|_{\Delta_{1\ldots5}} \longrightarrow 0 \longrightarrow
$$
$$
\longrightarrow A_{5,2}\Big|_{\Delta} \xrightarrow{\alpha_5} A_{6,2} \longrightarrow 0
$$

which is exact out of the diagonal. In the same way we can prove that the complexes $''\mathcal{E}_1^{\bullet,q}$ for q even are:

$$
''\mathcal{E}_1^{\bullet,-4} \simeq 0 \longrightarrow A_{3,4}\Big|_{\Delta_{1\ldots4}} \xrightarrow{(\alpha_3,\nu)} A_{4,4} \oplus A_{3,4}\Big|_{\Delta_{1\ldots5}} \xrightarrow{(-\nu,\alpha_3|_{\Delta_{1\ldots5}})} A_{4,4}\Big|_{\Delta_{1\ldots5}} \longrightarrow 0 \longrightarrow
$$
$$
\longrightarrow A_{5,4}\Big|_{\Delta} \xrightarrow{\alpha_5} A_{6,4} \longrightarrow 0
$$

$$
\begin{aligned}
''\mathcal{E}_1^{\bullet,-6} &\simeq 0 \longrightarrow A_{4,6} \xrightarrow{-\nu} A_{4,6}\Big|_{\Delta_{1\ldots5}} \longrightarrow 0 \longrightarrow A_{5,6}\Big|_{\Delta} \xrightarrow{\alpha_5} A_{6,6} \longrightarrow 0 \\
''\mathcal{E}_1^{\bullet,-8} &\simeq 0 \longrightarrow A_{5,8}\Big|_{\Delta} \xrightarrow{\alpha_5} A_{6,8} \longrightarrow 0 \\
''\mathcal{E}_1^{\bullet,-10} &\simeq A_{6,10}
\end{aligned}
$$

All these complexes are exact out of the diagonal. The spectral sequence looks like:

$$
\begin{array}{l}
\boldsymbol{''\mathcal{E}_1^{2,-2}} \twoheadrightarrow {''\mathcal{E}_1^{3,-2}} \longrightarrow 0 \longrightarrow {''\mathcal{E}_1^{5,-2}} \longrightarrow {''\mathcal{E}_1^{6,-2}} \longrightarrow {''\mathcal{E}_1^{7,-2}} \longrightarrow 0 \longrightarrow {'\mathcal{E}_1^{9,-2}} \longrightarrow {''\mathcal{E}_1^{10,-2}} \\
\qquad\qquad 0 \longrightarrow {''\mathcal{E}_1^{5,-4}} \longrightarrow {''\mathcal{E}^{6,-4}} \longrightarrow {''\mathcal{E}_1^{7,-4}} \longrightarrow 0 \longrightarrow {''\mathcal{E}_1^{9,-4}} \longrightarrow {''\mathcal{E}_1^{10,-4}} \\
\qquad\qquad\qquad\qquad \boldsymbol{''\mathcal{E}^{6,-6}} \twoheadrightarrow {''\mathcal{E}_1^{7,-6}} \longrightarrow 0 \longrightarrow {''\mathcal{E}_1^{9,-6}} \longrightarrow {''\mathcal{E}_1^{10,-6}} \\
\qquad\qquad\qquad\qquad\qquad\qquad 0 \longrightarrow {''\mathcal{E}_1^{9,-8}} \longrightarrow {''\mathcal{E}_1^{10,-8}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{''\mathcal{E}_1^{10,-10}}
\end{array}
$$

where we indicate with a bold character the degrees where the complexes are not exact. As a consequence the terms $''\mathcal{E}_2^{p,q}$ vanish except for

- $''\mathcal{E}_2^{0,0} \simeq {''\mathcal{E}_\infty^{0,0}}$;
- $''\mathcal{E}_2^{2,-2}$, $''\mathcal{E}_2^{6,-6}$, $''\mathcal{E}_2^{10,-10}$.

The last three ones are torsion terms lying on the diagonal. As a consequence of corollary 4.10:

$$\mathbf{R}\mu_*(\Lambda^2 E^{[6]}) \simeq (\Lambda^2 \mathcal{C}_E{}^\bullet)^G \ .$$

4.9 The general case

4.9.1 The symmetric power

Let X a smooth quasi-projective surface. Let E a line bundle over X. Consider the resolution $\mathcal{C}_E^\bullet$ of the image $\mathbf{\Phi}(E^{[n]}) \simeq p_*(E_B^{[n]})$ we found in the previous chapter. By corollary 4.9 the term $\mathbf{\Phi}(S^2 E^{[n]})$ is quasi-isomorphic to the term $'E_\infty^{0,0}$ of the spectral sequence:

$$'E_1^{p,q} \simeq \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}_E^i, \mathcal{C}_E^j)^H$$

abutting to $H^{p+q}(S^2_L\mathcal{C}_E{}^\bullet)$. Our aim is to prove (36) for all $n \geq 0$. This can be achieved by proving it on every affine open set of S^nX of the form S^nU, with U affine open subset of X, since such open subsets cover the entire symmetric variety S^nX, by lemma 1.27. This amounts to considering the G-invariant sections of ${}'E_1^{p,q}$ on every G-invariant affine open set $U^n \subseteq X^n$. In order to do this we will apply the (exact) functor $\Gamma^G_{U^n}$ of G-invariant sections over U^n to the spectral sequence ${}'E_1^{p,q}$, and we get another spectral sequence:

$$'\mathcal{E}_1^{p,q} = H^0(U^n, \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i_E, \mathcal{C}^j_E))^P$$

abutting to the G-equivariant cohomology $H^*_G(U^n, S^2_L\mathcal{C}^\bullet_E) \simeq H^*(U^{[n]}, S^2E^{[n]})$, where $U^{[n]}$ indicates the inverse image $\mu^{-1}(S^nU)$ of the quotient $S^nU \simeq U^n/G$ for the Hilbert-Chow morphism.

We summarize and prove in the following lemma the properties 1), 2), 3) used to get the result for $n = 6$. This lemma completes Danila's lemma in the form of lemma 4.20.

Lemma 4.23. *Suppose that $J, K \subseteq \{1 \dots, n\}$, $|J| = j + 1$, $|K| = k + 1$, $j \leq k$. Then the sheaf*

$$\mathrm{Tor}_q(E_J, E_K)$$

has no $P_{I,J}$-invariant sections, except for:

a) q is even, $j = k$ is nonzero even and $|J \cap K| = j$, or $j+1$. In these cases we have the isomorphism:

$$H^0(U^n, \mathrm{Tor}_q(E_J, E_K))^{P_{J,K}} \xrightarrow{\simeq} H^0(U^n, \Lambda^q N^*_{J\cap K} \otimes E^{\otimes^2}_{J\cup K})^{P_{J,K}} \ ;$$

b) q is even, $k = j + 1$ and $J \subseteq K$. We then have the isomorphism:

$$H^0(U^n, \mathrm{Tor}_q(E_J, E_K))^{P_{J,K}} \xrightarrow{\simeq} H^0(U^n, \Lambda^q N^*_J \otimes E^{\otimes^2}_K)^{G_{I,J}} \ ;$$

c) $q = 0$, $j = k = 0$.

Proof. We begin by studying the $P_{J,K}$ action on $\mathrm{Tor}_q(E_J, E_K)$ for q even. We know by remark 4.19 that if $|J| \neq |K|$, then $P_{J,K} = G_{J,K}$. Therefore we will concentrate on the case $|J| = |K|$, leaving the treatment of the $G_{J,K}$ action afterwords. In the case $|J| = |K|$ the stabilizer $P_{J,K}$ is:

$$P_{J,K} = \langle G_{J,K}, \tau\sigma_{J,K} \rangle,$$

where $\sigma_{J,K}$ is an involution of $\{1, \dots, n\}$ fixing $J \cap K$ and such that $\sigma(J \setminus (J \cap K)) = \sigma(K \setminus (J \cap K))$. If $J = K$ this can be taken to be the identity, but if $J \neq K$ we can set $\sigma_{J,K} = (j, k)$ where $j \in J \setminus (J \cap K)$, $k \in K \setminus (J \cap K)$. Consider first the case $J = K$. The transposition of factors τ acts on $\mathrm{Tor}_q(E_J, E_J)$ with the sign $(-1)^{q+j^2}$. Therefore there are no invariants if $|J| = j + 1$ is even. In the case $J \neq K$ the element $\tau\sigma_{I,J}$ acts with the sign $(-1)^{2q+j^2}$ because τ acts in any case with the sign $(-1)^{q+j^2}$ by definition of the action of the symmetric group on the tensor power of a complex, and $\sigma_{J,K} = (j, k)$ acts with the sign $(-1)^q$, by lemma 4.13. Again there are no invariants if j is odd. To resume,

$$\mathrm{Tor}_q(E_J, E_K)^{P_{I,J}} = \begin{cases} \mathrm{Tor}_q(E_J, E_K)^{G_{I,J}} & \text{if } |J| = |K| \text{ is odd or } |J| \neq |K| \\ 0 & \text{if } |J| = |K| \text{ is even} \end{cases}$$

We pass now to the $G_{I,J}$ action. Suppose first $|J \cap K| \geq 1$. Then

$$\mathrm{Tor}_q(E_J, E_K) = \Lambda^q N^*_{I\cap J} \otimes E^{\otimes^2}_{I\cup J} \ .$$

The stabilizer $G_{J,K}$ is isomorphic to

$$G_{J,K} \simeq \mathfrak{S}(J \cap K) \times \mathfrak{S}(J \setminus (J \cap K)) \times \mathfrak{S}(K \setminus (J \cap K)) \times \mathfrak{S}(\overline{J \cup K}) \ .$$

An element $(\sigma, \tau_1, \tau_2, \zeta)$ acts on the fiber of $\mathrm{Tor}_q(E_J, E_K)$ over a point $x \in \Delta_{I\cup J}$ like

$$\Lambda^q(\rho_{|J\cap K|}(\sigma) \otimes \mathbb{C}^2)\,\mathrm{sgn}(\sigma)^2\,\mathrm{sgn}(\tau_1)\,\mathrm{sgn}(\tau_2)$$

where $\rho_{|I\cap J|}$ is the standard representation of the group $\mathfrak{S}(I \cap J)$. If one of the groups $\mathfrak{S}(J \setminus (J \cap K))$, $\mathfrak{S}(K \setminus (J\cap K))$ is nontrivial, there are transpositions $\tau_1 \in \mathfrak{S}(J \setminus (J\cap K))$, $\tau_2 \in \mathfrak{S}(K\setminus(J\cap K))$ preventing $G_{J,K}$-invariants. Therefore to have invariants we have to set

$$|J \setminus (J \cap K)| \leq 1 \qquad ; \qquad |K \setminus (J \cap K)| \leq 1 \,. \tag{52}$$

In this case the action is reduced to the representation:

$$\Lambda^q(\rho_{|J\cap K|} \otimes \mathbb{C}^2) \,;$$

therefore there are invariants only if $|J \cap K| \geq 2$ and q even by lemma 4.14 or $|J \cap K| \leq 1$, with the conditions (52). The latter case forces q to be zero to have invariants, because $\mathrm{Tor}_q(E_J, E_K) = 0$ if $|J \cap K| \leq 1$, by remark 4.17.

The case $|J \cap K| \geq 2$*,* q *even.* There remain two subcases:

1. $|J| = |K|$ nonzero odd;
2. $|J| \neq |K|$.

In the case $|J| = |K|$ nonzero odd, the conditions (52) imply:

$$|J \setminus (J \cap K)| = |K \setminus (J \cap K)| \leq 1 \,.$$

Therefore

- if $|J \setminus (J \cap K)| = 0$ then $J = K$ and $|J \cap K| = j + 1$
- if $|J \setminus (J \cap K)| = 1$ then $|J \cap K| = j$

and we are in case $a)$ of the statement.
In case $|J| \neq |K|$, we have $|J \setminus (J\cap K)| \neq |K \setminus (J \cap K)|$ which yields:

$$|J \setminus (J \cap K)| = 0 \qquad ; \qquad |K \setminus (J \cap K)| = 1 \,.$$

this means $J \subseteq K$, and $k = j + 1$ and we are in case b).

The case $|J \cap K| = 1$*,* $q = 0$*.* This case splits in the two subcases $|J| = |K|$ and $|J| \neq |K|$. If $|J| = |K|$ we can have $J = K$, $|J| = 1$ or $J \neq K$, $|J \cap K| = 1$, $|J| = |K| = 2$ (case a)).
If $|J| \neq |K|$ we have $J \subseteq K$, $|K| = 2$, $|J| = 1$ (case b)).

The case $|J \cap K| = 0$*,* $q = 0$*.* Conditions (52) imply $|J| = |K| = 1$ which is $j = k = 0$ (case c)). In this case

$${}'\mathcal{E}_1^{0,0} \simeq H^0(U^n, E_1 \otimes E_2)^{P_{\{12\}}} \simeq H^0(S^2U \times S^{n-2}U, E \boxtimes E/\mathfrak{S}_2 \boxtimes \mathcal{O}_{S^{n-2}X}) \,.$$

□

We will denote the last space of sections $A_{0,0}$, and the vector bundle $E \boxtimes E/\mathfrak{S}_2 \boxtimes \mathcal{O}_{S^{n-2}X}$ on $S^2X \times S^{n-2}X$ with $\mathcal{A}_{0,0}$.

Corollary 4.24. *The term* ${}'\mathcal{E}_1^{p,-q}$ *is zero except in the following cases:*

a) q *is even,* $q \leq 2n-2$*,* $p \equiv 0 \mod 4$ *and* $p \leq q$*. We set* $p = 2j$*. Then* ${}'\mathcal{E}_1^{p,-q}$ *is isomorphic to*

$${}'\mathcal{E}_1^{p,-q} \simeq A_{j+1,q} \oplus A_{j,q}\Big|_{\Delta_{\{1,\dots,j+2\}}} ;$$

b) q is even, p is odd, $p \geq q$. We set $p = 2j-1$. Then ${}'\mathcal{E}_1^{p,-q}$ is isomorphic to

$$
{}'\mathcal{E}_1^{p,-q} \simeq A_{j,q}\Big|_{\Delta_{\{1,\ldots,j+1\}}} .
$$

Proof. It suffices to recall the preceding lemma, Danila's lemma, and the definition of $A_{j,q}$.

□

The differentials. We now pass to study the differentials of the spectral sequence ${}'\mathcal{E}_1^{p,-q}$. We just proved that ${}'\mathcal{E}_1^{p,-q} = 0$ if q is odd or if $p \equiv 2 \mod 4$. Therefore, for q even and $p = 4s$, we can consider the subcomplex $K_{4s,q}^{\bullet}$ of ${}'\mathcal{E}_1^{\bullet,-q}$ (centered in degree $4s$) given by:

$$
0 \longrightarrow {}'\mathcal{E}_1^{4s-1,-q} \longrightarrow {}'\mathcal{E}_1^{4s,-q} \longrightarrow {}'\mathcal{E}_1^{4s+1,-q} \longrightarrow 0 .
$$

By the last corollary it is:

$$
0 \longrightarrow A_{2s,q}\Big|_{\Delta_{\{1,\ldots,2s+1\}}} \longrightarrow A_{2s+1,q} \oplus A_{2s,q}\Big|_{\Delta_{\{1,\ldots,2s+2\}}} \longrightarrow A_{2s+1,q}\Big|_{\Delta_{\{1,\ldots,2s+2\}}} \longrightarrow 0 .
$$

As a consequence the complex ${}'\mathcal{E}_1^{\bullet,-q}$ is the direct sum:

$$
{}'\mathcal{E}_1^{\bullet,-q} = \bigoplus_{q \leq 4s \leq 2n-2} K_{4s,q}^{\bullet} .
$$

We can prove the following proposition for q even:

Proposition 4.25. *Let $n \in \mathbb{N}$, $n \geq 2$. Let $q \in 2\mathbb{N}$. Then*

1) *For $q < 4s \leq 2n-2$ the complex $K_{4s,q}^{\bullet}$ is acyclic.*

2) *For $0 < q = 4s \leq 2n-2$, the complex $K_{4s,q}^{\bullet}$ is reduced to 2-term complex:*

$$
0 \longrightarrow A_{2s+1,q} \longrightarrow A_{2s+1,q}\Big|_{\Delta_{\{1,\ldots,2s+2\}}} \longrightarrow 0 .
$$

and is exact in degree different from $4s$. For $0 = q = s$ the complex $K_{0,0}^{\bullet}$ is reduced to the 2-terms complex:

$$
0 \longrightarrow A_{1,0} \oplus A_{0,0} \longrightarrow A_{1,0}\Big|_{\Delta_{\{1,2\}}} \longrightarrow 0
$$

and is exact in degree different from 0.

3) *If n is odd and $4s = 2n-2$, the diagonal $\Delta_{\{1,\ldots,2s+1\}}$ is the small one Δ. For $q < 2n-2$ the complex $K_{4s,q}^{\bullet}$ is reduced to the acyclic 2-terms complex:*

$$
0 \longrightarrow A_{2s,q}\Big|_{\Delta} \longrightarrow A_{2s+1,q} \longrightarrow 0 .
$$

For $q = 2n-2$, the complex $K_{2n-2,q}^{\bullet}$ is reduced to the only term $\det N_{\Delta}^*$, placed in degree $2n-2$.

Proof. 1). By Danila's lemma, for $q < 4s \leq 2n-2$ the complex $K_{4s,q}^{\bullet}$ is the complex of $G_{\{1,\ldots,2s+1\}}$-invariant sections of the complex:

$$
0 \longrightarrow \bigoplus_{\substack{J \subseteq \{1,\ldots 2s+1\} \\ |J|=2s}} \Lambda^q N_J^*\Big|_{\Delta_{\{1,\ldots,2s+1\}}} \longrightarrow \Lambda^q N_{\{1,\ldots,2s+1\}}^* \bigoplus \bigoplus_{\substack{J \subseteq \{1,\ldots 2s+1\} \\ |J|=2s}} \Lambda^q N_J^*\Big|_{\Delta_{\{1,\ldots,2s+2\}}} \longrightarrow
$$

$$
\longrightarrow \Lambda^q N_{\{1,\ldots,2s+1\}}^*\Big|_{\Delta_{\{1,\ldots,2s+2\}}} \longrightarrow 0
$$

which is in turn induced by the complex of H-invariant sheaves:

$$0 \longrightarrow \mathrm{Tor}_q(\mathcal{C}^{2s-1}, \mathcal{C}^{2s}) \longrightarrow \mathrm{Tor}_q(\mathcal{C}^{2s}, \mathcal{C}^{2s}) \longrightarrow \mathrm{Tor}_q(\mathcal{C}^{2s}, \mathcal{C}^{2s+1}) \longrightarrow 0\ .$$

We are exactly in the situation of lemma (4.22). As a consequence the first component of the first map on $K^\bullet_{4s,q}$ is the isomorphism α_{2s} and the second component is the restriction ν (up to a sign). The second map is easily the couple $(\pm\nu, \alpha_{2s}\big|_{\Delta_{\{1,2s+2\}}})$. Therefore the complex is acyclic.

The cases 2) and 3) can be easily obtained as limit cases of the preceding, taking into account the vanishing terms.

□

Corollary 4.26. *The spectral sequence* $'\mathcal{E}_1^{p,q}$ *degenerates at level* $'\mathcal{E}_2$. *In particular* $'\mathcal{E}_2^{p,q} = 0$ *except for* $p+q=0$, $p \equiv 0 \mod 4$, $0 \leq p \leq 2n-2$. *In particular* $'\mathcal{E}_\infty^{0,0} = {}'\mathcal{E}_2^{0,0}$.

The main consequence of the corollary is that formula (36) holds on every open affine set of the symmetric variety S^nX of the form S^nU, with U affine open set in X. This yields its validity on all the symmetric variety S^nX, for every quasi-projective surface X, by lemma 1.27. We then have proved the following generalization of a Danila-Brion formula [23]:

Theorem 4.27. *Let X a smooth quasi-projective surface and E a line bundle on X. The image $\mu_*(S^2E^{[n]})$ of the double symmetric power of a tautological vector bundle $E^{[n]}$ for the Hilbert-Chow morphism μ is quasi-isomorphic to the complex of G-invariants of the (non-derived) symmetric power $S^2\mathcal{C}^\bullet_E$:*

$$\mathbf{R}\mu_*(S^2E^{[n]}) \simeq (S^2\mathcal{C}^\bullet_E)^G\ .$$

Remark 4.28. The conclusions of proposition 4.25 allow us to say actually something more. It turns out that $'\mathcal{E}_1^{2,0} = 0 = [(S^2\mathcal{C}^\bullet_E)^G]^2$. Therefore, if we denote with $\tau_{\leq l}$ the truncation functor, we get the simplified formula:

$$\mathbf{R}\mu_*(S^2E^{[n]}) \simeq \tau_{\leq 1}(S^2\mathcal{C}^\bullet_E)^G\ . \tag{53}$$

We remark that the complex

$$\tau_{\leq 1}(S^2\mathcal{C}^\bullet_E)^G : 0 \longrightarrow (S^2\mathcal{C}^0_E)^G \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^G \longrightarrow 0$$

is a 2-terms complex which is acyclic in degree $\neq 0$.

4.9.2 The exterior power.

We use the same notations of the symmetric power case. By corollary 4.10 the term $\mathbf{\Phi}(\Lambda^2E^{[n]})$ is quasi-isomorphic to the term $''E_\infty^{0,0}$ of the spectral sequence:

$$''E_1^{p,q} = \bigoplus_{i+j=p} (\mathrm{Tor}_{-q}(\mathcal{C}^i_E, \mathcal{C}^j_E) \otimes \varepsilon_H)^H$$

where ε_H is the alternating representation of H. To prove (37), it suffices to show that it holds on every affine open set of the symmetric variety of the form S^nU, with U affine open set in X, by lemma 1.27. This can be done at the sections level over S^nX, or, equivalently, considering the G-invariant sections on an G-invariant affine open set $U^n \subset X^n$. Applying the functor of invariant sections $\Gamma^G_{U^n}$ we get another spectral sequence:

$$''\mathcal{E}_1^{p,q} = H^0(U^n, \bigoplus_{i+j=p} \mathrm{Tor}_{-q}(\mathcal{C}^i_E, \mathcal{C}^j_E) \otimes \varepsilon_H)^P$$

abutting to the G-equivariant cohomology $H^*_G(U^n, \Lambda^2_L\mathcal{C}^\bullet_E) \simeq H^*(U^{[n]}, \Lambda^2E^{[n]})$. The equivalent of lemma 4.23 for the exterior power is the following:

Lemma 4.29. *Suppost that $J, K \subseteq \{1\dots,n\}$, $|J| = j+1$, $|K| = k+1$, $j \leq k$. Then the sheaf*

$$\mathrm{Tor}_q(E_J, E_K) \otimes \varepsilon_H$$

has no $P_{I,J}$-invariant sections, except for:

a) q is even, $j = k$ is nonzero odd and $|J \cap K| = j$, or $j+1$. In these cases we have the isomorphism:

$$H^0(U^n, \mathrm{Tor}_q(E_J, E_K) \otimes \varepsilon_H)^{P_{J,K}} \xrightarrow{\simeq} H^0(U^n, \Lambda^q N^*_{J\cap K} \otimes E^{\otimes^2}_{J\cup K})^{P_{J,K}} \ ;$$

b) q is even, $k = j+1$ and $J \subseteq K$. We then have the isomorphism:

$$H^0(U^n, \mathrm{Tor}_q(E_J, E_K) \otimes \varepsilon_H)^{P_{J,K}} \xrightarrow{\simeq} H^0(U^n, \Lambda^q N^*_J \otimes E^{\otimes^2}_K)^{G_{I,J}} \ ;$$

c) $q = 0$, $j = k = 0$.

Proof. The proof goes exactly as in the case of lemma 4.23, except for the fact that we are twisting by the alternant representation ε_H of H. The same computations we did in that proof show that if $K = J$, the transposition of factors acts on $\mathrm{Tor}_q(E_J, E_J)$ with the sign: $(-1)^{q+j^2+1}$. If $K \neq J$, but $|K| = |J|$, then the element $\tau\sigma_{J,K}$ acts with the sign: $(-1)^{2q+j^2+1}$. Therefore if q is even, we do not have invariants for j even.

□

Let $\tilde{A}_{0,0}$ the space of invariant sections of $E_1 \otimes E_2 \otimes \varepsilon_H$ on U^n for $P_{\{1,2\}}$:

$$\tilde{A}_{0,0} \simeq H^0(U^n, E_1 \otimes E_2 \otimes \varepsilon_H)^{P_{\{1,2\}}} \ .$$

Corollary 4.30. *The term ${}'\mathcal{E}_1^{p,-q}$ is zero except in the following cases:*

a) $p = q = 0$. Then ${}''\mathcal{E}_\infty^{0,0} \simeq \tilde{A}_{0,0}$.

b) q is even, $p \equiv 2 \mod 4$, and $p \geq q$. We set $p = 2j$. Then

$${}''\mathcal{E}_1^{p,-q} \simeq A_{j+1,q} \oplus A_{j,q}\Big|_{\Delta_{\{1,\dots,j+2\}}} \ .$$

c) $q$is even, p is odd, $p \geq q$. We set $p = 2j-1$. Then

$${}''\mathcal{E}_1^{p,-q} \simeq A_{j,q}\Big|_{\Delta_{\{1,j+1\}}} \ .$$

□

The differentials. Since ${}''\mathcal{E}_1^{p,-q} = 0$ if q is odd and $p \equiv 2 \mod 4$, we can consider the following subcomplex $K^\bullet_{4s+2,q}$ of ${}''\mathcal{E}_1^{\bullet,-q}$, centered in degree $4s+2$:

$$0 \longrightarrow {}''\mathcal{E}^{4s+1,-q} \longrightarrow {}''\mathcal{E}_1^{4s+2,-q} \longrightarrow {}''\mathcal{E}_1^{4s+3,-q} \longrightarrow 0 \ .$$

By the previous corollary it is:

$$0 \longrightarrow A_{2s+1,q}\Big|_{\Delta_{\{1,\dots,2s+2\}}} \longrightarrow A_{2s+2,q} \oplus A_{2s+1,q}\Big|_{\Delta_{\{1,2s+3\}}} \longrightarrow A_{2s+2,q}\Big|_{\Delta_{\{1,\dots,2s+3\}}} \longrightarrow 0 \ .$$

The complex $''\mathcal{E}_1^{\bullet,-q}$ is, for $q > 0$ even, the direct sum:

$$''\mathcal{E}_1^{\bullet,-q} \simeq \bigoplus_{q \leq 4s+2 \leq 2n-2} K^{\bullet}_{4s+2,q} \; ;$$

for $q = 0$ we have:

$$''\mathcal{E}_1^{\bullet,0} \simeq {}''\mathcal{E}_1^{0,0} \bigoplus_{0 \leq 4s+2 \leq 2n-2} K^{\bullet}_{4s+2,0} \; .$$

The differentials are clearly the natural morphisms induced by the complexes of H-invariant sheaves:

$$0 \longrightarrow \operatorname{Tor}_q(\mathcal{C}^{2s+1}, \mathcal{C}^{2s+2}) \longrightarrow \operatorname{Tor}_q(\mathcal{C}^{2s+2}, \mathcal{C}^{2s+2}) \longrightarrow \operatorname{Tor}_q(\mathcal{C}^{2s+2}, \mathcal{C}^{2s+3}) \longrightarrow 0 \; .$$

By lemma 4.22 we get the following analogue to proposition 4.25:

Proposition 4.31. *Let $n \in \mathbb{N}$, $n \geq 2$. Let $q \in 2\mathbb{N}$. Then:*

1. *For $q < 4s+2 < 2n-2$ the complex $K^{\bullet}_{4s+2,q}$ is acyclic.*
2. *If $0 < q = 4s+2 \leq 2n-2$ the complex $K^{\bullet}_{4s+2,q}$ is reduced to the 2-terms complex:*

$$0 \longrightarrow A_{2s+2,q} \longrightarrow A_{2s+2,q}\Big|_{\Delta_{\{1,\dots,2s+3\}}} \longrightarrow 0$$

   which is exact in degree different from $4s+2$.

3. *If n is even and $4s+2 = 2n-2$ then the diagonal $\Delta_{\{1,\dots,2s+2\}}$ is the small diagonal. The complex $K^{\bullet}_{4s+2,q}$ is reduced to the 2-terms complex:*

$$0 \longrightarrow A_{2s+1,q}\Big|_{\Delta} \longrightarrow A_{2s+2,q} \longrightarrow 0$$

   which is exact. If n is even and $q = 2n-2$ the complex $K^{\bullet}_{4s+2,q}$ is reduced to the only term: $\det N^*_{\Delta}$ placed in degree $2n-2$.

□

Corollary 4.32. *The spectral sequence $''\mathcal{E}_1^{p,q}$ degenerates at level $''\mathcal{E}_2$. In particular $''\mathcal{E}_2^{p,q} = 0$ except for $p+q = 0$, $p \equiv 2 \mod 4$, $0 \leq p \leq 2n-2$ or for $p = q = 0$. In particular $''\mathcal{E}_{\infty}^{0,0} = {}''\mathcal{E}_1^{0,0}$.*

□

As in the case of the symmetric power, this corollary implies that formula (37) holds on any affine open set of the symmetric variety of the form S^nU, with U affine open set in X; hence it holds globally on S^nX, for every quasiprojective surface X, by lemma 1.27. Moreover, by the last corollary, we can identify the direct image $\Lambda^2 E^{[n]}$ of the exterior power of the tautological bundle $E^{[n]}$ with the term $''\mathcal{E}_1^{0,0}$. We have then just proved:

Theorem 4.33. *Let X a smooth quasi-projective surface and E a line bundle on X. The image $\mu_*(\Lambda^2 E^{[n]})$ of the double exterior power of a tautological vector bundle $E^{[n]}$ for the Hilbert-Chow morphism μ is quasi-isomorphic to the sheaf of G-invariants of the exterior power $\Lambda^2\mathcal{C}^0_E$:*

$$\mathbf{R}\mu_*(\Lambda^2 E^{[n]}) \simeq (\Lambda^2 \mathcal{C}^{\bullet}_E)^G \simeq (\Lambda^2 \mathcal{C}^0_E)^G \; .$$

Putting together theorems 4.27 and 4.33 we get:

Theorem 4.34. *Let X a smooth quasi-projective surface and E a line bundle on X. Then the derived direct image $\mathbf{R}\mu_*(E^{[n]} \otimes E^{[n]})$ of the double tensor power of the tautological bundle $E^{[n]}$ for the Hilbert-Chow morphism μ is quasi-isomorphic to the two terms complex:*

$$0 \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^0_E)^G \xrightarrow{d} (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^G \longrightarrow 0$$

acyclic in degree higher than zero, where the morphism d is given by $d = \mathrm{id} \otimes d^0_{\mathcal{C}^{\bullet}_E}$.

4.10 Cohomology

The aim of this section is the application of the theorems 4.27, 4.33 to the computation of the cohomology of the Hilbert scheme with values in the double symmetric and exterior power of a tautological vector bundle $E^{[n]}$ associated to a line bundle E on the surface X. We will prove that for every quasi-projective surface:

$$H^*(X^{[n]}, S^2E^{[n]}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus S^2H^*(X,E) \otimes S^{n-2}H^*(X, \mathcal{O}_X)$$

where $\mathcal{J}$ is the ideal of cohomology classes in $S^{n-1}H^*(X, \mathcal{O}_X)$ vanishing on the subscheme $\{a\} \times S^{n-2}X \hookrightarrow S^{n-1}X$. For $n = 2, 3$ this result has already been obtained by Danila [24]. Moreover for every quasi-projective surface:

$$H^*(X^{[n]}, \Lambda^2 E^{[n]}) \simeq \Lambda^2 H^*(X, E) \otimes S^{n-2}H^*(X, \mathcal{O}_X) \; .$$

As a consequence of these two formulas we will get:

Theorem 4.35. *Let X be a smooth quasi-projective surface. The cohomology of the double tensor power ${E^{[n]}}^{\otimes^2}$ of a tautological vector bundle $E^{[n]}$ on the Hilbert scheme, associated to a line bundle E on the surface X, is given by:*

$$H^*(X^{[n]}, {E^{[n]}}^{\otimes^2}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus H^*(X,E)^{\otimes^2} \otimes S^{n-2}H^*(\mathcal{O}_X) \; .$$

4.10.1 Cohomology of the symmetric product of a tautological vector bundle

We know from the previous section that the image of the symmetric power of a tautological bundle for the Hilbert-Chow morphism is:

$$\mathbf{R}\mu_*(S^2E^{[n]}) \simeq (S^2\mathcal{C}^\bullet_E)^G \simeq \tau_{\leq 1}(S^2\mathcal{C}^\bullet_E)^G$$

where $\tau_{\leq 1}(S^2\mathcal{C}^\bullet_E)^G$ indicates the complex $(S^2\mathcal{C}^\bullet_E)^G$ truncated in degree ≤ 1. The truncated complex is exactly:

$$0 \longrightarrow (S^2\mathcal{C}^0_E)^G \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^G \longrightarrow 0 \; .$$

As a consequence we have the short exact sequence:

$$0 \longrightarrow \mu_*(S^2E^{[n]}) \longrightarrow (S^2\mathcal{C}^0_E)^G \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^G \longrightarrow 0 \; .$$

and the associated long exact cohomology sequence:

$$\begin{aligned} \ldots \longrightarrow H^i(X^{[n]}, S^2E^{[n]}) \longrightarrow H^i_G(X^n, S^2\mathcal{C}^0_E) \longrightarrow \\ \longrightarrow H^i_G(X^n, \mathcal{C}^0_E \otimes \mathcal{C}^1_E) \longrightarrow H^{i+1}(X^{[n]}, S^2E^{[n]}) \longrightarrow \ldots \end{aligned} \tag{54}$$

Now

$$S^2\mathcal{C}^0_E \simeq \bigoplus_{i=1}^n E_i^{\otimes^2} \oplus \bigoplus_{i<j} E_i \otimes E_j$$

$$\mathcal{C}^0_E \otimes \mathcal{C}^1_E \simeq \bigoplus_{1\leq i,j \leq n} E^{\otimes^2}\Big|_{\Delta_{ij}} \oplus \bigoplus_{\substack{i\neq j,k \\ j<k}} E_i \otimes E\Big|_{\Delta_{jk}}$$

Therefore:

$$\begin{aligned} H^*_G(X^n, S^2\mathcal{C}^0_E) &\simeq \big(\bigoplus_{i=1}^n H^*(X^n, E_i^{\otimes^2})\big)^G \oplus \big(\bigoplus_{i<j} H^*(X^n, E_i \otimes E_j)\big)^G \\ &\simeq \left(\left[H^*(E^{\otimes^2}) \otimes H^*(\mathcal{O}_X)^{\otimes^{n-1}}\right]^n\right)^G \oplus \left(\left[H^*(E)^{\otimes^2} \otimes H^*(\mathcal{O}_X)^{n-2}\right]^{\binom{n}{2}}\right)^G \\ &\simeq H^*(E^{\otimes^2}) \otimes S^{n-1}H^*(\mathcal{O}_X) \bigoplus S^2H^*(E) \otimes S^{n-2}H^*(\mathcal{O}_X) \end{aligned}$$

applying Kunneth formula and taking G-invariants, while

$$H^*_G(X^n, \mathcal{C}^0_E \otimes \mathcal{C}^1_E) \simeq \Big(\bigoplus_{1 \leq i,j \leq n} H^*(X^n, E^{\otimes^2}\Big|_{\Delta_{ij}}) \Big)^G$$

since the sheaf

$$\bigoplus_{\substack{i \neq j,k \\ j<k}} E_i \otimes E\Big|_{\Delta_{jk}}$$

has no G-invariants. Therefore

$$\begin{aligned} H^*_G(X^n, \mathcal{C}^0_E \otimes \mathcal{C}^1_E) &\simeq \Big(\big[H^*(E^{\otimes^2})^2 \otimes H^*(\mathcal{O}_X)^{\otimes^{n-2}}\big]^{\binom{n}{2}} \Big)^G \\ &\simeq H^*(E^{\otimes^2}) \otimes S^{n-2} H^*(\mathcal{O}_X) \end{aligned}$$

The differential

$$d^0 : (S^2 \mathcal{C}^0_E)^G \longrightarrow (\mathcal{C}^0_E \otimes \mathcal{C}^1_E)^G$$

induces a morphism:

$$H^*(E^{\otimes^2}) \otimes S^{n-1} H^*(\mathcal{O}_X) \bigoplus S^2 H^*(E) \otimes S^{n-2} H^*(\mathcal{O}_X) \longrightarrow H^*(E^{\otimes^2}) \otimes S^{n-2} H^*(\mathcal{O}_X) \,. \tag{55}$$

We will now prove that the morphism (55) is surjective; hence the long exact cohomology sequence splits. We remark that the second component of (55) is the canonical coupling

$$S^2 H^*(E) \longrightarrow H^*(E^{\otimes^2})$$

tensorized by the identity.

Lemma 4.36. *Let F a line bundle on X. Consider, for $k \in \mathbb{N}^*$, the embedding:*

$$X \times S^{k-1} X \hookrightarrow X \times S^k X \tag{56}$$

given by $(x, z) \longmapsto (x, x + z)$. The restriction morphism:

$$D : H^*(F) \otimes S^k H^*(\mathcal{O}_X) \longrightarrow H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X)$$

induced by this embedding is given, for $\alpha \in H^*(F)$ and $u_i \in H^*(\mathcal{O}_X)$, $i = 1, \dots, k$, homogeneous of degree p_i, by the formula:

$$\alpha \otimes u_1 \dots u_k \longrightarrow \frac{1}{k} \sum_{i=1}^k (-1)^{(\sum_{j<i} p_j) p_i} \alpha u_i \otimes u_1 \dots \hat{u_i} \dots u_k \,.$$

Proof. The embedding (56) is induced by the embedding:

$$\begin{aligned} X \times X^{k-1} &\hookrightarrow X \times X^k \,. \\ (x, z_1, \dots, z_{k-1}) &\longmapsto (x, x, z_1, \dots, z_{k-1}) \end{aligned}$$

In cohomology the induced morphism is given by:

$$\alpha \otimes u_1 \otimes \dots \otimes u_k \longrightarrow \alpha u_1 \otimes u_2 \otimes \dots \otimes u_k \,.$$

The canonical projection $X^k \longrightarrow S^k X$ identifies $u_1 \dots u_k \in H^*(S^k X) \simeq S^k H^*(\mathcal{O}_X)$ with the element:

$$\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon_{\sigma, p_1, \dots, p_k} \sigma(u_1 \otimes \dots \otimes u_k)$$

of $H^*(\mathcal{O}_{X^k}) \simeq H^*(\mathcal{O}_X)^{\otimes^k}$, where $\varepsilon_{\sigma, p_1, \dots, p_k}$ is a sign depending on the permutation σ and on the respective degrees of u_i and it is characterized, in the graduated algebra $\mathbb{C}[u_1, \dots, u_k]$ (where $\deg(u_i) = p_i$), by the relation: $\sigma(u_1 \otimes \dots \otimes u_k) = \varepsilon_{\sigma, p_1, \dots, p_k} u_1 \dots u_k$. For the transposition $\tau_{1,i}$ we have $\varepsilon_{\tau_{1,i}, p_1, \dots, p_k} = (-1)^{(\sum_{j<i} p_j) p_i}$. The formula follows.

$\square$

Lemma 4.37. *The restriction morphism* $D : H^*(F) \otimes S^k H^*(\mathcal{O}_X) \longrightarrow H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X)$ *is surjective and has a canonical section.*

Proof. Let $u_i \in H^*(\mathcal{O}_X)$ of degree p_i, $i = 1, \dots, k$, $\alpha \in H^*(F)$. Then by lemma (4.36) the morphism D is given by:

$$D(\alpha \otimes u_1 \dots u_k) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{(\sum_{j<i} p_j) p_i} \alpha u_i \otimes u_1 \dots \hat{u_i} \dots u_k \ .$$

Let λ_j be the morphism:

$$\begin{aligned} \lambda_j : H^*(F) \otimes S^j H^*(\mathcal{O}_X) &\longrightarrow H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X) \\ \alpha \otimes u_1 \dots u_j &\longmapsto \alpha \otimes \underbrace{1 \dots 1}_{k-j-1\text{-fois}} .u_1 \dots u_j \end{aligned}$$

We have $\lambda_{k-1} = \mathrm{id}$, $\lambda_0 = 0$. Let W_i the image of λ_i. We have the filtration:

$$\{0\} = W_0 \subsetneq W_1 \subsetneq \dots \subsetneq W_{k-2} \subsetneq W_{k-1} = H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X) \ .$$

Let now σ be the morphism:

$$\begin{aligned} H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X) &\longrightarrow H^*(F) \otimes S^k H^*(\mathcal{O}_X) \\ \alpha \otimes u_1 \dots u_{k-1} &\longmapsto \alpha \otimes 1.u_1 \dots u_{k-1} \end{aligned}$$

We have the following relation:

$$\begin{aligned} D(\sigma(\lambda_j(\alpha \otimes u_1 \dots u_j))) &= D(\alpha \otimes \underbrace{1 \dots 1}_{k-j\text{-fois}} .u_1 \dots u_j) \\ &= \frac{k-j}{k} \alpha \otimes \underbrace{1 \dots 1}_{k-j-1\text{-fois}} .u_1 \dots u_j + \sum_{h=1}^{j} C_h^j \alpha u_h \otimes \underbrace{1 \dots 1}_{k-j-1\text{-fois}} .u_1 \dots \hat{u_h} \dots u_j \\ &= \frac{k-j}{k} \lambda_j(\alpha \otimes u_1 \dots u_j) + v \end{aligned}$$

where $v \in W_{j-1}$, for some rational constants C_h^j. This means that, indicated with Ψ the endomorphism $D \circ \sigma$ of $H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X)$ we have:

$$\left(\Psi - \frac{k-j}{k} \right) (W_j) \subseteq W_{j-1}$$

which implies:

$$\prod_{j=0}^{k-1} \left(\Psi - \frac{k-j}{k} \right) = 0 \ .$$

In other words there exist $a_1, \dots a_k \in \mathbb{Q}$, with $a_k \neq 0$ such that:

$$\Psi^k + a_1 \Psi^{k-1} + \dots + a_{k-1} \Psi + a_k = \Psi(\Psi^{k-1} + a_1 \Psi^{k-2} + \dots + a_{k-1}) + a_k = 0 \ ,$$

that is, Ψ is invertible. Therefore D is surjective and has a canonical section.

$\square$

A consequence of this lemma is that the kernel of D is isomorphic to a direct factor of the image of σ. The next lemma allows us to characterize such a direct factor.

Lemma 4.38. *Let $a \in X$ a point of X. Consider the morphism:*

$$\mathrm{id} \otimes \nu : H^*(F) \otimes S^k H^*(\mathcal{O}_X) \longrightarrow H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X)$$

where ν is the morphism induced in cohomology by the inclusion $S^{k-1}X \hookrightarrow S^k X$ given by $z \longrightarrow a+z$. Therefore $\mathrm{id} \otimes \nu$ is surjective and its kernel is isomorphic to the kernel of D.

Proof. The morphism $\mathrm{id} \otimes \nu$ is given by:

$$\mathrm{id} \otimes \nu(\alpha \otimes u_1 \dots u_k) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{(\sum_{j<i} p_j) p_i} \alpha u_i(a) \otimes u_1 \dots \hat{u_i} \dots u_k \,.$$

We know that $u_h(a) = 0$ if $\deg u_h > 0$. Therefore, if we denote with $\tilde{\Psi}$ the endomorphism $\mathrm{id} \otimes \nu \circ \sigma$ of $H^*(F) \otimes S^{k-1} H^*(\mathcal{O}_X)$, we have:

$$\begin{aligned} \tilde{\Psi}(\lambda_j(\alpha \otimes u_1 \dots u_j)) &= (\mathrm{id} \otimes \nu)(\alpha \otimes \underbrace{1 \dots 1}_{k-j\text{-fois}} .u_1 \dots u_j) \\ &= \frac{k-j}{k} \lambda_j(\alpha \otimes u_1 \dots u_j) + v \end{aligned}$$

where $v \in W_{j-1}$. Therefore $\left(\tilde{\Psi} - \frac{k-j}{k}\right)(W_j) \subseteq W_{j-1}$ and we have again for $\tilde{\Psi}$ the relation:

$$\prod_{j=0}^{k-1} \left(\tilde{\Psi} - \frac{k-j}{k}\right) = 0$$

which implies that $\tilde{\Psi}$ is invertible, that $\mathrm{id} \otimes \nu$ is surjective and that $\mathrm{Im}\, \sigma$ is a direct factor of $\ker(\mathrm{id} \otimes \nu)$.

$\square$

Lemma 4.39. *Let V, W, Z three vector spaces, not necessarily of finite dimension, over a field k. Let*

$$F = (f, g) : V \oplus W \longrightarrow Z$$

a linear map such that the component f is surjective. Then $\ker F \simeq \ker f \oplus W$.

Proof. Let $K = \ker f$ and $\tilde{V}$ a supplementary of K in V. In the decomposition:

$$K \oplus \tilde{V} \oplus W \longrightarrow Z$$

F can be written as $(0, a, b)$, with a invertible. Therefore $F(x, y, w) = 0$ if and only if $ay + bw = 0$, that is, if and only if, $y = -a^{-1}bw$. Therefore $\ker F \simeq K \oplus W$.

$\square$

Applying the lemmas to the morphism (55) we get:

Theorem 4.40. *Let X a smooth quasi-projective surface. Let a a point in X. Let $\mathcal{J}$ the kernel of the morphism:*

$$S^{n-1} H^*(\mathcal{O}_X) \simeq H^*(S^{n-1}X) \longrightarrow H^*(\{a\} \times S^{n-2}X) \simeq S^{n-2} H^*(\mathcal{O}_X)$$

induced by the morphism:

$$\begin{aligned} S^{n-2}X &\longrightarrow S^{n-1}X \\ x &\longmapsto a + x \end{aligned}$$

We have the isomorphism of graded modules:

$$H^*(X^{[n]}, S^2 E^{[n]}) \simeq H^*(X, E^{\otimes^2}) \otimes \mathcal{J} \bigoplus S^2 H^*(X, E) \otimes S^{n-2} H^*(X, \mathcal{O}_X) \,.$$

4.10.2 Cohomology of the exterior power of a tautological vector bundle

Let X a smooth quasi projective surface and E a line bundle on X. The cohomology of the double exterior power $\Lambda^2 E^{[n]}$ of the tautological vector bundle $E^{[n]}$ is much simpler, since we know that:

$$\mathbf{R}\mu_*(\Lambda^2 E^{[n]}) \simeq (\Lambda^0 \mathcal{C}^0_E)^G \ .$$

Now

$$\Lambda^2 \mathcal{C}^0_E \simeq (\mathcal{C}^0_E \otimes \mathcal{C}^0_E \otimes \varepsilon_H)^H$$

We have

$$\mathcal{C}^0_E \otimes \mathcal{C}^0_E \otimes \varepsilon_H \simeq \bigoplus_{i=1}^n \left(E_i^{\otimes^2} \otimes \varepsilon_H\right) \bigoplus_{i,j} E_i \otimes E_j \otimes \varepsilon_H$$

and

$$\begin{aligned} H^*_G(X^n, \Lambda^2 \mathcal{C}^0_E) &\simeq H^*(X^n, \bigoplus_{i=1}^n E_i^{\otimes^2} \otimes \varepsilon_H)^{G\times H} \bigoplus_{i,j} H^*(X^n, \bigoplus_{i,j} E_i \otimes E_j \otimes \varepsilon_H)^{G\times H} \\ &\simeq H^*(X^n, E_1 \otimes E_2 \otimes \varepsilon_H)^{P_{\{12\}}} \\ &\simeq \Lambda^2 H^*(E) \otimes S^{n-2} H^*(\mathcal{O}_X) \end{aligned}$$

Therefore we get:

Theorem 4.41. *The cohomology of the double exterior power $\Lambda^2 E^{[n]}$ of a tautological vector bundle $E^{[n]}$ on the Hilbert scheme $X^{[n]}$, associated to the line bundle E on the smooth quasi-projective surface X, is given by the isomorphism of graded modules:*

$$H^*(X^{[n]}, \Lambda^2 E^{[n]}) \simeq \Lambda^2 H^*(E) \otimes S^{n-2} H^*(\mathcal{O}_X) \ .$$

□

Perturbations of the metric in Seiberg-Witten equations

Introduction

In the late '80's Donaldson [28], [29] built the first differential invariants for compact simply connected 4-manifolds. These kinds of invariants allow to make the distinction between manifolds which are homeomorphic but not diffeomorphic. For example, one can prove that the smooth quintic in $\mathbb{P}^3_{\mathbb{C}}$ and the manifold $9\mathbb{P}^2_{\mathbb{C}}\sharp 44\overline{\mathbb{P}^2_{\mathbb{C}}}$ are homeomorphic but not diffeomorphic. Donaldson invariants are polynomial invariants:

$$q_d : H_2(M,\mathbb{Z}) \times \cdots \times H_2(M,\mathbb{Z}) \longrightarrow \mathbb{Q}$$

built by means of the $SU(2)$-gauge theory of instantons, or anti-self-dual connections. In other words, fixed a $\mathcal{C}^\infty$-hermitian vector bundle E of rank 2, with trivial determinant, consider the space $\mathcal{S}$ of $SU(2)$-connections A satisfying the anti-self-dual (ASD) condition:

$$F_A^+ = 0\ . \tag{1}$$

The moduli space of instantons M_E is the quotient:

$$M_E = \mathcal{S}/\mathcal{G}$$

where $\mathcal{G}$ is the group of automorphism of E. It is always possible to give M_E a structure of complex analytic space, but there are no reasons why M_E should be smooth. To ensure that M_E is a smooth manifold, one needs to prove that the $\mathcal{G}$-action is free, and that the space of solutions $\mathcal{S}$ is cut out transversally by the equations (1), hence being a smooth Banach submanifold of the (Banach) affine space of all the $SU(2)$-connections $\mathcal{A}$. While the first task is simple to solve (considering the action of a reduced group $\bar{\mathcal{G}} = \mathcal{G}/C(G)$ and gettind rid of the reducible connections by a change of metric) the second is highly non-trivial and constitutes one of the fundamental steps in the construction of instantons moduli spaces. The problem was solved by Freed and Uhlenbeck [46] who considered perturbations of the equations (1) of the form:

$$F_A^{+,g} = 0 \tag{2}$$

where the metric g on the manifold M is seen as an additional parameter. The two authors prove that 0 is a regular value for the application $(A,g) \longmapsto F_A^{+,g}$; consequently, the space of solutions $\mathcal{S}$ can be given the structure of smooth Banach manifold. A standard application of the Sard-Smale theorem then yields that for a generic $\mathcal{C}^k$-metric h in $\mathrm{Met}(M)$ the moduli space of instantons M_E^h, relative to the metric h, is a smooth manifold. This fundamental fact, together with the (difficult) existence of a compactification (done by Donaldson [29], Uhlenbeck [114], [115]), allows the construction of Donaldson polynomial invariants.

In october 1994, Seiberg and Witten (see [104], [105], [119]) built another kind of differential invariants, numerical invariants, based on a much simpler $U(1)$-gauge theory, which can interpreted from the point of view of quantum field theory as a "dual" of Donaldson theory. On the ground of deep physical considerations, Witten predicted that Seiberg-Witten invariants would be able to seize all the richness and subtility of Donaldson's invariants; furthermore he precisely conjectured that Donaldson's polynomials could actually be expressed in terms of Seiberg-Witten invariants. Witten conjecture is on the way of being proved, along ideas of Pidstrigach and Tyurin, after a long and technical work by Okonek, Teleman [99], [100], [108] and most of all by Feehan-Leness [38], [39], [40], [42], [41]. Seiberg-Witten invariants are built from Seiberg-Witten equations: once fixed a $Spin^c$-structure on the compact orientable riemannian 4-manifold (M,g) of spinor bundle $W \simeq W_+ \oplus W_-$ and of fundamental unitary line bundle $L \simeq \det W_+$, the equations read:

$$D_A\psi = 0 \tag{3a}$$

$$F_A^+ = [\psi^* \otimes \psi]_0 \tag{3b}$$

where A is a unitary connection on L, ψ is a positive spinor $\psi \in \Gamma(W_+)$, and $[\psi^* \otimes \psi]_0$ is the traceless part of the operator $\psi^* \otimes \psi \in \mathfrak{u}(W_+)$, hence $[\psi^* \otimes \psi]_0 \in i\mathfrak{su}(W_+) \simeq i\Lambda^2_+ T^*M$. The gauge group is here $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$ and it acts on the solutions via $(A, \psi) \longmapsto ((g^2)^*A, g\psi)$. The group acts freely on the solutions of equations (3a), (3b) of the form (A, ψ), with $\psi \neq 0$, which are said irreducible monopoles. The moduli space of Seiberg-Witten monopoles is the quotient:

$$\mathcal{M}^{SW} = \mathcal{S}/\mathcal{G}$$

where $\mathcal{S}$ is the space of solutions to the Seiberg-Witten equations. To guarantee that the moduli space is smooth we have to ensure that the $\mathcal{G}$-action is free (which can be done as for instantons, by a change the metric preventing reducible monopoles) and that the space of solutions $\mathcal{S}$ is a Banach submanifold of all the configuartion space $\mathcal{A}_L \times \Gamma(W_+)$, that is, cut out transversally by equations (3a), (3b). The second problem is commonly solved by a perturbation of the equations of the kind:

$$D_A\psi = 0 \tag{4a}$$

$$F_A^+ = [\psi^* \otimes \psi]_0 + \eta \tag{4b}$$

where η is an arbitrary imaginary self dual 2-form $\eta \in iA^2_+(M)$. In this way we can get the wanted transversality and the smoothness of the moduli spaces $\mathcal{M}^{SW}_\eta$ of solutions to equations (4a), (4b) for a generic 2-form $\eta \in iA^2_+(M)$. Even though this perturbation is very simple, it does not seem the most natural, nor the most geometric; as we saw previously, in Donaldson theory the transversality of equations is achieved by the perturbation just of the metric, procedure which allows at the same time to get rid of the reducible connections. The perturbation of the metric in Donaldson theory has a deeper geometric meaning; on the other hand the 2-form η lacks any geometric or physical interpretation. Moreover, the behaviour of Seiberg-Witten equations under variations of the metric is interesting in its own, although not so much is known. The only reference about perturbations of the metric in Seiberg-Witten the literature in an article by Eichhorn and Friedrich [31], where the two authors claim that they proved a transversality result for generic metrics, but a careful reading of the proof reveals several mistakes which cannot be straightforwardly corrected.

We proposed ourselves to clarify the question. The first difficulty we meet is the variation of the Dirac operator corresponding to a variation of the metric: the question has already been studied by Bourguignon and Gauduchon ([12], [11]). The two authors build isomorphisms (identifications) between different spinor bundles associated with different metrics, thus succeeding in comparing different Dirac operators living in different spinor bundles. We decided to take another approach, which we now explain. Giving a *Spin*^{c}-structure on a compact riemannian 4-manifold M is equivalent to giving a spin representation (W, ρ), that is, the data of an hermitian vector bundle W on M and a bundle map:

$$\rho : TM \longrightarrow \mathrm{End}(W)$$

such that $\rho(x)^* = -\rho(x)$, $\rho(x)^2 = -g(x, x)$, for all $x \in TM$ (cf. [81], [38], [40])). We definitely fix a spinor bundle W on the riemannian 4-manifold (M, g) and in correspondence to a change of the metric ($g_t = \varphi_t^* g$, $\varphi_t \in \mathrm{Aut}(TM)$) we change, in an obvious way, the Clifford multiplication by means of the diagram:

$$\begin{tikzcd}
(TM, g) \arrow[r, "\rho"] & \mathrm{End}(W) \\
(TM, g_t) \arrow[u, "\varphi_t"] \arrow[ur, "\rho_t"'] &
\end{tikzcd}$$

The couple (W, ρ_t) given by the same spinor bundle W, with the new Clifford multiplication ρ_t becomes a new $Spin^c$-structure for the new riemannian manifold (M, g_t). It is evident that, in this way, it is *inevitable* to change the Clifford multiplication in order to change the metric. So, what does it mean to perturb the metric *only*, if we are obliged to change Clifford multiplication any time we want to change the metric? To answer this question we are induced to study the relations between Clifford multiplications (or $Spin^c$-structures, since the spinor bundle is fixed) and metrics. It turns out that if we fix the spinor bundle W and we take the set of all the compatible couples (g, ρ):

$$\Xi = \{(g, \rho) \mid g \in \mathrm{Met}(M)\,,\ \rho : TM \longrightarrow \mathrm{End}(W)\ , \rho(u)^* = \rho(-u)\,,\ \rho(u)^2 = -g(u,u)\}$$

then Ξ is a $\mathcal{C}^\infty(M, PU(W))$-fibration $\Xi \longrightarrow \mathrm{Met}(M)$ on the space of metrics on which $\mathrm{Aut}(TM)$ acts. In this setting the concept of perturbing the metric alone corresponds, in a weak way, to choosing variations of the $Spin^c$ structure transversal to the vertical distribution, that is, we need the notion of a connection over this fibration. Now there is a natural connection, the horizontal distribution in a point (g, ρ) being given by the tangent space to the image of the section $\sigma(g, \rho)$, transversal to the fiber Ξ_g:

$$\begin{array}{ccc} \mathrm{Sym}^+(TM, g) & \xrightarrow{\sigma(g,\rho)} & \Xi \\ \varphi & \longmapsto & (\varphi^* g, \rho \circ \varphi) \end{array}$$

where $\mathrm{Sym}^+(TM, g)$ denotes the positive symmetric automorphisms of the tangent bundle with respect to the metric g. In other words $H_{(g,\rho)} = T_{(g,\rho)} \mathrm{Im}\sigma(g, \rho)$. This connection clarifies the concept of perturbation of the metric alone in a stronger sense. We define Seiberg-Witten equations and consequently a Seiberg-Witten moduli space $\mathcal{M}$ parametrized by Ξ, whose fiber over a point $\xi = (g, \rho)$ is the standard Seiberg-Witten moduli space $\mathcal{M}^{SW}_{g,\rho}$ associated to the $Spin^c$-structure given by the couple (g, ρ). We prove that the group of unitary automorphisms of the spinor bundle acts on the fibration Ξ (in a vertical way), on the solutions of the parametrized Seiberg-Witten equations and hence on the moduli space $\mathcal{M}$; in the case M is simply connected this action is transitive on the fibres: as a consequence two Seiberg-Witten moduli spaces for two different Clifford multiplications over the same metric are isomorphic:

$$\mathcal{M}^{SW}_{(g,\rho)} \simeq \mathcal{M}^{SW}_{(g,\rho')}\ .$$

We use variations of the $Spin^c$ structure tangent to the natural horizontal distribution to compute the variation of the Seiberg-Witten equations. In particular, the variation of the Dirac operator we obtain in this way is the same of Bourguignon and Gauduchon. We compute the differential $D\mathbb{F}$ to the perturbed Seiberg-Witten functional (in terms of variations of the unitary connection A, the spinor φ and the metric g) and its (formal) adjoint $D\mathbb{F}^*$, and we study the kernel equations $D\mathbb{F}^* u = 0$. Proving a vanishing theorem for the solutions of the kernel equations is equivalent to proving transversality of Seiberg-Witten equations for generic metrics. In the general case the equations are intricate and we still do not have the answer.

When M is a Kähler complex surface with canonical line bundle K_M, Seiberg-Witten equations have an interpretations in terms of holomorphic couples (∂_A, α), where ∂_A is a holomorphic $(0,1)$-semiconnection on a line bundle N such that $K_M^* \otimes N^{\otimes^2} \simeq L$, and α is a holomorphic section of (N, ∂_A). This facts allows a drastic simplification of the Seiberg-Witten equations and consequently of our question. After interpreting all the preceding objects in the context of complex geometry, and thanks to a splitting of the symmetric endomorphisms with respect to the metric into hermitian and anti-hermitian ones, the kernel equations become extremely simpler and we get that Seiberg-Witten equations are transversal for a generic hermitian metric sufficiently close to the Kähler metric g. We precisely proved:

Theorem 0.42. *Let (M, g, J) a Kähler surface. Let N a hermitian line bundle on M such that $2\deg(N) - \deg(K) < 0$. Consider the canonical $Spin^c$-structure on M twisted by the hermitian line bundle N. For* *a generic metric h in a small open neighbourhood of $g \in \mathrm{Met}(M)$ the Seiberg-Witten moduli space $\mathcal{M}_h^{SW}$ is smooth. Actually, the statement holds for a generic hermitian metric h in a small open neighbourhood of g.*

We find a counterexample which clarifies that it is necessary to go out of the Kähler class of metrics to obtain transversality.

1 Preliminaries and notations

In this introductory section we will recall briefly the framework of Seiberg-Witten equations, the definitions of a *Spin*^{c}-structure, of a spin bundle and how the spinorial connection, the Dirac operator and finally the Seiberg-Witten equations are constructed. Moreover we will fix the notations we will be using throughout this part.

1.1 Connections over principal fibre bundles

Let $\pi : P \longrightarrow M$ a principal fibre bundle of structural group G over a manifold M. A *connection* over the bundle P is a G-equivariant subbundle H of the tangent bundle TP, complementary to the vertical tangent space $V = \ker \pi_*$: in other words, $T_pP = V_p \oplus H_p$, and $H_{pg} = (R_g)_* H_p$, where $R_g : P \longrightarrow P$ is the automorphism of P given by the action of the element $g \in G$. Such notion of connection is equivalent to the data of an equivariant $\mathfrak{g}$-valued 1-form $\omega \in A^1(P, \mathfrak{g})$ (equivariant because it must satisfy: $R_g^* \omega = \mathrm{ad} g^{-1} \omega$, where $\mathrm{ad} : G \longrightarrow \mathfrak{g}$ is the adjoint representation) such that, when we identify the tangent space to the fibre V_p with the Lie algebra $\mathfrak{g}$ by the isomorphism $\mathfrak{g} \in A \longmapsto A^* \in V_p$ associating to an element A of the Lie algebra the fundamental vertical vector field A^*, we have $\omega(A^*) = A$, for all $A \in \mathfrak{g}$. A connection on a principal bundle P induces a splitting $TP \simeq V \oplus H \simeq V \oplus \pi^* TM$. We denote again with $H : TP \longrightarrow H$ the projection on the horizontal space. For a good treatment of connections on principal bundles see [76], [92].

Let us recall briefly how to pass from a connection on a principal bundle P to a connection on an associated vector bundle E. Let V a fixed vector space and $\rho : G \longrightarrow GL(V)$ a representation of G in $GL(V)$. It is well known that the quotient of $P \times V$ for the action of G given by $(p, v)g := (pg, \rho(g^{-1})v)$ is isomorphic to a vector bundle, which we indicate with $E \simeq P \times_\rho V$. We have a commutative diagram:

$$\begin{tikzcd} P \times V \arrow[r] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ E \arrow[r, "p"] & M \end{tikzcd}$$

We remark that the pull back $\pi^* E$ of the bundle E on P is canonically isomorphic to the trivial bundle $P \times V$. We recall that if $\sigma \in A^r(M, E)$ is an E-valued r-form on M the pull back $\pi^* \sigma \in A^r(P, \pi^* E) \simeq A^r(P, V)$ is an horizontal and G-equivariant r-form: in other words $\pi^* \sigma = H^* \pi^* \sigma$, and $R_g^* \pi^* \sigma = \rho(g^{-1}) \pi^* \sigma$; we call such a form *tensorial* of type (ρ, V). There is a bijection (cf [76]) between r-tensorial forms of type (ρ, V) and E-valued r forms in $A^r(M, E)$. Once this is explained, let σ be a connection 1-form on the bundle P, and ϕ a section in $\Gamma(E)$. It is easy to see (cf [76], [103]) that the 1 form $d\pi^* \phi + \sigma(\pi^* \phi)$ coincides exactly with the horizontal part $H^* d\pi^* \phi$ of the differential $d\pi^* \phi$ and it is tensorial of type (ρ, V), then there exists a unique 1-form $\nabla \phi \in A^1(M, E)$ such that:

$$\pi^*(\nabla \phi) = d\pi^* \phi + \sigma(\pi^* \phi) \quad . \tag{5}$$

The form $\nabla\phi$ is called the covariant derivative of the section ϕ on E, and proves the existence of a vector bundle connection $\nabla : \Gamma(E) \longrightarrow A^1(M,E)$. With this characterization of vector bundle connections induced from connections on principal bundles, it is easy to prove the following properties:

Lemma 1.1 (General principle)**.** *(cf [92], [98], [8]). Let P a principal G-bundle and V a G-vector space. If $v \in V$ is fixed by the G-action, there is a naturally induced section $\tilde{v}$ of $E = P \times_G V$, such that for any covariant derivative ∇_σ on E induced by a G connection σ on P, $\nabla_\sigma \tilde{v} = 0$.*

Lemma 1.2. *(cf [92]) Let P and Q two principal fibre bundles over a manifold M with structural group G, and let $f : P \longrightarrow Q$ an isomorphism. Let σ a connection 1 form on Q. Then $f^*\sigma$ is a connection 1-form on P. Let now V a G-vector space and $E = P \times_G V$, $F = Q \times_G V$ the associated vector bundles, and let again $f : E \longrightarrow F$ the isomorphism of vector bundles induced by the isomorphism $f : P \longrightarrow Q$. Then the covariant derivative $\nabla_{f^*\sigma}$ on E is exactly $f^{-1}\nabla_\sigma f$, where ∇_σ is the covariant derivative induced on F by σ.*

1.2 The group $Spin^c$, Clifford algebras and spin representations

Our main references for material about Spin Geometry are Lawson-Michelsohn [82], Morgan [91], Nicolaescu [98]. We recall that the group $Spin(n)$ is the universal covering of the group $SO(n)$. Let $Ad : Spin(n) \longrightarrow SO(n)$ the double covering map. The group $Spin^c(n)$ is then defined as the quotient: $Spin^c(n) := Spin(n) \times_{\pm 1} U(1)$. The group $Spin^c(n)$ is a double covering of the product $SO(n) \times U(1)$: we indicate with μ the covering map:

$$\begin{aligned} \mu : Spin^c(n) &\xrightarrow{2:1} SO(n) \times U(1) \\ [\alpha, \lambda] &\longmapsto (Ad(\alpha), \lambda^2) \end{aligned}$$

and μ_1, μ_2 the two components. More precisely we have the following diagram:

$$\begin{tikzcd} Spin(n) \times U(1) \arrow[d, "2:1"'] \arrow[dr, "4:1", "\nu"'] & \\ Spin(n) \times_{\pm 1} U(1) \arrow[r, "\mu", "2:1"'] & SO(n) \times U(1) \end{tikzcd} \tag{6}$$

We now recall the definition of Clifford algebra of an euclidian vector space (E, g) that is a vector space E with a given scalar product g.

Definition 1.3. The Clifford algebra $Cl(E)$ of the euclidian vector space (E, g) is the quotient of the tensor algebra $T(E)$ by the nonhomogeneous two-sided ideal generated by the elements of the form $x \otimes x + g(x, x)$, $x \in E$.

The Clifford algebra $Cl(E)$ is an associative algebra with a natural injection $E \hookrightarrow Cl(E)$; it is characterized by the following universal property:

Proposition 1.4. *For every unitary algebra A and for every linear map $f : E \longrightarrow A$ such that $f(x)^2 = -g(x, x)$ there exists a unique homomorphism of unitary algebras $\varphi : Cl(E) \longrightarrow A$ such that the diagram*

$$\begin{tikzcd} E \arrow[d, hook] \arrow[dr, "f"] & \\ Cl(E) \arrow[r, "\varphi"'] & A \end{tikzcd}$$

commutes.

If $x, y \in E$, then, in the Clifford algebra $xy + yx = -2g(x, y)$. This rule allows us to find a basis for the Clifford algebra in the following way: if $e_i, 1 \leq i \leq n$ is an orthonormal basis of (E, g), then $Cl(E)$ is generated by the e_i with the rule $e_ie_j + e_je_i = -2\delta_{ij}$; therefore an orthonormal basis for the vector space underlying $Cl(E)$ is given by $e_{i_1} \cdot \dots \cdot e_{i_p}$, $1 \leq i_1 < \dots < i_p \leq n$. As a consequence $\dim_{\mathbb{R}} Cl(E) = 2^n$ and $Cl(E)$ splits in a direct sum of vector spaces: $Cl(E) = \oplus_{i=0}^n Cl_i(E)$. Despite this splitting, the Clifford algebra is not $\mathbb{Z}$-graded, but only $\mathbb{Z}/2\mathbb{Z}$-graded: indeed the position: $Cl_+(E) = \oplus_{m \text{ even}} Cl_m(E)$, $Cl_-(E) = \oplus_{m \text{ odd}} Cl_m(E)$ defines a consistent $\mathbb{Z}_2$-grading: $Cl(E) = Cl_+(E) \oplus Cl_-(E)$. The even part $Cl_+(E)$ is a subalgebra of $Cl(E)$ and $Cl_-(E)$ is a $Cl_+(E)$-module. We define now the complexified Clifford algebra as $Cl^c(E) := Cl(E) \otimes_{\mathbb{R}} \mathbb{C}$: as well as the real Clifford algebra, the complexified one is $\mathbb{Z}_2$ graded.

We can identify $Spin(n)$ as the subgroup of $Cl_+(E)^*$ generated by elements $x \in E$, $g(x, x) = 1$. In this identification the Lie algebra $\mathfrak{spin}(n)$ of $Spin(n)$ coincides with $Cl_2(E)$. In the same way we can identify the group $Spin^c(E)$ as the subgroup of $Cl^c_+(E)^*$ generated by elements $x \in E$, $g(x, x) = 1$. In the identification the Lie algebra $\mathfrak{spin}^c(n)$ is isomorphic to $i\mathbb{R} \oplus Cl_2(E)$. Going back to diagram (6), and taking the differentials at the unity, we obtain the following diagram of Lie algebras:

$$\begin{tikzcd}
\mathfrak{spin}(n) \oplus i\mathbb{R} \simeq Cl_2(T) \oplus i\mathbb{R} \arrow[d, "="'] \arrow[dr, "d\nu", "\simeq"'] & \\
\mathfrak{spin}^c(n) \arrow[r, "d\mu"'] & \mathfrak{so}(n) \oplus i\mathbb{R}
\end{tikzcd} \tag{7}$$

where the morphism $d\nu : i\mathbb{R} \oplus Cl_2(E) \longrightarrow i\mathbb{R} \oplus \mathfrak{so}(n)$ is given by: $d\nu(e_ie_j, \lambda) = (2E_{ij}, 2\lambda)$ and where E_{ij} denotes the skew-symmetric matrix with -1 in the (i, j)-place and 1 in the (j, i) place.

1.3 $Spin^c$ and Clifford representations

Definition 1.5. A complex spin representation of the euclidian vector space (E, g) is the data of a hermitian vector space S and of a linear map $f : E \longrightarrow \mathrm{End}(S)$ such that:
(1) $f(x) = -g(x, x)$
(2) $f(x)^* = f(-x)$.

On $\mathrm{End}(S)$ we fix the hermitian metric given by $(a, b) = 1/\dim S \cdot \mathrm{tr}(ab^*)$ so that any spin representation (S, f) is an isometry. Two spin representations (S, f) and (S', f') are isomorphic if there is an hermitian isometry $\beta : S \longrightarrow S'$ such that $f'(x) = \beta f(x) \beta^{-1}$.

Proposition 1.6. *If E is an even dimensional euclidian vector space, there exists a unique (up to isomorphism) irreducible spin representation (S, f). Any such representation has dimension 2^m, if $\dim_{\mathbb{R}} E = 2m$.*

If E is a $2m$-dimensional euclidian vector space, an irreducible complex spin representation is obtained (cf. [82]) by identifying E with $\mathbb{C}^m$ (by means of an orthogonal complex structure J) and considering the $\mathbb{R}$-linear map:

$$\begin{aligned} \rho : \mathbb{C}^m &\longrightarrow \mathrm{End}(\Lambda^*\mathbb{C}^m, \Lambda^*\mathbb{C}^m) \\ x &\longmapsto x \wedge (\cdot) - x \lrcorner (\cdot) \end{aligned} \tag{8}$$

If (S, f) is an irreducible spin representation, by the universal property and dimension counting we have an isomorphism of algebras $Cl^c(E) \simeq \mathrm{End}_{\mathbb{C}}(S)$, which is compatible with the hermitian structures of the two members. An irreducible spin representation (S, f) gives rise to a representation of the group

$Spin^c(E)$, by composition: $Spin^c(E) \hookrightarrow Cl^c(E) \longrightarrow \mathrm{End}_{\mathbb{C}}(S)$. Suppose now that E is oriented and $e_1, \ldots, e_n$ is an oriented orthonormal basis. Let $\omega_{\mathbb{C}}$ the complex volume element, defined by $\omega_{\mathbb{C}} := i^{[n+1/2]} e_1 \ldots e_n \in Spin^c(E) \cap Cl^c_n(E)$. It is easy to see that $\omega_{\mathbb{C}}^2 = 1$ and that $\omega_{\mathbb{C}}$ is independent of the chosen orthonormal basis. Let $f(\omega_{\mathbb{C}})$ the induced map $f(\omega_{\mathbb{C}}) : S \longrightarrow S$ and let S_+ and S_- the ± 1 eigenspaces.

Proposition 1.7. *Let E an oriented even dimensional euclidian vector space. The $Spin^c$ representation induced by an irreducible spin representation (S, f) of E splits in the direct sum of two irreducible $Spin^c$-representations S_+ and S_- of dimension 2^{m-1}.*

Given an irreducible spin representation (S, f) of an oriented even dimensional vector space E we have an induced isomorphism:

$$f : E \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{C}}(S_+, S_-) \quad ,$$

or analogously, $f : E \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{C}}(S_-, S_+)$. Moreover we have an embedding: $Spin^c(E) \longrightarrow U(S_+) \times U(S_-)$, which, for $\dim_{\mathbb{R}} E \geq 4$ induces an embedding: $Spin^c(E) \longrightarrow SU(S_+) \times SU(S_-)$. For $\dim_{\mathbb{R}} E = 4$ the preceding is an isomorphism. Taking Lie algebras, for dimension 4 we have an isomorphism:

$$\gamma : \Lambda^2 E \simeq \mathfrak{spin}(E) \xrightarrow{\simeq} \mathfrak{su}(S_+) \oplus \mathfrak{su}(S_-) \quad . \tag{9}$$

It is easy to see that

$$\gamma(v \wedge w) = \frac{1}{2}[\rho(v), \rho(w)] = \frac{1}{2}(\rho(v)\rho(w) - \rho(w)\rho(v))$$

and therefore $\gamma(e_i \wedge e_j) = \rho(e_i e_j)$. In other words γ is induced by ρ via the identification $Cl_2(E) \simeq \Lambda^2 E$. For this reason, if there is no risk of confusion, we will indicate the map (9) with ρ instead of γ. Let $*$ the Hodge star on ΛE. The Hodge star action on $\Lambda^2 E$ commutes with the action of $f(\omega_{\mathbb{C}})$ on $\mathrm{End}_{\mathbb{C}}(S)$, so that $f(*\sigma) = f(\omega_{\mathbb{C}})f(\sigma)$, for all $\sigma \in \Lambda^2 E$. This implies that (9) splits as follows $\gamma = \gamma_+ \oplus \gamma_-$, $\gamma_+ : \Lambda^2_+ E \xrightarrow{\simeq} \mathfrak{su}(S_+)$, $\gamma_- : \Lambda^2_- : \xrightarrow{\simeq} \mathfrak{su}(S_-)$. We remark that given an irreducible spin representation (S, f) of E, the group $\mathrm{Aut}_{Spin^c}(S, f)$, defined as:

$$\mathrm{Aut}_{Spin^c}(S, f) = \{\theta \in SO(g), \zeta \in U(S) \mid f(\theta(v)) = \zeta f(v) \zeta^{-1}\}$$

is isomorphic to $Spin^c(E)$, via the map:

$$\begin{aligned} Spin^c(E) &\longrightarrow \mathrm{Aut}_{Spin^c}(S, f) \\ \sigma &\longrightarrow (\mu_1(\sigma), f(\sigma)) \end{aligned} \tag{10}$$

1.4 $Spin^c$-structures

Let M a manifold and (E, g) an euclidian vector bundle of even rank ($\mathrm{rk}(E) = 2m$). Let L a hermitian line bundle on M. Let $P_{SO(g)} \xrightarrow{\pi_1} M$ and $P_{U(1)} \xrightarrow{\pi_2} M$ the principal bundles of orthonormal and hermitian frames for E and L, respectively. Consider the fibered product of $P_{SO(g)}$ and $P_{U(1)}$ over M: $P_{SO(g)} \times_M P_{U(1)} \simeq \pi_1^* P_{U(1)} \simeq \pi_2^* P_{SO(g)}$. It is a principal fibre bundle over M of structural group $SO(2m) \times U(1)$. We recall the covering map $\mu = (\mu_1, \mu_2) : Spin^c(2m) \longrightarrow SO(2m) \times U(1)$.

Definition 1.8. A $Spin^c$-structure on the euclidian vector bundle (E, g) of determinant line bundle L is a principal fibre bundle P_{Spin^c} of structural group $Spin^c(2m)$, which is an equivariant double covering $\xi : P_{Spin^c} \longrightarrow P_{SO(g)} \times_M P_{U(1)}$ over the fibered product $P_{SO(g)} \times_M P_{U(1)}$, in the sense that $\xi(pg) = \xi(p)\mu(g)$, for all $p \in P_{Spin^c}$, $g \in Spin^c(2m)$.

Actually, to give a $Spin^c$-structure on the bundle E it is sufficient to give a $Spin^c$-principal fibre bundle P_{Spin^c}, and a μ_1-equivariant map : $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$, satisying $\alpha(pg) = \alpha(p)\mu_1(g)$. We will automatically have a $Spin^c$ structure of determinant line bundle $L \simeq P_{Spin^c} \times_{\mu_2} \mathbb{C}$. Summarizing we have a diagram:

$$\begin{tikzcd}
P_{Spin^c(2m)} \arrow[dr, "\xi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& P_{SO(2m)} \times_M P_{U(1)} \arrow[r] \arrow[d] & P_{SO(2m)} \arrow[d, "\pi_1"] \\
& P_{U(1)} \arrow[r, "\pi_2"] & P_{SO(2m)}
\end{tikzcd}$$

in which the two projections α and β can be considered as quotient projections with respect to $U(1)$ and $SO(2m)$ respectively.

From the central exact sequence of groups:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^c(2m) \longrightarrow SO(2m) \times U(1) \longrightarrow 1$$

we get a long exact sequence of pointed sets in nonabelian cohomology:

$$H^1(M, \mathbb{Z}_2) \longrightarrow H^1(M, Spin^c(2m)) \longrightarrow H^1(M, SO(2m)) \times H^1(M, U(1)) \xrightarrow{\delta} H^2(M, \mathbb{Z}_2)$$

where the Bockstein morphism δ is given by $\delta(E, L) = \overline{c_1(L)} + w_2(E)$. Therefore we have:

Theorem 1.9. *Let (E, g) an oriented euclidian vector bundle on M. There exists a $Spin^c$-structure on the vector bundle (E, g) if and only if the second Stiefel-Whitney class $w_2(E) \in H^2(M, \mathbb{Z}_2)$ lifts to a class of integral cohomology in $H^2(M, \mathbb{Z})$. If M is simply connected there is at most one such lifting.*

Classes in $H^2(M, \mathbb{Z}_2)$ admitting a lifting to an integral class in $H^2(M, \mathbb{Z})$ can be characterized as being orthogonal to the torsion subgroup T of $H_2(M, \mathbb{Z})$ with respect to the coupling: $H^2(M, \mathbb{Z}_2) \times H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Z}_2$. Now, for the tangent bundle of a compact oriented 4-manifold, a theorem by Wu states that $w_2(TM)x = x^2 \mathrm{mod}\ 2$ in $\mathbb{Z}_2$ (cf. [29]) for every class $x \in H_2(M, \mathbb{Z}_2)$; then $w_2(TM)$ is orthogonal to the torsion subgroup and hence can be lifted to an integral class. As a consequence there always exists a $Spin^c$ structure on the tangent bundle of a compact oriented 4-manifold.

1.5 Spinors

Let $E \longrightarrow M$ be an even rank ($\mathrm{rk}(E) = 2m$) oriented euclidian vector bundle on the manifold M admitting a $Spin^c$-structure P_{Spin^c} of determinant line bundle L. Let us consider an irreducible spin representation (S, f) of $\mathbb{R}^{2m}$, splitting in two irreducible $Spin^c(2m)$ representations $(S, f) = (S_+, f_+) \oplus (S_-, f_-)$. We will call the vector bundle $W := P_{Spin^c} \times_f S$ the *bundle of spinors*. It splits in the direct sum $W = W_+ \oplus W_-$, where $W_+ := P_{Spin^c} \times_{f_+} S_+$, $W_- = P_{Spin^c} \times_{f_-} S_-$ are called *bundles of half spinors*. Let $Cl(E) := P_{SO(2m)} \times_{SO(2m)} Cl(\mathbb{R}^{2m})$ the bundle of Clifford algebras associated to E. The projection $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$ induces a well defined isomorphism:

$$\begin{aligned} P_{Spin^c} \times_{\mu_1} \mathbb{R}^{2m} &\longrightarrow E \\ [p, v] &\longmapsto \alpha(p)v \end{aligned} \tag{11}$$

and hence a Clifford multiplication

$$\begin{aligned} \rho : E \xrightarrow{\simeq} P_{Spin^c} \times_{\mu_1} \mathbb{R}^{2m} &\longrightarrow P_{Spin^c} \times_{Spin^c} \mathrm{End}(S) \\ \alpha(p)v \longmapsto [p, v] &\longmapsto [p, f(v)] \end{aligned} \tag{12}$$

The map ρ induces isomorphisms:

$$\rho_+ : E \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{Hom}(W_+, W_-) \tag{13}$$

or, analogously $\rho_- : E \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{Hom}(W_-, W_+)$. Moreover it extends to an isomorphism:

$$\rho : Cl^c(E) \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$$

called the Clifford multiplication. Moreover, we have an isomorphism:

$$\gamma = \gamma_+ \oplus \gamma_- : \Lambda^2_+ E \oplus \Lambda^2_- E \longrightarrow \mathfrak{su}(W_+) \oplus \mathfrak{su}(W_-) \quad .$$

Since $\det f_\pm(x) = \mu_1(x)^{2^{m-2}}$, $\Lambda^{2^{m-1}} W_\pm \simeq L^{\otimes^{2^{m-2}}}$, and $\det W \simeq L^{\otimes^{2^{m-1}}}$.

1.6 The $Spin^c$ connection

Let (E, g) be an even rank ($\mathrm{rk}E = 2m$) oriented euclidian vector bundle on a manifold M, and let ∇^E be a given $SO(g)$-connection on E: and let $\omega_g \in A^1(P_{SO(g)}, \mathfrak{so}(2m))$ the corresponding equivariant $\mathfrak{so}(2m)$-valued 1-form on the $SO(2m)$- principal bundle of orthonormal frames $P_{SO(g)}$ of (E, g). Suppose that E has a $Spin^c$-structure $P_{Spin^c} \xrightarrow{\alpha} P_{SO(g)}$ of determinant line bundle L, and suppose given an hermitian connection A on L. We indicate again with $A \in A^1(P_{U(1)}, \mathfrak{u}(1))$ the connection 1-form on the principal bundle of orthonormal hermitian frames $P_{U(1)}$ of L. Then we can lift the connection ω_g on $P_{SO(g)}$ and the connection A on $P_{U(1)}$ to a connection on the $Spin^c(2m)$-bundle $P_{Spin^c(2m)}$ in the following way: consider the two projections $\alpha : P_{Spin^c(2m)} \longrightarrow P_{SO(g)}$ and $\beta : P_{Spin^c(2m)} \longrightarrow P_{U(1)}$. We recall the morphism of Lie algebras (7) : $d\mu : \mathfrak{spin}^c(2m) \longrightarrow \mathfrak{so}(2m) \oplus i\mathbb{R}$. We define the $Spin^c$- connection form $\Omega_{\alpha,A}$ as:

$$\Omega_{\alpha,A} := (d\mu)^{-1}(\alpha^* \omega_g + \beta^* A) \in A^1(P_{Spin^c(2m)}, \mathfrak{spin}^c(2m)) \; . \tag{14}$$

It is easy to see that $\Omega_{\alpha,A}$ is a $Spin^c(2m)$-equivariant form on $P_{Spin^c(2m)}$, hence it defines a $Spin^c$ connection on $P_{Spin^c(2m)}$. It follows that $\Omega_{\alpha,A}$ induces a connection ∇^W_A on the bundle of spinors, called the *spinorial connection*. It follows from the general principle 1.1 and by the definition of the Clifford multiplication (12) that ρ is parallel with respect to the Levi-Civita connection on $Cl(E)$ and to the spinorial connection on W. Therefore:

$$\nabla^W(\rho(x)\psi) = \rho(\nabla^{LC} x)\psi + \rho(x)\nabla^W \psi$$

for each $x \in Cl(E)$, $\psi \in W$.

1.7 The Dirac operator

Let (M, g) a $2m$-dimensional oriented riemannian manifold with as $Spin^c$-structure (such that its tangent bundle has a $Spin^c$-structure) of determinant line bundle L. Let $P_{SO(g)}$ the principal bundle of orthonormal frames of (TM, g). On $P_{SO(g)}$ we consider the Levi-Civita connection, that is the unique torsion-free $SO(2m)$-connection. Consider a $U(1)$ connection A on L. Let W the bundle of spinors for the tangent bundle, associated to the $Spin^c$-structure. The Clifford multiplication establishes an isomorphism: $\rho : Cl^c(TM) \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$. We remark that the metric g provides an isomorphism between TM and T^*M; under this isomorphism, a $Spin^c$-structure for TM will also be a $Spin^c$-structure for T^*M, whose associated Clifford multiplication is $\rho \circ g^{-1} : Cl^c(T^*M) \xrightarrow{g^{-1}} Cl^c(TM) \xrightarrow{\rho} \mathrm{End}(W)$.

We now lift the Levi-Civita connection on (TM, g) and the unitary connection A on L to the $Spin^c(2m)$ connection $\Omega_{\alpha,A}$ on $P_{Spin^c(2m)}$ which induces the spinorial connection ∇^W_A on W. Let $\tilde{\rho} : T^*M \otimes W \longrightarrow W$ the evaluation map induced by the Clifford multiplication on T^*M.

Definition 1.10. The *Dirac operator* is the first order differential operator: $D_A : \Gamma(W) \longrightarrow \Gamma(W)$ given by the composition

$$D_A := \tilde{\rho} \circ \nabla_A^W : \Gamma(W) \xrightarrow{\nabla_A^W} \Gamma(T^*M \otimes W) \xrightarrow{\tilde{\rho}} \Gamma(W) .$$

The Dirac operator splits according to the splitting of the spinorial connection: $D_A = D_A^+ \oplus D_A^- : \Gamma(W_+) \oplus \Gamma(W_-) \longrightarrow \Gamma(W_-) \oplus \Gamma(W_+)$. The Dirac operator is an elliptic, formally self-adjoint differential operator; in the previous splitting D_A^+ and D_A^- are formal adjoints of one another. The symbol of the Dirac operator is the Clifford multiplication: $\sigma(D_A)(\xi, \psi) = \rho(\xi)\psi$ for all $\xi \in T^*M$, $\psi \in W$.

1.8 The Seiberg-Witten equations

From now on M will always denote a compact oriented 4-manifold. We will suppose M equipped with a given metric g. Such a riemannian manifold always admits a $Spin^c$ structure. Let us fix one of determinant line bundle L. Let W the bundle of spinors for the tangent bundle (TM, g) and let $\rho : Cl^c(TM) \longrightarrow \mathrm{End}(W)$ the Clifford multiplication. We recall that, as in (9), we have an isomorphism $\gamma : \Lambda^2 T^*M \longrightarrow \mathfrak{su}(W_+) \oplus \mathfrak{su}(W_-)$ carrying $\Lambda^2_+ T^*M$ on $\mathfrak{su}(W_+)$ and Λ^2_- on $\mathfrak{su}(W_-)$. Now, for each $\sigma \in W_+$, it is easy to see, taking an orthonormal basis for W_+, that the traceless part $[\sigma^* \otimes \sigma]_0$ of $\sigma^* \otimes \sigma$ is in $i\mathfrak{su}(W_+)$. We are ready to write the Seiberg-Witten equations for a couple of unknowns (A, ψ), where A is a hermitian connection on L and ψ is a section of W_+:

$$D_A \psi = 0 \tag{15a}$$

$$\rho(F_A^+) = [\psi^* \otimes \psi]_0 \tag{15b}$$

In the equations D_A is the Dirac operator associated to the Levi-Civita connection on TM and the hermitian connection A on L. $F_A \in A^2(M, i\mathbb{R})$ is the imaginary curvature 2-form of the connection A and F_A^+ denotes its self-dual part. If we indicate with F (or with $F^{g,\rho}$ when we want to emphasize the metric and the Clifford multiplication) the map:

$$\begin{aligned} F^{g,\rho} : \mathcal{A}_L^{U(1)} \times \Gamma(W_+) &\longrightarrow \Gamma(W_-) \times A^2_+(M, i\mathbb{R}) \\ (A, \psi) &\longmapsto (D_A\psi, \rho(F_A^+) - [\psi^* \otimes \psi]_0) \end{aligned}$$

the Seiberg-Witten equation can be written simply as $F^{g,\rho}(A, \psi) = 0$. We will call $F^{g,\rho}$ the *Seiberg-Witten functional* and $\mathcal{C} = \mathcal{A}_L^{U(1)} \times \Gamma(W_+)$ the *configuration space*. The space of solutions to Seiberg-Witten equations is the zero set $Z(F^{g,\rho})$ of the functional $F^{g,\rho}$. A solution to Seiberg-Witten equations (15) is called a *monopole*. A monopole (A, ψ) is said *irreducible* if $\psi \neq 0$, *reducible* otherwise. Until now we have worked in the smooth category. To be able to give a manifold structure to the solutions it is better to work with Hilbert or Banach spaces (where the Implicit Function Theorem can be applied). Therefore we will complete the space of configurations $\mathcal{C}$ in the Sobolev norm $|| \; ||_{2,k}$. We will indicate with $\mathcal{C}^2_k = (\mathcal{A}_L^{U(1)})^2_k \times \Gamma^2_k(W_+)$ the Sobolev completion. We will consider the Seiberg-Witten functional as a map of Hilbert manifolds:

$$(F^{g,\rho})^2_k : (\mathcal{A}_L^{U(1)})^2_k \times \Gamma^2_k(W_+) \longrightarrow \Gamma^2_{k-1}(W_-) \times A^2_+(M, i\mathbb{R})^2_{k-1} .$$

Actually it is not so important what k to use, provided that it is sufficiently large; in any case the moduli space is made of smooth objects (cf. [91]).

The space of configurations and the space of solutions possess a natural $\mathcal{C}^\infty(M, S^1)$ action. To define the Seiberg-Witten moduli space we have to cut out the space of solution by the action of the gauge group $\mathcal{G} := \mathcal{C}^\infty(M, S^1)$. The $\mathcal{C}^\infty(M, S^1)$-action is given by:

$$\begin{aligned} \mathcal{A}_L^{U(1)} \times \Gamma(W_+) \times \mathcal{C}^\infty(M, S^1) &\longrightarrow \mathcal{A}_L^{U(1)} \times \Gamma(W_+) \\ (A, \psi, \lambda) &\longmapsto ((\lambda^2)^*A, \lambda^{-1}\psi) \end{aligned}$$

As usual to be able to make differential considerations we will take the Sobolev completion of this action. Therefore we will use the group $\mathcal{G}^2_{k+1} = L^2_{k+1}(M, S^1)$ instead of $\mathcal{C}^\infty(M, S^1)$. The zero set of $F^{g,\rho}$ is preserved by this action and hence we can define the Seiberg-Witten moduli space as the quotient

$$\mathcal{M}^{SW}_g := Z(F^{g,\rho})/\mathcal{G}$$

or as $Z((F^{g,\rho})^2_k)/\mathcal{G}^2_{k+1}$, if we put Sobolev indices. We will also indicate with $\mathcal{B}$ the quotient $\mathcal{C}/\mathcal{G}$ of all the configuration space by the action of $\mathcal{G}$.

1.9 Kähler surfaces

In this subsection we will briefly recall the particular form of Seiberg-Witten equations on a Kähler manifold and their properties.

Definition 1.11. (cf. [103]) A manifold of even dimension $2m$ is said to be *almost hermitian* if the principal bundle $P_{GL(2m,\mathbb{R})}$ of linear frames admits a reduction $P_{U(m)} \hookrightarrow P_{GL(2m,\mathbb{R})}$ of the structural group to $U(m)$. An almost hermitian manifold is said to be a *Kähler manifold* if there exists a torsion free connection on $P_{U(m)}$. Equivalently, the manifold M is Kähler if its holonomy group is contained in $U(m)$.

As a consequence an almost hermitian manifold is a Riemannian manifold (because of the injection $U(m) \hookrightarrow SO(2m)$), an almost complex manifold (because $U(m) \hookrightarrow GL(m, \mathbb{C})$), and an almost symplectic manifold (because $U(m) \hookrightarrow Sp(2m, \mathbb{R})$) and all the structures are compatible. In other words, the almost hermitian structure induces a metric g, an almost complex structure J, a non degenerate 2-form ω intertwined by the relations: $g(X, JY) = \omega(X, Y)$, $g(JX, JY) = g(X, Y)$. If M is Kähler the torsion free connection on $P_{U(m)}$ induces clearly the Levi-Civita connection on $P_{SO(2m)}$ and it descends from the relation (cf. [77]), valid for an arbitrary almost hermitian manifold

$$4g(\nabla^{LC}_X Y, Z) = 6d\omega(X, JY, JZ) - 6d\omega(X, Y, Z) + g(N_J(Y, Z), JZ) = 0 \tag{16}$$

that the Nijenhuis tensor N_J vanishes, since by the general principle 1.1, $\nabla^{LC}\omega = 0$ (and $d\omega = 0$) and $\nabla^{LC} J$ =0. By Newlander-Nirenberg theorem [97], the almost complex structure J is integrable.

We will now sketch how an almost hermitian structure on M induces a canonical $Spin^c$ structure on the manifold M. We recall that for an even dimensional vector space E a complex spin representation can be obtained by the $\mathbb{R}$-linear map : $\rho : \mathbb{C}^n \longrightarrow \mathrm{End}(\Lambda^*\mathbb{C}^n, \Lambda^*\mathbb{C}^n)$, defined by $\rho(x) = x \wedge (\cdot) - x \lrcorner (\cdot)$, identifying E with $\mathbb{C}^n$ and then by extending ρ to $E \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. We have $S_+ = \Lambda^{\mathrm{even}}_{\mathbb{C}} E$ and $S_- = \Lambda^{\mathrm{odd}}_{\mathbb{C}} E$. We now have the following representation of $U(m)$ (cf. [91]):

Lemma 1.12. *The monomorphism of Lie groups $U(m) \hookrightarrow SO(2m) \times S^1$ given by $f \longmapsto (f, \det f)$ lifts to a monomorphism: $U(m) \hookrightarrow Spin^c(2m)$ such that the diagram:*

$$\begin{tikzcd} & Spin^c(2m) \arrow[d, "\mu"] \\ U(m) \arrow[ur] \arrow[r, hook] & SO(2m) \times S^1 \end{tikzcd}$$

is commutative.

Therefore we have a diagram

$$\begin{tikzcd} U(m) \arrow[r, hook] \arrow[dr, "\det"'] & Spin^c(2m) \arrow[d, "\mu_2"] \\ & S^1 \end{tikzcd}$$

As a consequence an almost hermitian structure on a manifold M induces a $Spin^c$ structure on its tangent bundle TM, called the *canonical $Spin^c$ structure* in the following way:

$$P_{Spin^c(2m)} := P_{U(m)} \times_{U(m)} Spin^c(2m)\ .$$

The following diagram summarizes the situation:

$$\begin{tikzcd}
 & P_{U(1)} \\
P_{U(m)} \arrow[ur, "\det"] \arrow[r, hook] \arrow[dr] & P_{Spin^c}(2m) \arrow[u, "\beta"'] \arrow[d, "\alpha"] \\
 & P_{SO(2m)}
\end{tikzcd}$$

Consider now the complexified tangent bundle $TM \otimes \mathbb{C}$ of the almost hermitian manifold M and the complex tangent bundle $T^{1,0}M$. We have the following: (cf. [91])

Proposition 1.13. *Let M an almost hermitian manifold M with almost complex structure J. The canonical $Spin^c$-structure on M has K_M^* as determinant line bundle. The spinor bundle is isomorphic to the exterior power $W \simeq \Lambda^* T^{1,0}M$, and the Clifford multiplication is given by:*

$$\begin{aligned} TM &\longrightarrow \mathrm{End}(\Lambda^*T^{1,0}M, \Lambda^*T^{1,0}M) \\ x &\longmapsto \sqrt{2}(x^{1,0} \wedge (\cdot) - x^{1,0} \lrcorner (\cdot)) \end{aligned}$$

Moreover the spinor bundle can be identified with the exterior algebra of all $(0,*)$-forms: $W \simeq \Lambda^{0,*}T^*M$. The Clifford multiplication is then given by:

$$\begin{aligned} T^*M &\longrightarrow \mathrm{End}(\Lambda^*T^{1,0}M, \Lambda^*T^{1,0}M) \\ x &\longmapsto \sqrt{2}(x^{0,1} \wedge (\cdot) - x^{0,1} \lrcorner (\cdot)) \end{aligned}$$

An easy consequence of the preceding proposition is that, when we complexify the Clifford multiplication we get the map:

$$\begin{aligned} T^*M \otimes \mathbb{C} &\longrightarrow \mathrm{End}(\Lambda^*T^{1,0}M, \Lambda^*T^{1,0}M) \\ z &\longmapsto \sqrt{2}(z^{0,1} \wedge (\cdot) - \overline{z^{1,0}} \lrcorner (\cdot)) \end{aligned} \tag{17}$$

We now pass to recall the Dirac operator on an almost hermitian manifold as done is [51] or in [98].

Definition 1.14. Let (M, g, J) an almost hermitian manifold and let ω the fundamental form. The *Lee* form θ is the real 1-form defined as :

$$\theta := \Lambda d\omega\ .$$

where Λ denotes the contraction with the fundamental form ω.

Remark 1.15. On an almost complex manifold (M, g, J) the Cauchy-Riemann operator on $\Lambda^{p,q}T^*M$ is defined as $\bar{\partial}\eta = (d\eta)^{p,q+1}$.

Let now M an almost hermitian 4-manifold.

Lemma 1.16. *Let (M, g, J) an almost hermitian 4-manifold. The Dirac operator for the canonical $Spin^c$ structure on M is the operator: $D : \Gamma(\Lambda^{0,*}T^*M) \longrightarrow \Gamma(\Lambda^{0,*}T^*M)$ given by:*

$$D : \sqrt{2}(\bar{\partial} + \bar{\partial}^*) \pm \frac{1}{4}\rho(\theta)$$

where the sign is negative for positive spinors and positive for negative ones.

The the complex bilinear form $g_{\mathbb{C}}$ obtained by extending g by $\mathbb{C}$ bilinearity provides a $\mathbb{C}$-linear isomorphism between $T^{1,0}M$ and $T^*M^{0,1}$ and between $T^{0,1}M$ and $T^*M^{1,0}$. The metric g induce also an hermitian metric on $TM \otimes \mathbb{C}$ by the position $h(Z, W) = g_{\mathbb{C}}(Z, \bar{W})$, for $Z, W \in T^*M \otimes \mathbb{C}$. The real hodge star $*$ on M is again compatible with $g_{\mathbb{C}}$:

$$\sigma \wedge *\tau = (\sigma, \tau)_{\mathbb{C}} \Phi$$

where $(\cdot, \cdot)_{\mathbb{C}}$ is the coupling on forms induced by $g_{\mathbb{C}}$ and Φ is the volume form on M. On an almost hermitian manifold we dispose also of the complex Hodge star $\sharp$, defined by means of the hermitian metric $\langle \cdot, \cdot \rangle_h$ induced by h:

$$\sigma \wedge \sharp\tau = \langle \sigma, \tau \rangle_h \Phi \ .$$

By the relations above we immediately get:

$$\sigma \wedge \sharp\tau = \langle \sigma, \tau \rangle_h \Phi = (\sigma, \bar{\tau})_{\mathbb{C}} = \sigma \wedge *\bar{\tau}$$

which implies $\sharp = \bar{*}$. On an almost hermitian manifold the complex self dual 2-forms for the real Hodge star $*$ decompose as follows:

$$\Lambda^2_+ T^*M \otimes \mathbb{C} \simeq \Lambda^{2,0}T^*M \oplus \Lambda^{2,0}T^*M \oplus \mathbb{C}\omega \ . \tag{18}$$

The real self-dual 2-forms $\Lambda^2_+T^*M$ are identified with the real part of the bundle (18) via the isomorphism:

$$\begin{aligned} \Lambda^{0,2}T^*M \oplus \mathbb{R}\omega &\longrightarrow \Lambda^2_+ T^*M \\ (\mu, \lambda) &\longmapsto \mu + \bar{\mu} + \lambda \ . \end{aligned}$$

We interpret now the morphism $\rho : \Lambda^2_+T^*M \longrightarrow \mathfrak{su}(W_+)$ on an almost hermitian manifold (cf. [91]):

Lemma 1.17. *The isomorphism: $\rho : \Lambda^2_+T^*M \longrightarrow \mathfrak{su}(W_+)$ induced by the canonical $Spin^c$ structure on an almost hermitian 4-manifold M is given by :*

$$\begin{aligned} \Lambda^{0,2}T^*M \oplus \mathbb{R}\omega &\longrightarrow \mathfrak{su}(\mathbb{C} \oplus \Lambda^{0,2}T^*M) \\ (\alpha, \lambda\omega) &\longrightarrow 2\begin{pmatrix} -i\lambda & -\alpha\lrcorner(\cdot) \\ \alpha \wedge (\cdot) & i\lambda \end{pmatrix} \end{aligned} \tag{19}$$

It extends to a real isomorphism:

$$\begin{aligned} \Lambda^{0,2}T^*M \oplus \Lambda^{2,0}T^*M \oplus \mathbb{C}\omega &\longrightarrow \mathrm{End}_0(\mathbb{C} \oplus \Lambda^{0,2}T^*M) \\ (\alpha, \beta, \lambda\omega) &\longrightarrow 2\begin{pmatrix} -i\lambda & -\bar{\beta}\lrcorner(\cdot) \\ \alpha \wedge (\cdot) & i\lambda \end{pmatrix} \end{aligned}$$

The canonical $Spin^c$ structure on an almost hermitian manifold has fundamental class $c = c_1(K_M^{-1}) = -K_M$. Let now change $Spin^c$-structure. Any other $Spin^c$ structure is obtained by twisting the canonical spinor bundle with an hermitian line bundle N. The corresponding spinor bundle will be $W \simeq \Lambda^{0,*}T^*M \otimes N$; the bundles of half spinors: $W_+ \simeq \Lambda^{\mathrm{even}}T^*M \otimes N$ and $W_- \simeq \Lambda^{\mathrm{odd}}T^*M \otimes N$. The determinant line bundle is $L \simeq K_M^{-1} \otimes N^{\otimes^2}$, so that the fundamental class of this new $Spin^c$-structure is $c = c_1(K_M^{-1} \otimes N^{\otimes^2}) = 2c_1(N) - c_1(K_M)$. For such a $Spin^c$ structure the Clifford multiplication:

$$\rho_N : T^*M \otimes \mathbb{C} \longrightarrow \mathrm{End}((\Lambda^{0,0}T^*M \oplus \Lambda^{0,2}T^*M) \otimes N, \Lambda^{0,1}T^*M \otimes N)$$

is given by $\rho_N := \rho \otimes \mathrm{id}_N$ and the corresponding isomorphism:

$$\gamma_N : \Lambda^{0,2}T^*M \oplus \mathbb{R}\omega \longrightarrow \mathfrak{su}((\mathbb{C} \oplus \Lambda^{0,2}T^*M) \otimes N)$$

is again obtained by twisting γ with the identity of N: $\gamma_N := \gamma \otimes \mathrm{id}_N$. The choice of an hermitian connection A on L by lemma 2.2 is equivalent to the choice of an hermitian connection A_0 on the line bundle N, since $L = N^{\otimes^2} \otimes K_M^{-1}$, and this is then sufficient to define a spinorial connection on $\Lambda^{0,*}T^*M \otimes N$: the resulting spinorial connection ∇_A^W is actually the tensor product of the connections ∇^{T^*M} and the connection A_0 on N. As a consequence the corresponding Dirac operator becomes:

$$D_A = \sqrt{2}(\bar{\partial}_{A_0} + \bar{\partial}^*_{A_0}) \pm \frac{1}{4}\rho_N(\theta)$$

and is obtained by coupling the $\bar{\partial} + \bar{\partial}^*$ operator on $\Lambda^{0,*}T^*M$ with the connection A_0 on the N-component (cf. [51], [9], [98], [91]).

2 Parametrized Seiberg-Witten equations

2.1 Vector bundle characterization of $Spin^c$-structures and connections

Let us begin with the following characterization of $Spin^c$-structure, which will be useful in the sequel. This point of view is close to that of Kronheimer-Mrowka (cf. [81]) or Feehan-Leness (cf. [38], [40]).

Proposition 2.1. *Let (E, g) an even rank oriented euclidian vector bundle on a manifold M. A $Spin^c$-structure on E is equivalent to an irreducible spin representation (W, ρ), that is, the data of an hermitian bundle W and a bundle map : $\rho : E \longrightarrow \mathrm{End}(W)$ such that for all $u \in E$,*
(1) $\rho(u)^* = \rho(-u)$
(2) $\rho(u)^2 = -g(u, u)\mathrm{id}_W$

Proof. We recall that given a $Spin^c$-structure $P_{Spin^c} \xrightarrow{\alpha} P_{SO(E,g)}$ we can form the bundle of spinors by means of an irreducible spin representation (S, f) for $\mathbb{R}^{2m}$: $f : \mathbb{R}^{2m} \longrightarrow \mathrm{End}(S)$ which induces a representation of the group $Spin^c(2m) \hookrightarrow U(S)$. The μ_1 invariance of the map α implies that the map (11) ($[p, v] \longmapsto \alpha(p)v$) gives an isomorphism $P_{Spin^c(2m)} \times_{\mu_1} \mathbb{R}^{2m} \simeq E$. Now taking the map (12), $\rho : E \simeq P_{Spin^c(2m)} \times_{\mu_1} \mathbb{R}^{2m} \longrightarrow P_{Spin^c(2m)} \times_{Spin^c(2m)} End(S)$, defined by $\alpha(p)v \longmapsto [p, f(v)]$, we get an irreducible spin representation for the bundle (E, g): $\rho : E \longrightarrow \mathrm{End}(W)$. The bundle properties (1) and (2) come directly from the corresponding properties for f. Indeed, if $u = \alpha(p)v$, then $\rho(u)^2 = [p, f(v)^2] = [p, -|v|^2\mathrm{id}_S] = -g(u, u)\mathrm{id}_W$ and $\rho(u)^* = [p, f(v)^*] = [p, -f(v)] = -\rho(u)$.

Conversely, let $f : \mathbb{R}^{2m} \longrightarrow End(S)$ a fixed irreducible spin representation for $\mathbb{R}^{2m}$. and let

$$\rho : (E, g) \longrightarrow \mathrm{End}(W)$$

an irreducible spin representation for the euclidian vector bundle (E, g). The set of couples (θ_x, ζ_x), such that $\theta_x : (\mathbb{R}^{2m}, (\cdot, \cdot)) \longrightarrow (E_x, g_x)$ is an orientation preserving isometry and $\zeta_x : (S, h_S) \longrightarrow (W_x, h_x)$ is an isometry and $\rho_x(\theta_x(v)) = \zeta_x f(v)\zeta_x^{-1}$, forms a principal fibre bundle of structural group $Spin^c(2m)$ because of the isomorphism (10). The projection

$$\begin{aligned} P_{Spin^c(2m)} &\longrightarrow P_{SO(E,g)} \\ (\theta, \zeta) &\longmapsto \theta \end{aligned}$$

is μ_1 equivariant and defines the $Spin^c$-structure. The morphism: $P_{Spin^c(2m)} \times_{\mu_1} \mathbb{R}^{2m} \longrightarrow E$ carrying $[(\theta_x, \zeta_x), v] \longmapsto \theta_x(v)$ is an isomorphism. Analogously we have an isomorphism: $P_{Spin^c} \times_{Spin^c} S \longrightarrow W$ carrying $[(\theta_x, \zeta_x), w] \longmapsto \zeta_x(w)$.

We want to show now that the two procedures are the inverse of one another. Suppose we start from a $Spin^c$ structure $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$ and we build a Clifford multiplication $\rho : E \longrightarrow \mathrm{End}(W)$ by means of an irreducible spin representation (S, f), as we have explained above. We then form the $Spin^c$ principal bundle $\tilde{P}_{Spin^c}$ of couples (θ_x, ζ_x), where θ_x is an orientation preserving isometry θ_x :

$\mathbb{R}^{2m} \longrightarrow E_x$, and $\zeta_x : S \longrightarrow W_x$ is an isometry such that $\rho(\theta_x(v)) = \zeta_x f(v) \zeta_x^{-1}$. Now, if $p \in P_{Spin_c}$, then the position $w \longmapsto [p, w]$ well defines an hermitian isometry from $S \longrightarrow W_x$, which we will call $\beta(p)$. We can see that $\rho(\alpha(p)v)[p, w] = [p, f(v)w] = \beta(p) f(v) \beta(p)^{-1}[p, w]$ for all $w \in S$. That is, for all $p \in P_{Spin^c}$ the couple $(\alpha(p), \beta(p))$ is in $\tilde{P}_{Spin^c}$. Therefore, we have an injective morphism of $Spin^c$-bundles :

$$\begin{array}{rcl} P_{Spin^c} & \longrightarrow & \tilde{P}_{Spin^c} \\ p & \longmapsto & (\alpha(p), \beta(p)) \end{array}$$

which is obviously an isomorphism, being injective and equivariant.

For the converse let us start with a Clifford representation $\rho : E \longrightarrow \mathrm{End}(W)$, satisfying (1) and (2), and form the $\tilde{P}_{Spin^c}$ bundle of couples (θ_x, ζ_x) satisfying $\rho(\theta_x(v)) = \zeta_x f(v) \zeta_x^{-1}$ as above. Then, by the procedure explained in the beginning, we get a Clifford multiplication

$$\begin{array}{ccccccc} E & \xrightarrow{\simeq} & \tilde{P}_{Spin^c} \times_{\mu_1} \mathbb{R}^n & \longrightarrow & P_{Spin^c} \times_{Spin^c} \mathrm{End}(S) & \xrightarrow{\simeq} & \mathrm{End}(W) \\ \theta_x(v) & \longmapsto & [(\theta_x, \zeta_x), v] & \longmapsto & [(\theta_x, \zeta_x), f(v)] & \longmapsto & \zeta_x f(v) \zeta_x^{-1} = \rho(\theta_x(v)) \end{array}$$

Therefore the Clifford multiplication associated to the structure $\tilde{P}_{Spin^c}$ is exactly ρ.

□

An isomorphism of irreducible spin representation (W, ρ), (W', ρ') of (E, g) is a couple (θ, ζ), $\theta \in SO(E, g)$, $\zeta : W \longrightarrow W'$ an isometry such that $\rho'(\theta(v)) = \zeta \rho(v) \zeta^{-1}$.

We see now how to characterize the spinorial connection in vector bundle terms. We will need the following lemma, in order to compare connections on the line bundle L and on $\det W \simeq L^{\otimes^2}$. We indicate with $A^{\otimes^2}$ the connection on $L^{\otimes^2}$ naturally induced by A, sometimes indicated with $A_{L^{\otimes 2}}$, that is $A \otimes \mathrm{id}_L + \mathrm{id}_L \otimes A$. See Teleman [109] for a more general argument.

Lemma 2.2. *Let L a hermitian line bundle on a manifold M. Let $\mathcal{A}_L^{U(1)}$ the affine space of hermitian connections on L, and $\mathcal{A}_{L^{\otimes 2}}^{U(1)}$ the affine space of hermitian connections on $L^{\otimes^2}$. The map $\mathcal{A}_L^{U(1)} \xrightarrow{\otimes^2} \mathcal{A}_{L^{\otimes 2}}^{U(1)}$ carrying a unitary connection A on its tensor square $A^{\otimes^2}$ is an isomorphisms of affine spaces, modelled on the isomorphism of vector spaces: $A^1(M, i\mathbb{R}) \longrightarrow A^1(M, i\mathbb{R})$ carrying a form ω on 2ω. Moreover, if A is a unitary connection on L and F_A is its curvature 2-form, $F_A \in A^2(M, i\mathbb{R})$, then $F_{A^{\otimes 2}} = 2F_A$, where $F_{A^{\otimes 2}}$ is the curvature 2-form of the connection $A^{\otimes^2}$ on $L^{\otimes^2}$.*

Proof. Let us fix an origin A_0 on the affine space $\mathcal{A}_L^{U(1)}$. Then $A_0^{\otimes^2}$ will be the corresponding origin in $\mathcal{A}_{L^{\otimes 2}}^{U(1)}$. Let now $A = A_0 + \omega \in \mathcal{A}_L^{U(1)}$. We have $A^{\otimes^2} - A_0^{\otimes^2} = 2\omega \in A^1(M, i\mathbb{R})$, indeed, fixing a unitary frame e on L, and $e \otimes e$ on $L^{\otimes^2}$, we get:

$$\begin{aligned} (A^{\otimes^2} - A_0^{\otimes^2}) e \otimes e &= A^{\otimes^2}(e \otimes e) - A_0^{\otimes^2}(e \otimes e) \\ &= Ae \otimes e + e \otimes Ae - A_0 e \otimes e - e \otimes A_0 e \\ &= (A - A_0) e \otimes e + e \otimes (A - A_0) e = \omega e \otimes e + e \otimes \omega e = 2\omega(e \otimes e) \end{aligned}$$

From the definition of $A^{\otimes^2} = A \otimes \mathrm{id}_L + \mathrm{id}_L \otimes A$ we see immediately that $F_{A^{\otimes 2}} = F_A \otimes \mathrm{id}_L + \mathrm{id}_L \otimes F_A$. On the unitary frame $e \otimes e$ we have $F_{A^{\otimes 2}}(e \otimes e) = F_A e \otimes e + e \otimes F_A e = 2F_A(e \otimes e)$.

□

An analogous result is valid for $L^{\otimes^n}$.

Proposition 2.3. *Let (E, g) an even rank oriented euclidian vector bundle on a manifold M with a $Spin^c$ structure of determinant line bundle L and let W the spinor bundle for the bundle E associated*

to the $Spin^c$-structure. Let ρ the Clifford multiplication. Let ∇^E an orthogonal connection on E. The spinorial connection ∇^W_A is a hermitian connection on W satisfying:
(1) $[\nabla^W_A, \rho(x)] = \rho(\nabla^E x)$ for all $x \in \Gamma(E)$
(2) ∇^W induces the connection $A^{\otimes^{2^{m-1}}}$ on $L^{\otimes^{2^{m-1}}}$.
If $\mathrm{rk}(E) = 4$ then the conditions (1) and (2) completely characterize ∇^W_A.

Proof. Let $\rho_0 : \mathbb{R}^{2m} \longrightarrow \mathrm{End}(S)$ the irreducible spin representation used to build the spinor bundle W. We recall that the Clifford multiplication ρ is obtained by "bundlelizing" ρ_0:

$$\rho : E \simeq P_{Spin^c(2m)} \times_{\mu_1} \mathbb{R}^{2m} \longrightarrow P_{Spin^c(2m)} \times_{Spin^c(2m)} \mathrm{End}(S) \simeq \mathrm{End}(W) .$$

In other words, $\rho \in E^* \otimes \mathrm{End}(W) \simeq P_{Spin^c(2m)} \times_{\mu_1 \otimes Spin^c(2m)} (\mathbb{R}^{2m})^* \otimes \mathrm{End}(S)$. From this point of view, ρ is the bundle version of $\rho_0 \in (\mathbb{R}^{2m})^* \otimes \mathrm{End}(S)$. Now ρ_0 is $\mu_1 \otimes Spin^c(2m)$- invariant: it follows from the general principle 1.1 that the corresponding tensor field ρ is parallel for any connection on $E^* \otimes \mathrm{End}(W)$ induced by a $Spin^c(2m)$-connection on $P_{Spin^c(2m)}$. Now the $Spin^c$ connection we have fixed on P_{Spin^c} induces the connection $\nabla_A^{E^*\otimes \mathrm{End}(W)}$. As a consequence, $\nabla_A^{E^*\otimes \mathrm{End}(W)} \rho = 0$. Recalling the definition of connection on a tensor product we get (1). It is clear that the connection induced on L is A, so (2) is immediate.

For the uniqueness, suppose first that we have two hermitian connections ∇^W and $\tilde{\nabla}^W$ satisfying (1) and (2). If $\mathrm{rk}(E) = 4$ a unitary connection on W satifying condition (1) determines an orthogonal connection on $\mathfrak{su}(W)$, induced by the connection on $\Lambda^2 E$ via the parallel isomorphism $\gamma : \Lambda^2 E \longrightarrow \mathfrak{su}(W)$, and a unitary connection on $\det W$. A choice of an orthogonal connection on $\mathfrak{su}(W)$ and a unitary connection on $\det W$ uniquely determines a unitary connection on W (cf. Feehan and Leness [38],[40]). Since the connections they induce on $\det W$ coincide by (2) and the connections induced on $\mathfrak{su}(W)$ are forced to coincide by (1) and by the parallel isomorphism γ, the two unitary connections ∇^W and $\tilde{\nabla}^W$ verifying (1) and (2) must coincide.

2.2 Changes of metric

Let (M, g) an oriented compact riemannian 4-manifold. Let $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$ a $Spin^c$-structure for the tangent bundle (TM, g). Let h another metric on TM and let $\varphi = \varphi^h_g \in \mathrm{Aut}(TM)$ such that $h = \varphi^* g$. Therefore φ induces an isometry $\varphi : (TM, \varphi^* g) \longrightarrow (TM, g)$. The inverse isometry φ^{-1} induces an isomorphism of orthonormal frames $\varphi^{-1} : P_{SO(g)} \longrightarrow P_{SO(h)}$. From the point of view of vector bundles let $\rho : (TM, g) \longrightarrow \mathrm{End}(W)$ the Clifford representation associated to the $Spin^c$-structure. We recall that it satisfies: $\rho(u)^* = -\rho(u)$, $\rho(u)^2 = -g(u, u)\mathrm{id}_W$. The composition:

$$\begin{tikzcd} (TM, g) \arrow[r, "\rho"] & \mathrm{End}(W) \\ (TM, \varphi^* g) \arrow[u, "\varphi"] \arrow[ur, "\rho\circ\varphi = \varphi^*\rho"'] & \end{tikzcd}$$

is again a Clifford representation, this time for the metric h. Indeed

$$\begin{aligned} (\varphi^*\rho)(u)^* &= \rho(\varphi(u))^* = -\rho(\varphi(u)) = -(\varphi^*\rho)(u) \\ (\varphi^*\rho)(u)^2 &= \rho(\varphi(u))^2 = -g(\varphi(u), \varphi(u)) = -h(u, u) \end{aligned}$$

This new Clifford representation is isomorphic to the old one associated to the metric g by the isomorphism (φ, id_W), so that the $Spin^c$-principal bundle $P^{\varphi^*\rho}_{Spin^c}$ associated to the Clifford representation $\varphi^*\rho$

is isomorphic to the $Spin^c$ principal bundle $P^{\rho}_{Spin^c}$ by the map:

$$\begin{array}{ccc} P^{\rho}_{Spin^c} & \xrightarrow{\varphi^{-1}} & P^{\varphi^*\rho}_{Spin^c} \\ (\theta,\zeta) & \longmapsto & (\varphi^{-1}\theta,\zeta) \end{array}$$

This map is clearly fibered over φ^{-1}:

$$\begin{tikzcd} P^{\rho}_{Spin^c} \arrow[r, "\varphi^{-1}"] \arrow[d, "\alpha"] & P^{\varphi^*\rho}_{Spin^c} \arrow[d, "\alpha"] \\ P_{SO(g)} \arrow[r, "\varphi^{-1}"] & P_{SO(h)} \end{tikzcd}$$

so that $P^{\varphi^*\rho}_{Spin^c} = \varphi^* P^{\rho}_{Spin^c}$. It is clear that $P^{\varphi^*\rho}_{Spin^c}$ is a $Spin^c$ structure for the tangent bundle (TM,h) with the metric h. Actually the map $\varphi^{-1}\circ\alpha : P^{\rho}_{Spin^c} \longrightarrow P_{SO(h)}$ is already a $Spin^c$ structure for the bundle (TM,h), indeed:

$$(\varphi^{-1}\circ\alpha)(pg) = \varphi^{-1}(\alpha(pg)) = \varphi^{-1}(\alpha(p)\mu_1(g)) = \varphi^{-1}(\alpha(p))\mu_1(g)$$

because α, being a $Spin^c$ structure for $P_{SO(g)}$, is μ_1 equivariant, and φ^{-1} is $SO(4)$ equivariant. Moreover the map $\varphi^{-1}\circ\alpha$ induces the Clifford representation $\varphi^*\rho$. Indeed, the Clifford representation associated to the $Spin^c$ -structure $\varphi^{-1}\circ\alpha : P^{\rho}_{Spin^c} \longrightarrow P_{SO(h)}$ is given by

$$\begin{array}{ccccc} TM & \xrightarrow{\simeq} & P^{\rho}_{Spin^c}\times_{\varphi^{-1}\circ\alpha}\mathbb{R}^{2m} & \longrightarrow & P^{\rho}_{Spin^c}\times_{Spin^c}\mathrm{End}(S)\simeq\mathrm{End}(W) \\ \varphi^{-1}(\alpha(p)v) & \longmapsto & [p,v] & \longmapsto & [p,f(v)] \end{array}$$

which is exactly $\varphi^*\rho = \rho\circ\varphi$, indeed, $\rho\circ\varphi(\varphi^{-1}(\alpha(p)v)) = \rho(\alpha(p)v) = [p,f(v)]$. As a consequence, when changing the metric we do not need to change nor the bundle of spinors, nor the principal $Spin^c$-bundle, but only the covering map:

$$\begin{tikzcd} P^{\rho}_{Spin^c} \arrow[r, "\alpha\times_M\beta"] \arrow[dr, "(\varphi^{-1}\circ\alpha)\times_M\beta"'] & P_{SO(g)}\times P_{U(1)} \arrow[d, "\varphi^{-1}\times_M \mathrm{id}_W"] \\ & P_{SO(h)}\times P_{U(1)} \end{tikzcd}$$

or, in other words, the Clifford multiplication, to obtain another $Spin^c$-structure for the new metric $h=\varphi^* g$.

2.3 Compatible Clifford multiplications

Let $\mathrm{Met}(M)$ the space of riemannian metrics over the manifold M: it is an open cone in the space of sections $\mathcal{C}^\infty(M,S^2T^*M)$ and once we fix a metric g it is parametrized by the space of positive symmetric automorphisms $\mathrm{Sym}^+(TM,g)$ of the tangent bundle with respect to g. For more informations about the structure of the space of Riemannian metrics on a manifold M see Freed and Groisser [45], Gil-Medrano and Michor [52]. What we will do is to fix a metric $g\in\mathrm{Met}(M)$ and consider the isomorphism $\mathrm{Sym}^+(TM,g)\simeq\mathrm{Met}(M)$. Now $\mathrm{Sym}^+(TM,g)$ (or better its space of sections, but, by abuse of notation, we will not write them differently) can always be completed in Sobolev norms to $\mathrm{Sym}^+(TM,g)^2_k$, with k sufficiently large. Therefore we will speak of the tangent space $T_g\mathrm{Met}(M)$, meaning the tangent space to the identity $T_{\mathrm{id}}\mathrm{Sym}^+(TM,g) \simeq \mathrm{sym}(TM,g)$, that is the space of sections of symmetric endomorphisms of TM with respect to g, completed in Sobolev norms if necessary.

Let $W \longrightarrow M$ a fixed hermitian rk $= 4$ vector bundle on the manifold M. Let $\mathcal{R}$ the set of representations $\rho : TM \longrightarrow \mathrm{End}(W)$ such that $\rho(u)^* = -\rho(u)$ for all $u \in TM$. Such a representation is said *compatible* with a metric g if, moreover, it satisfies the other condition $\rho(u)^2 = -g(u,u)\mathrm{id}_W$, for all $u \in TM$. Let now be Ξ the set of compatible couples (g,ρ):

$$\Xi := \{(g,\rho) | g \in \mathrm{Met}(M)\, , \rho \in \mathcal{R}\, , \rho(u)^2 = -g(u,u)\ \forall u \in TM\ \} \ .$$

The next proposition gives the structure of Ξ. We recall that the group $PU(V)$, for an hermitian vector space V, is defined as the quotient $PU(V) := U(V)/U(1) \simeq SU(V)/\pm 1$.

Proposition 2.4. *The projection $p : \Xi \longrightarrow \mathrm{Met}(M)$ gives on Ξ the structure of a principal fibration of structural group $\mathcal{C}^\infty(M, PU(W))$.*

Proof. Let $g \in \mathrm{Met}(M)$ and let ρ, $\rho' \in \Xi_g := p^{-1}(g)$ two representations compatible with the metric g. On every point $x \in M$ ρ_x and ρ'_x are two irreducible spin representations of the euclidian vector space (T_xM, g_x):

$$\begin{tikzcd} (T_xM, g_x) \arrow[r, "\rho_x"] \arrow[dr, "\rho'_x"] & \mathrm{End}(W_x) \\ & \mathrm{End}(W_x) \end{tikzcd}$$

But by proposition 1.6 every two such representations are isomorphic: it follows that there exists $f \in U(W_x)$ such that

$$\rho'_x(v_x) = f\rho_x(v_x)f^{-1} \quad .$$

Now $f_x\rho_x(v_x)f_x^{-1} = \rho_x(v_x)$ for all v_x if and only if $f_x \in U(1)$. Therefore over each point $x \in M$ there is a $PU(W_x)$-bundle of possible representations with the metric g_x. This implies that the representations ρ, ρ' differ globally by a section $f \in \mathcal{C}^\infty(M, PU(W))$, where $PU(W)$ denotes the bundle of groups $\coprod_{x \in M} PU(W_x)$.

□

Proposition 2.5. *The position:*

$$\begin{aligned} \mathrm{Aut}(TM) \times \Xi &\longrightarrow \Xi \\ (\varphi, (g,\rho)) &\longmapsto (\varphi^* g, \rho \circ \varphi) \end{aligned}$$

defines a free action of $\mathrm{Aut}(TM)$ on Ξ. The restriction of this action to $SO(TM,g)$ for a fixed metric $g \in \mathrm{Met}(M)$ acts vertically on the fiber Ξ_g.

Proof. The proof is almost evident. The fact that if $(g,\rho) \in \Xi_g$ then $(\varphi^* g, \varphi^*\rho)$ is in $\Xi_{\varphi^* g}$ has actually already been proven in subsection 2.2. It remains to prove that the action is free. Suppose that $(\varphi^* g, \varphi^* \rho) = (g,\rho)$. Then $\varphi^*\rho(x) = \rho(x)$ for all $x \in TM$. Then $\rho(\varphi(x)) = \rho(x)$, which is equivalent to $\rho(\varphi(x) - x) = 0$ for all $x \in TM$. But then $0 = \rho(\varphi(x) - x)^2 = -g(\varphi(x) - x, \varphi(x) - x) = |\varphi(x) - x|^2$ which implies $\varphi(x) = x$ for all $x \in TM$, that is $\varphi = \mathrm{id}_{TM}$.

□

Remark 2.6. We have just seen that changes of the *Spin*^{c} structure in the fibre Ξ_g do not change the metric and correspond to changes in the Clifford multiplication. Changing the metric means, in a weak way, changing the *Spin*^{c} structure associated to (g,ρ) according to directions transversal to the fibre Ξ_g and varying as many parameters as the space of metrics. In other words, we need a distribution H of $T\Xi$ such that: $T_{(g,\rho)}\Xi \simeq T_{(g,\rho)}\Xi_g \oplus H_{(g,\rho)}$ and $H_{(g,\rho)} \simeq T_g\mathrm{Met}(M) \simeq \mathcal{C}^\infty(M, S^2T^*M)$. It means that we need to fix an isomorphism :

$$T\Xi \simeq V \oplus p^*T\mathrm{Met}(M)$$

where V is the vertical distribution. In particular a connection on Ξ is sufficient for this purpose.

Proposition 2.7. *For each $(g,\rho) \in \Xi$, the section:*

$$\begin{array}{rcl} \mathrm{Sym}^+(TM,g) & \xrightarrow{\sigma(g,\rho)} & \Xi \\ \varphi & \longmapsto & (\varphi^*g, \rho\circ\varphi) \end{array}$$

is transversal to the fibre Ξ_g and the tangent space to its image in the point (g,ρ):

$$H_{(g,\rho)} := T_{(g,\rho)}\mathrm{Im}\,\sigma(g,\rho)$$

defines naturally an equivariant horizontal distribution of $T\Xi$ and hence a connection on Ξ.

Proof. The fact that the section $\sigma(g,\rho)$ is transversal to the fibre and that $T_{(g,\rho)}\Xi \simeq T_{(g,\rho)}\Xi_g \oplus H_{(g,\rho)}$ is evident. To show that the distribution H actually defines a connection it remains to prove that it is $\mathcal{C}^\infty(M, PU(W))$-equivariant. But it follows immediately that the $\mathcal{C}^\infty(M, PU(W))$-action and the $\mathrm{Aut}(TM)$-action commute. Indeed let $f_x \in PU(W_x)$ and let $\tilde{f}_x$ a lifting to $U(W_x)$. Now ρ_x is the punctual Clifford multiplication $\rho_x : T_xM \longrightarrow \mathrm{End}(W_x)$. Let $\varphi_x \in \mathrm{Aut}(T_xM)$. Let us denote with $I_{\tilde{f}}$ the inner automorphism $I_{\tilde{f}} : \mathrm{End}(W) \longrightarrow \mathrm{End}(W)$ associated to $\tilde{f}$: it is clear that $I_{\tilde{f}}$ depends only on the $PU(W_x)$-class of $\tilde{f}$, that is f. We have the following diagram:

$$\begin{tikzcd}
(T_xM, g_x) \arrow[dr, "\rho_x"] \arrow[ddrr, bend left, "I_f(\rho_x)"] & & \\
& \mathrm{End}(W_x) \arrow[dr, "I_f"] & \\
(T_xM, \varphi_x^*g_x) \arrow[uu, "\varphi_x"] \arrow[ur, "\varphi_x^*\rho_x"'] \arrow[rr, "\varphi_x^*I_f(\rho_x)=I_f(\varphi_x^*\rho_x)"'] & & \mathrm{End}(W_x)
\end{tikzcd}$$

The fact that it commutes means exactly that $\varphi_x^*I_f(\rho_x) = I_f(\varphi_x^*\rho_x)$; this proves the commutation of the global actions. Differentiating the the commutation formula for $\varphi \in \mathrm{Sym}^+(TM,g)$, we immediately get the $\mathcal{C}^\infty(M, PU(W))$-invariance of the distribution H.

□

Since the choice of a metric $g \in \mathrm{Met}(M)$ allows us to identify $\mathrm{Sym}^+(TM,g) \simeq \mathrm{Met}(M)$ with the map: $\varphi \longmapsto \varphi^*g$, we have:

Lemma 2.8. *The choice of an element $\xi \in \Xi$ determines a trivialisation:*

$$\begin{array}{rcl} \mathrm{Met}(M) \times \mathcal{C}^\infty(M, PU(W)) \simeq \mathrm{Sym}^+(TM,g) \times \mathcal{C}^\infty(M, PU(W)) & \longrightarrow & \Xi \\ (\varphi, f) & \longmapsto & f^*\varphi^*\xi \end{array}$$

It is evident that we have a $\mathcal{C}^\infty(M, U(W))$-action on Ξ, because of the map

$$\mathcal{C}^\infty(M, U(W)) \longrightarrow \mathcal{C}^\infty(M, PU(W)) \ ;$$

in general this induced action is not transitive on the fibers because of the obstruction :

$$\mathcal{C}^\infty(M, U(W)) \longrightarrow \mathcal{C}^\infty(M, PU(W)) \xrightarrow{\delta} H^1(M, S^1) \quad . \tag{20}$$

The following lemma gives some more information on the Bockstein operator δ, which allows us to prove the transitivity of the $\mathcal{C}^\infty(M, U(W))$-action when the manifold M is simply connected.

Lemma 2.9. *The Bockstein operator $\delta : \mathcal{C}^\infty(M, PU(W)) \longrightarrow H^1(M, S^1)$ takes its values in the torsion subgroup of $H^1(M, S^1)$ given by the image of $H^1(M, \mu_4)$ in $H^1(M, S^1)$, where μ_4 indicates the subgroup of S^1 of 4-roots of unity.*

Proof. A section $f \in \mathcal{C}^\infty(M, PU(W))$ can be lifted locally to $U(W)$; therefore let $\{U_i\}_{i\in I}$ a covering of M such that for each $i \in I$ there exists a local lifting $\tilde{f}_i \in \mathcal{C}^\infty(U_i, \underline{U}(W))$ of the section f. On the intersection $U_{ij} := U_i \cap U_j$ two liftings $\tilde{f}_i$ and $\tilde{f}_j$ differ by an element $\lambda_{ij} \in \mathcal{C}^\infty(U_{ij}, S^1)$: $\tilde{f}_i = \lambda_{ij}\tilde{f}_j$. The cocycle $\{\lambda_{ij}\}_{ij}$ constitutes the image of the section f by the Bockstein operator $\delta(f)$, that is, the obstruction to a global lifting of the section f. Now both $\tilde{f}_i$, $\tilde{f}_j$ are in $\mathcal{C}^\infty(U_{ij}, U(W))$. As a consequence $\det \tilde{f}_i = \lambda_{ij}^4 \det \tilde{f}_j$, which implies $\{\lambda_{ij}^4\}_{ij} = \{\det \tilde{f}_i \det \tilde{f}_j^{\,-1}\}_{ij} = 0$. It means that the cocycle $\{\lambda_{ij}\}_{ij}$ takes its values in the kernel of the map $H^1(M, S^1) \longrightarrow H^1(M, S^1)$ induced by the short exact sequence:

$$0 \longrightarrow \mu_4 \longrightarrow S^1 \xrightarrow{z^4} S^1 \longrightarrow 0$$

which is precisely the image of $H^1(M, \mu_4)$ in $H^1(M, S^1)$.

$\square$

Corollary 2.10. *If M is simply connected any section f of $\mathcal{C}^\infty(M, PU(W))$ lifts to a section $\tilde{f} \in \mathcal{C}^\infty(M, U(W))$.*

Proof. If the manifold M is simply connected, $H_1(M, \mathbb{Z}) = 0$, then the exact sequence of the universal coefficient theorem

$$0 \longrightarrow \operatorname{Ext}^1_M(H_0(M, \mathbb{Z}), \mathbb{Z}_4) \longrightarrow H^1(M, \mathbb{Z}_4) \longrightarrow \operatorname{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}_4) \longrightarrow 0$$

implies the isomorphism: $\operatorname{Ext}^1_M(H_0(M, \mathbb{Z}), \mathbb{Z}_4) \simeq H^1(M, \mathbb{Z}_4)$. But $H_0(M, \mathbb{Z}) \simeq \mathbb{Z}$ and $\operatorname{Ext}^1_M(\mathbb{Z}, \mathbb{Z}_4) = 0$. This implies that in the sequence (20) the Bockstein operator is the zero map, and the projection

$$\mathcal{C}^\infty(M, U(W)) \longrightarrow \mathcal{C}^\infty(M, PU(W)) \longrightarrow 0$$

is surjective.

$\square$

Corollary 2.11. *If the manifold M is simply connected the group $\mathcal{C}^\infty(M, U(W))$ acts transitively on the fibers of Ξ with stabilizer $\mathcal{C}^\infty(M, S^1)$.*

Corollary 2.12. *If M is simply connected then $\operatorname{Aut}(TM) \times \mathcal{C}^\infty(M, U(W))$ acts transitively on Ξ with stabilizer $\{1\} \times \mathcal{C}^\infty(M, S^1)$.*

2.4 Parametrized Seiberg-Witten equations

In this subsection we will define Seiberg-Witten equations parametrized by the space Ξ. The first thing to do is to fix a suitable space of configurations for the equations we are going to write. Let $\xi = (g, \rho) \in \Xi$. $\rho : TM \longrightarrow \mathrm{End}(W)$ is a Clifford representation compatible with the metric g. Let ω_g the volume element on the manifold M for the metric g. The involution $\rho(\omega_g)$ allows to define the bundles of half-spinors W_+^ξ and W_-^ξ as the eigenbundles of $\rho(\omega_g)$ for ± 1, respectively. We have: $W \simeq W_+^\xi \oplus W_-^\xi$. We have to pay attention to the fact that varying $\xi \in \Xi$, the involution $\rho(\omega_g)$ may vary and W_+^ξ and W_-^ξ may change. The determinant line bundle L^ξ is always a square root of $\det W$, $(L^\xi)^{\otimes^2} \simeq \det W$ and it is isomorphic to $L^\xi \simeq \det W_+^\xi \simeq \det W_-^\xi$. The main problem here is to define a suitable space of $U(1)$ connections on L^ξ, the difficulty being due to the fact that L^ξ varies with $\xi \in \Xi$. To overcome this obstacle, we will use lemma 2.2 to perform actually a change of variables. Indeed for all ξ the affine space of $U(1)$ connections on L^ξ is isomorphic, via the tensor square, to the space of $U(1)$ connections on $\det W$. By this lemma and by proposition 2.3 the spinorial connection is determined by an g-orthogonal connection on TM, and by a unitary connection on $\det W$. So it is the same (up to taking the square, or square root to pass from one to the other) to fix the unknown connections in $\mathcal{A}_{L^\xi}^{U(1)}$ or in $\mathcal{A}_{\det W}^{U(1)}$, but the latter has the advantage that it does not change when we vary ξ; hence it is natural to take $\mathcal{A}_{\det W}^{U(1)}$ as the space of unknown $U(1)$ connections. To avoid cumbersome notations, we will adopt a different notation for the spinorial connection and the Dirac operator: once we fix an unitary connection A on $\det W$, $\nabla_A^{W,\rho,g}$ will indicate the spinorial connection on W, that is, the only hermitian connection on W for which ρ is parallel (always with respect to the Levi-Civita connection on TM) and which coincides with A on $\det W$ (and *not* with $A^{\otimes^2}$!). $D_A^{\rho,g}$ is the Dirac operator built using *this* spinorial connection $\nabla_A^{W,\rho,g}$. As a consequence, F_A is the curvature 2-form of the connection A on $\det W$, thus being twice the curvature 2-form of its square root $\sqrt{A}$ on L^ξ, $F_A = 2F_{\sqrt{A}}$; this explains the factor 1/2 in the second equation below. Now we are ready to define parametrized Seiberg-Witten equations. Let $\xi = (g, \rho) \in \Xi$ and let $\pi_+^\xi : \mathrm{End}(W) \longrightarrow \mathfrak{su}(W_+^\xi)$ the orthogonal projection on the (real) bundle of traceless anti-hermitian endomorphisms of W_+^ξ, and, if $\psi \in W$, let us denote with $\psi_{+,\xi}$ the positive part of the spinor ψ: $\psi_{+,\xi} = -1/2(\rho(\omega_g)\psi - \psi)$. The *space of configurations* $\mathfrak{C}$ is given by $\mathfrak{C} := \mathcal{A}_{\det W}^{U(1)} \times \Gamma(W) \times \Xi$. The *parametrized Seiberg-Witten equations* for the unknowns $(A, \psi, (g, \rho)) \in \mathfrak{C}$ are:

$$\rho(\omega_g)\psi = -\psi \tag{21a}$$

$$D_A^{g,\rho}\psi_{+,\xi} = 0 \tag{21b}$$

$$\frac{\rho(F_A^{+,g})}{2} = \pi_+^\xi[\psi^* \otimes \psi] \tag{21c}$$

We recall that we do not fix the decomposition $W = W_+ \oplus W_-$, and that in general W_+^ξ and W_-^ξ may vary with ξ. The first equation is therefore necessary to guarantee that, for each $(g, \rho) \in \Xi$, a spinor solution ψ is in W_+^ξ. We call *parametrized Seiberg Witten functional* the map:

$$\begin{aligned} \mathbb{F} : \mathcal{A}_{\det W}^{U(1)} \times \Gamma(W) \times \Xi &\longrightarrow \Gamma(W) \times \Gamma(W) \times i\mathfrak{su}(W) \\ (A, \psi, (g, \rho)) &\longmapsto \left(\rho(\omega_g)\psi + \psi, D_A^{g,\rho}\psi_{+,\xi}, \frac{\rho(F_A^{+,g})}{2} - \pi_+^\xi[\psi^* \otimes \psi]\right) \end{aligned}$$

Consider the trivial bundle over Ξ: $\Xi \times \Gamma(W) \times \Gamma(W) \times i\mathfrak{su}(W)$. The functional $\mathbb{F}$ take its values in the subbundle whose fiber over ξ is $\Gamma(W_-^\xi) \times \Gamma(W_-^\xi) \times i\mathfrak{su}(W_-^\xi)$, that is the kernel of the bundle map: $\Xi \times \Gamma(W) \times \Gamma(W) \times i\mathfrak{su}(W) \longrightarrow \Xi \times \Gamma(W) \times \Gamma(W) \times i\mathfrak{su}(W)$ given by: $(\xi, \sigma, \chi, h) \longmapsto (\xi, \rho(\omega)\sigma - \sigma, \rho(\omega_g)\chi - \chi, *_g h - h)$. We denote with $Z(\mathbb{F})$ the zero set of $\mathbb{F}$, or the space of solutions to (21).

Let us consider the projection $p : Z(\mathbb{F}) \longrightarrow \Xi$. It is clear that the fiber of p over (g, ρ), $p^{-1}(g, \rho) \simeq Z(F^{g,\rho})$ is the space of solutions of standard Seiberg-Witten equations with metric g and Clifford representation ρ. Our aim is to define a big moduli space $\mathcal{M}$ for the equations (21) parametrized by Ξ, that is, with projection $\pi : \mathcal{M} \longrightarrow \Xi$ such that each fiber of π, $\pi^{-1}(g, \rho)$ is isomorphic to the standard Seiberg-Witten moduli space $\mathcal{M}^{SW}_{g,\rho}$ for fixed metric g and fixed Clifford representation ρ. To do this we need a $\mathcal{C}^\infty(M, S^1)$ action on $\mathfrak{C}$, inducing the standard action on each fibre. Let us define the right $\mathcal{C}^\infty(M, S^1)$-action in the obvious way:

$$\begin{aligned} \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W) \times \Xi \times \mathcal{C}^\infty(M, S^1) &\longrightarrow \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W) \times \Xi \\ (A, \psi, (g, \rho), \lambda) &\longmapsto ((\lambda^4)^* A, \lambda^{-1}\psi, (g, \rho)) \end{aligned}$$

The projection

$$\mathcal{A}^{U(1)}_{\det W} \times \Gamma(W) \times \Xi \longrightarrow \Xi \tag{22}$$

is clearly invariant for the $\mathcal{C}^\infty(M, S^1)$-action, so the action is fiberwise. We note that if we take a connection $A^{\otimes^2}$ on $\det W$, the connection $(\lambda^4)^* A^{\otimes^2} = ((\lambda^2)^* A)^{\otimes^2}$, so that on every fiber the action coincides with the standard action for Seiberg-Witten equations. As the action is fiberwise and in each fiber the zero set $Z(F^{g,\rho})$ of Seiberg-Witten equations is preserved by the action, we have an induced action on the zero set $Z(\mathbb{F})$:

$$Z(\mathbb{F}) \times \mathcal{C}^\infty(M, S^1) \longrightarrow Z(\mathbb{F}) \ .$$

We can therefore pass to the quotient $Z(\mathbb{F})/\mathcal{C}^\infty(M, S^1)$.

Definition 2.13. The *parametrized Seiberg-Witten moduli space* $\mathcal{M}$ is the quotient of the space of solutions of the parametrized Seiberg-Witten equations (21) by the gauge group $\mathcal{C}^\infty(M, S^1)$.

The $\mathcal{C}^\infty(M, S^1)$ invariance of the map (22) together with the fact that $\mathcal{C}^\infty(M, S^1)$ preserves $Z(\mathbb{F})$ implies that the map $Z(\mathbb{F}) \longrightarrow \Xi$ is $\mathcal{C}^\infty(M, S^1)$-invariant and hence, taking the quotient, it descends to a projection:

$$\pi : Z(\mathbb{F})/\mathcal{C}^\infty(M, S^1) \simeq \mathcal{M} \longrightarrow \Xi \quad .$$

The situation is summarized by the diagram :

$$\begin{tikzcd}
Z(F^{g,\rho}) \arrow[rr, hook] \arrow[dd, "p"'] \arrow[dr] & & Z(\mathbb{F}) \arrow[dd, "p"'] \arrow[dr] & \\
& \mathcal{M}^{SW}_{g,\rho} \arrow[rr, hook] \arrow[dl, "\pi"] & & \mathcal{M} \arrow[dl, "\pi"] \\
(g, \rho) \arrow[rr, hook] & & \Xi &
\end{tikzcd}$$

where the horizontal maps are embeddings of topological spaces. In particular, the fiber of the projection $\pi : \mathcal{M} \longrightarrow \Xi$ over a point (g, ρ) is the standard Seiberg-Witten moduli space $\mathcal{M}^{SW}_{g,\rho}$ for a fixed metric g and a Clifford multiplication ρ.

We will show now that we have a $\mathcal{C}^\infty(M, U(W))$-action on the space of configuration $\mathfrak{C}$ that preserves the space of solutions $Z(\mathbb{F})$. Moreover the restriction of this $\mathcal{C}^\infty(M, U(W))$ action to $\mathcal{C}^\infty(M, S^1)$ coincides with the $\mathcal{C}^\infty(M, S^1)$ action defined above; since $\mathcal{C}^\infty(M, S^1)$ is the center of $\mathcal{C}^\infty(M, U(W))$, the $\mathcal{C}^\infty(M, U(W))$-action we will define commutes with the $\mathcal{C}^\infty(M, S^1)$-action defined above, thus descending to a nontrivial $\mathcal{C}^\infty(M, U(W))$ action on the moduli space $\mathcal{M}$. We define the right $\mathcal{C}^\infty(M, U(W))$-action on the space of configurations $\mathfrak{C}$ by the following position:

$$\begin{aligned} \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W) \times \Xi \times \mathcal{C}^\infty(M, U(W)) &\longrightarrow \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W) \times \Xi \\ (A, \psi, (g, \rho), f) &\longmapsto ((\det f)^* A, f^{-1}\psi, (g, f^* \rho)) \end{aligned}$$

We can define again a natural right $\mathcal{C}^\infty(M, U(W))$ action on the space $i\mathfrak{su}(W) \times \Gamma(W) \times \Gamma(W)$ as follows:

$$\begin{array}{rcl} i\mathfrak{su}(W) \times \Gamma(W) \times \Gamma(W) \times \mathcal{C}^\infty(M, U(W)) & \longrightarrow & i\mathfrak{su}(W) \times \Gamma(W) \times \Gamma(W) \\ (h, \psi, \varphi, f) & \longmapsto & (f^*h, f^{-1}\psi, f^{-1}\varphi) \end{array}$$

If $\xi = (g, \rho)$ we will write $f^*\xi$ for $(g, f^*\rho) = (g, f^{-1}\rho f)$. The next proposition includes all the remarks made here above:

Proposition 2.14. *The parametrized Seiberg-Witten functional* $\mathbb{F} : \mathfrak{C} \longrightarrow i\mathfrak{su}(W) \times \Gamma(W) \times \Gamma(W)$ *is* $\mathcal{C}^\infty(M, U(W))$*-equivariant:*

$$\mathbb{F}((A, \psi, (g, \rho)) \cdot f) = \mathbb{F}(\det f^*A, f^{-1}\psi, (g, f^*\rho)) = \mathbb{F}(A, \psi, (g, \rho)) \cdot f$$

Proof. First of all we prove that $f^{-1}(\psi_{+,\xi}) = (f^{-1}\psi)_{+,f^*\xi}$. Indeed $(f^{-1}\psi)_{+,f^*\xi} = -1/2((f^*\rho)(\omega_g)f^{-1}\psi - f^{-1}\psi) = 1/2(f^{-1}\rho(\omega)ff^{-1}\psi - f^{-1}\psi) = -1/2f^{-1}(\rho(\omega_g)\psi - \psi) = f^{-1}(\psi_{+,\xi})$ We consider now the connection $\nabla^{W,g,f^*\rho}_{(\det f)^*A}$: by proposition 2.3 it is characterized by the property that $f^*\rho : TM \longrightarrow \mathrm{End}(W)$ is parallel, that is $[\nabla^{W,g,f^*\rho}_{(\det f)^*A}, f^*\rho(x)] = f^*\rho(\nabla^{LC}x)$, and by the property that $\nabla^{W,g,f^*\rho}_{(\det f)^*A}$ induces the connection $(\det f)^*A$ on $\det W$. We will show that $f^{-1}\nabla^{W,g,\rho}_A f$ verifies these properties, thus proving that it coincides with $\nabla^{W,g,f^*\rho}_{(\det f)^*A}$. For the first property:

$$\begin{aligned} [f^{-1}\nabla^{W,g,\rho}_A f, (f^*\rho)(x)] &= [f^{-1}\nabla^{W,g,\rho}_A f, f^{-1}\rho(x)f] = f^{-1}[\nabla^{W,g,\rho}_A, \rho(x)]f \\ &= f^{-1}\rho(\nabla^{LC}x)f = f^*\rho(\nabla^{LC}x) \quad . \end{aligned}$$

For the second it is evident to see that the connection $f^{-1}\nabla^{W,g,\rho}_A f$ induces the connection $\det f^{-1}A \det f = (\det f)^*A$ on $\det W$. Once this is done, the Dirac operator

$$\begin{aligned} D^{g,f^*\rho}_{(\det f)^*A} &= f^*\rho \circ \nabla^{W,g,f^*\rho}_{(\det f)^*A} = f^{-1}\rho f \circ f^{-1}\nabla^{W,g,\rho}_A f \\ &= f^{-1}\rho\nabla^{W,g,\rho}_A f = f^{-1}D^{g,\rho}_A f \quad . \end{aligned}$$

We pass now to the second equation. The curvature 2-form is $F_{(\det f)^*A} = (\det f)^{-1} \circ F_A \circ \det f$, but F_A is a tensor, that is, $\mathcal{C}^\infty(M, \mathbb{C})$-linear, so $F_{(\det f)^*A} = F_A$. We remark that $\pi^\xi_+[\psi^* \otimes \psi] = [\psi^*_{+,\xi} \otimes \psi_{+,\xi}]_0$. Now, for any spinor ψ, we remark that $(f^{-1}\psi)^* \otimes f^{-1}\psi$ is the composition: $f^{-1}(\psi^* \otimes \psi)(f^{-1})^*$, but, since $f \in \mathcal{C}^\infty(M, U(W))$, then $ff^* = 1$, so that $(f^{-1})^* = f$, and then $(f^{-1}\psi)^* \otimes f^{-1}\psi = f^{-1}(\psi^* \otimes \psi)f$. Moreover $\mathrm{tr}\, f^{-1}(\psi^* \otimes \psi)f = \mathrm{tr}\, \psi^* \otimes \psi$, hence $[(f^{-1}\psi)^* \otimes f^{-1}\psi]_0 = f^{-1}[\psi^* \otimes \psi]_0 f$. Finally $\pi^{f^*\xi}_+[(f^{-1}\psi)^* \otimes (f^{-1}\psi)] = [(f^{-1}\psi)^*_{+,f^*\xi} \otimes (f^{-1}\psi)_{+,f^*\xi}]_0 = [(f^{-1}\psi_{+,\xi})^* \otimes (f^{-1}\psi_{+,\xi})]_0 = f^{-1}[\psi^*_{+,\xi} \otimes \psi_{+,\xi})]_0 f = f^{-1}\pi^\xi_+[\psi^* \otimes \psi]f$. As a consequence :

$$\begin{aligned} \mathbb{F}((A, \psi, (g, \rho)) \cdot f) =& \mathbb{F}(\det f^*A, f^{-1}\psi, (g, f^*\rho)) \\ =& (D^{g,f^*\rho}_{(\det f)^*A}(f^{-1}\psi)_{+,f^*\xi}, \frac{f^*\rho(F^{+,g}_{(\det f)^*A})}{2} - \pi^{f^*\xi}_+[(f^{-1}\psi)^* \otimes (f^{-1}\psi)], f^*\rho(\omega_g)f^{-1}\psi + f^{-1}\psi) \\ =& (f^{-1}D^{g,\rho}_A ff^{-1}(\psi_{+,\xi}), \frac{f^{-1}\rho(F^{+,g}_A)f}{2} - f^{-1}\pi^\xi_+[\psi^* \otimes \psi]f, f^{-1}\rho(\omega_g)ff^{-1}\psi + f^{-1}\psi) \\ =& (f^{-1}D^{g,\rho}_A\psi_{+,\xi}, f^{-1}(\frac{(\rho(F^{+,g}_A)f}{2} - \pi^\xi_+[\psi^* \otimes \psi])f, f^{-1}(\rho(\omega_g)\psi + \psi)f) \\ =& (D^{g,\rho}_A\psi_{+,\xi}, \frac{\rho(F^{+,g}_A)}{2} - \pi^\xi_+[\psi^* \otimes \psi], \rho(\omega_g)\psi + \psi) \cdot f = \mathbb{F}(A, \psi, (g, \rho)) \cdot f \end{aligned}$$

as required.

□

Corollary 2.15. *The space of solutions to the parametrized Seiberg-Witten equations is invariant under the action of* $\mathcal{C}^\infty(M, U(W))$*.*

Remark 2.16. It is evident that the $\mathcal{C}^\infty(M, U(W))$ action, restricted to $\mathcal{C}^\infty(M, S^1)$, coincides with the $\mathcal{C}^\infty(M, S^1)$ action defined in the beginning of the subsection. It is clear that the two actions commute because $\mathcal{C}^\infty(M, S^1)$ is the center on $\mathcal{C}^\infty(M, U(W))$.

Proposition 2.17. *There is a $\mathcal{C}^\infty(M, U(W))$ action on the moduli space $\mathcal{M}$.*

Proof. Let $f \in \mathcal{C}^\infty(M, U(W))$ and let $f : Z(\mathbb{F}) \longrightarrow Z(\mathbb{F})$ the automorphism associated to f. The fact that the $\mathcal{C}^\infty(M, U(W))$ action commutes with the $\mathcal{C}^\infty(M, S^1)$ action means exactly that for each such f the automorphism f is $\mathcal{C}^\infty(M, S^1)$ equivariant, and then it descends to a map $f : \mathcal{M} \longrightarrow \mathcal{M}$. It is clear that the identity induces the identity of the quotients. Now, if $f,\ f' \in \mathcal{C}^\infty(M, U(W))$, we have a commutative diagram:

$$\begin{tikzcd} Z(\mathbb{F}) \arrow[r, "f"] \arrow[rr, bend right, "f'f"'] & Z(\mathbb{F}) \arrow[r, "f'"] & Z(\mathbb{F}) \end{tikzcd} \quad .$$

Passing to the quotient, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[rr, bend right, "f'f"'] & \mathcal{M} \arrow[r, "f'"] & \mathcal{M} \end{tikzcd}$$

that proves the associativity for the action on the moduli space.

□

Corollary 2.18. *Let $f \in \mathcal{C}^\infty(M, U(W))$. We have $\mathcal{M}^{SW}_{g,f^*\rho} \simeq \mathcal{M}^{SW}_{g,\rho}$*

Proof. It is clear that the map $f : Z(\mathbb{F}) \longrightarrow Z(\mathbb{F})$ is fibered over the map $f : \Xi \longrightarrow \Xi$:

$$\begin{tikzcd} Z(\mathbb{F}) \arrow[r, "f"] \arrow[d, "p"'] & Z(\mathbb{F}) \arrow[d, "p"'] \\ \Xi \arrow[r, "f"] & \Xi \end{tikzcd}$$

and all the maps are $\mathcal{C}^\infty(M, S^1)$-equivariant. Passing to the quotient we get a commutative diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{M} \arrow[d, "\pi"'] \\ \Xi \arrow[r, "f"] & \Xi \end{tikzcd}$$

that is, the map $f : \mathcal{M} \longrightarrow \mathcal{M}$ is fibered over $f : \Xi \longrightarrow \Xi$. As a consequence, it exchanges the fibers of the projection π and induces an isomorphism $\pi^{-1}(g, \rho) \simeq \mathcal{M}^{SW}_{g,\rho} \xrightarrow{f} \mathcal{M}^{SW}_{g,f^*\rho} \simeq \pi^{-1}(g, f^*\rho)$

□

Corollary 2.19. *If M is simply connected and ρ, ρ' are two Clifford multiplications*

$$\rho, \rho' : (TM, g) \longrightarrow \mathrm{End}(W)$$

compatible with the same metric g, then $\mathcal{M}^{SW}_{g,\rho} \simeq \mathcal{M}^{SW}_{g,\rho'}$.

Proof. In the case M is simply connected, the group $\mathcal{C}^\infty(M, U(W))$ acts transitively on each fibre Ξ_g of Ξ, that is, any two Clifford representations $\rho, \rho' : (TM, g) \longrightarrow \mathrm{End}(W)$ compatible with the same metric g differ by the action of an element $f \in \mathcal{C}^\infty(M, U(W))$, $\rho' = f^*\rho$. Then we conclude by the preceding corollary.

□

2.5 Compatible Clifford representations with fixed decomposition

Let (M, g) a riemannian 4-manifold with a given Clifford representation $\rho : TM \longrightarrow \mathrm{End}(W)$ compatible with the metric. We have seen in the preceding subsections that a change in the metric $g \longmapsto \varphi^* g$ along the horizontal distribution (that is, by means of a positive symmetric automorphism $\varphi \in \mathrm{Sym}^+(TM)$) corresponds to a change in the Clifford representation $\rho \longmapsto \rho \circ \varphi$. Consider now the diagram of Clifford representations induced by the isometry $\varphi : (TM, \varphi^* g) \longrightarrow (TM, g)$:

$$\begin{tikzcd} Cl(TM, g) \arrow[r] & \mathrm{End}(W) \\ Cl(TM, \varphi^* g) \arrow[u, "\varphi"] \arrow[ur, "\varphi^*\rho"'] & \end{tikzcd}$$

The volume element for the new metric $\omega_{\varphi^* g}$ is exactly the image of the volume element ω_g for the map φ^{-1}, or $\varphi(\omega_{\varphi^* g}) = \omega_g$. As a consequence $(\varphi^*\rho)(\omega_{\varphi^* g}) = \rho(\omega_g)$: the image of the volume elements ω_g and $\omega_{\varphi^* g}$ remains the same: this means also that the decomposition of the bundle of spinors W in the direct sum of bundles of half spinors $W = W_+ \oplus W_-$ does not change. Conversely, if for two representations ρ, ρ' compatible with the metrics g,h, respectively, the decomposition of W in half spinors is fixed $W = W_+ \oplus W_-$, then the images of the volume elements are the same: $\rho(\omega_g) = \rho'(\omega_h)$. For this reason when changing the metric in a horizontal way, we can restrict our attention to the space of Clifford representations with fixed decomposition in half spinors, or, equivalently, with a fixed compatible unitary involution in $\mathrm{End}(W)$ playing the role of all volume elements. More precisely we can restrict us to bundle maps:

$$\rho_+ : TM \longrightarrow \mathrm{Hom}(W_+, W_-)$$

such that $\rho_+(x)^* \rho_+(x) = g(x, x)\mathrm{id}_{W_+}$ for all $x \in TM$. We can then build a map

$$\begin{aligned} \rho : TM &\longrightarrow \mathrm{Hom}(W_+ W_-) \oplus \mathrm{Hom}(W_-, W_+) \\ x &\longmapsto (\rho_+(x), \rho_-(x)) \end{aligned}$$

where $\rho_-(x) = -\rho_+(x)^*$. Such a map ρ satisfies the properties: $\rho(x)^2 = -g(x, x)\mathrm{id}_W$ and $\rho(x)^* = -\rho(x)$, in other words $\rho \in \Xi_g$. Let $\mathcal{R}_+$ the space of bundle maps $\rho_+ : TM \longrightarrow \mathrm{Hom}(W_+, W_-)$, and let Ξ_+ the set of compatible couples $\Xi_+ = \{(g, \rho_+) \mid g \in \mathrm{Met}(M)\ ,\ \rho_+ \in \mathcal{R}_+ \mid \rho_+^*(x)\rho_+(x) = g(x, x)\mathrm{id}_W\}$. It is easy to see that $\mathcal{R}_+$ is a subspace of $\mathcal{R}$ via the injection: $\rho_+ \longmapsto \rho$, and Ξ_+ is naturally a subfibration of Ξ, via the embedding $(g, \rho_+) \longmapsto (g, \rho)$.

Proposition 2.20. *(1) The projection $\Xi_+ \longrightarrow \mathrm{Met}(M)$ is naturally a principal $\mathcal{C}^\infty(M, P(U(W_+) \times U(W_-)))$ subfibration of Ξ.*
(2) There is a natural connection on Ξ_+ compatible with the natural connection on Ξ.
(3) The $\mathrm{Aut}(TM)$ action on Ξ preserves the subfibration Ξ_+.
(4) There is a fiberwise $\mathcal{C}^\infty(M, U(W_+) \times U(W_-))$ action on Ξ_+, which is transitive on the fibres if M is simply connected.

Proof. The proof of this proposition is analogous to the proof of correspondent propositions for Ξ. The injection $\Xi_+ \hookrightarrow \Xi$ carrying ρ_+ to $\rho = \rho_+ \oplus \rho_-$ is clearly equivariant and fibered over the identity of $\mathrm{Met}(M)$. If I is the unitary involution in $U(W)$ image in $\mathrm{End}(W)$ of all the elements $\rho(\omega_g)$ coming from a (g, ρ_+) in Ξ_+, Ξ_+ is identified to the space Ξ^I of all compatible representation $(g, \rho) \in \Xi$ such that $\rho(\omega_g) = I$. As we have remarked in the discussion at the beginning of this subsection the group $\mathrm{Aut}(TM)$ acts on Ξ without altering the volume element, that is, it induces an action on Ξ_+. The horizontal distribution H on Ξ_+ is again given in a point $\xi = (g, \rho)$ by the tangent space to the section $\sigma(g, \rho)$:

$\mathrm{Sym}^+(TM,g) \longrightarrow \Xi_+$ sending $\varphi \longmapsto \varphi^*\xi$, that is, $H_\xi = T_\xi \mathrm{Im}\,\sigma(g,\rho)$. If $(f_+,f_-) \in \mathcal{C}^\infty(M,U(W_+)\times U(W_-))$, the $\mathcal{C}^\infty(M,U(W_+)\times U(W_-))$ action on Ξ_+ is given by $((f_+,f_-),(g,\rho_+)) \longmapsto (g, f_-^{-1}\rho_+ f_+)$ The fourth statement is a consequence of the fact that, analogously to proposition 2.9, the Bockstein operator

$$\mathcal{C}^\infty(M,U(W_+)\times U(W_-)) \xrightarrow{\delta} \mathcal{C}^\infty(M,P(U(W_+)\times U(W_-))) \xrightarrow{\delta} H^1(M,S^1)$$

takes its values in a torsion subgroup of $H^1(M,S^1)$, which is zero if M is simply connected.

□

Let $W = W_+ \oplus W_-$ a decomposition given by a fixed volume element $I = \rho(\omega_g)$, $(g,\rho) \in \Xi$. The restriction of equations (21) to $\mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times \Xi^I$ gives rise to a system of Seiberg-Witten equations parametrized by Ξ_+, when we identify Ξ^I with Ξ_+:

$$D^{g,\rho}_A \psi = 0 \tag{23a}$$

$$\frac{\rho(F^{+,g}_A)}{2} = [\psi^* \otimes \psi]_0 \tag{23b}$$

for unknowns $(A,\psi,\xi) \in \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times \Xi_+$. In the same way the restriction of the functional $\mathbb{F}$ to $\mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times \Xi^I$ gives a $\Gamma(W_+) \times i\mathfrak{su}(W_+)$-valued functional $\mathbb{F}_+$. This functional is equivariant for the action of the group $\mathcal{C}^\infty(M,U(W_+)\times U(W_-))$ induced by the restriction of the corresponding action of the group $\mathcal{C}^\infty(M,U(W))$. We can consider the moduli space $\mathcal{M}_+$ of solutions to equations (23) defined as $\mathcal{M}_+ \simeq Z(\mathbb{F}_+)/\mathcal{C}^\infty(M,S^1)$. It is clearly fibered over Ξ_+. The fibres $\mathcal{M}_{+,g,\rho_+}$ are clearly isomorphic to the fibres $\mathcal{M}^{SW}_{g,\rho}$, for $\rho = \rho_+ \oplus \rho_-$, and therefore to standard Seiberg-Witten moduli spaces for a fixed metric g and Clifford representation ρ. The following proposition summarizes the properties of $\mathcal{M}_+$ corresponding to the analogous properties of $\mathcal{M}$.

Proposition 2.21. *There is a nontrivial $\mathcal{C}^\infty(M,U(W_+)\times U(W_-))$ action on the moduli space $\mathcal{M}_+$, such that the projection $\mathcal{M}_+ \longrightarrow \Xi_+$ is equivariant. If $f \in \mathcal{C}^\infty(M,U(W_+)\times U(W_-))$, then $\mathcal{M}^{g,f^*\rho_+}_+ \simeq \mathcal{M}^{g,\rho_+}_+$. If M is simply connected, and ρ_+, ρ'_+ are two Clifford representations compatible with the same metric and with the same decomposition in half spinors, then $\mathcal{M}^{g,\rho_+}_+ \simeq \mathcal{M}^{g,\rho'_+}_+$.*

3 Variation of the Seiberg-Witten equations

In the previous section we have built the setting of parametrized Seiberg-Witten equations which gives sense to the study of perturbations of the metric alone. We have seen that the perturbations we are interested in correspond to horizontal variations of Seiberg-Witten equations for a natural connection on the space of parameters. In this section we perform the computation of the variation of Seiberg-Witten equations corresponding to such directions. The more interesting part of this computation is the variation of the Dirac operator. Our method allows to consider different Dirac operators $D^{g,\rho}_A$ and $D^{\varphi^*g,\varphi^*\rho}_A$ for two different metrics as acting on the space of sections of the same bundle of spinors, fixed once for all. Our result coincides with that obtained by Bourguignon and Gauduchon in [12] and in [11].

Let (M,g) a riemannian 4-manifold and $P_{SO(g)}$ the principal $SO(4)$ bundle of oriented g-orthonormal frames. Let us fix on the riemannian 4-manifold (M,g) a $Spin^c$-structure $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$. Let W be the bundle of spinors for (TM,g) associated to the $Spin^c$ structure α and let $\rho : TM \longrightarrow \mathrm{End}(W)$ the Clifford representation associated to α. Let $W = W_+ \oplus W_-$ be the decomposition in bundles of half-spinors given by the volume element $\rho(\omega_g)$. Let $\mathrm{Met}(M)$ be the space of metrics on M. Consider now the fibration $\Xi_+ \longrightarrow \mathrm{Met}(M)$ of representations $\rho_+ : TM \longrightarrow \mathrm{Hom}(W_+,W_-)$ compatible with some metric $h \in \mathrm{Met}(M)$, that is $\rho_+(x)^*\rho_+(x) = h(x,x)\mathrm{id}_W$ for some $h \in \mathrm{Met}(M)$. The $Spin^c$-structure

α corresponds to the point $\xi = (g, \rho)$ in the fibration $\Xi_+ \subseteq \Xi$. Consider the parametrized Seiberg Witten equations (23) for unknowns $(A, \psi, \xi) \in \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times \Xi_+$, $\xi = (g, \rho)$:

$$D^{g,\rho}_A \psi = 0 \tag{24}$$

$$\frac{\rho(F^{+,g}_A)}{2} = [\psi^* \otimes \psi]_0 \quad . \tag{25}$$

Let $H \leqslant T\Xi_+$ the horizontal distribution on Ξ_+ defining the natural connection: $H_\xi = T_\xi \mathrm{Im}\, \sigma(g, \rho)$. The purpose of this section is to compute the variation of the parametrized Seiberg-Witten equations (24) in correspondence of a horizontal variation of the parameter ξ: this variation corresponds to a "pure" perturbation of the metric, by our discussion on remark 2.6.

3.1 Variation of the Dirac operator

3.1.1 Variation of the spinorial connection

The purpose of this subsection is to study how the $Spin^c$ connection on the bundle of spinors W varies when changing the metric. Let $\alpha : P_{Spin^c} \longrightarrow P_{SO(g)}$ our $Spin^c$ structure on the manifold M, as discussed above. Let $h \in \mathrm{Met}(M)$ another metric and $\varphi \in \mathrm{Sym}(TM, g)$ the only symmetric automorphism of TM with respect to g such that $h = \varphi^* g$. Let $P_{SO(h)}$ the principal fibre bundle of h-orthonormal frames of TM. The isometry $\varphi : (TM, \varphi^* g) \longrightarrow (TM, g)$ lifts to an isomorphism of principal $SO(4)$-bundles : $\varphi : P_{SO(h)} \longrightarrow P_{SO(g)}$ which can be further lifted to an automorphism of the $GL_+(4)$-bundle of oriented frames of TM, $P_{GL_+(4)}$. As discussed in subsection 2.2, when changing the metric we do not need to change the principal bundle P_{Spin^c}, nor the bundle of spinors W: we can take as $Spin^c$ structure for the euclidian vector bundle (TM, h) the composition map: $\varphi^{-1} \circ \alpha$:

$$\begin{tikzcd}
P_{Spin^c} \arrow[d, "\alpha"'] \arrow[dr, "\varphi^{-1}\circ\alpha"] & \\
P_{SO(g)} \arrow[r, "\varphi^{-1}"] & P_{SO(h)}
\end{tikzcd}$$

which corresponds to the following change in the Clifford representation:

$$\begin{tikzcd}
(T, g) \arrow[r, "\rho"] & \mathrm{End}(W) \\
(T, h) \arrow[u, "\varphi"] \arrow[ur, "\rho\circ\varphi"'] &
\end{tikzcd}$$

Let $P_{U(1)}$ the fundamental $U(1)$ bundle associated to the $Spin^c$ structure, and let $P_{Spin^c} \xrightarrow{\beta} P_{U(1)}$ the projection. The following diagram summarizes the situation:

$$\begin{tikzcd}
P_{U(1)} & P_{Spin^c(4)} \arrow[l, "\beta"'] \arrow[d, "\alpha"'] \arrow[dr, "\varphi^{-1}\circ\alpha"] & \\
& P_{SO(4)}(g) \arrow[r, "\varphi^{-1}"] \arrow[d, hook] & P_{SO(4)}(h) \arrow[d, hook] \\
& P_{GL_+(4)} \arrow[r, "\varphi^{-1}"] & P_{GL_+(4)}
\end{tikzcd}$$

Let now $\omega_g \in A^1(P_{SO(4)}(g), \mathfrak{so}(4))$ the Levi-Civita connection 1-form on $P_{SO(g)}$, that is the only torsion free $SO(4)$ connection. Let $A \in A^1(P_U(1), \mathfrak{u}(1))$ a unitary connection 1-form on $P_{U(1)}$. The spinorial connection 1-form $\Omega_{\alpha,A}$ on P_{Spin^c} is obtained by pulling back to P_{Spin^c} the forms ω_g and A: $\Omega_{\alpha,A} := (d\mu)^{-1}(\alpha^*\omega_g + \beta^*A) \in A^1(P_{Spin^c}, \mathfrak{spin}^c(4))$. When we change the metric using the symmetric automorphism φ, we have to lift the Levi-Civita connection ω_h by means of the new projection $\varphi^{-1} \circ \alpha$; as a consequence the new spinorial connection 1 form is

$$\begin{aligned}\Omega_{\varphi^{-1}\circ\alpha,A} &= (d\mu)^{-1}((\varphi^{-1}\circ\alpha)^*\omega_h + \beta^*A) \\ &= (d\mu)^{-1}((\alpha^*\circ(\varphi^{-1})^*\omega_h + \beta^*A) \quad .\end{aligned}$$

Let now g_t a differentiable path of metrics in $\mathrm{Met}(M)$ of the form: $g_t = g(1+ts) = g + tk$, where s is a symmetric endomorphism with respect to g, and $k \in S^2T^*M$, $k = gs$. Let φ_t the positive symmetric automorphism of the tangent bundle (with respect to g) such that $g_t = \varphi_t^* g$; we will indicate φ_t^{-1} with ϕ_t. We can write $\phi_t = (1+ts)^{-\frac{1}{2}}$. Let ω_t the Levi Civita connection 1 forms for the metric g_t, $\omega_t \in A^1(P_{SO(g_t)}, \mathfrak{so}(4))$. We obtain a differentiable path of spinorial connections on P_{Spin^c}:

$$\Omega_{t,A} = \Omega_{\phi_t\circ\alpha,A} = (d\mu)^{-1}((\phi_t\circ\alpha)^*\omega_t + \beta^*A) = (d\mu)^{-1}((\alpha^*\circ\phi_t^*\omega_t + \beta^*A) \in A^1(P_{Spin^c}, \mathfrak{spin}^c(4)) \ .$$

Let us compute now the derivative of this connection form in the vector space $A^1(P_{Spin^c}, \mathfrak{spin}^c(4))$:

$$\begin{aligned}\Omega' = \frac{d}{dt}\Omega_t \mid_{t=0} &= \frac{d}{dt}(d\mu)^{-1}(\alpha^*\phi_t^*\omega_t + \beta^*A) \mid_{t=0} \\ &= (d\mu)^{-1}(\alpha^*\frac{d}{dt}\phi_t^*\omega_t \mid_{t=0}) \\ &= (d\mu)^{-1}(\alpha^*[r^*\omega_g + \omega']) \quad .\end{aligned}$$

We remark that $r^*\omega_g + \omega'$ belongs to the vector space $A^1(P_{SO(4)}(g), \mathfrak{so}(4))$, and therefore Ω' can be identified with an element of $A^1(P_{Spin^c(4)}, \mathfrak{so}(4))$. We remark furthermore that ω' does not take its values in $\mathfrak{so}(4)$, because the different connection forms ω_t live on different principal bundles $P_{SO(g_t)}$. However, we can think the ω_t as connection 1- forms on $P_{GL_+(4)}$, hence as elements in $A^1(P_{GL_+(4)}, \mathfrak{gl}(4))$. Therefore ω' makes sense as an element in $A^1(P_{GL_+(4)}, \mathfrak{gl}(4))$. It is thanks to the corrective term $r^*\omega_g$ that we can lift the derivative to P_{Spin^c} and obtain a $\mathfrak{spin}^c(4)$-valued 1 form.

We are going to prove that the form $r^*\omega_g + \omega'$ is a tensorial form of type $(\mathrm{ad}, \mathfrak{so}(4))$: as a consequence it is the pull back of a vector bundle valued 1 form $\dot\omega_M \in A^1(M, \mathrm{End}(W))$ on the manifold M. We begin with the following lemma:

Lemma 3.1. *Let Q a principal fibre bundle on a manifold M of structural group G. Let $\mathfrak{g}$ its Lie algebra. Let $(-\varepsilon, \varepsilon) \ni t \longmapsto \omega_t$ a differentiable path of connection 1-forms. Then the derivative:*

$$\omega' := \frac{d}{dt}\omega_t \mid_{t=0}$$

is a tensorial 1-form on Q of type $(\mathrm{ad}, \mathfrak{g})$.

Proof. To prove that the derivative ω' is a tensorial form of type $(\mathrm{ad}, \mathfrak{g})$ we have to show that
(1) ω' is ad-equivariant, that is $R_g^*\omega' = \mathrm{ad}(g^{-1})\omega'$
(2) ω' is horizontal, that is, ω' vanishes on the vertical distribution ($\omega'(V) = 0$ if V is a vertical vector field).
For the first:

$$R_g^*\omega' == R_g^*\frac{d}{dt}\omega_t = \frac{d}{dt}R_g^*\omega_t = \frac{d}{dt}\mathrm{ad}(g^{-1})\omega_t = \mathrm{ad}(g^{-1})\frac{d}{dt}\omega_t = \mathrm{ad}(g^{-1})\omega' \quad .$$

For the second, let $A \in \mathfrak{g}$ and let A^* the fundamental vector field on Q associated to A (cf [76]). The value of an arbitrary connection form on a fundamental vector field A^* is exactly A, therefore, for all

t, $\omega_t(A^*) = A$. Taking the derivative at the point $t = 0$ we get $\omega'(A^*) = 0$ for all fundamental vector fields A^*, $A \in \mathfrak{g}$. Now it is simple to conclude, remembering that any vertical vector field is generated by fundamental vector fields A^*.

□

Corollary 3.2. *The forms Ω' and ω' are tensorial of type $(\mathrm{ad}, \mathfrak{spin}^c)$ and $(\mathrm{ad}, \mathfrak{gl}(4))$ respectively. Ω' can be identified with a tensorial form of type $(\mathrm{ad}, \mathfrak{so}(4))$.*

We remark that $\phi_t^*\omega_t$ are connection 1-forms on $A^1(P_{SO(g)}, \mathfrak{so}(4))$ for all t, therefore we get:

Corollary 3.3. *The 1-form $r^*\omega_g + \omega' \in A^1(M, \mathfrak{so}(4))$ is a tensorial form.*

Let us now indicate with $\dot{\omega}_P = d/dt(\phi_t^*\omega_t)|_{t=0}$. We have seen by lemma 3.1 that $\dot{\omega}_P$ is a tensorial 1-form of type $(\mathrm{ad}, \mathfrak{so}(4))$ on $P_{SO(g)}$. As a consequence there is a unique 1-form $\dot{\omega}_M \in A^1(M, \mathfrak{so}(T))$ realizing $\dot{\omega}_P$: $\pi^*(\dot{\omega}_M) = \dot{\omega}_P$. Let us now compute the variation of the spinorial connection form on the vector bundle level. Let ∇_t^W the spinorial connection induced by $\Omega_{t,A}$ on W. As seen in subsection 1.1, it is characterized by the formula:

$$\pi^*(\nabla_t^W \phi) = d\pi^*\phi + (d\mu)^{-1}(\Omega_{t,A})(\pi^*\phi) \quad .$$

Differentiating both terms in $t = 0$ we get:

$$\pi^*(\dot{\nabla}^W \phi) = (d\mu)^{-1}(\dot{\Omega})(\pi^*\phi) \quad .$$

Now we know that $\dot{\Omega} = \alpha^*\dot{\omega}_P$, where $\dot{\omega}_P \in A^1(P_{SO(g)}, \mathfrak{so}(4))$ so, actually:

$$\pi^*(\dot{\nabla}^W \phi) = (d\mu_1)^{-1}(\alpha^*\dot{\omega}_P)(\pi^*\phi) \quad .$$

We remark that $(d\mu_1)^{-1} : \mathfrak{so}(4) \simeq \Lambda^2\mathbb{R}^4 \longrightarrow \mathfrak{spin}(4) \simeq \mathfrak{su}(S_+) \times \mathfrak{su}(S_-)$ coincides with $(1/2)\rho_0 : \Lambda^2\mathbb{R}^4 \longrightarrow \mathfrak{su}(S_+) \times \mathfrak{su}(S_-)$. Moreover $\dot{\omega}_P = \pi^*\dot{\omega}_M$, so that

$$(d\mu_1)^{-1}(\alpha^*\dot{\omega}_P) = \frac{1}{2}\rho_0(\alpha^*\pi^*\dot{\omega}_M) = \frac{1}{2}\pi^*\rho(\dot{\omega}_M)$$

where we have used the diagram:

$$\begin{tikzcd} P \times \mathrm{End}(S) \arrow[r, "\pi"] \arrow[d] & \mathrm{End}(W) \arrow[d] \\ P \arrow[r, "\pi"] & M \end{tikzcd}$$

Therefore

$$\pi^*(\dot{\nabla}^W \phi) = \frac{1}{2}(\pi^*\rho(\dot{\omega}_M))(\pi^*\phi) = \pi^*\left(\frac{\rho(\dot{\omega}_M)}{2}\phi\right)$$

and this implies

$$\dot{\nabla}^W \phi = \frac{\rho(\dot{\omega}_M)}{2}\phi \quad .$$

We now want to find out what $\dot{\omega}_M$ is in vector bundle terms. By lemma 1.2 we know that the connection form ω_t is associated to the connection $\phi_t^{-1}\nabla_t^{LC}\phi_t$ on the tangent bundle TM, where ∇_t^{LC} is the Levi-Civita connection form the metric g_t. Therefore the derivative $\dot{\omega}_P$ is the tensorial 1 form on $P_{SO(g)}$ which is associated to the derivative of the path of connections $\phi_t^{-1}\nabla_t^{LC}\phi_t$ on TM, which is $\dot{\nabla}^{LC} + \nabla^{LC}r$, where $r = -d\phi_t/dt|_{t=0} = d\varphi_t/dt|_{t=0}$. We have indicated with $\dot{\nabla}^{LC}$ the variation of the Levi-Civita connection ∇_t^{LC} on TM, for the path of metrics g_t. As a consequence we can establish:

$$\dot{\omega}_M = \dot{\nabla}^{LC} - \nabla^{LC}r \quad . \tag{26}$$

3.1.2 Variation of the Levi-Civita connection

We are now going to compute the variation $\dot{\nabla}^{LC}$ of the Levi-Civita connection correspondent to a variation of the metric. Let g_t a path of metrics $g_t = g + tk = g(1+ts)$, with s a symmetric endomorphism of TM with respect to the metric g. Let ∇^t the Levi-Civita connection for the metric g_t. It is characterized by the property of being compatible with the metric:

$$dg_t(\xi,\eta) = g_t(\nabla^t\xi,\eta) + g_t(\xi,\nabla^t\eta)$$

and the property of being torsion-free:

$$\nabla^t_\xi\eta - \nabla^t_\eta\xi = [\xi,\eta] \quad . \tag{27}$$

We rewrite the first condition:

$$dg((1+ts)\xi,\eta) = g((1+ts)\nabla^t\xi,\eta) + g((1+ts)\xi,\nabla^t\eta) \quad .$$

Taking the derivative with respect to t we get:

$$dg(s\xi,\eta) = g(s\nabla\xi,\eta) + g(\nabla^{'}\xi,\eta) + g(s\xi,\nabla\eta) + g(\xi,\nabla^{'}\eta)$$

therefore we can write:

$$g(\nabla(s\xi),\eta) + g(s\xi,\nabla\eta) = g(s\nabla\xi,\eta) + g(\nabla^{'}\xi,\eta) + g(s\xi,\nabla\eta) + g(\xi,\nabla^{'}\eta)$$

which gives:

$$g((\nabla s)\xi,\eta) = g(\nabla^{'}\xi,\eta) + g(\xi,\nabla^{'}\eta) \quad . \tag{28}$$

Now differentiating in t the torsion-free condition (27) we get:

$$\nabla^{'}_\xi\eta = \nabla^{'}_\eta\xi \quad .$$

We evaluate the condition (28) on a vector field θ:

$$g((\nabla_\theta s)\xi,\eta) = g(\nabla^{'}_\theta\xi,\eta) + g(\xi,\nabla^{'}_\theta\eta) \quad .$$

Finally we obtain:

$$\begin{aligned} 2g(\nabla^{'}_\theta\xi,\eta) = \quad & g(\nabla^{'}_\theta\xi,\eta) + g(\xi,\nabla^{'}_\theta\eta) - g(\nabla^{'}_\eta\xi,\theta) \\ & -g(\xi,\nabla^{'}_\eta\theta) + g(\nabla^{'}_\xi\eta,\theta) + g(\eta,\nabla^{'}_\xi\theta) \quad = g((\nabla_\theta s)\xi,\eta) - g((\nabla_\eta s)\xi,\theta) + g((\nabla_\xi s)\eta,\theta) \quad . \end{aligned}$$

which implies that the variation of the Levi-Civita connection is:

$$g(\nabla^{'}_\theta\xi,\eta) = \frac{1}{2}\left[g((\nabla_\theta s)\xi,\eta) - g((\nabla_\eta s)\xi,\theta) + g((\nabla_\xi s)\eta,\theta)\right] \quad . \tag{29}$$

Fix now a local orthonormal frame in TM, $e_1,\ldots,e_4$ and let $e^1,\ldots,e^4$ the dual orthonormal frame on T^*M. We will indicate with τ^k_{ij} the components of the tensor $\dot{\nabla}$ with respect to the frame e_i, and with c^k_{ij} the component of the tensor ∇s, that is:

$$\tau^k_{ij} = g(\dot{\nabla}_{e_i}e_j,e_k)\ , \quad c^k_{ij} = g((\nabla_{e_i}s)e_j,e_k) \quad .$$

We remark that the tensor c^k_{ij} is symmetric in j and k, since the Levi-Civita connection preserves the bundle of symmetric automorphisms. Therefore from (29) we get:

$$\tau^k_{ij} = \frac{1}{2}[c^k_{ij} - c^i_{kj} + c^i_{jk}]\ .$$

We remark that τ^k_{ij} is symmetric in i and j. It is now simple to compute the tensor $\dot{\omega}_M = \dot{\nabla} - \nabla r = \dot{\nabla} - 1/2\nabla s$: if we indicate its components with $\dot{\omega}^k_{ij}$, we have:

$$\dot{\omega}^k_{ij} = \frac{1}{2}[c^i_{jk} - c^i_{kj}]$$

where we remark that $\dot{\omega}_{ij}$ is skew-symmetric in j and k, and hence belongs to $A^1(M,\mathfrak{so}(T))$ as expected.

3.1.3 Variation of the Dirac operator

We pass now to study the variation of the Dirac operator. We recall that for each metric g_t we chose as Clifford multiplication the map:

$$\rho_t := \rho \circ \varphi_t^{-1} : (TM, g_t) \longrightarrow \mathrm{End}(W) \quad . \tag{30}$$

Let now $\nabla_A^{W,t}$ the spinorial connection on the spinor bundle W with respect to the Levi-Civita connection ∇^t for the metric g_t and the unitary connection A. The Dirac operator $D_A^{\xi_t} = D_A^{g_t,\rho_t}$ for the *Spin*^{c}-structure $\xi_t = (g_t, \rho_t)$ given by (30) is:

$$D_A^{\xi_t} = D_A^{g_t,\rho_t} = \tilde{\rho}_t \circ \nabla_A^{W,t} : \Gamma(W) \longrightarrow \Gamma(W) \quad .$$

We remark that the different Dirac operators $D_A^{g_t,\rho_t}$ act on the same space of sections $\Gamma(W)$: it is now easy to compute the derivative in $t = 0$:

$$\frac{d}{dt} D_A^{\xi_t}|_{t=0} = \frac{d}{dt}[\tilde{\rho}_t \circ \nabla_A^{W,t}] = \frac{d}{dt}\tilde{\rho}_t|_{t=0} \circ \nabla_A^{W,t} + \tilde{\rho} \circ \dot{\nabla}^W \quad .$$

Now $d/dt\ \rho_t|_{t=0} = -\tilde{\rho} \circ r = -1/2\ \tilde{\rho} \circ s$. We need now the definition of the trace and divergence of a symmetric tensor.

Definition 3.4. Let g a riemannian metric on the tangent bundle TM. Let σ a 2-tensor. We define the *metric trace* $\mathrm{tr}_{\,g}(\sigma)$ as the trace of σ when it is identified with an element of $T^*M \otimes TM$, by means of the identification between the tangent and the contangent bundles provided by the metric g.
The *divergence* of σ is the differential 1-form defined by:

$$(\mathrm{div}\, \sigma)(Y) := \mathrm{tr}_{\,g}[X \longmapsto (\nabla_X \sigma)(Y^\flat, -)] \qquad \text{if } \sigma \in TM \otimes TM$$

or

$$(\mathrm{div}\, \sigma)(Y) := \mathrm{tr}_{\,g}[X \longmapsto (\nabla_X \sigma)(Y, -)] \qquad \text{if } \sigma \in T^*M \otimes TM, \text{ or } \sigma \in T^*M \otimes T^*M \; .$$

If $\sigma \in S^2T^*M \simeq \mathrm{sym}\,(TM, g)$, e_i is a local orthonormal frame for TM then $\mathrm{div}\, \sigma = \sum_i (\nabla_{e_i}\sigma)e_i$.

Let us now compute $(\tilde{\rho} \circ \dot{\nabla}^W)\phi$ for a spinor ϕ. From (26) we know

$$\dot{\nabla}^W \phi = \frac{\rho(\dot{\omega}_M)}{2}\phi \quad .$$

We know that

$$\begin{aligned} \dot{\omega}_M &= \sum_{ijk} (\dot{\omega}_M)_{ij}^k e^i \otimes e^j \otimes e_k = \frac{1}{2}\sum_{ijk}[c_{jk}^i - c_{kj}^i] e^i \otimes e^j \otimes e_k \\ &= \frac{1}{2}\sum_{ijk}[c_{jk}^i - c_{kj}^i] e^i \otimes (e^j \wedge e_k) = \frac{1}{4}\sum_{ijk}[c_{jk}^i - c_{kj}^i] e^i \otimes E_j^k \quad . \end{aligned}$$

Therefore

$$\dot{\nabla}^W \phi = \frac{\rho(\dot{\omega}_M)}{2}\phi = \frac{1}{8}\sum_{ijk}[c_{jk}^i - c_{kj}^i] e^i \otimes \rho(e_j e_k)\phi$$

When we apply once more the Clifford multiplication we get

$$\tilde{\rho}(\dot{\nabla}^W \phi) = \frac{1}{8}\sum_{ijk}[c_{jk}^i - c_{kj}^i]\rho(e^i)\rho(e_j e_k)\phi = \frac{1}{8}\sum_{ijk}[c_{jk}^i - c_{kj}^i]\rho(e_i e_j e_k)\phi$$

and recalling that $(\dot{\omega}_M)_{ij}^k = \tau_{ij}^k - 1/2\ c_{ij}^k$ we get:

$$\tilde{\rho}(\dot{\nabla}^W \phi) = \frac{1}{8}\sum_{ijk}[2\tau_{ij}^k - c_{ij}^k]\rho(e_i e_j e_k)\phi = \frac{1}{4}\sum_{ij}\tau_{ii}^j \rho(e_j)\phi - \frac{1}{8}\sum_{ij} c_{ji}^i \rho(e_k)\phi \quad .$$

Now $\tau_{ii}^j = 1/2[c_{ii}^j - c_{ji}^i + c_{ij}^i]$, and recalling that $c_{ij}^k = g((\nabla_{e_i} s)e_j, e_k)$, we have $\sum_{ij} \tau_{ii}^j = \mathrm{div}\, s - 1/2 d\mathrm{tr}\, s$ and $\sum_{ij} c_{ji}^i = d\mathrm{tr}\, s$. Finally we have:

$$\tilde{\rho}(\dot{\nabla}^W \phi) = \frac{1}{4}\rho(\mathrm{div}\, s - d\mathrm{tr}\, s)\phi$$

As a consequence we get the theorem:

Theorem 3.5. *The variation of the Dirac operator $D_A^\xi : \Gamma(W) \longrightarrow \Gamma(W)$ associated to the spin representation $\xi = (g, \rho)$, corresponding to the variation of the metric g along the direction $s \in \mathrm{sym}(TM, g)$, is given by*

$$\frac{d}{dt} D_A^{\xi_t}|_{t=0} = -\frac{1}{2}\,\tilde{\rho} \circ s \circ \nabla^W + \frac{1}{4}\rho(\mathrm{div}\, s - d\mathrm{tr}\, s) \tag{31}$$

This variation coincides with the one found by Gauduchon and Bourguignon in [12], and Bourguignon in [11].

3.2 Variation of the Seiberg-Witten equations

We come now back to equations (24):

$$\begin{aligned} D_A^\xi \psi =& 0 \\ \frac{\rho(F_A^{+,g_\xi})}{2} =& [\psi^* \otimes \psi]_0 \quad . \end{aligned}$$

in the unknowns $(A, \psi, \xi) \in \mathcal{A}_{\det W}^{U(1)} \times \Gamma(W_+) \times \Xi_+$ with $\xi = (g_\xi, \rho_\xi)$. Let $\mathbb{F}_+ : \mathcal{A}_{\det W}^{U(1)} \times \Gamma(W_+) \times \Xi_+ \longrightarrow \Gamma(W_+) \times i\mathfrak{su}(W_+)$ the parametrized Seiberg-Witten functional. The purpose of this subsection is to compute the variation of equations (24) for a variation of the parameter ξ along the horizontal direction, or in an equivalent manner, the partial differential $\partial \mathbb{F}_+/\partial\xi(\dot{\xi})$ for a variation $\dot{\xi}$ of the parameter ξ in the horizontal distribution H_ξ. Since the horizontal distribution H is isomorphic to the pull back $\pi^* T\mathrm{Met}(M)$, and therefore $H_\xi \simeq T_{g_\xi}\mathrm{Met}(M)$ and since, the choice of the point ξ allows to identify the space of positive symmetric automorphisms $\mathrm{Sym}^+(TM, g)$ to the space of metrics $\mathrm{Met}(M)$ via the map $\varphi \longmapsto \varphi^* g$, it is the same to take the partial differential $\partial\tilde{\mathbb{F}}_+/\partial\varphi$ at the identity of the composed functional :

$$\tilde{\mathbb{F}}_+ : \mathcal{A}_{\det W}^{U(1)} \times \Gamma(W_+) \times \mathrm{Sym}^+(TM, g) \longrightarrow \Gamma(W_-) \times i\mathfrak{su}(W_+)$$

defined by $\tilde{\mathbb{F}}_+(A, \psi, \varphi) = \mathbb{F}_+(A, \psi, \varphi^*\xi)$. The partial differential $\partial\tilde{\mathbb{F}}_+/\partial\varphi$ is then a map:

$$\frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi} : T_{\mathrm{id}}\mathrm{Sym}^+(TM, g) \longrightarrow \Gamma(W_-) \oplus i\mathfrak{su}(W_+) \quad .$$

The first component of this differential has actually already been computed in subsection 3.1.3. Indeed if $s = \dot{\varphi} \in \mathrm{sym}\,(TM, g)$,

$$\frac{\partial\tilde{\mathbb{F}}_{+,1}}{\partial\varphi}(A, \psi, \mathrm{id})(s) = -\tilde{\rho} \circ s \circ \nabla^W \psi + \frac{1}{2}\rho(\mathrm{div}\, s - d\mathrm{tr}\, s)\psi \quad .$$

We will now compute the partial derivative $\partial\tilde{\mathbb{F}}_{+,2}/\partial\varphi$ of the second component, that is the variation of the second equation in (24). We begin by writing the second equation in another form. The self-dual part of the curvature F_A can be written:

$$F_A^{+,g} = P^{+,g} F_A = \frac{(*_g + 1)}{2} F_A \quad .$$

Let now $h = \varphi^* g$ another metric, for $\varphi \in \mathrm{Sym}(TM, g)$, and φ orientation preserving. The fact that φ is an orientation-preserving isometry between (TM, h) and (TM, g) implies the following commutative diagram of morphisms of vector bundles:

$$\begin{tikzcd}
(\Lambda^2 T^*, g) \arrow[r, "*_g"] \arrow[d, "\Lambda^2\varphi^*"'] & (\Lambda^2 T^*, g) \arrow[d, "\Lambda^2\varphi^*"'] \\
(\Lambda^2 T^*, \varphi^* g) \arrow[r, "*_{\varphi^* g}"] & (\Lambda^2 T^*, \varphi^* g)
\end{tikzcd}$$

We deduce that $*_{\varphi^* g} = \Lambda^2\varphi^* \circ *_g \circ (\Lambda^2\varphi^*)^{-1}$ and therefore

$$\begin{aligned}
F_A^{+,\varphi^* g} &= P^{+,\varphi^* g} F_A = \left(\frac{*_{\varphi^* g} + 1}{2}\right) F_A = \left(\frac{\Lambda^2\varphi^* \circ *_g \circ (\Lambda^2\varphi^*)^{-1}}{2} + 1\right) F_A \\
&= \Lambda^2\varphi^* \circ \left(\frac{*_{\varphi^* g} + 1}{2}\right) \circ (\Lambda^2\varphi^*)^{-1} F_A = \Lambda^2\varphi^* \circ P^{+,g} \circ (\Lambda^2\varphi^*)^{-1} F_A \quad .
\end{aligned}$$

Now the Clifford representation for the metric $\varphi^* g$ is given by $\rho \circ \varphi$, which acts on 2-forms $\Lambda^2 T^* M$ as $\rho \circ (\Lambda^2\varphi^*)^{-1}$, therefore

$$\begin{aligned}
\rho^{\varphi^*\xi}(F_A^{+,\varphi^*\xi}) &= \rho \circ (\Lambda^2\varphi^*)^{-1} \circ \Lambda^2\varphi^* \circ P^{+,g} \circ (\Lambda^2\varphi^*)^{-1} F_A \\
&= \rho(P^{+,g} \circ (\Lambda^2\varphi^*)^{-1} F_A) \quad .
\end{aligned}$$

The second equation then becomes:

$$\frac{1}{2}\rho(P^{+,g} \circ (\Lambda^2\varphi^*)^{-1} F_A) = [\psi^* \otimes \psi]_0 \quad .$$

Suppose now given a path of metrics $g_t = \varphi_t^* g$, $\varphi_t \in \mathrm{Sym}^+(TM, g)$, and let $s = d\varphi_t/dt|_{t=0}$. The variation of the second equation in correspondence of the variation $s \in \mathrm{sym}\,(TM, g)$ is

$$\frac{\partial \tilde{\mathbb{F}}_{+,2}}{\partial \varphi}(A, \psi, \mathrm{id})(s) = -\frac{1}{2}\rho(P^{+,g} i(s^*) F_A)$$

where $i(s^*)$ is the derivation of degree 0 on the exterior algebra $\Lambda T^* M$ that coincides with s^* on T^*; in other words $i(s^*)$ acts on the wedge product of two 1-forms $\tau, \sigma \in \Gamma(T^*)$ as $i(s*)(\tau \wedge \sigma) = (s^*\tau) \wedge \sigma + \tau \wedge (s^*\sigma)$.

3.3 The perturbed Seiberg-Witten operator

We will call *perturbed Seiberg-Witten operator* in the point (A, ψ) the full differential of the map $\tilde{\mathbb{F}}_+$:

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+ : T_A \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times T_{\mathrm{id}}\mathrm{Sym}^+(TM, g) \longrightarrow \Gamma(W_-) \times i\mathfrak{su}(W_+)$$

To simplify the notations we will indicate the variation $(\dot{A}, \dot{\psi}, \dot{\varphi})$ with $(\tau, \phi, s) \in T_A \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+) \times T_{\mathrm{id}}\mathrm{Sym}^+(TM, g) \simeq A^1(M, i\mathbb{R}) \times \Gamma(W_+) \times \mathrm{sym}\,(TM, g)$. In the sequel we will identify by means of the metric g the space of sections $\mathrm{sym}\,(TM, g)$ of symmetric endomorphisms of TM with respect to the metric and the space of sections of symmetric covariant 2-tensors $\Gamma(S^2 T^* M)$. We review now how to compute the rest of the differential $D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+$ (cf [91]).

Let $\tau \in A^1(M, i\mathbb{R})$. If we change the connection A by $A + \tau$, the spinorial connection 1-form on P_{Spin^c} will change as follows:

$$\Omega_{\alpha, A+\tau} = \Omega_{\alpha, A} + \frac{1}{2}(d\mu_2)^{-1}(\beta^*\tau) = \Omega_{\alpha, A} + \frac{1}{4}\pi^*\tau\,.$$

It means that the corresponding connection on W is $\nabla^W_{A+\tau}\phi = \nabla^W\phi + (1/4)\,\tau\otimes\phi$: as a consequence the corresponding Dirac operator is

$$D^\xi_{A+\tau} = D^\xi_A + \frac{1}{4}\rho^\xi(\tau) \quad .$$

It follows that

$$\frac{\partial\tilde{\mathbb{F}}_{+,1}}{\partial A}(A,\psi,\mathrm{id})(\tau) = \frac{1}{4}\rho(\tau)\psi \quad .$$

We clarify that the factor 1/4 instead of the more usual 1/2 is due to the fact that we have fixed the unknown $U(1)$ connections on $\det W \simeq L^{\otimes^2}$ instead of on L. The curvature $F_{A+\tau}$ is easily $F_A + d\tau$, therefore $F^{+,g}_{A+\tau} = F^{+,g}_A + d^+\tau$ and hence

$$\frac{\partial\tilde{\mathbb{F}}_{+,2}}{\partial A}(A,\psi,\mathrm{id})(\tau) = \frac{\rho(d^+\tau)}{2} \quad .$$

The derivative $\partial\tilde{\mathbb{F}}_{+,1}/\partial\psi(A,\psi,\mathrm{id})(\phi)$ is immediately computed as being:

$$\frac{\partial\tilde{\mathbb{F}}_{+,1}}{\partial\psi}(A,\psi,\mathrm{id})(\phi) = D^\xi_A\phi \quad .$$

The derivative of the quadratic term $\psi^*\otimes\psi$ with respect to a variation ϕ is $\phi^*\otimes\psi + \psi^*\otimes\phi$: hence

$$\frac{\partial\tilde{\mathbb{F}}_{+,2}}{\partial\psi}(A,\psi,\mathrm{id})(\phi) = -[\phi^*\otimes\psi + \psi^*\otimes\phi]_0 \quad .$$

In the sequel we will definitely identify the space $i\Gamma(\mathfrak{su}(W_+))$ of traceless hermitian endomorphisms of W_+ with the space of imaginary self-dual 2-forms $A^2(M,i\mathbb{R})$.

We are now ready to write down the full differential of the Seiberg-Witten functional. We have proved the

Proposition 3.6. *The perturbed Seiberg-Witten operator on the point* (A,ψ,id)*:*

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+ : A^1(M,i\mathbb{R})\times\Gamma(W_+)\times\mathrm{sym}\,(TM,g) \longrightarrow \Gamma(W_-)\times A^2_+(M,i\mathbb{R})$$

is given by :

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+(\tau,\phi,s) = \begin{pmatrix} \frac{1}{4}\rho(\tau)\psi + D_A\phi - \rho\circ s\circ\nabla^W\psi + \frac{1}{2}\rho(\mathrm{div}\, s - d\mathrm{tr}\, s)\psi \\ \frac{1}{2}d^+\tau - [\phi^*\otimes\psi + \psi^*\otimes\phi]_0 - \frac{1}{2}P^{+,g}i(s^*)F_A \end{pmatrix}$$

We want now to understand better the term $P^{+,g}i(s^*)F_A$. The traceless 2 symmetric tensors $S^2_0T^*M$ are identified with the traceless symmetric endomorphisms of TM with respect to g, $\mathrm{sym}_0(TM,g)$.

Lemma 3.7. *The bundle* $\mathrm{sym}_0(TM,g)$ *of traceless symmetric endomorphisms of* TM *with respect to the metric* g*, is isomorphic to the bundle* $\mathrm{Hom}(\Lambda^2_-T^*M,\Lambda^2_+T^*M)$ *of homomorphisms between the antiself-dual* 2*-forms* $\Lambda^2_-T^*M$ *and the self-dual* 2*-forms* $\Lambda^2_+T^*M$ *by means of the map:*

$$\begin{aligned} \delta : S^2_0T^*M \simeq \mathrm{sym}_0(TM,g) &\longrightarrow \mathrm{Hom}(\Lambda^2_-T^*M,\Lambda^2_+T^*M) \\ s_0 &\longmapsto P^{+,g}i(s_0^*)|_{\Lambda^2_-} \end{aligned}$$

Proof. It is sufficient to prove the lemma for vector spaces. Let (E,g) an oriented euclidian vector space of dimension 4. Let $Sym^+(E,g)$ the orientation preserving automorphisms of E, symmetric with respect to the metric g. Let $\mathrm{Gr}(3,\Lambda^2E)$ the grassmannian of subspaces of dimension 3 in Λ^2E and let ω_E a volume element in Λ^4E. Consider the map:

$$\begin{aligned} \xi : Sym^+(E,g) &\longrightarrow \mathrm{Gr}(3,\Lambda^2E) \\ \varphi &\longmapsto \Lambda^2_{-,\varphi^*g} = (\Lambda^2\varphi^{-1})(\Lambda^2_{-,g}) \end{aligned}$$

where $\Lambda^2_{-,\varphi^* g}$ is the vector subspace of bivectors in $\Lambda^2 E$ which are anti self-dual with respect to $\varphi^* g$. We remark that this map is constant on the orbits of the group $\mathbb{R}_+$ and induces an open embedding of the quotient $Sym^+(E,g)/\mathbb{R}_+$ onto the open set of $\mathrm{Gr}(3,\Lambda^2 E)$ where the quadratic form

$$\wedge : \Lambda^2 E \times \Lambda^2 E \longrightarrow \Lambda^4 E \simeq \mathbb{R}\omega_E$$

is negative defined. The tangent map to the identity gives an isomorphism:

$$\delta : \mathrm{sym}\,(E,g)/\mathbb{R} \longrightarrow \mathrm{Hom}(\Lambda^2_- E, \Lambda^2_+ E) \quad .$$

Since the map ξ in the local chart $\mathrm{Hom}(\Lambda^2_- E, \Lambda^2_+ E)$ of the grassmannian is given by $\varphi \longmapsto P^{+,g}\Lambda^2\varphi^{-1}|_{\Lambda^2_-}$ the image $\delta(s)$ of a traceless symmetric endomorphism is easily seen as being the morphism $s \longrightarrow -P^{+,g} i(s)|_{\Lambda^2_+}$. We now equip $\Lambda^2 E$ with the standard metric and $\mathrm{Hom}(\Lambda^2_- E, \Lambda^2_+ E)$ with the metric $(u,v) = 1/2\,\mathrm{tr}\,(uv^*)$. The tangent space to the quotient at id, $\mathrm{sym}\,(E,g)/\mathbb{R}$, is then naturally equipped with the metric $(\bar{u},\bar{v}) = 2\mathrm{tr}\,(uv) - 1/2(\mathrm{tr}\,u)(\mathrm{tr}\,v)$. We can identify the quotient $\mathrm{sym}\,(E,g)/\mathbb{R}$ with $\mathrm{sym}_0(E,g)$, which comes equipped with the metric induced from $\mathrm{sym}\,(E,g)$, that is $(u,v) = 2\mathrm{tr}\,(uv)$.

It is now easy to see that δ is an isometry. Let $s \in \mathrm{sym}_0(E,g)$ and let e_i an orthonormal basis of E for the metric g for which s is diagonal: $s(e_i) = \lambda_i e_i$. Since s is traceless, $\sum_i \lambda_i = 0$. Let $e_i \wedge e_j$ the corresponding basis for $\Lambda^2 E$. On the basis element $e_i \wedge e_j$ $i(s)$ acts in the following way:

$$i(s)(e_i \wedge e_j) = \lambda_i(e_i \wedge e_j) + \lambda_j(e_i \wedge e_j) = (\lambda_i + \lambda_j)(e_i \wedge e_j) \quad .$$

Let now ω_1^-, ω_2^-, ω_3^- the basis of $\Lambda^2_- E$ given by:

$$\omega_1^- = e_1 \wedge e_2 - e_3 \wedge e_4\,, \quad \omega_2^- = e_1 \wedge e_3 + e_2 \wedge e_4\,, \quad \omega_3^- = e_1 \wedge e_4 - e_2 \wedge e_3$$

We have

$$\begin{aligned}
i(s)\omega_1^- &= (\lambda_1 + \lambda_2) e_1 \wedge e_2 - (\lambda_3 + \lambda_4) e_3 \wedge e_4 \\
i(s)\omega_2^- &= (\lambda_1 + \lambda_3) e_1 \wedge e_3 + (\lambda_2 + \lambda_4) e_2 \wedge e_4 \\
i(s)\omega_3^- &= (\lambda_1 + \lambda_4) e_1 \wedge e_4 - (\lambda_2 + \lambda_3) e_2 \wedge e_3
\end{aligned}$$

and expressing $e_i \wedge e_j$ in terms of ω_i^- we get:

$$\begin{aligned}
i(s)\omega_1^- &= (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\omega_1^+ \\
i(s)\omega_2^- &= (\lambda_1 + \lambda_3 - \lambda_2 - \lambda_4)\omega_2^+ \\
i(s)\omega_3^- &= (\lambda_1 + \lambda_4 - \lambda_2 - \lambda_3)\omega_3^+
\end{aligned}$$

and with respect to the norms taken on $\mathrm{sym}_0(E,g)$ and on $\mathrm{Hom}(\Lambda^2_- E, \Lambda^2_+ E)$, we get that the norm of $\delta(f)$ is exactly the norm of $s \in \mathrm{sym}_0(E,g)$: $||\delta(s)||^2 = 2\sum_i \lambda_i^2 = 2\mathrm{tr}\, s^2 = ||s||^2$. This proves that δ is an isometry, with the chosen norms.

□

With an analogue computation we can establish:

Lemma 3.8. *The map $f \longrightarrow i(f)$ from $\mathrm{sym}\,(TM,g) \longrightarrow End(\Lambda^2 TM)$ induces an embedding of the bundle of homotheties of TM into the bundle of homotheties of $\Lambda^2_\pm T^*M$.*

Proof. The bundle of diagonal endomorphism of TM is isomorphic to $\mathcal{C}^\infty(M,\mathbb{R})$. If $\lambda \in \mathcal{C}^\infty(M,\mathbb{R})$ then $i(\lambda \mathrm{id})\omega_i^\pm = 4\lambda\omega_i^\pm$.

□

As a consequence an element $s \in \mathrm{sym}\,(TM, g)$ acts on the form F_A in the following way:

$$\begin{aligned} i(s)F_A &= i(s)F_A^+ + i(s)F_A^- \\ &= \delta(s_0)F_A^+ + i(\frac{\mathrm{tr}\, s}{4}\mathrm{id})F_A^+ + \delta(s_0)F_A^- + i(\frac{\mathrm{tr}\, s}{4}\mathrm{id})F_A^- \\ &= \delta(s_0)F_A^+ + (\mathrm{tr}\, s)F_A^+ + \delta(s_0)F_A^- + (\mathrm{tr}\, s)F_A^- \quad . \end{aligned}$$

Therefore

$$P^{+,g}i(s)F_A = (\mathrm{tr}\, s)F_A^+ + \delta(s_0)F_A^- \ .$$

We can now rewrite the Seiberg-Witten operator splitting the space $\mathrm{sym}\,(TM, g)$ in $\mathrm{sym}\,(TM, g) \simeq \mathcal{C}^\infty(M, \mathbb{R}) \oplus \mathrm{sym}_0(TM, g)$ and using the isomorphism δ defined above:

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+(\tau, \phi, f, s_0) = \begin{pmatrix} \frac{1}{4}\rho(\tau)\psi + D_A\phi - fD_A\psi + \frac{3}{2}\rho(df)\psi - \rho \circ s_0 \circ \nabla^W\psi + \frac{1}{2}\rho(\mathrm{div}\, s_0)\psi \\ \frac{1}{2}d^+\tau - [\phi^* \otimes \psi + \psi^* \otimes \phi]_0 - 4fF_A^+ - \delta(s_0)F_A^- \end{pmatrix}$$

If (A, ψ, id) is a zero of $\tilde{\mathbb{F}}_+$, $\tilde{\mathbb{F}}_+(A, \psi, \mathrm{id}) = 0$, then, in particular $D_A\psi = 0$ and the Seiberg-Witten operator simplifies to:

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+(\tau, \phi, f, s_0) = \begin{pmatrix} \frac{1}{4}\rho(\tau)\psi + D_A\phi + \frac{3}{2}\rho(df)\psi - \rho \circ s_0 \circ \nabla^W\psi + \frac{1}{2}\rho(\mathrm{div}\, s_0)\psi \\ \frac{1}{2}d^+\tau - [\phi^* \otimes \psi + \psi^* \otimes \phi]_0 - 4fF_A^+ - \delta(s_0)F_A^- \end{pmatrix} . \tag{32}$$

The operator we have just computed takes into account the most general perturbation of the metric. It is interesting to restrict us to more special perturbation, like, for example, conformal perturbations. A conformal change of metric is always given by $g \longmapsto e^{2f}g$ for $f \in \mathbb{C}^\infty(M, \mathbb{R})$. Let now $g_t = e^{2tf}g$ a conformal deformation of the metric. We have $e^{2tf}g = \varphi_t^* g$, with $\varphi_t = (1 + e^{2tf})^{-1/2}$. Then $d\varphi_t/dt|_{t=0} = -f$. Therefore the Seiberg-Witten operator for a conformal perturbation of the metric becomes:

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+(\tau, \phi, f) = \begin{pmatrix} \frac{1}{4}\rho(\tau)\psi + D_A\phi + \frac{3}{2}\rho(df)\psi \\ \frac{1}{2}d^+\tau - [\phi^* \otimes \psi + \psi^* \otimes \phi]_0 - 4fF_A^+ \end{pmatrix} . \tag{33}$$

4 The question of transversality

In this section we will take up the discussion of the transversality of the Seiberg-Witten functional F with perturbations of the metric or, said another way, the transversality of the perturbed Seiberg-Witten functional $\tilde{\mathbb{F}}_+$ introduced in the previous chapter. Proving that the functional $\tilde{\mathbb{F}}_+$ is transversal, that is, proving that its differential $D\tilde{\mathbb{F}}_+$ is surjective, guarantees by the implicit function theorem that the space of solutions forms a smooth Hilbert manifold and it is the first step in order to obtain a smooth Seiberg-Witten moduli space.

The question of transversality with perturbations of the metric alone for Seiberg-Witten equations has already been taken up by Eichhorn and Friedrich in [31] and by Friedrich in [49], but, to our point of view, with not convincing arguments. The authors claim that they prove a generic metrics transversality theorem for Seiberg-Witten equations, but, as we will see, their proof is based on some false statements.

After setting up the functional machinery, we will compute the adjoint of the perturbed Seiberg-Witten operator and we will find equations for its kernel. A non trivial solution to these kernel equations represents an obstruction to transversality. Therefore proving the generic metrics transversality theorem amounts to proving a vanishing theorem for the solutions of these kernel equations.

4.1 Elliptic differential operators

In this subsection we will recall some facts about elliptic differential operators. Our main references for this material are [118], [8] and [79].

Definition 4.1. Let E and F two euclidian vector bundles on a compact manifold M and let $\langle\cdot,\cdot\rangle_E$, $\langle\cdot,\cdot\rangle_F$ the metrics on E and F respectively. Let $L : \Gamma(E) \longrightarrow \Gamma(F)$ a linear differential operator of order $k \in \mathbb{N}$. A differential operator $L^* : \Gamma(F) \longrightarrow \Gamma(E)$ is said to be the *formal adjoint* of L if

$$\int_M \langle Lu, v\rangle_F = \int_M \langle u, L^*v\rangle_E$$

for every $u \in \Gamma(E)$, $v \in \Gamma(F)$.

It is well known (cf [118], [8]) that the formal adjoint of a differential operator is unique, that any differential operator admits a formal adjoint, and that the symbol of the formal adjoint L^* is the adjoint of the symbol of the operator $L : \sigma(L^*)(x,\xi) = (-1)^k\sigma(L)(x,\xi)^*$ for $x \in M$, $\xi \in T^*M - \{0\}$, k the order of L. Let $\pi : T^*M \setminus \{0\} \longrightarrow M$ the projection.

Definition 4.2. Let $E_1, \ldots, E_r$ vector bundles on a manifold M. A complex of differential operators $L_i : \Gamma(E_i) \longrightarrow \Gamma(E_{i+1})$:

$$0 \longrightarrow \Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \ldots\ldots\Gamma(E_{r-1}) \xrightarrow{L_{r-1}} \Gamma(E_r) \longrightarrow 0$$

is called *elliptic* if it induces an exact sequence of vector bundles:

$$0 \longrightarrow \pi^*E_0 \xrightarrow{\sigma(L_0)} \pi^*E_1 \xrightarrow{\sigma(L_1)} \ldots\ldots\pi^*E_{r-1} \xrightarrow{\sigma(L_{r-1})} \pi^*E_r \longrightarrow 0$$

at the symbol level. A differential operator $L : \Gamma(E) \longrightarrow \Gamma(F)$ is called elliptic if the complex

$$0 \longrightarrow \Gamma(E) \longrightarrow \Gamma(F) \longrightarrow 0$$

induced by L is elliptic.

Proposition 4.3. *Let E, F and G three euclidian vector bundles on a compact oriented manifold M and $L : \Gamma(E) \longrightarrow \Gamma(F)$, $\Lambda : \Gamma(F) \longrightarrow \Gamma(G)$ two differential operators of the same order, such that $\Lambda \circ L = 0$. Then :*
(1) the operator $LL^* + \Lambda^*\Lambda$ is elliptic if and only if the symbol sequence

$$\pi^*E \xrightarrow{\sigma(L)} \pi^*F \xrightarrow{\sigma(\Lambda)} \pi^*G \tag{34}$$

is exact;
(2) the operator $L^* \oplus \Lambda : \Gamma(F) \longrightarrow \Gamma(E) \oplus \Gamma(G)$ is elliptic if and only if the sequence

$$0 \longrightarrow \pi^*E \xrightarrow{\sigma(L)} \pi^*F \xrightarrow{\sigma\Lambda} \pi^*G \longrightarrow 0 \tag{35}$$

is exact.

Proof. The symbol of the operator $P = LL^* + \Lambda^*\Lambda$ is, up to a sign, $\sigma(P) = \sigma(L)\sigma(L)^* + \sigma(\Lambda)^*\sigma(\Lambda)$. We easily see that $\ker\sigma(P) = \ker\sigma(\Lambda) \cap \ker\sigma(L)^*$. Now the hypothesis implies that $\operatorname{Im}\sigma(L) \subseteq \ker\sigma(\Lambda)$, so the cohomology H_x on the symbol level on a point $x \in M$:

$$E_x \xrightarrow{\sigma(L)_x} F_x \xrightarrow{\sigma(\Lambda)_x} G_x$$

is given by $H_x = \ker\sigma(P)_x$. Now P is elliptic if and only if $\ker\sigma(P)_x = 0$ and this happens if and only if $H_x = 0$, that is, if and only if the sequence (34) is exact.

For the second statement, the symbol of the differential operator $Q = L^* \oplus \Lambda$ is $\sigma(L)^* \oplus \sigma(\Lambda)$: the operator Q is elliptic if and only if $\sigma(Q)$ is an isomorphism. Since the hypothesis always implies that $\operatorname{Im}\sigma(L) \subseteq \ker\sigma(\Lambda)$, the condition that $\sigma(Q)$ is an isomorphism is easily equivalent to the fact that the symbol sequence (35) is exact.

□

The proposition motivates the next definition:

Definition 4.4. A linear differential operator $L : \Gamma(E) \longrightarrow \Gamma(F)$ between two euclidian vector bundles is called *underdetermined elliptic* if LL^* is elliptic, *overdetermined elliptic* if L^*L is elliptic. Equivalently, L is underdetermined elliptic if its symbol $\sigma(L)$ is surjective, overdetermined if its symbol is injective.

Let E an euclidian vector bundle on a compact oriented manifold M and let $\Gamma(E)$ its space of $\mathcal{C}^\infty$ sections. Let ∇ be a metric connection on E. If $f \in \Gamma(E)$ we define the *Sobolev norm* as: $||f||_{2,p} = \left(\sum_{i=0}^p \int_M |\nabla^i f|^2 d\mathrm{vol}_g\right)^{1/2}$. We denote with $\Gamma^2_p(E)$ the completion of $\Gamma(E)$ in the norm $||\ ||_{2,p}$. Let now $P : \Gamma(E) \longrightarrow \Gamma(F)$ a linear differential operator of order k. For all $p \geq k$ it induces a bounded operator of Hilbert spaces :

$$P_p : \Gamma^2_p(E) \longrightarrow \Gamma^2_{p-k}(F) \quad .$$

The fundamental result we will use is the following (it is nothing but one of the many versions of elliptic regularity theorem):

Theorem 4.5. *Let $P : \Gamma(E) \longrightarrow \Gamma(F)$ a linear differential operator of order k and let*

$$P_p : \Gamma^2_p(E) \longrightarrow \Gamma^2_{p-k}(F)$$

be its extension to Sobolev completions, for $p \geq k$. Then :
(1) If P is underdetermined elliptic, then for all $p \geq k$, $\ker(P^*)_p$ is finite dimensional and $\ker(P^*)_p = \ker P \subseteq \Gamma(F)$. Moreover $\Gamma(F) = P\Gamma(E) \oplus \ker P^*$ and $\Gamma^2_p(F) = P_p\Gamma^2_{p+k}(E) \oplus \ker P^*$ and the direct sums are L^2-orthogonal. In particular an underdetermined elliptic operator has closed range.
(2) If P is overdetermined elliptic, then for all p $\ker P_p$ is finite dimensional and $\ker P_p = \ker P \subset \Gamma(E)$.

For the proof see [118] or [8].

Remark 4.6. By the preceding proposition proving the surjectivity of the Sobolev extension of an underdetermined elliptic operator $P_p : \Gamma^2_p(E) \longrightarrow \Gamma^2_{p-k}(F)$ is equivalent to proving that if $u \in \Gamma(F)$ satisfies the equation $P^*u = 0$ then $u = 0$, that is a vanishing statement for a smooth solution u of the equation $P^*u = 0$.

Another fundamental result we will use is the following Unique Continuation Principle (see [46]) for solutions of elliptic differential equations of second order.

Theorem 4.7 (Unique Continuation Principle)**.** *Let M be a differential manifold and E a vector bundle on M. Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ a linear elliptic differential operator of second order with scalar symbol. Then, if $v \in \Gamma(E)$ is a solution of the second order differential equation $Pu = 0$ that vanishes on a nonempty open set of M, then v vanishes everywhere on M.*

For the proof see [1].

Corollary 4.8. *Let M be a differential manifold, E a vector bundle on M. Let $\pi : T^*M \setminus \{0\} \longrightarrow M$ the projection. Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ a linear elliptic differential operator of second order with principal symbol of the form:*

$$\sigma(P) = f\pi^*(A)$$

where $A \in \mathrm{Aut}(E)$ is an automorphism of E and $f \in \mathcal{C}^\infty(T^*M \setminus \{0\})$ is a pointwisely nonzero function. Then the Unique Continuation Principle applies to the operator P.

Proof. Consider the operator $A^{-1}P$. This is a linear elliptic differential operator of the second order with scalar symbol, and it has the same solutions of P. The Unique Continuation Principle applies then to solutions of $A^{-1}P$, hence of those of P.

□

4.2 The adjoint of the perturbed Seiberg-Witten operator

Let us consider the perturbed Seiberg-Witten functional $\tilde{\mathbb{F}}_+$ and its extension to Sobolev completions in the norm $|| \,||_{2,p}$:

$$(\tilde{\mathbb{F}}_+)^2_p : (\mathcal{A}^{U(1)}_{\det W})^2_p \times \Gamma^2_p(W_+) \times \mathrm{Sym}^+(TM,g)^2_p \longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma^2_{p-1}(W_-) \quad . \tag{36}$$

Our aim is to prove that its differential at a solution (A,ψ,id)

$$(D\tilde{\mathbb{F}}_+)^2_p : A^1(M,i\mathbb{R})^2_p \times \Gamma^2_p(W_+) \times \mathrm{sym}\,(TM,g)^2_p \longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma^2_{p-1}(W_-)$$

is surjective. By the implicit function theorem this guarantees that the zero set of the map $(\tilde{\mathbb{F}}_+)^2_p$ is a smooth Hilbert manifold. Then we could use the Sard-Smale theorem (cf [107]) to prove that for generic metrics the space of solutions $Z(F^{\varphi^* g,\varphi^*\rho})^2_p$ is a smooth Hilbert manifold. Now the fact that the $\mathcal{C}^\infty(M,S^1)$ action preserves the solutions of Seiberg-Witten equations (or in other words preserves the zeros of the Seiberg-Witten functional F) implies that at a tangential level (that is differentiating) we have the following *complex of deformations*:

$$0 \longrightarrow \Gamma(i\mathbb{R}) \longrightarrow \Gamma(\Lambda^1 T^*M \otimes i\mathbb{R}) \oplus \Gamma(W_+) \xrightarrow{D_{(A,\psi)}F} \Gamma(\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus \Gamma(W_-) \longrightarrow 0$$

where the first arrow is the differential of the $\mathcal{C}^\infty(M,S^1)$ action. This complex of differential operators happens to be elliptic (cf [91]). In particular the operator $D_{(A,\psi)}F$ is underdetermined elliptic. This means that $D_{(A,\psi)}F$ and all its Sobolev extensions $D_{(A,\psi)}F^2_p$ have closed range and finite dimensional cokernel by the elliptic regularity theorem 4.5. Moreover for all $p \geq 1$, $\mathrm{coker}\,(D_{(A,\psi)}F)^2_p = \mathrm{coker}\,D_{(A,\psi)}F = \ker D_{(A,\psi)}F^*$. Now the perturbed Seiberg-Witten operator $D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+$ is the sum

$$D_{(A,\psi,id)}\tilde{\mathbb{F}}_+ = D_{(A,\psi)}F^{g,\rho} + \frac{\partial \tilde{\mathbb{F}}_+}{\partial \varphi} \ ,$$

and hence its symbol $\sigma(D\tilde{\mathbb{F}}_+)$ is the sum of symbols: $\sigma(D\tilde{\mathbb{F}}_+) = \sigma(DF) + \sigma(\partial\tilde{\mathbb{F}}_+/\partial\varphi)$: therefore $\sigma(D\tilde{\mathbb{F}}_+)$ is surjective and the perturbed Seiberg-Witten operator $D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+$ is underdetermined elliptic. As a consequence of remark 4.6 and of theorem 4.5 proving the surjectivity of $(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^2_p$ is equivalent to proving that $\ker(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^* = 0$.

4.2.1 Computation of the adjoint operator

In this subsection we are going to compute the adjoint differential operator:

$$(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^* : A^2_+(M,i\mathbb{R}) \oplus \Gamma(W_-) \longrightarrow A^1(M,i\mathbb{R}) \oplus \Gamma(W_+) \oplus \Gamma(S^2T^*M) \quad .$$

In the sequel, when it will not cause confusion, we will drop the indication of the point (A,ψ,id) and we will simply write $D\tilde{\mathbb{F}}_+$ for $D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+$. We begin by recalling the L^2_0 norms with respect to which we are going to compute the adjoint differential operator.

On the bundles $T^*M \otimes i\mathbb{R}$ the norm is the standard norm induced by the metric g. We recall that on $T^*M^{\otimes^m}$ the metric g induces the metric $\langle x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m\rangle = m! \prod_{i=0}^m (x_i,y_i)$. With respect to this metric the decomposition $T^*M \otimes T^*M \simeq S^2T^*M \oplus \Lambda^2 T^*M$ is an orthogonal direct sum. We will take on S^2T^*M and on Λ^2T^*M the metrics induced by the metric on $T^*M \otimes T^*M$. In this way $e_i \otimes e_j = e_ie_j \oplus e_i \wedge e_j$ and $||e_i \otimes e_j|| = 2$, $||e_ie_j|| = 1 = ||e_i \wedge e_j||$. In other words the metric induced by $T^*M \otimes T^*M$ on $\mathrm{sym}\,(TM,g)$ is $(s,t) = 2\mathrm{tr}\,(st)$.

By lemma 3.7 we can identify $S^2_0T^*M$ with $\mathrm{Hom}(\Lambda^2_-T^*M, \Lambda^2_+T^*M)$ by means of the isomorphism δ, defined by $\delta(s) = P^{+,g}i(s^*)|_{\Lambda^2_+T^*M}$. We recall that δ is an isometry if we take on $\mathrm{Hom}(\Lambda^2_-T^*M, \Lambda^2_+T^*M)$ the metric $(u,v) = 1/2\mathrm{tr}\,uv^*$.

On $\Gamma(W_+)$ and on $\Gamma(W_-)$ we take the real part of the hermitian metric and on $\mathrm{Hom}(W_+, W_-)$ we will take the scalar product $(u, v) = 1/2\mathrm{tr}\,(uv^*)$. In this way the Clifford multiplication ρ will be an isometry.

Finally the real part of the hermitian metric on $\mathrm{End}(W)$ given by $(A, B) = 1/4\mathrm{tr}\,(AB^*)$ induces an orthogonal sum $\mathfrak{u}(W) \simeq i\mathbb{R} \oplus \mathfrak{su}(W)$. We will take on $i\mathfrak{su}(W)$ the real inner product induced by the real inner product just defined on $\mathrm{End}(W)$. In this way the isomorphism $\rho : \Lambda^2 T^*M \longrightarrow i\mathfrak{su}(W)$ is an isometry. In the sequel we will indicate with $(\cdot, \cdot)$ the real inner products and with $\langle \cdot, \cdot \rangle$ the hermitian ones.

We begin with the following lemma:

Lemma 4.9. *Let (M, g) a compact oriented riemannian manifold . When we identify tangent and cotangent bundle by means of the metric g, the formal adjoint of the first order differential operator*

$$\mathrm{div}\ : \Gamma(S^2T^*M) \longrightarrow \Gamma(TM)$$

is the differential operator $X \longmapsto (-1/2)L_X g$.

Proof. Let $X \in \Gamma(TM)$ a vector field. Let $k \in S^2T^*M$. We will first prove that pointwisely:

$$(L_X g, k) + 2(X, \mathrm{div}\, k) = 2\mathrm{div}\, k(X) \quad . \tag{37}$$

The two members are first order differential operators in the variable k. We will prove first that their symbols coincide. The symbol of a differential operator P of order j, as defined in [8], is

$$\sigma(P, \sigma)_{x_0}(\xi) = \frac{i^j}{j!} P(f^j \sigma)(x_0)$$

where f is a real function $f \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $f(x_0) = 0$, $df(x_0) = \xi \in T^*_{\xi_0}M$. Now take such a function f. Since $\mathrm{div}\,(fk) = k(df^\sharp) + f\mathrm{div}\, k$, we have

$$2(X, \mathrm{div}\,(fk))(x_0) = 2(X, k(\xi^\sharp)(x_0) = 2k(\xi^\sharp, X)(x_0) \quad .$$

Now

$$2\mathrm{div}\,(fk(X))(x_0) = 2g(\xi, k(X)) = 2k(\xi^\sharp, X) \quad .$$

Hence the two symbols coincide. Therefore the difference $F_X(k) := 2\mathrm{div}\, k(X) - 2(X, \mathrm{div}\, k)$ is a 0 order differential operator in k, that is, a tensor, or $\mathcal{C}^\infty(M, \mathbb{R})$-linear. It suffices then to verify that $F_X(k) = (L_X g, k)$ in a point p. Taking normal coordinates $x_1, \ldots, x_n$ (see [76], section III.8) centered in the point p, it suffices to take $k = dx_i dx_j = 1/2(dx_1 \otimes dx_2 + dx_2 \otimes dx_1)$. We immediately see that $(\mathrm{div}\, k)(p) = 0$ so

$$F_X(k)(p) = \mathrm{div}\, \left(X^i \frac{\partial}{\partial x^j} + X^j \frac{\partial}{\partial x^i} \right)(p) = \left(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^j}{\partial x^i} \right)(p) \, .$$

Let us now compute $(L_X g, k)$. We have:

$$L_X g(p) = \sum_{ij} (L_X g_{ij})(p) dx_i dx_j + \sum_{ij} g_{ij}(p)(L_X dx_i) dx_j + \sum_{ij} g_{ij}(p) dx_i (L_X dx_j) \, .$$

Since we chose normal coordinates centered in p, $(L_X g_{ij})(p) = X(g_{ij})(p) = 0$ and $g_{ij}(p) = \delta_{ij}$,

$$L_X g(p) = \sum_i (L_X dx_i) dx_i + \sum_i dx_i (L_X dx_i) \, .$$

By the Lie-Cartan formula $L_X = di_X + i_X d$ we immediately see that $L_X dx_i = dX^i$. As a consequence:

$$L_X g(p) = \frac{1}{2} \sum_{ij} \left(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^j}{\partial x^i} \right) dx_i dx_j$$

and its scalar product with $k = dx_i dx_j$ is

$$(L_X g, k) = \left(\frac{\partial X^i}{\partial x^j} + \frac{\partial X^j}{\partial x^i} \right)$$

which is exactly $F_X(k)$. We have then established 37. Integrating on the manifold and recalling that, by Green theorem (see [76]), the integral of the divergence of a vector field on a compact oriented riemannian manifold vanishes, we get:

$$(L_X g, k)_{L^2} = \int_M (L_X g, k)\omega_g = -2 \int_M (X, \operatorname{div} k)\omega_g = -2(X, \operatorname{div} k)_{L^2}$$

which establishes the lemma.

□

The isomorphism $\rho : TM \otimes \mathbb{C} \longrightarrow \mathrm{Hom}(W_+, W_-)$ allows us to define a complex conjugation in $\mathrm{Hom}(W_+, W_-)$ in an obvious way, and hence a real and imaginary part for elements of $\mathrm{Hom}(W_+, W_-)$. We will set: $\overline{\rho(\sigma)} := \rho(\bar{\sigma})$ and hence $\mathrm{Re}\, \rho(\sigma) = \rho(\mathrm{Re}\, \sigma)$, $\mathrm{Im}\rho(\sigma) = \rho(\mathrm{Im}\sigma)$, for $\sigma \in T^*M \otimes \mathbb{C}$.

We are now ready to compute the operator $(D\tilde{\mathbb{F}}_+)^* = DF^* + (\partial\tilde{\mathbb{F}}_+/\partial\varphi)^*$. Let us begin with $(DF)^*$.

Lemma 4.10. *The adjoint of the map: $j_\psi : A^1(M, \mathbb{C}) \longrightarrow \Gamma(W_-)$ defined by $j_\psi(\sigma) = \rho(\sigma)\psi$ is given by the map: $j_\psi^* : \Gamma(W_-) \longrightarrow A^1(M, \mathbb{C})$ defined by $j_\psi^*(\chi) = 2\psi^* \otimes \chi$.*

Proof. It is always true that, for an hermitian map A between two vector spaces one has: $\langle Ax, y \rangle = \mathrm{tr}\,(A \circ y^* \otimes x)$, where $y^* = \langle -, y \rangle$. Therefore :

$$\langle \rho(\sigma)\psi, \chi \rangle \quad = \quad \mathrm{tr}\,(\rho(\sigma) \circ \chi^* \otimes \psi) = \mathrm{tr}\,(\rho(\sigma) \circ (\psi^* \otimes \chi)^*) = 2\langle \rho(\sigma), \psi^* \otimes \chi \rangle_{\mathrm{Hom}(W_+, W_-)}$$

because of the choice of the hermitian metric in $\mathrm{Hom}(W_+, W_-)$. Since ρ is an isometry with respect to the taken norms, identifying $\mathrm{Hom}(W_+, W_-)$ with $T \otimes \mathbb{C}$ we get :

$$\langle \rho(\sigma)\psi, \chi \rangle = 2\langle \sigma, \psi^* \otimes \chi \rangle_{T^* \otimes \mathbb{C}}$$

□

Lemma 4.11. *The adjoint of the operator $d^+ : A^1(M, i\mathbb{R}) \longrightarrow A^2_+(M, i\mathbb{R})$ is the operator*

$$d^* : A^2_+(M, i\mathbb{R}) \longrightarrow A^1(M, i\mathbb{R})\ .$$

Proof. $(d^+\tau, \theta)_{A^2_+(M,i\mathbb{R})} = (d^+\tau, \theta)_{A^2(M,i\mathbb{R})}$ because $\Lambda^2 T^*M \simeq \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$ is an orthogonal direct sum. Then

$$(d^+\tau, \theta)_{A^2_+(M,i\mathbb{R})} = (d\tau, \theta)_{A^2(M,i\mathbb{R})} = (\tau, d^*\theta)_{A^1(M,i\mathbb{R})}$$

□

Since the Dirac operator splits in $D_A = D_A^+ \oplus D_A^-$ and D_A^+ and D_A^- are the adjoint of one another we immediately get:

Lemma 4.12. *The adjoint of the operator $D_A : \Gamma(W_+) \longrightarrow \Gamma(W_-)$ is the operator*

$$D_A : \Gamma(W_-) \longrightarrow \Gamma(W_+)\ .$$

We now compute the adjoint of the term $q_\psi(\phi) = [\phi^* \otimes \psi + \psi^* \otimes \phi]_0$:

Lemma 4.13. *The adjoint of the map: $q_\psi : \Gamma(W_+) \longrightarrow A^2_+(M, i\mathbb{R})$ defined by $q_\psi(\phi) = [\phi^* \otimes \psi + \psi^* \otimes \phi]_0$ is the operator $q_\psi^* : A^2_+(M, i\mathbb{R}) \longrightarrow \Gamma(W_+)$ given by: $q_\psi^*(\theta) = -1/2\ \rho(\theta)\psi$.*

Proof. We begin by proving that if $f \in i\mathfrak{su}(W_+)$, that is, if f is a traceless hermitian endomorphism of W_+, then for all $\varphi \in W_+$ we have pointwisely:

$$\langle f, (\varphi^* \otimes \varphi)_0 \rangle = \frac{1}{4} \langle f\varphi, \varphi \rangle \quad . \tag{38}$$

Indeed, we can always suppose $|\varphi| = 1$; so we can consider an orthonormal basis ψ_1, ψ_2 of W_+ with $\psi_1 = \varphi$. We can express f in this basis as

$$f = \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix} \quad .$$

Then $f\varphi = a$ and

$$(\varphi^* \otimes \varphi)_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Then, recalling that the metric on $i\mathfrak{su}(W_+)$ is given by $1/2 \mathrm{tr}\, uv^*$, we get (38). Now differentiating (38) with respect to φ and taking the real part we get:

$$\begin{aligned} \mathrm{Re}\, \langle f, [\phi^* \otimes \psi + \psi^* \otimes \varphi]_0 \rangle &= \frac{1}{4} \mathrm{Re}\, [\langle f\phi, \psi \rangle + \langle f\psi, \phi \rangle] \\ &= \frac{1}{2} \mathrm{Re}\, \langle f\psi, \phi \rangle \end{aligned}$$

because $\langle f\phi, \psi \rangle = \langle \phi, f^*\psi \rangle = \langle \phi, f\psi \rangle$. Taking now $f = \rho(\theta)$ and identifying $i\mathfrak{su}(W_+)$ with $\Lambda^2_+ T^*M$ we get the lemma.

□

We now compute the adjoint $(\partial\tilde{\mathbb{F}}_+/\partial\varphi)^*$ of the partial differential that gives the contribution of the metric.

Lemma 4.14. *The adjoint of the operator* $s \longrightarrow P^{+,g} i(s^*)|_{\Lambda^2_-} : \Gamma(S^2T^*) \longrightarrow A^2_+(M, i\mathbb{R})$ *is the linear map:* $A^2_+(M, i\mathbb{R}) \longrightarrow \Gamma(S^2T^*)$ *given by* $\theta \longrightarrow 1/4(F^+_A, \theta)g + 2(F^-_A)^* \otimes \theta$ *where the term* $(F^-_A)^* \otimes \theta$ *is seen in* $\mathrm{Hom}(\Lambda^2_- T^*M, \Lambda^2_+ T^*M) \simeq S^2_0 T^*M$.

Proof. By lemmas 3.7 and 3.8 we have: $P^{+,g} i(s^*) F_A = (\mathrm{tr}\, s) F^+_A + \delta(s^*_0) F^-_A$. Now

$$\delta : S^2_0 T^*M \longrightarrow \mathrm{Hom}(\Lambda^2_- T^*M, \Lambda^2_+ T^*M)$$

is an isometry, with the chosen metrics. Then

$$\begin{aligned} (P^{+,g} i(s^*) F_A, \theta) &= ((\mathrm{tr}\, s) F^+_A, \theta) + (\delta(s^*_0) F^-_A, \theta) \\ &= (\mathrm{tr}\, s)(F^+_A, \theta) + 2(\delta(s^*_0), (F^-_A)^* \otimes \theta) \\ &= \frac{1}{2}(F^+_A, \theta)(s, g) + 2(s^*_0, (F^-_A)^* \otimes \theta) \\ &= \frac{1}{2}(F^+_A, \theta)(s, g) + 2(s, (F^-_A)^* \otimes \theta) \end{aligned}$$

□

Finally the following three lemmas allow us to compute the adjoint $(\partial\tilde{\mathbb{F}}_+/\partial\varphi)^*$. We denote with $(\nabla\psi)^*$ the linear map $TM \longrightarrow W^*_+$ defined by $X \longrightarrow \langle -, \nabla\psi \rangle$, with $\mathrm{Re}\, (\nabla\psi^* \otimes \chi)$ the 2-tensor $(X, Y) \longmapsto (\mathrm{Re}\, (\nabla_X \psi^* \otimes \chi), Y)_{TM}$ and with $\mathrm{SymRe}\, (\nabla\psi^* \otimes \chi)$ the symmetric part of $\mathrm{Re}\, (\nabla\psi^* \otimes \chi)$.

Lemma 4.15. *Th adjoint of the map:* $\mathrm{sym}\, (TM, g) \longrightarrow \Gamma(W_-)$ *defined by* $s \longmapsto -\rho \circ s^* \circ \nabla\psi^*$ *is the map:* $\Gamma(W_-) \longrightarrow \Gamma(\mathrm{sym}\, (TM, g))$ *defined by:* $\chi \longmapsto -\mathrm{SymRe}\, \nabla\psi^* \otimes \chi$.

Proof. Taking a local orthormal frame $e^1, \dots, e^4$ of T^*M it is easy to see that $\rho \circ s^* \circ \nabla\psi$ is given by $\sum_i \rho(s^* \otimes \mathrm{id})(e^i \otimes \nabla_{e^i}\psi) = \sum_i \rho(s^* e^i)\nabla_{e^i}\psi$. In the same way we can see that SymRe $\nabla\psi^* \otimes \chi$ is given by $\sum_i e^i \otimes \mathrm{Re}\, \nabla_{e^i}\psi^* \otimes \chi$. Therefore

$$\begin{aligned}
\mathrm{Re}\,\langle \rho \circ s^* \circ \nabla\psi^*, \chi\rangle &= \sum_i \mathrm{Re}\,\langle \rho(s^* e^i)\nabla_{e^i}\psi, \chi\rangle = 2\sum_i (s^* e^i, \mathrm{Re}\,\nabla_{e^i}\psi^* \otimes \chi)_{T^*M} \\
&= \sum_i (s^*, e_i \otimes \mathrm{Re}\,\nabla_{e_i}\psi^* \otimes \chi\rangle_{T^*\otimes T} = (s^*, \mathrm{Sym}(\sum_i e_i \otimes \mathrm{Re}\,\nabla_{e_i}\psi^* \otimes \chi))_{\mathrm{sym}\,(T^*M,g)} \\
&= (s, \mathrm{SymRe}\,\nabla\psi^* \otimes \chi)_{\mathrm{sym}\,(TM,g)}
\end{aligned}$$

□

Lemma 4.16. *The adjoint of the differential operator* $\Gamma(S^2T^*M) \longrightarrow \Gamma(W_-)$ *defined as* $s \longmapsto \rho(d\mathrm{tr} s)\psi$ *is given by the differential operator* $\chi \longmapsto 1/4\, d^*\mathrm{Re}\,(\psi^* \otimes \chi)g$.

Proof. The operator $s \longrightarrow \rho(d\mathrm{tr}\, s)\psi$ is the composition $j_\psi \circ i_\mathbb{R} \circ d \circ \mathrm{tr}$, where $i_\mathbb{R}$ is the injection

$$i_\mathbb{R} : A^1(M, \mathbb{R}) \longrightarrow A^1(M, \mathbb{C}) \; .$$

As a consequence its adjoint is $\mathrm{tr}^* \circ d^* \circ i_\mathbb{R}^* \circ j_\psi^*$. Then

$$\mathrm{tr}^* \circ d^* \circ i_\mathbb{R}^* \circ j_\psi^*(\chi) = 2\mathrm{tr}^* \circ d^*\mathrm{Re}\,(\psi^* \otimes \chi) = \frac{d^*\mathrm{Re}\,(\psi^* \otimes \chi)}{2} g$$

since the adjoint of $i_\mathbb{R}$ is exactly the real part Re .

□

Lemma 4.17. *The adjoint of the differential operator* $\Gamma(S^2T^*M) \longrightarrow \Gamma(W_-)$ *defined as* $s \longmapsto \rho(\mathrm{div} s)\psi$ *is the differential operator:* $\chi \longrightarrow 1/2\, L_{\mathrm{Re}\,(\psi^*\otimes\chi)} g$.

Proof. The differential operator $s \longmapsto \rho(\mathrm{div}\, s)\psi$ is the composition: $j_\psi \circ i_\mathbb{R} \circ \mathrm{div}$. Therefore its adjoint is

$$\chi \longrightarrow \mathrm{div}^* \circ i_\mathbb{R}^* \circ j_\psi^*(\chi) = -\frac{1}{2} L_{\mathrm{Re}\,(\psi^*\otimes\chi)} g \quad .$$

□

We are finally ready to write down the adjoint of the perturbed Seiberg-Witten operator:

Proposition 4.18. *The adjoint* $(D\tilde{\mathbb{F}}_+)^*$ *of the perturbed Seiberg-Witten operator in the point* (A, ψ, id), $D\tilde{\mathbb{F}}_+$, *is given by the differential operator:* $\Gamma(W_-) \oplus A_+^2(M, i\mathbb{R}) \longrightarrow A^1(M, i\mathbb{R}) \oplus \Gamma(W_+) \oplus \Gamma(S^2T^*M)$ *given by:*

$$(D\tilde{\mathbb{F}}_+)^*(\chi, \theta) = \begin{pmatrix} d^*\theta + 2i\mathrm{Im}(\psi^* \otimes \chi) \\ \\ D_A\chi - \frac{1}{2}\rho(\theta)\psi \\ \\ -\mathrm{sym}\,\mathrm{Re}\,(\nabla\psi^* \otimes \chi) - \dfrac{1}{2} L_{\mathrm{Re}\,(\psi^*\otimes\chi)} g - \dfrac{d^*\mathrm{Re}\,(\psi^* \otimes \chi)}{2} g - \dfrac{1}{4}(F_A^+, \theta)g - (F_A^-)^* \otimes \theta \end{pmatrix}$$

4.3 The kernel equations $(D\tilde{\mathbb{F}}_+)^*(\chi, \theta) = 0$

We have seen in subsection 4.1 that the cokernel of the operator $(D\tilde{\mathbb{F}}_+)_p^2$ coincides with the kernel of the formal adjoint $(D\tilde{\mathbb{F}}_+)^*$. From now on we will always consider the differential $D\tilde{\mathbb{F}}_+$ and its formal adjoint over a point (A, ψ, g) solution to the parametrized Seiberg-Witten equations (23) and we will always suppose that the monopole (A,ψ) is irreducible, that is $\psi \neq 0$. In particular this means that we will always suppose that $D_A\psi = 0$; moreover, by the unique continuation principle we can argue that ψ never vanishes on an open subset; therefore $M - Z(\psi) = \{x \in M | \psi(x) \neq 0\}$ is dense in M. From what we have said a solution u to the equation $(D\tilde{\mathbb{F}}_+)^* u = 0$ represents an obstruction to the transversality of the map $(\tilde{\mathbb{F}}_+)^2_p$ on the point (A,ψ,id). In this subsection we begin the study of the equation $(D\tilde{\mathbb{F}}_+)^*(\chi,\theta) = 0$ and we will prove some properties of the possible solutions. Unluckily in the general case we do not get to prove a general vanishing theorem for these solutions.

Let us write down the equations for the kernel of $(D\tilde{\mathbb{F}}_+)^*$:

$$d^*\theta + 2i\mathrm{Im}(\psi^* \otimes \chi) = 0 \tag{39a}$$

$$D_A\chi - \frac{1}{2}\rho(\theta)\psi = 0 \tag{39b}$$

$$-\mathrm{sym}\,\mathrm{Re}\,(\nabla\psi^* \otimes \chi) - \frac{1}{2}L_{\mathrm{Re}\,(\psi^*\otimes\chi)}g - \frac{d^*\mathrm{Re}\,(\psi^* \otimes \chi)}{2}g - \frac{1}{4}(F_A^+,\theta)g - (F_A^-)^* \otimes \theta = 0 \tag{39c}$$

where (A,ψ,id) satisfies : $\tilde{\mathbb{F}}_+(A,\psi,\mathrm{id}) = 0$ and $\psi \neq 0$, that is (A,ψ) is an irreducible monopole. We will first analyse equations (39a) and (39b).

Lemma 4.19. *Let $\varphi \in \Gamma(W_+)$ a positive spinor and $\zeta \in \Gamma(W_-)$ a negative spinor. Then*

$$\mathrm{Re}\,(\varphi^* \otimes \zeta)\varphi = \frac{|\varphi|^2}{2}\zeta\ , \quad \mathrm{Im}(\varphi^* \otimes \zeta)\varphi = \frac{|\varphi|^2}{2i}\zeta \quad .$$

Proof. It is clear that it is sufficient to prove the lemma in each point. Now, for all $x \in M$, $\rho_x : T_xM \otimes \mathbb{C} \longrightarrow \mathrm{Hom}(W_{+,x}, W_{-,x})$ induces an irreducible spin representation and we know by proposition 1.6 that all irreducible spin representations are isomorphic, so it is sufficient to prove the result of the lemma for a two dimensional complex hermitian vector space E equipped with the irreducible spin representation (8), $E \longrightarrow \mathrm{End}(\Lambda^*_{\mathbb{C}}E, \Lambda^*_{\mathbb{C}}E)$ given by $x \longrightarrow x \wedge (\cdot) - x \lrcorner (\cdot)$. We have $W_+ \simeq \mathbb{C} \oplus \Lambda^2_{\mathbb{C}}E$ and $W_- \simeq E$. Let us take an hermitian basis e_1, e_2 on E. Then e_1, ie_1, e_2, ie_2 will be an oriented orthonormal basis for the underlying real euclidien space $E_{\mathbb{R}}$. An orthonormal basis for W_+ is $1, e_1 \wedge e_2$ and an orthonormal basis for W_- is obviously e_1, e_2. Then we have :

$$\rho(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \rho(ie_1) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$
$$\rho(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho(ie_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Extending now ρ to $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$ linearity, we easily get for $a,b,c,d \in \mathbb{C}$,

$$\rho(ae_1 + b(ie_1) + ce_2 + d(ie_2)) = \begin{pmatrix} a+ib & c-id \\ c+id & -a+ib \end{pmatrix} \ , \quad \rho(\bar{a}e_1 + \bar{b}(ie_1) + \bar{c}e_2 + \bar{d}(ie_2)) = \begin{pmatrix} \bar{a}+i\bar{b} & \bar{c}-i\bar{d} \\ \bar{c}+i\bar{d} & -\bar{a}+i\bar{b} \end{pmatrix}$$

which means that if

$$\rho(Z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ then

$$\overline{\rho(Z)} := \rho(\bar{Z}) = \begin{pmatrix} -\bar{\delta} & \bar{\gamma} \\ \bar{\beta} & -\bar{\alpha} \end{pmatrix} \quad ;$$

therefore

$$\mathrm{Re}\,\rho(Z) = \frac{1}{2}\rho(Z+\bar{Z}) = \frac{1}{2}\begin{pmatrix} \alpha-\bar{\delta} & \beta+\bar{\gamma} \\ \gamma+\bar{\beta} & \delta-\bar{\alpha} \end{pmatrix} \ , \quad \mathrm{Im}\rho(Z) = \frac{1}{2i}\rho(Z-\bar{Z}) = \frac{1}{2i}\begin{pmatrix} \alpha+\bar{\delta} & \beta-\bar{\gamma} \\ \gamma-\bar{\beta} & \delta+\bar{\alpha} \end{pmatrix} \ .$$

Let now $\varphi = \varphi_1 + \varphi_2 e_1 \wedge e_2$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2$. Then

$$\varphi^* \otimes \zeta = (\bar{\varphi_1}, \bar{\varphi_2}) \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \bar{\varphi_1}\zeta_1 & \bar{\varphi_2}\zeta_1 \\ \bar{\varphi_1}\zeta_2 & \bar{\varphi_2}\zeta_2 \end{pmatrix} , \quad \mathrm{Re}\,(\varphi^* \otimes \zeta) = \frac{1}{2} \begin{pmatrix} \bar{\varphi_1}\zeta_1 - \varphi_2\bar{\zeta_2} & \bar{\varphi_2}\zeta_1 + \varphi_1\bar{\zeta_2} \\ \bar{\varphi_1}\zeta_2 + \varphi_2\bar{\zeta_1} & \bar{\varphi_2}\zeta_2 - \varphi_1\bar{\zeta_1} \end{pmatrix} .$$

As a consequence

$$\mathrm{Re}\,(\varphi^* \otimes \zeta)\varphi = \frac{|\varphi_1|^2 + |\varphi_2|^2}{2} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \frac{|\varphi|^2}{2}\zeta .$$

Now, since $\varphi^* \otimes \zeta = \mathrm{Re}\,(\varphi^* \otimes \zeta) + i\mathrm{Im}(\varphi^* \otimes \zeta)$ from the evaluation of $\varphi^* \otimes \zeta$ and $\mathrm{Re}\,(\varphi^* \otimes \zeta)$ on φ we immediately get

$$\mathrm{Im}(\varphi^* \otimes \zeta)\varphi = \frac{|\varphi|^2}{2i}\zeta .$$

$\square$

Lemma 4.20. *The bundle map: $j_\psi : \Lambda^2 T^*M \otimes i\mathbb{R} \longrightarrow W_+$ defined by $j_\psi(\theta) = \rho(\theta)\psi$ is injective if $\psi \neq 0$.*

Proof. We know that $\Lambda^2 T^*M \otimes i\mathbb{R} \simeq i\mathfrak{su}(W_+)$. We place ourselves in a neighbourhood U of a point p such that $\psi(x) \neq 0$ for all $x \in U$. Shrinking U if necessary we can always find a local orthonormal frame ψ_1, ψ_2 on W_+ such that $\psi_1 = \psi/|\psi|$. An element $\rho(\theta)$ can be written, with respect to the basis ψ_1, ψ_2, as:

$$\rho(\theta) = \begin{pmatrix} \alpha & b \\ \bar{b} & -\alpha \end{pmatrix} \quad \alpha \in \mathbb{R}\,,\ \beta \in \mathbb{C}\,.$$

Now $\rho(\theta)\psi = |\psi|(\alpha, -\bar{b})$, which is zero if and only if $\alpha = 0 = b$.

$\square$

Since the operator $D\tilde{\mathbb{F}}_+$ is underdetermined elliptic, by proposition 4.3 its adjoint $(D\tilde{\mathbb{F}}_+)^*$ is overdetermined elliptic and $\ker(D\tilde{\mathbb{F}}_+)^* = \ker D\tilde{\mathbb{F}}_+(D\tilde{\mathbb{F}}_+)^*$. Now, always by proposition 4.3, $D\tilde{\mathbb{F}}_+(D\tilde{\mathbb{F}}_+)^*$ is elliptic.

Lemma 4.21. *The Unique Continuation Principle applies to the operator $DF \circ DF^*$, where F is the Seiberg-Witten functional.*

Proof. By lemma 4.8 it suffices to show that the operator $DF \circ DF^*$ has scalar symbol, up to an automorphism of $i\Lambda^2_+ T^*M \oplus W_-$. We recall that F is the functional:

$$\begin{aligned} F : \mathcal{A}_L \times \Gamma(W_+) &\longrightarrow iA_2^+(M) \times \Gamma(W_-) \\ (A, \psi) &\longrightarrow (F_A^+ - [\psi^* \otimes \psi]_0, D_A\psi) \end{aligned}$$

The differential DF is of the form:

$$DF_{(A,\psi)}(\tau, \phi) = \begin{pmatrix} d^+\tau & -q_\psi(\phi) \\ \frac{1}{2}\rho(\tau)\psi & +D_A\phi \end{pmatrix} \qquad \tau \in iA^1(M),\ \phi \in \Gamma(W_+)\,,$$

where the terms out of the diagonal are of order zero. The operator DF has then the same symbol of the operator:

$$P := \begin{pmatrix} d^+ & 0 \\ 0 & D_A \end{pmatrix}$$

whose adjoint is

$$P^* = \begin{pmatrix} d^* & 0 \\ 0 & D_A \end{pmatrix} .$$

The symbol of the operator $DF \circ DF^*$ is then the same of the operator

$$PP^* = \begin{pmatrix} d^+d^* & 0 \\ 0 & D_A^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Delta & 0 \\ 0 & D_A^2 \end{pmatrix}$$

because $d^+d^* = 1/2\Delta$ on self-dual 2-forms. Now, by Weitzenböck formula, D_A^2 has the same symbol of the operator $\nabla_W \circ \nabla_W^*$, which is a laplacian, and then has scalar symbol

$$\sigma(\nabla_W \circ \nabla_W^*)(\xi)(\varphi) = -|\xi|^2\varphi \; .$$

We deduce that the principal symbol of the operator $DF \circ DF^*$ is

$$\sigma(DF \circ DF^*)(\theta, \varphi) = -|\xi|^2 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \; .$$

□

Proposition 4.22. *Let (χ, θ) a solution to (39a), (39b) and suppose that $\chi = 0$ on a nonempty open set U. Then (χ, θ) vanish identically on M. The same statement holds if $\theta = 0$ on a nonempty open set.*

Proof. If $\chi = 0$ on U then equations (39a), (39b) on U become:

$$d^*\theta = 0$$
$$\rho(\theta)\psi = 0$$

By a preceding lemma, the operator $\theta \longmapsto \rho(\theta)\psi$ is injective if $\psi \neq 0$. Therefore $\theta = 0$ on $U - Z(\psi)$ which is a nonempty open set because $M - Z(\psi)$ is dense. We then conclude by the unique continuation principle, since $(\chi, \theta) = 0$ on the nonempty open subset $U - Z(\psi)$.

If $\theta = 0$ on U then equations (39a), (39b) on U become:

$$\mathrm{Im}(\psi^* \otimes \chi) = 0$$
$$D_A\chi = 0$$

Then lemma 4.19 implies $\chi = 0$ on $U - Z(\psi)$ and we conclude again by the unique continuation principle.

□

Now we pass to analyse the third equation (39c).

Lemma 4.23. *Let $\varphi \in \Gamma(W_+)$ a positive spinor and $\zeta \in \Gamma(W_-)$ a negative spinor. Then*

$$2\mathrm{div}\,(\varphi^* \otimes \zeta) = -\langle D_A\zeta, \varphi\rangle + \langle \zeta, D_A\varphi\rangle \; .$$

In particular for a monopole (A, ψ) we get :

$$\mathrm{div}\,\mathrm{Re}\,(\psi^* \otimes \chi) = -1/2\,\mathrm{Re}\,\langle \psi, D_A\chi\rangle \; , \quad \mathrm{div}\,\mathrm{Im}(\varphi^* \otimes \chi) = -1/2\,\mathrm{Im}\langle \psi, D_A\chi\rangle \; .$$

Proof. We begin by remarking that the two sides are first order differential operators in the variable ζ with the same symbol, since $-\langle \rho(\xi)\zeta, \varphi\rangle = 2\langle \varphi^* \otimes \zeta, \rho(\xi)\rangle$ because $\rho(\xi)^* = -\rho(\xi)$. There difference is therefore $\mathcal{C}^\infty$ linear. We will now see that they coincide in every point p. Taking an adapted frame e_i for T^*M centered in the point p (that is a frame e_i that $(\nabla e_i)(p) = 0$) we can write:

$$\begin{aligned} -\langle D_A\zeta, \varphi\rangle + \langle \zeta, D_A\varphi\rangle \; (p) &= \sum_i \left[\langle \rho(e_i)\nabla_{e_i}\zeta, \varphi\rangle - \langle \zeta, \rho(e_i)\nabla_{e_i}\varphi\rangle\right](p) \\ &= -\sum_i d_i\langle \rho(e_i)\zeta, \varphi\rangle(p) \\ &= 2d_i\sum_i\langle \varphi^* \otimes \zeta, e_i\rangle(p) = 2\mathrm{div}\,(\varphi^* \otimes \zeta) \end{aligned}$$

where we indicated with $d_if = (df)(e_i)$, for a complex function f on M.

□

We will now apply the trace operator to the third equation (39c).

Lemma 4.24. *Let* (A,ψ) *a monopole. Then* $\mathrm{tr}\,\mathrm{SymRe}\,(\nabla\psi^*\otimes\chi)=0$*,* $\mathrm{tr}\,L_{\mathrm{Re}\,(\psi^*\otimes\chi)}g = 2\mathrm{div}\,\mathrm{Re}\,(\psi^*\otimes\chi)$*.*

Proof. Let us prove the first. Taking a local orthonormal frame e_i in TM and its dual e^i in T^*M we obtain:

$$\begin{aligned}
\mathrm{tr}\,\mathrm{SymRe}\,(\nabla\psi^*\otimes\chi) &= \mathrm{tr}\,\mathrm{Re}\,(\nabla\psi^*\otimes\chi) = \sum_i \mathrm{tr}\,e^i\otimes\mathrm{Re}\,(\nabla_i\psi^*\otimes\chi)\\
&= \sum_i \mathrm{Re}\,\langle e^i, \mathrm{Re}\,(\nabla_i\psi^*\otimes\chi)\rangle = \sum_i \mathrm{Re}\,\langle e^i, \nabla_i\psi^*\otimes\chi\rangle\\
&= \frac{1}{2}\sum_i \mathrm{Re}\,\langle e^i\nabla_i\psi^*,\chi\rangle = \mathrm{Re}\,\langle D_A\psi,\chi\rangle
\end{aligned}$$

because (A,ψ) is a monopole. For the second identity, it is easy to see (in normal coordinates) that $L_Xg = 2\mathrm{Sym}\nabla X^\flat$, where $X^\flat$ is the 1 form $X^\flat = g(-,X)$. Therefore $\mathrm{tr}\,L_Xg = 2\mathrm{tr}\,\mathrm{Sym}\nabla X^\flat = 2\mathrm{tr}\,\nabla X^\flat = 2\mathrm{div}\,X$.

□

Proposition 4.25. *If* (χ,θ) *is a solution of the equations (39)* $(D\tilde{\mathbb{F}}_+)^*(\chi,\theta)=0$*, then*

$$(\theta, F_A^+) = 0 = \langle D_A\chi,\psi\rangle = 0 = \mathrm{div}\,(\psi^*\otimes\chi)$$

Proof. Applying the trace operator to the equation (39c) we get, by lemma 4.24:

$$3\mathrm{div}\,\mathrm{Re}\,(\psi^*\otimes\chi) + (F_A^+,\theta) = 0\ . \tag{40}$$

Applying the operator d^* to the first equation we immediately get $d^*\mathrm{Im}(\psi^*\otimes\chi) = -\mathrm{div}\,\mathrm{Im}(\psi^*\otimes\chi) = 0$ which means that, by lemma 4.23, $\mathrm{Im}\langle\psi, D_A\chi\rangle = 0$. Now taking the scalar product with χ in the second equation we get:

$$\langle\psi, D_A\chi\rangle - \frac{1}{2}\langle\psi,\rho(\theta)\psi\rangle = 0$$

which becomes

$$\langle\psi, D_A\chi\rangle - \langle\psi^*\otimes\psi,\rho(\theta)\rangle = 0$$

but $\langle\psi^*\otimes\psi,\rho(\theta)\rangle = \langle[\psi^*\otimes\psi]_0,\rho(\theta)\rangle$ and $[\psi^*\otimes\psi]_0 = F_A^+/2$, because (A,ψ) is a monopole, then we can write:

$$\langle\psi, D_A\chi\rangle - \frac{1}{2}\langle F_A^+,\theta\rangle = 0\ . \tag{41}$$

Since $\langle\psi, D_A\chi\rangle$ is real, we have $\mathrm{div}\,\mathrm{Re}\,(\psi^*\otimes\chi) = -1/2\,\langle\psi, D_A\chi\rangle$ by lemma 4.23, therefore equation 40 becomes:

$$-\frac{3}{2}\langle\psi, D_A\chi\rangle + (F_A,\theta) = 0$$

which coupled with (41) gives the result.

□

Remark 4.26. We could have remarked that the trace of equation (39c) carries the contribution of conformal perturbation of the metric. The kernel of the adjoint to the Seiberg-Witten operator with conformal perturbations (33) corresponds to equation (39a), (39b) and the trace of (39c).

We can now simplify equations (39).

Corollary 4.27. *The equations (39), $(D\tilde{\mathbb{F}}_+)^*(\chi,\theta)=0$ on the monopole (A,ψ) are equivalent to:*

$$d^*\theta + 2i\mathrm{Im}(\psi^*\otimes\chi) = 0 \tag{42a}$$

$$D_A\chi - \frac{1}{2}\rho(\theta)\psi = 0 \tag{42b}$$

$$-\mathrm{sym}\,\mathrm{Re}\,(\nabla\psi^*\otimes\chi) - \frac{1}{2}L_{\mathrm{Re}\,(\psi^*\otimes\chi)}g - (F_A^-)^*\otimes\theta = 0 \tag{42c}$$

$$(\theta, F_A^+) = 0 \tag{42d}$$

$$\mathrm{div}\,(\psi^*\otimes\chi) = 0 \tag{42e}$$

4.4 Eichhorn and Friedrich's argument

We now explain why we find Eichhorn and Friedrich's treatment of the question of generic metrics transversality for Seiberg-Witten equations unsatisfactory, and finally incorrect. The two authors in [31] and Friedrich alone in [49] have a direct approach of the problem, that is, they try to prove directly that the differential $D\mathbb{F}$ of the perturbed Seiberg-Witten functional is surjective. A first source of unclearness is that they never give a precise expression of the variation of the Dirac operator, which we have seen as being a fundamental difficulty in the question. No mention is made about the term $\rho\circ s\circ\nabla\psi$. In [49], Friedrich deals with the variation of the Dirac equation with respect to conformal perturbations ($\in \mathcal{C}^\infty(M,\mathbb{R})$) independently from the variation of the curvature equation with respect to volume preserving perturbations ($\in \mathrm{sym}_0(TM,g)$), but this is not allowed, since the differential

$$\frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi} : \mathcal{C}^\infty(M,\mathbb{R})\oplus\mathrm{sym}_0(TM,g) \longrightarrow \Gamma(W_-)\oplus A^2_+(M,i\mathbb{R})$$

is not in diagonal form, nor in (lower or upper) triangular form, as we have seen. Thanks to this uncorrect argument, which is explained only in [49], in both publications the authors get to the two independent conditions:

$$\langle\frac{d}{dg}(*_g)(s_0)F_A,\theta\rangle = 0 \qquad \forall s_0\in\mathrm{sym}_0(TM,g)$$

$$\langle\rho(df)\psi,\chi\rangle = 0 \qquad \forall f\in\mathcal{C}^\infty(M,\mathbb{R})$$

which are to be satisfied by an element (χ,θ) in the cokernel of $D\tilde{\mathbb{F}}$. They claim that these two conditions are sufficient to determine the vanishing of χ and θ. This is not true, as we shall see. Indeed, if it is true that the first condition is equivalent to the equation $(F_A^-)^*\otimes\theta = 0$, and this implies the vanishing of θ if $F_A^-\neq 0$, on the other hand the second condition is equivalent to $d^*\mathrm{Re}\,(\psi^*\otimes\chi)=0$, which means only that the vector field $\mathrm{Re}\,(\psi^*\otimes\chi)$ has zero divergence. Therefore for every zero divergence nontrivial real vector field X on the manifold M, if we take $\chi=\rho(X)\psi$, then $\chi\neq 0$, but χ verifies the condition above. Actually the two authors restrict theirselves to perturbations of the metric s with $\mathrm{div}\,s=0$, in order to get rid of the term $\rho(\mathrm{div}\,s)\psi$ in the variation of the Dirac operator. But, if we want a zero-divergence tensor of the form $f\mathrm{id}_T$, this implies $df=0$ and the second condition is not significative, that is, all χ verify it.

5 Transversality over Kähler monopoles

In this section we will specialize our study of the perturbed Seiberg-Witten functional in the simpler case of Kähler surfaces. It is well known that on a Kähler surface Seiberg-Witten equations simplify dramatically: as a consequence our study of equations (42) will be greatly simplified in the Kähler context as well. The main motivation for this simplification is that the solutions of Seiberg-Witten equations on a Kähler surface define *holomorphic pairs* $(\bar{\partial}_A, \alpha)$, where the $\bar{\partial}_A$ is a holomorphic structure on a line bundle N and α is a holomorphic section of $(N, \bar{\partial}_A)$. This fact allows the algebro-geometric interpretation of the Seiberg-Witten moduli space as the moduli space of holomorphic pairs $(\bar{\partial}_A, \alpha)$, or with the Hilbert scheme of effective divisors D on M with fixed fundamental class $c_1(D)$=$c_1(N)$ (see [119], [99], [91], [47], [48]).

Another main simplification is due to the fact that, thanks to the presence of an (almost) complex structure J on the manifold M, we can split the space of symmetric 2-tensors sym (TM, g) in J-linear and J-antilinear. As a consequence we will be able to split the third of the kernel equations (42c) in simpler equations, thus succeeding in extracting significant information from (42c), and in proving the main theorem of this work: the transversality of the Seiberg-Witten functional over an irreducible monopole with *hermitian* perturbations. This means that for a generic hermitian metric h sufficiently close to the Kahler metric g of the manifold the Seiberg-Witten moduli space will be smooth. We can show with a counterexample that we cannot hope to get transversality remaining in the Kähler class of g.

5.1 Kähler monopoles

In this subsection we will write Seiberg-Witten equations on a Kähler surface and we will prove that a Kähler monopole can always be identified with a holomorphic pair $(\bar{\partial}_A, \alpha)$, where $\bar{\partial}_A$ represents a holomorphic structure on the line bundle N and α is a holomorphic section of $(N, \bar{\partial}_A)$. We will sketch the proof that the Seiberg-Witten moduli space is isomorphic to the moduli space $\mathbb{M}_N$ of holomorphic pairs. We obtain two important consequences: the first is the structure of Seiberg-Witten moduli space as a fibration over the moduli space of holomorphic structures on N, with fibre $|N_A| \simeq \mathbb{P}(H^0(M, N_A))$, the complete linear system of divisors linearly equivalent to N_A; on the other hand the deformation complex for $\mathbb{M}_N$ allows us to identify the cohomology group $H^1(D, N_A|_D)$ as the obstruction to the transversality of the Seiberg-Witten functional in the point (A, α), where $D = Z(\alpha)$. In the end of this subsection we use this fact to provide a counterexample which shows that we cannot hope to obtain transversality deforming the Kähler metric remaining in the Kähler class, and that it is necessary to change the metric with perturbations more general than Kähler ones.

Let us begin by writing the Seiberg-Witten equations on a Kähler surface. A spinor $\psi \in W_+ \simeq A^{0,0}(N) \oplus A^{0,2}(N)$ can be written as a couple $\psi = (\alpha, \beta)$, $\alpha \in A^{0,0}(N)$, $\beta \in A^{0,2}(N)$. The element $\psi^* \otimes \psi$ is the element of $\mathrm{End}(W_+)$ of the form :

$$\psi^* \otimes \psi = \begin{pmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \alpha\bar{\beta} & |\beta|^2 \end{pmatrix} .$$

Therefore

$$[\psi^* \otimes \psi]_0 = \begin{pmatrix} \frac{|\alpha|^2 - |\beta|^2}{2} & \bar{\alpha}\beta \\ \alpha\bar{\beta} & \frac{|\beta|^2 - |\alpha|^2}{2} \end{pmatrix} .$$

comes from the element $(\bar{\alpha}\beta/2 + i\omega(|\alpha|^2 - |\beta|^2)/4)$ in the isomorphism (19). We will denote with A the unitary connection on N and with A_0 the unitary connection on $L = N^{\otimes^2} \otimes K_M^{-1}$. We remark that the curvature F_{A_0} on L is $2F_A - F_{A_{K_M}}$, since A_0 is the connection $A^2 \otimes A_{K_M^{-1}}$, hence we can express everything in terms of the unitary connection A on N. We remark also that the hermitian connection on K_M is holomorphic, then $F_{A_0}^{0,2} = 2F_A^{0,2}$. We will denote with Λ the contraction with the Kähler form. We are now ready to write Seiberg-Witten equations on a Kähler surface:

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0 \tag{43a}$$

$$F_A^{0,2} = \frac{\bar{\alpha}\beta}{2} \tag{43b}$$

$$2\Lambda F_A^{1,1} - F_{K_M}^{1,1} = i\frac{|\alpha|^2 - |\beta|^2}{4} \tag{43c}$$

We will denote the Seiberg-Witten functional for these equations with F_N and with $\mathcal{C}_N = \mathcal{A}^{U(1)}(N) \times A^{0,0}(N) \times A^{0,2}(N)$ the configuration space for Seiberg-Witten equations on a Kähler surface with $Spin^c$ structure given by the spinor bundle $\Lambda^{0,*}T^*M \otimes N$. We have immediately the following involution between the configuration space $\mathcal{C}_N$ and $\mathcal{C}_{N^*\otimes K_M}$:

Proposition 5.1. *The conjugation $(A, \alpha, \beta) \longmapsto (A^* \otimes A_{K_M}, \sharp\beta, \sharp\alpha)$ given by the complex hodge star $\sharp$ induces an isomorphism between the configuration space $\mathcal{C}_N$ and the configuration space $\mathcal{C}_{N^*\otimes K_M}$.*

Proof. The proof of the proposition is immediate. The only thing to verify is that $(A^* \otimes A_{K_M})^* \otimes A_{K_M} = A$, but this is evident.

□

Definition 5.2. Let L a line bundle on a Kähler surface and let ω its Kähler form. The degree of the bundle L is the real number:

$$J(L) = \int_M c_1(L) \wedge \omega = \frac{i}{2\pi} \int_M F_A \wedge \omega = c_1(L).[\omega] \, .$$

We will now sketch the proof that a solution to Seiberg-Witten equations (43) determines a holomorphic couple $(\bar\partial_A, \alpha)$, where $\bar\partial_A$ is a holomorphic structure on N and α is a holomorphic section of $N_A := (N, \bar\partial_A)$.

Proposition 5.3. *Let N a line bundle on the Kähler surface M. Let (A, α, β) a solution of the Seiberg-Witten equations (43), that is a zero of the functional F_N. Then*
(1) If $2J(N) - J(K_M) < 0$ then $\beta = 0$. As a consequence the semiconnection $\bar\partial_A$ is integrable ($F_A^{0,2} = 0$) and defines a holomorphic structure on N; moreover α is a holomorphic section of N_A;
(2) If $2J(N) - J(K_M) > 0$ then $\alpha = 0$ and the semiconnection $\bar\partial_{A^*\otimes A_{K_M}}$ defines an holomorphic structure on $N^* \otimes K_M$; moreover β is a holomorphic section of $N_A^* \otimes K_M$;
(3) If $2J(N) = J(K)$ then $\alpha = 0 = \beta$ and A is an anti self-dual connection (or an abelian istanton) on N.

Proof. Let us prove the first statement. This is readily done by applying the $\bar\partial_A$-operator to the Dirac equation obtaining

$$F_A^{0,2}\alpha + \Delta_{\bar\partial_A}\beta = 0$$

and by plugging in the second equation,

$$\Delta_{\bar\partial_A}\beta + \frac{|\alpha|^2\beta}{4} = 0$$

which yields, after taking the scalar product with β and integrating on the manifold:

$$||\bar\partial_A^*\beta||_2^2 + \frac{1}{4}|| \, |\alpha||\beta| \, ||_2^2 = 0 \, .$$

This implies the overdetermined elliptic equations $\bar\partial_A^*\beta = 0$ and $\bar\partial_A\alpha = 0$, and $|\alpha||\beta| = 0$. By lemma 4.5 and theorem 4.7 we get that $\alpha = 0$ identically or $\beta = 0$ identically. Now we can express the degree $J(N^{\otimes^2} \otimes K_M) = 2J(N) - J(K)$ as a function of the integral of $|\alpha|^2 - |\beta|^2$ by means of the third equation, and the hypothesis of the negativity of $J(N^{\otimes^2} \otimes K_M)$ allows us to conclude stating that β have to vanish. As a consequence $\bar\partial_A$ is a holomorphic structure on N and α is a holomorphic section.

Suppose now $2J(N) - J(K) > 0$. Then $2J(N^* \otimes K) - J(K) < 0$ and we use statement (1) and proposition 5.1 to prove that $\sharp\alpha = 0$ and $\bar\partial_{A^*\otimes A_{K_M}}$ defines a holomorphic connection on $N^* \otimes K_M$. We only need to remark that $\bar\partial_{A^*\otimes A_{K_M}}$ induces on $N^* \otimes K_M$ the holomorphic structure $(\bar\partial_A)^* \otimes \bar\partial_{K_M}$, that is $(N^* \otimes K_M)_{A^*\otimes A_{K_M}}$ is exactly $N_A^* \otimes K_M$.

□

5.1.1 Holomorphic pairs

We will now describe briefly the moduli space of holomorphic pairs $\mathbb{M}_N$.

Definition 5.4. (cf. [75]) A semiconnection $\bar{\partial}_A$ of type $(0,1)$ on a vector bundle E on a complex manifold M is said a *holomorphic structure* if $\bar{\partial}_A^2 = 0$.

We will denote with $\mathcal{H}(E)$ the set of holomorphic structures on E and with E_A the holomorphic vector bundle induced on E by the holomorphic structure $\bar{\partial}_A$. Two holomorphic structures $\bar{\partial}_A$ and $\bar{\partial}_B$ are *gauge equivalent* if there exists $g \in \mathrm{Aut}(E)$ such that $g^{-1}\bar{\partial}_A g = \bar{\partial}_B$, or equivalently if there exists $g \in \mathrm{Aut}(E)$ such that $g : E_A \longrightarrow E_B$ is an isomorphism of holomorphic vector bundles. If L is a line bundle the set $\mathcal{H}(L)$ is simple to describe. Indeed let $\bar{\partial}_A \in \mathcal{H}(L)$. Then any other semiconnection $\bar{\partial}_A + \tau$, $\tau \in A^{0,1}(M, \mathbb{C})$ is holomorphic if and only if $\bar{\partial}\tau = 0$. Therefore $\mathcal{H}(L)$ is an affine space with underlying vector space the space of $\bar{\partial}$-closed $(0,1)$-forms $Z^{0,1}(M, \mathbb{C})$. The moduli space of holomorphic structures on L is then the quotient $\mathcal{M}(L) \simeq \mathcal{H}(L)/\mathcal{C}^\infty(M, \mathbb{C}^*)$, since $\mathrm{Aut}(L) \simeq \mathcal{C}^\infty(M, \mathbb{C}^*)$. It easy to see that for a line bundle L the moduli space $\mathcal{M}(L)$ is always smooth (cf. Kobayashi [75]) and its tangent space $T_{\bar{\partial}_A}\mathcal{M}(L) \simeq H^1(M, \mathbb{C})$. If M is simply connected $\mathcal{M}(L)$ is then a point, or equivalently there is only one holomorphic structure up to gauge equivalence.

We pass now to talk about holomorphic pairs $(\bar{\partial}_A, \alpha)$, where $\bar{\partial}_A \in \mathcal{H}(L)$, $\alpha \in H^0(M, L_A)$. Two holomorphic pairs $(\bar{\partial}_A, \alpha)$, $(\bar{\partial}_B, \beta)$ are said *gauge equivalent* if there exists $g \in \mathrm{Aut}(L)$ such that $\bar{\partial}_B = g^{-1}\bar{\partial}_A g$ and $\beta = g\alpha$. The space of holomorphic pairs is clearly the zero set of the map

$$\begin{aligned} \mathcal{F} : \bar{\mathcal{A}}(L) \times \Gamma(L) &\longrightarrow A^{0,2}(M) \times A^{0,1}(L) \\ (\bar{\partial}_A, \alpha) &\longrightarrow (-\bar{\partial}_A^2, \bar{\partial}_A \alpha) \end{aligned}$$

and the moduli space of holomorphic pairs $\mathbb{M}_L$ is the quotient $\mathbb{M}_L \simeq Z(\mathcal{F})/\mathcal{C}^\infty(M, \mathbb{C}^*)$. Let us study the deformation complex for this moduli space, and its Zariski tangent space. Consider first the morphism G of trivial bundles given by

$$\begin{aligned} Z(\mathcal{F}) \times A^{0,2}(M) \times A^{0,1}(L) &\xrightarrow{G} Z(\mathcal{F}) \times A^{0,2}(L) \\ ((\bar{\partial}_A, \alpha), \sigma, \beta) &\longmapsto ((\bar{\partial}_A, \alpha), \sigma\alpha + \bar{\partial}_A\beta) \end{aligned}$$

It is clear that $\mathcal{F}$ takes its values in the kernel of G. Therefore $(\bar{\partial}_A, \alpha)$ will be a regular point if $D_{(\bar{\partial}_A,\alpha)}\mathcal{F}$ is onto $\ker G$. The differential of $\mathcal{F}$ in the point $(\bar{\partial}_A, \alpha)$ is $D_{(\bar{\partial}_A,\alpha)}\mathcal{F} = (-\bar{\partial}\omega, \omega\alpha + \bar{\partial}_A\beta)$ and the deformation complex $\mathcal{K}^\bullet$ is given by:

$$0 \longrightarrow \mathcal{C}^\infty(M, \mathbb{C}) \longrightarrow A^{0,1}(M) \oplus \Gamma(L) \xrightarrow{D\mathcal{F}} A^{0,2}(M) \oplus A^{0,1}(L) \xrightarrow{G} A^{0,2}(L) \longrightarrow 0$$

where the first arrow is given by the linearization of the $\mathcal{C}^\infty(M, \mathbb{C}^*)$-action : $f \longmapsto (-\bar{\partial}f, f\alpha)$. We recall that given two complexes

$$K_1^\bullet := \ldots \longrightarrow K_1^p \longrightarrow K_1^{p+1} \longrightarrow K_1^{p+2} \longrightarrow \ldots$$

and

$$K_2^\bullet := \ldots \longrightarrow K_2^p \longrightarrow K_2^{p+1} \longrightarrow K_2^{p+2} \longrightarrow \ldots$$

and given a morphism of complexes $f^\bullet : K_1^\bullet \longrightarrow K_2^\bullet$, the mapping cone of f, denoted with $M(f)^\bullet$ is defined by $M(f)^i = K_1^{i+1} \oplus K_2^i$ with differential

$$d^i_{M(f)^\bullet} = \begin{pmatrix} -d^{i+1}_{K_1^\bullet} & 0 \\ f^{i+1} & d^i_{K_2^\bullet} \end{pmatrix} .$$

By definition we have an exact sequence

$$0 \longrightarrow K_2^\bullet \longrightarrow M(f)^\bullet \longrightarrow K_1^\bullet[1] \longrightarrow 0$$

where $K_1^\bullet[1]^i = K_1^{i+1}$. Now if we set $K_2 = A^{0,\bullet}(L)$ and $K_1^\bullet = A^{0,\bullet}(M)$ the deformation complex $\mathcal{K}^\bullet$ can naturally be identified with the mapping cone of the map: $\alpha : A^{0,\bullet}(M) \longrightarrow A^{0,\bullet}(L)$. As a consequence we have an exact sequence of complexes:

$$0 \longrightarrow A^{0,\bullet}(L) \longrightarrow \mathcal{K}^\bullet \longrightarrow A^{0,\bullet}(M)[1] \longrightarrow 0$$

which induces a long exact sequence in cohomology:

$$\ldots \longrightarrow H^i(K_2^\bullet) \longrightarrow H^i(\mathcal{K}^\bullet) \longrightarrow H^i(K_1^\bullet[1]) \longrightarrow \ldots \,. \tag{44}$$

By Dolbeault theorem the i-th cohomology group H^i of the complex $A^{0,\bullet}(M)$ is exactly the sheaf cohomology $H^i(M, \mathcal{O}_M)$ and the cohomology of the complex $A^{0,\bullet}(L)$ is $H^i(M, L)$. Therefore the long exact sequence (44) becomes:

$$\ldots \longrightarrow H^i(M, L) \longrightarrow H^i(\mathcal{K}^\bullet) \longrightarrow H^{i+1}(M, \mathcal{O}_M) \xrightarrow{\alpha} H^{i+1}(M, L) \longrightarrow \ldots$$

We remark that the multiplication by $\alpha : \mathcal{O}_M \longrightarrow L$ gives rise to an exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_M \xrightarrow{\alpha} L \longrightarrow L_D \longrightarrow 0 \tag{45}$$

where $D = Z(\alpha)$ is the divisor defined by the section α. The sequence (45) induces a long exact cohomology sequence:

$$\ldots \longrightarrow H^i(M, \mathcal{O}_M) \longrightarrow H^i(M, L) \longrightarrow H^i(D, L_D) \longrightarrow H^{i+1}(M, \mathcal{O}) \longrightarrow \ldots \tag{46}$$

and comparing (44) and (46) we get:

$$H^i(\mathcal{K}^\bullet) \simeq H^i(D, L_D)\,. \tag{47}$$

This implies $H^0(\mathcal{K}^\bullet) \simeq H^0(D, L_D)$, $H^1(\mathcal{K}^\bullet) \simeq H^1(D, L_D)$ and $\chi(L_D) \simeq h^0(L_D) - h^1(L_D)$. The Zariski tangent space of the moduli space of holomorphic pairs is isomorphic to $H^0(\mathcal{K}^\bullet) \simeq H^0(D, L_D)$ and the expected dimension of the moduli space is $\chi(L_D)$. Finally the differential $D_{(\bar\partial_A, \alpha)}\mathcal{F}$ is surjective and the moduli space is smooth at $(\bar\partial_A, \alpha)$ of the expected dimension if and only if $H^1(D, L_D) = 0$. By means of the Riemann-Roch formula (cf. Kobayashi [75]) for curves we can compute the expected dimension $\chi(L_D)$:

$$\begin{aligned} \chi(L_D) &= \chi(L) - \chi(\mathcal{O}_M) \\ &= \chi(\mathcal{O}_M) + \frac{1}{2}(c_1(L)^2 - c_1(L)c_1(K_M)) - \chi(\mathcal{O}_M) \\ &= \frac{1}{2}(c_1(L)^2 - c_1(L)c_1(K_M))\,. \end{aligned}$$

Consider now the projection

$$\begin{aligned} Z(\mathcal{F}) &\longrightarrow \mathcal{H}(L) \\ (\bar\partial_A, \alpha) &\longmapsto \bar\partial_A \end{aligned}$$

By construction of the $\mathcal{C}^\infty(M, \mathbb{C}^*)$-action on the two spaces it is immediate to see that the projection is equivariant, thus inducing a projection:

$$\pi : \mathbb{M}_L \longrightarrow \mathcal{M}(L)\,.$$

It is easy to see that the fiber $\pi^{-1}(\bar\partial_A)$ can be identified with the complete linear system $\mathbb{P}(H^0(M, L_A))$ of divisors linearly equivalent to L_A. If the manifold is simply connected $\mathcal{M}(L)$ is a point and the moduli space of holomorphic pairs is isomorphic to the projective space $\mathbb{P}(H^0(M, L_A))$.

We will now sketch the proof that the Seiberg-Witten moduli space of Kählerian monopoles $\mathcal{M}_N^{SW}$ is isomorphic to the moduli space of holomorphic pairs $\mathbb{M}_N$. We first remark that we chose the couples (A,ψ) as unknowns for the Seiberg-Witten functional F_N, where A is a connection on N (and not on $L = N^{\otimes^2} \otimes K_M^{-1}$ as it is usually done). The induced action of the gauge group $\mathcal{C}^\infty(M,\mathbb{C}^*)$ on $\mathcal{A}_N^{U(1)}$ is simply given by: $g.A = gAg^{-1}$. Indeed the action on the connection $A^{\otimes^2} \otimes A_{K_M^{-1}}$ on L is $g.(A^{\otimes^2} \otimes A_{K_M^{-1}}) = A^{\otimes^2} \otimes A_{K_M^{-1}} - 2(dg)g^{-1} = (A-(dg)g^{-1})^{\otimes^2} \otimes A_{K_M^{-1}}$ by lemma 2.2. This means that the induced action on $\mathcal{A}_N^{U(1)}$ is $g.A = gAg^{-1}$.

Proposition 5.5. *Let N a holomorphic line bundle on the Kähler surface M. Let $\mathcal{M}_N^{SW}$ the Seiberg-Witten moduli space of Kählerian monopoles. If $J(N^{\otimes^2} \otimes K_M^{-1}) < 0$ then $\mathcal{M}_N^{SW}$ is isomorphic to the moduli space of holomorphic pairs $\mathbb{M}_N$.*

Proof. Consider the map:

$$\begin{aligned} j : Z(F_N) &\hookrightarrow Z(\mathcal{F}) \\ (A,\alpha) &\longmapsto (\bar\partial_A,\alpha) \end{aligned}$$

that associates to a Kähler monopole (A,α) the holomorphic pair $(\bar\partial_A,\alpha)$. This injection is clearly equivariant for the $\mathcal{C}^\infty(M,S^1)$-action. Indeed let $g \in \mathcal{C}^\infty(M,S^1)$. Then

$$g.(A,\alpha) = (A-(dg)g^{-1}, g\alpha)\ ;$$

the $(0,1)$ part of $A-(dg)g^{-1}$ is $\bar\partial_A - (\bar\partial g)g^{-1}$ and hence $\bar\partial_{g.A} = g\bar\partial_A g^{-1} = g.\bar\partial_A$ which implies that the map j is equivariant. To prove that j induces an isomorphism between the quotients one proves that the $\mathcal{C}^\infty(M,\mathbb{C}^*)$ orbit of an element $(\bar\partial_A,\alpha)$ intersects $Z(F_N)$ along exactly one $\mathcal{C}^\infty(M,S^1)$-orbit. The proof consists in expressing an element $g \in \mathcal{C}^\infty(M,\mathbb{C}^*)$ as a product $g = u\theta$, where $u \in \mathcal{C}^\infty(M,\mathbb{R}^+)$ and $\theta \in \mathcal{C}^\infty(M,S^1)$. With a classical argument by Bradlow and Kazdan-Warner (cf. [14], [73], [47]) one can see that the $\mathcal{C}^\infty(M,\mathbb{R}^+)$ orbit of the element $(\bar\partial_A,\alpha)$ meets $Z(F_N)$ in exactly one point if and only if the hypothesis on the negative degree $J(N^{\otimes^2} \otimes K_M^{-1})$ is satisfied. Acting with $\mathcal{C}^\infty(M,S^1)$ will then produce $\mathcal{C}^\infty(M,S^1)$ equivalent points, that is, the searched $\mathcal{C}^\infty(M,S^1)$ orbit in $Z(F_N)$.

□

The interrelation between Seiberg-Witten theory and complex geometry is extremely rich. Some years before the coming out of Seiberg-Witten equations, Bradlow ([14]) proved the isomorphism between the moduli space of holomorphic pairs on a line bundle $\mathcal{M}(L)$ and the moduli space of vortex equations, which were precursors of Seiberg-Witten equations on Kähler surfaces. His point of view has been taken up by Okonek and Teleman in [99], who show the links between vortex equations, moduli spaces of stable pairs and coupled Seiberg-Witten equations, thus extending Bradlow's work to non abelian monopoles. Friedman and Morgan in [48] prove the isomorphism of real analytic spaces between the Seiberg-Witten moduli space and the Hilbert scheme $\mathrm{Div}_+^w(M)$ of effective divisors D of fundamental class $c_1(D) = c_1(N) = w$.

Corollary 5.6. *Let N a line bundle on a Kähler surface M, with $c_1(N) = w$. Let $c = 2w - k$ the Chern class of the line bundle $L = N^{\otimes^2} \otimes K_M^{-1}$. Let $W \simeq \Lambda^{0,*}T^*M \otimes N$ the spinor bundle associated to the $Spin^c$ structure of determinant line bundle L and let $k = c_1(K)$.*
(1) If $ck < 0$ the moduli space of Seiberg-Witten monopoles $\mathcal{M}_N^{SW}$ is isomorphic to the Hilbert scheme $\mathrm{Div}_+^w(M)$ of effective divisors of fundamental class w. It is a fibration over the moduli space of holomorphic structures $\mathcal{M}(N)$ on N with fibre $\mathcal{M}_{\bar\partial_A}^{SW} \simeq \mathbb{P}(H^0(M,N_A))$, the complete linear system of divisors linearly equivalent to N_A.

If $ck > 0$, $\mathcal{M}_{N^*\otimes K}^{SW}$ is isomorphic to the Hilbert scheme $\mathrm{Div}_+^{k-w}(M)$ of effective divisors of fundamental class $k - w$. It is a fibration over the moduli space of holomorphic structures of $N^* \otimes K_M$ with fibre over

$\bar{\partial}_{A^*\otimes K_M}$ *the complete linear system* $\mathbb{P}(H^0(M, N_A^* \otimes K_M))$ *of divisors linearly equivalent to* $N_A^* \otimes K_M$.
(2) The expected dimension in the point D *of* $\mathcal{M}_N^{SW}$ *is*

$$\chi(N_D) = \frac{1}{8}(c^2 - k^2) = \frac{1}{8}(c^2 - (2\chi + 3\tau)) .$$

(3) $\mathcal{M}_N^{SW}$ *is smooth of the expected dimension in the point* D *if and only if* $H^1(M, N_D) = 0$*, or if* $D = 0$.

Proof. By the preceding proposition we only need to prove the dimension formula. By what we have said about the moduli space of holomorphic couples

$$\begin{aligned}\chi(N_D) &= \frac{1}{2}(c_1(N)^2 - c_1(N)k) \\ &= \frac{1}{2}(w^2 - wk)\end{aligned}$$

but $w = (c+k)/2$; then

$$\begin{aligned}\chi(N_D) &= \frac{1}{2}\left(\frac{(c+k)^2}{4} - \frac{(c+k)k}{2}\right) \\ &= \frac{1}{2}\left(\frac{c^2-k^2}{4}\right) = \frac{1}{8}(c^2 - (2\chi + 3\tau)) .\end{aligned}$$

The last equality follows from Noether formula (cf. [54])

$$\chi(\mathcal{O}_M) = \frac{c_1(M)^2 + c_2(M)}{12}$$

and recalling that $\chi(\mathcal{O}_M) = -k^2/2$.

□

5.1.2 A counterexample

In this subsection we will use our knowledge of the obstruction to transversality of Seiberg-Witten functional F_N in cohomological terms (i.e. the group $H^1(M, N_D)$) to prove that Kähler perturbations of the metric do not suffice to obtain transversality. Let M be a smooth projective surface of degree 7 in the projective space $\mathbb{P}^3_{\mathbb{C}}$. We then have (cf. Hartshorne [65] or Griffiths-Harris [54]): $K_M = \mathcal{O}_M(3)$. M is simply connected. Let us take $N = \mathcal{O}_M(1)$.

$$J(N^{\otimes^2} \otimes K^{-1}) = J(\mathcal{O}_M(2) \otimes \mathcal{O}_M(-3)) = J(\mathcal{O}_M(-1)) = -1 < 0 .$$

We have just proved that the moduli space of Seiberg-Witten monopoles associated to the *Spin*^{c} structure $\Lambda^{0,*}T^*M \otimes \mathcal{O}_M(1)$ is isomorphic to $\mathbb{P}(H^0(M, \mathcal{O}_M(1)))$, that is the complete linear system of divisors linearly equivalent to $\mathcal{O}_M(1)$, or, said another way, the hyperplane sections. We have $\dim \mathbb{P}(H^0(M, \mathcal{O}_M(1))) = 3$, $\dim H^0(D, N_D) = \dim H^0(D, \mathcal{O}_D(1)) = 3$ and the expected dimension $\chi(N_D) = (1/8)(7 - 63) = -7$. Therefore $\dim H^1(D, N_D) = 10$: the Seiberg-Witten moduli space is not smooth of the expected dimension. We now try to obtain transversality by changing the metric in the following way. The metric g, the complex structure J and the symplectic form ω are linked by the relation:

$$g(X, Y) = \omega(JX, Y) .$$

We then can change the metric by fixing ω in the preceding relation and varying the complex structure $J \longmapsto J_t$ (among the *tame* complex structures with respect to ω, that is $\omega(J_tX, X) \gneq 0$), but we impose that the complex structure J_t remains integrable. A new metric g_t will be defined as $g_t(X, Y) = \omega(J_tX, Y)$. Since ω is fixed, and the triple (g_t, J_t, ω) is again an hermitian triple, then the relation (16) implies that the metric g_t is again Kähler. Moreover (M, J_t) is a clearly a deformation of the complex manifold (M, J) (cf. Kodaira-Morrow [78]). Now we can take the versal deformation (M, J_t) of (M, J). We know that a versal deformation exists and that a deformation of a Kähler manifold is again Kähler (for small t). Moreover the versal deformation contains all possible deformations of (M, J). In other words, with a versal deformation we will realize all possible Kähler metrics g_t in the way described above. Let now $X \longrightarrow S$ the versal deformation of the surface M: it is a family of complex surfaces parametrized by a complex manifold S. Let $s_0 \in S$. We will assume that $M \simeq X_{s_0}$. Let $w \in H^2(X, \mathbb{Z})$ and $w_s = j_s^* w \in H^2(X_s, \mathbb{Z})$, with j_s the embedding of the fiber over the point s: $j_s : X_s \hookrightarrow X$. Let $\mathcal{H} = \mathrm{Div}^w(X/S) \longrightarrow S$ the relative Hilbert scheme of couples (s, D), where $s \in S$ and $D \in \mathrm{Div}^{w_s}(X_s)$ is a divisor of fundamental class w_s. We have the exact sequence of locally free sheaves over D:

$$0 \longrightarrow N_{D/X_s} \longrightarrow N_{D/X} \longrightarrow N_{X^s/X} \longrightarrow 0$$

where in general $N_{Y/Z}$ indicates the normal sheaf of Y in Z. Now $N_{X^s/X} \simeq O_D(T_s S)$. Therefore we get a Bockstein morphism:

$$T_s S \longrightarrow H^1(D, N_{D/X_s}) .$$

On the other hand we have an exact sequence on X_s:

$$0 \longrightarrow TX_s \longrightarrow TX\Big|_{X_s} \longrightarrow N_{X^s/X} \longrightarrow 0$$

from which we get another Bockstein morphism:

$$T_s S \longrightarrow H^1(X_s, TX_s)$$

which is nothing but the morphism of infinitesimal deformation of Kodaira-Spencer. We get a commutative diagram:

$$\begin{tikzcd} T_s S \arrow[r] \arrow[dr] & H^1(X_s, TX_s) \arrow[d] \\ & H^1(D, N_{D/X^s}) \end{tikzcd}$$

where the vertical arrow is induced by the differential $d\alpha : TX\big|_D \longrightarrow N_D$ and α is a section of N such that $Z(\alpha) = D$. Since we are going to deal with the versal deformation the horizontal arrow in the diagram above (the Kodaira-Spencer map) is an isomorphism. The map

$$H^1(M, TM) \longrightarrow H^1(D, N_D) \tag{48}$$

is the algebraic-geometric interpretation of the contribution of the variation of the metric (in terms of variation of complex structures) to the surjectivity of the Seiberg-Witten functional and corresponds to the map induced by the partial differential $\partial\tilde{\mathbb{F}}_+/\partial\varphi$ restricted to Kähler variations of the metric followed by the projection onto the cokernel $\operatorname{coker} DF$ of the Seiberg-Witten operator DF. In other words we have transversality with Kähler perturbations of the metric if and only if the map (48) is surjective.

The map (48) is surjective if and only if the transposed map:

$$H^1(D, N_D)^* \longrightarrow H^1(M, TM)^*$$

is injective. By Serre duality and by the adjunction formula $N_D^* \simeq K_M\big|_D \otimes K_D^*$ this is equivalent to the injectivity of the map:

$$H^0(D, K_M\big|_D) \longrightarrow H^1(M, \Omega^1_M(K_M)) . \tag{49}$$

Consider now the restriction map:

$$H^0(M, K_M) \longrightarrow H^0(D, K_M\big|_D) \; ; \tag{50}$$

by the long cohomology sequence associated to the short exact sequence:

$$0 \longrightarrow K_M(-D) \longrightarrow K_M \longrightarrow K_M\big|_D \longrightarrow 0$$

it is surjective if and only if $h^1(M, K_M(-D)) = h^1(M, \mathcal{O}_M(D)) = 0$. Now the short exact sequence:

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0$$

and the fact that $H^1(M, \mathcal{O}_M) = 0$ (by Hodge decomposition and by Lefschetz theorem) induce the following exact sequence:

$$0 \longrightarrow H^1(M, \mathcal{O}_M(D)) \longrightarrow H^1(D, \mathcal{O}_D(D)) \longrightarrow H^2(M, \mathcal{O}_M) \longrightarrow H^2(M, \mathcal{O}_M(D)) \longrightarrow 0 \; . \tag{51}$$

We have:

$$\begin{aligned} h^2(\mathcal{O}_M) =& h^0(K_M) = h^0(\mathcal{O}_M(3)) = h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = \binom{3+3}{3} = 20 \\ h^2(\mathcal{O}_M(D)) =& h^2(\mathcal{O}_M(1)) = h^0(\mathcal{O}_M(2)) = h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{3} = 10 \; . \end{aligned}$$

Knowing that, by Riemann-Roch and by Bertini theorem,

$$\begin{aligned} \chi(\mathcal{O}_D(D)) =& 1 - g(D) + \deg\mathcal{O}_D(D) = 1 - \frac{(7-1)(7-2)}{2} + 7 = -7 \\ h^0(\mathcal{O}_D(D)) =& h^0(\mathcal{O}_D(1)) = h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = \binom{2+1}{2} = 3 \end{aligned}$$

which implies $h^1(\mathcal{O}_D(D)) = 10$, we get from (51)

$$h^1(\mathcal{O}_M(D)) = h^1(\mathcal{O}_D(D)) - h^2(\mathcal{O}_M) + h^2(\mathcal{O}(D)) = 10 - 20 + 10 = 0 \; .$$

This means that the restriction map (50) is surjective. Therefore the composed linear map:

$$H^0(M, K_M) \longrightarrow H^1(M, \Omega^1_M(K_M))$$

is given by the cup product with the Chern class $c_1(D)$. Since $K_M = 3D$ we see that the multiplication by the Chern class is zero in cohomology. As a consequence the map (49) is zero and therefore (48) is the zero map, which is not surjective(!).

5.2 Transversality over a Kähler monopole

In this subsection we will prove the main theorem of this part, the transversality of the perturbed Seiberg-Witten functional on Kähler monopoles. Actually we will obtain the result varying the Kähler metric with hermitian perturbations, that is, perturbations that preserve the complex structure. As a consequence, when we consider the Seiberg-Witten moduli space on a Kähler manifold, this result implies that the moduli space is a smooth compact manifold for a generic *hermitian* metric sufficiently close to the given Kähler metric.

The key of the proof is the decomposition of 2-symmetric tensors S^2T^*M in J-hermitian and J-antihermitian. This allows to isolate the contribution coming from hermitian perturbations and to split the third equation (42c) in two simpler equations, the first of which gives the result.

Remark 5.7. We will deal with the case $J(N^2 \otimes K_M^{-1}) < 0$ since by proposition 5.1 the conjugation $\sharp$ exchanges the solutions of F_N and of $F_{N^* \otimes K_M}$, and $J(N^* \otimes K_M) < 0$ if and only if $J((N^* \otimes K_M)^2 \otimes K_M^{-1}) = J(N^{-2} \otimes K_M) > 0$. The case $J(N^2 \otimes K_M^{-1}) > 0$ can be treated as the first after setting $\tilde{N} = N^* \otimes K_M$.

Let M a Kähler surface, $\Lambda^{0,*}T^*M \otimes N$ the spinor bundle on M with $J(N^2 \otimes K_M^{-1}) < 0$. Let $(A, \alpha, 0) \in \mathcal{A}^{U(1)}(N) \times A^{0,0}(N)$ a monopole, that is a solution of Seiberg-Witten equations (43). We proved in proposition 5.3 that such a monopole satisfies :

$$\bar{\partial}_A \alpha = 0 \tag{52a}$$

$$F_A^{0,2} = 0 \tag{52b}$$

$$2\Lambda F_A^{1,1} - \Lambda F_{K_M}^{1,1} = \frac{i}{4}|\alpha|^2 \tag{52c}$$

Let now interpret the kernel equations (42) in the Kähler context. We will identify 1-forms in $A^1(M, i\mathbb{R})$ with $(0,1)$-forms in $A^{0,1}(M)$ via the isomorphism $\sigma \longmapsto \sigma - \bar{\sigma}$. We have:

Lemma 5.8. *Let $\chi \in A^{0,1}(N)$ a negative spinor. Then the differential form $\xi = \bar{\alpha} \otimes \chi$ is given by: $\xi = \frac{1}{\sqrt{2}}\bar{\alpha}\chi$.*

Proof. The differential form ξ is characterized by the fact that $\rho(\xi)\alpha = |\alpha|^2\chi$ and $\rho(\xi)\beta = 0$ for all $\beta \in A^{0,2}(N)$. Now

$$\rho(\xi) = \sqrt{2}(\xi^{0,1} \wedge \cdot - \overline{\xi^{1,0}} \lrcorner \cdot) .$$

It follows that $\sqrt{2}\xi^{0,1}\alpha = |\alpha|^2\chi$ and $\overline{\xi^{1,0}} \lrcorner \beta = 0$ for all $\beta \in A^{0,2}(N)$. Therefore $\xi^{1,0} = 0$ and $\xi^{0,1} = 1/\sqrt{2}\bar{\alpha}\chi$.

□

Let now consider an imaginary self dual form $\theta \in A_+^2(M, i\mathbb{R})$. We know that such a form can be written as $\theta = i\lambda\omega + \mu + \bar{\mu}$, with $\lambda \in \mathbb{R}$, $\mu \in A^{0,2}(M, \mathbb{C})$. We recall that $\rho(\theta)$ is the endomorphism in $i\mathfrak{su}(W_+) \simeq \mathbb{R} \oplus \Lambda^{0,2}$ given by

$$(\mu, \lambda) \longmapsto 2 \begin{pmatrix} \lambda & \mu \lrcorner \cdot \\ \mu \wedge \cdot & -\lambda \end{pmatrix} .$$

The first equation (42), $d^*\theta + 2i\mathrm{Im}(\alpha^* \otimes \chi) = 0$ becomes:

$$\bar{\partial}^*\mu + i\bar{\partial}^*(\lambda\omega) - \partial^*\bar{\mu} + i\partial^*(\lambda\omega) + \frac{1}{\sqrt{2}}(\bar{\alpha}\chi - \alpha\bar{\chi}) = 0$$

and with the identification with $A^{0,1}$:

$$\bar{\partial}^*\mu + i\bar{\partial}^*(\lambda\omega) + \frac{1}{\sqrt{2}}\bar{\alpha}\chi = 0 . \tag{53}$$

The second equation easily gives the two equations in $A^{0,0}(N) \oplus A^{0,2}(N)$:

$$\sqrt{2}\bar{\partial}^*\chi - \lambda\alpha = 0$$

$$\sqrt{2}\bar{\partial}\chi - \mu\alpha = 0$$

Equation (42d), $(\theta, F_A^+) = 0$ gives immediately $\lambda|\alpha|^2 = 0$ and knowing that $M - Z(\alpha)$ is a dense open set, we get $\lambda = 0$. Moreover equation (42e) becomes: $\bar{\partial}^*(\bar{\alpha}\chi) = 0$; we remark that this condition can be obtained applying the operator $\bar{\partial}^*$ in (53), and thus it is not independent from the others. Therefore equations (42a) (42b) (42d) (42e) are equivalent to:

$$\bar{\partial}^*\mu + \frac{1}{\sqrt{2}}\bar{\alpha}\chi = 0 \tag{54a}$$

$$\sqrt{2}\bar{\partial}_A\chi - \mu\alpha = 0 \tag{54b}$$

$$\bar{\partial}_A^*\chi = 0 \tag{54c}$$

$$\lambda = 0 \tag{54d}$$

Consider now the mapping cone $M((1/\sqrt{2})\alpha)^\bullet$ of the morphism $(1/\sqrt{2})\alpha : A^{0,\bullet}(M,\mathbb{C}) \longrightarrow A^{0,\bullet}(M,N)$. As we saw in subsection 5.1.1 it is the elliptic complex:

$$0 \longrightarrow A^{0,0}(M) \longrightarrow A^{0,1}(M)\oplus\Gamma(L) \xrightarrow{DF} A^{0,2}(M)\oplus A^{0,1}(L) \xrightarrow{G} A^{0,2}(L) \longrightarrow 0 \tag{55}$$

where $DF(\omega,\sigma) = (-\bar\partial\omega, (1/\sqrt{2})\omega\alpha + \bar\partial_A\sigma)$ and $G(\mu,\chi) = ((1/\sqrt{2})\mu\alpha + \bar\partial_A\chi)$. If we form the laplacian

$$P = DF^* \oplus G : A^{0,2}(M)\oplus A^{0,1}(L) \longrightarrow A^{0,1}(M)\oplus\Gamma(L)\oplus A^{0,2}(L)$$

the equations $Pu = 0$ are exactly equations (54). Now by Hodge theorem for elliptic complexes, the space of harmonic solutions $\ker P$ is isomorphic to the first cohomology group of the complex (55) : $\ker P \simeq H^1(M((1/\sqrt{2})\alpha)^\bullet)$ which, in turn, is isomorphic to $H^1(D,N_D)$, where $D = Z(\alpha)$ and N_D is the restriction of N to D and is identified with the normal bundle of D in M. Now equations (54) are exactly the equations of the kernel of the perturbed Seiberg-Witten operator restrained to conformal perturbations. When we apply this reasonment to a Kähler surface M with a line bundle N such that for some section α of N, $H^1(D,N_D)\neq 0$ (for example a smooth algebraic surface of degree 7 in the complex projective 3 space), we get a counterexample of the fact that conformal perturbations help in obtaining transversality. In such an example conformal perturbations reduce by no means the obstruction $H^1(D,N_D)$. Therefore we have to proceed in analysing the equation (42c).

5.2.1 A decomposition for the symmetric 2-tensors

In order to extract significant and useful information from equation (42c) we will decompose the bundle of symmetric 2-tensors S^2T^*M in hermitian (or compatible with the complex structure J) and antihermitian. This decomposition corresponds to the decomposition of symmetric endomorphisms $\mathrm{sym}\,(TM,g)$ of TM with respect to the metric g in J-linear and J-antilinear.

We begin by taking an euclidian $2n$-vector space (E,g) with a complex structure J compatible with the metric. We extend J to $E\otimes\mathbb{C}$ as usual by $\mathbb{C}$-linearity: we obtain again an antiinvolution $J : E\otimes\mathbb{C} \longrightarrow E\otimes\mathbb{C}$. We denote with $E^{1,0}$ and with $E^{0,1}$ the eigenspaces of i and $-i$, respectively. Therefore $E^{1,0} = \{X - iJX,\ x\in E\}$; $E^{0,1} = \{X + iJX,\ x\in E\}$. We will take $\Lambda^{1,0}E = (E^{0,1})^\perp$ in $E^*\otimes\mathbb{C}$ and $\Lambda^{0,1}E = (E^{1,0})^\perp$. Let $f\in\mathrm{End}(E)$. We will say that f is J-linear if $Jf = fJ$ and that f is antilinear if $fJ = -Jf$. We will denote with $\mathrm{End}(E,J)$ the space of J-linear endomorphisms of E, and $\overline{\mathrm{End}(E,J)}$ the space of J-antilinear ones. It is clear that we have an isomorphism:

$$\begin{aligned}\mathrm{End}(E) &\longrightarrow \mathrm{End}(E,J)\oplus\overline{\mathrm{End}(E,J)}\\ f &\longmapsto (f + JfJ, f - JfJ)\,.\end{aligned}$$

From another point of view an endomorphism $f\in\mathrm{End}(E)$ extends by $\mathbb{C}$-linearity to a unique endomorphism in $\mathrm{End}_{\mathbb{C}}(E\otimes\mathbb{C})$ such that $f(\bar z) = \overline{f(z)}$. With respect to the decomposition $E\otimes\mathbb{C} = E^{1,0}\oplus E^{0,1}$ the endomorphism f writes:

$$f = \begin{pmatrix} a & \bar b\\ b & \bar a\end{pmatrix}\,.$$

We have immediately that f is J-linear if and only if $b = 0$, and f is J-antilinear if and only if $a = 0$. Now let us take the symmetric 2-tensors S^2E^* and define the space of *hermitian* 2-tensors $S^{1,1}E^*$ as

$$S^{1,1}E^* := \{s\in S^2E^* |\ s(JX,JY) = s(X,Y)\,, \forall\, X,Y\,\in\, E\}\,,$$

and the space of antihermitian 2-tensors

$$AH(E) = \{s\in S^2E^* |\ s(JX,JY) = -s(X,Y)\,, \forall\, X,Y\,\in\, E\}\,.$$

It is clear that we have an isomorphism $S^2E^* \simeq S^{1,1}E^*\oplus AH(E)$ sending $s\longmapsto (s_J, s_{\bar J})$, where $s_J = s(J\cdot,J\cdot) + s(\cdot,\cdot)$ and $s_{\bar J} = s(\cdot,\cdot) - s(J\cdot,J\cdot)$. We have the following lemma:

Lemma 5.9. *Let $s \in S^2E^*$. Then*
(1) s can be extended uniquely (by $\mathbb{C}$-bilinearity) to a complex symmetric bilinear form in $S^2(E\otimes\mathbb{C})$ such that $s(\bar{Z},\bar{W}) = \overline{s(Z,W)}$ for all Z,$W \in E\otimes\mathbb{C}$;
(2) $s \in S^{1,1}E^*$ if and only if $s(Z,W) = 0$ for all Z,$W \in E^{1,0}$. In this case the position $(Z,W) \longmapsto s(Z,\bar{W})$ defines an hermitian form on $E^{1,0}$. Therefore $S^{1,1}E^* \simeq \mathrm{Herm}(E^{1,0})$;
(3) $s \in AH(E)$ if and only if $s(Z,\bar{W}) = 0$ for all Z,$W \in E^{1,0}$. Such an s defines naturally a quadratic form $\in S^2(E^{1,0}) = S^{2,0}E$. Therefore $AH(E) \simeq S^{2,0}E$.

Proof. The first statement is clear: when we extend a symmetric tensor to the complexified it is clearly invariant by conjugation. We remark that a 2-tensor in $S^2(E\otimes\mathbb{C})$ invariant by conjugation is determined by its restriction to $E^{1,0}\times E^{1,0}$ and to $E^{1,0}\times E^{0,1}$. The restriction to $E^{1,0}\times E^{1,0}$ will clearly be a complex symmetric bilinear form and the position $(Z,W) \longmapsto s(Z,\bar{W})$ will clearly define an hermitian form in $\mathrm{Herm}(E^{1,0})$. Conversely a 2-symmetric tensor invariant by conjugation is completely determined by the values it takes on the real vectors and thus comes from an element of S^2E^*. We have then proved that the symmetric 2-tensors S^2E^* are isomorphic to $S^2E^* \simeq \mathrm{Herm}(E^{1,0}) \oplus \mathrm{Quad}(E^{1,0})$. To prove the lemma it remains to show that, in this identification, $S^{1,1}E^* \subseteq \mathrm{Herm}(E^{1,0})$ and $AH(E) \subseteq \mathrm{Quad}(E^{1,0})$. But it is clear that the extension by $\mathbb{C}$-bilinearity of an element in $S^{1,1}E^*$ will satisfy $s(Z,W) = 0$ for all Z,$W \in E^{1,0}$, since:

$$s(X - iJX, Y - iJY) = s(X,Y) - is(JX,Y) - is(X,JY) - s(JX,JY) = 0$$

because $s(JX,Y) = -s(X,JY)$. Therefore $S^{1,1}E^* \subseteq \mathrm{Herm}(E^{1,0})$. Analogously we can prove that $AH(E) \subseteq \mathrm{Quad}(E^{1,0})$ and therefore $S^{1,1}E^* \simeq \mathrm{Herm}(E^{1,0})$; $AH(E) \simeq \mathrm{Quad}(E^{1,0})$.

□

Let us consider now the symmetric endomorphisms $\mathrm{sym}\,(E,g)$ of E with respect to the metric g. We define the space of *hermitian* endomorphisms of (E,g,J) as $\mathfrak{u}(E,J) := \mathrm{sym}\,(E,g) \cap \mathrm{End}(E,J)$ and the space of *antihermitian* endomorphisms as $\mathfrak{su}(E,J) := \mathrm{sym}\,(E,J) \cap \overline{\mathrm{End}(E,J)}$. Now it is clear that in the identification $S^2E^* \simeq \mathrm{sym}\,(E,g)$ provided by the metric, $\mathfrak{u}(E,J) \simeq S^{1,1}E^*$, $\mathfrak{su}(E,J) \simeq AH(E)$. The following lemma is the analogous to the preceding for symmetric endomorphisms:

Lemma 5.10. *Let $f \in \mathrm{End}(E)$ and let*

$$f = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$$

its $\mathbb{C}$-linear extension to an endomorphism of $E\otimes\mathbb{C}$. Then:
(1) f is symmetric with respect to g if and only if the form $(Z,W) \longmapsto g(a(Z),\bar{W})$ is an hermitian form in $\mathrm{Herm}(E^{1,0})$ and the form $(Z,W) \longmapsto g(b(Z),W)$ is a complex quadratic form in $\mathrm{Quad}(E^{1,0})$.
(2) $f \in \mathfrak{u}(E,J)$ if and only if $b = 0$; $f \in \mathfrak{su}(E,J)$ if and only if $a = 0$.

Proof. f is symmetric with respect to g if and only if $g(f(X),Y) = g(X,f(Y))$, for all $X,Y \in E$: the real bilinear form $(X,Y) \longmapsto g(f(X),Y)$ is therefore symmetric, and lemma 5.9 implies the first statement.

Let now f in $\mathfrak{u}(E,J)$. By definition, f preserves $E^{1,0}$ and $E^{0,1}$ and therefore $b = 0$. The real bilinear form $g(f(\cdot),\cdot)$ is in $S^{1,1}E^*$ since for all X,$Y \in E$ $g(f(JX),JY) = g(Jf(X),JY) = g(f(X),Y)$ and then by lemma 5.9 the position $(Z,W) \longmapsto g(f(Z),\bar{W})$ defines an hermitian form in $\mathrm{Herm}(E^{1,0})$. Analogously, if $f \in \mathfrak{su}(E,J)$ then it is clear that $a = 0$ and the bilinear form $g(f(\cdot),\cdot)$ is in $AH(E)$. Then by lemma 5.9, the position $(Z,W) \longrightarrow g(f(Z),W)$ defines a complex quadratic form on $E^{1,0}$.

□

Lemma 5.11. *The space of the real hermitian tensors $S^{1,1}E^*$ is isomorphic to the real $(1,1)$-forms $\Lambda^{1,1}_{\mathbb{R}}$.*

Proof. We associate to $s \in S^{1,1}E^*$ the 2-form $\varphi \in \Lambda^2 E^*$ defined as: $\varphi(X,Y) := s(X, JY)$. It is clear that φ is skew-symmetric, since $\varphi(X,Y) = s(X,JY) = -s(JX,Y) = -s(Y,JX) = -\varphi(Y,X)$, and that it is a $(1,1)$-form: if $Z,\ W \in E^{1,0}$ then $\varphi(Z,W) = s(Z,JW) = is(Z,W) = 0$ by lemma 5.9. That this correspondence defines an isomorphism is trivial.

□

Remark 5.12. If φ is the real $(1,1)$ form associated to a real hermitian endomorphism $s \in S^{1,1}E^*$, then, taken a complex basis $Z_1, \ldots, Z_n$ of $E^{1,0}$ and a dual basis $\xi^1, \ldots, \xi^n$ in $\Lambda^{1,0}E$ and setting $s_{i,\bar{j}} = s(Z_i, \bar{Z}_j)$, we have

(1) $s_{i,\bar{j}} = \overline{s_{j,\bar{i}}}$

(2) $\varphi = -2i \sum_{jk} s_{j,\bar{k}} \xi^j \wedge \bar{\xi}^k$

See [77] for details.

We consider now the real tangent bundle TM of an almost hermitian manifold M with its almost complex structure J and we define the bundles $\mathrm{End}(TM,J)$ and $\overline{\mathrm{End}(TM,J)}$ of J-linear and J-antilinear endomorphisms of TM, respectively; the bundles $\mathfrak{u}(TM,J)$ and $\mathfrak{su}(TM,J)$ of real hermitian and antihermitian endomorphisms of TM, and finally the bundles $S^{1,1}T^*M$ and $AH(TM)$ of hermitian and antihermitian 2 tensors on TM. Global versions of lemmas 5.9, 5.10 and 5.11 are valid on TM. Let us now take M an hermitian manifold (we assume now that the complex structure J is integrable), and let $z_1, \ldots, z_n$ local complex coordinates in M. The extension of f to $TM \otimes \mathbb{C}$ decomposes in

$$f = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} .$$

We can write $a = \sum_{ij} a_{ij} dz_i \otimes \partial/\partial z_j$ and $b = \sum_{ij} b_{ij} dz_i \otimes \partial/\partial \bar{z}_j$. We can use the complex bilinear form $g_{\mathbb{C}}$ to identify $TM \otimes \mathbb{C}$ and $T^*M \otimes \mathbb{C}$ (remembering that the complex bilinear form $g_{\mathbb{C}}$ exchanges $TM^{1,0}$ with $\Lambda^{0,1}T^*M$ and $TM^{0,1}$ with $\Lambda^{1,0}T^*M$. The J-linear endomorphism a is then identified with a section $\tilde{a}$ of $\Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M$ and writes as: $\tilde{a} = \sum_{ij} \tilde{a}_{i\bar{j}} dz_i \otimes d\bar{z}_j$. The J-antilinear endomorphism b can be identified with a section $\tilde{b} \in S^{2,0}T^*M = S^2(\Lambda^{1,0}T^*M)$, $\tilde{b} = \sum_{ij} \tilde{b}_{ij} dz_i \otimes dz_j$. By the preceding lemmas 5.9 and 5.10 and we have $\tilde{a}_{i,\bar{j}} = \overline{\tilde{a}_{j,\bar{i}}}$, and $\tilde{b}_{ij} = \tilde{b}_{ji}$. Moreover the remark 5.12 implies that the real $(1,1)$ form associated to $\tilde{a}$ is $-2i \sum_{ij} \tilde{a}_{i,\bar{j}} dz_i \wedge d\bar{z}_j$.

We can now make the identification 3.7, $\mathrm{sym}_0(TM,g) \simeq \mathrm{Hom}(\Lambda^2_- T^*M, \Lambda^2_+ T^*M)$ more precise. We identify $\Lambda^2_- T^*M$ with $\Lambda^{1,1}_{\omega^\perp,\mathbb{R}}$, that is with the real $(1,1)$-forms othogonal to the Kahler form ω, and $\Lambda^2_+ T^*M$ with $\Lambda^{0,2}T^*M \oplus \mathbb{R}\omega$. If $f \in \mathrm{sym}\,(TM,g)$ let $a(f) \in \mathrm{End}(TM^{1,0})$ and $b(f) \in \mathrm{Hom}(TM^{1,0}, TM^{0,1})$ the components of the extension of f to $TM \otimes \mathbb{C}$ as seen in lemma 5.10. With this notations we have:

Lemma 5.13. *For all $f \in \mathfrak{u}_0(T,J)$ then*

$$\delta(f)\Lambda^{1,1}_{\omega^\perp} \subseteq \mathbb{R}\omega .$$

Therefore the isometry $\delta : \mathrm{sym}_0(TM,g) \longrightarrow \mathrm{Hom}(\Lambda^2_- T^*M, \Lambda^2_+ T^*M)$ splits as :

$$\begin{array}{ccc} \mathrm{sym}_0(TM,g) \simeq \mathfrak{u}_0(TM,J) \oplus \mathfrak{su}(TM,J) & \longrightarrow & \mathrm{Hom}(\Lambda^{1,1}_{\omega^\perp,\mathbb{R}}, \Lambda^{0,2}T^*M) \oplus \mathrm{Hom}(\Lambda^{1,1}_{\omega^\perp,\mathbb{R}}, \mathbb{R}\omega) \\ (s,t) & \longmapsto & (\ \delta(\bar{b}(t)^*)\ ,\ \delta(s^*)\ ) \end{array}$$

Proof. It is clear that the derivation $i(s^*)$ induced by an element $s \in \mathfrak{u}_0(TM,J)$ preserves the spaces $\Lambda^{1,1}T^*M$, $\Lambda^{2,0}T^*M$ and $\Lambda^{0,2}T^*M$ since s is J-linear. Therefore for such s, $\delta(s^*)\Lambda^{1,1}_{\omega^\perp} \subseteq \Lambda^{1,1}$ but we know by lemma 3.7 that $\delta(s^*)\Lambda^2_- T^*M \subseteq \Lambda^2_+ T^*M$; as a result $\delta(s^*)\Lambda^{1,1}_{\omega^\perp} \subseteq \Lambda^{0,2}T^*M \oplus \mathbb{R}\omega$ and hence $\delta(s^*)\Lambda^{1,1}_{\omega^\perp} \subseteq \mathbb{R}\omega$. If $t \in \mathfrak{su}(TM,J)$, t is J antilinear, as a consequence its extension to $TM \otimes \mathbb{C}$ exchanges $TM^{1,0}$ and $TM^{0,1}$, and hence $i(t^*)\Lambda^{1,1} \subseteq \Lambda^{0,2}TM \oplus \Lambda^{2,0}TM$. We can write $t^* = b(t)^* + \bar{b}(t)^*$, $b(t)^*$:

$\Lambda^{0,1}T^*M \longrightarrow \Lambda^{1,0}T^*M$, and $\bar{b}(t)^* : \Lambda^{1,0}T^*M \longrightarrow \Lambda^{0,1}T^*M$. Therefore $i(b(t)^*)\Lambda^{1,1}T^*M \subseteq \Lambda^{2,0}T^*M$ and $i(\bar{b}(t)^*)\Lambda^{1,1}T^*M \subseteq \Lambda^{0,2}T^*M$. Therefore in the splitting

$$\mathrm{Hom}(\Lambda^2_- T^*M, \Lambda^2_+ T^*M) \simeq \mathrm{Hom}(\Lambda^{1,1}_{\omega^\perp,\mathbb{R}}, \Lambda^{0,2}T^*M) \oplus \mathrm{Hom}(\Lambda^{1,1}_{\omega^\perp,\mathbb{R}}, \mathbb{R}\omega)$$

an element $f = s + t$ acts as $\delta(\bar{b}(t)^*) \oplus \delta(s^*)$.

□

5.2.2 The main theorem

In this subsection we continue the computation of kernel equations (42) in the Kähler context. We are going to interpret the contribution of the metric (equation (42c), $(\partial\tilde{\mathbb{F}}_+/\partial\varphi)^*(\chi,\theta) = 0$) for a Kähler surface on an irreducible monopole $(A,\alpha,0)$. We recall that in general the partial differential $\partial\tilde{\mathbb{F}}_+/\partial\varphi(A,\psi,\mathrm{id})(s)$ is

$$\frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi}(A,\psi,\mathrm{id})(s) = \begin{pmatrix} -\frac{1}{2}\rho(\mathrm{div}\, s) - \rho\circ s\circ\nabla\psi \\ \delta(s)F_A^- \end{pmatrix}$$

where we have taken $s \in \mathrm{sym}_0(TM,g)$. Now $\nabla\alpha = \partial\alpha + \bar{\partial}\alpha = \partial\alpha$ since by (52) α is holomorphic. Moreover if $s \in \mathfrak{u}_0(TM,J)$, s leaves $TM^{1,0}$ invariant; hence by definition of the Clifford multiplication ρ on an hermitian surface:

$$\begin{aligned} \rho\circ s\circ\partial\alpha &= \sum_i \rho(a(s)^*(dz_i)\otimes\partial_{A,i}\alpha) \\ &= \sqrt{2}\sum_i [(a(s)^*(dz_i))^{0,1}\wedge\partial_{A,i}\alpha + \overline{(a(s)^*(dz_i))^{1,0}}\lrcorner\partial_{A,i}\alpha] = 0 \end{aligned}$$

because $a(s)^*(dz_i) \in \Lambda^{1,0}T^*M$ and because of (17). Therefore $\rho\circ(s+t)\circ\partial\alpha = \rho\circ\bar{b}^*(t)\circ\partial\alpha$. Seeing $\bar{b}^*$ in $A^{0,1}(TM^{1,0})$ we can further interpret $\rho\circ\bar{b}^*\circ\partial\alpha$ as $\sqrt{2}\bar{b}^*\lrcorner\partial\alpha$, where the last expression means the duality contraction between $TM^{1,0}$ and $\Lambda^{1,0}T^*M$, followed by multiplication by the form component in $\Lambda^{0,1}T^*M$. From what has been said until now the operator $\partial\tilde{\mathbb{F}}_+/\partial\varphi\,(A,\psi,\mathrm{id})$ can be written on Kähler surface as:

$$\begin{aligned} \partial\tilde{\mathbb{F}}_+/\partial\varphi(A,\alpha,0,\mathrm{id}) : \mathfrak{u}_0(TM,J)\oplus\mathfrak{su}(TM,J) &\longrightarrow A^{0,1}(N)\oplus A^{0,2}(M)\oplus\Gamma(\mathbb{R})\omega \\ (s,t) &\longmapsto \begin{pmatrix} -\frac{1}{\sqrt{2}}(\mathrm{div}\,(s+t))^{0,1}\alpha - \rho\circ\bar{b}^*(t)\circ\partial\alpha \\ \delta(\bar{b}(t)^*)F_A^- \\ \delta(s^*)F_A^- \end{pmatrix} \end{aligned}$$

We will now better interpret the term $(\mathrm{div}\,(s+t))^{0,1}\alpha$.

Lemma 5.14. *Let $s \in \mathrm{sym}\,(TM,g)$ a symmetric endomorphism. In the identification $TM \simeq T^*M$ provided by the metric we have*

$$\mathrm{div}\, s = -\nabla^* s\ .$$

Proof. We compute the symbol of the two differential operators in order to verify that the two symbols are equal in a point p. We use an adapted orthonormal frame e_i in the point p. Let e^i the dual frame, s_{ij} the symmetric endomorphism $e^i\otimes e_j + e^j\otimes e_i$ and f a function such that $f(p) = 0$ and $df(p) = \xi$. Then

$$\mathrm{div}\, f(s_{ij}) = \xi_j e^i + \xi_i e^j$$

and

$$\nabla^*(f s_{ij}) = -(\xi, e^i)e_j + -(\xi, e^j)e_i\ ,$$

which proves that the two operators have the same symbol. Now, it is clear that they coincide in p, because they are both zero on all the elements of the form s_{ij}.

$\square$

Let now $s+t$ in $\mathrm{sym}\,(TM,g)$, $s\in\mathfrak{u}(TM,J)$, $t\in\mathfrak{su}(TM,J)$. In the decomposition $TM^{1,0}\oplus TM^{0,1}$ $(s+t)$ can be written as:

$$s+t=\begin{pmatrix} a & \bar{b} \\ b & \bar{a}\end{pmatrix} \quad .$$

Therefore $\mathrm{div}\,(s+t)=\mathrm{div}\,(a+\bar{a}+b+\bar{b})$. Now a belongs to $\Lambda^{1,0}T^*M\otimes TM^{1,0}$. Identifying tangent and cotangent bundle by means of the complex bilinear form $g_{\mathbb{C}}$, a can be regarded as an element of $\Lambda^{1,0}T^*M\otimes\Lambda^{0,1}T^*M$. Analogously b can be seen as an element of $\Lambda^{1,0}T^*M\otimes\Lambda^{1,0}T^*M$, $\bar{a}\in\Lambda^{0,1}T^*M\otimes\Lambda^{1,0}T^*M$, $\bar{b}\in\Lambda^{0,1}T^*M\otimes\Lambda^{0,1}T^*M$. The Levi-Civita connection ∇ is compatible with the complex structure ($\nabla J=0$) and therefore preserves the type decomposition $TM^{1,0}\oplus TM^{0,1}$ and $\Lambda^{1,0}T^*M\oplus\Lambda^{0,1}T^*M$. As a consequence the connection ∇ induces differential operators (connections):

$$\nabla:\Gamma(\Lambda^{1,0}T^*M)\longrightarrow\Gamma(T^*M\otimes\Lambda^{1,0}T^*M)\ ,\quad \nabla:\Gamma(\Lambda^{0,1}T^*M)\longrightarrow\Gamma(T^*M\otimes\Lambda^{0,1}T^*M)\ .$$

Moreover we can split the connection ∇ according to types: $\nabla=\nabla^{1,0}+\nabla^{0,1}$. For brevity's sake we indicate $\nabla^{1,0}$ with D and $\nabla^{0,1}$ with $\bar{D}$. As a consequence:

$$\begin{aligned}\mathrm{div}\,(s+t) &= -\nabla^*(s+t)\\ &= -(D+\bar{D})^*(a+\bar{a}+b+\bar{b})\\ &= -D^*a-\bar{D}^*\bar{a}-D^*b-\bar{D}^*\bar{b}\ .\end{aligned}$$

When we take the $(0,1)$-component of $\mathrm{div}\,(s+t)$ we get

$$\mathrm{div}\,(s+t)^{0,1}=-D^*a-\bar{D}^*\bar{b}$$

We can finally write the partial differential $\partial\tilde{\mathbb{F}}_+/\partial\varphi(A,\alpha,0,\mathrm{id})(s,t)$ for $s\in\mathfrak{u}(TM,J)$, $t\in\mathfrak{su}(TM,J)$:

$$\frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi}(A,\alpha,0,\mathrm{id})(s,t)=\begin{pmatrix}\frac{1}{\sqrt{2}}(D^*a(s))\alpha+\frac{1}{\sqrt{2}}(\bar{D}^*\bar{b}(t))-\sqrt{2}\bar{b}(t)^*\lrcorner\partial\alpha\\ \delta(\bar{b}(t)^*)F_A^-\\ \delta(s^*)F_A^-\end{pmatrix}\tag{56}$$

We are now ready to prove the main theorem. Let $U^+(TM,J)=\mathrm{Sym}^+(TM,g)\cap\mathrm{End}(TM,J)$.

Theorem 5.15. *Let M a Kähler surface, g its Kähler metric. Let N a complex line bundle on M such that $2\deg(N)-\deg(K)<0$. Consider the Spin^c-structure on M whose spinor bundle is $W=\Lambda^{0,*}T^*M\otimes N$. Consider the perturbed Seiberg-Witten functional (36):*

$$\begin{aligned}(\tilde{\mathbb{F}}_+)^2_p:(\mathcal{A}^{U(1)}_{\det W})^2_p\times\Gamma^2_p(W_+)\times\mathrm{Sym}^+(TM,g)^2_p &\longrightarrow \Gamma^2_{p-1}(W_-)\times i\mathfrak{su}(W_+)^2_{p-1}\\ (A,\psi,\varphi) &\longmapsto (D_A^{\varphi^*g,\varphi^*\rho}\psi,\frac{(\varphi^*\rho)}{2}(F_A^{+,\varphi^*g})-[\psi^*\otimes\psi]_0)\end{aligned}$$

Any zero of $\tilde{\mathbb{F}}_+$ of the form (A,ψ,id) is a regular point for $\tilde{\mathbb{F}}_+$.

Proof. To prove that a zero of $\tilde{\mathbb{F}}_+$ of the form (A,ψ,id) is a regular point for $\tilde{\mathbb{F}}_+$ it is sufficient to prove that the differential at the point (A,ψ,id)

$$(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^2_p:A^1(M,i\mathbb{R})^2_p\times\Gamma^2_p(W_+)\times\mathrm{sym}\,(TM,g)^2_p\longrightarrow\Gamma^2_{p-1}(W_-)\times i\mathfrak{su}(W_+)^2_{p-1}$$

is surjective. By what we have said in remark 4.6 and by theorem 4.5 proving the surjectivity of $(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^2_p$ at a point (A,ψ,id) is equivalent to proving the surjectivity of

$$D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+:A^1(M,i\mathbb{R})\times\Gamma(W_+)\times\mathrm{sym}\,(TM,g)\longrightarrow\Gamma(W_-)\times i\mathfrak{su}(W_+)$$

Now by the discussion in subsection 4.2, proving the surjectivity of $D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+$ is equivalent to proving that $\ker(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^* = 0$. In subsection 4.3 we have proved that an element $(\chi,\theta) \in \Gamma(W_-)\times A^2_+(M,i\mathbb{R})$ satisfies $(D_{(A,\psi,\mathrm{id})}\tilde{\mathbb{F}}_+)^*(\chi,\theta) = 0$ if and only if it is a solution of equations (42). In the previous subsections of this section we have proved that equations (42a), (42b), (42d), (42e) on a Kähler manifold are equivalent to the system:

$$\begin{gathered}
\bar{\partial}^*\mu + \frac{1}{\sqrt{2}}\bar{\alpha}\chi = 0 \\
\sqrt{2}\bar{\partial}_A\chi - \mu\alpha = 0 \\
\bar{\partial}^*_A\chi = 0 \\
\lambda = 0
\end{gathered}$$

for $\chi \in A^{0,1}(N)$ and $\theta = i\lambda\omega + \mu - \bar{\mu} \in \mathbb{R}\omega \oplus A^{0,2}(M) \oplus A^{2,0}(M)$ and where the spinor $\psi = (\alpha, 0) \in A^{0,0}(N) \oplus A^{0,2}(N)$. We remark that the equations above take already into account the contribution of conformal perturbations of the metric. Let us now interpret equation (42c) in the context of Kähler geometry. It corresponds to

$$\left(\frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi}|_{\mathrm{sym}_0(TM,g)}\right)^*(\chi,\theta) = 0\ .$$

Thanks to the splitting $S^2_0T^*M \simeq \mathfrak{u}_0(TM,J) \oplus \mathfrak{su}(TM,J)$ we can define the two differential operators

$$P_1 := \frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi}|_{\mathfrak{u}_0(TM,J)}\ ,\quad P_2 := \frac{\partial\tilde{\mathbb{F}}_+}{\partial\varphi}|_{\mathfrak{su}(TM,J)}\ .$$

Therefore equation (42c) is equivalent to the two equations

$$\begin{gathered}
P_1^*(\chi,\theta) = 0 \\
P_2^*(\chi,\theta) = 0\ .
\end{gathered}$$

By the computation made in this subsection we can express the operator P_1 as:

$$P_1(s) = \begin{pmatrix} \frac{1}{2}(D^*a(s))\alpha \\ 0 \\ \delta(s)F_A^- \end{pmatrix}\ .$$

Therefore its adjoint is easily

$$P_1^*(\chi,\theta) = \mathrm{herm}[D(\bar{\alpha}\chi)] - \mathrm{Re}\,\mathrm{tr}\,D(\bar{\alpha}\chi) + (F_A^-)^*\otimes\lambda\omega\ .$$

Now $\mathrm{herm}[D(\bar{\alpha}\chi)] = D(\bar{\alpha}\chi) + \bar{D}(\alpha\bar{\chi})$ and $\mathrm{Re}\,\mathrm{tr}\,D(\bar{\alpha}\chi) = i(\partial^*(\bar{\alpha}\chi) - \bar{\partial}^*(\alpha\bar{\chi}))$. Therefore the equation $P_1^*(\chi,\theta) = 0$ becomes:

$$D(\bar{\alpha}\chi) + \bar{D}(\alpha\bar{\chi}) - i(\partial^*(\bar{\alpha}\chi) - \bar{\partial}^*(\alpha\bar{\chi})) + (F_A^-)^*\otimes\lambda\omega = 0\ ,$$

and identifying the first terms with real $(1,1)$-forms by lemma 5.11 it becomes:

$$\partial(\bar{\alpha}\chi) + \bar{\partial}(\alpha\bar{\chi}) - i(\partial^*(\bar{\alpha}\chi) - \bar{\partial}^*(\alpha\bar{\chi})) + (F_A^-)^*\otimes\lambda\omega = 0\ .$$

It is now easy to see that equations (42a), (42b), (42d), (42e) coupled with $P_1^*(\chi,\theta) = 0$ admit no nontrivial solutions. Indeed suppose that (χ, θ) is a solution of

$$\bar{\partial}^* \mu + \frac{1}{\sqrt{2}} \bar{\alpha} \chi = 0 \tag{58a}$$

$$\sqrt{2} \bar{\partial}_A \chi - \mu \alpha = 0 \tag{58b}$$

$$\bar{\partial}_A^* \chi = 0 \tag{58c}$$

$$\lambda = 0 \tag{58d}$$

$$\partial(\bar{\alpha}\chi) + \bar{\partial}(\alpha\bar{\chi}) - i(\partial^*(\bar{\alpha}\chi) - \bar{\partial}^*(\alpha\bar{\chi})) = 0 \tag{58e}$$

where we did not write the term $(F_A^-)^* \otimes \lambda\omega$ in the last equation because $\lambda = 0$. The first equation implies that $\bar{\partial}^*(\bar{\alpha}\chi) = 0$ and therefore $\partial^*(\alpha\bar{\chi}) = 0$. As a consequence the system (58) becomes equivalent to

$$\bar{\partial}^* \mu + \frac{1}{\sqrt{2}} \bar{\alpha} \chi = 0$$

$$\sqrt{2} \bar{\partial}_A \chi - \mu \alpha = 0$$

$$\bar{\partial}_A^* \chi = 0$$

$$\lambda = 0$$

$$\partial(\bar{\alpha}\chi) + \bar{\partial}(\alpha\bar{\chi}) = 0$$

We now apply the operator ∂^* to the last equation, obtaining

$$\Delta_{\partial}(\bar{\alpha}\chi) + \partial^* \bar{\partial}(\alpha\bar{\chi}) = 0 \; .$$

On a Kähler surface one has the Kähler identity $\partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$, hence $\partial^*\bar{\partial}(\alpha\bar{\chi}) = -\bar{\partial}\partial^*(\alpha\bar{\chi}) = 0$. As a consequence $\Delta_{\partial}(\bar{\alpha}\chi) = 0$, which means that $\bar{\alpha}\chi$ is Δ_{∂}-harmonic. But, again, on a Kähler manifold $\Delta_{\partial} = \Delta_{\bar{\partial}}$ and hence $\bar{\alpha}\chi$ is $\Delta_{\bar{\partial}}$-harmonic. This implies $\bar{\partial}(\bar{\alpha}\chi) = 0$. Applying the operator $\bar{\partial}$ to the first equation we get $\Delta_{\bar{\partial}}\mu = 0$, which implies $\bar{\partial}^*\mu = 0$. As a consequence $\bar{\alpha}\chi = 0$ and $\chi = 0$. From the second equation (or from lemma 4.22) we get $\mu = 0$. Therefore if there is a solution to 58, it must be necessarily zero.

□

Remark 5.16. We can reobtain theorem 5.15 by means of Gauduchon result (cf. [51], Corollaire 3) on the form of the Dirac operator on an almost hermitian 4-manifold for the canonical $Spin^c$-structure twisted by a line bundle N, which we recalled in subsection 1.9:

$$D_A = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*) - \frac{1}{4}\rho(\theta)$$

where θ is the Lee form. We place ourselves on a Kähler surface (M, g, J) and we perturb the metric $g_t = g + tk$ imposing that the new metric g_t remains hermitian with respect to the complex structure J, that is:

$$g_t(JX, JY) = g_t(X, Y) \tag{60}$$

for all X,$Y \in TM$. In this way we obtain a family (M, g_t, J) of hermitian structures on M. Taking the derivative of (60) in $t = 0$ we get $k(JX, JY) = k(X, Y)$, which means that the variation k is in $S^{1,1}T^*M$. We remark that the bundle of spinors remains the same, since J is fixed, but the Clifford multiplication changes. In any case the Dirac operator for the canonical $Spin^c$-structure on (M, g_t, J) is given by:

$$D_A^t = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*) - \frac{1}{4}\rho(\theta_t)$$

where now $\theta_t = \Lambda_t \omega_t$ is the Lee form of the non closed fundamental form ω_t. We remark that $\omega_0 = \omega$ is the original Kähler form of (M, g, J), which is closed, and that the Dolbeault operator $\bar{\partial}_A$ is fixed. We now take the initial Dirac equation $\bar{\partial}_A \alpha = 0$ on the Kähler surface and we perturb the metric with hermitian variations. As a result we obtain a family of Dirac equations:

$$\sqrt{2}(\bar{\partial}_A \alpha) - \frac{1}{4}\rho(\theta_t)\alpha = 0$$

and taking the derivative in $t = 0$ we get the variation of the Dirac operator applied to the spinor $(\alpha, 0)$ with respect to hemitian perturbations:

$$\frac{d}{dt} D^t_A \alpha|_{t=0} = -\frac{1}{4}\rho(\dot{\theta})\alpha \ .$$

We have:

$$\begin{aligned} \dot{\theta} = \frac{d}{dt}\Lambda_t d\omega_t|_{t=0} &= \dot{\Lambda} d\omega_0 + \Lambda d\dot{\omega} \\ &= \Lambda dk \end{aligned}$$

where we indicated again with k the real $(1,1)$-form associated to the hermitian tensor $k \in S^{1,1}T^*M$. Therefore the variation of the Dirac operator applied to the spinor $(\alpha, 0)$ is $-1/4\,(\Lambda k)^{0,1}\alpha$. But

$$\begin{aligned} (\Lambda dk)^{0,1} &= \Lambda \bar{\partial} k \\ &= [\Lambda, \bar{\partial}]k + \bar{\partial}\langle k, \omega\rangle \\ &= -i\partial^* k + \bar{\partial} \mathrm{tr}\, k \ . \end{aligned}$$

Now, when we take the adjoint to this operator we get the contribution

$$\partial(\bar{\alpha}\chi) + \bar{\partial}(\alpha\bar{\chi}) - i[\bar{\partial}^*(\bar{\alpha}\chi) - \partial^*(\alpha\bar{\chi})]\omega = 0$$

for the kernel equations. We can then proceed as in the preceding argument.

We now come to the geometric meaning of theorem 5.15. We need to recall the slice theorem for standard Seiberg-Witten moduli spaces (cf [91]). Let $\mathcal{C} = \mathcal{A}^{U(1)}_{\det W} \times \Gamma(W_+)$ be the standard Seiberg-Witten configuration space, $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$ the gauge group, and let $\mathcal{C}^2_p$ and $\mathcal{G}^2_{p+1}$ be their Sobolev completions.

Proposition 5.17. *There are local slices for the action of $\mathcal{G}^2_{p+1}$ on $\mathcal{C}^2_p$, that is, for each point $x \in \mathcal{C}^2_p$ there is an open neighbourhood of the point x and a smooth Hilbert submanifold S_x of this neighbourhood invariant under the stabilizer* $\mathrm{Stab}(x)$ *of x such that the natural map:*

$$S_x \times_{\mathrm{Stab}(x)} \mathcal{G}^2_{p+1} \longrightarrow \mathcal{C}^2_p$$

is a diffeomorphism onto a neighbourhood of the orbit through x.

The slice S for the action of $\mathcal{G}^2_{p+1}$, through a point (A, ψ) solution to the Seiberg-Witten equations, such that $\psi \neq 0$, is easily given by means of the map $\Upsilon_{(A,\psi)} : \mathcal{C}^2_p \longrightarrow \mathcal{C}^\infty(M, i\mathbb{R})$ defined as $\Upsilon_{(A,\psi)}(A', \varphi) = (D\gamma_{A,\psi})^*(A' - A, \varphi)$ where $\gamma_{A,\psi}$ denotes the $\mathcal{G}^2_{p+1}$-action through (A, ψ): $\gamma_{A,\psi}(g) = (g^{-2}Ag^2, g\psi)$, and thus $(D\gamma_{A,\psi})^*$ is then the adjoint of the differential of the action. We clearly have $D\Upsilon_{(A,\psi)} = (D\gamma_{A,\psi})^*$ which is underdetermined elliptic and surjective if $\psi \neq 0$ because $\ker D\gamma_{A,\psi} = \mathrm{Stab}(x) = \{0\}$; therefore $(D\gamma_{A,\psi})^*$ is surjective by theorem 4.5. Therefore by Implicit Function Theorem there is an open neighbourhood $U_{A,\psi}$ of (A, ψ) in $\mathcal{C}^2_p$ such that $Z(\Upsilon) \cap U_{A,\psi}$ is a smooth Hilbert manifold $S_{A,\psi}$. A direct application of the slice theorem gives a corresponding slice theorem for the action of $\mathcal{G}^2_{p+1}$ on $\mathfrak{C}^2_p = \mathcal{C}^2_p \times \mathrm{Sym}^+(TM, g)^2_p$ or for $\tilde{\mathfrak{C}}^2_p := \mathcal{C}^2_p \times U^+(TM, J)^2_p$. In particular the slice for the $\mathcal{G}^2_{p+1}$-action on $\mathfrak{C}^2_p$ (or $\tilde{\mathfrak{C}}^2_p$) on a point (A,ψ,φ) can be given (since $\mathcal{G}^2_{p+1}$ acts trivially on the second factor) by $S_{A,\psi} \times B(\varphi,\varepsilon)^2_p$, where $B(\varphi,\varepsilon)^2_p$ is the open ball of ray ε centered in φ in $\mathrm{Sym}^+(TM,g)^2_p$ (or in $U^+(TM,J)^2_p$). We know now from proposition 2.14 and 2.20 that the map:

$$\tilde{\mathbb{F}}_+ : \mathfrak{C}^2_p = \mathcal{C}^2_p \times \mathrm{Sym}^+(TM,g)^2_p \longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma(W_-)^2_{p-1}$$

is $\mathcal{G}^2_{p+1}$ equivariant. This means that if we consider the Hilbert vector bundle:

$$\mathcal{E} := \mathfrak{C}^2_p \times_{\mathcal{G}^2_{p+1}} (A^2_+(M,i\mathbb{R})^2_{p-1} \oplus \Gamma(W_-)^2_{p-1}) \longrightarrow \mathfrak{B}^2_p = \mathfrak{C}^2_p/\mathcal{G}^2_{p+1} = \mathcal{B}^2_p \times \mathrm{Sym}^+(TM,g)^2_p$$

the map $\tilde{\mathbb{F}}_+$ defines, by passing to the quotient, a section:

$$\Psi : \mathfrak{B}^2_p \longrightarrow \mathcal{E}$$

of the Hilbert vector bundle $\mathcal{E}$ on $\mathfrak{B}^2_p$ whose zero set is exactly the moduli space $\mathcal{M}_+$ considered in subsection 2.5. We can say the same for the restriction of $\tilde{\mathbb{F}}_+$ to $\tilde{\mathfrak{C}}^2_p = \mathcal{C}^2_p \times U^+(TM,J)^2_p$. We are now ready to prove:

Theorem 5.18. *Let (M,g,J) a Kähler surface. Let N a hermitian line bundle on M such that $2\deg(N) - \deg(K) < 0$. Consider the canonical Spin^c-structure on M twisted by the hermitian line bundle N. There exists $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that for a generic $\mathcal{C}^\infty$ metric $h = \varphi^* g$, $\varphi \in B(\mathrm{id},\varepsilon) \subseteq \mathrm{Sym}^+(TM,J)$, the Seiberg-Witten moduli space $\mathcal{M}^{SW}_h$ is smooth. Actually, the statement holds for a generic $\mathcal{C}^\infty$ hermitian metric $h = \varphi^* g$, $\varphi \in B(\mathrm{id},\varepsilon) \subseteq U^+(TM,J)$.*

Proof. The proof consists in finding the suitable smooth Hilbert manifold to which apply the Sard-Smale theorem. The existence of the slice for $\mathcal{B}^2_p$ provides a local model for $\mathfrak{B}^2_p$: if $x \in \mathcal{C}^2_p$ is a point with $\mathrm{Stab}(x) = \{1\}$, then the map:

$$\begin{aligned} S_{A,\psi} \times B(\varphi,\varepsilon)^2_p &\longrightarrow \mathcal{B}^2_p \times \mathrm{Sym}^+(TM,g)^2_p \\ (s,\phi) &\longmapsto ([s],\phi) \end{aligned} \tag{61}$$

is a diffeomorphism onto an open neighbourhood of $([A,\psi],\varphi)$. The vector bundle section Ψ can be seen locally as

$$\begin{aligned} S_{A,\psi} \times B(\varphi,\varepsilon)^2_p &\longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma(W_-)^2_{p-1} \\ (A',\psi',\varphi') &\longmapsto \tilde{\mathbb{F}}_+(A',\psi',\varphi')\ . \end{aligned} \tag{62}$$

Let now place ourselves on a Kähler monopole (A,ψ,id). Remembering how the slice $S_{A,\psi}$ has been built we immediately get $T_{A,\psi}S_{A,\psi} = \ker(D\gamma_{A,\psi})^* = (\mathrm{Im}D_{A,\psi}\gamma)^\perp$. Now $\mathrm{Im}D_{A,\psi}\gamma \subseteq \ker D\tilde{\mathbb{F}}_+$. Therefore theorem 5.15 tells that for a zero of $\tilde{\mathbb{F}}_+$ of the form (A,ψ,id) the differential $D_{A,\psi,\mathrm{id}}\Psi$ of the section Ψ is surjective. By the Implicit Function Theorem this means that there exists a neighbourhood $W_{A,\psi,\mathrm{id}}$ of (A,ψ,id) in $S_{A,\psi} \times B(\mathrm{id},\varepsilon)^2_p$ such that $Z(\Psi) \cap W_{A,\psi,\mathrm{id}}$ is a smooth Hilbert manifold and for all $x \in W_{A,\psi,\mathrm{id}} \cap Z(\Psi)$ the differential $D_x\Psi$ is surjective. We can always suppose that the neighbourhood $W_{A,\psi,\mathrm{id}}$ is of the form $V_{A,\psi} \times B(\mathrm{id},\varepsilon(A,\psi,\mathrm{id}))^2_p$, where $V_{A,\psi}$ is an open neighbourhood of (A,ψ) in $S_{A,\psi}$. Such neighbourhood defines by the diffeomorphism (62) an open neighborhood $\overline{W}_{A,\psi,\mathrm{id}}$ of the point $([A,\psi],\mathrm{id}) \in \mathfrak{B}^2_p$. In particular this proves firstly that every point $([A,\psi],\mathrm{id}) \in \mathcal{M}^{SW}_g = \pi^{-1}(\mathrm{id})$ is smooth as a point of $\mathcal{M}_+$; secondly that the moduli space $\mathcal{M}^{SW}_g$ viewed as the fiber $\pi^{-1}(\mathrm{id})$ of the projection $\pi : \mathcal{M}_+ \longrightarrow \mathrm{Met}(M)^2_p$ can be covered by the open set

$$\mathcal{U} = \cup_{([A,\psi],\mathrm{id}) \in \mathcal{M}^{SW}_g} \overline{W}_{A,\psi,\mathrm{id}}\ .$$

Now it is a fundamental result of standard Seiberg-Witten theory that the moduli space $\mathcal{M}^{SW}_g$ is compact (cf [91]). As a consequence a finite number of neighborhoods $\overline{W}_{A,\psi,\mathrm{id}}$ suffices to cover $\mathcal{M}^{SW}_g$. This implies in particular that there exists an ε such that $\pi^{-1}(B(\mathrm{id},\varepsilon)_p^2) \subseteq \mathcal{U}$. By construction of the neighbourhoods $W_{A,\psi,\mathrm{id}}$ we have that $\mathcal{W}^\varepsilon_{\mathrm{id}} := Z(\Psi) \cap \pi^{-1}(B(\mathrm{id},\varepsilon)_p^2)$ is a smooth Hilbert manifold and for each $x \in \mathcal{W}^\varepsilon_{\mathrm{id}}$ $D_x\Psi$ is surjective. This is the smooth Hilbert manifold we will use to apply Sard-Smale theorem. The rest of the proof is now standard matter. In each point of $\mathcal{W}^\varepsilon_{\mathrm{id}}$ the tangent space is given by:

$$
\begin{aligned}
T_{(A,\psi,\varphi)}\mathcal{W}^\varepsilon_{\mathrm{id}} &= \ker D_{(A,\psi,\varphi)} = \ker D_{(A,\psi,\varphi)}\tilde{\mathbb{F}}_+|_{\ker D_{(A,\psi,\varphi)}\gamma^*} \\
&= \{(X,Y) \in \ker(D_{(A,\psi,\varphi)}\gamma)^* \oplus \mathrm{sym}\,(TM,g)_p^2 \,|D_{A,\psi}F^{\varphi^* g}(X) + \partial\tilde{\mathbb{F}}_+/\partial\varphi(Y) = 0\} \,.
\end{aligned}
$$

Let now consider the projection

$$
\begin{aligned}
\pi : \mathcal{W}^\varepsilon_{\mathrm{id}} &\longrightarrow B(\mathrm{id},\varepsilon)_p^2 \\
(X,Y) &\longmapsto Y
\end{aligned}
$$

Since by construction $D_{(A,\psi,\varphi)}\tilde{\mathbb{F}}_+|_{\ker D_{(A,\psi,\varphi)}\gamma^*}$ is surjective for all $(A,\psi,\varphi) \in \mathcal{W}^\varepsilon_{\mathrm{id}}$ it is immediate to see that $D\pi$ is surjective at a point (A,ψ,φ) if and only if $D_{(A,\psi)}F^{\varphi^* g}$ is surjective. Therefore if φ is a regular value for π, the fiber $\pi^{-1}(\varphi) = \mathcal{M}^{SW}_{\varphi^* g}$ will be a smooth manifold. Moreover one can see that $\mathrm{coker}\, D_{(A,\psi,\varphi)}\pi = \mathrm{coker}\, D_{(A,\psi)}F^{\varphi^* g}$ and $\ker D_{(A,\psi,\varphi)}\pi = \ker D_{(A,\psi)}F^{\varphi^* g}$. Therefore the kernel and cokernel of $D\pi$ have finite dimension, hence π is a smooth Fredholm map of paracompact Hilbert manifolds. In particular Sard-Smale theorem applies and we get that the regular values of π form a second category (G_δ) set in $B(\mathrm{id},\varepsilon)_p^2$. Actually, shrinking the ball $B(\mathrm{id},\varepsilon)_p^2$ if necessary, since π has compact fibers, the regular values are a dense open set Ω_p^2 in $B(\mathrm{id},\varepsilon)_p^2$. Therefore for a dense open set of metrics g' in $B(\mathrm{id},\varepsilon)_p^2$ the moduli space $\mathcal{M}^{SW}_{g'}$ is smooth. Since $\Omega_p^2 \cap \mathrm{Sym}^+(TM,J)$ is dense in $B(\mathrm{id},\varepsilon) := B(\mathrm{id},\varepsilon)_p^2 \cap \mathrm{Sym}^+(TM,J)$, we can deduce that for a generic $\mathcal{C}^\infty$ metric $h = \varphi^* g$, $\varphi \in B(\mathrm{id},\varepsilon) \subseteq \mathrm{Sym}^+(TM,g)$, the moduli space $\mathcal{M}^{SW}_h$ is smooth.

We carried out our discussion assuming the surjectivity of

$$
D_{(A,\psi,\varphi)}\tilde{\mathbb{F}}_+ : T_{(A,\psi)}\mathcal{C}_p^2 \times \mathrm{sym}\,(TM,g)_p^2 \longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma(W_-)^2_{p-1} \,.
$$

Theorem 5.15 actually states something stronger: the surjectivity of

$$
D_{(A,\psi,\varphi)}\tilde{\mathbb{F}}_+ : T_{(A,\psi)}\mathcal{C}_p^2 \times \mathfrak{u}(TM,J)_p^2 \longrightarrow A^2_+(M,i\mathbb{R})^2_{p-1} \times \Gamma(W_-)^2_{p-1} \,.
$$

Carrying out the discussion with this stronger hypothesis we can choose $U^+(TM,J)^2_p$ as parameter space and the ball $B(\mathrm{id},\varepsilon)_p^2 \in U^+(TM,J)^2_p$. With exactly the same proof we get the stronger result that for a generic $\mathcal{C}^\infty$ *hermitian* metric $h = \varphi^* g$, $\varphi \in B(\mathrm{id},\varepsilon) \subseteq U^+(TM,J)$ the moduli space $\mathcal{M}^{SW}_h$ is smooth.

□

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