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La mesure de Mahler d'une famille de polynômes exacts

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Mathematics knows no races or geographic boundaries;
for mathematics, the cultural world is one country.

DAVID HILBERT

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Titre: La mesure de Mahler d'une famille de polynômes exacts

Résumé

Dans cette thèse, nous étudions la suite de mesures de Mahler d'une famille de polynômes à deux variables exacts et réguliers, que nous notons $P_d := \sum_{0 \leq i+j \leq d} x^i y^j$. Elle n'est bornée ni en volume, ni en genre de la courbe algébrique sous-jacente. Nous obtenons une expression pour la mesure de Mahler de P_d comme somme finie de valeurs spéciales du dilogarithme de Bloch-Wigner. Nous utilisons SageMath pour approximer $m(P_d)$ pour $1 \leq d \leq 1000$. En recourant à trois méthodes différentes, nous prouvons que la limite de la suite de mesures de Mahler de cette famille converge vers $\frac{9}{2\pi^2}\zeta(3)$. De plus, nous calculons le développement asymptotique de la mesure de Mahler de P_d et prouvons que sa vitesse de convergence est de $O(\frac{\log d}{d^2})$. Nous démontrons également une généralisation du théorème de Boyd-Lawton, affirmant que les mesures de Mahler multivariées peuvent être approximées en utilisant les mesures de Mahler de dimension inférieure. Enfin, nous prouvons que la mesure de Mahler de P_d pour d arbitraire peut être écrite comme une combinaison linéaire de fonctions L associées à un caractère de Dirichlet primitif impair. Nous calculons finalement explicitement la représentation de la mesure de Mahler de P_d en termes de fonctions L, pour $1 \leq d \leq 6$.

Mots clés

Mesure de Mahler, polynôme, polynôme exact, polynôme régulier, dilogarithme de Bloch-Wigner, développement asymptotique, fonctions L de Dirichlet, caractère de Dirichlet.

Title: The Mahler measure of a family of exact polynomials

Abstract

In this thesis we investigate the sequence of Mahler measures of a family of bivariate regular exact polynomials, called $P_d := \sum_{0 \le i+j \le d} x^i y^j$, unbounded in both degree and the genus of the algebraic curve. We obtain a closed formula for the Mahler measure of P_d in terms of special values of the Bloch–Wigner dilogarithm. We approximate $m(P_d)$, for $1 \le d \le 1000$, with arbitrary precision using SageMath. Using 3 different methods we prove that the limit of the sequence of the Mahler measure of this family converges to $\frac{9}{2\pi^2}\zeta(3)$. Moreover, we compute the asymptotic expansion of the Mahler measure of P_d which implies that the rate of the convergence is $O(\frac{\log d}{d^2})$. We also prove a generalization of the theorem of the Boyd-Lawton which asserts that the multivariate Mahler measures can be approximated using the lower dimensional Mahler measures. Finally, we prove that the Mahler measure of P_d, for arbitrary d can be written as a linear combination of L-functions associated with an odd primitive Dirichlet character. In addition, we compute explicitly the representation of the Mahler measure of P_d in terms of L-functions, for $1 \le d \le 6$.

Key words

Mahler measure, polynomial, exact polynomial, regular polynomial, Bloch–Wigner dilogarithm, asymptotic expansion, Dirichlet L-functions, Dirichlet characters.

Contents

CHAPTER I

Introduction

I.1. Historical context and important results around the Mahler measure

In[1] 1933, Derrick Henry Lehmer [Leh33] searched for large primes among the prime factors of Pierce numbers $\Delta_n(P) := \prod_{j=1}^{d}(\alpha_j^n - 1)$, where P is a monic polynomial with integer coefficients and $P(x) = \prod_{j=1}^{d}(x - \alpha_j)$ is its complex factorization. He proved that $\lim_{n\to\infty} \sqrt[n]{\Delta_n}$ exists and is equal to $\prod_{j=1}^{d} \max\{1, |\alpha_j|\}$ (see Proposition II.1.14). Lehmer denoted the limit by $M(P)$. Later, Kurt Mahler extended the definition of $M(\cdot)$, to arbitrary non zero univariate polynomials as follows:

$$M(P) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where, $P(x) = a_d \prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{C}[x]$. The Mahler measure of the non zero constant polynomial $P(x) = a_0$ is $|a_0|$ and the Mahler measure of zero is defined to be 1. He successfully extended the definition to the multivariable polynomials (see Eq. (I.1.1)) and $M(P)$ is named the Mahler measure of P in honor of Kurt Mahler. Following a theorem of Kronecker and Dedekind, one can prove that the Mahler measure of a univariate polynomial with integer coefficients is an algebraic integer (see Section II.1). A property of the Mahler measure, implied directly by the definition, is that the Mahler measure of polynomials with integer coefficients is greater than or equal to 1. Kronecker's Theorem [Kro57] characterizes the univariate polynomials with integer coefficients and Mahler measure equal to 1: those are essentially the product of cyclotomic polynomials up to monomial factors (see Proposition II.1.16). Lehmer argued that to obtain large primes from the factorization of $\Delta_n(P)$, it is advantageous that this sequence increases very slowly. In other words, we want $M(P)$ as small as possible. Inspired by this, he asked a question in 1933 which is still open and became known as the Lehmer conjecture:

CONJECTURE. [Lehmer Conjecture (1933)] There is a constant $C > 1$, such that for any $P \in \mathbb{Z}[x]$ if $M(P) > 1$, then $M(P) \geq C$.

The smallest Mahler measure known until now is for the degree 10 polynomial, $P_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, already discovered by Lehmer [Leh33]. Breusch [Bre51] and (independently) Smyth [Smy71] proved the Lehmer conjecture for a non reciprocal polynomial P (a polynomial P is reciprocal if $P(x) = \pm x^{\deg(P)} P(1/x)$, see Definition II.1.19):

PROPOSITION ([Smy71]). *Let P be a non reciprocal irreducible polynomial $P(x) \in \mathbb{Z}[x]$, then $M(P) \geq x_0 = 1.32471795\ldots$, the real zero of the polynomial $x^3 - x - 1$.*

For other partial solutions to the Lehmer conjecture we refer to [BM71],[Ste78],[CS82] and [Lou83]. This conjecture has importance in many different domains in Mathematics such as Analytic Number Theory, Hyperbolic Geometry, Ergodic Theory, etc. For instance, an answer to the Lehmer conjecture would affect many different conjectures such as Boyd conjecture (Conjecture II.4.23), Salem conjecture (Conjecture II.4.12), Short Geodesic Conjecture for arithmetic

[1] The historical part of this introduction is inspired by [BL13, BZ20, Boy81b].

hyperbolic 2-orbifolds ([MR03, Theorem 12.3.4]), Schmidt conjecture on the periodic points of the β-transformation, in the case that β is a Salem number ([DK02, Conjecture 28.1]), etc. There is also the Schinzel and Zassenhaus conjecture, which was recently solved by Dimitrov [Dim19] and if Lehmer's conjecture is true, then it gives the universal constant for this already solved conjecture (Theorem II.4.5).

As we mentioned, Kurt Mahler [Mah62b], in 1960, generalized the definition of the Mahler measure to multivariate polynomials. The idea of this generalization comes from Jensen's equality:

Proposition ([Jen00]). *For any $\alpha \in \mathbb{C}$ we have $\int_0^1 \log|e^{2\pi i t} - \alpha| dt = \max\{0, \log|\alpha|\}$.*

By applying Jensen's equality for a non zero polynomial $P(x) = a_d \prod_{j=1}^d (x - \alpha_j)$, Mahler proved the following equalities:

$$m(P) := \log(M(P)) = \log|a_d| + \sum_{j=1}^{d} \log\max\{|\alpha_j|, 1\} = \frac{1}{2\pi i}\int_0^1 \log|P(x)|\frac{dx}{x}.$$

Mahler generalized the definition of logarithmic Mahler measure to multivariate polynomials, using the integral formula:

Definition. For a non zero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$, the logarithmic Mahler measure is defined by:

$$m(P) = \frac{1}{(2\pi i)^n}\int \cdots \int_{|x_1|=\cdots=|x_n|=1} \log|P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}. \tag{I.1.1}$$

In fact, $m(P)$ is the arithmetic mean of $\log|P|$ over the n-dimensional unit torus $\mathbb{T}^n$. Mahler [Mah62a] proved that for any Laurent polynomial P, the Mahler measure $m(P)$ exists as an improper Riemann integral. Moreover, if P has integer coefficients, then $m(P) \geq 0$. Furthermore, for a nonzero polynomial P he defined

$$M(P) := \exp(m(P)).$$

From the definition we conclude that the Mahler measure is a kind of height function for polynomials. In fact, Mahler constructed this object because he was looking for inequalities between the heights (such as length, $L(P)$, or height, $H(P)$, see Eqs. (II.1.1) and (II.1.2)) of a product of polynomials and the heights of the factors.
The computation of the Mahler measure of a polynomial in 2 or more variables in general is a hard question. The simplest evaluation in 2 or 3 variables are the following famous examples due to Smyth (1981):

Example. We have:

- [Smy81a]
$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$
where $\chi_{-3}(n) = (\frac{-3}{n})$ is the odd quadratic Dirichlet character of modulus 3.
- [Smy81a, Boy81b]
$$m(1 + x + y + z) = \frac{7}{2\pi^2}\zeta(3),$$
where ζ is the Riemann Zeta function.

The above examples illustrate an important application of Mahler measure in Analytic Number Theory. To see more examples of computations of Mahler measures of multivariate polynomials, we refer to Bertin [Ber08, Ber01a, Ber04b, Ber04a], Boyd [Boy98, Boy81b, Boy02], Boyd and Rodriguez-Villegas [BRVD03, BRV02], Condon [Con03], Rodriguez-Villegas [RV99], Smyth [Smy81b, Smy02], Bertin and Zudilin [BZ17, BZ16], Lalín [Lal06b, Lal03, Lal06a, Lal07, DL07]. It is not always possible to compute a closed formula for the Mahler measure of multivariate polynomials. However, there exists a class of bivariate polynomials, called regular exact, for which the Mahler measure can be expressed as a finite sum. A bivariate polynomial P is called exact if the differential 1-form

$$\eta := \log|y| d \arg x - \log|x| d \arg y,$$

restricted to the smooth zeros of P, is exact. There is also a generalization of the notion of exactness for multivariate polynomials, called k-exact, for $k \in \mathbb{Z}_{\geq 1}$ (see [Den97, Mai00, Lal16, GL21]). It is interesting that the Mahler measure of some exact polynomials links to special values of L-function or ζ-function. Despite the existence of a closed formula to compute the Mahler measure of regular exact polynomials, such as [GM21] or [BRV02, BRVD03, Lal07, Lal08] there is no general algorithm to recognize the exactness of polynomials. For instance, there is a finite number of exact bivariate polynomials of genus $g \leq 1$ with Newton polygon of bounded area (see [GM21]).

One may ask about the links between the Mahler measure of univariate and multivariate polynomials. In other words, the question is to compute the Mahler measure of multivariate polynomial using certain sequences of univariate polynomials. This question was answered by Boyd and Lawton and it became one of the most important results in this area, called the theorem of Boyd-Lawton. Historically, the idea of this theorem came to Boyd during his research on the Lehmer conjecture. As we know, the set of the values of the Mahler measure of polynomials with integer coefficients is a subset of $[1, \infty)$. Following Boyd [Boy81b], we set

$$L := \{M(P) | P \in \mathbb{Z}[x]\},$$

so if the Lehmer conjecture is proved, then 1 cannot be a limit point of L (x is a limit point of $L \subset [1, \infty)$ if every neighbourhood of x with respect to the Euclidean topology inherited from $\mathbb{R}$ to $[1, \infty)$ also contains a point of L other than x itself). One can see that if 1 is a limit point of L, then $L^{(k)} = [1, \infty)$, for every k (see Proposition II.4.6). Hence, to provide a positive answer to the Lehmer question it would suffice to show that L is nowhere dense in $[1, \infty)$ or only $\min L^{(k)} > 1$, for some k (here $L^{(1)}$ is the set of the limit points of L and $L^{(k)}$ is the set of k-th derived set of L). The study of two remarkable subsets of L, namely the set of Pisot numbers (i.e. a real algebraic integer $\alpha > 1$, such that all of its algebraic conjugates (except itself) have absolute value less than one), denoted by S, and Salem numbers (i.e. a real algebraic integer $\alpha > 1$ whose conjugate roots all have absolute value less than or equal to 1, and at least one of which has absolute value exactly 1), denoted by T, and the work done by Salem [Sal44], Siegel [Sie44] and Smyth [Smy71], inspired Boyd to continue in this direction. We recall that the Mahler measure of an algebraic number is defined as the Mahler measure of its minimal polynomial. Salem [Sal44] proved that S is closed [2]. Hence, $\min S > 1$, since $1 \notin S$. However, we do not have enough information about T. According to [Sal45], if $a > 1$ is in T, then its minimal polynomial is reciprocal. Because of the success of S in contrast to T it is natural to consider the following set

$$L_0 := \{M(P) | P \in \mathbb{Z}[x] \text{ and } P \text{ is non reciprocal }\}.$$

[2] In this thesis, the topology that we consider is the Euclidean topology that is inherited from $\mathbb{R}$.

Then, Smyth [Smy71] proved that:

$$\min S = \min L_0 = x_0,$$

where x_0 is the real zero of $x^3 - x - 1$. Then, as we mentioned, the Lehmer conjecture for non reciprocal polynomials is solved. It is conjectured that the Mahler measure of Lehmer's 10 degree polynomial, P_L, is the smallest Salem number, and the answer to the Lehmer conjecture:

$$\min L = M(P_L) = \min T.$$

By analogy with S, it could be possible that L_0 or even L is a closed set. However, it seems highly unlikely. Indeed, Boyd suggested to study some larger sets:

$$L^\sharp := \{M(P) | P \in \mathbb{Z}[x_1, \dots, x_n],\ n \geq 1\},$$
$$L_0^\sharp := \{M(P) | P \text{ is non reciprocal},\ P \in \mathbb{Z}[x_1, \dots, x_n],\ n \geq 1\}.$$

For the definition of multivariate reciprocal polynomials we refer to Definition II.4.14. One may think that the two new sets $L^\sharp$ and $L_0^\sharp$ are much bigger than L and L_0. However, Boyd noticed that $L^\sharp \subseteq \bar{L}$. This is due to the fact that the Mahler measure of a multivariable polynomial can be obtained as the limit of a suitable sequence of Mahler measures of univariate polynomials (see Section II.4.3). Some special cases of this theorem were proved in [Boy81b]. The most general case, as we already mentioned, known as the Theorem of Boyd-Lawton, has been proved later by Lawton in [Law83]. In other words, $L^\sharp$ is bigger than L, but not much larger. As we mentioned, the Mahler measure of a polynomial with integer coefficients is an algebraic integer. On the other side, we have the example of Smyth $m(1+x+y+z) = \frac{7}{2\pi}\zeta(3)$, which is probably transcendental. Then, since $L \subset L^\sharp \subseteq \bar{L}$, L could not be closed. This further clarifies the reason for which Boyd studied $L^\sharp$ and stated the following conjectures:

Conjecture. (Boyd Conjectures)

(1) $L^\sharp$ is closed.

(2) $L_0^\sharp$ is closed.

We notice that Boyd Conjecture ($L^\sharp = \overline{L^\sharp}$), implies the Lehmer conjecture (see Section II.4.2). In order to progress towards this conjecture, it is important to study more examples of the Mahler measure of multivariate polynomials. This thesis undertakes the systematic study of a family of such examples.

I.2. Content of the thesis

In this thesis, we study the Mahler measure of a family of exact bivariate polynomials, $P_d(x,y) := \sum_{0 \leq i+j \leq d} x^i y^j$, with $d \in \mathbb{Z}_{\geq 1}$, suggested by François Brunault. This family contains infinitely many polynomials, whose (total) degree and genus (of the associated algebraic curve) are unboundedly increasing. The study of this family allows us to illustrate the Boyd conjecture, to generalize Boyd-Lawton theorem, and to exhibit new examples of links between Mahler measure and special values of L-functions. Each polynomial P_d is regular exact (see Theorem III.1.18). Thus, we can apply the following closed formula from [GM21] to express its Mahler measure. The toric points of P are points (x,y) such that $|x| = |y| = 1$ and $P(x,y) = 0$. Roughly speaking a polynomial P is called regular if $\frac{x\partial_x P}{y\partial_y P} \notin \mathbb{R}$ at each toric points (see Definition III.2.11 for the precise definition).

Proposition ([GM21]). *For a regular exact monic polynomial $P \in \mathbb{C}[x,y]$ we have:*

$$m(P) = \frac{1}{2\pi} \sum_{(x,y) \text{ is a toric point of } P} \epsilon(x,y) V(x,y).$$

Here, $\epsilon(x,y)$ is a sign; and V is a volume function associated to P (i.e. a primitive for the differential form η restricted to the smooth zeros of P). See Section III.2 for precise definitions.

Chapter III of this thesis is devoted to obtaining an explicit and effective expression for $m(P_d)$ for arbitrary d. In Theorem III.1.18 we prove that P_d is exact and we compute a volume function in terms of the Bloch-Wigner dilogarithm (see Definition III.1.7):

$$V(x,y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)].$$

For the case $d = 1$ it is a classical fact that $P(x,y) = x + y + 1$ is exact and a volume function is $-D(-x)$. In Proposition III.2.9 we compute the set of the toric points of P_d. It is the union $U_{d+1} \cup U_{d+2}$ where:

$$U_{d+1} := \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1, x \neq y\},$$
$$U_{d+2} := \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1, x \neq y\}.$$

Finally, in Proposition III.2.15 we determine $\epsilon(x,y)$ at each toric point of P_d, for every d. Thus, we can apply the formula in [GM21] to each P_d. In Proposition III.2.18, we obtain the following closed formula to compute $m(P_d)$.

$$\begin{aligned} 2\pi m(P_d) =& \frac{2}{(d+1)} \sum_{\substack{(x,y)\in U_{d+2} \\ \text{with } \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)] \\ & - \frac{2}{d+2} \sum_{\substack{(x,y)\in U_{d+1} \\ \text{with } \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)]. \end{aligned}$$

We can approximate the values of $m(P_d)$, with arbitrary precision, for $1 \leq d \leq 1000$ using SageMath. Fig. 1 shows the graph of $m(P_d)$.

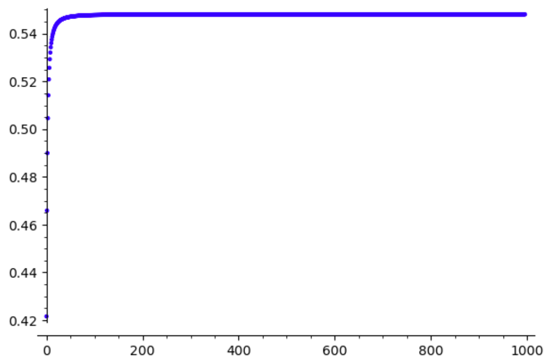

Figure 1. The graph of $m(P_d)$, for $1 \leq d \leq 1000$.

The figure indicates a limit for $m(P_d)$. The aim of Chapter IV is to determine this limit. In Chapter IV, we provide two different methods for computing this limit, and in Chapter V, we also give a third method for achieving this goal. During my work, the first method that I applied was based on Riemann sum technics and error estimations. More precisely, we simplify the volume function at toric points by writing it in terms of the values of a new function, $\mathrm{vol} : \mathbb{R}^2 \to \mathbb{R}$, defined by $\mathrm{vol}(\theta, \alpha) := D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})$. In Theorem IV.2.5, using this new function, we write another closed formula of $m(P_d)$:

THEOREM. *For every $d \in \mathbb{Z}_{\geq 1}$ we have:*

$$2\pi m(P_d) = \frac{2}{d+1} \sum_{0<k<k'\leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right) - \frac{2}{d+2} \sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right).$$

One advantage of the new formula of $m(P_d)$ is that we can write

$$m(P_d) = \frac{(d+2)^2}{(d+1)} R_{d+2} - \frac{(d+1)^2}{(d+2)} R_{d+1},$$

where R_d is the following Riemann sum of vol on the triangle with vertices $(0,0), (0,2\pi)$, and $(2\pi, 0)$, denoted by T:

$$R_d := \frac{4\pi^2}{d^2} \sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d}, \frac{2(k'-k)\pi}{d}\right).$$

Since vol is continuous, the Riemann sums R_{d+1} and R_{d+2} converge to the integral of vol over T. However, the coefficients multiplying the Riemann sums in the expression of $m(P_d)$ depend on d and go to infinity. Therefore, we need to estimate the errors $E(d) = |\iint_T \mathrm{vol}\, dA - R_d|$ to compute the limit. The analytical study of vol shows that it is concave and positive on T, see Lemmas IV.3.2 and IV.3.6. Finally, in Lemma IV.4.1, by taking advantage of the concavity of vol we prove that $E(d) = o(1/d)$. As we can write:

$$2\pi m(P_d) = \frac{3d^2+8d+7}{4\pi^3(d^2+3d+2)} \iint_T \mathrm{vol}(\theta,\alpha) dA + \frac{(d+1)^2}{2\pi^2(d+2)} E(d+1) - \frac{(d+2)^2}{2\pi^2(d+1)} E(d+2),$$

using that $E(d) = o(1/d)$ and $\iint_T \mathrm{vol}(\theta,\alpha)dA = 6\pi\zeta(3)$ (see Lemma IV.2.6), we prove in Theorem IV.1.1 the following:

THEOREM ([Meh21, BGMP22]). *We have:*

$$\lim_{d\to\infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3).$$

The presence of $\zeta(3)$ in the limit is striking and reminds us of the famous examples of Smyth.

After I computed the above limit, during a collaboration with Brunault, Guilloux and Pengo, we realized that the value of the limit, $\frac{9}{2\pi^2}\zeta(3)$, is itself a Mahler measure:

THEOREM I.2.1 (D'Andrea, Lalín [DL07]). *Let $P_\infty := (1-x)(1-y) - (1-z)(1-w) \in \mathbb{C}[x,y,z,w]$. Then the following equality holds:*

$$m(P_\infty) = \frac{9}{2\pi^2}\zeta(3).$$

That implies that $\lim_{d\to\infty} m(P_d) = m(P_\infty)$. We wondered if this equality has links to Boyd-Lawton theorem. We found the following expression for P_d:

$$P_d(x,y) = \frac{P_\infty(x^{d+2}, y, x, y^{d+2})}{(1-x)(1-y)(x-y)},$$

and since the Mahler measure of the denominator is zero we have:

$$m(P_d(x,y)) = m(P_\infty(x^{d+2}, y, x, y^{d+2})). \tag{I.2.1}$$

In fact, if the Theorem of Boyd-Lawton could be generalized to sequences of bivariate polynomials (instead of univariate), this would prove directly that $\lim_{d\to\infty} m(P_d) = m(P_\infty)$. Indeed, we prove this generalization in our article [BGMP22] in collaboration with Brunault, Guilloux and Pengo. In fact, we prove that the Mahler measure of a multivariable polynomial can be approximated by lower-dimensional Mahler measures. Let us clarify which sequences of lower dimensional polynomials give rise to such approximations for P. Let $A = (a_{ij})$ be a matrix in $\mathbb{Z}^{m\times n}$, and $P \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. We define the m-variable polynomial P_A as follows: $P_A(z_1, \dots, z_m) := P(z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \dots,\ z_1^{a_{1,n}} \cdots z_m^{a_{m,n}})$. To describe that a sequence of matrices goes to infinity, we consider the function $\rho : \mathbb{Z}^{m\times n} \mapsto \mathbb{Z}_{\geq 1}$, defined by:

$$\rho(A) := \min\{\|\mathbf{v}\|_\infty : \mathbf{v} \in \mathbb{Z}^{n\times 1} \setminus \{\mathbf{0}\},\ A \cdot \mathbf{v} = \mathbf{0}\}.$$

Theorem ([BGMP22] Generalization of Boyd-Lawton's theorem). *For an n-variable non zero Laurent polynomial P, for any sequence of matrices $A_d \in \mathbb{Z}^{m_d\times n}$ such that* $\lim_{d\to\infty} \rho(A_d) = \infty$*, we have:*

$$\lim_{d\to\infty} m(P_{A_d}) = m(P).$$

To prove this theorem, we note that for an n-variable polynomial P, the Mahler measure $m(P)$ is the integral of $\log|P|$ with respect to the probability Haar measure $\mu = \mu_n := \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ on the real n-dimensional torus. Let now A_d be a matrix $(a_{ij}) \in \mathbb{Z}^{m\times n}$. We consider the push forward μ_{A_d} of μ along the map $\mathbb{T}^m \to \mathbb{T}^n$ given by $(z_1, \dots, z_m) \mapsto (z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \dots,\ z_1^{a_{1,n}} \cdots z_m^{a_{m,n}})$. Then, one can see that $m(P_{A_d})$ is the integral of $\log|P|$ with respect to μ_{A_d} and we need to prove that it converges to the integral of $\log|P|$ with respect to μ, for every non zero Laurent polynomial. In Lemma II.4.21, we recall the classic fact that, if the condition $\rho(A_d) \to \infty$ is satisfied, then $(\mu_{A_d})_{d\in\mathbb{Z}_{\geq 1}}$ converges to μ weakly. However, the weak convergence of the measures is not sufficient since the function $\log|P|$ can have value ∞. We circumvent this difficulty by using uniform L^2 estimates for $\log|P_{A_d}|$, borrowed from [DH19, Lemma A.3] (see Proposition II.4.22 and Section II.4.4). Therefore, proof of $m(P_d) \to \frac{9}{2\pi^2}\zeta(3)$ is a corollary of the generalization of Boyd-Lawton by considering $P_\infty = (1-x)(1-y)-(1-z)(1-w)$ as a limit polynomial and the sequence $A_d := \begin{pmatrix} d+2 & 0 & 1 & 0 \\ 0 & 1 & 0 & d+2 \end{pmatrix}$ (see Lemma IV.1.3). We notice that this new proof is much shorter than our first proof. However, it is a kind of reverse engineering, because the choice of the limit polynomial P_∞ necessitates the value of its Mahler measure. Without the exact value of the limit of $m(P_d)$, and without the work of D'Andrea and Lalín, guessing a candidate as a limit polynomial a priori is somehow impossible.

In [BGMP22] we also have a discussion about the error terms. We provide an upper bound for the rate of the convergence of the sequence $m(P(A_d))$ in terms of the number of variables of P, the number of non zero coefficients of P, the diameter of the Newton polygon of P and $\rho(A)$. Since the main objective of this thesis is to study the sequence P_d, we do not cover this general discussion. However, we provide the rate of convergence of the sequence $m(P_d)$. In fact, in Chapter V, we are able to determine the full asymptotic expansion of $m(P_d) - m(P_\infty)$.

Theorem ([BGMP22]). *The asymptotic expansion of $m(P_d) - m(P_\infty)$ is as follows:*

$$m(P_d) - m(P_\infty) = \frac{1}{(d+1)(d+2)} \left[-\frac{\log(d)}{2} + \sum_{j=0}^{2m-3} \frac{\alpha_j}{d^j} \right] + O(\frac{1}{d^{2m-1}}) \quad \textit{for all} \quad m \geq 2.$$

where the coefficients $\alpha_j \in \mathbb{R}$ are defined as:

$$\alpha_0 := 6\big(\zeta'(-1) - \zeta'(-2)\big) + \frac{\log(2\pi)}{2} - 1,$$
$$\alpha_j := \frac{12 \cdot (-1)^j}{j(j+1)} \sum_{t=0}^{\lfloor j/2 \rfloor} \binom{j+1}{2t} \cdot \frac{(2^{j+1-2t}-1)(2t-1)}{(2t+1)(2t+2)} \cdot B_{2t+2} \cdot \zeta(2t) \qquad (j \geq 1).$$

Here, B_n denotes the n-th Bernoulli number.

To prove the above theorem, we first use another closed formula for $m(P_d)$, as a sum of the values of dilogarithm (see Theorem V.1.1):

THEOREM. *For every $d \in \mathbb{Z}_{\geq 1}$ we have:*

$$\text{(I.2.2)} \qquad 2\pi m(P_d) = \frac{1}{d+1} S_{d+2} - \frac{1}{d+2} S_{d+1}, \quad \textit{with} \quad S_d := 3 \sum_{1 \leq k \leq d-1} (d-2k) D((e^{\frac{2\pi}{d} i})^k).$$

The above closed formula can still be interpreted as a linear combination of Riemann sums but this time for the univariate function $f(x) = (1-2x)D(e^{2\pi x i})$. Therefore, to compute the asymptotic expansion we can use the Euler-Maclaurin formulas for univariate functions. However, the function f has logarithmic singularities at 0 and 1 due to the dilogarithm function (see Lemma V.1.6). Hence, we need to use the Euler-Maclaurin summation formulas for both smooth and singular functions (see [SI88]). The details of this expansion can be found in Proposition V.2.3 and Proposition V.3.2.
The asymptotics of this expansion is given by :

$$m(P_d) = \frac{9}{2\pi^2}\zeta(3) - \frac{\log d}{2(d+1)(d+2)} + O\left(\frac{1}{d^2}\right).$$

The above equation gives both the limit (this is the third method to prove the limit) and the rate of the convergence of $(m(P_d))_{d \in \mathbb{Z}_{\geq 1}}$.

In Chapter VI of this thesis we explore the links between the Mahler measure of P_d and special values of L-functions, which is an ongoing project in collaboration with Bertin. The reason that we suspected the existence of such links is the existence of a closed formula for both $m(P_d)$ and $L'(\chi, -1)$, where χ is an odd primitive Dirichlet character, as a linear combination of dilogarithm at certain roots of unity. More precisely, from [Gra81], we have:

PROPOSITION ([Gra81]). *Let $-f$ be a fundamental discriminant and $\chi_{-f} := \left(\frac{-f}{\cdot}\right)$ be the odd quadratic Dirichlet character of conductor f. Then we have:*

$$\text{(I.2.3)} \qquad L'(\chi_{-f}, -1) = \frac{f^{\frac{3}{2}}}{4\pi} L(\chi_{-f}, 2) = \frac{f}{4\pi} \sum_{n=1}^{f} \chi_{-f}(n) D(\zeta_f^n),$$

where ζ_f is a primitive f-th root of unity.

Furthermore, the example of Smyth, $m(x+y+1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$, gives a striking link between P_1 and $L'(\chi_{-3}, -1)$. By computing the formula for $m(P_2)$ and the formula for $L'(\chi_{-4}, -1)$, we obtain a formula for $m(P_2)$ in terms of L-functions in Proposition VI.1.4:

$$m(P_2) = L'(\chi_{-4}, -1) - \frac{L'(\chi_{-3}, -1)}{2} = \frac{2}{\pi} L(\chi_{-4}, 2) - \frac{3\sqrt{3}}{8\pi} L(\chi_{-3}, 2).$$

After observing these links between $m(P_1)$, $m(P_2)$ and L-functions, we are interested in writing $m(P_d)$ explicitly in terms of L-functions, for $d \geq 2$. We hope this ongoing project will shed some light on the Chinburg conjecture:

Conjecture (Chinburg conjecture). [Ray87, Page 697] For every odd quadratic character $\chi_{-f} := \left(\frac{-f}{\cdot}\right)$, there exists a non-zero polynomial $P_f(x,y)$ with integer coefficients, such that $\frac{m(P_f)}{L'(\chi_{-f},-1)}$ is a rational number.

Smyth's formula for the Mahler measure of P_1, provides an example for the case $f=3$. Ray [Ray87] was able to construct polynomials $P_f(x,y)$, for $f=3,4,7,8,20$ and 24. In Chapter VI, we show that $P_1P_2^2$, verifies the Chinburg conjecture for $f=4$. However, to write $m(P_d)$ for $3 \leq d \leq 6$ in terms of L-functions, we also need L-functions associated with complex odd primitive Dirichlet characters. For instance, let χ_{-3}, χ_{-4} be respectively the odd quadratic characters of conductor 3 and 4, and χ^i and χ^{-i} be the following odd primitive Dirichlet characters of conductor 5:

m	0	1	2	3	4
$\chi^i(m)$	0	1	i	$-i$	-1
$\chi^{-i}(m)$	0	1	$-i$	i	-1

Then, in Propositions VI.3.1 and VI.3.2 we prove the following equalities:

$$m(P_3) = \frac{3(3-i)}{20}L'(\chi^i,-1) + \frac{3(3+i)}{20}L'(\chi^{-i},-1) - \frac{3}{5}L'(\chi_{-4},-1),$$

$$m(P_4) = \frac{-3+i}{10}L'(\chi^i,-1) + \frac{-3-i}{10}L'(\chi^{-i},-1) + \frac{16}{5}L'(\chi_{-3},-1).$$

It seems necessary to introduce χ^i and χ^{-i}, as our formulas for $m(P_3)$ and $m(P_4)$ contain values of the dilogarithm at 5-th roots of unity, while 5 is not a fundamental discriminant.
In Propositions VI.3.3 and VI.3.4, we also describe $m(P_5)$ and $m(P_6)$ in terms of L-functions. Moreover, in Theorem VI.0.1, we prove the following result:

Theorem. *Let $d \in \mathbb{Z}_{\geq 1}$. For every odd primitive Dirichlet character χ of conductor k, such that $k|d$, there exists a coefficient $C_{k,\chi} \in \mathbb{Q}(e^{\frac{2\pi i}{\phi(k)}}) \subset \mathbb{Q}(e^{\frac{2\pi i}{\phi(d)}})$ such that:*

$$m(P_d) = \sum_{k|(d+1)(d+2)} \sum_{\chi \text{ odd primitive mod } k} C_{k,\chi} L'(\chi,-1).$$

To prove the above theorem we use Eq. (I.2.2) and 3 principal arguments: First, for any primitive odd (non principal) Dirichlet character of conductor k, denoted by χ, we have the following formula for $L'(\chi,-1)$, which is a generalization of Eq. (I.2.3):

$$L'(\chi,-1) = \frac{k}{4\pi}\sum_{m=1}^{k-1}\chi(m)D({\zeta_k}^m) = \frac{-ik\tau(\chi)}{4\pi}L(\bar{\chi},2).$$

Here, $\tau(\chi) = \sum_{1\leq a\leq k}\chi(a)e^{\frac{2\pi i a}{k}}$ is a Gauss sum (see Page 697 [Ray87]). The second argument is that the primitive characters of modulus $k|d$ generate the space of periodic functions of period d. The last argument is that every imprimitive Dirichlet character of modulus d and conductor $k|d$ is induced by a uniquely determined primitive character of modulus k.

CHAPTER II

Different aspect of the Mahler measure

This chapter aims to introduce the Mahler measure and some of its applications in Number Theory. As we have already seen in the introduction, the main results of this thesis are about the Mahler measure of a family of two variable polynomials called P_d. Before introducing the Mahler measure of multivariable polynomials, we introduce the definition of the Mahler measure for a univariate polynomial. We bring attention to some important computational examples done by Smith. We see that the values of the Mahler measure of certain multivariate polynomials are linked to special values of L-functions, which illustrates an important application of the Mahler measure. In the last section of this chapter, we will introduce some important information about the sequence of Mahler measures such as Boyd-Lawton's theorem and its generalization. In future chapters, these results will be used to compute the limit of the sequence of the Mahler measure of P_d.

II.1. Introduction to Mahler measure

Historically, Derrick Henry Lehmer [Leh33] introduced the definition of univariate Mahler measure (for the monic polynomials with integer coefficients) in his research for discovering large prime numbers. However, the Mahler measure was named after Kurt Mahler who successfully extended this definition to arbitrary univariate polynomials as well as multiple variables. Before introducing the definition of univariate Mahler measure we recall some basic definitions and terminology about polynomials.

II.1.1. Basic definitions about polynomials. Mahler measure is a kind of height function for polynomials. In the most general sense, the height of a polynomial is a quantity by which we measure the complexity of the polynomial P. There are several different types of height functions. The simplest heights take into account the size of the coefficients of a polynomial. For instance for a polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ the height is defined as

$$H(P) := \max_{0 \le j \le d} |a_j|, \tag{II.1.1}$$

the length of P is defined as

$$L(P) := |a_d| + \cdots + |a_0|, \tag{II.1.2}$$

which are examples of height functions. Another way to define the height is to consider the absolute value of the roots of the polynomial P and define the Mahler measure.

Definition II.1.1 ([Mah60])**.** The Mahler measure $M(P)$ of a non zero polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ in $\mathbb{C}[x]$ is defined as:

$$M(P) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\},$$

where P factorizes over $\mathbb{C}$ as $P(x) = a_d(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d)$.

We notice that in this definition, an empty product is assumed to be 1, so the Mahler measure of the non zero constant polynomial $P(x) = a_0$ is $|a_0|$. In addition $M(0)$ is defined to be one.

For number theorists, the most useful height functions are the three discussed above. We will see some relations between these height functions applied to a fixed polynomial (see Proposition II.2.7). The quantity $M(P)$ for polynomials in one variable occurs naturally in many problems of number theory or dynamical systems.

Definition II.1.2. A polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ of degree d is called **monic** if its leading coefficient is $a_d = 1$.

According to Definition II.1.1 the Mahler measure of $C \cdot P$, for $C \in \mathbb{C}$ is proportional to the one of P by the proportionality constant $|C|$. Therefore, to compute $M(P)$ for an arbitrary polynomial P it suffices to compute the Mahler measure of the primitive part of the polynomial. In what follows we recall the necessary terminology in this regard.

Let R be a unique factorization domain. The **content** of a polynomial $P(x)$ with coefficients in R is the greatest common divisor of its coefficients and, as such, is defined up to multiplication by a unit. It is denoted by $C(P)$.

Definition II.1.3. The polynomial P is called **primitive**, if the content of P is a unit.

As we have already mentioned, for an arbitrary polynomial, we only compute the Mahler measure of its primitive part. However, we can simplify the computation even more. According to Definition II.1.1, for a reducible polynomial $P = HG$, we have $M(P) = M(H)M(G)$. Therefore, it suffices to study the Mahler measure of the irreducible polynomials in the factorization of P. An irreducible polynomial is a polynomial that cannot be factored into the product of two non-constant polynomials. The property of irreducibility depends on the nature of the coefficients that are accepted for the possible factors, that is, the field or ring to which the coefficients of the polynomial and its possible factors are supposed to belong. Since we often work with polynomials with integer coefficients, we directly define this notion for $P \in \mathbb{Z}[x]$.

Definition II.1.4. If $P \in \mathbb{Z}[x]$ is not primitive or can be represented as the product of two polynomials in $\mathbb{Z}[x]$ of degree strictly less than $\deg(P)$, then P is called **reducible**; otherwise it is **irreducible**.

II.1.2. Kronecker's theorem. We have introduced the Mahler measure and we can compute it for a polynomial if we know the values of its roots. According to Definition II.1.1 the Mahler measure of a monic polynomial whose roots are only roots of unity is equal to 1. Moreover, if we focus on polynomials with integer coefficients the value of their Mahler measure is greater than or equal 1. In this section, we introduce Kronecker's theorem, which characterizes the irreducible monic polynomials with integer coefficients whose Mahler measure equals 1. We recall some necessary definitions about algebraic numbers which are needed to state Kronecker's theorem.

Definition II.1.5. An **algebraic number** over $\mathbb{Q}$ is any complex number (including real numbers) that is a root of a non-zero univariate polynomial with rational coefficients. The set of all algebraic numbers is denoted by $\bar{\mathbb{Q}}$.

We notice that, in the above definition, we can equivalently say a is an algebraic number if it is the root of a polynomial with integer coefficients. All integers and rational numbers are examples of algebraic numbers. The real and complex numbers which are not algebraic, such as π and e, are called transcendental numbers. Since a number a is algebraic if it is a root of a polynomial $P(x) \in \mathbb{Z}[x]$, one can find a polynomial P of the lowest degree, such that a is a root of P, so we have the following definition.

Definition II.1.6. Let a be an algebraic number, an irreducible polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(a) = 0$, is called a **minimal polynomial**, associated to a.

After this definition we may ask about the uniqueness of such a polynomial. Indeed, it is unique up to sign. In the sequel ,we may use the word “the minimal polynomial”, but it refers to the uniqueness up to a sign. The above definition implies that the minimal polynomial is irreducible and primitive, but not necessarily monic. If in addition the minimal polynomial of a is monic, then it belongs to a special family of algebraic numbers, introduced in the following definition;

Definition II.1.7. If the minimal polynomial of α is monic (up to a sign), then α is said to be an **algebraic integer**.

For instance, 5 and $\sqrt{2}$ are algebraic integers while $\frac{5}{3}$ is not an algebraic integer. More generally, the only algebraic integers which are found in the set of rational numbers are the integers. Moreover, the square root $\sqrt{n}$ of a non negative integer n is an algebraic integer, but is irrational unless n is a perfect square. In Example II.1.11 we will introduce an important family of algebraic integers, which are not necessarily real. One can verify the following observations which introduce some important properties of algebraic integers and allow us to construct new algebraic integers. For a proof of the following observation see for instance [AM69, Corollary 5.3].

Observation II.1.8. *The set of all algebraic integers is closed under addition, subtraction and multiplication and is therefore a commutative subring of the complex numbers.*

The following observation gives us a sufficient condition to have an algebraic integer; See [BZ20] for more information about Observation II.1.9.

Observation II.1.9. *Every root of a monic polynomial with integer coefficients is itself an algebraic integer. Since, if $Q(a) = 0$ for some monic polynomial which is not necessarily minimal, we consider the decomposition of Q into irreducible factors. They are polynomials with degrees less than that of Q in $\mathbb{Z}[x]$ and they are monic. Therefore, any irreducible monic polynomial in the decomposition of Q which vanishes at a gives us the minimal polynomial of a (since all the factors in the decomposition of Q are monic with integer coefficients, so such a minimal polynomial is unique). In particular, every root of a monic polynomial whose coefficients are integers is itself an algebraic integer.*

Observation II.1.9 and Observation II.1.8 imply that the Mahler measure of a monic polynomial with integer coefficients is an algebraic integer. Moreover we have the following classical result which dates back to Dedekind and Kronecker. See [Edw13, Part 0] or [DD04] for a proof.

Corollary II.1.10. *Let $P \in \mathbb{Z}[x]$, then $M(P)$ is an algebraic integer.*

Example II.1.11. [BZ20] The roots of unity are algebraic integers. Indeed, $\zeta_n^k = e^{\frac{2\pi k i}{n}}$ with $k \in \mathbb{Z}_{\geq 1}$ is a root of the monic polynomial $x^n - 1$, which thanks to Observation II.1.9 we conclude that they are algebraic integers.

We mention the last definition before Kronecker's Theorem.

Definition II.1.12. Let $P = a_d \prod_{1 \leq j \leq d}(x - \alpha_j)$, with the leading coefficient equal to a_d, be the minimal polynomial of an algebraic number α. All zeros $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ of P, are called

algebraic conjugates of α. Furthermore, if all the roots are real, α is called a **totally real algebraic number**.

We can now state and prove Kronecker's Theorem. Here we exhibit the proof that appears in [BZ20, Proposition 1.1].

KRONECKER'S THEOREM ([Kro57]). *If α is a non-zero algebraic integer such that all its conjugates, including α, are inside the unit disc, then α is a root of unity.*

PROOF. Let P is the minimal polynomial of α and $n = \deg(P)$. Since α is an algebraic integer, so P can be chosen monic. We suppose that $P = \prod_{1 \leq j \leq n}(x - \alpha_j)$ and $\alpha = \alpha_1$. The set of all monic polynomials of degree n with integer coefficients having all their roots in the unit disc is finite. To see this, we write;

$$P = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = \prod_{j=1}^{n}(x - \alpha_j),$$

where $a_{n-j} \in \mathbb{Z}$, for $1 \leq j \leq n$. Using the fact that $|\alpha_j| \leq 1$, for $1 \leq j \leq n$ we have:

$$\begin{aligned}
|a_{n-1}| &= |\alpha_1 + \cdots + \alpha_n| \leq n = \binom{n}{1}, \\
|a_{n-2}| &= |\sum_{1 \leq j < k \leq n} \alpha_j \alpha_k| \leq \binom{n}{2}, \\
&\vdots \\
|a_0| &= |\alpha_1 \alpha_2 \cdots \alpha_n| \leq \binom{n}{n}.
\end{aligned}$$

Since the a_{n-j} are integers, each a_{n-j} is limited to at most $2\binom{n}{j} + 1$ values and therefore the total number of the polynomials that satisfy the hypothesis of the theorem is finite. We define;

$$P_1 = P = \prod_{j=1}^{n}(x - \alpha_j),$$

and for $k \geq 2$,

$$P_k = \prod_{j=1}^{n}(x - \alpha_j^k).$$

We claim that $P_k(x)$ has integer coefficients;
To see this, let $P(y) = \prod_{j=1}^{n}(y - \alpha_j)$ and $Q_k(y) = x - y^k$ and $P, Q_k \in \mathbb{Z}[x][y]$. Therefore, Eq. (VII.1.1) in Appendix implies that:

$$\operatorname{Res}(P, Q) = (1)^{\deg(Q)} \prod_{j=1}^{n} Q(\alpha_j) = \prod_{j=1}^{n}(x - \alpha_j^k) = P_k(x).$$

On the other hand the coefficient of the resultant are polynomial functions of the coefficients of P and Q_d. Since P and Q_k have coefficients in $\mathbb{Z}[x]$, so does $P_k = \operatorname{Res}(P, Q_k)$. It is also clear that all the roots of P_k are in the unit disc. Therefore, the number of such P_k is finite. Thus, we must have $P_t = P_k$ for some $t \leq k$. Since the set of the roots of P_t is $\{\alpha_1^t, \ldots, \alpha_n^t\}$ and for P_k is $\{\alpha_1^k, \ldots, \alpha_n^k\}$, then the two sets must be equal up to a permutation:

$$\alpha_r^t = \alpha_{\sigma(r)}^k,$$

for $r = 1, 2, \ldots, n$. Let m be the order of σ in S_n (i.e. the order m is the smallest non zero (positive) integer such that $\sigma^{(m)}(r) = r$, for all r such that $1 \leq r \leq n$). Thus, for an arbitrary root of $P(x)$, denoted by α_r we have;

$$\alpha_r^{t^m} = \alpha_r^{tt^{m-1}} = \alpha_{\sigma(r)}^{kt^{m-1}} = \alpha_{\sigma(r)}^{tkt^{m-2}} = \alpha_{\sigma(\sigma(r))}^{k^2t^{m-2}} = \cdots = \alpha_{\sigma^{(m)}(r)}^{k^mt^{m-m}} = \alpha_r^{k^m}.$$

Therefore, we have,

$$\alpha_r^{t^m - k^m} = 1,$$

which implies that α_r is a root of unity, for each $1 \leq r \leq n$.

□

According to Definition II.1.1 we know that the Mahler measure of a polynomial with integer coefficients whose roots are roots of unity is equal to 1. In the following section by using Kronecker's Theorem we prove that if $P \in \mathbb{Z}[x]$ and $M(P) = 1$ then P is monic and its nonzero roots are only roots of unity. More precisely the polynomials with integer coefficients and Mahler measure equal to 1 are the product of monomials and cyclotomic polynomials, introduced as follows:

Definition II.1.13. For any positive integer n the **n-th cyclotomic polynomial**, $\phi_n(x)$ is given by:

$$\phi_n(x) = \prod_{\substack{k=1 \\ \gcd(n,k)=1}}^{n} (x - \zeta_n^k),$$

where $\zeta_n^k = e^{\frac{2k\pi i}{n}}$.

One can observe that if $\gcd(n, k) = 1$ the minimal polynomial of $\zeta_n^k = e^{\frac{2k\pi i}{n}}$ is the n-th cyclotomic polynomial.

II.1.3. Mahler measure of univariate polynomials and Lehmer Conjecture. In this section we come back to the definition of the Mahler measure (univariate case) to introduce some of its important properties. In the end of this section we state the well known conjecture of Lehmer which still remains open. Moreover, we give information about some particular cases of polynomials for which the conjecture is answered.

As we have already mentioned the Mahler measure of monic univariate polynomials with integer coefficients was first defined by Lehmer during his research for discovering large primes. Let us briefly explain about his research and the history behind this definition.

As we know the Mersenne primes are the prime numbers of the form $2^p - 1$ for some prime p. However, the primality of n does not guarantee $2^n - 1$ to be prime. Many fundamental questions about Mersenne primes remain unresolved. It is not even known whether the set of Mersenne primes is finite or infinite. The largest known prime number, is $2^{82,589,933} - 1$, which is also a Mersenne prime. This is not a coincidence, in fact, there are special primality tests allowing to verify the primality of numbers arising as values of cyclotomic polynomials. For instance, Pierce [Pie16] constructed the sequence $\Delta_n(P)$, associated with a monic polynomial $P \in \mathbb{Z}[x]$ with $P(x) = \prod_{j=1}^{d}(x - \alpha_j)$ as its complex factorization, and defined:

$$\Delta_n(P) := \prod_{j=1}^{d} (\alpha_j^n - 1).$$

Then, he looked for primes among the factorization of $\Delta_n(P)$. Equivalently we can define $\Delta_n(P) := \mathrm{Res}(P, Q_n)$, where $Q_n(x) = x^n - 1$. Since α_j are algebraic integers it is easy to see as before that $\Delta_n(P) \in \mathbb{Z}$. Note that if $P(x) = x - 2$, we get the Mersenne sequence

$\Delta_n(P) = 2^n - 1$, which if n is prime may give a Mersenne prime. For an arbitrary polynomial P, Pierce observed that if $p \in \mathbb{Z}_{\geq 1}$ is a prime number then Δ_p is often a prime. In 1933, Lehmer [Leh33] studied the growth of $\{\Delta_n(P)\}_{n=1}^{\infty}$ and proved the following concerning the growth of this sequence as $n \to \infty$:

Proposition II.1.14 ([Leh33]). *Let $P \in \mathbb{Z}[x]$ be the monic polynomial with complex factorization $P(x) = \prod_{j=1}^{d}(x - \alpha_j)$ and $P(0)P(1) \neq 0$ then:*

$$\limsup_{n\to\infty} \sqrt[n]{|\Delta_n(P)|} = \prod_{j=1}^{d} \max\{1, |\alpha_j|\}.$$

Moreover, if the sequence $\Delta_n(P)$, $n = 1, 2, \ldots$ is not periodic, then the absolute value of $\Delta_n(P)$ unboundedly increases with n.

Proof. Since $P(0)P(1) \neq 0$, in particular $P(1) \neq 0$, then the proof of the limit follows from the fact that $\limsup_{n\to\infty} |\alpha^n - 1|^{\frac{1}{n}} = \max\{1, |\alpha|\}$, for $\alpha \neq 1$. For the second part of the proposition, we first suppose that P is irreducible, then it is the minimal polynomial of all its roots, so for the sequel of the proof we use the proof of Kronecker's Theorem. As we have seen in the proof of Kronecker's Theorem, for a monic polynomial $P(x) = \prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{Z}[x]$ we defined the polynomial $P_n(x) = \prod_{j=1}^{d}(x - \alpha_j^n)$ and we proved that $P_n \in \mathbb{Z}[x]$, so $P_n(1) \in \mathbb{Z}$. We notice that $|\Delta_n(P)| = |P_n(1)|$, for every $n \in \mathbb{Z}_{\geq 1}$. Then, $(|\Delta_n(P)|)_{n\in\mathbb{Z}_{\geq 1}}$ is a sequence of integers. Thus, if the sequence is bounded, then the limit superior is 1, and from the first part of the theorem, for every $1 \leq j \leq d$ we have $|\alpha_j| \leq 1$. Using Kronecker's Theorem all the roots of P are roots of unity, so the polynomial $P(x)$ is a cyclotomic polynomial and $|\Delta_n(P)| = |P_n(1)|$ is periodic. Finally, if P is not irreducible, we decompose it to the irreducible factors. Then, the mentioned proof is applicable to each irreducible factor, and by using the property that $\Delta_n(P_1P_2) = \Delta_n(P_1)\Delta_n(P_2)$, for every P_1, P_2 monic with integer coefficients, the proof is complete. ☐

We come back to the definition of the Mahler measure for an arbitrary univariate polynomial $P(x) = a_n(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$, proposed by Mahler, Definition II.1.1, which is:

$$M(P) = |a_n| \prod_{|\alpha_i|\geq 1} |\alpha_i| = |a_n| \prod_{i=1}^{n} \max\{1, |\alpha_i|\}, \tag{II.1.3}$$

In the above definition if P is monic, then the value of $M(P)$ and the limit of the sequence $|\Delta_n(P)|^{1/n}$ coincide. In fact, Lehmer called the above limit a measure of P denoted by $M(P)$. Later, Mahler generalized the definition to arbitrary polynomials, and this is the reason that $M(P)$ is called Mahler measure. At the end of this section, in order to give the intuition behind the Lehmer conjecture we come back to Lehmer's research for large primes, but let us first mention some important properties of the Mahler measure which are needed for the sequel.

Fact II.1.15. *For the Mahler measure of a univariate polynomial we have the following:*

(1) For $P \in \mathbb{C}[x]$, where $P \neq 0$ we have $M(P) > 0$.
(2) For $P \in \mathbb{Z}[x]$, we have $M(P) \geq 1$, since the leading coefficient of P belongs to $\mathbb{Z}$.
(3) For $P, Q \in \mathbb{C}[x]$, we have $M(P.Q) = M(P)M(Q)$, in particular for $C \in \mathbb{C}$, we have $M(CP) = |C|M(P)$.
(4) We can even extend the definition from polynomials to rational functions (if $M(Q) \neq 0$) by setting $M(\frac{P}{Q}) = \frac{M(P)}{M(Q)}$.
(5) Let P be a cyclotomic polynomial. It is monic and all its roots are the roots of unity, which implies $M(P) = 1$.

(6) *Let P be the product of cyclotomic polynomials and $\pm x^n$, for some $n \in \mathbb{Z}_{\geq 1}$, then we have $M(P) = 1$.*
(7) *Let $P \in \mathbb{Z}[x]$, then $M(P)$ is an algebraic integer (see Corollary II.1.10).*

We justified most of the above properties in the previous section. The following proposition together with the sixth property in Fact II.1.15 characterize the non zero polynomials with integer coefficients and the Mahler measure equal to 1.

Proposition II.1.16. *If $Q \in \mathbb{Z}[x] \setminus \{0\}$, and $M(Q) = 1$, then $Q = \pm x^n P(x)$, where $P(x)$ is a product of cyclotomic polynomials and $n \in \mathbb{N}$ (in this thesis we consider $0 \in \mathbb{N}$).*

Proof. The key tool to prove the proposition is Kronecker's Theorem:
Since, $M(Q) = 1$ and $Q \in \mathbb{Z}[x]$, we conclude that Q is monic and all its roots must have modulus less than or equal to 1. If Q vanishes at 0 with multiplicity $n \geq 0$, then we can decompose Q as $Q = \pm x^n P(x)$, where $P \in \mathbb{Z}[x]$ is monic and $n \in \mathbb{N}$. Therefore, an arbitrary root of P, named a, is non zero with $|a| \leq 1$ and according to Observation II.1.9, a is an algebraic integer. Thus, according to Kronecker's Theorem, all the algebraic conjugates of a, which contains a as well are roots of unity and they all have modulus equal to 1. Hence, P is the product of cyclotomic polynomials. $\square$

We come back to Lehmer's research to produce large prime numbers. In Proposition II.1.14 we discussed the growth of $|\Delta_n(P)|$. One can show that if $m|n$ then $\Delta_m(P)|\Delta_n(P)$. Thus, we may look at $|\frac{\Delta_n(P)}{\Delta_1(P)}|$, for any n or $|\frac{\Delta_n(P)}{\Delta_2(P)}|$, for n even. To obtain large primes from the factorization of $\Delta_n(P)$, Lehmer [Leh33] suggested to have the increase of the sequence very slowly. In other words, to have $M(P)$ as small as possible. According to Proposition II.1.16, $M(P) > 1$, for $P \in \mathbb{Z}[x] \setminus \{0\}$ monic, irreducible and non-cyclotomic. Then, Lehmer asked for the greatest bound for $M(P)$ from below (greater than one). This question is yet open and called the Lehmer conjecture:

Conjecture II.1.17. [Lehmer Conjecture (1933)] There is a constant $C > 1$, such that for any $P \in \mathbb{Z}[x]$ if $M(P) > 1$, then $M(P) \geq C$.

According to Proposition II.1.16, the condition $M(P) > 1$ in the above conjecture is equivalent to the property that P is not a product of cyclotomic polynomials (up to a monomial factor). There is a stronger version of this conjecture which proposes that the constant C in the above conjecture is itself the Mahler measure of a polynomial with integer coefficients. The best guess until now is Lehmer's guess [Leh33], which is the Mahler measure of the following polynomial;

$$M(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 1.17628\ldots.$$

Since 1933, this value is still the smallest value of $M(P) > 1$ and it is widely believed to be the least possible value. Of course, the Lehmer Conjecture is known to be true for polynomials with bounded degree.

Proposition II.1.18. *For every $d \in \mathbb{Z}_{>0}$ there exists $\mu_d > 1$ such that if $P \in \mathbb{Z}[x]$ is a monic, irreducible and non-cyclotomic polynomial of degree d, then $M(P) \geq \mu_d$.*

Proof. [BZ20] Since $M(x^d - 2) = 2$, it is sufficient to show that there are finitely many monic polynomials $P \in \mathbb{Z}[x]$ of degree d for which $M(P) < 2$, then μ_d is the minimum of $M(P)$ over the finite set. For a polynomial $P = \prod_{j=1}^{d}(x - \alpha_j)$, the bound $M(P) = \prod_{1 \leq j \leq d} \max\{1, |\alpha_j|\} < 2$ implies $|\alpha_j| \leq 2$, for $j = 1, \ldots, d$. As the coefficients of P are integers, the same idea as the proof of Kronecker's theorem proves that the set of such polynomials is finite. $\square$

Lehmer's conjecture for polynomials with bounded degree has been intensely studied. One can find some of the lower bounds in [Dob79, BM71, Ste78, CS82, Lou83].

There is a family of polynomials called non reciprocal polynomials, for which the Lehmer Conjecture was solved by Breusch [Bre51] and (independently) by Smyth [Smy71]:

Definition II.1.19. Let $P \in \mathbb{C}[x]$ is a polynomial of degree d, write its reciprocal (reflected) polynomial P^* as follows;

$$P^*(x) := x^d P\left(\frac{1}{x}\right).$$

The polynomial P is called **reciprocal** if $P^* = \pm P$, and non reciprocal otherwise.

Proposition II.1.20 (Smyth[Smy71], Breusch [Bre51])**.** *If the Mahler measure of an irreducible polynomial $P(x) \in \mathbb{Z}[x]$ satisfies $M(P) < x_0 = 1.32471795\ldots$, the real zero of the polynomial $x^3 - x - 1$, then $P(x)$ is reciprocal.*
Furthermore if $M(P) \leq c = x_0 + 10^{-4}$, then either $P(x)$ has a zero $x_0^{\frac{1}{m}}$ for some $m \in \mathbb{Z}_{\geq 1}$ or $P(x)$ is reciprocal.

For more information about Smyth's theorem see [Smy71, Pages 170-175] or [BZ20, Theorem 2.1]. In fact the constant $c = x_0 + 10^{-4}$ in the statement of Proposition II.1.20 was later improved to $c = 1.32497826\ldots$, the largest real zero of $4x^8 - 5x^6 - 2x^4 - 5x^2 + 4$ in [DD04]. In Section II.4.2 we come back to this theorem after introducing Boyd's conjecture and we give more information about the Lehmer conjecture and the links between these two statements.

II.1.4. Jensen's formula and logarithmic the Mahler measure. In this section, using Jensen's equality we state another definition of the Mahler measure of a univariate polynomial by computing an integral over the unit circle. This integral representation of the univariate Mahler measure formula gives us the opportunity to generalize the definition to multivariate polynomials.
Let us fix the following notation, which will be used in the sequel;

Notation II.1.21. For any $\alpha \in \mathbb{C}$ we define:

$$\log^+ |\alpha| = \log \max\{1, |\alpha|\}.$$

Jensen's formula ([Jen00]). *For any $\alpha \in \mathbb{C}$ we have:*

$$\int_0^1 \log |e^{2\pi i t} - \alpha| dt = \log^+ |\alpha|.$$

Proof. [BZ20] Jensen's formula is trivially true for $\alpha = 0$ (the two sides of the equality are equal to 0), so we suppose that $|\alpha| > 0$. Let us replace the real integral $\int_0^1 \log |\alpha - e^{2\pi i t}| dt$ with the complex integral using the change of variables $x = e^{2\pi i t}$. Therefore $2\pi i e^{2\pi i t} dt = dx$ and we have:

$$\int_0^1 \log |\alpha - e^{2\pi i t}| dt = \frac{1}{2\pi i} \oint_{|x|=1} \log |x - \alpha| \frac{dx}{x} = \Re\left(\frac{1}{2\pi i} \oint_{|x|=1} \log(x - \alpha) \frac{dx}{x}\right).$$

The proof is divided into three cases;

- First case : $|\alpha| > 1$) In this case the function $f(x) = \log(x - \alpha)$ is analytic inside the unit disc $|x| < 1$ and on its boundary, hence Cauchy's theorem implies that the latter integral evaluates to $f(0) = \Re(\log \alpha) = \log |\alpha|$.

- Second case : $|\alpha| = 1$) In this case we have an improper integral. We replace the integration path around $x = \alpha$ with an arc C of radius ϵ, where $0 < \eta < \frac{1}{2}$, centered at this point lying entirely inside the disc $|x| \leq 1$, and use Cauchy's theorem for the newer contour as well as the estimate
$$\left|\frac{1}{2\pi i}\int_C \log(x-\alpha)\frac{dx}{x}\right| \leq \epsilon \max_{x:|x-\alpha|=\epsilon} \frac{|\log(x-\alpha)|}{|x|} \leq 2\epsilon|\log\epsilon| \to 0 \quad \text{as } \epsilon \to 0\ .$$
- Third case : $|\alpha| < 1$) In this case $\log|x-\alpha| = \log|1-\alpha/x|$ on the contour of integration $|x| = 1$, so that
$$\begin{aligned}\int_0^1 \log|e^{2\pi it}-\alpha|dt &= \Re\left(\frac{1}{2\pi i}\oint_{|x|=1}\log\left(1-\frac{\alpha}{x}\right)\frac{dx}{x}\right)\\ &= \Re\left(\frac{1}{2\pi i}\oint_{|y|=1}\log\left(1-\alpha y\right)\frac{dy}{y}\right) = 0\end{aligned}$$
Notice that, by changing variables we reverse the orientation which flips the sign of the result also $\frac{dx}{x} = -\frac{dy}{y}$. To see the last equality, we notice that $y^{-1}\log(1-\alpha y)$ has a removable singularity at $y = 0$ and no other singularities within the disc $|y| \leq 1$. Therefore the integral is zero.

$\square$

In [Mah60], Mahler replaced the polynomial $(x-\alpha)$ in Jensen's formula with an arbitrary polynomial $P(x) = a_d\prod_{j=1}^d(x-\alpha_j) \in \mathbb{C}[x]$ and got the following result;
$$\begin{aligned}\int_0^1 \log|P(e^{2\pi it})|dt &= \frac{1}{2\pi i}\oint_{|x|=1}\log|a_d|\frac{dx}{x} + \sum_{j=1}^d \frac{1}{2\pi i}\oint_{|x|=1}\log|x-\alpha_j|\frac{dx}{x}\\ &= \log|a_d| + \sum_{j=1}^d \log^+|\alpha_j|.\end{aligned}$$
Comparing this result with the Definition II.1.1 which sets $M(P) = |a_d|\prod_{i=1}^n \max\{1,|\alpha_i|\}$, we conclude: $\log M(P) = \int_0^1 \log|P(e^{2\pi it})|dt$. In other words we have:

$$M(P) = \exp\left(\int_0^1 \log|P(e^{2\pi it})|dt\right). \tag{II.1.4}$$

The previous equation leads to the definition of the logarithmic Mahler measure of P, introduced by Kurt Mahler:

Definition II.1.22. Let $P(x) \in \mathbb{C}[x]$ is a non zero polynomial. The **logarithmic Mahler measure** of P, denoted by $m(P)$ is defined as:

$$m(P) = \log M(P) = \int_0^1 \log|P(e^{2\pi it})|dt. \tag{II.1.5}$$

In some references, the definition is extended to all polynomials by setting $m(0) = \infty$. We recall that for the zero polynomial it is defined that $M(0) = 1$. Moreover, there is no harm in allowing P to be a Laurent polynomial, since these can be converted to ordinary polynomials by multiplication by a monomial in x and the logarithmic Mahler measure of a monomial is 0. Some of the important properties of logarithmic Mahler measure are stated in the following fact:

Fact II.1.23. *For the logarithmic Mahler measure of a univariate polynomial we have:*

(1) For $P \in \mathbb{Z}[x]$*, a non zero polynomial, we have seen* $M(P) \geq 1$ *which implies* $m(P) \geq 0$.

(2) For $P, Q \in \mathbb{C}[x]$, we have $m(P.Q) = m(P) + m(Q)$, in particular for $c \in \mathbb{C} \setminus \{0\}$, we have $m(cP) = \log|c| + m(P)$.
(3) For a cyclotomic polynomial denoted by P, since $M(P) = 1$, we have $m(P) = 0$.
(4) Let P be the product of cyclotomic polynomials, then, we have $m(P) = 0$, and in addition, for any $n \in \mathbb{N}$ we have $m(\pm x^n P) = 0$.

Thanks to Proposition II.1.16, we conclude the following:

Corollary II.1.24 (Theorem 1.33 [EW99])**.** *If $Q \in \mathbb{Z}[x]$ and $m(Q) = 0$, then $Q = \pm x^n P(x)$, where P is a product of cyclotomic polynomials.*

The fourth property of $m(P)$, mentioned in Fact II.1.23 together with Corollary II.1.24 characterize the univariate polynomial with *integer* coefficients and the logarithmic Mahler measure equal to zero. This gives an analogue for Proposition II.1.16, for the logarithmic Mahler measure to be equal to 0. One may ask about the characterization of $m(P) = 0$ for $P(x) \in \mathbb{C}[x]$. We can not answer this question in general, but there is a family for which we can answer this question, called unit-monic. A non zero polynomial $P \in \mathbb{C}[x]$ is said to be **unit-monic** if $P(x) = a_d x^d + \cdots + a_0$ has $|a_d| = |a_0| = 1$.

Proposition II.1.25 ([EW99], Lemma 3.12)**.** *If $P \in \mathbb{C}[x]$ is unit-monic then $m(P) = 0$ if and only if all zeros of P lie on the unit circle.*

Proof. To prove this proposition, we only need to use the complex analogue of Kronecker's theorem for unit-monic polynomials which asserts: If all the roots of P are inside the closed unit disc, then they all have modulus of 1.
The proof of this version of Kronecker's theorem is simple. Let $P(x) = a_d x^d + \cdots + a_0$, the product of the roots of P is equal to $\frac{a_0}{a_d} = 1$. Since all the roots of P have modulus less than or equal to 1, we conclude that all the roots of P have the modulus equal to 1.
To prove the proposition, we notice that if all zeros of P lie on the unit circle then clearly $m(P) = 0$ and on the other side the analogue of Kronecker's theorem completes the proof. □

We can translate the Lehmer Conjecture in the setting of the logarithmic Mahler measure as well. It states that there exists a constant $C > 0$, such that for every polynomial $P \in \mathbb{Z}[x]$ with $m(P) > 0$ we have $m(P) \geq C$.

Remark II.1.26. In the cases that we work with $m(\cdot)$ rather than $M(\cdot)$, the adjective "logarithmic" is often dropped since it is clear from the context that which one is used.

II.2. Multivariate Mahler measures

The aim of this section is to introduce the generalization proposed by Mahler of the definition of univariate Mahler measures to multivariate. We will see some examples of the evaluation of Mahler measure of multivariate polynomials by Smyth which their Mahler measure link to the special values of L-function. This is an important application of the Mahler measure in Number Theory.

II.2.1. Mahler measure, multivariate polynomials. Mahler generalized the definition of the logarithmic Mahler measure to non zero polynomial in several variables by using the integral representation of the logarithmic Mahler measure introduced in Eq. (II.1.5) as follows;

Definition II.2.1 ([Mah62b]). For a non zero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ define:

$$m(P) = \int \cdots \int_{[0,1]^n} \log |P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| dt_1 \cdots dt_n \tag{II.2.1}$$

$$= \frac{1}{(2\pi i)^n} \int \cdots \int_{|x_1| = \cdots = |x_n| = 1} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}. \tag{II.2.2}$$

If in the integral in Eq. (II.2.1) we do the change of variables, $e^{2\pi i t_j} = x_j$, for $1 \leq j \leq n$ we have the complex integral in Eq. (II.2.2). In fact $m(P)$ is the arithmetic mean of $\log |P|$ over the n-dimensional complex torus $\mathbb{T}^n$. Moreover, based on the relation between $M(P)$ and $m(P)$ in the univariate case, Mahler defined the Mahler measure of multivariate polynomials;

Definition II.2.2. For a non zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ define:

$$M(P) = \exp(m(P)),$$

and the Mahler measure of the zero polynomial is set to be one.

Some of the important properties of the Mahler measure of a multivariate polynomial are listed in the following fact:

Fact II.2.3. *For the Mahler measure of a multivariate polynomial we have the following:*

(1) For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we have $M(P) \geq 0$.
(2) For $P, Q \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we have $M(PQ) = M(P)M(Q)$ and $m(PQ) = m(P) + m(Q)$. By using this property one can extend the definition of the Mahler measure to rational functions (i.e. If $M(Q) \neq 0$ then $M(\frac{P}{Q}) = \frac{M(P)}{M(Q)}$ and $m(\frac{P}{Q}) = m(P) - m(Q)$).

In Eq. (II.2.1) if the polynomial P vanishes on the torus $\mathbb{T}^n$, the integrand can have singularities. Mahler [Mah62a] proved the existence of $m(P)$ for any Laurent polynomial $P(x_1, \dots, x_n)$. We provide a proof using the method in [EW99]:

Proposition II.2.4 ([Mah62a]). *The expression $m(P)$ in Eq.* (II.2.1) *always exists as an improper Riemann integral. Moreover, if P has integer coefficients, then $m(P) \geq 0$.*

PROOF. [EW99, Lemma 3.7] Let us first prove the existence of $m(P)$ by induction over the number of the variables of P. For the one variable case ($n = 1$), since $M(P) > 0$ for $P \neq 0$, so $M(P)$ exists. For proving the existence of $m(P)$ for P an arbitrary polynomial in n variables, we write $m(P)$ as a limit of an increasing sequence which is bounded above, so it converges and the limit is $m(P)$. To do so let $l(P)$ be the logarithm of the sum of the absolute values of the coefficients of P. According to the definition of $m(P)$ and the triangle inequality we have, $m(P) \leq l(P)$. We write P as a polynomial in x_1 with coefficients in $\mathbb{C}[x_2, \dots, x_n]$

$$P(x_1, \dots, x_n) = a_d(x_2, \dots, x_n)x_1^d + \cdots + a_0(x_2, \dots, x_n).$$

By choosing some suitable algebraic functions $g_1, \dots, g_j$, we factorize P as follows:

$$P(x_1, \dots, x_n) = a_d(x_2, \dots, x_n) \prod_{j=1}^{d} (x_1 - g_j(x_2, \dots, x_n)).$$

If we prove that $m(a_d(x_2, \dots, x_n))$ and $\sum_{j=1}^{d} m(x_1 - g_j(x_2, \dots, x_n))$ exist then, by using the properties of Mahler measure, $m(P) = m(a_d(x_2, \dots, x_n)) + \sum_{j=1}^{d} m(x_1 - g_j(x_2, \dots, x_n))$ exists as well. By the inductive hypothesis, $m(a_d(x_2,\dots,x_n))$ exists. Moreover we have:

$$\begin{aligned} m(P) &= m(a_d(x_2,\dots,x_n)) + \sum_{j=1}^{d} m(x_1 - g_j(x_2,\dots,x_n)) \\ &= m(a_d(x_2,\dots,x_n)) + \sum_{j=1}^{d} \int_{[0,1]^{n-1}} \underbrace{\int_0^1 (\log|e^{2\pi i\theta_1} - g_j(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n})|d\theta_1}_{\dagger} d\theta_2\cdots d\theta_n. \end{aligned}$$

By applying Jensen's formula to the indicated integral we have $\dagger = \log^+|g_j(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n})|$. For each $N \in \mathbb{Z}_{\geq 1}$, define

$$a_N = m(a_d(x_2,\dots,x_n)) + \sum_{j=1}^{d} \int_{[0,1]^{n-1}} \int_{X_N} \log^+|g_j(e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n})|d\theta_2\cdots d\theta_n,$$

where $X_N := \{(\theta_2,\dots,\theta_n) \in [0,1]^{n-1} \mid |g_j(\theta_2,\dots,\theta_n)| \leq N\}$. Then for each N, a_N exists since the integrand is continuous. Moreover, a_N is an increasing sequence verifying $a_N \leq m(P) < l(P)$, so it is bounded above by $l(P)$. Therefore a_N converges to $m(P)$.

In this second step we prove $m(P) \geq 0$ if P has integer coefficients. We again argue by induction, and for $n = 1$, it is already proved. Thus, we suppose that the proposition is true for all polynomial in $n-1$ variables. For an n variable polynomial P with integer coefficients $a_N \geq 0$, since the integrand is non negative, also by the inductive hypothesis $m(a_d(x_2,\dots,x_n)) \geq 0$, so the proof is complete.

□

The computation of the Mahler measure of multivariate polynomials is complicated since we do not have the analogue of Jensen's formula. In this section, we state some important examples of the Mahler measure of multivariable polynomials without providing the computation. The examples are due to Smyth and the values of their Mahler measures are related to special values of L-functions. Hence, we first introduce Dirichlet characters and Dirichlet L-functions and we then provide the computation of their Mahler measure in Section II.3.4.

Proposition II.2.5 ([Smy81a]). *For the polynomial $2+x+y$ we have:*

$$m(2+x+y) = \log 2.$$

[EW99].

$$\begin{aligned} m(2+x+y) &= \int_0^1\int_0^1 \log|e^{2\pi it} + e^{2\pi is} + 2|dtds \overset{[1]}{=} \int_0^1 \log^+|e^{2\pi is}+2|ds \\ &\overset{[2]}{=} \int_0^1 \log|e^{2\pi is}+2|ds \overset{[3]}{=} \log 2. \end{aligned}$$

In [1] we used the Jensen's formula for $\int_0^1 \log|e^{2\pi it}+e^{2\pi is}+2|dt$ and in [3] for $\int_0^1 \log|e^{2\pi is}+2|ds$. In [2] we used the fact that, for any $s \in [0,1]$ we have $|e^{2\pi is}+2| \geq 1$. □

The following evaluations, due Smyth, illustrate the link between the Mahler measure of certain multivariate polynomials and special values of L-functions.

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2), \tag{II.2.3}$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3). \tag{II.2.4}$$

We refer to Section II.3.4 for the proofs of the above equalities.

There are some properties of the Mahler measure which are true in the univariate case as well as multivariate one. For instance, there is an analogous to Corollary II.1.24 in the multivariate setting. The following proposition is due to Smyth; see [Smy81a] for the proof.

Proposition II.2.6 ([Smy81a],Theorem 1). *For any primitive polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $m(P)$ is zero if and only if P is a monomial times a product of cyclotomic polynomials evaluated on monomials. (i.e. $P(x) = x_1^{a_1} \cdots x_n^{a_n} \phi(x_1^{b_1} \cdots x_n^{b_n})$, where $a_i, b_i \in \mathbb{Z}$, for $1 \leq i \leq n$ and ϕ is univariate cyclotomic polynomial.)*

We have mentioned that there is a relation between the Mahler measure and the other height functions, introduced in Section II.1.1. Using the integral definition of the Mahler measure mentioned above we conclude the following proposition and for more information see [BZ20].

Proposition II.2.7 ([EW99, BZ20]). *For a polynomial $P(x) = a_d x^d + \cdots + a_0 \in \mathbb{C}[x]$, of degree d we have:*

$$M(P) \leq L(P) \leq 2^d M(P).$$

Moreover we have :

$$H(P) \leq \binom{d}{\lfloor \frac{d}{2} \rfloor} M(P), \quad M(P) \leq \sqrt{d+1}\ H(P).$$

There is also the analogous of Proposition II.2.7 for the multivariate setting:

Proposition II.2.8 ([Mah62a]). *For a polynomial $P \in \mathbb{C}[x_1, \ldots, x_k]$, we have*

$$M(P) \leq L(P) \leq 2^{d_1 + \cdots + d_k} M(P),$$

where $d_1, \ldots d_k$ are degrees of P with respect to the corresponding variables $x_1, \ldots, x_k$.

One can find the analogous of the above proposition for the link between $M(P)$ and $H(P)$ in [Mah62a, Exercise 3.2].

On the other side there are some properties of the univariate Mahler measure which can not be generalized to the multivariate one. For instance, in Corollary II.1.10, we have seen that the Mahler measure $M(P)$ of a univariate polynomial $P \in \mathbb{Z}[x]$ is an algebraic integer. However, for a multivariate polynomial $P(x_1, \ldots, x_n)$ even with integer coefficients it seems unlikely that $M(P)$ is an algebraic number. For example in the example of Smyth (II.2.4) we have $m(1 + x + y + z) = \frac{7}{2\pi^2}\zeta(3)$, and it is most probably transcendental.

II.2.2. Affine transformation and invariance of the Mahler measure. In the previous sections we have seen some important properties and definitions about the Mahler measure. Here, we introduce one more property, namely, its invariance under bijective affine transformations. This property is needed for our future computations. Its relevance is that if $m(P)$ is hard to compute, there might be a bijective affine transformation which sends the polynomial P to Q, for which computing $m(Q)$ is easier. In this section, first, we explain the action of affine transformations on polynomials and then, we prove the theorem of invariance.

Definition II.2.9. In the finite-dimensional case, an **affine transformation** from $\mathbb{R}^m$ to $\mathbb{R}^n$ is the composition of two functions: a translation and a linear map. In other words an affine transformation f acting on a vector $x \in \mathbb{R}^m$ can be represented as $f(x) = Ax + b$, where $A \in \mathbb{R}^{m \times n}$ is the matrix that represents the linear map and the vector $b \in \mathbb{R}^n$ represents the translation.

Let us first fix the following notation;

Notation II.2.10. Given a k-tuple $\mathbf{j} = (j_1, \dots, j_k) \in \mathbb{Z}^k$, the monomial $x_1^{j_1} \cdots x_k^{j_k} \in \mathbb{C}[x_1, \dots, x_k]$ is denoted by $\mathbf{x}^{\mathbf{j}}$.

In the following, we define the action of the group of bijective affine transformations of $\mathbb{Z}^k$ over the set of the Laurent polynomials.

Definition II.2.11. Let $P = \sum_{j \in \mathbb{Z}^k} a_j \mathbf{x}^{\mathbf{j}}$ be a Laurent polynomial and $g : \mathbb{Z}^k \to \mathbb{Z}^k$ is a bijective affine transformation, define $gP := \sum_{j \in \mathbb{Z}^k} a_j \mathbf{x}^{g\mathbf{j}}$.

The goal of this section is to prove that $m(gP) = m(P)$, for any bijective affine transformation $g : \mathbb{Z}^k \mapsto \mathbb{Z}^k$.

Definition II.2.12 ([BZ20])**.** The group $GL_k(\mathbb{Z})$ naturally acts on $\mathbb{T}^k$ by coordinate transformation as follows:

$$g = (g_{jl})_{1 \le j,l \le k} \in GL_k(\mathbb{Z}), \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \mathbb{T}^k \mapsto \begin{pmatrix} x_1^{g_{11}} x_2^{g_{12}} \cdots x_k^{g_{1k}} \\ \vdots \\ x_1^{g_{k1}} x_2^{g_{k2}} \cdots x_k^{g_{kk}} \end{pmatrix} \in \mathbb{T}^k.$$

In fact g defines an automorphism of $\mathbb{T}^k$ to itself, since it is inversible. To see that, we show that $g^{-1} \circ g\mathbf{x} = \mathbf{x}$, where $g^{-1} = (a_{jl})_{1 \le j,l \le k} \in GL_k(\mathbb{Z})$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \xrightarrow{g} \begin{pmatrix} x_1^{g_{11}} x_2^{g_{12}} \cdots x_k^{g_{1k}} \\ \vdots \\ x_1^{g_{k1}} x_2^{g_{k2}} \cdots x_k^{g_{kk}} \end{pmatrix} \xrightarrow{g^{-1}} \begin{pmatrix} (x_1^{g_{11}} x_2^{g_{12}} \cdots x_k^{g_{1k}})^{a_{11}} \cdots (x_1^{g_{k1}} x_2^{g_{k2}} \cdots x_k^{g_{kk}})^{a_{1k}} \\ \vdots \\ (x_1^{g_{11}} x_2^{g_{12}} \cdots x_k^{g_{1k}})^{a_{k1}} \cdots (x_1^{g_{11}} x_2^{g_{12}} \cdots x_k^{g_{1k}})^{a_{kk}} \end{pmatrix}.$$

which will be :

$$\begin{pmatrix} x_1^{g_{11}a_{11} + \cdots + g_{k1}a_{1k}} \cdots x_k^{g_{1k}a_{11} \cdots + g_{kk}a_{1k}} \\ \vdots \\ x_1^{g_{11}a_{k1} + \cdots + g_{11}a_{kk}} \cdots x_k^{g_{1k}a_{k1} + \cdots + g_{1k}a_{kk}} \end{pmatrix} \stackrel{[1]}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}.$$

In [1] we use that the powers of the coefficients are exactly the coefficients of the matrix $g^{-1}g = \mathrm{Id}_k$.

We have seen the action of the group $GL_k(\mathbb{Z})$ on two sets. First, the set of a Laurent polynomials belongs to $\mathbb{C}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$ and second, on $\mathbb{T}^k$. In the following easily verified lemma, we see the link between these two actions. For more information see page [BZ20, Section 3.2].

Lemma II.2.13 ([BZ20])**.** *Let $g \in GL_k(\mathbb{Z})$, and $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$ a Laurent polynomial, the two actions, defined in Definition II.2.12 and Definition II.2.11 are compatible in the following sense;*

$$(gP)(\boldsymbol{x}) = P({}^t g \boldsymbol{x}),$$

for all $\boldsymbol{x} = (x_1, \dots, x_k) \in \mathbb{T}^k$.

We notice that in the above lemma ${}^t g$ is the transpose of the matrix g. We are now able to prove the theorem of invariance of the Mahler measure under a bijective affine transformations.

Proposition II.2.14. *Let $g \in GL_k(\mathbb{Z})$ and P a Laurent polynomial in $\mathbb{C}[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$. We have the following equality:*

$$m(gP) = m(P).$$

Proof. The proof is an application of change of variable formula for integrals. Indeed, we have:

$$\begin{aligned} m(gP) = &\frac{1}{(2\pi i)^k} \int \cdots \int_{|x_1| = \cdots = |x_n| = 1} \log |gP(x_1, \dots, x_k)| \frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k} \overset{[1]}{=} \\ &\frac{1}{(2\pi i)^k} \int \cdots \int_{|x_1| = \cdots = |x_n| = 1} \log |P({}^t g(x_1, \dots, x_k))| \frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k} \overset{[2]}{=} \\ &|J| \frac{1}{(2\pi i)^k} \int \cdots \int_{|y_1| = \cdots = |y_n| = 1} \log |P(y_1, \dots, y_k)| \frac{dy_1}{y_1} \cdots \frac{dy_k}{y_k} = m(P) \end{aligned}$$

In [1], we used Lemma II.2.13 and in [2] we applied the changes of variable $(y_1, \dots, y_k) = ({}^t g(x_1, \dots, x_k))$. Moreover, the region of the new integral in the third line is the action of the invertible matrix g on the torus $\mathbb{T}^k$, which is $|y_1| = \cdots = |y_k| = 1$ (since g is an automorphism of the torus). We notice that in the last integral, the absolute value of the Jacobian is one :

$$|J| = \left| \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \vdots & & \\ \frac{\partial x_k}{\partial y_1} & \cdots & \frac{\partial x_k}{\partial y_k} \end{pmatrix} \right| = \left| \begin{pmatrix} g_{11} \frac{y_1}{x_1} & \cdots & g_{1k} \frac{y_1}{x_k} \\ \vdots & & \\ g_{k1} \frac{y_k}{x_1} & \cdots & g_{kk} \frac{y_k}{x_k} \end{pmatrix} \right| = \left| \prod_{i=1}^k \frac{1}{x_i} \prod_{j=1}^k y_j \det(g) \right| = 1.$$

The last equality follows from the fact that $g \in GL_K(\mathbb{Z})$, so $|\det(g)| = 1$, and all the x_i and y_j have modulus equal to one. □

We finish this part by mentioning an special and important case of the previous proposition:

Corollary II.2.15. *For any Laurent polynomial $P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we have $m(P(x_1, \dots, x_n)) = m(P(x_1^{-1}, \dots, x_n^{-1}))$.*

II.3. Special values of L-function and Mahler measure

In the previous section, we have seen examples of Smyth relating the Mahler measure to special values of L-functions. In this section we will prove these evaluations. Before stating the proofs, we need to introduce some prerequisites about Dirichlet characters and Dirichlet L-functions.

II.3.1. Dirichlet characters. Dirichlet characters are an important notion in analytic Number Theory and related branches of mathematics. By Fourier theory for finite abelian groups, any function on $\frac{\mathbb{Z}}{n\mathbb{Z}}$ can be written as a linear combination of characters. Moreover, Dirichlet L-functions are defined by using Dirichlet characters. Thanks to the completely multiplicative property of Dirichlet characters, Dirichlet L-series admit Euler products. In this section, we introduce necessary information about Dirichlet characters in order to introduce Dirichlet L-functions in the following section. There are two points of view for defining Dirichlet characters. The first one is that they are homomorphisms of certain groups (see Definition II.3.1). The second one which we will use, is that Dirichlet characters are multiplicative arithmetic functions (see Definition II.3.2).

Definition II.3.1. A **Dirichlet character** modulo $k \in \mathbb{Z}_{\geq 1}$ is a multiplicative homomorphism χ from the group of units in the ring of integers modulo k (i.e. $\left(\frac{\mathbb{Z}}{k\mathbb{Z}}\right)^*$) to the group of non-zero complex numbers (i.e. $\mathbb{C}^*$).

However, we prefer to consider the extended Dirichlet character from $\mathbb{Z}$ to $\mathbb{C}$ by defining $\chi(a) = 0$, if a is not coprime to k and taking advantage of the periodicity of χ. A motivation of this extension is to write down the formula for the Dirichlet L-series associated to χ in the way that we do not have to specifically restrict to integers that are coprime with k. Let us introduce the second point of view on the definition of Dirichlet characters.

Definition II.3.2. A function χ from $\mathbb{Z}$ to the set of complex numbers $\mathbb{C}$ is a Dirichlet character if :

- There exists a positive integer k called modulus such that:
  (1) For all n, if $\gcd(n, k) > 1$, then $\chi(n) = 0$; if $\gcd(n, k) = 1$ then $\chi(n) \neq 0$.
  (2) $\chi(n) = \chi(n + k)$ for all integers n.
- χ is completely multiplicative (i.e. $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n).

Several properties can be deduced from this definition. For instance for any Dirichlet character χ, we have $\chi(0) = 0$ and $\chi(1) = 1$. Moreover, let χ be a character of modulus k and a be any integer with $\gcd(a, k) = 1$. Euler's theorem asserts that $a^{\phi(k)} \equiv 1 \pmod k$ (where $\phi(k)$ is the totient function). Therefore $(\chi(a))^{\phi(k)} = \chi(a^{\phi(k)}) = \chi(1) = 1$. In other words, for all a relatively prime to k, $\chi(a)$ is a $\phi(k)$-th complex root of unity.

We notice that a character can be considered with different modulus. For instance a character with modulus 9 can be seen as a character with modulus 27 as well. In fact there is a notion of induced characters which further clarifies this situation. We will explain it in Definition II.3.13. In the following, we see some simple examples of characters:

Example II.3.3. The unique character of modulus 1, called the **trivial** character. It has the value 1 everywhere except at zero.

Example II.3.4. A character that assumes the value 1 for arguments coprime to its modulus and otherwise 0 is called **principal**.

As we have seen the values of Dirichlet characters are roots of unity, so they can be real or complex numbers.

Definition II.3.5. A character is **real** if all its values are real.

According to the multiplicative property of Dirichlet characters we have $1 = \chi(1) = \chi(-1)\chi(-1)$. Thus, for a Dirichlet character χ the value $\chi(-1)$ can be either 1 or -1 and based on this evaluation it is called odd or even. Odd Dirichlet characters are important for our future computations for the link between the Mahler measure and L-functions.

Definition II.3.6. A Dirichlet character χ is said to be **odd** if $\chi(-1) = -1$ and **even** if $\chi(-1) = 1$.

In the following, we introduce quasi periods of a Dirichlet character and its conductor which leads to the definition of primitive character.

Definition II.3.7. Let χ be a character of modulus k and q a positive number smaller than k. Then, q is called a **quasi period** for χ, if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod q$ and $\gcd(mn, k) = 1$.

Let χ be a Dirichlet character of modulus k, and q an arbitrary quasi period of χ. It is not necessarily true that $q|k$. However, we will prove that if q is the smallest quasi period of χ, then it is a divisor of k.

Definition II.3.8. The smallest quasi period is called the **conductor** of the character.

To prove that the conductor divides the modulus of a character we prove the following lemma:

Lemma II.3.9. *Let χ be a character of modulus k and $g = \gcd(k, q)$ where q is a quasi period of χ. Then, g is a quasi period of χ.*

PROOF. To prove that g is a quasi period of χ, we show that for every $m, n \in \mathbb{Z}$ such that $m \equiv n \bmod g$ and $\gcd(mn, k) = 1$, we have $\chi(m) = \chi(n)$. Since $g = \gcd(k, q)$, Euclid's algorithm implies that there exist $x_1, y_1 \in \mathbb{Z}$ such that $g = kx_1 + qy_1$. Moreover, $m \equiv n \bmod g$ so we have $m - n = gt$, for some $t \in \mathbb{Z}$. Thus, we conclude that there exist $x, y \in \mathbb{Z}$ such that $m - n = kx + qy$ (let $x = tx_1$ and $y = ty_1$). Then, we have:

$$\chi(m) = \chi(m - kx) = \chi(n + qy) = \chi(n).$$

The last equity follows from the fact that q is a quasi period of χ. Thus, g is also a quasi period for χ. □

A direct result of the previous lemma is that for a non principal Dirichlet character of modulus k we have $\gcd(k, q) > 1$, for every quasi period q of χ. Since, otherwise, according to Lemma II.3.9, 1 would be a quasi period of χ. In other words $\chi(m) = \chi(n) = \chi(1) = 1$, for every $m, n \in \mathbb{Z}$ with $\gcd(mn, k) = 1$, which means that χ is the principal character which is a contradiction. For more information about the proof of the following lemma see [Ove14, Section 3.7].

Lemma II.3.10. *The conductor of a Dirichlet character divides the modulus of the character.*

PROOF. Let χ be a Dirichlet character of modulus k and conductor c then according to Lemma II.3.9 we have $q = \gcd(c, k)$ is a quasi period of χ dividing c. Thus, it is smaller than or equal to c, but the conductor c is the smallest quasi period of χ, so we have $c = \gcd(c, k)$. In other words we have $c|k$. □

To clarify the difference between the modulus, the quasi periods, and the conductor of a Dirichlet character we consider the following example;

Example II.3.11. Let χ be the following character of modulus 9:

m	0	1	2	3	4	5	6	7	8
$\chi(m)$	0	1	-1	0	1	-1	0	1	-1

We consider χ as a character with modulus 9. However, it can be considered as a character with modulus 3^n for $n \geq 1$. Then, 3 and 6 are quasi periods for this character. Since 3 is the smallest quasi period, it is the conductor.

As we have seen in the previous example the modulus and the conductor are not equal and here we define a special family of characters for which conductor and the modulus are equal.

Definition II.3.12. A Dirichlet character whose modulus is its conductor is called **primitive**. Those who are not primitive are called **imprimitive**.

We explain more about "primitive" and "imprimitive" Dirichlet characters after introducing induced characters.

Definition II.3.13. Let χ^* be a Dirichlet character modulus q and k such that $q|k$. Define a Dirichlet character χ as follows:

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if} \gcd(n,k) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, χ of modulus $q \in \mathbb{Z}_{\geq 1}$ is called **induced character** from χ^*.

Using the above definition a Dirichlet character is primitive if it is not induced by any character other than itself; for instance, the Dirichlet character introduced in Example II.3.11 is not primitive and it is induced by the following primitive character with modulus (= conductor) 3:

m	0	1	2
$\chi(m)$	0	1	-1

In Section VI.4, we come back to the notion of induced character and we will see that every Dirichlet character is induced by a uniquely determined primitive Dirichlet character.

In the sequel of this section we recall the necessary information for defining an important family of real primitive characters, called quadratic. These characters are important to introduce the evaluation of the Mahler measure of the examples of Smyth. To define quadratic characters, we need to define Jacobi, Legendre and Kronecker symbols.

Definition II.3.14. The **Legendre symbol** modulo an odd prime number p is denoted by $(\frac{\cdot}{p})$ and is a multiplicative function with values 1, -1, 0. The value of the Legendre symbol at a is denoted by $(\frac{a}{p})$ and is defined by:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod p, \\ 1 & \text{if } a \not\equiv 0 \pmod p \text{ and there exists an } x \in \mathbb{Z}\ ,\ a \equiv x^2 \pmod p, \\ -1 & \text{if } a \not\equiv 0 \pmod p \text{ and there is no } x \in \mathbb{Z} \text{ with, } a \equiv x^2 \pmod p. \end{cases}$$

Note from the above that $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p$.

By construction, the Legendre symbol $(\frac{\cdot}{p})$ is a Dirichlet character of modulus p.
The generalization of this definition for an arbitrary odd integer n instead of a prime number p is the following:

Definition II.3.15. For any positive odd integer n, the **Jacobi symbol** $(\frac{\cdot}{n})$ is defined as the product of the Legendre symbols corresponding to the prime factors of n and its value at any integer a is computed by:

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1} \left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}.$$

Where, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Moreover, following the normal convention for the empty products we set, $(\frac{a}{1}) = 1$.

The Jacobi symbol is multiplicative by construction. One can verify that the Jacobi symbol $\left(\frac{\cdot}{n}\right)$ is a Dirichlet character of modulus $|n|$ whose value at a is $\left(\frac{a}{n}\right)$. Another question is whether $\left(\frac{m}{\cdot}\right)$ is a Dirichlet character as well. It is completely multiplicative by the definition. An immediate difficulty is that the Jacobi symbol does not admit even values for the denominator. We can formally circumvent this problem by defining the values of $\left(\frac{m}{0}\right)$ and $\left(\frac{m}{2}\right)$ and then extending to all integers by multiplicativity. Obviously, to have a Dirichlet character for the symbol $\left(\frac{m}{\cdot}\right)$ we need to define $\left(\frac{m}{0}\right) = 0$. However, defining $\left(\frac{m}{2}\right)$ is more subtle. The following is the most complete generalization of the Legendre symbol for the case where n is an arbitrary integer.

Definition II.3.16. Let a be an integer and $n = up_1^{e_1} \cdots p_k^{e_k}$, where $u = \pm 1$. The **Kronecker symbol** $\left(\frac{a}{n}\right)$ is defined by:

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{e_i}.$$

For odd p_i, $\left(\frac{a}{p_i}\right)$ is the Legendre symbol and for $p_i = 2$:

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \equiv \pm 1 \pmod 8, \\ -1 & \text{if } a \equiv \pm 3 \pmod 8. \end{cases}$$

Moreover;

$$\left(\frac{a}{-1}\right) = \begin{cases} 1 & \text{if } a \geq 0, \\ -1 & \text{if } a < 0. \end{cases}$$

A fundamental property of the Kronecker symbol is the quadratic reciprocity law, explained in the following proposition. We refer to [Vek19] for a minimalist proof.

Proposition II.3.17. *For any nonzero integers m and n we have:*

$$\left(\frac{m}{n}\right) = \sigma(m,n)(-1)^{\frac{m_1-1}{2}\frac{n_1-1}{2}} \left(\frac{n}{m}\right), \tag{II.3.1}$$

where $m = \pm 2^{v_2(m)} p_1^{v_{p_1}(m)} \cdots p_n^{v_{p_n}(m)}$ and $n = \pm 2^{v_2(n)} p_1^{v_{p_1}(n)} \cdots p_n^{v_{p_n}(n)}$ and m_1 and n_1 are respectively the largest odd factors of m and n (i.e. $m_1 = \frac{m}{2^{v_2(m)}}$ and $n_1 = \frac{n}{2^{v_2(n)}}$) and

$$\sigma(m,n) = \begin{cases} -1 & \textit{if both } m, n < 0, \\ 1 & \textit{otherwise.} \end{cases}$$

The Kronecker symbol as well as the Jacobi symbol and the Legendre symbol are multiplicative. Indeed, by construction $\left(\frac{\cdot}{n}\right)$ is a Dirichlet character of modulus n. To answer the question that when the Kronecker symbol $\left(\frac{m}{\cdot}\right)$ is a Dirichlet character, we first define quadratic characters. After introducing quadratic characters and necessary prerequisites about fundamental discriminants, we will see that the Kronecker symbol $\left(\frac{m}{\cdot}\right)$ is a Dirichlet character if m is a fundamental discriminant. However, if m is not a fundamental discriminant, we do not have any certain answer.

II.3.2. Quadratic characters. The quadratic characters are an important family of real Dirichlet characters, since any real primitive character is either the principal or a quadratic character. Moreover, the Mahler measures of the famous examples of Smyth are linked to special values of Dirichlet character associated with quadratic characters. The information and discussions around quadratic characters in this section comes from [Gol].

Definition II.3.18. [MV06] A character of modulus k is **quadratic** if it has order 2 in the group of characters of modulus k (i.e. $\chi^2 = 1$ and χ is not principal).

From the above definition we realize that a quadratic character takes only the values ± 1 and -1 at least once. Thus, a quadratic character is a real and non-principal character. To answer the question of when the Kronecker symbol is a Dirichlet character we need to introduce the fundamental discriminant of a quadratic number field. To do so we first introduce quadratic number fields.

Definition II.3.19. A **quadratic number field** is a field F (inside $\mathbb{C}$) such that F has dimension 2 as a vector space over $\mathbb{Q}$.

[LZ21, Proposition 6.46] proves that for a field extension F of $\mathbb{Q}$ of degree 2 there is a unique squarefree n such that F takes the form:

$$F = \mathbb{Q}(\sqrt{n}) = \{a + b\sqrt{n} : a, b \in \mathbb{Q}\}, \quad n \in \mathbb{Z} \setminus \{0, 1\} \text{ squarefree.}$$

If n is positive then F is a **real quadratic number field**, and if n is negative then F is an **imaginary quadratic number field**. The discriminant of a number field is one of the most basic invariants of a number field. It is proportional to the squared volume of the fundamental domain of the ring of integers and it regulates which primes are ramified. This definition of the discriminant of a field involves many new definitions which are not necessary for the sequel of this thesis. Thus, we explain a method to compute the discriminant of a quadratic field, which [Coh10, Definition 5.1.2] considered as a definition.

Definition II.3.20. The **discriminant** of a quadratic field $F = \mathbb{Q}(\sqrt{n})$ denoted by D_F is computed by:

$$D_F = \begin{cases} n & \text{if } n = 1 \pmod 4, \\ 4n & \text{if } n = 2, 3 \pmod 4). \end{cases}$$

From the above definition we conclude that not every integer can be a discriminant of a quadratic field.

Definition II.3.21. A **fundamental discriminant** is any integer which is the discriminant of some quadratic extensions of $\mathbb{Q}$.

The set of all fundamental discriminants of magnitude smaller than 16 is as follows:

$$\{-15, -11, -8, -7, -4, -3, 5, 8, 12, 13\}.$$

The following theorem characterizes the quadratic characters. Moreover, it answers the question of when the Kronecker symbol $\left(\frac{m}{\cdot}\right)$ defines a Dirichlet character. To see the proof of the theorem see section 2.2.4 of [Coh07] or [MV06, Theorem 9.13].

Theorem II.3.22. *For any fundamental discriminant D, $\left(\frac{D}{\cdot}\right)$ is a primitive quadratic character and in this case it is denoted by χ_D. Conversely, given any primitive quadratic character χ, there exists a unique fundamental discriminant D such that $\chi = \chi_D$.*

Moreover, quadratic characters corresponding to real quadratic fields (i.e. $D > 0$) are even and those correspond to imaginary quadratic fields (i.e. $D < 0$) are odd. Indeed, one can observe that:

$$\chi_D(-1) = \left(\frac{D}{-1}\right) = \begin{cases} 1 & \text{if } D > 0, \\ -1 & \text{if } D < 0. \end{cases}$$

A simple example of quadratic characters is as follows:

Example II.3.23. Since -3 is a fundamental discriminant, χ_{-3} is a quadratic character and its values are as follows:

$$\chi_{-3}(n) = \left(\frac{-3}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3, \\ \text{-1} & \text{if } n \equiv 2 \bmod 3, \\ 0 & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

Remark II.3.24. We notice that the theorem says nothing about $(\frac{m}{\cdot})$ where m is not a fundamental discriminant. Such $(\frac{m}{\cdot})$ might be primitive Dirichlet characters (e.g. $(\frac{2}{\cdot})$), imprimitive characters (e.g. $(\frac{4}{\cdot})$), or even not a character at all (e.g. $(\frac{3}{\cdot})$).

Example II.3.25. It is easy to check that both of $(\frac{\pm 2}{\cdot})$ are primitive characters (mod 8):

$$\left(\frac{2}{n}\right) = \begin{cases} 0 & \text{if } n \equiv 0, \pm 2, 4 \pmod 8, \\ 1 & \text{if } n \equiv \pm 1 \pmod 8, \\ -1 & \text{if } n \equiv \pm 3 \pmod 8. \end{cases}$$

$$\left(\frac{-2}{n}\right) = \begin{cases} 0 & \text{if } n \equiv 0, \pm 2, 4 \pmod 8, \\ -1 & \text{if } n \equiv 1, 3 \pmod 8, \\ 1 & \text{if } n \equiv -1, -3 \pmod 8. \end{cases}$$

This does not contradict the classification of primitive real characters, since each of these characters can be written in terms of a fundamental discriminant: $\chi_{\pm 2} = \chi_{\pm 8}$.

As a simple example consider the real principle Dirichlet character mod 1 introduced in Example II.3.4. It can be defined by the Kronecker symbol as χ_1. We do not consider 1 as a fundamental discriminant, since it is the discriminant of $\mathbb{Q}$ which is a trivial extension of $\mathbb{Q}$ (of degree 1). Thus, by our definition it is not a quadratic extension, but some references consider it as degenerate extension of $\mathbb{Q}$. Then we should consider Theorem II.3.22 for fundamental discriminants not equal to 1, otherwise χ_1 is not a quadratic character. However, it is the unique primitive principal character, as it induces all other principal characters.

Example II.3.26. Since -4 is a fundamental discriminant, then according to Theorem II.3.22 χ_{-4} is a real non principal primitive character mod 4 as follows:

$$\chi_{-4}(n) = \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if } n \equiv 0, 2 \pmod 4, \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Thus, it is an odd Dirichlet character. On the other side $(\frac{4}{\cdot})$ is computed as follows:

$$\left(\frac{4}{n}\right) = \begin{cases} 0 & \text{if } n \equiv 0, 2 \pmod 4, \\ 1 & \text{if } n \equiv 1, 3 \pmod 4, \\ 0 & \text{if } n \equiv 0, 2 \pmod 4. \end{cases}$$

From the above computation we realize that $(\frac{4}{\cdot})$ is the Dirichlet character which has value one everywhere, so it is the principal one. In this case although 4 is not a fundamental discriminant, still $(\frac{4}{\cdot})$ defines a character (but it is not quadratic).

Example II.3.27. The Kronecker symbol $(\frac{3}{\cdot})$ does not define a Dirichlet character, since it is not a 3-periodic function. To see that we compute $(\frac{3}{n})$ for $1 \leq n \leq 7$;

n	1	2	3	4	5	6	7
$(\frac{3}{n})$	1	-1	0	1	-1	0	-1

Therefore, it is not 3-periodic.

II.3.3. Dirichlet L-functions. In this section, we introduce some important functions in analytic Number Theory such as the Gamma function, the Riemann Zeta function, and Dirichlet L-functions. Moreover, thanks to the Gamma function we prove some important functional equations for L-functions. These equations are not only needed for introducing Smyth's examples, (see Section II.3.4) but also in Chapter VI of this thesis, where we will use them to relate the Mahler measure of P_d to values of L-functions.

II.3.3.1. *The Gamma function.* The Gamma function is the extension of the factorial function to complex numbers.

Definition II.3.28. For a complex number with real part strictly positive the following integral converges absolutely and defines $\Gamma(z)$:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx.$$

In the following we see some properties of $\Gamma(z)$, for the proofs see Section VII.2 in Appendix.

- For every z with real part strictly positive we have $\Gamma(1+z) = z\Gamma(z)$ (see Lemma VII.2.1 in Appendix).
- For $n \in \mathbb{Z}_{\geq 2}$, we have $\Gamma(n) = (n-1)!$ (see Lemma VII.2.2 in Appendix).
- We have the following equation called Legendre duplication formula : $\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z)$ (see Lemma VII.2.3 in Appendix).

II.3.3.2. *The L-function.* Dirichlet L-functions are important in additive Number Theory. For instance, they were used to prove Dirichlet's theorem and have a close connection with modular forms. A special type of Dirichlet L-function is the Riemann Zeta function which will be introduced in the next section. There are interesting conjectures around L-functions such as the generalized Riemann hypothesis which is one of the most important conjectures in mathematics. It conjectures that neither the Riemann Zeta function nor any Dirichlet L-function has a zero with real part larger than $\frac{1}{2}$. Computing special values of L-functions is one of the interesting research work in analytic Number Theory. The examples of Smyth which will be introduced in Section II.3.4 illustrate the link between Mahler measures of some particular polynomials and special values of L-functions. Thus, one of the applications of computing Mahler measures is to further explore this link. The aim of this section is to introduce L-functions and their functional equations, which will be useful for our future computations in Chapter VI.

Definition II.3.29. A Dirichlet L-series is a function of the form

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Here, χ is a Dirichlet character and s a complex variable with real part greater than 1. By analytic continuation, this function can be extended to a meromorphic function on the whole complex plane, still called a Dirichlet **L-function** and also denoted by $L(\chi, s)$.

As we have already mentioned, we are searching for some crucial functional equations for primitive Dirichlet L-functions. In fact these equations relate $L(\chi, s)$ and $L(\bar{\chi}, 1-s)$ (where

$\bar{\chi}$ is the complex conjugate of χ). These equations involve a factor of a Gauss sum which we introduce as follows:

Definition II.3.30. Let χ be a primitive character of modulus k and $n \in \mathbb{Z}$. The Gauss sums associated to χ is defined as follows;

$$\tau_n(\chi) := \sum_{1 \le a \le k} \chi(a) e^{\frac{2\pi i n a}{k}}.$$

Moreover, when $n = 1$, we generally abbreviate it as $\tau(\chi)$;

$$\tau(\chi) = \sum_{1 \le a \le k} \chi(a) e^{\frac{2\pi i a}{k}}.$$

If the Dirichlet character χ is primitive, then we have the following correspondence between $\tau(\chi)$ and $\tau_n(\chi)$, proved in [Elk, Equation 5]:

Lemma II.3.31. *Let χ be a primitive character of modulus k and $n \in \mathbb{Z}$. Then*

$$\tau_n(\chi) = \bar{\chi}(n)\tau(\chi).$$

As we have seen in Section II.3.2, the real primitive characters are either principal or quadratic, and the quadratic ones are those which correspond to a quadratic number field. In other words, for a quadratic character χ there exist a quadratic number field with discriminant D and for $n \in \mathbb{Z}_{\ge 1}$ we have $\chi(n) = (\frac{D}{n})$. In this case, we may denote χ by χ_D. The Gauss sum $\tau(\chi)$ associated to the quadratic Dirichlet character can be simply determined without any computation, only by using the discriminant of the associated number field. The following lemma clarifies this point;

Lemma II.3.32 ([Apo76], Theorem 9.13)**.** *The Gauss sum associated to the quadratic Dirichlet character χ_D is:*

$$\tau(\chi_D) := \begin{cases} i\sqrt{|D|} & \text{if } D < 0, \\ \sqrt{D} & \text{if } D > 0. \end{cases}$$

The last step before stating the functional equation for the L-function associated to a primitive Dirichlet character is to introduce the function $\xi(\chi, s)$.

Definition II.3.33. Let χ is a primitive Dirichlet character of modulus k. The function $\xi(\chi, s)$ is defined by :

$$\xi(\chi, s) := (\frac{\pi}{k})^{\frac{-(s+a)}{2}} \Gamma(\frac{s+a}{2}) L(\chi, s),$$

where a is an integer depending on the parity of χ, defined as follows:

$$a := \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

From the above definition we can find $\xi(\chi, s)$ in terms of $\xi(\bar{\chi}, 1-s)$ for every primitive Dirichlet character. For the proof and further information corresponding to the following proposition see [Elk, Equation 12].

Proposition II.3.34. *Let χ is a primitive Dirichlet character of modulus k, then we have the following equation:*

$$\xi(\chi, s) = \frac{\tau(\chi)}{i^a \sqrt{k}} \xi(\bar{\chi}, 1-s). \tag{II.3.2}$$

We have all the necessary prerequisites to find a functional equation relating $L(\chi, s)$ and $L'(\bar{\chi}, 1-s)$. We assume $s = 2$, since this is the version that we will use in Section II.3.4 and later in Chapter VI. We notice that following proposition concerns all the primitive Dirichlet characters, needed for the computations of Chapter VI, but in Corollary II.3.36 we have a particular case of this proposition for quadratic characters, needed for the computations in Section II.3.4. For more information about the following proposition see [Ray87, Page 697].

Proposition II.3.35. *Let χ be an odd primitive Dirichlet character of conductor k, we have:*

$$L(\chi, 2) = \frac{4\pi}{ik^2}\tau(\chi)L'(\bar{\chi}, -1). \tag{II.3.3}$$

PROOF. Since χ is odd we let $a = 1$ in Eq. (II.3.2) and by applying Proposition II.3.34 we have:

$$\xi(\chi, s) = \left(\frac{\pi}{k}\right)^{-(s+1)/2}\Gamma((s+1)/2)L(\chi, s) = \frac{\tau(\chi)}{i\sqrt{k}}\xi(\bar{\chi}, 1-s)$$

In the previous equation again we use II.3.2 for $\xi(\bar{\chi}, 1-s)$ and we have:

$$\begin{aligned}\left(\frac{\pi}{k}\right)^{-(s+1)/2}\Gamma((s+1)/2)L(\chi, s) &= \frac{\tau(\chi)}{i\sqrt{k}}(\frac{\pi}{k})^{-(1-s+1)/2}\Gamma((1-s+1)/2)L(\bar{\chi}, 1-s) \\ &= \frac{\tau(\chi)}{i\sqrt{k}}(\frac{\pi}{k})^{-(2-s)/2}\frac{2-s}{2}\frac{\Gamma((2-s)/2)L(\bar{\chi}, 1-s)}{2-s}2.\end{aligned}$$

In the R.H.S of the last equation we use the properties of the Gamma function explained in Appendix (see Lemmas VII.2.1, VII.2.2 and VII.2.3 in Section VII.2 in Appendix) and we have:

$$\left(\frac{\pi}{k}\right)^{-(s+1)/2}\Gamma((s+1)/2)L(\chi, s) = \frac{\tau(\chi)}{i\sqrt{k}}(\frac{\pi}{k})^{-(2-s)/2}\Gamma(1+\frac{(2-s)}{2})\frac{L(\bar{\chi}, -1+2-s)}{2-s}2.$$

By letting $s \to 2$ we have the following equation:

$$L(\chi, 2) = \frac{4\pi}{ik^2}\tau(\chi)L'(\bar{\chi}, -1).$$

□

We notice that for real characters, $\chi = \bar{\chi}$. In particular by using Lemma II.3.32 for quadratic characters we conclude:

Corollary II.3.36. *Let $\chi_D(n) = \left(\frac{D}{n}\right)$ be the odd quadratic character of modulus $|D| = -D$, then we have:*

$$L(\chi_D, 2) = \frac{4\pi}{|D|^{\frac{3}{2}}}L'(\chi_D, -1).$$

PROOF. The proof is the conclusion from Proposition II.3.35 and Lemma II.3.32 since we have :

$$L(\chi_D, 2) = \frac{4\pi}{iD^2}i\sqrt{|D|}L'(\chi_D, -1) = \frac{4\pi}{|D|^{\frac{3}{2}}}L'(\chi_D, -1).$$

□

II.3.3.3. *The Zeta function.* As we already mentioned in the previous section, a special type of Dirichlet L-functions is the Riemann Zeta function. In fact if in the definition of $L(\chi, s)$ we fix the Dirichlet character to be the trivial character, then we have the Riemann Zeta function.

Definition II.3.37. The **Riemann Zeta** function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$. When $\Re(s) = \sigma > 1$, the function can be written as a converging summation or integral:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x,$$

where Γ is the gamma function. The Riemann Zeta function is defined for other complex values via analytic continuation of the function defined for $\sigma > 1$.

Euler considered the above series for positive integer values of s, and later Chebyshev extended the definition to $\Re(s) > 1$. The above series is a prototypical Dirichlet series that converges absolutely to an analytic function for s such that $\sigma > 1$ and diverges for all other values of s. Riemann showed that the function defined by the series on the half-plane of convergence can be continued analytically to all complex values $s \neq 1$. For $s = 1$, the series is the harmonic series which diverges to ∞, and

$$\lim_{s \to 1} (s-1)\zeta(s) = 1.$$

Thus, the Riemann Zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.
Concerning special values of the Zeta function, Euler proved that $\zeta(2) = \frac{\pi^2}{6}$ (see [Der03, Page 64]), which can be generalized to all the positive even ζ-values (see [Ayo74]), using Bernoulli numbers as follows:

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}. \tag{II.3.4}$$

In particular, all even ζ-values are transcendental (see Fact VII.7.6). For non positive integers, one has:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1},$$

for $n \geq 0$ (using the convention that $B_1 = \frac{-1}{2}$ see Section VII.7.1 and Remark VII.7.4). In particular, ζ vanishes at the negative even integers because $B_m = 0$ for all odd m other than 1 (see Fact VII.7.3). These are the so-called "trivial zeros" of the Zeta function. Bernhard Riemann (1859) conjectured that the Riemann Zeta function has its zeros only at the negative even integers and complex numbers with the real part $\frac{1}{2}$. This is called the Riemann hypothesis and is still unsolved. Many consider it to be the most important open problem in pure mathematics, and it is one of the Millennium Prize Problems. It is of great interest in number theory because it implies results about the distribution of prime numbers (see for instance [MS16]). For odd positive integers, no simple expression is known, and the nature of the odd ζ-values remains mysterious. It is known that $\zeta(3) \notin \mathbb{Q}$ ([VdPA79]), and that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational [Zud04]. In fact, infinitely many odd zeta values are irrational [Riv00, FSZ19], but yet it is not proved that a single odd ζ-value is transcendental.

The derivative of the zeta function at the negative even integers is given by [Apo85]:

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1). \tag{II.3.5}$$

The first few values of which are $\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}, \zeta'(-4) = \frac{3}{4\pi^4}\zeta(5), \ldots$.

II.3.4. The Mahler measure, some examples linked to L-functions. As promised, in this section we introduce some famous examples of the computation of the Mahler measure for multivariable polynomials. The first computation of the Mahler measure of multivariate polynomial is due to Smyth (1981).

Proposition II.3.38 ([Smy81a]). *We have*

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

where $\chi_{-3}(n) = (\frac{-3}{n})$ is the odd quadratic Dirichlet character of modulus 3, defined in Example II.3.23.

PROOF. We have:

$$\begin{aligned}
m(1+x+y) &= \int_0^1 \int_0^1 \log|1+e^{2\pi it}+e^{2\pi is}|dtds \overset{[1]}{=} \int_0^1 \log^+|1+e^{2\pi is}|ds \\
&= \int_0^1 \log\max\{1, |1+e^{2\pi is}|\}ds = \int_{\frac{-1}{3}}^{\frac{1}{3}} \log|1+e^{2\pi is}|ds \\
&= \int_{\frac{-1}{3}}^{\frac{1}{3}} \Re\left(\log(1+e^{2\pi is})\right) ds = \Re \int_{\frac{-1}{3}}^{\frac{1}{3}} \log(1+e^{2\pi is}) ds \\
&= \Re \int_{\frac{-1}{3}}^{\frac{1}{3}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{2\pi int}}{n} dt = \Re \sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^{2\pi int}}{2\pi in^2} \Big|_{t=\frac{-1}{3}}^{t=\frac{1}{3}} \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(\frac{2\pi n}{3})}{n^2}.
\end{aligned}$$

In [1], we used Jensen's formula. Then we used the fact that $|1+e^{2\pi is}| = 2\cos(\pi s)$ for $s \in [\frac{-1}{3}, \frac{1}{3}]$ we have $|1+e^{2\pi is}| \geq 1$. In the last line we used the Taylor's series of $\log(1+e^{2\pi is})$, which converges uniformly on $[\frac{-1}{3}, \frac{1}{3}]$ so we can change the order of the sum and the integral. Now the last step of the proof needs the equality $\sin(\frac{2\pi n}{3}) = \frac{\sqrt{3}}{2}\chi_{-3}(n)$, and certain properties of Dirichlet characters (see [EW99, Lemma 3.6]);

$$\begin{aligned}
&\frac{\sqrt{3}}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \chi_{-3}(n) = \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty} \frac{\chi_{-3}(2n-1)}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{\chi_{-3}(2n)}{(2n)^2}\right) \\
&\frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2} - 2\sum_{n=1}^{\infty} \frac{\chi_{-3}(2n)}{(2n)^2}\right) \overset{[2]}{=} \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2} - \frac{1}{2}\chi_{-3}(2)\sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2}\right) \\
&= \frac{\sqrt{3}}{2\pi}\frac{3}{2}\sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2} = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2).
\end{aligned}$$

In [2] we use the multiplicativity of the Dirichlet character. □

One may compute the above evaluation in terms of the special values $L'(\chi_{-3}, -1)$, so by using Corollary II.3.36 we have the following corollary:

Corollary II.3.39. *We have the following equalities:*

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1).$$

Before stating the second example let us recall an important property of the Mahler measure which is needed for future computations.

Lemma II.3.40. *For $P(x) = ax + b$ we have $M(ax+b) = |a| \max\left\{\left|\frac{b}{a}\right|, 1\right\} = \max\{|a|, |b|\}$, then by using Jensen's formula we have:*

$$m(ax+b) = \log \max\{|a|, |b|\} = \int_0^1 \log |ae^{2\pi i\theta} + b| d\theta.$$

Proposition II.3.41 ([Smy81a], [Boy81b])**.** *We have*

$$m(1 + x + y + z) = \frac{7}{2\pi^2}\zeta(3),$$

where ζ is the Riemann Zeta function.

PROOF. In order to simplify the question we use the monomial transformation $(x, y, z) \mapsto (x, yz, z)$, then according to Proposition II.2.14 we have $m(1+x+y+z) = m(1+x+(1+y)z)$;

$$\begin{aligned} m(1 + x + (1 + y)z) &= \int_0^1 \int_0^1 \int_0^1 \log |1 + e^{2\pi i\theta_1} + (1 + e^{2\pi i\theta_2})e^{2\pi i\theta_3}| d\theta_1 d\theta_2 d\theta_3 \\ &= \int_0^1 \int_0^1 \int_0^1 \log |1 + e^{2\pi i\theta_1} + (1 + e^{2\pi i\theta_2})e^{2\pi i\theta_3}| d\theta_3 d\theta_1 d\theta_2. \end{aligned}$$

Now that we changed the order of the integral we can use Lemma II.3.40;

$$m(1 + x + (1 + y)z) = \int_0^1 \int_0^1 \log \max\{|1 + e^{2\pi i\theta_1}|, |1 + e^{2\pi i\theta_2}|\} d\theta_1 d\theta_2.$$

We notice that $|1 + e^{2\pi i\theta}| = |2\cos(\pi\theta)|$ and, for $0 \le \theta \le \frac{1}{2}$ the function $\cos(\pi\theta)$ is increasing. But when $\frac{1}{2} \le \theta \le 1$ it is decreasing so by using the change of variables $t = 2\pi\theta_1$ and $s = 2\pi\theta_2$ we have an increasing function in the region of integral:

$$\begin{aligned} m(P(x, y, z)) &= \int_0^1 \int_0^1 \log \max\{|1 + e^{2\pi i\theta_1}|, |1 + e^{2\pi i\theta_2}|\} d\theta_1 d\theta_2 \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \max\{|1 + e^{it}|, |1 + e^{is}|\} dt ds \\ &= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \max\{\log |1 + e^{it}|, \log |1 + e^{is}|\} dt ds. \end{aligned}$$

The last equality is based on the fact that $\log |x|$ is a decreasing function. Now let $F(t, s) := |1 + e^{it}| = 2\cos(\frac{t}{2})$, and $0 \le t, s \le \pi$. The function $F(t, s)$ is a symmetric function, so instead of computing the integral over the square $[0, \pi] \times [0, \pi]$, we can compute the integral over the triangle as following:

$$\begin{aligned} m(P(x, y, z)) =& \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \max\{\log |1 + e^{it}|, \log |1 + e^{is}|\} dt ds \\ =& \frac{2}{\pi^2} \int_0^{\pi} \int_s^{\pi} \max\{\log |1 + e^{it}|, \log |1 + e^{is}|\} dt ds \\ \overset{[3]}{=}& \frac{2}{\pi^2} \int_0^{\pi} \int_s^{\pi} \log |1 + e^{is}| dt ds = \frac{2}{\pi^2} \int_0^{\pi} (\pi - s) \log |1 + e^{is}| ds \\ =& \frac{2\pi}{\pi^2} \int_0^{\pi} \log |1 + e^{is}| ds + \frac{-2}{\pi^2} \int_0^{\pi} s \log |1 + e^{is}| ds \\ =& m(1 + z) + \frac{-2}{\pi^2} \int_0^{\pi} s \log |1 + e^{is}| ds = \frac{-2}{\pi^2} \int_0^{\pi} s \log |1 + e^{is}| ds. \end{aligned}$$

In [3] we use the fact that $\cos(t/2)$ is decreasing in the interval $[0,\pi]$. Moreover thanks to Jensen's formula we can replace the integral with $m(1+z)$ which is equal to zero. We again use the Taylor's series of $\log|1+e^{is}|$:

$$\log|1+e^{is}| = \Re\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{isn}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(sn)}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{e^{isn}+e^{-isn}}{2}\right).$$

We continue the computation of the integral;

$$\begin{aligned} m(P(x,y,z)) =& \frac{-2}{\pi^2}\int_0^{\pi} s\log|1+e^{is}|ds = \frac{-2}{\pi^2}\int_0^{\pi} s\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{e^{isn}+e^{-isn}}{2}\right)ds \\ =& \frac{-1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi} s(e^{isn}+e^{-isn})ds. \end{aligned}$$

We compute each of the integrals of $\int_0^{\pi} se^{isn}ds$ and $\int_0^{\pi} se^{-isn}ds$ by integrating by parts.

$$\int_0^{\pi} se^{isn}ds = \frac{se^{isn}}{in}|_0^{\pi} - \frac{e^{isn}}{n^2}|_0^{\pi} = \frac{\pi(-1)^n}{in} + \frac{(-1)^n}{n^2} - \frac{1}{n^2},$$

$$\int_0^{\pi} se^{-isn}ds = \frac{-se^{-isn}}{in}|_0^{\pi} + \frac{e^{-isn}}{n^2}|_0^{\pi} = \frac{-\pi(-1)^n}{in} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}.$$

Then we have:

$$\begin{aligned} m(P(x,y,z)) =& \frac{-1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi} s(e^{isn}+e^{-isn})ds \\ =& \frac{-1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{2(-1)^n}{n^2}-\frac{2}{n^2}\right). \end{aligned}$$

For $n=2k$ we have $\frac{2(-1)^n}{n^2}-\frac{2}{n^2}=0$ and for $n=2k+1$ we have $\frac{2(-1)^n}{n^2}-\frac{2}{n^2}=\frac{-4}{n^2}$:

$$m(P(x,y,z)) = \frac{-1}{\pi^2}\sum_{k=0}^{\infty}\frac{4}{(2k+1)^3}.$$

Now by using the formula for special values of Riemann Zeta function $\zeta(3) = \frac{8}{7}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}$ we have:

$$m(P(x,y,z)) = \frac{-1}{\pi^2}\sum_{k=0}^{\infty}\frac{4}{(2k+1)^3} = \frac{7}{2\pi^2}\zeta(3).$$

□

II.4. Topology on the set of Mahler measure values

As we have seen in Fact II.1.15 the set of the values of the Mahler measure of univariate polynomials with integer coefficients is a subset of $[1,\infty)$. Let us denote this set by L. Boyd [Boy81b] was interested to know that if 1 is the limit point of L. In fact, this question is equivalent to the Lehmer conjecture. To answer this question, he attempted to characterize the limit points of L. There exists a notion of the Mahler measure of algebraic number. He studied the set of Mahler measures of Salem and Pisot numbers (see Section II.4.2) which are remarkable subsets of L to have more information about L and its limit points. However, later he suggested to study a larger set, namely the set of Mahler measures of multivariate polynomials with integer coefficients. He was interested in knowing if this set is closed in order to answer Lehmer's conjecture. However, answering this question seems also impossible and became a new conjecture, called Boyd's conjecture. His efforts in this direction together with the work of Lawton led to one of the most important theorems in this area, called the Theorem of Boyd-Lawton, which will be announced in this section. Moreover, in the end of this section we state a generalization of the Theorem of Boyd-Lawton.

II.4.1. The Mahler measure of algebraic numbers, Schinzel and Zassenhaus question. One of the interesting topic in Number Theory is studying the solutions of Diophantine equations. A Diophantine equation is a polynomial equation over $\mathbb{Z}$ in n variables in which we look for integer solutions. Thus, the solutions of Diophantine equations are algebraic integers. For studying the solutions it is natural to associate to such solutions a "size", a "complexity" or a "measure". An algebraic number a is described uniquely (up to sign) by its minimal polynomial, $P_a \in \mathbb{Z}[x]$. It is logical to establish several notions of "size" on P_a. For instance we can compute $M(P_a)$, which does not depend on the sign of the polynomial. In Section II.3.4, we have seen examples that connect the Mahler measure and special values. One of the interesting research topics in this area is investigating Diophantine properties for special values of L-functions (for instance one can see [PP20, PTW20]).
In this section, we introduce the definition of the Mahler measure for algebraic numbers. We present an important, recently solved conjecture which is due to Schinzel and Zassenhaus and its link to the Lehmer conjecture.

Definition II.4.1. The Mahler measure and the logarithmic Mahler measure of an algebraic number a are defined as the corresponding quantities for a minimal polynomial $P_a(x) \in \mathbb{Z}[x]$ of a.

We again notice that the minimal polynomial in $\mathbb{Z}[x]$ is unique up to sign, but the above objects are well defined, since $M(P_a) = M(-P_a)$. In the sequel, we use the word "the minimal polynomial" of a since the goal is to compute the Mahler measure and considering P_a or $-P_a$ does not change the result.

Let us introduce some important properties of the Mahler measure of algebraic numbers in the following fact:

Fact II.4.2. *For the Mahler measure of algebraic numbers we have the following:*

- *For an algebraic number a, $M(a) \geq 1$, since according to the definition, the minimal polynomial of a has integer coefficients.*
- *$M(0) = 1$, since the Mahler measure of $P(x) = 0$ is equal to 1.*
- *For $a \neq 0$ if $M(a) = 1$ then a is a root of unity (see Proposition II.1.16).*

In Conjecture II.1.17 we introduced the Lehmer conjecture. Suppose now that $P \in \mathbb{Z}[x]$ is reducible and its factorization to irreducible polynomials on $\mathbb{Z}[x]$ is $P = \prod_{1\leq i\leq n} P_i$ where $P_i \in \mathbb{Z}[x]$ are irreducible for $1 \leq i \leq n$. Thus, thanks to the multiplicativity of the Mahler measure and the fact that the Mahler measure of a polynomial with integer coefficients is greater than or equal to one, we have $M(P) \geq M(P_i)$ for $1 \leq i \leq n$. Therefore, stating the Lehmer conjecture for irreducible polynomials on $\mathbb{Z}[x]$ is equivalent to the actual form of the conjecture. Then, according to the definition of minimal polynomial, any irreducible polynomial in $\mathbb{Z}[x]$ is the minimal polynomial of all its roots. Thus, in the statement of Lehmer conjecture instead of considering the Mahler measure of irreducible (non cyclotomic) polynomials with integer coefficients we consider the Mahler of algebraic integers (which are neither roots of unity nor zero), so we have the following point of view of the Lehmer conjecture:

CONJECTURE (Lehmer conjecture). Does there exist a constant C such that $M(a) \geq C > 1$ for any $a \in \overline{\mathbb{Q}}$ which is not 0 and not a root of unity?

By considering the above point of view for the Lehmer conjecture, we can ask the same question for the Mahler measure of Pisot or Salem numbers, which will be introduced in Section II.4.2. We will see that the restriction of the Lehmer conjecture to Pisot numbers is already solved, by Salem [Sal44]. However, the restriction to Salem numbers is itself a conjecture, known as the Salem conjecture. Consider the other point of view of the Lehmer conjecture introduced in Conjecture II.1.17. The conjecture can be restricted to reciprocal or non reciprocal polynomials. For the non reciprocals the conjecture was already solved by Breusch [Bre51] and (independently) Smyth Proposition II.1.20.

Definition II.4.3. Let α be a nonzero algebraic integer of degree d, with conjugates $\alpha_1 = \alpha, \ldots, \alpha_d$ and minimal polynomial P_a. The **house** of α is defined as the maximum modulus of its conjugates (including α itself) and denoted by $\lceil\!\overline{\alpha}\!\rceil$.

Suppose that the minimal polynomial of α has degree d, with $r > 0$ roots of modulus greater than 1. Obviously, we have the following inequality:

$$M(\alpha)^{\frac{1}{d}} \leq M(\alpha)^{\frac{1}{r}} \leq \lceil\!\overline{\alpha}\!\rceil \leq M(\alpha). \tag{II.4.1}$$

It is clear that $\lceil\!\overline{\alpha}\!\rceil \geq 1$ and from the Kronecker's theorem $\lceil\!\overline{\alpha}\!\rceil = 1$ if and only if α is a root of unity.
As we have already mentioned, the set of all algebraic integers is a commutative subring of the complex numbers. The invertible elements of this subring are called algebraic units. In other words an algebraic number α is called an algebraic unit (or simply a unit) if α and α^{-1} are algebraic integers. Let $P_a(x)$ be the monic minimal polynomial of degree n associated to a. Then, its reciprocal $P_a^*(x) = x^n P_a(1/x)$ (see Definition II.1.19) is the monic minimal polynomial associated to a^{-1}. According Corollary II.2.15, for P_a we have $M(P_a^*) = M(P_a)$, which implies $M(\alpha) = M(\alpha^{-1})$. According to the inequality II.4.1, for the algebraic unit a we have $\lceil\!\overline{\alpha}\!\rceil \leq M(a)$. On the other side, for a^{-1} we have $\lceil\!\overline{\alpha^{-1}}\!\rceil \leq M(a^{-1})$, and since $M(\alpha) = M(\alpha^{-1})$ we conclude that $\max\{\lceil\!\overline{\alpha^{-1}}\!\rceil, \lceil\!\overline{\alpha}\!\rceil\} \leq M(a)$.
In 1965 Schinzel and Zassenhaus [SZ65] proved the following proposition:

Proposition II.4.4 ([SZ65])**.** *Let $\alpha \neq 0$ be an algebraic integer that is not a root of unity. If $2s$ of its conjugates are nonreal, then*

$$\lceil\!\overline{\alpha}\!\rceil \geq 1 + 4^{-s-2}.$$

See [SZ65] for the proof of the above proposition. For any a that satisfies the condition in the above proposition by applying Eq. (II.4.1) we have the same lower bound for $M(a)$ in the Lehmer problem. Later, Schinzel-Zassenhaus-Dimitrov conjectured that a much stronger bound should hold (see [SZ65, Equation (3)]). Recently, the conjecture was shown to be true and the value of the constant c is given by Dimitrov according to the preprint [Dim19]. Thus, we call it the Schinzel-Zassenhaus-Dimitrov Theorem:

Theorem II.4.5 (Schinzel-Zassenhaus-Dimitrov Theorem). *Let $\alpha \neq 0$ be an algebraic integer that is not a root of unity and its minimal polynomial is of degree d, then for some absolute constant $c > 0$ we have;*

$$\lceil\!\overline{\alpha}\!\rceil \geq 1 + \frac{c}{d}.$$

We notice that a positive answer to Lehmer's conjecture would imply the above proposition. To see this, suppose that the constant $C > 1$ is the answer to Lehmer's question. Thus, for any algebraic integer α which is not a root of unity we have $M(\alpha) \geq C > 1$. According to inequality II.4.1 we have $\lceil\!\overline{\alpha}\!\rceil \geq M(\alpha)^{\frac{1}{d}}$, which implies $\log \lceil\!\overline{\alpha}\!\rceil \geq \frac{\log(M(\alpha))}{d}$. Since α is not a root of unity, then $\lceil\alpha\rceil > 1$ and we have $\lceil\alpha\rceil \geq \log\lceil\alpha\rceil + 1 \geq 1 + \frac{\log(M(\alpha))}{d} \geq 1 + \frac{\log(C)}{d}$. Thus, the universal constant in Schinzel and Zassenhaus Theorem would be $\log C$.

II.4.2. Towards Boyd conjecture. In this section we follow Boyd's steps [Boy81b] for solving the Lehmer conjecture and introduce his intuition to state Boyd conjecture.
As we mentioned, we consider

$$L := \{M(P)|P(x) \in \mathbb{Z}[x]\},$$

and a negative answer to the Lehmer conjecture (the constant $C > 1$ does not exists) is equivalent to the property that 1 is a limit point of L (x is a limit point of $L \subset [1, \infty)$ if every neighbourhood of x with respect to the Euclidean topology inherited from $\mathbb{R}$ to $[1, \infty)$ also contains a point of L other than x itself). We notice that since the Mahler measure is multiplicative, if 1 is a limit point of L, then $L^{(1)}$ of L (i.e. the set of all limits points of L) and all the successive derived sets $L^{(k)}$ will be the set $[1, \infty)$ (see Proposition II.4.6). Before proving Proposition II.4.6 we announce the following easily verified result in topology:
Let Y is a metric space and $X \subset Y$. Suppose that for any arbitrary $\epsilon > 0$, X is ϵ-dense in Y, then X is dense in Y. We remind that for the metric spaces Y and $X \subset Y$, the set X is called ϵ-dense in Y (for a given $\epsilon > 0$) if for any $y \in Y$, there exists $x \in X$ such that the distance between x and y is smaller than ϵ.

Proposition II.4.6 ([Boy81b])**.** *If* 1 *is a limit point of* L*, then for any* $k \in \mathbb{Z}_{\geq 1}$ *we have* $L^{(k)} = [1, \infty)$.

Proof. Using the multiplicativity of the Mahler measure, $M(P \cdot Q) = M(P) \cdot M(Q)$, the set L is a semigroup. This implies that if $a \in L$ then for all $n > 1$ we have $a^n \in L$. Suppose that 1 is a limit point of L, then we prove that L is dense in $[1, \infty)$. To prove that, we prove that L is ϵ-dense in $[1, \infty)$ for any $\epsilon > 0$. Let us choose an arbitrary $\epsilon > 0$. Our goal is to prove that L is ϵ-dense in $[1, \infty)$ for the fixed ϵ. We fix an arbitrary $N > 0$, it suffices to prove that $L \cap [1, N]$ is ϵ-dense in $[1, N]$. We suppose that 1 is a limit point of L, then for any $0 < \eta < \epsilon/N$ there exists a polynomial with Mahler measure equal to $1 + \eta$, (i.e. $1 + \eta \in L \cap [1, N]$). Since L is a semigroup, $\{(1 + \eta)^k | k \geq 0\} \subseteq L$, which implies that $\{(1 + \eta)^k | k \geq 0\} \cap [1, N] \subseteq L \cap [1, N]$. Thus, for any k such that $(1 + \eta)^k \leq N$ we have:

$$\left|(1 + \eta)^{k+1} - (1 + \eta)^k\right| = (1 + \eta)^k \eta < \epsilon.$$

Let us now explain how the above property implies that $L \cap [1, N]$ is ϵ-dense in $[1, N]$. For an arbitrary point $y \in [1, N]$, we consider the maximum value of $k \in \mathbb{Z}_{\geq 1}$ such that $(1 + \eta)^k \leq y$, then $(1+\eta)^{k+1} > y$ and according to the above explanation $y - \epsilon < (1+\eta)^{k+1} - \epsilon < (1+\eta)^k \leq y$. We consider $x = (1 + \eta)^k \in L \cap [1, N]$. Then, according to the previous inequalities we have $x \in (y - \epsilon, y)$. Therefore, $L \cap [1, N]$ is ϵ-dense in $[1, N]$ and, as we already mentioned, since $N > 0$ is arbitrary, L is ϵ-dense in $[1, \infty)$. Also since ϵ is arbitrary, L is dense in $[1, \infty)$. In other words $L^{(1)} = [1, \infty)$, therefore for any $k \in \mathbb{Z}_{\geq 1}$ we also have $L^{(k)} = [1, \infty)$.

□

According to the above proposition if 1 is a limit point of L then L is not closed. This is simply because the polynomials in the definition of L have integer coefficients, so L is a countable set. However, from Proposition II.4.6 we have $L^{(1)} = [1, \infty)$, which implies that $L \subset \overline{L}$, so L is not closed. Therefore, to provide a positive answer to the Lehmer conjecture (i.e. there is a polynomial with the smallest Mahler measure greater than one), it would suffice to show that L is nowhere dense or only that there exist a $k \in \mathbb{Z}_{\geq 1}$, for which $\min L^{(k)} > 1$. In the following, by introducing two remarkable subsets of L, the set of Pisot and Salem numbers, we extract more information about L.

Pisot and Salem numbers. The set of Pisot numbers is a remarkable subset of algebraic integers which were discovered by A. Thue in 1912 and rediscovered by G. H. Hardy in 1919 within the context of Diophantine approximation. However, they became widely known after the publication of Charles Pisot's dissertation in 1938.

Definition II.4.7. A Pisot-Vijayaraghavan or simply **Pisot** number is a real algebraic integer $a > 1$, such that all of its algebraic conjugates (except itself) have absolute value less than one.

Let us fix the following notation for the set of Pisot-Vijayaraghavan numbers.

Notation II.4.8. The set of all Pisot numbers is denoted by S.

Let α be a Pisot number and let $P_\alpha = z^d + \cdots + a_0 \in \mathbb{Z}[x]$ be its minimal polynomial. We have $M(\alpha) = M(P_\alpha) = \alpha$ by definition. Hence, $\alpha \in L$, which proves that $S \subset L$. Therefore, P_α has one root with absolute value greater than 1 and $d-1$ roots with absolute value smaller than 1. Let $P^*_\alpha(z) = z^d P_\alpha(z^{-1})$ be the reciprocal polynomial of P_α, defined in Definition II.1.19 whose roots are the reciprocal roots of P_α. Hence, $P^*_\alpha(z)$ has one root with absolute value smaller than 1 and $d-1$ roots with absolute value greater than 1. Thus, $P_\alpha \neq \pm P^*_\alpha$ unless possibly for the case $d = 2$. Thus, apart from a few easily handled exceptions, the (monic) minimal polynomials satisfied by the elements of S are non-reciprocal. Salem proved that S is closed [Sal44]. Thus, $\bar{S} = S$ and since S is countable, therefore it is nowhere dense in $[1, \infty)$. Moreover, since S is closed, $\min S \in S$ which implies $\min S > 1$ (since $1 \notin S$). Let us denote $\min S$ by α_0. Siegel [Sie44] proved that α_0 is $1.32471\ldots$ which is the real zero of $z^3 - z - 1$. Thus, as we mentioned, the restriction of the Lehmer conjecture to Pisot numbers is solved. Dufresnoy and Pisot [DP55] proved that $\min S' = \frac{\sqrt{5}+1}{2} = 1.61803\cdots$, and Grandet and Hugot [GH65] proved that $\min S^{(2)} = 2$. Moreover, for an arbitrary k according to [Sal44, DP53, Boy79] we have $S^{(k)} \neq \varnothing$ for all finite k and in fact $\min S^{(k)} > \sqrt{k}$, but $S^{(\omega)} := \cap_{k\in\mathbb{Z}_{\geq 1}} S^{(k)} = \varnothing$. These properties indicate that S is a remarkable subset of L, so perhaps L may have a similar structure.

There is another important subset of L, called the set of Salem numbers.

Definition II.4.9. A **Salem** number is a real algebraic integer $\alpha > 1$ whose conjugate roots all have absolute value less than or equal to 1, and at least one of which has absolute value exactly 1.

We notice that, similar to the case of Pisot numbers for α a Salem number with minimal polynomial P_α, we have $M(\alpha) = M(P_\alpha) = \alpha$. We fix the following notation for the set of Salem numbers.

Notation II.4.10. The set of Salem numbers[1] is denoted by T.

The condition of having at least one algebraic conjugate of absolute value 1 in the definition of Salem numbers, forces P_α to be reciprocal (see [Sal45]). As a connection between the set of Salem numbers and Pisot numbers we mention that $S \subset T^{(1)}$ (see [Sal45]). In contrast with S, it is not known whether T is dense in $[1, \infty)$. Because of the success with S as opposed to T it is natural to focus on the following set:

[1]We notice that the notations for the set of Pisot numbers, S, and the set of Salem numbers, T, are chosen following Boyd [Boy81b]. Even if this choice of notation is unfortunate we preferred to conserve it because it has become standard in the literature.

Notation II.4.11. We set the following notation:

$$L_0 = \{M(P)|P \in \mathbb{Z}[x] \text{ is non reciprocal}\}.$$

Then, as we mentioned in Proposition II.1.20, Smyth and independently Breusch [Bre51] proved the Lehmer conjecture for non reciprocal polynomials:

$$\min L_0 = \alpha_0 = \min S.$$

However, as we mentioned, the Lehmer conjecture restricted to Salem numbers is itself a conjecture, called Salem conjecture. One can find more information in this regard, for instance in [Sal63, Ber86] or [MR03, Page 378]:

Conjecture II.4.12 (Salem Conjecture [Sal63])**.** There exists a constant $C_s > 1$ such that if α is a Salem number, then $\alpha \geq C_s$.

We notice that if the constant C in Lehmer conjecture exists, then it verifies the Salem conjecture as well. The Salem conjecture asserts that Salem numbers cannot be too close to 1. Moreover, if the constant in Salem conjecture, C_s, exists it may not be necessarily a Salem number, but one can ask if the smallest Salem number exists. The best guess until now is the largest real root of Lehmer's 10 degree polynomial, denoted by P_L, introduced in Section II.1.3 (i.e. $\alpha_1 = 1.17628\ldots$). It is conjectured that the smallest Salem number and the answer to the Lehmer conjecture is α_1. In other words, it is conjectured that:

$$\min L = \alpha_1 = \min T.$$

Later, Boyd suggested to study some larger sets $L^\sharp$ and $L_0^\sharp$ which respectively contain L and L_0. In the next section, we clarify more the reasons for studying these larger sets.

Definition II.4.13. We consider the following sets:

$$L^\sharp = \{M(P)|P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n], n \geq 1\},$$
$$L_0^\sharp := \{M(P)|P \text{ is non reciprocal and } P \in \mathbb{Z}[x_1, \ldots, x_n], n \geq 1\}.$$

The definition of reciprocal polynomial in the multivariable case is explained as follows:

Definition II.4.14. An n variable polynomial P, is called **reciprocal** if the following equation holds for some positive integer exponents $d_1, \ldots, d_n$:

$$x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} P(1/x_1, 1/x_2, \ldots, 1/x_n) = \pm P(x_1, x_2, \ldots, x_n),$$

and otherwise it is called, non reciprocal.

In the following section, we will see that $L^\sharp$ is contained in the closure of L [Boy81b, Theorem 1]).

II.4.3. Theorem of Boyd-Lawton. Boyd observed that for a two-variable polynomial $P \in \mathbb{C}[z_1, z_2]$ the following equality holds:

$$\lim_{n\to\infty} M(P(z_1, z_1^n)) = M(P(z_1, z_2)).$$

However, the proof he proposed in [Boy81b, Appendix 3] is *only valid* for polynomials that have no root on the torus. Similarly, for a multivariate polynomial $P(z_1, \ldots, z_n)$ with complex coefficients that has no root on the torus in [Boy81b, Appendix 4] he proved the following:

$$\lim_{r_1\to\infty} \cdots \lim_{r_n\to\infty} M(P(z_1^{r_1}, z_1^{r_2}, \ldots, z_1^{r_n})) = M(P(z_1, \ldots, z_n)),$$

where the limit is taken with all the exponents going to infinity independently. The fact that the above equalities are valid for arbitrary polynomials was conjectured by Boyd and was proved by Lawton [Law83]. To state the full theorem we need to introduce an additional notation. Given a vector of integers $\mathbf{r} = (r_1, \ldots, r_n)$, we denote by $\mu(\mathbf{r})$ the following:

$$\mu(\mathbf{r}) := \min\{||\mathbf{m}||_\infty : \mathbf{m} \in \mathbb{Z}^n, \mathbf{m} \neq 0 \ \text{ and } \ \mathbf{m}.\mathbf{r} = 0\},$$

where, for a vector $\mathbf{m} = (m_1, \ldots, m_n)$, we have $||\mathbf{m}||_\infty = \max\{|m_j| : 1 \leq j \leq n\}$. The statement reads:

Theorem II.4.15 (Theorem of Boyd-Lawton[Boy81b, Law83]). *Let $P(z_1, \ldots, z_n)$ be a multivariate polynomial with complex coefficients. Then, the following limit is valid:*

$$\lim_{\mu(\mathbf{r})\to\infty} M(P(z_1^{r_1}, \ldots, z_1^{r_n})) = M(P(z_1, \ldots, z_n)).$$

For the proof of this Theorem one can see [Law83]. Moreover, in the following section we will state a generalization of this theorem and its proof. This theorem points out the advantage of studying the larger sets, $L^\sharp$ and $L_0^\sharp$. More precisely, according to Corollary II.1.10, the Mahler measure of a polynomial with integer coefficients is an algebraic integer. However, in the example of Smyth we have $m(1 + x + y + z) = \frac{7}{2\pi}\zeta(3)$, which is probably transcendental. In this case, the fact that $L \subseteq L^\sharp \subseteq \bar{L}$, hints that L may not be closed. Thus, the set $L^\sharp$ is a natural object to study.
We notice that, although Boyd's method did not solve Lehmer's conjecture, we have the important theorem of Boyd-Lawton, and thanks to this theorem this conjecture in the several-variable case reduces to the one-variable case.

II.4.4. Generalization of the theorem of Boyd-Lawton. This section is part of the common article in collaboration with Brunault, Guilloux and Pengo [BGMP22] about the generalization of the theorem of Boyd-Lawton. We begin with some definitions and customary notation before stating the theorem. For all the necessary prerequisites regarding measure theory see Section VII.3 in Appendix.

First of all we notice that in all over this thesis $\mathbb{T}^n := (S^1)^n$ is the n-dimensional real-analytic torus.

Notation II.4.16. Let $n \in \mathbb{Z}_{\geq 1}$, then a point $(z_1, \ldots, z_n) \in \mathbb{T}^n$ is denoted by $\underline{z}_n$.

Let B be a Borel set of $\mathbb{T}^n$ and define the following measure on $\mathbb{T}^n$:

$$\mu_n(B) = \frac{1}{(2\pi i)^n} \int_B \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}. \tag{II.4.2}$$

It is easy to see that $\mu_n(\mathbb{T}^n) = 1$, so μ_n is a probability measure on $\mathbb{T}^n$. Moreover, one can verify that μ_n is regular and invariant under the action of $\mathbb{T}^n$ (translation). Thus, it is the probability Haar measure on $\mathbb{T}^n$ (see Definition VII.3.5 for more information about Haar measure). In order to define a probability measure associated with an arbitrary matrix $A = (a_{i,j}) \in \mathbb{Z}^{m\times n}$ we need the following definition:

Definition II.4.17. Let $A = (a_{i,j}) \in \mathbb{Z}^{m\times n}$ be an arbitrary matrix and define the following map:

$$\begin{aligned} \phi\colon \mathbb{T}^m &\to \mathbb{T}^n \\ (z_1, \ldots, z_m) &\mapsto (z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \ldots, z_1^{a_{1,n}} \cdots z_m^{a_{m,n}}) \end{aligned}$$

The above map is indeed a group homomorphism, and in the sequel for convenience $\phi(\underline{z}_m)$ is denoted by $\underline{z}_m^A$.

We have introduced the notation $\underline{z}_m^A$ for an arbitrary Matrix. In particular, we consider vectors $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$ as column matrices, so that $\underline{z}_n^v = z_1^{v_1} \dots z_n^{v_n}$ is a monomial. For every $n \in \mathbb{Z}_{\geq 1}$, we define the probability Haar measure on $\mathbb{T}^n$, associated with the matrix $\mathrm{Id_n}$ to be μ_n. It can be denoted by μ_{Id_n}. More generally, for every matrix $A = (a_{i,j}) \in \mathbb{Z}^{m\times n}$ we define the probability measure μ_A on $\mathbb{T}^n$, as the push-forward of μ_m along the group homomorphism $\phi : \mathbb{T}^m \to \mathbb{T}^n$. In other words, for B a Borel set of $\mathbb{T}^n$ we have

$$\mu_A(B) := \mu_m(\phi^{-1}(B)).$$

Before continuing, let us introduce the last notation we need in this section.

Notation II.4.18. For every Laurent polynomial $P \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ and every $A \in \mathbb{Z}^{m\times n}$ we write $P_A(\underline{z}_m) := P(\underline{z}_m^A)$.

In this setting the logarithmic Mahler measure of the n-variable Laurent polynomial $P(\underline{z}_n)$ is the integration of $\log|P|$ over $\mathbb{T}^n$ with respect to the probability Haar measure μ_n (i.e. $m(P) := \int \cdots \int_{\mathbb{T}^n} \log|P(\underline{z}_n)| d\mu_n(\underline{z}_n)$). In fact the generalization of the theorem of Boyd-Lawton explains that $m(P)$ can be approximated by lower-dimensional Mahler measures $m(P_A)$ for some matrices $A \in \mathbb{Z}^{m\times n}$. To explain which sequences of matrices give rise to such approximations, we need the following definition.

Definition II.4.19. Fix two integers $m, n \in \mathbb{Z}_{\geq 1}$. We define the function $\rho \colon \mathbb{Z}^{m\times n} \to \mathbb{Z}_{\geq 1}$ by:

$$\rho(A) := \min\{\|\mathbf{v}\|_\infty \colon \mathbf{v} \in \mathbb{Z}^{n\times 1} \setminus \{\mathbf{0}\},\ A \cdot \mathbf{v} = \mathbf{0}\}.$$

We are now ready to state the generalization of the theorem of Boyd-Lawton.

Theorem II.4.20 (Generalization of Boyd-Lawton's theorem [BGMP22])**.** *Let $n \in \mathbb{Z}_{\geq 1}$ be an integer, and $P(\underline{z}_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \setminus \{0\}$ be a non-zero Laurent polynomial. Then, for every sequence of matrices $A_d \in \mathbb{Z}^{m_d \times n}$ such that* $\lim_{d\to+\infty} \rho(A_d) = +\infty$, *we have that*

$$\lim_{d\to+\infty} m(P_{A_d}) = m(P).$$

The idea of the proof of Theorem II.4.20 is to relate the growth of $\rho(A_d)$ to the weak convergence of the push-forward measures μ_{A_d}, which is done in [BGMP22] and we explain again here. Let us define the following function which we will need later.

For every integrable function $f \colon \mathbb{T}^n \to \mathbb{C}$, and every vector $v \in \mathbb{Z}^n$, viewed as a row matrix, we denote by:

$$c_f(v) := \int_{\mathbb{T}^n} \frac{f(\underline{z}_n)}{\underline{z}_n^v} d\mu_n(\underline{z}_n)$$

the corresponding Fourier coefficient. In particular, if $P \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is a Laurent polynomial, then $P(\underline{z}_n) = \sum_{v\in\mathbb{Z}^n} c_P(v) \cdot \underline{z}_n^v$.

We recall some terminology concerning the Newton polytope of a polynomial. For more information in this regard see Section VII.5 in the Appendix. Fix a Laurent polynomial $P \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. We denote by $N_P \subseteq \mathbb{R}^n$ its **Newton polytope**, which is the convex hull in $\mathbb{R}^n$ of the (**support** set) $\mathrm{supp}(P) := \{v \in \mathbb{Z}^n \colon c_P(v) \neq 0\}$. The number of non-zero monomials appearing in P is denoted by $k(P) := |\mathrm{supp}(P)|$. Moreover, the **diameter** of P, denoted by $\mathrm{diam}(P)$, is the smallest $d \in \mathbb{N}$ such that N_P is contained inside a translate of $[0, d]^n$.

Lemma II.4.21. *Fix $n \in \mathbb{Z}_{\geq 1}$, and let $A_d \in \mathbb{Z}^{m_d\times n}$ be a sequence of integral matrices with fixed number of columns, such that $\rho(A_d) \to +\infty$ as $d \to +\infty$. Then the sequence of measures μ_{A_d} on $\mathbb{T}^n$ converges weakly to the measure μ_{Id_n}.*

Proof. This result is classic. We follow the lines of [Boy81a, Lemma 1], which treats the case when $m_d = 1$ for every d. By the definition of weak convergence and push-forward of measures, and by Stone–Weierstrass theorem (see Appendix Section VII.3.1), it is sufficient to prove that

$$\lim_{d\to+\infty}\left(\int_{\mathbb{T}^{m_d}} Q(\underline{z}_{m_d}^{A_d})\, d\mu_{m_d}(\underline{z}_{m_d})\right) = \int_{\mathbb{T}^n} Q(\underline{z}_n)\, d\mu_n(\underline{z}_n), \tag{II.4.3}$$

for every Laurent polynomial $Q(\underline{z}_n) \in \mathbb{C}[z_1^{\pm1}, \ldots, z_n^{\pm1}]$. For a matrix $A \in \mathbb{Z}^{m\times n}$ and $v \in \mathbb{Z}^n$, a direct computation proves that we have $P(\underline{z}_m^A) = \sum_{v\in\underline{z}^n} c_P(v)\underline{z}_m^{A.v^t}$. We see now immediately that for every d

$$\int_{\mathbb{T}^{m_d}} Q(\underline{z}_{m_d}^{A_d})\, d\mu_{m_d}(\underline{z}_{m_d}) = \sum_{v\in\mathbb{Z}^n} c_Q(v)\cdot \int_{\mathbb{T}^{m_d}} \underline{z}_{m_d}^{A_d\cdot v^t} d\mu_{m_d}(\underline{z}_{m_d}) = \sum_{\substack{v\in\mathbb{Z}^n\\ A_d\cdot v=0}} c_Q(v).$$

Now, the function c_Q has a finite support in $\mathbb{Z}^n$. Let R be the maximum of $\|v\|_\infty$ for v in this support. If $\rho(A_d) > R$, then the only vector v of $\ker(A_d) \cap \mathbb{Z}^n$ which may verify $c_Q(v) \neq 0$ is the null vector 0. In this case, we have that

$$\sum_{\substack{v\in\mathbb{Z}^n\\ A_d\cdot v=0}} c_Q(v) = c_Q(0) = \int_{\mathbb{T}^n} Q(\underline{z}_n)\, d\mu_{\mathrm{Id}_n}(\underline{z}_n),$$

which shows (II.4.3) (in fact, the sequence on the left is eventually constant) under the assumption that $\lim_{d\to+\infty} \rho(A_d) = +\infty$. □

Our goal is to prove that $m(P_{A_j}) = \int_{\mathbb{T}^n} \log|P| d\mu_{A_j}$ converges to $m(P) = \int_{\mathbb{T}^n} \log|P| d\mu_n$. The weak convergence of measures is defined as the convergence of integrals of any continuous function. But, here the integrand is $\log|P|$, which is singular. In fact, $\log|P|$ is continuous and accepts the values $+\infty$. For the continuous functions which may accept the value $+\infty$ (see Lemma VII.3.8 in Appendix) the uniform estimates on L^2-norms are enough to guarantee that the weak-convergence of measures implies the convergence of integrals. The following estimate is essentially obtained by Dimitrov and Habegger in [DH19, Appendix A], where they deal with Lawton's theorem and improves the rate of convergence, see also [Hab17].

Proposition II.4.22. *Let $n, k \in \mathbb{Z}_{\geq 1}$ be two integers. Then, there exists a constant $C > 0$ such that for all $P(\underline{z}_n) \in \mathbb{C}[z_1^{\pm1}, \ldots, z_n^{\pm1}]$ a Laurent polynomial, with k non vanishing coefficients whose maximum modulus of coefficient is one, the following holds:*

For every $m \leq n$ and every matrix $A = (a_{i,j}) \in \mathbb{Z}^{m\times n}$ with $\rho(A) > \mathrm{diam}(P)$,

$$||\log|P|||_{2,\mu_A}^2 := \int_{\mathbb{T}^m} |\log|P(\underline{z}_m^A)||^2 d\mu_A \leq C \quad \textit{and} \quad ||\log|P|||_{2,\mu_n}^2 \leq C.$$

Proof. As we mentioned, for $P(\underline{z}_n) = \sum_{v\in\mathbb{Z}^n} c_P(v)\cdot \underline{z}_n^v$ and $v \in \mathbb{Z}^n$, we have $P(\underline{z}_m^A) = \sum_{v\in\underline{z}^n} c_P(v)\underline{z}_m^{A.v^t}$. Two vectors v and w contribute non trivially to the same monomial in the above sum if and only if $c_P(v) \neq 0$, $c_P(w) \neq 0$ and $A.v^t = A.w^t$ or equivalently $v - w \in \ker(A) \cap \mathbb{Z}^n$. By definition of $\mathrm{diam}(P)$, in this case, we have $\|v - w\|_\infty \leq \mathrm{diam}(P)$. We see that if $\rho(A) > \mathrm{diam}(P)$ the only possibility is $v - w = 0$. In other terms, each monomial of P_A comes from a single monomial of $P(\underline{z}_m^A)$ with the same coefficient and no compensations.

So, for any A with $\rho(A) > \mathrm{diam}(P)$, the polynomial $P_A = P(\underline{z}_m^A)$ has $m \leq n$ variables, k non vanishing coefficients and the max of the coefficients is 1. The same is true as well for P itself. Our proposition comes then directly from the estimates of Dimitrov and Habegger. They show in [DH19, Lemma A.3] that there exist numbers $C_{l,k} > 0$, for l, k positive integers such that for any polynomial Q over $\mathbb{C}$ in l variables, with k non vanishing coefficients and the maximum of the coefficients equal to 1, we have:

$$\int_{\mathbb{T}^l} (\log |Q|)^2 d\mu_l \leq C_{l,k}.$$

Thanks to the above explanation this bound applies to both $Q = P_A$ and to $Q = P$. Thus, we can take C equal to the maximum on $m \leq n$ of the $C_{m,k}$. $\square$

Theorem II.4.20 is now an easy consequence:

Proof of Theorem II.4.20. Let $n \in \mathbb{Z}_{\geq 1}$ be an integer, and $P(\underline{z}_n) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial. Fix a sequence of matrices $A_d \in \mathbb{Z}^{m_d \times n}$ such that $\lim_{d\to+\infty} \rho(A_d) = +\infty$.

We first make an easy reduction: up to multiplying P by a constant a, we may assume that the maximum of the coefficients is 1. Indeed, we have, for all $a \in \mathbb{C}^*$, $m(aP) = \log |a| + m(P)$ and $m(aP(\underline{z}_{m_d}^{A_d})) = \log |a| + m(P(\underline{z}_{m_d}^{A_d}))$. So the problem of convergence is equivalently solved for P or aP. Then, for any j, we have

$$m(P(\underline{z}_{m_d}^{A_d})) = \int_{\mathbb{T}^{m_d}} \log |P(\underline{z}_{m_d}^{A_d})| d\mu_{m_d} = \int_{\mathbb{T}^n} \log |P| d\mu_{A_d}.$$

Let $d_0 \in \mathbb{Z}_{\geq 1}$ be such that for $d \geq d_0$ we have $\rho(A_d) \geq \operatorname{diam}(P)$. From Proposition II.4.22, we know that the function $\log |P|$ is uniformly L^2 for the family $\{\mu_{A_d}, d \geq d_0\} \cup \{\mu_n\}$. From Lemma II.4.21, we know the weak-convergence of μ_{A_d} to μ_n. We conclude with Lemma VII.3.8:

$$\lim_{d\to+\infty} m(P(\underline{z}_{m_d}^{A_d})) = \lim_{d\to+\infty} \int_{\mathbb{T}^n} \log |P| d\mu_{A_d} = \int_{\mathbb{T}^n} \log |P| d\mu_n = m(P).$$

$\square$

In Chapter III we will introduce a family of polynomial called P_d, for which we compute the Mahler measure. Moreover, in Chapter IV we compute the limit of $(m(P_d))_{d\in\mathbb{Z}_{\geq 1}}$ using three methods. We will see that the easiest method is applying Theorem II.4.20 to this family, following [BGMP22].

In [BGMP22], we also provide an upper bound for the rate of the convergence of the sequence $m(P(\underline{z}_{m_d}^{A_d}))$ in terms of the number of variables of P, the number of non zero coefficients of P, $\operatorname{diam}(P)$ and $\rho(A)$. We will not cover this result in this thesis. Indeed, we focus on the family of Mahler measures $m(P_d)$ and this rate of convergence holds no meaningful information about this sequence. We will obtain in Chapter V the actual rate of convergence for $m(P_d)$ which is much better than the general upper bound.

II.4.5. Boyd conjecture. As we mentioned, the fact that $L^\sharp$ is contained in the closure of L and the existence of the examples such as Proposition II.3.41, convinced Boyd that L may not be closed. This is the reason for which Boyd conjectured the following statements which are still open:

Conjecture II.4.23 ([Boy81b])**.** $L^\sharp$ is a closed set.

Conjecture II.4.24 ([Boy81b])**.** $L_0^\sharp$ is a closed set.

The first conjecture is often called Boyd conjecture. If the Boyd conjecture is proved, then the answer to the Lehmer question is positive. To clarify this, we first recall that Lehmer's question is equivalent to the fact that 1 is not a limit point of L. Let us now work with the contrapositive proposition. Suppose that 1 is a limit point of L, then according to Proposition II.4.6, L is dense in $[1, \infty)$. Since $L \subset L^\sharp$, so $L^\sharp$ is dense as well. Thus, $L^\sharp$ can not be closed, since otherwise $[1, \infty) = \overline{L^\sharp} = L^\sharp$, while $L^\sharp$ is countable which is a contradiction.

CHAPTER III

The family P_d of Polynomials

In this chapter we introduce the class of exact regular bivariate polynomials. The main interest of studying exact regular polynomials is the existence of a closed formula [GM21] to compute their Mahler measure. The advantage of the formula is that instead of computing a double integral we compute a finite sum. As examples of regular exact polynomials, we introduce a family of polynomials, denoted by P_d, which was introduced to us by Brunault. We compute each term in the formula of Mahler measure for P_d. At the end of this chapter, we can compute $m(P_d)$ for any d, with arbitrary precision and it gives us an idea about the behavior of $m(P_d)$ when d goes to infinity.

III.1. Exactness of the polynomials P_d

In this section, we introduce the polynomials P_d and we prove that they are exact. We notice that despite the existence of a closed formula to compute the Mahler measure of regular exact polynomials, recognizing the exactness of a polynomial (with genus greater than zero) is not always possible either. For instance, there is a finite number of exact bivariate polynomials of genus $g \leq 1$ with Newton polygon of bounded area (see [GM21]). Before starting this section let us fix the following conventions:

Convention III.1.1. In this thesis an algebraic variety refers to the general definition of quasi affine varieties. Also an algebraic curve is a 1-dimensional quasi affine algebraic variety (see Section VII.4 in the Appendix for the necessary definitions in this regard).

Convention III.1.2. We assume that $\log z$ is $\log_e z$ or $\ln z$, unless we mention the base of the logarithm.

Convention III.1.3. The argument $\arg x$ is a multi-valued function on $\mathbb{C}^*$. The restriction of $\arg x$ to the principle branch $(-\pi, \pi]$ is denoted by $\operatorname{Arg} x$.

In order to define exact polynomials we need to introduce the following differential form:

Definition III.1.4. The real differential 1-form η on $\mathbb{C}^{*2}$ is defined by

$$\eta = \log|y|\ d\arg(x) - \log|x|\ d\arg(y).$$

We note that the real differential 1-form $d\arg x$ is actually the imaginary part of the complex differential 1-form $d\log x = \frac{dx}{x}$. Consider the multi-valued function $\arg x$. We restrict it to the principal branch to have the function $\operatorname{Arg} x$. However, we still need to remove the negative part of the real axis from $(-\pi, \pi]$ to have a continuous function. On the contrary $d\arg x$ is defined in $\mathbb{C}^*$, since $\arg x = \operatorname{Arg} x + 2k\pi$, with $k \in \mathbb{Z}$, so the constant $2k\pi$ disappears in $d\arg x$.

Proposition III.1.5. *Let $P \in \mathbb{C}[x, y]$ and C be the algebraic curve defined by*

$$C = \{(x, y) \in \mathbb{C}^{*2} | P(x, y) = 0, dP(x, y) \neq 0\}.$$

Then, η restricted to C is a closed 1-form.

Proof. The algebraic curve C has complex dimension one, so the complex differential 2-form $\left(\frac{dx}{x} \wedge \frac{dy}{y}\right)$ on C is zero. Therefore, the imaginary part of this form is zero. We prove $d\eta = -Im\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = 0$ by computing the both sides of the equality:

$$\begin{aligned} d\eta &= d\log|y| \wedge d\arg(x) + \log|y| \wedge d(d\arg(x)) - d\log|x| \wedge d\arg(y) - \log|x| \wedge d(d\arg(x)) \\ &= d\log|y| \wedge d\arg(x) - d\log|x| \wedge d\arg(y). \end{aligned}$$

For computing the imaginary part of $\frac{dx}{x} \wedge \frac{dy}{y}$, we use $\frac{dx}{x} = d\log x$ and $\log x = \log|x| + i\arg x$, so we have:

$$\begin{aligned} Im(\frac{dx}{x} \wedge \frac{dy}{y}) &= Im\left((d\log|x| + id\arg(x)) \wedge (d\log|y| + id\arg(y))\right) \\ &= d\log|x| \wedge d\arg(y) + d\arg(x) \wedge d\log|y| \\ &= d\log|x| \wedge d\arg(y) - d\log|y| \wedge d\arg(x) \\ &= -(d\log|y| \wedge d\arg(x) - d\log|x| \wedge d\arg(y)). \end{aligned}$$

So $d\eta = -Im\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = 0$, consequently η is closed. □

We notice that in the previous proposition, P is a polynomial which is a holomorphic function and the condition $dP(x,y) \neq 0$, in the definition of the algebraic curve C, is translated as follows:

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy \neq 0 \Leftrightarrow \frac{\partial P}{\partial y} \neq 0 \text{ } or \text{ } \frac{\partial P}{\partial x} \neq 0.$$

Following Proposition III.1.5 one may ask about the exactness of $\eta|_C$. In general, the answer is that $\eta|_C$ is not always exact, but this question leads to the definition of exact polynomials.

Definition III.1.6. A polynomial $P \in \mathbb{C}[x,y]$ is called **exact** if the form η restricted to the algebraic curve C (which depends on P) is exact. In this case, any primitive for $\eta|_C$ is called a **Volume function** associated with the exact polynomial P.

To see a simple example of an exact polynomial we need to define the following classic function.

Definition III.1.7. The **Bloch-Wigner** dilogarithm function $D(z)$ is defined by:

$$D(z) = \mathrm{Im}(Li_2(z)) + \mathrm{Arg}(1-z)\log|z|,$$

where $Li_2(z)$ is the following function:

$$Li_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \qquad \text{for } z \in \mathbb{C} \setminus [1,\infty).$$

We briefly summarize some important properties of this function. For more information see [Zag07].

Fact III.1.8. *The function $D(z)$ is real analytic on $\mathbb{C}$ except at the two points 0 and 1, where it is continuous but not differentiable. Moreover, we have:*

(1) $D(\bar{z}) = -D(z)$.

(2) If $|z| = 1$, $D(z) = D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$, *in particular* $D(e^{k\pi i}) = 0$.

(3) Let $n \in \mathbb{Z}_{\geq 1}$ and ζ_n a primitive n-th root of unity, then

$$D(z^n) = n \sum_{j=0}^{n-1} D(\zeta_n^j z).$$

The last property above is called the distribution relation for dilogarithm and is mentioned in [BRV02].

The link between the differential of D and η is well known, see [BL13] or [BZ20, Theorem 7.2] for more information;

Fact III.1.9. *$-D(z)$ is a primitive for η restricted to $\{(z, 1-z) \in \mathbb{C}^{*2}\}$, i.e.,*

$$-dD(z) = \eta(z, 1-z).$$

PROOF. The proof is simply a computation as follows.

$$\begin{aligned}
dD(z) &= d(Im(Li_2(z)) + \arg(1-z)\log|z|) \\
&= Im(d(Li_2(z))) + d\arg(1-z)\log|z| + \arg(1-z)d\log|z| \\
&= -Im\left(\log(1-z)\frac{dz}{z}\right) + d\arg(1-z)\log|z| + \arg(1-z)d\log|z| \\
&= -Im(\log(1-z)d\log z) + d\arg(1-z)\log|z| + \arg(1-z)d\log|z| \\
&= -Im(\log(1-z)(d\log|z| + id\arg(z))) + d\arg(1-z)\log|z| + \arg(1-z)d\log|z| \\
&= -Im((\log|1-z| + i\arg(1-z))(d\log|z| + id\arg(z))) \\
&\quad + d\arg(1-z)\log|z| + \arg(1-z)d\log|z| \\
&= \log|z|d\arg(1-z) - \log|1-z|d\arg(z) \\
&= -\eta(z, 1-z).
\end{aligned}$$

□

In the following we introduce a simple example of an exact polynomial:

Example III.1.10. The polynomial $P_1(x, y) = x + y + 1$ is exact and a volume function is $-D(-x)$; (To see the proof see [Lal07] or Example III.1.17 in the following.)

We generalize the first example P_1 to a family of exact polynomials, called P_d :

Notation III.1.11. For $d \geq 1$ the polynomial

$$P_d(x, y) := \sum_{0 \leq i+j \leq d} x^i y^j,$$

is denoted by P_d.

The best way to prove the exactness of η restricted to the curve C associated with P_d is by an abstract algebraization of η. We refer to [BZ20, Section 7.2] for more information in this regard). Consider the multiplicative group ${K_d}^*$ of the field $K_d = \mathrm{Frac}\, \frac{\bar{\mathbb{Q}}[x,y]}{<P_d>}$ as a $\mathbb{Z}$-module. The second exterior product of ${K_d}^*$ is ${K_d}^* \wedge {K_d}^*$. Note that the associated group operation in ${K_d}^*$ and ${K_d}^* \wedge {K_d}^*$ are respectively multiplication and addition. We recall some basic properties of the wedge product, which will be used in our computations:

(1) $\forall f \in {K_d}^* :\ f \wedge f = 0.$
(2) $\forall f, g \in {K_d}^* :\ f \wedge g = -(g \wedge f).$
(3) $\forall f, g, h \in {K_d}^* :\ (fg) \wedge h = f \wedge h + g \wedge h.$

(4) $\forall f, g, h \in {K_d}^* : \ f \wedge (gh) = f \wedge g + f \wedge h$.
(5) $\forall f, g \in {K_d}^*, n \in \mathbb{Z} : \ n(f \wedge g) = f^n \wedge g = f \wedge g^n$.

Consider the alternating bi-linear map $\imath : {K_d}^* \times {K_d}^* \to \Omega^1_C$ defined by:

$$\begin{aligned} \imath \colon {K_d}^* \times {K_d}^* &\to \Omega^1_C \\ (f, g) &\mapsto \log|g| d \arg f - \log|f| d \arg g, \end{aligned}$$

where C is the curve of P_d minus the set of zeros and poles of f and g. Moreover, Ω^1_C is the $\mathbb{C}$-vector space of smooth differential one-forms on C. According to the universal property of the exterior product, there is a unique morphism of $\mathbb{Z}$-modules $\bar{\imath} : {K_d}^* \wedge {K_d}^* \to \Omega^1_C$ such that the following diagram commutes.

$$\begin{tikzcd} {K_d}^* \wedge {K_d}^* \arrow[dr, "\bar{\imath}"] & \\ {K_d}^* \times {K_d}^* \arrow[u, "\bigwedge"] \arrow[r, "\imath"'] & \Omega^1_C \end{tikzcd}$$

where $\bigwedge$ is defined by:

$$\begin{aligned} \bigwedge \colon {K_d}^* \times {K_d}^* &\to {K_d}^* \wedge {K_d}^* \\ (f, g) &\mapsto f \wedge g \end{aligned}$$

Note that according to the definitions of $\imath(f, g)$ and η we have $\eta_{(f,g)} = \imath(f, g)$.

By using the the universal property of the wedge product we verify in the next lemma that torsion elements of ${K_d}^* \wedge {K_d}^*$ belong to the kernel of $\bar{\imath}$. The proof of Lemmas III.1.12 and III.1.16, Example III.1.13, and Proposition III.1.14 are according to the properties of the wedge product and the definition of η.

Lemma III.1.12. *If $f \wedge g = f' \wedge g'$, then $\imath(f, g) = \imath(f', g')$. Moreover, if a finite sum with integer coefficients $\sum_{i=1}^n \epsilon_i f_i \wedge g_i = 0$, then $\sum_{i=1}^n \epsilon_i \imath(f_i, g_i) = 0$. In particular, the torsion elements of ${K_d}^* \wedge {K_d}^*$ are sent to zero by $\bar{\imath}$.*

Proof. The first part is clear by the universal property. For the second part, $\bar{\imath}$ is a morphism of $\mathbb{Z}$-module, so $\sum_{i=1}^n \epsilon_i \imath(f_i, g_i) = \sum_{i=1}^n \epsilon_i \bar{\imath}(f_i \wedge g_i) = \bar{\imath}(\sum_{i=1}^n \epsilon_i f_i \wedge g_i) = 0$. Finally, if $\sum_{i=1}^n \epsilon_i f_i \wedge g_i$ is a torsion element in ${K_d}^* \wedge {K_d}^*$, there is an integer n such that $n\left(\sum_{i=1}^n \epsilon_i f_i \wedge g_i\right) = 0$. Thus, $\bar{\imath}\left(n(\sum_{i=1}^n \epsilon_i f_i \wedge g_i)\right) = n\left(\bar{\imath}(\sum_{i=1}^n \epsilon_i f_i \wedge g_i)\right) = 0$. Hence, the differential form $\bar{\imath}\left(\sum_{i=1}^n \epsilon_i f_i \wedge g_i\right)$ is a torsion element in the $\mathbb{C}$-vector space Ω^1_C, so $\bar{\imath}\left(\sum_{i=1}^n \epsilon_i f_i \wedge g_i\right) = 0$. $\square$

Example III.1.13. For all $g \in K_d^*$, $g \wedge -1$ is a torsion element in ${K_d}^* \wedge {K_d}^*$.

Proof. We have $g \wedge 1 = g \wedge (1 \cdot 1) = g \wedge 1 + g \wedge 1$, so $g \wedge 1 = 0$. Also we have $0 = g \wedge 1 = (g \wedge (-1)(-1)) = (g \wedge -1) + (g \wedge -1) = 2.(g \wedge -1)$, so $g \wedge -1$ is a torsion element. $\square$

The following theorem gives us an algorithm to compute a volume function for an exact polynomial. We refer to [BZ20, Chapter 7] or [BRVD03, Page 6] for a similar proof of the following theorem.

Proposition III.1.14. *If $x, y \in {K_d}^*$ and $x \wedge y = \sum_{i=1}^n z_i \wedge (1 - z_i)$ modulo some torsion elements in ${K_d}^* \wedge {K_d}^*$, then $(-\sum_{i=1}^n D(z_i))$ is a primitive form for η restricted to the curve C associated with $P_d(x, y)$.*

Proof. We have:

$$\bar{\imath}(x \wedge y) = \bar{\imath}\left(\sum_{i=1}^{n} z_i \wedge (1 - z_i) + \sum_{i=1}^{n} f_i \wedge g_i\right), \quad \text{where } f_i \wedge g_i \text{ are torsion elements.}$$

Since $\bar{\imath}$ is a morphism of abelian groups, and $\sum_{i=1}^{n} f_i \wedge g_i$ is a torsion element, by Lemma III.1.12 and Example III.1.10, we have:

$$\begin{aligned}\eta(x,y) = \imath(x,y) = \bar{\imath}(x \wedge y) &= \bar{\imath}\left(\sum_{i=1}^{n}(z_i \wedge (1 - z_i)) + \sum_{i=1}^{n} f_i \wedge g_i\right) \\ &= \sum_{i=1}^{n} \bar{\imath}\,(z_i \wedge (1 - z_i)) = \sum_{i=1}^{n} \imath(z_i, 1 - z_i) = -(\sum_{i=1}^{n} dD(z_i)) = d(-\sum_{i=1}^{n} D(z_i)).\end{aligned}$$

□

Remark III.1.15. Since the $\wedge$ computation for finding a volume function does not depend on the torsion elements, in the sequel of this section we use the notation $\doteq$ to denote equality up to torsion elements. For example, according to Example III.1.13, for all f, g we have $(-f)\wedge(-g) \doteq f \wedge (-g) \doteq (-f) \wedge g \doteq f \wedge g$.

In the following lemma, we state two equalities needed for proving the exactness of P_d.

Lemma III.1.16. *We have the following equalities:*

$$x \wedge (1 - x/y) - y \wedge (1 - y/x) \doteq x/y \wedge (1 - x/y) - x \wedge y, \tag{III.1.1}$$

$$y \wedge (1 - (y/x)^{d+1}) - x \wedge (1 - (x/y)^{d+1}) \doteq (y/x) \wedge (1 - (y/x)^{d+1})) + (d+1)x \wedge y. \tag{III.1.2}$$

Proof. We just prove the first equality, the proof for the second one is similar. By replacing x with $\frac{x}{y}.y$ we have:

$$\begin{aligned}x \wedge (1 - x/y) - y \wedge (1 - y/x) &= x/y \wedge (1 - x/y) + y \wedge (1 - x/y) - y \wedge (1 - y/x) \\ &= x/y \wedge (1 - x/y) + y \wedge \frac{1 - x/y}{1 - y/x} = x/y \wedge (1 - x/y) + y \wedge \frac{\frac{y-x}{y}}{\frac{x-y}{x}} = x/y \wedge (1 - x/y) + y \wedge \frac{-x}{y} \\ &= x/y \wedge (1 - x/y) + y \wedge -1 + y \wedge x - y \wedge y \doteq x/y \wedge (1 - x/y) - x \wedge y.\end{aligned}$$

□

We recover:

Example III.1.17. The polynomial P_1 is exact and a volume function is $-D(-x)$.

Proof. In [Lal07, Section 3] or [BZ20, Chapter 7], there are similar methods to prove the exactness and to compute the Mahler measure of P_1. We prove it as well. We notice that in $K_1 := \operatorname{Frac} \frac{\bar{\mathbb{Q}}[x,y]}{<P_1>}$ we have $1 + x = -y$. It yields:

$$x \wedge y \doteq (-x) \wedge (-y) = (-x) \wedge (1 - (-x)).$$

Then according to Proposition III.1.14, $-D(-x)$ is a volume function and P_1 is exact. □

The previous example generalizes to the whole family. The following proof was suggested by Brunault, and appeared in [Meh21, Theorem 2.1]:

Theorem III.1.18. *For all $d \geq 1$, P_d is an exact polynomial, and for $d \geq 2$ a volume function, denoted by V, is defined as follows:*

$$V(x,y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)].$$

Proof. In Example III.1.17, we proved that P_1 is exact with volume function $-D(-x)$. For $d \geq 2$ we have the following equations:

$$P_d(x,y) = P_{d-1}(x,y) + y^d(\frac{1-(x/y)^{d+1}}{1-(x/y)}) \text{ and } P_d(x,y) = yP_{d-1}(x,y) + (\frac{1-x^{d+1}}{1-x}).$$

Therefore, at smooth zeros of P_d, we have:

$$0 = P_d(x,y) = P_{d-1}(x,y) + y^d(\frac{1-(x/y)^{d+1}}{1-(x/y)}) = yP_{d-1}(x,y) + (\frac{1-x^{d+1}}{1-x}).$$

In other words, we have:

$$P_{d-1}(x,y) = -y^d(\frac{1-(x/y)^{d+1}}{1-(x/y)}) = -1/y(\frac{1-x^{d+1}}{1-x}), \tag{III.1.3}$$

hence,

$$y^{d+1} = \frac{1-x^{d+1}}{1-x}\frac{1-(x/y)}{1-(x/y)^{d+1}}. \tag{III.1.4}$$

Instead of $x \wedge y$, we compute $\frac{1}{d+1}x \wedge y^{d+1}$. Moreover, we substitute Eq. (III.1.4) in $\frac{1}{d+1}x \wedge y^{d+1}$ and we have :

$$x \wedge y = \frac{1}{d+1}\left(x \wedge (1-x^{d+1}) - x \wedge (1-x) + x \wedge (1-x/y) - x \wedge (1-(x/y)^{d+1})\right). \tag{III.1.5}$$

Since P_d for $d \geq 1$ is a symmetric polynomial, we can switch x and y in Eq. (III.1.5) and we have:

$$y \wedge x = \frac{1}{d+1}\left(y \wedge (1-y^{d+1}) - y \wedge (1-y) + y \wedge (1-y/x) - y \wedge (1-(y/x)^{d+1})\right). \tag{III.1.6}$$

By subtracting Eq. (III.1.6) from Eq. (III.1.5), we have:

$$\begin{aligned} 2(d+1)(x \wedge y) = {} & x \wedge (1-x^{d+1}) - y \wedge (1-y^{d+1}) - x \wedge (1-x) + y \wedge (1-y) \\ & + x \wedge (1-x/y) - y \wedge (1-y/x) - x \wedge (1-(x/y)^{d+1}) + y \wedge (1-(y/x)^{d+1}). \end{aligned} \tag{III.1.7}$$

By replacing Eq. (III.1.2) and Eq. (III.1.1) in Eq. (III.1.7) and simplifying (based on Lemma III.1.16), we have :

$$\begin{aligned} &(d+2)(x \wedge y) \doteq 1/(d+1)x^{d+1} \wedge (1-x^{d+1}) - 1/(d+1)y^{d+1} \wedge (1-y^{d+1}) - x \wedge (1-x) + \\ &y \wedge (1-y) + x/y \wedge (1-x/y) + 1/(d+1)(y/x)^{d+1} \wedge (1-(y/x)^{d+1})). \end{aligned}$$

In other words, we have:

$$\begin{aligned} (x \wedge y) \doteq {} & \frac{1}{(d+1)(d+2)}\left(x^{d+1} \wedge (1-x^{d+1}) - y^{d+1} \wedge (1-y^{d+1}) + (y/x)^{d+1} \wedge (1-(y/x)^{d+1})\right) \\ & + \frac{1}{d+2}\left(y \wedge (1-y) - x \wedge (1-x) + x/y \wedge (1-x/y)\right). \end{aligned}$$

Based on Proposition III.1.14 the volume function is:

$$V(x,y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)];$$

which proves the exactness of P_d. □

III.2. A closed formula to express the Mahler measure of P_d as a finite sum

The sequel of Chapter III explains the work done in [Meh21] in detail. As we have already mentioned, there is a closed formula in [GM21] to compute the Mahler measure of a family of exact polynomials as follows:

$$m(P) = \frac{1}{2\pi} \sum \epsilon(x, y) V(x, y). \tag{III.2.1}$$

The summation is on the set of toric points of the monic polynomial P (see Definition III.2.1); $\epsilon(x, y)$ is the opposite of the sign of the imaginary part of $\frac{x\partial_x P}{y\partial_y P}$ at a toric point (x, y); and V is a volume function.

There is a slight assumption on the polynomial necessary to apply the formula. We assume that $\frac{x\partial_x P}{y\partial_y P}$ is not real at each toric point. We will explain more about this assumption, called *regularity* of the polynomial in the next section. Since P_d is exact and regular we may apply the formula to compute $m(P_d)$. To do so, we need to compute the toric points of P_d and ϵ at each toric point.

III.2.1. Toric points of P_d. The aim of this section is to compute the set of the toric points of P_d. Let us start with the definition of toric points.

Definition III.2.1. The set of **toric points** of $P \in \mathbb{C}[x, y]$ is defined by:

$$\{(x, y) \in \mathbb{C}^{*2} | P(x, y) = 0, |x| = |y| = 1\}.$$

Let $P(x, y) \in \mathbb{C}[x, y]$. Here we introduce $P^*(x, y) = P(1/x, 1/y)$ which is a bivariate Laurent polynomial. We prove that the set of toric points of $P \in \mathbb{R}[x, y]$ and P^* are equal. This property helps us find the set of the toric points of P_d.

Proposition III.2.2 ([BZ20], Exercise 3.1.b)**.** *If $P(x, y) \in \mathbb{R}[x, y]$ the set of toric points of $P(x, y)$ and $P^*(x, y)$ are equal.*

Proof. For polynomials with real coefficients we have $\overline{P(x, y)} = P(\overline{x}, \overline{y})$. Let $|x| = 1$ then $\overline{x} = \frac{1}{x}$, so we have:

$$\begin{aligned}
(x, y) \text{ is a toric point of } P(x, y) &\Leftrightarrow P(x, y) = 0, |x| = |y| = 1 \\
&\Leftrightarrow \overline{P(\overline{x}, \overline{y})} = 0, |x| = |y| = 1 \\
&\Leftrightarrow P(\overline{x}, \overline{y}) = 0, |x| = |y| = 1 \\
&\Leftrightarrow P(\frac{1}{x}, \frac{1}{y}) = 0, |x| = |y| = 1 \\
&\Leftrightarrow P^*(x, y) = 0, |x| = |y| = 1 \\
&\Leftrightarrow (x, y) \text{ is a toric point of } P^*(x, y).
\end{aligned}$$

$\square$

According to the previous proposition if (x, y) is a toric point of P then we have $P(x, y) = P^*(x, y) = 0$. Therefore, if (x, y) is a toric point of P_d we have $P_d(x, y) + P_d^*(x, y) = 0$. Similarly for any $(i, j) \in \mathbb{N}^2$ we have:

$$P_d(x, y) + x^i y^j P_d^*(x, y) = 0. \tag{III.2.2}$$

We are searching for i, j such that $x^i y^j P_d^*(x, y)$ does not have any denominator, also $P_d(x, y) + x^i y^j P_d^*(x, y)$ factorizes as the product of two univariate polynomials. Let the univariate polynomials in the factorization be denoted by $H(x)$ and $Q(y)$. We want to find (i, j) such that we have:

$$P_d(x,y) + x^i y^j P_d^*(x,y) = H(x)Q(y). \tag{III.2.3}$$

The advantage of having such a equation is that if (x,y) is a toric point of P_d, then the L.H.S is zero which implies that either x is a root of H or y is a root of Q. Therefore, to find toric points of the bivariate polynomial P_d, we simply find the roots of two univariate polynomials H and Q and we choose the roots x and y for which (x,y) is a toric point. One can easily see that for an arbitrary d, a good candidate to have such a factorization might be $i = d+1, j = d$.

Lemma III.2.3. *We have the following equality:*

$$P_d(x,y) + x^{d+1} y^d P_d^*(x,y) = \frac{x^{d+2}-1}{x-1} \; \frac{y^{d+1}-1}{y-1}. \tag{III.2.4}$$

Proof.

$$\begin{aligned}
&P_d(x,y) = \\
&(1 + x + \cdots + x^d) + y(1 + x + \cdots + x^{d-1}) + \cdots + y^{d-1}(1+x) + y^d = \\
&\frac{x^{d+1}-1}{x-1} + \frac{x^d - 1}{x-1} y + \cdots + \frac{x^2-1}{x-1} y^{d-1} + \frac{x-1}{x-1} y^d = \\
&\frac{(x^{d+1} + yx^d + \cdots + y^{d-1}x^2 + y^d x) - (1 + y + \cdots + y^d)}{x-1}.
\end{aligned}$$

Since ${P_d}^*(x,y) = P_d(1/x, 1/y)$,

$${P_d}^*(x,y) = \frac{(1/x^{d+1} + 1/yx^d + \cdots + 1/y^{d-1}x^2 + 1/y^d x) - (1 + 1/y + \cdots + 1/y^d)}{1/x - 1},$$

and:

$$x^{d+1} y^d P_d^*(x,y) = \frac{(x^{d+2} + x^{d+2}y + \cdots + x^{d+2}y^d) - (x^{d+1} + yx^d + \cdots + y^{d-1}x^2 + y^d x)}{x-1}.$$

So we have:

$$P_d(x,y) + x^{d+1} y^d P_d^*(x,y) = \frac{(x^{d+2} + x^{d+2}y + \cdots + x^{d+2}y^d) - (1 + y + \cdots + y^d)}{x-1}.$$

Therefore:

$$P_d(x,y) + x^{d+1} y^d P_d^*(x,y) = \frac{x^{d+2}-1}{x-1} \; \frac{y^{d+1}-1}{y-1}.$$

□

The previous lemma gives us more information about the set of the toric points of P_d;

Lemma III.2.4. *The toric points of $P_d(x,y)$ are contained in:*

$$\{(x,y) \in \mathbb{C}^{*2} \mid x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1\} \cup$$
$$\{(x,y) \in \mathbb{C}^{*2} \mid x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1\}$$

Proof. If (x,y) is a toric point of P_d, then Eq. (III.2.2) and Eq. (III.2.4) leads to:

$$\frac{x^{d+2}-1}{x-1} = 0 \quad \text{or} \quad \frac{y^{d+1}-1}{y-1} = 0. \tag{III.2.5}$$

The polynomial $P_d(x,y)$ is a symmetric polynomial, so $P_d(x,y) = P_d(y,x)$. Thus, we switch x and y, so (y,x) is a toric point as well as (x,y). Hence,

$$\frac{y^{d+2}-1}{y-1} = 0 \text{ or } \frac{x^{d+1}-1}{x-1} = 0, \tag{III.2.6}$$

Therefore, according to Eq. (III.2.5) and Eq. (III.2.6) there are 4 possibilities:

(1) $x^{d+2} = 1, x \neq 1, y^{d+2} = 1, y \neq 1$.
(2) $x^{d+2} = 1, x^{d+1} = 1, x \neq 1$, which is not compatible.
(3) $y^{d+1} = 1, y^{d+2} = 1, y \neq 1$, which is not compatible.
(4) $y^{d+1} = 1, y \neq 1, x^{d+1} = 1, x \neq 1$.

□

The set of the toric points of $P_d(x,y)$ is subset of the set introduced in Lemma III.2.4. However, by computing the toric points of $P_d(x,y)$ for some small values of d, we realize that these two sets are not equal. In the following we search for a suitable subset of the set in Lemma III.2.4 which is equal to the set of toric points of P_d. The following example introduce the set of the toric points of P_1 and P_2.

Example III.2.5. We have the toric points of $P_1(x,y)$ and $P_2(x,y)$ computed by Maple:

- The toric points of P_1 are the following third roots of unity:
$$\{(e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}), (e^{\frac{4\pi}{3}i}, e^{\frac{2\pi}{3}i})\}.$$
- The toric points of P_2 are the following third and fourth roots of unity:
$$\{(e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}), (e^{\frac{4\pi}{3}i}, e^{\frac{2\pi}{3}i}), (-1, e^{\frac{6\pi}{4}i}), (e^{\frac{6\pi}{4}i}, -1), (e^{\frac{2\pi}{4}i}, -1), (-1, e^{\frac{2\pi}{4}i}), (e^{\frac{2\pi}{4}i}, e^{\frac{6\pi}{4}i}), (e^{\frac{6\pi}{4}i}, e^{\frac{2\pi}{4}i})\}.$$

We should remark that the values of the coordinates of toric points of P_1 and P_2 have been first computed approximately by Maple. Then, since they are $d+1$ or $d+2$ roots of unity we concluded the associated exact values.
In Example III.2.5 we observed that there is no symmetric pairs among the toric points (i.e. (x,x)), for the both examples $P_1(x,y)$ and $P_2(x,y)$. This motivated us prove the following lemma.

Lemma III.2.6. *If (x,y) is a toric point of $P_d(x,y)$, then $x \neq y$:*

PROOF. Let x be a $(d+1)$ or $(d+2)$ root of unity. We prove by contradiction that $P_d(x,x)$ is not equal to zero. Thus, suppose we have:

$$0 = P_d(x,x) = \sum_{0 \le i+j \le d} x^{i+j} = \sum_{0 \le k \le d} (k+1)x^k = \left(\frac{d}{dx}\sum_{k=0}^{d} x^{k+1}\right).$$

Therefore, x is a root of $\frac{d}{dx}\left(\sum_{k=0}^{d} x^{k+1}\right)$. The Gauss-Lucas theorem asserts that the zeroes of the derivative of a polynomial have to lie in the convex hull of the zeros of the polynomial itself. On the other side,

$$\sum_{k=0}^{d} x^{k+1} = \frac{x^{d+2}-1}{x-1}.$$

Since the two polynomials $\sum_{k=0}^{d} x^{k+1}$ and $P_d(x,x)$ are coprime to each other (see Remark III.2.7), x is strictly inside the convex hull of $(d+2)$-roots of unity. Therefore, $|x| < 1$, which contradicts the fact that x is a root of unity. Hence, there is no symmetric pair (x,x) in the set of toric points of P_d. □

Remark III.2.7. In the proof of the previous lemma we claim that the two polynomials $\sum_{k=0}^{d} x^{k+1} = \frac{x^{d+2}-1}{x-1}$ and $P_d(x,x) = \frac{d}{dx}\left(\sum_{k=0}^{d} x^{k+1}\right) = \frac{d}{dx}\left(\frac{x^{d+2}-1}{x-1}\right)$ are coprime to each other, but we need to clarify this fact. To do so instead of considering the quotient $\frac{x^{d+2}-1}{x-1}$ it is sufficient to show $x^{d+2}-1$ and $\frac{d}{dx}(x^{d+2}-1)$ are coprime. It follows from the Bezout relation:

$$\frac{1}{d+2}\left(\frac{d}{dx}(x^{d+2}-1)\right)x - (x^{d+2}-1) = \frac{x}{d+2}(d+2)x^{d+1} - x^{d+2} + 1 = 1.$$

Thus, they are coprime and they do not have any common roots.

Finally we are able to introduce the set of the toric points of $P_d(x,y)$. For the convenience, we fix the following notations:

Notation III.2.8. Let U_{d+1} and U_{d+2} are the following set:

$$U_{d+1} := \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1\ , x \neq y\},$$
$$U_{d+2} := \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1, x \neq y\}.$$

Proposition III.2.9. *[Meh21, Proposition 3.1] The set of toric pints of $P_d(x,y)$ is:*

$$U_{d+1} \cup U_{d+2}.$$

PROOF. From Lemma III.2.4 and Lemma III.2.6, we know that the set of toric points of P_d is included in $U_{d+1} \cup U_{d+2}$. To prove the reverse, we notice that for $(x,y) \in U_{d+1} \cup U_{d+2}$ we have $|x| = |y| = 1$, so we just prove $P_d(x,y) = 0$. To do so, we consider two cases:

- Case 1) $(x,y) \in U_{d+1} = \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+1} = y^{d+1} = 1, x \neq 1, y \neq 1\ , x \neq y\}$:

$$\begin{aligned} P_d(x,y) &= (1 + x + \cdots + x^d) + y(1 + x + \cdots + x^{d-1}) + \cdots + y^{d-1}(1+x) + y^d \\ &= \frac{(x^{d+1} + yx^d + \cdots + y^{d-1}x^2 + y^d x) - (1 + y + \cdots + y^d)}{x-1}. \end{aligned}$$

  Since, y is a $d+1$ root of unity, $(1 + y + \cdots + y^d)$ is equal to zero. Also, $0 = 1 - 1 = x^{d+1} - y^{d+1} = (x-y)(x^d + x^d y + \cdots + y^d)$, but $y \neq x$, so $(x^d + x^d y + \cdots + y^d) = 0$. Hence, $P_d(x,y) = 0$.
- Case 2) $(x,y) \in U_{d+2} = \{(x,y) \in \mathbb{C}^{*2} \mid x^{d+2} = y^{d+2} = 1, x \neq 1, y \neq 1, x \neq y\}$:
  $P_d(x,y)$, for $d \geq 1$, is symmetric so we have:

$$xP_d(x,y) + 1 + y + \cdots + y^{d+1} = P_{d+1}(x,y) = P_{d+1}(y,x) = yP_d(x,y) + 1 + x + \cdots + x^{d+1}.$$

  By subtracting $P_{d+1}(y,x)$ from $P_{d+1}(x,y)$, the following equation holds for any (x,y):

$$(x-y)P_d(x,y) + \frac{y^{d+2}-1}{y-1} - \frac{x^{d+2}-1}{x-1} = 0. \tag{III.2.7}$$

  Since for any point in U_{d+2} we have $y^{d+2} - 1 = x^{d+2} - 1 = 0$ and $x \neq y$, we conclude that $P_d(x,y) = 0$.

□

We also find some information about the number of toric points of each P_d, since the toric points are the (x,y) where x,y are roots of unity. The number of toric points is $|U_{d+1}| + |U_{d+2}| = d(d-1) + (d+1)d = 2d^2$.

More generally the set of toric points of a polynomial is finite. This can be proved using the information in [GM21]. Thus, using Eq. (III.2.1) instead of computing an integral for $m(P_d)$, we compute a finite sum.

III.2.2. Computing ϵ at toric points. In the previous sections we found a volume function associated to P_d and the set of toric points of P_d. In this section, to complete our information for applying [GM21] to P_d, we compute $\epsilon(x,y)$ at toric points of P_d. Let us rewrite the formula again:

$$m(P) = \frac{1}{2\pi}\sum \epsilon(x,y)V(x,y). \tag{III.2.8}$$

We recall that the summation is on the set of toric points of P, $\epsilon(x,y)$ is the opposite of the sign of the imaginary part of $\frac{x\partial_x P}{y\partial_y P}$ at toric point (x,y), and V is a volume function.
The necessary condition on P to apply this formula is that, for each toric point of P, the fraction $\frac{x\partial_x P}{y\partial_y P}$, should not be real. This property leads to the definition of regular polynomials. To introduce regular polynomials we need the following definition (for more information see [GM21]).

Definition III.2.10. The **logarithmic Gauss map** $\gamma : C \to \mathbb{P}^1(\mathbb{C})$ is defined by $\gamma(x,y) = [x\partial_x P, y\partial_y P]$.

Using the logarithmic Gauss map we define a regular polynomial.

Definition III.2.11. An exact polynomial $P(x,y)$ is called **regular** if for each toric point (x,y) we have $\gamma(x,y) \notin \mathbb{P}^1(\mathbb{R})$.

From the previous definition, we conclude that $\gamma(x,y)$ is a point in projective plane. Thus, for a regular polynomial P we have $y\partial_y P|_{(x,y)} \neq 0$ if and only if $x\partial_x P|_{(x,y)} \neq 0$, since otherwise if for example $y\partial_y P\mid_{(x,y)} = 0 \quad \text{and} \quad x\partial_x P\mid_{(x,y)} \neq 0$, then:

$$[x\partial_x P, y\partial_y P] = [1, \frac{y\partial_y P}{x\partial_x P}] = [1,0] \notin \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}),$$

which is a contradiction. Thus, for a regular polynomial P without loss of generality one can assume that one of the following conditions holds: $y\partial_y P|_{(x,y)} \neq 0$ or $\frac{y\partial_y P}{x\partial_x P} \neq 0$. We assume $y\partial_y P|_{(x,y)} \neq 0$, so the property $\gamma(x,y) = [x\partial_x P, y\partial_y P] = [\frac{x\partial_x P}{y\partial_y P}, 1] \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$, is equivalent to $\frac{x\partial_x P}{y\partial_y P} \notin \mathbb{R}$.

Example III.2.12. The following table shows $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$ at the toric points of P_1 and P_2.

- $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$ at the toric points of P_1:

$(x,y) \in U_3$	$\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$	$\epsilon(x,y)$
$(e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i})$	$-$	$+$
$(e^{\frac{4\pi}{3}i}, e^{\frac{2\pi}{3}i})$	$+$	$-$

- $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$ at toric points of P_2:

$(x,y) \in U_3$	$\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$	$\epsilon(x,y)$	$(x,y) \in U_4$	$\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$	$\epsilon(x,y)$
			$(e^{\frac{4\pi}{4}i}, e^{\frac{6\pi}{4}i})$	$-$	$+$
$(e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i})$	$+$	$-$	$(e^{\frac{6\pi}{4}i}, e^{\frac{4\pi}{4}i})$	$+$	$-$
			$(e^{\frac{2\pi}{4}i}, e^{\frac{4\pi}{4}i})$	$-$	$+$
			$(e^{\frac{4\pi}{4}i}, e^{\frac{2\pi}{4}i})$	$+$	$-$
$(e^{\frac{4\pi}{3}i}, e^{\frac{2\pi}{3}i})$	$-$	$+$	$(e^{\frac{2\pi}{4}i}, e^{\frac{6\pi}{4}i})$	$-$	$+$
			$(e^{\frac{6\pi}{4}i}, e^{\frac{2\pi}{4}i})$	$+$	$-$

To find $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$ for arbitrary P_d we define the following map which associates each toric point of P_d with a point in $\mathbb{R}^2$;

Definition III.2.13. The map Ω is defined over $\bigcup_{d\geq 1} U_d$ as follows: if $(x,y) = (e^{\frac{2\pi li}{d}}, e^{\frac{2\pi ki}{d}}) \in U_d$, then $\Omega((x,y)) = (l,k)$.

According to the definition of U_d, if $\Omega((x,y)) = (l,k)$, then $l \neq k$. Thus, we have the following convention:

Convention III.2.14. We say that $\Omega((x,y)) = (l,k)$ is above the diagonal if $l < k$ and below the diagonal if $k < l$.

As before, to find ϵ at each toric point of P_d, for arbitrary d, we start with computing it for the small values of d. Figs. 1 to 3 show the image of Ω over the set of the toric points of P_1 and P_2. The points in red are associated with the toric points with positive $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$ and the points in blue are associated with the toric points with negative sign.

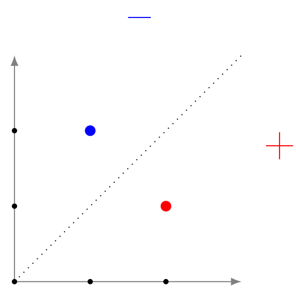

FIGURE 1. The points in U_3 and the associated $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_1}{y\partial_y P_1}))$.

This explanation leads us to the following proposition:

Proposition III.2.15. *[Meh21, Proposition 3.2] Let $d \geq 1$, for the polynomial $P_d(x,y)$, ϵ at each toric point is determined as follows;*

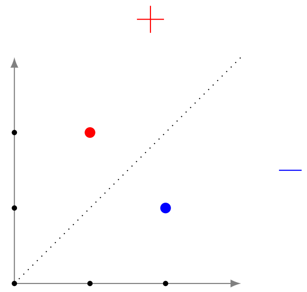

FIGURE 2. The points in U_3 and the associated $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_2}{y\partial_y P_2}))$.

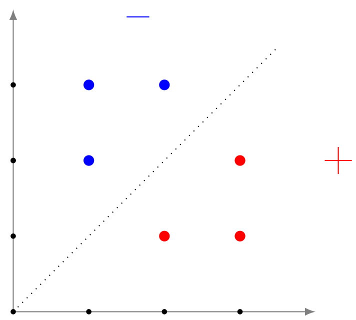

FIGURE 3. The points in U_4 and the associated $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_2}{y\partial_y P_2}))$.

- *For* $(x, y) \in U_{d+1}$*:*
  - *If* $\Omega((x, y))$ *is above the diagonal, then* $\epsilon(x, y) < 0$.
  - *If* $\Omega((x, y))$ *is below the diagonal, then* $\epsilon(x, y) > 0$.
- *For* $(x, y) \in U_{d+2}$*:*
  - *If* $\Omega((x, y))$ *is above the diagonal, then* $\epsilon(x, y) > 0$.
  - *If* $\Omega((x, y))$ *is below the diagonal, then* $\epsilon(x, y) < 0$.

PROOF. We find $\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P}{y\partial_y P}))$. Recall that $\epsilon(x, y)$ is its opposite! As we saw in the proof of Proposition III.2.9, at any point (x, y) Eq. (III.2.7) is satisfied:

$$0 = (x - y)P_d(x, y) + \frac{y^{d+2} - 1}{y - 1} - \frac{x^{d+2} - 1}{x - 1}.$$

Let $Q(x, y) = (x - 1)(y - 1)(x - y)$. For all $(x, y) \in \mathbb{C}^2$ we have this equality of polynomials:

$$P_d(x, y)Q(x, y) = (x^{d+2} - 1)(y - 1) - (y^{d+2} - 1)(x - 1).$$

We apply ∂_x and ∂_y to both sides:

$$\partial_x P_d(x, y)Q(x, y) + \partial_x Q(x, y)P_d(x, y) = (d + 2)(y - 1)x^{d+1} - (y^{d+2} - 1), \tag{III.2.9}$$

$$\partial_y P_d(x, y)Q(x, y) + \partial_y Q(x, y)P_d(x, y) = (x^{d+2} - 1) - (d + 2)(x - 1)y^{d+1}. \tag{III.2.10}$$

We divide Eq. (III.2.9) by Eq. (III.2.10), so for all the $(x, y) \in \mathbb{C}^2$ we have:

$$\frac{\partial_x P_d(x, y)Q(x, y) + \partial_x Q(x, y)P_d(x, y)}{\partial_y P_d(x, y)Q(x, y) + \partial_y Q(x, y)P_d(x, y)} = \frac{(d + 2)(y - 1)x^{d+1} - (y^{d+2} - 1)}{(x^{d+2} - 1) - (d + 2)(x - 1)y^{d+1}}. \tag{III.2.11}$$

We evaluate the previous equation at toric points and we consider two cases:

- Case 1) $(x,y) \in U_{d+1}$:
$$\frac{\partial_x P_d(x,y)}{\partial_y P_d(x,y)} = -\frac{y-1}{x-1}, \text{ so } \frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)} = \frac{-x(1-y)}{y(1-x)}.$$
- Case 2) $(x,y) \in U_{d+2}$:
$$\frac{\partial_x P_d(x,y)}{\partial_y P_d(x,y)} = -\frac{x^{d+1}(y-1)}{y^{d+1}(x-1)}, \text{ so } \frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)} = -\frac{1-y}{1-x}.$$

We now discuss each case with respect to $\Omega((x,y))$:

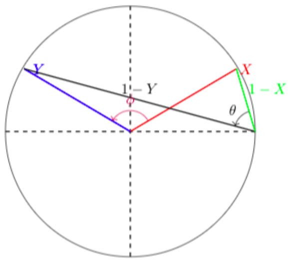

FIGURE 4

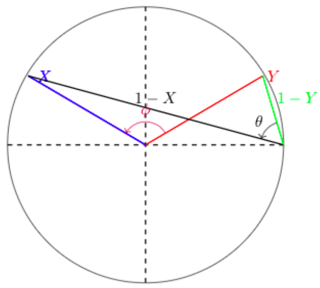

FIGURE 5

- Case 1) $(x,y) \in U_{d+1}$: We have two possible cases for $\Omega((x,y)) = (a,b)$:
  (1) If $\Omega((x,y))$ is above the diagonal, or equivalently $b > a$ (see Fig. 4), we have:

$$\text{(III.2.12)} \qquad \frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)} = \frac{-x}{y}\,\frac{1-y}{1-x} = -e^{-i\phi}re^{i\theta}.$$

  In the last equality in Eq. (III.2.12), we used the suitable polar representations according to Fig. 4, where $\frac{x}{y} = e^{-i\phi}$, with $0 < \phi < 2\pi$ and $\frac{1-y}{1-x} = re^{i\theta}$, with $r > 0$, $0 < \theta < \pi$. We notice that ϕ and θ are respectively central and inscribed angles with the same intercepted arc in the circle, so $\phi = 2\theta$. Therefore, we have:
$$\operatorname{Sgn}(\operatorname{Im}(\frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)})) = -\operatorname{Sgn}(\operatorname{Im}(re^{-i\phi/2})) = \operatorname{Sgn}(\sin(\frac{\phi}{2})),$$
  since $0 < \phi < 2\pi$, the sign is positive.
  (2) If $\Omega((x,y))$ is below the diagonal, or equivalently $a > b$ (see Fig. 5), we have:
$$\operatorname{Sgn}(\operatorname{Im}(\frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)})) = -\operatorname{Sgn}(\operatorname{Im}(e^{i\phi/2})) = \operatorname{Sgn}(-\sin(\frac{\phi}{2})),$$
  so, the sign is negative.

- Case 2) $(x,y) \in U_{d+2}$: Let $\omega((x,y)) = (a,b)$.

  (1) If $\Omega((x,y))$ is above the diagonal, or equivalently $b > a$ (see Fig. 4), we have:

$$\operatorname{Sgn}(\operatorname{Im}(\frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)})) = \operatorname{Sgn}(\operatorname{Im}(-\frac{1-y}{1-x})) = \operatorname{Sgn}(\operatorname{Im}(-re^{i\theta})) = \operatorname{Sgn}(-\sin(\theta)) = \operatorname{Sgn}(\sin(\frac{\phi}{2})),$$

  since $0 < \phi < 2\pi$, the sign is negative.

(2) If $\Omega((x,y))$ is below the diagonal, or equivalently $a > b$ (see Fig. 5), we have:

$$\mathrm{Sgn}(\mathrm{Im}(\frac{x\partial_x P_d(x,y)}{y\partial_y P_d(x,y)})) = \mathrm{Sgn}(\mathrm{Im}(-\frac{1-y}{1-x})) = \mathrm{Sgn}(\mathrm{Im}(-re^{-i\theta})) = \mathrm{Sgn}(-\sin(-\theta)) = \mathrm{Sgn}(\sin(\frac{\phi}{2})),$$

since $0 < \phi < 2\pi$, the sign is positive.

□

An immediate result from the previous proposition is:

Corollary III.2.16. *The family $P_d(x,y)$ is regular for each d.*

Remark III.2.17. In Proposition III.2.15, we write x, y as suitable powers of the first primitive root of unity. We can not chose any other primitive root of unity and conclude the same results. The reason is clear by looking at the figures. Consider z' as another primitive root of unity which is not the first one, so if $x = z'^a, y = z'^b$ and $a < b$. Then even by restricting the argument in $[0, 2\pi]$ we can not necessarily conclude that y is located after x on the unit circle (with counterclockwise orientation).

III.2.3. A closed formula for the Mahler measure of P_d. In the previous sections we found a volume function associated to P_d, the set of toric points of P_d, and ϵ evaluated at toric points. In this section, using this information we present a closed formula for $m(P_d)$ in terms of the values of dilogarithm. In Theorem IV.2.5 and Theorem V.1.1 we see other closed formulas for $m(P_d)$.

Proposition III.2.18. *[Meh21, Proposition 4.3] Let $d \in \mathbb{Z}_{\geq 2}$, so the closed formula for the Mahler measure of P_d is as follows:*

$$2\pi m(P_d) =$$

$$\frac{2}{(d+1)} \sum_{\substack{(x,y)\in U_{d+2} \\ \text{with } \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)] - \frac{2}{d+2} \sum_{\substack{(x,y)\in U_{d+1} \\ \text{with } \epsilon(x,y)>0}} [D(x) - D(y) - D(x/y)],$$

where U_{d+1} or U_{d+2} are the set of the $d+1$ and $d+2$ toric points of P_d, computed in Proposition III.2.9.

Proof. We have computed the volume function in Theorem III.1.18 and it is

$$V(x,y) = \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)].$$

The toric points and ϵ evaluated at toric points are computed respectively in Proposition III.2.9 and Proposition III.2.15. Then, thanks to the properties of the dilogarithm $D(z) = -D(\bar{z})$ and $D(1) = 0$, at $(x,y) \in U_{d+1}$ we have

$$V(x,y) = \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)],$$

also at $(x,y) \in U_{d+2}$ we have

$$V(x,y) = \frac{1}{(d+1)}[D(x) - D(y) - D(x/y)].$$

Moreover, for a point $(x,y) \in U_{d+1}$ with $\epsilon(x,y) > 0$ we have $(y,x) \in U_{d+1}$ with $\epsilon(y,x) < 0$ and thanks to the properties of the dilogarithm, $V(x,y) = -V(y,x)$. The same is true for a toric point in U_{d+2} and this completes the proof. □

The case of $m(P_1)$ is not covered in the above proposition so let us do it separately. This case was first computed by Smyth [Smy81a]. Here, we give another method to do so.

Example III.2.19. We have $m(P_1) = \frac{1}{\pi}D(e^{\frac{\pi}{3}i})$, which is approximately 0.32.

PROOF. [Lal07] According to Example III.1.17, $V(x, y) = -D(-x)$ is a volume function for P_1. By using Proposition III.2.9 the set of the toric points of P_1 is $U_2 \cup U_3 = U_3$ (note that $U_2 = \varnothing$). Proposition III.2.15 gives the values for $\epsilon(x, y)$. Then, we have the following table:

$(x, y) \in U_3$	$\epsilon(x, y)$	$V(x, y)$
$(e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i})$	$+$	$-D(-e^{\frac{2\pi}{3}i})$
$(e^{\frac{4\pi}{3}i}, e^{\frac{2\pi}{3}i})$	$-$	$-D(-e^{\frac{4\pi}{3}i})$

We notice that for ω on the unite circle we have $D(\bar{\omega}) = -D(\omega)$, so we have:

$$2\pi m(P_1) = \sum_{(x,y)\in U_3} \epsilon(x, y)V(x, y) = (-D(-e^{i\frac{2\pi}{3}})) - (-D(-e^{i\frac{4\pi}{3}})) = D(e^{i\frac{\pi}{3}}) - D(e^{-i\frac{\pi}{3}}) = 2D(e^{i\frac{\pi}{3}}).$$

Therefore, $m(P_1) = \frac{1}{\pi}D(e^{i\frac{\pi}{3}})$. □

We apply Proposition III.2.18 to $m(P_2)$ and we write it as a sum of the values of dilogarithm at specific roots of unity.

Example III.2.20. We have $m(P_2) = \frac{1}{2\pi}(\frac{3}{2}D(e^{i\frac{4\pi}{3}}) + 4D(e^{i\frac{\pi}{2}}))$, which is approximately 0.42.

According to Proposition III.2.9 the set of toric points of P_2 is $U_3 \cup U_4$. We have ϵ at each toric point by looking at the table in Example III.2.12. According to Proposition III.2.18 we have:

$$2\pi m(P_2)$$
$$= \frac{2}{3} \sum_{\substack{(x,y)\in U_3 \\ \text{with } \epsilon(x,y)>0}} \epsilon(x, y)(D(x) - D(y) - D(x/y)) + \frac{1}{2} \sum_{\substack{(x,y)\in U_4 \\ \text{with } \epsilon(x,y)>0}} \epsilon(x, y)(D(x) - D(y) - D(x/y)).$$

After a computation and using the properties of Dilogarithm we have:

$$m(P_2) = \frac{1}{2\pi}(\frac{3}{2}D(e^{i\frac{4\pi}{3}}) + 4D(e^{i\frac{\pi}{2}})).$$

Thanks to Proposition III.2.18 we can compute the Mahler measure of P_d, for every $d \geq 1$ using SageMath or Maple with arbitrary precision in a very efficient way. We did it using SageMath for $1 \leq d \leq 1000$ and the graph of $m(P_d)$ is represented in Fig. 6.

The figure hints to the existence of a limit for $m(P_d)$. In order to find the limit, in the next chapter we study more about the volume function associated with P_d.

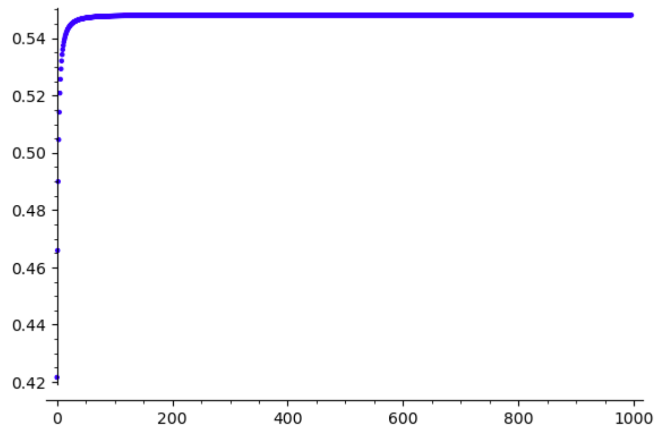

FIGURE 6. The graph of $m(P_d)$, for $1 \leq d \leq 1000$.

CHAPTER IV

The convergence of $(m(P_d))_{d\in\mathbb{Z}_{\geq 1}}$

In the previous chapter, we found an explicit formula for $m(P_d)$ and using SageMath we computed it for $1 \leq d \leq 1000$. Fig. 6 hinted to an asymptotic behavior for $m(P_d)$. In this section, using two different methods we prove that $\lim_{d\to\infty} m(P_d) = \frac{9\zeta(3)}{2\pi^2}$. The first method that we explain in Section IV.1 is an application of the generalization of the theorem of Boyd-Lawton, introduced in Theorem II.4.20. However, to apply Theorem II.4.20 we need a candidate for the limit polynomial, called P_∞. In sections IV.2 to IV.4 of this thesis, we explain a computational method to represent a P_∞. This method is based on Riemann sum of a bivariate function and an error estimation. Historically, this computational method was the first method that we used to prove $\lim_{d\to\infty} m(P_d) = \frac{9\zeta(3)}{2\pi^2}$. Later, during a collaboration with Brunault, Guilloux and Pengo we proved Theorem II.4.20. Furthermore, thanks to the work of D'Andrea and Lalín [DL07], we concluded that for $P_\infty(x, y, z, w) := (x-1)(y-1)-(z-1)(w-1)$, we have $m(P_\infty) = \lim_{d\to\infty} m(P_d)$. This gave us the opportunity to have a short proof for the limit, using the generalization of the theorem of Boyd-Lawton. We explain the computational method precisely to cover all the work done during this thesis. However, we will explain a third and computationally easier method in Chapter V. It is based on Riemann sums of a univariate function. Moreover, by applying Euler-Maclaurin formula we have simultaneously an asymptotic expansion for $m(P_d)$.

IV.1. A proof using the generalization of Boyd-Lawton

In this section we use the generalized Boyd-Lawton theorem to prove Theorem IV.1.1. In this section we use some of the notation, introduced in Section II.4.4, in particular in II.4.16 and II.4.18.

Theorem IV.1.1 ([BGMP22][Meh21])**.** *For every $d \in \mathbb{Z}_{\geq 1}$, let $P_d(\underline{z}_2) := \sum_{0\leq i+j\leq d} z_1^i z_2^j \in \mathbb{C}[\underline{z}_2]$. Then the following equality*

$$\lim_{d\to+\infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3), \tag{IV.1.1}$$

holds, where $\zeta(s)$ denotes Riemann's Zeta function.

The fist step of the proof using the generalization of Boyd-Lawton is to interpret the right hand side of (IV.1.1) as a Mahler measure of a polynomial. To do so, we use a theorem of D'Andrea and Lalín (see [DL07, Theorem 7]).

Theorem IV.1.2 (D'Andrea, Lalín [DL07]). *Let $P_\infty(\underline{z}_4) := (1-z_1)(1-z_2)-(1-z_3)(1-z_4) \in \mathbb{C}[\underline{z}_4]$. Then the following equality*

$$m(P_\infty) = \frac{9}{2\pi^2}\zeta(3)$$

holds.

To apply Theorem II.4.20, we need to present a suitable sequence of matrices A_d that relates the polynomials P_d appearing in Theorem IV.1.1 to P_∞ featured in Theorem IV.1.2.

Lemma IV.1.3 ([BGMP22]). *For every $d \in \mathbb{Z}_{\geq 1}$, set* $A_d := \begin{bmatrix} d+2 & 0 & 1 & 0 \\ 0 & 1 & 0 & d+2 \end{bmatrix} \in \mathbb{Z}^{2\times 4}$.

Then the following equality: $P_d(\underline{z}_2)\,(1-z_1)\,(1-z_2)\,(z_1-z_2) = P_\infty(\underline{z}_2^{A_d})$ holds in the polynomial ring $\mathbb{C}[\underline{z}_2]$.

Proof. With a simple computation we have $\underline{z}_2^{A_d} = (z_1^{d+2}, z_2, z_1, z_2^{d+2})$, and $P_\infty(\underline{z}_2^{A_d}) = (1-z_1^{d+2})(1-z_2) - (1-z_1)(1-z_2^{d+2})$. Moreover, we have the following simplification:

$$\begin{aligned} P_\infty(\underline{z}_2^{A_d}) &= (1-z_1^{d+2})(1-z_2) - (1-z_1)(1-z_2^{d+2}) \\ &= z_1^{d+2}(z_2-1) + z_2^{d+2}(1-z_1) + (1-z_2) - (1-z_1) \\ &= z_1^{d+2}(z_2-1) + z_2^{d+2}(1-z_1) + (z_1-z_2). \end{aligned}$$

Since $P_d(z_1, z_2)$, for $d \geq 1$, is symmetric, we have:

$z_1P_d(z_1,z_2)+1+z_2+\cdots+z_2^{d+1} = P_{d+1}(z_1,z_2) = P_{d+1}(z_2,z_1) = z_2P_d(z_1,z_2)+1+z_1+\cdots+z_1^{d+1}$.

By subtracting $P_{d+1}(z_2, z_1)$ from $P_{d+1}(z_1, z_2)$, and using the properties of geometric series we have:

$$\begin{aligned} P_d(\underline{z}_2) = P_d(z_1, z_2) &= \frac{z_1^{d+2}-1}{(z_1-1)(z_1-z_2)} - \frac{z_2^{d+2}-1}{(z_2-1)(z_1-z_2)} \\ &= \frac{z_1^{d+2}(z_2-1) + z_2^{d+2}(1-z_1) + (z_1-z_2)}{(1-z_1)(1-z_2)(z_1-z_2)}, \end{aligned}$$

which concludes the proof. □

Theorem IV.1.1 follows easily:

Proof of Theorem IV.1.1. [BGMP22] First of all, Lemma IV.1.3 shows that $m(P_d) = m(P_\infty(\underline{z}_2^{A_d}))$ for every $d \in \mathbb{Z}_{\geq 1}$. Moreover, the rank of A_d is 2 (two linearly independent columns) and it has 4 rows. According to the rank theorem, the kernel of A_d is of dimension 2. It is easy to observe that the two vectors $[-1, 0, d+2, 0]^t$ and $[0, d+2, 0, -1]^t$ belong to the kernel of A_d and they have integer coefficients, so we have:

$$\{\mathbf{v} \in \mathbb{Z}^{4\times 1} \mid A_d \cdot \mathbf{v} = \mathbf{0}\} = \left\langle \begin{pmatrix} -1 \\ 0 \\ d+2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d+2 \\ 0 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

Let us show that we have $\rho(A_d) = d+2$. Consider any vector in the $\mathbb{Z}$-module generated by $[-1, 0, d+2, 0]^t$ and $[0, d+2, 0, -1]^t$. It is of the form $c_1[-1, 0, d+2, 0]^t + c_2[0, d+2, 0, -1]^t = [-c_1, c_2(d+2), c_1(d+2), -c_2]^t$ with $c_1, c_2 \in \mathbb{Z}$. Since c_1, c_2 are integers $||[-c_1, c_2(d+2), 0, c_1(d+2), -c_2]^t||_\infty \geq (d+2)$ as soon as $(c_1, c_2) \neq (0, 0)$. This implies that $\rho(A_d) = d+2$ for every $d \in \mathbb{Z}_{\geq 1}$. In particular $\lim_{d\to+\infty} \rho(A_d) = +\infty$. Thus, Theorem II.4.20 implies that

$$\lim_{d\to+\infty} m(P_d) = \lim_{d\to+\infty} m(P_\infty(\underline{z}_2^{A_d})) = m(P_\infty),$$

and we can conclude that (IV.1.1) holds by using Theorem IV.1.2. □

We note, once more, that the above proof of Theorem IV.1.1 is short, and is started by interpreting the limit value, announced in the R.H.S of (IV.1.1) as a Mahler measure. However, we did not provide any clues as to why $\frac{9}{2\pi^2}\zeta(3)$ was set as the limit beforehand. To clarify more, we mention that we took advantage of the direct proof, explained in the next sections. In fact, it seems impossible to guess the limit and a limit polynomial only by considering the sequence $m(P_d)$. Thus, the proof using the generalization of the Boyd-Lawton is possible thanks to the information that comes from the direct method.

IV.2. Towards a direct proof of the limit using Riemann sum techniques

In the sequel of this chapter we are going to prove Theorem IV.1.1 using a direct computational method. This section explains the work done in [Meh21] in detail. We introduce a new concave function, called vol. The values of the volume function at toric points of P_d can be written in terms of the values of vol. Then, we recompute $m(P_d)$ in terms of the values of vol. These computations lead to writing $m(P_d)$ as a difference of two expressions, each of them being proportional to a Riemann sum of vol over a triangle. We replace the volume function by vol because vol is concave. As we recalled in Section VII.6 in the Appendix, we have certain under estimators and upper estimators for concave functions. This property of concave functions is the key point to find the limit.

IV.2.1. Replacing Volume function by vol. The aim of this section is to introduce vol and prove its important properties. Let us start with reformulating the volume function at toric points. After a computation, the volume function at $(x, y) \in U_{d+1}$ can be rewritten as follows:

$$\begin{aligned} V(x,y) =& \frac{1}{(d+1)(d+2)}[D(y^{d+1}) - D(x^{d+1}) - D((y/x)^{d+1})] + \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)] \\ =& \frac{1}{(d+2)}[D(x) - D(y) - D(x/y)]. \end{aligned} \tag{IV.2.1}$$

Likewise, at a point $(x, y) \in U_{d+2}$ we have:

$$V(x,y) = \frac{1}{(d+1)}[D(x) - D(y) - D(x/y)]. \tag{IV.2.2}$$

Following the above computation we introduce vol:

Definition IV.2.1. The function vol : $[0, 2\pi] \times [0, 2\pi] \mapsto \mathbb{R}$ is defined by;

$$\mathrm{vol}(\theta, \alpha) := D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha}).$$

According to the computation of the volume function at toric point and the definition of vol one can conclude the following relations:

(1) For $(x, y) \in U_{d+1}$, we have $x = e^{\frac{2k\pi i}{d+1}}$, $y = e^{\frac{2k'\pi i}{d+1}}$, where $0 < k < k' < d + 1$ and by replacing in Eq. (IV.2.1) we have:

$$V(e^{\frac{2k\pi i}{d+1}}, e^{\frac{2k'\pi i}{d+1}}) = \frac{1}{d+2} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) = -V(e^{\frac{2k'\pi i}{d+1}}, e^{\frac{2k\pi i}{d+1}}).$$

(2) For $(x, y) \in U_{d+2}$, we have $x = e^{\frac{2k\pi i}{d+2}}$ and $y = e^{\frac{2k'\pi i}{d+2}}$, where $0 < k < k' < d + 2$ and by replacing in Eq. (IV.2.2) *we have:*

$$V(e^{\frac{2k\pi i}{d+2}}, e^{\frac{2k'\pi i}{d+2}}) = \frac{1}{d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right) = -V(e^{\frac{2k'\pi i}{d+2}}, e^{\frac{2k\pi i}{d+2}}).$$

This new function vol has many important properties which we can use to find the limit of $m(P_d)$. For instance it is concave and it has positive values on the triangle with vertices $\{(0, 0), (0, 2\pi), (2\pi, 0)\}$. On the contrary it is convex and has negative values on the triangle with vertices $\{(2\pi, 2\pi), (0, 2\pi), (2\pi, 0)\}$. Before proving these properties, let us fix the following notation:

Notation IV.2.2. The triangles with vertices $\{(0,0), (0,2\pi), (2\pi, 0)\}$, and $\{(2\pi, 2\pi), (0, 2\pi), (2\pi, 0)\}$ are respectively denoted by T and T^*.

In the first step, we prove the positivity of vol on T.

Lemma IV.2.3 ([Meh21])**.** *The function* $\mathrm{vol}(\theta, \alpha)$ *is positive on* T.

Proof. According to the properties of dilogarithm (see Fact III.1.8) vol is real analytic everywhere except at (θ, α) where $e^{i\theta} = 1$, $e^{i\alpha} = 1$ or $e^{i(\theta+\alpha)} = 1$. We notice that at each boundary point of T, one of the following conditions is satisfied:

(1) The point (θ, α) is on $\theta = 0$. Hence, $\mathrm{vol}(0, \alpha) = D(e^{i0}) - D(e^{i(0+\alpha)}) + D(e^{i\alpha}) = D(1) + D(e^{i\alpha}) - D(e^{i\alpha}) = 0$.
(2) The point (θ, α) is on $\alpha = 0$, so again $\mathrm{vol}(\theta, \alpha) = 0$.
(3) The point (θ, α) is on $\theta+\alpha = 2\pi$. Hence, $\mathrm{vol}(\theta, \alpha) = D(e^{i\theta}) - D(e^{i(2\pi)}) + D(e^{i(2\pi-\theta)}) = D(e^{i\theta}) - D(e^{i\theta}) = 0$. Notice that $D(\bar{z}) = -D(z)$.

Therefore, vol vanishes on the boundary of T. Thus, we check the sign of vol, at inner points of T, where the function is differentiable. To do so, first, we find the critical points of vol. In other words we search for (θ_0, α_0) which satisfies the following:

$$\frac{\partial \,\mathrm{vol}}{\partial \theta}|_{(\theta_0, \alpha_0)} = \frac{\partial \,\mathrm{vol}}{\partial \alpha}|_{(\theta_0, \alpha_0)} = 0.$$

To solve the above differential system of equations, first, we compute $\frac{\partial \,\mathrm{vol}}{\partial \theta}$:

$$\frac{\partial \,\mathrm{vol}}{\partial \theta} = \frac{\partial D(e^{i\theta})}{\partial \theta} - \frac{\partial D(e^{i(\theta+\alpha)})}{\partial \theta}.$$

We compute $\frac{\partial D(e^{i\theta})}{\partial \theta}$, using the fact that $-dD(z) = \eta(z, 1-z)$ or equivalently $dD(z) = \eta(1-z, z)$. Let $Z(\theta) = e^{i\theta}$ and $z_0 = Z(\theta_0) = e^{i\theta_0}$:

$$\frac{\partial D(e^{i\theta})}{\partial \theta}|_{(\theta_0,\alpha_0)} = dD|_{z_0}(\frac{d}{d\theta}e^{i\theta}|_{\theta_0}) = \eta(1-z_0, z_0)(\frac{d}{d\theta}e^{i\theta}|_{\theta_0}) = -\log|1-e^{i\theta_0}| \; (d\arg_{z_0}(\frac{d}{d\theta}e^{i\theta}|_{\theta_0})).$$

$$-\log|1-e^{i\theta_0}|(\frac{d}{d\theta}\arg(e^{i\theta})|_{\theta_0}) = -\log|1-e^{i\theta_0}|(\frac{d}{d\theta}\theta|_{\theta_0}) = -\log|1-e^{i\theta_0}|(1|_{\theta_0}) = -\log|1-e^{i\theta_0}|.$$

In the same way, we compute the other partial derivatives, and have:

$$\frac{\partial D(e^{i\alpha})}{\partial \alpha} = -\log|1-e^{i\alpha}| \quad , \quad \frac{\partial D(e^{i(\theta+\alpha)})}{\partial \alpha} = \frac{\partial D(e^{i(\theta+\alpha)})}{\partial \theta} = -\log|1-e^{i(\theta+\alpha)}|.$$

Thus, the critical points are obtained by solving the following:

$$\frac{\partial \,\mathrm{vol}}{\partial \theta} = \log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\theta}| = \frac{\partial \,\mathrm{vol}}{\partial \alpha} = \log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\alpha}| = 0.$$

Therefore, we have:

$$\log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\alpha}| = \log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\theta}| = 0.$$

Since only the solutions of the system inside T are considered, we have that $0 < \theta < 2\pi$, $0 < \alpha < 2\pi$ and $0 < \alpha+\theta < 2\pi$. Hence, the unique critical point corresponds to $\theta = \alpha = 2\pi/3$. Note that $\mathrm{vol}(2\pi/3, 2\pi/3) = 3D(e^{\frac{2\pi}{3}i})$ is approximately 2.03. Thus we continue the proof by contradiction.

Suppose there exists $(\theta_0, \alpha_0) \in T^\circ$, with vol $(\theta_0, \alpha_0) < 0$. Therefore, there exists a minimum, denoted by (θ_1, α_1) where $\mathrm{vol}(\theta_1, \alpha_1) < 0$. Note that vol is differentiable inside T, so the minimum is another critical point inside T, which obviously is different from $(2\pi/3, 2\pi/3)$, but this is a contradiction. Hence, vol is positive inside T. $\square$

In fact, one can easily prove that $(2\pi/3, 2\pi/3)$ is the unique maximum of vol over T. In the following proposition, we prove the concavity of vol.

Proposition IV.2.4 ([Meh21])**.** *The function* $\mathrm{vol}(\theta,\alpha)$ *is concave on* T*.*

PROOF. According to Lemma VII.6.4, we only need to compute the Hessian matrix of vol, and prove that is negative definite. We compute all the partial derivatives in the Hessian matrix of vol, using the computations in Lemma IV.2.3 and we have:

$$\frac{\partial\,\mathrm{vol}}{\partial\theta} = \log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\theta}|,$$
$$\frac{\partial\,\mathrm{vol}}{\partial\alpha} = \log|1-e^{i(\theta+\alpha)}| - \log|1-e^{i\alpha}|,$$
$$\frac{\partial^2\,\mathrm{vol}}{\partial\theta^2} = \frac{\partial\log|1-e^{i(\theta+\alpha)}|}{\partial\theta} - \frac{\partial\log|1-e^{i\theta}|}{\partial\theta}.$$

Since $0 \le \theta \le 2\pi$, so we have $|1-e^{i\theta}| = 2\sin(\theta/2)$. Therefore, we have $\frac{\partial\log|1-e^{i\theta}|}{\partial\theta} = \frac{\partial\log(2\sin(\theta/2))}{\partial\theta} = \frac{1}{2}\cot(\frac{\theta}{2})$. Then, the Hessian matrix of vol is:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2\,\mathrm{vol}}{\partial\theta^2} & \frac{\partial^2\,\mathrm{vol}}{\partial\theta\partial\alpha} \\ \\ \frac{\partial^2\,\mathrm{vol}}{\partial\alpha\partial\theta} & \frac{\partial^2\,\mathrm{vol}}{\partial\alpha^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\theta}{2}) & \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) \\ \\ \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) & \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\alpha}{2}) \end{bmatrix}.$$

The symmetric (2×2) Hessian matrix is negative definite if and only if $D_1 < 0$ and $D_2 > 0$, where $D_i, (i = 1, 2)$ are leading principal minors. Then we compute the minors (inside T).

- Computation of D_1: $D_1 = \frac{1}{2}\cot(\frac{\theta+\alpha}{2}) - \frac{1}{2}\cot(\frac{\theta}{2})$.

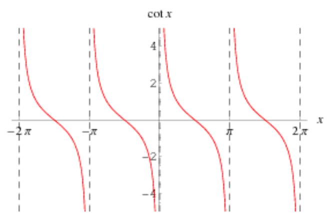

FIGURE 1. The graph of the cot(x).

Since $\cot(x)$ is decreasing between $[0,\pi]$ and $\alpha > 0$, we have $\frac{\theta}{2}, \frac{\theta+\alpha}{2} \in [0,\pi]$. Hence, $\cot(\frac{\theta+\alpha}{2}) < \cot(\frac{\theta}{2})$, and $D_1 < 0$.

- Computation of D_2:

$$
\begin{aligned}
D_2 &= Det(\mathbf{H})\\
&= \frac{1}{4}\left(\cot^2(\frac{\theta+\alpha}{2}) - \cot(\frac{\theta}{2})\cot(\frac{\theta+\alpha}{2}) - \cot(\frac{\theta+\alpha}{2})\cot(\frac{\alpha}{2}) + \cot(\frac{\theta}{2})\cot(\frac{\alpha}{2}) - \cot^2(\frac{\theta+\alpha}{2})\right)\\
&= \frac{1}{4}\left(\cot(\frac{\theta}{2})\cot(\frac{\alpha}{2}) - \cot(\frac{\theta+\alpha}{2})(\cot(\frac{\theta}{2}) + \cot(\frac{\alpha}{2}))\right)\\
&\overset{[1]}{=} \frac{1}{4}\left(\cot(\frac{\theta}{2})\cot(\frac{\alpha}{2}) - \frac{\cot(\frac{\theta}{2})\cot(\frac{\alpha}{2}) - 1}{(\cot(\frac{\theta}{2}) + \cot(\frac{\alpha}{2}))}(\cot(\frac{\theta}{2}) + \cot(\frac{\alpha}{2}))\right)\\
&= \frac{1}{4}.
\end{aligned}
$$

In [1] we used the trigonometric equality $\cot(\alpha+\beta) = \frac{\cot(\alpha)\cot(\beta)-1}{\cot(\alpha)+\cot(\beta)}$. Therefore, we have $D_2 = \frac{1}{4} > 0$ and consequently $\mathrm{vol}(\theta,\alpha)$ is concave inside T.

□

By looking at Fig. 2 we realize that vol is convex and has negative values in T^*. The proof is easy. It is sufficient to associate to each $(\theta',\alpha') \in T^*$ another point (θ,α) inside T, where $\theta = 2\pi - \theta'$ and $\alpha = 2\pi - \alpha'$. Since $-\mathrm{vol}(\theta',\alpha') = \mathrm{vol}(\theta,\alpha)$, vol is convex and has negative values in T^*.

IV.2.2. Recomputing $m(P_d)$ in terms of vol. In this section, using the relation between the values of the volume function at toric points and vol, we recompute $m(P_d)$ in terms of vol. It leads to writing $m(P_d)$ as a difference of two Riemann sums of vol over T, with certain coefficients.

Theorem IV.2.5 ([Meh21])**.** *We have:*

$$
2\pi m(P_d) = \frac{-2}{d+2}\sum_{0<k<k'\le d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1}\sum_{0<k<k'\le d+1}\mathrm{vol}\left(\frac{2k\pi}{d+2},\frac{2(k'-k)\pi}{d+2}\right).
$$

Proof. We use the closed formula for computing the Mahler measure [GM21];

$$
m(P_d) = \frac{1}{2\pi}\sum_{(x,y)\in U_{d+1}\cup\, U_{d+2}} \epsilon(x,y)V(x,y).
$$

We break the sum into the two summations over $d+1$, and $d+2$ toric points. Proposition III.2.15 gives the value of $\epsilon(x,y)$ at each toric point. Using Definition IV.2.1 we have:

(IV.2.3)

$$
2\pi m(P_d) = \frac{-1}{d+2}\sum_{0<k<k'\le d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right) - \frac{1}{d+2}\sum_{0<k<k'\le d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)
$$

(IV.2.4)

$$
+\frac{1}{d+1}\sum_{0<k<k'\le d+1}\mathrm{vol}\left(\frac{2k\pi}{d+2},\frac{2(k'-k)\pi}{d+2}\right) + \frac{1}{d+1}\sum_{0<k<k'\le d+1}\mathrm{vol}\left(\frac{2k\pi}{d+2},\frac{2(k'-k)\pi}{d+2}\right)
$$

(IV.2.5)

$$
\overset{[1]}{=} \frac{-2}{d+2}\sum_{0<k<k'\le d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1}\sum_{0<k<k'\le d+1}\mathrm{vol}\left(\frac{2k\pi}{d+2},\frac{2(k'-k)\pi}{d+2}\right).
$$

□

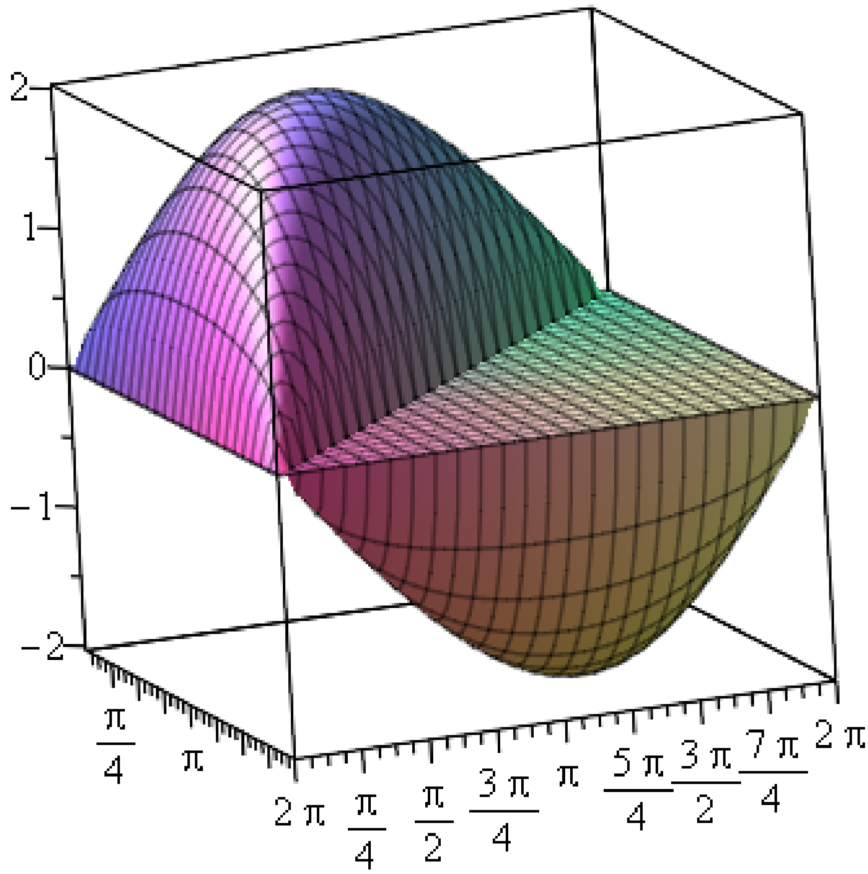

FIGURE 2. The graph of $\operatorname{vol}(\theta, \alpha)$, implemented by Maple.

In [1] we use the property that vol inherits from volume function: $\operatorname{vol}(\theta, \alpha) = -\operatorname{vol}(\alpha, \theta)$. In Eq. (IV.2.5), each summation is proportional to a Riemann sum of vol over T. We thus need to compute $\iint_T \operatorname{vol}(\theta, \alpha) dA$, with dA the euclidean measure on T for our future computations.

Lemma IV.2.6 ([Meh21])**.** *We have:*

$$\iint_T \operatorname{vol}(\theta, \alpha) dA = 6\pi\zeta(3),$$

where ζ is the Riemann Zeta function.

PROOF. In this proof, we use the formula, $D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$ (see Definition III.1.7).

$$\begin{aligned}
\iint_T \mathrm{vol}(\theta,\alpha)dA &= \int_0^{2\pi}\int_0^{2\pi-\alpha} \mathrm{vol}(\theta,\alpha)d\theta d\alpha \\
&= \int_0^{2\pi}\int_0^{2\pi-\alpha} D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})d\theta d\alpha \\
&= \int_0^{2\pi}\int_0^{2\pi-\alpha} (\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2} + \sum_{n=1}^{\infty}\frac{\sin(n\alpha)}{n^2} - \sum_{n=1}^{\infty}\frac{\sin(n(\theta+\alpha))}{n^2})d\theta d\alpha \\
&\overset{[1]}{=} \sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi-\alpha} \frac{\sin(n\theta)+\sin(n\alpha)-\sin(n(\theta+\alpha))}{n^2}d\theta d\alpha \\
&= \sum_{n=1}^{\infty}\int_0^{2\pi} [\frac{\cos(n(\theta+\alpha))-\cos(n\theta)+n\theta\sin(n\alpha)}{n^3}]_0^{2\pi-\alpha}d\alpha \\
&= \sum_{n=1}^{\infty}\int_0^{2\pi} \frac{2-2\cos(n\alpha)+n(2\pi-\alpha)\sin(n\alpha)}{n^3}d\alpha \\
&= \sum_{n=1}^{\infty}[\frac{2n\alpha-2\sin(n\alpha)}{n^4}]_0^{2\pi} + \sum_{n=1}^{\infty}\int_0^{2\pi}\frac{(2\pi-\alpha)\sin(n\alpha)}{n^2}d\alpha \\
&= 4\pi\sum_{n=1}^{\infty}\frac{1}{n^3} + \sum_{n=1}^{\infty}[\frac{-2\pi\cos(n\alpha)}{n^3}]_0^{2\pi} + \sum_{n=1}^{\infty}\int_0^{2\pi}\frac{-\alpha\sin(n\alpha)}{n^2}d\alpha \\
&= 4\pi\sum_{n=1}^{\infty}\frac{1}{n^3} + \sum_{n=1}^{\infty}\left[\frac{\alpha\cos(n\alpha)}{n^3}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\cos(n\alpha)}{n^3}d\alpha \\
&= 4\pi\sum_{n=1}^{\infty}\frac{1}{n^3} + \sum_{n=1}^{\infty}\frac{2\pi}{n^3} = 6\pi\sum_{n=1}^{\infty}\frac{1}{n^3} = 6\pi\zeta(3).
\end{aligned}$$

where [1] is because $\sum_{n=1}^{\infty}\frac{\sin(x)}{n^2}$ is uniformly convergent, and the summation and the integration commute. □

We have already mentioned that the Mahler measure of exact polynomials may be related to special values of L-function. The appearance of Riemann Zeta function in the above computation comes from exactness of P and the fact that the volume function is a linear combination of dilogarithm function. By proving Theorem IV.1.1, we exhibit an example of the existence of the Riemann Zeta function in the limit of a sequence of Mahler measures which is an interesting phenomenon.

IV.3. Estimation of the error terms for the Riemann sums

In Theorem IV.2.5 we computed $m(P_d)$ as a difference of Riemann sums. To explain more about the Riemann sums, let us rewrite the formula in Theorem IV.2.5 here again:

$$2\pi m(P_d) = \frac{-2}{d+2}\sum_{0<k<k'\leq d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1}\sum_{0<k<k'\leq d+1}\mathrm{vol}\left(\frac{2k\pi}{d+2},\frac{2(k'-k)\pi}{d+2}\right).$$

Then, we define the Riemann sum [1] of the two variable function vol over the triangle T as:

$$R_d := \frac{4\pi^2}{d^2} \sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d}, \frac{2(k'-k)\pi}{d}\right). \tag{IV.3.1}$$

Thus, the series appear on the R.H.S of the above equation for $m(P_d)$ are respectively $\frac{(d+1)^2}{(d+2)}R_{d+1}$ and $\frac{(d+2)^2}{(d+1)}R_{d+2}$. Since vol is continuous, the Riemann sums R_{d+1} and R_{d+2} converge to the integral of vol over T. The sequence of the errors between the value of the integral and the Riemann sums (i.e. $E(d) = |\int_T \mathrm{vol} - R_d|$) goes to zero when d goes to infinity. However, the coefficients of the Riemann sums in the equation of $m(P_d)$ depend on d and by moving d to infinity the error terms multiplying these coefficients (i.e. $\frac{(d+1)^2}{(d+2)}E(d+1)$ and $\frac{(d+2)^2}{(d+1)}E(d+2)$) may not converge to zero anymore. In this section, we are going to find a lower and an upper bound for the Riemann sums. In the next section using the properties proved in this section and studying the error terms we will prove that by sending d to infinity, $E(d)$ goes to zero faster than $1/d$. The computation of the limit of the sequence $m(P_d)$ will follow.

IV.3.1. A lower bound for the Riemann sum. In this section, first, we introduce a subpartition of T, and we define a Riemann sum of $\mathrm{vol}(\theta, \alpha)$ over this subpartition. Then, using concavity of vol, we exhibit a lower bound for the Riemann sum. We notice that all we need is about the $d+1$ and $d+2$ Riemann sums of vol. Thus, in the rest we do all the computations for the $d+1$-Riemann sums, then with a change of variable we conclude the same results for $d+2$-Riemann sums.

Observation IV.3.1. ***Square subpartition**:*
*Consider the set of the points $(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1})$ with $0 < k < k' < d+1$ inside T. For (x, y) in the defined set, consider the square with side $\frac{2\pi}{d+1}$ such that (x, y) is at the center of the square. The union of the squares is called **(d+1)-square subpartition** of T which does not cover all T. The number of such squares is $\sum_{i=1,\dots,d-1} i = \frac{d(d-1)}{2}$ and the area of each square is $\frac{4\pi^2}{(d+1)^2}$. The set difference of T and the (d+1)-square subpartition is called Blue part. The area of the Blue part part is:*

$$2\pi^2 - \left(\frac{4\pi^2}{(d+1)^2}\frac{d(d-1)}{2}\right) = 2\pi^2\frac{3d+1}{(d+1)^2}.$$

The 8-square subpartition of T is shown in Fig. 3. Then, we consider the $d+1$- Riemann sum of vol*, introduced in Eq.* (IV.3.1)*, denoted by R_{d+1}. Let us fix the notation $E(d)$ for the difference between the value of the integral and R_d for a fixed d. For instance, for the $d+1$-square subpartition we have:*

$$E(d+1) = \left| \iint_T \mathrm{vol}(\theta,\alpha)dA - \frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right)\right|.$$

We also introduce another notation:

$$\mathcal{E}(d+1) := \iint_{\text{Blue part}} \mathrm{vol}(\theta,\alpha)dA.$$

In the following lemma, we compute a lower bound for the Riemann sum.

[1]The notation of d- Riemann sum of vol, used in [Meh21] is S_d, but here for the coherence with the rest of the thesis, we changed to R_d.

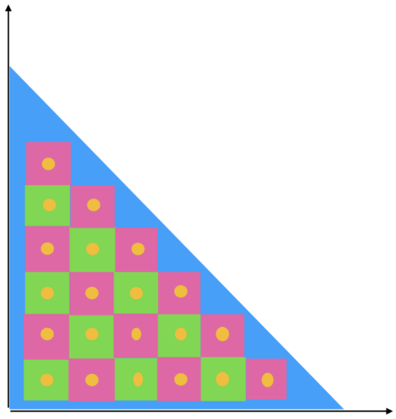

FIGURE 3. The 8-square subpartition of T.

Lemma IV.3.2 ([Meh21]). *We have $E(d+1) \leq \mathcal{E}(d+1)$. Moreover,*

$$\iint\limits_{T} \mathrm{vol}(\theta,\alpha)dA \leq \mathcal{E}(d+1) + \frac{4\pi^2}{(d+1)^2} \sum_{0<k<k'\leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right). \tag{IV.3.2}$$

PROOF. According to Observation IV.3.1, for a fixed d, T is partitioned into $\frac{(d-1)(d-2)}{2}$ squares and the blue part. The central points of the squares are denoted by (θ^*, α^*). The function vol is concave and differentiable inside T, so in particular it is concave on each square. According to Proposition VII.6.5 and Notation VII.6.6 in the Appendix, for an arbitrary, fixed square, the tangent plane to the graph of vol at (θ^*, α^*), denoted by $\mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*)$, is above the graph for all (θ, α) in the square, so we have:

$$\mathrm{vol}(\theta,\alpha) \leq \mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*).$$

The above inequality leads to a lower bound for the Riemann sums over the square. To see that consider the rectangular cuboid with the square as its base and bounded above by the tangent plane of $\mathrm{vol}(\theta,\alpha)$, at (θ^*, α^*). Let us denote the integral of vol over the square with $\iint\limits_{\square} \mathrm{vol}(\theta,\alpha)dA$. Then, the volume of this rectangular cuboid is greater than $\iint\limits_{\square} \mathrm{vol}(\theta,\alpha)dA$. Hence, we have:

$$\iint\limits_{\square} \mathrm{vol}(\theta,\alpha)dA \leq \iint\limits_{\square} \mathrm{Tang}_{\mathrm{vol}}(\theta^*, \alpha^*)dA = \frac{4\pi^2}{(d+1)^2}\mathrm{vol}(\theta^*, \alpha^*).$$

Therefore, we have:

$$\sum_{all\ squares\ inside\ T} \iint\limits_{\square} \mathrm{vol}(\theta,\alpha)dA \leq \sum_{0<k<k'\leq d} \frac{4\pi^2}{(d+1)^2}\mathrm{vol}(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}),$$

which is equivalent to the following:

$$\iint_T \mathrm{vol}(\theta,\alpha)dA - \sum_{all\ squares\ inside\ T} \iint_{\square} \mathrm{vol}(\theta,\alpha)dA \geq$$
$$\iint_T \mathrm{vol}(\theta,\alpha)dA - \sum_{0<k<k'\leq d} \frac{4\pi^2}{(d+1)^2}\mathrm{vol}(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}).$$

Thus, $E(d+1) \leq \mathcal{E}(d+1)$. Moreover, we have:

$$\iint_T \mathrm{vol}(\theta,\alpha)dA \leq \mathcal{E}(d+1) + \frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\leq d}\mathrm{vol}(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}).$$

□

IV.3.2. An upper bound for the Riemann sum. In this section, we define a partition of T that helps us find an upper bound for the Riemann sum of vol introduced in the previous section. To do so, we need to introduce a triangular partition of T.

Observation IV.3.3. ***Triangular partition****:*
The triangle T is partitioned into the smaller triangles belonging to $T_1 \cup T_2$, where T_1 and T_2 are the following collection of triangles:

$$T_1 := \bigcup_{i=0}^{d+1}\bigcup_{j=0}^{d+1-i}\left\{\left[\left(i\frac{2\pi}{d+1}, j\frac{2\pi}{d+1}\right), \left(i\frac{2\pi}{d+1}, (j+1)\frac{2\pi}{d+1}\right), \left((i+1)\frac{2\pi}{d+1}, j\frac{2\pi}{d+1}\right)\right]\right\},$$

$$T_2 := \bigcup_{i=1}^{d}\bigcup_{j=1}^{d+1-i}\left\{\left[\left((i-1)\frac{2\pi}{d+1}, j\frac{2\pi}{d+1}\right), \left(i\frac{2\pi}{d+1}, j\frac{2\pi}{d+1}\right), \left(i\frac{2\pi}{d+1}, (j-1)\frac{2\pi}{d+1}\right)\right]\right\}.$$

In the above definitions $[(i_1,j_1),(i_2,j_2),(i_3,j_3)]$ denotes the triangle with vertices $(i_1,j_1),(i_2,j_2)$ and (i_3,j_3). The figure for the 2-triangular partition is shown in Fig. 4; indeed, the pink and green triangles respectively belong to T_1 and T_2.

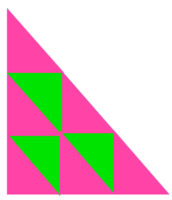

FIGURE 4. The 2-triangular partitions of T.

Definition IV.3.4. The vertices of small triangles, defined in Observation IV.3.3 that are not located on the boundary of T are called **inner** vertices. The set of all these inner vertices in d-th triangular partition is denoted by $In_d(T)$.

The following fact leads to an important correspondence between the triangular partition, and the square subpartition. The proof is elementary.

Fact IV.3.5. *Each inner vertex of a small triangle, in the d-triangular partition, is a central point of a unique square in the d-square subpartition.*

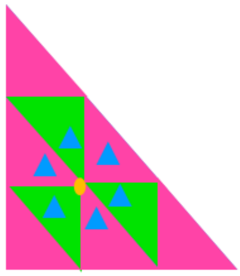

FIGURE 5. The yellow inner vertex is shared between the six triangles marked in blue.

We have seen that vol is a concave function, so when it is restricted to the triangle $[a,b,c]$ according to Proposition VII.6.8 in the Appendix, there exists a unique affine function called χ, such that $\mathrm{vol}(a)=\chi(a), \mathrm{vol}(b)=\chi(b)$, and $\mathrm{vol}(c)=\chi(c)$. Moreover, for any point $(\theta,\alpha)\in[a,b,c]$ we have $\chi(\theta,\alpha)\le \mathrm{vol}(\theta,\alpha)$. Hence, we have:

$$\iint_{[a,b,c]} \chi(\theta,\alpha)dA \le \iint_{[a,b,c]} \mathrm{vol}(\theta,\alpha)dA.$$

It is time to compute an upper bound for the Riemann sums.

Lemma IV.3.6 ([Meh21])**.** *We have the following upper bound for the Riemann sum:*

$$\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'<d}\mathrm{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)\le\iint_T \mathrm{vol}(\theta,\alpha)dA. \tag{IV.3.3}$$

PROOF. We know that:

$$\sum_{[b,c,d]\in T_2}\iint_{[b,c,d]}\mathrm{vol}(\theta,\alpha)dA+\sum_{[a,b,c]\in T_1}\iint_{[a,b,c]}\mathrm{vol}(\theta,\alpha)dA=\iint_T \mathrm{vol}(\theta,\alpha)dA.$$

Let us denote an arbitrary triangle in T_1 by $[a,b,c]$ and an arbitrary triangle in T_2 by $[b,c,d]$. By applying Lemma VII.6.9 in Appendix to each of the triangle in the last equality we have:

$$\sum_{[b,c,d]\in T_2}\mathrm{area}[b,c,d]\left(\frac{1}{3}\mathrm{vol}(d)+\frac{1}{3}\mathrm{vol}(b)+\frac{1}{3}\mathrm{vol}(c)\right)+$$
$$\sum_{[a,b,c]\in T_1}\mathrm{area}[a,b,c]\left(\frac{1}{3}\mathrm{vol}(a)+\frac{1}{3}\mathrm{vol}(b)+\frac{1}{3}\mathrm{vol}(c)\right)\le\iint_T \mathrm{vol}(\theta,\alpha)dA.$$

The triangles all have the same area, $\frac{2\pi}{(d+1)^2}$, so we factor it in the above summations. Notice that for each vertex a the number of times that $\mathrm{vol}(a)$ appears in the summation depends on its location. As we already mentioned, vol is zero on the boundary of T, so, we just consider inner vertices. Let a be an inner vertex. It appears in exactly 6 triangles, marked in blue in Fig. 5. Hence, we have:

$$\sum_{[b,c,d]\ \in T_2} \text{area}[b,c,d]\left(\frac{1}{3}\,\text{vol}(d)+\frac{1}{3}\,\text{vol}(b)+\frac{1}{3}\,\text{vol}(c)\right)+$$
$$\sum_{[a,b,c]\ \in T_1} \text{area}[a,b,c]\left(\frac{1}{3}\,\text{vol}(a)+\frac{1}{3}\,\text{vol}(b)+\frac{1}{3}\,\text{vol}(c)\right)=$$
$$\frac{4\pi^2}{(d+1)^2}\sum_{a\in In_{d+1}(T)}\frac{6}{3}\,\text{vol}(a)=\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'<d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right).$$

In the last equality we used Fact IV.3.5, that any inner vertex corresponds to a central point. Finally, we have the upper bound:

$$\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'<d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)\leq\iint_T\text{vol}(\theta,\alpha)dA.$$

□

IV.4. Direct proof of convergence

In this section, using Lemmas IV.3.2 and IV.3.6 we prove that $E(d)$ goes to zero faster than $1/d$. This result leads to the limit of $(m(P_d))_{d\in\mathbb{Z}_{\geq 1}}$, which is the objective of this chapter.

Lemma IV.4.1 ([Meh21])**.** *We have the following equality:*

$$E(d)=o(\frac{1}{d}).$$

Proof. We use the lower and upper bounds IV.3.2 and IV.3.3, found respectively in Lemma IV.3.2 and Lemma IV.3.6 and we have:

$$\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\leq d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)\leq\iint_T\text{vol}(\theta,\alpha)dA$$
$$\leq\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\leq d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)+\mathcal{E}(d+1).$$

Therefore, we conclude;

$$0\leq\iint_T\text{vol}(\theta,\alpha)dA-\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\leq d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)$$
$$\leq\mathcal{E}(d+1)\leq\text{Max}\,.\,\text{area}(\text{Blue part}),$$

where Max is the maximum of vol on the Blue part of the triangle. The area of the blue part is $2\pi^2\frac{3d+1}{(d+1)^2}$, so by the definition of $E(d+1)$ we have:

$$E(d+1)=\iint_T\text{vol}(\theta,\alpha)dA-\frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\leq d}\text{vol}\left(\frac{2k\pi}{d+1},\frac{2(k'-k)\pi}{d+1}\right)\leq 2\pi^2\frac{3d+1}{(d+1)^2}\text{Max}\,.$$

If d goes to infinity the points inside the blue part are approaching the boundary of T, where the values of vol are zero. Hence, the Maximum of vol in the blue part goes to zero as well. Therefore, we have $dE(d+1)\xrightarrow{d\to\infty}0$. In other words $E(d)=o(\frac{1}{d})$. □

We may now prove Theorem IV.1.1:

Proof. [Meh21] By using Theorem IV.2.5 we have:

$$2\pi m(P_d) = \frac{-2}{d+2}\sum_{0<k<k'\le d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1}\sum_{0<k<k'\le d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right).$$

In order to find $\lim_{d\to\infty} m(P_d)$, we compute the limit of the R.H.S. We have, by definition,

$$\iint\limits_T \mathrm{vol}(\theta,\alpha)dA = \frac{4\pi^2}{(d+1)^2}\sum_{0<k<k'\le d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) + E(d+1).$$

Hence, we have:

$$\frac{-2}{d+2}\sum_{0<k<k'\le d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) = \frac{-(d+1)^2}{2\pi^2(d+2)}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA + \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1).$$

Similarly:

$$\frac{2}{d+1}\sum_{0<k<k'\le d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right) = \frac{(d+2)^2}{2\pi^2(d+1)}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA - \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2).$$

This gives:

$$\begin{aligned}
2\pi m(P_d) =& \frac{(d+2)^2}{2\pi^2(d+1)}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA - \frac{(d+1)^2}{2\pi^2(d+2)}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA \\
&+ \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1) - \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2) \\
=& \frac{3d^2+8d+7}{4\pi^3(d^2+3d+2)}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA + \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1) - \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2).
\end{aligned}$$

According to Lemma IV.4.1, $E(d) = o(\frac{1}{d})$. Hence, $\lim_{d\to\infty} \frac{(d+1)^2}{2\pi^2(d+2)}E(d+1) = \lim_{d\to\infty} \frac{(d+2)^2}{2\pi^2(d+1)}E(d+2) = 0$. Therefore, based on Lemma IV.2.6 we have:

$$\lim_{d\to\infty} m(P_d) = \frac{3}{4\pi^3}\iint\limits_T \mathrm{vol}(\theta,\alpha)dA = \frac{9}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{9}{2\pi^2}\zeta(3).$$

□

As we mentioned in the introduction of this chapter, the computational method that we explained uses Riemann sums of two variable functions. It has difficult computations, but in Chapter V, we will present a third and easier method using one variable Riemann sums which also gives a rate of the convergence simultaneously.

CHAPTER V

Asymptotic expansion and rate of the convergence of $m(P_d)$

This chapter is part of the common article in collaboration with Brunault, Guilloux and Pengo [BGMP22]. Theorem II.4.20 is a generalization of the theorem of Boyd-Lawton to multivariate case. According to this theorem, the Mahler measure of a multivariate polynomial can be computed as a limit of the Mahler measure of certain sequence of lower dimensional Mahler measures. One may ask about the possible rates of convergence of these sequences. In other words, if we denote the limit polynomial by P_∞ and the sequence of polynomial by P_{A_d}, we are searching for an estimate for $|m(P_{A_d}) - m(P_\infty)|$ (the notation used in this chapter is from Chapter II). In Section IV.1, we can see that the rate of the convergence depends on P_∞. For instance, in [BGMP22], there are examples of P_∞ where any sequence $m(P_{A_d})$ converges exponentially fast to the limit $m(P_\infty)$; and other examples for which $m(P_{A_d}) - m(P_\infty) \equiv c(d)/d^{\frac{3}{2}}$, where c is a real valued function of d. There is also the sequence $P_{A_d} := P_d$, for which we will prove $|m(P_d) - m(P_\infty)| = O(\frac{\log d}{d^2})$. It is interesting that until now we have 3 different types rates of convergence. In this chapter, we study the asymptotic expansion of $|m(P_d) - m(P_\infty)|$ which gives us the rate of convergence of this sequence towards P_∞. A direct conclusion from the asymptotic expansion is that for d big enough the sequence $m(P_d)$ is increasing. We notice that the computation of the asymptotic expansion gives us another method to prove $\lim_{d\to\infty} m(P_d) = \frac{9\zeta(3)}{2\pi^2}$. To write the asymptotic expansion of $m(P_d)$, we first recompute it in terms of the values of dilogarithm at roots of unity. This step simplifies the computation with respect to the previous chapter. Indeed we will be able to use Euler–Maclaurin summation formula, which provides expressions for the error for Riemann sums of a one variable function in terms of the higher derivatives of the function. The existence of the Euler–Maclaurin formula for certain types of singular functions is the key tool for our computations.
We notice that the work exposed in this chapter covers [BGMP22, Theorem 5.1]. However, we provide a more complete and detailed proof of this theorem for the reader.

V.1. Rewriting $m(P_d)$ in terms of dilogarithm

In Theorem IV.2.5 in Section IV.2.2 we recompute $m(P_d)$ in terms of vol as follows:

(V.1.1)

$$2\pi m(P_d) = \frac{-2}{d+2} \sum_{0<k<k'\leq d} \operatorname{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1} \sum_{0<k<k'\leq d+1} \operatorname{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right),$$

where $\operatorname{vol}(\theta,\alpha) = D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})$. The above formula helped us to find the limit of $m(P_d)$. However, working with this formula has it own difficulties. Since the analysis of a bivariate function is less precise than that of a univariate function. For this reason we rewrite this formula.

Theorem V.1.1 ([BGMP22]). *Let $d \in \mathbb{Z}_{\geq 1}$, the closed formula for $m(P_d)$ in terms of dilogarithm evaluated at certain roots of unity is as follows:*

$$2\pi m(P_d) = \frac{1}{d+1} \sum_{1 \leq k \leq d+2} (3d - 6k + 6) D(e^{\frac{2k\pi}{d+2} i}) - \frac{1}{d+2} \sum_{1 \leq k \leq d+1} (3d - 6k + 3) D(e^{\frac{2k\pi}{d+1} i}).$$

PROOF. We replace the function $\mathrm{vol}(\theta, \alpha) = D(e^{i\theta}) - D(e^{i(\theta+\alpha)}) + D(e^{i\alpha})$ in the equation Eq. (V.1.1) and let $\zeta_1 = e^{\frac{2\pi i}{d+1}}$ and $\zeta_2 = e^{\frac{2\pi i}{d+2}}$, so we have;

$$\frac{-2}{d+2} \sum_{0<k<k' \leq d} \mathrm{vol}\left(\frac{2k\pi}{d+1}, \frac{2(k'-k)\pi}{d+1}\right) + \frac{2}{d+1} \sum_{0<k<k' \leq d+1} \mathrm{vol}\left(\frac{2k\pi}{d+2}, \frac{2(k'-k)\pi}{d+2}\right) =$$
$$\frac{-2}{d+2} \sum_{0<k<k' \leq d} \left(D(\zeta_1^k) - D(\zeta_1^{k'}) + D(\zeta_1^{k'-k})\right) + \frac{2}{d+1} \sum_{0<k<k' \leq d+1} \left(D(\zeta_2^k) - D(\zeta_2^{k'}) + D(\zeta_2^{k'-k})\right).$$

The rest of the proof consists of counting the coefficients of $D(\zeta_1^i)$ and $D(\zeta_2^j)$ for any $1 \leq i \leq d+1$ and $1 \leq j \leq d+2$. We notice that in the computation we used the property of dilogarithm that $D(\bar{z}) = -D(z)$.

□

We simplify the formula for $m(P_d)$, using the following notation:

Notation V.1.2.

$$S_d := 3 \sum_{1 \leq k \leq d-1} (d - 2k) D((e^{\frac{2\pi}{d} i})^k).$$

Therefore, the formula in Theorem V.1.1 can be written as follows:

$$2\pi m(P_d) = \frac{1}{d+1} S_{d+2} - \frac{1}{d+2} S_{d+1}.$$

Using this new formula, we write $m(P_d)$ as a linear combination of 1-dimensional Riemann sums of certain univariable functions. Then, to find the asymptotic expansion of $|m(P_d) - m(P_\infty)|$, we apply the Euler–Maclaurin summation formula for each of the univariate functions.

V.1.1. Computing $m(P_d)$ in terms of 1-dimensional Riemann sums. In this section, by leveraging the one variable closed formula of $m(P_d)$, introduced in Theorem V.1.1, $m(P_d)$ can be written as a linear combination of Riemann sums of a one variable function, called $f(x)$. We define f, such that S_d is proportional to a Riemann sum of f.

Notation V.1.3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = (1 - 2x) D(e^{2\pi x i})$. We denote by S_d^T the Riemann sum

$$S_d^T(f) := \sum_{1 \leq k \leq d-1} \frac{1}{d} (1 - 2\frac{k}{d}) D(e^{\frac{2k\pi}{d} i}).$$

Since $f(0) = f(1) = 0$, (see Fig. 1), $S_d^T(f)$ is the Riemann sum associated with the Trapezoid method over $[0, 1]$. Let us compute the integral of f, which is needed for the rest of our computation:

Lemma V.1.4. *We have the following evaluation:*

$$\int_0^1 f(x) dx = \frac{\zeta(3)}{\pi}.$$

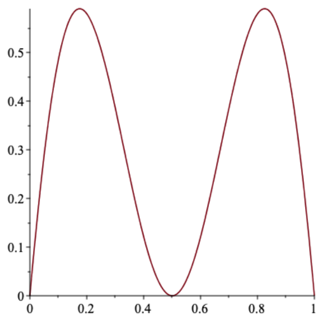

FIGURE 1. The plot of $f(x) = (1-2x)D(e^{2\pi x i})$.

PROOF. To compute the above integral we use that $D(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$. The series is uniformly convergent and we can change the order of integration and summation.

$$\int_0^1 (1-2x)D(e^{2\pi i x})dx = \int_0^1 \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2} dx - 2\int_0^1 x \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2} dx$$
$$\overset{[1]}{=} -2\sum_{n=1}^{\infty} \int_0^1 \frac{x\sin(2\pi n x)}{n^2} dx \overset{[2]}{=} \sum_{n=1}^{\infty} \frac{1}{n^3 \pi} = \frac{\zeta(3)}{\pi}.$$

In [1] we used that $\int_0^1 \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2} dx = 0$ and in [2] we compute the integral by integration by parts. ☐

We have $S_d = 3d^2 S_d^T(f)$. Thus, in the formula of $m(P_d)$, we replace the sums S_d with S_d^T and we have:

$$m(P_d) = \frac{3}{2\pi}\left(\frac{(d+2)^2}{d+1} S_{d+2}^T(f) - \frac{(d+1)^2}{d+2} S_{d+1}^T(f)\right). \tag{V.1.2}$$

The situation is similar to the previous chapter, but for the univariate function f. We use the Euler–Maclaurin summation formula to approximate the error between the value of the integral of f and its Riemann sums in terms of the higher derivatives $f^{(k)}(x)$ evaluated at the endpoints of the interval of integration. For the necessary information about the Euler–Maclaurin summation formula, see Section VII.7 of the Appendix. To simplify the calculation, we fix the following notation for the error terms:

Notation V.1.5. For an arbitrary function $k(x)$, with $k(0) = k(1) = 0$, the error between the trapezoid Riemann sum of k and the value of the integral of k is denoted by:

$$E_d(k) := \int_0^1 k(x)dx - S_d^T(k) = \int_0^1 k(x)dx - \sum_{i=1}^{d-1} \frac{1}{d} k(\frac{i}{d}).$$

Using Eq. (V.1.2) and the previous notation we can rewrite the formula of $m(P_d)$ in terms of the value of the integral of $f(x)$ and $E_{d+1}(f)$ and $E_{d+2}(f)$;

$$\begin{aligned} m(P_d) &= \frac{3}{2\pi}\left(\frac{(d+2)^2}{d+1} S_{d+2}^T(f) - \frac{(d+1)^2}{d+2} S_{d+1}^T(f)\right) \\ &= \frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2} E_{d+1}(f) - \frac{(d+2)^2}{d+1} E_{d+2}(f)\right) + \frac{3}{2\pi}(\frac{3d^2+9d+7}{d^2+3d+2})\int_0^1 f(x)dx. \end{aligned}$$

In the above computation by using Lemma V.1.4 we replace the value of the integral, namely $\frac{\zeta(3)}{\pi}$, and we have:

$$m(P_d) - \frac{9\zeta(3)}{2\pi^2} = \frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2}E_{d+1}(f) - \frac{(d+2)^2}{d+1}E_{d+2}(f)\right) + \frac{3\zeta(3)}{2\pi^2(d+1)(d+2)}. \tag{V.1.3}$$

Thanks to the above equality we can compute the asymptotic expansion of $|m(P_d)-m(P_\infty)|$, from the asymptotic expansion of R.H.S by applying Euler-Maclaurin formula to f. But, $f(x)$ has singularities, so we can not apply the normal Euler-Maclaurin formula to estimate the error $E_d(f)$. There is an extension of the Euler-Maclaurin formula to a function with logarithmic singularities at one of the end points of their domain. So to compute the asymptotic expansion of f, first, we should find the type of the singularities of $f(x)$. The singularities of $f(x)$ come from those of the dilogarithm, so we find the singularities of the dilogarithm on $[0,1]$.

Lemma V.1.6. *The function $x \mapsto D(e^{2\pi xi})$ has logarithmic singularities at 0 and 1 on the interval $[0,1]$:*

Proof. This lemma is classic. It comes from [BL13] or [BZ20, Theorem 7.2]:

$$\begin{aligned}\frac{\partial}{\partial_x}D(e^{2\pi xi}) &= -\eta_{(e^{2\pi xi},1-e^{2\pi xi})} = -2\pi\log|1-e^{2\pi xi}| = -2\pi\log|1-\sum_{n=0}^{\infty}\frac{(2\pi xi)^n}{n!}| \\ &= -2\pi\log(2\pi x) - 2\pi\log|\sum_{n=0}^{\infty}\frac{(2\pi xi)^n}{(n+1)!}|.\end{aligned}$$

When $x \to 0^+$ the differential of dilogarithm function is equivalent to the function $-2\pi\log(2\pi x)$. In addition, one can verify that $f(x)$ has logarithmic singularity at 1. $\square$

Thus, $f(x)$ has logarithmic singularities at the both end points of the interval. In the second step, we write $f(x)$ as a summation of its smooth and its singular parts then we study each part separately. To find the smooth part of $f(x)$ we remove the singularities from $D(e^{2\pi ix})$. The function $x \mapsto D(e^{2\pi ix}) + 2\pi x\log(2\pi x) - 2\pi(1-x)\log(2\pi(1-x))$ is smooth on $[0,1]$. Let the singular part of $f(x)$ be denoted by $F(x) := (1-2x)\left(-2\pi x\log(x) + 2\pi(1-x)\log(1-x)\right)$, so $f(x) - F(x)$ is a smooth function on $[0,1]$, which is denoted by $G(x)$, and is defined as follows:

$$G(x) = (1-2x)\left[D(e^{2\pi ix}) + 2\pi x\log x - 2\pi(1-x)\log(1-x)\right].$$

Since $f(x) = F(x) + G(x)$, by using the linearity of integrals and their corresponding Riemann sums, the following equation holds for the errors:

$$E_d(f) = E_d(F) + E_d(G). \tag{V.1.4}$$

In the following we separate the computation into 2 parts, smooth and singular parts, and we use the associated Euler-Maclaurin summation formula. We notice that in the future computation for $E_d(G)$ and $E_d(F)$, Bernoulli numbers will appear. To see the definition of Bernoulli numbers, based on the convention in this thesis, $B_1 = \frac{1}{2}$, see Section VII.7.1 and Remark VII.7.4.

V.2. Applying the Euler-Maclaurin formula to a smooth function

In this section, we study $E_d(G) = \int_0^1 G(x) - S_d^T(G)$, where G is the smooth part of f defined as $G(x) = (1-2x)\left[D(e^{2\pi ix}) + 2\pi x\log x - 2\pi(1-x)\log(1-x)\right]$. One can easily check that $G(x) = G(1-x)$, thus we have: $G^{(n)}(x) = (-1)^n G^{(n)}(1-x)$, in particular $G^{(n)}(0) = (-1)^n G^{(n)}(1)$. By considering this relation and by applying the Euler-Maclaurin formula (see the formula VII.7.1 in Proposition VII.7.1 in Appendix) to $G(x)$ we have the following formula for $E_d(G)$:

$$E_d(G) = \int_0^1 G(x)dx - \frac{1}{d}\sum_{j=0}^{d} G(\frac{j}{d}) = 2\sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(0)\left(\frac{1}{d}\right)^{2\mu} + O(\frac{1}{d^{2m}}). \tag{V.2.1}$$

To estimate $E_d(G)$ we need to compute $G^{(\mu)}(0)$, which itself depends on the values of the successive derivatives of another function, called L, defined by $L(y) := \log \sum_{n=0}^{\infty} |\frac{(yi)^n}{(n+1)!}|$. Thus, we state the following lemma which will help us in our future computation:

Lemma V.2.1. *Let $\mu \in \mathbb{Z}_{\geq 1}$. The successive derivatives of $L(y)$ at zero are as follows:*

$$L^{(\mu)}(0) = \begin{cases} 0 & \text{if } 2 \nmid \mu, \\ \frac{(-1)^{\frac{\mu}{2}} B_\mu}{\mu} & \text{if } 2 \mid \mu, \end{cases} \tag{V.2.2}$$

where B_μ is the μ-th Bernoulli number.

Proof. According to Lemma V.1.6 we have:

$$L(y) = \log \sum_{n=0}^{\infty} |\frac{(yi)^n}{(n+1)!}| = \log |\frac{e^{iy}-1}{iy}| = \frac{1}{2}\log \frac{|e^{iy}-1|^2}{y^2} = \frac{1}{2}\log\left(\frac{\sin(\frac{y}{2})}{\frac{y}{2}}\right)^2 = \log \frac{\sin(\frac{y}{2})}{\frac{y}{2}}.$$

So first we compute the asymptotic expansion of $\log(\frac{\sin x}{x})$ and then, with a change of variables, we have the one of $L(y)$;

$$\begin{aligned} \log(\frac{\sin x}{x}) &\overset{[1]}{=} \log \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2\pi^2}\right) \\ &\overset{[2]}{=} -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{n\pi}\right)^{2k} \overset{[3]}{=} -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{k}\left(\frac{x}{n\pi}\right)^{2k} \\ &= -\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}} x^{2k}. \end{aligned}$$

We notice that in [1] we used the Weierstrass factorization of $(\frac{\sin x}{x})$. In [2] the power series converges when $|x| < \pi$. In [3], it is legal to change the order of summation since all terms have the same sign. Thus, using a change of variables, we have:

$$L(y) = \log\left(\frac{\sin(y/2)}{y/2}\right) = -\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\pi^{2k}}(\frac{y}{2})^{2k}. \tag{V.2.3}$$

Thus, by replacing the following equation, mentioned in Eq. (II.3.4):

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k}(2\pi)^{2k}}{2(2k)!}$$

the lemma is proved. ∎

In the following lemma we deduce the values of higher derivatives of G:

Lemma V.2.2. *We have $G'(0) = 4\pi + 6\pi \log 2\pi$ and the following equality holds for every $\mu \geq 2$:*

$$G^{(\mu)}(0) = 2\mu(2\pi)^{\mu-1} L^{(\mu-2)}(0) + 4\mu\pi(\mu-3)! - (2\pi)^{\mu} L^{(\mu-1)}(0) - 2\pi(\mu-2)!$$

Proof. To simplify the computation let, $I(x) := D(e^{2\pi i x}) + 2\pi x \log x - 2\pi(1-x)\log(1-x)$, so $G(x) = (1-2x)I(x)$. We have:

$$G^{(\mu)} = (-2\mu)I^{(\mu-1)} + (1-2x)I^{(\mu)}, \text{ and } G^{(\mu)}(0) = (-2\mu)I^{(\mu-1)}(0) + I^{(\mu)}(0).$$

Therefore, to compute $G^{(\mu)}(0)$ we compute the successive derivatives of I at zero and we notice that $I(0) = 0$. According to Lemma V.1.6 we have:

$$I'(x) = 4\pi - 2\pi\log 2\pi - 2\pi \log \sum_{n=0}^{\infty} |\frac{(2\pi x i)^n}{(n+1)!}| + 2\pi\log(1-x).$$

Thus, $I'(0) = 4\pi - 2\pi\log(2\pi)$. Moreover, according to the definition of L we have:

$$I'(x) = 4\pi - 2\pi\log 2\pi - 2\pi L(2\pi x) + 2\pi\log(1-x),$$

and

$$I^{(2)}(x) = -(2\pi)^2 L'(2\pi x) - \frac{2\pi}{(1-x)}.$$

Consequently, for $\mu \geq 2$ we have:

$$I^{(\mu)}(x) = -(2\pi)^{\mu} L^{(\mu-1)}(2\pi x) - \frac{2\pi(\mu-2)!}{(1-x)^{\mu-1}}, \quad I^{(\mu)}(0) = -(2\pi)^{\mu} L^{(\mu-1)}(0) - 2\pi(\mu-2)!.$$

Hence, $G'(0) = -2I(0) + I'(0) = 4\pi - 2\pi\log 2\pi$ and for $\mu \geq 2$ we have:

$$G^{(\mu)}(0) = 2\mu(2\pi)^{\mu-1}L^{(\mu-2)}(0) + 4\mu\pi(\mu-3)! - (2\pi)^{\mu}L^{(\mu-1)}(0) - 2\pi(\mu-2)!$$

□

We know that $E_d(G) = 2\sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} G^{(2\mu-1)}(0)\left(\frac{1}{d}\right)^{2\mu} + O(\frac{1}{d^{2m}})$. Using the previous lemma we are able to estimate $E_d(G)$.

Proposition V.2.3. *Let r_μ be as follows:*

(V.2.4)
$$r_\mu := \frac{2\pi B_{2\mu}(2\mu-4)!}{(2\mu)!}(2\mu + 1 + 2(2\mu-3)\zeta(2\mu-2)),$$

then we have:

$$E_d(G) = \frac{4\pi - 2\pi\log(2\pi)}{6d^2} + 2\sum_{\mu=2}^{m-1} r_\mu \left(\frac{1}{d}\right)^{2\mu} + O(\frac{1}{d^{2m}}).$$

Proof. To estimate $E_d(G)$, we need the values of $G^{2\mu-1}(0)$, for $\mu \geq 1$. Lemma V.2.2 shows that $G^{2\mu-1}(0)$ is a linear combination of the values of $L^{2\mu-3}(0)$ and $L^{2\mu-2}(0)$. According to Eq. (V.2.2), for all $\mu \geq 2$ we have $L^{2\mu-3}(0) = 0$, and $L^{2\mu-2}(0) = \frac{(-1)^{\mu-1}B_{2\mu-2}}{2\mu-2}$. After computing the values of $G^{2\mu-1}(0)$ and doing all the simplifications, we have :

(V.2.5)
$$E_d(G) = \frac{4\pi - 2\pi\log(2\pi)}{6d^2} + 2\sum_{\mu=2}^{m-1} \frac{2\pi B_{2\mu}(2\mu-4)!}{(2\mu)!}(2\mu + 1 + 2(2\mu-3)\zeta(2\mu-2))\left(\frac{1}{d}\right)^{2\mu} + O(\frac{1}{d^{2m}}).$$

We notice that all the coefficients of $\left(\frac{1}{d}\right)^{2\mu}$ in the summation in V.2.5 are rational numbers multiplying to some powers of 2π. This coefficient is denoted by r_μ (see Eq. (V.2.4)). We have:

$$E_d(G) = \frac{4\pi - 2\pi\log(2\pi)}{6d^2} + 2\sum_{\mu=2}^{m-1} r_\mu \left(\frac{1}{d}\right)^{2\mu} + O(\frac{1}{d^{2m}}).$$

□

According to Eq. (V.1.2), $m(P_d) = \frac{3}{2\pi}(\frac{(d+2)^2}{d+1}S^T_{d+2}(f) - \frac{(d+1)^2}{d+2}S^T_{d+1}(f))$ and $E_d(f) = E_d(G) + E_d(F)$. We compute the contribution of G to $m(P_d)$:

Lemma V.2.4. *We have the following equation;*

$$\begin{aligned}
&\frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2}E_{d+1}(G) - \frac{(d+2)^2}{d+1}E_{d+2}(G)\right)\\
&= \frac{\log(2\pi)-2}{2(d+1)(d+2)} + \frac{3}{(d+1)(d+2)}\sum_{\mu=2}^{m-1}\frac{r_\mu}{\pi}\left(\frac{1}{(d+1)^{2\mu-3}} - \frac{1}{(d+2)^{2\mu-3}}\right)\\
&+ O(\frac{1}{d^{2m-1}}).
\end{aligned} \tag{V.2.6}$$

Here r_μ is a rational combination of certain powers of 2π, defined in Eq. (V.2.4)*.*

Proof. The proof is only a computation using the previous proposition:

$$\begin{aligned}
&\frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2}E_{d+1}(G) - \frac{(d+2)^2}{d+1}E_{d+2}(G)\right) = \frac{\log(2\pi)-2}{2(d+1)(d+2)}\\
&+\frac{3}{2\pi}\frac{(d+1)^2}{d+2}\left(2\sum_{\mu=2}^{m-1} r_\mu\left(\frac{1}{d+1}\right)^{2\mu} + O(\frac{1}{(d+1)^{2m}})\right)\\
&-\frac{3}{2\pi}\frac{(d+2)^2}{d+1}\left(2\sum_{\mu=2}^{m-1} r_\mu\left(\frac{1}{d+2}\right)^{2\mu} + O(\frac{1}{(d+2)^{2m}})\right)\\
&= \frac{\log(2\pi)-2}{2(d+1)(d+2)} + \frac{3}{(d+1)(d+2)}\sum_{\mu=2}^{m-1}\frac{r_\mu}{\pi}\left(\frac{1}{(d+1)^{2\mu-3}} - \frac{1}{(d+2)^{2\mu-3}}\right) + O(\frac{1}{d^{2m-1}}).
\end{aligned}$$

□

V.3. Applying the extended Euler-Maclaurin formula to a function with singularities

In this section, we focus on the singular part of f which is $F(x) = -2\pi x\log x + 4\pi x^2\log x - 2\pi(1-x)\log(1-x) + 4\pi(1-x)^2\log(1-x)$. To simplify the computation, let $K(x) := -2\pi x\log x + 4\pi x^2\log x$, thus $F(x) = K(x) + K(1-x)$. We notice that to compute $E_d(F(x))$, we only need to compute $E_d(K(x))$. This is because $E_d(K(x)) = E_d(K(1-x))$, so $E_d(F(x)) = 2E_d(K(x))$ (do the change of variable $y = 1-x$, so we have $\int_0^1 K(1-x)dx = \int_0^1 K(y)dy$ also $\frac{1}{d}\sum_{j=0}^{d} K(1-\frac{j}{d}) = \frac{1}{d}\sum_{j=0}^{d} K(\frac{j}{d})$). Therefore, we apply the extended Euler-Maclaurin formula (see the formula in Proposition VII.7.7 in the Appendix) to the function $K(x)$ with logarithmic singularity. According to the terminology used in Proposition VII.7.7, the smooth part of $K(x)$ is $-2\pi x +$

$4\pi x^2$. Here, this smooth part is denoted by $h(x) : -2\pi x + 4\pi x^2$, and we have $h^{(j)} = 0$, for $j > 2$. Then, according to formula VII.7.1 in the Appendix, for $d \in \mathbb{Z}_{\geq 1}$ we have:

$$\begin{aligned} E_d(K) = &- \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} K^{2\mu-1}(1) \left(\frac{1}{d^{2\mu}}\right) \\ &+ \sum_{\mu=0}^{2m-1} [\zeta'(-\mu) + \zeta(-\mu)\log(d)] \frac{h^{\mu}(0)}{\mu!} \left(\frac{1}{d^{\mu+1}}\right) + O(\frac{1}{d^{2m}}), \end{aligned}$$

Notice that for $\mu > 2$ we have $h^{(\mu)}(0) = 0$. We replace the following information: $h(0) = 0$, $h'(0) = -2\pi$, $h''(0) = 8\pi$, $\zeta(-2) = 0$, $\zeta(-1) = \frac{-1}{12}$, and we get:

(V.3.1)

$$E_d(K) = - \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} K^{(2\mu-1)}(1) \left(\frac{1}{d^{2\mu}}\right) + \frac{\pi}{6}\frac{\log d}{d^2} + \frac{-2\pi\zeta'(-1)}{d^2} + \frac{4\pi\zeta'(-2)}{d^3} + O(\frac{1}{d^{2m}}).$$

In the following lemma we compute the $K^{(2\mu-1)}$ in order to estimate $E_d(K)$.

Lemma V.3.1. *We have for every $\mu \geq 3$:*

$$K^{(\mu)}(1) = 2\pi(-1)^{\mu-3}(\mu-3)!(\mu+2), \quad \textit{for } \mu \geq 3.$$

PROOF. The proof is just a simple computation of the derivatives of $K(x)$;

$$\begin{aligned} &K(x) = (-2\pi x + 4\pi x^2)\log x,\ K(1) = 0, \\ &K'(x) = (-2\pi + 8\pi x)\log x + (-2\pi + 4\pi x),\ K'(1) = 2\pi, \\ &K^{(2)}(x) = 8\pi \log x - \frac{2\pi}{x} + 12\pi, \\ &\vdots \\ &K^{(\mu)}(x) = 8\pi\frac{(-1)^{\mu-3}(\mu-3)!}{x^{\mu-2}} - 2\pi\frac{(-1)^{\mu-2}(\mu-2)!}{x^{\mu-1}} \quad \text{,for } \mu \geq 3. \end{aligned}$$

Thus, we have:

$$K^{(\mu)}(1) = 8\pi(-1)^{\mu-3}(\mu-3)! - 2\pi(-1)^{\mu-2}(\mu-2)! = 2\pi(-1)^{\mu-3}(\mu-3)!(\mu+2), \quad \text{for } \mu \geq 3.$$

□

Proposition V.3.2. *Let t_μ be defined as follows:*

(V.3.2)
$$t_\mu := \frac{2\pi B_{2\mu}(2\mu-4)!(2\mu+1)}{(2\mu)!},$$

then we have:

(V.3.3)
$$E_d(F) = -2\sum_{\mu=2}^{m-1} t_\mu \left(\frac{1}{d^{2\mu}}\right) + \frac{\pi}{3}\frac{\log d}{d^2} - \frac{\pi + 12\pi\zeta'(-1)}{3d^2} + \frac{8\pi\zeta'(-2)}{d^3} + O(\frac{1}{d^{2m}}).$$

PROOF. To prove the above equality, it is sufficient to replace the values of $K^{(2\mu-1)}(1)$ in Eq. (V.3.1) from Lemma V.3.1. Since $B_2 = 1/6$, we have:

$$\begin{aligned} E_d(K) = &- \sum_{\mu=2}^{m-1} \frac{B_{2\mu}}{(2\mu)!} 2\pi(2\mu-4)!(2\mu+1) \left(\frac{1}{d^{2\mu}}\right) \\ &+ \frac{\pi}{6}\frac{\log d}{d^2} + \frac{-\pi - 12\pi\zeta'(-1)}{6d^2} + \frac{4\pi\zeta'(-2)}{d^3} + O(\frac{1}{d^{2m}}). \end{aligned}$$

We also replace $E_d(F) = 2E_d(K)$ and the notation t_μ, introduced in Eq. (V.3.2) in the formula of $E_d(F)$ and the proposition is proved. $\square$

The contribution of F to $m(P_d)$ is given by the following lemma:

Lemma V.3.3. *We have the following equality:*

$$\frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2}E_{d+1}(F) - \frac{(d+2)^2}{d+1}E_{d+2}(F)\right)$$
$$= \frac{-\log d}{2(d+1)(d+2)} + \frac{12\zeta'(-1)}{2(d+1)(d+2)} - \frac{1}{2(d+1)(d+2)}\sum_{j=1}^{2m-3}\frac{1}{d^j}\frac{(-1)^{j+1}(2^{j+1}-1)}{j(j+1)}$$
$$+\frac{3}{(d+1)(d+2)}\sum_{\mu=2}^{m-1}\frac{t_\mu}{\pi}\left(\frac{1}{(d+2)^{2\mu-3}} - \frac{1}{(d+1)^{2\mu-3}}\right) + O(\frac{1}{d^{2m-1}}).$$

PROOF. The proof is only a computation:

$$\frac{3}{2\pi}\left(\frac{(d+1)^2}{d+2}E_{d+1}(F) - \frac{(d+2)^2}{d+1}E_{d+2}(F)\right) =$$
$$\frac{1}{2}\frac{(d+1)\log(d+1)-(d+2)\log(d+2)}{(d+2)(d+1)} + \frac{1+12\zeta'(-1)}{2(d+1)(d+2)}$$
$$+\frac{3}{(d+1)(d+2)}\sum_{\mu=2}^{m-1}\frac{t_\mu}{\pi}\left(\frac{1}{(d+2)^{2\mu-3}} - \frac{1}{(d+1)^{2\mu-3}}\right) + O(\frac{1}{d^{2m-1}}).$$

After doing the computation and replacing the power series of $\log(1+x) = \sum_{j\geq 1}\frac{(-1)^{j+1}x^j}{j}$ (for x in the neighborhood of zero) in the logarithmic terms in the R.H.S of the above equality, we have:

$$\frac{(d+1)\log(d+1)-(d+2)\log(d+2)}{2(d+2)(d+1)} = -\frac{\log d + 1 + \sum_{j\geq 1}(\frac{1}{d})^j\frac{(-1)^{j+1}(2^{j+1}-1)}{j(j+1)}}{2(d+1)(d+2)}.$$

We replace the above equality in the previous one and after a computation the lemma is proved.

$\square$

V.4. The rate of the convergence and the asymptotic behavior of $m(P_d)$

In this section we compute the asymptotic expansion of $|m(P_d) - m(P_\infty)|$ using the results obtained in the previous sections. This asymptotic expansion indeed gives the rate of convergence of $(m(P_d))_{d\in\mathbb{Z}_{\geq 1}}$.

According to Eq. (V.1.3), Lemma V.2.4 and Lemma V.3.3 we conclude:

Theorem V.4.1 ([BGMP22], Theorem 5.1)**.** *The asymptotic expansion of $m(P_d) - m(P_\infty)$ is as follows:*

$$\text{(V.4.1)}\quad m(P_d)-m(P_\infty) = \frac{1}{(d+1)(d+2)}\left[-\frac{\log(d)}{2} + \sum_{j=0}^{2m-3}\frac{\alpha_j}{d^j}\right] + O(\frac{1}{d^{2m-1}})\quad \textit{for all}\quad m\geq 2.$$

where the coefficients $\alpha_j \in \mathbb{R}$ are defined as:

$$\alpha_0 := 6\big(\zeta'(-1) - \zeta'(-2)\big) + \frac{\log(2\pi)}{2} - 1$$

$$\alpha_j := \frac{12 \cdot (-1)^j}{j(j+1)} \sum_{t=0}^{\lfloor j/2 \rfloor} \binom{j+1}{2t} \cdot \frac{(2^{j+1-2t}-1)(2t-1)}{(2t+1)(2t+2)} \cdot B_{2t+2} \cdot \zeta(2t) \qquad (j \geq 1).$$

Here, B_n denotes the n-th Bernoulli number.

Proof. According to Eq. (V.1.3) and the equations, obtained in Lemmas V.2.4 and V.3.3, and the equation $\zeta'(-2) = \frac{-\zeta(3)}{4\pi^2}$ (see Eq. (II.3.5)) we have the following equality:

$$\begin{aligned} m(P_d) &= \frac{9\zeta(3)}{2\pi^2} - \frac{\log d}{2(d+1)(d+2)} + \frac{\log(2\pi) - 2 + 12\zeta'(-1) - 12\zeta'(-2)}{2(d+1)(d+2)} \\ &+ \frac{1}{2(d+1)(d+2)} \sum_{j=1}^{2m-3} \frac{(-1)^j(2^{j+1}-1)}{j(j+1)} \frac{1}{d^j} \\ &+ \frac{3}{(d+1)(d+2)} \sum_{\mu=2}^{m-1} \frac{(r_\mu - t_\mu)}{\pi} \left(\frac{1}{(d+1)^{2\mu-3}} - \frac{1}{(d+2)^{2\mu-3}} \right) + O(\frac{1}{d^{2m-1}}), \end{aligned}$$

where according to Eq. (V.2.4) and Eq. (V.3.2) we have:

$$\frac{r_\mu - t_\mu}{\pi} = \frac{2B_{2\mu}\zeta(2\mu-2)}{\mu(2\mu-1)(2\mu-2)}.$$

In the sequel of the computation the goal is to find the coefficient of $(\frac{1}{d})^j$. In the above formula for $m(P_d)$ we have two power series on $\frac{1}{d}$. For $j \in \mathbb{Z}_{\geq 1}$, sum of the coefficient of $(\frac{1}{d})^j$ in the two series gives the coefficient of $(\frac{1}{d})^j$ in $m(P_d)$. We start by simplifying the second series. One can verify that the following equality holds:

$$\sum_{\mu=2}^{m-1} \left(\frac{1}{(d+1)^{2\mu-3}} - \frac{1}{(d+2)^{2\mu-3}} \right) = \sum_{\mu=2}^{m-1} (2\mu-3) \left(\sum_{1 \leq k \leq 2m-2\mu} \frac{\binom{k+2\mu-4}{2\mu-3}(-1)^{k+1}(2^k-1)}{kd^{k+2\mu-3}} \right) + O(\frac{1}{d^{2m-1}}).$$

Let $a_{2\mu} := \frac{B_{2\mu}\zeta(2\mu-2)}{2\mu(2\mu-1)(2\mu-2)}$. We have:

$$\begin{aligned} m(P_d) - \frac{9\zeta(3)}{2\pi^2} &= -\frac{\log d}{2(d+1)(d+2)} + \frac{\log(2\pi) - 2 + 12\zeta'(-1) - 12\zeta'(-2)}{2(d+1)(d+2)} \\ &+ \frac{1}{2(d+1)(d+2)} \sum_{j=1}^{2m-3} \frac{(-1)^j(2^{j+1}-1)}{j(j+1)} \frac{1}{d^j} \\ &+ \frac{12}{(d+1)(d+2)} \sum_{\mu=2}^{m-1} \sum_{k=1}^{2m-2\mu} a_{2\mu}(2\mu-3) \frac{\binom{k+2\mu-4}{2\mu-3}(-1)^{k+1}(2^k-1)}{kd^{k+2\mu-3}} + O(\frac{1}{d^{2m-1}}), \end{aligned}$$

Let $j := k + 2\mu - 3$, so we have:

(V.4.2)

$$\sum_{\mu=2}^{m-1} \sum_{k=1}^{2m-2\mu} a_{2\mu}(2\mu-3) \frac{\binom{k+2\mu-4}{2\mu-3}(-1)^{k+1}(2^k-1)}{kd^{k+2\mu-3}} = \sum_{j=2}^{2m-3} \sum_{k=1}^{j-1} a_{j-k+3}(j-k) \frac{\binom{j-1}{j-k}(-1)^{k+1}(2^k-1)}{kd^j}$$

We notice that the number of indexes of the two series in the left and right hand sides of the above equation are not equal, but this is not actually a problem, since according to Fact VII.7.3 in the Appendix the odd Bernoulli numbers (except B_1) are equal to zero. For instance consider the index $(j,k)=(3,1)$ in the series in the R.H.S. There is no index (μ,k) on the L.H.S which under the change of variables $j=k+2\mu-3$ maps to $(3,1)$. However, the coefficient associated with this index in the R.H.S is multiplied by B_5, which is zero and this resolves the problem. Note that applying the change of variables $j=k+2\mu-3$ gives $(-1)^{k+1}=(-1)^{j-2\mu+4}=(-1)^j$. Then, we replace $(-1)^{k+1}$ by $(-1)^j$ in Eq. (V.4.2) and since the index of the first series is k, $(-1)^j$ can be written outside of the first series. Then, after simplifying the R.H.S of the above equality we have:

$$\begin{aligned}m(P_d)-\frac{9\zeta(3)}{2\pi^2}&=-\frac{\log d}{2(d+1)(d+2)}+\frac{\log(2\pi)-2+12\zeta'(-1)-12\zeta'(-2)}{2(d+1)(d+2)}\\&+\frac{1}{2(d+1)(d+2)}\sum_{j=1}^{2m-3}\frac{(-1)^j(2^{j+1}-1)}{j(j+1)}\frac{1}{d^j}\\&+\frac{12}{(d+1)(d+2)}\sum_{j=2}^{2m-3}(-1)^j\sum_{k=1}^{j-1}a_{j-k+3}\binom{j-1}{j-k}\frac{(2^k-1)(j-k)}{k}(\frac{1}{d})^j+O(\frac{1}{d^{2m-1}}).\end{aligned}$$

In the series in the third line of the above equality we can start the index of j from 1 since, for $j=1$ the index k of the interior summation will be $1\le k\le 0$ which means the summation is empty and equals zero. Let us fix the value of j and using the second and third summations in the above formula, which have both index j between 1 to $2m-3$. We replace the value of a_{j-k+3} in the above equation and compute the coefficient of $\frac{1}{(d+1)(d+2)}\frac{1}{d^j}$:

(V.4.3)

$$\frac{(-1)^j(2^{j+1}-1)}{2j(j+1)}+\underbrace{12(-1)^j\sum_{k=1}^{j-1}\frac{B_{j-k+3}\zeta(j-k+1)}{(j-k+3)(j-k+2)(j-k+1)}\binom{j-1}{j-k}\frac{(2^k-1)(j-k)}{k}}_{\dagger}.$$

In the second summation above, the Bernoulli numbers with odd index are equal to zero. Thus, we do another change of variables $2t:=j-k+1$ and we have:

$$\begin{aligned}\dagger&=12(-1)^j\sum_{t=1}^{[j/2]}\frac{B_{2t+2}\zeta(2t)}{(2t+2)(2t+1)(2t)}\binom{j-1}{2t-1}\frac{(2^{j-2t+1}-1)(2t-1)}{(j-2t+1)}\\&=\frac{12(-1)^j}{j(j+1)}\sum_{t=1}^{[j/2]}\frac{B_{2t+2}\zeta(2t)}{(2t+2)(2t+1)(2t)}\binom{j+1}{2t}\frac{(2^{j-2t+1}-1)(2t-1)(2t)(j+1-2t)}{(j-2t+1)}\\&=\frac{12(-1)^j}{j(j+1)}\sum_{t=1}^{[j/2]}\frac{B_{2t+2}\zeta(2t)}{(2t+2)(2t+1)}\binom{j+1}{2t}(2^{j-2t+1}-1)(2t-1).\end{aligned}$$

In the above summation, the index t starts from 1, but let us put $t=0$ in that summation, then according to the fact that $\zeta(0)=\frac{-1}{2}$ and $B_2=\frac{1}{6}$ we have $\frac{(-1)^j(2^{j+1}-1)}{2j(j+1)}$, which is exactly the first term in the summation V.4.3. Therefore to simplify the computation, instead of the summation V.4.3 we can write:

$$\frac{12(-1)^j}{j(j+1)}\sum_{t=0}^{[j/2]}\frac{B_{2t+2}\zeta(2t)}{(2t+2)(2t+1)}\binom{j+1}{2t}(2^{j-2t+1}-1)(2t-1).$$

Therefore, for the asymptotic expansion, we have:

$$
\begin{aligned}
m(P_d) - \frac{9\zeta(3)}{2\pi^2} &= -\frac{\log d}{2(d+1)(d+2)} + \frac{\log(2\pi) - 2 + 12\zeta'(-1) - 12\zeta'(-2)}{2(d+1)(d+2)} \\
&+ \frac{1}{(d+1)(d+2)} \sum_{j=1}^{2m-3} \frac{12(-1)^j}{j(j+1)} \sum_{t=0}^{[j/2]} \frac{B_{2t+2}\zeta(2t)}{(2t+2)(2t+1)} \binom{j+1}{2t} (2^{j-2t+1} - 1)(2t-1) \frac{1}{d^j} \\
&+ O(\frac{1}{d^{2m-1}}).
\end{aligned}
$$

In the above equation, let the coefficient of $\frac{1}{d^j}$ be denoted by α_j. Consider the coefficient $\frac{\log(2\pi)-2+12\zeta'(-1)-12\zeta'(-2)}{2(d+1)(d+2)}$ as α_0. Then, by factoring $\frac{1}{(d+1)(d+2)}$ from the coefficients we have Eq. (V.4.1). □

Remark V.4.2. The asymptotic expansion Eq. (V.4.1) has been checked numerically using the PARI/GP program Asympraw available at [BC21]. Moreover, in Fig. 2 we present the graph of $(m(P_\infty) - m(P_d)) \frac{d^2}{\log d}$, for $1 \leq d \leq 1000$, implemented by SageMath.

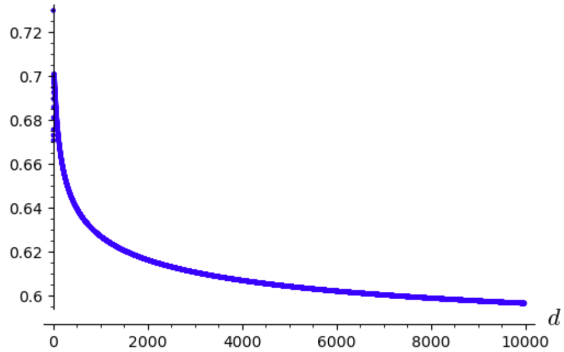

FIGURE 2. The graph of $(m(P_\infty) - m(P_d)) \frac{d^2}{\log d}$, for $1 \leq d \leq 1000$.

A direct result of the previous theorem is as follows:

Corollary V.4.3.

$$m(P_d) - m(P_\infty) = -\frac{\log(d)}{2(d+1)(d+2)} + O(\frac{1}{d^2}).$$

Of course, one can considerably simplify the computation if we are only interested in the rate of the convergence.

In the last section of Chapter II we noticed that in [BGMP22, Theorem 4.1] we provide an upper bound for the error terms in the generalization of the theorem of Boyd-Lawton. The error bound in certain cases is optimal, but for P_d our direct computation implies that $|m(P_d) - m(P_\infty)|$ is of $O(\frac{\log d}{d^2})$, which is a better error bound.

CHAPTER VI

Mahler measure of P_d and L-functions

This chapter is an ongoing project in collaboration with Marie-José Bertin. In Chapter II we have seen some important examples of computations of the Mahler measure connected to special values of L-function. For instance, in Corollary II.3.39 we have seen the evaluation of the Mahler measure of $P(x, y) = x + y + 1$ by Smyth:

$$m(x+y+1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1). \tag{VI.0.1}$$

Here χ_{-3} is the odd quadratic character of modulus 3 (for more information about χ_{-3} see Example II.3.23). In fact $P(x, y) = x + y + 1$ is the polynomial P_1 in the P_d family. We are interested in finding some links between special values of L-functions and the Mahler measure of other polynomials in the P_d family. As we have already mentioned, understanding the link between the values of the Mahler measure of polynomials and special values of L-functions is one of the applications of the Mahler measure in Number Theory. Moreover, it may help progress towards Chinburg's conjecture:

Conjecture. (Chinburg's conjecture [Ray87, Page 697]) For every odd quadratic character $\chi_{-f} := \left(\frac{-f}{\cdot}\right)$, there exists a non-zero polynomial $P_f(x, y)$ with integer coefficients, for which $\frac{m(P_f)}{L'(\chi_{-f},-1)}$ is a rational number.

Smyth's example VI.0.1 satisfies the case $f = 3$. Ray [Ray87] was able to construct polynomials $P_f(x, y)$, for $f = 3, 4, 7, 8, 20$ and 24. Boyd and Rodriguez-Villegas [BRVD03] constructed examples for $f = 3, 4, 7, 8, 11, 15, 20, 24, 35, 39, 55$ and 84. Recently in [HL19] examples for the conductors $f = 23, 303, 755$ is provided. The list of the results that we mentioned above of course does not cover all the result around this conjecture and there are many other mathematicians who have worked and have results on the Chinburg conjecture. In this chapter, using P_1 and P_2 we introduce another polynomial for the case $f = 4$. We believe that the systematic study of $m(P_d)$ can lead to other new results in this direction.
Moreover, we prove that $m(P_d)$, for every d, can be written as a linear combination of L-functions with coefficients in number fields.

Theorem VI.0.1. *Let $d \in \mathbb{Z}_{\geq 1}$, for every odd primitive Dirichlet character χ of conductor k, such that $k|(d+1)(d+2)$, there exists a coefficient $C_{k,\chi} \in \mathbb{Q}(e^{\frac{2\pi i}{\phi(k)}}) \subset \mathbb{Q}(e^{\frac{2\pi i}{\phi((d+1)(d+2))}})$ such that:*

$$m(P_d) = \sum_{k|(d+1)(d+2)} \sum_{\chi \text{ odd primitive mod } k} C_{k,\chi} L'(\chi, -1).$$

We present here the explicit representation for $1 \leq d \leq 6$. In our future work, we will write an algorithm to compute $m(P_d)$ in terms of L-functions, for every d. Because of the properties of the dilogarithm this representation is not necessarily unique. We will discuss this in the last section of this chapter.

We notice that in all this chapter the information related to Dirichlet characters such as parity, conductor, or being primitive or imprimitive is determined using [LMF22].

VI.1. L-functions of quadratic characters and the Mahler measure of P_d

In Section II.3.1, we saw the definition and some important properties of Dirichlet characters, specially the quadratic characters. According to Corollary II.3.36, if $-f<0$ is a fundamental discriminant, then for the odd quadratic Dirichlet character of conductor f, denoted by $\chi_{-f}(n)=\left(\frac{-f}{n}\right)$ we have;

$$L'(\chi_{-f},-1)=\frac{f^{\frac{3}{2}}}{4\pi}L(\chi_{-f},2).$$

Notation VI.1.1. Boyd [Boy98] used the following notation:

$$d_f:=L'(\chi_{-f},-1)=\frac{f^{\frac{3}{2}}}{4\pi}L(\chi_{-f},2).$$

Grayson [Gra81] in his computations (suggested by Bloch) proved that we can express $L'(\chi_{-f},-1)$ or equivalently $L(\chi_{-f},2)$ directly in terms of dilogarithm at certain roots of unity:

Proposition VI.1.2 ([Gra81])**.** *Let $-f$ be a fundamental discriminant and $\chi_{-f}:=\left(\frac{-f}{\cdot}\right)$ be the odd quadratic Dirichlet character of conductor f. Then we have:*

$$\frac{f^{\frac{3}{2}}}{4\pi}L(\chi_{-f},2)=L'(\chi_{-f},-1)=\frac{f}{4\pi}\sum_{n=1}^{f}\chi_{-f}(n)D(\zeta_f^n),$$

where ζ_f is a primitive f-th root of unity.

On the other side in Notation V.1.2, we expressed $m(P_d)$ as sum of the values of dilogarithm at certain roots of unity, which we recall here:

$$2\pi m(P_d)=\frac{1}{d+1}S_{d+2}-\frac{1}{d+2}S_{d+1},\quad \text{with}\quad S_d:=3\sum_{1\le k\le d-1}(d-2k)D((e^{\frac{2\pi}{d}i})^k).$$

Therefore, one may ask about the possibility of writing $m(P_d)$ as sum of $L'(-1,\chi_{-f})$ for some f. We have seen the computation of Smyth [Smy81b]. The next proposition recovers the example of Smyth, mentioned in Corollary II.3.39, using the closed formula for $m(P_1)$.

Proposition VI.1.3. *We have the following equality:*

$$m(P_1)=L'(\chi_{-3},-1)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2).$$

Proof. By applying the formula of $m(P_d)$ introduced in Notation V.1.2, for $d=1$ we have $m(P_1)=\frac{1}{\pi}D(e^{\frac{\pi}{3}i})$. Let us recall about χ_{-3} introduced in Example II.3.23.

m	0	1	2
$\chi_{-3}(m)$	0	1	-1

Then, according to Proposition VI.1.2 for χ_{-3} we have:

$$\begin{aligned}L'(\chi_{-3},-1)&=\frac{3}{4\pi}(\chi_{-3}(1)D(e^{\frac{2\pi}{3}i})+\chi_{-3}(2)D(e^{\frac{4\pi}{3}i}))=\frac{3}{4\pi}(D(e^{\frac{2\pi}{3}i})-D(e^{\frac{4\pi}{3}i}))=\frac{3}{4\pi}(2D(e^{\frac{2\pi}{3}i}))\\&=\frac{3}{2\pi}(D(e^{\frac{2\pi}{3}i})).\end{aligned}$$

In order to complete the proof, we use the distribution relation for dilogarithm (see Fact III.1.8), which is $D(z^n) = n\sum_{j=0}^{n-1} D((e^{\frac{2\pi i}{n}})^j z)$. We get:

$$D(e^{\frac{2\pi}{3}i}) = D((e^{\frac{\pi}{3}i})^2) = 2(D(-e^{\frac{\pi}{3}i}) + D(e^{\frac{\pi}{3}i})) \overset{[1]}{=} 2(D(e^{\frac{4\pi}{3}i}) + D(e^{\frac{\pi}{3}i})).$$

In [1] we use $-e^{\frac{\pi}{3}i} = e^{i\pi}e^{\frac{\pi}{3}i} = e^{\frac{4\pi}{3}i}$. As $e^{\frac{4\pi}{3}i} = \overline{e^{\frac{2\pi}{3}i}}$, we have $D(e^{\frac{2\pi}{3}i}) = 2(-D(e^{\frac{2\pi}{3}i}) + D(e^{\frac{\pi}{3}i}))$, which implies:

$$D(e^{\frac{2\pi}{3}i}) = \frac{2}{3}D(e^{\frac{\pi}{3}i}). \tag{VI.1.1}$$

This completes the proof. $\square$

We can proceed our analysis with P_2:

Proposition VI.1.4. *We have the following equalities:*

$$m(P_2) = L'(\chi_{-4}, -1) - \frac{L'(\chi_{-3}, -1)}{2} = \frac{2}{\pi}L(\chi_{-4}, 2) - \frac{3\sqrt{3}}{8\pi}L(\chi_{-3}, 2).$$

Proof. By applying the formula for $m(P_d)$ for $d = 2$ we have $m(P_2) = \frac{1}{2\pi}\left(\frac{1}{3}S_4 - \frac{1}{4}S_3\right) = \frac{1}{2\pi}(\frac{3D(e^{\frac{4\pi}{3}i})}{2} + 4D(e^{\frac{\pi}{2}i}))$. According to the computation done in Proposition VI.1.3 we have $\frac{1}{2\pi}(\frac{3D(e^{\frac{4\pi}{3}i})}{2}) = \frac{-L'(\chi_{-3},2)}{2}$. Consider the character χ_{-4}, introduced in Example II.3.26:

m	0	1	2	3
$\chi_{-4}(m)$	0	1	0	-1

Then, we compute $L'(\chi_{-4}, -1)$ as follows:

$$\begin{aligned} L'(\chi_{-4}, -1) &= \frac{4}{4\pi}\left(\chi_{-4}(1)D(e^{\frac{2\pi}{4}i}) + \chi_{-4}(2)D(e^{\frac{4\pi}{4}i}) + \chi_{-4}(3)D(e^{\frac{6\pi}{4}i})\right) \\ &= \frac{1}{\pi}\left(D(e^{\frac{2\pi}{4}i}) - D(e^{\frac{6\pi}{4}i})\right) = \frac{2}{\pi}D(i). \end{aligned}$$

Therefore, we have:

$$m(P_2) = L'(\chi_{-4}, -1) - \frac{L'(\chi_{-3}, -1)}{2}.$$

Then, by applying Corollary II.3.36 one can prove the second equality announced in the proposition. $\square$

Grayson in [Gra81, Page 699] gives a method for producing a polynomial associated to f answering Chinburg conjecture, for $f = 3, 4, 7, 8, 20$ and 24. By applying his method for $f = 4$, the polynomial that we get is $y^2x^2 + 2yx^2 + y^2 + x^2 - 2y + 1$. Here, using the link between the Mahler measure of P_d polynomials and L-functions, we give another candidate for $f = 4$.

Proposition VI.1.5. *Let $f = 4$, then the polynomial $P_2^2P_1 \in \mathbb{Z}[x, y]$ verifies:*

$$\frac{m(P_2^2P_1)}{L'(\chi_{-4}, -1)} = 2.$$

Proof. Propositions VI.1.3 and VI.1.4 imply that $m(P_2) = L'(\chi_{-4}, -1) - \frac{m(P_1)}{2}$. Hence, we have $m(P_2^2P_1) = 2m(P_2) + m(P_1) = 2L'(\chi_{-4}, -1)$.

$\square$

VI.2. Connections between L-functions with non-real primitive characters and the dilogarithm

We tried to write $m(P_3)$ in terms of L-functions associated with real quadratic Dirichlet character and failed. Indeed we have $m(P_3) = \frac{1}{2\pi}\left(\frac{1}{2}\left(9D(e^{i\frac{2\pi}{5}}) + 3D(e^{i\frac{4\pi}{5}})\right)\right) + \frac{-6}{5\pi}D(i)$, written as the combination of dilogarithm at 4-th and 5-th roots of unity, but -5 is not a fundamental discriminant. In order to solve the problem we are going to write $m(P_3)$ in terms of any possible primitive odd (non principal) Dirichlet characters, not necessarily real. In order to write $m(P_d)$ in terms of values of L or L' we search for an analogue of Proposition VI.1.2 for an arbitrary primitive odd (non principal) Dirichlet character. We notice that, contrary to the real (quadratic) case, we may have more than 1 primitive non principal odd Dirichlet character of conductor f. We recall Proposition II.3.35 which is a classical result on the connection between L and L':

For χ an odd primitive Dirichlet character of conductor k we have:

$$L(\chi, 2) = \frac{4\pi}{ik^2}\tau(\chi)L'(\bar{\chi}, -1).$$

Here, $\tau(\chi) = \sum_{1\leq a\leq k} \chi(a)e^{\frac{2\pi ia}{k}}$.

The first step for obtaining the analogue of Proposition VI.1.2 is the following result, announced in [BRV02, Equation 11]: [1]

Proposition VI.2.1. *Let χ be a complex odd primitive Dirichlet character of conductor k. We have:*

$$L(\chi, 2) = i\tau(\bar{\chi})^{-1} \sum_{1\leq a\leq k} \overline{\chi(a)} D(e^{\frac{2\pi ia}{k}}). \tag{VI.2.1}$$

Proof. The general relation between the polylogarithm $Li_m(z)$ and $L(\chi, m)$ [ZG00, Page 9], is given by:

For χ a primitive Dirichlet character of conductor k, $L(\chi, m) = \tau(\bar{\chi})^{-1} \sum_{1\leq a\leq k} \overline{\chi(a)} Li_m(e^{\frac{2\pi ia}{k}})$.

Using the definition of the Bloch Wigner dilogarithm, $D(z) = Im(Li_2(z)) + \arg(1-z)\log|z|$ and since $\log|e^{\frac{2\pi ia}{k}}| = 0$ we have:

$$D(e^{\frac{2\pi ia}{k}}) = Im(Li_2(e^{\frac{2\pi ia}{k}})) = \frac{Li_2(e^{\frac{2\pi ia}{k}}) - \overline{Li_2(e^{\frac{2\pi ia}{k}})}}{2i}.$$

[1] There is a typographical error in [BRV02, Equation 11]: since the character is odd, there should be a coefficient i.

We can now compute the R.H.S of Eq. (VI.2.1) for $m = 2$ and χ a primitive odd Dirichlet character of conductor k:

$$
\begin{aligned}
i\tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)} D(e^{\frac{2\pi i a}{k}}) &= i\tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)} \left(\frac{Li_2(e^{\frac{2\pi i a}{k}}) - \overline{Li_2(e^{\frac{2\pi i a}{k}})}}{2i} \right) \\
&= \frac{1}{2} \left(\tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)} Li_2(e^{\frac{2\pi i a}{k}}) - \tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)}\, \overline{Li_2(e^{\frac{2\pi i a}{k}})} \right) \\
&\overset{[1]}{=} \frac{1}{2} \left(\tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)} Li_2(e^{\frac{2\pi i a}{k}}) + \overline{\tau(\chi)^{-1}} \overline{\sum_{1 \leq a \leq k} \chi(a) Li_2(e^{\frac{2\pi i a}{k}})} \right) \\
&= \frac{1}{2} \left(L(\chi, 2) + \overline{L(\overline{\chi}, 2)} \right) \overset{[2]}{=} L(\chi, 2).
\end{aligned}
$$

In [1] we used the fact that $\tau(\bar{\chi}) = \chi(-1)\overline{\tau(\chi)}$, which can be verified as follows;

$$
\tau(\bar{\chi}) = \sum_{a \bmod k} \bar{\chi}(a) e^{\frac{2\pi i a}{k}} = \overline{\sum_{a \bmod k} \chi(a) e^{\frac{-2\pi i a}{k}}} \overset{[3]}{=} \overline{\tau(\chi)} \chi(-1).
$$

In [3], we used Lemma II.3.31, for $n = -1$, and since χ is an odd Dirichlet character, so we have $\chi(-1) = -1$. For [2] we used that $\overline{L(\chi, s)} = L(\bar{\chi}, s)$, since s is real. $\square$

Recall now from Proposition II.3.35 that $L(\chi, 2) = \frac{4\pi}{ik^2}\tau(\chi)L'(\bar{\chi}, -1)$. Hence, we have $L(\chi, 2) = \frac{4\pi}{ik^2}\tau(\chi)L'(\bar{\chi}, -1) = i\tau(\bar{\chi})^{-1} \sum_{1 \leq a \leq k} \overline{\chi(a)} D(e^{\frac{2\pi i a}{k}})$, or equivalently:

$$
L(\bar{\chi}, 2) = \frac{4\pi}{ik^2}\tau(\bar{\chi})L'(\chi, -1) = i\tau(\chi)^{-1} \sum_{1 \leq a \leq k} \chi(a) D(e^{\frac{2\pi i a}{k}}).
$$

The following lemma is a generalization of Lemma II.3.32 to every primitive character:

Lemma VI.2.2 ([Lan13], Page 84). *If χ is a primitive Dirichlet character of conductor k, then*

$$
|\tau(\chi)|^2 = k.
$$

This leads to the following:

Corollary VI.2.3 ([Ray87], Page 697). *For a primitive odd (non principal) Dirichlet character of conductor k, we have:*

$$
L'(\chi, -1) = \frac{k}{4\pi} \sum_{m=1}^{k-1} \chi(m) D({\zeta_k}^m) = \frac{-ik\tau(\chi)}{4\pi} L(\bar{\chi}, 2).
$$

We notice that if χ is an odd quadratic character χ_{-f}, then the above equation yields the equation mentioned in Proposition VI.1.2 (see Corollary II.3.36 for more information).

We extend Boyd's notation to:

Notation VI.2.4. Let χ be an odd Dirichlet character of modulus k, then:

$$
d_\chi := \frac{k}{4\pi} \sum_{m=1}^{k-1} \chi(m) D({\zeta_k}^m).
$$

According to Corollary VI.2.3, if χ is an odd primitive character d_χ is $L'(\chi,-1)$. Indeed, if $k=f$, where $-f$ is a fundamental discriminant, then $\chi(m)=\chi_{-f}(m)=\left(\frac{-f}{m}\right)$ is the quadratic Dirichlet character of conductor f. In this case d_f and d_χ are the same number.
We have the following classical proposition about induced Dirichlet characters that will be useful for the proof of Theorem VI.0.1. For the definition of induced character, see Definition II.3.7.

Proposition VI.2.5 ([Ove14], Page 91)**.** *Let χ be a Dirichlet character modulus k with conductor c. Then there exists a unique primitive character χ^* modulus c that induces χ. Moreover, χ is odd if and only if χ^* is odd.*

Proof. Since c is the conductor of χ, Lemma II.3.10 implies that $c|k$. Let $n\in\mathbb{Z}$. If $\gcd(n,k)=1$, then $\gcd(n,c)=1$, and we define $\chi^*(n)$ to be $\chi(n)$. If $\gcd(n,k)>1$, but $\gcd(c,k)=1$, then we chose any $t\in\mathbb{Z}$ for which $\gcd(n+tc,k)=1$ and we define $\chi^*(n)=\chi(n+tc)$. Note that such an integer exists, for it suffices to have $\gcd(n+tc,r)=1$, where $r:=\prod_{\substack{p^a|k\\ p\nmid c}}p^a$. Moreover, we note that although there are many possible choices of t, there is only one value of $\chi(n+tc)$, when $\gcd(n+tc,k)=1$. We extend this definition of χ^* by defining $\chi^*(n)=0$, when $\gcd(n,c)>1$. Then χ^* is a Dirichlet character modulus c. If χ_0 denotes the principal character modulus k, then for every $n\in\mathbb{Z}_{\geq1}$ we have $\chi(n)=\chi_0(n)\chi^*(n)$, so χ^* induces χ. It is clear that χ^* has no quasi period less than c, since otherwise so would χ, which contradicts minimality. Moreover, according to the equation $\chi(n)=\chi_0(n)\chi^*(n)$ we have χ is odd if and only if χ^* is odd. □

An important remark is that $d_\chi=d_{\chi^*}$:

Lemma VI.2.6. *Let χ be a character of modulus k and conductor c, which is induced by the primitive Dirichlet character χ^* (of conductor c). Then, we have:*

$$d_{\chi^*}=d_\chi.$$

Proof. Since the conductor of χ is c, and $c|k$, we suppose that $k=qc$, so we have:

$$\begin{aligned}d_\chi=\frac{k}{4\pi}\sum_{j=1}^{k-1}\chi(j)D(\zeta_k{}^j)&\overset{[1]}{=}\frac{k}{4\pi}\sum_{0\leq l<c}\ \sum_{0\leq m<q}\chi(l+mc)D((e^{\frac{2\pi i}{k}})^{l+mc})\\&\overset{[2]}{=}\frac{k}{4\pi}\sum_{0\leq l<c}\chi^*(l)\sum_{0\leq m<q}D(e^{\frac{2\pi li}{k}}e^{\frac{2\pi mi}{q}}).\end{aligned}$$

In [1], we used that $[0,k-1]=\bigcup_{0\leq l<c}\{l+mc\mid 0\leq m<q\}$. In [2], we used the equality $\chi(l+mc)=\chi^*(l)$. The distribution relation for the dilogarithm (see Fact III.1.8) gives that: $\sum_{0\leq m<q}D(e^{\frac{2\pi li}{k}}e^{\frac{2\pi mi}{q}})=\frac{1}{q}D(e^{\frac{2\pi li}{c}})$. We conclude :

$$d_\chi=\frac{k}{4\pi}\sum_{0\leq l<c}\chi^*(l)\frac{1}{q}D(e^{\frac{2\pi li}{c}})=d_{\chi^*}.$$

Using Lemma VI.2.6 and Corollary VI.2.3 we can proceed with our study of $m(P_d)$. □

VI.3. L-functions with primitive odd characters and Mahler measure of P_d

Corollary VI.2.3 gives us the opportunity to write $m(P_d)$ for $d\geq3$ in terms of L-functions. Since $2\pi m(P_d)=\frac{1}{d+1}S_{d+2}-\frac{1}{d+2}S_{d+1}$, where $S_d:=3\sum_{1\leq k\leq d-1}(d-2k)D((e^{\frac{2\pi}{d}i})^k)$, for every $d\in\mathbb{Z}_{\geq1}$ we have a coefficient of S_{d+2} in two successive Mahler measures $m(P_d)$ and $m(P_{d+1})$. In the following computations, we compute S_{d+2} in terms of L-functions only for $m(P_d)$ and we reuse this information for $m(P_{d+1})$.

Let us begin with P_3:

Proposition VI.3.1. *We have the following equality:*

$$m(P_3) = \frac{3(3-i)}{20}L'(\chi^i,-1) + \frac{3(3+i)}{20}L'(\chi^{-i},-1) - \frac{3}{5}L'(\chi_{-4},-1),$$

where χ^i and χ^{-i} are both primitive (non principal) odd complex Dirichlet characters of conductor 5 defined as follows:

m	0	1	2	3	4
$\chi^i(m)$	0	1	i	$-i$	-1
$\chi^{-i}(m)$	0	1	$-i$	i	-1

Proof. According to Notation V.1.2 we have:

$$m(P_3) = \frac{1}{2\pi}\left(\frac{1}{4}S_5 - \frac{1}{5}S_4\right) = \frac{1}{2\pi}\left(\frac{1}{2}\left(9D(e^{i\frac{2\pi}{5}}) + 3D(e^{i\frac{4\pi}{5}})\right)\right) + \frac{-6}{5\pi}D(i).$$

Using the computation of $L'(\chi_{-4},-1)$, in Proposition VI.1.4, we have $\frac{-3}{5}L'(\chi_{-4},-1) = \frac{-6}{5\pi}D(i)$. The only odd primitive non principal characters of conductor 5 are χ^i and χ^{-i}. Thus, we compute d_{χ^i} and $d_{\chi^{-i}}$ as follows:

$$\begin{aligned} d_{\chi^i} &= \frac{5}{4\pi}\left(\chi^i(1)D(e^{\frac{2\pi}{5}i}) + \chi^i(2)D(e^{\frac{4\pi}{5}i}) + \chi^i(3)D(e^{\frac{6\pi}{5}i}) + \chi^i(4)D(e^{\frac{8\pi}{5}i})\right) \\ &= \frac{5}{2\pi}\left(D(e^{\frac{2\pi}{5}i}) + iD(e^{\frac{4\pi}{5}i})\right), \\ d_{\chi^{-i}} &= \frac{5}{4\pi}\left(\chi^{-i}(1)D(e^{\frac{2\pi}{5}i}) + \chi^{-i}(2)D(e^{\frac{4\pi}{5}i}) + \chi^{-i}(3)D(e^{\frac{6\pi}{5}i}) + \chi^{-i}(4)D(e^{\frac{8\pi}{5}i})\right) \\ &= \frac{5}{2\pi}\left(D(e^{\frac{2\pi}{5}i}) - iD(e^{\frac{4\pi}{5}i})\right). \end{aligned}$$

We are searching for the suitable combination of $d_{\chi^{-i}}$ and d_{χ^i} which leads to $\frac{1}{2\pi}\left(\frac{1}{2}\left(9D(e^{i\frac{2\pi}{5}}) + 3D(e^{i\frac{4\pi}{5}})\right)\right)$ in the formula of $m(P_3)$. In other words we are searching for C_1 and C_2, where $C_1 d_{\chi^i} + C_2 d_{\chi^{-i}} = \frac{1}{4\pi}\left(9D(e^{i\frac{2\pi}{5}}) + 3D(e^{i\frac{4\pi}{5}})\right)$. Then we have:

$$\frac{5}{2\pi}\left(C_1 D(e^{\frac{2\pi}{5}i}) + iC_1 D(e^{\frac{4\pi}{5}i})\right) + \frac{5}{2\pi}\left(C_2 D(e^{\frac{2\pi}{5}i}) - iC_2 D(e^{\frac{4\pi}{5}i})\right) = \frac{1}{4\pi}\left(9D(e^{i\frac{2\pi}{5}}) + 3D(e^{i\frac{4\pi}{5}})\right).$$

We solve the following system of equations:

$$\begin{cases} 5C_1 + 5C_2 = \dfrac{9}{2}, \\ 5iC_1 - 5iC_2 = \dfrac{3}{2}. \end{cases}$$

The unique solution to the system is $C_1 = \frac{9-3i}{20}$ and $C_2 = \frac{9+3i}{20}$. Therefore, by applying Corollary VI.2.3 we have

$$\frac{S_5}{2\pi} = \frac{3(3-i)}{5}L'(\chi^i,-1) + \frac{3(3+i)}{5}L'(\chi^{-i},-1). \tag{VI.3.1}$$

Then, we conclude:

$$m(P_3) = \frac{3(3-i)}{20}L'(\chi^i,-1) + \frac{3(3+i)}{20}L'(\chi^{-i}-1) - \frac{3}{5}L'(\chi_{-4},-1).$$

$\square$

We notice that for a primitive odd Dirichlet character of conductor k, thanks to the functional equation Corollary VI.2.3 and the facts that $L(\bar{\chi}, 2) = \overline{L(\chi, 2)}$, and $\tau(\bar{\chi}) = -\overline{\tau(\chi)}$, we can conclude the following relation:

$$L'(\chi, -1) = \overline{L'(\bar{\chi}, -1)}.$$

Thus, according to Proposition VI.3.1 we have $m(P_3) = \frac{3}{10}\Re\left((3+i)L'(\chi^{-i}, -1)\right) - \frac{3}{5}L'(\chi_{-4}, -1)$. Then, thanks to Proposition VI.1.5 we have:

$$m(P_3) = \frac{3}{10}\Re\left((3+i)L'(\chi^{-i}, -1)\right) - \frac{3}{10}m(P_2^2 P_1).$$

This implies the following equality, reminiscent of Chinburg conjecture:

$$3\Re\left((3+i)L'(\chi^{-i}, -1)\right) = m(P_1^3 P_2^6 P_3^{10}).$$

By following the same process, we compute $m(P_d)$, for $4 \leq d \leq 6$ in terms of L-functions.

Proposition VI.3.2. *We begin as usual:*

$$m(P_4) = \frac{-3+i}{10}L'(\chi^i, -1) + \frac{-3-i}{10}L'(\chi^{-i}, -1) + \frac{16}{5}L'(\chi_{-3}, -1),$$

where χ^i and χ^{-i} are the primitive odd Dirichlet character of conductor 5 introduced in Proposition VI.3.1.

Proof. According to Notation V.1.2 we have :

$$m(P_4) = \frac{1}{2\pi}\left(\frac{1}{5}S_6 - \frac{1}{6}S_5\right) = \frac{1}{2\pi}\left(-3D(e^{\frac{2\pi i}{5}}) - D(e^{\frac{4\pi i}{5}}) + \frac{2}{5}\left(12D(e^{\frac{i2\pi}{6}}) + 6D(e^{\frac{i4\pi}{6}})\right)\right).$$

We already computed the representation of S_5 in terms of L-function in Eq. (VI.3.1). Thus, we only need to compute the representation of $S_6 = 24D(e^{\frac{\pi i}{3}}) + 12D(e^{\frac{2\pi i}{3}})$. According to Proposition VI.1.3 we have $L'(\chi_{-3}, -1) = \frac{3}{2\pi}D(e^{\frac{2\pi}{3}i})$. Moreover, using Eq. (VI.1.1) in Proposition VI.1.3, we have $\frac{12}{5\pi}D(e^{\frac{\pi}{3}i}) = \frac{18}{5\pi}D(e^{\frac{2\pi}{3}i}) = \frac{12}{5}L'(\chi_{-3}, -1)$. Therefore, by applying Corollary VI.2.3, we have:

$$m(P_4) = \frac{-3+i}{10}L'(\chi^i, -1) + \frac{-3-i}{10}L'(\chi^{-i}, -1) + \frac{16}{5}L'(\chi_{-3}, -1).$$

$\square$

Note that we got the equation

$$\frac{S_6}{2\pi} = 16L'(\chi_{-3}, -1) \tag{VI.3.2}$$

for our future computation. We notice that there is only one odd Dirichlet character of modulus 6, which is not primitive. It is induced by χ_{-3}. This explains the appearance of $d_{\chi_{-3}}$ and $L'(\chi_{-3}, -1)$.

As before, we can also write:

$$m(P_4^5/P_1^{16}) = \Re((-3+i)L'(\chi^i, -1))).$$

However, P_4^5/P_1^{16} is a rational function and not a polynomial. In the following we do the same process for $m(P_5)$, though the computations become more and more complicated:

Proposition VI.3.3. *We have the following equality:*

$$m(P_5) = \frac{-16}{7}L'(\chi_{-3},-1) + \frac{1}{3}L'(\chi^1,-1) + \frac{4-2\sqrt{3}i}{21}L'(\chi^2,-1) + \frac{4+2\sqrt{3}i}{21}L'(\chi^3,-1).$$

Where, χ^i for $1 \le i \le 3$ are the odd primitive Dirichlet characters mod 7 introduced in the following table (with $\omega = e^{\frac{i\pi}{3}}$);

m	0	1	2	3	4	5	6
$\chi^1(m)$	0	1	1	-1	1	-1	-1
$\chi^2(m)$	0	1	ω^2	ω	$-\omega$	$-\omega^2$	-1
$\chi^3(m)$	0	1	$-\omega$	$-\omega^2$	ω^2	ω	-1

Proof. By considering Notation V.1.2, for $d = 5$ we have:

$$m(P_5) = \frac{1}{2\pi}\left(\frac{1}{6}S_7 - \frac{1}{7}S_6\right) = \frac{1}{2\pi}\left(\frac{2}{7}\left(-12D(e^{\frac{2\pi i}{6}}) - 6D(e^{\frac{4\pi i}{6}})\right) + 5D(e^{\frac{i2\pi}{7}}) + 3D(e^{\frac{i4\pi}{7}}) + D(e^{\frac{i6\pi}{7}})\right).$$

We use Eq. (VI.3.2), so we only compute S_7 in terms of L-functions. We compute d_{χ^i}, for $1 \le i \le 3$.

$$\begin{aligned}
d_{\chi^1} &= \frac{7}{4\pi}\left(D(e^{\frac{2\pi}{7}i}) + D(e^{\frac{4\pi}{7}i}) - D(e^{\frac{6\pi}{7}i}) + D(e^{\frac{8\pi}{7}i}) - D(e^{\frac{10\pi}{7}i}) - D(e^{\frac{12\pi}{7}i})\right)\\
&= \frac{7}{2\pi}\left(D(e^{\frac{2\pi}{7}i}) + D(e^{\frac{4\pi}{7}i}) - D(e^{\frac{6\pi}{7}i})\right),\\
d_{\chi^2} &= \frac{7}{4\pi}\left(D(e^{\frac{2\pi}{7}i}) + \omega^2D(e^{\frac{4\pi}{7}i}) + \omega D(e^{\frac{6\pi}{7}i}) - \omega D(e^{\frac{8\pi}{7}i}) - \omega^2D(e^{\frac{10\pi}{7}i}) - D(e^{\frac{12\pi}{7}i})\right)\\
&= \frac{7}{2\pi}\left(D(e^{\frac{2\pi}{7}i}) + \omega^2D(e^{\frac{4\pi}{7}i}) + \omega D(e^{\frac{6\pi}{7}i})\right),\\
d_{\chi^3} &= \frac{7}{4\pi}\left(D(e^{\frac{2\pi}{7}i}) - \omega D(e^{\frac{4\pi}{7}i}) - \omega^2D(e^{\frac{6\pi}{7}i}) + \omega^2D(e^{\frac{8\pi}{7}i}) + \omega D(e^{\frac{10\pi}{7}i}) - D(e^{\frac{12\pi}{7}i})\right)\\
&= \frac{7}{2\pi}\left(D(e^{\frac{2\pi}{7}i}) - \omega D(2e^{\frac{4\pi}{7}i}) - \omega^2D(e^{\frac{6\pi}{7}i})\right).
\end{aligned}$$

We need to compute C_i, for $1 \le i \le 3$ such that $C_1d_{\chi^1} + C_2d_{\chi^2} + C_3d_{\chi^3} = \frac{1}{2\pi}(5D(e^{\frac{2\pi}{7}i}) + 3D(e^{\frac{4\pi}{7}i}) + D(e^{\frac{6\pi}{7}i}))$. We solve the following system of equations:

$$\begin{cases}7C_1 + 7C_2 + 7C_3 = 5,\\ 7C_1 + 7\omega^2C_2 - 7C_3\omega = 3,\\ -7C_1 + 7\omega C_2 - 7C_3\omega^2 = 1.\end{cases}$$

and the system has a unique solution. Then, we have $m(P_5) = \frac{-16}{7}L'(\chi_{-3},-1) + \frac{1}{3}L'(\chi^1,-1) + \frac{4-2\sqrt{3}i}{21}L'(\chi^2,-1) + \frac{4+2\sqrt{3}i}{21}L'(\chi^3,-1)$. We notice that the last computation indeed gives:

$$\text{(VI.3.3)} \qquad \frac{S_7}{2\pi} = 2L'(\chi^1,-1) + \frac{8-4\sqrt{3}i}{7}L'(\chi^2,-1) + \frac{8+4\sqrt{3}i}{7}L'(\chi^3,-1).$$

□

The last computation that we present here is for $m(P_6)$.

Proposition VI.3.4. *Let χ^i, for $1 \le i \le 3$ are the primitive odd Dirichlet characters of modulus 7, introduced in Proposition VI.3.3. We have the following equality:*

$$\begin{aligned} m(P_6) =& \frac{-1}{4} L'(\chi^1, -1) + \frac{-2+\sqrt{3}i}{14} L'(\chi^2, -1) + \frac{-2-\sqrt{3}i}{14} L'(\chi^3, -1) \\ &+ \frac{15}{14} L'(\chi_{-4}, -1) + L'(\chi_{-8}, -1). \end{aligned}$$

PROOF. We have:

$$\begin{aligned} m(P_6) = \frac{1}{2\pi}\left(\frac{1}{7}S_8 - \frac{1}{8}S_7\right) &= \frac{1}{2\pi}\left(\frac{1}{4}\left(-15D(e^{\frac{2\pi i}{7}}) - 9D(e^{\frac{4\pi i}{7}}) - 3D(e^{\frac{6\pi i}{7}})\right)\right) \\ &+ \frac{1}{2\pi}\left(\frac{2}{7}\left(18D(e^{\frac{i2\pi}{8}}) + 12D(e^{\frac{i4\pi}{8}}) + 6D(e^{\frac{i6\pi}{8}})\right)\right). \end{aligned}$$

Thanks to Eq. (VI.3.3), we focus on S_8. Let us consider the table associated to the Dirichlet character of modulus 8 :

n	1	3	5	7
$\chi^1(n)$	1	1	1	1
$\chi^2(n)$	1	-1	1	-1
$\chi^3(n)$	1	-1	-1	1
$\chi_{-8}(n)$	1	1	-1	-1

Between all the characters of modulus 8, only $\chi^2(n)$ and $\chi_{-8}(n)$ are odd, but the conductor of $\chi^2(n)$ is 4. Therefore, there are no additional primitive Dirichlet characters apart from the quadratic one (see Example II.3.25 for more information about χ_{-8}). We compute $L'(\chi_{-8}, 2)$:

$$L'(\chi_{-8}, 2) = \frac{8}{4\pi}\left(D(e^{\frac{2\pi}{8}i}) + D(e^{\frac{6\pi}{8}i}) - D(e^{\frac{10\pi}{8}i}) - D(e^{\frac{14\pi}{8}i})\right) = \frac{4}{\pi}\left(D(e^{\frac{2\pi}{8}i}) + D(e^{\frac{6\pi}{8}i})\right).$$

In $L'(\chi_{-8}, 2)$, we only have the coefficients of $D(e^{\frac{2\pi}{8}i})$ and $D(e^{\frac{6\pi}{8}i})$, but for $m(P_6)$ we need the coefficients of $D(e^{\frac{4\pi}{8}i}) = D(e^{\frac{\pi}{2}i}) = D(i)$ as well. Thus, we use $L'(\chi_{-4}, 2) = \frac{2}{\pi}D(i)$. We notice that the coefficient of $D(e^{\frac{2\pi}{8}i})$ and $D(e^{\frac{6\pi}{8}i})$ in d_8 are the same, but their coefficients in $m(P_6)$ are not equal. Thus, to write $\frac{2}{7}\left(18D(e^{\frac{i2\pi}{8}}) + 12D(e^{\frac{i4\pi}{8}}) + 6D(e^{\frac{i6\pi}{8}})\right)$ in the formula of $m(P_6)$ in terms of $L'(\chi_{-8}, 2)$ and $L'(\chi_{-4}, 2)$, we recompute $L'(\chi_{-8}, 2)$ only in terms of $D(i)$ and $D(e^{\frac{6\pi}{8}i})$, which gives us the possibility to choose the coefficient of $L'(\chi_{-8}, 2)$ and $L'(\chi_{-4}, 2)$. To do so, we use the properties of the dilogarithm such as $D(z) = -D(\bar{z})$ and the distribution formula of the dilogarithm:

$$\begin{aligned} D(i) = D((e^{\frac{\pi}{4}i})^2) &= 2(D(e^{\frac{\pi}{4}i}) + D(-e^{\frac{\pi}{4}i})) = 2(D(e^{\frac{\pi}{4}i}) + D(e^{\frac{5\pi}{4}i})) \\ &= 2(D(e^{\frac{\pi}{4}i}) - D(e^{\frac{3\pi}{4}i})) = 2(D(e^{\frac{2\pi}{8}i}) - D(e^{\frac{6\pi}{8}i})). \end{aligned}$$

Equivalently we have $\frac{D(i)}{2} + D(e^{\frac{6\pi}{8}i}) = D(e^{\frac{2\pi}{8}i})$. Therefore, we have:

$$\begin{aligned} L'(\chi_{-8}, -1) &= \frac{4}{\pi}\left(D(e^{\frac{2\pi}{8}i}) + D(e^{\frac{6\pi}{8}i})\right) = \frac{4}{\pi}\left(\frac{D(i)}{2} + D(e^{\frac{6\pi}{8}i}) + D(e^{\frac{6\pi}{8}i})\right) \\ &= \frac{4}{\pi}\left(\frac{D(i)}{2} + 2D(e^{\frac{6\pi}{8}i})\right). \end{aligned}$$

Moreover, in the formula for $m(P_6)$, we may write:

$$\frac{1}{7\pi}\left(18(\frac{D(i)}{2} + D(e^{\frac{6\pi}{8}i})) + 12D(i) + 6D(e^{\frac{i6\pi}{8}})\right) = \frac{1}{7\pi}\left(21D(i) + 24D(e^{\frac{i6\pi}{8}})\right).$$

We search for a_1, a_2 such that we have $a_1 L'(\chi_{-4}, 2) + a_2 L'(\chi_{-8}, 2) = \frac{1}{7\pi}\left(21D(i) + 24D(e^{\frac{i6\pi}{8}})\right)$, or in explicit terms:

$$a_1 \frac{2}{\pi} D(i) + a_2 \frac{4}{\pi}\left(\frac{D(i)}{2} + 2D(e^{\frac{6\pi}{8}i})\right) = \frac{1}{7\pi}\left(21D(i) + 24D(e^{\frac{i6\pi}{8}})\right).$$

Again by solving a linear system, we have the unique solution $a_2 = \frac{3}{7}$, and $a_1 = \frac{15}{14}$ which completes the proof of the proposition. $\square$

We notice that in the above proposition we had only χ_{-8} as an odd primitive Dirichlet character of conductor 8. On the other hand χ^2 is induced by χ_{-4} and this is the reason that we have $L'(\chi_{-4}, -1)$ in the representation of $m(P_6)$.

Remark VI.3.5. The values of $m(P_d)$, for $1 \le d \le 6$ in terms of L-functions may be numerically verified using SageMath.

To visualize some of the interesting values of the Mahler measures of certain polynomials generated by P_d's, we provide a table in Observation VI.3.6. The second column of the table provides information in terms of linear combinations of L-functions. As we mentioned in Notation VI.1.1, Boyd [Boy98] uses the notation d_f in his computations. Following Boyd, in the third column of the table, we have information in terms of linear combinations of d_f's.

Observation VI.3.6. *Let χ_{-f} be an odd quadratic Dirichlet character, and $d_f = L'(\chi_{-f}, -1)$. Using the computation done in this chapter we conclude the following table:*

$m(P_1)$	$L'(\chi_{-3}, -1)$	d_3
$m(P_2)$	$L'(\chi_{-4}, -1) - \frac{L'(\chi_{-3},-1)}{2}$	$d_4 - \frac{d_3}{2}$
$m(P_1P_2^2)$	$2L'(\chi_{-4}, -1)$	$2d_4$
$m(P_3^2P_4^3)$	$\frac{6}{5}(8L'(\chi_{-3}, -1) - L'(\chi_{-4}, -1))$	$\frac{6}{5}(8d_3 - d_4)$
$m(P_5^3P_6^4)$	$\frac{6}{7}(-8L'(\chi_{-3}, -1) + 5L'(\chi_{-4}, -1) + 2L'(\chi_{-8}, -1))$	$\frac{6}{7}(-8d_3 + 5d_4 + 2d_8)$

VI.4. Writing $m(P_d)$ in terms of L-functions and some perspective

In the previous sections, we have seen a representation of $m(P_d)$ in terms of L-functions, for $1 \leq d \leq 6$. In this section, we prove that $m(P_d)$, for all d can be written as a linear combination of L-functions. To do so, instead of considering Dirichlet characters of modulus d as a function over over $\frac{\mathbb{Z}}{d\mathbb{Z}}$, we consider them as functions over the set of d-th roots of unity. In the following, we explain this process more precisely.

Let $d \in \mathbb{Z}_{\geq 1}$, then we set $U_d := \frac{\mathbb{Z}}{d\mathbb{Z}}$ and $U_d^* = \left(\frac{\mathbb{Z}}{d\mathbb{Z}}\right)^*$, the group of invertible elements of the ring U_d. It is known [Kou19, Chapter 10] that Dirichlet characters form a basis for the $\mathbb{C}$-vector space $\mathbb{C}^{U_d^*}$ (the space of the functions from U_d^* to $\mathbb{C}$). Consequently, the odd Dirichlet characters mod d generate the odd d-periodic functions (i.e. f is a d-periodic function and $f(d-x) = -f(x)$ for every $x \in U_d^*$). As we have already mentioned, for a Dirichlet character modulo d we can consider this function over U_d, by defining its value on $U_d \setminus U_d^*$ to be zero. Let $\hat{U}_d$ denotes the set of the d-th roots of unity. Then, $\hat{U}_d$ and U_d are in bijection, so we can consider the Dirichlet character over this set. In other words, for $k \in \mathbb{Z}_{\geq 1}$ and a Dirichlet character modulo d we define $\chi((e^{\frac{2\pi i}{d}})^k) := \chi(k)$. We notice that U_d^* is mapped to the set of primitive d-th roots of unity, denoted by V_d. In other words, for $z \in \hat{U}_d$, we have $\chi(z) \neq 0$ if and only if $z \in V_d$. Moreover, the property of being odd d-periodic on $\hat{U}_d$ is simply $\chi(e^{\frac{2k\pi i}{d}}) = -\chi(e^{-\frac{2k\pi i}{d}})$. The following lemmas are needed for the future.

Lemma VI.4.1. *Let $f : \hat{U}_d \to \mathbb{Z}$ be an odd d-periodic function, k be an integer such that $k|d$, V_k be the set of k-th roots of unity, and 1_{V_k} be the characteristic function. Then $f_k : \hat{U}_d \to \mathbb{Z}$, defined by $f_k := 1_{V_k} f$ is an odd function.*

Proof. According to the definition, f_k is non zero on V_k. Thus, we only prove that for every $z \in V_k$ we have $f_k(\bar{z}) = -f(z)$. We notice that if $\gcd(l,k) = 1$, then $\gcd(k, l-k) = 1$. Hence, if $z \in V_k$, then $\bar{z} \in V_k$. In other words, $1_{V_k}(\bar{z}) = 1_{V_k}(z)$.

$$f_k(\bar{z}) = 1_{V_k}(\bar{z}) f(\bar{z}) \overset{[1]}{=} 1_{V_k}(z)(-f(z)) = -f_k(z).$$

In [1] we used that f is odd and $1_{V_k}(\bar{z}) = 1_{V_k}(z)$. $\square$

Lemma VI.4.2. *Let $f : \hat{U}_d \to \mathbb{Z}$ be an odd d-periodic function, k be an integer such that $k|d$. Moreover, let $\hat{f}_k : \hat{U}_k \to \mathbb{Z}$ be defined as follows:*

$$\hat{f}_k(z) = \begin{cases} 0 & \text{if } \hat{U}_k \setminus V_k \\ f(z) & \text{if } z \in V_k. \end{cases}$$

Then, $\hat{f}_k$ is an odd k-periodic function and can be written uniquely in terms of odd Dirichlet character of modulus k.

Proof. Since $\hat{f}_k$ is zero on $\hat{U}_k \setminus V_k$, we only need to prove that it is odd on V_k. We notice that $\hat{f}_k|_{V_k} = f_k$ According to Lemma VI.4.1, f_k is odd. Thus, $\hat{f}_k$ is an odd function and since it is defined over $\hat{U}_k$ it is k-periodic and vanishes outside V_k. Thus, due to the fact that odd Dirichlet characters form a basis for odd k-periodic function, it can be uniquely written as a linear combination of odd Dirichlet character modulus k. $\square$

We now come back to our problem of writing $m(P_d)$ in terms of L-functions. According to Notation V.1.2 we have $2\pi m(P_d) = \frac{1}{d+1} S_{d+2} - \frac{1}{d+2} S_{d+1}$, where $S_d := 3\sum_{1 \leq k \leq d-1} (d - 2k) D((e^{\frac{2\pi}{d} i})^k)$. Let us prove the following proposition concerning S_d:

Proposition VI.4.3. *Let $d \in \mathbb{Z}_{\geq 1}$, and $S_d = 3\sum_{1\leq k\leq d-1}(d-2k)D((e^{\frac{2\pi}{d}i})^k)$. Then, for every odd primitive Dirichlet character χ of conductor k, such that $k|d$, there exists a coefficient $C_{k,\chi} \in \mathbb{Q}(e^{\frac{2\pi i}{\phi(k)}}) \subset \mathbb{Q}(e^{\frac{2\pi i}{\phi(d)}})$ such that:*

$$S_d = \sum_{k|d} \sum_{\chi \text{odd primitive mod } k} C_{k,\chi} L'(\chi, -1).$$

Proof. We consider S_d as an inner product of two vectors in $\mathbb{C}^{d-1}$ as follows:

$$S_d = 3\left\langle [(d-2), \ldots, (2-d)], \left[D(e^{\frac{2\pi i}{d}}), \ldots, D(e^{\frac{2\pi i(d-1)}{d}})\right]\right\rangle.$$

In Corollary VI.2.3 we have seen the link between d_χ and L-functions for the odd primitive characters. Thus, writing S_d in terms of d_χ with odd primitive characters leads to the representation in terms of L-functions. We can similarly consider d_χ as an inner product of two vectors in $\mathbb{C}^{d-1}$ as follows:

$$d_\chi = \frac{d}{4\pi}\left\langle \left[\chi(e^{\frac{2\pi i}{d}}), \ldots, \chi(e^{\frac{2\pi i(d-1)}{d}})\right], \left[D(e^{\frac{2\pi i}{d}}), \ldots, D(e^{\frac{2\pi i(d-1)}{d}})\right]\right\rangle.$$

Due to the linearity of the inner product, it suffices to write the function $f : \hat{U}_d \to \mathbb{Z}$, defined by $f(e^{\frac{2\pi ki}{d}}) := d-2k$, for $0 \leq k \leq d-1$ as a linear combination of odd primitive characters. Indeed, f is an odd periodic function over $\hat{U}_d$, since $f(e^{\frac{2\pi(d-k)i}{d}}) = d-2(d-k) = 2k-d = -f(e^{\frac{2\pi ki}{d}})$. However, we have $f(z) \neq 0$ even for z who are not d-th primitive root of unity. Thus we can not write f as a linear combination of odd Dirichlet characters modulus d. To solve this problem, we write $f = \sum_{k|d} f_k$, with $f_k := 1_{V_k} f$. We notice that $\hat{U}_d = \bigcup_{k|d} V_k$ and it is indeed a disjoint union. Let us extend each f_k to $\hat{f}_k : \hat{U}_k \to \mathbb{Z}$ in the same way as we did in Lemma VI.4.2. Therefore, thanks to Lemma VI.4.2 for each $k|d$, $\hat{f}_k$ can be represented uniquely as a linear combination of odd Dirichlet characters modulus k. The same representation can be considered for f_k, since they have the same support. This implies that S_d can be written as a linear combination $\sum_{k|d}\sum_{\chi\text{odd of mod } k} C_{k,\chi} d_\chi$. Concerning the number field containing the coefficients $C_{k,\chi}$ we recall that $[d-2, \ldots, 2-d]$ is a vector with integer coefficients and it is written as a linear combination of Dirichlet characters modulus k, with $k|d$. On the other side, the values of Dirichlet characters modulus k belong to the cyclotomic field $\mathbb{Q}[e^{\frac{2\pi i}{\phi(k)}}]$ (see the discussion after Definition II.3.2). Due to the multiplicativity of Euler's function (see [Dez21, Page 13]), for any k where $k|d$ we have $\phi(k)|\phi(d)$, so $\mathbb{Q}[e^{\frac{2\pi i}{\phi(k)}}] \subseteq \mathbb{Q}[e^{\frac{2\pi i}{\phi(d)}}]$. Using a simple argument of linear algebra we conclude that $[d-2, \ldots, 2-d]$ is written as a linear combination of Dirichlet characters with coefficients belonging to $\mathbb{Q}[e^{\frac{2\pi i}{\phi(d)}}]$. Finally, in the representation of S_d in terms of d_χ we replace each induced character χ with the odd associated primitive Dirichlet character χ^*, and according to Lemma VI.2.6, we have $d_\chi = d_{\chi^*}$. Thus, S_d can be written as a linear combination of d_χ associated with odd primitive Dirichlet character. Using Corollary VI.2.3 we have a representation in terms of Dirichlet L-functions. □

The above proposition is the key to prove Theorem VI.0.1.

Proof. Proof of Theorem VI.0.1: We have $2\pi m(P_d) = \frac{1}{d+1}S_{d+2} - \frac{1}{d+2}S_{d+1}$, and according to Proposition VI.4.3 we can write S_{d+1} and S_{d+2} in terms of L-functions associated with odd primitive characters of conductor k where respectively $k|d+1$ or $k|d+2$. This completes the proof. □

For any $d \in \mathbb{Z}_{\geq 1}$, $m(P_d)$ can be written as a linear combination of Dirichlet L-functions associated with odd primitive Dirichlet characters of conductor k, where $k|d+1$, or $k|d+2$.

We notice that such representation is not unique. In fact, one can write $f = [d-2, \ldots, 2-d]$ as a linear combination of odd primitive Dirichlet characters uniquely. However, in the formula of S_d we compute the inner product of f with $V_D := \left[D(e^{\frac{2\pi i}{d}}), \ldots, D(e^{\frac{2\pi i(d-1)}{d}})\right]$, which changes the situation. More precisely, for any $V_{D^\perp}$ perpendicular to V_D, we have $S_d = 3\langle f + V_{D^\perp}, V_D\rangle = 3\langle f, V_D\rangle$.

VI.4.1. Future projects. One of our future goals is to inspect the orthogonal complement of the subspace generated by V_D, using the properties of the dilogarithm. We hope in this way to simplify the representation of $m(P_d)$ in terms of L-functions. We will continue to explore and improve the formulas giving $m(P_d)$ in terms of L-functions, both experimentally (by computing explicit representations for higher values of d) and theoretically (by understanding $V_{D^\perp}$ and a possible choice of a better vector in $f + V_{D^\perp}$). Moreover, as we know for every $d \in \mathbb{Z}_{\geq 1}$, S_{d+2} appears in the two successive Mahler measures $m(P_d)$ and $m(P_{d+1})$. In our examples we have computed S_d, for $3 \leq d \leq 8$ in terms of L-functions. We are interested in knowing more about the coefficients of the L-functions appearing in each S_d and how can we interpret these numbers. Answering this question may help us obtain a general formula for the coefficients in the representation of $m(P_d)$ in terms of L-functions. Moreover, a general formula for $m(P_d)$ leads to extension of the table in Observation VI.3.6. Thus, we discover more connections between the d_f's, and Mahler measures of polynomials generated by P_d's that shed light on Chinburg's Conjecture. Lately, we mention that Chinburge in [Chi84] announced a weaker version of Chinburg's conjecture, in which $P_f \in \mathbb{Q}(x, y)$. We believe that P_d-family provides infinitely many partial solutions for the weak version of Chinburg's conjecture. However, a proof is not immediate and is part of the future projects.

CHAPTER VII

Appendix

This chapter contains the preliminary information, such as basic definitions, customary terminologies, theorems, proofs, etc. , that we need to complete the information given in the initial chapters of this thesis. For the convenience of the reader, the Appendix is divided into 6 sections, and each section is devoted entirely to a certain topic.

VII.1. Resultant

The aim of this section is to recall some tools that we use for the proof of Kronecker's theorem. In mathematics, the resultant of two polynomials is a polynomial expression of their coefficients, which is equal to zero if and only if the polynomials have a common root (possibly in a field extension), or, equivalently, a common factor (over their field of coefficients).

Definition VII.1.1. Let A and B be two univariate polynomials over a field or over a commutative ring defined as follows: $A(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$ and $B(x) = b_0x^e + b_1x^{e-1} + \cdots + b_e$ such that $a_0 \neq 0$ and $b_0 \neq 0$. The resultant of A and B is the determinant of the following Matrix:

$$\begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & a_{d-1} & \cdots & \vdots & b_e & b_{e-1} & \cdots & \vdots \\ 0 & a_d & \ddots & \vdots & 0 & b_e & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{vmatrix}$$

which has e columns of a_i and d columns of b_j (the fact that the first column of a's and the first column of b's have the same length, that is $d = e$, is here only for simplifying the display of the determinant).

If the coefficients of the polynomials belong to an integral domain, then

$$\text{(VII.1.1)} \qquad \operatorname{res}(A,B) = a_0^e b_0^d \prod_{\substack{1\le i\le d \\ 1\le j\le e}} (\lambda_i - \mu_j) = a_0^e \prod_{i=1}^d B(\lambda_i) = (-1)^{de} b_0^d \prod_{j=1}^e A(\mu_j),$$

where $\lambda_1, \dots, \lambda_d$ and $\mu_1, \dots, \mu_e$ are respectively the roots, counted with their multiplicities, of A and B in any algebraically closed field containing the integral domain.

Notation VII.1.2. In many applications of the resultant, the polynomials depend on several variables and may be considered as univariate polynomials in one of their variables, with polynomials in the other variables as coefficients. For instance, we used such argument in the proof of Kronecker's theorem.

VII.2. Gamma function

The aim of this section is to recall some properties of the Gamma function given in Definition II.3.28, which are needed for Proposition II.3.35. In fact the following lemma was used for proving the functional equation of the L-function. The proofs are computational and easily verified, and can be found in many references (for instance see [Bon17, Chapter 1]). For the convenience of the reader we recall some short proofs here.

Lemma VII.2.1. *For every z with real part strictly positive we have $\Gamma(1+z) = z\Gamma(z)$.*

PROOF. Using integration by parts, we have:

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty x^z e^{-x}\,dx = \Big[-x^z e^{-x}\Big]_0^\infty + \int_0^\infty z x^{z-1} e^{-x}\,dx \\ &= \lim_{x\to\infty}\left(-x^z e^{-x}\right) - \left(-0^z e^{-0}\right) + z\int_0^\infty x^{z-1}e^{-x}\,dx.\end{aligned}$$

If $x \to \infty$, then $x^z e^{-x} \to 0$, so we have $\Gamma(z+1) = z\int_0^\infty x^{z-1}e^{-x}\,dx = z\Gamma(z)$.

$\square$

As we have already mentioned, the Gamma function is the extension of the factorial function to complex numbers. To visualize this fact, in the following lemma we prove that $\Gamma(n) = (n-1)!$ for every positive integer $n \geq 2$.

Lemma VII.2.2. *Let $n \in \mathbb{Z}_{\geq 2}$, we have $\Gamma(n) = (n-1)!$*

PROOF. We prove this by induction. For $n = 2$ we have $\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = \lim_{x\to\infty}\left(-e^{-x}\right) - \left(-e^{-0}\right) = 1$. Suppose that for every $k < n$ we have $\Gamma(k) = (k-1)!$, so by using Lemma VII.2.1 we have $\Gamma(n) = n\Gamma(n-1)$, so $\Gamma(n) = (n-1)!$. $\square$

Lemma VII.2.3. *We have the following equation called Legendre duplication formula :*

$$\Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\ \sqrt{\pi}\ \Gamma(2z).$$

PROOF. Suppose that $\Re(z_1), \Re(z_2) > 0$ and let $B(z_1,z_2) := \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ be the Beta function. One can check that $B(z_1,z_2) = \int_0^1 u^{z_1-1}(1-u)^{z_2-1}du$. Let $z_1 = z_2 = z$ and we have $B(z,z) =$

$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \int_0^1 u^{z-1}(1-u)^{z-1}du$. Applying the change of variables $u = \frac{1+x}{2}$ we have: $\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \frac{1}{2^{2z-1}}\int_{-1}^1 (1-x^2)^{z-1}dx$. Using the property of the definite integral of even functions we have :

$$2^{2z-1}\Gamma(z)\Gamma(z) = 2\Gamma(2z)\int_0^1 (1-x^2)^{z-1}dx. \tag{VII.2.1}$$

In the integral definition of $B(z_1, z_2)$ we use the change of variable $u = x^2$, so after a computation we have $B(z_1, z_2) = \int_0^1 x^{2z_1-2}(1-x^2)^{z_2-1}2xdx$. We replace $z_1 = \frac{1}{2}$ and $z_2 = z$ in this equality and we have:

$$B(\frac{1}{2}, z) = 2\int_0^1 (1-x^2)^{z-1}dx. \tag{VII.2.2}$$

Combining results Eq. (VII.2.1) and Eq. (VII.2.2) we have $2^{2z-1}\Gamma(z)\Gamma(z) = \Gamma(2z)B(\frac{1}{2}, z) = \Gamma(2z)\frac{\Gamma(1/2)\Gamma(z)}{\Gamma(\frac{1}{2}+z)}$. Therefore, $\Gamma(\frac{1}{2}+z)\Gamma(z) = 2^{1-2z}\Gamma(\frac{1}{2})\Gamma(2z)$. Using the integral definition of the Gamma function we have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and we conclude:

$$\Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\ \sqrt{\pi}\ \Gamma(2z).$$

□

VII.3. Probability Haar Measure and convergence of measures

In this section, we introduce the necessary prerequisites to work with measure and specially probability Haar measures. These notations are necessary in order to assert Generalized Boyd-Lawton's theorem.

A Measure is a kind of function that is defined over a set with certain properties called σ-algebra. A σ-algebra on a set X is a collection Σ of subsets of X that includes the empty subset, is closed under complement, and is closed under countable unions and countable intersections. Measures are generalizations of length, area, and volume, but are useful for much more abstract and irregular sets than intervals in $\mathbb{R}$ or balls in $\mathbb{R}^n$. One might expect to define a generalized measuring function μ^* on $\mathbb{R}^+ \cup \{\infty\}$ that fulfills some requirements that already length, area, volume or other sort of measures respect. For instance, translation invariance, countable additivity etc. However, by this kind of generalization we can not apply μ^* to any arbitrary set. In other words, there are some sets who are non-measurable. That is where the definition of an outer measure arises.

Definition VII.3.1. Given a set X, let $P(X)$ denote the collection of all subsets of X, including the empty set. An outer measure on X is a set function: $\mu^* : P(X) \to [0, \infty]$ such that $\mu^*(\varnothing) = 0$, and it is countably sub additive (i.e. for arbitrary subsets $A, B_1, B_2, \ldots$ of X, if $A \subseteq \bigcup_{j=1}^{\infty} B_j$ then $\mu^*(A) \le \sum_{j=1}^{\infty}\mu^*(B_j)$).

The purpose of constructing an outer measure on all subsets of X is to extract a class of subsets (to be called measurable) that satisfy the countable additivity property. Let us recall the mathematical definition of measure.

Definition VII.3.2. Let X be a set and Σ a σ-algebra over X. A function μ from Σ to the extended real number line is called a measure if it satisfies the following properties:

- Non-negativity: For all E in Σ, we have $\mu(E) \ge 0$.
- Null empty set: $\mu(\varnothing) = 0$.
- Countable additivity (or σ-additivity): For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ, we have $\mu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty}\mu(E_k)$.

The pair (X, Σ) is called a measurable space and the members of Σ are called measurable sets. If (X, Σ_X) and $(Y, \Sigma_Y)$are two measurable spaces, then a function $f : X \to Y$ is called measurable if for every Y-measurable set $B \in \Sigma_Y$, the inverse image is X-measurable (i.e.:$f^{(-1)}(B) \in \Sigma_X$).
There are some types of measures which are used in this thesis such as probability measure, Lebesgue measure and Haar measure. Let us start with the definition of probability measure.

Definition VII.3.3. A probability measure is a measure with total measure one (i.e. $\mu(X) = 1$).

After defining probability measures we define the Lebesgue measure. The Lebesgue measure, named after the French mathematician Henri Lebesgue, is the standard way of assigning a measure to subsets of n-dimensional Euclidean space. For $n = 1, 2$, or 3, it coincides with the standard measure of length, area, or volume. Before defining Lebesgue measure we define the following: For any interval $I = [a, b]$ (or $I = (a, b)$) in $\mathbb{R}$, let $\ell(I) = b - a$ denotes its length.

Definition VII.3.4. For any subset $E \subseteq \mathbb{R}$, the Lebesgue outer measure $\mu^*(E)$ is defined as an infimum:

$$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

Some sets E satisfy the Carathéodory criterion, which requires that for every $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

The set of all such E forms a σ-algebra. For any such E, its Lebesgue measure is defined to be its Lebesgue outer measure: $\mu(E) = \mu^*(E)$.

A set E that does not satisfy the Carathéodory criterion is not Lebesgue-measurable. Non-measurable sets do exist; an example is the Vitali sets. A closed interval $[a, b]$ of real numbers is Lebesgue-measurable, and its Lebesgue measure is the length $b - a$. The open interval (a, b) has the same measure, since the difference between the two sets consists only of the end points a and b and has measure zero.

The last type of measure that we need to introduce is the Haar measure, introduced by Alfréd Haar in 1933. It assigns an "invariant measure" to subsets of locally compact topological groups, consequently defining an integral for functions on those groups. To define this measure, we first define the Borel algebra of a locally compact Hausdorff topological group. Let G be a locally compact Hausdorff topological group. The σ-algebra generated by all the open subsets of G is called the Borel algebra. An element of the Borel algebra is called a Borel set. If g is an element of G and S is a subset of G, then we define the left and right translates of S by g as follows:

- Left translate: $gS = \{g \cdot s \,:\, s \in S\}$.
- Right translate: $Sg = \{s \cdot g \,:\, s \in S\}$.

We can now define a left Haar measure (for more information see [Coh80, Section 9.2]):

Definition VII.3.5. Let μ be a measure with the following properties:

- The measure μ is invariant by left-translation: $\mu(gS) = \mu(S)$, for every $g \in G$ and all Borel sets $S \subseteq G$.
- The measure μ is finite on every compact set: $\mu(K) < \infty$, for all compact $K \subseteq G$.

- The measure μ is outer regular on Borel sets $S \subseteq G$:
  $\mu(S) = \inf\{\mu(U) : S \subseteq U, U \text{ open}\}$.
- The measure μ is inner regular on open sets $U \subseteq G$:
  $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$.

Then, μ is called a left Haar measure on G.

There is, up to a positive multiplicative constant, a unique nontrivial left Haar measure μ on the Borel subsets of G. Moreover, it can be shown as a consequence of the above properties that $\mu(U) > 0$ for every non-empty open subset $U \subseteq G$. In particular, if G is compact then $\mu(G)$ is finite and positive, so we can uniquely specify a left Haar measure on G by adding the normalization condition $\mu(G) = 1$. In complete analogy, one can also prove the existence and uniqueness of a right Haar measure on G. The two measures need not coincide. However, according to definition of the measure, introduced in Eq. (II.4.2) the left and right Haar measures for $\mathbb{T}^n$ are equal, so is called Haar measure.

VII.3.1. Weakly convergence of Measures. There are different concepts of convergence for the measures. The one that we need in this thesis is weakly convergence of the measures.

Definition VII.3.6. Let X be a metric space with Borel σ-algebra Σ . A bounded sequence of positive probability measures $(\mu_n)_{n\in\mathbb{Z}_{\geq 1}}$ on (X, Σ) is said to converge weakly to the finite positive measure μ (denoted $\mu_n \Rightarrow \mu$) if for all bounded, continuous functions f we have:

$$\int f d\mu_n \to \int f d\mu.$$

The Weierstrass approximation Theorem (see Proposition VII.3.7) states that every continuous function defined on a compact Hausdorff space can be uniformly approximated as closely as desired by a polynomial function. This implies that if X is a compact Hausdorff space, to verify $\mu_n \Rightarrow \mu$, we only need to verify $\int f d\mu_n \to \int f d\mu$, where f is a polynomial.
Let us explain more about the Stone–Weierstrass theorem and its application in recognizing the weakly convergence of a sequence of measures. To do so, first, we need to know about separating subsets and unital*-algebra: A set S containing functions from a set D to a set C is called a separating set for D or said to separate the points of D if for any two distinct elements x and y of D, there exists a function f in S, such that $f(x) \neq f(y)$. The complex unital*-algebra generated by S consists of all those functions that can be obtained from the elements of S by throwing in the constant function 1 and adding them, multiplying them, conjugating them, or multiplying them with complex scalars, and repeating finitely many times. Let X be a compact Hausdorff space, the set of continuous complex-valued functions on X together with the supremum norm $||f|| = \sup_{x\in X} |f(x)|$, is a Banach algebra, (that is, an associative algebra and a Banach space such that $||fg|| \leq ||f||.||g||$ for all f, g) and is denoted by $C(X, \mathbb{C})$. We recall that a Banach space is a normed space that is complete in the metric induced by the norm. The following is a classical theorem of analysis that can be found in most analysis textbooks.

Proposition VII.3.7. *Stone–Weierstrass Theorem Let X be a compact Hausdorff space and let S be a separating subset of $C(X, \mathbb{C})$. Then, the complex unital*-algebra generated by S is dense in $C(X, \mathbb{C})$.*

It is a classical fact that the set of all polynomial functions forms a subalgebra of $C(X, \mathbb{C})$ that is dense in $C(X, \mathbb{C})$. Thus, in a compact Hausdorff metric space X to verify $\mu_n \Rightarrow \mu$ we only need to verify $\int f d\mu_n \to \int f d\mu$, where f is a polynomial. The aim of this section is to provide the prerequisites for the proof of a higher dimensional analogue of Boyd-Lawton theorem. In fact, in Theorem II.4.20 we prove that the sequence of integrals of $|\log(P_{A_d})|$ converges to the integral of $|\log(P)|$ (under some hypothesis). We know that weakly convergence of measure means the convergence of integral of continuous functions, but $|\log(P_{A_d})|$ is not necessarily a continuous function. This is a function which continues every where except at the points which are the roots of P_{A_d}. In these points the value of $|\log(P_{A_d})|$ is $+\infty$. One can see for instance [Rud87, Chapter 1] as a reference.

Lemma VII.3.8. *Let μ_k be a sequence of probability measure on $\mathbb{T}^n$ weak-converging to some probability measure μ_∞. Let $f : \mathbb{T}^n \to \mathbb{R} \cup \{+\infty\}$ be a continuous function. Suppose moreover, that it is uniformly L^2 for μ_k and μ_∞ (i.e. there exists $C > 0$ such that all these L^2 norms are less than C). Then, we have the convergence $\int_{\mathbb{T}^n} f d\mu_k \to \int_{\mathbb{T}^n} f d\mu$.*

PROOF. Fix $\varepsilon > 0$ and let $\lambda = \frac{C}{\varepsilon}$. Define the set $S_\lambda = \{t \in \mathbb{T}^n | |f(t)| > \lambda\}$. The L^2 bounds yields, for any $\mu = \mu_1, \ldots, \mu_k, \ldots, \mu_\infty$:

$$\left|\int_{S_\lambda} (f-\lambda) d\mu\right| \le 2\int_{S_\lambda} |f| d\mu \le 2\int_{S_\lambda} |f| \frac{|f|}{\lambda} d\mu \le \frac{2}{\lambda}\int |f|^2 d\mu \le \frac{2C}{\lambda} = 2\varepsilon. \tag{VII.3.1}$$

Now, let $\tilde{f}$ be the function $\min(f, \lambda)$. This function is bounded from above by λ and is continuous. Moreover, we have $f = \tilde{f} + (f - \lambda) \times \mathbf{1}_{S_\lambda}$. We can write, for all k:

$$\left|\int f d\mu_k - \int f d\mu_\infty\right| \le \left|\int \tilde{f} d\mu_k - \int \tilde{f} d\mu_\infty\right| + \left|\int_{S_\lambda} (f-\lambda) d\mu_k\right| + \left|\int_{S_\lambda} (f-\lambda) d\mu_\infty\right|$$

The last two terms are boubded by 2ε by (VII.3.1). For k big enough, by the convergence $\mu_k \Rightarrow \mu_\infty$, the first one is less than ε. Hence, we have proven that for k big enough, we have:

$$\left|\int f d\mu_k - \int f d\mu_\infty\right| \le 5\varepsilon.$$

This gives the convergence of integrals. □

VII.4. Quasi affine varieties and algebraic curves

In this section we recall some classical definitions in algebraic geometry which provide a complementary information for Convention III.1.1 in Chapter III. The main reference of this section is [CLO15]. Algebraic varieties are the central objects of study in algebraic geometry. In classical algebraic geometry (that is, the part of algebraic geometry in which one does not use schemes, which were introduced by Grothendieck around 1960) an algebraic variety is defined as the set of solutions of a system of polynomial equations over the real or complex numbers. In this section we assume that K is an algebraic closed field, in particular in this thesis we assume $K = \mathbb{C}$. To introduce affine varieties we need the following definition:

Definition VII.4.1. Given a field K and a positive integer n, we define the **n-dimensional affine space** over K to be the set

$$K^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in K\}.$$

We can now define affine varieties.

Definition VII.4.2. Let K be a field, and let $f_1, \ldots, f_s$ be polynomials in $K[x_1, \ldots, x_n]$. Then we set

$$V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in K^n | f_i(a_1, \ldots, a_n) = 0, \text{ for } 1 \le i \le s\}.$$

We call $V(f_1, \ldots, f_s)$ the **affine variety** defined by $f_1, \ldots, f_s$.

Conventions regarding the definition of an affine algebraic variety differ slightly. For example, some references consider the above as the definition of algebraic set. Furthermore, a nonempty affine algebraic set is called irreducible if it cannot be written as the union of two proper algebraic subsets. Then in some references, an affine algebraic variety require to be irreducible. However, in this thesis we only work with Definition VII.4.2. Using affine varieties we can define a natural topology on K^n. This topology is called the Zariski topology, introduced by Oscar Zariski. The Zariski topology is primarily defined by its closed sets. It is very different from topologies which are commonly used in the real or complex analysis. In particular, it is not Hausdorff.

Definition VII.4.3. Given a field K and a positive integer n, the **Zariski topology** of K^n is the topology whose closed sets are the affine algebraic varieties.

We again recall that in the above definition we consider Definition VII.4.2 for affine algebraic varieties. Moreover, any algebraic variety is equipped with a topology induced from the Zariski topology of K^n. Finally we are able to introduce quasi affine varieties.

Definition VII.4.4. A **quasi affine** variety is a Zariski open subset of an affine algebraic variety.

The final goal of this section is to introduce algebraic curves. To achieve this goal, we first explain about the dimension of an affine variety. The dimension of an affine algebraic variety may be defined in various equivalent ways. Here, we provide one which is valid simultaneously for both affine and quasi affine variety:

Definition VII.4.5. Let $V \in K^n$ be an affine (or quasi affine) variety, the dimension of V is the maximal length d of the chains $V_0 \subset V_1 \subset \cdots \subset V_d$ of distinct nonempty irreducible closed subsets of V.

As we mentioned in Convention III.1.1 in Chapter III, in this thesis varieties refer to quasi affine varieties. Moreover, an algebraic curve is a 1-dimensional quasi affine variety.

VII.5. Newton polytopes

In this section we define the Newton polytope, one of the important tools in studying the Mahler measure theory. Using the properties of the Newton polytope of a polynomial we obtain information about the topology of the algebraic curve associated to the polynomial (e.g genus of the algebraic curve). In Chapter III we studied the Mahler measure of a family of bivariate polynomials called $P_d(x, y)$. We use the notion of Newton polygon there to find the toric points of P_d. Moreover, to compute the genus of the algebraic curve associated with P_d we take advantage of the properties of its Newton polygon. Before starting we recall Notation II.2.10, where a monomial $x_1^{j_1} \cdots x_k^{j_k}$ is denoted by x^J for $J = (j_1, \ldots, j_k) \in \mathbb{Z}^k$. Thus, according to this notation a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$ can be written as $P = \sum_{J \in \mathbb{Z}^k} c_J x^J$, with $c_J \in \mathbb{C}$.

Definition VII.5.1. Let $P = \sum_{J \in \mathbb{Z}^k} c_J x^J \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$ with $C_J \in \mathbb{C}$ be a Laurent polynomial. The **Newton polytope** N_P of P is the convex hull of the exponents of the monomials appearing in P.

The N_P of a polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]$ is a convex polytope in $\mathbb{R}^k$ whose vertices lie in the lattice $\mathbb{Z}^k$. If $k = 2$ it is called **Newton polygon**.

Example VII.5.2. The Newton polygon of $P(x, y) = 1 + x + y$ is the following triangle;

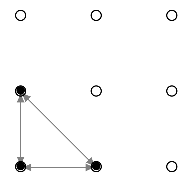

FIGURE 1. Newton polygon of $P(x, y) = x + y + 1$.

In Section II.2.2 we defined the action of the group of the bijective affine transformation over the set of the Laurent polynomials. From the definition, one can easily prove that if $P(x_1, \ldots, x_n)$ is a Laurent polynomial and $g : \mathbb{Z}^k \to \mathbb{Z}^k$ is a bijective affine map, then $N_{gP} = g(N_P)$, where $g(N_P) = \{g(x) | x \in N_P\}$. In [Bak94] Baker's theorem gives an upper-bound for the genus of a plane curve. Let us fix the following notation for the sequel:

Notation VII.5.3. Let $P(x, y)$ be a Laurent polynomial, so the number of interior lattice points of N_P in $\mathbb{Z}^2$ is denoted by h and the genus of the algebraic curve associated with $P(x, y) = 0$ is denoted by g.

According to [Bak94] the genus of the curve $P(x, y) = 0$ does not exceed h.

Definition VII.5.4 ([RV99])**.** Let $P(x, y) = \sum_{(m,n)\in\mathbb{Z}^2} c_{(m,n)} x^m y^n \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, so a side of N_P is denoted by τ and to emphasize on the fact that τ belongs to N_P we may write $\tau < N_P$. We parameterize a side clockwise around N_P and in such a way that $\tau_{(0)}, \tau_{(1)}, \ldots$ are the consecutive lattice points in τ. To every side we associate a one-variable polynomial, called **side polynomial**, denoted by $P_\tau(t)$, defined as follows;

$$P_\tau(t) := \sum_{k=0}^{\infty} c_{\tau(k)} t^k \in \mathbb{C}[t], \quad \tau < N_p,$$

and the above sum is naturally finite.

We notice that all roots of $P_\tau(t)$ are nonzero. Using the properties of Side polynomials associated to N_P one may extract information about the genus of the algebraic curve associated to P. The following lemma is a classical result going back to Baker [Bak94] in 1893.

Lemma VII.5.5 ([Bak94])**.** *For a polynomial $P(x, y)$ under a generic condition that every side polynomial of P has distinct roots, we have $g = h$.*

For example all curves whose polygons are represented in Fig. 2 (from [BZ20]) have genus 1 for a generic set of coefficients a, b, c. Another important example is the $P_d(x, y) = \sum_{0\leq i+j\leq d} x^i y^j$ family. We have already found the Newton polygon of $P_1 = x + y + 1$ in Example VII.5.2. We do the same for arbitrary P_d;

Example VII.5.6. For an arbitrary d, the Newton polygon of P_d is the triangle with vertices $\{(0, 0), (0, d), (d, 0)\}$, shown in Fig. 3 (the figure is T_4);

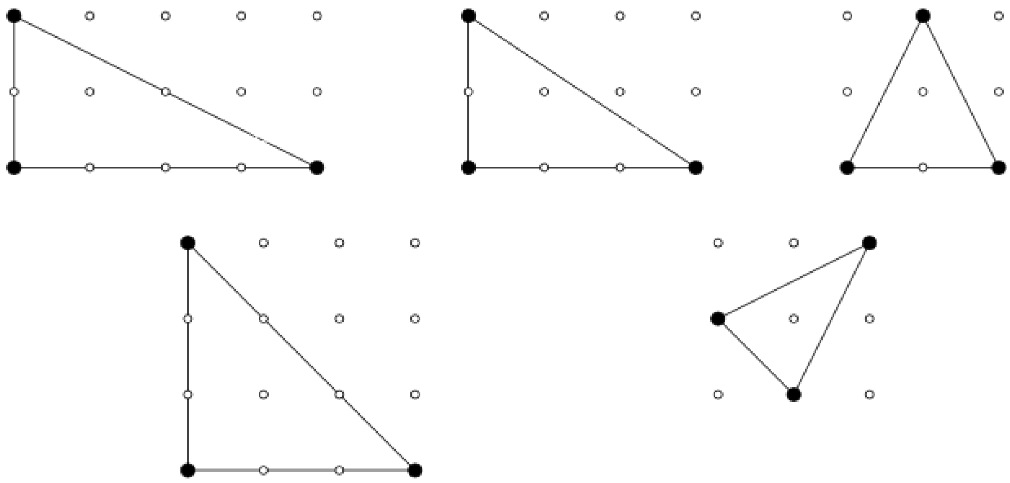

FIGURE 2. [BZ20] Newton polygons for the polynomials $P(x,y) = a + bx^4 + cy^2, a + bx^3 + cy^2, a + bx^2 + cxy^2, a + bx^3 + cy^3$ and $ax + by + cx^2y^2$.

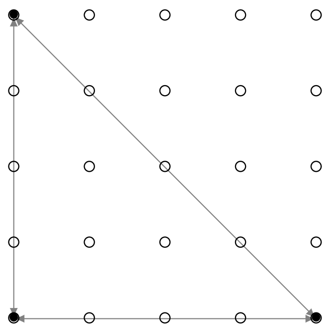

FIGURE 3. **Newton polygon of P_4.**

Notation VII.5.7. A triangle with vertices $\{(0,0),(0,d),(d,0)\}$, denoted by T_d.

The two following examples of Newton polygons can be easily verified by the reader and are needed for the computation of the toric points of P_d in Chapter III.

Example VII.5.8. The Newton polytope of a polynomial in one variable is a line. However, the multiplication of two univariate polynomials $H(x)$ and $Q(y)$ is a polynomial in two variables whose Newton polygon is a rectangle. Here we explain its construction by the following figure (see Fig. 4). Suppose that the red and blue points are respectively the non zero coefficients of H and Q in Fig. 4 then the Newton polygon is the black rectangle.

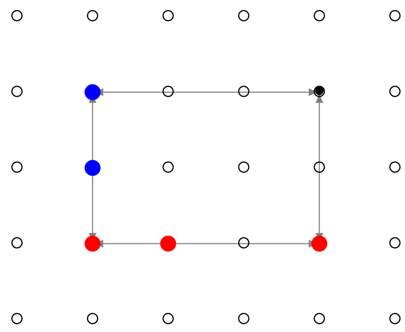

FIGURE 4. **Newton polygon of $H(x)Q(y)$.**

Let $P(x, y)$ be a polynomial in two variables. In Proposition III.2.2 we have already seen the definition of $P^*(x, y)$ which is $P(1/x, 1/y)$. We found the Newton polygon of P_d in Example VII.5.6, which according to Notation VII.5.7 is denoted by T_d. In the following we find the Newton polygon of P_d^*.

Example VII.5.9. The Newton polygon of $P_d^*(x, y)$ is the triangle with vertices $\{(0, 0), (-d, 0), (0, -d)\}$, denoted by T_d^*. (The Fig. 5 is T_4^*)

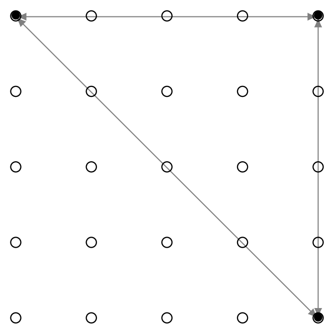

FIGURE 5. **Newton polygon of P_4^*.**

Example VII.5.10. The Newton polygon of the product of $x^i y^j$ and $P_d^*(x, y)$ is the triangle T_d^* which is translated respectively by i steps in the horizontal axis, and j steps in the vertical axis.

In Eq. (III.2.3) we searched for a good candidate for i and j such that $P_d(x, y)+x^i y^j P_d^*(x, y)$ factorizes into two univariate polynomials H and Q. The lemmas and examples mentioned in this Appendix helped us to compare the Newton polygons of the two sides of the equality and guess a good candidate. Since the Newton polygon of the R.H.S is a rectangle, to provide the same situation for the L.H.S we searched for i, j such that $x^i y^j P_d^*(x, y)$ does not have any denominator. Finally, in Lemma III.2.3, we proved that if $i = d + 1$ and $j = d$, the desired properties are verified.

We have seen many different examples of Newton polygones, especially related to $P_d(x, y)$. According to Lemma VII.5.5, in some generic cases the genus of the algebraic curve can be computed using the interior lattice point of the Newton polygon. One may ask the natural question about the genus of P_d.

Proposition VII.5.11. *The genus of P_d is $\frac{(d-1)(d-2)}{2}$.*

PROOF. The number of lattice points inside N_{P_d} is $\frac{(d-1)(d-2)}{2}$. To see that let $d > 3$ and count all the integer points (i,j) inside N_{P_d} (for $d = 1, 2$ is clear). They are the points (i,j) for which $i + j = k$, $2 \leq k \leq d-1$ with $i > 0, j > 0$. Thus, the number of the solutions is $\binom{k-2+(2-1)}{2-1} = k-1$, so we have $\sum_{k=1}^{d-1}(k-1) = \frac{(d-1)(d-2)}{2}$.
The polynomial associated with each face of N_{P_d} is in fact $1 + t + t^2 + \cdots + t^d$, which is equal to $\frac{t^{d+1}-1}{t-1}$ and all its roots are roots of unity and they are distinct. Therefore according to Lemma VII.5.5, the genus of P_d is equal to the number of the interior lattice points which is $\frac{(d-1)(d-2)}{2}$. □

An important remark from the previous observation is that when d goes to infinity the genus of P_d goes to infinity as well. A polynomial whose Newton polytope has all the side polynomials with only roots of unity is called **tempered**, so P_d is a tempered polynomial.

VII.6. A local estimator of the Concave and Convex functions

One of the main results of this thesis is the computation of the limit of $m(P_d)$, which is done in Chapter III. The computation of the limit is done in Chapter IV (also it is a partial result of Chapter V). In Chapter IV, we represented two different methods to do so. A direct computational method and a short method based on the generalization of the theorem of Boyd-Lawton. To follow the computations done in the direct method we need some prerequisites about concave and convex functions. This section is devoted to introducing these prerequisites. The domain of a convex or a concave function is a convex set. Thus, to begin this section we introduce the concept of a convex combination of a set of finite points which let us define convex sets.

Definition VII.6.1. A convex combination of the points $X_1, \ldots, X_n \in X$ is a point of the form $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$ where $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

The adjective convex is used in the above definition and is also used to describe a set or even a function. Using the previous definition, a set C of a vector space S is convex if the affine combination $(1-t)x + ty$ belongs to C, for all x and y in C, and t in the interval $[0,1]$. Since the affine combination of two points is the line segment connecting them, one can define a convex set equivalently as follows:

Definition VII.6.2. Let S be a vector space or an affine space over the real numbers. A subset C of S is convex if, for all x and y in C, the line segment connecting x and y is included in C.

According to the above definition the empty set and the whole space are convex. A Singleton is convex. Obviously the set which contains x, y and the line segment connecting x and y is convex. Triangles and squares are important example of convex sets which we used the most. In fact the concave functions (or even affine see Proposition VII.6.8) that were studied in Chapter IV are defined over squares or triangles. Let us recall the definition of convex and concave functions as well.

Definition VII.6.3. Let $f : X \to \mathbb{R}$ be a multivariate function where $X \subseteq \mathbb{R}^n$ is a convex set:

- f is called convex if:
$$\forall x_1, x_2 \in X, \forall t \in [0,1] : \qquad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$
- f is called strictly convex if:
$$\forall x_1 \neq x_2 \in X, \forall t \in (0,1) : \qquad f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

- A function f is said to be (strictly) concave function if $-f$ is (strictly) convex.

In fact, the above inequalities can be easily generalized to any convex combinations of points. For twice differentiable functions we have a criterion to check their concavity, which is explained in the following lemma. For a proof a see [QHA13].

Lemma VII.6.4 ([QHA13]). *Let $X \subseteq \mathbb{R}^n$ is an open convex set and $f : X \to \mathbb{R}$ is a twice differentiable function. Then, f is concave if the Hessian matrix of f is negative semi-definite for all values of $x_1, x_2, \ldots, x_n \in X$.*

As a simple example of a concave function consider $f(x, y) = y - x^2$. The Fig. 6 shows the graph of f. As we can see in Fig. 6 the tangent plane at a point is above the graph of f.

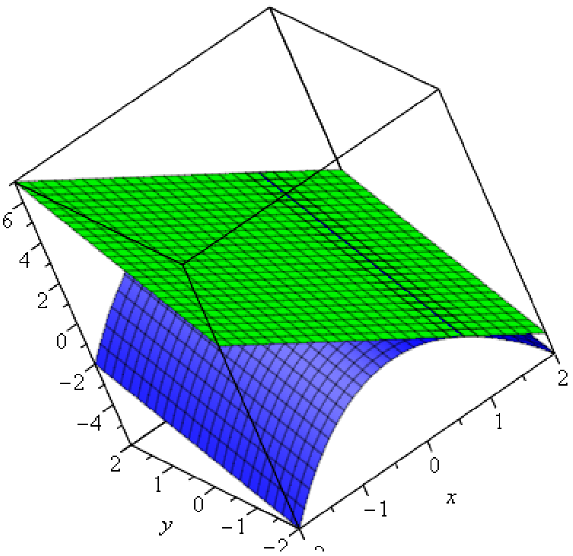

FIGURE 6. The tangent plane of the concave function $f(x, y) = y - x^2$ is above the graph of f.

This is not surprising and even if we change the point again the tangent plane of f is above the graph. This is an important property of concave functions.

Proposition VII.6.5. *[BA13, Page 81 Theorem 1] Let f be a differentiable concave function, so the tangent plane to the graph of f at (x_0, y_0) is above its graph.*

PROOF. Since $f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$ is the equation of the tangent plane of f at (x_0, y_0), we prove that for any (x, y) in the domain of f we have:

$$f(x, y) \leq f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0).$$

To prove that we define a new function $g(\lambda)$ as follows:

$$g(\lambda) := f((1 - \lambda)(x_0, y_0) + \lambda(x, y)).$$

Since f is concave, g is as well, and we have $g(\lambda) \geq (1 - \lambda)f(x_0, y_0) + \lambda f(x, y)$, which implies:

$$g(\lambda) \geq f(x_0, y_0) + \lambda(f(x, y) - f(x_0, y_0)).$$

As we can see $g(0) = f(x_0, y_0)$, therefore:

$$g(\lambda) - g(0) \geq \lambda(f(x, y) - f(x_0, y_0)).$$

In other words for all $\lambda \in (0, 1]$ we have:

$$\frac{g(\lambda) - g(0)}{\lambda - 0} \geq f(x, y) - f(x_0, y_0).$$

Thus when $\lambda \to 0$ we have:

$$g'(0) = \lim_{\lambda \to 0} \frac{g(\lambda) - g(0)}{\lambda - 0} \geq f(x, y) - f(x_0, y_0).$$

Let $X(\lambda) = x_0 + \lambda(x - x_0)$ and $Y(\lambda) = y_0 + \lambda(y - y_0)$, so $X(0) = x_0$ and $Y(0) = y_0$. Then by using the chain rule for computing $g'(\lambda)$, we have:

$$\begin{aligned}
g(\lambda) &= f((x_0 + \lambda(x - x_0), y_0 + \lambda(y - y_0)), \\
g'(\lambda) &= \frac{\partial f}{\partial x}|_{(X(\lambda), Y(\lambda))}(x - x_0) + \frac{\partial f}{\partial y}|_{(X(\lambda), Y(\lambda))}(y - y_0), \\
g'(0) &= \frac{\partial f}{\partial x}|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0, y_0)}(y - y_0).
\end{aligned}$$

then we have:

$$f(x, y) \leq f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0, y_0)}(y - y_0).$$

□

We took advantage of this property of concave functions in Chapter IV to introduce an upper estimator for vol which is a concave differentiable function inside T. For convenience, let us fix the following notation for the equation of the tangent plane.

Notation VII.6.6. Let $f(x, y)$ be a differentiable function, so the equation of the tangent plane of f at (x_0, y_0) is denoted by $\text{Tang}_f(x_0, y_0)$.

In Definition II.2.9 affine transformations were introduced. In the following lemma we characterize them using the concave and convex functions.

Lemma VII.6.7. *[HW11, Theorem 10.2.1] The transformation f is affine if and only if it is both convex and concave.*

Proof. In order to show that a convex and concave function f is an affine function, it is sufficient to show that $g(X) := f(X) - f(0)$ (which is also both convex and concave, and satisfies $g(0) = 0$) is a linear transformation. For $t > 0$ we can write $X = \frac{1}{t}.(tX) + (1 - \frac{1}{t})0$, so we have:

$$g(X) = g(\frac{1}{t}(tX) + (1 - \frac{1}{t})0) \overset{[1]}{=} \frac{1}{t}g(tX) + (1 - \frac{1}{t})g(0) = \frac{1}{t}g(tX).$$

In [1] we used the fact that g is concave so it should satisfy the inequality in Definition VII.6.3. It is also convex hence it satisfies the reverse inequality as well so we have:

$$tg(X) = g(tX).$$

The last equality, $tg(X) = g(tX)$, shows that g is linear, hence f is affine.
For the reverse we consider an affine function $A(x) = Mx + B$, where M and B are matrices.

Since the multiplication of matrices is distributive with respect to addition, we have:

$$A(\sum_{i=1}^{n}\lambda_i X_i) = M\sum_{i=1}^{n}\lambda_i X_i + B = \sum_{i=1}^{n} M\lambda_i X_i + B = \sum_{i=1}^{n}\lambda_i M X_i + B \overset{[2]}{=}$$
$$\sum_{i=1}^{n}\lambda_i M X_i + \sum_{i=1}^{n}\lambda_i B = \sum_{i=1}^{n}\lambda_i(MX_i + B) = \sum_{i=1}^{n}\lambda_i A(X_i).$$

In [2] we used that $\sum_{i=1}^{n}\lambda_i = 1$. The last equality shows that A is a concave and a convex function. $\square$

In Proposition VII.6.5, we introduced an upper estimator for a concave function at certain point, namely the tangent plane at that point. In Proposition VII.6.8 we introduce a local under estimator for a concave function by using affine functions. To visualize the following proposition see Fig. 7. We can find the graph of an affine function (the pink triangle) which is an under estimator for the concave function $f(x,y) = y - x^2$. For more information about this regard see [Mur10, Theorem 2.1].

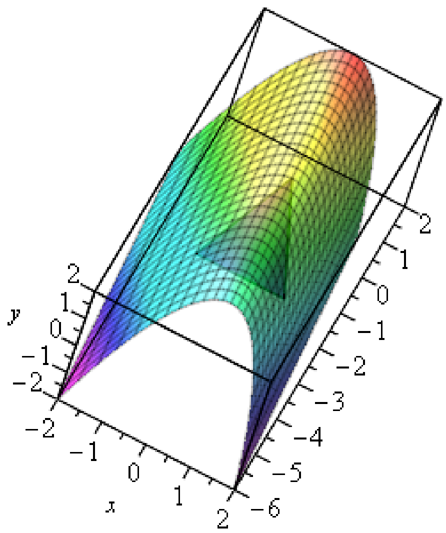

FIGURE 7. The affine under estimator of $f(x,y) = y - x^2$.

Proposition VII.6.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be concave, and such that the triangle with vertices a, b and c is contained in the domain of f. There exists a unique affine function $A : \mathbb{R}^2 \to \mathbb{R}$, such that $A(a) = f(a), A(b) = f(b), A(c) = f(c)$ and for any point p in the triangle with vertices a, b, c we have :*

$$A(p) \leq f(p).$$

PROOF. For the moment suppose that the affine function A with $A(a) = f(a), A(b) = f(b), A(c) = f(c)$ exists. Since a triangle is a convex set, for any point p inside the triangle abc, there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+$ such that $\lambda_1+\lambda_2+\lambda_3 = 1$ and $p = \lambda_1 a+\lambda_2 b+\lambda_3 c$. According to the concavity of f we have $f(p) = f(\lambda_1 a+\lambda_2 b+\lambda_3 c) \geq \lambda_1 f(a)+\lambda_2 f(b)+\lambda_3 f(c)$. Moreover, A is affine and according to Lemma VII.6.7 we have $A(p) = A(\lambda_1 a+\lambda_2 b+\lambda_3 c) = \lambda_1 A(a)+\lambda_2 A(b)+\lambda_3 A(c)$. Since $A(a) = f(a), A(b) = f(b)$ and $A(c) = f(c)$ we have:

$$A(p) \leq f(p).$$

To prove the existence and the uniqueness of the affine function $A : \mathbb{R}^2 \to \mathbb{R}$ we introduce the matrix $M := [m_{11} m_{12}]$ and the constant C such that:

$$A(x, y) = [m_{11} m_{12}] \begin{bmatrix} x \\ \\ y \end{bmatrix} + C = m_{11}x + m_{12}y + C.$$

Therefore $A(x, y)$ is the equation of a plane in $\mathbb{R}^3$. The conditions $A(a) = f(a), A(b) = f(b)$ and $A(c) = f(c)$ are equivalent to the fact that the plane is passing through the points $(a, f(a)), (b, f(b))$ and $(c, f(c))$, and a, b, c are not collinear. Therefore we can uniquely determine this plane. $\square$

In Definition IV.2.1 we introduced the function vol, which is concave over $T = T_1 \cup T_2$. For the definition of T_1 and T_2 we refer to Lemma IV.3.6. We finish this appendix by proving the following lemma which is needed for the computation done in Observation IV.3.3. The proof is just a computation and can be basically deduced from Proposition VII.6.8 and properties of affine function.

Lemma VII.6.9. *Let χ be the affine under estimator of the concave function* $\mathrm{vol}(\theta, \alpha)|_{T_1 \cup T_2}$ *on the triangle $[a, b, c] \in T_1 \cup T_2$, explained in Proposition VII.6.8, then we have:*

$$\iint\limits_{[a,b,c]} \chi(\theta, \alpha) dA = \mathrm{area}[a, b, c] \left(\frac{1}{3} \mathrm{vol}(a) + \frac{1}{3} \mathrm{vol}(b) + \frac{1}{3} \mathrm{vol}(c) \right).$$

PROOF. We assume that $[a, b, c]$ belongs to T_1 and prove the above lemma (the other case is similar). The vertices a, b, c are the points in $\mathbb{R}^2$ with the following coordinates:

- $a = (\theta_0, \alpha_0)$ and $\chi(\theta_0, \alpha_0) = \mathrm{vol}(a)$,
- $b = (\theta_0, \alpha_0 + \frac{2\pi}{d+1})$ and $\chi(\theta_0, \alpha_0 + \frac{2\pi}{d+1}) = \mathrm{vol}(b)$,
- $c = (\theta_0 + \frac{2\pi}{d+1}, \alpha_0)$ and $\chi(\theta_0 + \frac{2\pi}{d+1}, \alpha_0) = \mathrm{vol}(c)$.

Then we have:

$$\iint\limits_{[a,b,c]} \chi(\theta, \alpha) dA = \int\limits_{\alpha_0}^{\alpha_0+\frac{2\pi}{d+1}} \int\limits_{\theta_0}^{\theta_0+\alpha_0+\frac{2\pi}{d+1}-\alpha} \chi(\theta, \alpha) d\theta d\alpha. \tag{VII.6.1}$$

We use a change of variables in order to simplify the computation of the integral. To do so, consider the following transformation;

- $\theta = f_1(s, t) = \theta_0 + \frac{2\pi}{d+1} s$,
- $\alpha = f_2(s, t) = \alpha_0 + \frac{2\pi}{d+1} t$.

The Jacobian of the transformation is as follows:

$$\frac{\partial(\theta, \alpha)}{\partial(s, t)} = \begin{pmatrix} \frac{\partial\theta}{\partial s} & \frac{\partial\theta}{\partial t} \\ \\ \frac{\partial\alpha}{s} & \frac{\partial\alpha}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{2\pi}{d+1} & 0 \\ \\ 0 & \frac{2\pi}{d+1} \end{pmatrix} = \frac{4\pi^2}{(d+1)^2}.$$

Notice that with this transformation the triangle with vertices $\{(0,0),(0,1),(1,0)\}$ denoted by $\triangle$ in the st-plane is sent to the triangle (a,b,c) in the $\theta\alpha$-plane. Therefore, we have:

$$\int\limits_{\alpha_0}^{\alpha_0+\frac{2\pi}{d+1}} \int\limits_{\theta_0}^{\theta_0+\alpha_0+\frac{2\pi}{d+1}-\alpha} \chi(\theta,\alpha)d\theta d\alpha = \iint\limits_{[a,b,c]} \chi(\theta,\alpha)dA \overset{[1]}{=} \iint\limits_{\triangle} \chi(f_1(s,t),f_2(s,t))\ |\frac{\partial(\theta,\alpha)}{\partial(s,t)}|\ \ d\bar{A}$$

$$= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \int\limits_0^{1-t} \chi(\theta_0+\frac{2\pi}{d+1}s, \alpha_0+\frac{2\pi}{d+1}t)dsdt.$$

Here, we used $d\bar{A}$ in st-integral above after [1] to denote that it will be in terms of ds and dt once we convert to two single integrals rather than $d\theta$ and $d\alpha$, which we used for dA. We notice that the point $(\theta_0+\frac{2\pi}{d+1}s, \alpha_0+\frac{2\pi}{d+1}t)$ can be written as the following affine combination of the points in $\mathbb{R}^2$: $(1-t-s)(\theta_0,\alpha_0)+t(\theta_0+\frac{2\pi}{d+1},\alpha_0)+s(\theta_0,\alpha_0+\frac{2\pi}{d+1})$. Then, by using the property of affine function from Lemma VII.6.7, we have:

$$\begin{aligned}\chi(\theta_0+\frac{2\pi}{d+1}s, \alpha_0+\frac{2\pi}{d+1}t) &= \chi((1-t-s)(\theta_0,\alpha_0)+t(\theta_0+\frac{2\pi}{d+1},\alpha_0)+s(\theta_0,\alpha_0+\frac{2\pi}{d+1}))\\ &= (1-t-s)\chi(\theta_0,\alpha_0)+t\chi(\theta_0+\frac{2\pi}{d+1},\alpha_0)+s\chi(\theta_0,\alpha_0+\frac{2\pi}{d+1})\\ &= (1-s-t)\operatorname{vol}(a)+t\operatorname{vol}(b)+s\operatorname{vol}(c).\end{aligned}$$

We compute the double integral:

$$\begin{aligned}&\frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \int\limits_0^{1-t} \chi(\theta_0+\frac{2\pi}{d+1}s, \alpha_0+\frac{2\pi}{d+1}t)dsdt\\ &= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \int\limits_0^{1-t} ((1-s-t)\operatorname{vol}(a)+t\operatorname{vol}(b)+s\operatorname{vol}(c))dsdt\\ &= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \int\limits_0^{1-t} ((1-t)\operatorname{vol}(a)+t\operatorname{vol}(b))dsdt + \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \int\limits_0^{1-t} s(\operatorname{vol}(c)-\operatorname{vol}(a))dsdt\\ &= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 [s((1-t)\operatorname{vol}(a)+t\operatorname{vol}(b))]_0^{1-t}dt + \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 [\frac{s^2}{2}(\operatorname{vol}(c)-\operatorname{vol}(a))]_0^{1-t}dt\\ &= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 ((1-t)^2\operatorname{vol}(a)+(1-t)t\operatorname{vol}(b))dt + \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 \frac{(1-t)^2}{2}(\operatorname{vol}(c)-\operatorname{vol}(a))dt\\ &\quad= \frac{4\pi^2}{(d+1)^2}\int\limits_0^1 (\frac{(1-t)^2}{2}\operatorname{vol}(a)+\frac{(1-t)^2}{2}\operatorname{vol}(c)+t(1-t)\operatorname{vol}(b))dt\\ &\quad= \frac{4\pi^2}{(d+1)^2}[\frac{1}{2}(t-t^2+\frac{t^3}{3})\operatorname{vol}(a)+\frac{1}{2}(t-t^2+\frac{t^3}{3})\operatorname{vol}(c)+(\frac{t^2}{2}-\frac{t^3}{3})\operatorname{vol}(b)]_0^1\\ &\quad= \frac{4\pi^2}{(d+1)^2}(\frac{1}{6}\operatorname{vol}(a)+\frac{1}{6}\operatorname{vol}(b)+\frac{1}{6}\operatorname{vol}(c))\\ &\quad= \operatorname{area}[a,b,c](\frac{1}{3}\operatorname{vol}(a)+\frac{1}{3}\operatorname{vol}(b)+\frac{1}{3}\operatorname{vol}(c)).\end{aligned}$$

Finally we have:

$$\iint\limits_{[a,b,c]} \chi(\theta,\alpha)dA = \operatorname{area}[a,b,c]\left(\frac{1}{3}\operatorname{vol}(a) + \frac{1}{3}\operatorname{vol}(b) + \frac{1}{3}\operatorname{vol}(c)\right).$$

□

VII.7. Extended Euler– Maclaurin formula

The aim of the last section of the Appendix is to introduce the basic definitions for presenting the Euler-Maclaurin summation formula for smooth and certain types of singular functions. In Chapter V we computed the asymptotic expansion of $m(P_d)$ using the Euler–Maclaurin method for functions with a singularity. In mathematics, the Euler-Maclaurin formula is a formula for the difference between an integral and a closely related sum. As we know a **Riemann sum** for a function is a certain kind of approximation of the integral of the function by a finite sum. Therefore, for a definite integral we can define a Riemann sum to approximate it and then by using the **Euler–Maclaurin formula** we can measure the error between the exact value of the integral and the value of Riemann sum. The **Euler–Maclaurin summation formula** is a method for generating an asymptotic expansion for a function. An **asymptotic expansion** of a given function f in a fixed neighborhood is a finite sum of functions which gives a good approximation of the behavior of f in the considered neighborhood. Since the sum is finite, the question of convergence does not arise. However, we sometimes speak of “asymptotic series” for a sum of an infinity of terms. This infinite sum is most often formal, because the series is generally divergent.

In Chapter V, using the Euler–Maclaurin summation formula we computed the asymptotic expansion of $m(P_d) - m(P_\infty)$. We basically refer to the article of Sidi and Israeli [SI88] for Euler–Maclaurin formulas. However their article is based on the work done by Navot in [Nav62] and [Nav61]. We notice that we introduce the formulas mentioned in [SI88] only for functions defined on $[0,1]$, with $f(0) = f(1) = 0$, since the function we work with in Chapter V has this property.

Proposition VII.7.1 ([SI88])**.** *Let $f(x)$ be $2m$ times differentiable on $[0,1]$ with $f(0) = f(1) = 0$, then*

$$\int_0^1 f(x)dx - \frac{1}{d}\sum_{j=0}^{d} f\left(\frac{j}{d}\right) = \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!}\left[f^{(2\mu-1)}(0) - f^{(2\mu-1)}(1)\right]\frac{1}{d^{2\mu}} + O\left(\frac{1}{d^{2m}}\right), \tag{VII.7.1}$$

where B_k is the kth Bernoulli number which will be introduced in Definition VII.7.2.

We do not need the explicit formula of the error terms in our computation, but for more information see [SI88, Theorem 1]. In the following, we introduce Bernoulli numbers and then we are able to introduce the Euler–Maclaurin formula for some special type of singular functions as well.

VII.7.1. Bernoulli numbers. Many characterizations of the Bernoulli numbers have been found in the last 300 years, and any could be used to introduce these numbers. Here we introduce only the recursive definition and some explicit formulas.

Recursive definition: Here we give a recursive definition for Bernoulli numbers.

Definition VII.7.2 ([Lar19])**.** Bernoulli numbers obey the sum formula:

$$B_0 = 0 \text{ and } \sum_{k=0}^{n} \binom{n+1}{k} B_k = n+1 \text{ for } n \geq 1. \tag{VII.7.2}$$

See [Lar19, Proposition 4.1] for more information. One of the most important facts about Bernoulli numbers is the following property that we use in our computations in Chapter V.

Fact VII.7.3. *[Mol12, Proposition 13.3.4] For every odd $n > 1$, $B_n = 0$.*

Moreover, for every even $n > 0$, B_n is negative if n is divisible by 4 and positive otherwise. The first few Bernoulli numbers used frequently in our computations are:

$$B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}.$$

Eq. (VII.7.2) is a valuable recurrence relation which proves one of the key properties of the Bernoulli numbers that they are rational.

Remark VII.7.4. Notice that there are two conventions for Bernoulli numbers used in the literature, denoted by B_n^- and B_n^+; they differ only for $n = 1$, where $B_1^- = -1/2$ and $B_1^+ = +1/2$. In this thesis, we work with B_n^+ and denoted by B_n, unless mention that the preference convention is changed.

Explicit formulas: Here we give an explicit definition of Bernoulli numbers:

Fact VII.7.5 ([Hig70])**.** *An explicit formula for the Bernoulli numbers is as follows:*

$$B_n^- = \sum_{k=1}^{n} \sum_{v=1}^{k} (-1)^v \binom{k}{v} \frac{v^n}{k+1}.$$

The Bernoulli numbers can be expressed in terms of the Riemann Zeta function. By means of the functional equation and the Gamma reflection formula the following relation can be obtained [Arf85]:

$$B_{2n} = \frac{(-1)^{n+1} 2 (2n)!}{(2\pi)^{2n}} \zeta(2n) \quad \text{for } n \geq 1.$$

Fact VII.7.6. *From the above equation, we conclude that the values of the Zeta function for positive even numbers are irrational, since Bernoulli numbers are rational and they are multiplied by powers of π and some non-zero rational numbers.*

VII.7.2. Euler– Maclaurin formula for the function with singularities.

Proposition VII.7.7 ([SI88])**.** *Let $f(x)$ be $2m$ times differentiable on $[0,1]$ and let $F(x) = \log(x) f(x)$ with $F(0) = F(1) = 0$, then*

$$\int_0^1 F(x)\ \mathrm{d}x - \frac{1}{d} \sum_{j=0}^{d} F(\frac{j}{d}) =$$

$$- \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{(2\mu)!} F^{(2\mu-1)}(1) \left(\frac{1}{d^{2\mu}}\right) + \sum_{\mu=0}^{2m-1} [\zeta'(-u) + \zeta(-\mu) \log(d)] \frac{f^{(\mu)}(0)}{\mu!} \left(\frac{1}{d^{\mu+1}}\right) + O(\frac{1}{d^{2m}}),$$

where $\zeta'(t) = d\zeta(t)/dt$.

For more information about the reminder see [SI88, Theorem 3]. With the notation used in the article of Sidi and Israeli, we only stated the theorem for the case $S = 0$.

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