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Tautological rings of moduli spaces of curves

à Michel qui était si amateur de Savoir

à Yvon qui m'aurait encouragé à fond!

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Résumé

Les espaces de modules de Riemann répondent au problème de la classification des surfaces de Riemann compactes d'un genre donné. Le sujet de cette thèse est la cohomologie de l'espace des modules des courbes d'un genre donné avec un certain nombre de points marqués. La description de cet anneau a été initiée par D. Mumford puis C. Faber avait proposé une description de l'anneau tautologique des espaces de modules sans points marqués. Une première source de relations provient des relations A. Pixton démontrées par A. Pixton, R. Pandharipande et D. Zvonkine mais on ne sait pas si elles sont complètes. Une autre source de relations utilisée dans ce travail sont les relations de A. Buryak, S. Shadrin et D. Zvonkine. Avant cette thèse, il y avait peu de résultats sur l'anneau tautologique d'espaces de modules de courbes avec un nombre quelconque de points marqués. Cette thèse donne une description complète des l'anneaux tautologiques des espaces de modules de courbes de genres 0, 1, 2, 3 et 4.

Un des résultats ayant demandé beaucoup de travail est le groupe de degré 2 de l'anneau tautologique des espaces de modules de courbes lisses de genre 4. Ce groupe demande un travail sur l'annulation de certaines classes tautologiques sur le bord de la compactification de Deligne-Mumford de l'espace des modules en plus d'un astucieux travail numérique.

L'espace des modules des courbes réelles de genre 0 et sa théorie de l'intersection sont également étudiés. On peut alors démontrer plusieurs résultats analogues à ceux obtenus dans le cas complexe comme l'équation de la corde. On démontre une formule donnant les nombres d'intersection.

Summary

The problem of the moduli spaces of compact Riemann surfaces is the problem of the classification of compact Riemann surfaces of a certain genus. The topic of this thesis is the cohomology of the moduli spaces of curves of a certain genus with marked points and more precisely its subbring called tautological ring. The description of the tautological ring has been initiated by D. Mumford, then C. Faber conjectured a description of the moduli space of curves without marked points. A source of tautological relations are Pixton's relations proven by A. Pixton, R. Pabndharipande and D. Zvonkine. Another source of relations are relations of A. Buryak, S. Shadrin and D. Zvonkine. Before this thesis, there were only few results on the tautological ring of curves with any number of marked points. This thesis gives a complete description of the tautological rings of moduli curves of genera 0, 1, 2, 3 and 4 with any number of marked points.

A result which needed a lot of work is the group of degree 2 of the tautological ring of the moudli space of smooth curves of genus 4. We need to work on the vanishing of some tautological classes on the boundary of the Deligne-Mumford compactification of the moduli space of curves and a clever numerical work.

The moduli space of real curves of genus 0 and its intersection theory are also studied. Then we can show several results which are analogous to results in the complex case like the string equation. One result of this thesis is a formula giving intersection numbers of products of xi classes.

Contents

1 Introduction

1.1 Tautological rings of moduli spaces

The main topic of this PhD is the cohomology ring of the moduli space $\mathcal{M}_{g,n}$ of genus g smooth curves with n marked points, more precisely, its subring called the *tautological ring* and denoted $R^*(\mathcal{M}_{g,n}) \subset H^{\text{even}}(\mathcal{M}_{g,n}, \mathbb{Q})$.

The study of the tautological ring has been initiated by D. Mumford in [30]. In 1999, C. Faber computed the tautological rings of the moduli space $\mathcal{M}_g$ of smooth curves without marked points for $g \leq 15$ [9]. Building on this work, Faber and Zagier conjectured a set of relations between the tautological classes of $\mathcal{M}_g$, namely the κ-classes; these were later proven to be true relations by R. Pandharipande and A. Pixton [31] using localization on the space of stable quotients.

A generalization of these relations to the space $\overline{\mathcal{M}}_{g,n}$ of stable curves was proposed by A. Pixton in [33]. These conjectural relations were proven to be true relations by R. Pandharipande, A. Pixton, and D. Zvonkine in [32]. The proof is based on the study of a cohomological field theory obtained from Witten's r-spin classes (see [36], [34], [3], [29], [10] for details about these classes). Relations on $\mathcal{M}_{g,n}$ can be obtained by restriction of these relations from $\overline{\mathcal{M}}_{g,n}$.

The tautological ring of $\mathcal{M}_{g,n}$ is generated by tautological classes $\psi_1, \dots, \psi_n \in R^1$ associated with the marked points and by the classes $\kappa_m \in R^m$ for $m \geq 1$ (see Sections 3.2.2, 3.2.3). Madsen and Weiss [27] proved that there are no relations between these classes in degree $d \leq g/3$. Moreover, for an integer d satisfying $1 \leq d \leq g/3$, we have $H^{2d}(\mathcal{M}_{g,n}, \mathbb{Q}) = R^d(\mathcal{M}_{g,n})$ and $H^{2d-1}(\mathcal{M}_{g,n}, \mathbb{Q}) = 0$.

Building on E. Looijenga's work [26], E. Ionel [18] proved the following vanishing property: $R^d = 0$ for $d \geq g \geq 1$. The rank of the group $R^{g-1}(\mathcal{M}_{g,n})$ is known to be equal to n for any $g \geq 2$. Moreover, the classes ψ_i^{g-1} form a basis of this space, and an explicit expression in this basis of all other elements is known. Following a conjecture by D. Zvonkine this was proved by A. Buryak, S. Shadrin, and D. Zvonkine in [2].

1.2 Main results

This thesis contains two main results:

- a complete description of the tautological ring $R^*(\mathcal{M}_{g,n})$ for $g \leq 4$ and any n;
- a computation of intersection numbers of Stiefel-Whitney classes on the *real* genus 0 moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$.

The rings $R^*(\mathcal{M}_{g,n})$ for $g \leq 4$. An element of the tautological ring can be shown to vanish by proving that it lies in the span of Pixton's tautological relations restricted to $\mathcal{M}_{g,n}$. On the other hand, since the top degree (that is, degree $g-1$) part of the tautological ring is explicitly known by [2], one can show that an element of $R^d(\mathcal{M}_{g,n})$ does not vanish by proving that its product with a well chosen element of $R^{g-1-d}(\mathcal{M}_{g,n})$ does not vanish in $R^{g-1}(\mathcal{M}_{g,n})$. Thus the tautological ring is in a way "bounded" from above and from below. It turns out that for $g \leq 4$ these bounds coincide with each other; in other words, every element that cannot be shown to vanish can be shown to not vanish and vice versa.

Proving this eventually boils down to computing ranks of large matrices. For instance, in the hardest case, $g = 4$, $d = 2$, one has to compute the rank of a matrix of size

$$\left(\frac{n(n+1)}{2} + 1\right) \times (n^2 + 1).$$

This involves the following steps. One needs to extract a well-chosen square matrix out of the rectangular matrix above; find a family of eigenvectors of this matrix (treated as the matrix of a linear map); find a family of invariant planes of the matrix; show that the determinant on each invariant plane does not vanish; determine the span of the eigenvectors and invariant planes; determine the action of the linear map in the quotient space by this span; show that the determinant of the linear map on the quotient space does not vanish; treat differently the cases where, due to a numerical exception, one of the invariant planes contains one of the eigenvectors. It would be interesting to find a geometric interpretation of all these eigenvectors and invariant planes, but so far we do not have any.

Denote by $r_g^d(n)$ the rank of $R^d(\mathcal{M}_{g,n})$.

Theorem 1. *We have*

$$\begin{array}{l} r_1^0(n) = 1, \\ r_2^0(n) = 1, \;\; r_2^1(n) = n, \\ r_3^0(n) = 1, \;\; r_3^1(n) = n+1, \;\; r_3^2(n) = n, \\ r_4^0(n) = 1, \;\; r_4^1(n) = n+1, \;\; r_4^2(n) = \frac{n(n+1)}{2} + 1, \;\; r_4^3(n) = n. \end{array}$$

As a by-product of our computations we also show that in genus 3 and degree 2, the relations of Buryak-Shadrin-Zvonkine are obtained from Pixton's relations by a change of variables whose coefficients are polynomials in n.

Intersection numbers on $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$. On the real moduli space written $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of genus 0 stable curves one defines n real line bundles L_i: they are the cotangent line bundles to the marked points. Denote by $\xi_i \in H^1(\overline{\mathcal{M}}_{0,n}, \mathbb{Z}/2\mathbb{Z})$ the first Stiefel-Whitney class of L_i. These classes are the real analogs of ψ-classes, except that they belong to H^1 instead of H^2 and are $\mathbb{Z}/2\mathbb{Z}$-valued rather than $\mathbb{Z}$-valued. The question is then to find the intersection number

$$\xi_1^{d_1} \dots \xi_n^{d_n} \in \mathbb{Z}/2\mathbb{Z}$$

for $d_1 + \cdots + d_n = n - 3$.

The intersection theory of the real genus 0 moduli space was extensively studied by P. Etingof, A. Henriques, J. Kamnitzer, and E. M. Rains [6]; however the natural question of finding the above intersection number was not addressed.

Write every integer d_i in the binary system and denote by $\delta_j^{(i)}$ the j-th digit of d_i from the end. For instance, if $d_i = 6$, we have $\delta_0^{(i)} = 0$, $\delta_1^{(i)} = 1$, $\delta_2^{(i)} = 1$, $\delta_j^{(i)} = 0$ for $j \geq 3$.

Theorem 2. *Let $d_1, \dots, d_n$ be nonnegative integers such that $\sum_{i=1}^n d_i = n - 3$. The intersection number*

$$\xi_1^{d_1} \dots \xi_n^{d_n} \in \mathbb{Z}/2\mathbb{Z}$$

is equal to 1 if and only if we have

$$\sum_{i=1}^{n} \delta_j^{(i)} \leq 1$$

for every $j \geq 0$. Otherwise the intersection number vanishes.

An equivalent way to formulate this theorem is to say that the intersection number is equal to 1 if there are no carryovers in the binary addition of the integers d_i, and vanishes as soon as there is at least one carryover.

The proof of this theorem is based on the string equation. The proof of the string equation for the ξ-classes is very similar to its proof in the classical case of ψ-classes; on the other hand, it is easy to see that the intersection numbers are uniquely determined by the string equation.

1.3 The organization of the thesis

In Sections 2 and 3 we introduce the moduli spaces of Riemann surfaces with marked points and their tautological rings. The next two sections contain the study of the tautological rings in low genus: $g \leq 3$ in Section 4 and $g = 4$ in Section 5. Finally, Section 6 contains the computations of intersection numbers of Stiefel-Whitney classes on real genus 0 moduli spaces.

2 Moduli spaces

2.1 Preliminaries

Definition 1. A *locally small category* is a class of objects $\mathcal{C}$ with a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ or $\mathrm{Hom}(X,Y)$ assigned to every pair of objects $X, Y \in \mathcal{C}$. For every triple of objects $X, Y, Z \in \mathcal{C}$, a composition map

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$

is given. The composition is associative. For any object X, an element $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ called the *identity* is distinguished. For any objects X, Y any $f \in \mathrm{Hom}(Y,X)$ and $g \in \mathrm{Hom}(X,Y)$ we have

- $\mathrm{id}_X \circ f = f$,
- $g \circ \mathrm{id}_X = g$.

All categories we consider will be locally small.

Definition 2. The *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ has the same objects as $\mathcal{C}$, but for two objects A, B we have

$$\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

Example 1. The first example of category is the category of sets **Set** whose objects are sets and for $X, Y \in \mathbf{Set}$ a homomorphism

$$X \to Y$$

is a map from X to Y.

Definition 3. For $\mathcal{C}, \mathcal{D}$ two categories, we call a functor from $\mathcal{C}$ to $\mathcal{D}$ a map

$$F : \mathcal{C} \to \mathcal{D}$$

such that an object $X \in \mathcal{C}$ is sent to an object $F(X) \in \mathcal{D}$ and for $Y \in \mathcal{C}$, a morphism $f \in \mathrm{Hom}(X,Y)$ is sent to a morphism $F(f) \in \mathrm{Hom}(F(X), F(Y))$ such that

- If $f = \mathrm{id}_X$, $F(f) = \mathrm{id}_{f(X)}$.
- For $Z \in \mathcal{C}$ and $g \in \mathrm{Hom}(Y, Z)$, $F(g \circ f) = F(g) \circ F(f)$.

Definition 4. For $\mathcal{C}, \mathcal{D}$ two categories and $F, G : \mathcal{C} \to \mathcal{D}$ two functors we call a *natural transformation* $\eta : F \to G$ a map from $\mathcal{C} \to \mathrm{Hom}_{\mathcal{D}}$ sending $X \in \mathcal{C}$ to a morphism $\eta_X : F(X) \to G(X)$ and such that, for $Y \in \mathcal{C}$ and $f \in \mathrm{Hom}(X, Y)$,

$$\eta_Y \circ F(f) = G(f) \circ \eta_X.$$

If, for every $X \in \mathcal{C}$, η_X is an isomorphism, then we call η a natural isomorphism or simply isomorphism of functors.

Definition 5. We denote by $Fct(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$, the natural transformations being the morphisms of the category.

Definition 6. An object I in a category $\mathcal{C}$ is an *initial object* if, for every $X \in \mathcal{C}$, $\mathrm{Hom}(I, X)$ has exactly one element.

Yoneda's lemma. Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$ and a functor $A \in Fct(\mathcal{C}^{op}, \mathbf{Set})$. Denote by h_X the functor sending an object $Y \in \mathcal{C}$ to the object $\mathrm{Hom}_{\mathcal{C}}(Y, X) \in \mathbf{Set}$. Then we have

$$\mathrm{Hom}_{Fct(\mathcal{C}^{op}, \mathbf{Set})}(h_X, A) \simeq A(X).$$

2.2 Fine and coarse moduli spaces

A moduli problem consists in the classification of objects (schemes, varieties, sheaves...) and the construction of a structured set representing the equivalence classes of the objects, the equivalence relation having to be defined. The structure of this set should reflect the evolution of the objects. We usually define the objects over a base topological space B as a map $\mathcal{F} \to B$ whose fibers are objects we want to classify and call this map a family of the objects of interest over B.

For example, suppose our objects are rational curves with quadruples of pairwise distinct numbered points. We define a family of such objects over a base scheme B as a map $p : \mathcal{F} \to B$ with four disjoint sections $\sigma_1, ..., \sigma_4$ such that, for $b \in B$, the fiber $p^{-1}(b)$ is isomorphic to $\mathbb{CP}^1$. Thus $(p^{-1}(b), \sigma_1(b), ..., \sigma_4(b))$ is a rational curve with four marked points. Two such families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there is an isomorphism $\phi : \mathcal{F} \to \mathcal{F}'$ such that $p' \circ \phi = p$.

The notion of pullback is crucial in the definition of equivalence between families, in order to compare families over different base spaces. The notion of equivalence we define should be stable by pullback. In other words, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two equivalent families over a scheme B and $f : B' \to B$ is a morphism, then $f^*\mathcal{F}_1$ and $f^*\mathcal{F}_2$ are equivalent families over B'.

Now the idea in the construction of a fine moduli space is to find a *universal family* $\mathcal{U} \to M$. This means that, for any family $\mathcal{F} \to B$, there exists a unique morphism $f : B \to M$ such that $\mathcal{F}$ is equivalent to $f^*\mathcal{U}$.

If such a family exists the space M is called a *fine moduli space* of the objects of interest. The points of M parametrize equivalence classes of objects, that is, any equivalence class of objects corresponds to a unique point $m \in M$.

In the language of categories, a moduli problem is a problem of representability of a functor from the category of schemes to the category of sets.

Let S be a scheme. The functor of points of S is defined as the functor

$$\begin{aligned} h_S : \mathbf{Sch} &\to \mathbf{Set} \\ X &\to \mathrm{Hom}(X, S). \end{aligned}$$

A functor F is said to be *representable* if there exists a scheme M such that $F \simeq h_M$. In this case, for any scheme X, the set of equivalence classes of families over X is in a one-to-one correspondence with the set of morphisms $X \to M$.

Now, families and pull-backs of families give rise to a contravariant functor $F : \mathbf{Sch} \to \mathbf{Set}$ which takes a scheme X to the set of equivalence classes of families over X and a morphism of schemes f to the pull-back morphism f^*. Suppose F is represented by a scheme M. Then, to any family $\mathcal{F}$ of objects over X corresponds a unique morphism $f : X \to M$. This is indeed a categorical way of saying what has been previously explained.

Fine moduli spaces do not always exist. For instance, in the case of Riemann surfaces, the existence of a fine moduli space is prevented by the automorphisms of Riemann surfaces. In such cases, the moduli functor is not representable, and we look should look for a different kind of representation where the object in question is not unique.

Definition 7. Let F be a moduli functor. A *coarse moduli space* for F is a pair (M, v) where M is a scheme and $v : F \to h_M$ is a natural transformation, moreover this pair should be initial among all such pairs.

By Yoneda's lemma, natural transformations $h_M \to h_{M'}$ are in one-to-one correspondence with morphisms $M \to M'$; thus any natural transformation $F \to h_{M'}$ factors through h_M.

2.3 Riemann surfaces

For a topological surface X, and $x \in X$, we call *chart* the data of

- a neighborhood U of x,
- a homeomorphic map ϕ from U to the unit disc of $\mathbb{C}$.

We say that two charts $(U, \phi), (U', \phi')$ are compatible if the maps $\phi' \circ \phi^{-1}$ and $\phi \circ (\phi')^{-1}$ are holomorphic on their domains of definition.

An *atlas* is a covering of X by compatible charts. Two atlases are *equivalent* if the charts are compatible.

Definition 8. A *Riemann surface* is a topological surface X endowed with an equivalence class of atlases.

A morphism between two Riemann surfaces X and X' is a map $f : X \to X'$ such that, for every $x \in X$ and two charts (U, ϕ) and (V, ψ) around x and $f(x)$, the map $\psi \circ f \circ \phi^{-1}$ is holomorphic.

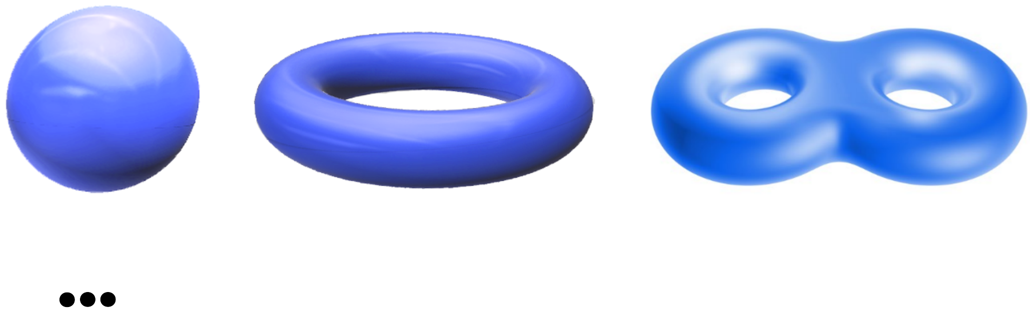
Theorem 3. *Up to isomorphism there exists exactly three connected and simply connected Riemann surfaces:*

- *the Riemann sphere,*
- *the complex plane*
- *or the unit disc in the complex plane.*

Every connected Riemann surface is the quotient of one of the above by a free proper holomorphic action of a discrete group.

From now on we will always assume that our Riemann surfaces are connected and compact unless specified otherwise. We will also refer to Riemann surfaces as *complex curves.* The *genus* of a complex curve is the genus of its underlying topological surface.

Any complex curve of genus 0 is isomorphic to the complex projective line (also called the Riemann sphere) denoted by $\mathbb{CP}^1$ or $\mathbb{P}^1$. A curve of genus 1 is called an *elliptic curve.*



Definition 9. For $n \in \mathbb{N}$, a *curve with n marked points* is a complex curve C with n pairwise distinct numbered points $x_1, \ldots, x_n \in C$. We will write such a curve $(C, x_1, \ldots, x_n)$.

Definition 10. Let $(C, x_1, ..., x_n)$ and $(C', x'_1, ..., x'_n)$ be two curves with n marked points. An *isomorphism* between these two marked curves is an isomorphism $\phi : C \to C'$ such that $\phi(x_i) = x'_i$ for all i.

Definition 11. A smooth curve $(C, x_1, ..., x_n)$ of genus g with n marked points is said to be *stable* if one of the following equivalent conditions is satisfied:

- $2g - 2 + n > 0$,
- $|\mathrm{Aut}(C, x_1, ..., x_n)| < \infty$.

We will prove the equivalence between the two conditions in Section 2.4.

Definition 12. A smooth curve $(C, x_1, ..., x_n)$ of genus g with n marked points is said to be *semi-stable* if

$$2g - 2 + n \geq 0.$$

From the equivalence between the stability conditions, the definition of semi-stable curve allow such a curve to have an infinite automorphism group.

Now we define the moduli space of smooth stable curves with marked points as a set.

Definition 13. Let $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$. We call the *moduli space of smooth curves of genus g with n marked points* the set

$$\mathcal{M}_{g,n} := \{\text{curves of genus } g \text{ with } n \text{ marked points}\}/\sim,$$

where $\sim$ is the relation of "being isomorphic".

Note that the stability condition excludes the pairs $(g,n) = (0,0), (0,1), (0,2)$ and $(1,0)$.

Example 2.

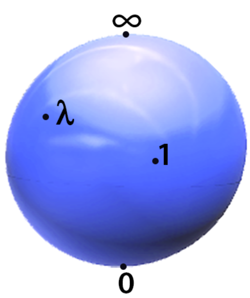
$\boldsymbol{\mathcal{M}_{0,3}}$ Let (C, x_1, x_2, x_3) be a stable curve of genus 0 with three marked points. We can always find an isomorphism

$$(C, x_1, x_2, x_3) \simeq (\mathbb{CP}^1, 0, 1, \infty).$$

Thus all curves of genus 0 with 3 marked points are isomorphic, so

$$\mathcal{M}_{0,3} = \{pt\}.$$

$\boldsymbol{\mathcal{M}_{0,4}}$ Let (C, x_1, x_2, x_3, x_4) be a curve of genus 0 with 4 marked points. There exists a unique isomorphism from C to $\mathbb{CP}^1$ sending x_1 to 0, x_3 to 1 and x_3 to ∞. The image of x_4 is sent to a $\lambda \in \mathbb{CP}^1$ which is unique for each isomorphism class and cannot be equal to $0, 1$ or ∞.



Thus we get

$$\mathcal{M}_{0,4} = \mathbb{CP}^1 \backslash \{0, 1, \infty\}.$$

$\boldsymbol{\mathcal{M}_{1,1}}$ An elliptic curve can be written $\mathbb{C}/L$, where L is a lattice, that is, a descrete additive subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$. Two elliptic curves $\mathbb{C}/L_1$ and $\mathbb{C}/L_2$ are isomorphic if and only if there exists a complex number $z \neq 0$ such that $L_1 = zL_2$.

Let us choose a direct basis (z_1, z_2) of L and denote $\tau = z_2/z_1 \in \mathbb{H}$. The group $\mathrm{SL}(2, \mathbb{Z})$ of direct basis changes acts on $\mathbb{H}$ via

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

The moduli space $\mathcal{M}_{1,1}$ is the quotient $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$.

The examples above indicate that each moduli space has a natural topology and that moduli spaces are not necessarily compact. Later we will describe their compatification $\overline{\mathcal{M}}_{g,n}$ using nodal stable curves.

2.4 The stability condition

We will prove the following theorem

Theorem 4. *For a compact Riemann surface of genus g, if we fix n points such that*

$$2g - 2 + n > 0$$

then the number of automphisms of the surface is finite.

2.4.1 The genus zero case

If we don't fix any point The group of automorphisms of the surface is the group of the Möbius transformations and is then infinite.

If we fix one point Suppose that a Möbius transformation $\dfrac{az+b}{cz+d}$ fixes z_1. Then

$$\begin{aligned}\frac{az_1+b}{cz_1+d} =& z_1\\ az_1+b =& cz_1^2+dz_1\\ cz_1^2+(d-a)z_1-b =& 0\end{aligned}$$

We can chose d and a in infinitely many ways to get the same quantity $d-a$. Thus the number of automorphisms of $\mathbb{CP}^1$ fixing z_1 is infinite.

If we fix two points Let z_1 and z_2 be two distinct points of $\mathbb{CP}^1$ that we want to be fixed. We suppose that $z_1 = \infty$, otherwise we get this situation applying the automorphism of $\mathbb{CP}^1$ defined by $z \to \dfrac{1}{z-z_1}$. Then we consider the homographies $z \to az+b$, thus we are looking for a and b in $\mathbb{C}$ such that $az_2+b = z_2$, this is equivalent to $\dfrac{b}{1-a} = z_2$. There is an infinite number of solutions, then the number of automorphism of $\mathbb{CP}^1$ fixing two points z_1 and z_2 is infinite.

If we fix three points We know that there is a unique automorphism fixing these three points.

2.4.2 The genus 1 case

If we don't mark any points. An elliptic curve is given by the quotient of $\mathbb{C}$ by a lattice L. The translations of $\mathbb{C}$ induce isomorphisms of the elliptic curve. Thus the automorphism group is infinite.

If one point is marked.

Proposition 1. *Let L and M be two lattices in $\mathbb{C}$ and let's write $X = \mathbb{C}/L$ and $Y = \mathbb{C}/M$ the corresponding elliptic curves. Let $f : X \to Y$ be a holomorphic nonconstant map. Then f is unramified.*

Proof. An elliptic curve has genus 1, applying the Riemann-Hurwitz formula we get

$$\deg f \times 0 + \sum_{p \in X} [\mathrm{mult}_p(f) - 1] = 0.$$

Thus $\sum_{p \in X} [\mathrm{mult}_p(f) - 1] = 0$, that is, for any point $p \in X$, we have $\mathrm{mult}_p(f) = 1$. Therefore f is unramified. □

Proposition 2. *Let L and M be two lattices in $\mathbb{C}$. Denote by $(X, x) = \mathbb{C}/L$ and $(Y, y) = \mathbb{C}/M$ the quotient elliptic curves with one marked point each, the marked point being the image of the lattice. Let $f : X \to Y$ be a nonconstant map such that $f(x) = y$. Then f is induced by a map $g : \mathbb{C} \to \mathbb{C}$ sending z to γz for some $\gamma \in \mathbb{C}^*$ such that $\gamma(L) \subset M$.*

Proof. We will denote $\pi_X : \mathbb{C} \to X$ and $\pi_Y : \mathbb{C} \to Y$ the natural projections. By Proposition 1, f is a nonramified covering, so $f \circ \pi_Y$ is also a nonramified covering and therefore can be lifted to a map g to the universal covering of Y. Since $f(x) = y$ we can choose g in such a way that $g(0) = 0$.

Since f is holomorphic and unramified, so is g. For any $l \in L$, we have $\pi_Y \circ g(l) = f \circ \pi_X(l) = 0$, so $g(l) \in M$ and, for any $z \in \mathbb{C}$, $f \circ \pi_X(z+l) = f \circ \pi_X(z) = \pi_Y \circ g(z+l)$, so $\pi_Y \circ g(z+l) = \pi_Y \circ g(z)$. Thus $g(z+l) = g(z)$ modulo M, we can write this as $g(z+l) - g(z) \in M$.

Now, for $l \in L$, define the function $\omega_l(z) = g(z+l) - g(z)$. Since g is holomorphic, ω_l is homolorphic, hence it is continuous. Since M is discrete, ω_l is constant. Thus, for any $z \in \mathbb{C}$, we have $\omega_l'(z) = g'(z+l) - g'(z) = 0$. So g' is invariant under translation by lattice points. We write P a fundamental domain of L. P is compact so $g'(P) = g'(\mathbb{C})$ is bounded. By Liouville's theorem, g' is constant. We obtain that, for any $z \in \mathbb{C}$, $g(z) = \gamma z + \delta$, for some complex numbers γ and δ. Since $g(0) = 0$, we have $\delta = 0$, so $g(z) = \gamma z$. $\square$

Proposition 3. *Defining f as before and denoting by g the function inducing f for a certain $\gamma \in \mathbb{C}$, f is an isomorphism if and only if $\gamma L = M$.*

Proof. Let's first suppose that f is an isomorphism. There exist a map $p : Y \to X$ such that $f \circ p = id_Y$ and $p \circ f = id_X$. That map is induced by a map $q : \mathbb{C} \to \mathbb{C}$ such that, for any $z \in \mathbb{C}$, $q(z) = \delta z$ for some complex number δ such that $\delta M \subset L$. We have $\delta = \gamma^{-1}$, because f and p are inverse to each other. So we have $\gamma\delta M \subset \gamma L \subset M$ which gives $M \subset L \subset M$, so $\gamma L = M$.
Let's suppose now that $\gamma L = M$. Let's write q the function $z \mapsto \gamma^{-1} z$, q induces a map $p : Y \to X$. We have

$$f \circ p \circ \pi_Y = f \circ \pi_X \circ q = \pi_Y \circ g \circ q = \pi_Y,$$

so $f \circ p = id_Y$. In the same way we obtain that $p \circ f = id_X$. Then f is an isomorphism. $\square$

To study the automorphism group of an elliptic curve we will need the two next results.

Lemma 1. *For an elliptic curve defined by a lattice L, an automorphism is given by a complex number γ. Then $||\gamma|| = 1$ and γ is a root of unity.*

Proof. Let $l \in L$, we write $R := ||l||$ and $B(0, R)$ the closed disc centered at the origin of radius R. $l \in B(0, R)$ so $L \cap B(0, R) \neq \emptyset$ and $B(0, R)$ is a compact subset of $\mathbb{C}$, so $L \cap B(0, R)$ is finite. Let ω' be an element of L of minimal length α. From the previous proposition, $\gamma L = L$, then there exist $\omega \in L$ sucht that $\gamma\omega = \omega'$. Since $\omega' \neq 0$, $\omega \neq 0$ and $||\gamma\omega|| = \alpha \Leftrightarrow ||\gamma|| = \frac{\alpha}{||\omega||}$. α is the minimum length of the nonzeroes elements of L so $||\gamma|| \leq 1$.

On the other hand, $\gamma\omega' \in L\backslash\{0\}$, so $||\gamma\omega'|| \geq \alpha$. Thus we get $||\gamma|| \geq \frac{\alpha}{||\omega'||} = 1$. Finally $||\gamma|| = 1$.
Let's consider $\omega \in L\backslash\{0\}$ and let's write $R := ||\omega||$ and $C(0, R)$ the circle centered at the origin of radius R. For any $k \in \mathbb{Z}$, $||\gamma^k|| = 1$, then $\gamma^k\omega \in C(0, R)$. $T := \{\gamma^k\omega, k \in \mathbb{Z}\}$. We have $T \subset L \cap C(0, R)$. L is discrete and $C(0, R)$ is compact
then $L \cap C(0, R)$ is finite, then T is also finite. Thus there exist $m, n \in \mathbb{Z}$ different such that $\gamma^m\omega = \gamma^n\omega$. Let's suppose that $n < m$ and write $k := m - n$. We have $\gamma^k = 1$. Then γ is a kth root of unity. □

Lemma 2. *If γ is a root of unity satisfying the equation*

$$z^2 - mz - n$$

then γ is a kth root of unity for $k = 3, 4$, or 6.

Proof. We suppose that γ is a primitive kth root of unity. Then $\mathbb{Q}(\gamma)$ is an extension of $\mathbb{Q}$ of degree $[\mathbb{Q}(\gamma) : \mathbb{Q}] = \phi(k)$, where ϕ is the Euler function. Since γ satisfies a quadratic equation with coefficients in $\mathbb{Q}$, $[\mathbb{Q}(\gamma) : \mathbb{Q}] \leq 2$. $k > 2$ so $\phi(k) > 1$, we obtain $\phi(k) = 2$). Hence $k = 3, 4$ or 6. □

Proposition 4. *$L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, $X = \mathbb{C}/L$, $f \in \mathrm{Aut}_0(X)$, γ inducing f, we chose an element $\omega \in L$ of minimal length. Then γ is of minimal length and L is generated by ω and $\gamma\omega$.*

Proof. For ω of minimal length and γ of length 1, $||\gamma^k\omega||$ has minimal length. It is now also clear that $\omega\mathbb{Z} + \gamma\omega\mathbb{Z} \subset L$. Si there exists $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ such that

$$\begin{cases} \omega = m_1\omega_1 + m_2\omega_2 \\ \gamma\omega = n_1\omega_1 + n_2\omega_2 \end{cases}$$

Let's write $v := (\omega, \gamma\omega) \in \mathbb{C}^2$ and $w := (\omega_1, \omega_2) \in \mathbb{C}^2$.
If v and w are not linearly independent,there exist $z_0 \in \mathbb{C}$ such

that $v = z_0\omega$, this is equivalent to

$$\begin{cases} \omega = z_0\omega_1 \\ \gamma\omega = z_0\omega_2 \end{cases}$$

Because ω_1 and ω_2 are linearly independent, these relations give $m_1 = n_2 = z_0$ and $n_1 = m_2 = 0$, and we obtain

$$||\omega|| = |n_1|||\omega_1|| \text{ and } ||\gamma\omega|| = |n_2|||\omega_2||.$$

By minimality of the lengths of ω and $\gamma\omega$, $m_1 = \pm 1$ and $n_2 = \pm 1$, so $\omega = \pm\omega$ and $\omega_2 = \pm\gamma\omega$.
If v and w are linearly independent, there is a matrix A such that $\det A \neq 0$ and $v = Aw$. We write

$$M := \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}.$$

We have $|\det A| \geq 1$. If $|\det A| > 1$, then $||w|| < ||v|| = \sqrt{2}||\omega||$. This is in contradiction with the fact that the length of ω is minimal, hence $|\det A| = 1$, so $\det A = \pm 1$ and the coefficients of A^{-1} are integers. Thus $\omega_1, \omega_2 \in \omega\mathbb{Z} + \gamma\omega\mathbb{Z}$.

In all cases, $L \subset \omega\mathbb{Z} + \gamma\omega\mathbb{Z}$ so $L = \omega\mathbb{Z} + \gamma\omega\mathbb{Z}$ hence ω and $\gamma\omega$ generate L. □

2.4.3 The genus 2 case

A theorem due to Hurwitz can be formulated as follows

Theorem 5. *X a Riemann surface of genus ≥ 2, then*

$$|\mathrm{Aut}(X)| \leq 84(g-1)$$

Proof. We have a projection $\pi : X \to \frac{X}{\mathrm{Aut}(X)}$. We are interested in the number of sheets of that covering, that number is the number of automorphisms. Let $q \in \frac{X}{\mathrm{Aut}(X)}$ and $p \in \pi^{-1}(q)$. For any $p'\pi^{-1}(q)$ there is a $g \in \mathrm{Aut}(X)$ such that $g(p') = p$. Now consider $h \in \mathrm{Stab}(p)$, then $g^{-1}hg(p') = p'$, hence $g\mathrm{Stab}(p)g^{-1} \subset \mathrm{Stab}(p')$. Conversely, for $f \in \mathrm{Stab}(p')$, the equality $gfg^{-1}(p) = p$ shows that $\mathrm{Stab}(p') \subset g\mathrm{Stab}(p)g^{-1}$ and thus $\mathrm{Stab}(p') = g\mathrm{Stab}(p)g^{-1}$. Due to this equality, all elements of the fiber of q has the same number of stabilizators, we write that number n_q and we are interested in $|\mathrm{Aut}(X) = n_q|\pi^{-1}(q)|$. Applying Riemann-Hurwitz formula to π we obtain the following equality

$$\chi(X) = |\mathrm{Aut}(X)| \cdot \chi\left(\frac{X}{\mathrm{Aut}(X)}\right) - \sum_{p \in X}(e_p - 1).$$

Here e_p denotes the ramification number of the point $p \in X$. If we denote by g the genus of X and g' the genus of $\frac{X}{\mathrm{Aut}(X)}$ the previous equality can be rewritten as

$$2 - 2g = |\mathrm{Aut}(X)| \cdot (2 - 2g') - \sum_{q \in X/\mathrm{Aut}(X)} |\pi^{-1}(q)|(n_q - 1),$$

which is equivalent to

$$2g - 2 = |\mathrm{Aut}(X)| \cdot \left[2g' - 2 + \sum_{q \in X/\mathrm{Aut}(X)} \left(1 - \frac{1}{n_q}\right)\right].$$

Denote by $C = 2g' - 2 + \sum_{q \in X/\mathrm{Aut}(X)} \left(1 - \frac{1}{n_q}\right)$ the factor in the right-hand side. Since $g \geq 2$, it follows that C is positive. We claim that the smallest possible value for this positive rational number is $\frac{1}{42}$. Indeed, if $g' \geq 2$ then $C \geq 2$. If $g' = 1$ then $C \geq \frac{1}{2}$, since every summand in the sum is at least $\frac{1}{2}$. Finally, if $g' = 0$, we need to find a collection of positive integers n_q such that $\sum \left(1 - \frac{1}{n_q}\right)$ exceeds 2 by the smallest possible amount. It is easy to see that the optimal collection is $(2, 3, 7)$. In this case we have

$$C = 2 \cdot 0 - 2 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{7}\right) = \frac{1}{42}.$$

Thus we have $2g - 2 \geq 42 \cdot |\mathrm{Aut}(X)|$, or

$$|\mathrm{Aut}(X)| \leq 84(g-1).$$

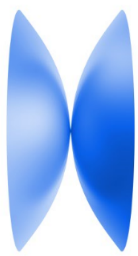
$\square$

Remark 1. The automorphism groups of curves up to genus 3 are described in [25]. In particular, the so-called *Klein quartic* is shown to be the unique genus 3 curve whose automorphism group has a cardinal equal to the Hurwitz bound $84(g-1) = 168$. The automorphism groups of curves of genus 4 have been described by the same authors in [23] and [24]. Later work based on these two articles is presented in [19].

2.5 Nodal curves

Definition 14. A node of a complex algebraic curve is a point which has a neighborhood which is isomorphic to a neighborhood of the origin in the analytic space given by the equation $xy = 0$ in $\mathbb{C}^2$.

We have the following picture of a node



Definition 15. A nodal curve is a complex algebraic curve whose only singularities are nodes.

Now the term *curve* can refer to a smooth curve or a nodal curve. The definition of the genus remains the same, however, we need to change a little bit the definitions of marked pointed curve and the stability condition.

Definition 16. For $n \in \mathbb{N}$ a curve with n marked points is a curve C with n of its smooth points $x_1, ..., x_n$ pairwise distinct.

Definition 17. For $g, n \in \mathbb{N}$, an n-pointed curve of genus g $(C, x_1, ..., x_n)$ is said to be stable if its automorphism group is finite. The equivalent condition has to be changed. Let's write $C_1, ..., C_r$, where $r \in \mathbb{N}$, the irreducible components of the curve. For any $i \in \mathbb{N}$ such that $1 \leq i \leq r$, let's write g_i the genus of C_i and

$$\begin{aligned} n_i = \#\{&\text{the number of marked points}\} \\ &+\#\{\text{the number of nodes on } C_i\}. \end{aligned}$$

The curve $(C, x_1, ..., x_n)$ is stable if, for any $i \in \mathbb{N}$ such that $1 \leq i \leq r$, we have $2g_i - 2 + n_i > 0$.

Example 3. The following curve is not stable

while this one is

We have a new way of calculating the genus of a curve.

Definition 18. With the same notations as the previous definition and denoting δ the number of nodes of $(C, x_1, ..., x_n)$, the genus of the curve is

$$g = \sum_{i=1}^{r} g_i - r + \delta + 1.$$

We have now a new moduli space

Definition 19. For $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$, we define the moduli space of stable nodal curves of genus g with n marked points the set

$$\overline{\mathcal{M}}_{g,n} := \{\text{stable nodal curves of genus} \\ g \text{ with } n \text{ marked points}\}/\sim .$$

When $n = 0$, we often write $\overline{\mathcal{M}}_g$. It is possible to simply talk about curve without precising whether the curve is stable, smooth or nodal; we will always deal with stable curve unless otherwise stated and hopefully the context makes clear whether the curve is smooth or can have nodes.

Example 4. $\overline{\boldsymbol{\mathcal{M}}}_{\mathbf{0,3}}$: We saw that $\mathcal{M}_{0,3} = \{pt\}$, then

$$\overline{\mathcal{M}}_{0,3} = \{pt\}.$$

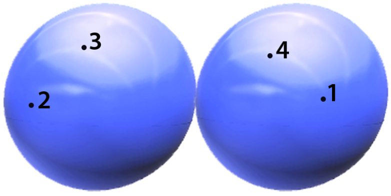
$\overline{\boldsymbol{\mathcal{M}}}_{\mathbf{0,4}}$: The problem here was that the marked points are not allowed to coincide. Let's see what happens while two marked points tend to each other. Let (C, x_1, x_2, x_3, x_4) be a 4-pointed curve of genus 0. There is an isomorphism

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{CP}^1, 0, 1, \infty, \lambda).$$

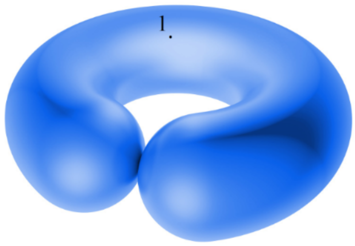
While $\lambda \to 0$, we obtain $(C, x_1, x_2, x_3, x_4 = x_1)$, a 4-pointed curve of genus 0 where the first and the fourth points coincide. Now, if we make the change of coordinates $z \mapsto \frac{z}{t}$, we have an isomorphism

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{CP}^1, 0, \frac{1}{\lambda}, \infty, 1).$$

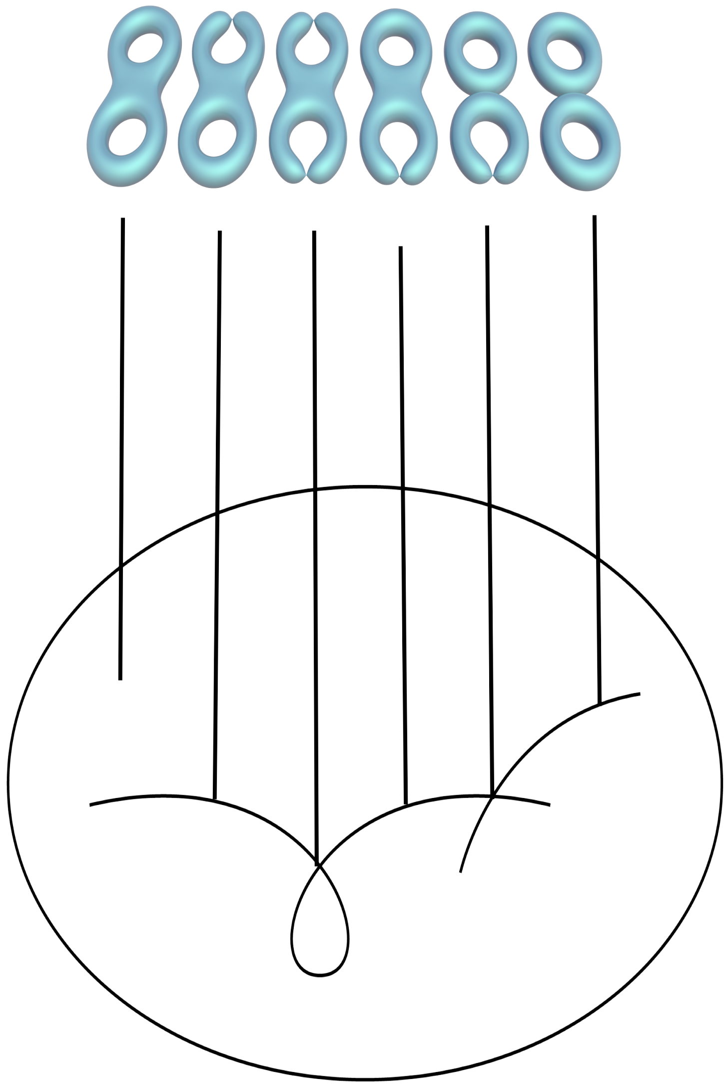
When $\lambda \to 0$, we obtain $(C, x_1, x_2, x_3 = x_2, x_4)$, so a curve with the second and the third points which coincide. The best way to consider these two cases at the same time is to chose the following curve as limit when $x_1 \to x_4$:



$\overline{\boldsymbol{\mathcal{M}}}_{\mathbf{1,1}}$: The problem was that we couldn't have $\tau = \infty$, we solve this problem by adding to $\mathcal{M}_{1,1}$ the class of the following curve.



We can think of the moduli space of curves of genus g as a space whose points represent curves which are representants of their isomorphism classes and that space contains loci representing curves with nodes, these loci can intersect other kind of loci of curves with nodes or intersect themselves and, in this case, presenting twice the same kind of nodes. The following image is a good representation for $g = 2$.



The universal curve is the set of all curves above the moduli space glued together.

The moduli space $\overline{\mathcal{M}}_{g,n}$ was first defined as a Deligne-Mumford stack as a compactification of $\mathcal{M}_{g,n}$ [4]. This notion will not be treated here, stacks are shortly presented in [22] Chapter 2 and the paper [11] tries to make the notion of Deligne-Mumford stack accessible. In [5], $\overline{\mathcal{M}}_{g,n}$ is defined as a Deligne-Mumford stack. $\overline{\mathcal{M}}_{g,n}$ also has a structure of analytic space [17].

We can also see $\overline{\mathcal{M}}_{g,n}$ as an orbifold of dimension $3g-3+n$ such that the stabilizer of a point $[(C, x_1, ..., x_n)]$ is the group of automorphisms of $(C, x_1, ..., x_n)$, see [16] Section 2.D. The universal curve $\overline{\mathcal{C}}_{g,n}$ is also a smooth compact orbifold, its dimension is $3g-2+n$.

Example 5. Let's return to the case of the moduli space of curves of genus 1 with one marked point. Such a curve is defined by a lattice L of $\mathbb{C}$ and the stabilizer of a lattice is the group of rotations of C that preserve L.

- For a generic L, the stabilizer is $\mathbb{Z}/2\mathbb{Z}$.
- For the lattice represented by $(1, i)$, the stabilizer is $\mathbb{Z}/4\mathbb{Z}$.
- For the lattice represented by $(1, \frac{1+i\sqrt{3}}{2})$, the stabilizer is $\mathbb{Z}/4\mathbb{Z}$.

At a point $[(C, x_1, ..., x_n)]$ corresponding to a curve without nontrivial automorphism, $\mathcal{M}_{g,n}$ is smooth. Around such a point $[(C, x_1, ..., x_n)]$, $\mathcal{M}_{g,n}$ is isomorphic to an open subset of $\mathbb{C}^{3g-3+n}$ quotiented by the group of automorphisms of $(C, x_1, ..., x_n)$ which is trivial... The singularities are the loci corresponding to curves with at least one nontrivial automorphism.

Proposition 5. *There is an orbifold morphism*

$$p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

such that any stable curve $(C, x_1, ..., x_n)$ is isomorphic to exactly one fiber of p, the stabilizer at a point $[(C, x_1, ..., x_n)] \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of $(C, x_1, ..., x_n)$ $\mathrm{Aut}(C)$ and $\mathcal{M}_{g,n}$ is an open dense suborbifold of $\overline{\mathcal{M}}_{g,n}$ whose preimage is $\mathcal{C}_{g,n}$.

The previous proposition defines $\mathcal{C}_{g,n}$ and $\overline{\mathcal{C}}_{g,n}$ which are the universal family for the classifications of, respectively, smooth curves and nodal curves. We then call them universal curves, which explains the notations.

We will see it as a projective algebraic variety, the following theorem allows us to do so.

Proposition 6. *(Deligne-Mumford-Knudsen [20],[21]) There exists a coarse moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves and these spaces are projective varieties.*

We have

$$dim_{\mathbb{C}}\overline{\mathcal{M}}_{g,n} = 3g - 3 + n.$$

Riemann knew that number [35].

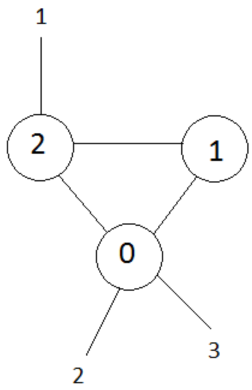
To each nodal curve we can associate its dual graph.

Definition 20. The *dual graph* associated to a curve is the graph such that

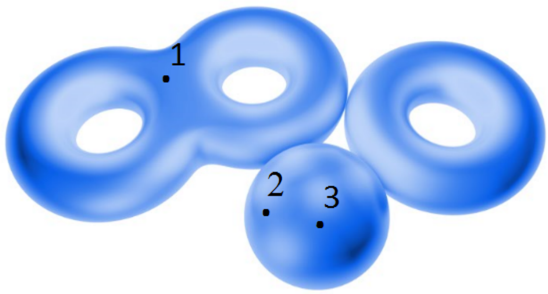
- its vertices correspond to the irreducible components of the curve;
- to each vertex one assigns the genus of the corresponding component;
- an edge between two vertices corresponds to a node between the components corresponding to the vertices, in particular, a loop corresponds to a self-intersection of a component;
- the graph has special half-edges called *legs* corresponding to the markings and attached to the vertices whose components contain the markings.

The graph is *stable* if the condition $2g(v) - 2 + n(v) > 0$ is satisfied for every vertex v. Here $g(v)$ is the genus assigned to v and $n(v)$ is the degree of v.

Example 6. The following graph



encodes the (topological type of the) following curve



For a given graph Γ, if $V(\Gamma)$ denotes the set of vertices of Γ, if g_v is the integer attached to the vertex $v \in V(\Gamma)$ and δ is the number of edges, we have the same formula as previous giving the genus of the corresponding curve

$$g = \sum_{v \in V(\Gamma)} g_v - \#V(\Gamma) + \delta + 1.$$

Dual graphs are of great use in the study of the moduli spaces of stable curves. They are an essential tool in the proof of Pixton's relations[32] which is a crucial result for the work presented here. A graph Γ corresponds to a stratum of the moduli space of curves, we define

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v);n(v)}.$$

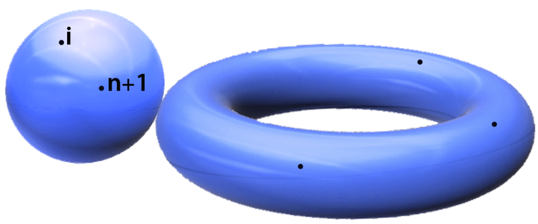
and we have a morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}.$$

2.6 Morphisms of moduli spaces $\overline{\mathcal{M}}_{g,n}$

Three types of morphisms between stable curves are interesting for us. An important operation on curves is the stabilization. Considering a semi-stable curve, the components keeping the curve from being stable are spheres with two special points, at least one of them being a node. If there are component with two nodes and no other special point, the stabilization contracts all these components to the nodes connecting them to a stable component. We are left with a curve with only the other possibility, the case where only one semistable component has a special point being a node and the other being a marked point. The contraction of the component to the node, which is no longer a node, becoming the marked point of the component, yields to a stable curve. That operation can be performed on families [1].

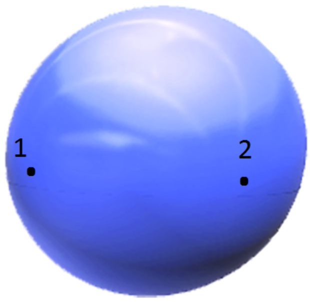
Forgetful map:$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.
The forgetful map is the map which forgets the last marked point and makes the new curve stable by contracting the unstable components to the nodes attaching them to the other components.



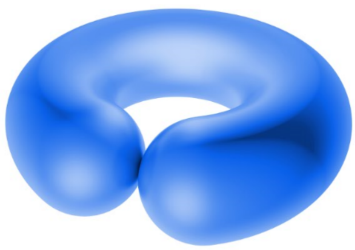
becomes

Attaching maps:
The attaching map of non separating kind is the map which glues two of the marked points, creating then a new handle to the curve. Starting from a curve of genus g with n marked points, we get a curve of genus $g+1$ with $n-2$ marked points by identifying the $(n-1)$th and the nth marked points together.

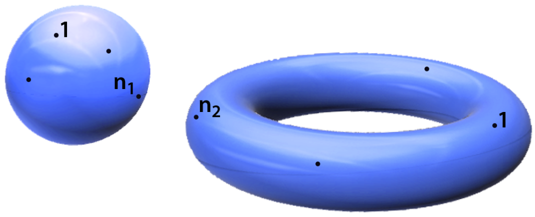
Example 7. For $g = 0$ and $n = 2$



becomes



The attaching map of separating kind glues the last marked points of two curves. Starting from a pair of curves, one of genus g_1 with n_1 marked points and a second one of genus g_2 with n_2 marked points, we get a curve of genus $g_1 + g_2$ with $n_1 + n_2$ relabeled marked points by identifying the n_1th point of the first curve with the n_2th marked point of the second curve.



is sent to

These operations can be performed on families of curves and thus we have two morphism between moduli spaces

$$q : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$$

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}.$$

More details about these properties are explained in [1], *Chap X*.
We have the following result about the forgetful map

Proposition 7. *The universal curve* $\pi : \overline{\mathcal{C}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ *and the forgetful map* $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ *are isomorphic as families over* $\overline{\mathcal{M}}_{g,n}$.

3 The tautological ring

3.1 Line bundles over $\overline{\mathcal{M}}_{g,n}$

The intersection theory of the moduli space of curves starts with the definitions of some vector bundles defined over it.

3.1.1 The relative dualizing sheaf

Over $\mathcal{C}_{g,n}$ we can define the line bundle $\mathcal{K}$ cotangent to the fibers of $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$. We can extend $\mathcal{K}$ to $\overline{\mathcal{C}}_{g,n}$. In the local picture $(x, y) \to xy$, the sections of $\mathcal{K}$ are generated by $\frac{dx}{x}$ and $\frac{dy}{y}$ modulo the relation

$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y} = 0.$$

The line bundle thus obtained is the *relative cotangent line bundle* over $\overline{\mathcal{C}}_{g,n}$.

3.1.2 Line bundles $\mathcal{L}_i$

Let i be an integer in $\{1, ..., n\}$, we have a section s_i of $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, sending $[(C, x_1, ..., x_n)] \in \overline{\mathcal{M}}_{g,n}$ to x_i. We define n line bundles $\mathcal{L}_1, ..., \mathcal{L}_n$ over $\overline{\mathcal{M}}_{g,n}$ such that $\mathcal{L}_i := s_i^*(\mathcal{K})$. Then the fiber over $[(C, x_1, ..., x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of C at the point x_i.

3.1.3 The Hodge bundle

Definition 21. We call *abelian differential* on a stable curve a meromorphic 1-form such that

- the only poles are at the nodes
- the poles are at most simple
- the residues of the poles on two branches of a node are opposite to each other.

Definition 22. The *Hodge bundle* over $\overline{\mathcal{M}}_{g,n}$ is the vector bundle Λ_g such that the fiber over a point $[(C, x_1, ..., x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the vector space of abelian differentials on the curve parametrized by $[(C, x_1, ..., x_n)]$.

These are also the sections of $\mathcal{K}$ [16] and we have

$$\Lambda_g := \pi_*(\mathcal{K}).$$

3.2 The tautological classes

We will study a subring of the cohomology ring of $\mathcal{M}_{g,n}$ which is called the tautological ring. For large g and n, this subring is smaller than the cohomology ring [15]. However, most of natural classes lie in the tautological ring and it is not easy to construct classes lying outside the tautological ring. Methods to find non-tautological classes can be found in [8] and [14].

We give now a first definition of the tautological ring.

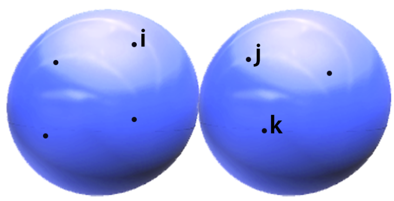
Definition 23. We denote by $R^*(\overline{\mathcal{M}}_{g,n})$ the smallest family of subrings of $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ stable by pullbacks and pushforwards along the forgetful and attaching maps. We call $R^*(\overline{\mathcal{M}}_{g,n})$ the *tautological ring* of the moduli space of curves.

3.2.1 The boundary classes

We present these classes of $H^*(\overline{\mathcal{M}}_{g,n})$ at first because we use them in the next paragraph, they are the classes of the strata corresponding to curves with nodes. In the Chow ring of $\overline{\mathcal{M}}_{g,n}$, the codimension of a stratum is given by the number of nodes of the curves it represents. Thus, in cohomology, a stratum representing curves with one node will lie in $H^2(\overline{\mathcal{M}}_{g,n})$ and the stratum of curves whose components have all genus 0 and three special points has codimension 0 .

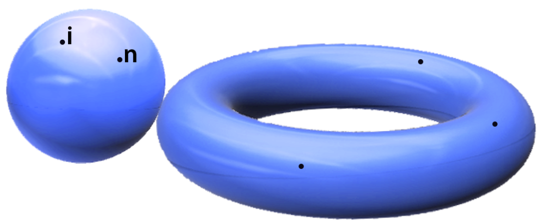
These classes lie in the tautological ring since they are obtained by pushforward of the fundamental class along gluing maps. There are particular type of boundary classes which are useful in properties of the tautological ring.

Example 8.
- In $\overline{\mathcal{M}}_{0,n}$, let $1 \leq i, j, k \leq n$ be distinct integers and let $\delta_{i|j,k}$ be the locus parametrizing curves of genus zero with node separating the ith marked point from the jth and the kth marked point:

  We denote $[\delta_{i|j,k}] \in H^2(\overline{\mathcal{M}}_{0,n})$ the Poincaré dual cohomology class of $\delta_{i|j,k}$.

- In $\overline{\mathcal{M}}_{g,n}$, for $1 \leq i \leq n-1$, we write $\delta_{(i,n)}$ the locus parametrizing nodal curves of genus g with a genus zero component with the marked points i and n only:

3.2.2 ψ-classes

For $i \in \{1, ..., n\}$, we define

$$\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

and present several properties of these classes.

Proposition 8. *Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then we have*

$$\pi^*(\psi_i) = \psi_i - [\delta_{(i,n+1)}].$$

In [36] we have the following proof of this result.

Proof. Let's write $\mathcal{K}_{n+1}$ the cotangent bundle along the fibers of $\overline{\mathcal{C}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{K}_n$ the cotangent bundle along the fibers of $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. We also write $x_1, ..., x_{n+1}$ disjoint sections of $\overline{\mathcal{C}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{L}_{n+1,(i)} = x_i^*(\mathcal{K}_{n+1})$ and $\mathcal{L}_{n,(i)} = x_i^*(\mathcal{K}_n)$.

Let $\psi : \Sigma' \to \Sigma$ be a map between curves and s be a local holomorphic differential on Σ. $\psi^*(s)$ vanishes on any component of Σ' which is mapped to a point in Σ.
Since a nonzero local section s of $\mathcal{K}_n$ determines a nonzero local section $x_i^*(s)$ of $\mathcal{L}_{n+1,(i)}$. The section $x_i^*\psi^*(s)$ vanishes on $\delta_{(i,n+1)}$.
We have $x_i^*\psi^* = \pi^*\sigma_i^*$, hence $\pi^*\sigma_i^*(s)$ vanishes on $\delta_{(i,n+1)}$.
We obtain

$$\mathcal{L}_{n+1,(i)} = \pi^*\mathcal{L}_{n,(i)} \otimes \mathcal{O}(\delta_{(i,n+1)}).$$

from that equality, we have

$$c_1(\mathcal{L}_{n+1,(i)}) = c_1(\pi^*(\mathcal{L}_{n+1,(i)})) + [\delta_{(i,n+1)}].$$

□

Proposition 9.

$$\psi_i\delta_{(i,n+1)} = \psi_{(i,n+1)}\delta_{(i,n+1)} = 0.$$

Proof. The line bundles $\mathcal{L}_i$ and $\mathcal{L}_{n+1}$ are trivial on $\delta_{(i,n+1)}$. □

Proposition 10. *For $i \neq j$,*

$$\delta_{(i,n+1)}\delta_{(j,n+1)} = 0.$$

Proof. The intersection of these divisors is empty. □

3.2.3 κ-classes

These classes were introduced and studied in Mumford's article [30]. They keep playing an essential role in the description of the tautological ring. They are defined as follows.
Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then

$$\kappa_i := p_*(\psi_{n+1}^{i+1}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}).$$

This definition can be generalized as follows.
Let $p : \overline{\mathcal{M}}_{g,n+l} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. For non-negative integers $e_1, \ldots, e_l$ we define

$$\kappa_{e_1,\ldots,e_l} = p_* \left(\psi_{n+1}^{e_1+1} \cdots \psi_{n+l}^{e_l+1}\right).$$

The κ-classes with multiple indices can be expressed as polynomials in the classes κ_i as follows:

$$\kappa_{e_1,\ldots,e_l} := \sum_{\tau \in S_l} \prod_{c \text{ cycle in } \tau} \kappa_{e_c},$$

where e_c is the sum of the e_j appearing in the cycle c.

Computations with κ-classes. Here we will perform several computations with κ-classes that we will use later. First, let us calculate $\kappa_{a,b}$. In order to stick with the previous notation, we write $e_1 := a$ ans $e_2 := b$ and we consider S_2, the symmetric group of order 2 whose elements are $(e_1\ e_2)$ and the identity id.

- $(e_1\ e_2)$ has only one cycle and the sum of the terms of this cycle is $e_1 + e_2 = a + b$. Thus the contribution of this permutation is κ_{a+b}.
- The identity has two cycles of size one, $id = (e_1)(e_2)$. For each of these cycles the sum of the terms is equal to its only element, $e_1 = a$ for the first cycle and $e_2 = b$ for the second one, hence the contribution of the identity is $\kappa_a \kappa_b$.

Finally, summing all contributions we obtain

$$\kappa_{a,b} = \kappa_a \kappa_b + \kappa_{a+b}.$$

From this, if $a = b = 1$, we deduce

$$\kappa_{1,1} = \kappa_1^2 + \kappa_2.$$

When one of the indices is zero, let's say $b = 0$. we have

$$\kappa_{a,0} = \kappa_a \kappa_0 + \kappa_a = (1 + \kappa_0)\kappa_a.$$

This formula will be of great use.

As another example, let's decompose $\kappa_{1,1,1}$. The formula gives

$$\kappa_{1,1,1} = \sum_{\tau \in S_3} \prod_{c \text{ cycle in } \tau} \kappa_{e_c}$$

The elements of S_3 are $(e_1\ e_2\ e_3)$, $(e_1\ e_3\ e_2)$, $(e_1\ e_2)$, $(e_1\ e_3)$, $(e_2\ e_3), id$, where id denotes the identity.

- $(e_1\ e_2\ e_3)$ has one cycle and the sum of its elements is $e_1 + e_2 + e_3 = 1 + 1 + 1 = 3$. Hence this permutation gives κ_3.
- $(e_1\ e_3\ e_2)$ also gives κ_3.
- $(e_1\ e_2)$ has two cycle, $(e_1\ e_2)$ and (e_3). The former one gives κ_2 since the sum of its elements is $e_1 + e_2 = 2$ and the latter one gives κ_1 since there is only one element corresponding to $e_3 = 1$. Finally, we make the product of the contributions of the cycles and we see that this permutation gives $\kappa_1\kappa_2$.
- $(e_1\ e_3)$ gives $\kappa_1\kappa_2$ by the same argument.
- $(e_2\ e_3)$ also gives $\kappa_1\kappa_2$.
- The identity is a composition of three cycles of size 1, the element of each of these cycles correspond to 1 then each of them gives κ_1. Hence the product gives κ_1^3.

Finally, we have to sum all these contributions and we obtain

$$\kappa_{1,1,1} = 2\kappa_3 + \kappa_1^3 + 3\kappa_1\kappa_2.$$

In the case of three indices we can calculate that, for a a natural number, we have

$$\kappa_{0,0,a} = (1+\kappa_0)(2+\kappa_0)\kappa_a.$$

The case of three indices gives a first example of partial decomposition. While decomposing $\kappa_{0,1,1}$ we get

$$\kappa_{0,1,1} = \kappa_0\kappa_1^2 + 2\kappa_1^2 + \kappa_0\kappa_2 + 2\kappa_2$$

we remark that this last expression is equal to $(2+\kappa_0)(\kappa_1^2+\kappa_2)$ and, having calculated $\kappa_{1,1} = \kappa_1^2 + \kappa_2$, we have

$$\kappa_{0,1,1} = (2+\kappa_0)\kappa_{1,1}.$$

This expression is useful in the study of the tautological relations. Instead of the completely decomposed expression, we will also prefer to write

$$\kappa_{0,0,1,1} = (3+\kappa_0)(2+\kappa_0)\kappa_{1,1}.$$

It can be useful later to rewrite products of κ's with simple terms. For example, we take a look at the product $\kappa_1\kappa_{1,1}$. Since we have $\kappa_{1,1} = \kappa_1^2 + \kappa_2$ then $\kappa_1\kappa_{1,1} = \kappa_1(\kappa_1^2+\kappa_2)$ and, from $\kappa_{1,1,1} = 2\kappa_3 + \kappa_1^3 + 3\kappa_1\kappa_2$, we have $\kappa_1^3 = \kappa_{1,1,1} - 2\kappa_3 - 3\kappa_1\kappa_2$. Now we get easily that $\kappa_{1,2} = \kappa_3 + \kappa_1\kappa_2$, thus $\kappa_1\kappa_2 = \kappa_{1,2} - \kappa_3$ so we get

$$\begin{aligned}
\kappa_1\kappa_{1,1} &= \kappa_1^3 + \kappa_1\kappa_2 \\
&= \kappa_{1,1,1} - 2\kappa_3 - 3\kappa_1\kappa_2 + \kappa_1\kappa_2 \\
&= \kappa_{1,1,1} - 2\kappa_3 - 2\kappa_1\kappa_2 \\
&= \kappa_{1,1,1} - 2\kappa_3 - 2(\kappa_{1,2} - \kappa_3) \\
&= \kappa_{1,1,1} - 2\kappa_{1,2}.
\end{aligned}$$

These examples are not chosen innocently, we will refer to them in some calculations, while studying the tautological ring of the moduli spaces of curves of genus 3 and 4.

3.2.4 λ-classes

We define the λ-classes as

$$\lambda_i = c_i(\Lambda_g) \in H^{2i}(\overline{\mathcal{M}}_{g,n}).$$

Since the Hodge bundle has rank g, for $i > g$, we have $\lambda_i = 0$. We have few properties on λ-classes which will be useful later in the study of $\mathbb{R}^2(\mathcal{M}_{4,n})$.

Proposition 11. *The class λ_g vanishes on the stratum of $\overline{\mathcal{M}}_{g,n}$ parametrizing curves with a nonseparating node.*

Proof. Consider the gluing map $q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$. The pull-back $q^*(\Lambda_g)$ is a vector bundle whose fibers are meromorphic 1-forms on genus $g-1$ curves with at most simple poles at marked points x_{n+1} and x_{n+2} and no other poles. The residue of the 1-form at x_{n+1} is then a surjective map from $q^*(\Lambda_g)$ to the trivial line bundle. Thus the top Chern class of $q^*(\Lambda_g)$ vanishes. □

Proposition 12. *The class $\lambda\lambda_{g-1}$ vanishes on the strata of $\overline{\mathcal{M}}_{g,n}$ parametrizing curves with a separating node whose two components have genus at least* 1*.*

Proof. For i an integer such that $1 \leq i \leq [\frac{g}{2}]$, let's write Δ_i the closure of the locus of irreducible singular curves consisting of one component of genus i and one of genus $g-i$. The Hodge bundle over Δ_i is the direct sum of the Hodge bundle over the moduli space of curves of genus i and the Hodge bundle over the moduli space of curves of genus $g-i$, this sum will be written $\Lambda_i \oplus \Lambda_{g-i}$. We have

$$\begin{aligned}\lambda_g\lambda_{g-1} &= c_g(\Lambda_g)c_{g-1}(\Lambda_g)\\ &= c_g(\Lambda_i \oplus \Lambda_{g-i})c_{g-1}(\Lambda_i \oplus \Lambda_{g-i})\\ &= c_g(\Lambda_i)c_g(\Lambda_{g-i})c_{g-1}(\Lambda_i)c_{g-1}(\Lambda_{g-i})\end{aligned}$$

From

$$\begin{aligned}c(\Lambda) &= c(\Lambda_i \oplus \Lambda_{g-i}) = c(\Lambda_i)c(\Lambda_{g-i})\\ &= \big(1 + c_1(\Lambda_i) + ... + c_i(\Lambda_i)\big)\big(1 + c_1(\Lambda_{g-i}) + ... + c_{g-i}(\Lambda_{g-i})\big)\end{aligned}$$

we deduce

$$\begin{aligned}c_{g-1}(\Lambda) &= \sum_{k+l=g-1} c_k(\Lambda_i)c_l(\Lambda_{g-i})\\ &= \sum_{k=0}^{g-1} c_k(\Lambda_i)c_{g-1-k}(\Lambda_{g-i})\\ &= \sum_{k=0}^{i} c_k(\Lambda_i)c_{g-1-k}(\Lambda_{g-i})\\ &= c_{i-1}(\Lambda_i)c_{g-i}(\Lambda_{g-i}) + c_i(\Lambda_i)c_{g-i-1}(\Lambda_{g-i})\end{aligned}$$

and

$$\begin{aligned}c_g(\Lambda) &= \sum_{p+q=g} c_p(\Lambda_i)c_q(\Lambda_{g-i})\\ &= \sum_{p=0}^{i} c_p(\Lambda_i)c_q(\Lambda_{g-i})\\ &= c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i}).\end{aligned}$$

Thus we have

$$\begin{aligned}\lambda_g\lambda_{g-1} &= c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i})\big[c_{i-1}(\Lambda_i)c_{g-i}(\Lambda_{g-i}) + c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i})\big]\\ &= c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i})c_{i-1}(\Lambda_i)c_{g-i}(\Lambda_{g-i})\\ &\quad + c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i})c_i(\Lambda_i)c_{g-i-1}(\Lambda_{g-i})\\ &= c_{i-1}(\Lambda_i)c_i(\Lambda_i)c_{g-i}(\Lambda_{g-i})^2\\ &\quad + c_i(\Lambda_i)^2c_{g-i-1}(\Lambda_{g-i})c_{g-i}(\Lambda_{g-i})\\ &= 0\end{aligned}$$

because $\lambda_g^2 = 0$, for any genus g.

$\square$

Further results on these classes and their products with κ-classes and ψ-classes can be found in [7] where the following proposition is proven.

Proposition 13. *If* $a_1, ..., a_n$ *are non-negative integers such that* $\sum_{i=1}^{n} a_i = 3g - 3 + n$, *we have*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} ... \psi_n^{a_n} \lambda_g = \binom{2g+n-3}{a_1, ..., a_n} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g.$$

3.3 The tautological ring of $\mathcal{M}_{g,n}$

Definition 24. The *tautological ring* $R^*(\mathcal{M}_{g,n})$ is the subring of $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$ generated by the classes $\psi_1, \ldots, \psi_n$ and $\kappa_1, \kappa_2, \ldots$.

The study of this ring boils down to describing the relations between the ψ- and κ-classes. In this section we describe the following results.

- Pixton's relations (more precisely, their restrictions from $\overline{\mathcal{M}}_{g,n}$ to $\mathcal{M}_{g,n}$): this is a family of relations among the ψ- and κ-classes that is conjectured to be complete.
- The vanishing, socle, and top intersection properties of the tautological rings: these properties completely describe the tautological ring in degrees $d \geq g - 1$.
- Mumford's stability: this is a claim that the tautological ring stabilizes to a free ring as $g \to \infty$.
- Mumford's formula for the λ-classes: this formula expresses the λ-classes in terms of ψ- and κ-classes.

3.3.1 Pixton's relations

In [33], A. Pixton conjectured a set of relations for the tautological ring of moduli spaces of stable nodal curves with marked points generalizing Faber's relations. These relations are proven to be true relations in [32] and they play a crucial role in the work presented here. We present the restriction of these relations to the moduli space of smooth stable curves with marked points.

We have the series

$$A := \sum_{n\geq 0} \frac{(6n)!}{(2n)!(3n)!} T^n = 1 + 60T + 27720T^2 ...$$

and

$$B := \sum_{n\geq 0} \frac{6n+1}{6n-1} \frac{(6n)!}{(2n)!(3n)!} T^n = -1 + 84T + 32760T^2 ...$$

For any natural number i not congruent to 2 modulo 3, we write

$$C_{3i} = T^i A$$
$$C_{3i+1} = T^i B.$$

For a power series S, $[S]_{T^n}$ denotes the nth coefficient. We transform the power series S variables T^n into a power series denoted $\{S\}$ in the variables $K_n T^n$, so we have

$$\{S\} := \sum_{n\geq 0} [S]_{T^n} K_n T^n.$$

Let l and $e_1, \ldots, e_l$ be a nonnegative integer and let κ denote the linear operator defined by

$$\kappa(K_{e_1}...K_{e_l}) = \kappa_{e_1,\ldots,e_l} = \sum_{\tau \in S_l} \prod_{c \text{ cycle in } \tau} \kappa_{e_c},$$

with e_c the sum of the e_j appearing in the cycle c.

Let σ be a partition with no part congruent to 2 modulo 3 and $a_1, ..., a_n$ positive integers not congruent to 2 modulo 3.

If

$$\begin{cases} 3d \geq g + 1 + \sum \sigma_i + \sum a_i \\ 3d \equiv g + 1 + \sum \sigma_i + \sum a_i \mod 2 \end{cases},$$

then

$$[\kappa(\exp(\{1-A\})\{C_{\sigma_1}\}...\{C_{\sigma_l}\}) \prod C_{a_i}(\psi_i T)]_{T^2} = 0.$$

3.3.2 Vanishing, socle, and top intersection for $R^*(\mathcal{M}_g)$

These properties were conjectured by C. Faber [9] and later proved by E. Looijenga [26], E. Getzler and R. Pandharipande [12]. Here we assume that $g \geq 2$.

Vanishing. We have $R^d(\mathcal{M}_g) = 0$ for $d \geq g - 1$.

Socle. The rank of $R^{g-2}(\mathcal{M}_g)$ is equal to 1 and $R^{g-2}(\mathcal{M}_g)$ is spanned by κ_{g-2}.

Top intersection. For positive integers $k_1, \ldots, k_m$ such that $\sum_{i=1}^m k_i = g - 2$, we have

$$\kappa_{k_1,\ldots,k_m} = \frac{(2g-3+m)!(2g-1)!!}{(2g-1)!\prod_{i=1}^m (2k_i+1)!!}\kappa_{g-2} \in R^{g-2}(\mathcal{M}_{g-2}).$$

3.3.3 Vanishing, socle, and top intersection for $R^*(\mathcal{M}_{g,n})$, Buryak-Shadrin-Zvonkine Relations

These properties were proved by E. Ionel [18], A. Buryak, S. Shadrin, and D. Zvonkine [2]. The tautological ring of $\mathcal{M}_{1,n}$ is isomorphic to $\mathbb{Q}$; thus we can assume that $g \geq 2$.

Vanishing. We have $R^d(\mathcal{M}_{g,n}) = 0$ for $d \geq g$.

Socle. The rank of $R^{g-1}(\mathcal{M}_{g,n})$ is equal to n and $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$.

Top intersection. For nonnegative integers $d_1, \ldots, d_n$ and positive integers $k_1, \ldots, k_m$ such that $\sum_{i=1}^n \psi_i + \sum_{i=1}^m k_i = g-1$, we have

$$\prod_{i=1}^n \psi_i^{d_i} \, \kappa_{k_1,\ldots,k_m} = \frac{(2g-1)!!}{\prod(2d_i+1)!! \prod(2k_j+1)!!} \frac{(2g-3+n+m)!}{(2g-2+n)!} \times \sum_{i=1}^n \frac{(2g-2+n)d_i + \sum k_j}{g-1} \psi_i^{g-1}.$$

Example 9. For $g=3$ we have

$$\begin{aligned}
\kappa_2 &= \sum_{i=1}^n \psi_i^2, \\
\kappa_{1,1} &= \frac{5}{3}(n+5) \sum_{i=1}^n \psi_i^2, \\
\kappa_1 \psi_i &= \frac{5}{6}(n+5)\psi_i^2 + \frac{5}{6} \sum_{j \neq i} \psi_j^2, \\
\psi_i \psi_j &= \frac{5}{6}(\psi_i^2 + \psi_j^2).
\end{aligned}$$

3.3.4 Stability

The stability property was first conjectured by D. Mumford and proved by Madsen and Weiss [27].

Denote by $\mathbb{Q}[\psi, \kappa]$ the graded ring of polynomials in variables $\psi_1, \ldots, \psi_n$ and $\kappa_1, \kappa_2, \ldots$. The grading is given by assigning degree 1 to each ψ_i and degree m to each κ_m. Denote by $\mathbb{Q}_d[\psi, \kappa]$ the degree d part of the ring.

Theorem 6. *For an integer d satisfying $1 \le d \le g/3$, we have*

$$H^{2d}(\mathcal{M}_{g,n}, \mathbb{Q}) = R^d(\mathcal{M}_{g,n}) = \mathbb{Q}_d[\psi, \kappa]$$

and

$$H^{2d-1}(\mathcal{M}_{g,n}, \mathbb{Q}) = 0.$$

3.3.5 Mumford's formula for the λ-classes

The formula expressing the λ-classes in terms of ψ-, κ-, and boundary classes in $\overline{\mathcal{M}}_{g,n}$ was obtained D. Mumford in [30] using the Grothendieck-Riemann-Roch formula. Here we present the restriction of Mumford's formula to $\mathcal{M}_{g,n}$.

Lemma 3. *Let $E \to B$ be a vector bundle. Denote by $\mathrm{ch}_j = \mathrm{ch}_j(E)$ its jth Chern character and by $c(E)$ its full Chern class. Then we have*

$$c(E) = \exp\left(\sum_{j\geq 1}(-1)^{j-1}(j-1)!\mathrm{ch}_j\right).$$

Proof. Denote by $r_1, \ldots, r_k$ the Chern roots of E. Then we have

$$c(E) = \prod_{i=1}^{k}(1+r_i).$$

Hence

$$c(E) = \sum_{i=1}^{k}\ln(1+r_i) = \sum_{i=1}^{k}\sum_{j\geq 1}(-1)^{j-1}\frac{r_i^j}{j}.$$

On the other hand, for each integer j, we have

$$\mathrm{ch}_j = \sum_{i=1}^{k}\frac{r_i^j}{j!}.$$

Comparing the two formulas we deduce the statement of the lemma. □

Example 10. We have

$$\begin{aligned}
c_1(E) &= \mathrm{ch}_1, \\
c_2(E) &= \frac{1}{2}\mathrm{ch}_1(\Lambda)^2 - \mathrm{ch}_2(\Lambda), \\
c_3(E) &= 2\mathrm{ch}_3(\Lambda) - \frac{1}{2}\mathrm{ch}_1(\Lambda)\mathrm{ch}_2(\Lambda) + \frac{1}{6}\mathrm{ch}_1(\Lambda)^3.
\end{aligned}$$

Definition 25. The Bernouilli numbers denoted B_k, for some nonnegative integer k, are defined by the following formula

$$\frac{1}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!} x^{2n}.$$

Then the Bernoulli numbers are zero for all odd indices greater than 1 and we have

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}...$$

Theorem 7 (Mumford's formula)**.** *Denote by $\Lambda \to \mathcal{M}_{g,n}$ the Hodge bundle. Then the following equalities hold in the cohomology of $\mathcal{M}_{g,n}$:*

$$\begin{aligned}\mathrm{ch}_0(\Lambda) =& g,\\ \mathrm{ch}_{2k}(\Lambda) =& 0,\\ \mathrm{ch}_{2k-1}(\Lambda) =& \frac{B_{2k}}{(2k)!}\big[\kappa_{2k-1} - \sum_{i=1}^{n} \psi_i^{2k-1}\big].\end{aligned}$$

A detailed proof of this theorem together with an introduction to the Grothendieck-Riemann-Roch formula can be found in [37].

Example 11. In $R^*(\mathcal{M}_{g,n})$, we have

$$\begin{aligned}\mathrm{ch}_1(\Lambda) &= \frac{1}{12}\big[\kappa_1 - \sum_{i=1}^{n} \psi_i\big]\\ \mathrm{ch}_3(\Lambda) &= \frac{-1}{720}\big[\kappa_3 - \sum_{i=1}^{n} \psi_i^3\big].\end{aligned}$$

Combining the theorem and the lemma we obtain expressions of each λ_i in terms of the ψ- and κ-classes.

Example 12.

$$\lambda_1 = \frac{1}{12}\left[\kappa_1 - \sum_{i=1}^{n}\psi_i\right]$$

$$\lambda_2 = \frac{1}{2}\left[\kappa_1 - \sum_{i=1}^{n}\psi_i\right]^2$$

$$\lambda_3 = -\frac{1}{360}\left[\kappa_3 - \sum_{i=1}^{n}\psi_i^3\right] + \frac{1}{72}\left[\kappa_1 - \sum_{i=1}^{n}\psi_i\right]^3.$$

3.3.6 The tautological ring of $\overline{\mathcal{M}}_{g,n}$

The tautological rings of $\overline{\mathcal{M}}_{g,n}$ form the smallest system of subrings of $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ stable under push-forwards by the gluing and forgetful maps. There is, however, a more explicit description of these rings.

Let Γ be a stable graph (see Definition 20). A *basic class* in $\overline{\mathcal{M}}_\Gamma$ is monomial

$$\gamma := \prod_v \prod_{i>0} \kappa_i[v]^{k_i[v]} \cdot \prod_h \psi_h^{d_h} \in H^*(\overline{\mathcal{M}}_\Gamma, \mathbb{Q}).$$

Here the first product goes over the vertices v of the graph and the class $\kappa_i[v] = \kappa_i \in H^*(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{Q})$. The second product goes over the half-edges h of the graph, including the legs.

Of course, if for some vertex v the total degree of the ψ-classes $\sum d_h$ over the half-edges attached to v together with the total degree $\sum ik_i[v]$ of the κ-classes exceeds the dimension $3g(v) - 3 + n(v)$, then the basic class vanishes.

The product of two basic classes can be described solely in terms of these dual graphs, see the appendix of [14].

The algebra thus obtained in denoted by $\mathcal{S}_{g,n}$. We have the natural morphism

$$q : \mathcal{S}_{g,n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

defined by

$$\gamma \mapsto \xi_{\Gamma*}(\gamma).$$

The *tautological ring of* $\overline{\mathcal{M}}_{g,n}$ is the image of this morphism; the *tautological relations* form its kernel.

3.4 Structure of the tautological ring

D. Mumford conjectured the structure of the cohomology ring of $\mathcal{M}_g$ with coefficients in $\mathbb{Q}$, when g tends to infinity [30]. It later have been proven by I. Madsen and M. Weiss [27] that Mumford's conjecture was true. The main description of $\mathcal{M}_g$ for a given g is due to C. Faber [9]. The study of the tautological ring lies in the search of dimensions of the tautological groups and in the search of relations between tautological class.

3.4.1 Elimination of the λ-classes

In his paper, D. Mumford started this work by expressing the λ-classes in terms of the κ-classes and boundary classes in $R^*(\overline{\mathcal{M}}_g)$. To do this, he uses the Grothendieck-Riemann-Roch formula. In $R^*(\overline{\mathcal{M}}_{g,n})$, we get expressions of the λ-classes in terms of κ-classes, boundary classes and ψ-classes and the restrictions of these expressions give expressions without boundary classes.

Definition 26. Let $V \to B$ be a vector bundle of rank k over a complex manifold B. We say that$V$ can be exhausted by line bundles if there is a line bundle L_1 of V, a line subbundle L_2 of $V_1 := V/L_1$, a line subbundle L_3 of $V_2 := V_1/L_2$ and so on until the quotient V_{k-1}/L_k is a line bundle for a certain k. In this case, the first Chern classes $r_i := c_1(L_i)$ are called Chern roots of V and we define its total Chern class of V by $c(V) := \prod_{i=1}^{k}(1+r_i)$ and its Chern character by $ch(V) := \sum_{i=1}^{k} e^{r_i}$.

Proposition 14. *We have the following identity*

$$\lambda = \exp\Big(\sum_{j\geq 1}(-1)^{j+1}(j-1)!ch_j(\Lambda)\Big).$$

Proof. This result is true for any vector bundle of rank k over a d-dimensional variety.
Since we have

$$\lambda := 1+\lambda_1+...+\lambda_g = 1+c_1(\Lambda)+c_2(\Lambda)+...+c_g(\Lambda) = c(\Lambda)$$

we can write

$$\ln(\lambda) = \sum_{i=1}^{k}\ln(1+r_i) = \sum_{i=1}^{k}\sum_{j=0}^{\infty}(-1)^j\frac{r_i^{j+1}}{j+1}.$$

On the other hand, since, for each integer j, $ch_j(\Lambda) = \sum_{i=1}^{k}\frac{r_i^j}{j!}$, we have

$$\begin{aligned}\sum_{j\geq 1}(-1)^{j+1}(j-1)!ch_j(\Lambda) &= \sum_{j\geq 1}(-1)^{j+1}(j-1)!\sum_{i=1}^{k}\frac{r_i^j}{j!}\\ &= \sum_{j\geq 1}\sum_{i=1}^{k}(-1)^{j+1}\frac{r_i^j}{j}.\end{aligned}$$

□

Example 13.

$$\begin{aligned}\lambda_1 &= ch_1(\Lambda)\\ \lambda_2 &= \frac{1}{2}ch_1(\Lambda)^2 - ch_2(\Lambda)\\ \lambda_3 &= 2ch_3(\Lambda) - \frac{1}{2}ch_1(\Lambda)ch_2(\Lambda) + \frac{1}{6}ch_1(\Lambda)^3.\end{aligned}$$

In [37], we can find details on the Grothendieck-Riemann-Roch theorem and how we can apply it to get the following result. Applying the Grothendieck-Riemann-Roch formula to the dualizing sheaf. Before let's define the Bernoulli numebers

Definition 27. The Bernouilli numbers denoted B_k, for some nonnegative integer k, are defined by the following formula

$$\frac{1}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!} x^{2n}.$$

Then the Bernoulli numbers are zero for all odd indices greater than 1 and we have

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}...$$

Proposition 15.

$$\begin{aligned} ch_0(\Lambda) - 1 =& g - 1, \\ ch_{2k}(\Lambda) =& 0, \\ ch_{2k-1}(\Lambda) =& \frac{B_{2k}}{(2k)!}\big[\kappa_{2k-1} - \sum_{i=1}^{n} \psi_i^{2k-1} + \delta_{2k-1}^{\Lambda}\big]. \end{aligned}$$

where δ_{2k-1}^{Λ} is push forward along π of the Poincaré dual cohomology class of the subvariety of $\overline{\mathcal{C}}_{g,n}$ consisting in the nodes of the singular fibers multiplied by a certain coefficient.

Example 14. In $R^*(\mathcal{M}_{g,n})$, we have

$$\begin{aligned} ch_1(\Lambda) &= \frac{1}{12}\big[\kappa_1 - \sum_{i=1}^{n} \psi_i\big], \\ ch_3(\Lambda) &= \frac{-1}{720}\big[\kappa_3 - \sum_{i=1}^{n} \psi_i^3\big] \end{aligned}$$

Combining these two results we obtain expressions of each λ_i in terms of the other tautological classes.

Example 15.

$$\lambda_1 = \frac{1}{12}\big[\kappa_1 - \sum_{i=1}^{n} \psi_i\big]$$
$$\lambda_2 = \frac{1}{2}\big[\kappa_1 - \sum_{i=1}^{n} \psi_i\big]^2$$
$$\lambda_3 = \frac{-1}{360}\big[\kappa_3 - \sum_{i=1}^{n} \psi_i^3\big] + \frac{1}{72}\big[\kappa_1 - \sum_{i=1}^{n} \psi_i\big]^3.$$

Hence, from now on, the λ-classes are not considered as generators of the tautological ring and we think of the tautological ring as the follows.

Definition 28. Let g, n be non-negative integers satisfying the stability condition. Then we right $R^*(\overline{\mathcal{M}}_{g,n})$ and call tautological ring of $\overline{\mathcal{M}}_{g,n}$ the ring generated by the κ-classes, the ψ-classes and the boundary classes.
In the same way, we write $R^*(\mathcal{M}_{g,n})$ and call tautological ring of $\mathcal{M}_{g,n}$ the ring generated by the κ-classes and the ψ-classes.

Then from here, the work on the tautologcial ring lies in finding relations between κ-classes, ψ-classes and boundary classes when we deal with the moduli space of nodal curves.

3.4.2 Faber's description of the tautological ring

In [9], C. Faber formulated two conjectures on the structure of the tautological ring of the moduli space of smooth curves of genus g.

Conjecture 1

- $R^*(\mathcal{M}_g)$ is Gorenstein with socle in degree $g-2$ and the pairing
$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(M_g)$$
is perfect.
- The classes $\kappa_1, ..., \kappa_{[g/3]}$ generate $R^*(\mathcal{M}_g)$ and there is no relation in degree $\leq [\frac{g}{3}]$.
- We have the two following formulas in degree $R^{g-2}(\mathcal{M}_g)$
  - In $R^{g-2}(\mathcal{M}_g)$, for $k_1, ..., k_m$ non-negative integers such that $\sum_{i=1}^m k_i = g-2$, we have
  $$\kappa_{k_1,...,k_m} = \frac{(2g-3+m)!(2g-1)!!}{(2g-1)!\prod_{i=1}^m (2k_i+1)!!}\kappa_{g-2}.$$
  - If $\mathcal{S}_k$ denotes the symmetric group of degree k and if, for $\sigma \in \mathcal{S}_k$ is composed by $\nu(\sigma)$ disjoint cycles $\alpha_1, ..., \alpha_{\nu(\sigma)}$, $|\alpha_i|$ denotes the sum of the elements in α_i of we write $\kappa_\sigma = \kappa_{|\alpha_1|}\kappa_{|\alpha_2|}...\kappa_{|\alpha_{\nu(\sigma)}|}$ and we have
  $$< \tau_{d_1+1}\tau_{d_2+1} \dots \tau_{d_k+1} >= \sum_{\sigma \in \mathcal{S}_k} \kappa_\sigma$$

Faber checked that this conjecture is true for $g \leq 15$. In the second conjecture, Faber proposed a set of relations between the tautological classes.

Conjecture 2 For $j \geq g$ and M a monomial in K_i and D_{ij}, we define I_g as the ideal of relations of the form
$$\pi_*(M \cdot (\mathbb{F}_{2g-1} - \mathbb{E})).$$
The conjecture is that $\mathbb{Q}[\kappa_1, ..., \kappa_{g-2}]/I_g$ is Gorenstein with socle in degree $g-2$, and then
$$R^*(\mathcal{M}_g) \simeq \mathbb{Q}[\kappa_1, ..., \kappa_{g-2}]/I_g.$$

A generalization of Conjecture 1 to the moduli space of curves with marked points is presented in [2]. In this paper, the following results are proven.

- For $i > g-1$, $R^i(\mathcal{M}_{g,n}) = 0$ and $R^{g-1}(\mathcal{M}_{g,n}) = \mathbb{Q}^n$ and is generated by the monomials ψ_i^{g-1} for $1 \leq i \leq n$.
- For $d_1, ..., d_n, k_1, ..., k_m$ non-negative integers such that $\sum_{j=1}^n k_j + \sum_{i=1}^m d_i = g-1$, we have

$$\begin{aligned}\prod_{i=1}^n \psi_i^{d_i} \cdot \kappa_{k_1,k_2,...,k_m} =& \frac{(2g-3+n+m)!}{(2g-2+n)!} \\ &\cdot \frac{(2g-1)!!}{\prod_{i=1}^n (2d_i+1)!! \prod_{j=1}^m (2k_j+1)!!} \\ &\cdot \sum_{i=1}^n \frac{(2g-2+n)d_i + \sum_{j=1}^m k_j}{g-1} \psi_i^{g-1}\end{aligned}$$

  which is proven later for $g = 3$.

4 The tautological rings of $\mathcal{M}_{1,n}$, $\mathcal{M}_{2,n}$, and $\mathcal{M}_{3,n}$

We study here the tautological rings of moduli spaces of smooth Riemann surfaces of genus $g \leq 3$ with n marked points.

Proposition 16.
Genus 1. *We have*

$$\begin{aligned} R^0(\mathcal{M}_{1,n}) &= \mathbb{Q}, \\ R^d(\mathcal{M}_{1,n}) &= 0 \quad \textit{for } d \geq 1. \end{aligned}$$

Genus 2. *We have*

$$\begin{aligned} R^0(\mathcal{M}_{2,n}) &= \mathbb{Q}, \\ R^1(\mathcal{M}_{2,n}) &= \mathbb{Q}\langle \psi_1, \dots, \psi_n \rangle, \\ R^d(\mathcal{M}_{2,n}) &= 0 \quad \textit{for } d \geq 2. \end{aligned}$$

The only relation of in degree 1 is $\kappa_1 = \sum_{i=1}^n \psi_i$.
Genus 3. *We have*

$$\begin{aligned} R^0(\mathcal{M}_{3,n}) &= \mathbb{Q}, \\ R^1(\mathcal{M}_{3,n}) &= \mathbb{Q}\langle \kappa_1, \psi_1, \dots, \psi_n \rangle, \\ R^2(\mathcal{M}_{3,n}) &= \mathbb{Q}\langle \psi_1^2, \dots, \psi_n^2 \rangle, \\ R^d(\mathcal{M}_{3,n}) &= 0 \quad \textit{for } d \geq 3. \end{aligned}$$

The relations in degree 2 are listed in Example 9.

Proof. The vanishing for $d \geq g$ is known by [18]. The rank and the relations in $R^{g-1}(\mathcal{M}_{g,n})$ are known by [2]. The group R^0 is always equal to $\mathbb{Q}$. The only remaining case is $R^1(\mathcal{M}_{3,n})$ where, by Mumford's stability [27] there are no relations. □

4.1 The top tautological group of $\mathcal{M}_{3,n}$

There are two famous families of relations in $R^2(\mathcal{M}_{3,n})$. One is due to Buryak-Shadrin-Zvonkine [2] (see Section 3.3.2). This family is known to be complete and is used to determine the rank of $R^2(\mathcal{M}_{3,n})$. The other family is that of Pixton's relations [33](see Section 3.3.1). The completeness of this family is, in general, not known. We prove it here for $g = 3$, degree 2 and any n.

Theorem 8. *Pixton's relations give a complete system of relations in $R^2(\mathcal{M}_{3,n})$ for all n. Buryak-Shadrin-Zvonkine's relations are linear combinations of Pixton's relations whose coefficients are polynomials in n.*

Proof. Pixton's relations are labeled by n-tuples $a_1, \cdots, a_n$ and a partition σ satisfying the conditions $3d \geq g+1+\sum_{i=1}^n a_i + |\sigma|$ and $g+1+\sum_{i=1}^n a_i + \sum \sigma_i \equiv 0 \pmod 2$ (see Section 3.3.1). In our case, these conditions become

$$\begin{cases} \sum_{i=1}^n a_i + \sum \sigma_i \leq 2 \\ \sum_{i=1}^n a_i + \sum \sigma_i \equiv 0 \mod 2 \end{cases}$$

This imply that there are 5 possibilities listed below.

1. all the a_i are zero and $\sigma = \emptyset$;
2. $a_k = 2$ for some k, the others a_i are zero and $\sigma = \emptyset$;
3. all the a_i are zero and $\sigma = \{2\}$;
4. $a_k = 1$ for some k, the other a_i are zero and $\sigma = \{1\}$;
5. $a_k = a_l = 1$ for some k and some l, the others a_i are zero and $\sigma = \emptyset$.

The corresponding Pixton's relations for these cases are:

$$(1) \qquad 35\sum_{i=1}^{n}\psi_i^2+6\sum_{i<j}\psi_i\psi_j-6\sum_{i=1}^{n}\kappa_1\psi_i-35\kappa_2+3\kappa_{1,1}=0,$$

$$(2)_k \qquad 35\sum_{i=1}^{n}\psi_i^2-45\psi_k^2-10\sum_{i=1,i\neq k}^{n}\psi_k\psi_i+6\sum_{i<j}\psi_i\psi_j+10\kappa_1\psi_k$$
$$-6\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i-35\kappa_2+3\kappa_{1,1}=0,$$

$$(3) \qquad 35\kappa_0\sum_{i=1}^{n}\psi_i^2+6\kappa_0\sum_{i<j}\psi_i\psi_j-10\sum_{i=1}^{n}\kappa_1\psi_i-6\sum_{i=1}^{n}\kappa_{0,1}\psi_i$$
$$-45\kappa_2-35\kappa_{0,2}+10\kappa_{1,1}+3\kappa_{0,1,1}=0,$$

$$(4)_k \qquad 35\kappa_0\sum_{i=1,i\neq k}^{n}\psi_i^2+6\kappa_0\sum_{i,j\neq k}\psi_i\psi_j-6\sum_{i=1,\neq k}^{n}\kappa_{0,1}\psi_i$$
$$-35\kappa_{0,2}+3\kappa_{0,1,1}=0,$$

$$(5)_{k,l} \qquad 35\sum_{i=1,i\neq k,l}^{n}\psi_i^2+6\sum_{i,j\neq k,l}\psi_i\psi_j-6\sum_{i=1,i\neq k,l}^{n}\kappa_1\psi_i$$
$$-35\kappa_2+3\kappa_{1,1}=0.$$

Eliminating the zeroes in the indices of the κ-classes we get

$$(1)\qquad 35\sum_{i=1}^{n}\psi_i^2+6\sum_{i<j}\psi_i\psi_j-6\sum_{i=1}^{n}\kappa_1\psi_i-35\kappa_2+3\kappa_{1,1}=0,$$

$$(2)_k\quad 35\sum_{i=1,i\neq k}^{n}\psi_i^2-45\psi_k^2-10\sum_{i=1,i\neq k}^{n}\psi_k\psi_i+6\sum_{i<j,i,j\neq k}\psi_i\psi_j$$
$$+10\kappa_1\psi_k-6\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i-35\kappa_2+3\kappa_{1,1}=0,$$

$$(3)\qquad 35\kappa_0\sum_{i=1}^{n}\psi_i^2+6\kappa_0\sum_{i<j}\psi_i\psi_j-(16+6\kappa_0)\sum_{i=1}^{n}\kappa_1\psi_i$$
$$-(80+35\kappa_0)\kappa_2+(16+3\kappa_0)\kappa_{1,1}=0,$$

$$(4)_k\quad 35\kappa_0\sum_{i=1,i\neq k}^{n}\psi_i^2+6\kappa_0\sum_{i,j\neq k}\psi_i\psi_j-6(1+\kappa_0)\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i$$
$$-35(1+\kappa_0)\kappa_2+3(2+\kappa_0)\kappa_{1,1}=0,$$

$$(5)_{k,l}\quad 35\sum_{i=1,i\neq k,l}^{n}\psi_i^2+6\sum_{i,j\neq k,l}\psi_i\psi_j-6\sum_{i=1,i\neq k,l}^{n}\kappa_1\psi_i$$
$$-35\kappa_2+3\kappa_{1,1}=0,$$

or, taking into account that $\kappa_0 = 2g - 2 + n = n + 4$,

$$(1) \qquad 35\sum_{i=1}^{n}\psi_i^2 + 6\sum_{i<j}\psi_i\psi_j - 6\sum_{i=1}^{n}\kappa_1\psi_i - 35\kappa_2 + 3\kappa_{1,1} = 0,$$

$$(2)_k \quad 35\sum_{i=1,i\neq k}^{n}\psi_i^2 - 45\psi_k^2 - 10\sum_{i=1,i\neq k}^{n}\psi_k\psi_i + 6\sum_{i<j,i,j\neq k}\psi_i\psi_j$$
$$+10\kappa_1\psi_k - 6\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i - 35\kappa_2 + 3\kappa_{1,1} = 0,$$

$$(3) \qquad 35(n+4)\sum_{i=1}^{n}\psi_i^2 + 6(n+4)\sum_{i<j}\psi_i\psi_j - (6n+40)\sum_{i=1}^{n}\kappa_1\psi_i$$
$$-(35n+220)\kappa_2 + (3n+28)\kappa_{1,1} = 0,$$

$$(4)_k \quad 35(n+4)\sum_{i=1,i\neq k}^{n}\psi_i^2 + 6(n+4)\sum_{i,j\neq k}\psi_i\psi_j$$
$$-6(n+5)\textstyle\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i - 35(n+5)\kappa_2 + 3(n+6)\kappa_{1,1} = 0,$$

$$(5)_{k,l} \quad 35\sum_{i=1,i\neq k,l}^{n}\psi_i^2 + 6\sum_{i,j\neq k,l}\psi_i\psi_j - 6\sum_{i=1,i\neq k,l}^{n}\kappa_1\psi_i$$
$$-35\kappa_2 + 3\kappa_{1,1} = 0.$$

We keep the same definitions of the indices k and l, for $g = 3$, there are four different types of Buryak-Shadrin-Zvonkine's relations (see Example 9):

$$(a)_{k,l} \quad \psi_k\psi_l = \frac{5}{6}(\psi_k^2 + \psi_l^2),$$

$$(b)_l \qquad \kappa_1\psi_l = \frac{5}{6}(n+5)\psi_l^2 + \frac{5}{6}\sum_{i=1,i\neq l}^{n} \psi_i^2,$$

$$(c) \qquad \kappa_2 = \sum \psi_i^2,$$

$$(d) \qquad \kappa_{1,1} = \frac{5}{3}(n+5)\sum \psi_i^2.$$

We have the following operations giving them from Pixton's relations

$$(a)_{k,l} = (1) + \frac{3}{2}\cdot(2)_k + \frac{3}{2}\cdot(2)_l - 4\cdot(5),$$

$$(b)_l = -\frac{2n+1}{3}\cdot(1) - (n-2)\cdot(2)_l - \sum_{i=1,i\neq l}^{n}(2)_i + \frac{8}{3}\sum_{i=1,i\neq l}^{n}(5)_{l,i},$$

$$(c) = \frac{7}{4}\cdot(1) - \frac{3}{16}\cdot(3) + \frac{3}{16}\sum_{i=1}^{n}(2)_i,$$

$$(d) = -\frac{2n^2+4n+33}{3}\cdot(1) - \frac{8n-9}{4}\sum_{i=1}^{n}(2)_i + \frac{7}{4}\cdot(3)$$
$$+ \frac{16}{3}\sum_{i<j}(5)_{i,j}.$$

Thus we obtained BSZ's relations as linear combinations of Pixton's relations with polynomial coefficients. In particular, this proves that Pixton's relations form a complete family in genus 3. □

5 The tautological ring of $\mathcal{M}_{4,n}$

Proposition 17. *We have*

$$\begin{aligned}
&R^0(\mathcal{M}_{4,n}) = \mathbb{Q},\\
&R^1(\mathcal{M}_{4,n}) = \mathbb{Q}\langle \kappa_1, \psi_1, \ldots, \psi_n\rangle,\\
&R^3(\mathcal{M}_{4,n}) = \mathbb{Q}\langle \psi_1^3, \ldots, \psi_n^3\rangle,\\
&R^d(\mathcal{M}_{4,n}) = 0 \quad \text{for } d \geq 3.
\end{aligned}$$

The Buryak-Shadrin-Zvonkine relations listed below form a complete family of relations is degree 3:

$$\begin{aligned}
\kappa_3 &= \sum_{i=1}^{n} \psi_i^3,\\
\kappa_{2,1} &= \frac{7}{3}(n+7)\sum_{i=1}^{n} \psi_i^3,\\
\kappa_{1,1,1} &= \frac{35}{9}(n+7)(n+8)\sum_{i=1}^{n} \psi_i^3,\\
\kappa_2\psi_k &= \frac{7}{9}(n+8)\psi_k^3 + \frac{14}{9}\sum_{i\neq k}^{n} \psi_i^3,\\
\kappa_{1,1}\psi_k &= \frac{35}{27}(n+7)(n+8)\psi_k^3 + \frac{70}{27}(n+7)\sum_{i\neq k} \psi_i^3,\\
\kappa_1\psi_k^2 &= \frac{7}{9}(2n+13)\psi_k^3 + \frac{7}{9}\sum_{i\neq k}^{n} \psi_i^3,\\
\kappa_1\psi_k\psi_l &= \frac{35}{27}(n+7)(\psi_k^3+\psi_l^3) + \frac{35}{27}\sum_{i\neq k,l}^{n} \psi_i^3,\\
\psi_k^2\psi_l &= \frac{14}{9}\psi_k^3 + \frac{7}{9}\psi_l^3,\\
\psi_k\psi_l\psi_p &= \frac{35}{27}(\psi_k^3+\psi_l^3+\psi_p^3).
\end{aligned}$$

Proof. The vanishing for $d \geq g$ is known by [18]. The rank and the relations in $R^{g-1}(\mathcal{M}_{g,n})$ are known by [2]. By Mumford's stability [27] there are no relations in $R^1(\mathcal{M}_{4,n})$. $\square$

It remains to study the group $R^2(\mathcal{M}_{4,n})$. This group is spanned by the tautological classes $\kappa_2, \kappa_{1,1}, \psi_i\kappa_1, \psi_i^2, \psi_i\psi_j$, where i and j are integers from 1 to n.

Theorem 9. *The classes $\psi_i\psi_j$, $1 \leq i < j \leq n$, ψ_i^2, $1 \leq i \leq n$, and $\kappa_{1,1}$ form a basis of $R^2(\mathcal{M}_{4,n})$.*

The proof of this theorem is the goal of this section.

5.1 Pixton's relations in genus 4

Pixton's relations between these classes are determined by integers $a_1, \ldots, a_n$ and a partition σ such that

$$3d \geq g + 1 + \sum_{i=1}^{n} a_i + |\sigma|$$

and the left-hand side has the same parity as the right-hand side. In our case these conditions boil down to

$$\sum_{i=1}^{n} a_i + |\sigma| = 1.$$

Thus we have the following possibilities

- $\sigma = \{1\}$ and $a_i = 0$ for all i.
- $\sigma = \emptyset$, $a_k = 1$ for some $1 \leq k \leq n$ and $a_i = 0$ for $i \neq k$.

In the first case, the relation is

$$\Big[\kappa(\exp(\{-60T - 27720T^2\}) \cdot \{-1 + 84T + 32760T^2\}) \\ \cdot \prod_{i=1}^{n}(1 + 60\psi_i T + 27720\psi_i^2 T^2)\Big]_{T^2}$$

which gives

$$\Big[\kappa(exp(1-60K_1T-27720K_2T^2+1800K_1^2T^2)$$
$$\cdot(-K_0+84K_1T+32760K_2T^2))\cdot\prod_{i=1}^{n}(1+60\psi_iT+27720\psi_i^2T^2)\Big]_{T^2}.$$

From this we obtain

$$\Big[(-\kappa_0+(84+60(1+\kappa_0))\kappa_1T+(32760+27720(1+\kappa_0))\kappa_2T^2$$
$$-(5040+1800(2+\kappa_0))\kappa_{1,1}T^2)\cdot\prod_{i=1}^{n}(1+60\psi_iT+27720\psi_i^2T^2)\Big]_{T^2}.$$

Extracting the coefficient of degree 2 and dividing it by 360 we get

$$(630-77\kappa_0)\kappa_2-(24+5\kappa_0)\kappa_{1,1}-77\kappa_0\sum_{i=1}^{n}\psi_i^2$$
$$+(24+10\kappa_0)\sum_{i=1}^{n}\kappa_1\psi_i-10\kappa_0\sum_{i<j}\psi_i\psi_j.$$

We see that the coefficients depend on n since $\kappa_0=2g-2+n$ $=6+n$.

In the second case, if σ is empty, one of the a_i's is 1 and the others are zero we get the following relation

$$\Big[\kappa(\exp(\{-60T-27720T^2\}))\cdot(-1+84\psi_kT+32760\psi_k^2T^2)$$
$$\cdot\prod_{i=1,i\neq k}^{n}(1+60\psi_iT+27720\psi_i^2T^2)\Big]_{T^2}$$

which gives

$$\Big[(1-60\kappa_1T-27720\kappa_2T^2+1800\kappa_{1,1}T^2)$$
$$\cdot(-1+84\psi_kT+32760\psi_k^2T^2)\cdot\prod_{i=1,i\neq k}^{n}(1+60\psi_iT+27720\psi_i^2T^2)\Big]_{T^2}.$$

Extracting the coefficient of T^2 and dividing by 360, we get

$$77\kappa_2 - 5\kappa_{1,1} + 91\psi_k^2 - 77\sum_{i=1,i\neq k}^{n}\psi_i^2 - 14\kappa_1\psi_k + 10\sum_{i=1,i\neq k}^{n}\kappa_1\psi_i$$
$$+14\sum_{i=1,i\neq k}^{n}\psi_k\psi_i - 10\sum_{i<j,i,j\neq k}\psi_i\psi_j.$$

5.2 Upper bound for the dimension

Let's write the matrix whose lines are composed by the coefficients of the classes $\kappa_2, \kappa_1\psi_1, ..., \kappa_1\psi_n$ in that order in the relations in the same order as previous. We see that, in the first line of this matrix, the coefficient of κ_2 can be written $1092 + 77n$ and that the coefficients $\kappa_1\psi_i$ can be written $84 + 10n$. For a fixed integer n the matrix will look like

$$\begin{pmatrix} 1092+77n & 84+10n & \cdots & \cdots & 84+10n \\ 77 & -14 & 10 & \cdots & 10 \\ \vdots & 10 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 10 \\ 77 & 10 & \dots & 10 & -14 \end{pmatrix}$$

We can divide the first column by 7 to obtain the following matrix

$$\begin{pmatrix} 156+11n & 84+10n & \cdots & \cdots & 84+10n \\ 11 & -14 & 10 & \cdots & 10 \\ \vdots & 10 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 10 \\ 11 & 10 & \dots & 10 & -14 \end{pmatrix}$$

Adding the first column multiplied by $-\dfrac{10}{11}$ to all the other columns we obtain

$$\begin{pmatrix} 156+11n & -\frac{636}{11} & \cdots & \cdots & -\frac{636}{11} \\ 11 & -24 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 11 & 0 & \dots & 0 & -24 \end{pmatrix}$$

Now we add all lines but the fisrt multiplied by $-\dfrac{53}{22}$ to the first line we get the following matrix

$$\begin{pmatrix} 156-\frac{31}{22}n & 0 & \cdots & \cdots & 0 \\ 11 & -24 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 11 & 0 & \dots & 0 & -24 \end{pmatrix}$$

The determinant of this matrix, $(-24)^n \cdot (156 - \dfrac{31}{22}n)$, which only vanishes for $n = \dfrac{31 \cdot 156}{22} \notin \mathbb{N}$, hence can not vanish for any integer n, this matrix is then invertible for all n and in the system of $n+1$ relations thus the classes $\kappa_2, \kappa_1\psi_1, ..., \kappa_1\psi_n$ can be expressed in terms of the classes $\kappa_{1,1}$ and $\psi_i\psi_j$, for $1 \leq i, j \leq n$.

5.3 The matrix M

We want to show that the classes $\kappa_{1,1}, \psi_i^2, \psi_i\psi_j$ don't have any relation between them. We consider now the classes $\lambda_4\lambda_3 \prod_{i=1}^n \psi_i$, $\lambda_4\lambda_3\kappa_1 \prod_{i=1, i\neq k}^n \psi_i$, for integers k such that $1 \leq k \leq n$, and $\lambda_4\lambda_3\psi_k^2 \prod_{i=1, i\neq k,l}^n \psi_i$, where $1 \leq k < l \leq n$.

Proposition 18. *These classes are defined on $\overline{\mathcal{M}}_{4,n}$ and vanish on its boundary.*

Proof. The part $\lambda_4\lambda_3$ makes these classes vanish on the whole boundary except on the curves having a separating node whith a branch of genus 0. The vanishing on these strata comes from the vanishing of $\prod_{i=1}^n \psi_i$, $\kappa_1 \prod_{i=1,i\neq k}^n \psi_i$ and $\psi_k \prod_{i=1,i\neq l}^n \psi_i$ on $\overline{\mathcal{M}}_{0,n+1}$. $\square$

In the following, we will use the following notations

- $\Psi_\emptyset := \prod_{i=1}^n \psi_i$.
- For k a positive integer such that $1 \leq k \leq n$, we write $\Psi_k := \prod_{i=1,i\neq k}^n \psi_i$.
- For k, l distinct positive integers such that $1 \leq k < l \leq n$, we write $\Psi_{\{k,l\}} := \psi_k^2 \prod_{i=1,i\neq k,l} \psi_i$.

We build now the matrix M composed by the intersection numbers given by the products classes of the two lists.

$$\begin{array}{cc}
\begin{array}{ccccccc} \lambda_4\lambda_3\Psi_\emptyset & \lambda_4\lambda_3\kappa_1\Psi_1 & \cdots & \lambda_4\lambda_3\kappa_1\Psi_n & \lambda_4\lambda_3\Psi_{\{1,2\}} & \cdots & \lambda_4\lambda_3\Psi_{\{n-1,n\}} \end{array} & \\
\left(\begin{array}{c} \\ M_{\alpha\beta} \\ \\ \\ \\ \\ \\ \end{array}\right) & \begin{array}{c} \kappa_{1,1} \\ \psi_1^2 \\ \vdots \\ \psi_n^2 \\ \psi_1\psi_2 \\ \vdots \\ \psi_{n-1}\psi_n \end{array}
\end{array}$$

where $M_{\alpha\beta} = \displaystyle\int_{\overline{\mathcal{M}}_{4,n}} \alpha \cdot \beta$. We are able to calculate these integrals using the following result.

Proposition 19. *Let* $\pi : \mathcal{M}_{g,n}^{rt} \to \mathcal{M}_g$ *be the forgetful map forgetting all marked points and let* $d_1, ..., d_n, k_1, ..., k_m$ *be non-negative integers. Then in* $R^{g-2}(\mathcal{M}_g)$, *we have*

$$\pi_*(\kappa_{k_1,...,k_m} \prod_{i=1}^{n} \psi_i^{d_i+1}) = \frac{(2g-3+n+m)!(2g-3)!!}{(2g-2)! \prod_{i=1}^{n}(2d_i+1)!! \prod_{i=1}^{m}(2k_i+1)!!} \kappa_{g-2}.$$

That result is proved in [2].

5.3.1 Coefficients

Let's number the lines of M in the following way: $\emptyset$ designates the first line, the n following lines will be denoted by an integer i from 1 to n and the other lines by $\{i, j\}$, where i and j are integers such that $1 \leq i < j \leq n$. We denote the columns of M in the same way. It might be useful to define the set of these coordinates, $\mathcal{E} := \{\emptyset, 1, ..., n, \{1, 2\}, ..., \{n-1, n\}\}$. Logically, for a column vector u of dimension $\frac{n(n+1)}{2} + 1$, for $k \in \mathcal{E}$ designating a line we denote u_k the entry of the vector u at the line k. In the case $k = \{i, j\}$, we may allow ourselves to write $u_{i,j}$, we simply never do this abuse for coefficients of matrices since something like $M_{1,2,3}$ would yield to a lot of confusion.

In the following, the letters k, l, p, q will always denote positive integers such that

$$\begin{aligned} 1 \leq k < l \leq n \\ 1 \leq p < q \leq n \end{aligned}$$

and $\alpha, \beta \in \mathcal{E}$, α standing for the lines and β for the columns.

The last proposition enables us to calculate the coefficients of M.

First line

- $\boldsymbol{\alpha = \beta = \emptyset}$

$$\begin{aligned}
M_{\emptyset\emptyset} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3 \cdot \prod_{i=1}^{n} \psi_i \cdot \kappa_{1,1} \\
&= \frac{(n+7)! \cdot 5!!}{6! \cdot \prod_{i=1}^{n} 1!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+7)! \cdot 5}{6! \cdot 3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+7)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

- $\boldsymbol{\alpha = \emptyset}$ **and** $\boldsymbol{\beta = p}$

$$\begin{aligned}
M_{\emptyset,p} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq p}^{n} \psi_i \cdot \kappa_{1,1} \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3(\kappa_{1,1,1} - 2\kappa_{1,2}) \cdot \prod_{i=1, i\neq p}^{n} \psi_i
\end{aligned}$$

because $\kappa_1\kappa_{1,1} = \kappa_{1,1,1} - 2\kappa_{1,2}$, this intersection number is then the sum of two terms which we calculate separately

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_{1,1,1} \cdot \prod_{i=1, i\neq p}^{n} \psi_i &= \frac{(n+8)! \cdot 5!!}{6! \cdot 3!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+8)!}{2^4 \cdot 3^4} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

The second term is

$$\begin{aligned}
-2 \cdot \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_{1,2} \cdot \prod_{i=1, i\neq p}^{n} \psi_i &= -2 \cdot \frac{(n+7)! \cdot 5!!}{6! \cdot 3!! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= -2 \cdot \frac{(n+7)!}{2^4 \cdot 3^3 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
M_{\emptyset,p} &= \frac{(n+8)!}{2^4 \cdot 3^4} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 - 2 \cdot \frac{(n+7)!}{3 \cdot 6!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(5n+34) \cdot (n+7)!}{2^4 \cdot 3^4 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 .
\end{aligned}
$$

- $\boldsymbol{\alpha = \emptyset}$ **and** $\boldsymbol{\beta = \{p, q\}}$

$$
\begin{aligned}
M_{\emptyset,\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4 \lambda_3 \psi_p^2 \cdot \prod_{i=1, i \neq p,q}^{n} \psi_i \cdot \kappa_{1,1} \\
&= \frac{(n+7)! \cdot 5!!}{6! \cdot 3!! \cdot \prod_{i=1,i\neq k,l}^{n} 1!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(n+7)! \cdot 3 \cdot 5}{6! \cdot 3 \cdot 3 \cdot 3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(n+7)!}{2^4 \cdot 3^4} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 .
\end{aligned}
$$

Lines of type $\boldsymbol{\alpha = k}$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = \emptyset}$

$$\begin{aligned}
M_{k,\emptyset} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3 \cdot \prod_{i=1}^{n} \psi_i \cdot \psi_k^2 \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3 \cdot \prod_{i=1, i\neq k}^{n} \psi_i \cdot \psi_k^3 \\
&= \frac{(n+5)! \cdot 5!!}{6! \cdot \prod_{i=1,i\neq k}^{n} 1!! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+5)! \cdot 5!!}{6! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+5)!}{2^4 \cdot 3^2 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = p}$
  - When $k \neq p$

$$\begin{aligned}
M_{k,p} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq p}^{n} \psi_i \cdot \psi_k^2 \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq p,k}^{n} \psi_i \cdot \psi_k^3 \\
&= \frac{(n+6)! \cdot 5!!}{6! \cdot \prod_{i=1,i\neq p,k}^{n} 1!! \cdot 3!! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+6)!}{2^4 \cdot 3^3 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

– When $k = p$

$$
\begin{aligned}
M_{k,k} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4 \lambda_3 \kappa_1 \cdot \prod_{i=1, i\neq k}^{n} \psi_i \cdot \psi_k^2 \\
&= \frac{(n+6)! \cdot 5!!}{6! \prod_{i=1, i\neq k}^{n} 1!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(n+6)! \cdot 5}{6! \cdot 3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(n+6)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 .
\end{aligned}
$$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = \{p, q\}}$

  – When $k \neq p, q$

$$
\begin{aligned}
M_{k,\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4 \lambda_3 \psi_p^2 \cdot \prod_{i=1, i\neq p,q}^{n} \psi_i \cdot \psi_k^2 \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4 \lambda_3 \psi_p^2 \psi_k^3 \cdot \prod_{i=1, i\neq k,p,q}^{n} \psi_i \\
&= \frac{(n+5)! \cdot 5!!}{6! \cdot 5!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 \\
&= \frac{(n+5)!}{2^4 \cdot 3^3 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 .
\end{aligned}
$$

– When $k = p$

$$M_{k,\{k,q\}} = \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1,i\neq p,q}^{n} \psi_i \cdot \psi_p^2$$

$$= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_{4,n}\lambda_3\psi_p^4 \cdot \prod_{i=1,i\neq p,q}^{n} \psi_i$$

$$= \frac{(n+5)! \cdot 5!!}{6! \cdot 7!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2$$

$$= \frac{(n+5)!}{2^4 \cdot 3^2 \cdot 5 \cdot 7} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.$$

– When $k = q$

$$M_{k,\{p,k\}} = \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1,i\neq p,q}^{n} \psi_i \cdot \psi_q^2$$

$$= \frac{(n+5)! \cdot 5!!}{6! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2$$

$$= \frac{(n+5)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.$$

Lines of type $\boldsymbol{\alpha = \{k,l\}}$

- $\boldsymbol{\alpha = \{k,l\}}$ **and** $\boldsymbol{\beta = \emptyset}$

$$M_{\{k,l\},\emptyset} = \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3 \cdot \prod_{i=1}^{n} \psi_i \cdot \psi_k\psi_l$$

$$= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3 \cdot \prod_{i=1,i\neq k,l}^{n} \psi_i \cdot \psi_k^2\psi_l^2$$

$$= \frac{(n+5)! \cdot 5!!}{6! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2$$

$$= \frac{(n+5)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.$$

- $\boldsymbol{\alpha = \{k,l\}}$ **and** $\boldsymbol{\beta = p}$
  - When $p \neq k, l$

$$\begin{aligned} M_{\{k,l\},p} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq p}^{n} \psi_i \cdot \psi_k\psi_l \\ &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq p,k,l}^{n} \psi_i \cdot \psi_k^2\psi_l^2 \\ &= \frac{(n+6)! \cdot 5!!}{6! \cdot 3!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\ &= \frac{(n+6)!}{2^4 \cdot 3^4} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2. \end{aligned}$$

  - When $p = k$ or $p = l$

$$\begin{aligned} M_{\{k,l\},k} = M_{\{k,l\},l} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq k}^{n} \psi_i \cdot \psi_k\psi_l \\ &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\kappa_1 \cdot \prod_{i=1, i\neq l}^{n} \psi_i\psi_l^2 \\ &= \frac{(n+6)! \cdot 5!!}{6! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\ &= \frac{(n+6)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2. \end{aligned}$$

- $\boldsymbol{\alpha = \{k,l\}}$ **and** $\boldsymbol{\beta = \{p,q\}}$

– When $\{k,l\} \cap \{p,q\}$ is empty

$$\begin{aligned}
M_{\{k,l\},\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1, i\neq p,q}^{n} \psi_i \cdot \psi_p\psi_q \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1, i\neq k,l,p,q}^{n} \psi_i \cdot \psi_k^2\psi_l^2 \\
&= \frac{(n+5)! \cdot 5!!}{6! \cdot 3!! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+5)!}{2^4 \cdot 3^4} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.
\end{aligned}$$

– When $\{k,l\} \cap \{p,q\} = \{p\}$

$$\begin{aligned}
M_{\{p,l\},\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1, i\neq p,q}^{n} \psi_i \cdot \psi_p\psi_l \\
&= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^3 \cdot \prod_{i=1, i\neq k,l,p,q}^{n} \psi_i \cdot \psi_l^2 \\
&= \frac{(n+5)! \cdot 5!!}{6! \cdot 3!! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\
&= \frac{(n+5)!}{2^4 \cdot 3^3 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2
\end{aligned}$$

and

$$M_{\{k,l\},\{p,k\}} = \frac{(n+5)!}{2^4 \cdot 3^3 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.$$

– When $\{k,l\} \cap \{p,q\} = \{q\}$

$$\begin{aligned} M_{\{q,l\},\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1, i\neq p,q}^{n} \psi_i \cdot \psi_q\psi_l \\ &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2\psi_l^2 \cdot \prod_{i=1, i\neq q,p,l}^{n} \psi_i \\ &= \frac{(n+5)! \cdot 5!!}{6! \cdot 3!! \cdot 3!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\ &= \frac{(n+5)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \end{aligned}$$

and

$$M_{\{k,q\},\{p,q\}} = \frac{(n+5)!}{2^4 \cdot 3^3} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2.$$

– When $\{k,l\} \cap \{p,q\} = \{p,q\}$

$$\begin{aligned} M_{\{p,q\},\{p,q\}} &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^2 \cdot \prod_{i=1, i\neq p,q}^{n} \psi_i \cdot \psi_p\psi_q \\ &= \int_{\overline{\mathcal{M}}_{4,n}} \lambda_4\lambda_3\psi_p^3 \cdot \prod_{i=1, i\neq p}^{n} \psi_i \\ &= \frac{(n+5)! \cdot 5!!}{6! \cdot 5!!} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2 \\ &= \frac{(n+5)!}{2^4 \cdot 3^2 \cdot 5} \cdot \int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2. \end{aligned}$$

5.3.2 Simplification of M, the matrix $\widehat{M}$

The integral $\int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2$ is non zero since, for any genus g, we have from [2]

$$\int_{\overline{\mathcal{M}}_g} \lambda_g\lambda_{g-1}\kappa_{g-2} = \frac{(-1)^{g-1}B_{2g}(g-1)!}{2^g(2g)!}.$$

Recalling that B_{2g} are the Bernouilli numbers given by

$$\frac{1}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!} x^{2n}.$$

We have $B_8 = \frac{-1}{30}$, so we can even calculate

$$\int_{\overline{\mathcal{M}}_4} \lambda_4 \lambda_3 \kappa_2 = \frac{-1}{2^{11}\cdot 3^2 \cdot 5^2 \cdot 7},$$

in $R^9(\overline{\mathcal{M}_4}) \simeq \mathbb{Q}$. In order to simplify the calculations, we define a matrix $\widehat{M}$ by dividing M by $\dfrac{(n+5)!}{2^4\cdot 3} \cdot \displaystyle\int_{\overline{\mathcal{M}}_4} \lambda_4\lambda_3\kappa_2$ and by the following operations on the columns and raws of M

- The line $\alpha = \emptyset$ is divided by $n+7$.
- The lines $\alpha = i$, for $1 \leq i \leq n$, are divided by $\frac{3}{7}$.
- The column $\beta = \emptyset$ is divided by 3.
- The column $\beta = i$, for $1 \leq i \leq n$, are divided by $n+6$.

Here are the coefficients of $\widehat{M}$.

First line

- $\boldsymbol{\alpha = \beta = \emptyset}$

$$\widehat{M}_{\emptyset,\emptyset} = 5(n+6).$$

- $\boldsymbol{\alpha = \emptyset}$ **and** $\boldsymbol{\beta = p}$

$$\widehat{M}_{\emptyset,k} = 5n+34.$$

- $\boldsymbol{\alpha = \emptyset}$ **and** $\boldsymbol{\beta = \{k,l\}}$

$$\widehat{M}_{\emptyset,k\{k,l\}} = 5(n+6).$$

Lines of type $\boldsymbol{\alpha = k}$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = \emptyset}$

$$\widehat{M}_{k,\emptyset} = 7.$$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = p}$
  - When $k \neq p$

$$\widehat{M}_{k,p} = 7.$$

  - When $k = p$

$$\widehat{M}_{k,k} = 35.$$

- $\boldsymbol{\alpha = k}$ **and** $\boldsymbol{\beta = \{p, q\}}$
  - When $k \neq p, q$

$$\widehat{M}_{k,\{p,q\}} = 7.$$

  - When $k = p$

$$\widehat{M}_{k,\{k,q\}} = 3.$$

  - When $k = q$

$$\widehat{M}_{k,\{p,k\}} = 35.$$

Lines of type $\boldsymbol{\alpha = \{k, l\}}$

- $\boldsymbol{\alpha = \{k, l\}}$ **and** $\boldsymbol{\beta = \emptyset}$

$$\widehat{M}_{\{k,l\},\emptyset} = 5.$$

- $\boldsymbol{\alpha = \{k, l\}}$ **and** $\boldsymbol{\beta = p}$
  - When $p \neq k, l$

$$\widehat{M}_{\{k,l\},p} = 5.$$

  - When $p = k$ (or $p = l$)

$$\widehat{M}_{\{k,l\},k} = 15.$$

- $\boldsymbol{\alpha = \{k, l\}}$ **and** $\boldsymbol{\beta = \{p, q\}}$
  - When $\{k, l\} \cap \{p, q\}$ is empty
$$\widehat{M}_{\{k,l\},\{p,q\}} = 5.$$
  - When $\{k, l\} \cap \{p, q\} = \{p\}$
$$\widehat{M}_{\{p,l\},\{p,q\}} = 3,$$
$$\widehat{M}_{\{k,p\},\{p,q\}} = 3.$$
  - When $\{k, l\} \cap \{p, q\} = \{q\}$
$$\widehat{M}_{\{q,l\},\{p,q\}} = 15,$$
$$\widehat{M}_{\{k,q\},\{p,q\}} = 15.$$
  - When $\{k, l\} \cap \{p, q\} = \{p, q\}$
$$\widehat{M}_{\{p,q\},\{p,q\}} = 9.$$

5.4 Rank of M

We look at M as the matrix of an endomorphism of a $\mathbb{Q}$-vector space E of dimension $N = 1 + n + n(n-1)/2$. A vector of E has coordinates α, β_i for $1 \leq i \leq n$, and γ_{ij} with $1 \leq i < j \leq n$.

Calculations via Maple shows that, for $n \leq 18$, the characteristic polynomial of $\widehat{M}$ is the product of degree 2 polynomials of multiplicity 1, a degree 3 polynomial of multiplicity 1 also (sometimes this ones splits into a two polynomials of degrees 1 and 2) and a polynomial of degree 1 with a multiplicity which is a function of n. The decomposition via Maple is made only if it leads to polynomials in $\mathbb{Z}[X]$. This justifies the strategy for the calculation of the rank.

The polynomials of degree 2 are associated to stable planes which we describe at first. Next we will look for the eigenvectors associated to the polynomial of degree 1 and we will finish by the study of the stable 3-dimensional space of $\widehat{M}$.

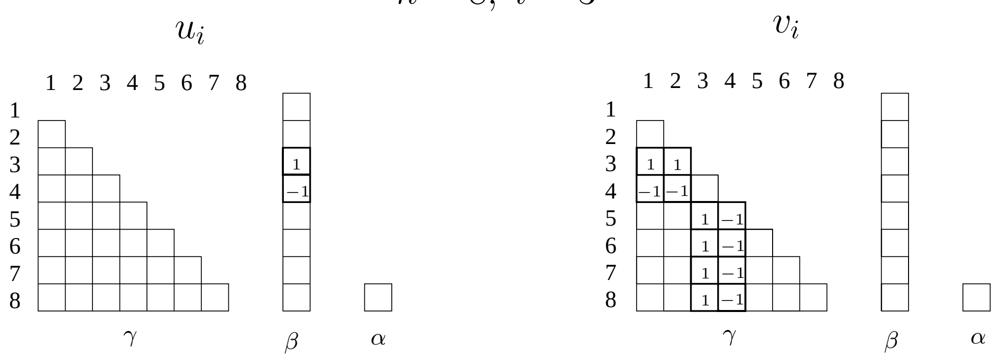
5.4.1 Basis of stable planes

For $1 \leq i \leq n-1$ introduce two vectors u_i and v_i in E. These vectors have the following nonzero coordinates:

$$
\begin{aligned}
&u_i : (\beta_i = 1, \beta_{i+1} = -1)\\
&v_i : (\gamma_{ik} = 1, \gamma_{i+1,k} = -1) \text{ for }, \quad k \neq i, i+1.
\end{aligned}
$$

All the coordinates that are not listed are equal to 0.

$$n = 8, \ i = 3$$



Proposition 20. *The plane spanned by u_i and v_i is invariant under the action of M for every i. Specifically, we have*

$$
\begin{aligned}
M(u_i) &= 28u_i + 10v_i\\
M(v_i) &= (32i - 24 - 4n)u_i + (12i - 12 - 2n)v_i
\end{aligned}
$$

Proof. We will calculate the products $M \cdot u_i$ and $M \cdot v_i$ to show that they are linear combinations of u_i and v_i.

Coordinate $\emptyset$ We have

$$
\begin{aligned}
\left(\widehat{M} \cdot u_i\right)_\emptyset &= \widehat{M}_{\emptyset,i} - \widehat{M}_{\emptyset,i+1}\\
&= (5n + 34) - (5n + 34)\\
&= 0.
\end{aligned}
$$

Coordinates k

- If $k < i$, we have

$$\begin{aligned}\left(\widehat{M} \cdot u_i\right)_k &= \widehat{M}_{k,i} - \widehat{M}_{k,i+1} \\ &= 7 - 7 \\ &= 0.\end{aligned}$$

- Similarly when $i + 1 < k$, we have

$$\begin{aligned}\left(\widehat{M} \cdot u_i\right)_k &= \widehat{M}_{i,k} - \widehat{M}_{i+1,k} \\ &= 7 - 7 \\ &= 0.\end{aligned}$$

- The cases $k = i$ and $k = i + 1$, we have

$$\begin{aligned}\left(\widehat{M} \cdot u_i\right)_i &= \widehat{M}_{i,i} - \widehat{M}_{i+1,i} \\ &= 35 - 7 \\ &= 28\end{aligned}$$

  and

$$\begin{aligned}\left(\widehat{M} \cdot u_i)\right)_{i+1} &= \widehat{M}_{i,i+1} - \widehat{M}_{i+1,i+1} \qquad &= 7 - 35 \\ &= -28.\end{aligned}$$

Coordinates $\{k, l\}$ Now let's calculate the coordinates of type $\{k, l\}$ of $\left(M \cdot u_i\right)$.

- For k and l distinct integers such that $1 \leq k < l \leq n$, none of them being equal to i or $i + 1$, we have

$$\begin{aligned}\left(\widehat{M} \cdot u_i)\right)_{\{k,l\}} &= \widehat{M}_{\{k,l\},i} - \widehat{M}_{\{k,l\},i+1} \\ &= 15 - 15 \\ &= 0\end{aligned}$$

- We also get

$$\big(M \cdot u_i\big)_{\{i,i+1\}} = \widehat{M}_{\{i,i+1\},i} - \widehat{M}_{\{i,i+1\},i+1} = 15 - 15 = 0.$$

- For $k < i$,

$$\begin{aligned} \big(M \cdot u_i\big)_{\{k,i\}} &= \widehat{M}_{\{k,i\},i} - \widehat{M}_{\{k,i\},i+1} = 15 - 5 = 10 \\ \big(M \cdot u_i\big)_{\{k,i+1\}} &= \widehat{M}_{\{k,i\},i+1} - \widehat{M}_{\{k,i\},i} = 5 - 15 = -10. \end{aligned}$$

- For $l > i + 1$,

$$\begin{aligned} \big(M \cdot u_i\big)_{\{i,l\}} &= \widehat{M}_{\{i,l\},a} - \widehat{M}_{\{i,l\},i+1} = 15 - 5 = 10 \\ \big(M \cdot u_i\big)_{\{i+1,l\}} &= \widehat{M}_{\{i,l\},i+1} - \widehat{M}_{\{i,l\},i} = 5 - 15 = -10. \end{aligned}$$

In conclusion, we see that

$$\widehat{M} \cdot u_i = 28u_i + 10v_i.$$

We calculate the coordinates of $M \cdot v_i$ in the same way and we get

Coordinate $\emptyset$ We have

$$\begin{aligned}\left(\widehat{M}\cdot v_i\right)_\emptyset &= \sum_{p=1}^{i-1}\widehat{M}_{\emptyset,\{p,i\}} + \sum_{p=i+2}^{n}\widehat{M}_{\emptyset,\{i,p\}}\\ &\quad - \sum_{p=1}^{i-1}\widehat{M}_{\emptyset,\{p,i+1\}} - \sum_{p=i+2}^{n}\widehat{M}_{\emptyset,\{i+1,p\}}\\ &= \sum_{p=1}^{i-1}5(n+6) + \sum_{p=i+2}^{n}5(n+6)\\ &\quad - \sum_{p=1}^{i-1}5(n+6) - \sum_{p=i+2}^{n}5(n+6)\\ &= 0.\end{aligned}$$

Coordinates $\boldsymbol{k}$

- For k a positive integer such that $k < i$, We have

$$\begin{aligned}\left(M\cdot v_i\right)_k &= \sum_{p=1}^{i-1}\widehat{M}_{k,\{p,i\}} + \sum_{p=i+2}^{n}\widehat{M}_{k,\{i,p\}} - \sum_{p=1}^{i-1}\widehat{M}_{k,\{p,i+1\}}\\ &\quad - \sum_{p=i+2}^{n}\widehat{M}_{k,\{i+1,p\}}\\ &= \sum_{p=1,p\neq k}^{i-1}\widehat{M}_{k,\{p,i\}} + \widehat{M}_{k\{k,i\}} + \sum_{p=i+2}^{n}\widehat{M}_{k,\{i,p\}}\\ &\quad - \sum_{p=1,p\neq k}^{i-1}\widehat{M}_{k,\{p,i+1\}} - \widehat{M}_{k,\{k,i+1} - \sum_{p=i+2}^{n}\widehat{M}_{k,\{i+1,p\}}\\ &= \sum_{p=1,p\neq k}^{i-1}7 + 3 + \sum_{p=i+2}^{n}7 - \sum_{p=1,p\neq k}^{i-1}7 - 3 - \sum_{p=i+2}^{n}7\\ &= 0.\end{aligned}$$

- Similarly, for k such that $i+1<k$, we have

$$\big(M \cdot v_i\big)_k = 0.$$

- For $k=i$ and $k=i+1$, we calculate

$$\begin{aligned}
\big(M \cdot v_i\big)_i &= \sum_{p=1}^{i-1} \widehat{M}_{i,\{p,i\}} + \sum_{p=i+2}^{n} \widehat{M}_{i,\{i,p\}} \\
&\quad - \sum_{p=1}^{i-1} \widehat{M}_{i,\{p,i+1\}} - \sum_{p=i+2}^{n} \widehat{M}_{i,\{i+1,p\}} \\
&= \sum_{p=1}^{i-1} 35 + \sum_{p=i+2}^{n} 3 - \sum_{p=1}^{i-1} 7 - \sum_{p=i+2}^{n} 7 \\
&= 35(i-1) + 3(n-i-1) \\
&\quad - 7(i-1) - 7(n-i-1) \\
&= 32i - 24 - 4n
\end{aligned}$$

and

$$\begin{aligned}
\big(M \cdot v_i\big)_{i+1} &= \sum_{p=1}^{i-1} \widehat{M}_{i+1,\{p,i\}} + \sum_{p=i+2}^{n} \widehat{M}_{i+1,\{i,p\}} \\
&\quad - \sum_{p=1}^{i-1} \widehat{M}_{i+1,\{p,i+1\}} - \sum_{p=i+2}^{n} \widehat{M}_{i+1,\{i+1,p\}} \\
&= \sum_{p=1}^{i-1} 7 + \sum_{p=i+2}^{n} 7 - \sum_{p=1}^{i-1} 35 - \sum_{p=i+2}^{n} 3 \\
&= 7(i-1) + 7(n-i-1) \\
&\quad - 35(i-1) - 3(n-i-1) \\
&= -32i + 24 + 4n.
\end{aligned}$$

Coordinates $\{k,l\}$

- For k and l distinct integers such that $1 \leq k < l \leq< i$, then

$$
\begin{aligned}
\left(M \cdot v_i\right)_{\{k,l\}} &= \sum_{p=1}^{i-1} \widehat{M}_{\{k,l\},\{p,i\}} + \sum_{p=i+2}^{n} \widehat{M}_{\{k,l\},\{i+1,p\}} \\
&\quad - \sum_{p=1}^{i-1} \widehat{M}_{\{k,l\},\{p,i+1\}} - \sum_{p=i+2}^{n} \widehat{M}_{\{k,l\},\{i+1,p\}} \\
&= \sum_{p=1,p\neq k,l}^{i-1} \widehat{M}_{\{k,l\},\{p,i\}} + \widehat{M}_{\{k,l\},\{k,i\}} + \widehat{M}_{\{k,l\},\{l,i\}} \\
&\quad + \sum_{p=i+2,p\neq k,l}^{n} \widehat{M}_{\{k,l\},\{i+1,p\}} - \sum_{p=1,p\neq k,l}^{i-1} \widehat{M}_{\{k,l\},\{p,i+1\}} \\
&\quad - \widehat{M}_{\{k,l\},\{k,i+1\}} - \widehat{M}_{\{k,l\},\{l,i+1\}} \\
&\quad - \sum_{p=i+2,p\neq k,l}^{n} \widehat{M}_{\{k,l\},\{i+1,p\}} \\
&= \sum_{p=1,p\neq k,l}^{i-1} 5 + 3 + 3 + \sum_{p=i+2,p\neq k,l}^{n} 5 \\
&\quad - \sum_{p=1,p\neq k,l}^{i-1} 5 - 3 - 3 - \sum_{p=i+2,p\neq k,l}^{n} 5 \\
&= 0.
\end{aligned}
$$

- In the same way, in all cases with $k, l \neq i, i+1$, we have

$$\left(M \cdot v_i\right)_{\{k,l\}} = 0.$$

- We have

$$
\begin{aligned}
\big(M \cdot v_i\big)_{\{i,i+1\}} &= \sum_{p=1}^{i-1} \widehat{M}_{\{i,i+1\},\{p,i\}} + \sum_{p=i+2}^{n} \widehat{M}_{\{i,i+1\},\{i+1,p\}} \\
&\quad - \sum_{p=1}^{i-1} \widehat{M}_{\{i,i+1\},\{p,i+1\}} - \sum_{p=i+2}^{n} \widehat{M}_{\{i,i+1\},\{i+1,p\}} \\
&= \sum_{p=1}^{i-1} 15 + \sum_{p=i+2}^{n} 15 - \sum_{p=1}^{i-1} 15 - \sum_{p=i+2}^{n} 15 \\
&= 0.
\end{aligned}
$$

- For $k < i$, we have

$$
\begin{aligned}
\big(M \cdot v_i\big)_{\{k,i\}} =& \sum_{p=1}^{i-1} \widehat{M}_{\{k,i\},\{p,i\}} + \sum_{p=i+2}^{n} \widehat{M}_{\{k,i\},\{i,p\}} \\
& - \sum_{p=1}^{i-1} \widehat{M}_{\{k,i\},\{p,i+1\}} - \sum_{p=i+2}^{n} \widehat{M}_{\{k,i\},\{i+1,p\}} \\
=& \sum_{p=1,p\neq k}^{i-1} 15 + 9 + \sum_{p=i+2}^{n} 3 \\
& - \sum_{p=1,p\neq k}^{i-1} 5 - 3 - \sum_{p=i+2}^{n} 5 \\
=& 15(i-2) + 9 + 3(n-i-1) \\
& - 5(i-2) - 3 - 5(n-i-1) \\
=& 12i - 12 - 2n.
\end{aligned}
$$

Similarly, for $k < i$ and $l > i + 1$, we have

$$\left(\widehat{M}\cdot v_i\right)_{\{k,i+1\}} = -12i+12+2n$$
$$\left(\widehat{M}\cdot v_i\right)_{\{i,l\}} = 12i-12-2n$$
$$\left(\widehat{M}\cdot v_i\right)_{\{i+1,l\}} = -12i+12+2n.$$

Gathering these results together we have

$$M\cdot v_i = (32i-24-4n)u_i + (12i-12-2n)v_i.$$

Thus the vectors u_i and v_i form a stable plane for M $\square$

We can form a base of E containing the pairs (u_i, v_i) for $1 \le i \le n-1$, the matrix M in this this base contains blocks as

$$\begin{pmatrix} 28 & 32i-24-4n \\ 10 & 12i-12-2n \end{pmatrix}$$

corresponding to the ith stable plane. The determinant of such a matrix is

$$-16(n-i+6).$$

This determinant vanishes for $i = n+6$, which never happens since $i < n$.

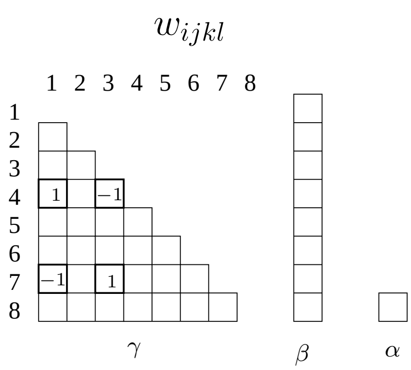
5.4.2 Eigenvectors of $\widehat{M}$

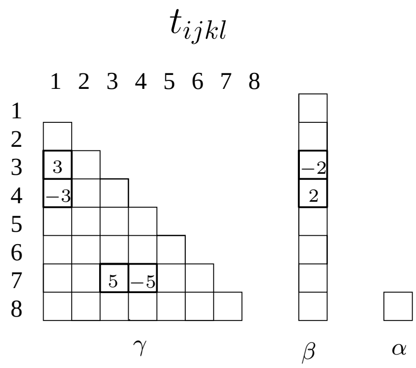
Further, for $1 \le i < j < k < l \le n$, introduce two vectors w_{ijkl} and t_{ijkl} in E. These vectors have the following nonzero coordinates:

$$w_{ijkl} : (\gamma_{ik} = 1, \gamma_{jl} = 1, \gamma_{il} = -1, \gamma_{jk} = -1),$$
$$t_{ijkl} : (\beta_j = 2, \beta_k = -2, \gamma_{ij} = -3, \gamma_{ik} = 3, \gamma_{jl} = -5, \gamma_{kl} = 5).$$

All the coordinates that are not listed are equal to 0.

$$n = 8,\ i = 1,\ j = 3,\ k = 4,\ l = 7$$





Proposition 21. *The vectors w_{ijkl} and t_{ijkl} are eigenvectors of M with eigenvalue -4 for any i, j, k, l.*

Proof. Let's calculate the products $M \cdot w_{ijkl}$ and $M \cdot t_{ijkl}$.

- The first coordinate of $M \cdot w_{ijkl}$ is

$$\begin{aligned}\left(\widehat{M} \cdot w_{ijkl}\right)_{\emptyset} &= \widehat{M}_{\emptyset,\{i,k\}} - \widehat{M}_{\emptyset,\{i,l\}} - \widehat{M}_{\emptyset,\{j,k\}} + \widehat{M}_{\emptyset,\{j,l\}} \\ &= 5(n+6) - 5(n+6) - 5(n+6) + 5(n+6) \\ &= 0.\end{aligned}$$

- For $p \neq i, j, k, l$, we have

$$\begin{aligned}\left(\widehat{M} \cdot w_{ijkl}\right)_{p} &= \widehat{M}_{p,\{i,k\}} - \widehat{M}_{p,\{i,l\}} - \widehat{M}_{p,\{j,k\}} + \widehat{M}_{p,\{j,l\}} \\ &= 7 - 7 - 7 + 7 \\ &= 0.\end{aligned}$$

- If $p = i$ or $p = j$, we have

$$\begin{aligned}\left(\widehat{M} \cdot w_{ijkl}\right)_{i} &= \widehat{M}_{i,\{i,k\}} - \widehat{M}_{i,\{i,l\}} - \widehat{M}_{i,\{j,k\}} + \widehat{M}_{i,\{j,l\}} \\ &= 3 - 3 - 7 + 7 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_j &= \widehat{M}_{j,\{i,k\}} - \widehat{M}_{j,\{i,l\}} - \widehat{M}_{j,\{j,k\}} + \widehat{M}_{j,\{j,l\}} \\ &= 7-7-3+3 \\ &= 0.\end{aligned}$$

- If $p=k$ or $p=l$, we have

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_j &= \widehat{M}_{k,\{i,k\}} - \widehat{M}_{k,\{i,l\}} - \widehat{M}_{k,\{j,k\}} + \widehat{M}_{k,\{j,l\}} \\ &= 35-7-35+7 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_l &= \widehat{M}_{l,\{i,k\}} - \widehat{M}_{l,\{i,l\}} - \widehat{M}_{l,\{j,k\}} + \widehat{M}_{l,\{j,l\}} \\ &= 7-35-7+35 \\ &= 0.\end{aligned}$$

- The coordinates $\{p,q\}$ with $p,q\neq i,j,k,l$ are

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_j &= \widehat{M}_{\{p,q\},\{i,k\}} - \widehat{M}_{\{p,q\},\{i,l\}} \\ &\quad - \widehat{M}_{\{p,q\},\{j,k\}} + \widehat{M}_{\{p,q\},\{j,l\}} \\ &= 5-5-5+5 \\ &= 0.\end{aligned}$$

- The coordinates $\{i,q\}$ whith $q\neq j,k,l$ and $\{j,q\}$ with $q\neq k,l$ are

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_j &= \widehat{M}_{\{i,q\},\{i,k\}} - \widehat{M}_{\{i,q\},\{i,l\}} \\ &\quad - \widehat{M}_{\{i,q\},\{j,k\}} + \widehat{M}_{\{i,q\},\{j,l\}} \\ &= 3-3-5+5 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_j &= \widehat{M}_{\{j,q\},\{i,k\}} - \widehat{M}_{\{j,q\},\{i,l\}}\\ &\quad - \widehat{M}_{\{j,q\},\{j,k\}} + \widehat{M}_{\{j,q\},\{j,l\}}\\ &= 5-5-3+3\\ &= 0.\end{aligned}$$

- We have

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_{\{i,j\}} &= \widehat{M}_{\{i,j\},\{i,k\}} - \widehat{M}_{\{i,j\},\{i,l\}}\\ &\quad - \widehat{M}_{\{i,j\},\{j,k\}} + \widehat{M}_{\{i,j\},\{j,l\}}\\ &= 3-3-15+15\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_{\{k,l\}} &= \widehat{M}_{\{k,l\},\{i,k\}} - \widehat{M}_{\{k,l\},\{i,l\}}\\ &\quad - \widehat{M}_{\{k,l\},\{j,k\}} + \widehat{M}_{\{k,l\},\{j,l\}}\\ &= 3-15-3+15\\ &= 0.\end{aligned}$$

- We have

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_{\{i,k\}} &= \widehat{M}_{\{i,k\},\{i,k\}} - \widehat{M}_{\{i,k\},\{i,l\}}\\ &\quad - \widehat{M}_{\{i,k\},\{j,k\}} + \widehat{M}_{\{i,k\},\{j,l\}}\\ &= 9-3-15+5\\ &= -4.\end{aligned}$$

Similarly we calculate

$$\begin{aligned}\left(\widehat{M}\cdot w_{ijkl}\right)_{\{i,l\}} &= 4\\ \left(\widehat{M}\cdot w_{ijkl}\right)_{\{j,k\}} &= 4\\ \left(\widehat{M}\cdot w_{ijkl}\right)_{\{j,l\}} &= -4\end{aligned}$$

Hence w_{ijkl} is an eigenvector of $\widehat{M}$ for the eigenvalue -4.

Let's calculate $\widehat{M}.t_{ijkl}$. The following coordinates are easy to calculate

- For $p \neq j, k$, $\left(\widehat{M}.t\right)_p = 0$.
- For $p, q \neq i, j, k, l$,

$$\begin{aligned}
\left(\widehat{M}.t_{ijkl}\right)_{\{p,q\}} &= \left(\widehat{M} \cdot t_{ijkl}\right)_{\{i,q\}} = \left(\widehat{M} \cdot t_{ijkl}\right)_{\{p,i\}} \\
&= \left(\widehat{M} \cdot t_{ijkl}\right)_{\{k,q\}} = \left(\widehat{M} \cdot t_{ijkl}\right)_{\{p,k\}} \\
&= \left(\widehat{M} \cdot t_{ijkl}\right)_{\{j,k\}} = \left(\widehat{M} \cdot t_{ijkl}\right)_{\{i,l\}} \\
&= \left(\widehat{M} \cdot t_{ijkl}\right)_{\{p,j\}} = \left(\widehat{M} \cdot t_{ijkl}\right)_{\{j,q\}} \\
&= \left(\widehat{M} \cdot t_{ijkl}\right)_{\{p,k\}} = \left(\widehat{M} \cdot t_{ijkl}\right)_{\{k,q\}} \\
&= 0
\end{aligned}$$

We have

$$\begin{aligned}
\left(\widehat{M} \cdot t_{ijkl}\right)_j =& - \left(\widehat{M} \cdot t_{ijkl}\right)_k \\
=& 2\widehat{M}_{j,j} - 2\widehat{M}_{j,k} - 3\widehat{M}_{j,\{i,j\}} + 3\widehat{M}_{j,\{i,k\}} \\
& - 5\widehat{M}_{j,\{j,l\}} + 5\widehat{M}_{j,\{k,l\}} \\
=& 2 \cdot 35 - 2 \cdot 7 - 3 \cdot 35 + 3 \cdot 7 - 5 \cdot 3 + 5 \cdot 7 \\
=& -8,
\end{aligned}$$

$$\begin{aligned}
\left(\widehat{M} \cdot t_{ijkl}\right)_{\{i,j\}} =& - \left(\widehat{M} \cdot t_{ijkl}\right)_{\{i,k\}} \\
=& 2\widehat{M}_{\{i,j\},j} - 2\widehat{M}_{\{i,j\},k} - 3\widehat{M}_{\{i,j\},\{i,j\}} + 3\widehat{M}_{\{i,j\},\{i,k\}} \\
& - 5\widehat{M}_{\{i,j\},\{j,l\}} + 5\widehat{M}_{\{i,j\},\{k,l\}} \\
=& 2 \cdot 15 - 2 \cdot 5 - 3 \cdot 9 + 3 \cdot 3 - 5 \cdot 3 + 5 \cdot 5 \\
=& 12
\end{aligned}$$

and

$$\begin{aligned}\left(\widehat{M}\cdot t_{ijkl}\right)_{\{j,l\}} =& -\left(\widehat{M}\cdot t_{ijkl}\right)_{\{k,l\}}\\ =&2\widehat{M}_{\{j,l\},j}-2\widehat{M}_{\{j,l\},k}-3\widehat{M}_{\{j,l\},\{i,j\}}+3\widehat{M}_{\{j,l\},\{i,k\}}\\ &-5\widehat{M}_{\{j,l\},\{j,l\}}+5\widehat{M}_{\{j,l\},\{k,l\}}\\ =&2\cdot 15-2\cdot 5-3\cdot 15+3\cdot 5-5\cdot 9+5\cdot 15\\ =&20\end{aligned}$$

Thus we have $\left(\widehat{M}\cdot t_{ijkl}\right)=-4t_{ijkl}$. □

5.4.3 The exceptional case

When n is congruent to 2 modulo 8, one of the stable planes contains an eigenvector for the eigenvalue -4. In this cases, that we called exceptional, we need to find a new vector in order to determine the rank of $\widehat{M}$. In this purpose, we introduce the following vector

$$z:(\gamma_{3m+1,n}=1,\gamma_{3m+2,n}=-1).$$

The other coordinates of z vanish.

Proposition 22. *If $n=8p+2$, the invariant plane spanned by u_{3p+1} and v_{3p+1} contains an eigenvector with eigenvalue -4. Moreover, the vector $z+\dfrac{m}{5m+7}u_{3m+1}$ is annihilated by $(M+4)^2$, but not by $M+4$.*

Proof. When n is congruent to 2 modulo 8, in other terms, when there exists $m\in\mathbb{N}$ such that $n=8m+2$, the stable plan spanned by u_{3m+1} and v_{3m+1} is associated to the matrix

$$\begin{pmatrix}28 & 32(3m+1)-24-4(8m+2)\\ 10 & 12(3m+1)-12-2(8m+2)\end{pmatrix}=\begin{pmatrix}28 & 64m\\ 10 & 20m-4\end{pmatrix}.$$

Its characteristic polynomial is

$$\lambda^2-(20m+24)\lambda-80m-112$$

which factorizes into

$$(\lambda + 4)(\lambda - 20m - 28).$$

Hence we have two eigenvectors.

For the eigenvalue -4 The system

$$\begin{cases} 28x - 64my = -4x \\ 10x(20m-4)y = -4y \end{cases} \Leftrightarrow x + 2my = 0$$

shows that $\mathcal{U} := 2mu_{3m+1} - v_{3m+1}$ is an eigenvector for the eigenvalue -4.

If we write $\mathcal{V}$ an eigenvector for the eigenvalue $20m + 28$, in the base $(\mathcal{U}, \mathcal{V})$, the block corresponding to the $(3m+1)$st stable plan is written

$$\begin{pmatrix} -4 & 0 \\ 0 & 20m+28 \end{pmatrix}$$

Vector of $Ker(M+4)^2 \backslash Ker(M+4)$ Now we consider the vector z.
We have

- $$\begin{aligned} \left(\widehat{M} \cdot z\right)_{3m+1} &= -\left(\widehat{M}.z\right)_{3m+2} \\ &= \widehat{M}_{3m+1,\{3m+1,n\}} - \widehat{M}_{3m+1,\{3m+2,n\}} \\ &= 3 - 7 = -4. \end{aligned}$$

- $$\begin{aligned} \left(\widehat{M} \cdot z\right)_{\{p,3m+1\}} &= \widehat{M}_{\{p,3m+1\},\{3m+1,n\}} - \widehat{M}_{\{p,3m+1\},\{3m+2,n\}} \\ &= 3 - 5 = -2. \end{aligned}$$

- $$\begin{aligned}\left(\widehat{M}\cdot z\right)_{\{3m+1,p\}} &= \widehat{M}_{\{3m+1,p\},\{3m+1,n\}} - \widehat{M}_{\{3m+1,p\},\{3m+2,n\}} \\ &= 3-5=-2.\end{aligned}$$

- $$\begin{aligned}\left(\widehat{M}\cdot z\right)_{\{3m+1,m+2\}} &= \widehat{M}_{\{3m+1,3m+2\},\{3m+1,n\}} \\ &\quad - \widehat{M}_{\{3m+1,m+2\},\{3m+2,n\}} \\ &= 3-3=0.\end{aligned}$$

- $$\begin{aligned}\left(\widehat{M}\cdot z\right)_{\{3m+1,n\}} &= \widehat{M}_{\{3m+1,n\},\{3m+1,n\}} - \widehat{M}_{\{3m+1,n\},\{3m+2,n\}} \\ &= 9-15=-6.\end{aligned}$$

Then $(\widehat{M}+4)\cdot z = -4u_{3m+1} - 2v_{3m+1}$.
Now let's write $\delta := \dfrac{m}{5m+7}$ and $\mathcal{Z} := z + \delta u_{3m+1}$. We have

$$\begin{aligned}(\widehat{M}+4)\cdot\mathcal{Z} &= (\widehat{M}+4)\cdot z + \delta(\widehat{M}+4)\cdot u_{3m+1} \\ &= -4u-3m+1-2v_{3m+1}+\delta(28u_{3m+1}+10v_{3m+1})+4u_{3m+1} \\ &= 28\delta u_{3m+1} + (10\delta-2)v_{3m+1} \\ &= \frac{-14}{5m+7}\mathcal{U}.\end{aligned}$$

Since $\mathcal{U}$ is an eigenvector associated to the eigenvalue -4, we have

$$(\widehat{M}+4)^2\cdot\mathcal{Z} = 0.$$

$\square$

5.4.4 The span of vectors $u, v, w, t, \mathcal{Z}$.

Proposition 23. *The vectors u_i, v_i, w_{ijkl}, t_{ijkl}, together with z in the case $n = 8p + 2$, span the codimension 3 subspace of E given by the equations*

$$\alpha = 0, \quad \sum_{i=1}^{n} \beta_i = 0, \quad \sum_{i<j} \gamma_{ij} = 0.$$

Proof. First of all, it is easy to see that all vectors u, v, w, t, z satisfy the three linear equations given above.

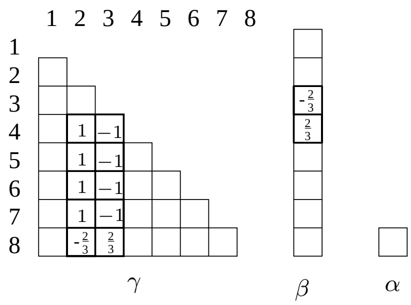
Now consider a vector s in E satisfying the three equations. We will show that we can make it vanish by adding an appropriate linear combination of vectors u_i, v_i, w_{ijkl}, t_{ijkl}, and z if $n = 8p + 2$.

For $1 \leq i \leq n-2$ let

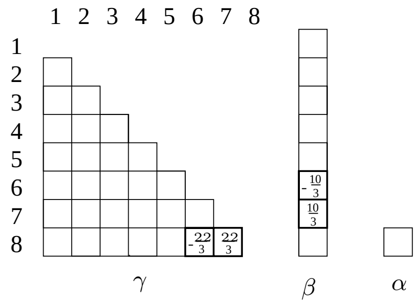
$$\tilde{v}_i = v_i - \frac{1}{3}\left(t_{1,i,i+1,n} + t_{2,i,i+1,n} + \cdots + t_{i-1,i,i+1,n}\right).$$

$n = 8$

$\widetilde{v}_2$



$\widetilde{v}_6$



This vector has the property that its only nonzero γ-coordinates have the form γ_{ij} or $\gamma_{i+1,j}$. Moreover, the coordinates with first index i add up to $(n-1-i) - \frac{5}{3}(i-1) = \frac{3n+2-8i}{3}$. Similary, its coordinates with first index $i+1$ add up to the opposite number $-\frac{3n+2-8i}{3}$.

Now we perform the **first step** that consists in annihilating the γ-coordinates of s. This is done using the following sequence of operations.

As a preparation we add to s a multiple of v_{n-1} so as to annihilate the sum $\sum_{i=1}^{n-1} \gamma_{in}$. We will not use the vector v_{n-1} again in the following operations, and one can easily check that in all other vectors u, v, w, t, z the sum of coordinates γ_{in} is equal to 0. Thus none of the following operations will change this sum, and it will remain equal to 0 all the time. Now we can start killing the coordinates in earnest.

First we add a multiple of t_{1234} to annihilate γ_{12}. Then we add a multiple of $v_1 = \tilde{v}_1$ so as to annihilate the sum $\sum_j \gamma_{1j}$. Note that this does not change γ_{12}. Now we use the vectors w_{12kl} to annihilate all the coordinates γ_{1j}. Note that this does not change either γ_{12} or $\sum_j \gamma_{1j}$.

We have now achieved the vanishing of all coordinates γ_{1j} . The next step is to kill the coordinates γ_{2j}, and then we continue to increase the first index of the γ-coordinates one by one. Assume that we have already achieved the vanishing of all γ-coordinates with first index less than i and let us do the coordinates γ_{ij} for j from $i+1$ to n. We assume that $i \leq n-3$.

First we use the vector $t_{i,i+1,i+2,i+3}$ to kill the coordinate $\gamma_{i,i+1}$. Second, we use the vector $\tilde{v}_i$ to annihilate the sum $\sum_{j=i+1}^{n} \gamma_{ij}$. This is always possible unless $3n+2-8i = 0$. In this case we have $n = 8p+2$, $i = 3p+1$, so we can use the vector z instead. Third, we use the vectors $w_{i,i+1,k,l}$ to kill all coordinates γ_{ij} for our given value of i. Note that all these operations do not change the coordinates $\gamma_{i',j}$ with $i' < i$.

Once we are finished with $i = n-3$ we are left with only three possibly nonzero γ-coordinates: $\gamma_{n-2,n-1}$, $\gamma_{n-2,n}$, and $\gamma_{n-1,n}$. Recall that as a preliminary step we have achieved the vanishing

$\sum \gamma_{in} = 0$; now this condition reads $\gamma_{n-2,n} = -\gamma_{n-1,n}$. Thus we can use the vector $\tilde{v}_{n-2}$ to kill both $\gamma_{n-2,n}$ and $\gamma_{n-1,n}$. Now the remaining coordinate $\gamma_{n-2,n-1}$ is automatically equal to 0, because the sum of all γ-coordinates of s was 0 from the start.

After getting rid of the γ-coordinates, the **second step** consists in eliminating the β-coordinates without changing the γ-coordinates. This step is much simpler: we just use the vectors u_i. $\square$

5.4.5 Complement subspace

We write F the span of $u, v, w, t, \mathcal{Z}$. If we write

$$\begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_n \\ \gamma_{1,2} \\ \vdots \\ \gamma_{n-1,n} \end{pmatrix}$$

any vector of E, F corresponds to the subspace of vectors satisfying the equations

$$\alpha = 0$$
$$\sum_{i=1}^{n} \beta_i = 0$$
$$\sum_{i<j} \gamma_{i,j} = 0.$$

We look now for a complement of F in E. Hence we introduce the vectors a, b and c of E with the following nonzero coordinates

$$a : (\alpha = 1),$$
$$b : (\beta_1 = 1),$$
$$c : (\gamma_{1,2} = 1).$$

We calculate the image of a. We have

$$\left(\widehat{M} \cdot a\right)_{\emptyset} = \widehat{M}_{\emptyset,\emptyset} = 5(n+6).$$

For $1 \leq i \leq n$, we have

$$\left(\widehat{M} \cdot a\right)_{i} = \widehat{M}_{i,\emptyset} = 7$$

and for $1 \leq i < j \leq n$, we have

$$\left(\widehat{M} \cdot a\right)_{\{i,j\}} = \widehat{M}_{\{i,j\},\emptyset} = 5.$$

Now we calculate the image of b. We have

$$\left(\widehat{M} \cdot b\right)_{\emptyset} = \widehat{M}_{\emptyset,1} = 5n + 34$$

and

$$\left(\widehat{M} \cdot b\right)_{1} = \widehat{M}_{1,1} = 35$$

For $2 \leq i \leq n$, we have

$$\left(\widehat{M} \cdot b\right)_{i} = \widehat{M}_{i,1} = 7$$

and

$$\left(\widehat{M} \cdot b\right)_{\{1,i\}} = \widehat{M}_{\{1,i\},1} = 15.$$

For $2 \leq i, j \leq n$, we have

$$\left(\widehat{M} \cdot b\right)_{\{i,j\}} = \widehat{M}_{\{i,j\},1} = 5.$$

Let's calculate the image of c. For the first coordinate we have

$$\left(\widehat{M} \cdot c\right)_{\{1,2\}} = \widehat{M}_{\emptyset,\{1,2\}} = 5(n+6),$$
$$\left(\widehat{M} \cdot c\right)_{1} = \widehat{M}_{1,\{1,2\}} = 3$$

and

$$\left(\widehat{M} \cdot c\right)_{2} = \widehat{M}_{2,\{1,2\}} = 35.$$

For $3 \leq i \leq n$, we have

$$\left(\widehat{M} \cdot c\right)_{i} = \widehat{M}_{i,\{1,2\}} = 7.$$

We have
$$\left(\widehat{M} \cdot c\right)_{\{1,2\}} = \widehat{M}_{\{1,2\},\{1,2\}} = 9.$$
For $2 \leq i \leq n$, we have
$$\left(\widehat{M} \cdot c\right)_{\{1,i\}} = \widehat{M}_{\{1,i\},\{1,2\}} = 3$$
and for $3 \leq i \leq n$, we have
$$\left(\widehat{M} \cdot c\right)_{\{2,i\}} = \widehat{M}_{\{2,i\},\{1,2\}} = 15.$$
For $3 \leq i < j \leq n$, we have
$$\left(\widehat{M} \cdot c\right)_{\{i,j\}} = \widehat{M}_{\{i,j\},\{1,2\}} = 5.$$

Modulo the subspace F, the images of these vectors only depend on the first coordinate, the sum of the n following coordinates and the sum of the others coordinates, hence on 3 numbers. We can write that, modulo F,

$$\left(\widehat{M}\cdot a\right) \sim \begin{pmatrix} \left(\widehat{M} \cdot a\right)_{\emptyset} \\ \sum_{i=1}^{n} \left(\widehat{M} \cdot a\right)_{i} \\ 0 \\ \vdots \\ 0 \\ \sum_{i<j} \left(\widehat{M} \cdot a\right)_{\{i,j\}} \\ \vdots \\ 0 \end{pmatrix}, \left(\widehat{M}\cdot b\right) \sim \begin{pmatrix} \left(\widehat{M} \cdot b\right)_{\emptyset} \\ \sum_{i=1}^{n} \left(\widehat{M} \cdot b\right)_{i} \\ 0 \\ \vdots \\ 0 \\ \sum_{i<j} \left(\widehat{M} \cdot b\right)_{\{i,j\}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{and } \left(\widehat{M}\cdot c\right) \sim \begin{pmatrix} \left(\widehat{M}\cdot c\right)_{\emptyset} \\ \sum_{i=1}^{n}\left(\widehat{M}\cdot c\right)_{i} \\ 0 \\ \vdots \\ 0 \\ \sum_{i<j}\left(\widehat{M}\cdot c\right)_{\{i,j\}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let's calculate these coordinates, we have

$$\begin{aligned} \left(\widehat{M}\cdot a\right) &= \widehat{M}_{\emptyset,\emptyset}a + \sum_{i=1}^{n}\widehat{M}_{i,\emptyset}b + \sum_{i<j}\widehat{M}_{\{i,j\},\emptyset}c \\ &= 5(n+6)a + 7nb + 5\binom{n}{2}c. \end{aligned}$$

$$\begin{aligned} \left(\widehat{M}\cdot b\right) &= \widehat{M}_{\emptyset,1}a + \sum_{i=1}^{n}\widehat{M}_{i,1}b + \sum_{i<j}^{n}\widehat{M}_{\{i,j\},1}c \\ &= (5n+34)a + \big(35 + 7(n-1)\big)b \\ &\quad + \Big(15(n-1) + 5((\binom{n}{2} - (n-1))\Big)c \\ &= (5n+34)a + (7n+28)b + \frac{5}{2}(n-1)(n+4)c. \end{aligned}$$

$$\begin{aligned}
\big(\widehat{M}\cdot c\big) =&\widehat{M}_{\emptyset,\{1,2\}}a+\sum_{i=1}^{n}\widehat{M}_{i,\{1,2\}}b+\sum_{i<j}\widehat{M}_{\{i,j\},\{1,2\}}c\\
=&5(n+6)a+\big(3+35+7(n-2)\big)b\\
&+\Big(9+3(n-2)+15(n-2)\\
&\quad+5\big(\binom{n}{2}-(n-1)-(n-2)\big)\Big)c\\
=&5(n+6)a+(7n+24)b+\big(\frac{5}{2}n^2+\frac{11}{2}n-12\big)c.
\end{aligned}$$

Hence we have the following block

$$\begin{pmatrix} 5(n+6) & 5n+34 & 5(n+6)\\ 7n & 7n+28 & 7n+24\\ \frac{5}{2}n(n-1) & \frac{5}{2}(n-1)(n+4) & \frac{5}{2}n^2+\frac{11}{2}n-12 \end{pmatrix}.$$

Its determinant is $-32(n+6)(2n+15)$ which never vanishes for $n\in\mathbb{N}$.

5.5 Conclusion

We have rewritten $\widehat{M}$ as a block upper triangular matrix whose block have been described in the previous sections. The determinant of $\widehat{M}$ is now easy to calculate through this new matrix. This last matrix is composed by $n-1$ blocks of size 2, a block of size 3 and blocks of size -4. The latter type of block appears $\frac{n(n-1)}{2}+n+1-\big[2(n-1)+3\big]=\frac{n(n-3)}{2}$ times. Then we have

$$\det\widehat{M}=\prod_{i=1}^{n-1}(-16(n-i+6))\cdot(-4)^{\frac{n(n-3)}{2}}\cdot(-32)(n+6)(2n+15)$$

which is better written

$$\det\widehat{M}=(-1)^{\frac{n(n-1)}{2}}2^{n^2+n+1}(2n+15)\frac{(n+6)!}{6!}.$$

In this form we see clearly that $\det \widehat{M}$ never vanishes, proving that the classes $\kappa_{1,1}, \psi_i^2, \psi_i\psi_j$ are independent. Since we know that the other classes lie in the span of these classes, they generate the degree 2 group of $R^*(\mathcal{M}_{4,n})$. Hence we have

$$R^2(\mathcal{M}_{4,n}) \simeq < \psi_1^2, ..., \psi_n^2, \kappa_{1,1} > .$$

6 The moduli space of real curves of genus zero

6.1 The moduli space of real curves

The complex conjugation acts on the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ of stable complex curves of genus zero with $n \geq 3$ marked points. The locus of fixed points under this action is the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of stable real curves. It is a compact connected smooth real manifold [13]. As in the complex case, we have

$$\overline{\mathcal{C}}_{0,n}(\mathbb{R}) \simeq \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}).$$

6.2 Stiefel-Whitney classes

6.2.1 The axiomatic approach

Definition 29. Let B be a CW-complex. To every real vector bundle E over B one assigns its *Stiefel-Whitney class* $w(E) \in H^*(B, \mathbb{Z}/2\mathbb{Z})$. The degree k part of $w(E)$ is called the kth Stiefel-Whitney class and denoted by $w_k(E)$. The Stiefel-Whitney class is defined via the following axioms.

- The degree 0 part $w_0(E)$ equals 1.
- The first Stiefel-Whitney class of the tautological line bundle over the real projective line is nonzero.
- For $k > \text{rank}(E)$ we have $w_k(E) = 0$.
- For any short exact sequence of vector bundles

  $$0 \to E_1 \to E_2 \to E_3 \to 0$$

  we have

  $$w(E_2) = w(E_1) \smile w(E_3),$$

  where $\smile$ denotes the cup product.

- For a continuous map $f : B' \to B$ we have

$$w(f^*(E)) = f^* w(E).$$

The existence and uniqueness of the Stiefel-Whitney class is proved in [28] Section 8.

To illustrate how these axioms work we prove two basic properties of the Stiefel-Whitney classes.

Proposition 24. *The Stiefel-Whitney class of a trivial vector bundle is equal to 1.*

Proof. If the base of the vector E bundle is a point, we have $w_0(E) = 1$ by Axiom 1 and $w_k(E) = 0$ for $k \geq 1$. Now it suffices to note that a trivial line bundle is always a pull-back from a point. $\square$

Proposition 25. *If $w_{\mathrm{rank}(E)} \neq 0$, then every section of the bundle vanishes at some point.*

Proof. A non-vanishing section spans a trivial line subbundle L of E. Thus we obtain a short exact sequence

$$0 \to L \to E \to E_1 \to 0$$

with $\mathrm{rank}(E_1) = \mathrm{rank}(E) - 1$. Since $w(L) = 1$, we see that $w_{\mathrm{rank}(E)} = w_{\mathrm{rank}}(E_1) = 0$. $\square$

Proposition 26. *Let B be a smooth compact real manifold, $L \to B$ a line bundles, and s a smooth section of L that intersects the zero section transversally. Denote by $Z \in B$ the zero locus of s. Then $w_1(L)$ is Poincaré dual to Z.*

Proof. It suffices to check that any 1-homology class represented by a smooth closed loop S in B has the same intersection index with $w_1(L)$ and with Z. By axiom 4, the intersection of $w_1(L)$ with S equals 0 if $L|_S$ is orientable and 1 if it is not. The intersection of S with Z is the number of (simple) zeros of the section s restricted to S. Now it suffices to note that on a circle any section of an orientable line bundle has an even number of zeros, while a section of a nonorientable line bundle has an odd number of zeros. $\square$

Proposition 27. *Let X be a smooth real algebraic manifold, $L \to X$ a real analytic line bundle and $D \subset X$ a divisor. Then we have*

$$w_1(L(D)) = w_1(L) + [D].$$

Proof. Consider a smooth section s of L that intersects the zero section transversally and does not indentically vanish on D. Denote by Z its vanishing locus. Then s also represents a smooth section of $L(D)$ with vanishing locus $Z \cup D$. Therefore $w_1(L(D)) = [Z] + [D] = c_1(L) + [D]$. $\square$

6.2.2 A construction of the first Stiefel-Whitney class of a line bundle

Since we are mostly interested in the first Stiefel-Whitney classes of line bundles let us sketch a construction here.

Denote by S^∞ the space of real sequences $x_1, x_2, \dots$ such that $\sum x_i^2 = 1$. This space inherits a topology from the space $L^2(\mathbb{Z}_+)$.

Lemma 4. *The space S^∞ is contractible.*

Proof. Consider, for $t \in [0,1]$, the map $f_t : S^\infty \to L^2(\mathbb{Z}_+)$ defined by $f_t(x_1, x_2, \dots) \mapsto (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$. Since f_t never vanishes, we can define the map

$$\frac{f_t}{|f_t|} : S^\infty \to S^\infty.$$

This is a homotopy of S^∞ to its equator $\{x_1 = 0\}$. Now let $g_t(0, x_1, x_2, \dots) = (1-t)(0, x_1, x_2, \dots) + t(1, 0, 0, \dots)$. Once again, g_t never vanishes, so we can define the map $\frac{g_t}{|g_t|}$, which is a homotopy of the equator to the constant map. Composing f and g we get a homotopy between the identity map from S^∞ to itself and the constant map. $\square$

Denote by $\mathbb{P}^\infty(\mathbb{R})$ the quotient of S^∞ by the central symmetry. Denote by ξ the generator of $H^1(\mathbb{P}^\infty(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

Proposition 28. $\mathbb{P}^\infty(\mathbb{R})$ *is an Eilenberg-MacLane space.*

Proof. S^∞ is a double cover of $\mathbb{P}^\infty(\mathbb{R})$.
S^∞ is contractible hence $\pi_1(\mathbb{P}^\infty(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ and

$$\pi_i(\mathbb{P}^\infty(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z} = \pi_i(S^\infty) = 0,$$

for $i > 1$. $\square$

It follows that any line bundle $L \to B$ is the pull-back of the tautological line bundle on $\mathbb{P}^\infty(\mathbb{R})$ under an appropriate map $f : B \to \mathbb{P}^\infty(\mathbb{R})$. The first Stiefel-Whitney class is then equal to $f^*(\xi)$.

6.3 The ξ-classes on $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$

Let $L_i \to \overline{\mathcal{M}}_{g,n}(\mathbb{R})$ be the cotangent line bundle to the ith marked point. We define $\xi_i := w_1(L_i)$.

Denote by $\delta_{(i,n+1)} \subset \overline{\mathcal{M}}_{g,n}(\mathbb{R})$ the divisor of stable curves containing a rational irreducible component with one node and two marked points with numbers i and $n+1$. Adapting a similar result on the space of complex curves [36], we can obtain the following proposition.

Proposition 29. *We have*

$$\pi^*(\xi_i) = \xi_i - [\delta_{(i,n+1)}].$$

Proof. Let's temporarily denote by $(C, x'_1, \ldots, x'_n)$ stable curves with n marked points and by $(C, x_1, \ldots, xn+1)$ stable curves with $n+1$ marked points. We also denote by L'_i the cotangent line bundle to x'_i and by L_i the cotangent line bundle to x_i.

There is a natural isomorphism between the line bundles L_i and L'_i outside of the divisor $\delta_{(i,n+1)}$. In order to understand their relashionship in the neighbordhood of the divisor, introduce local coordinates $t_1, \ldots, t_{n-3}$ on $\overline{\mathcal{M}}_{0,n}$ in the neighborhood of some point $p \in \overline{\mathcal{M}}_{0,n}$. Let C_p be the corresponding stable genus 0 curve and z the local coordinate on C_p at the neighborhood of the marking x'_i. The 1-form dz determines a nonzero section of L'_i in the neighborhood of p. Let $(t_1, \ldots, t_{n-3}, t)$ is a family of local coordinates on $\overline{\mathcal{M}}_{0,n+1}$, where t is the z-coordinate of the $(n+1)$st marked point. The divisor $\delta_{(i,n+1)}$ in these local coordiantes is given by $t = 0$. The local coordinate on the stable curve at the neighborhood of the point x_i is given by $w = z/t$. Thus $dz = tdw$ is a section of L_i with a simple vanishing along $\delta_{(i,n+1)}$. In other words, we have $L_i = L'_i(\delta_{(i,n+1)})$. By Proposition 27 it follows that

$$w_1(L_i) = w_1(\pi^*(L'_i)) + [\delta_{(i,n+1)}].$$

□

Denote by ξ_i the class ξ_i on $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and by ξ'_i the pullback from the class ξ_i on $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$. Since the bundles $\mathcal{L}_i$ and $\mathcal{L}_{n+1}$ are trivial over $\delta_{(i,n+1)}$, we have

$$\xi_i \delta_{(i,n+1)} = \xi_{n+1} \delta_{(i,n+1)} = 0.$$

From this we get

$$\begin{aligned} \xi_i^d - (\xi'_i)^d &= (\xi_i - \xi'_i)(\xi_i^{d-1} + \cdots + (\xi'_i)^{d-1}) \\ &= \delta_{(i,n+1)} (\xi'_i)^{d-1}. \end{aligned}$$

We will use this equality later in the form

$$\xi_i^d = (\xi_i')^d + \delta_{(i,n+1)}(\xi_i')^{d-1}.$$

Further, since the intersection between $\delta_{(i,n+1)}$ and $\delta_{(j,n+1)}$ is empty for $i \neq j$, we have

$$\delta_{(i,n+1)}\delta_{(j,n+1)} = 0.$$

6.4 String equation

Now we can calculate $\displaystyle\int_{\overline{\mathcal{M}}_{0,n+1}} \xi_1^{d_1} \dots \xi_n^{d_n}$. We have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{0,n+1}} \xi_1^{d_1} \dots \xi_n^{d_n} \\
&\quad = \int_{\overline{\mathcal{M}}_{0,n+1}} \left((\xi_1')^{d_1} + \delta_{(i,n+1)}.(\xi_1')^{d_1-1}\right) \dots \left((\xi_n')^{d_n} + \delta_{(n,n+1)}.(\xi_n')^{d_1}\right) \\
&\quad = \int_{\overline{\mathcal{M}}_{0,n+1}} (\xi_1')^{d_1} \dots (\xi_n')^{d_n} \\
&\qquad + \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{0,n+1}} (\xi_1')^{d_1} \dots \delta_{(i,n+1)}(\xi_i')^{d_i-1} \dots (\xi_n')^{d_n}.
\end{aligned}$$

Let see the terms of this sum of integrals one by one. We have $\int_{\overline{\mathcal{M}}_{0,n+1}}(\xi_1')^{d_1} \dots (\xi_n')^{d_n} = 0$ because $(\xi_1')^{d_1} \dots (\xi_n')^{d_n}$ is a pullback from $\overline{\mathcal{M}}_{0,n}$. For any i,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{0,n+1}} (\xi_1')^{d_1} \dots \delta_{(i,n+1)}(\xi_i')^{d_i-1} \dots (\xi_n')^{d_n} \\
&\qquad = \int_{\delta_{(i,n+1)}} (\xi_1')^{d_1} \dots (\xi_i')^{d_i-1} \dots (\xi_n')^{d_n} \\
&\qquad = \int_{\pi^*(\delta_{(i,n+1)})} \pi^*((\xi_1')^{d_1} \dots (\xi_i')^{d_n} \dots (\xi_n')^{d_n}) \\
&\qquad = \int_{\overline{\mathcal{M}}_{0,n}} (\xi_1')^{d_1} \dots (\xi_i')^{d_i-1} \dots (\xi_n')^{d_n}.
\end{aligned}$$

We thus have shown that the string equation is true for ξ classes. Explicitly we have

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \xi_1^{d_1} \dots \xi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{0,n}} (\xi_1')^{d_1} \dots (\xi_i')^{d_i - 1} \dots (\xi_n')^{d_n}.$$

The string equation is sufficient to calculate all intersection numbers $\int_{\overline{\mathcal{M}}_{0,n}(\mathbb{R})} \xi_1^{d_1} \dots \xi_n^{d_n}$ using the initial value

$$\int_{\overline{\mathcal{M}}_{0,3}(\mathbb{R})} 1 = 1.$$

Indeed, it follows from the dimension constraint $\sum_{i=1}^{n} d_i = \dim \overline{\mathcal{M}}_{0,n} = n - 3$ that at least one of the integers d_i vanishes.

6.5 Computing the intersection number

Write every integer d_i in the binary system and denote by $\delta_j^{(i)}$ the j-th digit of d_i from the end. For instance, if $d_i = 6$, we have $\delta_0^{(i)} = 0$, $\delta_1^{(i)} = 1$, $\delta_2^{(i)} = 1$, $\delta_j^{(i)} = 0$ for $j \geq 3$.

Theorem 10. *Let $d_1, \dots, d_n$ be nonnegative integers such that $\sum d_i = n - 3$. The intersection number*

$$\xi_1^{d_1} \dots \xi_n^{d_n} \in \mathbb{Z}/2\mathbb{Z}$$

is equal to 1 if and only if we have

$$\sum_{i=1}^{n} \delta_j^{(i)} \leq 1$$

for every $j \geq 0$. Otherwise the intersection number vanishes.

An equivalent way to formulate this theorem is to say that the intersection number is equal to 1 if there are no carryovers in the binary addition of the integers d_i, and vanishes as soon as there is at least one carryover.

Proof. The string equation has the same form for the intersection numbers as for the multinomial coefficients

$$\binom{n-3}{d_1, \ldots, d_n},$$

except that the addition takes place modulo 2. Moreover, the initial value

$$\int_{\overline{\mathcal{M}}_{0,3}(\mathbb{R})} 1 = 1$$

coincides with

$$\binom{0}{0,0,0} = 1.$$

It follows that

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \xi_1^{d_1} \ldots \xi_n^{d_n} = \frac{(n-3)!}{d_1! \ldots d_n!} \mod 2.$$

In order to determine the parity of this multinomial coefficient we first prove the following statement. Write a positive integer d in binary system and denote its digits by δ_j. Then we have

$$\mathrm{val}_2(d!) = \sum_{j \geq 0} (2^j - 1)\delta_j,$$

where $\mathrm{val}_2(\mathrm{d!})$ is the largest power of two that divides $d!$. This formula can be easily deduced from

$$\mathrm{val}_2(d!) = \left\lfloor \frac{d}{2} \right\rfloor + \left\lfloor \frac{d}{4} \right\rfloor + \left\lfloor \frac{d}{8} \right\rfloor + \ldots.$$

First suppose that the binary addition $d_1 + \cdots + d_n$ has no carryovers. In this case the binary valuation of $d_1! \cdots d_n!$ is given by the same sum of terms of the form $2^j - 1$ and the binary valuation of $(d_1 + \cdots + d_n)!$. Thus their quotient is odd.

Now note that $2^{j+1} - 1$ exceeds $2 \cdot (2^j - 1)$ exactly by 1. Thus, every time there is a carryover in the binary addition $d_1 + \cdots + d_n$, the binary valuation of $(d_1+\cdots+d_n)!$ gains an extra 1 compared to the binary valuation of $d_1!\cdots d_n!$. Therefore, as soon as there is at least one carryover, the multinomial coefficient is even. $\square$

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