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A Deligne–Riemann–Roch isometry for flat unitary vector bundles on modular curves

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Un gouvernement passe et tombe, un peuple grandit, resplendit, puis décroît, qu'importe ! les vérités de la science se transmettent, s'accroissent toujours, font toujours plus de lumière et plus de certitude. Le recul d'un siècle ne compte pas, la marche en avant reprend quand même, l'humanité va au savoir, malgré les obstacles. Objecter qu'on ne saura jamais tout est une sottise, il s'agit de savoir le plus possible, pour arriver au plus de bonheur possible.

Émile Zola, *Travail*

Governments rise and fall, peoples grow, shine, and decline, none of it matters ! the truths of science are passed on, ever increased, always bringing more light and certainty. A delay of a century is inconsequential, the march forward resumes anyway, humanity strives toward knowledge, in spite of any obstacle. To say one might not ever know everything is absurd, the point is to know the most, to achieve the most happiness.

Émile Zola, *Work*

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[1] de la William !
[2] ¿ dónde te daré que no te duela ?

À ma grand-mère,
j'ai trouvé ...

Une isométrie de Deligne-Riemann-Roch pour les fibrés vectoriels plats unitaires sur des courbes modulaires

Résumé

L'objectif de cette thèse est de définir et d'étudier une métrique de Quillen sur une courbe modulaire ayant des cusps pour la métrique de Poincaré, munie d'un fibré vectoriel holomorphe plat unitaire. Dans le cas compact, étudié par Bismut-Gillet-Soulé et Deligne, cette métrique de Quillen est une modification de la métrique L^2 par le déterminant du laplacien de Dolbeault agissant sur les sections. Une de ses propriétés fondamentales est qu'elle satisfait une isométrie de type Riemann-Roch. Sous nos hypothèses, cette construction ne peut être faite, en raison de la singularité aux cusps de la métrique de Poincaré sur la courbe modulaire, et de la métrique canonique sur le fibré. Un premier essai de définition, réalisé par Takhtajan et Zograf pour des fibrés d'endomorphismes en remplaçant le déterminant du laplacien de Dolbeault par la valeur en 1 de la dérivée de la fonction zeta de Selberg, a mené à une formule de courbure, mais pas à une isométrie. La métrique de Quillen définie dans ce texte généralise celle donnée par Takhtajan et Zograf aux fibrés plats unitaires. Elle aura cependant l'avantage crucial de rentrer dans le cadre d'une isométrie, similaire à celle étudiée par Deligne, et de mener à un théorème de Riemann-Roch arithmétique.

Afin de contourner les singularités des métriques, nous utiliserons des méthodes dues à Freixas i Monplet et von Pippich. Ces dernières sont principalement constituées d'outils de chirurgie analytique, tels que la troncature des métriques et des formules de recollement de Mayer-Vietoris, pour réduire l'étude des déterminants de laplaciens à des calculs de type global, pour lequel nous utiliserons la formule des traces de Selberg, et de type local, qui nécessitera l'introduction de fonctions spéciales.

Mots-clés

Isométrie de Deligne-Riemann-Roch, métrique de Quillen, torsion analytique holomorphe, théorie d'Arakelov, théorème de Riemann-Roch arithmétique, Courbe modulaire, groupe fuchsien, fibré vectoriel plat unitaire, chirurgie analytique, formule de Mayer-Vietoris, formule des traces de Selberg, fonction hypergéométrique, sommation de Ramanujan.

A Deligne-Riemann-Roch isometry for flat unitary vector bundles on modular curves

Abstract

The purpose of this thesis is to define and study a Quillen metric on a modular curve with cusps and the Poincaré metric, endowed with a flat unitary holomorphic vector bundle. In the compact case studied by Bismut-Gillet-Soulé and Deligne, this is given by a modification of the L^2-metric using the determinant of the Dolbeault Laplacian acting on sections. One of its fundamental properties is that it fits into a Riemann-Roch type isometry. Under our hypotheses, this definition cannot be applied, since the Poincaré metric on a modular curve, and the canonical metric on the vector bundle present singularities at each cusp. A first attempt was made by Takhtajan and Zograf, for endomorphism bundles, using the first order derivative ar 1 of the Selberg zeta function. They proved a curvature formula, but not an isometry. The Quillen metric we will define in this text will generalize the one defined by Takhtajan and Zograf to flat unitary vector bundles, and will also involve derivatives of the Selberg zeta function. However, it will have the decisive advantage of satisfying an isometry, similar to the one studied by Deligne, leading to an arithmetic Riemann-Roch formula.

To circumvent the metric singularities, we will use methods developped by Freixas i Montplet and von Pipppich. They primarily involve the use of analytic surgery methods, taking the form of truncation of metrics and Mayer-Vietoris glueing formulae, to reduce the study of determinants of Laplacians to computations of two natures: global, for which we use the Selberg trace formula, and local, requiring us to introduce special functions.

Keywords

Deligne-Riemann-Roch isometry, *Quillen metric*, *holomorphic analytic torsion*, *Arakelov geometry*, *arithmetic Riemann-Roch theorem*, *modular curve*, *Fuchsian group*, *flat unitary vector bundle*, *analytic surgery*, *Mayer-Vietoris formula*, *Selberg trace formula*, *hypergeometric function*, *Ramanujan summation.*

Contents

Introduction

The main focus of this thesis is to derive a Deligne-Riemann-Roch isometry for arithmetic surfaces, whose incarnation at every complex place is a modular curve, endowed with particular vector bundles. Before we dive into this pressentation, let us see a brief history of the various Riemann-Roch theorems, and their place in complex geometry, differential geometry, and Arakelov geometry.

Theorem 1 (Riemann-Roch)**.** *Let X be a compact Riemann surface, and L be a holomorphic line bundle on X. Denote by ω_X the holomorphic cotangent bundle on X. We have the following equality of integers*

$$h^0(X,L) - h^1(X,L) \quad = \quad h^0(X,L) - h^0\left(X, \omega_X \otimes L^{-1}\right) \quad = \quad \deg L + 1 - g \ ,$$

where g is the genus of X, and h^i denotes the dimension of a cohomology space.

As written above, this theorem relates a fundamental topological invariant, the *genus*, to quantities defined in complex geometry, taking advantage of the fact that Riemann surfaces live in both worlds. We will often refer to theorems which, as a generalization or in a similar way, link two domains which are *a priori* unrelated, as *Riemann-Roch type theorems.* In the 1950s, two such results emerged: a complex geometric version, and a relative algebraic geometric version.

Theorem 2 (Hirzebruch-Riemann-Roch)**.** *Let X be a compact, complex manifold, and E be a holomorphic vector bundle on X. Denote by $\chi(X,E)$ the holomorphic Euler characteristic of E, i.e. the alternating sum of the dimensions of the coherent cohomology spaces of E. We have*

$$\chi(X,E) \quad = \quad \int_X \mathrm{ch}(E)\,\mathrm{td}(X) \ ,$$

where $\mathrm{ch}(E)$ is the Chern character of E, and $\mathrm{td}(X)$ is the Todd class of the tangent bundle of X.

Theorem 3 (Grothendieck-Riemann-Roch)**.** *Let $f : X \longrightarrow S$ be a smooth and projective morphism of smooth complex quasi-projective algebraic varieties, and E be a vector bundle on X. Denoting by T_f the relative tangent bundle, we have the equality in the de Rham cohomology of S*

$$\mathrm{ch}(R^\bullet f_* E) \quad = \quad f_*\left(\mathrm{ch}(E)\,\mathrm{td}(T_f)\right) \ .$$

This last equality in cohomological in nature, though one can introduce a smooth Kähler metric g on X and a smooth hermitian metric h on E to lift

$$\mathrm{ch}(R^\bullet f_* E) - f_*\left(\mathrm{ch}(E)\,\mathrm{td}(T_f)\right)$$

to a differential form, which must be exact, meaning it has a d-primitive. The dd^c-lemma then states that a dd^c-primitive exists, though it does not indicate how to find one. A canonical representative, in the sense of [21], can be found, using the work of Bismut, Gillet, and Soulé from [9, 10, 11], as well as that of Bismut and Köhler from [12], though the latter assume that the higher direct images $R^i f_* E$ are locally free. This form, well-defined up to $\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}$, is called the *holomorphic* *analytic torsion form*. The notion of torsion originated in 1935 from the work of Reidemeister, and was greatly developped by Ray and Singer, first in the real case as the *analytic torsion* in [78], and then in [79] as the holomorphic analytic torsion form in the complex case. It has been studied extensively. Taking the $(1,1)$-part of this lift of the Grothendieck-Riemann-Roch is of particular importance. On the left-hand side, we find the Knudsen-Mumford *determinant line bundle* defined in [64]. It is an algebraic line bundle on S, whose fiber at a point $s \in S$ is given by

$$\lambda(E)_s = \bigotimes_i \left(\det H^i(X_s, E_s)\right)^{(-1)^i} .$$

To lift the degree 2 part of the Grothendieck-Riemann-Roch theorem to the level of $(1,1)$-forms, we need to define a metric on $\lambda(E)$ from the metrics g and h. Using Hodge theory, and the identifications of the Dolbeault cohomology spaces to the appropriate spaces of harmonic forms, we can define the L^2-metric on $\lambda(E)$. However, this metric is not satisfactory, as it does not vary smoothly in family. This is not surprising, since the L^2-metric only depends on kernels of Laplacians, and not on their full spectra. To fill this gap, we consider the $(0,0)$-part of the holomorphic analytic torsion, given by

$$\log T(g,h) = \sum_{q \geqslant 0} (-1)^q \, q \log \det{}' \Delta^{(0,q)} ,$$

where $\Delta^{(0,q)}$ is the Dolbeault Laplacian acting on $(0,q)$-forms with values in E, and $\det'$ denotes the *modified determinant*, which is built from the strictly positive eigenvalues through a zeta regularization process. The renormalization of the L^2-metric, defined as

$$\|\cdot\|_Q = T(g,h)^{1/2} \, \|\cdot\|_{L^2}$$

produces a metric on the determinant line bundle, called the *Quillen metric*, which varies smoothly in family. The study of the curvature of this line bundle equipped with this metric has been of extreme importance, and is in particular made in [11], leading to the following result.

Theorem 4. *We have the equality of* $(1,1)$*-forms*

$$c_1\left(\lambda_X(E), \|\cdot\|_Q\right) = f_*\left(\mathrm{ch}(E,h) \cdot \mathrm{td}(T_X, g)\right)^{(1,1)} ,$$

where $\mathrm{ch}(E,h)$ *and* $\mathrm{td}(T_X, g)$ *are the Chern-Weil representatives.*

In order for this result to make sense, it should be stressed that $f : X \longrightarrow S$ must be proper, and that the metrics g and h have to be smooth. When the metrics are not smooth, nothing can be said *a priori*, even when the base is a point, but advances have been made. For instance, Takhtajan and Zograf defined in [93] a Quillen metric on a modular curve arising from a Fuchsian group of the first kind Γ without torsion with the endomorphism bundle of a stable parabolic vector bundle coming from a unitary representation ρ of Γ. Their Quillen metric is defined as

$$\|\cdot\|_Q = \left(Z'(1, \Gamma, \mathrm{Ad}\,\rho)\right)^{-1/2} \|\cdot\|_{L^2} ,$$

where $\mathrm{Ad}\,\rho$ is the adjoint representation of ρ, which induces the bundle $\mathrm{End}\,E$, and Z denotes the Selberg zeta function. In the process, they derived the following curvature formula, where $\widetilde{\Omega}$ is the Kähler form on the moduli space of stable parabolic bundles, and δ is the cuspidal defect.

Theorem 5. *We have the equality of* $(1,1)$*-forms*

$$c_1\left(\lambda_X(\mathrm{End}\,E), \|\cdot\|_Q\right) = -\frac{1}{2\pi^2}\widetilde{\Omega} + \delta .$$

This result has recently been extended to account for the presence of elliptic points in [94]. Takhtajan and Zograf further studied in [93] the moduli space of stable parabolic vector bundles, extending the work of Mehta and Seshadri from [69]. These curvature formulae are not the only interesting results to get using Quillen metrics. As a matter of fact, Deligne proved a functorial Riemann-Roch theorem, in the form of an isomorphism compatible with base change and exact sequences, for families of compact Riemann surfaces and vector bundles, both endowed with smooth metrics. Using the work of Bismut, Gillet, and Soulé from [9, 10, 11] and of Bismut and Lebeau from [13], this isomorphism becomes an isometry up to explicit factor. Furthermore, when Riemann surfaces are considered, the definition of the Quillen metric can be simplified to

$$\|\cdot\|_Q \quad = \quad \left(\det{}' \Delta\right)^{-1/2} \|\cdot\|_{L^2} \ ,$$

where Δ is the Dolbeault Laplacian acting on sections of E. It should be noted that this Laplacian appears instead of the one acting on $(0,1)$-forms because both have the same determinant.

Theorem 6 (Deligne-Riemann-Roch isometry)**.** *Let $\mathcal{X} \longrightarrow \mathcal{S}$ be a proper, smooth scheme morphism of relative dimension 1, with geometrically connected fibers, and $\mathcal{E}$ a vector bundle over $\mathcal{X}$ of rank r. There is a functorial isomorphism of line bundles over the base $\mathcal{S}$, canonical up to sign*

$$\lambda_{\mathcal{X}/\mathcal{S}}\left(\mathcal{E}\right)^{12} \quad \simeq \quad \left\langle \omega_{\mathcal{X}/\mathcal{S}}, \omega_{\mathcal{X}/\mathcal{S}} \right\rangle^r \left\langle \det \mathcal{E}, \det \mathcal{E} \otimes \omega_{\mathcal{X}/\mathcal{S}}^{-1} \right\rangle^6 IC_2\left(\mathcal{X}/\mathcal{S}, \mathcal{E}\right)^{-12} \ .$$

When both $\mathcal{X}$ and $\mathcal{E}$ are endowed with smooth metrics, this isomorphism becomes an isometry, up to an explicit factor $c\left(g,r\right)$ depending on the genus of the fibers of $\mathcal{X}/\mathcal{S}$ and on the rank of $\mathcal{E}$.

We have used two intersection bundles here, namely the Deligne pairing, and the IC_2 bundle. Both notions represent functorial lifts to the level of line bundles of the integration along the fibers of polynomials in Chern classes. This is extensively studied by Elkik in [40, 41], and will be further explained in sections 5.1.2 and 5.1.3. This theorem was obtained with the aim of getting a deeper understanding of *Arakelov geometry*, and of the arithmetic Riemann-Roch theorem of Gillet-Soulé, which had been obtained but not published at that time. This relatively new theory was developped by Arakelov in [2, 3] from *arithmetic intersection theory*, which generalizes classical intersection theory. These fields of study were greatly expanded by Gillet and Soulé in [51], leading to the proof, in [52], of their Riemann-Roch type theorem. The right setting is that of *arithmetic varieties*, which are regular, quasi-projective, flat schemes over $\operatorname{Spec}\mathbb{Z}$, although there are slight variations on this definition according to the reference. We will denote by R the R-genus of Gillet-Soulé.

Theorem 7 (Arithmetic Riemann-Roch)**.** *Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a projective, flat morphism of arithmetic varieties which is generically smooth and $\overline{\mathcal{E}}$ be a Hermitian vector bundle on $\mathcal{X}$. Consider a Kähler metric g on $\mathcal{X}_{\mathbb{C}}$. We have the following equality*

$$\widehat{c_1}\left(\lambda_{\mathcal{X}/\mathcal{S}}\left(\overline{\mathcal{E}}\right)_Q\right) \quad = \quad f_*\left(\widehat{\operatorname{ch}}\left(\overline{\mathcal{E}}\right)\widehat{\operatorname{td}}\left(\overline{T_f}\right)\right)^{(1)} - \left[\left(0, \operatorname{ch}\left(\mathcal{E}_{\mathbb{C}}\right)\operatorname{td}\left(\mathcal{E}_{\mathbb{C}}\right)R\left(T_{f_{\mathbb{C}}}\right)\right)\right]^{(1)} \quad \in \quad \widehat{CH}^1\left(\mathcal{S}\right)_{\mathbb{Q}} ,$$

where $\widehat{\operatorname{ch}}\left(\mathcal{E},h\right)$ the arithmetic Chern character of $\left(\mathcal{E},h\right)$, and by $\widehat{\operatorname{td}}\left(\overline{T_f}\right)$ is the arithmetic Todd class of the tangent complex endowed with g.

This type of theorem holds in a more general setting than Deligne's result, though it is less refined, as it does not give information on the determinant line bundle itself. In a similar fashion, Köhler and Roessler proved an arithmetic Lefshetz fixed point theorem in [65]. This result was one of the motivations behind this thesis, as will be explained later.

So far, we have only dealt with smooth metrics, with the exception of the curvature formula obtained by Takhtajan and Zograf. However, arithmetic intersection theory has been extended, so as to take into account some singularity on the metrics. This was done by Burgos, Kramer, and

Kühn in [22], and by Bost in [14]. Throughout these works, the question of obtaining Riemann-Roch type theorems was left unresolved. This was partially answered by Freixas i Montplet, first alone in [45] and then with von Pippich in [47]. In this last article, whose methods will be followed in this thesis, the case of the trivial bundle is dealt with. It should also be mentioned that in their respective theses [29, 56], De Gaetano and Hahn derived close results, using methods, unrelated to those used here, which find their origin in the work of Jorgenson and Lundelius in [62]. Some advances for more general surfaces with vector bundles have also been made by Finski, for instance in [42, 43]. In the following, the Fuchsian groups of the first kind we work with are without torsion.

1 Survey of the thesis

The purpose of this thesis is to prove a Deligne-Riemann-Roch isometry for a particular class of arithmetic surfaces $\mathcal{X}$ over $\mathcal{S} = \operatorname{Spec} \mathcal{O}_K$, where K is a number field. We assume that the structure morphism $\mathcal{X} \longrightarrow \mathcal{S}$ has disjoint sections $\sigma_1, \ldots, \sigma_h$ such that for every embedding $\tau : K \hookrightarrow \mathbb{C}$, the non-compact Riemann surface $\mathcal{X}_\tau \setminus \{\sigma_1(\tau), \ldots, \sigma_h(\tau)\}$ is a modular curve, defined from a Fuchsian group of the first kind Γ_τ. We will further work with a vector bundle $\mathcal{E}$ of rank r defined by unitary representations ρ_τ scalar at the cusps for every τ. This will be detailed below.

Theorem 8. *We have an isometry of line bundles over* $\mathcal{S}$

$$
\begin{aligned}
&\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})_Q^{12m^2r^{m-1}} \\
&\quad \simeq \left\langle \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}) \right\rangle^{m^2r^m} \left\langle (\det \mathcal{E})^m(\mathcal{D}'), (\det \mathcal{E})^m(\mathcal{D}') \right\rangle^{6r^{m-2}(1-m(1-r^m))} \\
&\qquad\qquad \left\langle (\det \mathcal{E})^m(\mathcal{D}'), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}) \right\rangle^{-6mr^{m-1}} IC_2\left(\mathcal{X}, \mathcal{E}^{\otimes m}(\mathcal{D}'')\right)^{-12m} \otimes \psi_\omega \otimes \psi_{\mathcal{E}} \\
&\qquad\qquad\quad \otimes \left(\mathcal{O}_{\mathcal{S}}, c(g,r)\right)^{m^2r^{m-1}} \otimes \mathcal{O}(\delta)^{m^2r^m},
\end{aligned}
$$

where g denotes the genus of any modular curve $X_\tau(\mathbb{C})$, and the metric on $\mathcal{O}_{\mathcal{S}}$ is $c(g,r)$ times the trivial metric, this last constant being given by $c(g,r) = r(1-g)(1-24\zeta'(-1))$.

1.1 Framework

Let Γ be a Fuchsian group of the first kind, which here means a finitely generated discrete subgroup of $PSL_2(\mathbb{R})$, whose action on the upper half-plane has a fundamental domain with finite volume for the Poincaré metric. We assume that Γ has no torsion, and further consider a unitary representation

$$\rho \;:\; \Gamma \longrightarrow U_r(\mathbb{C})$$

of Γ. The trivial bundle $\mathbb{H} \times \mathbb{C}^r$ of rank r on $\mathbb{H}$ is then endowed with an action of Γ, given by

$$\gamma \cdot (z, v) = (\gamma \cdot z, \rho(\gamma) v)$$

and the quotient $E = \Gamma \backslash (\mathbb{H} \times \mathbb{C}^r)$ is a unitary flat holomorphic vector bundle over the modular curve $\Gamma \backslash \mathbb{H}$. Let us now describe how we can extend E over the compactified modular curve. We note that this amounts to extending E over the cusps. Let p be one of them. We can describe an open neighborhood $U_{p,\varepsilon}$ of p as a quotient $\langle T \rangle \backslash (\mathbb{R} \times]a(\varepsilon), +\infty[)$, where T is the translation and

$$a(\varepsilon) = \tfrac{1}{2\pi} \log \varepsilon^{-1}$$

endowed with the Poincaré metric g. We can also make the identification

$$(U_{p,\varepsilon}, g) \simeq \left(D^\times(0, \varepsilon), \tfrac{|dz|^2}{|z|^2 (\log|z|)^2} \right),$$

where $D^{\times}(0,\varepsilon)$ is the punctured disk of radius ε. For each cusp p, we denote by γ_p a generator of the stabilizer Γ_p of p in Γ. The matrix $\rho(\gamma_p)$ being unitary, it can be diagonalized in an orthonormal basis. We fix such a basis $(e_{p,j})_j$ and write

$$\rho(\gamma_p)\, e_{p,j} \quad = \quad e^{2i\pi\alpha_{p,j}} e_{p,j} \quad ,$$

with $\alpha_{p,j}$ being defined modulo 1. Choosing lifts of these real numbers, we set

$$s_{p,j} \quad : \quad z \quad \longmapsto \quad e^{2i\pi\alpha_{p,j} z} e_{p,j} \quad .$$

We can use a Deligne extension to get a holomorphic vector bundle E over the compactified modular curve. The canonical hermitian metric on $\mathbb{C}^r$ then induces a metric on E over $U_{p,\varepsilon}$, given by

$$\|s_{p,j}\|_z^2 \quad = \quad |z|^{2\alpha_{p,j}} \quad ,$$

which is not smooth at p unless all $\alpha_{p,j}$ vanish. This extension and the associated metric, which is called the *canonical metric* h, depend on the choices of lifts. The singularities of both the Poincaré metric and the canonical metric prevent us from defining the Quillen metric, and from applying the Deligne-Riemann-Roch isometry from [32].

1.2 Truncation of the metrics

To circumvent the singularity of the metrics at the cusps, we will take a closer look at the situation. For every cusp p, under the identification of $U_{p,\varepsilon}$ with the punctured disk of radius ε, we note that every metric considered here depends only on the modulus of the local coordinate z. We can define the truncated Poincaré metric g_ε and the truncated canonical metric h_ε as on the drawing below.

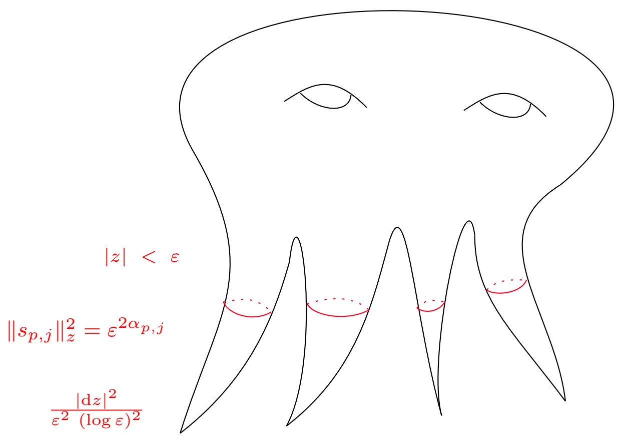


Figure 1 – Truncation of the metrics on a modular curve

Outside the open subsets $U_{p,\varepsilon}$, the metrics g_ε and h_ε are given by g and h respectively. Near the cusp p, however, we have replaced g and h by the constant metrics described on the drawing. The metrics thus obtained are only piecewise smooth, and have a global Sobolev H^1-regularity. Taking smooth approximations g_k and h_k of g and h, we have the following isometry

$$\lambda_X(E)_{Q,k}^{12} \quad \simeq \quad \langle \omega_{X,\varepsilon,k}, \omega_{X,\varepsilon,k}\rangle^r \left\langle \det E_{\varepsilon,k}, \det E_{\varepsilon,k} \otimes \omega_{X,\varepsilon,k}^{-1} \right\rangle^6 IC_2(X, E_{\varepsilon,k})^{-12} \quad ,$$

where the various bundles have been equipped with the appropriate smooth approximations of the truncated metrics. The limit as k goes to infinity can be taken on the right-hand side, which means that an *ε-truncated Quillen metric* can be defined on the determinant line bundle, yielding the *truncated Deligne-Riemann-Roch isometry*

$$\lambda_X(E)^{12}_{Q,\varepsilon} \quad \simeq \quad \langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \rangle^r \left\langle \det E_\varepsilon, \det E_\varepsilon \otimes \omega_{X,\varepsilon}^{-1} \right\rangle^6 IC_2(X, E_\varepsilon)^{-12} \ .$$

We now want to let ε go to 0^+, which requires a lot of care, since we want to get a precise control on the metrics on either side.

1.3 Regularization of the isomorphism

The first step required to take the limit of the truncated isometry is to regularize it, This amounts to modifying the isometry so as to extract the singularity as ε goes to 0^+ and set it aside.

The Deligne pairings. We begin by dealing with the Deligne pairings, which are the first two factors on the right-hand side of the isometry. Denoting by D the divisor given by the sum of the cusps, and using the metric bimultiplicativity of the Deligne pairing, we get an isometry

$$\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \rangle \quad \simeq \quad \langle \omega_X(D)_\varepsilon, \omega_X(D)_\varepsilon \rangle \otimes \bigotimes_{p \text{ cusp}} \left(\omega_{X,p}, |\cdot|_p \right)^{-1} \otimes \left(\mathbb{C}, R_\omega(\varepsilon) |\cdot| \right) \ ,$$

where $|\cdot|$ denotes the usual modulus on $\mathbb{C}$, and $|\cdot|_p$ is the metric on $\omega_{X,p}$ defined by $|\mathrm{d}z|_p = 1$. This last choice of metric on $\omega_{X,p}$ originated in Wolpert's work, more precisely in [102], and was named the *Wolpert metric* by Freixas i Montplet in [45]. The singular term $R_\omega(\varepsilon)$ is defined by

$$\log R_\omega(\varepsilon) \quad = \quad -h \log 2 - h \log \varepsilon - 2h \log \log \varepsilon^{-1} \ ,$$

with h denoting the number of cusps. The advantage is that the metric on $\omega_X(D)$ induced by the Poincaré metric on ω_X and the trivial metric on $\mathcal{O}_X(D)$ has an H^1-regularity, since $\omega_X(D)$ is generated around p by $\mathrm{d}z/z$, and we have $\|\mathrm{d}z/z\|_z = -\log|z|$. This manipulation amounts to allowing some singularity, *i.e.* simple poles, in order to remove the same amount of singularity from the metric. We then have the following convergence

$$\langle \omega_X(D)_\varepsilon, \omega_X(D)_\varepsilon \rangle \quad \underset{\varepsilon \to 0^+}{\longrightarrow} \quad \langle \omega_X(D), \omega_X(D) \rangle \ .$$

This allows for the regularization of the first Deligne pairing in the isometry, and something similar can be done for the second one, assuming all $\alpha_{p,j}$ are rational numbers.

The IC_2 bundle. We proceed to regulare the factor $IC_2(X, E_\varepsilon)$. Since we cannot twist E by a line bundle to allow different poles in multiple directions, we need to assume that ρ is scalar at the cusps, which means that all $\alpha_{p,j}$ are equal and rational, with $\alpha_{p,j} = \widetilde{m}_p/m$. Using the compatibility of the IC_2 bundle with tensor products, we have an isometry

$$\begin{aligned} & IC_2(X, E_\varepsilon)^{2m^2 r^{m-1}} \\ & \quad \simeq \quad IC_2\left(X, E^{\otimes m}(D'')_\varepsilon\right)^{2m} \left\langle (\det E)^m (D')_\varepsilon, (\det E)^m (D')_\varepsilon \right\rangle^{r^{m-2}(m(1-r^m)+r-1)} \\ & \qquad \otimes \bigotimes_{p \text{ cusp}} \left(\det E_p, \|\cdot\|_p \right)^{m\widetilde{m}_p r^{m-1}(m(r^m-1)+2(1-r))} \otimes \bigotimes_{p \text{ cusp}} \left(\omega_{X,p}, |\cdot|_p \right)^{-\widetilde{m}_p{}^2 r^m (1-r)} \\ & \qquad \otimes \left(\mathbb{C}, R_{IC_2}(\varepsilon) |\cdot| \right), \end{aligned}$$

where, for each cusp p, we have considered the Wolpert metric on $\omega_{X,p}$, and the metric on $\det E_p$ is defined by $\|s_{p,1} \wedge \cdots \wedge s_{p,r}\|_p = 1$. Furthermore, the singular factor is given by

$$\log R_{IC_2}(\varepsilon) \quad = \quad \left(\sum_{p \text{ cusp}} \widetilde{m_p}^2\right) r^m (1-r) \log \varepsilon \ .$$

All other factors converge as ε goes to 0^+, since the metrics on $E^{\otimes m}(D'')$ and on $(\det E)^m(D')$ induced by the canonical metric on E and the trivial metrics on $\mathcal{O}_X(D')$ and on $\mathcal{O}_X(D'')$ are smooth on the compactified modular curve.

1.4 Analytic surgery

The Quillen metric for smooth approximations g_k and h_k, which coincide with g and h except near the cusps, of the truncated metrics is defined as

$$\|\cdot\|_{Q,k} \quad = \quad \left(\det{}' \Delta_{\overline{\partial}_E}^{g_k,h_k}\right)^{-1/2} \|\cdot\|_{L^2,k} \ ,$$

using the Dolbeault Laplacian for these smooth metrics. The convergence of the right-hand side of the isometry implies that the Quillen metric converges as k goes to infinity, to the ε-truncated Quillen metric. Since the modular curve has finite volume, the L^2-metric also converges, which means we must have convergence in k of the modified determinants. This limit should be studied using spectral geometry. We achieve this goal by using two glueing formulae of Mayer-Vietoris type. The first one, proved by Burghelea, Friedlander, and Kappeler in [20] holds for smooth metrics, and yields

$$\det{}' \Delta_{\overline{\partial}_E}^{g_k,h_k} \quad = \quad \frac{V_{g_k}^d}{\ell^d} \det{}' N_{\overline{\partial}_E,\varepsilon}^{g_k,h_k} \det \Delta_{\overline{\partial}_E,X_\varepsilon}^{g_k,h_k} \prod_{p \text{ cusp}} \det \Delta_{\overline{\partial}_E,p,\varepsilon}^{g_k,h_k} \ ,$$

where X_ε is the compactified modular curve from which we have removed every $U_{p,\varepsilon}$, every Laplacian on the right-hand side is with Dirichlet boundary condition, and N is a Dirichlet-to-Neumann jump operator. The second holds for the singular metrics g ans h, is proved in chapter 2, and reads

$$\det{}'(\Delta_E, \Delta_{E,\text{cusp},\varepsilon}) \quad = \quad \frac{V_g^d}{\ell^d} \det{}' N_{E,\varepsilon}^{g,h} \det \Delta_{E,X_\varepsilon}^{g,h} \ ,$$

where the Laplacians are associated to the Chern connection, and the left-hand side is a modified relative determinant, which can be thought of as "well-defined quotients of ill-defined determinants", and are defined by Müller in [73]. Proving this last comparison is actually done in several steps: the first one involves the formula with a parameter μ and a constant, the second is the computation of the constant by using asymptotic expansions as μ goes to infinity. In these formulae, we have denoted by V the volume of X, and by ℓ the total length of all boundaries $\partial U_{p,\varepsilon}$. The link between both results is made by using the flatness of E over X_ε and relating the Chern and Dolbeault Laplacians using the Bochner-Kodaira-Nakano identity. The convergence in k, followed by the divergence in ε can then be studied, although they involve a *truncation constant* which can so far only be computed when all $\alpha_{p,j}$ vanish. There are leads to fill that gap, using Wentworth's ideas from [101], and the extension of some of them by Kokotov in [66].

1.5 Computation of determinants

After using both Mayer-Vietoris formulae, the modified relative determinant of Δ_E and $\Delta_{E,\text{cusp},\varepsilon}$ is left, and be computed asymptotically as ε goes to 0^+, by introducing an auxiliary Laplacian Δ_ε.

On the modular curve

The first relative determinant we can deal with is that of $\Delta_E + \mu$ and $\Delta_\varepsilon + \mu$, using the Selberg trace formula. We prove that the constant term in the asymptotic expansion as μ goes to infinity vanishes, and compute the modified relative determinant for $\mu = 0$. It is given by

$$\begin{aligned}
&\log\det{}'\left(\Delta_E, \Delta_\varepsilon\right) \\
&\quad = \tfrac{1}{2}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right) + \log Z^{(d)}\left(1\right) + \log\left(d\,!\right) - \tfrac{1}{2}rh\log 2 + \tfrac{1}{2}k\left(\Gamma,\rho\right)\log 2 \\
&\qquad + \tfrac{r\,\mathrm{Vol}\,F}{2\pi}\left[2\zeta'\left(-1\right) + \tfrac{1}{2}\log 2\pi - \tfrac{1}{4}\right] - \tfrac{1}{2}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right).
\end{aligned}$$

We can observe in this formula that the first order derivative of the Selberg zeta function used by Takhtajan and Zograf in [93], and by Freixas i Montplet and von Pippich in [47] will be replaced by the first non-zero derivative of the Selberg zeta function. Every result pertaining to this paragraph can be generalized to all Fuchsian groups of the first kind, even those with torsion.

Around a cusp

The other relative determinant we must take care of is that of $\Delta_{E,\mathrm{cusp},\varepsilon} + \mu$ and $\Delta_\varepsilon + \mu$ for μ real and positive. In chapter 4, we compute the asymptotic expansion as μ goes to infinity, and then the asymptotic expansion for $\mu = 0$ as ε goes to 0^+ of the logarithm of this determinant. To do that, we use explicit computations on model cusps, hypergeometric functions, and the Ramanujan summation process. Rather than presenting computations without context, let us see one of the difficulties on a toy example. Consider the function

$$s \longmapsto \sum_{k\geqslant 1}\tfrac{1}{k^{s+1}}\sqrt{k+\mu}$$

for $\mu \geqslant 0$, which is well-defined and holomorphic on the half-plane $\mathrm{Re}\, s > 1/2$. We will need to prove that functions of this type have a holomorphic continuation to a neighborhood of 0, and compute the value of this continuation at 0 as μ goes to infinity. The typical method to prove that such continuations exist is to use a Taylor expansion, and to write

$$\sum_{k\geqslant 1}\tfrac{1}{k^{s+1}}\sqrt{k+\mu} = \zeta\left(s+\tfrac{1}{2}\right) + \tfrac{1}{2}\sum_{k\geqslant 1}\tfrac{1}{k^{s+1/2}}\int_0^{\mu/k}\tfrac{\mathrm{d}t}{\sqrt{1+t}}\ .$$

The Riemann zeta function above can be extended near the origin, and the remainder already induces a holomorphic function there. However, improving the convergence in k, so as to get an absolutely convergent remainder increases the divergence in μ, and we cannot recover an asymptotic expansion in μ of the continuation evaluated at 0. This can be solved using the *Ramanujan summation process*, presented by Candelpergher in [23]. We prove in chapter 4 that we have

$$\mathrm{Fp}_{\mu=+\infty}\log\det{}'\left(\Delta_{E,\mathrm{cusp},\varepsilon}+\mu, \Delta_\varepsilon+\mu\right) = 0\ ,$$

where Fp denotes the finite part, meaning the constant term in an asymptotic expansion. We also prove that we have, as ε goes to 0^+ when μ vanishes,

$$\begin{aligned}
\log\det\left(\Delta_{E,\mathrm{cusp},\varepsilon}, \Delta_\varepsilon\right) &= 2\pi a\left(\varepsilon\right)\sum_{p\ \text{cusp}}\sum_{j=k_p+1}^{r}\alpha_{p,j}^2 - 2\pi a\left(\varepsilon\right)\sum_{p\ \text{cusp}}\sum_{j=k_p+1}^{r}\alpha_{p,j} + \tfrac{1}{3}\pi hra\left(\varepsilon\right) \\
&\quad + \tfrac{1}{2}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right) - \tfrac{1}{2}\sum_{p\ \text{cusp}}\sum_{j=k_p+1}^{r}\log\sin\pi\alpha_{p,j} - \tfrac{1}{2}hr\log 2 + o\left(1\right).
\end{aligned}$$

1.6 A Deligne-Riemann-Roch isometry

Proving theorem 8 is now simply a matter of putting all previous results together. Indeed, the isomorphism part of it is given by Deligne's result, and getting an isometry can be done by studying the case where S is a point, X is a modular curve, and E is a vector bundle such as described above. First, we use the regularization of all three factors on the right-hand side of the truncated isometry. We use the combined singular term, denoted by $R(\varepsilon)$ in this introduction, to see that we can define the *singular determinant* as

$$\det{}' \Delta_{\overline{\partial}_E} = \lim_{\varepsilon \to 0^+} \tfrac{1}{R(\varepsilon)} \det{}' \Delta_{\overline{\partial}_E}^{g_\varepsilon, h_\varepsilon} ,$$

the determinant on the right-hand side above being the limit of the determinants for the smooth approximations. It should be understood that this quantity is *not the determinant of an operator*, as the Poincaré metric on X and the canonical metric on E are singular, but an *ad-hoc* definition. Using the glueing formulae discussed in the paragraph on analytic surgery, we can then restate the limit more conveniently as

$$\begin{aligned} \log \det{}' \Delta_{\overline{\partial}_E} = {} & C_E + \tfrac{1}{2} k(\Gamma, \rho) \log 2 + \log Z^{(d)}(1) + \log (d!) + \tfrac{r}{3} [g_\Gamma - 1] \log 2 \\ & + \tfrac{1}{2\pi} rV \left[2\zeta'(-1) + \tfrac{1}{2} \log 2\pi - \tfrac{1}{4}\right] + \tfrac{1}{2} \Big(\sum_{p \text{ cusp}} \sum_{j=1}^{r} \alpha_{p,j} \Big) \log 2, \end{aligned}$$

where g_Γ is the genus of X, and C_E is the truncation constant. As a consequence of the isometry, we can get an arithmetic Riemann-Roch formula. The sections σ_i have been assumed to be disjoint for simplification only.

2 Potential applications and future work

Computation of the truncation constant

A part of theorem 8 that still remains to be determined is the truncation constant C_E. So far, it can only be computed when all $\alpha_{p,j}$ vanish, in which case the metric on E is smooth, by using a direct generalization of the Polyakov formula presented in [76], and the same type of arguments as those developped in [47] to deal with the jump operator. It should be noted that the conformal properties of Dirichlet-to-Neumann operators have been studied in [55] by Guillarmou and Guillopé. However, due to the absence of Polyakov formulae for vector bundles and Dirichlet boundary conditions, and due to the complicated nature of the jump operator when the metric on the vector bundle changes, the constant C_E cannot yet be computed in general. Here are two possible leads towards achieving that goal.

1. The first one is to use a Mayer-Vietoris formula to regroup the terms appearing in the truncation constant to obtain a determinant on the whole modular curve, and then use another such formula proved by Wentworth in [101] to break it apart again, this time with *Alvarez boundary conditions*. As explained by Wentworth, these boundary conditions have the advantage that a Polyakov formula can be found. Using the methods for finding asymptotics of determinants of jump operators presented by Wentworth in [101] and expanded by Kokotov in [66], it stands to reason that C_E could be computed in general.

2. The most satisfying way to solve this problem would be to find a Polyakov formula for vector bundles and Dirichlet boundary conditions. Unfortunately, it does not appear that one can easily be found. Even if such a formula was obtained, one would still need to use the methods of Wentworth and Kokotov to deal with the jump operators.

Elliptic points

In this thesis, all Fuchsian groups of the first kind are assumed to be without torsion, which means that the modular curve has no elliptic points. The reason for that choice is not very deep. The computations of relative determinants using the Selberg trace formula have already been made, though not stated here, even if Γ has elliptic elements, but computations similar to the ones made in chapter 4 would have to be performed around elliptic points. The same methods, in particular the Ramanujan summation process, are expected to work, but this will require some time.

Curvature formulae and moduli spaces

Following [93], where Takhtajan and Zograf define a Quillen metric on modular curves and certain endomorphism bundles in order to get a curvature formula, it would be interesting to derive similar results in the more general setting described in this thesis. This would also lead to a deeper understanding of the moduli space of parabolic bundles.

Applications in Arakelov geometry and analytic number theory

The first consequence of the Deligne-Riemann-Roch isometry proved in this thesis is an arithmetic Riemann-Roch theorem. This formula is stated under the assumption that the sections σ_i are disjoint, but generalizing that would only be a matter of taking intersections at finite places into account. As a result, in the same manner as [47, Sec. 10.2], one could hope to derive an explicit formula for the first non-zero derivative of the Selberg zeta function in certain cases. It should be noted that the behavior of the Selberg zeta function and its derivatives at 1 is important in the work of Jorgenson and Kramer, for instance in [61].

Another interesting consequence in Arakelov geometry would be an arithmetic Lefschetz formula, of the same type as the one proved by Köhler and Roessler in [65]. The original aim of this thesis was actually to get this formula. It only evolved into finding an isometry when it was realized that there were essentially no added difficulty.

Modular forms with Nebentypus

Let $N \in \mathbb{N}$ be an integer, and consider for Γ a congruence subgroup, *i.e.* a subgroup of $SL_2(\mathbb{Z})$ containing a subgroup

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), \quad a \equiv d \equiv 1 \mod N, \quad b \equiv c \equiv 0 \mod N \right\}.$$

Both are identified to their image in $PSL_2(\mathbb{Z})$. Let χ be a character of Γ, which we assume is non-trivial. A *modular form with Nebentypus* χ of *weight* $2k$ with respect to Γ is a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ which is holomorphic, including at the cusps, and such that we have

$$f(\gamma \cdot z) = \chi(\gamma)(cz+d)^{2k} f(z)$$

for any element γ of Γ. Such a function can be identified to a global section of the line bundle $\omega^k L$ on X, where ω is the holomorphic cotangent bundle, and L is the line bundle induced by χ. We consider the case $k = 1$. By Serre duality, we can identify this space of global sections $H^0(X, \omega L)$ to $H^1\left(X, L^{-1}\right)^\vee$. Furthermore, since χ is non-trivial, the space $H^0\left(X, L^{-1}\right)$ equals zero. Applying the arithmetic Riemann-Roch theorem could provide a way to compute the Petersson norm of these modular forms in terms of special values of Selberg zeta functions.

Higher dimensions

A natural question also arises from the considerations made in this thesis: are there generalizations to higher dimensions? The notion of modular curves would have to be replaced by that of *Hilbert modular varieties*. Even the case of Hilbert modular surfaces is vastly more complicated. The aim would not be to get a Deligne-Riemann-Roch isometry in this case, as such a result does not exist for smooth metrics in relative dimension 2, but obtaining an arithmetic Riemann-Roch theorem could be envisioned. Compared to the case of modular curves, there is an added complication right from the beginning, which is that Hilbert modular surfaces are not smooth objects. One would first need to resolve the singularities, using the work of Hirzebruch in [59]. The truncation of the metrics, which we perform here, has no analog for surfaces, and there does not appear to be a simple way around that. It is still a very interesting open question, and it has been conjectured by Freixas i Montplet, for instance in [46, 48], that special values of Shimizu L-functions and their derivatives should appear.

Relations to other approaches

As we have already mentioned, several results related to the work presented in this thesis have been obtained, by Hahn, De Gaetano, and Finski, among others. Though the methods they use bear no link the ones used here, it would be interesting to see if combining these different approaches could yield better results.

Chapter 1

Modular curves and unitary representations of Fuchsian groups

In this first chapter, the aim will be to present the situation we will be dealing with in this text, *i.e.* that of *modular curves*, of *Fuchsian groups of the first kind*, and of unitary representations of such groups. The notations defined here will be adopted throughout the rest of this document.

1.1 Fuchsian groups

This section is devoted to introduce the notions of Fuchsian groups of the first kind, and of modular curves. There are several possible references for a more comprehensive presentation. For instance, one could follow Venkov's or Iwaniec's work, in [60, 97, 98]. Although the context is slightly different, as it only deals with Fuchsian groups of a stronger arithmetic nature, the reader is also referred to [34]. As we will see, Fuchsian groups are a particular type of subgroups of $PSL_2(\mathbb{R})$, which will later be divided into *two kinds*. We will take some time to review these notions.

Definition 1.1.1. The *special linear group* $SL_2(\mathbb{R})$ is defined as

$$SL_2(\mathbb{R}) \quad = \quad \left\{ M \ = \ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \, , \ \ \det M \ = \ ad - bc \ = \ 1 \right\} \ .$$

The *standard topology* on $SL_2(\mathbb{R})$ is the topology inherited through its canonical inclusion in $\mathbb{R}^4$ obtained by sending a matrix as above to (a, b, c, d).

Proposition 1.1.2. *The center of the special linear group is given by*

$$Z(SL_2(\mathbb{R})) \quad = \quad \{-I_2, \, I_2\} \ .$$

The standard topology on $SL_2(\mathbb{R})$ *is compatible with the action on it by the center.*

Definition 1.1.3. The *projective special linear group* $PSL_2(\mathbb{R})$ is defined as the quotient

$$PSL_2(\mathbb{R}) \quad = \quad Sl_2(\mathbb{R}) \, / \, \{\pm I_2\} \ .$$

of $SL_2(\mathbb{R})$ by its center. The *standard topology* on $PSL_2(\mathbb{R})$ is defined as the quotient topology induced by the standard topology on $SL_2(\mathbb{R})$.

Definition 1.1.4. A subgroup Γ of $PSL_2(\mathbb{R})$ is said to be *Fuchsian* if it is discrete.

Example 1.1.5. The projective special linear group $PSL_2(\mathbb{Z})$ is Fuchsian.

Definition 1.1.6. A non-trivial element γ of $PSL_2(\mathbb{R})$ is said to be:

- *elliptic* if we have $0 \leqslant (\mathrm{Tr}\,\gamma)^2 < 4$;
- *parabolic* if we have $(\mathrm{Tr}\,\gamma)^2 = 4$;
- *hyperbolic* if we have $4 < (\mathrm{Tr}\,\gamma)^2$.

This classification is also used for any subgroup of $PSL_2(\mathbb{R})$, in particular for Fuchsian groups.

Remark 1.1.7. The square trace of any element of $PSL_2(\mathbb{R})$ is well-defined, as such elements can be lifted to the special linear group up to sign.

1.1.1 Action on the Riemann sphere and the upper half-plane

One of the defining aspects of the group $PSL_2(\mathbb{R})$ is given by its action on the upper half-plane, and more generally on $\mathbb{P}^1(\mathbb{C})$, which is the compactification of $\mathbb{C}$ by adjunction of an infinity point. Unless otherwise specified, we will denote by Γ a subgroup of $PSL_2(\mathbb{R})$.

Definition 1.1.8. The *upper half-plane* $\mathbb{H}$ is defined as

$$\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im}\, z > 0\} .$$

It is endowed with the restriction of the canonical topology on $\mathbb{C}$.

Proposition 1.1.9. *The Riemann sphere can be partitionned as*

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{H} \sqcup \overline{\mathbb{H}} \sqcup \mathbb{P}^1(\mathbb{R}) ,$$

with $\mathbb{P}^1(\mathbb{R})$ being $\mathbb{R} \sqcup \{\infty\}$, and ∞ denoting the infinity point added to define $\mathbb{P}^1(\mathbb{C})$.

Proposition 1.1.10. *The group $PSL_2(\mathbb{R})$ acts on the Riemann sphere by*

$$\begin{array}{ccc} PSL_2(\mathbb{R}) & \longrightarrow & \mathfrak{S}\left(\mathbb{P}^1(\mathbb{C})\right) \\ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \longmapsto & \left[z \longmapsto \gamma \cdot z = \dfrac{az+b}{cz+d} \right] \end{array} ,$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$. This action preserves $\mathbb{H}$, as well as $\overline{\mathbb{H}}$ and $\mathbb{P}^1(\mathbb{R})$, and thus induces one on the upper half-plane $\mathbb{H}$.

Proposition 1.1.11. *A subgroup Γ of $PSL_2(\mathbb{R})$ is Fuchsian if and only if its action on $\mathbb{H}$ is properly discontinuous, i.e. if for any compact $K \subset \mathbb{H}$, the set $\{\gamma \in \Gamma,\ \gamma K \cap K \neq \emptyset\}$ is finite.*

From now on, we consider a Fuchsian group Γ.

Definition 1.1.12. A point $z \in \mathbb{P}^1(\mathbb{C})$ is said to be:

- *elliptic* with respect to Γ if it is fixed by an elliptic element of Γ;
- *parabolic* with respect to Γ if it is fixed by a parabolic element of Γ;
- *hyperbolic* with respect to Γ if it is fixed by a hyperbolic element of Γ.

Remark 1.1.13. The mention "with respect to Γ" will be dropped when no confusion can arise.

Remark 1.1.14. Let $\gamma \in \Gamma$, which we write, up to sign, as a matrix

$$\gamma \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ .$$

of determinant 1. A fixed point $z \in \mathbb{P}^1(\mathbb{C})$ is then a solution of the equation

$$cz^2 + (d-a)\,z - b \quad = \quad 0 \ .$$

The solutions of this equation are studied by looking at the discriminant

$$D \quad = \quad (d-a)^2 + 4bc \quad = \quad (d+a)^2 + 4\,(bc - ad) \quad = \quad (d+a)^2 - 4 \ .$$

The number and location of such solutions are then entirely characterized by the trace of the element γ, which gives the following proposition.

Proposition 1.1.15. *An element $\gamma \in \Gamma$ is:*

- *elliptic if and only if it has two fixed points in $\mathbb{P}^1(\mathbb{C})$, one in $\mathbb{H}$ and another one in $\overline{\mathbb{H}}$;*
- *parabolic if and only if it has exactly one fixed point in $\mathbb{P}^1(\mathbb{R})$;*
- *hyperbolic if and only if it has exactly two fixed points in $\mathbb{P}^1(\mathbb{R})$.*

Example 1.1.16. Let us study the parabolic points for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{C})$. For a parabolic element γ of this group, the discriminant D of the equation seen above is

$$D \quad = \quad (\operatorname{Tr}\gamma)^2 - 4 \quad = \quad 0 \ ,$$

which means that γ has a single fixed point in $\mathbb{P}^1(\mathbb{R})$, given by

$$z \quad = \quad \frac{a-d}{2c} \quad \in \quad \mathbb{P}^1(\mathbb{Q}) \ .$$

Conversely, any point $z \in \mathbb{P}^1(\mathbb{Q})$ can be sent to ∞ by means of an element g of $PSL_2(\mathbb{Z})$. Having

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \infty \quad = \quad \infty \ ,$$

we see that ∞, and thus z, is a parabolic point for the action of $PSL_2(\mathbb{Z})$. The set of parabolic points for this group is therefore $\mathbb{P}^1(\mathbb{Q})$.

Remark 1.1.17. The result presented above for $PSL_2(\mathbb{Z})$ still holds for other Fuchsian groups of great arithmetic interest, like congruence subgroups. However, the same cannot be said for the more general groups we study here.

Proposition 1.1.18. *A point $z \in \mathbb{P}^1(\mathbb{C})$ is of at most one of the three types described above.*

Proposition 1.1.19. *Let $z \in \mathbb{P}^1(\mathbb{C})$. The stabilizer Γ_z of z in Γ is*

- *finite and cyclic if z is elliptic;*
- *infinite and cyclic if z is parabolic.*

Proposition 1.1.20. *The group Γ is without torsion if and only if it has no elliptic elements.*

Remark 1.1.21. The stabilizer of ∞ in $PSL_2(\mathbb{Z})$ is generated by the translation

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} .$$

Proposition 1.1.22. *Let p be a parabolic point for Γ. There exists an element g_p in $PSL_2(\mathbb{R})$ such that we have $g_p \cdot \infty = p$ and*

$$\Gamma_p = g_p \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle g_p^{-1} .$$

Remark 1.1.23. This proposition will be crucial in this document, especially in chapter 4, as it will allow us to make explicit computations. We will always denote by g_p an element of $PSL_2(\mathbb{R})$ such as in this proposition.

Remark 1.1.24. Note that, in proposition 1.1.22, we get exactly the translation T, and not some power of it. If Γ is a congruence subgroup, we can require that g_p belongs to $PSL_2(\mathbb{Z})$ at the cost of getting only a power of T.

1.1.2 Limit sets and the two kinds of Fuchsian groups

As we have already mentioned, Fuchsian groups are divided into two kinds, based on the nature of their *limit sets*. The topology considered here is the usual one on $\mathbb{P}^1(\mathbb{C})$.

Proposition 1.1.25. *The set of accumulation points in $\mathbb{P}^1(\mathbb{C})$ of an orbit $\Gamma \cdot z$ with $z \in \mathbb{H}$ is included in $\mathbb{P}^1(\mathbb{R})$, and is independant of z.*

Definition 1.1.26. The *limit set* of the Fuchsian group Γ is defined as

$$\Lambda(\Gamma) = \overline{\Gamma \cdot z} \setminus \Gamma \cdot z ,$$

meaning the set of accumulation points of $\Gamma \cdot z$.

Definition 1.1.27. The group Γ is said to be *Fuchsian of the first kind* if the limit set $\Lambda(\Gamma)$ is the whole real projective line $\mathbb{P}^1(\mathbb{R})$. It is said to be *Fuchsian of the second kind* otherwise.

Example 1.1.28. The group $PSL_2(\mathbb{Z})$ is the most classical example of Fuchsian group of the first kind. Some of its subgroups are also of the first kind. For instance, *principal congruence subgroups*

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) , \ a \equiv d \equiv 1 \mod N, \ b \equiv c \equiv 0 \mod N \right\} ,$$

or rather their image in $PSL_2(\mathbb{Z})$, are Fuchsian of the first kind, where $N \in \mathbb{N}^*$ is a strictly positive integer. In the same spirit, the image in $PSL_2(\mathbb{Z})$ of the groups

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) , \ a \equiv d \equiv 1 \mod N, \ c \equiv 0 \mod N \right\},$$

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) , \ c \equiv 0 \mod N \right\}$$

are also Fuchsian of the first kind, and are the most important such groups after the principal congruence subgroups. More generally, every *congruence group*, *i.e.* every subgroup Γ of $PSL_2(\mathbb{Z})$ containing any $\Gamma(N)$ is Fuchsian of the first kind.

Proposition 1.1.29. *Let Γ' be a normal subgroup of Γ. The limit sets of Γ and Γ' are equal.*

1.1.3 The Poincaré metric

The aim of this paragraph is to present a way to measure distances and volumes related to the upper half-plane, called a metric.

Definition 1.1.30. The *Poincaré metric* on $\mathbb{H}$ is given in the canonical coordinates by

$$\mathrm{d}s^2_{\text{hyp}} \quad = \quad \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \quad = \quad \frac{|dz|^2}{(\operatorname{Im} z)^2} \quad .$$

Remark 1.1.31. This Poincaré metric was defined using a notation which we must make clear. For any point $z = (x, y)$ of $\mathbb{H}$, we can consider the tangent space of $\mathbb{H}$ at z, which is isomorphic to $\mathbb{C}$. In the canonical basis of this tangent space, given by the derivations $\partial/\partial x$ and $\partial/\partial y$, we consider the inner-product given by the matrix

$$\begin{bmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{bmatrix}.$$

The volume form associated to this metric is given by

$$\omega_{\text{hyp}} \quad = \quad \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} \quad .$$

The measure on $\mathbb{H}$ defined by this is thus a density measure with respect to the restriction of the Lebesgue measure on $\mathbb{C}$.

Proposition 1.1.32. *The Poincaré metric is invariant by the action of* $PSL_2(\mathbb{R})$.

1.1.4 Fundamental domains

As for every group action, the shape and properties of fundamental domains can be studied, and yields interesting results here, as it allows us to give a more practical separation for Fuchsian groups of the first and second kind. It will also be the occasion to clear up some misconception.

Definition 1.1.33. A subset F of $\mathbb{H}$ is said to be a *fundamental domain* for the action of Γ on $\mathbb{H}$ if the orbit $\Gamma \cdot z$ of any point $z \in \mathbb{H}$ intersects F in one and only one point.

Example 1.1.34. For some explicit Fuchsian groups, a fundamental domain can be computed explicitely. For instance, here is the usual fundamental domain for the action of $PSL_2(\mathbb{Z})$.

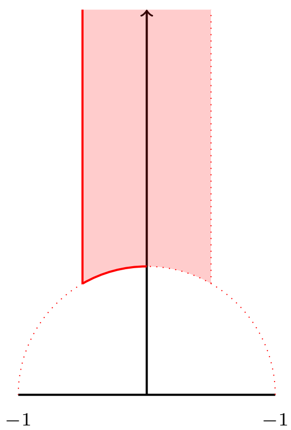


Figure 1.1 – A fundamental domain for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$

As we will see shortly, the fact that $PSL_2(\mathbb{Z})$ is Fuchsian of the first type made it possible to find a fundamental domain whose boundary is *polygonal*, in the sense that its sides are pieces of geodesics for the Poincaré metric. It should be noted, in this case, that ∞ is a vertex of said polygon, but does not belong to the fundamental domain, as it is not a point in $\mathbb{H}$.

Proposition 1.1.35. *Two fundamental domains for the action of* Γ *on* $\mathbb{H}$ *have the same volume.*

Definition 1.1.36. The *covolume* of the Fuchsian group Γ is defined to be the volume of any of the fundamental domains for its action on the upper half-plane.

Definition 1.1.37. The *extended upper half-plane* $\mathbb{H}^*$ is defined as

$$\mathbb{H}^* \quad = \quad \mathbb{H} \sqcup \{p \text{ parabolic for } \Gamma\} \ .$$

Remark 1.1.38. It can be seen that Γ preserves the set of parabolic points. Thus, its action on the Riemann sphere induces one on $\mathbb{H}^*$. The group Γ also preserves the set of elliptic points.

Proposition 1.1.39. *If the action of* Γ *on* $\mathbb{H}$ *has a fundamental domain with finite volume for the Poincaré metric, then* Γ *is Fuchsian of the first kind.*

Remark 1.1.40. Using this proposition, we can see once again that $PSL_2(\mathbb{Z})$ is Fuchsian of the first kind. Indeed, denoting by F the fundamental domain from this example, we have

$$\operatorname{Vol} F \quad = \quad \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{+\infty} \frac{\mathrm{d}y\mathrm{d}x}{y^2} \quad = \quad \int_{-1/2}^{1/2} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad = \quad [\arcsin x]_{-1/2}^{1/2} \quad = \quad \frac{\pi}{3} \ .$$

The same domain would of course not have finite volume for the Lebesgue measure.

Remark 1.1.41. The converse to the proposition above is false. Let Γ be a Fuchsian group of the first kind without torsion and with finite covolume. We now consider a normal subgroup Γ' with infinite index of Γ. The limit sets of Γ' and Γ being equal, as Γ' is normal in Γ, the group Γ' is also Fuchsian of the first kind. Since it has infinite index in Γ, any fundamental domain must have infinite volume.

Remark 1.1.42. It is often implicitely assumed that Fuchsian groups of the first kind are finitely generated, as in [60, Prop 2.3] or in [97, Thm 1.2.1]. Even though this is actually not automatic, it will be in our framework.

Proposition 1.1.43. *Assume* Γ *is finitely generated. Then* Γ *is of the first kind if and only if it has finite covolume. In this case, there are finitely many orbits of parabolic and elliptic points.*

From now on, we will work with a finitely generated Fuchsian group of the first kind Γ.

Definition 1.1.44. The group Γ is said to be *cocompact* if its action on $\mathbb{H}$ has a compact fundamental domain.

Example 1.1.45. The group $PSL_2(\mathbb{Z})$ is not cocompact, as ∞ does not belong to the fundamental domain. It is actually a more general result, which constitutes the next proposition.

Proposition 1.1.46. *The group* Γ *is cocompact if and only if there are no parabolic elements.*

Theorem 1.1.47. *The group* Γ *is entirely characterized by the datum of a finite set of generators, comprised of* $2s$ *hyperbolic elements* $H_{1,i}$ *and* $H_{2,i}$*, of* r *parabolic elements* P_k*, and of* n *elliptic elements* E_j*. The latter are of finite order, and all these elements are linked by the relation*

$$[H_{1,i}, H_{2,i}] \dots [H_{1,s}, H_{2,s}] \, P_1 \dots P_r E_1 \dots E_n \quad = \quad I_2 \ .$$

Here, we have denoted by $[\cdot,\cdot]$ *the commutator of two matrices. Furthermore, one can choose a fundamental domain for the action of* Γ *on* $\mathbb{H}^*$ *which is a closed polygon, in the sense that its sides are pieces of geodesics for the Poincaré metric.*

Remark 1.1.48. This theorem is stated at the beginning of [97, Sec. 1.2], and its first part can be reformulated to say that Γ is the quotient of a free group of finite rank by the appropriated relations between the generators.

1.2 Modular curves

Let Γ be a Fuchsian group of the first kind. In the last section, we saw what some fundamental domains for the action of Γ on $\mathbb{H}^*$ may look like. As for every group action, any fundamental domain is in bijection with the set of orbits, also called the quotient space. In our case, this last set will be of a great interest.

Theorem 1.2.1. *The quotient space $\Gamma\backslash\mathbb{H}$ can be endowed with a structure of Riemann surface. It is compact if and only if there are no parabolic points.*

Remark 1.2.2. As we will see later in this section, this Riemann surface can always be compactified by adjunction of a finite number of points, called the *cusps*. These play the role of "infinity points", in the same spirit as the compactification of the complex plane into the Riemann sphere.

1.2.1 Topology

In order to define a structure of Riemann surface on $\Gamma\backslash\mathbb{H}$, we must first define a topology on it. Even though the context is slightly more general, we follow the presentation from [34, Sec. 2.1].

Definition 1.2.3. The *quotient topology* on $\Gamma\backslash\mathbb{H}$ is the finest topology making continuous the canonical projection

$$\begin{array}{rcccl} \pi & : & \mathbb{H} & \longrightarrow & \Gamma\backslash\mathbb{H} \\ & & z & \longmapsto & \Gamma\cdot z \end{array}.$$

In other words, a subset U of $\Gamma\backslash\mathbb{H}$ is open if $\pi^{-1}(U)$ is open in $\mathbb{H}$.

Proposition 1.2.4. *Let z and w be two points in $\mathbb{H}$. There exist two open neighborhoods U of z and V of w such that for any $\gamma\in\Gamma$, having $\gamma U\cap V\neq\emptyset$ implies that we have $\gamma z = w$.*

Remark 1.2.5. In particular, for any non-elliptic point z, there exists an open neighborhood U of z which is disjoint from its image by any $\gamma\in\Gamma$.

Proposition 1.2.6. *The space $\Gamma\backslash\mathbb{H}$ is Hausdorff.*

1.2.2 Charts

The next step which must be taken in order to turn $\Gamma\backslash\mathbb{H}$ into a Riemann surface is to define *charts* around the image $\pi(z)$ of each point in the upper half-plane. we follow [34, Sec. 2.2] here.

Around the image of points with trivial stabilizers. The easiest type of points to deal with is the image of non-elliptic points, which have trivial stabilizer in Γ. Let $z\in\mathbb{H}$ be such a point, and U be an open neighborhood of z in $\mathbb{H}$ satisfying the statement made in remark 1.2.5. The restriction of the canonical projection

$$\pi \quad : \quad U \quad \longrightarrow \quad \pi(U)$$

is then a homeomorphism, and its inverse yields a chart around $\pi(z)$.

Around the image of elliptic points. Dealing with elliptic points is slightly more complicated. Let z be such a point. Let U be an open neighborhood of z in $\mathbb{H}$ such that for every element $\gamma \in \Gamma$, we have

$$\gamma U \cap U \neq \emptyset \quad \Longrightarrow \quad \gamma \in \Gamma_z \ .$$

The image of U by the projection π is an open neighborhood of $\pi(z)$, given explicitely by

$$\pi(U) \quad = \quad \Gamma_z \backslash U \ .$$

We now consider the transformation

$$\begin{array}{cccc} \delta_z & : & \mathbb{P}^1(\mathbb{C}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ & & w & \longmapsto & \dfrac{w-z}{w-\overline{z}} \end{array} ,$$

which sends z to 0 and its complex conjugate $\overline{z}$ to ∞, and denote by m the order of the stabilizer of z in Γ. Up to reducing U, we assume it is a euclidean disk of radius $\varepsilon^{1/m} |\mathrm{Im}\, z|$, with $\varepsilon > 0$ small enough. We then have, for every $w \in U$,

$$|\delta_z(w)| \quad = \quad \frac{|w-z|}{|w-\overline{z}|} \quad \leqslant \quad \varepsilon^{1/m} \ .$$

The open neighborhood $\delta_z(U)$ of 0 in $\mathbb{C}$ is then the open disk $D\left(0, \varepsilon^{1/m}\right)$ of radius $\varepsilon^{1/m}$. Furthermore, we see that conjugation by δ_z transforms the stabilizer Γ_z into the group μ_m generated by the rotation of the complex plane around the origin, with angle $2\pi/m$. Thus δ_z induces a homeomorphism

$$\delta_z \quad : \quad \pi(U) \quad \longrightarrow \quad \mu_m \backslash D\left(0, \varepsilon^{1/m}\right) \ .$$

The m-th power application

$$\begin{array}{cccc} \rho & : & \mathbb{P}^1(\mathbb{C}) & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \\ & & z & \longmapsto & z^m \end{array} .$$

then identifies the quotient $\mu_m \backslash D\left(0, \varepsilon^{1/m}\right)$ to the open disk $D(0, \varepsilon)$, thus giving an open chart around $\pi(z)$.

Remark 1.2.7. By extension, the image of elliptic points in $\Gamma \backslash \mathbb{H}$, which are in finite number, are also called elliptic points.

Modular curves We have now seen that the quotient space $\Gamma \backslash \mathbb{H}$ is a Hausdorff space endowed with an atlas. This yields the following theorem.

Theorem 1.2.8. *The space* $\Gamma \backslash \mathbb{H}$ *is a Riemann surface, called a modular curve.*

Remark 1.2.9. The term “modular curve” is traditionally reserved for the quotients by Fuchsian groups of the first kind included in $PSL_2(\mathbb{Z})$. For lack of a better term, this is extended to the more general case studied here of Fuchsian groups of the first kind included in $PSL_2(\mathbb{R})$. This choice is for instance made by Zagier in [103].

Remark 1.2.10. The modular curve is compact if and only if there are no parabolic points.

1.2.3 Compactification

In general, the modular curve $\Gamma\backslash\mathbb{H}$ is not compact, as there may be parabolic points. We will now see how to add them, so as to get a compact Riemann surface. As was already mentioned, the action of Γ on $\mathbb{P}^1(\mathbb{C})$ preserves the set of parabolic points, and thus acts on it, with a finite number of orbits, as Γ is of the first kind. This results in an action of Γ on

$$\mathbb{H}^* \quad = \quad \mathbb{H} \sqcup \{p \text{ parabolic}\} \ ,$$

We will consider the quotient set

$$\Gamma\backslash\mathbb{H}^* \quad = \quad (\Gamma\backslash\mathbb{H}) \sqcup (\Gamma\backslash \{p \text{ parabolic}\}) \ ,$$

i.e. the modular curve $\Gamma\backslash\mathbb{H}$ to which we have added finitely many points.

Definition 1.2.11. The elements added to $\Gamma\backslash\mathbb{H}$ to get $\Gamma\backslash\mathbb{H}^*$ are called the *cusps*.

Topology. The first step towards studying $\Gamma\backslash\mathbb{H}^*$ is to extend the topology already defined on the modular curve $\Gamma\backslash\mathbb{H}$. We follow [34, Sec. 2.4]. We begin by defining open neighborhoods of each parabolic point in $\mathbb{H}^*$. Let $p \in \mathbb{P}^1(\mathbb{R})$ be a parabolic point for Γ. We consider $g_p \in PSL_2(\mathbb{R})$ such that we have

$$\begin{cases} p & = \quad g_p \cdot \infty \\ \Gamma_p & = \quad g_p \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle g_p^{-1} \quad = \quad g_p \langle T \rangle g_p^{-1} \end{cases} .$$

We will thus only define open neighborhoods of ∞ in $\mathbb{H} \sqcup \{\infty\}$. For any $0 < \varepsilon < 1$, we set

$$a(\varepsilon) \quad = \quad \frac{1}{2\pi} \log \varepsilon^{-1} \ .$$

The topology on $\mathbb{H} \sqcup \{\infty\}$ is then defined as being generated by the topology on $\mathbb{H}$ and sets of the form $\mathbb{R} \times]a(\varepsilon), +\infty[\sqcup \{\infty\}$. Therefore, we can define the topology on $\mathbb{H}^*$ as being generated by the topology on $\mathbb{H}$ and the sets $g_p(\mathbb{R} \times]a(\varepsilon), +\infty[\sqcup \{\infty\})$.

Proposition 1.2.12. *The quotient space $\Gamma\backslash\mathbb{H}^*$, endowed with the quotient topology, is Hausdorff.*

Remark 1.2.13. The inclusion of $\mathbb{H}^*$ in the Riemann sphere is not continuous.

Charts. Now that we have made $\Gamma\backslash\mathbb{H}^*$ a Hausdorff topological space, we need to define an atlas over it. The one already defined for the modular curve $\Gamma\backslash\mathbb{H}$ can be used, and we only need to define charts around each cusp. We will use the last paragraph for that. Up to reducing ε, we have

$$\pi(g_p(\mathbb{R} \times]a(\varepsilon), +\infty[)) \quad \simeq \quad \Gamma_p\backslash(g_p(\mathbb{R} \times]a(\varepsilon), +\infty[))$$

as a homeomorphism. The transformation g_p further induces a homeomorphism

$$g_p \quad : \quad \langle T \rangle \backslash (\mathbb{R} \times]a(\varepsilon), +\infty[) \quad \longrightarrow \quad \Gamma_p\backslash(g_p(\mathbb{R} \times]a(\varepsilon), +\infty[)) \ .$$

The action of T on $\mathbb{R} \times]a(\varepsilon), +\infty[$ is quite simple, and we note that we have a homeomorphism

$$\langle T \rangle \backslash (\mathbb{R} \times]a(\varepsilon), +\infty[) \quad \simeq \quad S^1 \times]a(\varepsilon), +\infty[\ ,$$

with S^1 being the unit circle, which is parametrized by $x \in [0, 1[$. This set is now homeomorphic to a punctured open disk of radius ε, using the application

$$\begin{array}{ccc} S^1 \times]a(\varepsilon), +\infty[& \longrightarrow & D^\times(0, \varepsilon) \\ (x, y) & \longmapsto & e^{2i\pi(x+iy)} \end{array} ,$$

the center of this disk corresponding to the cusp. This provides a chart around the cusp $\pi(p)$. This open neighborhood of p, or rather its image in $\Gamma\backslash\mathbb{H}^*$, is denoted by $U_{p,\varepsilon}$.

Remark 1.2.14. Depending on the situation, it will be better to see $U_{p,\varepsilon}$ as $S^1 \times]a(\varepsilon), +\infty[$, or a the punctured disk $D(0, \varepsilon)$. We will do so without explicit mention of the homeomorphisms.

Compact modular curves. Using the last two paragraphs, the quotient space $\Gamma\backslash\mathbb{H}^*$ is a Hausdorff topological space endowed with an atlas, hence a Riemann surface. There is one crucial advantage with regard to the modular curve $\Gamma\backslash\mathbb{H}$.

Theorem 1.2.15. *The space $X = \Gamma\backslash\mathbb{H}^*$ is a compact Riemann surface.*

Remark 1.2.16. It is common to also call $\Gamma\backslash\mathbb{H}^*$ a modular curve, with context making clear whether we consider the compactification, or the quotient $\Gamma\backslash\mathbb{H}$.

Definition 1.2.17. The *punctured modular curve* associated to Γ is defined to be

$$Z \quad = \quad (\Gamma\backslash\mathbb{H}^*) \setminus (\{p \text{ cusp}\} \sqcup \{q \text{ elliptic point}\}) \quad = \quad (\Gamma\backslash\mathbb{H}) \setminus \{q \text{ elliptic point}\} \ ,$$

i.e. the compactified modular curve X from which we have removed both cusps and elliptic points.

Remark 1.2.18. The reason why this punctured modular curve is an important object will be made clear shortly, and has to do with the nature of the Poincaré metric near those points.

1.2.4 The Poincaré metric on a modular curve

As mentioned in proposition 1.1.32, the Poincaré metric on $\mathbb{H}$ is invariant by the action of the group $PSL_2(\mathbb{R})$, and thus by that of Γ. This means that it induces a metric on the punctured modular curve Z, as metrics may have problems at fixed points.

Around cusps. We have seen that natural open neighborhoods of a cusp p are homeomorphic, and actually diffeomorphic, to open neighborhoods $S^1 \times]a(\varepsilon), +\infty[$ of ∞ in $\mathbb{H}$. Since the Poincaré metric is invariant by the action of the transformation g_p, this homeomorphism between a neighborhood of p and one of ∞ sends the Poincaré metric on itself. This is not the complete picture of what happens around a cusp, as we can also see $U_{p,\varepsilon}$ as the punctured disk $D^\times(0, \varepsilon)$, using the application

$$\begin{array}{ccc} S^1 \times]a(\varepsilon), +\infty[& \longrightarrow & D^\times(0, \varepsilon) \\ (x, y) & \longmapsto & e^{2i\pi(x+iy)} \end{array} ,$$

with S^1 being once again parametrized by $x \in [0, 1[$.

Proposition 1.2.19. *The application above induces an isometry*

$$\left(S^1 \times]a(\varepsilon), +\infty[\,, \ \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \right) \quad \simeq \quad \left(D^\times, \ \frac{|\mathrm{d}z|^2}{(|z| \log |z|)^2} \right) \ .$$

Remark 1.2.20. Using the description provided in this proposition, we see that the Poincaré metric has a *singularity* at every cusp, since such a point corresponds to the center of the punctured disk $D^{\times}(0,\varepsilon)$, and we have

$$\lim_{|z|\to 0} \frac{1}{\left(|z|\log|z|\right)^2} \quad = \quad +\infty \ .$$

This represents a loss of control regarding the way to measure distances when reaching the cusp.

Around elliptic points. Let $q \in \Gamma\backslash\mathbb{H}$ be an elliptic point. Having previously described canonical open neighborhoods $U_{q,\varepsilon}$ of q which are diffeomorphic, to disk of radii ε, we can use these applications to give the Poincaré metric on this last disk.

Proposition 1.2.21. *We have an isometry*

$$\left(U_{q,\varepsilon}\setminus\{q\}\,,\ \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}\right) \quad \simeq \quad \left(D^{\times}(0,\varepsilon)\,,\ \frac{4\,|\mathrm{d}z|^2}{m^2\,|z|^{2-2/m}\left(1-|z|^{2/m}\right)^2}\right) \ .$$

Remark 1.2.22. The integer m, which is the order of the stabilizer of any lift of q to $\mathbb{H}$, being at least 2, the Poincaré metric also has a singularity at the elliptic points.

Graphic representation of a modular curve. A modular curve such as studied in this document can be represented in the following way.

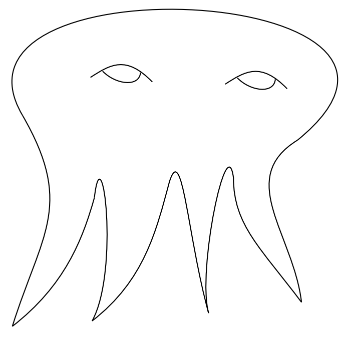


Figure 1.2 – An example of modular curve

The four points at the bottom of this drawing represent the cusps. It should be noted that they are smooth points, in the sense of differential geometry. They are manifested in this way to emphasize the singularity of the Poincaré metric at these points.

1.3 Unitary representations of Fuchsian groups

There are a few more crucial notions we must see before diving into the heart of this text, meaning that of unitary representations of Fuchsian groups of the first kind, their associated flat unitary holomorphic vector bundles, and their extensions to the compactified modular curve. For simplicity, we will make an assumption, which will hold in everything that follows: the Fuchsian group Γ has no torsion, though it is expected that most of this text can be extended to this case.

Definition 1.3.1. A *unitary representation of rank* r of Γ is the datum of a group morphism

$$\rho \ : \ \Gamma \ \longrightarrow \ U_r(\mathbb{C}) \ ,$$

where $U_r(\mathbb{C})$ denotes the space $r \times r$ complex unitary matrices, *i.e.* matrices $U \in M_r(\mathbb{C})$ such that we have ${}^t\overline{U}U = I_r$.

Remark 1.3.2. There is a well-known result in linear algebra which we will use in the following: any unitary matrix can be diagonalized in an orthonormal basis. The next proposition is a direct consequence of this fact.

From now on, we consider such a representation ρ of the fixed Fuchsian group Γ.

Proposition 1.3.3. *Let p be a cusp, and γ_p be a generator of the stabilizer of p in Γ. There exists an orthonormal basis $e_{p,1}, \ldots, e_{p,r}$ of $\mathbb{C}^r$ in which the matrix $\rho(\gamma_p)$ can be written as*

$$\rho(\gamma_p) \quad \sim \quad \begin{bmatrix} \ddots & & \\ & e^{2i\pi\alpha_{p,j}} & \\ & & \ddots \end{bmatrix} ,$$

where each $\alpha_{p,j}$ is a real number, well-defined modulo 1.

Definition 1.3.4. The representation ρ, which is often called the *monodromy*, is said to satisfy the "*finite monodromy at the cusps*" hypothesis if every $\alpha_{p,j}$ is rational.

1.3.1 Flat unitary vector bundles

In this paragraph, we will see that to every representation ρ as above, we can attach a flat unitary holomorphic vector bundle E over the punctured modular curve Z.

Proposition 1.3.5. *The action of Γ on $\mathbb{H}$ and the unitary representation ρ induce an action on the trivial vector bundle of rank r by*

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \mathfrak{S}(\mathbb{H} \times \mathbb{C}^r) & & \\ \gamma & \longmapsto & [(z, v) & \longmapsto & (\gamma \cdot z, \rho(\gamma) v)] \end{array} .$$

Proposition 1.3.6. *The quotient $E = \Gamma \backslash (\mathbb{H} \times \mathbb{C}^r)$ is a flat unitary holomorphic vector bundle over the punctured modular curve Z.*

Proposition 1.3.7. *A section of E over Z can be identified to a function $f : Z \longrightarrow \mathbb{C}^r$ verifying*

$$f(\gamma \cdot z) \quad = \quad \rho(\gamma) f(z)$$

for every point z of Z. Similarly, a section of E over an open neighborhood $U_{p,\varepsilon}$ of a cusp p can be identified to a function $g : \mathbb{R} \times \left]a(\varepsilon), +\infty\right[$ with

$$g(x+1, y) \quad = \quad \rho(\gamma_p) f(x, y) \ .$$

Proof. We will only write the details for the first part of this proposition, as the second one is completely similar. By definition, a section s of E over Z is a function

$$\begin{array}{cccc} s \ : & \mathbb{H} & \longrightarrow & \mathbb{H} \times \mathbb{C}^r \\ & z & \longmapsto & (z, f(z)) \end{array}$$

which is compatible with the actions of Γ on $\mathbb{H}$ and on $\mathbb{H}\times\mathbb{C}^r$. This gives

$$(\gamma\cdot z, f(\gamma\cdot z)) = s(\gamma\cdot z) = \gamma\cdot s(z) = \gamma\cdot(z, f(z)) = (\gamma\cdot z, \rho(\gamma) f(z)) \ ,$$

and completes the proof.

□

Definition 1.3.8. A section s of E over Z is saif to be a *constant section* if it can be identified, under the last proposition to a constant vector-valued function

$$f \ : \ \mathbb{H} \longrightarrow \mathbb{C}^r$$

which is compatible with the representation ρ.

Remark 1.3.9. Such a function being constant and compatible with ρ, its constant value must belong to the space $(\mathbb{C}^r)^{\Gamma}$ of fixed vectors. In particular, there can be no constant section if for every cusp p, no $\alpha_{p,j}$ vanishes modulo 1.

1.3.2 Extension over the cusps

The vector bundle E attached to such a representation ρ has only been defined over the punctured modular curve, which here is $\Gamma\backslash\mathbb{H}$, as we assumed there were no elliptic points. It will be important in what follows to be able to extend these bundles over the whole compactified modular curve. This extension is not unique, inasmuch as it depends on a choice of branch of logarithm. This extension is known as Deligne's canonical extension, for which the reader is referred to [31, 77].

Theorem 1.3.10. *For any choice of lift $\alpha_{p,j}$ to $\mathbb{Q}$, where p runs through the cusps of X and j is an integer between 1 and r, there is a canonical way to extend the flat unitary holomorphic vector bundle E to X. These extensions are pairwise non-isomorphic.*

Proof. The vector bundle E being already defined over Z, we note that it is enough to extend it from every open neighborhood $U_{p,\varepsilon}$ to the associated cusp. Let p be one, and consider an orthonormal basis $(e_{p,j})_{j=1}^r$ of $\mathbb{C}^r$ such that we have

$$\rho(\gamma_p)\, e_{p,j} = e^{2i\pi\alpha_{p,j}} e_{p,j} \ .$$

Here, the rational numbers $\alpha_{p,j}$ are only each defined modulo 1. We consider particular lifts, still denoted the same way. Using the description of $E_{|U_{p,\varepsilon}}$ as a vector bundle over $S^1\times\,]a(\varepsilon), +\infty[$, we have a decomposition

$$E_{|U_{p,\varepsilon}} = \bigoplus_{j=1}^{r} L_{p,j}$$

where the line bundle $L_{p,j}$ can be defined as

$$L_{p,j} = \langle T\rangle \backslash ((\mathbb{R}\times\,]a(\varepsilon), +\infty[)\times\mathbb{C}e_{p,j}) \ .$$

Here, T is the translation $(x,y)\longmapsto(x+1,y)$, and the action of the group it generates on the trivial bundle $(\mathbb{R}\times\,]a(\varepsilon), +\infty[)\times\mathbb{C}$ is given by the character

$$\begin{array}{rcccl} \rho_{p,j} & : & \langle T\rangle & \longrightarrow & \mathbb{C}^* \\ & & T & \longmapsto & e^{2i\pi\alpha_{p,j}} \end{array} .$$

We can now see $L_{p,j}$ as a free sheaf of rank 1, as it is trivialized by the section

$$s_{p,j} \quad : \quad (x,y) \quad \longmapsto \quad e^{2i\pi\alpha_{p,j}(x+iy)} e_{p,j} \quad .$$

Seeing now $U_{p,\varepsilon}$ as the punctured disk $D^{\times}(0,\varepsilon)$, we have an isomorphism of sheaves

$$L_{p,j} \quad \simeq \quad \mathcal{O}_{D^{\times}(0,\varepsilon)} s_{p,j} \quad ,$$

where $\mathcal{O}_{D^{\times}(0,\varepsilon)}$ is the sheaf of holomorphic functions on the punctured disk of radius ε. The extension is then defined as

$$L_{p,j} \quad \simeq \quad \mathcal{O}_{D(0,\varepsilon)} s_{p,j} \quad ,$$

and we extend E by taking the direct sum of these free sheaves.

□

Remark 1.3.11. The extension of E defined in the proof above depends on the lifts $\alpha_{p,j}$ as the sections $s_{p,j}$ used to trivialize $L_{p,j}$ do. The resulting vector bundle E is in general no longer flat.

Definition 1.3.12. For any extension of the vector bundle E as in theorem 1.3.10, the choices of lifts $\alpha_{p,j}$ are called the *weights* of the vector bundle.

Proposition 1.3.13. *Consider an extension E of a vector bundle associated to a unitary representation ρ. We have an isomorphism*

$$H^0(X,E) \quad \simeq \quad (\mathbb{C}^r)^{\Gamma} \quad ,$$

where the right-hand side denotes the space of fixed vectors, and the one on the left is the 0-th Dolbeault cohomology space, describing global holomorphic sections.

Proof. This result constitutes proposition 1.2 of [69].

□

1.3.3 Singular metric on the extensions

In this last part, we will see that any of the previously defined extension has a canonical metric defined on it, which is in general singular, in a way to be defined. Let E be such an extension, with weights $\alpha_{p,j}$.

Proposition 1.3.14. *The canonical hermitian metric on $\mathbb{C}^r$ induces a hermitian metric on E over Z, so that the sections $s_{p,j}$ form an orthogonal frame, with*

$$\|s_{p,j}\|_z^2 \quad = \quad |z|^{2\alpha_{p,j}} \quad .$$

Proof. To prove this reuslt, recall that we have defined $s_{p,j}$ by

$$s_{p,j} \quad : \quad (x,y) \quad \longmapsto \quad e^{2i\pi\alpha_{p,j}(x,y)} \quad ,$$

which then gives

$$|s_{p,j}(x,y)|^2 \quad = \quad \left|e^{2i\pi(x+iy)}\right|^{2\alpha_{p,j}} \quad .$$

Going from the interpretation of $U_{p,\varepsilon}$ as the product $S^1 \times]a(\varepsilon), +\infty[$ to the one as the punctured disk $D^\times(0,\varepsilon)$ then yields the result.

$\square$

Remark 1.3.15. We say that this metric is *in general* singular at the cusps, since we have

$$\lim_{z\to 0} \|s_{p,j}\|_z^2 \quad = \quad \begin{cases} 0 & \text{if} \quad \alpha_{p,j} > 0 \\ 1 & \text{if} \quad \alpha_{p,j} = 1 \\ +\infty & \text{if} \quad \alpha_{p,j} < 0 \end{cases} ,$$

though some authors prefer the term *degeneration* when the limit is zero. We will only use the term "singularity" to simplify. The only way not to have any singularity is for every weight $\alpha_{p,j}$ to vanish. This does not necessarily imply that E is the trivial bundle of rank r however.

Definition 1.3.16. This metric on E is called the *canonical metric.*

Chapter 2

Analytic surgery

This chapter is devoted to the study of a set of techniques regrouped under the term “analytic surgery”. The objective is to understand how determinants of certain Laplacians vary when either the object on which they act, or the metric by which they are defined are themselves variable.

2.1 Background in differential and complex geometry

Before concerning ourselves with the specific situation we are studying in this text, we will first review some facts about differential and complex geometry. The reader is referred to [6, 7] among others, for a more detailed introduction to these notions.

2.1.1 Connections on the tangent bundle

Let M be a Riemannian manifold, whose tangent bundle is denoted by TM. The cotangent bundle of M, denoted by T^*M, is the dual bundle of TM.

Definition 2.1.1. Let U be an open subset of M. A *smooth vector field* over U is a smooth section of the tangent bundle TM over U. Namely, it is the datum, for every point $x \in U$, of a tangent vector $X_x \in TM_x$ of M at x, depending smoothly on the point.

Definition 2.1.2. A *smooth metric* g on the manifold M is the datum, for every point $x \in M$, of an inner product g_x on the tangent space TM_x, so that the function

$$\begin{array}{rccl} g(X,Y) & : & U & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & g_x(X_x, Y_x) \end{array}$$

is smooth for every open subset U of M, as well as every smooth vector fields X and Y over U.

Definition 2.1.3. Let U be an open subset of M. A *smooth differential* 1*-form* over U is a smooth section ω of the cotangent bundle T^*M. In other words, it is the datum, for every point $x \in U$, of a linear form ω_x on the tangent space TM_x, varying smoothly with respect to the point.

Remark 2.1.4. Similarly, a *smooth differential* k*-form* on U is a smooth section of the k-th exterior power $\Lambda^k T^*M$ of the cotangent bundle.

Definition 2.1.5. Let U be an open subset of M. For any smooth differential 1-form ω on U and any smooth vector field X on U, we denote by $\omega(X)$ the smooth function on U defined by

$$\begin{array}{rcccl} \omega(X) & : & U & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \omega_x(X_x) \end{array}.$$

Definition 2.1.6. A *connection* on M is a collection of linear operators

$$\nabla_U \quad : \quad \mathcal{C}^\infty (U, TM) \quad \longrightarrow \quad \mathcal{C}^\infty (U, T^*M \otimes TM)$$

which are compatible with restrictions, and satisfy Leibniz's rule, meaning such that we have

$$\nabla_U (fX) \quad = \quad f \nabla_U X + df \otimes X \ ,$$

for any smooth function f on U, and any smooth vector field X over U.

Remark 2.1.7. Out of convenience, all these operators ∇_U are denoted by ∇. Given the requirement that ∇ be compatible with restrictions, it can readily be interpreted as a morphism of sheaves, with the added condition that it satisfies Leibniz's rule.

Definition 2.1.8. A connection ∇ on M is said to be *compatible* with a metric g on M if we have

$$dg (X, Y) \quad = \quad g (\nabla X, Y) + g (X, \nabla Y) \ ,$$

for any smooth vector fields X and Y over any open subset U, where the inner-products g_x have been extended so as to have

$$g (X, \alpha \otimes Y) \quad = \quad \alpha g (X, Y) \ ,$$

for any smooth differential form $\alpha \in \mathcal{C}^\infty (U, T^*M)$.

Definition 2.1.9. Let X be a smooth vector field over an open subset U of M. We define the contraction operator ι_X to be

$$\begin{array}{cccccc} \iota_X & : & \mathcal{C}^\infty (U, T^*M) & \longrightarrow & \mathcal{C}^\infty (U) & \\ & & \omega & \longmapsto & \omega (X) & . \end{array}$$

Definition 2.1.10. Let X be a smooth vector field over U. We define the *covariant derivative* with respect to the vector field X to be

$$\nabla_X \quad = \quad \iota_X \circ \nabla \quad : \quad \mathcal{C}^\infty (U, TM) \quad \longrightarrow \quad \mathcal{C}^\infty (U, TM) \ .$$

Proposition-Definition 2.1.11. *Assume M is endowed with a smooth metric g. There exists a unique connection ∇ which is:*

1. *compatible with g;*
2. *without torsion, i.e. such that we have*

$$\nabla_X^{(M,g)} Y - \nabla_Y^{(M,g)} X \quad = \quad [X, Y] \ ,$$

   where the term on the right-hand side denotes the Lie bracket of any two smooth vector fields.

This connection is called the Levi-Civita connection associated to (M, g), and denoted by $\nabla^{(M,g)}$, or by ∇^g if no confusion on the manifold can arise.

Remark 2.1.12. One of the aims of this chapter being the study of operators as the metrics themselves change, it will be important to keep an explicit mention of them in the notations.

Definition 2.1.13. For any smooth function φ and any smooth metric g on M, we denote by g_φ the smooth metric on M defined by

$$g_\varphi \quad = \quad e^{2\varphi} g \ .$$

The metrics g and g_φ are said to be *conformally equivalent*, or simply *conformal*.

Proposition 2.1.14. *Let φ be a smooth function on M, and g be a smooth metric on M. The Levi-Civita connections on M respectively associated to the metrics g_φ and g are related by*

$$\nabla^{g_\varphi}_X Y \quad = \quad \nabla^g_X Y + X(\varphi) Y + Y(\varphi) X - g(X,Y) \operatorname{grad}(\varphi) \ .$$

Proof. This result is a consequence of a computation made using the Koszul formula for the Levi-Civita connection. The reader is referred to [8, Thm 1.159].

□

2.1.2 Connections on holomorphic vector bundles

From now on, we will assume that M is a complex manifold, above which we consider a holomorphic vector bundle E of rank r. The notions presented below, similar in aspect to those presented in the last section, are presented so as to insist on the differences between the real and complex cases.

Definition 2.1.15. A *smooth hermitian metric* h on E is the datum, for every point x of M, of a hermitian product h_x on the complex vector space E_x, so that the function

$$\begin{array}{rcccl} h(s,t) & : & U & \longrightarrow & \mathbb{C} \\ & & x & \longmapsto & h_x(s_x, t_x) \end{array}$$

is smooth for every smooth sections $s, t \in \mathcal{C}^\infty(U, E)$ of E over U.

Definition 2.1.16. Let U be an open subset of M. A smooth differential 1-form over U with values in E is a smooth section ω of the tensor product $T^*M \otimes E$. Similarly, we can define smooth $(1,0)$-forms and smooth $(0,1)$-forms, both with values in E.

Definition 2.1.17. A *connection* on E is a collection of linear operators

$$\nabla_U \quad : \quad \mathcal{C}^\infty(U, E) \quad \longrightarrow \quad \mathcal{C}^\infty(U, T^*M \otimes E)$$

which are compatible with restrictions, and satisfy Leibniz's rule, meaning such that we have

$$\nabla_U(fs) \quad = \quad f \nabla_U s + df \otimes s \ ,$$

for any smooth function f on U, and any smooth section s of E over U.

Remark 2.1.18. Again, mention of the open subset U will be omitted, and such a connection will be denoted by ∇. A definition using the language of sheaves could be made.

Definition 2.1.19. Let h be a smooth hermitian metric on E. We say that a connection ∇ on E is *compatible* with h if we have

$$dh(s,t) \quad = \quad h(\nabla s, t) + h(s, \nabla t)$$

for every smooth sections s and t of E over any open subset U of M.

Remark 2.1.20. In the definition above, the hermitian products h_x have extended, so as to have

$$h(s, \alpha \otimes t) \quad = \quad \overline{\alpha} h(s, t) \ .$$

The difference between this definition, and the one that was given in the last section is that hermitian products are sesquilenar, and not bilinear.

Proposition-Definition 2.1.21. *Assume E is endowed with a smooth hermitian metric h. There exists a unique connection ∇ on E such that:*

1. *it is compatible with the metric h;*
2. *its $(0,1)$-part, meaning its composition with the projection on the space of $(0,1)$-forms with values in E, coincides with the Dolbeault operator $\overline{\partial}_E$.*

This connection is called the Chern connection associated to (E, h) and denoted by $\nabla^{E,h}$.

Remark 2.1.22. The $(1,0)$-part of this connection is denoted by $\nabla^{E,h}_{1,0}$.

Remark 2.1.23. The reader is referred to [99] for information on the Dolbeault operator $\overline{\partial}_E$.

Remark 2.1.24. The fact that the $(0,1)$-part of the Chern connection is prescribed to be the Dolbeault operator, which does not depend on the metric h, means that the $(1,0)$-part bears the information related to h.

Definition 2.1.25. For any smooth metric h on E, and any smooth section ψ of the endomorphism bundle $\operatorname{End} E$, we denote by h_ψ the smooth hermitian metric on E defined by

$$h_\psi(s, t) \quad = \quad h\left(e^\psi s, e^\psi t\right)$$

for any smooth sections s and t of E over U.

Remark 2.1.26. These changes of metric on the holomorphic vector bundle E may at times be called "conformal" changes, so as to mirror the changes of metric on the tangent bundle. We will now see how the Chern connection varies under such changes of metrics on E.

Proposition 2.1.27. *Let ψ be a smooth section of the endomorphism bundle $\operatorname{End} E$, and h be a smooth hermitian metric on E. The Chern connections associated to (E, h_ψ) and (E, h) respectively are related by*

$$\nabla^{E,h_\psi} \quad = \quad e^{-\psi} \left(e^{-\psi}\right)^* \nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi + \overline{\partial}_E \ ,$$

where the adjunction is taken pointwise with respect to the hermitian product h_x.

Proof. Let s and t be smooth sections of E over an open subset U of M. We have

$$\begin{aligned}
h_\psi & \left(\left(e^{-\psi} \left(e^{-\psi}\right)^* \nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi + \overline{\partial}_E\right) s, t\right) + h_\psi \left(s, \left(e^{-\psi} \left(e^{-\psi}\right)^* \nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi + \overline{\partial}_E\right) t\right) \\
&= h\left(e^\psi \overline{\partial}_E s, e^\psi t\right) + h\left(\nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi s, t\right) + h\left(e^\psi s, e^\psi \overline{\partial}_E t\right) + h\left(s, \nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi t\right) \\
&= h\left(\overline{\partial}_E s, \left(e^\psi\right)^* e^\psi t\right) + h\left(\nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi s, t\right) \\
&\qquad + h\left(\left(e^\psi\right)^* e^\psi s, \overline{\partial}_E t\right) + h\left(s, \nabla^{E,h}_{1,0} \left(e^\psi\right)^* e^\psi t\right) \\
&= \partial h\left(s, \left(e^\psi\right)^* e^\psi t\right) + \overline{\partial} h\left(\left(e^\psi\right)^* e^\psi s, t\right) \\
&= dh_\psi(s, t) \,.
\end{aligned}$$

Having obtained the compatibility of the connection we study here with the metric h_ψ, we now note that its $(0,1)$-part is given by the Dolbeault operator. The unicity in the definition of the Chern connection then yields the result.

$\square$

Remark 2.1.28. Considering a metric on the manifold is not needed to define the Chern connection, which is why we have not considered any conformal change on the tangent bundle. This will not be the case in the next section, where we consider *formal adjoints*.

2.1.3 Formal adjoints

We will now investigate the conformal behavior of the formal adjoints of the $(1,0)$ and $(0,1)$-parts of the Chern connection. This time, we also need to take into account a change of metric on the manifold to get the full picture. The term *formal adjoint* refers to the adjoint for the L^2-hermitian product on global sections of E, as explained in [99]. From now on, and unless otherwise specified, we assume M to be a Riemann surface.

Proposition 2.1.29. *Let φ be a smooth function on M, and ψ be a smooth section of the endomorphism bundle* $\operatorname{End} E$*. The formal adjoint of the $(1,0)$-part of the Chern connection, relative to the metrics g_φ on M and h_ψ on E, is related to that linked to the metrics g on M and h on E by*

$$\left(\nabla_{1,0}^{E,h_\psi}\right)^*_{g_\varphi,h_\psi} \quad = \quad e^{-2\varphi}\left(\nabla_{1,0}^{E,h}\right)^*_{g,h} \quad .$$

Proof. Let s be a smooth sections of E and t be a smooth differential $(1,0)$-form with values in E. Assuming both these sections have compact support in the interior of the manifold. We have

$$\begin{aligned}
\left\langle \nabla_{1,0}^{(E,h_\psi)} s, t\right\rangle_{\substack{T^*M\otimes E, g_\varphi^*\otimes h_\psi \\ M,g_\varphi}} \quad &= \quad \left\langle e^\psi\, \nabla_{1,0}^{(E,h_\psi)} s, e^\psi t\right\rangle_{\substack{T^*M\otimes E, g^*\otimes h \\ M,g}} \\
= \quad \left\langle \cancel{\left(e^{-\psi}\right)^*}\, \nabla_{1,0}^{(E,h)}\, \left(e^\psi\right)^*\, e^\psi s, \cancel{e^\psi} t\right\rangle_{\substack{T^*M\otimes E, g^*\otimes h \\ M,g}} \quad &= \quad \left\langle s, e^{-2\varphi}\, \left(\nabla_{1,0}^{(E,h)}\right)^*\, t\right\rangle_{\substack{E,h_\psi \\ M,g_\varphi}} .
\end{aligned}$$

This completes the proof of the proposition.

$\square$

Proposition 2.1.30. *Let φ be a smooth function on M, and ψ be a smooth section of the endomorphism bundle* $\operatorname{End} E$*. The formal adjoint of the Dolbeault operator, relative to the metrics g_φ on M and h_ψ on E, is related to that linked to the metrics g on M and h on E by*

$$\left(\overline{\partial}_E\right)^*_{g_\varphi,h_\psi} \quad = \quad e^{-2\varphi} e^{-\psi}\, \left(e^{-\psi}\right)^*\, \left(\overline{\partial}_E\right)^*_{g,h}\, \left(e^\psi\right)^*\, e^\psi \quad .$$

Proof. Let s be a smooth sections of E and t be a smooth differential $(0,1)$-form with values in E. Assuming both these sections have compact support in the interior of the manifold, we have

$$\begin{aligned}
\left\langle \overline{\partial}_E s, t\right\rangle_{\substack{T^*M\otimes E, g_\varphi^*\otimes h_\psi \\ M,g_\varphi}} \quad &= \quad \left\langle e^\psi\, \overline{\partial}_E s, e^\psi t\right\rangle_{\substack{T^*M\otimes E, g^*\otimes h \\ M,g}} \quad = \quad \left\langle s, \left(\overline{\partial}_E\right)^*_{(M,g),(E,h)}\, \left(e^\psi\right)^*\, e^\psi t\right\rangle_{\substack{E,h \\ M,g}} \\
= \quad &\left\langle s, e^{-2\varphi} e^{-\psi}\, \left(e^{-\psi}\right)^*\, \left(\overline{\partial}_E\right)^*_{(M,g),(E,h)}\, \left(e^\psi\right)^*\, e^\psi t\right\rangle_{\substack{E,h_\psi \\ M,g_\varphi}} .
\end{aligned}$$

This yields the required formula.

$\square$

2.1.4 Laplacians

Having seen how both the holomorphic and the antiholomorphic parts of the Chern connection behave under conformal changes of metric on the Riemann surface M and the vector bundle E, we turn our attention to the Laplacians built from these components.

Chern Laplacian. Among the Laplacians we will define, the most natural is the one corresponding to the usual Laplacian for the trivial bundle. We will call it the *Chern Laplacian*, as it is built from the Chern connection. Let g and h be smooth metrics on M and E, respectively.

Definition 2.1.31. The *Chern Laplacian* is defined by

$$\Delta_E^{g,h} \quad = \quad \left(\nabla^{(E,h)}\right)^*_{g,h} \nabla^{(E,h)} \quad ,$$

meaning it is the *connection Laplacian* associated to the Chern connection, acting on smooth sections of E, which are compactly supported in the interior of M.

Proposition 2.1.32. *As an operator acting on smooth sections of E, whose compact support is included in M, the Chern Laplacian is a symmetric positive operator, whose L^2-adjoint has the following domain*

$$H^2\left((M,g),(E,h),\Delta_E^{g,h}\right) \quad = \quad \left\{f \in L^2\left((M,g),(E,h)\right),\ \ \Delta_E^{g,h} f \in L^2\left((M,g),(E,h)\right)\right\} \ ,$$

the Laplacian appearing on the right-hand side being considered in the distributional sense. The Friedrichs extension of the Chern Laplacian yields a definite-positive self-adjoint operator defined on the intersection of Sobolev spaces

$$\operatorname{Dom} \Delta_E^{g,h} \quad = \quad H^2\left((M,g),(E,h),\Delta_E^{g,h}\right) \cap H_0^1\left((M,g),(E,h),\Delta_E^{g,h}\right) \ .$$

If M is without boundary, this domain is reduced to the appropriate H^2-space.

Remark 2.1.33. If the boundary of M is non-empty, the Laplacian defined above is said to be with *Dirichlet boundary conditions*, insofar as it acts on functions vanishing on ∂M.

Remark 2.1.34. The notation used here for the Sobolev spaces is quite cumbersome. This is due to the fact that we may allow M to be non-compact, which means that every elliptic operator tends to define a Sobolev space. Under certain assumptions, comparisons can be made between these spaces. This is the aim of [39, Sec. 1.3].

Holomorphic Laplacian. Instead of considering the full Chern connection in the definition of the Chern Laplacian, we can define a Laplacian by only taking the $(1,0)$-part of the Chern connection, as well as its formal adjoint. Let g and h be smooth metrics on M and E, respectively.

Definition 2.1.35. The *holomorphic Laplacian* is defined by

$$\Delta_{E,1,0}^{g,h} \quad = \quad \left(\nabla_{1,0}^{(E,h)}\right)^*_{g,h} \nabla_{1,0}^{(E,h)} \quad ,$$

acting on smooth sections of E, which are compactly supported in the interior of M.

Proposition 2.1.36. *As an operator acting on smooth sections of E, whose compact support is included in M, the holomorphic Laplacian is a symmetric positive operator, whose L^2-adjoint has the following domain*

$$H^2\left((M,g),(E,h),\Delta_{E,1,0}^{g,h}\right) \quad = \quad \left\{f \in L^2\left((M,g),(E,h)\right),\ \ \Delta_{E,1,0}^{g,h} f \in L^2\left((M,g),(E,h)\right)\right\} \ ,$$

the Laplacian appearing on the right-hand side being considered in the distributional sense. The Friedrichs extension of the holomorphic Laplacian yields a definite-positive self-adjoint operator defined on the intersection of Sobolev spaces

$$\operatorname{Dom} \Delta^{g,h}_{E,1,0} \quad = \quad H^2\left((M,g)\,,(E,h)\,,\Delta^{g,h}_{E,1,0}\right) \cap H^1_0\left((M,g)\,,(E,h)\,,\Delta^{g,h}_{E,1,0}\right) \ .$$

If M is without boundary, this domain is reduced to the appropriate H^2-space.

Dolbeault Laplacian. The third Laplace-type operator we consider here is built from the last remaining piece of the Chern connection, which coincide by definition with the Dolbeault operator. Accordingly, it will be called the *Dolbeault Laplacian*, though it is sometimes referred to as the *anti-holomorphic Laplacian.* Let g and h be smooth metrics on M and E, respectively.

Definition 2.1.37. The *Dolbeault Laplacian* is defined by

$$\Delta^{g,h}_{\overline{\partial}_E} \quad = \quad \left(\overline{\partial}_E\right)^*_{g,h} \overline{\partial}_E \ ,$$

acting on smooth sections of E, which are compactly supported in the interior of M.

Proposition 2.1.38. *As an operator acting on smooth sections of E, whose compact support is included in M, the Dolbeault Laplacian is a symmetric positive operator, whose L^2-adjoint has the following domain*

$$H^2\left((M,g)\,,(E,h)\,,\Delta^{g,h}_{\overline{\partial}_E}\right) \quad = \quad \left\{f \in L^2\left((M,g)\,,(E,h)\right),\ \ \Delta^{g,h}_{\overline{\partial}_E} f \in L^2\left((M,g)\,,(E,h)\right)\right\} \ ,$$

the Laplacian appearing on the right-hand side being considered in the distributional sense. The Friedrichs extension of the Dolbeault Laplacian yields a definite-positive self-adjoint operator defined on the intersection of Sobolev spaces

$$\operatorname{Dom} \Delta^{g,h}_{\overline{\partial}_E} \quad = \quad H^2\left((M,g)\,,(E,h)\,,\Delta^{g,h}_{\overline{\partial}_E}\right) \cap H^1_0\left((M,g)\,,(E,h)\,,\Delta^{g,h}_{\overline{\partial}_E}\right) \ .$$

If M is without boundary, this domain is reduced to the appropriate H^2-space.

Conformal changes. We will, in this paragraph, see how both holomorphic and the Dolbeault Laplacians vary under conformal changes of metrics on the Riemann Surface M and the holomorphic vector bundle E. Let g and h be smooth metrics on M and E, respectively.

Proposition 2.1.39. *Let φ be a smooth function on M, and ψ be a smooth section of* $\operatorname{End} E$. *The holomorphic Laplacian associated to the metrics g_φ on M and h_ψ on E is given by*

$$\Delta^{g_\varphi,h_\psi}_{E,1,0} \quad = \quad e^{-2\varphi}\left(\nabla^{(E,h)}_{1,0}\right)^*_{g,h} e^{-\psi}\ \left(e^{-\psi}\right)^*\ \nabla^{(E,h)}_{1,0}\ \left(e^{\psi}\right)^*\ e^{\psi} \ .$$

Proof. This formula is obtained bu putting together propositions 2.1.27 and 2.1.29. ∎

Proposition 2.1.40. *Let φ be a smooth function on M, and ψ be a smooth section of* $\operatorname{End} E$. *The Dolbeault Laplacian associated to the metrics g_φ on M and h_ψ on E is given by*

$$\Delta^{g_\phi,h_\psi}_{\overline{\partial}_E} \quad = \quad e^{-2\varphi}e^{-\psi}\ \left(e^{-\psi}\right)^*\ \left(\overline{\partial}_E\right)^*_{g,h}\ \left(e^{\psi}\right)^*\ e^{\psi}\,\overline{\partial}_E \ .$$

Proof. This formula is obtained by putting together propositions 2.1.27 and 2.1.30. ∎

Remark 2.1.41. When no change of metric has been effected on the vector bundle E, one notes that these two results agree with the better known formulae related to variations of Laplacians under conformal changes of metrics on the Riemann surface. Here, as we will later see, the presence of factors lodged between the formal adjoints and the relevant parts of the Chern connections, although not surprising, will complicate our study a great deal.

Proposition 2.1.42. *The kernel of the Dolbeault Laplacian is invariant under conformal changes of metrics on M and E.*

Proof. Let u be in the kernel of the Dolbeault Laplacian $\Delta_{\overline{\partial}_E}^{g_\varphi,h_\psi}$. We have

$$e^{-2\varphi}e^{-\psi}\left(e^{-\psi}\right)^*\left(\overline{\partial}_E\right)_{g,h}^*\left(e^{\psi}\right)^* e^{\psi}\,\overline{\partial}_E\,u \quad = \quad 0 \ ,$$

which then gives

$$\left(\overline{\partial}_E\right)_{g,h}^*\left(e^{\psi}\right)^* e^{\psi}\,\overline{\partial}_E\,u \quad = \quad 0 \ .$$

Taking the L^2-product of this section with u, we get

$$0 \quad = \quad \left\langle\left(\overline{\partial}_E\right)_{g,h}^*\left(e^{\psi}\right)^* e^{\psi}\,\overline{\partial}_E\,u, u\right\rangle_{L^2((M,g),(E,h))} \quad = \quad \left\|e^{\psi}\,\overline{\partial}_E u\right\|_{L^2((M,g),(E,h))}^2 .$$

This proves that the section $e^{\psi}\,\overline{\partial}_E u$ vanishes, and the same must be true of $\overline{\partial}_E u$. The Dolbeault operator being independant of the metrics, we have the result.

□

Remark 2.1.43. This result depends entirely on how the Dolbeault Laplacian varies under conformal changes of metrics and does not hold, in general, for the holomorphic or the Chern Laplacian.

Relations between these Laplacians. The three Laplace-type operators we have just defined being built out of the Chern connection, it is natural to wonder what relations might exist between them. In the case of the trivial bundle, this amounts to using the well-known *Kähler identities.* We will use extensions of the Kähler identities, as well as the *Bochner-Kodaira-Nakano* formula. For more information on this matter, the reader is referred to [33, Sec. 7.1].

Proposition 2.1.44. *The Chern, holomorphic, and Dolbeault Laplacians are related by*

$$\Delta_E^{g,h} \quad = \quad \Delta_{E,1,0}^{g,h} + \Delta_{\overline{\partial}_E}^{g,h} \ .$$

Proposition 2.1.45 (Bochner-Kodaira-Nakano identity)**.** *The holomorphic and Dolbeault Laplacians are related by*

$$\Delta_{\overline{\partial}_E}^{g,h} \quad = \quad \Delta_{E,1,0}^{g,h} + \left[i\Theta\left(E\right),\Lambda\right] \ ,$$

where Λ is the formal adjoint of the operator L taking a differential form u to $\omega\wedge u$, with ω being the Kähler form, and $\Theta\left(E\right)$ is the curvature form of (E,h).

Proof. This is theorem 1.2 and corollary 1.3 of [33, Sec. 7.1].

□

Corollary 2.1.46. *The Chern and Dolbeault Laplacians are related by*

$$2\Delta_{\overline{\partial}_E}^{g,h} \quad = \quad \Delta_E^{g,h} + \left[i\Theta\left(E\right),\Lambda\right] \ .$$

In particular, if the hermitian bundle (E,h) is flat, then the formula above becomes

$$2\Delta^{g,h}_{\partial_E} = \Delta^{g,h}_{E} \ .$$

2.1.5 Boundary trace operator

We will now define the *boundary trace operator*, which is an extension of the notion of restriction to the boundary of a manifold. In the case of the trivial bundle, this notion is very well-known. Since it is less common for sections of a vector bundle, we will spend time reviewing it. In this section, we consider a Riemann surface M, not necessarily compact, with a smooth boundary ∂M, and a holomorphic vector bundle E over M.

Definition 2.1.47. We denote by ι the inclusion of ∂M in M. The *restriction* of the holomorphic vector bundle E to ∂M is defined by the pullback

$$E_{|\partial M} = \iota^* E \ .$$

Remark 2.1.48. Pulling back E to ∂M has the effect of removing the complex structure from the vector bundle, as the boundary ∂M is not a complex manifold. Since ι is the inclusion, the restriction of E is the vector bundle over ∂M whose fiber over $x \in \partial M$ is given by E_x.

Definition 2.1.49. The *boundary trace operator* is defined as

$$\begin{array}{rccc} \gamma_{\partial M} : & \mathcal{C}^\infty(M,E) & \longrightarrow & \mathcal{C}^\infty\left(\partial M, E_{|\partial M}\right) \\ & s & \longmapsto & s_{|\partial M} \end{array} ,$$

where $s_{|\partial M}$ denotes the restriction of a smooth section.

Proposition 2.1.50. *The operator $\gamma_{\partial M}$ is surjective, and continuous when the norm on $\mathcal{C}^\infty(M,E)$ is the H^1-norm, and the norm on $\mathcal{C}^\infty\left(\partial M, E_{|\partial M}\right)$ is the L^2-norm, relatively to smooth metrics g on M and h on E. It thus extends to a continuous surjective operator*

$$\gamma_{\partial M} : H^1\left((M,g),(E,h),\nabla^{(E,h)}\right) \longmapsto L^2\left((\partial M,g),\left(E_{|\partial M},h\right)\right) \ .$$

Proof. The continuity of the boundary trace operator $\gamma_{\partial M}$ for the norms mentioned in this proposition stems directly from Stokes' formula.

□

Remark 2.1.51. The notion of Sobolev spaces can be defined using either connections or elliptic operators. It can be noted that the Sobolev spaces associated to the Chern connection or the Chern Laplacian are the same.

Remark 2.1.52. We will later need to consider, in some sense, the adjoint operator of $\gamma_{\partial M}$. Using the proposition above, this would yield an operator

$$\gamma^*_{\partial M} : L^2((M,g),(E,h)) \longmapsto H^1\left((M,g),(E,h),\nabla^{(E,h)}\right) ,$$

which is not to our advantage, as we prefer to work with L^2-adjunction. We could consider $\gamma_{\partial M}$ as a densely defined operator between the appropriate L^2-spaces, whose domain would be the Sobolev space written above. This construction certainly fits into the definition of the adjoint operator for a densely defined operator between Hilbert spaces. However, since the boundary trace $\gamma_{\partial M}$ is not continuous for the L^2-norms, we would have no control over the domain of the adjoint.

Proposition 2.1.53. *The kernel of the boundary trace operator $\gamma_{\partial M}$ is given by*

$$\ker \gamma_{\partial M} \quad = \quad H_0^1\left((M,g),(E,h),\nabla^{(E,h)}\right) \ ,$$

i.e. the H^1-closure of the space of smooth, compactly supported in the interior of M, sections.

2.1.6 Poisson operator

The last kind of operator we will need to define in this section is the so-called *Poisson operator*. Their aim is to provide a way to extend sections of the restricted bundle $E_{|\partial M}$ defined over the compact boundary ∂M of a (possibly noncompact) Riemann surface M.

Poisson operator for the Chern Laplacian. The idea behind this first type of Poisson operator is to take a section s of $E_{|\partial M}$ over ∂M and to solve the following *Dirichlet problem*

$$\begin{cases} \left(\Delta_E^{g,h}+z\right)v & = \quad 0 \\ \gamma_{\partial M}\, v & = \quad s \end{cases} ,$$

for sections v of E over M which are regular enough, where the first equality is to be understood in a distributional sense. The resolution of this problem will be done in a way similar to the one presented by Burghelea, Friedlander, and Kappeler in [20, Sec. 2.7].

Definition 2.1.54. Let z be a complex number lying in $\mathbb{C}\setminus\mathbb{R}_-$. We define the application Φ_z by

$$\begin{array}{rcccl} \Phi_z & : & H^2\big((M,g),(E,h),\Delta_E^{g,h}\big) & \longrightarrow & L^2\left((M,g),(E,h)\right)\oplus\ H^{3/2}\left((\partial M,g),\left(E_{|\partial M},h\right)\right) \\ & & v & \longmapsto & \left(\left(\Delta^{(M,g),(E,h)}+z\right)v,\ \gamma_{\partial M}\, v\right) \end{array} .$$

As can be gathered from the statement of the Dirichlet problem we aim to solve, the first step will be to prove that every Φ_z is bijective.

Proposition 2.1.55. *For any complex number $z\in\mathbb{C}\setminus\mathbb{R}_-$, the application Φ_z is an isomorphism.*

Proof. The fact that Φ_z is linear directly stems from the definition. We now note that Φ_z is injective, as an H^2-section v sent to 0 by Φ_z solves the problem

$$\begin{cases} (\Delta_E^{g,h}+z)v & = \quad 0 \\ \gamma_{\partial M}\, v & = \quad 0 \end{cases} .$$

Such a section then belongs to the intersection of Sobolev spaces

$$D \quad = \quad H^2((M,g),(E,h),\Delta_E^{g,h})\cap H_0^1\left((M,g),(E,h),\nabla^{(E,h)}\right) \ ,$$

which is the domain of the Chern Laplacian with Dirichlet boundary conditions. The operator

$$\Delta_E^{g,h}+z \quad : \quad D \quad \longrightarrow \quad L^2\left((M,g),(E,h)\right)$$

being invertible, the section v considered above has to be zero, which yields the injectivity of Φ_z. We now move to prove the surjectivity of this application. To do that, we consider an element

$$(u,w) \quad \in \quad L^2\left((M,g),(E,h)\right)\oplus H^{3/2}\left((\partial M,g),\left(E_{|\partial M},h\right)\right) \ .$$

In order to prove that this pair is reached by Φ_z, we first need to consider an extension $\widetilde{w}$ of w to the whole Riemann surface M, meaning a section $\widetilde{w} \in H^2\big((M,g),(E,h),\Delta_E^{g,h}\big)$ such that we have

$$\gamma_{\partial M}\,\widetilde{w} \quad = \quad w \ .$$

This is possible, since the boundary trace operator is surjective. We can then consider the image of $\widetilde{w}$ by $\Delta + z$, which yields a section

$$(\Delta + z)\ \widetilde{w} \quad \in \quad L^2\left((M,g),(E,h)\right) \ .$$

The distributional Laplacian considered here was denoted by Δ instead of $\Delta_E^{g,h}$ so as to differentiate it from the Chern Laplacian with Dirichlet boundary condition. We can then take the inverse image of the section written above by $\Delta_E^{g,h} + z$, which yields a section

$$\left(\Delta_E^{g,h} + z\right)^{-1}\ (\Delta + z)\ \widetilde{w} \quad \in \quad H^2((M,g),(E,h),\Delta_E^{g,h}) \cap H_0^1\left((M,g),(E,h),\nabla^{(E,h)}\right).$$

We then set

$$\widetilde{v} \quad = \quad \widetilde{w} - \left(\Delta_E^{g,h} + z\right)^{-1}\ (\Delta + z)\ \widetilde{w} \quad \in \quad H^2\big((M,g),(E,h),\Delta_E^{g,h}\big) \ .$$

This section satisfies

$$(\Delta + z)\ \widetilde{v} \quad = \quad (\Delta + z)\,\widetilde{w} - (\Delta + z)\ \left(\Delta_E^{g,h} + z\right)^{-1}\ (\Delta + z)\ \widetilde{w} \quad = \quad 0,$$

and its image by the boundary trace operator $\gamma_{\partial M}$ yields w, which means going from $\widetilde{w}$ to $\widetilde{v}$ has not caused the boundary value to be changed. We further note that no loss of regularity has occured, and that we have actually gained information on the image by $\Delta + z$, which is relevant here. However, this element $\widetilde{v}$ is not the one whose image Φ_z will give (u, w). To remedy that problem, we set

$$v \quad = \quad \widetilde{v} + \left(\Delta_E^{g,h} + z\right)^{-1} u \quad \in \quad H^2((M,g),(E,h),\Delta_E^{g,h}) \ .$$

This modification of $\widetilde{v}$ has still not modified the boundary value, as the inverse image is taken relatively to the Chern Laplacian with Dirichlet boundary condition, which means that the resulting section vanishes on ∂M. The difference is that we now have

$$(\Delta + z)\, v \quad = \quad (\Delta + z)\ \widetilde{v} + (\Delta + z)\ \left(\Delta_E^{g,h} + z\right)^{-1}\ u \quad = \quad u \ .$$

The image of v by Φ_z now yields the pair (u, w), which is exactly what remained to be proved.

$\square$

Definition 2.1.56. Let $z \in \mathbb{C} \setminus \mathbb{R}_-$. The *Poisson operator* $P(z)$ is defined to be the restriction of the isomorphism Φ_z^{-1} to the subspace $\{0\} \oplus H^{3/2}\left((\partial M, g), \left(E_{|\partial M}, h\right)\right)$. This yields a linear operator defined between Sobolev spaces

$$P(z) \quad : \quad H^{3/2}\left((\partial M, g), \left(E_{|\partial M}, h\right)\right) \quad \longrightarrow \quad H^2((M,g),(E,h),\Delta_E^{g,h}) \ .$$

Proposition 2.1.57. *The family of Poisson operators $P(z)$, depending on $z \in \mathbb{C} \setminus \mathbb{R}_-$, is weakly continuous for the L^2-norms.*

Proof. Let f and g be elements of $H^{3/2}\left(\left(\partial M, g\right), \left(E_{|\partial M}, h\right)\right)$ and $L^2\left(\left(M, g\right), \left(E, h\right)\right)$, respectively. Now, let z be a complex number in $\mathbb{C} \setminus \mathbb{R}_-$, and h be another complex number of modulus small enough so that we have $z + h \in \mathbb{C} \setminus \mathbb{R}_-$. We have

$$\begin{aligned}
&\left\langle \left(\Delta + (z+h)\right)\left(P(z+h) - P(z)\right) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\quad = \left\langle \left(\Delta + (z+h)\right) P(z+h) f, g\right\rangle_{L^2((M,g),(E,h))} - \left\langle \left(\Delta + z\right) P(z) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\qquad - h \left\langle P(z) f, g\right\rangle_{L^2((M,g),(E,h))} .
\end{aligned}$$

Taking instead of g the particular function $\left(\Delta + z\right)\left(P(z+h) - P(z)\right) f$, we get

$$\left\| \left(\Delta + (z+h)\right)\left(P(z+h) - P(z)\right) f\right\|^2_{L^2((M,g),(E,h))} = |h|^2 \left\| P(z) f\right\|^2_{L^2((M,g),(E,h))} .$$

Noting that $P(z+h) - P(z)$ is H^2 and satisfies the Dirichlet boundary condition, we see that the image of this section by the distributional Laplacian $\Delta + (z+h)$ is the same as its image by the Laplacian with Dirichlet boundary condition $\Delta_E^{g,h} + (z+h)$. The latter is invertible, and its bounded inverse induces a holomorphic family of operators, in the sense of [63, Sec. 7.11]. We get

$$\begin{aligned}
&\left\| \left(P(z+h) - P(z)\right) f\right\|^2_{L^2((M,g),(E,h))} \\
&\qquad \leq C \left\| \left(\Delta_E^{g,h} + (z+h)\right)\left(P(z+h) - P(z)\right) f\right\|^2_{L^2((M,g),(E,h))} \\
&\qquad \leq C |h|^2 \left\| P(z) f\right\|^2_{L^2((M,g),(E,h))} ,
\end{aligned}$$

where $C > 0$ is a constant, independant of h. This completes the proof, as we have

$$\lim_{h \to 0} \left\| \left(P(z+h) - P(z)\right) f\right\|_{L^2((M,g),(E,h))} = 0 .$$

$\square$

Proposition 2.1.58. *The family of Poisson operators*

$$P(z) \quad : \quad H^{3/2}\left(\left(\partial M, g\right), \left(E_{|\partial M}, h\right)\right) \longrightarrow L^2\left(\left(M, g\right), \left(E, h\right)\right)$$

depending on $z \in \mathbb{C} \setminus \mathbb{R}_-$ is holomorphic, with respect to the L^2-norms.

Proof. Using the same notations as in the proof of the last proposition, we have

$$\begin{aligned}
&\tfrac{1}{h} \left\langle \left(\Delta + z\right)\left(P(z+h) - P(z)\right) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\quad = \tfrac{1}{h} \Big[\left\langle \left(\Delta + (z+h)\right) P(z+h) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\qquad\quad - h \left\langle P(z+h) f, g\right\rangle_{L^2((M,g),(E,h))} - \left\langle \left(\Delta + z\right) P(z) f, g\right\rangle_{L^2((M,g),(E,h))} \Big] \\
&\quad = - \left\langle P(z+h) f, g\right\rangle_{L^2((M,g),(E,h))} \underset{h \to 0}{\longrightarrow} - \left\langle P(z) f, g\right\rangle_{L^2((M,g),(E,h))}
\end{aligned}$$

by a previously shown continuity. Using the fact that we have

$$\begin{aligned}
&\tfrac{1}{h} \left\langle \left(\Delta_E^{g,h} + z\right)\left(P(z+h) - P(z)\right) f, \left(\Delta_E^{g,h} + \overline{z}\right)^{-1} g\right\rangle_{L^2((M,g),(E,h))} \\
&\qquad = \tfrac{1}{h} \left\langle \left(\Delta_E^{g,h} + \overline{z}\right)^{-1} \left(\Delta_E^{g,h} + z\right)\left(P(z+h) - P(z)\right) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\qquad = \tfrac{1}{h} \left\langle \left(P(z+h) - P(z)\right) f, g\right\rangle_{L^2((M,g),(E,h))} ,
\end{aligned}$$

we now deduce that we have

$$\begin{aligned}
&\lim_{h\to 0} \tfrac{1}{h} \left\langle \left(P\left(z+h\right) - P\left(z\right)\right) f, g\right\rangle_{L^2((M,g),(E,h))} \\
&\quad = \lim_{h\to 0} \tfrac{1}{h} \left\langle \left(\Delta_E^{g,h} + z\right) \left(P\left(z+h\right) - P\left(z\right)\right) f, \left(\Delta_E^{g,h} + \overline{z}\right)^{-1} g\right\rangle_{L^2((M,g),(E,h))} \\
&\quad = -\left\langle P\left(z\right) f, \left(\Delta_E^{g,h} + \overline{z}\right)^{-1} g\right\rangle_{L^2((M,g),(E,h))} \\
&\quad = -\left\langle \left(\Delta_E^{g,h} + z\right)^{-1} P\left(z\right) f, g\right\rangle_{L^2((M,g),(E,h))} .
\end{aligned}$$

This proves the proposition, and further gives the formula

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\left(\Delta_E^{g,h} + z\right)^{-1} P\left(z\right) .$$

□

Remark 2.1.59. Each and every one of the results given above can be extended to complex numbers z with $-z$ not in the spectrum of the Laplacian with Dirichlet boundary condition $\Delta_E^{g,h}$.

Proposition 2.1.60. *The Poisson operator $P\left(z\right)$ takes a smooth section of E over ∂M to a smooth section of E over M.*

Proof. This result stems from the elliptic regularity of the Chern Laplacian, and the bijectivity of the application Φ_z, which must then induce an application between the spaces of smooth sections.

□

Poisson operator for the Dolbeault Laplacian. We will need another Poisson operator, this time related to the Dolbeault Laplacian. Every statement made in the last paragraph can be adapted, using the Dolbeault Laplacian instead of the Chern Laplacian. We get an operator

$$P_{\overline{\partial}_E}\left(z\right) \quad : \quad H^{3/2}\left(\left(\partial M, g\right), \left(E_{|\partial M}, h\right)\right) \quad \longrightarrow \quad H^2(\left(M, g\right), \left(E, h\right), \Delta_{\overline{\partial}_E}^{g,h})$$

which solves the same kind of Dirichlet problem as before, with the Dolbeault Laplacian being used, and not the Chern Laplacian.

2.2 Application to the case of modular curves

We now aim to apply the constructions detailed in the previous section to the specific case we are concerned with. This will be the occasion of making notations clear for what will follow. Let X be a compactified modular curve arising from a Fuchsian group of the first kind Γ, and E be a flat unitary holomorphic vector bundle of rank r coming from a unitary representation ρ.

2.2.1 Laplacians

We will first concern ourselves with the definition of the various Laplacians we need, for the Chern connection and the Dolbeault operator, and associated to several regions of the modular curve. When such a part of X has a boundary, we will impose Dirichlet boundary conditions.

On the punctured modular curve. We begin this list of Laplacians with those defined on the *punctured modular curve*, meaning the open subset $Z = X \setminus \{\text{cusps}\}$ built by removing the cusps. The reason we have to consider Z instead of X is that the metrics we will deal with may have singularities at the cusps. Let g be a smooth metric on Z, and h be a smooth Hermitian metric on the vector bundle E over Z. We attach to the latter the Chern connection $\nabla^{(E,h)}$.

Definition 2.2.1. The *Chern Laplacian on the punctured modular curve* Z is defined as the Laplacian given on Z by definition 2.1.31. It yields a self-adjoint positive operator

$$\Delta_E^{g,h} \quad : \quad H^2\left((Z,g),(E,h),\Delta^{g,h}\right) \quad \longrightarrow \quad L^2\left((Z,g),(E,h)\right) \ .$$

Definition 2.2.2. The *Dolbeault Laplacian on the punctured modular curve* Z is defined as the Laplacian given on Z by definition 2.1.37. It yields a self-adjoint positive operator

$$\Delta_{\overline{\partial}_E}^{g,h} \quad : \quad H^2\left((Z,g),(E,h),\Delta^{g,h}\right) \quad \longrightarrow \quad L^2\left((Z,g),(E,h)\right) \ .$$

Remark 2.2.3. When the metrics g and h are smooth on the whole modular curve X, we can replace Z by X in the definitions above. In the case where g is the Poincaré metric on Z and h is the canonical metric on E over Z, meaning the metric induced by the canonical hermitian product on $\mathbb{C}^r$, the Chern Laplacian is denoted by Δ_E and its Dolbeault counterpart by $\Delta_{\overline{\partial}_E}$.

On the compact part. Recall that we have defined the compact part of the modular curve to be X from which we have removed a disk of radius $\varepsilon > 0$ around each cusp and elliptic point. The resulting set is denoted by X_ε, and is a compact Riemann surface with smooth boundary.

Definition 2.2.4. The *Chern Laplacian on the compact part* X_ε is defined as the Laplacian with Dirichlet boundary condition on X_ε given by definition 2.1.31. It is a self-adjoint positive operator

$$\begin{aligned}\Delta_{E,\varepsilon,0}^{g,h} \quad : \quad H^2\left((X_\varepsilon,g),(E,h),\Delta^{g,h}\right) \cap H_0^1\left((X_\varepsilon,g),(E,h),\Delta^{g,h}\right)& \\ \longrightarrow \quad L^2\left((X_\varepsilon,g),(E,h)\right).&\end{aligned}$$

Definition 2.2.5. The *Dolbeault Laplacian on the compact part* X_ε is the Laplacian with Dirichlet boundary condition on X_ε given by definition 2.1.37. It is a self-adjoint positive operator

$$\begin{aligned}\Delta_{\overline{\partial}_E,\varepsilon,0}^{g,h} \quad : \quad H^2\left((X_\varepsilon,g),(E,h),\Delta^{g,h}\right) \cap H_0^1\left((X_\varepsilon,g),(E,h),\Delta^{g,h}\right)& \\ \longrightarrow \quad L^2\left((X_\varepsilon,g),(E,h)\right).&\end{aligned}$$

Remark 2.2.6. When g is the Poincaré metric on X_ε and h is the canonical metric on E, the Chern Laplacian above is denoted by $\Delta_{E,\varepsilon,0}$, and the Dolbeault Laplacian by $\Delta_{\overline{\partial}_E,\varepsilon,0}$.

Around the cusps. Let p be a cusp. Using previously introduced notations, we denote by $U_{p,\varepsilon}$ the open neighborhood of p in X which is isometric to a disk of radius ε.

Definition 2.2.7. The *Chern Laplacian near the cusp* p is defined as the Laplacian with Dirichlet boundary condition given on $U_{p,\varepsilon}$ by definition 2.1.31. It yields a self-adjoint positive operator

$$\begin{aligned}\Delta_{E,p,\varepsilon}^{g,h} \quad : \quad H^2\left((U_{p,\varepsilon},g),(E,h),\Delta^{g,h}\right) \cap H_0^1\left((U_{p,\varepsilon},g),(E,h),\Delta^{g,h}\right)& \\ \longrightarrow \quad L^2\left((U_{p,\varepsilon},g),(E,h)\right).&\end{aligned}$$

Definition 2.2.8. The *Dolbeault Laplacian near the cusp* p X_ε is the Laplacian with Dirichlet boundary condition given on $U_{p,\varepsilon}$ by definition 2.1.37. It yields a self-adjoint positive operator

$$\begin{aligned}\Delta_{\overline{\partial}_E,\varepsilon,0}^{g,h} \quad : \quad H^2\left((U_{p,\varepsilon},g),(E,h),\Delta^{g,h}\right) \cap H_0^1\left((U_{p,\varepsilon},g),(E,h),\Delta^{g,h}\right)& \\ \longrightarrow \quad L^2\left((U_{p,\varepsilon},g),(E,h)\right).&\end{aligned}$$

Remark 2.2.9. In case g is the Poincaré metric on X_ε and h is the flat metric on E coming from the canonical hermitian metric on $\mathbb{C}^r$, the Chern Laplacian above is denoted by $\Delta_{E,p,\varepsilon}$, and the Dolbeault Laplacian by $\Delta_{\overline{\partial}_E,p,\varepsilon}$.

2.2.2 Boundary trace operators

We will now apply the constructions pertaining to the boundary trace operators to the situation we are concerned with. To begin, let us note that we have

$$\Sigma_\varepsilon \quad = \quad \partial X_\varepsilon \quad = \quad \left(\bigsqcup_{p \text{ cusp}} \underbrace{\partial U_{p,\varepsilon}}_{\Sigma_{p,\varepsilon}} \right) .$$

Definition 2.2.10. The *boundary trace operator on the compact part* $\gamma_{\varepsilon,0}$ is defined as the operator given on X_ε by definition 2.1.49 and proposition 2.1.50. It yields a surjective continuous operator

$$\gamma_{\varepsilon,0} \quad : \quad H^1\left((X_\varepsilon, g), (E,h), \Delta^{g,h}\right) \quad \longmapsto \quad L^2\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) .$$

Remark 2.2.11. The partition of Σ_ε considered above then gives an orthogonal decomposition

$$L^2\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \quad = \quad \left(\bigoplus_{p \text{ cusp}} L^2\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \right)$$

Thus, the boundary trace $\gamma_{\varepsilon,0}$ gives a section defined on each connected component of Σ_ε, corresponding to each cusp.

Definition 2.2.12. The *boundary trace operator near the cusp* p, denoted by $\gamma_{p,\varepsilon}$, is defined as the operator given on $U_{p,\varepsilon}$ by definition 2.1.49 and proposition 2.1.50. It yields a surjective continuous linear operator

$$\gamma_{p,\varepsilon} \quad : \quad H^1\left((U_{p,\varepsilon}, g), (E,h), \Delta^{g,h}\right) \quad \longmapsto \quad L^2\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) .$$

Definition 2.2.13. We denote by γ_ε the *glued boundary trace operator*, given by the collection of operators $\gamma_{p,\varepsilon}$, and $\gamma_{\varepsilon,0}$.

2.2.3 Poisson operators

We now turn our attention to clearly defining the various Poisson operators we will later use, as well as some of the properties that will be required.

On the compact part. The first Poisson operator we will be concerned with is the one defined on the compact part of the modular curve X_ε. As before, we will need a version of the Poisson operators for the Chern Laplacian, and one related to the Dolbeault Laplacian.

Definition 2.2.14. For $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, the *Poisson operator on the compact part* $P_{\varepsilon,0}^{g,h}(z)$ is the operator on X_ε given by definition 2.1.56. It yields an operator

$$P_{\varepsilon,0}^{g,h}(z) \quad : \quad H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \quad \longrightarrow \quad H^2\left((X_\varepsilon, g), (E,h), \Delta^{g,h}\right) .$$

Similarly, the Poisson operator on X_ε associated to the Dolbeault operator is an operator

$$P_{\overline{\partial}_E,\varepsilon,0}^{g,h}(z) \quad : \quad H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \quad \longrightarrow \quad H^2\left((X_\varepsilon, g), (E,h), \Delta^{g,h}\right) .$$

Around the cusps. We now move on to defining the Poisson operators near a cusp p.

Definition 2.2.15. For $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, the *Poisson operator near the cusp* p, denoted by $P_{E,p,\varepsilon}^{g,h}(z)$, is the operator on $U_{p,\varepsilon}$ given by definition 2.1.56. It yields an operator

$$P_{p,\varepsilon}^{g,h}(z) \quad : \quad H^{3/2}\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \quad \longrightarrow \quad H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \ .$$

Similarly, the Poisson operator on $U_{p,\varepsilon}$ associated to the Dolbeault operator is an operator

$$P_{\overline{\partial}_E,p,\varepsilon}^{g,h}(z) \quad : \quad H^{3/2}\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \quad \longrightarrow \quad H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \ .$$

Glued on the whole modular curve. Finally, we can glue all the Poisson operators defined so far into a global one, whose aim is to extend a section defined on Σ_ε. As we will see, there are some subtleties regarding global regularity, as well as the number of times each connected component of the hypersurface Σ_ε is actually to be considered.

Definition 2.2.16. The *glued Poisson operator* for the Chern Laplacian is defined as the operator

$$\begin{aligned} P_{\varepsilon}^{g,h}(z) \quad : \quad & H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \oplus \bigoplus_{p \text{ cusp}} H^{3/2}\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \\ & \longrightarrow \quad H^2\left((X_\varepsilon, g), (E, h), \Delta^{g,h}\right) \oplus \bigoplus_{p \text{ cusp}} H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \end{aligned}$$

given by $P_{\varepsilon,0}^{g,h}(z)$ on the compact part, and $P_{p,\varepsilon}^{g,h}(z)$ near every cusp, for every $z \in \mathbb{C} \setminus \mathbb{R}_-^*$.

Remark 2.2.17. As mentioned before, we could define this glued Poisson operator for any complex number z such that $-z$ is not in the spectrum of any Laplacian with Dirichlet boundary condition we have considered so far. This is the reason we worked with $z \in \mathbb{C} \setminus \mathbb{R}_-$. For technical reasons, we will actually need at some point to use these results for z in a neighborhood of 0. It will not matter how small that open neighborhood is.

Remark 2.2.18. The Sobolev space, or rather the sum of such spaces, on which $P_\varepsilon^{g,h}(z)$ is defined can actually be understood as

$$\begin{aligned} & H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \oplus \bigoplus_{p \text{ cusp}} H^{3/2}\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \\ & \qquad = \quad \left(H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right)\right)^{\oplus 2} \end{aligned}$$

since the connected component of Σ_ε are pairwise disjoint. The same cannot be said for the sum of Sobolev space in which this glued Poisson operator takes values, as H^2-distributions cannot in general be glued into a global H^2-section, or even into an H^1 section. Under certain circumstances, we will be able to get some regularity, which we will see below.

Definition 2.2.19. The *glued Poisson operator* for the Dolbeault Laplacian is defined as was done in definition 2.2.16, using the Dolbeault Laplacians instead of their Chern counterparts.

Definition 2.2.20. We define the *sum operator* as

$$\begin{aligned} \sigma \quad : \quad & H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \oplus \bigoplus_{p \text{ cusp}} H^{3/2}\left((\Sigma_{p,\varepsilon}, g), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \\ & \qquad \longrightarrow \quad H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \\ & \left(u, (u_p)_p\right) \quad \longmapsto \quad \left(u_{|\Sigma_{p,\varepsilon}} + u_p\right)_p . \end{aligned}$$

Proposition 2.2.21. *The L^2-adjoint of the sum operator is the doubling operator*

$$\begin{aligned}\sigma^* \quad : \quad & H^{3/2}\left(\left(\Sigma_\varepsilon, g\right), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \\ & \qquad\qquad \longrightarrow \quad H^{3/2}\left(\left(\Sigma_\varepsilon, g\right), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \oplus \bigoplus_{p \text{ cusp}} H^{3/2}\left(\left(\Sigma_{p,\varepsilon}, g\right), \left(E_{|\Sigma_{p,\varepsilon}}, h\right)\right) \\ u \qquad & \qquad\qquad \longmapsto \quad \left(u, \left(u_{|\Sigma_{p,\varepsilon}}\right)_p\right) .\end{aligned}$$

Remark 2.2.22. One can note that these operators, for related reasons, are used by Burghelea, Friedlander, and Kappeler in [20], when the double cover is considered.

Proposition 2.2.23. *For any section f of $H^{3/2}\left(\left(\Sigma_\varepsilon, g\right), \left(E_{|\Sigma_\varepsilon}, h\right)\right)$, we have*

$$P_\varepsilon^{g,h}\left(z\right)\sigma^* f \quad \in \quad H^1\left(\left(Z, g\right), \left(E, h\right), \Delta^{g,h}\right) \ .$$

Proof. This result is a consequence of the jump formula, or rather its direct generalization to the sort of distributions we consider here.

□

Remark 2.2.24. The presence of the doubling operator σ^* means we are actually considering the most "natural" Poisson operator, which is the one that extends a single section defined on Σ_ε to all sides of the hypersurface. How far this section is from being H^2 will actually be quantified by the *jump operator*, also called the *Dirichlet-to-Neumann operator*.

2.2.4 Normal derivatives

We will now define the notion of *normal derivative* along the hypersurface Σ_ε, which intuitively is

$$\partial_n s \quad = \quad \langle \nabla s, n \rangle \ ,$$

with n being "the" normal unit vector, relatively to Σ_ε. However, the definition of this vector field is not entirely obvious, and can only be done close to the hypersurface in question. In order to avoid using notions such as tubular neighborhoods and parallel transport in a general setting, we will use the fact that the situation we study is explicit.

Around cusps. Let p be a cusp. We will define two normal unit vector fields relatively to the connected component $\Sigma_{p,\varepsilon}$ of Σ_ε, pointing to either side of it. For that, we will define the normal unit vector field on $U_{p,2\varepsilon}$, relatively to either side of $\Sigma_{p,\varepsilon}$. As usual, we see the neighborhood $U_{p,2\varepsilon}$ of p as the product $S^1 \times \left]a\left(2\varepsilon\right), +\infty\right[$, endowed with the Poincaré metric.

Proposition 2.2.25. *The normal unit vector field pointing outward on $U_{p,\varepsilon}$ relatively to $\Sigma_{p,\varepsilon}$ is*

$$n_p^+ \quad = \quad \left(0, y\right)$$

at the point $\left(x, y\right) \in S^1 \times \left]a\left(\varepsilon\right), +\infty\right[$. Furthermore, the normal unit vector field pointing inward on $U_{p,2\varepsilon} \setminus U_{p,\varepsilon}$ relatively to $\Sigma_{p,\varepsilon}$ is, at the point $\left(x, y\right) \in S^1 \times \left]a\left(2\varepsilon\right), a\left(\varepsilon\right)\right[$,

$$n_p^- \quad = \quad \left(0, -y\right) \ .$$

Proof. This result stems directly from a computation using the explicit description of the open neighborhood $U_{p,2\varepsilon}$ of p in Z.

□

Using the explicit local description of the Chern connection, we can now define more explicitely the notion of normal derivative near a cusp. Recall that $U_{p,\varepsilon}$ can also be seen as the quotient of the product $\mathbb{R} \times \left]a(\varepsilon), +\infty\right[$ by the action of the stabilizer of ∞. Sections of E over $U_{p,\varepsilon}$ can be identified to functions defined on $\mathbb{R} \times \left]a(\varepsilon), +\infty\right[$ and compatible with said action.

Proposition 2.2.26. *Let s be a smooth, compactly supported section of E over $U_{p,\varepsilon}$, seen as a vector-valued function $s : \mathbb{R} \times \left]a(\varepsilon), +\infty\right[\longrightarrow \mathbb{C}^r$ compatible with the representation. The normal derivative of s is given by*

$$\partial_{n_p^+} s \quad = \quad y \, \frac{\partial s}{\partial y} \ .$$

Proposition 2.2.27. *The normal derivative operator $\partial_{n_p^+}$ extends to a linear operator*

$$\partial_{n_p^+} \quad : \quad H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \quad \longrightarrow \quad H^1\left((U_{p,\varepsilon}, g), (E, h), \nabla^{(E,h)}\right) \ .$$

Proposition 2.2.28. *Let s be a smooth, compactly supported section of E over $U_{p,\varepsilon}$, seen as a vector-valued function $s : \mathbb{R} \times \left]a(2\varepsilon), a(\varepsilon)\right[\longrightarrow \mathbb{C}^r$ compatible with the representation. The normal derivative of s is given by*

$$\partial_{n_p^-} s \quad = \quad -y \, \frac{\partial s}{\partial y} \ .$$

Proposition 2.2.29. *The normal derivative operator $\partial_{n_p^-}$ extends to a linear operator*

$$\partial_{n_p^-} \quad : \quad H^2\left((U_{p,2\varepsilon} \setminus U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \quad \longrightarrow \quad H^1\left((U_{p,2\varepsilon} \setminus U_{p,\varepsilon}, g), (E, h), \nabla^{(E,h)}\right) \ .$$

Remark 2.2.30. Getting an H^1-regularity in the normal derivative was important, as we will later compose the resulting operator with the appropriate boundary traces.

Glued normal derivative. Much like we did for the Poisson operators, we will define a glued version of the normal derivative operators.

Definition 2.2.31. The *glued normal derivative* is defined as the operator

$$\begin{aligned} \partial_n \quad : \quad & H^2\left((X_\varepsilon, g), (E, h), \Delta^{g,h}\right) \oplus \bigoplus_{p \text{ cusp}} H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \\ & \longrightarrow \quad \bigoplus_{p \text{ cusp}} H^2\left((U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \oplus H^2\left((U_{p,2\varepsilon} \setminus U_{p,\varepsilon}, g), (E, h), \Delta^{g,h}\right) \ . \\ & \left(s, (s_p)_p\right) \quad \longmapsto \quad \left(\partial_{n_p^+} s_{|U_{p,2\varepsilon} \setminus U_{p,\varepsilon}}, \partial_{n_p^-} s_p\right)_p \end{aligned}$$

Remark 2.2.32. In the same way as before, this glued normal derivative operator can be composed with the glued boundary trace operator, and then with the sum operator in a transparent way.

Remark 2.2.33. The notion of normal derivatives has only been definied when g is the Poincaré metric, and h is the canonical metric on E. Only its behavior near the hypersurface Σ_ε is important, as it will always be composed with the boundary trace. Thus, the normal derivative operators we could define for metrics g and h coinciding with the Poincaré metric on X and the canonical metric on E near Σ_ε coincide with $\partial_{n_p^+}$ and $\partial_{n_p^-}$ near Σ_ε.

2.2.5 Jump operator

In remark 2.2.24, we saw that composing the glued Poisson operator $P_\varepsilon^{g,h}(z)$ with the doubling operator σ^* yields a global H^1-regularity, not an H^2-one. It is then natural to quantify this gap.

Using the jump formula, this is done by looking at the first order derivatives and at the way they behave when crossing Σ_ε. The value on Σ_ε being prescribed and with sufficient regularity, there can be no jump in tangential derivatives, which means we need only look at the normal derivatives. Let g be a smooth metric on Z, and h be a smooth metric on E over Z, which coincide with the Poincaré metric and the canonical metric on E except possibly on open subsets $U_{p,\eta}$ with $\eta < \varepsilon$.

Definition 2.2.34. The *jump operator*, also called the *Dirichlet-to-Neumann operator*, associated to the Chern Laplacian, is defined for $z \in \mathbb{C} \setminus \mathbb{R}_-^*$ by

$$\begin{array}{rcccc} N_{E,\varepsilon}^{g,h}(z) & : & H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) & \longrightarrow & H^{1/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \\ & & f & \longmapsto & -\sigma\, \gamma_\varepsilon\, \partial_n\, P_\varepsilon^{g,h}(z)\, \sigma^* f \end{array} .$$

Remark 2.2.35. The reason why we used the normal derivative operator ∂_n, and not one defined using the metrics g and h, is that it is composed with the boundary trace operator γ_ε.

Remark 2.2.36. The sign convention chosen in the definition above will be justified later, but it amounts to inverting the direction of the normal unit vector field.

Proposition 2.2.37. *For any $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, the jump operator induces an operator*

$$N_{E,\varepsilon}^{g,h}(z) \quad : \quad \mathcal{C}^\infty(\Sigma_\varepsilon, E) \quad \longrightarrow \quad \mathcal{C}^\infty(\Sigma_\varepsilon) \ .$$

Proof. This is a direct consequence of proposition 2.1.60. □

Proposition 2.2.38. *For any complex number $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, the restriction of the jump operator to smooth sections of E over Σ_ε induces a holomorphic family of pseudo-differential operators of order 1 and weight 2 with respect to the parameter z.*

Proof. As indicated in [20, Sec. 3.13], this proposition can be seen as resulting from more general results about Green operators, and more importantly theorem 3.3.2 and corollary 2.7.8 of [54]. □

Definition 2.2.39. The *jump operator*, also called the *Dirichlet-to-Neumann operator*, associated to the Dolbeault Laplacian, is defined for $z \in \mathbb{C} \setminus \mathbb{R}_-^*$ by

$$\begin{array}{rcccc} N_{\overline{\partial}_E,\varepsilon}^{g,h}(z) & : & H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) & \longrightarrow & H^{1/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right) \\ & & f & \longmapsto & -\sigma\, \gamma_\varepsilon\, \partial_n\, P_{\overline{\partial}_E,\varepsilon}^{g,h}(z)\, \sigma^* f \end{array} .$$

Remark 2.2.40. There is an analog of propositions 2.2.37 and 2.2.38 for the jump operator related to the Dolbeault Laplacian.

Remark 2.2.41. Even though these jump operators are defined on Σ_ε, and the normal derivative operators were only considered for metrics which coincide with the canonical ones around Σ_ε, the fact that they involve the Poisson operators means that they depend on the metrics everywhere.

Proposition 2.2.42. *Let f be an element of $H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right)$. We have*

$$P_\varepsilon^{g,h}(z)\, \sigma^* f \ \in \ H^2\left((Z, g), (E, h), \Delta^{g,h}\right) \quad \Longleftrightarrow \quad N_{E,\varepsilon}^{g,h}(z)\, f \ = \ 0 \ .$$

Proof. Once again, this is a direct consequence of the jump formula. □

We have an analogous result for the jump operator associated to the Dolbeault Laplacian.

Proposition 2.2.43. *Let f be an element of $H^{3/2}\left((\Sigma_\varepsilon, g), \left(E_{|\Sigma_\varepsilon}, h\right)\right)$. We have*

$$P^{g,h}_{\overline{\partial}_{E},\varepsilon}(z)\,\sigma^* f \;\in\; H^2\left((Z,g),(E,h),\Delta^{g,h}\right) \quad\Longleftrightarrow\quad N^{g,h}_{\overline{\partial}_{E},\varepsilon}(z)\,f \;=\; 0\ .$$

The main result about jump operators, which will be crucial in what follows, is the following.

Theorem 2.2.44. *For any complex number $z \in \mathbb{C}\setminus\mathbb{R}_-$, we have*

$$N^{g,h}_{E,\varepsilon}(z)^{-1} \quad=\quad \tfrac{1}{4}\sigma\gamma_\varepsilon\left(\Delta^{g,h}_E + z\right)^{-1/2}\left(\sigma\gamma_\varepsilon\left(\Delta^{g,h}_E + \overline{z}\right)^{-1/2}\right)^* \ ,$$

where the adjunction is taken with respect to the inner product of $L^2\left((Z,g),(E,h)\right)$.

Remark 2.2.45. In this theorem, we had to restrict ourselves to $z \in \mathbb{C}\setminus\mathbb{R}_-$, which means we had to exclude 0, since we need $\Delta_E + z$ to be invertible.

Definition 2.2.46. For any smooth section $u \in \mathcal{C}^\infty(\Sigma_\varepsilon, E)$, we define the distribution $u\otimes\delta_{\Sigma_\varepsilon}$ by

$$\begin{array}{rccl} u\otimes\delta_{\Sigma_\varepsilon} & : & \mathcal{C}^\infty_0(Z,E) & \longrightarrow \quad \mathbb{C} \\ & & \varphi & \longmapsto \quad \int_{\Sigma_\varepsilon} h_z\left(u(z),\varphi(z)\right)\,\mathrm{d}\mu_g(z) \end{array} .$$

Lemma 2.2.47. *For any complex number $z \in \mathbb{C}\setminus\mathbb{R}_-$, and any $u \in \mathcal{C}^\infty(\Sigma_\varepsilon, E)$ we have the following equality of distributions on Z, where $\Delta^{g,h}$ is the distributional Chern Laplacian,*

$$\left(\Delta^{g,h} + z\right)P^{g,h}_\varepsilon(z)\,\sigma^* u \quad=\quad N^{g,h}_{E,\varepsilon}(z)\,u\otimes\delta_{\Sigma_\varepsilon}\ .$$

Proof. We first note that both distributions in the equality we wish to prove vanish when evaluated on smooth sections whose compact support stays away from the hypersurface Σ_ε. Thus, we only need to work with smooth sections compactly supported in an open neighborhood $U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}$ of the connected component $\Sigma_{p,\varepsilon}$. We have the following local decomposition

$$E \quad=\quad \bigoplus_{j=1}^{r} L_j\ ,$$

over $U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}$, which gives an orthogonal decomposition

$$\mathcal{C}^\infty_0\left(U_{p,2\varepsilon}\setminus U_{p,\eta}, E\right) \quad=\quad \bigoplus_{j=1}^{r}\mathcal{C}^\infty_0\left(U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}, L_j\right)\ .$$

It is worth recalling that L_j is a line bundle related to the diagonalization of $\rho(\gamma_p)$, where γ_p is a generator of the stabilizer of p in Γ. We then need to prove that we have

$$\left(\left(\Delta^{g,h}+z\right)T_{P^{g,h}_\varepsilon(z)\sigma^* u}\right)(\varphi) \quad=\quad \left(N^{g,h}_{E,\varepsilon}(z)\,u\otimes\delta_{\Sigma_\varepsilon}\right)(\varphi) \quad \text{for any} \quad \varphi\in\mathcal{C}^\infty_0\left(U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}, L_j\right)$$

We consider such a φ, identified with a smooth function on $\mathbb{R}\times]a(2\varepsilon), a(\eta)[$, compactly supported in the second variable, which is compatible with the representation ρ in the following way

$$\varphi(x+1,y) \quad=\quad e^{2i\pi\alpha_{p,j}}\varphi(x,y)\ .$$

Here, we denoted by T a distribution associated to an L^2-section, for clarity. We now set

$$v \quad=\quad \left(P^{g,h}_\varepsilon(z)\,\sigma^* u\right)_{|U_{p,2\varepsilon}\setminus U_{p,\varepsilon}}\ , \qquad\qquad w \quad=\quad \left(P^{g,h}_\varepsilon(z)\,\sigma^* u\right)_{|\overline{U_{p,\varepsilon}}\setminus U_{p,\eta}}\ ,$$

and we make the same type of identifications on v and w as we did on φ. Both functions v and w are then sections which are smooth up until $\Sigma_{p,\varepsilon}$. We get

$$N_{E,\varepsilon}^{g,h}(z)\,u \quad = \quad a(\varepsilon)\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\ ,$$

these functions having been restricted to $\Sigma_{p,\varepsilon}$. Hence, the result we need to prove becomes

$$\left(\left(\Delta^{g,h}+z\right)T_{(v,w)}\right)(\varphi) \quad = \quad a(\varepsilon)\left(\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\otimes\delta_{\Sigma_{p,\varepsilon}}\right)(\varphi)\ ,$$

where (v,w) denotes the function on $U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}$ obtained by glueing v and w. The argument we use here is slightly adapted from [68, Lem A.1]. We set

$$\begin{array}{rccl}\Psi & : & \mathbb{R}\times\left]a(2\varepsilon),a(\eta)\right[& \longrightarrow\ \mathbb{C}\\ & & (x,y) & \longmapsto \begin{cases}\frac{\partial v}{\partial y}(x,y) & \text{if}\quad y\leq a(\varepsilon)\\ \frac{\partial w}{\partial y}(x,y)+\frac{\partial v}{\partial y}(x,a(\varepsilon))-\frac{\partial w}{\partial y}(x,a(\varepsilon)) & \text{if}\quad y>a(\varepsilon)\end{cases}\ .\end{array}$$

We note that Ψ, like v and w, is compatible with ρ, so it induces a section of L_j over $U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}$. It is further continuous, smooth on $\mathbb{R}\times\left]a(2\varepsilon),a(\varepsilon)\right]$ and $\mathbb{R}\times\left[a(\varepsilon),a(\eta)\right[$, but not smooth on the totality of the set on which it is defined. Moving on to the computation at hand, we have

$$\begin{aligned}
\left(\left(\Delta^{g,h}+z\right)T_{(v,w)}\right)(\varphi) \quad &= \quad T_{(v,w)}\left(\left(\Delta^{g,h}+\overline{z}\right)\varphi\right)\\
= \quad & -\left(\left(y^2\frac{\partial^2}{\partial x^2}-z\right)T_{(v,w)}\right)(\varphi)+\int_0^1\int_{a(2\varepsilon)}^{a(\varepsilon)}\frac{\partial v}{\partial y}\cdot\frac{\partial\overline{\varphi}}{\partial y}\,\mathrm{d}y\,\mathrm{d}x+\int_0^1\int_{a(\varepsilon)}^{a(\eta)}\frac{\partial w}{\partial y}\cdot\frac{\partial\overline{\varphi}}{\partial y}\,\mathrm{d}y\,\mathrm{d}x\\
& -\int_0^1\left(\cancel{\left[v(x,y)\frac{\partial\overline{\varphi}}{\partial y}\right]_{a(2\varepsilon)}^{a(\varepsilon)}}+\cancel{\left[w(x,y)\frac{\partial\overline{\varphi}}{\partial y}\right]_{a(\varepsilon)}^{a(\eta)}}\right)\mathrm{d}x\\
= \quad & -\left(\left(y^2\frac{\partial^2}{\partial x^2}-z\right)T_{(v,w)}\right)(\varphi)-\int_0^1\int_{a(2\varepsilon)}^{a(\varepsilon)}\frac{\partial^2 v}{\partial y^2}\,\overline{\varphi(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
& +\int_0^1\left(\left[\frac{\partial v}{\partial y}\overline{\varphi(x,y)}\right]_{a(2\varepsilon)}^{a(\varepsilon)}+\left[\frac{\partial w}{\partial y}\overline{\varphi(x,y)}\right]_{a(\varepsilon)}^{a(\eta)}\right)-\int_0^1\int_{a(\varepsilon)}^{a(\eta)}\frac{\partial^2 w}{\partial y^2}\,\overline{\varphi(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
= \quad & -\left(\left(y^2\frac{\partial^2}{\partial x^2}-z\right)T_{(v,w)}\right)(\varphi)-\int_0^1\int_{a(2\varepsilon)}^{a(\varepsilon)}\frac{\partial^2 v}{\partial y^2}\,\overline{\varphi(x,y)}\,\mathrm{d}y\,\mathrm{d}x-\int_0^1\int_{a(\varepsilon)}^{a(\eta)}\frac{\partial^2 w}{\partial y^2}\,\overline{\varphi(x,y)}\,\mathrm{d}y\,\mathrm{d}x\\
& +\int_0^1\left(\frac{\partial v}{\partial y}(x,a(\varepsilon))-\frac{\partial w}{\partial y}(x,a(\varepsilon))\right)\overline{\varphi(x,a(\varepsilon))}\,\mathrm{d}x\\
= \quad & T_{-\left(y^2\frac{\partial^2}{\partial x^2}-z\right)(v,w)-y^2\frac{\partial\Psi}{\partial y}}(\varphi)+\left(a(\varepsilon)\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\otimes\delta_{\Sigma_{p,\varepsilon}}\right)(\varphi)\,.
\end{aligned}$$

We will now prove that the first term on the right-hand side above vanishes. For that, we note that, if the support of φ is included in $U_{p,2\varepsilon}\setminus\overline{U_{p,\varepsilon}}$, we have

$$\underbrace{\left(\left(\Delta^{g,h}+z\right)T_{(v,w)}\right)(\varphi)}_{=0} \quad = \quad T_{-\left(y^2\frac{\partial^2}{\partial x^2}-z\right)(v,w)-y^2\frac{\partial\Psi}{\partial y}}(\varphi)+\underbrace{\left(a(\varepsilon)\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\otimes\delta_{\Sigma_{p,\varepsilon}}\right)(\varphi)}_{=0}\,.$$

Furthermore, if the support of φ is included in $U_{p,\varepsilon}\setminus\overline{U_{p,\eta}}$, we have

$$\underbrace{\left(\left(\Delta^{g,h}+z\right)T_{(v,w)}\right)(\varphi)}_{=0} \quad = \quad T_{-\left(y^2\frac{\partial^2}{\partial x^2}-z\right)(v,w)-y^2\frac{\partial\Psi}{\partial y}}(\varphi)+\underbrace{\left(a(\varepsilon)\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\otimes\delta_{\Sigma_{p,\varepsilon}}\right)(\varphi)}_{=0}\,.$$

This means that the function

$$\begin{array}{ccc} \mathbb{R}\times\left]a\left(2\varepsilon\right),a\left(\eta\right)\right[& \longrightarrow & \mathbb{C} \\ (x,y) & \longmapsto & -\left(y^2\frac{\partial^2}{\partial x^2}+z\right)(v,w)-y^2\frac{\partial\Psi}{\partial y} \end{array},$$

which is square-integrable, vansihes almost everywhere. Thus, the associated distribution equals zero. Therefore, we have

$$\left(\left(\Delta^{g,h}+z\right)T_{(v,w)}\right)(\varphi) \quad = \quad a\left(\varepsilon\right)\left(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial y}\right)\otimes\delta_{\Sigma_{p,\varepsilon}}$$

as distributions on $U_{p,2\varepsilon}\setminus\overline{U_{p,\eta}}$. This completes the proof of the lemma.

□

Proof of theorem 2.2.44. The first point to note is that, since the punctured modular curve Z is without boundary, the operator

$$\Delta_E^{g,h}+\overline{z} \quad : \quad \mathcal{C}_0^\infty\left(Z,E\right) \longrightarrow \mathcal{C}_0^\infty\left(Z,E\right)$$

has a dense image. Let now φ be a smooth, compactly supported section of E over Z, which is in the image of the operator above. We have

$$\begin{aligned} \left(P_\varepsilon^{g,h}\left(z\right)\sigma^*u\right)(\varphi) &= \left(\left(\Delta^{g,h}+z\right)P_\varepsilon^{g,h}\left(z\right)\sigma^*u\right)\left(\left(\Delta_E^{g,h}+\overline{z}\right)^{-1}\varphi\right) \\ &= \left\langle N_{E,\varepsilon}^{g,h}\left(z\right)u,\tfrac{1}{2}\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1}\varphi\right\rangle_{L^2(\Sigma_\varepsilon,E)} \\ &= \left\langle N_{E,\varepsilon}^{g,h}\left(z\right)u,\tfrac{1}{2}\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\varphi\right\rangle_{L^2(\Sigma_\varepsilon,E)} \\ &= \tfrac{1}{2}\left\langle\left(\Delta_E^{g,h}+z\right)^{-1/2}\left(\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\right)^*N_{E,\varepsilon}^{g,h}\left(z\right)u,\varphi\right\rangle_{L^2(Z,E)}. \end{aligned}$$

This gives the following equality of distributions

$$P_\varepsilon^{g,h}\left(z\right)\sigma^*u \quad = \quad \tfrac{1}{2}\left(\Delta_E^{g,h}+z\right)^{-1/2}\left(\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\right)^*N_{E,\varepsilon}^{g,h}\left(z\right)u\ ,$$

and using the definition of the Poisson operators, we get

$$\begin{aligned} u \quad = \quad \tfrac{1}{2}\sigma\sigma^*u &= \tfrac{1}{2}\sigma\gamma_\varepsilon P_\varepsilon^{g,h}\left(z\right)\sigma^*u \\ &= \tfrac{1}{4}\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+z\right)^{-1/2}\left(\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\right)^*N_{E,\varepsilon}^{g,h}\left(z\right)u. \end{aligned}$$

The proof of the theorem is now almost complete. For any complex number $z\in\mathbb{C}\setminus\mathbb{R}_-$, we set

$$\begin{array}{cccc} A_z & : & L^2\left(\Sigma_\varepsilon,E\right) & \longrightarrow & L^2\left(\Sigma_\varepsilon,E\right) \\ & & v & \longmapsto & \tfrac{1}{4}\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+z\right)^{-1/2}\left(\sigma\gamma_\varepsilon\left(\Delta_E^{g,h}+\overline{z}\right)^{-1/2}\right)^*v \end{array},$$

which is a bounded operator, of adjoint given by $A_{\overline{z}}$. When restricted to smooth functions, it further yields a surjective operator

$$A_z \quad : \quad \mathcal{C}^\infty\left(\Sigma_\varepsilon,E\right) \longrightarrow \mathcal{C}^\infty\left(\Sigma_\varepsilon,E\right)$$

as it is right-invertible by the computation we have seen earlier in this proof. Going back to A_z seen as an operator between L^2-spaces, we see that its image is dense in L^2 for every complex number $z \in \mathbb{C} \setminus \mathbb{R}_-$. We then have

$$(\ker A_z)^{\perp} \quad = \quad \overline{\operatorname{Im} A_{\overline{z}}} \quad = \quad L^2\left(\Sigma_\varepsilon, E\right) \ ,$$

which means that A_z is injective for all considered z, and thus gives the bijectivity of the operator

$$A_z \quad : \quad \mathcal{C}^\infty\left(\Sigma_\varepsilon, E\right) \quad \longrightarrow \quad \mathcal{C}^\infty\left(\Sigma_\varepsilon, E\right) \ .$$

The jump operators, restricted to smooth sections, are then invertible, seen as operators

$$N_{E,\varepsilon}^{g,h}(z) \quad : \quad \mathcal{C}^\infty\left(\Sigma_\varepsilon, E\right) \quad \longrightarrow \quad \mathcal{C}^\infty\left(\Sigma_\varepsilon, E\right) \ .$$

Their inverse is then precisely given by A_z, which completes the proof of the theorem.

□

Corollary 2.2.48. *For any real number $\mu > 0$, the operator A_μ is self-adjoint positive-definite, seen between L^2-spaces.*

Proof. Let $\mu > 0$ be a strictly positive real number. The fact that A_μ is self-adjoint stems from the fact that it is bounded and symmetric. We will therefore only prove that it is positive-definite. Let φ be an element of $L^2\left(\Sigma_\varepsilon, E\right)$. We have

$$\begin{aligned}
\langle A_\mu \varphi, \varphi \rangle_{L^2(\Sigma_\varepsilon, E)} \quad &= \quad \tfrac{1}{4}\left\langle \sigma\gamma_\varepsilon\big(\Delta_E^{g,h} + \mu\big)^{-1/2}\left(\sigma\gamma_\varepsilon\big(\Delta_E^{g,h} + \mu\big)^{-1/2}\right)^{*}\varphi, \varphi \right\rangle_{L^2(\Sigma_\varepsilon, E)} \\
&= \quad \tfrac{1}{4}\left\|\left(\sigma\gamma_\varepsilon\big(\Delta_E^{g,h} + \mu\big)^{-1/2}\right)^{*}\varphi\right\|^2_{L^2(Z,E)} .
\end{aligned}$$

□

Remark 2.2.49. Every result stated in this section admits a completely similar counterpart for the jump operator associated to the Dolbeault Laplacian. The analog of theorem 2.2.44 is stated below for further reference.

Theorem 2.2.50. *For any $z \in \mathbb{C} \setminus \mathbb{R}_-$, and any smooth section u of E over Σ_ε, we have*

$$N_{\overline{\partial}_E,\varepsilon}^{g,h}(z)^{-1}\, u \quad = \quad \tfrac{1}{4}\sigma\gamma_\varepsilon\big(\Delta_{\overline{\partial}_E}^{g,h} + z\big)^{-1/2}\left(\sigma\gamma_\varepsilon\big(\Delta_{\overline{\partial}_E}^{g,h} + \overline{z}\big)^{-1/2}\right)^{*} u \ ,$$

where the adjunction is taken with respect to the inner product of $L^2\left((Z,g), (E,h)\right)$.

2.3 Mayer-Vietoris formula in the compact case

This section is devoted to the statement of the first of several analytical formulae we will later need. Except for the content of the last paragraph, everything here can be found in [20]. The aim of the formula presented here is to relate the determinants of various Dolbeault Laplacians defined so far, when the metrics on the modular curve X and the vector bundle E are assumed to be smooth. This comparaison will also involve a jump operator.

2.3.1 Parameter-dependent formula

We will work step by step in this paragraph, presenting first the *Mayer-Vietoris formula* for the Dolbeault Laplacian with parameter $\mu > 0$. This result will involve a constant, the computation of which will be taken care of subsequently. In this section, we assume that the metrics g on X and h on E are smooth, and coincide with the Poincaré metric on X and the canonical hermitian metric on E, respectively. We make the following definition in order to shorten the statement of the main result of this section.

Definition 2.3.1. The Dolbeault Laplacian with Dirichlet boundary condition along Σ_ε is the operator defined by the following glueing

$$\Delta^{g,h}_{\overline{\partial}_E,\varepsilon} \quad = \quad \left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon,0}, \left(\Delta^{g,h}_{\overline{\partial}_E,p,\varepsilon} \right)_{p \text{ cusp}} \right)$$

of the various Dolbeault Laplacians with Dirichlet boundary conditions.

Theorem 2.3.2. *For every real number $\mu > 0$, we have*

$$\frac{\det \left(\Delta^{g,h}_{\overline{\partial}_E} + \mu \right)}{\det \left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon} + \mu \right)} \quad = \quad \det N^{g,h}_{\overline{\partial}_E,\varepsilon} \left(\mu \right) \ .$$

Proof. Using theorem A from [20, Sec. 3.19], there is a constant $K > 0$ such that we have

$$\frac{\det \left(\Delta^{g,h}_{\overline{\partial}_E} + \mu \right)}{\det \left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon} + \mu \right)} \quad = \quad K \det N^{g,h}_{\overline{\partial}_E,\varepsilon} \left(\mu \right) \ .$$

Taking the logarithm of both sides, we now get

$$\log \det \left(\Delta^{g,h}_{\overline{\partial}_E} + \mu \right) - \log \det \left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon} + \mu \right) \quad = \quad \log K + \log \det N^{g,h}_{\overline{\partial}_E,\varepsilon} \left(\mu \right) \ ,$$

and the constant K can be computed by looking at the constant terms in the various asymptotic expansions as μ goes to infinity. Using the information found in [20, Sec. 3.12], we then note that we have $K = 1$, This completes the proof.

$\square$

2.3.2 Coefficient for $\mu = 0$

We now wish to let μ go to 0^+ in theorem 2.3.2. The apparent problem is that the Dolbeault Laplacian on the whole modular curve and the jump operator may have non-zero kernels, which means that the limit of both sides of this theorem may vanish. To remedy that, we will once again take the logarithm of each side of the theorem above, and then compare the constant coefficients in the asymptotic expansions as μ goes to 0, which is how the *modified determinant* is defined.

Proposition-Definition 2.3.3. *As μ goes to 0^+, we have the following asymptotic expansion*

$$\log \det \left(\Delta^{g,h}_{\overline{\partial}_E} + \mu \right) \quad = \quad d \log \mu + \log \det{}' \, \Delta^{g,h}_{\overline{\partial}_E} + o\left(1\right) \ ,$$

where d is the dimension of the kernel of the Dolbeault Laplacian, and $\det'$ *is the modified determinant of an operator, i.e. the determinant of its restriction to the orthogonal of the kernel.*

Remark 2.3.4. The prime used in the notation of the determinant will always stand for the modifed determinant, built from the spectral zeta function involving only strictly positive eigenvalues.

We now need to define a similar notion of modified determinant for the jump operator

$$N^{g,h}_{\overline{\partial}_E,\varepsilon} \quad = \quad N^{g,h}_{\overline{\partial}_E,\varepsilon}(0) \ .$$

Proposition 2.3.5. *The kernel of the jump operator $N^{g,h}_{\overline{\partial}_E,\varepsilon}$ is of dimension d.*

Proof. We will prove that the kernel of this jump operator is isomorphic to that of the Dolbeault Laplacian. First, recall that we have

$$f \in \ker N^{g,h}_{\overline{\partial}_E,\varepsilon} \quad \Longleftrightarrow \quad P^{g,h}_{\varepsilon}\sigma^* f \in H^2(X,E) \ .$$

We can then apply the Dolbeault Laplacian to this Poisson extension, and the result vanishes. This gives an a linear application

$$\begin{array}{ccc} \ker N^{g,h}_{\overline{\partial}_E,\varepsilon} & \longrightarrow & \ker \Delta^{g,h}_{\overline{\partial}_E} \\ f & \longmapsto & P^{g,h}_{\varepsilon}\sigma^* f \end{array} ,$$

which is injective, as one can see by applying the operator $\sigma\gamma_\varepsilon$. Conversely, a section $u \in H^2(X,E)$ in the kernel of the Dolbeault Laplacian is a smooth section, by elliptic regularity, and thus is H^2. Its restriction to Σ_ε then produces a section which is in the kernel of the jump operator. This completes the proof of the proposition.

□

Proposition 2.3.6. *For any real number $\mu > 0$, let $\alpha_1(\mu)$, ..., $\alpha_{d+1}(\mu)$ be the first $d+1$ eigenvalues of the jump operator with parameter μ in ascending order. These functions $\alpha_i(\mu)$ are continuous near 0^+, and we have*

$$\lim_{\mu\to 0^+} \alpha_i(\mu) \quad \begin{cases} = & 0 \quad \text{if} \quad i \leq d \\ > & 0 \quad \text{if} \quad i = d+1 \end{cases} .$$

Proof. This proposition is a consequence of the continuity of the family of jump operators.

□

Proposition 2.3.7. *As μ goes to 0^+, we have the following asymptotic expansion*

$$\log\det N^{g,h}_{\overline{\partial}_E,\varepsilon}(\mu) \quad = \quad \sum_{i=1}^{d} \log\alpha_i(\mu) + \log\det{}' N^{g,h}_{\overline{\partial}_E,\varepsilon} + o(1) \ ,$$

where $\det'$ *is the modified determinant.*

Remark 2.3.8. Note that the definition of the modified determinant of the jump operator does not take into account the asymptotic behavior of the eigenvalues that will collapse to 0 as μ goes to 0^+. This will be the object of the next proposition.

Proposition 2.3.9. *Let ψ_1, ..., ψ_d be an orthonormal basis of the kernel of the Dolbeault Laplacian, which consists of smooth sections of E over X. As μ goes to 0^+, we have the asymptotic expansion, where the determinant equals 1 by convention if the integer d is 0,*

$$\sum_{i=1}^{d} \log\alpha_i(\mu) \quad = \quad d\log\mu - \log\det\Big(\langle\psi_i,\psi_j\rangle_{L^2(\Sigma_\varepsilon,E)}\Big)_{1\leq i,j\leq d} + o(1) \ .$$

Remark 2.3.10. It should be noted that the smoothness of the sections ψ_j in this proposition is a consequence of elliptic regularity. This is important, as it allows us to restrict them to the hypersurface Σ_ε in the most natural way. The argument that follows relies on theorem 2.2.50, and is related to the proof of theorem B^* in [20, Sec. 4.9].

Proof of proposition 2.3.9. Using the fact that the Dolbeault Laplacian is a self-adjoint operator, which can then be diagonalized, we consider an orthonormal family $(\psi_j)_j$ of eigensections of this operator, whose first d terms are the sections already considered, with ψ_j associated to the eigenvalue λ_j. We now recall that, using theorem 2.2.50, we can extend, for any $\mu > 0$, the inverse of the jump operator restricted to smooth section as a bounded operator

$$N^{g,h}_{\overline{\partial}_E,\varepsilon}(\mu)^{-1} \quad : \quad L^2(\Sigma_\varepsilon, E) \quad \longrightarrow \quad L^2(\Sigma_\varepsilon, E) \ .$$

For any section $\omega \in L^2(\Sigma_\varepsilon, E)$, we now have

$$\begin{aligned}
\left(\sigma\gamma_\varepsilon\left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon}+\mu\right)^{-1/2}\right)^*\omega &= \sum_j \left\langle \left(\sigma\gamma_\varepsilon\left(\Delta^{g,h}_{\overline{\partial}_E,\varepsilon}+\mu\right)^{-1/2}\right)^*\omega, \psi_j \right\rangle_{L^2(X,E)} \psi_j \\
&= 2\sum_j \frac{1}{\sqrt{\lambda_j+\mu}} \langle \omega, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)} \psi_j .
\end{aligned}$$

Using the formula obtained for the inverse of the jump operator, we get

$$\mu N^{g,h}_{\overline{\partial}_E,\varepsilon}(\mu)^{-1}\omega \quad = \quad \sum_{j=1}^{d} \langle \omega, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)} \psi_j + \sum_{j>d} \frac{\mu}{\lambda_j+\mu} \langle \omega, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)} \psi_j .$$

This operator acting on ω can be seen as a continuous, bounded perturbation of

$$\begin{aligned}
P \quad : \quad L^2(\Sigma_\varepsilon, E) \quad &\longrightarrow \quad L^2(\Sigma_\varepsilon, E) \\
\omega \qquad &\longmapsto \quad \sum_{j=1}^{d} \langle \omega, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)} \psi_j
\end{aligned} \ .$$

Denoting by $\beta_1, \ldots, \beta_d$ the non-zero eigenvalues of P, we then have $\mu\alpha_j(\mu)^{-1} = \beta_j + o(\mu)$ as μ goes to 0^+, for every integer $j \in [\![1, d]\!]$. We finally note that we have

$$\sum_{j=1}^{d} \log \beta_j \quad = \quad \log\det\left(\langle \psi_i, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)}\right)_{1\le i,j\le d} \ ,$$

which completes the proof of the proposition, since we have, as μ goes to 0^+,

$$\sum_{j=1}^{d} \log \alpha_j(\mu) \quad = \quad d\log\mu - \log\det\left(\langle \psi_i, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)}\right)_{1\le i,j\le d} + o(1) \ .$$

□

Theorem 2.3.11. *We have the following comparison of (modified) determinants*

$$\frac{\det' \Delta^{g,h}_{\overline{\partial}_E}}{\det \Delta^{g,h}_{\overline{\partial}_E,\varepsilon}} \quad = \quad \det\left(\langle \psi_i, \psi_j \rangle_{L^2(\Sigma_\varepsilon,E)}\right)^{-1}_{1\le i,j\le d} \det' N^{g,h}_{\overline{\partial}_E,\varepsilon} \ ,$$

where the determinant equals 1 *by convention if the integer* d *is* 0.

Proof. We begin by using theorem 2.3.2, which gives, after taking the logarithm of both sides,

$$\log\det\left(\Delta_{\overline{\partial}_E}^{g,h}+\mu\right)-\log\det\left(\Delta_{\overline{\partial}_{E,\varepsilon}}^{g,h}+\mu\right) \quad = \quad \log\det N_{\overline{\partial}_{E,\varepsilon}}^{g,h}(\mu) \ .$$

The glued Dolbeault Laplacian with Dirichlet condition along Σ_ε being invertible, we have

$$\lim_{\mu\to 0^+} \log\det\left(\Delta_{\overline{\partial}_{E,\varepsilon}}^{g,h}+\mu\right) \quad = \quad \log\det\Delta_{\overline{\partial}_{E,\varepsilon}}^{g,h} \ ,$$

and we can then use propositions 2.3.3, 2.3.7, and 2.3.9, to get the following comparison of asymptotic expansions, as μ goes to 0^+,

$$\begin{aligned} & d\log\mu+\log\det{}'\,\Delta_{\overline{\partial}_E}^{g,h}-\log\det\Delta_{\overline{\partial}_{E,\varepsilon}}^{g,h} \\ & \qquad = \quad d\log\mu+\log\det{}'\,N_{\overline{\partial}_{E,\varepsilon}}^{g,h}-\log\det\left(\langle\psi_i,\psi_j\rangle_{L^2(\Sigma_\varepsilon,E)}\right)_{1\le i,j\le d}+o(1)\,. \end{aligned}$$

The divergent terms cancelling each other out, we get the theorem. □

Using proposition 2.1.42, we can now compute the determinant that appears in the theorem above.

Proposition 2.3.12. *The sections ψ_1, …, ψ_d are constant sections of E over Z, meaning that they can be identified to constant vector-valued functions $\mathbb{H}\longrightarrow\mathbb{C}^r$ which are compatible with the representation ρ. In particular, the constant value taken by any of these function is a vector fixed by the representation.*

Proof. This result comes from propositions 2.1.42 and 2.1.45, as well as proposition 1.2 of [69]. □

Remark 2.3.13. Let s and t be constant sections of E over Z. Assuming that the metric h coincides, near every cusp, with the canonical metric in directions corresponding to vanishing weights, the following function is constant

$$\begin{array}{rcccl} \langle\psi,\chi\rangle_E & : & Z & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & h_z(\psi(z),\chi(z)) \end{array} .$$

Proposition 2.3.14. *For any integers j and k between 1 and d, we have*

$$\langle\psi_j,\psi_k\rangle_{L^2(\Sigma_\varepsilon,E)} \quad = \quad \tfrac{\ell}{V_g}\delta_{jk} \ ,$$

where δ_{ij} denotes the Kronecker symbol, and V_g, ℓ denote the volume of Z and the length of Σ_ε for the metric g, respectively.

Remark 2.3.15. We assumed that the metric g coincided with the Poincaré metric except on some small open neighborhoods of cusps and elliptic points of radii strictly smaller than ε, which means that the length ℓ of Σ_ε is independant of g.

Proof of proposition 2.3.14. We first recall that ψ_1, …, ψ_d form an L^2-orthonormal basis of the kernel of the Dolbeault Laplacian, which gives

$$\langle\psi_j,\psi_k\rangle_{L^2(\Sigma_\varepsilon,E)} \quad = \quad \ell\,\langle\psi_j,\psi_k\rangle_E \quad = \quad \tfrac{\ell}{V_g}\int_Z\,\langle\psi_j,\psi_k\rangle_{E_z}\,\mathrm{d}\mu_g(z) \quad = \quad \tfrac{\ell}{V_g}\delta_{jk} \ .$$

□

Theorem 2.3.16. *We have the following comparison of (modified) determinants*

$$\frac{\det' \Delta^{g,h}_{\overline{\partial}_E}}{\det \Delta^{g,h}_{\overline{\partial}_{E,\varepsilon}}} \quad = \quad \frac{V_g^d}{\ell^d} \det' N^{g,h}_{\overline{\partial}_{E,\varepsilon}} \quad .$$

2.4 Mayer-Vietoris formula in the singular case

The purpose of this section is to get a *singular Mayer-Vietoris formula*, relating determinants of Laplacians, in a way to be defined, associated to the Poincaré metric g on Z and the canonical metric h on E over Z. The presence of singularities of the metrics prohibits the use of [20]. We will instead draw inspiration from [25], though the proof below is different in many aspects.

2.4.1 Parameter-dependent formula

As stated above, the Mayer-Vietoris formula we wish to prove aims to relate determinants of Laplacians for the Poincaré metric on Z and the canonical metric on E. However, such determinants are not defined, due to the presence of singularities. We will make use of the notion of *relative determinant*, presented in [73], which gives meaning to certain well-defined quotients of two ill-defined determinants. Following the last section, we will first prove the Mayer-Vietoris formula with parameter $\mu > 0$, which will involve a constant. The computation of this constant will again be done by finding asymptotics expansions as μ goes to infinity. The results of [20] cannot be used for that, and these problems will constitute one of the aims of chapters 3 and 4.

Determinant of the jump operator. Even though this type of determinant has already been used, let us recall how the determinant of the jump operator is defined. The spectral zeta function of the jump operator associated to the metrics g and h is defined on $\operatorname{Re} s > 1$ by

$$\zeta_{N_{E,\varepsilon}(z)} \quad : \quad s \quad \longmapsto \quad \operatorname{Tr}\left(N_{E,\varepsilon}(z)^{-s}\right) \quad ,$$

where the complex powers are defined on this half-plane by

$$N_{E,\varepsilon}(z)^{-s} \quad = \quad \int_{\mathcal{C}} \lambda_\vartheta^{-s} \left(N_{E,\varepsilon}(z) - \lambda\right)^{-1} \, đ\lambda \quad ,$$

and the contour $\mathcal{C}$ is associated to the spectral cut $\vartheta = \pi$. For a more exhaustive review, the reader is referred to [85, Sec. 1.4.2]. The meromorphic continuation of this spectral zeta function to the whole complex plane is holomorphic around $s = 0$, and the determinant is given by

$$\log\det N_{E,\varepsilon}(z) \quad = \quad -\zeta'_{N_{E,\varepsilon}(z)}(0) \quad .$$

As a function of z, the spectral zeta function $\zeta_{N_{E,\varepsilon}(z)}(s)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$.

Relative determinant. In this section, we will need to use a *relative determinant*. This notion was developped in [73], and can be thought of as a well-defined quotient of two potentially undefined determinants. We will have to use several results which proved in chapters 3 and 4.

Definition 2.4.1. The Chern Laplacian with Dirichlet boundary condition along Σ_ε is the operator defined by the following glueing of the various Chern Laplacians with Dirichlet boundary conditions

$$\Delta_{E,\varepsilon} \quad = \quad \left(\Delta_{E,\varepsilon,0}, \left(\Delta_{E,p,\varepsilon}\right)_{p \text{ cusp}}\right) \quad ,$$

for the Poincaré metric on Z and the canonical metric on E.

Proposition 2.4.2. *The relative spectral zeta function*

$$\zeta_{\Delta_E+z,\Delta_{E,\varepsilon}+z} \quad : \quad s \quad \longmapsto \quad \mathrm{Tr}\left((\Delta_E+z)^{-s}-(\Delta_{E,\varepsilon}+z)^{-s}\right)$$

is defined and holomorphic on the half-plane $\mathrm{Re}\, s > 1$. *It has a meromorphic continuation to a domain of* $\mathbb{C}$ *which contains* 0 *and is holomorphic at that point.*

Definition 2.4.3. The *relative determinant* associated to $\Delta_E + z$ and $\Delta_{E,\varepsilon} + z$ is defined as

$$\log\det(\Delta_E+z,\Delta_{E,\varepsilon}+z) \quad = \quad -\zeta'_{\Delta_E+z,\Delta_{E,\varepsilon}+z}(0) \ .$$

Mayer-Vietoris formula with parameter. We can now state the Mayer-Vietoris formula for the Chern Laplacian associated to the Poincaré metric and the canonical metric on E.

Theorem 2.4.4. *There exists a constant* $K > 0$ *such that, for any* $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, *we have*

$$\det(\Delta_E+z,\Delta_{E,\varepsilon}+z) \quad = \quad K \det N_{E,\varepsilon}(z) \ .$$

The proof of this result will be done step by step, beginning with the following proposition.

Proposition 2.4.5. *For any complex numbers* s *such that* $\mathrm{Re}\, s > 1$ *and* $z \in \mathbb{C} \setminus \mathbb{R}_-^*$, *we have*

$$\frac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad -s\,\mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-s-1}\,\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right) \ .$$

Proof. By continuity of the trace and the integral, we first note that we have

$$\frac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad \mathrm{Tr}\left(\int_{\mathcal{C}} \lambda_\vartheta^{-s}\,\frac{\mathrm{d}}{\mathrm{d}z}\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1}\,đ\lambda\right)$$

on $\{s \in \mathbb{C},\ \mathrm{Re}\, s > 1\}$. The derivative of the resolvent of the jump operator is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1} \quad = \quad \left(N_{E,\varepsilon}(z)-\lambda\right)^{-1}\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1} \ ,$$

We then get, with $[\cdot,\cdot]$ being the commutator of two operators,

$$\begin{aligned}\frac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad & \mathrm{Tr}\left(\int_{\mathcal{C}} \lambda_\vartheta^{-s}\,\left(N_{E,\varepsilon}(z)-\lambda\right)^{-2}\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\,đ\lambda\right) \\ & + \mathrm{Tr}\left(\int_{\mathcal{C}} \lambda_\vartheta^{-s}\,\left[\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1}\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z},\,\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1}\right]\,đ\lambda\right),\end{aligned}$$

Since the two operators involved in this commutator are pseudo-differential of order -2 and -1 respectively, we can interchange trace and integral, yielding

$$\frac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad \mathrm{Tr}\left(\int_{\mathcal{C}} \lambda_\vartheta^{-s}\,\left(N_{E,\varepsilon}(z)-\lambda\right)^{-2}\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\,đ\lambda\right) \ .$$

We can extract the derivative from the integral, and then integrate by parts, which yields

$$\begin{aligned}\frac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad &= \quad -s\,\mathrm{Tr}\left[\left(\int_{\mathcal{C}} \lambda_\vartheta^{-s-1}\,\left(N_{E,\varepsilon}(z)-\lambda\right)^{-1}\,đ\lambda\right)\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right] \\ &= \quad -s\,\mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-s-1}\,\frac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right).\end{aligned}$$

$\square$

Corollary 2.4.6. *The determinant of the jump operator is given by*

$$\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z) \quad = \quad \mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-1}\,\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right) \ .$$

Proof. Recall that proposition 2.4.5 yields

$$\tfrac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad -s\,\mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-s-1}\,\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right) \ .$$

We note that the family in s of operators considered in the trace, without the factor s, is a gauging, in the sense of [85, Sec. 1.5.6.1], of a pseudo-differential operator of order -2 acting on the compact manifold Σ_ε of dimension 1, which is trace class. The function

$$s \quad \longmapsto \quad \mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-s-1}\,\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right)$$

is then holomorphic around 0, which means that we have

$$\tfrac{\partial}{\partial s}_{|s=0}\,\tfrac{\partial}{\partial z}\zeta_{N_{E,\varepsilon}(z)}(s) \quad = \quad -\,\mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-1}\,\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right) \ .$$

Interchanging the two derivatives then gives the corollary. □

The following theorem is the most important point of this section, as it allows us to go from an operator acting on the compact curve Σ_ε, in which case being of trace class is very well understood, to an operator acting on the non-compact punctured modular curve Z.

Theorem 2.4.7. *For any complex number $z \in \mathbb{C}\setminus\mathbb{R}_-$, we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z) \quad = \quad \mathrm{Tr}\left[(\Delta_E + z)^{-1} - (\Delta_{E,\varepsilon} + z)^{-1}\right] \ .$$

Proof. Using corollary 2.4.6, we have

$$-\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z) \quad = \quad -\,\mathrm{Tr}\left(N_{E,\varepsilon}(z)^{-1}\,\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z}\right) \ .$$

The first step towards deriving the theorem will be to compute the derivative of the jump operator, using its definition, as well as the formula yielding the derivative of the jump operator. We have

$$N_{E,\varepsilon}(z) \quad = \quad -\sigma\,\gamma_\varepsilon\,\partial_n\,P_\varepsilon(z)\,\sigma^* \ ,$$

where the notation for the glued Poisson operator has been lightened, as we only consider it for the Poincaré metric on Z and the canonical metric on E, and we then have

$$\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z} \quad = \quad -\sigma\,\gamma_\varepsilon\,\partial_n\,\tfrac{\mathrm{d}P_\varepsilon(z)}{\mathrm{d}z}\sigma^* \quad = \quad \sigma\,\gamma_\varepsilon\,\partial_n\,(\Delta_{E,\varepsilon}+z)^{-1}\,P_\varepsilon(z)\,\sigma^* \ .$$

Since the operator $(\Delta_E + z)^{-1}$ sends an L^2-section to an H^2-one, which has then no jump in normal derivative at the hypersurface Σ_ε, we have $\sigma\,\gamma_\varepsilon\,\partial_n\,(\Delta_E - z)^{-1}\,P_\varepsilon(z)\,\sigma^* = 0$. Thus, we can add this to the derivative of the jump operator, giving

$$\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z} \quad = \quad -\sigma\,\gamma_\varepsilon\,\partial_n\,\left[(\Delta_E + z)^{-1} - (\Delta_{E,\varepsilon} + z)^{-1}\right]P_\varepsilon(z)\,\sigma^* \ .$$

The advantage of having added this term is that we can now express this difference of resolvent in another manner. More precisely, we have

$$\left(\Delta_E + z\right)^{-1} - \left(\Delta_{E,\varepsilon} + z\right)^{-1} \quad = \quad P_\varepsilon\left(z\right) \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} \ ,$$

and inserting this in the previous equality yields

$$\begin{aligned}
\tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z} \quad &= \quad -\sigma\, \gamma_\varepsilon\, \partial_n \left[P_\varepsilon\left(z\right) \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} \right] P_\varepsilon\left(z\right) \sigma^* \\
&= \quad \tfrac{1}{2} \left[-\sigma\, \gamma_\varepsilon\, \partial_n\, P_\varepsilon\left(z\right) \sigma^* \right] \sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} P_\varepsilon\left(z\right) \sigma^* \\
&= \quad \tfrac{1}{2} N_{E,\varepsilon}\left(z\right) \sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} P_\varepsilon\left(z\right) \sigma^*.
\end{aligned}$$

This computation holds as operators between the appropriate Hilbert spaces. Restricting to smooth sections, we can now invert the jump operator, thus giving

$$N_{E,\varepsilon}\left(z\right)^{-1} \tfrac{\mathrm{d}N_{E,\varepsilon}(z)}{\mathrm{d}z} \quad = \quad \tfrac{1}{2} \sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} P_\varepsilon\left(z\right) \sigma^* \ ,$$

as operators acting on smooth sections of E over Σ_ε. Combining the result obtained so far, we get

$$\tfrac{\mathrm{d}}{\mathrm{d}z} \log\det N_{E,\varepsilon}\left(z\right) \quad = \quad \tfrac{1}{2} \operatorname{Tr} \Big(\sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} P_\varepsilon\left(z\right) \sigma^* \Big) \ .$$

The idea is now to interchange two groups of operators within the last trace, which will naturally give rise to an operator acting on smooth sections of E over the punctured modular curve Z. Unfortunately, the fact that Z is in general non-compact (it would be if and only if there were no cusps) implies that some care must be taken. Though this operator will not be considered outside this proof, we will need to consider the Laplacian $\Delta_{\Sigma_\varepsilon,E}$ associated to the Poincaré metric restricted to Σ_ε and the canonical metric h over the restriction of E to Σ_ε. We have

$$\tfrac{\mathrm{d}}{\mathrm{d}z} \log\det N_{E,\varepsilon}\left(z\right) \quad = \quad \tfrac{1}{2} \operatorname{Tr} \Big(\Big[\left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{3/4} \sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} \Big] \Big[P_\varepsilon\left(z\right) \sigma^* \left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{-3/4} \Big] \Big) \, .$$

We will now work towards interchanging the two operators appearing in the trace above, which is delicate, as non-compactness is involved. To that effect, we begin by noting that the operator

$$\left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{3/4} \sigma \gamma_\varepsilon \left(\Delta_E + z\right)^{-1} \quad : \quad L^2\left(Z, E\right) \quad \longrightarrow \quad L^2\left(\Sigma_\varepsilon, E\right)$$

is bounded. We will now prove that we have

$$P_\varepsilon\left(z\right) \sigma^* \left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{-3/4} \quad \in \quad S_1 \left(L^2\left(\Sigma_\varepsilon, E\right), L^2\left(Z, E\right) \right) \ ,$$

where this space denotes the first Schatten class, and the operator considered here is bounded. The first step towards that goal is to compute the L^2-adjoint of this last operator. Let u and v be two smooth sections of E over Σ_ε. Using the proof of theorem 2.2.44, we have

$$\begin{aligned}
&\Big\langle P_\varepsilon\left(z\right) \sigma^* \left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{-3/4} u, v \Big\rangle_{L^2(Z,E)} \\
&\quad = \quad \tfrac{1}{2} \Big\langle \left(\Delta_E + z\right)^{-1/2} \Big(\sigma \gamma_\varepsilon \left(\Delta_E + \overline{z}\right)^{-1/2} \Big)^* N_{E,\varepsilon}\left(z\right) \left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{-3/4} u,\ v \Big\rangle_{L^2(Z,E)} \\
&\quad = \quad \tfrac{1}{2} \Big\langle u,\ \left(\Delta_{\Sigma_\varepsilon,E} + 1\right)^{-3/4} N_{E,\varepsilon}\left(\overline{z}\right) \sigma \gamma_\varepsilon \left(\Delta_E + \overline{z}\right)^{-1} v \Big\rangle_{L^2(Z,E)} \, .
\end{aligned}$$

By density, this proves that we have

$$\left(P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\right)^* \quad = \quad \tfrac{1}{2}\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4} N_{E,\varepsilon}(\overline{z})\,\sigma\gamma_\varepsilon\left(\Delta_E+\overline{z}\right)^{-1} \ .$$

To prove that the operator we consider belongs to the first Schatten class, we note that we have

$$\begin{aligned}
&\left(P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\right)^* \left(P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\right)\\
&\quad = \tfrac{1}{4}\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4} N_{E,\varepsilon}(\overline{z})\,\sigma\gamma_\varepsilon\left(\Delta_E+\overline{z}\right)^{-1}\left(\sigma\gamma_\varepsilon\left(\Delta_E+\overline{z}\right)^{-1}\right)^* N_{E,\varepsilon}(z)\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\\
&\quad = \tfrac{1}{4}\underbrace{\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}}_{\text{order } -3/2}\ \underbrace{N_{E,\varepsilon}(\overline{z})}_{\text{order } 1}\ \underbrace{\frac{\mathrm{d}}{\mathrm{d}z}\left(N_{E,\varepsilon}(z)^{-1}\right)}_{\text{order } -3}\ \underbrace{N_{E,\varepsilon}(z)}_{\text{order } 1}\ \underbrace{\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}}_{\text{order } -3/2}.
\end{aligned}$$

This operator, restricted to smooth sections of E over Σ_ε, then induces a pseudo-differential operator of order -4. Its square root is then of order -2, and acts on (sections of a vector bundle over) a manifold of dimension 1, so is trace class. The operator $P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}$ is then in the first Schatten class $S_1\left(L^2(\Sigma_\varepsilon,E)\,,L^2(Z,E)\right)$, showing that the operator

$$\begin{aligned}
P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\ \left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{3/4}\sigma\gamma_\varepsilon\left(\Delta_E+z\right)^{-1} \quad &= \quad P_\varepsilon(z)\,\sigma^*\sigma\gamma_\varepsilon\left(\Delta_E+z\right)^{-1}\\
= \quad 2P_\varepsilon(z)\,\gamma_\varepsilon\left(\Delta_E+z\right)^{-1} \quad &= \quad 2\left(\left(\Delta_E+z\right)^{-1}-\left(\Delta_{E,\varepsilon}+z\right)^{-1}\right)
\end{aligned}$$

is also trace class as a linear endomorphism of $L^2(Z,E)$. This completes the proof, as we have

$$\begin{aligned}
&\mathrm{Tr}\left(\left(\Delta_E+z\right)^{-1}-\left(\Delta_{E,\varepsilon}+z\right)^{-1}\right)\\
&\qquad = \quad \tfrac{1}{2}\,\mathrm{Tr}\left(\left[\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{3/4}\sigma\gamma_\varepsilon\left(\Delta_E+z\right)^{-1}\right]\ \left[P_\varepsilon(z)\,\sigma^*\left(\Delta_{\Sigma_\varepsilon,E}+1\right)^{-3/4}\right]\right)\\
&\qquad = \quad \tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z)\,.
\end{aligned}$$

□

Proof of theorem 2.4.4. Using theorem 2.4.7, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z) \quad = \quad \mathrm{Tr}\left(\left(\Delta_E+z\right)^{-1}-\left(\Delta_{E,\varepsilon}+z\right)^{-1}\right) \ .$$

In order to prove the theorem, we will show that the right-hand side above equals the log-derivative of the appropriate relative determinant. We have

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det\left(\Delta_E+z,\Delta_{E,\varepsilon}+z\right) \quad &= \quad -\,\mathrm{Fp}_{s=0}\ \tfrac{\partial}{\partial s}\,\tfrac{\partial}{\partial z}\,\zeta_{\Delta_E+z,\Delta_{E,\varepsilon}+z}(s)\\
= \quad \tfrac{\partial}{\partial s}_{|s=0}\ \left[s\zeta_{\Delta_E+z,\Delta_{E,\varepsilon}+z}(s+1)\right] \quad &= \quad \mathrm{Tr}\left(\left(\Delta_E+z\right)^{-1}-\left(\Delta_{E,\varepsilon}+z\right)^{-1}\right).
\end{aligned}$$

where we have denoted by Fp the *finite part* at a point of a meromorphic function, *i.e.* the constant term in its Laurent expansion at that point. The two equalities stated finally give

$$\tfrac{\mathrm{d}}{\mathrm{d}z}\log\det\left(\Delta_E+z,\Delta_{E,\varepsilon}+z\right) \quad = \quad \tfrac{\mathrm{d}}{\mathrm{d}z}\log\det N_{E,\varepsilon}(z) \ ,$$

and integrating with respect to z yields the full result.

□

Computation of the constant K. As we did in the last section, the computation of the constant K in the Mayer-Vietoris formula will be done by taking the logarithm of both sides and looking at the constant terms in the asymptotic expansion as the parameter $\mu = z$ goes to infinity through strictly positive real values. The difference here is that, although results stated [20] can be applied to deal with the jump operator and the part of the relative determinant having to do with the Laplacian with Dirichlet boundary condition on the compact part X_ε, the presence of cusps brings difficulties. We will also denote by Fp the constant term in the various asymptotic expansions as μ goes to infinity.

Theorem 2.4.8. *As μ goes to infinity, we have*

$$\mathrm{Fp}_{\mu=+\infty} \log\det N_{E,\varepsilon}(\mu) \quad = \quad 0 \ .$$

Proof. Since the jump operator acts on a compact manifold, this theorem is a direct consequence of the work of Burghelea, Friedlander, and Kappeler, more specifically [20, Sec. A.17].

□

Theorem 2.4.9. *As μ goes to infinity, we have*

$$\mathrm{Fp}_{\mu=+\infty} \log\det\left(\Delta_E + \mu, \Delta_{E,\varepsilon} + \mu\right) \quad = \quad 0 \ .$$

Proof. This theorem is a direct consequence of theorems 3.5.6 and 4.9.1.

□

Comparing these asymptotic expansions in theorem 2.4.4, we get the following result, which is called the *Mayer-Vietoris formula with parameter.*

Theorem 2.4.10. *In theorem 2.4.4, the constant K equals* 1*, and we have, for every $z \in \mathbb{C} \setminus \mathbb{R}_-$,*

$$\det\left(\Delta_E + z, \Delta_{E,\varepsilon} + z\right) \quad = \quad \det N_{E,\varepsilon}(z) \ .$$

2.4.2 Coefficient for $\mu = 0$

Much like we did for the Mayer-Vietoris formula in the case of smooth metrics and the Dolbeault operator, we now want to let $\mu = z$ go to 0^+ in theorem 2.4.10. This will result in considering, once again, modified determinants. We denote by V the volume of the modular curve, by ℓ the length of Σ_ε, and by d the dimension of the kernel of the Laplacian Δ_E.

Theorem 2.4.11. *We have*

$$\det{}'\left(\Delta_E, \Delta_{E,\varepsilon}\right) \quad = \quad \tfrac{V^d}{\ell^d} \det{}' N_{E,\varepsilon} \ .$$

Proof. The same argument as the one we used to get theorem 2.3.16 can be applied here.

□

Remark 2.4.12. In theorem 2.3.16, note that the length of Σ_ε did not depend on the metrics, as it was assumed that they coincided with the Poincaré metric on X and the canonical metric on E near the hypersurface. This part of the coefficient is thus the same in both Mayer-Vietoris formulae. However, the volume heavily depends on the metric chosen on the modular curve.

2.5 Determinants for the truncated metrics

In this last section related to analytic surgery, we will aim to give meaning and derive a formula yielding the determinant of the Dolbeault Laplacian associated to the truncated metrics g_ε on the modular curve and h_ε on E, which have been defined in the last chapter. Throughout the following, we will denote by g the Poincaré metric on the punctured modular curve Z and by h the canonical metric on E, meaning the one built from the canonical hermitian metric on $\mathbb{C}^r$.

2.5.1 Mayer-Vietoris formula for the truncated metrics

The first step in obtaining the required formula is to define what "the determinant of the Dolbeault Laplacian associated to the truncated metrics" means, since these metrics are not smooth.

Remark 2.5.1. As previously explained in definition 2.1.13, for any smooth function φ on Z, we denote by g_φ the metric on Z defined by

$$g_\varphi \quad = \quad e^{2\varphi} g \ .$$

Furthermore, recalling definition 2.1.25, for any smooth section ψ of End E, we denote by h_ψ the metric on E over Z given for any smooth sections s and t by

$$h_\psi \left(s, t\right) \quad = \quad h\left(e^{\psi} s, e^{\psi} t\right) \ .$$

We can extend these definitions to the case where ϕ and ψ only have an H^1-regularity. Such metrics on the tangent bundle have been extensively studied by Bost in [14], although the Sobolev space is there denoted by L_1^2.

Remark 2.5.2. From now on, we denote by φ and ψ the function on Z and the section of $\operatorname{End} E$ over Z, respectively, such that we have $g_\varepsilon = g_\varphi$ and $h_\varepsilon = h_\psi$. The metrics g and h being singular, these sections φ and ψ must be as well at the cusps (unless, for ψ, if every $\alpha_{p,j}$ vanishes).

Definition 2.5.3. We say that a sequence g_k of smooth metrics on X converges in H^1-norm to the truncated metric g_ε if there is a sequence of H^1-functions φ_k converging to 0 for that norm such that we have

$$g_k \quad = \quad e^{2\varphi_k} g_\varepsilon \ .$$

Remark 2.5.4. A similar definition can be made for the metrics on the vector bundle E.

For the remainder of this section, we consider sequences $(g_k)_k$ and $(h_k)_k$ of smooth metrics converging to the truncated metrics g_ε and h_ε, respectively, with $g_k = g_\varepsilon$ and $h_k = h_\varepsilon$ on the compact part X_ε of Z. As a consequence of the Deligne-Riemann-Roch isometry, we have the following result, which constitutes proposition-definition 5.3.11.

Proposition-Definition 2.5.5. *We define the (modified) determinant of the Dolbeault Laplacian for the truncated metrics to be given by the followilimit*

$$\det{}' \Delta_{\bar\partial_E}^{g_\varepsilon, h_\varepsilon} \quad = \quad \lim_{k \to +\infty} \det{}' \Delta_{\bar\partial_E}^{g_k, h_k} \quad ,$$

which is well-defined and independant of the sequences g_k and h_k.

Definition 2.5.6. The *truncation function* $C(\varepsilon)$ is defined by

$$C\left(\varepsilon\right) \quad = \quad \lim_{k \to +\infty} \left[\left(\prod_{p \text{ cusp}} \det \Delta_{\bar\partial_E, p, \varepsilon}^{g_k, h_k} \right) \cdot \det{}' N_{\bar\partial_E, \varepsilon}^{g_k, h_k} \right] \ .$$

Remark 2.5.7. This function is well-defined since, by proposition-definition 2.5.5, the limit does not depend on the chosen approximations, as long as they coincide with g and h on the compact part X_ε. If this last requirement is not assumed, we need to add the determinants on X_ε.

Theorem 2.5.8. *We have the following comparison of (modified) determinants*

$$\det{}' \Delta_{\overline{\partial}_E}^{g_\varepsilon, h_\varepsilon} \quad = \quad C(\varepsilon)\, \frac{V_{g_\varepsilon}^d}{\ell^d} \det \Delta_{\overline{\partial}_E, \varepsilon, 0}^{g,h} \quad .$$

2.5.2 Combination of Mayer-Vietoris formulae

It is now time to combine all the Mayer-Vietoris formulae in order to get an expression for the determinant of the Dolbeault Laplacian associated to the truncated metrics on X and on E. We will first decompose further the relative determinant we used in the singular Mayer-Vietoris formula.

Remark 2.5.9. The singular Mayer-Vietoris formula can then be restated as

$$\det{}' \left(\Delta_E, \Delta_{E,\mathrm{cusp},\varepsilon}\right) \quad = \quad \frac{V^d}{\ell^d} \left(\det{}' N_{E,\varepsilon}\right) \left(\det \Delta_{E,\varepsilon,0}\right) \quad .$$

Looking at both Mayer-Vietoris formulae, we note that the Laplacians on X_ε are not the same. One is attached to the Chern connection, the other to the Dolbeault operator. Fortunately, the hermitian metric h on E over X_ε is flat, which gives $\Delta_{E,\varepsilon,0} = 2\Delta_{\overline{\partial}_E,\varepsilon,0}$, using proposition 2.1.45.

Proposition 2.5.10. *On the compact part X_ε, the determinants of the Dolbeault and Chern Laplacians with Dirichlet boundary conditions are related by*

$$\det \Delta_{\overline{\partial}_E,\varepsilon,0} \quad = \quad 2^{\frac{r}{6}[h+2g_\Gamma-2]} \det \Delta_{E,\varepsilon,0} \quad ,$$

where h is the number of cusps and g_Γ is the genus of X.

Proof. We will work directly on the spectral zeta function. Using transparent notations, we have

$$\zeta\left(\Delta_{\overline{\partial}_E,\varepsilon,0},\ s\right) \quad = \quad \zeta\left(\tfrac{1}{2}\Delta_{E,\varepsilon,0},\ s\right) \quad = \quad 2^s\, \zeta\left(\Delta_{E,\varepsilon,0},\ s\right) \quad ,$$

which yields the following equality of determinants

$$\det \Delta_{\overline{\partial}_E,\varepsilon,0} \quad = \quad 2^{-\zeta(\Delta_{E,\varepsilon,0},\ 0)} \det \Delta_{E,\varepsilon,0} \quad .$$

Since the spectral zeta function is the Mellin transform of the heat trace, the special value above is the constant term in the small time asymptotic of the heat trace. Using [50, Thm 3.4.1], we have

$$\zeta\left(\Delta_{E,\varepsilon,0},\ 0\right) \quad = \quad \tfrac{r}{24\pi} \left[\ \int_{X_\varepsilon} \tau + 2 \int_{\partial X_\varepsilon} k \right] \quad ,$$

where τ denotes the scalar curvature of X_ε, and k is the geodesic curvature of ∂X_ε. The scalar curvature τ being twice the Gaussian curvature K, we thus get, using the Gauß-Bonnet theorem,

$$\zeta\left(\Delta_{E,\varepsilon,0},\ 0\right) \quad = \quad \tfrac{r}{12\pi} \left[\ \int_{X_\varepsilon} K + \int_{\partial X_\varepsilon} k \right] \quad = \quad \tfrac{1}{6} r \chi\left(X_\varepsilon\right) \quad ,$$

where χ denotes the Euler characteristic. It can be computed in terms of the genus of the modular curve, since there are no elliptic points and the genera of X_ε and Z are the same. We have

$$\chi\left(X_\varepsilon\right) \quad = \quad 2 - 2g_\Gamma - h \quad ,$$

with h being the number of cusps, which is the same as the number of connected components of the boundary ∂X_ε. Putting these results together, we get

$$\det \Delta_{\overline{\partial}_E,\varepsilon,0} \quad = \quad 2^{\frac{r}{6}[h+2g_\Gamma-2]} \det \Delta_{E,\varepsilon,0} \quad .$$

This completes the proof of the proposition.

□

Theorem 2.5.11. *We have the following equality of determinants*

$$\det{}' \Delta_{\overline{\partial}_E}^{g_\varepsilon,h_\varepsilon} \quad = \quad \frac{C(\varepsilon)}{\det{}' N_{E,\varepsilon}} \cdot 2^{\frac{r}{6}[h+2g_\Gamma-2]} \cdot \frac{V_{g_\varepsilon}^d}{V^d} \cdot \det{}' \left(\Delta_E, \Delta_{E,cusp,\varepsilon}\right) \quad .$$

Remark 2.5.12. The behavior of the first factor as ε goes to 0^+ on the right-hand side of the theorem above is only fully understood when the canonical metric h on E is smooth at the cusps, which happens if and only if all the weights $\alpha_{p,j}$ vanish. This is because we do not have to change the metric h in this case, and the conformal behavior of the determinants of Laplacians and of the jump operators are known. The Deligne-Riemann-Roch isometry will however give the divergence of this factor.

Theorem 2.5.13. *The modified relative determinant of Δ_E and $\Delta_{E,cusp,\varepsilon}$ is asymptotically given by, as ε goes to 0^+,*

$$\begin{aligned} &\log \det{}' \left(\Delta_E, \Delta_{E,cusp,\varepsilon}\right) \\ &\quad = \quad \log \varepsilon^{-1} \sum_{p\ cusp} \sum_{j=k_p+1}^{r} \left(\alpha_{p,j} - \alpha_{p,j}^2\right) - \tfrac{1}{6} r h \log \varepsilon^{-1} + \tfrac{1}{2} k\left(\Gamma,\rho\right) \log 2 + \log Z^{(d)}\left(1\right) \\ &\qquad\qquad + \log\left(d!\right) + \tfrac{1}{2\pi} r V \left[2\zeta'\left(-1\right) + \tfrac{1}{2}\log 2\pi - \tfrac{1}{4}\right] + o\left(1\right). \end{aligned}$$

Proof. This is a combination of theorems 3.5.10 and 4.9.2, remembering that we have

$$a\left(\varepsilon\right) \quad = \quad \tfrac{1}{2\pi} \log \varepsilon^{-1} \quad .$$

□

Corollary 2.5.14. *As ε goes to 0^+, the determinant of the Dolbeault Laplacian for the truncated metrics is given by*

$$\begin{aligned} \log \det{}' \Delta_{\overline{\partial}_E}^{g_\varepsilon,h_\varepsilon} \quad = \quad & \log \tfrac{C(\varepsilon)}{\det{}' N_{E,\varepsilon}} + \log \varepsilon^{-1} \sum_{p\ cusp} \sum_{j=k_p+1}^{r} \left(\alpha_{p,j} - \alpha_{p,j}^2\right) - \tfrac{1}{6} r h \log \varepsilon^{-1} \\ & + \tfrac{1}{2} k\left(\Gamma,\rho\right)\log 2 + \log Z^{(d)}\left(1\right) + \log\left(d!\right) + d \log \tfrac{V_{g_\varepsilon}}{V} \\ & + \tfrac{1}{2\pi} r V \left[2\zeta'\left(-1\right) + \tfrac{1}{2}\log 2\pi - \tfrac{1}{4}\right] + \tfrac{r}{6}\left[h + 2g_\Gamma - 2\right]\log 2 + o\left(1\right). \end{aligned}$$

Proof. This is a direct consequence of theorems 2.5.11 and 2.5.13.

□

Chapter 3

Determinants on a modular curve: the Selberg trace formula

The purpose of this chapter is to compute the modified relative determinant

$$\det{}'(\Delta_E, \Delta_\varepsilon)$$

this notion having already been studied, where Δ_ε is a one-dimensional Laplacian defined near each cusp, and trivially extended to X, as was done in [47, Sec. 2.1.3 & 8]. As before, we denote by Δ_E the Laplacian associated to the Poincaré metric on Z and the canonical metric on E. In order to deal with the structure of E over X, we will need to use the *Selberg trace formula*, which has been studied extensively in literature. The reader is referred to [97, 98], where Venkov presents everything we need. This chapter will also follow the work of Fischer, which can be found in [44]. Many other references exist, sometimes dealing with a situation that is not general enough for our purposes, but which may still be of use to the reader, see for example [57, 58, 60].

3.1 Description of the framework

3.1.1 Hyperbolic Laplacian Δ_E

Identification between Laplacians. As explained in the last chapter, a unitary representation

$$\rho \quad : \quad \Gamma \quad \longrightarrow \quad U_r(\mathbb{C}) \ ,$$

where Γ is the Fuchsian group of the first kind defining the modular curve X, induces a holomorphic flat vector bundle E of rank r over the punctured modular curve Z, for which we can consider its Chern connection ∇^E and its associated Laplacian

$$\Delta_E \quad : \quad \mathcal{C}_0^\infty(Z, E) \quad \longrightarrow \quad \mathcal{C}_0^\infty(Z, E)$$

acting on smooth compactly supported sections of E over Z. The Friedrichs extension process then yields an L^2-selfadjoint positive-definite operator

$$\Delta_E \quad : \quad H^2(Z, E) \quad \longrightarrow \quad L^2(Z, E) \ .$$

In order to fully apply the results regarding the Selberg trace formula given for instance by Venkov and Fischer, we will need to interpret this Laplacian differently. The resulting operator, being naturally identified with the Laplacian we have considered thus far, will be denoted in the same way. This shift of definition will be done throughout the rest of this text.

Definition 3.1.1. Let F be a fundamental domain for the action of Γ on $\mathbb{H}$. We define $\mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$ to be the space of smooth vector-valued functions $f : \mathbb{H} \longrightarrow \mathbb{C}^r$ which are square-integrable on F with respect to the Poincaré metric, and which are compatible with ρ, that is satisfying

$$f(\gamma \cdot z) \quad = \quad \rho(\gamma) f(z)$$

for every element γ of Γ and every point z in $\mathbb{H}$.

Remark 3.1.2. The complex vector space $\mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$ can be endowed with the hermitian product

$$(f, g) \quad = \quad \int_F (f(z), g(z))_{\mathbb{C}^r} \, \mathrm{d}\mu(z) \ ,$$

where μ is associated to the Poincaré metric.

Remark 3.1.3. The Laplacian associated to the Poincaré metric acts on $\mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$ by

$$\Delta f \quad = \quad -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f \ .$$

The resulting operator $\Delta : \mathcal{C}_\rho^\infty(F, \mathbb{C}^r) \longrightarrow \mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$ yields a selfadjoint positive-definite operator, after Friedrichs extension, whose domain is denoted by D, and is still denoted by Δ. When acting on $\mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$, both this Laplacian Δ and Δ_E, coincide in a natural manner, and we identify them, even after the extension.

Kernel of the resolvent. Before moving on to the definition of the auxiliary Laplacian, a quick remark is in order here, regarding the kernel of the resolvent of Δ_E. Denoting by $k(\cdot, \cdot, s)$ the kernel of the operator $(\Delta - s)^{-1}$, the kernel of $(\Delta_E - s)^{-1}$, seen as acting $\mathcal{C}_\rho^\infty(F, \mathbb{C}^r)$, is given by

$$k_{\Gamma,\rho}(z, z', s) \quad = \quad \sum_{\gamma \in \Gamma} \rho(\gamma) k(z, \gamma z', s) \ ,$$

for z and z' in the chosen fundamental domain F. This series is furthermore absolutely convergent on the half-plane $\mathrm{Re}\, s > 1$.

Remark 3.1.4. In a way, this new kernel is an average of k, though it does not involve any coefficients, as they are implicitely included in the domain on which the operators act. To see that more clearly, let us make an informal computation, where Γ is temporarily assumed to be finite, even though that cannot be the case, and r equals 1. We will see that the Laplacian Δ_E can be thought of as an average, in a more classical way, of "twisted" Laplacians. We have

$$\begin{aligned}
\left((\Delta_E - s)^{-1} f \right)(z) &= \int_F k_{\Gamma,\rho}(z, z', s) f(z') \, \mathrm{d}z' \\
&= \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \int_F k_{\Gamma,\rho}(z, \gamma z', s) f(\gamma z') \, \mathrm{d}z' \\
&= \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \int_{\gamma F} k_{\Gamma,\rho}(z, z', s) f(z') \, \mathrm{d}z' \\
&= \frac{1}{\#\Gamma} \int_{\mathbb{H}} k_{\Gamma,\rho}(z, z', s) f(z') \, \mathrm{d}z' \\
&= \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \rho(\gamma) \int_{\mathbb{H}} k(z, \gamma z', s) f(z') \, \mathrm{d}z' \\
&= \left(\frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \rho(\gamma) \left((\Delta - s)^{-1} \gamma^* \right) f \right)(z) \,.
\end{aligned}$$

This allows us, at least formally, to make the following identification

$$(\Delta_E - s)^{-1} \quad = \quad \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} \rho(\gamma) \left((\Delta - s)^{-1} \gamma^* \right) \ ,$$

and the operators appearing on the right hand side are resolvent of Laplacians which have been twisted by the isometry induced by the elements of the group Γ. Such twists, in different contexts, have been studied by Donnelly and Patodi in [35] and [36], and by Köhler and Roessler in [65].

The assumption that Γ should be finite cannot hold. However, one can imagine that the kernel of ρ is of finite index in Γ. As noted by Iwaniec in [60], near the end of section 2.1, a subgroup of finite index of a fuchsian group of the first kind is itself fuchsian of the first kind, which means that one can construct a modular curve out of $\ker\rho$. Similar considerations as those made above then justify the fact that the Laplacian associated to ρ on $\Gamma\backslash\mathbb{H}$ can be thought of as an average of twisted Laplacians on the "intermediary" modular curve $\ker\rho\backslash\mathbb{H}$.

Proposition 3.1.5. *The kernel of the operator* $(\Delta - s)^{-1}$ *on* $\mathbb{H}$ *is given by*

$$k(z,z',s) \quad = \quad \sigma(z,z')^{-s} \cdot \frac{1}{4\pi} \cdot \frac{\Gamma(s)^2}{\Gamma(2s)} F\left(s,s,2s,\frac{1}{\sigma(z,z')}\right) \ ,$$

where the function σ *is defined by*

$$\sigma(z,z') \quad = \quad \frac{|z-z'|^2}{4\operatorname{Im} z \operatorname{Im} z'} \ ,$$

and F *stands for the modified Bessel functions of the second kind.*

3.1.2 Auxiliary Laplacian Δ_ε

As will be proved in this chapter, the previously constructed Laplacian Δ_E does not behave well in general, insofar as it can have an absolutely continuous spectrum. This prohibits the definition of the determinant through a ζ-regularization process. For this reason, we will not define the determinant of Δ_E, but rather the relative determinant of Δ_E and of another operator Δ_ϵ yet to be introduced. The introduction of this operator is based on [47, Sec. 2.1.3], the main difference being the higher dimension of the vector space we consider, as well as the presence of a representation. In order to be coherent with the notations used by Venkov in [97, Sec. 1.2], we will first set $V = \mathbb{C}^r$. Let p be a cusp, and γ_p be a generator of the stabilizer Γ_p of p in Γ.

Definition 3.1.6. The fixed subspace V_p of V is defined as

$$V_p \quad = \quad \{v \in V,\ \rho(\gamma_p)(v) = v\} \ .$$

Remark 3.1.7. This subspace does not depend on the choice of the generator γ_p.

Definition 3.1.8. We define the auxiliary Laplacian Δ_ε acting on the space $\mathcal{C}_0^\infty(]a(\varepsilon),+\infty[,V_p)$ of smooth, compactly supported functions in the interval $]a(\varepsilon),+\infty[$ with values in V_p to be

$$\Delta_\varepsilon \quad = \quad -y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} \ .$$

Remark 3.1.9. This operator, after having undergone the Friedrichs extension process, yields a self-adjoint positive-definite operator

$$\Delta_\varepsilon \quad : \quad H^2(]a(\varepsilon),+\infty[,V_p) \cap H_0^1(]a(\varepsilon),+\infty[,V_p) \quad \longrightarrow \quad L^2(]a(\varepsilon),+\infty[,V_p) \ ,$$

where self-adjointness is to be understood in the L^2-sense, and the intersection appearing in the domain of this operator reflects the Dirichlet boundary condition $y = a(\varepsilon)$.

Proposition 3.1.10. *The kernel of the heat operator $e^{-t\Delta_\varepsilon}$ on $]a(\varepsilon), +\infty[^2$ is given by*

$$K_\varepsilon(y_1, y_2, t) = \frac{1}{\sqrt{4\pi t}} e^{-t/4} \sqrt{y_1 y_2} \left[e^{-\log(y_1/y_2)^2/(4t)} - e^{-\left(\log(y_1 y_2) - \log\left(a(\varepsilon)^2\right)\right)^2/(4t)} \right] .$$

Remark 3.1.11. When V is of dimension 1 and the representation is trivial, this result is referenced in [47], near the beginning of section 8.1, as well as in [1], at the start of section 1.5.3, where Aldana refers the reader to either Carslaw's and Jaeger's work in [26], or, for a more detailed solution that is applicable to the current situation, to Müller's computation in [72, Sec. 6]. The proof is absolutely similar here, as can be noted after having chosen an orthonormal basis of V_p.

Definition 3.1.12. The operator Δ_ε is extended as an operator acting on functions defined on the product $\mathbb{R} \times]a(\varepsilon), +\infty[$ with values in V_p by trivial exntesion in the first variable. We further extend it to functions with values in V after defining by 0 its image on functions taking values in the orthogonal $V_p^\perp$ of V_p in V.

Definition 3.1.13. The operator Δ_ε is extended over the whole punctured modular curve Z by requiring it to act as 0 outside the considered open neighborhoods of cusps.

Remark 3.1.14. The kernel of this last Laplacian can be obtained from proposition 3.1.10, and is in particular zero except when both variables are near a certain cusp.

3.2 The continuous spectrum of the Laplacian Δ_E

The Laplacian Δ_E having an absolutely continuous spectrum, we cannot define its determinant. To remedy that, we will study this part of the spectrum further, using Eisenstein series.

3.2.1 Eisenstein series

Following [97, Sec. 3.1], and definition 1.5.3 of [44], we have the following definition.

Definition 3.2.1. Let p be a cusp, and v be a vector belonging to V_p. We define the *Eisenstein series* associated to p and v by

$$E(z, s, p, v) = \sum_{\gamma \in \Gamma_p \backslash \Gamma} \operatorname{Im}\left(g_p^{-1} \gamma z\right)^s \rho(\gamma)^{-1} v ,$$

where g_p is an element of $\mathrm{PSL}_2(\mathbb{R})$ sending both the cusp p and its stabilizer to the infinity, and the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the unit translation.

Proposition 3.2.2. *For any p and v as above, as well as any $z \in \mathbb{H}$, the function*

$$s \longmapsto E(z, s, p, v)$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1$, *has a meromorphic continuation to $\mathbb{C}$, whose potential poles' location and order do not depend on z. Moreover, this function has no pole of real part $1/2$.*

Proposition 3.2.3. *For any cusp p and any $v \in V_p$, the Eisenstein series $E(z, s, p, v)$ is absolutely convergent in the half-plane* $\operatorname{Re} s > 1$ *for any point z of the upper half-plane $\mathbb{H}$, and hence holomorphic in s there. As a function of z, it satisfies the following properties:*

1. *for every $\gamma \in \Gamma$, we have $E(\gamma z, s, p, v) = \rho(\gamma) E(z, s, p, v)$;*

2. the function $E(z,s,p,v)$ is smooth on a fundamental domain F for the action of Γ on $\mathbb{H}$;

3. we have $\Delta E(z,s,p,v) = s(1-s)\, E(z,s,p,v)$ as distributions.

Remark 3.2.4. The second condition above does not imply that Eisenstein series are smooth in z over the whole upper half-plane, as there can be problems at boundary points of F. As we will later see, these series are not square-integrable, which means that the third condition above does not mean that $s(1-s)$ is an eigenvalue of the Laplacian Δ_E, as those must be positive, and $s(1-s)$ covers a good portion of the complex plane.

Remark 3.2.5. There is an important point worth noting here. Though the Einstein series are defined relatively to a cusp, the invariance condition (the second point in the proposition above) states that they each induce functions on open neighborhoods of the other cusps. This can be summed up by saying that "cusps have a long range influence". A similar observation is made by Bunke, albeit in a different context, in [19], right before theorem 3.6.

Definition 3.2.6. Let p be a cusp. We let $e_{p,1}, \ldots, e_{p,r}$ be a basis of V which diagonalizes $\rho(\gamma_p)$, meaning that for every integer j between 1 and r, we have

$$\rho(e_{p,j}) \quad = \quad \begin{cases} e_{p,j} & \text{for} \quad j \in [\![1, k_p]\!] \\ e^{2i\pi\alpha_{p,j}} e_{p,j} & \text{for} \quad j \in [\![k_p+1, r]\!] \end{cases},$$

where each real number $\alpha_{p,j}$ belongs to $]0,1[$.

Remark 3.2.7. We will now follow closely Fischer's presentation, detailed in [44, Sec. 1.5].

Definition 3.2.8. For any cusp p, and every integer j between 1 and k_p, we set

$$\begin{array}{rccc} E_{p,j}(\cdot,s) & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & E(z,s,p,e_{p,j}) \end{array}.$$

Proposition 3.2.9. *Let p and q be two cusps, which may or may not be distinct, and j be an integer between 1 and k_p. The restricted Eisenstein series*

$$\begin{array}{ccc} \mathbb{R} \times]a(\varepsilon), +\infty[& \longrightarrow & \mathbb{C} \\ z & \longmapsto & E_{p,j}(g_q z, s) \end{array}$$

is then 1-periodic in the first variable, and its zeroth order Fourier coefficient is given by

$$\mathfrak{u}_{p,j,q}(y,s) \quad = \quad \begin{cases} \mathfrak{p}_{p,j,q}(s)\, y^{1-s} + \delta_{p,q}\, y^s\, e_{p,j} & \text{if} \quad k_p > 0 \\ 0 & \text{if} \quad k_p = 0 \end{cases},$$

for any y belonging to $]a(\varepsilon), +\infty[$ and any s a point where $E_{p,j}(z,\cdot)$ is holomorphic.

Proof. This is proposition 1.5.6 of [44], for which Fischer refers the reader to [58]. □

Remark 3.2.10. It should be noted that the Kronecker symbol $\delta_{p,q}$ is to be understood as a condition on whether or not the cusps p and q are the same.

3.2.2 Scattering matrix

Having defined the Eisenstein series associated to each cusp, we can now turn to the notion of the *scattering matrix*, which measures the influence that cusps have on each other. Once again, the notations and results that follow are taken from Fischer's work, more precisely from [44, Sec. 1.5], the only difference being that the notations for cusps and integers are interchanged from Fischer's.

Definition 3.2.11. For any two cusps p and q, and any integers $j \in [\![1, k_p]\!]$ and $l \in [\![1, k_q]\!]$, we set

$$\varphi_{(p,j),(q,l)}(s) \quad = \quad \langle e_{q,l}, \, \mathfrak{p}_{p,j,q}(s) \rangle \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical hermitian product on $V = \mathbb{C}^r$.

Definition 3.2.12. For any point of holomorphy s of every ϕ function from definition 3.2.11, we define the *scattering matrix* to be

$$\Phi(s) \quad = \quad \left[\varphi_{(p,j),(q,l)} \right] \ ,$$

where (p, j) is the row index, to be read in the lexicographical order, and (q, l) is the column index.

Remark 3.2.13. It is implicitely assumed here that the cusps of the modular curve are ordered.

Proposition 3.2.14. *For any s as above, we have*

$${}^t\Phi(\overline{s}) \quad = \quad \overline{\Phi(s)} \ .$$

Definition 3.2.15. For any s as above, the *determinant of the scattering matrix* is denoted by

$$\varphi(s) \quad = \quad \det \Phi(s) \ .$$

Proposition 3.2.16. *For any $z \in \mathbb{H}$, we have the functional equation*

$$E_{p,j}(z, 1-s) \quad = \quad \sum_{q \ cusp} \sum_{l=1}^{k_p} \varphi_{(p,j),(q,l)}(1-s) \, E_{q,l}(z, s) \ ,$$

where s is a point at which each and every one of the terms above is holomorphic.

Corollary 3.2.17. *For any s as in the last proposition, we have*

$$\Phi(1-s) \, \Phi(s) \quad = \quad I \ ,$$

where I denotes the identity matrix. Furthermore, for every $t \in \mathbb{R}$, we have

$$\left| \Phi\left(\tfrac{1}{2} + it\right) \right| \quad = \quad 1 \ .$$

3.2.3 Maaß-Selberg relations

To conclude this section on Eisenstein series, we must see how their L^2-norms behave. As already stated, these series are not square-integrable over a fundamental domain F for the action of Γ on $\mathbb{H}$, and we will therefore need to apply some truncation, in order to give meaning to an integral. For this section, we will follow Venkov's work, presented in [97, Sec. 3.2].

Definition 3.2.18. Let F be a fundamental domain for the action of Γ on $\mathbb{H}$. For any $\varepsilon > 0$, and any cusp $p \in F$, we set $F_p(\varepsilon) = g_p(\mathbb{R} \times]a(\varepsilon), +\infty[) \cap F$ and write F as the disjoint union

$$F \quad = \quad F_0(\varepsilon) \quad \sqcup \quad \bigsqcup_{\substack{p\in F \\ \text{cusp}}} F_p(\varepsilon).$$

Remark 3.2.19. This partition reflects the consideration of a small enough open neighborhood in F of every cusp belonging to the considered fundamental domain. The part "away from the cusps", denoted by $F_0(\varepsilon)$ is defined so that we have the decomposition of F stated above.

Definition 3.2.20. For any cusp p, and any $j \in [\![1, k_p]\!]$, the *truncated Eisenstein series* $\widetilde{E}_{p,j}$ is defined as

$$\widetilde{E}_{p,j,\varepsilon}(z,s) \quad = \quad \begin{cases} E_{p,j}(z,s) & \text{for} \quad z \in F_0(\varepsilon) \\ E_{p,j}(z,s) - \mathfrak{u}_{p,j,q}\left(\operatorname{Im} g_q^{-1} z, s\right) & \text{for} \quad z \in F_q(\varepsilon) \end{cases}.$$

Remark 3.2.21. As can be seen in the definition of these truncated Eisenstein series, the point is to remove the zeroth-order Fourier coefficient around each cusp, while doing nothing away from each cusp of F. The result yields a square-integrable function on F. We further have

$$\int_F \left\|\widetilde{E}_{p,j,\varepsilon}(z,s)\right\|^2 \mathrm{d}\mu(z) \quad = \quad \int_{F_0(\varepsilon)} \left\|E_{p,j}(z,s)\right\|^2 \mathrm{d}\mu(z) \;,$$

where μ denotes the measure associated to the Poincaré metric. In order to consider the integral of an Eisenstein series, truncation has to be performed, either on the domain, which results in considering $F_0(\varepsilon)$ instead of F, or on the series, which means integrating $\widetilde{E}_{p,j}$ and not $E_{p,j}$.

Theorem 3.2.22 (Maaß-Selberg relations). *Let p be a cusp and j be an integer between 1 and k_p. We have, for any non-real complex number s of real part strictly larger than $1/2$,*

$$\begin{aligned}\int_F \left\|\widetilde{E}_{p,j,\varepsilon}(z,s)\right\|^2 d\mu(z) \quad &= \quad \tfrac{1}{s+\bar{s}-1}\Big(a(\varepsilon)^{s+\bar{s}-1} - \sum_{\substack{q\in F \\ cusp}} \sum_{l=1}^{k_q} \left|\varphi_{(p,j),(q,l)}(s)\right|^2 a(\varepsilon)^{1-s-\bar{s}}\Big) \\ &\quad + \tfrac{1}{s-\bar{s}}\left(a(\varepsilon)^{s-\bar{s}} \phi_{(p,j),(p,j)}(s) - a(\varepsilon)^{\bar{s}-s} \overline{\phi_{(p,j),(p,j)}(s)}\right).\end{aligned}$$

Remark 3.2.23. The Maaß-Selberg relations can be extended to all complex number s of real part larger than or equal to $1/2$ by noting that problems on the right hand side can be deal with by interpreting the appropriate terms as difference quotients. We will state these extensions below.

Corollary 3.2.24. *For any real number $s > 1/2$, we have*

$$\int_F \left\|\widetilde{E}_{p,j,\varepsilon}(z,s)\right\|^2 d\mu(z) \quad = \quad \tfrac{1}{2s-1}\Big(a(\varepsilon)^{2s-1} - \sum_{\substack{q\in F \\ cusp}} \sum_{l=1}^{k_q} \left|\varphi_{(p,j),(q,l)}(s)\right|^2 a(\varepsilon)^{1-2s}\Big) + 2\log(a(\varepsilon))$$

Corollary 3.2.25. *For any non-zero real number r, we have*

$$\begin{aligned}&\int_F \left\|\widetilde{E}_{p,j,\varepsilon}\left(z, \frac{1}{2}+ir\right)\right\|^2 d\mu(z) \\ &\quad = \quad 2\log(a(\varepsilon)) - \sum_{\substack{q\in F \\ cusp}} \sum_{l=1}^{k_q} \varphi'_{(p,j),(q,l)}\left(\tfrac{1}{2}+ir\right) \varphi_{(q,l),(p,j)}\left(\tfrac{1}{2}-ir\right) \\ &\qquad + \tfrac{1}{2ir}\left(a(\varepsilon)^{2ir} \varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right) - a(\varepsilon)^{-2ir} \varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)\right).\end{aligned}$$

Corollary 3.2.26. *We have*

$$\int_F \left\| \widetilde{E}_{p,j,\varepsilon}\left(z, \frac{1}{2}\right) \right\|^2 \, d\mu(z) \quad = \quad 4\log\left(a\left(\varepsilon\right)\right) - \sum_{\substack{q\in F\\ cusp}} \sum_{l=1}^{k_q} \varphi'_{(p,j),(q,l)}\left(\tfrac{1}{2}+ir\right)\varphi_{(q,l),(p,j)}\left(\tfrac{1}{2}-ir\right) \; .$$

3.3 The Selberg trace formula

Now that we have stated all preliminary results required in this chapter, we can proceed to the main ingredient we will use, namely the Selberg trace formula. We will follow [44] here. In order to be consistent with the situation studied in this text, we will assume that the Fuchsian group of the first kind Γ has no elliptic element.

3.3.1 Spectral expansion for the resolvent

The first step in getting the most general version of the Selberg trace formula we will need here, we first need to have a spectral expansion theorem for the resolvent associated to the Laplacian Δ_E.

Theorem 3.3.1. *Let s be a complex number whose real part is strictly larger than 1, such that the spectrum of Δ_E does not contain $\lambda = s(1-s)$. For any function f defined on $\mathbb{H}$ which is both square-integrale over F and compatible with the representation ρ, we have*

$$\left(\Delta_E - \lambda\right)^{-1} f \quad = \quad \int_F k_{\Gamma,\rho}\left(\cdot, z, \lambda\right) f\left(z\right) \; d\mu\left(z\right) \; .$$

The function $k_{\Gamma,\rho}(\cdot,\cdot,\lambda)$ is called the resolvent kernel of Δ_E assocated to the parameter λ.

We will now use this kernel $k_{\Gamma,\rho}$, also called a *Green's function*, to give a slightly more explicit integral representation, which will separate the influence of the discrete spectrum from that of the absolutely continuous part, the latter being fully described by the Eisenstein series.

Theorem 3.3.2. *Let $\left(\lambda_j\right)_{j\geq 0}$ denote the eigenvalues of Δ_E belonging to the discrete part of its spectrum, repeated with multiplicity. We further consider a Hilbert basis $\left(f_j\right)_{j\geq 0}$ associated to this sequence. Let then s and α be two complex numbers of real part strictly larger than 1. We then have, for every point z of F,*

$$\begin{aligned}
&\sum_{j\geq 0} \left(\tfrac{1}{\lambda_j - s(1-s)} - \tfrac{1}{\lambda_j - \alpha(1-\alpha)}\right) \left|f_j\left(z\right)\right|^2 \\
&\quad = \quad (s-\alpha) \int_F \operatorname{Tr}\left(k_{\Gamma,\rho}\left(z,w,s\right) k_{\Gamma,\rho}\left(w,z,\alpha\right)\right) \; d\mu\left(w\right) \\
&\qquad\qquad - \sum_{\substack{p\in F\\ cusp}} \sum_{j=1}^{k_p} \tfrac{1}{4\pi} \int_{-\infty}^{+\infty} \left(\tfrac{1}{1/4+t^2-s} - \tfrac{1}{1/4+t^2-\alpha}\right) \left\|E_{p,j}\left(z, \tfrac{1}{2}+it\right)\right\|^2 \; dt,
\end{aligned}$$

where the series on the left hand side is absolutely convergent.

Remark 3.3.3. This theorem can be restated in the following way. For any two points z_1 and z_2 in F, we have

$$\begin{aligned}
&\sum_{j\geq 0} \left(\tfrac{1}{\lambda_j - s(1-s)} - \tfrac{1}{\lambda_j - \alpha(1-\alpha)}\right) \left\langle f_j\left(z_1\right), f_j\left(z_2\right)\right\rangle \\
&= \; - \sum_{\substack{p\in F\\ \text{cusp}}} \sum_{j=1}^{k_p} \tfrac{1}{4\pi} \int_{-\infty}^{+\infty} \left(\tfrac{1}{\frac{1}{4}+t^2-s(1-s)} - \tfrac{1}{\frac{1}{4}+t^2-\alpha(1-\alpha)}\right) \left\langle E_{p,j}\left(z_1, \tfrac{1}{2}+it\right), E_{p,j}\left(z_2, \tfrac{1}{2}+it\right)\right\rangle \; \mathrm{d}t \\
&\qquad + (s-\alpha) \int_F \operatorname{Tr}\left(k_{\Gamma,\rho}\left(z_1,w,s\right) k_{\Gamma,\rho}\left(w,z_2,\alpha\right)\right) \; \mathrm{d}\mu\left(w\right).
\end{aligned}$$

It can also be integrated over F, which makes the functions f_j disappear. We have

$$\sum_{j\geq 0}\left(\frac{1}{\lambda_j-s(1-s)}-\frac{1}{\lambda_j-\alpha(1-\alpha)}\right)$$
$$= (s-\alpha)\int_F\int_F \operatorname{Tr}\left(k_{\Gamma,\rho}(z,w,s)\,k_{\Gamma,\rho}(w,z,\alpha)\right)\,\mathrm{d}\mu(w)\,\mathrm{d}\mu(z)$$
$$-\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\frac{1}{4\pi}\int_F\int_{-\infty}^{+\infty}\left(\frac{1}{\frac14+t^2-s(1-s)}-\frac{1}{\frac14+t^2-\alpha(1-\alpha)}\right)\left\|E_{p,j}\left(z,\tfrac12+it\right)\right\|^2\,\mathrm{d}t.$$

As we can gather from this last formulation of the spectral expansion theorem, the second term can be computed by using the appropriate Maaß-Selberg relations, after having itnerchanged the two integrals. The first term will require some more work, and is the object of the next proposition.

Proposition 3.3.4. *For any s and α as before, we have*

$$(s-\alpha)\int_F \operatorname{Tr}\left(k_{\Gamma,\rho}(z,w,s)\,k_{\Gamma,\rho}(w,z,\alpha)\right)\,d\mu(w)$$
$$=\lim_{w\to z}\operatorname{Tr}\left(k_{\Gamma,\rho}(z,w,s)-k_{\Gamma,\rho}(z,w,\alpha)\right).$$

Proof. This result stems from the formulae corresponding to the appropriate resolvents.

□

Remark 3.3.5. By definition of $k_{\Gamma,\rho}$, we have

$$\lim_{w\to z}\left(k_{\Gamma,\rho}(z,w,s)-k_{\Gamma,\rho}(z,w,\alpha)\right) = \lim_{w\to z}\sum_{\gamma\in\Gamma}\operatorname{Tr}\rho(\gamma)\left(k(z,\gamma w,s)-k(z,\gamma w,\alpha)\right)$$
$$=\lim_{w\to z}\sum_{\{\gamma\}_\Gamma}\sum_{g\in\Gamma_\gamma\backslash\Gamma}\rho(\gamma)\left(k\left(z,g\gamma g^{-1}w,s\right)-k\left(z,g\gamma g^{-1}w,\alpha\right)\right),$$

where $\{\gamma\}_\Gamma$ is the conjugacy class of an element γ of Γ, and Γ_γ is the centralizer of the element γ.

Corollary 3.3.6. *For any α and α' as above, we have*

$$(s-\alpha)\int_F \operatorname{Tr}\left(k_{\Gamma,\rho}(z,w,s)\,k_{\Gamma,\rho}(w,z,\alpha)\right)\,d\mu(w)$$
$$=\lim_{w\to z}\sum_{\{\gamma\}_\Gamma}\sum_{g\in\Gamma_\gamma\backslash\Gamma}\operatorname{Tr}\rho(\gamma)\left(k\left(z,g\gamma g^{-1}w,s\right)-k\left(z,g\gamma g^{-1}w,\alpha\right)\right).$$

Remark 3.3.7. Using the last corollary, we note that the first sum can be broken into four parts: one that deals with the (conjugacy class of the) identity, another one with conjugacy classes of hyperbolic elements, and a third one with those of parabolic elements. Since we assumed Γ to be without torsion, there are elliptic elements.

Proposition 3.3.8. *For any element γ of Γ which is not the identity, the series*

$$\sum_{\{\gamma\}_\Gamma}\sum_{g\in\Gamma_\gamma\backslash\Gamma}\rho(\gamma)\,k\left(z,g\gamma g^{-1}w,\alpha\right)$$

converges uniformly as a function of w in an open neighborhood of almost (in the sense of the Poincaré measure) every z.

Remark 3.3.9. The limit in corollary 3.3.6, for any γ, can then be computed almost everywhere term by term. This is enough, as we only require an equality of square-integrable functions.

3.3.2 Contribution of the identity

We begin the study of the sums appearing in corollary 3.3.6 with the one associated to the identity. As stated before, the limit cannot be computed term by term.

Theorem 3.3.10. *The contribution from the identity in corollary 3.3.6 is given by*

$$\lim_{w\to z} \operatorname{Tr}\left(k\left(z,w,s\right)-k\left(z,w,\alpha\right)\right) \quad = \quad -\tfrac{n}{2\pi}\left(\psi\left(s\right)-\psi\left(\alpha\right)\right) \ ,$$

where ψ denotes the digamma function.

Remark 3.3.11. For reasons that will be clear later, we need to interpret the right hand side in this computation as a logarithmic derivative, after having integrated over the fundamental domain.

Proposition-Definition 3.3.12. *Denote by G the Barnes G-function. The function*

$$\begin{array}{rcccl} \Xi_I & : & \mathbb{C}\setminus\left]-\infty,0\right] & \longrightarrow & \mathbb{C} \\ & & s & \longmapsto & \exp\left[\frac{r\operatorname{Vol}F}{2\pi}\left(s\log\left(2\pi\right)+s\left(1-s\right)+\log\Gamma\left(s\right)-2\log G\left(s+1\right)\right)\right] \end{array} ,$$

is holomorphic, nowhere vanishing, and its logarithmic derivative satisfies

$$\tfrac{\Xi_I'}{\Xi_I}\left(s\right) \quad = \quad -\left(2s-1\right)\operatorname{Vol}F\cdot\tfrac{n}{2\pi}\psi\left(s\right) \ .$$

This derivative has a meromorphic continuation to $\mathbb{C}$ whose poles are located at negative integers.

Proposition 3.3.13. *The function Ξ_I satisfies the following asymptotic expansion*

$$\log\Xi_I\left(\nu\right) \quad = \quad \tfrac{r\operatorname{Vol}F}{2\pi}\left[-\nu^2\log\nu+\tfrac{1}{2}\nu^2+\nu\log\nu-\tfrac{1}{3}\log\nu+\tfrac{1}{2}\log\left(2\pi\right)-2\zeta'\left(-1\right)\right]+O\left(\tfrac{1}{\nu}\right)$$

as ν goes to infinity through strictly positive values.

Proof. Going back to the definition of Ξ_I, we have, for every real number $\nu>1$,

$$\begin{aligned} \log\Xi_I\left(\nu\right) \quad &= \quad \tfrac{r\operatorname{Vol}F}{2\pi}\Big[\cancel{\nu\log\left(2\pi\right)}+\cancel{\nu}-\nu^2+\left(\tfrac{1}{2}\log\left(2\pi\right)+\left(\nu-\tfrac{1}{2}\right)\log\nu-\cancel{\nu}+O\left(\tfrac{1}{\nu}\right)\right) \\ &\qquad -2\left(\zeta'\left(-1\right)+\cancel{\tfrac{1}{2}\nu\log\left(2\pi\right)}+\left(\tfrac{1}{2}\nu^2-\tfrac{1}{12}\right)\log\nu-\tfrac{3}{4}\nu^2+O\left(\tfrac{1}{\nu^2}\right)\right)\Big] \\ &= \quad \tfrac{r\operatorname{Vol}F}{2\pi}\left[-\nu^2\log\nu+\tfrac{1}{2}\nu^2+\nu\log\nu-\tfrac{1}{3}\log\nu+\tfrac{1}{2}\log\left(2\pi\right)-2\zeta'\left(-1\right)\right]+O\left(\tfrac{1}{\nu}\right). \end{aligned}$$

Here, we used the asymptotic expansions for the Gamma function and the Barnes G-function. ∎

3.3.3 Contribution of hyperbolic elements

We now move to the first non-trivial conjugacy class in Γ, which is that of hyperbolic elements. As we will see, this is the contribution about which we have the least information. This fact wuill lead to the definition of the so-called *Selberg zeta function*. As before, the results that follow are taken rather directly from Fischer's work. For more details, the reader is therefore refered to [44].

Proposition-Definition 3.3.14. *Let $\{\gamma\}_\Gamma$ be a hyperbolic conjugacy class in Γ. We define $N\left(\gamma\right)$ to be the only real number strictly larger than 1 such that we have*

$$\gamma \quad = \quad A^{-1}\begin{bmatrix} N\left(\gamma\right)^{1/2} & 0 \\ 0 & N\left(\gamma\right)^{-1/2} \end{bmatrix} A$$

for some matrix A in $PSL_2(\mathbb{R})$. We then define $N_0(\gamma) > 1$ to be the only real number such that

$$\gamma_0 \quad = \quad A^{-1} \begin{bmatrix} N_0(\gamma)^{1/2} & 0 \\ 0 & N_0(\gamma)^{-1/2} \end{bmatrix} A$$

generates the centralizer of γ in Γ. We call γ_0 the primitive hyperbolic element associated to γ.

Remark 3.3.15. It should be noted in the result above that we have $N(\gamma_0) = N_0(\gamma)$.

Theorem 3.3.16. *The integrated contribution of hyperbolic classes in corollary 3.3.6 is given by*

$$\begin{aligned} \sum_{\{\gamma\}_{\Gamma,hyp}} \sum_{g \in \Gamma_\gamma \backslash \Gamma} \operatorname{Tr} \rho(\gamma) \int_F \left(k\left(z, g\gamma g^{-1} z, s\right) - k\left(z, g\gamma g^{-1} z, \alpha\right)\right) \, d\mu(z) \\ = \tfrac{1}{2s-1} \sum_{\{\gamma_0\}_{\Gamma,hyp}} \sum_{m \geq 1} \operatorname{Tr} \rho(\gamma_0)^n \cdot \log N(\gamma_0) \cdot \tfrac{N(\gamma_0)^{-ms}}{1-N(\gamma_0)^{-m}} \\ -\tfrac{1}{2\alpha-1} \sum_{\{\gamma_0\}_{\Gamma,hyp}} \sum_{m \geq 1} \operatorname{Tr} \rho(\gamma_0)^n \cdot \log N(\gamma_0) \cdot \tfrac{N(\gamma_0)^{-m\alpha}}{1-N(\gamma_0)^{-m}}. \end{aligned}$$

Proposition-Definition 3.3.17. *The Selberg zeta function*

$$\begin{array}{rcccl} Z & : & \{s \in \mathbb{C}, \ \operatorname{Re} s > 1\} & \longrightarrow & \mathbb{C} \\ & & s & \longmapsto & \prod_{\{\gamma\}_{\Gamma,hyp}} \prod_{m \geq 0} \det\left(I_n - \rho(\gamma) N_0(\gamma)^{-s-m}\right) \end{array}$$

is holomorphic, and its logarithmic derivative satisfies, on this half-plane,

$$\tfrac{Z'}{Z}(s) \quad = \quad \sum_{\{\gamma_0\}_{\Gamma,hyp}} \sum_{m \geq 1} \operatorname{Tr} \rho(\gamma_0)^m \cdot \log N(\gamma_0) \cdot \tfrac{N(\gamma_0)^{-ms}}{1-N(\gamma_0)^{-m}} \quad .$$

Remark 3.3.18. It is always implicitely assume that γ_0 is the primitive hyperbolic element in the considered conjugacy class. Furthermore, the real number $N(\gamma_0)$ can be seen to be the length of the closed geodesic associated to γ_0.

Remark 3.3.19. As a consequence of the Selberg trace formula, we will see that the Selberg zeta function has a meromorphic continuation to $\mathbb{C}$, for whom 1 is a zero of order the dimension of the kernel of the hyperbolic Laplacian Δ_E.

Proposition 3.3.20. *The Selberg zeta function satisfies*

$$\log Z(\nu) \quad = \quad o(1)$$

as ν goes to infinity by strictly positive real values.

Proof. This computation is in [44], right before lemma 3.2.6. For any real number $\nu > 1$, we have

$$\log Z(\nu) \quad = \quad -\sum_{\{\gamma_0\}_{\Gamma,\text{hyp}}} \sum_{m \geq 1} \operatorname{Tr} \rho(\gamma_0)^m \cdot \tfrac{\cancel{\log N(\gamma_0)}}{1-N(\gamma_0)^{-m}} \cdot \tfrac{1}{m\cancel{\log N(\gamma_0)}} \cdot N(\gamma_0)^{-ms} \quad .$$

We can bound this logarithm, by writing

$$|\log Z(\nu)| \quad \leq \quad \sum_{\{\gamma_0\}_{\Gamma,\text{hyp}}} \sum_{m \geq 1} \underbrace{\left|\operatorname{Tr}\left(\rho(\gamma_0)^m\right)\right|}_{\leq n} \cdot \left|\tfrac{N(\gamma_0)^{-ms}}{1-N(\gamma_0)^{-1}}\right| \quad \leq \quad Cm(\Gamma)^{-\nu} \quad ,$$

where $C > 0$ is a constant, and we have defined $m(\Gamma) = \min\{N(\gamma), \ \gamma \in \Gamma \ \text{ hyperbolic}\}$, which is strictly larger than 1. This completes the proof.

$\square$

Remark 3.3.21. The crucial point above is to use either Venkov's lemma 4.4.1 from [97], or Fischer's lemma 2.2.2 from [44], which state that the number of primitive hyperbolic conjugacy classes $\{\gamma_0\}$ with $N(\gamma_0) \leq x$ is bounded by Cx with $C > 0$ a constant.

3.3.4 Contribution of parabolic elements

Having dealt with the contribution from the identity and the hyperbolic classes, we turn our attention to the parabolic classes.

Theorem 3.3.22. *The integrated contribution of parabolic elements in corollary 3.3.6 is given by*

$$\begin{aligned}
&\sum_{\{\gamma\}_{\Gamma,par}} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \operatorname{Tr}\rho(\gamma) \int_F \left(k\left(z, g\gamma g^{-1}z, s\right) - k\left(z, g\gamma g^{-1}z, \alpha\right)\right) \, d\mu(z) \\
&\quad = \sum_{\substack{p\in F \\ cusp}} \sum_{j=1}^{k_p} \tfrac{1}{4\pi} \int_F \int_{-\infty}^{+\infty} \left(\tfrac{1}{(s-1/2)^2+t^2} - \tfrac{1}{(\alpha-1/2)^2+t^2}\right) \left\|E_{p,j}\left(z, \tfrac{1}{2}+it\right)\right\|^2 \, dt \, d\mu(z) \\
&\qquad + \tfrac{1}{4\pi} \int_{-\infty}^{+\infty} \left(\tfrac{1}{(s-1/2)^2+t^2} - \tfrac{1}{(\alpha-1/2)^2+t^2}\right) \tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right) \, dt \\
&\qquad + \tfrac{1}{2s-1}\Big(-nh\log 2 - \log \prod_{\substack{p\in F \\ cusp}} \prod_{j=k_p+1}^{n} \sin(\pi\alpha_{p,j}) - k(\Gamma,\rho)\,\psi\left(s+\tfrac{1}{2}\right) \\
&\qquad\qquad + \tfrac{1}{2s-1} \operatorname{Tr}\left(I_{k(\Gamma,\rho)} - \Phi\left(\tfrac{1}{2}\right)\right)\Big) \\
&\qquad - \tfrac{1}{2\alpha-1}\Big(-nh\log 2 - \log \prod_{\substack{p\in F \\ cusp}} \prod_{j=k_p+1}^{n} \sin(\pi\alpha_{p,j}) - k(\Gamma,\rho)\,\psi\left(\alpha+\tfrac{1}{2}\right) \\
&\qquad\qquad + \tfrac{1}{2\alpha-1} \operatorname{Tr}\left(I_{k(\Gamma,\rho)} - \Phi\left(\tfrac{1}{2}\right)\right)\Big).
\end{aligned}$$

Proposition-Definition 3.3.23. *The function*

$$\begin{aligned}
\Xi_{par} \ : \ \mathbb{C} \setminus \left]-\infty, \tfrac{1}{2}\right] &\longrightarrow \mathbb{C} \\
s &\longmapsto 2^{-nhs} \prod_{\substack{p\in F \\ cusp}} \prod_{j=k_p+1}^{n} \left(\sin(\pi\alpha_{p,j})\right)^{-s} \\
&\qquad \cdot \left(s-\tfrac{1}{2}\right)^{\frac{1}{2}\operatorname{Tr}\left(I_{k(\Gamma,\rho)} - \Phi\left(\frac{1}{2}\right)\right)} \cdot \Gamma\left(s+\tfrac{1}{2}\right)^{-k(\Gamma,\rho)}
\end{aligned}$$

is holomorphic, and its logarithmic derivative satisfies, on the appropriate domain

$$\begin{aligned}
\frac{\Xi'_{par}}{\Xi_{par}}(s) &= -nh\log 2 - \log \prod_{\substack{p\in F \\ cusp}} \prod_{j=k_p+1}^{n} \sin(\pi\alpha_{p,j}) - k(\Gamma,\rho)\,\psi\left(s+\tfrac{1}{2}\right) \\
&\qquad + \tfrac{1}{2s-1} \operatorname{Tr}\left(I_{k(\Gamma,\rho)} - \Phi\left(\tfrac{1}{2}\right)\right).
\end{aligned}$$

Remark 3.3.24. One of the main differences between this section and the work of Fischer in [44] lies in the result above, as the reader will note that the function Ξ_{par} we just defined is simpler than the one used in corollary 2.4.22 of [44]. The reason for this change is that the need to express

$$\tfrac{1}{4\pi} \int_{-\infty}^{+\infty} \left(\tfrac{1}{(s-1/2)^2+t^2} - \tfrac{1}{(\alpha-1/2)^2+t^2}\right) \tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right) \, \mathrm{d}t$$

as a logarithmic derivative is not as high. It will still be needed at some point however, and we can note from corollary 2.4.22 of [44] that the term above is the logarithmic derivative of a function $\widetilde{\Xi_{\mathrm{par}}}$ defined on a domain D.

Proposition 3.3.25. *The function Ξ_{par} satisfies the following asymptotic expansion*

$$\begin{aligned}\log \Xi_{par}(\nu) &= -k(\Gamma,\rho)\,\nu\log\nu - \Big(nh\log 2 - k(\Gamma,\rho) + \sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j})\Big)\nu \\ &\quad +\tfrac{1}{2}\left(k(\Gamma,\rho) - \operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\right)\log\nu - \tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) + O\left(\tfrac{1}{\nu}\right)\end{aligned}$$

as ν goes to infinity by taking strictly positive values.

Proof. By definition, we have, for every real number $\nu > 1$,

$$\begin{aligned}\log \Xi_{\mathrm{par}}(\nu) &= -nh\log(2)\,\nu - \nu\log\Big(\prod_{\substack{p\in F\\ \text{cusp}}}\prod_{j=k_p+1}^{n}\sin(\pi\alpha_{p,j})\Big) \\ &\quad +\tfrac{1}{2}\operatorname{Tr}\left(I_{k(\Gamma,\rho)} - \Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu-\tfrac{1}{2}\right) - k(\Gamma,\rho)\log\Gamma\left(\nu+\tfrac{1}{2}\right) \\ &= -\Big(hn\log 2 + \sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j})\Big)\nu + \tfrac{1}{2}\left(k(\Gamma,\rho) - \operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\right)\log\nu \\ &\quad -k(\Gamma,\rho)\left(\tfrac{1}{2}\log 2\pi + \nu\log\nu - \nu + O\left(\tfrac{1}{\nu}\right)\right) \\ &= -k(\Gamma,\rho)\,\nu\log\nu - \Big(hn\log 2 - k(\Gamma,\rho) + \sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j})\Big)\nu \\ &\quad +\tfrac{1}{2}\left(k(\Gamma,\rho) - \operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\right)\log\nu - \tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) + O\left(\tfrac{1}{\nu}\right).\end{aligned}$$

Once again, we have here used the asymptotic expansion of the Gamma function.

□

3.3.5 Resolvent trace formula

Now that we have computed every contribution in the integrated form of corollary 3.3.6, we can put all these results together to state the following resolvent trace formula. Though it appears to be only a special case of the more general Selberg trace formula, and as noted by Fischer in [44], right before theorem 2.5.1, the two results are actually equivalent.

Theorem 3.3.26 (Resolvent trace formula)**.** *For any complex numbers s and α of real part strictly larger than 1, we have*

$$\begin{aligned}&\sum_{j\geq 0}\left(\tfrac{1}{\lambda_j - s(1-s)} - \tfrac{1}{\lambda_j-\alpha(1-\alpha)}\right) - \tfrac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\tfrac{1}{(s-1/2)^2+t^2} - \tfrac{1}{(\alpha-1/2)^2+t^2}\right)\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right)\,dt \\ &\quad= \tfrac{1}{2s-1}\left[\tfrac{\Xi_I'}{\Xi_I}(s) + \tfrac{Z'}{Z}(s) + \tfrac{\Xi_{par}'}{\Xi_{par}}(s)\right] - \tfrac{1}{2\alpha-1}\left[\tfrac{\Xi_I'}{\Xi_I}(\alpha) + \tfrac{Z'}{Z}(\alpha) + \tfrac{\Xi_{par}'}{\Xi_{par}}(\alpha)\right] \\ &\quad= -\operatorname{Vol}F\cdot\tfrac{n}{2\pi}\psi(s) + \tfrac{1}{2s-1}\cdot\tfrac{Z'}{Z}(s) - \tfrac{1}{2s-1}rh\log 2 - \tfrac{1}{2s-1}\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j}) \\ &\qquad -\tfrac{1}{2s-1}k(\Gamma,\rho)\,\psi\left(s+\tfrac{1}{2}\right) + \tfrac{1}{(2s-1)^2}\left(k(\Gamma,\rho) - \operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right) \\ &\qquad +\operatorname{Vol}F\cdot\tfrac{n}{2\pi}\psi(\alpha) - \tfrac{1}{2\alpha-1}\cdot\tfrac{Z'}{Z}(\alpha) + \tfrac{1}{2\alpha-1}rh\log 2 + \tfrac{1}{2\alpha-1}\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j}) \\ &\qquad +\tfrac{1}{2\alpha-1}k(\Gamma,\rho)\,\psi\left(\alpha+\tfrac{1}{2}\right) - \tfrac{1}{(2\alpha-1)^2}\left(k(\Gamma,\rho) - \operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\end{aligned}$$

Remark 3.3.27. The statement of the resolvent trace formula above slightly differs from the one made by Fischer in theorem 2.5.1 of [44], insofar as we chose not to explicitely write the Selberg zeta function, and the part involving the logarithmic derivative of the determinant of the scattering matrix has been left intact on the left hand side.

Proposition-Definition 3.3.28. *The function*

$$\begin{array}{cccc} \Xi & : & \{s \in \mathbb{C},\ \operatorname{Re} s > 1\} & \longrightarrow & \mathbb{C} \\ & & s & \longmapsto & \Xi_I(s) \cdot Z(s) \cdot \Xi_{par}(s) \cdot \widetilde{\Xi_{par}}(s) \end{array}$$

satisfies the following functional equation $\Xi(s) = \Xi(1-s)$. The function Ξ thus has a holomorphic continuation to the whole complex plane. Consequently, the Selberg zeta function has a meromorphic continuation to $\mathbb{C}$, which is still denoted by Z.

3.3.6 The Selberg trace formula

The aim now is to state the apparently more general version of the Selberg trace formula, as is done by Fischer in [44, Chap. 4]. This will require us to use special kinds of functions.

Definition 3.3.29. Let $\delta > 0$. We denote by $\mathcal{D}_\delta$ the space of functions

$$h \ : \ \left\{z \in \mathbb{C},\ |\operatorname{Im} z| < \tfrac{1}{2} + \delta\right\} \ \longrightarrow \ \mathbb{C} \ ,$$

which are holomorphic, and satisfy the following two conditions:

1. for every z in the appropriate domain, we have $h(-z) = h(z)$;
2. we have $|h(z)| = O\big(|\operatorname{Re} z|^{-2-\delta}\big)$ as $|\operatorname{Re} z|$ goes to infinity.

Remark 3.3.30. For any such function h, we can consider the Fourier transform

$$\begin{array}{cccc} g & : & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & & x & \longmapsto & \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(u)\, e^{-ixu}\, \mathrm{d}u \end{array} .$$

Remark 3.3.31. As is commonly done when presenting the Selberg trace formula, we will reindex the eigenvalues λ_j belonging to the discrete spectrum of the Laplacian Δ_E as

$$\lambda_j \ = \ \tfrac{1}{4} + r_j^2 \ .$$

Unlike what is done in [97], we will not consider here both real numbers r_j giving the same λ_j, which is why there is a difference by a factor 2 between this formula and what Venkov presents.

Theorem 3.3.32 (Selberg trace formula)**.** *Let $\delta > 0$ and h be function in D_δ. We have*

$$\begin{aligned} &\sum_{j \geq 0} h(r_j) - \tfrac{1}{4\pi}\int_{-\infty}^{+\infty} h(r)\ \tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2} + ir\right)\ dr \\ &= \ \operatorname{Vol} F \cdot \tfrac{n}{4\pi}\int_{-\infty}^{+\infty} rh(r)\operatorname{th}(r)\ dr - \tfrac{1}{2\pi}k(\Gamma,\rho)\int_{-\infty}^{+\infty} h(r)\,\psi(1+ir)\ dr \\ &\quad + \sum_{\{\gamma_0\}_{\Gamma,hyp}} \sum_{m=1}^{+\infty} \operatorname{Tr}\rho\left((\gamma_0)^m\right) \cdot \tfrac{\log N(\gamma_0)}{N(\gamma_0)^{m/2} - N(\gamma_0)^{-m/2}} \cdot g(\log N(\gamma)) \\ &\quad + \tfrac{1}{4}h(0)\left(k(\Gamma,\rho) - \operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\right) - g(0)\left(rh\log 2 + \sum_{\substack{p \in F \\ cusp}} \sum_{j=k_p+1}^{r} \log\sin(\pi\alpha_{p,j})\right), \end{aligned}$$

Remark 3.3.33. The resolvent trace formula can be recovered by considering for h the function

$$h(r) = \frac{1}{(s-1/2)^2+r^2} - \frac{1}{(\alpha-1/2)^2+r^2}.$$

3.4 Relative spectral zeta function

We can now study the relative spectral zeta function associated to the operators Δ_E and Δ_ε. This is done by taking advantage of the fact that zeta functions are *Mellin transforms* of heat traces. We will also adapt an argument presented by Efrat in [37].

3.4.1 Relative trace of heat operators

Theorem 3.4.1. *For every $t>0$, the difference $e^{-t\Delta_E} - e^{-t\Delta_\varepsilon}$ is trace class, and we have*

$$\begin{aligned}\operatorname{Tr}\left(e^{-t\Delta_E} - e^{-t\Delta_\varepsilon}\right) &= \sum_{j\geq 0} e^{-\left(\frac14+r_j^2\right)t} - \frac{1}{4\pi}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\operatorname{Tr}\left(\Phi'\left(\tfrac12+ir\right)\Phi\left(\tfrac12-ir\right)\right)\,dr \\ &\quad +\frac14 k(\Gamma,\rho)\,e^{-t/4} + \frac14 e^{-t/4}\operatorname{Tr}\Phi\left(\tfrac12\right) + \frac{1}{\sqrt{4\pi t}}e^{-t/4}k(\Gamma,\rho)\log a(\varepsilon).\end{aligned}$$

Proof. In order to prove this result, we would like to begin by writing the following

$$\begin{aligned}\operatorname{Tr}\left(e^{-t\Delta_E} - e^{-t\Delta_\varepsilon}\right) &= \int_F \Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\int_F \left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2 \mathrm{d}\mu(z)\,\mathrm{d}r \\ &\quad -K_\varepsilon(z,z,t)\Bigg]\,\mathrm{d}\mu(z) + \sum_{j\geq 0} e^{-\left(\frac14+r_j^2\right)t}.\end{aligned}$$

However, we would first need to prove that the difference of the heat operators is trace class, *i.e.* that what appears in the integral is integrable over F. We will prove that the following limit

$$\lim_{\eta\to 0}\int_{F_0(\eta)}\Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}r - K_\varepsilon(z,z,t)\Bigg]\mathrm{d}\mu(z)$$

exists, and actually compute it. For any real number η such that we have $0<\eta<\varepsilon$, we have

$$\begin{aligned}&\int_{F_0(\eta)}\Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}r - K_\varepsilon(z,z,t)\Bigg]\mathrm{d}\mu(z) \\ &= \sum_{\substack{q\in F\\ \text{cusp}}}\int_{F_q(\varepsilon)\setminus F_q(\eta)}\Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}r - K_\varepsilon(z,z,t)\Bigg]\mathrm{d}\mu(z) \\ &\quad +\int_{F_0(\varepsilon)}\Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}r\Bigg]\mathrm{d}\mu(z).\end{aligned}$$

We will now compute each of the integrals appearing in the first term. For any cusp q, we have

$$\begin{aligned}&\int_{F_q(\varepsilon)\setminus F_q(\eta)}\Bigg[\frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}r - K_\varepsilon(z,z,t)\Bigg]\mathrm{d}\mu(z) \\ &= \frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac14+r^2\right)t}\int_{F_q(\varepsilon)\setminus F_q(\eta)}\left\|E_{p,j}\left(z,\tfrac12+ir\right)\right\|^2\mathrm{d}\mu(z)\,\mathrm{d}r \\ &\quad -\int_{F_q(\varepsilon)\setminus F_q(\eta)} K_\varepsilon(z,z,t)\,\mathrm{d}\mu(z)\end{aligned}$$

and we can now truncate each Eisenstein series to get

$$\begin{aligned}
&\int_{F_q(\varepsilon)\setminus F_q(\eta)} \left\|E_{p,j}\left(z,\tfrac{1}{2}+ir\right)\right\|^2 \mathrm{d}\mu(z) \\
&\quad= \int_{F_q(\varepsilon)\setminus F_q(\eta)} \left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2 \mathrm{d}\mu(z) + \left(\sum_{k=1}^{k_q} \left|\varphi_{(p,j),(q,k)}\left(\tfrac{1}{2}+ir\right)\right|^2\right) \int_{a(\varepsilon)}^{a(\eta)} \tfrac{1}{y}\,\mathrm{d}y \\
&\qquad +2\delta_{p,q}\,\mathrm{Re}\left[\,\phi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)\int_{a(\varepsilon)}^{a(\eta)} y^{-1-2ir}\,\mathrm{d}y\,\right] + \delta_{p,q}\int_{a(\varepsilon)}^{a(\eta)} \tfrac{1}{y}\,\mathrm{d}y.
\end{aligned}$$

This decomposition works because the truncated Einsenstein series are L^2-orthogonal to the zeroth order coefficient in the Fourier decomposition of the full Eisenstein series. Furthermore, the limit

$$\lim_{\eta\to 0}\left[\int_{F_q(\varepsilon)\setminus F_q(\eta)} \left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2 \mathrm{d}\mu(z)\right]$$

exists. It is also important to be aware that the open neighborhood of any cusp is always thought of as a product $\mathbb{R}\times\,]a,+\infty[$, where a is associated to either ε or η in the usual manner, according to whether we consider $F_q(\varepsilon)$ or $F_q(\eta)$. The remaining integrals can be computed, and we have

$$\sum_{\substack{q\in F\\ \text{cusp}}}\sum_{k=1}^{k_q}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p} \left|\varphi_{(p,j),(q,k)}\left(\tfrac{1}{2}+ir\right)\right|^2 = \mathrm{Tr}\left(\Phi\left(1-\left(\tfrac{1}{2}+ir\right)\right)\Phi\left(\tfrac{1}{2}+ir\right)\right) = k(\Gamma,\rho)\ .$$

Moving on to the study of the auxiliary Laplacian, we note that we have

$$\int_{F_q(\varepsilon)\setminus F_q(\eta)} K_\varepsilon(z,z,t)\,\mathrm{d}\mu(z) = k_q\frac{1}{\sqrt{4\pi t}}e^{-t/4}\int_{a(\varepsilon)}^{a(\eta)} y\,\mathrm{d}\mu(z) - k_q\frac{1}{\sqrt{4\pi}}\int_0^{\frac{1}{\sqrt{t}}\log\left(\frac{a(\eta)}{a(\varepsilon)}\right)} \exp\left(-x^2\right)\mathrm{d}x.$$

The first term of the above will be compensated by others, and we have

$$\lim_{\eta\to 0}\left[k_q\frac{1}{\sqrt{4\pi}}\int_0^{\frac{1}{\sqrt{t}}\log\left(\frac{a(\eta)}{a(\varepsilon)}\right)} \exp\left(-x^2\right)\,\mathrm{d}x\right] = k_q\frac{1}{\sqrt{4\pi}}\int_0^{+\infty}\exp\left(-x^2\right)\,\mathrm{d}x = \tfrac{1}{4}k_q\ .$$

Hence, we have

$$\begin{aligned}
&\int_{F_0(\eta)}\left[\tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\left\|E_{p,j}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}r - K_\varepsilon(z,z,t)\right]\mathrm{d}\mu(z) \\
&= \tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\int_{F_0(\eta)}\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}\mu(z)\,\mathrm{d}r \\
&\quad +\tfrac{1}{4\pi}\log\left(\tfrac{a(\varepsilon)}{a(\eta)}\right)k(\Gamma,\rho)\sqrt{\tfrac{\pi}{t}}e^{-t/4} - k(\Gamma,\rho)\frac{1}{\sqrt{4\pi t}}e^{-t/4}\log\left(\tfrac{a(\varepsilon)}{a(\eta)}\right) \\
&\quad +\tfrac{1}{4\pi}\log\left(\tfrac{a(\varepsilon)}{a(\eta)}\right)k(\Gamma,\rho)\sqrt{\tfrac{\pi}{t}}e^{-t/4} + k(\Gamma,\rho)\frac{1}{\sqrt{4\pi}}e^{-t/4}\int_0^{\frac{1}{\sqrt{t}}\log\left(\frac{a(\eta)}{a(\varepsilon)}\right)}\exp\left(-x^2\right)\,\mathrm{d}x \\
&\quad +\tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\tfrac{1}{2ir}\Big[\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a(\eta)^{-2ir} \\
&\qquad\qquad -\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a(\eta)^{2ir}\Big]\,\mathrm{d}r \\
&\quad +\tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\tfrac{1}{2ir}\Big[\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a(\varepsilon)^{-2ir} \\
&\qquad\qquad -\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a(\varepsilon)^{2ir}\Big]\,\mathrm{d}r
\end{aligned}$$

It should be pointed out that the only divergent terms as η goes to 0 are those appearing on the second and third line of the last formula above, which cancel each other. This means that the auxiliary Laplacian Δ_ε has indeed the "same type" of absolutely continuous spectrum as the hyperbolic Laplacian Δ_E. We now have

$$\lim_{\eta\to 0}\left[\ \frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\int_{F_0(\eta)}\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}\mu\left(z\right)\mathrm{d}r\ \right]$$

$$= \frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\int_{F}\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}\mu\left(z\right)\mathrm{d}r,$$

which we will compute shortly using the appropriate Maaß-Selberg relations, and more precisely corollary 3.2.25. As has already been pointed out, we have

$$\lim_{\eta\to 0}\left[\ k\left(\Gamma,\rho\right)\frac{1}{\sqrt{4\pi}}e^{-t/4}\int_0^{\frac{1}{\sqrt{t}}\log\left(\frac{a(\eta)}{a(\varepsilon)}\right)}\exp\left(-x^2\right)\ \mathrm{d}x\ \right]\ =\ \frac{1}{4}k\left(\Gamma,\rho\right)e^{-t/4}\ .$$

We now need to study the following term

$$\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\frac{1}{2ir}\left[\ \varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a\left(\eta\right)^{-2ir}-\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a\left(\eta\right)^{2ir}\right]\ \mathrm{d}r$$

as η goes to 0, which will require us to use a trick presented by Müller in [72], p.274. We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\frac{1}{2ir}\left[\ \varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a\left(\eta\right)^{-2ir}-\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a\left(\eta\right)^{2ir}\right]\ \mathrm{d}r\\
&\quad=\ \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\frac{1}{2ir}\left(\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)-\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)\right)\cos\left(2r\log a\left(\eta\right)\right)\ \mathrm{d}r\\
&\qquad-\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\frac{1}{2r}\left(\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)+\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)\right)\sin\left(2r\log a\left(\eta\right)\right)\ \mathrm{d}r.
\end{aligned}$$

We can now compute the limit of the first term above and show that it vanishes, using the Riemann-Lebesgue lemma. For the second term, we use results about mollifiers, to get

$$\lim_{\eta\to 0}\left[\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\frac{1}{2ir}\left[\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a\left(\eta\right)^{-2ir}-\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a\left(\eta\right)^{2ir}\right]\mathrm{d}r\right]$$

$$=\ \pi e^{-t/4}\,\mathrm{Tr}\,\Phi\left(\tfrac{1}{2}\right).$$

This proves that the following integral exists and is given by

$$\begin{aligned}
&\int_F\ \left[\ \frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}r-K_\varepsilon\left(z,z,t\right)\ \right]\mathrm{d}\mu\left(z\right)\\
&\quad=\ \frac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\int_F\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2\mathrm{d}\mu\left(z\right)\mathrm{d}r+\frac{1}{4}k\left(\Gamma,\rho\right)e^{-t/4}\\
&\qquad+2e^{-t/4}\,\mathrm{Tr}\,\Phi\left(\tfrac{1}{2}\right)+\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\cdot\frac{1}{2ir}\left[\ \varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)a\left(\varepsilon\right)^{-2ir}\right.\\
&\qquad\qquad\left.-\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right)a\left(\varepsilon\right)^{2ir}\right]\ \mathrm{d}r.
\end{aligned}$$

As anticipated, we will compute the first term using the relevant Maaß-Selberg relations. We have

$$\begin{aligned}
&\int_F \left\| \widetilde{E}_{p,j,\varepsilon}\left(z, \tfrac{1}{2}+ir\right)\right\|^2 \mathrm{d}\mu(z) \\
&\quad = \quad 2\log(a(\varepsilon)) - \sum_{\substack{q\in F\\ \text{cusp}}} \sum_{l=1}^{k_q} \varphi'_{(p,j),(q,l)}\left(\tfrac{1}{2}+ir\right)\varphi_{(q,l),(p,j)}\left(\tfrac{1}{2}-ir\right) \\
&\qquad\qquad + \tfrac{1}{2ir}\left(a(\varepsilon)^{2ir}\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}+ir\right) - a(\varepsilon)^{-2ir}\varphi_{(p,j),(p,j)}\left(\tfrac{1}{2}-ir\right)\right).
\end{aligned}$$

We now get a more explicit version of the above formula, as we have, after having made the appropriate cancellation,

$$\begin{aligned}
&\int_F \left[\tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\left\|\widetilde{E}_{p,j,\varepsilon}\left(z,\tfrac{1}{2}+ir\right)\right\|^2 \mathrm{d}r - K_\varepsilon(z,z,t)\right]\mathrm{d}\mu(z) \\
&\quad = \quad -\tfrac{1}{4\pi}\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\sum_{\substack{q\in F\\ \text{cusp}}}\sum_{l=1}^{k_q}\varphi'_{(p,j),(q,l)}\left(\tfrac{1}{2}+ir\right)\varphi_{(q,l),(p,j)}\left(\tfrac{1}{2}-ir\right)\mathrm{d}r \\
&\qquad\qquad +\tfrac{1}{4}k(\Gamma,\rho)e^{-t/4} + \tfrac{1}{4}e^{-t/4}\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right) + \tfrac{1}{2\pi}\log(a(\varepsilon))\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=1}^{k_p}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\mathrm{d}r \\
&\quad = \quad -\tfrac{1}{4\pi}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\operatorname{Tr}\left(\Phi'\left(\tfrac{1}{2}+ir\right)\Phi\left(\tfrac{1}{2}-ir\right)\right)\mathrm{d}r \\
&\qquad\qquad +\tfrac{1}{4}k(\Gamma,\rho)e^{-t/4} + \tfrac{1}{4}e^{-t/4}\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right) + \tfrac{1}{\sqrt{4\pi t}}e^{-t/4}k(\Gamma,\rho)\log a(\varepsilon).
\end{aligned}$$

This completes the proof of the theorem.

□

Lemma 3.4.2. *For any real number r, we have*

$$\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right) \quad = \quad \operatorname{Tr}\left(\Phi'\left(\tfrac{1}{2}+ir\right)\Phi\left(\tfrac{1}{2}-ir\right)\right) \; .$$

Proof. This is a consequence of Jacobi's formula related to the computation of the derivative of the determinant of a family of matrices.

□

We note that, in this last theorem 3.4.1, we left a sum uncomputed. Fortunately, this can be resolved by using the Selberg trace formula (see theorem 3.3.32) for the function defined by

$$h(r) \quad = \quad e^{-\left(\frac{1}{4}+r^2\right)t} \; .$$

Corollary 3.4.3. *For every real number $t>0$, we have*

$$\begin{aligned}
&\operatorname{Tr}\left(e^{-t\Delta_E} - e^{-t\Delta_\varepsilon}\right) \\
&\quad = \quad \tfrac{1}{\sqrt{4\pi t}}e^{-t/4}k(\Gamma,\rho)\log a(\varepsilon) + \operatorname{Vol}F\cdot\tfrac{n}{4\pi}\int_{-\infty}^{+\infty} re^{-\left(\frac{1}{4}+r^2\right)t}\operatorname{th}(r)\; dr \\
&\qquad\qquad + \sum_{\{\gamma_0\}_{\Gamma,hyp}}\sum_{m=1}^{+\infty}\operatorname{Tr}\left(\rho(\gamma_0)^m\right)\cdot\tfrac{\log N(\gamma_0)}{N(\gamma_0)^{m/2}-N(\gamma_0)^{-m/2}}\cdot g(m\log N(\gamma_0)) \\
&\qquad\qquad -\tfrac{1}{\sqrt{4\pi t}}e^{-t/4}\left(rh\log 2 + \sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)\right) \\
&\qquad\qquad -\tfrac{1}{2\pi}k(\Gamma,\rho)\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\psi(1+ir)\; dr.
\end{aligned}$$

Proof. This corollary is obtained by plugging the Selberg trace formula (theorem 3.3.32) into theorem 3.4.1, and by noting that some terms cancel each other out.

□

3.4.2 Relative spectral zeta function

Mellin transform. The first point in this subsection is to recall a few facts about the Mellin transform, which can also be found in [7, Sec. 9.6], whose notations will be adopted in this paragraph. This particular tool will help us recover the spectral zeta function from the relative heat trace, which we have already computed.

Definition 3.4.4. Let f be smooth function on $]0,+\infty[$, which behaves well at both extremities. The *Mellin transform* of f is defined as

$$M\left[f\right](s) \quad = \quad \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1} f\left(t\right)\, \mathrm{d}t \ .$$

Remark 3.4.5. The definition above was intentionally vague as far as the hypotheses on f are concerned. We now need to see what conditions at 0 and $+\infty$ we need to impose so that this definition makes sense. These will be slightly different than those proposed by Berline, Getzler and Vergne in [7], although the proofs will be very much the same.

Proposition 3.4.6. *Assume f satisfies the following asymptotic estimate as t goes to 0^+*

$$f\left(t\right) \quad = \quad A\cdot\frac{1}{t} + B\cdot\frac{\log t}{\sqrt{t}} + C\cdot\frac{1}{\sqrt{t}} + D + O\left(\sqrt{t}\right) \ ,$$

where A, B, C, D are real constants, and satisfies the bound, as t goes to infinity,

$$\left|f\left(t\right)\right| \quad \leq \quad Ke^{-\lambda t} \ ,$$

where K and λ are strictly positive constants. The Mellin transform

$$s \quad \longmapsto \quad M\left[f\right](s)$$

is then well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1$*. Furthermore, it has a meromorphic continuation to the half-plane* $\operatorname{Re} s > -1/2$*, which is holomorphic at* 0*.*

Proof. We first note that the bound at infinity satisfied by f proves that the function

$$s \quad \longmapsto \quad \frac{1}{\Gamma(s)}\int_1^{+\infty} t^{s-1} f\left(t\right)\, \mathrm{d}t$$

is entire, that is holomorphic on $\mathbb{C}$. We will now work with an integral between 0 and 1, and we write the function f on this interval as

$$f\left(t\right) \quad = \quad A\cdot\frac{1}{t} + B\cdot\frac{\log t}{\sqrt{t}} + C\cdot\frac{1}{\sqrt{t}} + D + g\left(t\right)$$

where g is a smooth function on $[0,1]$ satisfying the bound

$$\left|g\left(t\right)\right| \quad \leq \quad K'\sqrt{t} \ ,$$

with K' being a strictly positive constant. We have, for any complex number s with $\operatorname{Re} s > 1$,

$$\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left[A\cdot\frac{1}{t} + B\cdot\frac{\log t}{\sqrt{t}} + C\cdot\frac{1}{\sqrt{t}} + D\right]\mathrm{d}t \quad = \quad \frac{1}{\Gamma(s)}\left[\frac{A}{s-1} - \frac{B}{(s-1/2)^2} + \frac{C}{s-1/2} + \frac{D}{s}\right]$$

which means that the function

$$s \quad \longmapsto \quad \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left[A \cdot \frac{1}{t} + B \cdot \frac{\log t}{\sqrt{t}} + C \cdot \frac{1}{\sqrt{t}} + D \right] \mathrm{d}t$$

is holomorphic on the half-plane $\operatorname{Re} s > 1$, and that it has a meromorphic continuation to the complex plane, which is holomorphic at 0, as the pole induced by the term D/s is canceled by the pole of $\Gamma(s)$. Finally, on the half-plane $\operatorname{Re} s > -1/2$, we have

$$\left| t^{s-1} g(t) \right| \quad \leq \quad t^{\operatorname{Re} s - 1/2} \quad ,$$

implying, using the dominated convergence theorem (in its holomorphic version), that the function

$$s \quad \longmapsto \quad \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} g(t) \, \mathrm{d}t$$

is holomorphic on every half-plane $\operatorname{Re} s > -1/2 + \delta$ for any real number $\delta > 0$, and thus on the half-plane $\operatorname{Re} s > -1/2$, which concludes the proof.

□

Example 3.4.7. For any strictly positive real number λ, the function $t \longmapsto e^{-\lambda t}$ is indeed smooth on $\mathbb{R}_+$, and satisfies both asymptotic conditions set forth in the proposition we have just seen. For any complex number s, we further have

$$\int_0^{+\infty} e^{-\lambda t} t^{s-1} \, \mathrm{d}t \quad = \quad \lambda^{-s} \int_0^{+\infty} e^{-x} x^{s-1} \, \mathrm{d}x \quad = \quad \lambda^{-s} \Gamma(s) \quad ,$$

which means that the Mellin transform of this function is given by

$$s \quad \longmapsto \quad \lambda^{-s} \quad .$$

This example illustrates why the Mellin transform allows us to recover the spectral zeta function of an operator (or the relative such function of two operators) from the heat trace (or the relative heat trace).

Asymptotics of the relative heat trace. Having recalled the definition of the Mellin transform, we now proceed to show that the relative heat trace with parameter $\mu > 0$, given by

$$\operatorname{Tr} \left(e^{-t(\Delta_E + \mu)} - e^{-t(\Delta_\varepsilon + \mu)} \right) \quad = \quad e^{-t\mu} \operatorname{Tr} \left(e^{-t\Delta_E} - e^{-t\Delta_\varepsilon} \right)$$

fits into the formalism from proposition 3.4.6. Note that it can be computed using corollary 3.4.3. The asymptotics as t goes to 0^+ were modified from those in [7] to actually reflect what we are going to find here. The versions of theorem 3.4.1 and corollary 3.4.3 using parameters are transparent, as they only involve multiplication by $e^{-t\mu}$. This factor will not play any role in the asymptotics for small time, and will be left untouched for them, as it will be needed later. We will also need to see what happens when the parameter μ equals 0, although the bound at infinity is not in general satisfied. We will need to slightly modify it, in order for the proper requirements to be met. The following definition is used so that the study that follows is the same regardless of this.

Definition 3.4.8. For any positive real number $\mu \geq 0$, we set

$$\delta_\mu \quad = \quad \dim \ker (\Delta_E + \mu) \quad .$$

Remark 3.4.9. The reader will note that we have $\delta_\mu = 0$ except (possibly) when μ equals 0. This definition is made to unify results.

We begin with the asymptotics "for large times".

Proposition 3.4.10. *The relative heat trace with parameters satisfies the following bound as t goes to infinity*

$$\left|\operatorname{Tr}\left(e^{-t(\Delta_E+\mu)} - e^{-t(\Delta_\varepsilon+\mu)}\right) - \delta_\mu\right| \quad \leq \quad Ce^{-(\beta+\mu)t} \ ,$$

where $\beta > 0$ is a strictly positive constant, independant of μ.

Proof. Using theorem 3.4.1, we have

$$\begin{aligned}
&\operatorname{Tr}\left(e^{-t(\Delta_E+\mu)} - e^{-t(\Delta_\varepsilon+\mu)}\right) - \delta_\mu \\
&\quad = \left(e^{-t\mu}\sum_{j\geq 0} e^{-\lambda_j t} - \delta_\mu\right) - \tfrac{1}{4\pi}e^{-t\mu}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\operatorname{Tr}\left(\Phi'\left(\tfrac{1}{2}+ir\right)\Phi\left(\tfrac{1}{2}-ir\right)\right)\,dr \\
&\qquad\qquad + \tfrac{1}{4}k(\Gamma,\rho)\,e^{-t(1/4+\mu)} + \tfrac{1}{4}e^{-t/4}\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right) + \tfrac{1}{\sqrt{4\pi t}}e^{-t(1/4+\mu)}k(\Gamma,\rho)\log a(\varepsilon)\,.
\end{aligned}$$

We first note that each and every one of the terms above, save for the first one, satisfy the required bound quite directly. Then, we see that we have

$$\begin{aligned}
\left|e^{-t\mu}\sum_{j\geq 0} e^{-\lambda_j t} - \delta_\mu\right| \quad &= \quad e^{-t\mu}\sum_{j\geq\delta_\mu} e^{-\lambda_j t} \quad \leq \quad e^{-t\mu}\Bigg(\sum_{j\geq\delta_\mu} e^{-(\lambda_j-\lambda_{\delta_\mu})t}\Bigg)e^{-\lambda_{\delta_\mu}t} \\
&\leq \quad e^{-(\lambda_{\delta_\mu}+\mu)t}\sum_{j\geq\delta_\mu+1} e^{-(\lambda_j-\lambda_{\delta_\mu})},
\end{aligned}$$

whenever t is larger than 1, and assuming that the sequence (λ_j) has been put in acsending order, so that $\lambda_{\delta_\mu}+\mu$ is the first non-zero eigenvalue of $\Delta_E+\mu$. This completes the proof. □

We now move to show that the relative heat trace with parameter $\mu \geq 0$ has an asymptotic expansion as t goes to 0^+ of the form presented in proposition 3.4.6. Unlike what we did above, we will not be using theorem 3.4.1 this time, but corollary 3.4.3. We will deal with each term that appears in this corollary separately, after having multiplied them by $e^{-t\mu}$. Similar results are presented by Venkov, but they will need to be (slightly) improved for our purposes.

Remark 3.4.11. In the various asymptotics as t goes to 0^+ that will follow this remark, we will leave the factor $e^{-t\mu}$ aside, and actually only perform the expansion on the relative heat trace without parameter. It must be noted that this is not an issue as far as the formalism regarding the Mellin transform is concerned. It is then implied that, when they are not explicit, the coefficients of the asymptotic expansions, as well as the remainders, are independant of μ.

Proposition 3.4.12. *As t goes to 0^+, the first term of corollary 3.4.3 satisfies*

$$\tfrac{1}{\sqrt{4\pi t}}e^{-t(1/4+\mu)}k(\Gamma,\rho)\log a(\varepsilon) \quad = \quad e^{-t\mu}\left(\tfrac{1}{2\sqrt{\pi}}k(\Gamma,\rho)\log a(\varepsilon)\cdot\tfrac{1}{\sqrt{t}} + O(t)\right) \ .$$

Proof. This proposition stems directly from the expansion at 0 of $e^{-t/4}$. □

Proposition 3.4.13. *As t goes to 0^+, the second term of corollary 3.4.3 satisfies*

$$\operatorname{Vol}F\cdot\tfrac{n}{4\pi}\int_{-\infty}^{+\infty} re^{-\left(\frac{1}{4}+\mu+r^2\right)t}\operatorname{th}(r)\ dr \quad = \quad e^{-t\mu}\left(\operatorname{Vol}F\cdot\tfrac{n}{4\pi}\cdot\tfrac{1}{t} + O(t)\right) \ .$$

Proof. We note that the core of the problem is to understand the behavior as t goes to 0^+ of

$$\int_{-\infty}^{+\infty} re^{-r^2 t}\,\mathrm{th}\,(r)\ \mathrm{d}r\ .$$

After having performed the change of variable $x = r\sqrt{t}$, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} re^{-r^2 t}\,\mathrm{th}\,(r)\ \mathrm{d}r &= \tfrac{2}{t}\int_{0}^{+\infty} xe^{-x^2}\,\mathrm{th}\left(\tfrac{x}{\sqrt{t}}\right)\ \mathrm{d}x \\
&= \tfrac{2}{t}\int_{0}^{+\infty} xe^{-x^2}\ \mathrm{d}x + \tfrac{2}{t}\int_{0}^{+\infty} xe^{-x^2}\cdot\left(\tfrac{1-e^{-2x/\sqrt{t}}}{1+e^{-2x/\sqrt{t}}}-1\right)\ \mathrm{d}x \\
&= \tfrac{1}{t}\left[-e^{-x^2}\right]_0^{+\infty} - \tfrac{4}{t}\int_{0}^{+\infty} xe^{-x^2}\tfrac{e^{-2x/\sqrt{t}}}{1+e^{-2x/\sqrt{t}}}\ \mathrm{d}x \\
&= \tfrac{1}{t} - \tfrac{4}{t}\int_{0}^{+\infty} xe^{-x^2}\tfrac{e^{-2x/\sqrt{t}}}{1+e^{-2x/\sqrt{t}}}\ \mathrm{d}x.
\end{aligned}$$

We will now prove that we have

$$\tfrac{4}{t}\int_{0}^{+\infty} xe^{-x^2}\tfrac{e^{-2x/\sqrt{t}}}{1+e^{-2x/\sqrt{t}}}\ \mathrm{d}x \quad = \quad O\,(t)$$

as t goes to 0^+. To do that, we note that a function study yields

$$\left|\tfrac{1}{t^2}e^{-2x/\sqrt{t}}e^{-x^2}\right| \quad \leq \quad \tfrac{16}{x^3}e^{-4-x^2}$$

on the interval $[1,+\infty[$. Combined to a bound on $[0,1]$, this allows us to use Lebesgue's dominated convergence theorem, which gives the required asymptotic behavior. This completes the proof. □

Proposition 3.4.14. *As t goes to 0^+, the third term of corollary 3.4.3 satisfies*

$$e^{-t\mu}\sum_{\{\gamma_0\}_{\Gamma,hyp}}\sum_{m=1}^{+\infty}\mathrm{Tr}\left(\rho\left(\gamma_0\right)^m\right)\cdot\tfrac{\log N(\gamma_0)}{N(\gamma_0)^{m/2}-N(\gamma_0)^{-m/2}}\cdot g\left(m\log N\left(\gamma_0\right)\right) \quad = \quad e^{-t\mu}O\,(t)\ .$$

Proof. Recall that the function g is defined as the Fourier transform of the function h, given by

$$g\,(u) \quad = \quad \exp\left(-\tfrac{t}{4}-\tfrac{u^2}{4t}\right)\ .$$

We now note that we have the following bound

$$\exp\left(-\tfrac{1}{4t}m^2\left(\left(\log N\left(\gamma_0\right)\right)^2-\left(\log m\left(\Gamma\right)\right)^2\right)\right) \quad \leq \quad \exp\left(-\tfrac{1}{4}m^2\left(\left(\log N\left(\gamma_0\right)\right)^2-\left(\log m\left(\Gamma\right)\right)^2\right)\right)$$

for t smaller than 1, which then gives

$$\begin{aligned}
g\left(m\log N\left(\gamma_0\right)\right) &= \exp\left(-\tfrac{t}{4}-\tfrac{1}{4t}m^2\left(\log m\left(\Gamma\right)\right)^2\right) \\
&\qquad\cdot\exp\left(-\tfrac{1}{4t}m^2\left(\left(\log N\left(\gamma_0\right)\right)^2-\left(\log m\left(\Gamma\right)\right)^2\right)\right) \\
&\leq \exp\left(-\tfrac{t}{4}-\tfrac{1}{4t}\left(\log m\left(\Gamma\right)\right)^2\right) \\
&\qquad\cdot\exp\left(-\tfrac{1}{4}m^2\left(\left(\log N\left(\gamma_0\right)\right)^2-\left(\log m\left(\Gamma\right)\right)^2\right)\right).
\end{aligned}$$

This, in turn, allows us to bound the hyperbolic contribution, and complete the proof, as we have

$$\begin{aligned}&\left|\sum_{\{\gamma_0\}_{\Gamma,\mathrm{hyp}}}\sum_{m=1}^{+\infty}\mathrm{Tr}\left(\rho\left(\gamma_0\right)^m\right)\cdot\tfrac{\log N(\gamma_0)}{N(\gamma_0)^{m/2}-N(\gamma_0)^{-m/2}}\cdot g\left(m\log N\left(\gamma_0\right)\right)\right|\\
&\quad\leq\ \exp\left(-\tfrac{t}{4}-\tfrac{1}{4t}\left(\log m\left(\Gamma\right)\right)^2\right)\sum_{\{\gamma_0\}_{\Gamma,\mathrm{hyp}}}\sum_{m=1}^{+\infty}\Big[\mathrm{Tr}\left(\rho\left(\gamma_0\right)^m\right)\tfrac{\log N(\gamma_0)}{N(\gamma_0)^{m/2}-N(\gamma_0)^{-m/2}}\\
&\qquad\qquad\cdot\exp\left(-\tfrac{1}{4}m^2\left(\left(\log N\left(\gamma_0\right)\right)^2-\left(\log m\left(\Gamma\right)\right)^2\right)\right)\Big].\end{aligned}$$

□

Proposition 3.4.15. *As t goes to 0^+, the fourth term of corollary 3.4.3 satisfies*

$$\begin{aligned}&\tfrac{1}{\sqrt{4\pi t}}e^{-t(1/4+\mu)}\Big(rh\log 2+\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)\Big)\\
&\qquad=\ e^{-t\mu}\Big[\tfrac{1}{\sqrt{4\pi t}}\Big(rh\log 2+\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)\Big)+O\left(\sqrt{t}\right)\Big].\end{aligned}$$

Proof. This proposition readily stems from the expansion of $e^{-t/4}$.

□

Proposition 3.4.16. *As t goes to 0^+, the fifth term of corollary 3.4.3 satisfies*

$$\tfrac{1}{2\pi}k\left(\Gamma,\rho\right)e^{-t\mu}\int_{-\infty}^{+\infty}e^{-\left(\frac14+r^2\right)t}\psi\left(1+ir\right)\,dr\ =\ e^{-t\mu}\left(A\cdot\tfrac{1}{\sqrt{t}}+B\cdot\tfrac{\log t}{\sqrt{t}}+C+O\left(\sqrt{t}\right)\right)\ ,$$

for some real constants A, B, and C, which are independant of μ.

Proof. We begin by using the asymptotic expansion for the digamma function, for which the reader is referred to formula 5.11.2 of [75]. We have

$$\psi\left(1+ir\right)\ =\ \log\left(1+ir\right)-\tfrac12\cdot\tfrac{1}{1+ir}+O\left(\tfrac{1}{r^2}\right)\ ,$$

as r goes to $\pm\infty$. We then have

$$\int_{-\infty}^{+\infty}e^{-\left(\frac14+r^2\right)t}\psi\left(1+ir\right)\,\mathrm{d}r\ =\ \int_{-\infty}^{+\infty}e^{-\left(\frac14+r^2\right)t}\left(\log\left(1+ir\right)-\tfrac12\cdot\tfrac{1}{1+ir}+\alpha\left(r\right)\right)\,\mathrm{d}r\ ,$$

where the function α is smooth on $\mathbb{R}$ and equals $O\left(1/r^2\right)$ as r goes to $\pm\infty$. We now treat every part of the above integral separately, starting with the remainder. We have

$$\int_{-\infty}^{+\infty}e^{-\left(\frac14+r^2\right)t}\alpha\left(r\right)\,\mathrm{d}r\ =\ e^{-t/4}\int_{-\infty}^{+\infty}\alpha\left(r\right)\,\mathrm{d}r+e^{-t/4}\int_{-\infty}^{+\infty}\left(e^{-r^2t}-1\right)\alpha\left(r\right)\,\mathrm{d}r\ .$$

We will now prove that the second integral above vanishes as t goes to 0^+. We have

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(e^{-\left(\frac14+r^2\right)t}-1\right)\alpha\left(r\right)\,\mathrm{d}r\\
&\qquad=\ e^{-t/4}\int_{\mathbb{R}\setminus]-1,1[}\left(e^{-r^2t}-1\right)\alpha\left(r\right)\,\mathrm{d}r+e^{-t/4}\int_{-1}^{1}\left(e^{-r^2t}-1\right)\alpha\left(r\right)\,\mathrm{d}r.\end{aligned}$$

The integral over $[-1,1]$ then satisfies

$$\left|\int_{-1}^{1}\left(e^{-r^2t}-1\right)\alpha\left(r\right)\,\mathrm{d}r\right|\ \leq\ 2t\int_{-1}^{1}\alpha\left(r\right)\,\mathrm{d}r\ ,$$

Using once again the bound above, this time for the integral over $\mathbb{R} \setminus]-1, 1[$, we get

$$\left|\left(e^{-r^2 t} - 1\right) \alpha\left(r\right)\right| \quad \leq \quad 2tC \cdot \tfrac{1}{r^2} \ ,$$

where $C > 0$ is a strictly positive real constant (without link to the C in the statement of the proposition). This proves that we have

$$\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \alpha\left(r\right) \, \mathrm{d}r \quad = \quad A + O\left(\sqrt{t}\right) \ ,$$

as t goes to 0^+. We can now move on to studying the integral

$$-\tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \cdot \tfrac{1}{1+ir} \, \mathrm{d}r \ .$$

We first note that, by using the change of variable $r \leftrightarrow -r$, we have

$$-\tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \cdot \tfrac{1}{1+ir} \, \mathrm{d}r \quad = \quad -\tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \cdot \tfrac{1}{1+r^2} \, \mathrm{d}r \ ,$$

and a similar argument as the one used above proves that we have

$$-\tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \cdot \tfrac{1}{1+ir} \, \mathrm{d}r \quad = \quad B + O\left(\sqrt{t}\right) \ ,$$

and the last term to be taken care of is

$$\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \log\left(1+ir\right) \, \mathrm{d}r \ .$$

Once again using the change of variable $r \leftrightarrow -r$, we see that we have

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \log\left(1+ir\right) \, \mathrm{d}r \quad &= \quad \tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t} \log\left(1+r^2\right) \, \mathrm{d}r \\ = \quad \tfrac{1}{2} \left[\int_0^1 e^{-r^2 t} \log\left(1+r^2\right) \, \mathrm{d}r + \int_1^{+\infty} e^{-r^2 t} \log\left(1+\tfrac{1}{r^2}\right) \, \mathrm{d}r + 2 \int_1^{+\infty} e^{-r^2 t} \log r \, \mathrm{d}r\right] .\end{aligned}$$

The first of these integrals can be seen to converge, as t goes to 0^+, since the integrand is uniformly bounded in t. More precisely, we have

$$\lim_{t \to 0^+} \int_0^1 e^{-r^2 t} \log\left(1+r^2\right) \, \mathrm{d}r \quad = \quad \int_0^1 \log\left(1+r^2\right) \, \mathrm{d}r + O\left(t\right) \ .$$

A similar result holds for the second integral. The last integral requires more care. We perform the change of variable $x = r^2 t$, which yields

$$\begin{aligned} 2 \int_1^{+\infty} e^{-r^2 t} \log r \, \mathrm{d}r \quad &= \quad \tfrac{1}{2\sqrt{t}} \int_t^{+\infty} \tfrac{e^{-x}}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x \\ &= \quad \tfrac{1}{2\sqrt{t}} \int_0^{+\infty} \tfrac{e^{-x}}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x + \tfrac{1}{2\sqrt{t}} \int_0^{t} \tfrac{e^{-x}}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x \end{aligned}$$

It remains to find the behavior of the second part of the above as $t > 0$ vanishes. We have

$$\begin{aligned} &\tfrac{1}{2\sqrt{t}} \int_0^{t} \tfrac{e^{-x}}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x \\ &\qquad = \quad \tfrac{1}{2\sqrt{t}} \int_0^{t} \tfrac{1}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x + \tfrac{1}{2\sqrt{t}} \int_0^{t} \tfrac{e^{-x}-1}{\sqrt{x}} \left(\log x - \log t\right) \, \mathrm{d}x, \end{aligned}$$

and an integration by parts allows us to compute the first of these two terms, as we have

$$\frac{1}{2\sqrt{t}}\int_0^t \frac{1}{\sqrt{x}}\left(\log x - \log t\right)\,\mathrm{d}x \quad = \quad \frac{1}{2\sqrt{t}}\left(\cancel{\left[2\sqrt{x}\left(\log x - \log t\right)\right]_0^t} - 2\int_0^t \frac{1}{\sqrt{x}}\,\mathrm{d}x\right) \quad = \quad -4 \ .$$

We further have

$$\left|\frac{1}{2\sqrt{t}}\int_0^t \frac{e^{-x}-1}{\sqrt{x}}\left(\log x - \log t\right)\,\mathrm{d}x\right| \quad \leq \quad \frac{1}{2\sqrt{t}}\int_0^t \sqrt{x}\left(\log t - \log x\right)\,\mathrm{d}x \quad = \quad \frac{1}{2\sqrt{t}}\cdot\frac{4}{9}t^{3/2} \quad = \quad \frac{2}{9}t \ ,$$

which then gives

$$\frac{1}{2\sqrt{t}}\int_0^t \frac{e^{-x}}{\sqrt{x}}\left(\log x - \log t\right)\,\mathrm{d}x \quad = \quad A\cdot\frac{1}{\sqrt{t}} + B\cdot\frac{\log t}{\sqrt{t}} - 4 + O\left(t\right)$$

for constants A and B which can be explicitely determined, though it is not needed. It should be noted that these constants have nothing to do with those bearing the same name which were used earlier in this proof. Putting all these results together, we have

$$\frac{1}{2\pi}k\left(\Gamma,\rho\right)e^{-t\mu}\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{4}+r^2\right)t}\psi\left(1+ir\right)\,\mathrm{d}r \quad = \quad e^{-t\mu}\left(A\cdot\frac{1}{\sqrt{t}} + B\cdot\frac{\log t}{\sqrt{t}} + C + O\left(\sqrt{t}\right)\right) \ ,$$

with A, B, and C three real constants, independant of μ, whose value is not required here. This completes the proof of the proposition.

$\square$

Proposition 3.4.17. *The relative heat trace with parameters satisfies, as t goes to 0^+, the following asymptotic expansion, with A, B, C, and D are real constants which are independant of μ,*

$$\mathrm{Tr}\left(e^{-t(\Delta_E+\mu)} - e^{-t(\Delta_\varepsilon+\mu)}\right) \quad = \quad e^{-t\mu}\left(A\cdot\frac{1}{t} + B\cdot\frac{\log t}{\sqrt{t}} + C\cdot\frac{1}{\sqrt{t}} + D + O\left(\sqrt{t}\right)\right) \ .$$

Relative spectral zeta function. The asymptotics for the relative heat trace we have proved in the last paragraph prove that the function

$$t \quad \longmapsto \quad \mathrm{Tr}\left(e^{-t(\Delta_E+\mu)} - e^{-t(\Delta_\varepsilon+\mu)}\right)$$

satisfy all hyotheses of proposition 3.4.6, which leads to the following definition.

Definition 3.4.18. The *relative spectral zeta function* associated to the Laplacians Δ_E and Δ_ε is defined for every $\mu \geq 0$ on the half-plane $\mathrm{Re}\, s > 1$ as

$$\zeta\left(\Delta_E,\Delta_\varepsilon,\mu,s\right) \quad = \quad \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\left(\mathrm{Tr}\left(e^{-t(\Delta_E+\mu)} - e^{-t(\Delta_\varepsilon+\mu)}\right) - \delta_\mu\right)\,\mathrm{d}t \ ,$$

and meromorphically extended to the half-plane $\mathrm{Re}\, s > -1/2$, which is holomorphic at 0.

Remark 3.4.19. It is worth noting here that this zeta function is smooth in μ as long as it stays away from 0. If δ_0 is different from 0, then this function is not even continuous in μ at 0.

Proposition 3.4.20. *For any complex number s whose real part is strictly larger than 1, we have*

$$\begin{aligned}\zeta\left(\Delta_E,\Delta_\varepsilon,\mu,s\right) \quad = \quad & \sum_{j\geq 0}\frac{1}{(\lambda_j+\mu)^s} - \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\cdot\frac{dr}{(1/4+r^2+\mu)^s} \\ & + \frac{1}{4}k\left(\Gamma,\rho\right)\cdot\frac{1}{(1/4+\mu)^s} + \frac{1}{4}\,\mathrm{Tr}\,\Phi\left(\frac{1}{2}\right)\cdot\frac{1}{(1/4+\mu)^s} \\ & + \frac{1}{2\sqrt{\pi}}k\left(\Gamma,\rho\right)\left(\frac{1}{4}+\mu\right)^{-s+1/2}\frac{\Gamma(s+1/2)}{(s-1/2)\Gamma(s)}\log a\left(\varepsilon\right).\end{aligned}$$

Proof. This proposition is obtained by plugging theorem 3.4.1 into definition 3.4.18. We will do explicitely part of the computation as a way to illustrate this. We have

$$\begin{aligned}
&\tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\left(\tfrac{1}{\sqrt{4\pi t}}k\left(\Gamma,\rho\right)e^{-(1/4+\mu)t}\log a\left(\varepsilon\right)\right)\,\mathrm{d}t \\
&\qquad = \tfrac{1}{2\sqrt{\pi}}\cdot\tfrac{1}{\Gamma(s)}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)\int_0^{+\infty} t^{s-3/2}e^{-(1/4+\mu)t}\,\mathrm{d}t \\
&\qquad = \tfrac{1}{2\sqrt{\pi}}\cdot\tfrac{1}{\Gamma(s)}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)\left(\tfrac{1}{4}+\mu\right)^{-s+1/2}\int_0^{+\infty} x^{(s-1/2)-1}e^{-x}\,\mathrm{d}x \\
&\qquad = \tfrac{1}{2\sqrt{\pi}}k\left(\Gamma,\rho\right)\left(\tfrac{1}{4}+\mu\right)^{-s+1/2}\tfrac{\Gamma(s+1/2)}{(s-1/2)\Gamma(s)}\log a\left(\varepsilon\right).
\end{aligned}$$

□

Remark 3.4.21. From now on, we will forget about δ_μ, which was introduced as a convenient way to regroup results valid for the cases $\mu > 0$ and $\mu = 0$. We set

$$d \quad = \quad \dim\ker\Delta_E \ ,$$

which is also, as indicated in remark 3.3.19, the order of 1 as a zero of the Selberg zeta function. The form taken by the spectral zeta function in proposition 3.4.20 is also reminiscent of the resolvent trace formula (see theorem 3.3.26), which we will now use, following a technique presented by Efrat in [37, Sec. 3]. We should point out that the notations used in theorem 3.3.26 will slightly change, as we replace the s that appears there by ν, so s can stand for the variable of the relative spectral zeta function defined above. We also set $\mu = \nu(\nu-1)$. The resolvent trace formula then reads

$$\begin{aligned}
&\sum_{j\geq 0}\left(\tfrac{1}{(\nu-1/2)^2+r_j^2}-\tfrac{1}{(\nu-1/2)^2+r_j^2}\right)-\tfrac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\tfrac{1}{(\alpha-1/2)^2+t^2}-\tfrac{1}{(\alpha-1/2)^2+t^2}\right)\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right)\,\mathrm{d}t \\
&\quad = \tfrac{1}{2\nu-1}\left[\tfrac{\Xi_I'}{\Xi_I}(\nu)+\tfrac{Z'}{Z}(\nu)+\tfrac{\Xi_{\mathrm{par}}'}{\Xi_{\mathrm{par}}}(\nu)\right]-\tfrac{1}{2\alpha-1}\left[\tfrac{\Xi_I'}{\Xi_I}(\alpha)+\tfrac{Z'}{Z}(\alpha)+\tfrac{\Xi_{\mathrm{par}}'}{\Xi_{\mathrm{par}}}(\alpha)\right]
\end{aligned}$$

where ν and α are compelx numbers of real part strictly larger than 1.

Theorem 3.4.22. *There exist real constants A and B such that we have*

$$\begin{aligned}
&-\tfrac{\partial}{\partial s}_{|s=0}\left(\sum_{j\geq 0}\tfrac{1}{\left((\nu-1/2)^2+r_j^2\right)^s}-\tfrac{1}{4\pi}\int_{-\infty}^{+\infty}\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\tfrac{dr}{\left((\nu-1/2)^2+r^2\right)^s}\right) \\
&\qquad = \log\Xi_I(\nu)+\log Z(\nu)+\log\Xi_{par}(\nu)+A\left(2\nu-1\right)^2+B,
\end{aligned}$$

where ν is a complex number of real part strictly larger than 1.

Proof. The idea behind the proof of this result is to differentiate with respect to the parameter ν. Since the appropriate functions are smooth, we will be able to interchange the various derivatives, meaning that we have

$$\begin{aligned}
&-\tfrac{\mathrm{d}}{\mathrm{d}\nu}\left[\tfrac{1}{2\nu-1}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\nu}\tfrac{\partial}{\partial s}_{|s=0}\left(\sum_{j\geq 0}\tfrac{1}{\left((\nu-1/2)^2+r_j^2\right)^s}-\tfrac{1}{4\pi}\int_{-\infty}^{+\infty}\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\tfrac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^s}\right)\right] \\
&\quad = \tfrac{\partial}{\partial s}_{|s=0}\left[\tfrac{\partial}{\partial\nu}\left(\tfrac{1}{2\nu-1}\left(-\left(2\nu-1\right)s\sum_{j\geq 0}\tfrac{1}{\left((\nu-1/2)^2+r_j^2\right)^{s+1}}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.+\left(2\nu-1\right)\tfrac{s}{4\pi}\int_{-\infty}^{+\infty}\tfrac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\tfrac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^{s+1}}\right)\right)\right].
\end{aligned}$$

We can then further compute the derivatives with respect to ν, which yields

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\frac{1}{2\nu-1}\cdot\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\partial}{\partial s}_{|s=0}\left(\sum_{j\geq 0}\frac{1}{\left((\nu-1/2)^2+r_j^2\right)^s}-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\frac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^s}\right)\right] \\
= \quad \frac{\partial}{\partial s}_{|s=0}\left[s\,(s+1)\,(2\nu-1)\sum_{j\geq 0}\frac{1}{\left((\nu-1/2)^2+r_j^2\right)^{s+2}}\right. \\
\left. -\,(2\nu-1)\,\frac{s(s+1)}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\frac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^{s+2}}\right] \\
= \quad (2\nu-1)\left[\sum_{j\geq 0}\frac{1}{\left((\nu-1/2)^2+r_j^2\right)^2}-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\frac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^2}\right].
\end{aligned}$$

This last expression can be linked to the left hand side of the resolvent trace formula, as we have

$$\begin{aligned}
\sum_{j\geq 0}\frac{1}{\left((\nu-1/2)^2+r_j^2\right)^2}-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\frac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^2} \\
= \quad \frac{1}{2\nu-1}\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\sum_{j\geq 0}\left(\frac{1}{(\nu-1/2)^2+r_j^2}-\frac{1}{(\nu-1/2)^2+r_j^2}\right)\right. \\
\left. -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{(\alpha-1/2)^2+t^2}-\frac{1}{(\alpha-1/2)^2+t^2}\right)\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+it\right)\,\mathrm{d}t\right] \\
= \quad \frac{1}{2\nu-1}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{2\nu-1}\left[\frac{\Xi_I'}{\Xi_I}(\nu)+\frac{Z'}{Z}(\nu)+\frac{\Xi_{\mathrm{par}}'}{\Xi_{\mathrm{par}}}(\nu)\right]\right)
\end{aligned}$$

This gives

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\frac{1}{2\nu-1}\cdot\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\partial}{\partial s}_{|s=0}\left(\sum_{j\geq 0}\frac{1}{\left((\nu-1/2)^2+r_j^2\right)^s}-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\varphi'}{\varphi}\left(\tfrac{1}{2}+ir\right)\cdot\frac{\mathrm{d}r}{\left((\nu-1/2)^2+r^2\right)^s}\right)\right] \\
= \quad \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{2\nu-1}\left[\frac{\Xi_I'}{\Xi_I}(\nu)+\frac{Z'}{Z}(\nu)+\frac{\Xi_{\mathrm{par}}'}{\Xi_{\mathrm{par}}}(\nu)\right]\right)
\end{aligned}$$

and integrating with respect to ν then yields the required result.

□

Remark 3.4.23. It should be noted in this proof that the parameter α does not really play a role, as a differentiation with respect to ν is considered. We have chosen to explicitely write it so as to recognize more clearly the resolvent trace formula. Although the constants often changed from one proposition to the next so far, they will now be fixed.

Corollary 3.4.24. *There exist real constants A and B such that we have*

$$\begin{aligned}
-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E,\Delta_\varepsilon,\nu\,(\nu-1)\,,s\right) \\
= \quad \log\Xi_I(\nu)+\log Z(\nu)+\log\Xi_{par}(\nu)+A\,(2\nu-1)^2+B \\
+\tfrac{1}{2}\left(k\,(\Gamma,\rho)+\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu-\tfrac{1}{2}\right)+k\,(\Gamma,\rho)\log a\,(\varepsilon)\left(\nu-\tfrac{1}{2}\right).
\end{aligned}$$

Proof. This statement is a direct consequence of theorem 3.4.22, the only computations to be made being the derivative at $s=0$ of the last three terms from proposition 3.4.20, for which one only needs to know that the Gamma function has a single pole at 0, with residue 1, and the digamma function has a single pole at 0, with residue -1.

□

We will now work on determining both constants A and B. The way to do that will be to find the asymptotic behavior as ν goes to infinity of the derivative at $s=0$ of the relative spectral zeta function in two different ways: the first one will use the corollary above, while the other one will require us to work with the asymptotics as t goes to 0^+ and as t goes to infinity of the relative heat trace, which, after undergoing the Mellin transform, will yield an asymptotic expansion as ν goes to infinity (through strictly positive real values) this time of the same zeta function, which will be far less explicit than the first one. By unicity, these two asymptotic expansions will tie together all relevant constants, and allow us to determine them fully.

Proposition 3.4.25. *The derivative at $s = 0$ of the relative spectral zeta function satisfies the following asymptotic behavior, as ν goes to infinity,*

$$\begin{aligned}
&-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E,\Delta_\varepsilon,\nu\left(\nu-1\right),s\right)\\
&\quad= -\tfrac{r\operatorname{Vol}F}{2\pi}\nu^2\log\nu+\left[\tfrac{r\operatorname{Vol}F}{4\pi}+4A\right]\nu^2+\left[\tfrac{r\operatorname{Vol}F}{2\pi}-k\left(\Gamma,\rho\right)\right]\nu\log\nu\\
&\qquad+\Bigg[k\left(\Gamma,\rho\right)-rh\log 2-\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)+k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)-4A\Bigg]\nu\\
&\qquad+\Big[k\left(\Gamma,\rho\right)-\tfrac{r\operatorname{Vol}F}{6\pi}\Big]\log\nu-\tfrac{1}{2}\log\left(2\pi\right)k\left(\Gamma,\rho\right)-\tfrac{1}{2}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)\\
&\qquad+\tfrac{r\operatorname{Vol}F}{2\pi}\left[\tfrac{1}{2}\log\left(2\pi\right)-2\zeta'\left(-1\right)\right]+A+B+o\left(1\right).
\end{aligned}$$

Proof. This computation is quite cumbersome, although one simply needs to put together results which have already been proved. As stated in corollary 3.4.24, we have

$$\begin{aligned}
&-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E,\Delta_\varepsilon,\nu\left(\nu-1\right),s\right)\\
&\quad= \log\Xi_I\left(\nu\right)+\log Z\left(\nu\right)+\log\Xi_{\text{par}}\left(\nu\right)+A\left(2\nu-1\right)^2+B\\
&\qquad+\tfrac{1}{2}\left(k\left(\Gamma,\rho\right)+\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu-\tfrac{1}{2}\right)+k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)\left(\nu-\tfrac{1}{2}\right).
\end{aligned}$$

Every asymptotic expansion in this proof will e as ν goes to infinity. Following proposition 3.3.13, the contribution of the identity satisfies the following asymptotic expansion

$$\log\Xi_I\left(\nu\right) = \tfrac{r\operatorname{Vol}F}{2\pi}\left[-\nu^2\log\nu+\tfrac{1}{2}\nu^2+\nu\log\nu-\tfrac{1}{3}\log\nu+\tfrac{1}{2}\log\left(2\pi\right)-2\zeta'\left(-1\right)\right]+O\left(\tfrac{1}{\nu}\right).$$

The contribution of the Selberg zeta function is stated in proposition 3.3.20, and we have

$$\log Z\left(\nu\right) = o\left(1\right).$$

We now recall that the contribution from parabolic elements is given by 3.3.25, and yields

$$\begin{aligned}
\log\Xi_{\text{par}}\left(\nu\right) &= -k\left(\Gamma,\rho\right)\nu\log\nu-\Bigg(nh\log 2-k\left(\Gamma,\rho\right)+\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)\Bigg)\nu\\
&\quad+\tfrac{1}{2}\left(k\left(\Gamma,\rho\right)-\operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\right)\log\nu-\tfrac{1}{2}\log\left(2\pi\right)k\left(\Gamma,\rho\right)+O\left(\tfrac{1}{\nu}\right).
\end{aligned}$$

Putting these results together, we get the proposition.

∎

As was indicated before this proposition, we will use the bound for large times of the relative heat trace with parameter $\mu>0$, which is

$$\left|\operatorname{Tr}\left(e^{-t\left(\Delta_E+\mu\right)}-e^{-t\left(\Delta_\varepsilon+\mu\right)}\right)\right| \leq Ce^{-\left(\beta+\mu\right)t},$$

where C and β are strictly positive real constants, and the asymptotics as t goes to 0^+, which read

$$\mathrm{Tr}\left(e^{-t(\Delta_E+\nu(\nu-1))}-e^{-t(\Delta_\varepsilon+\nu(\nu-1))}\right) = e^{-t\nu(\nu-1)}\left(D\cdot\tfrac{1}{t}+E\cdot\tfrac{\log t}{\sqrt{t}}+G\cdot\tfrac{1}{\sqrt{t}}+H+\xi(t)\right) ,$$

with D, E, G, H real constants independant of ν, and ξ is a smooth function on $\mathbb{R}_+$, which does not depend on the parameter ν, and satisfies

$$|\xi(t)| \leq K\sqrt{t} .$$

around 0. Following Efrat's observation in [37], proposition 1, we see that the bound at infinity for the relative heat trace show that ξ is bounded at infinity. Thus, we can bound the absolute value of $\xi(t)$ by $K\sqrt{t}$, with a possibly different constant K, still independant of ν, on $\mathbb{R}_+$.

Proposition 3.4.26. *The derivative at $s=0$ of the relative spectral zeta function satisfies the following asymptotic behavior*

$$\begin{aligned}
-\tfrac{\partial}{\partial s}_{|s=0}&\zeta(\Delta_E,\Delta_\varepsilon,\nu(\nu-1),s)\\
&= -2D\nu^2\log\nu+D\nu^2+2\left[D-2\sqrt{\pi}E\right]\nu\log\nu+2\sqrt{\pi}\left[E(2-2\log 2-\gamma)+G\right]\nu\\
&\qquad +2\left[\sqrt{\pi}E+H\right]\log\nu-\tfrac{1}{2}D+\sqrt{\pi}\left[(2\log 2+\gamma)E-G\right]+o(1).
\end{aligned}$$

Proof. We note that the relative spectral zeta function can be written as

$$\begin{aligned}
\zeta(\Delta_E,\Delta_\varepsilon,\nu(\nu-1),s) = \tfrac{1}{\Gamma(s)}\Big[&D\int_0^{+\infty} t^{s-2}e^{-t\nu(\nu-1)}\,\mathrm{d}t+E\int_0^{+\infty} t^{s-3/2}(\log t)\,e^{-t\nu(\nu-1)}\,\mathrm{d}t\\
&+G\int_0^{+\infty} t^{s-3/2}e^{-t\nu(\nu-1)}\,\mathrm{d}t+H\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\,\mathrm{d}t\\
&+\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\xi(t)\,\mathrm{d}t\Big].
\end{aligned}$$

We will begin by dealing with the term involving the remainder ξ. The function

$$s \longmapsto \tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\xi(t)\,\mathrm{d}t$$

is holomorphic around $s=0$, and its derivative at this point is given by

$$\tfrac{\partial}{\partial s}_{|s=0}\left[\tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\,\mathrm{d}t\right] = \int_0^{+\infty}\tfrac{1}{t}e^{-t\nu(\nu-1)}\xi(t)\,\mathrm{d}t ,$$

since the Gamma function has a single pole at 0, and we have

$$\begin{aligned}
\left|\int_0^{+\infty}\tfrac{1}{t}e^{-t\nu(\nu-1)}\xi(t)\,\mathrm{d}\nu\right| &\leq \int_0^{+\infty}\tfrac{1}{t}e^{-t\nu(\nu-1)}\,|\xi(t)|\,\mathrm{d}t\\
&\leq K\int_0^{+\infty}\tfrac{1}{\sqrt{t}}e^{-t\nu(\nu-1)}\,\mathrm{d}t\\
&\leq \tfrac{K}{\sqrt{\nu(\nu-1)}}\Gamma\left(\tfrac{1}{2}\right).
\end{aligned}$$

This proves that this derivative goes to 0 as ν goes to infinity through strictly positive real values. We can now move on to the other terms of the small time asymptotic expansion of the relative heat trace, which we will compute explicitely. We have

$$\tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\,\mathrm{d}t = \tfrac{1}{\Gamma(s)}\cdot(\nu(\nu-1))^{-s}\int_0^{+\infty} x^{s-1}e^{-x}\,\mathrm{d}x = (\nu(\nu-1))^{-s} .$$

This term is then indeed holomorphic around 0, and its derivative at that point equals

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}e^{-t\nu(\nu-1)}\,\mathrm{d}t\right] \quad = \quad -\log\left(\nu\left(\nu-1\right)\right) \quad = \quad -2\log\nu + o\left(1\right),$$

as ν goes to infinity. We move on to the next term. We have

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-3/2}e^{-t\nu(\nu-1)}\,\mathrm{d}t \quad &= \quad \frac{1}{\Gamma(s)}\cdot\left(\nu\left(\nu-1\right)\right)^{-s+1/2}\int_0^{+\infty} x^{s-3/2}e^{-x}\,\mathrm{d}x \\ &= \quad \frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot\left(\nu\left(\nu-1\right)\right)^{-s+1/2}.\end{aligned}$$

Here again, this term is holomorphic around 0, and its derivative there is given by

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-3/2}e^{-t\nu(\nu-1)}\,\mathrm{d}t\right] \quad = \quad \sqrt{\nu\left(\nu-1\right)}\Gamma\left(-\tfrac{1}{2}\right) \quad = \quad -2\sqrt{\pi}\left(\nu-\tfrac{1}{2}\right)+o\left(1\right)$$

as ν goes to infinity. We now deal with the next integral. We have

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-3/2}\left(\log t\right)e^{-t\nu(\nu-1)}\,\mathrm{d}t \quad = \quad \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{x^{s-3/2}}{(\nu(\nu-1))^{s-1/2}}\left(\log x-\log\left(\nu\left(\nu-1\right)\right)\right)e^{-x}\,\mathrm{d}x\ ,$$

after having performed the change of variable $x = t\nu\left(\nu-1\right)$. We further note that the integral representation of the Gamma function can be differentiated, to yield

$$\Gamma'\left(s-\tfrac{1}{2}\right) \quad = \quad \int_0^{+\infty} x^{s-3/2}\left(\log x\right)e^{-x}\,\mathrm{d}x\ ,$$

which means that we have

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-3/2}\left(\log t\right)e^{-t\nu(\nu-1)}\,\mathrm{d}t \\ &\quad = \quad \frac{1}{\Gamma(s)}\left(\nu\left(\nu-1\right)\right)^{-s+1/2}\left[\int_0^{+\infty} x^{s-3/2}\left(\log x\right)e^{-x}\,\mathrm{d}x\right. \\ &\qquad\qquad\qquad\qquad\qquad \left. -\left(\int_0^{+\infty} x^{s-3/2}e^{-x}\,\mathrm{d}x\right)\log\left(\nu\left(\nu-1\right)\right)\right] \\ &\quad = \quad \frac{1}{\Gamma(s)}\left(\nu\left(\nu-1\right)\right)^{-s+1/2}\left[\Gamma'\left(s-\tfrac{1}{2}\right)-\Gamma\left(s-\tfrac{1}{2}\right)\log\left(\nu\left(\nu-1\right)\right)\right] \\ &\quad = \quad \frac{1}{\Gamma(s)}\left(\nu\left(\nu-1\right)\right)^{-s+1/2}\Gamma\left(s-\tfrac{1}{2}\right)\left[\psi\left(s-\tfrac{1}{2}\right)-\log\left(\nu\left(\nu-1\right)\right)\right].\end{aligned}$$

This term is holomorphic around $s = 0$, and we have

$$\begin{aligned}&\frac{\partial}{\partial s}_{|s=0}\left[\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-3/2}\left(\log t\right)e^{-t\nu(\nu-1)}\,\mathrm{d}t\right] \\ &\quad = \quad \left(\nu\left(\nu-1\right)\right)^{1/2}\Gamma\left(-\tfrac{1}{2}\right)\left[\psi\left(-\tfrac{1}{2}\right)-\log\left(\nu\left(\nu-1\right)\right)\right] \\ &\quad = \quad -2\sqrt{\pi}\left(\nu\left(\nu-1\right)\right)^{1/2}\left[2-2\log 2-\gamma-\log\left(\nu\left(\nu-1\right)\right)\right] \\ &\quad = \quad -2\sqrt{\pi}\nu\left(1-\tfrac{1}{\nu}\right)^{1/2}\left[2-2\log 2-\gamma-2\log\nu-\log\left(1-\tfrac{1}{\nu}\right)\right] \\ &\quad = \quad -2\sqrt{\pi}\nu\left[1-\tfrac{1}{2}\cdot\tfrac{1}{\nu}+O\left(\tfrac{1}{\nu^2}\right)\right]\left[2-2\log 2-\gamma-2\log\nu+\tfrac{1}{\nu}+O\left(\tfrac{1}{\nu^2}\right)\right] \\ &\quad = \quad -2\sqrt{\pi}\nu\left[\left(2-2\log 2-\gamma\right)-2\log\nu+\frac{\log\nu}{\nu}+\left(\cancel{1}-\cancel{1}+\log 2+\tfrac{\gamma}{2}\right)\cdot\tfrac{1}{\nu}+O\left(\frac{\log\nu}{\nu^2}\right)\right] \\ &\quad = \quad -2\sqrt{\pi}\left[-2\nu\log\nu+\left(2-2\log 2-\gamma\right)\nu+\log\nu+\log 2+\tfrac{\gamma}{2}+o\left(1\right)\right]\end{aligned}$$

as ν goes to infinity. We finally move on to the last term. We have

$$\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-2}e^{-t\nu(\nu-1)}\mathrm{d}t \quad = \quad \frac{1}{\Gamma(s)}\left(\nu\left(\nu-1\right)\right)^{-s+1}\int_0^{+\infty} x^{s-2}e^{-x}\mathrm{d}x \quad = \quad \frac{1}{s-1}\left(\nu\left(\nu-1\right)\right)^{-s+1}.$$

This term is then holomorphic, and we have

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}\left[\frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-2}e^{-t\nu(\nu-1)}\,\mathrm{d}t\right] &= -\nu\left(\nu-1\right)+\nu\left(\nu-1\right)\log\left(\nu\left(\nu-1\right)\right)\\ &= 2\nu^2\log\nu-\nu^2-2\nu\log\nu+\tfrac{1}{2}+o\left(1\right).\end{aligned}$$

Putting these pieces together, we then find that we have

$$\begin{aligned}&-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E,\Delta_\varepsilon,\nu\left(\nu-1\right),s\right)\\ &\quad= -2D\nu^2\log\nu+D\nu^2+2\left[D-2\sqrt{\pi}E\right]\nu\log\nu+2\sqrt{\pi}\left[E\left(2-2\log2-\gamma\right)+G\right]\nu\\ &\qquad+2\left[\sqrt{\pi}E+H\right]\log\nu-\tfrac{1}{2}D+\sqrt{\pi}\left[\left(2\log2+\gamma\right)E-G\right]+o\left(1\right)\end{aligned}$$

as ν goes to infinity through strictly positive real values, which is exactly what we aimed to prove. □

Remark 3.4.27. Unlike what we saw for the small time asymptotics of the relative heat trace, it was important here to express the asymptotic expansion as ν goes to infinity in terms of the previously introduced constants, in order not to introduce any unnecessary constants.

We will now compare the asymptotic expansions for the derivative at $s=0$ of the relative spectral zeta function as ν goes to infinity through strictly positive values obtained in propositions 3.4.25 and 3.4.26, which are equal by unicity. This will allow us to determine both constants A and B in proposition 3.4.25.

Theorem 3.4.28. *The derivative at $s=0$ of the relative spectral zeta function satisfies*

$$\begin{aligned}&-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E,\Delta_\varepsilon,\nu\left(\nu-1\right),s\right)\\ &\quad= \log\Xi_I\left(\nu\right)+\log Z\left(\nu\right)+\log\Xi_{par}\left(\nu\right)+\tfrac{1}{2}\left(k\left(\Gamma,\rho\right)+\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu-\tfrac{1}{2}\right)\\ &\qquad+k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)\left(\nu-\tfrac{1}{2}\right)+\tfrac{1}{2}\log\left(2\pi\right)k\left(\Gamma,\rho\right)+\tfrac{r\operatorname{Vol}F}{2\pi}\left[2\zeta'\left(-1\right)-\tfrac{1}{2}\log2\pi\right]\\ &\qquad-\tfrac{r\operatorname{Vol}F}{8\pi}+\tfrac{1}{2}\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)+\tfrac{1}{2}rh\log2.\end{aligned}$$

Proof. From propositions 3.4.25 and 3.4.26, we infer that we have

$$\begin{aligned}&-\tfrac{r\operatorname{Vol}F}{2\pi}\nu^2\log\nu+\left[\tfrac{r\operatorname{Vol}F}{4\pi}+4A\right]\nu^2+\left[\tfrac{r\operatorname{Vol}F}{2\pi}-k\left(\Gamma,\rho\right)\right]\nu\log\nu\\ &\quad+\Big[k\left(\Gamma,\rho\right)-rh\log2-\sum_{\substack{p\in F\\ \text{cusp}}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right)+k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)-4A\Big]\nu\\ &\qquad-\tfrac{1}{2}\log\left(2\pi\right)k\left(\Gamma,\rho\right)-\tfrac{1}{2}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)+\tfrac{r\operatorname{Vol}F}{2\pi}\left[\tfrac{1}{2}\log\left(2\pi\right)-2\zeta'\left(-1\right)\right]+A+B\\ &= -2D\nu^2\log\nu+D\nu^2+2\left[D-2\sqrt{\pi}E\right]\nu\log\nu+2\sqrt{\pi}\left[E\left(2-2\log2-\gamma\right)+G\right]\nu\\ &\qquad+2\left[\sqrt{\pi}E+H\right]\log\nu-\tfrac{1}{2}D+\sqrt{\pi}\left[\left(2\log2+\gamma\right)E-G\right]+o\left(1\right),\end{aligned}$$

which, by unicity of the various coefficients, leads to several equations. The first we will use require us to identify the coefficients in front of $\nu^2 \log \nu$. It yields

$$D = \tfrac{r \operatorname{Vol} F}{4\pi} \ .$$

Using the coefficients associated to ν^2 now gives

$$D = \tfrac{r \operatorname{Vol} F}{4\pi} + 4A \ ,$$

and the value of D, which we have just determined, gives $A = 0$. Determining B is slightly more difficult, as we will need to compute both E and G first. To determine E, we identify the coefficients in front of $\nu \log \nu$, which gives

$$2D - 4\sqrt{\pi}E = \tfrac{r \operatorname{Vol} F}{2\pi} - k(\Gamma, \rho) \ ,$$

and using the value of D, which we have just computed, we get

$$E = \tfrac{1}{4\sqrt{\pi}} k(\Gamma, \rho) \ .$$

We now turn our attention to G, for which we identify the coefficents associated to ν, yielding

$$\begin{aligned} & 2\sqrt{\pi}E(2 - 2\log 2 - \gamma) + 2\sqrt{\pi}G \\ & \qquad = k(\Gamma, \rho) - rh \log 2 - \sum_{\substack{p \in F \\ \text{cusp}}} \sum_{j=k_p+1}^{r} \log \sin(\pi \alpha_{p,j}) + k(\Gamma, \rho) \log a(\varepsilon) - 4A. \end{aligned}$$

Using the values of A and E, we get

$$\begin{aligned} G = & \tfrac{1}{2\sqrt{\pi}} (k(\Gamma, \rho) - rh) \log 2 - \tfrac{1}{2\sqrt{\pi}} \sum_{\substack{p \in F \\ \text{cusp}}} \sum_{j=k_p+1}^{r} \log \sin(\pi \alpha_{p,j}) \\ & + \tfrac{1}{4\sqrt{\pi}} k(\Gamma, \rho)\gamma + \tfrac{1}{2\sqrt{\pi}} k(\Gamma, \rho) \log a(\varepsilon) \,. \end{aligned}$$

We can finally compute the most important constant, namely B. Identifying the constant coefficients above asymptotic expansion, we have

$$\begin{aligned} & -\tfrac{1}{2}D + \sqrt{\pi}\left[(2\log 2 + \gamma)E - G\right] \\ & \quad = -\tfrac{1}{2}\log(2\pi) k(\Gamma, \rho) - \tfrac{1}{2} k(\Gamma, \rho) \log a(\varepsilon) + \tfrac{r \operatorname{Vol} F}{2\pi} \left[\tfrac{1}{2}\log(2\pi) - 2\zeta'(-1)\right] + A + B \end{aligned}$$

and using the values of the various constants already computed, we get

$$\begin{aligned} B = & \tfrac{1}{2}\log(2\pi) k(\Gamma, \rho) + \tfrac{r \operatorname{Vol} F}{2\pi} \left[2\zeta'(-1) - \tfrac{1}{2}\log 2\pi - \tfrac{1}{4}\right] \\ & + \tfrac{1}{2} \sum_{\substack{p \in F \\ \text{cusp}}} \sum_{j=k_p+1}^{r} \log \sin(\pi \alpha_{p,j}) + \tfrac{rh}{2} \log 2 \end{aligned}$$

This completes the proof of the proposition.

$\square$

3.5 Relative determinant

We now have everything we need to compute the relative determinant (with parameter $\nu > 1$) associated to the operators $\Delta_E + \nu(\nu-1)$ and $\Delta_\varepsilon + \nu(\nu-1)$, defined as

$$\det\left(\Delta_E + \nu(\nu-1), \Delta_\varepsilon + \nu(\nu-1)\right) = \exp\left[-\frac{\partial}{\partial s}_{|s=0}\zeta\left(\Delta_E, \Delta_\varepsilon, \nu(\nu-1), s\right)\right].$$

Theorem 3.5.1. *For any real number $\nu > 1$, we have*

$$\begin{aligned}
&\log\det\left(\Delta_E + \nu(\nu-1), \Delta_\varepsilon + \nu(\nu-1)\right) \\
&\quad= \log \Xi_I(\nu) + \log Z(\nu) + \log \Xi_{par}(\nu) + \tfrac{1}{2}\left(k(\Gamma,\rho) + \operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu - \tfrac{1}{2}\right) \\
&\qquad + k(\Gamma,\rho)\log a(\varepsilon)\left(\nu - \tfrac{1}{2}\right) + \tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) + \tfrac{r \operatorname{Vol} F}{2\pi}\left[2\zeta'(-1) - \tfrac{1}{2}\log 2\pi\right] \\
&\qquad - \tfrac{r \operatorname{Vol} F}{8\pi} + \tfrac{1}{2}\sum_{\substack{p\in F \\ cusp}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right) + \tfrac{1}{2} r h \log 2.
\end{aligned}$$

Proof. This result is a restatement of theorem 3.4.28. $\square$

Asymptotic study. In this paragraph, we give the asymptotic expansion of the relative determinant associated to $\Delta_E + \mu$ and $\Delta_\varepsilon + \mu$, as $\mu > 0$ goes to infinity. Although this is close to the asymptotic study as ν goes to infinity we have already seen, the two are not completely the same.

Remark 3.5.2. For any real numbers $\nu > 1$ and $\mu > 0$, we have

$$\mu = \nu(\nu-1) \iff \nu = \tfrac{1+\sqrt{1+4\mu}}{2}.$$

We can then rewrite theorem 3.5.1 in the following way

$$\begin{aligned}
&\log\det\left(\Delta_E + \mu, \Delta_\varepsilon + \mu\right) \\
&\quad= \log \Xi_I\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right) + \log Z\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right) + \log \Xi_{\mathrm{par}}\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right) \\
&\qquad + \tfrac{1}{2}\left(k(\Gamma,\rho) + \tfrac{1}{2}\operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\tfrac{1}{4} + \mu\right) + k(\Gamma,\rho)\log a(\varepsilon)\sqrt{\tfrac{1}{4} + \mu} + \tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) \\
&\qquad + \tfrac{r \operatorname{Vol} F}{2\pi}\left[2\zeta'(-1) - \tfrac{1}{2}\log 2\pi\right] - \tfrac{r \operatorname{Vol} F}{8\pi} + \tfrac{1}{2}\sum_{\substack{p\in F \\ \mathrm{cusp}}}\sum_{j=k_p+1}^{r}\log\sin\left(\pi\alpha_{p,j}\right) + \tfrac{1}{2} r h \log 2.
\end{aligned}$$

We will now reformulate the asymptotic expansions (as μ goes to infinity) of the Selberg zeta function and the various Xi functions given in propositions 3.3.13, 3.3.20, 3.3.25.

Proposition 3.5.3. *As μ goes to infinity, the Selberg zeta function satisfies*

$$\log Z\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right) = o(1).$$

Proposition 3.5.4. *The contribution from the identity satisfies*

$$\log \Xi_I\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right) = \tfrac{r \operatorname{Vol} F}{2\pi}\left[-\tfrac{1}{2}\mu\log\mu + \tfrac{1}{2}\mu - \tfrac{1}{6}\log\mu + \tfrac{1}{4} + \tfrac{1}{2}\log 2\pi - 2\zeta'(-1)\right] + o(1)$$

as μ goes to infinity.

Proposition 3.5.5. *The contribution from parabolic elements satisfies*

$$\begin{aligned}
&\log \Xi_{par}\left(\tfrac{1+\sqrt{1+4\mu}}{2}\right)\\
&= -\tfrac{1}{2}k(\Gamma,\rho)\sqrt{\mu}\log\mu - \Big(hr\log 2 - k(\Gamma,\rho) + \sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j})\Big)\sqrt{\mu}\\
&\quad -\tfrac{1}{4}\operatorname{Tr}\left(\Phi\left(\tfrac{1}{2}\right)\right)\log\mu - \tfrac{1}{2}hr\log 2 - \tfrac{1}{2}\sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j})\\
&\quad -\tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) + o(1).
\end{aligned}$$

as μ goes to infinity.

Theorem 3.5.6. *As μ goes to infinity, we have the following asymptotic expansion*

$$\begin{aligned}
&\log\det(\Delta_E+\mu,\Delta_\varepsilon+\mu)\\
&= -\tfrac{r\operatorname{Vol}F}{4\pi}\mu\log\mu + \tfrac{r\operatorname{Vol}F}{4\pi}\mu - \tfrac{1}{2}k(\Gamma,\rho)\sqrt{\mu}\log\mu\\
&\quad -\Big(hr\log 2 - k(\Gamma,\rho) + \sum_{\substack{p\in F\\ cusp}}\sum_{j=k_p+1}^{r}\log\sin(\pi\alpha_{p,j}) + k(\Gamma,\rho)\log a(\varepsilon)\Big)\sqrt{\mu} + o(1).
\end{aligned}$$

Modified relative determinant. The aim of this last paragraph is to compute the modified relative determinant of the operators Δ_E and Δ_ε, defined as

$$\det{}'(\Delta_E,\Delta_\varepsilon) = \exp\left[-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E,\Delta_\varepsilon,s)\right],$$

where the relative spectral zeta function $\zeta(\Delta_E,\Delta_\varepsilon,s)$ is given on the half-plane $\operatorname{Re}s>1$ by

$$\zeta(\Delta_E,\Delta_\varepsilon,s) = \tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\left(\operatorname{Tr}\left(e^{-t\Delta_E}-e^{-t\Delta_\varepsilon}\right)-d\right)\,dt,$$

where d denotes the dimension of the kernel of Δ_E, and by its holomorphic continuation on a neighborhood of 0. The formalism of the Mellin transform presented earlier in this section applies here. We will first relate the derivative at $s=0$ of the relative spectral zeta function defined above to that of the version with parameter $\nu(\nu-1)$.

Proposition 3.5.7. *We have*

$$-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E,\Delta_\varepsilon,s) = \lim_{\nu\to1}\left[-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E,\Delta_\varepsilon,\nu(\nu-1),s) - d\log(\nu-1)\right].$$

Proof. For any real number $\nu>1$, we have

$$\begin{aligned}
&-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E,\Delta_\varepsilon,\nu(\nu-1),s) - d\log(\nu-1) - d\log\nu\\
&= -\tfrac{\partial}{\partial s}_{|s=0}\left[\zeta(\Delta_E,\Delta_\varepsilon,\nu(\nu-1),s) - d(\nu(\nu-1))^{-s}\right]\\
&= -\tfrac{\partial}{\partial s}_{|s=0}\left[\tfrac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\left(\operatorname{Tr}\left(e^{-t(\Delta_E+\nu(\nu-1))}-e^{-t(\Delta_\varepsilon+\nu(\nu-1))}\right)-de^{-t\nu(\nu-1)}\right)\,dt\right],
\end{aligned}$$

and we can take the limit as ν goes to 1, which yields the result. To actually do that, we only need to note that this limit, on the right hand side, can be interchanged with the derivative at $s=0$, and then with the integral.

□

Looking at this proposition, we see that the right hand side can be computed precisely using what we have done so far. We will first need a few preliminary results.

Lemma 3.5.8. *The* Ξ*-function associated to the identity satisfies*

$$\log \Xi_I(1) \quad = \quad \tfrac{r \operatorname{Vol} F}{2\pi} \log 2\pi \ .$$

Lemma 3.5.9. *The* Ξ*-function associated to the parabolic elements satisfies*

$$\begin{aligned} \log \Xi_{par}(1) \quad = \quad & -rh \log 2 - \sum_{\substack{p \in F \\ cusp}} \sum_{j=k_p+1}^{r} \log \sin(\pi \alpha_{p,j}) \\ & -\tfrac{1}{2}\left(k(\Gamma,\rho) - \operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log 2 + k(\Gamma,\rho)\log 2 - \tfrac{1}{2}k(\Gamma,\rho)\log \pi . \end{aligned}$$

Theorem 3.5.10. *The modified relative determinant associated to* Δ_E *and* Δ_ε *is given by*

$$\begin{aligned} & \log \det{}'(\Delta_E, \Delta_\varepsilon) \\ & \quad = \tfrac{1}{2}k(\Gamma,\rho)\log a(\varepsilon) + \log Z^{(d)}(1) + \log(d\,!) - \tfrac{1}{2}rh\log 2 + \tfrac{1}{2}k(\Gamma,\rho)\log 2 \\ & \qquad + \tfrac{r \operatorname{Vol} F}{2\pi}\left[2\zeta'(-1) + \tfrac{1}{2}\log 2\pi - \tfrac{1}{4}\right] - \tfrac{1}{2}\sum_{\substack{p \in F \\ cusp}} \sum_{j=k_p+1}^{r} \log \sin(\pi\alpha_{p,j}) . \end{aligned}$$

Proof. Using proposition 3.5.7, recall that we have

$$-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E, \Delta_\varepsilon, s) \quad = \quad \lim_{\nu \to 1}\left[-\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E, \Delta_\varepsilon, \nu(\nu-1), s) - d\log(\nu - 1)\right],$$

and we now note that we have

$$\begin{aligned} & -\tfrac{\partial}{\partial s}_{|s=0}\zeta(\Delta_E, \Delta_\varepsilon, \nu(\nu-1), s) - d\log(\nu-1) \\ & \quad = \log \Xi_I(\nu) + \log Z(\nu) + \log \Xi_{\mathrm{par}}(\nu) + \tfrac{1}{2}\left(k(\Gamma,\rho) + \operatorname{Tr}\Phi\left(\tfrac{1}{2}\right)\right)\log\left(\nu - \tfrac{1}{2}\right) \\ & \qquad + k(\Gamma,\rho)\log a(\varepsilon)\left(\nu - \tfrac{1}{2}\right) + \tfrac{1}{2}\log(2\pi)\,k(\Gamma,\rho) + \tfrac{r \operatorname{Vol} F}{2\pi}\left[2\zeta'(-1) + \tfrac{1}{2}\log 2\pi\right] \\ & \qquad - \tfrac{r \operatorname{Vol} F}{8\pi} + \tfrac{1}{2}\sum_{\substack{p \in F \\ \mathrm{cusp}}} \sum_{j=k_p+1}^{r} \log\sin(\pi\alpha_{p,j}) - d\log(\nu-1) . \end{aligned}$$

Before taking the limit as ν goes to 1, we need to remember that 1 is a zero of order d of the Selberg zeta function, which means that we have

$$\lim_{\nu \to 1}\left[\log Z(\nu) - d\log(\nu-1)\right] \quad = \quad \lim_{\nu \to 1}\log\left(\tfrac{Z(\nu)}{(\nu-1)^d}\right) \quad = \quad \log Z^{(d)}(1) - \log(d\,!) \ ,$$

which is the logarithm of the first non-zero derivative of the Selberg zeta function at 1. Every other term of the above has a well-defined limit as ν goes to 0, and we can use the lemmas stated before this theorem to get the full result.

□

Chapter 4

Determinants around a cusp

In chapter 2, we applied several analytical techniques whose aim was to study the determinant of the Dolbeault Laplacian on the modular curve, attached to the truncated metrics on the tangent bundle and on the flat vector bundle, and defined in an *ad hoc* manner. There were, however, two results that had to be left for later considerations. The first one required us to know that we had

$$\mathrm{Fp}_{\mu=+\infty} \log\det\left(\Delta_E + \mu, \Delta_{E,\mathrm{cusp},\varepsilon} + \mu\right) \quad = \quad 0 \ ,$$

where Fp stands for the "finite part", that is the constant term in an asymptotic expansion. The second one called for the computation of the relative modified determinant

$$\det{}'\left(\Delta_E, \Delta_{E,\mathrm{cusp},\varepsilon}\right) \ ,$$

at least asymptotically, as ε goes to 0^+. In order to begin both these studies using the Selberg trace formula, we had to introduce an auxiliary Laplacian Δ_ε in chapter 3. Using the fact that relative determinants, which can be thought of as "well-defined quotients of potentially ill-defined determinants", behave nicely with respect to the introduction of an operator, the two problems we were concerned with were each broken down into two pieces. What remains to be done is therefore to prove that we have

$$\mathrm{Fp}_{\mu=+\infty} \log\det\left(\Delta_{E,\mathrm{cusp},\varepsilon} + \mu, \Delta_\varepsilon + \mu\right) \quad = \quad 0$$

and to (asymptotically) compute the relative modified determinant $\det'\left(\Delta_{E,\mathrm{cusp},\varepsilon}, \Delta_\varepsilon\right)$. These problems being of a local nature around cusps, we will take advantage of the explicit description of the chosen open neighborhood $U_{p,\varepsilon}$ of a cusp p, and of the decomposition

$$E_{|U_{p,\varepsilon}} \quad = \quad \bigoplus_{j=1}^{r} L_{p,j}$$

of the vector bundle over that open subset. More precisely, we have, for any real number $\mu \geqslant 0$,

$$\begin{aligned} &\log\det\left(\Delta_{E,\mathrm{cusp},\varepsilon} + \mu, \Delta_\varepsilon + \mu\right) \\ &\qquad = \sum_{p \text{ cusp}} \left[\sum_{j=1}^{k_p} \log\det\left(\Delta_{L_{p,j},\varepsilon} + \mu, \Delta_\varepsilon + \mu\right) + \sum_{j=k_p+1}^{r} \log\det\left(\Delta_{L_{p,j},\varepsilon} + \mu\right) \right], \end{aligned}$$

with $\Delta_{L_{p,j},\varepsilon}$ being the Laplacian with Dirichlet boundary condition on $U_{p,\varepsilon}$. The fact that the auxiliary Laplacian is only needed for integers j between 1 and k_p for each cusp has already been seen in the last chapter, and will be adressed again in this one. We need one more observation. Let p be a cusp and j be an integer between 1 and r. If we have $j > k_p$, then the eigenvalues and eigenfunctions of $\Delta_{L_{p,j,\varepsilon}}$ can be found by solving the following spectral problem

$$\begin{cases} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi & = & \lambda\psi \\ \int_F |\psi|^2 & < & +\infty \\ \psi(x+1,y) & = & e^{2i\pi\alpha_{p,j}}\psi(x,y) \\ \psi(x,a(\varepsilon)) & = & 0 \end{cases},$$

where $U_{p,\varepsilon}$ has been identified to $S^1\times]a(\varepsilon),+\infty[$ and sections of $L_{p,j}$ over $U_{p,\varepsilon}$ to functions defined on $\mathbb{R}\times]a(\varepsilon),+\infty[$ which are compatible with the representation. When j lies between 1 and k_p, however, the fact that we need to consider the auxiliary Laplacian means we have to work some more to find the right spectral problem. For that, we note that we have

$$L^2(\mathbb{R}\times]a(\varepsilon),+\infty[) = L^2_0(\mathbb{R}\times]a(\varepsilon),+\infty[)\overset{\perp}{\bigoplus}L^2\left(]a(\varepsilon),+\infty[,-y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right),$$

where the first term on the right-hand side denotes the L^2-space, with the added condition

$$\int_{S^1}\psi(x,y)\ \mathrm{d}x = 0 \quad \text{for almost every } y\geqslant a\ .$$

The associated Laplacian is called the *pseudo-Laplacian*, first considered in [27, 28]. Furthermore, the 1-dimensional Laplacian attached to the second term above is precisely the auxiliary Laplacian. The appropriate relative determinant can thus be computed through the spectral problem

$$\begin{cases} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi & = & \lambda\psi \\ \int_F |\psi|^2 & < & +\infty \\ \psi(x+1,y) & = & \psi(x,y) \\ \psi(x,a(\varepsilon)) & = & 0 \\ \int_{S^1}\psi(x,y)\ \mathrm{d}x & = & 0 & \text{for almost every } y\geqslant a \end{cases}$$

with the same identifications as before. We will study both these spectral problems simultaneously.

4.1 Spectral problem around cusps

As we saw above, the point of this chapter is to study the following spectral problem

$$\begin{cases} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi & = & \lambda\psi \\ \int_F |\psi|^2 & < & +\infty \\ \psi(x+1,y) & = & e^{i\omega}\psi(x,y) \\ \psi(x,a) & = & 0 \\ \int_{S^1}\psi(x,y)\ \mathrm{d}x & = & 0 & \text{for almost every } y\geqslant a \text{ if } \omega=0 \end{cases}$$

where $a > 0$ is a strictly positive real number, and $\omega \in [0, 2\pi[$ is to be thought of as $2\pi\alpha_{p,j}$. We first note that, using the change of functions

$$\varphi(x,y) \quad = \quad e^{-i\omega x}\psi(x,y) \ ,$$

solving the spectral problem considered here is equivalent to solving

$$\begin{cases} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi & = & \left(\lambda - y^2\omega^2\right)\varphi + 2i\omega y^2\frac{\partial\varphi}{\partial x} & \\ \int_F |\varphi|^2 & < & +\infty & \\ \varphi(x+1,y) & = & \varphi(x,y) & \\ \varphi(x,a) & = & 0 & \\ \int_{S^1} \varphi(x,y)\ \mathrm{d}x & = & 0 & \text{for almost every } y \geqslant a \text{ if } \omega = 0. \end{cases}$$

By elliptic regularity, solutions to either of these problems will be smooth functions. This last reformulation is easier to work with, as its solutions are required to be periodic in the first variable. Writing such a solution as a sum of its Fourier series

$$\varphi(x,y) \quad = \quad \sum_{k\in\mathbb{Z}} a_k(y)\, e^{2ik\pi x} \ ,$$

the partial differential equation defining the spectral problem then becomes

$$\left[y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \lambda - y^2(2\pi k+\omega)^2\right] a_k(y) \quad = \quad 0$$

for every integer k, with the exception of $k = 0$ should ω vanish. This exception comes from the added condition related to the pseudo-Laplacian. For such integers and parameters, we set

$$C_{\omega,k} \quad = \quad |2\pi k+\omega| \ .$$

Taking into account the integrability over a fundamental domain, the solutions are given by

$$a_k(y) \quad = \quad y^{1/2}K_{s-1/2}(C_{\omega,k}y) \ ,$$

where the possible values for $\lambda = s(1-s)$ are determined by the boundary condition

$$K_{s-1/2}(C_{\omega,k}a) \quad = \quad 0 \ .$$

The functions K used here are known as the *modified Bessel functions of the second kind.* The reader is referred to [74, 75] for more information on them.

4.2 Zeros of modified Bessel functions of the second kind

In the last section, the spectral problem we considered was solved using modified Bessel functions of the second kind. However, the possible eigenvalues could not be precisely determined beyond the fact that they should be compatible with the boundary condition

$$K_{s-1/2}(C_{\omega,k}a) \quad = \quad 0 \ ,$$

where we have written $\lambda = s(1-s)$. Before we can get information on the spectral zeta function associated to these eigenvalues, we need to investigate their distribution across real numbers, which leads us to studying these Bessel functions in more details.

Proposition 4.2.1. *Let x be a strictly positive real number. The modified Bessel function of the second kind*

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \nu & \longmapsto & K_{i\nu}(x) \end{array},$$

seen as a function of its order, has zeros of order at most 1, all of them real (should they exist).

Proof. The argument below is an adaptation of the one given by Saharian in [82]. We begin by noting that, for real numbers t and u, with $u > 0$, we have $K_t(u) \in \mathbb{R}$. The Schwarz reflexion principle, coupled with known properties of Bessel functions, then states that we have

$$\overline{K_{i\nu}(u)} \quad = \quad K_{-i\overline{\nu}}(u) \quad = \quad K_{i\overline{\nu}}(u) \ .$$

Using the differential equation satisfied by these Bessel functions, we now have

$$\left\{ \begin{array}{rcl} uK'_{i\overline{\nu}}(u) & = & \left(u^2 - \overline{\nu}^2\right) K_{i\overline{\nu}}(u) - u^2 K''_{i\overline{\nu}}(u) \\ uK'_{i\nu}(u) & = & \left(u^2 - \nu^2\right) K_{i\nu}(u) - u^2 K''_{i\nu}(u) \end{array} \right. ,$$

where it should be noted that differentiation of the Bessel functions is taken with respect to the argument, and not the order. Hence, we have

$$\begin{array}{l} u\left[K_{i\nu}(u) K'_{i\overline{\nu}}(u) - K_{i\overline{\nu}}(u) K'_{i\nu}(u)\right] \\ \quad = \quad \left(u^2 - \overline{\nu}^2\right) K_{i\nu}(u) K_{i\overline{\nu}}(u) - u^2 K_{i\nu}(u) K''_{i\overline{\nu}}(u) - \left(u^2 - \nu^2\right) K_{i\overline{\nu}}(u) K_{i\nu}(u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + u^2 K_{i\overline{\nu}}(u) K''_{i\nu}(u) \\ \quad = \quad \left(\nu^2 - \overline{\nu}^2\right) K_{i\nu}(u) K_{i\overline{\nu}}(u) + u^2\left[K_{i\overline{\nu}}(u) K''_{i\nu}(u) - K_{i\nu}(u) K_{i\overline{\nu}}(u)\right]. \end{array}$$

If the complex number ν is neither real nor purely imaginary, we have, for any $v > 0$,

$$\begin{array}{rcl} \frac{1}{v} K_{i\nu}(v) K_{i\overline{\nu}}(v) & = & -\frac{1}{\overline{\nu}^2 - \nu^2}\left[K_{i\nu}(v) K'_{i\overline{\nu}}(v) - K_{i\overline{\nu}}(v) K'_{i\nu}(v)\right] \\ & & \qquad + \frac{v}{\overline{\nu}^2 - \nu^2}\left[K_{i\overline{\nu}}(v) K''_{i\nu}(v) - K_{i\nu}(v) K_{i\overline{\nu}}(v)\right]. \end{array}$$

Integrating for v between 0 and u, and then integrating by parts yields

$$\begin{array}{rcl} \int_0^u \frac{1}{v}\left|K_{i\nu}(v)\right|^2 \,\mathrm{d}v & = & \int_0^u \frac{1}{v} K_{i\nu}(v) K_{i\overline{\nu}}(v) \,\mathrm{d}v \\ & = & -\frac{u}{\overline{\nu}^2 - \nu^2}\left[K_{i\nu}(u) K'_{i\overline{\nu}}(u) - K_{i\overline{\nu}}(u) K'_{i\nu}(u)\right]. \end{array}$$

If ν was a complex number such that we had $\overline{\nu}^2 - \nu^2 \neq 0$, *i.e.* not real or purely imaginary, and such that we had $K_{i\nu}(u) = 0$, then, using previously mentioned symmetries and the equality above, we would have

$$\int_0^u \frac{1}{v}\left|K_{i\nu}(v)\right|^2 \,\mathrm{d}v \quad = \quad -\frac{u}{\overline{\nu}^2 - \nu^2}\left[K_{i\nu}(u) K'_{i\overline{\nu}}(u) - K_{i\overline{\nu}}(u) K'_{i\nu}(u)\right] \quad = \quad 0 \ .$$

This would be absurd, since the Bessel function, seen as a function of its order, is entire and non-zero. Hence, the complex number ν such that we have $K_{i\nu}(u) = 0$ can only be real or purely imaginary. However, they cannot be purely imaginary, by known properties of the Bessel functions.

Therefore, we are left with a discrete set of real numbers ν such that we have $K_{i\nu}(u) = 0$. We now want to prove that these zeros, if they exist, can only be simple. To do that, we go back to the formula above, which holds away from both axes, and we apply Taylor-Young's formula to compute the limit of the right-hand side as ν goes to a non-zero real number. We get

$$\int_0^u \frac{1}{v} |K_{i\nu}(v)|^2 \, \mathrm{d}v \quad = \quad \frac{u}{2i\nu} \left[K_{i\nu}(u) \frac{\partial}{\partial \alpha}_{|\alpha = i\nu} K'_\alpha(u) - K'_{i\nu}(u) \frac{\partial}{\partial \alpha}_{|\alpha = i\nu} K_\alpha(u) \right] \; .$$

If ν was a zero of order at least 2, then the right-hand side above would vanish, which would be absurd by the same argument as before. This completes the proof of the proposition.

□

Remark 4.2.2. In the context provided by the last section, for any integer k, including $k = 0$ unless ω vanishes, the function

$$s \quad \longmapsto \quad K_{s-1/2}(C_{\omega,k} a)$$

can only vanish on a discrete subset of $\{1/2 + ir, \; r \in \mathbb{R}^*\}$, and these zeros cannot be more than simple. This shows that the spectral problem we consider gives rise to a discrete set of eigenvalues which can be written as

$$\lambda_{k,j} \quad = \quad \frac{1}{4} + r_{k,j}^2 \; ,$$

where $r_{k,j}$ is a strictly positive real number. However, we lack information on the multiplicities of these eigenvalues.

4.3 Weyl's law

Having obtained the first important result regarding the distribution the eigenvalues for the spectral problem we consider, we turn our attention to the second step, which is knowing how these eigenvalues asymptotically behave with respect to k and j. This will enable us to define the associated spectral zeta function on a certain half-plane, which is essential before we can properly study its properties. In the following, we will need to assume that ω can be written as

$$\omega \quad = \quad 2\pi \frac{m}{l} \; ,$$

where m and l are both non-zero integers, and l is non-zero. This condition will in practice be satisfied, using the "finite monodromy at the cusps" hypothesis. We will see that the apparently added condition in the case $\omega = 0$ is actually implicitely present in every situation. Consider a solution ψ to the spectral problem in its first formulation. We have, for almost every $y \geqslant a$,

$$\int_0^l \psi(x,y) \, \mathrm{d}x \quad = \quad \sum_{t=0}^{l-1} \int_t^{t+1} \psi(x,y) \, \mathrm{d}x \quad = \quad \sum_{t=0}^{l-1} \left(e^{2i\pi \frac{m}{l}} \right)^t \int_0^1 \psi(x,y) \, \mathrm{d}x \quad = \quad 0 \; .$$

The fact that this equals zero is the added condition if ω vanishes, and is automatic otherwise. The eigenvalues associated to the spectral problem we consider are then in particular eigenvalues of the pseudo-Laplacian with Dirichlet boundary condition on $[0,n] \times [a, +\infty[$, with periodicity in the first variable. Identifying $[0,n]$ with periodicity to S^1, we are led to prove that the pseudo-Laplacian with Dirichlet boundary condition on $S^1 \times [a, +\infty[$ has no essential spectrum, and that the eigenvalue counting function satisfies a Weyl-type bound. For that, we will use a variation of Colin de Verdière's arguments, presented in [27, 28]. We will need to use both the pseudo-Laplacian with Dirichlet and with Neumann boundary conditions. Let us quickly review how they are defined in this precise situation, using some of the language from [24, Sec. I.5]. In the following, we denote by Λ an interval included in $[a, +\infty[$.

Definition 4.3.1. The *pseudo-Laplacian with Neumann boundary condition* H_Λ^N on $S^1 \times \Lambda$ is defined as the self-adjoint positive-definite operator associated to the sesquilinear form

$$Q_\Lambda^N \quad : \quad (u, v) \quad \longrightarrow \quad \int_{S^1 \times \Lambda} \nabla u\,(x, y)\, \overline{\nabla v\,(x, y)}\, \mathrm{d}x\, \mathrm{d}y$$

defined on the following domain

$$\mathcal{D}\left(Q_\Lambda^N\right) \quad = \quad \left\{f \in H^1\left(S^1 \times \Lambda\right),\ \pi_* f = 0\right\}\ .$$

Here, we have denoted by $\pi : S^1 \times \Lambda \longrightarrow \Lambda$ the canonical projection, and by $\pi_* f$ the constant coefficient in the Fourier expansion of f with respect to the first variable.

Definition 4.3.2. The *pseudo-Laplacian with Dirichlet boundary condition* H_Λ^D on $S^1 \times \Lambda$ is defined as the self-adjoint positive-definite operator associated to the closure of

$$Q_\Lambda^D \quad : \quad (u, v) \quad \longrightarrow \quad \int_{S^1 \times \Lambda} \nabla u\,(x, y)\, \overline{\nabla v\,(x, y)}\, \mathrm{d}x\, \mathrm{d}y$$

defined on the following domain

$$\mathcal{D}\left(Q_\Lambda^D\right) \quad = \quad \left\{f \in \mathcal{C}_0^\infty\left(S^1 \times \Lambda\right),\ \pi_* f = 0\right\}\ .$$

Remark 4.3.3. The domain of the closure of this last sesquilinear form is $H_0^1\left(S^1 \times \Lambda\right) \cap \{\pi_* f = 0\}$.

Remark 4.3.4. Both these constructions can be considered when Λ is a finite or countable reunion of disjoint intervals of $[a, +\infty[$.

Remark 4.3.5. In the following, we will compare self-adjoint operators using the partial order $\leqslant$ defined by comparing the values and domains of their associated quadratic forms. For more information on this order relation, the reader is referred to [63, Sec. VI.2.5]. Since the quadratic forms attached to the pseudo-Laplacian with either Dirichlet or Neumann boundary conditions are the same, comparing these operators is only a matter of inclusion of domains.

Proposition 4.3.6. *Assume we have $\Lambda = \overline{\Lambda_1} \cup \overline{\Lambda_2}$, and that the measure of $\overset{\circ}{\Lambda} \setminus (\Lambda_1 \cup \Lambda_2)$ vanishes, where Λ_1 and Λ_2 are open. We then have the following comparisons*

$$\begin{aligned} 0 \ &\leqslant\ H_\Lambda^D &&\leqslant\ H_{\Lambda_1 \cup \Lambda_2}^D, \\ 0 \ &\leqslant\ H_{\Lambda_1 \cup \Lambda_2}^N &&\leqslant\ H_\Lambda^N. \end{aligned}$$

Proof. The comparison of pseudo-Laplacians with Dirichlet boundary conditions stems from the fact two H_0^1 functions on Λ_1 and Λ_2 respectively can be glued into an H_0^1 function on Λ. For the Neumann boundary condition, the comparison is reversed, since an H^1 function on Λ can be restricted to two H^1 functions on Λ_1 and Λ_2, respectively.

□

Remark 4.3.7. The comparison of pseudo-Laplacians with Neumann boundary conditions still holds when Λ is split into a countable number of Λ_i as in the proposition above, since restriction of H^1 functions still gives H^1 functions. However, we canot glue (in general) a countable set of H_0^1 functions into a global H_0^1 function, as such a glued function may not even be L^2.

Proposition 4.3.8 (Dirichlet-Neumann bracketing)**.** *The pseudo-Laplacians with Dirichlet and Neumann boundary conditions, respectively, can be compared in the following way*

$$0 \leqslant H_{\Lambda}^{N} \leqslant H_{\Lambda}^{D} .$$

Proof. This result is a direct consequence of the inclusion of H_0^1 in H^1.

□

We can now apply these tools to get a Weyl bound in our situation. Let $\delta > 0$ be a strictly positive real number. For any integer $n \in \mathbb{N}^*$, we set

$$a_n = a + n\delta ,$$

and we split the intervall $[a, +\infty[$ according to these steps. Using the results above, we then have

$$0 \leqslant H^N_{\bigcup\limits_n S^1 \times]a_n, a_{n+1}[} \leqslant H^N_{S^1 \times [a,+\infty[} \leqslant H^D_{S^1 \times [a,+\infty[} .$$

Denoting by μ_k the k-th element of the spectrum counted with multiplicity, we get

$$\mu_k \left(H^D_{S^1 \times [a,+\infty[} \right) \geqslant \mu_k \left(H^N_{S^1 \times [a,+\infty[} \right) \geqslant \mu_k \left(H^N_{\bigcup\limits_n S^1 \times]a_n, a_{n+1}[} \right) .$$

By the max-min principle, the sequence $(\mu_k)_k$ associated to a positive self-adjoint operator stops at the infimum of the essential spectrum.

Definition 4.3.9. Let H be a positive self-adjoint operator and $\lambda \geqslant 0$ be a positive real number. The *spectrum counting function* is defined by

$$N(H, \lambda) = \# \{k,\ \mu_k \leqslant \lambda\} .$$

Remark 4.3.10. In the setting of the definition above, we have $N(H, \lambda) = +\infty$ for any real number λ which at least equals the infimum of the essential spectrum.

Proposition 4.3.11. *There exists a real constant $C > 0$ such that, for any $\lambda \geqslant 0$, we have*

$$N\left(H^D_{S^1 \times [a,+\infty[}, \lambda \right) \leqslant C\lambda .$$

Proof. Using the inequalities above, for any $\lambda \geqslant 0$, we have

$$\begin{aligned} N\left(H^D_{S^1 \times [a,+\infty[}, \lambda \right) \leqslant N\left(H^N_{S^1 \times [a,+\infty[}, \lambda \right) &\leqslant N\left(H^N_{\bigcup\limits_n S^1 \times]a_n, a_{n+1}[}, \lambda \right) \\ &\leqslant \sum_n N\left(H^N_{S^1 \times]a_n, a_{n+1}[}, \lambda \right). \end{aligned}$$

We will now compare the terms involved in these inequalities to the ones associated to the Laplacian on $S^1 \times [a, +\infty[$ for the euclidean metric, which can then be precisely evaluated, using lemma 4.1 of [28]. We have

$$\frac{\int_{S^1 \times]a_n, a_{n+1}[} |\nabla f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_{S^1 \times]a_n, a_{n+1}[} |f(x,y)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}y}{y^2}} \geqslant a_n^2 \frac{\int_{S^1 \times]a_n, a_{n+1}[} |\nabla f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_{S^1 \times]a_n, a_{n+1}[} |f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y} ,$$

which gives, for the counting functions

$$N\left(H^N_{S^1\times]a_n,a_{n+1}[},\lambda\right) \quad \leqslant \quad N\left(\Delta^N_{S^1\times]a_n,a_{n+1}[},\frac{\lambda}{a_n^2}\right) \ ,$$

where Δ^N here denotes the pseudo-Laplacian with Neumann boundary condition for the euclidean metric. Using Colin de Verdière's estimates, we now have

$$\begin{array}{rcll} N\left(\Delta^N_{S^1\times]a_n,a_{n+1}[},\frac{\lambda}{a_n^2}\right) & \leqslant & \frac{\delta}{4\pi}\cdot\frac{\lambda}{a_n^2}+\frac{\sqrt{\lambda}}{\pi a_n} & \text{if } \lambda\geqslant 4\pi^2a_n^2=4\pi^2\left(a+n\delta\right)^2 \\ & = & 0 & \text{otherwise.} \end{array}$$

In particular, we can set an integer M large enough so that the left-hand side above vanishes for any integer $n\geqslant M$. This proves that, for fixed λ and δ, we have

$$\begin{array}{rcl} \sum\limits_n N\left(H^N_{S^1\times]a_n,a_{n+1}[},\lambda\right) & \leqslant & \sum\limits_{n=0}^{M-1} N\left(\Delta^N_{S^1\times]a_n,a_{n+1}[},\frac{\lambda}{a_n^2}\right) \\ & \leqslant & \sum\limits_{n=0}^{M-1}\left(\frac{\delta}{4\pi}\cdot\frac{\lambda}{a_n^2}+\frac{\sqrt{\lambda}}{\pi a_n}\right) \quad \leqslant \quad \frac{\delta\lambda}{4\pi}\sum\limits_{n=0}^{M-1}\frac{1}{(a+n\delta)^2}+\frac{\sqrt{\lambda}}{\pi}\sum\limits_{n=0}^{M-1}\frac{1}{(a+n\delta)}. \end{array}$$

It remains to study both these finite sums. We start with the first one. We have

$$\frac{\delta\lambda}{4\pi}\sum_{n=0}^{M-1}\frac{1}{(a+n\delta)^2} \quad \leqslant \quad \frac{\delta\lambda}{4\pi a^2}+\frac{\lambda}{4\pi}\int_a^{+\infty}\frac{\mathrm{d}y}{y^2} \quad = \quad \left[\frac{\delta}{4\pi a^2}+\frac{1}{4\pi a}\right]\lambda \ .$$

We can now move on to the second term. We have

$$\sum_{n=1}^{M-1}\frac{1}{a+n\delta} \quad = \quad \frac{1}{\delta}\sum_{n=1}^{M-1}\int_{a_{n-1}}^{a_n}\frac{\mathrm{d}y}{a+n\delta} \quad \leqslant \quad \frac{1}{\delta}\log\left(\frac{\sqrt{\lambda}}{2\pi a}\right) \ .$$

Putting these results together yields the proposition.

□

4.4 The spectral zeta function and its integral representation

4.4.1 Definition of the zeta function

Unless otherwise specified, we will denote by ω a real number lying in $[0,2\pi[$, and by μ a strictly positive real number. As previously explained in this chapter, the spectral problem

$$\left\{\begin{array}{lcll} -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi & = & \lambda\psi & \\ \int_F\ |\psi|^2 & < & +\infty & \\ \psi\left(x+1,y\right) & = & e^{i\omega}\psi\left(x,y\right) & \\ \psi\left(x,a\right) & = & 0 & \\ \int_{S^1}\ \psi\left(x,y\right)\ \mathrm{d}x & = & 0 & \text{for almost every } y\geqslant a \ \text{ if } \ \omega\ =\ 0 \end{array}\right.$$

gives rise to a discrete set of strictly positive real numbers λ, temporarily denoted by $\left\{\lambda_j\right\}_{j\geqslant 1}$.

Under the finite monodromy at the cusps hypothesis, which was made in the last section, every real number λ_j appears m_j times in the aforementioned sequence, and the growth of that sequence is controlled by the Weyl bound. The idea is now to consider the spectral zeta function with parameter $\mu \geqslant 0$ associated with this sequence, which we define below.

Proposition-Definition 4.4.1. *Let $\omega \in [0, 2\pi[$ and $\mu \geqslant 0$ be two real numbers. The spectral zeta function with parameters μ and ω, defined as*

$$\zeta_\omega(\mu, s) \quad = \quad \sum_{j=1}^{+\infty} \frac{1}{(\lambda_j + \mu)^s}$$

is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1$*.*

Proof. Both parts of the above proposition stem directly from the Weyl type law previously shown. Rearranging the sequence $(\lambda_j)_j$ in ascending order yields, according to the notations of this section,

$$\lambda_1 \quad = \quad \ldots \quad = \quad \lambda_{m_1} \quad < \quad \lambda_{m_1+1} \quad = \quad \ldots \quad = \quad \lambda_{m_1+m_2} \quad < \quad \ldots$$

the Weyl bound can be expressed as follows

$$\beta_j \quad \coloneqq \quad \sum_{k=1}^{j} m_k \quad \leqslant \quad C\lambda_a \quad \text{for any } a \in [\![\beta_{j-1}+1,\ \beta_j]\!] \ ,$$

where the integers α_j have been defined thus for convenience. Note that λ_a is constant when the integer a is chosen as indicated. The sequence $(\lambda_j)_j$ being made of strictly positive real numbers, the estimate can be restated as

$$\frac{1}{\lambda_a} \quad \leqslant \quad \frac{C}{\alpha_j} \quad \leqslant \quad \frac{C}{a} \quad \text{for any } a \in [\![\beta_{j-1}+1,\ \beta_j]\!]$$

We then have, on the half-plane $\operatorname{Re} s > 1$,

$$\left| \sum_{j=1}^{+\infty} \frac{1}{(\lambda_j + \mu)^s} \right| \quad \leqslant \quad \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^{\operatorname{Re} s}} \quad = \quad \sum_{j=1}^{+\infty} \sum_{a=\alpha_{j-1}+1}^{\alpha_j} \frac{1}{\lambda_a^{\operatorname{Re} s}} \ .$$

We can now apply the key estimate, which, since the real part of s is strictly positive, gives

$$\sum_{j=1}^{+\infty} \sum_{a=\alpha_{j-1}+1}^{\alpha_j} \frac{1}{\lambda_a^{\operatorname{Re} s}} \quad \leqslant \quad \sum_{j=1}^{+\infty} \sum_{a=\alpha_{j-1}+1}^{\alpha_j} \frac{C}{a^{\operatorname{Re} s}} \quad \leqslant \quad C \sum_{j=1}^{+\infty} \frac{1}{j^{\operatorname{Re} s}} \ .$$

This last series being absolutely convergent on the half-plane $\operatorname{Re} s > 1$, the result is proved. $\square$

Remark 4.4.2. Using results pertaining to modified Bessel functions of the second kind, we can rearrange the set of eigenvalues $\{\lambda_j\}_j$ as

$$\{\lambda_j,\ j \geqslant 1\} \quad = \quad \left\{ \tfrac{1}{4} + r_{k,j}^2,\ k \in \mathbb{Z},\ j \geqslant 1 \right\} \ ,$$

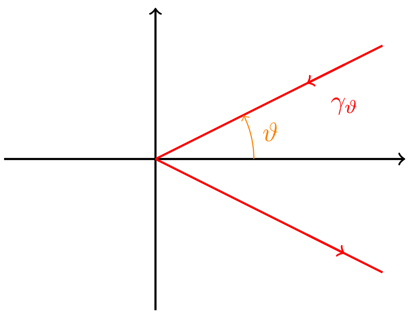
where, for every integer $k \in \mathbb{Z}$, the real numbers $r_{k,j}$ are the (simple) zeros of the modified Bessel function of the second kind $\nu \longmapsto K_{i\nu}(C_{\omega,k}a)$ as a function of its purely imaginary order.

Proposition 4.4.3. *Let ω and μ be real numbers, with ω lying in $[0, 2\pi[$ and μ strictly positive. In the half-plane* $\operatorname{Re} s > 1$*, we have*

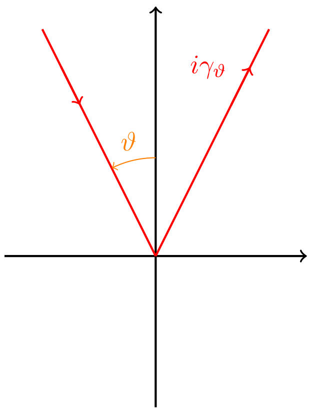
$$\zeta_\omega(\mu, s) \quad = \quad \begin{cases} \frac{1}{2i\pi} \sum\limits_{k\in\mathbb{Z}} \int_{i\gamma_\vartheta} \left(\frac{1}{4} - t^2 + \mu\right)^{-s} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k} a\right) \, dt & \text{if} \quad \omega \neq 0 \\ \frac{1}{2i\pi} \sum\limits_{k\in\mathbb{Z}\setminus\{0\}} \int_{i\gamma_\vartheta} \left(\frac{1}{4} - t^2 + \mu\right)^{-s} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k} a\right) \, dt & \text{if} \quad \omega = 0 \end{cases} ,$$

where the contour of integration γ_ϑ is defined so as to include the set of positive real numebers $\mathbb{R}_+$.

Remark 4.4.4. The very statement above calls for the definition of some contour γ_ϑ, and the consideration of its rotated version $i\gamma_\vartheta$. While the proof works for any contour that includes $\mathbb{R}_+$, we will only work with such γ_ϑ as specified below.



(a) Integration contour γ_ϑ



(b) Rotated integration contour $i\gamma_\vartheta$

Figure 4.1 – Integration contours

Proof of proposition 4.4.3. We assume, for simplicity, that ω is not zero, keeping in mind that the only difference in dealing with this case is that we must remove the case $k = 0$ from consideration. We now use remark 4.4.2. For every integer $k \in \mathbb{Z}$, the zeros of the function $\nu \longmapsto K_{i\nu}\left(C_{\omega,k} a\right)$ are simple, and denoted by $r_{k,j}$. The Cauchy formula then states that we have

$$\sum_{j\geqslant 1} \frac{1}{\left(\frac{1}{4} + r_{k,j}^2 + \mu\right)^s} \quad = \quad \frac{1}{2i\pi} \int_{\gamma_\vartheta} \left(\frac{1}{4} + r^2 + \mu\right)^{-s} \frac{\partial}{\partial r} \log K_{ir}\left(C_{\omega,k} a\right) \, \mathrm{d}r \quad ,$$

where s is such that we have $\operatorname{Re} s > 1$, and the contour γ_ϑ is given by

$$\gamma_\vartheta \quad = \quad \left[\infty\, e^{i\vartheta}, 0\right] \cup \left[0, \infty\, e^{-i\vartheta}\right] \quad ,$$

with ϑ being strictly between 0 et $\pi/2$. The reader is referred to figure 4.1a for more clarity on this. It is also worth noting that we have implicitely used the last proposition in the equality above, which implies convergence for the series appearing on the left hand side, and further gives

$$\zeta_\omega(\mu, s) \quad = \quad \sum_{k\in\mathbb{Z}} \sum_{j\geqslant 1} \frac{1}{\left(\frac{1}{4} + r_{k,j}^2 + \mu\right)^s} \quad = \quad \frac{1}{2i\pi} \sum_{k\in\mathbb{Z}} \int_{\gamma_\vartheta} \left(\frac{1}{4} + r^2 + \mu\right)^{-s} \frac{\partial}{\partial r} \log K_{ir}\left(C_{\omega,k} a\right) \, \mathrm{d}r.$$

A change of variable $t = ir$, which in particular rotates the contour of integration (see figure 4.1b) then gives the required formula, which completes the proof of the proposition.

$\square$

4.4.2 Letting ϑ go to $\frac{\pi}{2}$

In the proposition above, any angle ϑ strictly between 0 and $\pi/2$ can be chosen. We now want to let ϑ go to $\pi/2$, though some care must be taken, as there are convergence problems.

Definition 4.4.5. For any integer $k \in \mathbb{Z}$, and ω, μ as before, we set

$$\begin{array}{cccc} f_{\mu,k} & : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & & t & \longmapsto & \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k} a\right) - \frac{t}{\sqrt{\frac{1}{4}+\mu}} \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}} \log K_t\left(C_{\omega,k} a\right) \end{array} .$$

The introduction of this function, which is similar to the one used in [47, *Sec.*6.1], will be justfied shortly. For now, we note that because of the equality

$$\int_{i\gamma_\vartheta} \left(\tfrac{1}{4} + \mu - t^2\right)^{-s} t \,\mathrm{d}t \quad = \quad 0 \ ,$$

which holds for any complex number whose real part is strictly larger than 1, we have

$$\int_{i\gamma_\vartheta} \left(\tfrac{1}{4} + \mu - t^2\right)^{-s} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k}\right) \,\mathrm{d}t \quad = \quad \int_{i\gamma_\vartheta} \left(\tfrac{1}{4} + \mu - t^2\right)^{-s} f_{\mu,k}\left(t\right) \,\mathrm{d}t \ .$$

It should be surprising that removing an integral which vanishes from an integral we want to study could prove useful. The idea is that taking the limit as ϑ goes to $\pi/2$ without this manipulation would involve two integrability conditions requiring us to have both Re < 1 and $\mathrm{Re}\, s > 1$. Using this function $f_{\mu,k}$ serves to move the condition $\mathrm{Re}\, s < 1$ slightly to the right, so that satisfying both conditions become the same as being in the strip $1 < \mathrm{Re}\, s < 2$. In order to prepare taking the limit mentioned above, we note that the main problem lies with the factor

$$\left(\tfrac{1}{4} + \mu - t^2\right)^{-s} \quad = \quad \exp\left(-s \log\left(\tfrac{1}{4} + \mu - t^2\right)\right)$$

that appears under the integral. Though the complex logarithm, to be understood as the principal branch thereof, makes perfect sense as long as we stay away from the half-line of negative real numbers. However, as can be seen below in figure 4.2, letting ϑ go to $\pi/2$ will cause part of the term appearing within the logarithm to collapse onto negative real numbers.

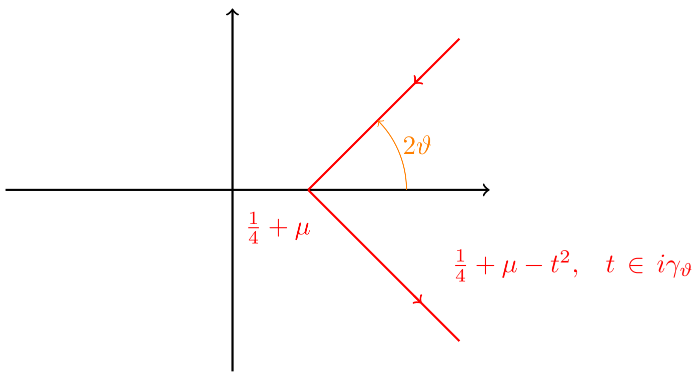


Figure 4.2 – Variation of t on the contour $i\gamma_\vartheta$

In order to solve that problem, we will split the contour $i\gamma_\vartheta$ into four parts, according to the values of $1/4+\mu-t^2$ when t runs through the contour. This can be seen in figure 4.3 below.

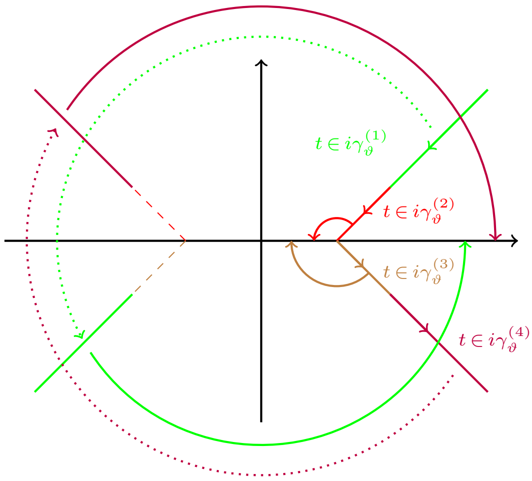


Figure 4.3 – Modification of $i\gamma_\vartheta$ and limit as ϑ goes to $\frac{\pi}{2}$

Definition 4.4.6. The four paths of integration $\gamma_\vartheta^{(1)}, \ldots, \gamma_\vartheta^{(4)}$ are defined as follows:

$$\begin{aligned} \gamma_\vartheta^{(1)} &= \left\{ t = re^{i\vartheta} \in \gamma_\vartheta,\ r \geqslant \tfrac{1}{4}+\mu \right\}, & \gamma_\vartheta^{(2)} &= \left\{ t = re^{i\vartheta} \in \gamma_\vartheta,\ r < \tfrac{1}{4}+\mu \right\}, \\ \gamma_\vartheta^{(3)} &= \left\{ t = re^{-i\vartheta} \in \gamma_\vartheta,\ r < \tfrac{1}{4}+\mu \right\}, & \gamma_\vartheta^{(4)} &= \left\{ t = re^{i\vartheta} \in \gamma_\vartheta,\ r \geqslant \tfrac{1}{4}+\mu \right\}. \end{aligned}$$

Remark 4.4.7. It is of course readily checked that we have

$$\gamma_\vartheta \quad = \quad \gamma_\vartheta^{(1)} \quad \cup \quad \gamma_\vartheta^{(2)} \quad \cup \quad \gamma_\vartheta^{(3)} \quad \cup \quad \gamma_\vartheta^{(4)} \quad ,$$

which means that we have, for any integer $k \in \mathbb{Z}$

$$\int_{i\gamma_\vartheta} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \quad = \quad \sum_{j=1}^{4} \int_{i\gamma_\vartheta^{(j)}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \ ,$$

with the exception of $k = 0$ should ω equal zero.

Going back to figure 4.3, we note that the parts $\gamma_\vartheta^{(2)}$ and $\gamma_\vartheta^{(3)}$ are easier to deal with, as letting ϑ go to $\pi/2$ there is not an issue. Indeed, we have

$$\begin{aligned} &\int_{i\gamma_\vartheta^{(2)}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \\ &\qquad + \int_{i\gamma_\vartheta^{(3)}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \end{aligned} \quad \underset{\vartheta \to \frac{\pi}{2}^-}{\longrightarrow} \quad \int_{-\sqrt{\frac{1}{4}+\mu}}^{\sqrt{\frac{1}{4}+\mu}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \quad = \quad 0 \ ,$$

where the last equality stems from the oddness of $f_{\mu,k}$. Thus $\gamma_\vartheta^{(1)}$ and $\gamma_\vartheta^{(4)}$ are the most interesting parts of the integration contour γ_ϑ are, and we have

$$\left(\tfrac{1}{4}+\mu-t^2\right)^{-s} \quad = \quad \begin{cases} e^{is\pi}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} & \text{on} \quad \gamma_\vartheta^{(1)} \\ e^{-is\pi}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} & \text{on} \quad \gamma_\vartheta^{(4)} \end{cases} .$$

This manipulation is represented on figure 4.3 by two dotted arcs. The point of doing this is that we can now let ϑ go to $\pi/2$ without having to go to the half-line of negative real numbers. It is also worth noting that the change of sign within the exponential comes from the choice of branch for the logarithm, and will be essential in what follows. We have

$$\begin{aligned}\int_{i\gamma_\vartheta^{(1)}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t &= e^{is\pi}\int_{i\gamma_\vartheta^{(1)}} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \\ \underset{\vartheta\to\frac{\pi}{2}^-}{\longrightarrow} & \ e^{is\pi}\int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t,\end{aligned}$$

for the first part of γ_ϑ, and

$$\begin{aligned}\int_{i\gamma_\vartheta^{(4)}} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t &= e^{is\pi}\int_{i\gamma_\vartheta^{(4)}} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t \\ \underset{\vartheta\to\frac{\pi}{2}^-}{\longrightarrow} & \ e^{-is\pi}\int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t\end{aligned}$$

for the last one. Putting these results together, and using the fact that the integral

$$\int_{i\gamma_\vartheta} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t$$

is constant in ϑ, we get the equality

$$\int_{i\gamma_\vartheta} \left(\tfrac{1}{4}+\mu-t^2\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t = 2i\sin(\pi s)\int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t\ .$$

One thing must yet be done before we can state the result we have in essence just proved, and that is saying for which complex numbers s these manipulations actually make sense. This will be the reason why we have introduced the function $f_{\mu,k}$, as was hinted at earlier. We have

$$\begin{aligned}\frac{1}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^s} f_{\mu,k}(t) &= \frac{\sqrt{\frac{1}{4}+\mu}\cdot\frac{1}{t}\frac{\partial}{\partial t}\log K_t(C_{\omega,k}a)-\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t(C_{\omega,k}a)}{t-\sqrt{\frac{1}{4}+\mu}} \\ &\qquad\cdot\frac{t}{\sqrt{\frac{1}{4}+\mu}}\cdot\frac{1}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{s-1}}\cdot\frac{1}{t+\sqrt{\frac{1}{4}+\mu}}.\end{aligned}$$

We can now recognize in this last factor a difference quotient, and we have

$$\begin{aligned}&\lim_{t\to\sqrt{\frac{1}{4}+\mu}}\left[\frac{\sqrt{\frac{1}{4}+\mu}\cdot\frac{1}{t}\frac{\partial}{\partial t}\log K_t(C_{\omega,k}a)-\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t(C_{\omega,k}a)}{t-\sqrt{\frac{1}{4}+\mu}}\right] \\ &\qquad = \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\left(\sqrt{\tfrac{1}{4}+\mu}\cdot\tfrac{1}{t}\tfrac{\partial}{\partial t}\log K_t(C_{\omega,k}a)\right).\end{aligned}$$

This means that the function $t\longmapsto\left(t^2-(1/4+\mu)\right)^{-s} f_{\mu,k}(t)$ is integrable at $\sqrt{1/4+\mu}$ if and only if we have $\operatorname{Re} s < 2$. The integrability condition at $+\infty$ has not changed, and is still $\operatorname{Re} s > 1$. We can summarize this discussion as follows.

Proposition 4.4.8. *On the strip $1 < \operatorname{Re} s < 2$, the spectral function ζ_ω is given by*

$$\zeta_\omega(\mu,s) = \begin{cases}\frac{\sin(\pi s)}{\pi}\sum\limits_{k\in\mathbb{Z}}\int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ dt & \text{if}\quad \omega\neq 0\\ \frac{\sin(\pi s)}{\pi}\sum\limits_{k\in\mathbb{Z}\setminus\{0\}}\int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ dt & \text{if}\quad \omega=0\end{cases}.$$

4.5 Splitting the interval of integration

As we saw in the last section, for every real number $\mu > 0$ and every ω in $[0, 2\pi[$, the spectral zeta function ζ_ω is given by an absolutely convergent sum of integrals on the strip $1 < \operatorname{Re} s < 2$. We will now carefully study these integrals and their sum.

Definition 4.5.1. Let μ and ω be as above, and $k \in \mathbb{Z}$ be an integer, which is not zero if ω vanishes. We define, on the strip $1 < \operatorname{Re} s < 2$, the integral $I_{\mu,k}$ to be

$$I_{\mu,k} = \frac{\sin(\pi s)}{\pi} \int_{\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4} + \mu\right)\right)^{-s} f_{\mu,k}(t) \, \mathrm{d}t \; .$$

Our main focus in the remainder of this chapter will be to prove that the sum of the integrals $I_{\mu,k}$ induces a holomorphic function around $s = 0$, and also to get a precise idea of the asymptotic behavior of the derivative at 0 of this continuation as μ goes to $+\infty$ on the one hand, and as a goes to $+\infty$ for $\mu = 0$ on the other. One of the key ingredients we will use is the binomial formula

$$\left(t^2 - \left(\frac{1}{4} + \mu\right)\right)^{-s} = \sum_{j=0}^{+\infty} \frac{(s)_j}{j!} \left(\frac{1}{4} + \mu\right)^j \cdot \frac{1}{t^{2s+2j}} \; ,$$

which holds for $t > \sqrt{1/4 + \mu}$, where the so-called *Pochhammer symbol* $(s)_j$ is defined by

$$(s)_j = \frac{\Gamma(s+j)}{\Gamma(s)}$$

for any integer $j \geqslant 0$ and any complex number s, using the fact that the Γ function has a single pole at every negative integer. However, it is not enough that this inequality is satisfied on the interval of integration we are dealing with, as we would also like to interchange the sum over j and the integral itself, which means we need to stay at a certain distance from this singularity before applying this formula. We will thus split the interval of integration, much in the fashion of [47], end of paragraph 6.1. Throughout the rest of this chapter, we will denote by $\delta > 0$ a strictly positive real number, for whom we will give bounds. Each of these bounds will be implicitly assumed, and δ will then be any stricly positive real number satisfying them all. We have

$$\left]\sqrt{\tfrac{1}{4}+\mu}, +\infty\right[= \begin{cases} \left]\sqrt{\tfrac{1}{4}+\mu},\ 2|k|^\delta \sqrt{\tfrac{1}{4}+\mu}\right[\ \sqcup \ \left[2|k|^\delta \sqrt{\tfrac{1}{4}+\mu},\ +\infty\right[& \text{when } k \neq 0 \\ \left]\sqrt{\tfrac{1}{4}+\mu},\ 2\sqrt{\tfrac{1}{4}+\mu}\right[\ \sqcup \ \left[2\sqrt{\tfrac{1}{4}+\mu},\ +\infty\right[& \text{when } k = 0 \end{cases} .$$

The fact that we do not need to introduce δ in the case $k = 0$, which is only to be considered if ω is not zero, stems from the fact that we can consider the associated term on its own, outside the sum over k, so there is no convergence on k to be improved. We can then split the integrals $I_{\mu,k}$ according to this splitting of the interval of integration. This is the purpose of the next definition.

Definition 4.5.2. For every $k \in \mathbb{Z}$, and every complex number s with $1 < \operatorname{Re} s < 2$, we set

$$L_{\mu,k}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^\delta\sqrt{\frac{1}{4}+\mu}} \left(t^2 - \left(\frac{1}{4} + \mu\right)\right)^{-s} f_{\mu,k}(t) \, \mathrm{d}t & \text{if } k \neq 0 \\ \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2\sqrt{\frac{1}{4}+\mu}} \left(t^2 - \left(\frac{1}{4} + \mu\right)\right)^{-s} f_{\mu,k}(t) \, \mathrm{d}t & \text{if } k = 0 \\ & \text{and } \omega \neq 0 \end{cases} ,$$

for the part of the integral that goes near the singularity, and

$$M_{\mu,k}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t & \text{if } k \neq 0 \\ \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t & \text{if } k = 0 \\ & \text{and } \omega \neq 0 \end{cases},$$

for the part that stays away from it.

Remark 4.5.3. The splitting for the case $k = 0$ can actually be performed at any point, since there is no series involved. It was actually only done as a way to render their study similar to that of the actual sum.

Remark 4.5.4. The splitting we have performed has been done so as to have

$$I_{\mu,k}(s) = L_{\mu,k}(s) + M_{\mu,k}(s)$$

in the appropriate strip. The study of $I_{\mu,k}$ is therefore reduced to that of $L_{\mu,k}$ and $M_{\mu,k}$.

4.6 Study of the integrals $L_{\mu,k}$

The first step in the study we must conduct is that of the (sums of) the integrals $L_{\mu,k}$, which have been defined above as

$$L_{\mu,k}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2k^\delta\sqrt{\frac{1}{4}+\mu}} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t & \text{if} \quad k \neq 0 \\ \dfrac{\sin(\pi s)}{\pi} \displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2\sqrt{\frac{1}{4}+\mu}} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t & \text{if} \quad k = 0 \ \text{ and } \ \omega \neq 0 \end{cases}.$$

The study will be led in a similar manner as section 6.2 of [47], the main differences being the parameters μ and ω, which we must keep, and the asymptotics as μ goes to infinity.

4.6.1 Global study

Similarly to what is done in (6.7) of [47], we begin with the following definition.

Definition 4.6.1. For any μ and ω as above, we define the function $F_{\mu,k}$ on $\mathbb{C}$ by

$$F_{\mu,k}(t) = \log K_t(C_{\omega,k}a) - \log K_{\sqrt{\frac{1}{4}+\mu}}(C_{\omega,k}a) - \frac{t^2-\left(\frac{1}{4}+\mu\right)}{2\sqrt{\frac{1}{4}+\mu}} \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}} \log K_t(C_{\omega,k}a) \ .$$

The main point of this section will be to prove the next result, which is done in the exact same way as that of Corollay 6.4 of [47]. Unlike what is done there, this result will not be sufficient for us. It is nevertheless a crucial step.

Proposition 4.6.2. *For any integer $k \neq 0$, we can write*

$$F_{\mu,k}(t) = \left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^2 R_{\mu,k}(t) \quad \textit{for } t \in \left[\sqrt{\tfrac{1}{4}+\mu}, 2|k|^\delta\sqrt{\tfrac{1}{4}+\mu}\right]$$

where the function $R_{\mu,k}$ is analytic in t and satisfies a bound of the type

$$|R_{\mu,k}| \quad \leqslant \quad C_\mu \cdot \tfrac{1}{|k|^{2-4\delta} a^2} \ ,$$

on the same interval, where the constant $C_\mu > 0$ depends only on μ, and not on either t or k.

Proof. We first note that the function $F_{\mu,k}$ has been defined so that we have

$$F_{\mu,k}\left(\pm\sqrt{\tfrac{1}{4}+\mu}\right) \quad = \quad F'_{\mu,k}\left(\pm\sqrt{\tfrac{1}{4}+\mu}\right) \quad = \quad 0 \ .$$

Since $F_{\mu,k}$ is an even and entire function in t, it is of the form $F_{\mu,k}(t) = h_{\mu,k}\left(t^2\right)$, where $h_{\mu,k}$ is entire and such that we have

$$h_{\mu,k}\left(\tfrac{1}{4}+\mu\right) \quad = \quad h'_{\mu,k}\left(\tfrac{1}{4}+\mu\right) \quad = \quad 0 \ .$$

The Taylor-Lagrange theorem then allows us to write

$$F_{\mu,k}(t) \quad = \quad h_{\mu,k}\left(t^2\right) \quad = \quad \tfrac{1}{2}\left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^2 \, h''_{\mu,k}\left(\xi^2_{\mu,t}\right) \ ,$$

where $\xi_{\mu,t}$ is some real number with $\sqrt{1/4+\mu} \leqslant \xi_{\mu,t} \leqslant 2\,|k|^\delta \sqrt{1/4+\mu}$. It is important to note that we do not know how $\xi_{\mu,t}$ depends on μ, t, or k. By differentiating $F_{\mu,k}$, we get

$$\begin{cases} F'_{\mu,k}(t) & = \quad 2t h'_{\mu,k}\left(t^2\right) \\ F''_{\mu,k}(t) & = \quad 2h'_{\mu,k}\left(t^2\right) + 4t^2 h''_{\mu,k}\left(t^2\right) \end{cases} ,$$

and these two equalities can be combined to yield

$$h_{\mu,k}\left(t^2\right) \quad = \quad \tfrac{1}{4t^2}\, F''_{\mu,k}(t) - \tfrac{1}{4t^3}\, F'_{\mu,k}(t) \quad = \quad \tfrac{1}{4t^2}\,\tfrac{\partial^2}{\partial t^2} \log K_t\left(C_{\omega,k}a\right) - \tfrac{1}{4t^3}\,\tfrac{\partial}{\partial t} \log K_t\left(C_{\omega,k}a\right)$$

Therefore, we have

$$F_{\mu,k}(t) \quad = \quad \left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^2 \left[\tfrac{1}{4t^2}\,\tfrac{\partial^2}{\partial t^2}_{|t=\xi_{\mu,t}} \log K_t\left(C_{\omega,k}a\right) - \tfrac{1}{4t^3}\,\tfrac{\partial}{\partial t}_{|t=\xi_{\mu,t}} \log K_t\left(C_{\omega,k}a\right)\right] \ .$$

For any real number ξ such that we have

$$\xi \quad \in \quad \left[\sqrt{\tfrac{1}{4}+\mu},\ 2\,|k|^\delta \sqrt{\tfrac{1}{4}+\mu}\right] \ ,$$

we denote by D_ξ the disk of the complex plane centered at ξ and of radius $1/4$. The modified Bessel function $K_\nu(z)$ being entire in ν for any positive real number z, the Cauchy formula gives

$$\tfrac{\partial}{\partial t}_{|t=\xi} \log K_t\left(C_{\omega,k}a\right) \quad = \quad \tfrac{1}{2i\pi K_\xi(C_{\omega,k}a)} \int_{\partial D_\xi} \tfrac{K_\nu(C_{\omega,k}a)}{(\nu-\xi)^2}\, \mathrm{d}\nu \ .$$

Using the appropriate asymptotics for the modified Bessel functions of the second kind, we get

$$\begin{aligned} \tfrac{\partial}{\partial t}_{|t=\xi} \log K_t\left(C_{\omega,k}a\right) \quad &= \quad \tfrac{1}{2i\pi}\,\tfrac{K_{1/2}(C_{\omega,k}a)}{K_\xi(C_{\omega,k}a)} \int_{\partial D_\xi} \tfrac{1}{(\nu-\xi)^2}\left(1 + \tfrac{a_1(\nu)}{C_{\omega,k}a} + \tfrac{a_2(\nu)}{(C_{\omega,k}a)^2}\rho_2\left(\nu, C_{\omega,k}a\right)\right) \mathrm{d}\nu \\ &= \quad \tfrac{K_{1/2}(C_{\omega,k}a)}{K_\xi(C_{\omega,k}a)} \left(\tfrac{\xi}{C_{\omega,k}a} + O\left(\tfrac{1}{|k|^{2-4\delta}a^2}\right)\right). \end{aligned}$$

For that last point, we have used the explicit expressions of a_1 and a_2, as well as the following estimate for the remainder ρ_2

$$\left|\rho_2\left(\nu, C_{\omega,k}a\right)\right| \quad \leqslant \quad 2\exp\left(\frac{\left|\nu^2-\frac{1}{4}\right|}{C_{\omega,k}a}\right) \quad \leqslant \quad 2\exp\left(\frac{\left(\frac{1}{4}+\mu\right)|k|^{2\delta}}{C_{\omega,k}a}\right) \;,$$

which is uniformly bounded in k, but not in μ. Similarly, we have

$$\frac{\partial^2}{\partial t^2}_{|t=\xi} \log K_t\left(C_{\omega,k}a\right) \quad = \quad \frac{K_{1/2}(C_{\omega,k}a)}{K_\xi(C_{\omega,k}a)}\left(1+O\left(\frac{1}{|k|^{2-4\delta}}\right)\right) \;,$$

which means that we have, still on the same interval,

$$R_{\mu,k}\left(t\right) \quad = \quad \frac{K_{1/2}(C_{\omega,k}a)}{K_\xi(C_{\omega,k}a)} \cdot O\left(\frac{1}{|k|^{2-4\delta}}\right) \;,$$

with an implicit constant depending only on μ, and not on k. Furthermore, the asymptotics for the modified Bessel functions of the second kind show that the first factor on the right-hand side above is bounded on the interval we consider, uniformly in k. This concludes the proof.

□

Remark 4.6.3. Note that a similar result holds when $k = 0$ and $\mu = 0$.

As we have stated before, the binomial formula will be an extremely important tool in the study of the integrals we are concerned with, and we split every integral into two parts so as to be able to use it on the one that remains far away from the singularity. Since the integrals $L_{\mu,k}$ involve a domain that goes up to this problematic point, the binomial formula should *a priori* be of no use to us. The next proposition will show that we can, using the previous result, further break apart $L_{\mu,k}$ into a part on which the binomial formula can actually be used, and a part which, even though we hold very little control over it, will not matter, as its derivative at $s = 0$ vanishes.

Proposition 4.6.4. *For any integer k, and μ, ω as before, we have, on the strip $1 < \operatorname{Re} s < 2$,*

$$\begin{aligned} L_{\mu,k}\left(s\right) \quad = \quad \frac{\sin(\pi s)}{\pi} & \left(\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s} F_{\mu,k}\left(2\,|k|^\delta\sqrt{\frac{1}{4}+\mu}\right)\right. \\ & \left. +2s\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^\delta\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,k}\left(t\right)\, dt\right), \end{aligned}$$

if k is different from 0, and

$$\begin{aligned} L_{\mu,0}\left(s\right) \quad = \quad \frac{\sin(\pi s)}{\pi} & \left(\frac{1}{3^s}\left(\frac{1}{4}+\mu\right)^{-s} F_{\mu,0}\left(2\sqrt{\frac{1}{4}+\mu}\right)\right. \\ & \left. +2s\int_{\sqrt{\frac{1}{4}+\mu}}^{2\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,0}\left(t\right)\, dt\right), \end{aligned}$$

for the case $k = 0$, should ω not equal zero.

Proof. We will only deal with the case $k \neq 0$, as the other one is similar. Throughout this proof, we will assume that s is a complex number such that we have $1 < \operatorname{Re} s < 2$. The idea is to perform an integration by parts on $L_{\mu,k}$. We have

$$\begin{aligned} L_{\mu,k}\left(s\right) \quad = \quad \frac{\sin(\pi s)}{\pi} & \left(\left[\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} F_{\mu,k}\left(t\right)\right]_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}\right. \\ & \left. +2s\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^\delta\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,k}\left(t\right)\, \mathrm{d}t\right), \end{aligned}$$

so the only thing that remains to be done is to compute the first term above. On the appropriate interval, we have, using proposition 4.6.2,

$$\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} F_{\mu,k}(t) \quad = \quad \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s+2} R_{\mu,k}(t) \quad \underset{t\to\sqrt{\frac{1}{4}+\mu}^{+}}{\longrightarrow} \quad 0,$$

since the real part of s is strictly smaller than 2. This completes the proof, as it yields

$$\begin{aligned} L_{\mu,k}(s) \quad = \quad \tfrac{\sin(\pi s)}{\pi} \Bigg(& \tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s} F_{\mu,k}\left(2|k|^{\delta}\sqrt{\tfrac{1}{4}+\mu}\right) \\ & +2s\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,k}(t)\ \mathrm{d}t\Bigg). \end{aligned}$$

□

Having broken $L_{\mu,k}$ into two parts, we will now study them separately. As indicated before this last proposition, the part that involves an integral will actually not play any role. Before moving on to these studies, let us name these parts of $L_{\mu,k}$ for clarity.

Definition 4.6.5. For any integer k, and any ω and μ as before, we set

$$A_{\mu,k}(s) \quad = \quad \begin{cases} \dfrac{\sin(\pi s)}{\pi}\dfrac{1}{4^s}\left(\dfrac{1}{4}+\mu\right)^{-s}\left(|k|^{2\delta}-\dfrac{1}{4}\right)^{-s} F_{\mu,k}\left(2|k|^{\delta}\sqrt{\dfrac{1}{4}+\mu}\right) & \text{if } k\neq 0 \\ \dfrac{\sin(\pi s)}{\pi}\dfrac{1}{3^s}\left(\dfrac{1}{4}+\mu\right)^{-s} F_{\mu,0}\left(2\sqrt{\dfrac{1}{4}+\mu}\right) & \text{if } k=0 \\ & \text{and } \omega\neq 0 \end{cases}$$

for the integrated part, and

$$B_{\mu,k}(s) \quad = \quad \begin{cases} 2s\,\dfrac{\sin(\pi s)}{\pi}\displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\dfrac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,k}(t)\ \mathrm{d}t & \text{if } k\neq 0 \\ 2s\,\dfrac{\sin(\pi s)}{\pi}\displaystyle\int_{\sqrt{\frac{1}{4}+\mu}}^{2\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\dfrac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,0}(t)\ \mathrm{d}t & \text{if } k=0 \\ & \text{and } \omega\neq 0 \end{cases}$$

for the other one. These functions are defined on the strip $1<\operatorname{Re}s<2$.

4.6.2 Study of the terms $B_{\mu,k}$

The purpose of this paragraph will be to prove the following result.

Proposition 4.6.6. *For any real number $\mu>0$, and any ω in the interval $[0,2\pi[$, the function*

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} B_{\mu,k}(s),$$

which is well-defined and holomorphic on the strip $1<\operatorname{Re}s<2$, has a holomorphic continuation to an open neighborhood of 0, and we have

$$\frac{\partial}{\partial s}_{|s=0} \sum_{|k|\geqslant 1} B_{\mu,k}(s) \quad = \quad 0.$$

Remark 4.6.7. The proposition above features an abuse of notation which will be repeated many times in this chapter. Indeed, the function

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} B_{\mu,k}(s) \quad ,$$

has very little chance of being already defined and holomorphic around $s = 0$, so the expression

$$\frac{\partial}{\partial s}_{|s=0} \quad \sum_{|k|\geqslant 1} B_{\mu,k}(s) \quad = \quad 0.$$

is to be understood as "the derivative at $s = 0$ of the continuation of", rather than "the derivative at $s = 0$ of". This has the advantage of making which term we are talking about, since there will be many of them.

Proof of proposition 4.6.6. For any non-zero integer k, and any real number

$$t \quad \in \quad \left]\sqrt{\tfrac{1}{4}+\mu},\ 2|k|^{\delta}\sqrt{\tfrac{1}{4}+\mu}\right[\quad ,$$

we can bound the term appearing in the integral defining $B_{\mu,k}$ using proposition 4.6.2, as we have

$$\left|\frac{t}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{s+1}} F_{\mu,k}(t)\right| \quad = \quad \left|\frac{t}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{s-1}} R_{\mu,k}(t)\right| \quad \leqslant \quad C_{\mu} \cdot \frac{t}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{\operatorname{Re}s-1}} \cdot \frac{1}{|k|^{2-4\delta}} \cdot \frac{1}{a^2}.$$

We now note that the right hand side of this inequality can be bounded uniformly in s on any strip

$$\alpha \quad < \quad \operatorname{Re}s \quad < \quad \beta \quad < \quad 2$$

with α and β being fixed, possibly negative, real numbers, using the following inequalities

$$\frac{1}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{\operatorname{Re}s-1}} \quad \leqslant \quad \begin{cases} \frac{1}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{\alpha-1}} & \text{if} \quad t^2-\left(\frac{1}{4}+\mu\right) > 1 \\ \frac{1}{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{\beta-1}} & \text{if} \quad t^2-\left(\frac{1}{4}+\mu\right) < 1 \end{cases} .$$

For any such α and β, the dominated convergence theorem proves that the function

$$s \quad \longmapsto \quad \int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} \; F_{\mu,k}(t)\ \mathrm{d}t$$

is holomorphic on the strip $\alpha < \operatorname{Re}s < \beta$, which means that, due to the randomness of α and β, it is holomorphic on the half-plane $\operatorname{Re}s < 2$, where we further have

$$\begin{aligned} &\left|\int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} \; F_{\mu,k}(t)\ \mathrm{d}t\right| \\ &\qquad \leqslant \quad \frac{C_{\mu}}{|k|^{2-4\delta}a^2} \int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-\operatorname{Re}s+1}\ \mathrm{d}t \\ &\qquad \leqslant \quad \frac{C_{\mu}}{|k|^{2-4\delta}a^2}\left[\frac{1}{2-\operatorname{Re}s}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-\operatorname{Re}s+2}\right]_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} \\ &\qquad \leqslant \quad \frac{C_{\mu}}{2-\operatorname{Re}s}\left(\tfrac{1}{4}+\mu\right)^{-\operatorname{Re}s+2} \cdot \frac{1}{|k|^{2-4\delta}a^2} \cdot \frac{1}{\left(4|k|^{2\delta}-1\right)^{\operatorname{Re}s-2}}. \end{aligned}$$

The dominated convergence theorem then proves that the function

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} \int_{\sqrt{\frac{1}{4}+\mu}}^{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}} t\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s-1} F_{\mu,k}(t)\ \mathrm{d}t$$

is well-defined and holomorphic on the strip

$$4-\tfrac{1}{2\delta} \quad < \quad \operatorname{Re} s \quad < \quad 2\ ,$$

which contains 0 if we have $0<\delta<1/8$, which we may assume. Hence the function

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} B_{\mu,k}(s)$$

is holomorphic around 0, and we have

$$\frac{\partial}{\partial s}_{|s=0} \sum_{|k|\geqslant 1} B_{\mu,k}(s) \quad = \quad 0,$$

because the term $B_{\mu,k}$ involves the product of a function which we have shown was holomorphic around 0 with the factor $s\sin(\pi s)$.

□

Remark 4.6.8. In the last proposition, we have written a sum over non-zero integers k as way to give a uniform result, which holds for every element ω of $[0,2\pi[$. The following proposition deals with that voluntarily created gap.

Proposition 4.6.9. *For any real number $\mu>0$, and any ω in $]0,2\pi[$, the function*

$$s \quad \longmapsto \quad B_{\mu,0}(s)\ ,$$

which is well-defined and holomorphic on the strip $1<\operatorname{Re} s<2$, has a holomorphic continuation to an open neighborhood of 0, and we have

$$\frac{\partial}{\partial s}_{|s=0} B_{\mu,0}(s) \quad = \quad 0.$$

Proof. This result can be proved using the same methods as those of the last proposition. The argument is actually simpler, as there are no series involved here, only an integral.

□

Remark 4.6.10. The reader will note that putting propositions 4.6.6 and 4.6.9 together yields a result for the full sum over k when ω does not equal zero. Even though there is no new result, we summarize that in the following proposition.

Proposition 4.6.11. *Let μ and ω be real numbers satisfying $\mu>0$ and $\omega\in[0,2\pi[$. The function*

$$s \quad \longmapsto \quad \begin{cases} \sum_{k\in\mathbb{Z}} B_{\mu,k}(s) & \textit{if} \quad \omega\neq 0 \\ \\ \sum_{|k|\geqslant 1} B_{\mu,k}(s) & \textit{if} \quad \omega=0 \end{cases}$$

is holomorphic on the strip

$$4 - \tfrac{1}{2\delta} \quad < \quad \mathrm{Re}\, s \quad < \quad 2 \ ,$$

which contains 0 *if we have* $0 < \delta \ < \ 1/8$*, and we have*

$$\begin{cases} \dfrac{\partial}{\partial s}_{|s=0} \displaystyle\sum_{k\in\mathbb{Z}} B_{\mu,k}\left(s\right) & = & 0 \quad \textit{if} \quad \omega = 0 \\ \dfrac{\partial}{\partial s}_{|s=0} \displaystyle\sum_{|k|\geqslant 1} B_{\mu,k}\left(s\right) & = & 0 \quad \textit{if} \quad \omega \neq 0 \end{cases} .$$

This concludes the study of $B_{\mu,k}$ as defined in defnition 4.6.5. We now move on to $A_{\mu,k}$.

4.6.3 Study of the terms $A_{\mu,k}$

The investigation of the behavior of series involving $A_{\mu,k}$ is, as we will see below, significantly more complicated. We begin by recalling the definition of $A_{\mu,k}$, as given in definition 4.6.5. We have

$$A_{\mu,k}\left(s\right) \quad = \quad \begin{cases} \dfrac{\sin\left(\pi s\right)}{\pi}\dfrac{1}{4^s}\left(\dfrac{1}{4}+\mu\right)^{-s}\left(|k|^{2\delta}-\dfrac{1}{4}\right)^{-s} F_{\mu,k}\left(2\,|k|^{\delta}\sqrt{\dfrac{1}{4}+\mu}\right) & \text{if } k\neq 0 \\ \dfrac{\sin\left(\pi s\right)}{\pi}\dfrac{1}{3^s}\left(\dfrac{1}{4}+\mu\right)^{-s} F_{\mu,0}\left(2\sqrt{\dfrac{1}{4}+\mu}\right) & \begin{array}{l}\text{if } k = 0 \\ \text{and } \omega\neq 0\end{array} \end{cases}$$

Even tough we will be able to use the binomial formula, this advantage will be canceled by the fact that the derivatives at 0 will not cancel, which means that we will need to precisely understand their asymptotic behavior, as μ goes to infinity for any a, and as a goes to infinity for $\mu = 0$. Let us state the first piece of the result we wish to prove.

Theorem 4.6.12. *Let* μ *and* ω *be real numbers satisfying* $\mu\ >\ 0$ *and* $\omega\ \in\ [0, 2\pi[$*. The function*

$$s \quad \longmapsto \quad \begin{cases} \sum_{k\in\mathbb{Z}} A_{\mu,k}\left(s\right) & \textit{if} \quad \omega\neq 0 \\ \sum_{|k|\geqslant 1} A_{\mu,k}\left(s\right) & \textit{if} \quad \omega = 0 \end{cases}$$

is well-defined and holomorphic on the strip $1 \ < \ \mathrm{Re}\, s \ < \ 2$*, and has a holomorphic continuation to an open neighborhood of* 0*. Its derivative at* $s = 0$ *satisfies, as* a *goes to infinity for* $\mu = 0$*,*

$$\frac{\partial}{\partial s}_{|s=0} \sum_{k\in\mathbb{Z}} A_{0,k}\left(s\right) \quad = \quad O\left(\frac{1}{a^2}\right) \quad \textit{if} \quad \omega\neq 0,$$

$$\frac{\partial}{\partial s}_{|s=0} \sum_{|k|\geqslant 1} A_{0,k}\left(s\right) \quad = \quad O\left(\frac{1}{a^2}\right) \quad \textit{if} \quad \omega = 0.$$

We will now move on to the proof of theorem 4.6.12. The idea is to use an argument which is similar to the one used to deal with $B_{\mu,k}$. Before proceeding to that, we need to perform a small computation, which is the object of the next proposition.

Proposition 4.6.13. *Let μ and ω be real numbers satisfying $\mu > 0$ and $\omega \in [0, 2\pi[$. For every integer k, the function $s \longmapsto A_{\mu,k}(s)$ is holomorphic on $\mathbb{C}$, and its derivative satisfies*

$$\begin{aligned} \tfrac{\partial}{\partial s} A_{\mu,k} \quad = \quad & \cos(\pi s) \tfrac{1}{4^s} \left(\tfrac14 + \mu\right)^{-s} \left(|k|^{2\delta} - \tfrac14\right)^{-s} F_{\mu,k}\left(2|k|^\delta \sqrt{\tfrac14 + \mu}\right) \\ & - \left[\tfrac{\sin(\pi s)}{\pi} \log\left((4\mu+1)\left(|k|^{2\delta} - \tfrac14\right)\right) \tfrac{1}{4^s} \left(\tfrac14 + \mu\right)^{-s} \right. \\ & \left. \cdot \left(|k|^{2\delta} - \ 14\right)^{-s} F_{\mu,k}\left(2|k|^\delta \sqrt{\tfrac14 + \mu}\right)\right], \end{aligned}$$

in the case $k \neq 0$, and, if ω is different from zero, we have

$$\begin{aligned} \tfrac{\partial}{\partial s} A_{\mu,0}(s) \quad = \quad & \cos(\pi s) \tfrac{1}{3^s} \left(\tfrac14 + \mu\right)^{-s} F_{\mu,0}\left(2\sqrt{\tfrac14 + \mu}\right) \\ & - \tfrac{\sin(\pi s)}{\pi} \log\left(3\left(\tfrac14 + \mu\right)\right) \tfrac{1}{3^s} \left(\tfrac14 + \mu\right)^{-s} F_{\mu,0}\left(2\sqrt{\tfrac14 + \mu}\right). \end{aligned}$$

Proof. The proof of this result directly stems from definition 4.6.5.

□

Proposition 4.6.14. *Let μ and ω be real numbers satisfying $\mu > 0$ and $\omega \in [0, 2\pi[$. The function*

$$s \quad \longmapsto \quad \sum_{|k| \geqslant 1} A_{\mu,k}(s)$$

induces a holomorphic function on the half-plane

$$\operatorname{Re} s \quad > \quad 2 - \frac{1}{4\delta} \quad ,$$

which contains 0 if we have $\delta < 1/8$. On this half-plane, we can further differentiate term by term, and the derivative at 0 satisfies, as a goes to infinity,

$$\frac{\partial}{\partial s}_{|s=0} \sum_{|k| \geqslant 1} A_{0,k}(s) \quad = \quad O\left(\frac{1}{a^2}\right).$$

Proof. For any non-zero integer k, proposition 4.6.2 yields

$$\left| \left(|k|^{2\delta} - \tfrac14\right)^{-s} F_{\mu,k}\left(2|k|^\delta \sqrt{\tfrac14 + \mu}\right) \right| \quad \leqslant \quad C_\mu \left(\tfrac14 + \mu\right) \cdot \tfrac{1}{16a^2} \cdot \tfrac{1}{|k|^{2-4\delta}} \left(|k|^{2\delta} - \tfrac14\right)^{2 - \operatorname{Re} s} \quad ,$$

which, by the dominated convergence theorem, proves that the sum of $A_{\mu,k}$ over $|k| \geqslant 1$ induces a holomorphic function on the half-plane $\operatorname{Re} s > 2 - 1/(4\delta)$, and we can further differentiate term by term. Evaluating the derivative of $A_{\mu,k}$ at $s = 0$ yields

$$\tfrac{\partial}{\partial s}_{|s=0} A_{\mu,k}(s) \quad = \quad F_{\mu,k}\left(2|k|^\delta \sqrt{\tfrac14 + \mu}\right) \quad ,$$

and we can now set $\mu = 0$ to get the asymptotic behavior as a goes to infinity. We have

$$\left| \frac{\partial}{\partial s}_{|s=0} A_{0,k}(s) \right| \quad \leqslant \quad \frac14 C_0 \frac{1}{16a^2} \cdot \frac{1}{|k|^{2-4\delta}} \left(|k|^{2\delta} - \frac14\right)^2 \quad ,$$

and this gives

$$\left|\frac{\partial}{\partial s}_{|s=0}\sum_{|k|\geqslant 1}A_{0,k}\left(s\right)\right| \;=\; \left|\sum_{|k|\geqslant 1}\frac{\partial}{\partial s}_{|s=0}A_{0,k}\left(s\right)\right| \;\leqslant\; \frac{C_0}{4}\frac{1}{16a^2}\sum_{|k|\geqslant 1}\frac{1}{|k|^{2-4\delta}}\left(|k|^{2\delta}-\frac{1}{4}\right)^2 \;.$$

Since the series on the right hand side is absolutely convergent, we get

$$\frac{\partial}{\partial s}_{|s=0}\sum_{|k|\geqslant 1}A_{0,k}\left(s\right) \;=\; O\left(\frac{1}{a^2}\right) \;.$$

□

As we did for $B_{\mu,k}$, we will need to deal with the case $k = 0$ to complete the picture.

Proposition 4.6.15. *For any real number $\mu > 0$, and any ω lying in the interval $]0,2\pi[$, the derivative of the function $A_{0,0}$ satisfies*

$$\frac{\partial}{\partial s}_{|s=0}A_{0,0}\left(s\right) \;=\; O\left(\frac{1}{a^2}\right) \;.$$

Proof. After evaluating at $s = 0$ the derivative of $A_{0,0}$, we get

$$\frac{\partial}{\partial s}_{|s=0}A_{\mu,0}\left(s\right) \;=\; F_{\mu,0}\left(2\sqrt{\frac{1}{4}+\mu}\right) \;,$$

which gives, at $\mu = 0$

$$\frac{\partial}{\partial s}_{|s=0}A_{0,0}\left(s\right) \;=\; F_{0,0}\left(1\right) \;,$$

and we can thus complete the proof of the theorem by making use of remark 4.6.3, which states that we can have a result similar to 4.6.2 in the case $k = 0$ by using similar methods.

□

Now that we have studied the regularity of the sum of $A_{\mu,k}$, and the asymptotics as a goes to infinity for $\mu = 0$, we turn to the asymptotics as μ goes to infinity, for every $a > 0$. Unlike what we did with theorem 4.6.12, these asymptotics will be too complicated to fully state right away. Instead, we will break $A_{\mu,k}$ into several pieces, some of which will be too complicated to study in details. Fortunately, these will cancel other complicated terms that will appear in the next section. Recalling definition 4.6.5, we see that we need information on the term involving $F_{\mu,k}$, which, according to definition 4.6.1, is given for every non-zero integer k by

$$\begin{aligned} F_{\mu,k}\left(2\left|k\right|^{\delta}\sqrt{\tfrac{1}{4}+\mu}\right) \;&=\; \log K_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}\left(C_{\omega,k}a\right)-\log K_{\sqrt{\frac{1}{4}+\mu}}\left(C_{\omega,k}a\right) \\ &\qquad -2\sqrt{\tfrac{1}{4}+\mu}\,\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right), \end{aligned}$$

and for $k = 0$, assuming we have $\omega \neq 0$, by

$$\begin{aligned} F_{\mu,0}\left(2\sqrt{\tfrac{1}{4}+\mu}\right) \;&=\; \log K_{2\sqrt{\frac{1}{4}+\mu}}\left(\omega a\right)-\log K_{\sqrt{\frac{1}{4}+\mu}}\left(\omega a\right) \\ &\qquad -2\sqrt{\tfrac{1}{4}+\mu}\,\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(\omega a\right). \end{aligned}$$

The main purpose of this section will be to study the difference

$$\log K_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}(C_{\omega,k}a) \quad - \quad \log K_{\sqrt{\frac{1}{4}+\mu}}(C_{\omega,k}a) \ ,$$

and its analog for the case $k = 0$, namely

$$\log K_{2\sqrt{\frac{1}{4}+\mu}}(\omega a) \quad - \quad \log K_{\sqrt{\frac{1}{4}+\mu}}(\omega a) \ ,$$

while the other terms, which are essentially going to cancel other terms we have yet to encounter, will be dealt with at the end of this section. The reader is, from now on, assumed to be familiar with modified Bessel functions of the second kind, and is referred to [74, 75] to that effect.

Proposition 4.6.16. *For every integer $k \neq 0$, any real numbers $\mu > 0$ and $\omega \in [0, 2\pi[$, we have*

$$\begin{aligned}
&\log K_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}(C_{\omega,k}a) \\
&= \ \tfrac{1}{2}\log\left(\tfrac{\pi}{2}\right) - \sqrt{(C_{\omega,k}a)^2 + (4\mu+1)\,|k|^{2\delta}} + |k|^\delta\,\sqrt{4\mu+1}\ \operatorname{Argsh}\left(\tfrac{|k|^\delta\sqrt{4\mu+1}}{C_{\omega,k}a}\right) \\
&\qquad -\tfrac{1}{4}\log\left((C_{\omega,k}a)^2 + (4\mu+1)\,|k|^{2\delta}\right) + \tfrac{1}{|k|^\delta\sqrt{4\mu+1}}\ U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right)\right) \\
&\qquad +\tfrac{1}{(4\mu+1)|k|^{2\delta}}\ \widetilde{\rho}_2\left(\sqrt{4\mu+1}\,|k|^\delta\,,\ \ \tfrac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right),
\end{aligned}$$

where U_1 and $\widetilde{\rho}_2$ are defined as in [74, 75].

Proposition 4.6.17. *For any real numbers $\mu > 0$ and $\omega \in]0, 2\pi[$, we have*

$$\begin{aligned}
\log K_{2\sqrt{\frac{1}{4}+\mu}}(\omega a) \ &= \ \tfrac{1}{2}\log\left(\tfrac{\pi}{2}\right) - \sqrt{(\omega a)^2 + (4\mu+1)} + \sqrt{4\mu+1}\ \operatorname{Argsh}\left(\tfrac{\sqrt{4\mu+1}}{\omega a}\right) \\
&\qquad -\tfrac{1}{4}\log\left((\omega a)^2 + (4\mu+1)\right) + \tfrac{1}{\sqrt{4\mu+1}}\ U_1\left(\tau\left(\tfrac{\omega a}{\sqrt{4\mu+1}}\right)\right) \\
&\qquad +\tfrac{1}{4\mu+1}\ \widetilde{\rho}_2\left(\sqrt{4\mu+1},\ \ \tfrac{\omega a}{\sqrt{4\mu+1}}\right),
\end{aligned}$$

where U_1 and $\widetilde{\rho}_2$ are defined as in [74, 75].

Proposition 4.6.18. *For every integer k, assuming ω not to be zero for the case $k = 0$, any real numbers $\mu > 0$ and $\omega \in [0, 2\pi[$, we have*

$$\begin{aligned}
\log K_{\sqrt{\frac{1}{4}+\mu}}(C_{\omega,k}a) \ &= \ \tfrac{1}{2}\log\left(\tfrac{\pi}{2}\right) - \sqrt{(C_{\omega,k}a)^2 + \tfrac{1}{4} + \mu} \ + \ \sqrt{\tfrac{1}{4}+\mu}\ \operatorname{Argsh}\left(\tfrac{\sqrt{1/4+\mu}}{C_{\omega,k}a}\right) \\
&\qquad -\tfrac{1}{4}\log\left((C_{\omega,k}a)^2 + \tfrac{1}{4} + \mu\right) \ + \ \tfrac{1}{\sqrt{1/4+\mu}}\ U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{1/4+\mu}}\right)\right) \\
&\qquad +\tfrac{1}{1/4+\mu}\ \widetilde{\rho}_2\left(\sqrt{\tfrac{1}{4}+\mu},\ \ \tfrac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right),
\end{aligned}$$

where U_1 and $\widetilde{\rho}_2$ are defined as in [74, 75].

Because the aim is to study the part of $A_{\mu,k}$ given by

$$\begin{cases} \frac{\sin(\pi s)}{\pi}\ \frac{1}{4^s}\ \left(\frac{1}{4}+\mu\right)^{-s}\ \left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\ \left(\log K_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}\left(C_{\omega,k}a\right)-\log K_{\sqrt{\frac{1}{4}+\mu}}\left(C_{\omega,k}a\right)\right)\ \text{if } k\neq 0 \\ \frac{\sin(\pi s)}{\pi}\ \frac{1}{3^s}\ \left(\frac{1}{4}+\mu\right)^{-s}\ \left(\log K_{2\sqrt{\frac{1}{4}+\mu}}\left(\omega a\right)-\log K_{\sqrt{\frac{1}{4}+\mu}}\left(\omega a\right)\right)\ \text{if } k=0 \text{ and } \omega\neq 0 \end{cases},$$

we will split each term according to the expressions given in propositions 4.6.16, 4.6.17, and 4.6.18. We will then prove that the sum over all integers k, with the exception of $k=0$ if ω vanishes, of these terms has a holomorphic continuation to an open neighborhood of 0, and give an asymptotic expansion of all derivatives at $s=0$ as μ goes to infinity. As forewarned in remark 4.6.7, we will often mean "derivative of the continuation of" by "derivative of" in order to be able to precisely keep track of which term we are dealing with. We denote by ω a real number in $[0,2\pi[$, and by μ a positive real number. As much as possible, we will deal with the case $k=0$ separately each time, and we will always assume that we have $\omega\neq 0$ when doing it.

First part. In this first part of the computation, we are going to prove the following proposition.

Proposition 4.6.19. *The function*

$$s \longmapsto \frac{\sin(\pi s)}{\pi}\cdot\frac{1}{4^{s+1}}\cdot\left(\frac{1}{4}+\mu\right)^{-s-1}\sum_{|k|\geqslant 1}\left[\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\cdot\frac{1}{|k|^{2\delta}}\right.$$
$$\left.\widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^\delta\,,\ \frac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right)\right]$$

is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \ >\ -\frac{1}{2\delta}\ ,$$

and its derivative at $s=0$ satisfies, as μ goes to infinity,

$$\frac{\partial}{\partial s}_{|s=0}\left(\frac{\sin(\pi s)}{\pi}\cdot\frac{1}{4^{s+1}}\cdot\left(\frac{1}{4}+\mu\right)^{-s-1}\sum_{|k|\geqslant 1}\left[\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\right.\right.$$
$$\left.\left.\cdot\frac{1}{|k|^{2\delta}}\ \widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^\delta\,,\ \frac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right)\right]\right)\ =\ o(1)\,.$$

Proof. The first point to note is that the asymptotic expansions of the modified Bessel functions of the second kind given in [74, 75] are actually different than those stated here. The difference is that we need to study the logarithm of such functions, and not the functions themselves. In order to prove this proposition, we will need to relate the polynomial U_1 and the remainder $\widetilde{\rho_2}$ to the polynomial u_1 and the remainder ρ_2 used by Olver in [74, 75]. We have

$$\begin{aligned}&\frac{1}{(4\mu+1)|k|^{2\delta}}\widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^\delta\,,\ \frac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right)\\ &\quad=\ \log\left(1-\frac{1}{\sqrt{4\mu+1}|k|^\delta}u_1\left(\tau\left(\frac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^\delta}\right)\right)+\frac{1}{(4\mu+1)|k|^{2\delta}}\rho_2\left(\sqrt{4\mu+1}\,|k|^\delta\,,\ \frac{C_{\omega,k}a}{|k|^\delta\sqrt{4\mu+1}}\right)\right)\\ &\qquad\qquad+\frac{1}{\sqrt{4\mu+1}|k|^\delta}u_1\left(\tau\left(\frac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^\delta}\right)\right).\end{aligned}$$

The polynomial u_1 is given by

$$u_1(t)\ =\ \frac{1}{24}\left(3t-5t^3\right)\ ,$$

and τ is the function $x \longmapsto \left(1+x^2\right)^{-1/2}$. This function is bounded by 1, and the remainder ρ_2 is uniformly bounded, so we can write the logarithm as the sum of its Taylor series, giving

$$\begin{aligned}
&\tfrac{1}{(4\mu+1)|k|^{2\delta}}\widetilde{\rho}_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\\
&= \sum_{n=1}^{+\infty}\tfrac{(-1)^{n+1}}{n}\left[\tfrac{1}{(4\mu+1)|k|^{2\delta}}\rho_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)-\tfrac{1}{\sqrt{4\mu+1}|k|^{\delta}}u_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)\right)\right]^n\\
&\qquad +\tfrac{1}{\sqrt{4\mu+1}|k|^{\delta}}u_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)\right)\\
&= \sum_{n=2}^{+\infty}\tfrac{(-1)^{n+1}}{n}\left[\tfrac{1}{(4\mu+1)|k|^{2\delta}}\rho_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)-\tfrac{1}{\sqrt{4\mu+1}|k|^{\delta}}u_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)\right)\right]^n\\
&\qquad +\tfrac{1}{(4\mu+1)|k|^{2\delta}}\rho_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right).
\end{aligned}$$

We need to work with the estimate on ρ_2 given by Olver, which is

$$\tfrac{1}{4\mu+1}\left|\rho_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right| \leqslant \tfrac{2}{4\mu+1}\exp\left[\tfrac{2}{\sqrt{4\mu+1}}\mathcal{V}_{0,\tau}(u_1)\right]\mathcal{V}_{0,\tau}(u_2)\ ,$$

where $\mathcal{V}_{0,\tau}$ denotes the total variation of a function on the interval

$$\left[0,\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)\right]\ ,$$

and u_2 is the polynomial

$$u_2(x) = \tfrac{1}{1152}\left(81x^2-462x^4+385x^6\right)\ .$$

We now note that the exponential in the last inequality above is uniformly bounded in μ and k, since the function τ is bounded by 1, giving

$$\tfrac{2}{\sqrt{4\mu+1}}\mathcal{V}_{0,\tau}(u_1) \leqslant 2\mathcal{V}_{0,1}(u_1)\ .$$

Furthermore, we have

$$\begin{aligned}
\mathcal{V}_{0,\tau}(u_2) &\leqslant \tfrac{1}{1152}\left(81\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^2+462\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^4+385\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^6\right)\\
&\leqslant \tfrac{81+462+385}{1152}\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^2 \leqslant \tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^2 \leqslant (4\mu+1)\tfrac{|k|^{2\delta}}{(C_{\omega,k}a)^2}.
\end{aligned}$$

This gives the following bound

$$\tfrac{1}{4\mu+1}\left|\rho_2\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right| \leqslant C\cdot\tfrac{|k|^{2\delta}}{(C_{\omega,k}a)^2}\ ,$$

where $C = 2\exp\left[2\mathcal{V}_{0,1}(u_1)\right]$ does not depend on any of the parameter. We also have

$$\begin{aligned}
\left|u_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)\right)\right| &\leqslant \tfrac{1}{24}\left(3\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)+5\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right)^3\right)\\
&\leqslant \tfrac{1}{3}\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{4\mu+1}|k|^{\delta}}\right) \leqslant \tfrac{1}{3}\sqrt{4\mu+1}\tfrac{|k|^{\delta}}{C_{\omega,k}a}.
\end{aligned}$$

Combining these estimates, we get

$$\begin{aligned}\tfrac{1}{(4\mu+1)|k|^{2\delta}}\left|\widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right| \quad &\leqslant \quad \tfrac{C}{(C_{\omega,k}a)^2}+\sum_{n=2}^{+\infty}\tfrac{1}{n}\left[\tfrac{C}{(C_{\omega,k}a)^2}+\tfrac{1}{3C_{\omega,k}a}\right]^n \\ &\leqslant \quad \tfrac{C}{(C_{\omega,k}a)^2}+\sum_{n=2}^{+\infty}\tfrac{1}{n}\left[\tfrac{C}{C_{\omega,k}a}+\tfrac{1}{3}\right]^n\cdot\tfrac{1}{(C_{\omega,k}a)^n},\end{aligned}$$

which holds for integers k large enough in absolute value, independantly of any parameter. For such integers $|k|\geqslant K_0$, we have

$$\tfrac{1}{(4\mu+1)|k|^{2\delta}}\left|\widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right| \quad \leqslant \quad \tfrac{1}{(C_{\omega,k}a)^2}\left[C+\tfrac{1}{2}\right]\ .$$

Hence the function

$$s \quad \longmapsto \quad \tfrac{1}{4\mu+1}\cdot\sum_{|k|\geqslant 1}\left[\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\cdot\tfrac{1}{|k|^{2\delta}}\widetilde{\rho_2}\left(\sqrt{4\mu+1}\,|k|^{\delta},\ \tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right]$$

is holomorphic on the half-plane $\operatorname{Re} s > -1/(2\delta)$, and the inequality above further allows us to use the dominated convergence theorem to prove that its value at $s=0$ vanishes as μ goes to infinity, since $\widetilde{\rho_2}$ is uniformly bounded. This completes the proof.

□

Second part. We now need to take care of the other term involving a remainder $\widetilde{\rho_2}$, which is the purpose of the following proposition.

Proposition 4.6.20. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s-1}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\widetilde{\rho_2}\left(\sqrt{\frac{1}{4}+\mu},\ \frac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right)$$

is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \quad > \quad -\tfrac{1}{2\delta}$$

and its derivative at $s=0$ satisfies

$$\tfrac{\partial}{\partial s}_{|s=0}\left(\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^s}\,\left(\tfrac{1}{4}+\mu\right)^{-s-1}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\widetilde{\rho_2}\left(\sqrt{\tfrac{1}{4}+\mu},\ \tfrac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right)\right) \quad = \quad o(1)$$

as μ goes to infinity.

Proof. The proof of this proposition is similar to that of the last one.

□

Third part. Having dealt with the terms involving remainders, we now turn our attention to the more complicated ones. In this part, we deal with the term indicated in the proposition below.

Proposition 4.6.21. *The function*

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\sqrt{(C_{\omega,k}a)^2+(4\mu+1)\,|k|^{2\delta}}$$

is well-defined and holomorphic on the half-plane $\mathrm{Re}\, s > 1/\delta$*, has a holomorphic continuation to a neighborhood of* 0*, and the derivative at* $s = 0$ *of this continuation satisfies*

$$\frac{\partial}{\partial s}_{|s=0} \left(-\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4} + \mu\right)^{-s} \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \frac{1}{4}\right)^{-s} \sqrt{(C_{\omega,k} a)^2 + (4\mu+1)|k|^{2\delta}} \right)$$
$$= -\frac{1}{4\pi a\delta}\mu - \frac{1}{16\pi a\delta} + \frac{\partial}{\partial s}_{|s=0} \left[-\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4} + \mu\right)^{-s} \cdot \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}} \right].$$

Remark 4.6.22. The last term in the asymptotics above is left uncomputed, because it will be canceled by another term that will appear in the next section. Since a derivative at $s = 0$ is considered, we will still need to prove that the term which is differentiated has a holomorphic continuation to a neighborhood of 0.

Proof of proposition 4.6.21. We first note that the function

$$s \longmapsto \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \frac{1}{4}\right)^{-s} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}}$$

is well-defined and holomorphic on the half-plane $\mathrm{Re}\, s > 1/\delta$. Using the binomial formula, we have

$$\sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \frac{1}{4}\right)^{-s} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}}$$
$$= \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta(s+j)}} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}},$$

the interchanging of both sums being possible as k is not zero, which means that we have

$$\frac{1}{4|k|^{2\delta j}} \leqslant \frac{1}{4}.$$

We can then proceed with the computation. We have, on the appropriate half-plane

$$\sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \frac{1}{4}\right)^{-s} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}}$$
$$= \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}}$$
$$+ \sum_{j\geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta(s+j)}} \sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}}.$$

We will now expand in $1/|k|$ the term involved in both sums above at a high enough order so that the remainder would induce a holomorphic function around 0. The finitely many other terms will be deal with using the Riemann zeta function. For every non-zero integer k, we have

$$\sqrt{(2\pi k + \omega)^2 a^2 + (4\mu+1)|k|^{2\delta}} = 2\pi a|k| \sqrt{1 + \frac{\omega}{\pi} \cdot \frac{1}{|k|} + \frac{\omega^2}{4\pi^2} \cdot \frac{1}{k^2} + \frac{4\mu+1}{4\pi^2 a^2} \cdot \frac{1}{|k|^{2-2\delta}}}$$
$$= 2\pi a|k| \left[\sqrt{1 + \frac{\omega}{\pi} \cdot \frac{1}{|k|} + \frac{\omega^2}{4\pi^2} \cdot \frac{1}{k^2} + \frac{4\mu+1}{4\pi^2 a^2} \cdot \frac{1}{|k|^{2-2\delta}}} - \left(1 + \frac{\omega}{2\pi} \cdot \frac{1}{|k|} + \frac{4\mu+1}{8\pi^2 a^2} \cdot \frac{1}{|k|^{2-2\delta}}\right) \right]$$
$$+ \left(1 + \frac{\omega}{2\pi} \cdot \frac{1}{|k|} + \frac{4\mu+1}{8\pi^2 a^2} \cdot \frac{1}{|k|^{2-2\delta}}\right).$$

We will now deal with both sums above separately. We have

$$
\begin{aligned}
&\sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}} \sqrt{\left(2\pi k+\omega\right)^2 a^2 + \left(4\mu+1\right)|k|^{2\delta}} \\
&\quad = \quad 2\pi a \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s-1}} \left[\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}} - \left(1+\frac{\omega}{2\pi}\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\frac{1}{|k|^{2-2\delta}}\right)\right] \\
&\qquad\qquad +2\pi a \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s-1}} \left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right) \\
&\quad = \quad 2\pi a \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s-1}} \left[\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}} - \left(1+\frac{\omega}{2\pi}\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\frac{1}{|k|^{2-2\delta}}\right)\right] \\
&\qquad\qquad +2\zeta\left(2\delta s-1\right)+\frac{\omega}{\pi}\zeta\left(2\delta s\right)+\frac{4\mu+1}{4\pi^2a^2}\zeta\left(2\delta\left(s-1\right)+1\right),
\end{aligned}
$$

where one should remember that we sum over both strictly positive and negative integers, producing a factor 2 in front of the Riemann zeta function. The function associated to the second part of the above can then be seen to admit a holomorphic continuation to a neighborhood of 0 (whose precise description is irrelevant here). For every non-zero integer k, we further have

$$
\begin{aligned}
&\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}} - \left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right) \\
&\qquad = \quad \frac{1}{\left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right)+\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}}} \\
&\qquad\qquad \cdot\left(1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}} - \left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right)^2\right).
\end{aligned}
$$

We now note that the first factor is bounded by 1, and thus will not matter much. The second factor will involve some explicit cancellations, as we have

$$
\begin{aligned}
&1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}} - \left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right)^2 \\
&\qquad = \quad \cancel{1}+\cancel{\frac{\omega}{\pi}\cdot\frac{1}{|k|}}+\cancel{\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}}+\cancel{\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}}-\cancel{1}-\cancel{\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}}-\frac{(4\mu+1)^2}{64\pi^4a^4}\cdot\frac{1}{|k|^{4-4\delta}}-\cancel{\frac{\omega}{\pi}\cdot\frac{1}{|k|}} \\
&\qquad\qquad -\cancel{\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}}-\frac{4\mu+1}{8\pi^3a^2}\omega\cdot\frac{1}{|k|^{3-2\delta}} \\
&\qquad = \quad -\frac{(4\mu+1)^2}{64\pi^4a^4}\cdot\frac{1}{|k|^{4-4\delta}}-\frac{4\mu+1}{8\pi^3a^2}\,\omega\cdot\frac{1}{|k|^{3-2\delta}},
\end{aligned}
$$

and the full sum yields

$$
\begin{aligned}
&\sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}} \sqrt{\left(2\pi k+\omega\right)^2 a^2 + \left(4\mu+1\right)|k|^{2\delta}} \\
&\qquad = \quad -2\pi a \sum_{|k|\geqslant 1} \left[\frac{1}{|k|^{2\delta s-1}} \frac{1}{\left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right)+\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}}}\right. \\
&\qquad\qquad\qquad \left.\cdot\left(\frac{(4\mu+1)^2}{64\pi^4a^4}\cdot\frac{1}{|k|^{4-4\delta}}+\frac{4\mu+1}{8\pi^3a^2}\,\omega\cdot\frac{1}{|k|^{3-2\delta}}\right)\right] \\
&\qquad\qquad +2\pi a\left(2\zeta\left(2\delta s-1\right)+\frac{\omega}{\pi}\zeta\left(2\delta s\right)+\frac{4\mu+1}{4\pi^2a^2}\zeta\left(2\delta\left(s-1\right)+1\right)\right).
\end{aligned}
$$

Since the first term induces an absolutely convergent series for s in a neighborhood of 0, the full sum induces a holomorphic function around the origing. Its derivative there, after multiplication by the factor left out at the beginning of the proof is not to be explicitely computed, as it will later be canceled. Similarly, we have, for the sum involving the integer j,

$$\begin{aligned}
&\sum_{j\geqslant 1} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \sum_{|k|\geqslant 1} \tfrac{1}{|k|^{2\delta(s+j)}} \sqrt{(2\pi k+\omega)^2 a^2 + (4\mu+1)\,|k|^{2\delta}} \\
&= \quad -2\pi a \sum_{j\geqslant 1} \left(\tfrac{(s)_j}{j!}\tfrac{1}{4^j} \sum_{|k|\geqslant 1} \left[\tfrac{1}{|k|^{2\delta(s+j)-1}} \frac{1}{\left(1+\frac{\omega}{2\pi}\cdot\frac{1}{|k|}+\frac{4\mu+1}{8\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}\right)+\sqrt{1+\frac{\omega}{\pi}\cdot\frac{1}{|k|}+\frac{\omega^2}{4\pi^2}\cdot\frac{1}{k^2}+\frac{4\mu+1}{4\pi^2a^2}\cdot\frac{1}{|k|^{2-2\delta}}}} \right.\right. \\
&\qquad\qquad \left.\left. \cdot\left(\tfrac{(4\mu+1)^2}{64\pi^4a^4}\cdot\tfrac{1}{|k|^{4-4\delta}} + \tfrac{4\mu+1}{8\pi^3a^2}\,\omega\cdot\tfrac{1}{|k|^{3-2\delta}} \right) \right] \right) \\
&\qquad +2\pi a \sum_{j\geqslant 1} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \left(2\zeta\left(2\delta\left(s+j\right)-1\right) + \tfrac{\omega}{\pi}\zeta\left(2\delta\left(s+j\right)\right) + \tfrac{4\mu+1}{4\pi^2a^2}\zeta\left(2\delta\left(s+j-1\right)+1\right) \right).
\end{aligned}$$

The first term above induce a holomorphic function around 0, whose value at $s = 0$ vanishes because of the Pochhammer symbol $(s)_j$. We need to be a bit more careful with the second term. To study it precisely, we need, as much as possible, to stay away from the pole of the Riemann zeta function, which here means we should assume that 2δ is not the inverse of an integer, so that we never have $2\delta j = 1$. Once this hypothesis is made, the function

$$s \quad\longmapsto\quad 2\pi a \sum_{j\geqslant 1} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \left(2\zeta\left(2\delta\left(s+j\right)-1\right) + \tfrac{\omega}{\pi}\zeta\left(2\delta\left(s+j\right)\right) \right)$$

is seen to be holomorphic around 0, and its value at $s = 0$ vanishes, because of the Pochhammer symbol $(s)_j$. The last part to be studied, namely the function

$$s \quad\longmapsto\quad 2\pi a \sum_{j\geqslant 1} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \cdot \tfrac{4\mu+1}{4\pi^2a^2}\zeta\left(2\delta\left(s+j-1\right)+1\right)$$

also induces a holomorphic function around 0, though its value there does not vanish, as, for $j = 1$, the Pochhamer symbol $(s)_1 = s$ can only cancel the pole that arises from the zeta function. The value at $s = 0$ for this term is given by

$$2\pi a \cdot \tfrac{1}{2} \cdot \tfrac{4\mu+1}{8\pi^2a^2} \cdot \tfrac{1}{2\delta} \quad=\quad \tfrac{1}{4\pi a\delta}\mu + \tfrac{1}{16\pi a\delta}\enspace .$$

This concludes the proof of the proposition.

□

Fourth part. We continue the study of the series associated with each term coming from proposition 4.6.16 with the one involving Argsh.

Proposition 4.6.23. *The function*

$$s \quad\longmapsto\quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{4^{s-\frac{1}{2}}}\left(\frac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\frac{1}{4}\right)^{-s} |k|^{\delta}\ \mathrm{Argsh}\left(\frac{|k|^{\delta}\sqrt{4\mu+1}}{C_{\omega,k}a}\right)\enspace ,$$

which is well-defined and holomorphic on the half-plane

$$\mathrm{Re}\, s \quad>\quad \tfrac{1}{2}\left(1+\tfrac{1}{\delta}\right)\enspace ,$$

has a holomorphic continuation to a neighborhood of 0*, and the derivative at* $s = 0$ *of this continuation satisfies*

$$\frac{\partial}{\partial s}_{|s=0} \left(\frac{\sin(\pi s)}{\pi} \, \frac{1}{4^{s-\frac{1}{2}}} \, \left(\tfrac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \tfrac{1}{4}\right)^{-s} \, |k|^{\delta} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{C_{\omega,k}a}\right)\right)$$

$$= \frac{\partial}{\partial s}_{|s=0} \left(\frac{\sin(\pi s)}{\pi} \, \frac{1}{4^{s-\frac{1}{2}}} \, \left(\tfrac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s-\delta}} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right)\right) + \frac{1}{2\pi a\delta}\mu + \frac{1}{8\pi a\delta}.$$

Proof. We first note that the function

$$s \;\longmapsto\; \sqrt{4\mu+1} \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \tfrac{1}{4}\right)^{-s} \, |k|^{\delta} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{C_{\omega,k}a}\right)$$

is indeed well-defined and holomorphic on the half-plane $\operatorname{Re} s > \frac{1}{2}\left(1+\frac{1}{\delta}\right)$. Using the binomial formula, we have

$$\sqrt{4\mu+1} \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \tfrac{1}{4}\right)^{-s} \, |k|^{\delta} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{C_{\omega,k}a}\right)$$

$$= \sqrt{4\mu+1} \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k|\geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right),$$

the interchanging of sums being possible for similar reasons as in the previous case. On the appropriate half-plane, we thus have

$$\sqrt{4\mu+1} \sum_{|k|\geqslant 1} \left(|k|^{2\delta} - \tfrac{1}{4}\right)^{-s} \, |k|^{\delta} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{C_{\omega,k}a}\right)$$

$$= \sqrt{4\mu+1} \sum_{|k|\geqslant 1} \frac{1}{|k|^{\delta(2s-1)}} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right)$$

$$+\sqrt{4\mu+1} \sum_{j\geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k|\geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \, \operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right).$$

We will begin by studying the first term above, which we must show has a holomorphic continuation to a neighborhood of 0. There will be no need for a computation of its value at $s = 0$ as μ goes to infinity, since this term will be cancelled by another one later. The computation that follows will rely on the fundamental theorem of calculus, *i.e.* a first order Taylor expansion. For any non-zero integer k, we have

$$\operatorname{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right) = \frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a} - \int_0^{\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}} \frac{x}{(1+x^2)^{3/2}} \left(\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a} - x\right) \mathrm{d}x$$

$$= \frac{\sqrt{4\mu+1}}{2\pi a|k|^{1-\delta}} \cdot \frac{1}{\left|1+\frac{\omega}{2\pi k}\right|} - \int_0^{\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}} \frac{x}{(1+x^2)^{3/2}} \left(\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a} - x\right) \mathrm{d}x$$

$$= \frac{\sqrt{4\mu+1}}{2\pi a|k|^{1-\delta}} + \frac{\sqrt{4\mu+1}}{2\pi a|k|^{1-\delta}} \cdot \left(\frac{1}{\left|1+\frac{\omega}{2\pi k}\right|} - 1\right) - \int_0^{\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a}} \frac{x}{(1+x^2)^{3/2}} \left(\frac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\, a} - x\right) \mathrm{d}x.$$

It will be more convenient at this point to separate the sum over k into two sums, bearing respectively over strictly positive and negative integers, recalling that we have $C_{\omega,k} = |2\pi k + \omega|$. As one might see in the computation above, we can achieve the same result as changing the sign of k by changing that of ω. For strictly positive integers, we have

$$\begin{aligned}
\mathrm{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\,a}\right) \quad &= \quad \tfrac{\sqrt{4\mu+1}}{2\pi a k^{1-\delta}} + \tfrac{\sqrt{4\mu+1}}{2\pi a k^{1-\delta}} \cdot \left(\tfrac{1}{1+\frac{\omega}{2\pi k}} - 1\right) \\
&\qquad - \int_0^{\frac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}} \tfrac{x}{(1+x^2)^{3/2}} \left(\tfrac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a} - x\right)\,\mathrm{d}x.
\end{aligned}$$

$$= \quad \tfrac{\sqrt{4\mu+1}}{2\pi a k^{1-\delta}} + \tfrac{\sqrt{4\mu+1}}{2\pi a k^{1-\delta}} \cdot \underbrace{\frac{1}{1+\frac{\omega}{2\pi k}}}_{<1} \cdot \left(\cancel{1} - \cancel{1} - \tfrac{\omega}{2\pi k}\right) - \int_0^{\frac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}} \tfrac{x}{(1+x^2)^{3/2}} \left(\tfrac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a} - x\right)\,\mathrm{d}x.$$

Taking the sum over strictly positive integers, with s in the appropriate half-plane, we get

$$\begin{aligned}
&\sqrt{4\mu+1} \sum_{k\geqslant 1} \tfrac{1}{|k|^{\delta(2s-1)}}\ \mathrm{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\,a}\right) \\
&= \quad \sqrt{4\mu+1}\left[\tfrac{\sqrt{4\mu+1}}{2\pi a} \sum_{k\geqslant 1} \tfrac{1}{k^{1+2\delta(s-1)}} - \tfrac{\sqrt{4\mu+1}}{2\pi a} \cdot \tfrac{\omega}{2\pi} \sum_{k\geqslant 1} \tfrac{1}{k^{2+2\delta(s-1)}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \right. \\
&\qquad \left. - \sum_{k\geqslant 1} \tfrac{1}{k^{\delta(2s-1)}} \int_0^{\frac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}} \tfrac{x}{(1+x^2)^{3/2}} \left(\tfrac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a} - x\right)\,\mathrm{d}x\right] \\
&= \quad \tfrac{4\mu+1}{2\pi a}\,\zeta\left(1+2\delta\left(s-1\right)\right) - \tfrac{4\mu+1}{2\pi a} \cdot \tfrac{\omega}{2\pi} \sum_{k\geqslant 1} \tfrac{1}{k^{2+2\delta(s-1)}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \\
&\qquad -\sqrt{4\mu+1} \sum_{k\geqslant 1} \tfrac{1}{k^{\delta(2s-1)}} \int_0^{\frac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}} \tfrac{x}{(1+x^2)^{3/2}} \left(\tfrac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a} - x\right)\,\mathrm{d}x.
\end{aligned}$$

It is now readily seen that the first term above has a holomorphic continuation around 0, using classical results about the Riemann zeta function. The second and third term also induce holomorphic function around the origin, this time because the series involved are absolutely convergent there. For this, we simply need to note that we have

$$\left|\int_0^{\frac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}} \tfrac{x}{(1+x^2)^{3/2}} \left(\tfrac{k^{\delta}\sqrt{4\mu+1}}{(2\pi k+\omega)\,a} - x\right)\,\mathrm{d}x\right| \quad \leqslant \quad \tfrac{1}{2} \cdot \left(\tfrac{\sqrt{4\mu+1}k^{\delta}}{(2\pi k+\omega)a}\right)^3.$$

We can deal with the sum over strictly negative integers in a similar fashion, by noticing that this change of sign amounts to (formally) changing the sign of ω. This proves that the function

$$s \quad \longmapsto \quad \sqrt{4\mu+1} \sum_{|k|\geqslant 1} \tfrac{1}{|k|^{\delta(2s-1)}}\ \mathrm{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\,a}\right)$$

has a holomorphic continuation to a neighborhood of 0. A precise control on its value at $s = 0$, or, equivalently, a control on the derivative at 0 of the function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi}\ \tfrac{1}{4^{s-\frac{1}{2}}} \left(\tfrac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1} \tfrac{1}{|k|^{\delta(2s-1)}}\ \mathrm{Argsh}\left(\tfrac{|k|^{\delta}\sqrt{4\mu+1}}{|2\pi k+\omega|\,a}\right)$$

is not needed, as there will later be a cancellation. We now turn to studying the function

$$s \quad \longmapsto \quad \sqrt{4\mu+1} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k| \geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \;\; \operatorname{Argsh}\left(\frac{|k|^\delta \sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right).$$

Using similar computations as before, we get, for the sum over strictly positive integers k,

$$\sqrt{4\mu+1} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{k \geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \;\; \operatorname{Argsh}\left(\frac{|k|^\delta \sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right)$$

$$= \quad \frac{4\mu+1}{2\pi a} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j}\, \zeta\left(2\delta\left(s+j-1\right)+1\right) - \frac{4\mu+1}{2\pi a} \cdot \frac{\omega}{2\pi} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{k \geqslant 1} \frac{1}{k^{2\delta(s+j-1)+2}} \cdot \frac{1}{1+\frac{\omega}{2\pi k}}$$

$$-\sqrt{4\mu+1} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{k \geqslant 1} \int_0^{\frac{k^\delta \sqrt{4\mu+1}}{(2\pi k+\omega)\, a}} \frac{x}{(1+x^2)^{3/2}} \left(\frac{k^\delta \sqrt{4\mu+1}}{(2\pi k+\omega)\, a} - x\right) \mathrm{d}x.$$

We now note that the second and third term above induce holomorphic functions around 0, whose value at $s = 0$ vanishes because of the Pochhammer symbol. The first series also induces a holomorphic function around 0, though we will have to be careful in evaluating its value there. For every integer $j \geqslant 2$, we have

$$2\delta\left(j-1\right)+1 \quad > \quad 1$$

which means that the sum over $j \geqslant 2$ induces a holomorphic function around 0, whose value at this point vanishes because of the Pochhammer symbol. However, something different occurs when j equals 1, as we stand too close to the pole of the Riemann zeta function. The corresponding term is given by

$$\frac{4\mu+1}{2\pi a} \cdot \frac{1}{4} \cdot s \cdot \zeta\left(2\delta s+1\right) \;\; .$$

It is still true that this term induces a holomorphic function around 0, though the factor s coming from the Pochhammer symbol here can only serve to cancel the pole of the Riemann zeta function, instead of making the whole value at 0 disappear. Using a Laurent expansion at 0, we see that the value at 0 of this function is given by

$$\frac{4\mu+1}{16\pi a\delta} \quad = \quad \frac{1}{4\pi a\delta}\, \mu + \frac{1}{16\pi a\delta}.$$

We can now deal with the sum over strictly negative integers k in a similar way, or simply by formally changing the sign of ω. This proves that the function

$$s \quad \longmapsto \quad \sqrt{4\mu+1} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k| \geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \;\; \operatorname{Argsh}\left(\frac{|k|^\delta \sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right).$$

has a holomorphic continuation to a neighborhood of 0, and that its value at $s = 0$, or, equivalently, that the derivative at $s = 0$ of the function

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \, \frac{1}{4^{s-\frac{1}{2}}} \left(\frac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{j \geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \sum_{|k| \geqslant 1} \frac{1}{|k|^{\delta(2s+2j-1)}} \;\; \operatorname{Argsh}\left(\frac{|k|^\delta \sqrt{4\mu+1}}{|2\pi k+\omega|\, a}\right)$$

is given by $\frac{1}{2\pi a\delta}\, \mu + \frac{1}{8\pi a\delta}$. This completes the proof of the proposition.

$\square$

Fifth part. We now move on to the next term from proposition 4.6.16.

Proposition 4.6.24. *The function*

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^{s+1}}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\,\log\left(\left(C_{\omega,k}a\right)^2+(4\mu+1)\,|k|^{2\delta}\right)\,,$$

which is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \quad > \quad \tfrac{1}{2\delta}$$

has a holomorphic continuation to a neighborhood of 0*, whose derivative there satisfies*

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left(-\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^{s+1}}\,\left(\tfrac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\,\log\left(\left(C_{\omega,k}a\right)^2+(4\mu+1)\,|k|^{2\delta}\right)\right)\\
&\qquad = \quad \tfrac{\partial}{\partial s}_{|s=0}\left(-\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^{s+1}}\,\left(\tfrac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\,\log\left(\left((2\pi k+\omega)\,a\right)^2+(4\mu+1)\,|k|^{2\delta}\right)\right).
\end{aligned}$$

Proof. We begin by noting that the function

$$s \quad \longmapsto \quad -\frac{1}{4}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\,\log\left(\left(C_{\omega,k}a\right)^2+(4\mu+1)\,|k|^{2\delta}\right)$$

is well-defined and holomorphic on the half-plane mentioned in the statement above. After applying the binomial formula on this open domain, we get

$$\begin{aligned}
&-\tfrac{1}{4}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\,\log\left(\left(C_{\omega,k}a\right)^2+(4\mu+1)\,|k|^{2\delta}\right)\\
&\qquad = \quad -\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)^2\,a^2+(4\mu+1)\,|k|^{2\delta}\right)\\
&\qquad\qquad -\tfrac{1}{4}\sum_{j\geqslant 1}\tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j}\sum_{|k|\geqslant 1}\tfrac{1}{k^{2\delta(s+j)}}\log\left((2\pi k+\omega)^2\,a^2+(4\mu+1)\,|k|^{2\delta}\right).
\end{aligned}$$

We will deal in detail with the first term above, the other one being similar. We have

$$\begin{aligned}
&-\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\log\left((2\pi k+\omega)^2\,a^2+(4\mu+1)\,|k|^{2\delta}\right)\\
&\qquad = \quad -\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\left(2\log(2\pi a)+2\log|k|+\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)\right)\\
&\qquad = \quad -\log(2\pi a)\,\zeta(2\delta s)-\zeta'(2\delta s)\\
&\qquad\qquad -\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\left(\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)-\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}\right),
\end{aligned}$$

where it is important to note that the term $\omega/(\pi k)$ introduced in the last sum is not to be compensated by anything, as we sum over both strictly positive and negative integers, which produces the appropriate change of sign. We then note that the first two terms above induce holomorphic functions around 0, following results pertaining to the Riemann zeta function. The third term requires more care. We have, for every strictly positive integer k,

$$\begin{aligned}
&\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)-\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}\\
&\quad= \int_0^{1/k}\left(\frac{\frac{\omega}{\pi}+\frac{\omega^2}{2\pi^2}t+\frac{4\mu+1}{2\pi^2a^2}(1-\delta)t^{1-2\delta}}{1+\frac{\omega}{\pi}t+\frac{\omega^2}{4\pi^2}t^2+\frac{4\mu+1}{4\pi^2a^2}t^{2-2\delta}}-\frac{\omega}{\pi}\right)\mathrm{d}t\\
&\quad= \int_0^{1/k}\underbrace{\frac{1}{1+\frac{\omega}{\pi}t+\frac{\omega^2}{4\pi^2}t^2+\frac{4\mu+1}{4\pi^2a^2}t^{2-2\delta}}}_{\leqslant 1}\left(\cancel{\tfrac{\omega}{\pi}}+\tfrac{\omega^2}{2\pi^2}t+\tfrac{4\mu+1}{2\pi^2a^2}(1-\delta)\,t^{1-2\delta}-\cancel{\tfrac{\omega}{\pi}}\right.\\
&\qquad\qquad\left.-\tfrac{\omega^2}{\pi^2}t-\tfrac{\omega^3}{4\pi^3}t^2-\tfrac{4\mu+1}{4\pi^3a^2}\omega t^{2-2\delta}\right)\mathrm{d}t.
\end{aligned}$$

This allows us to properly bound the difference studied here, as we have

$$\begin{aligned}
&\left|\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)-\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}\right|\\
&\qquad\leqslant \tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{k^{2-2\delta}}+\tfrac{4\mu+1}{4\pi^3a^2}\cdot\tfrac{1}{3-2\delta}\cdot\tfrac{1}{k^{3-2\delta}}+\tfrac{\omega^3}{12\pi^3}\cdot\tfrac{1}{k^3},
\end{aligned}$$

with a similar estimate for strictly negative integers. This means that the function

$$s\ \longmapsto\ -\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\left(\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)-\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}\right)$$

is holomorphic around 0. Therefore, the function

$$s\ \longmapsto\ -\tfrac{1}{4}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta s}}\log\left((2\pi k+\omega)^2a^2+(4\mu+1)\,|k|^{2\delta}\right)$$

has a holomorphic continuation to a neighborhood of 0. A precise control on its value there is not needed, as there will later be a cancellation. Using similar computations, we have, on the appropriate half-plane

$$\begin{aligned}
&-\tfrac{1}{4}\sum_{j\geqslant 1}\tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j}\sum_{|k|\geqslant 1}\tfrac{1}{k^{2\delta(s+j)}}\log\left((2\pi k+\omega)^2a^2+(4\mu+1)\,|k|^{2\delta}\right)\\
&\quad= -\log(2\pi a)\sum_{j\geqslant 1}\tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j}\,\zeta\left(2\delta\left(s+j\right)\right)-\sum_{j\geqslant 1}\tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j}\,\zeta'\left(2\delta\left(s+j\right)\right)\\
&\qquad-\tfrac{1}{4}\sum_{j\geqslant 1}\tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j}\sum_{|k|\geqslant 1}\tfrac{1}{|k|^{2\delta(s+j)}}\left(\log\left(1+\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}+\tfrac{\omega^2}{4\pi^2}\cdot\tfrac{1}{k^2}+\tfrac{4\mu+1}{4\pi^2a^2}\cdot\tfrac{1}{|k|^{2-2\delta}}\right)-\tfrac{\omega}{\pi}\cdot\tfrac{1}{k}\right),
\end{aligned}$$

and, as before, these terms induce holomorphic functions around 0, the only difference being that their value at this point vanishes, because of the Pochhammer symbol. This concludes the proof of this proposition.

□

Sixth part. Unlike what we did for the last terms, we will take care of the polynomial terms, associated to U_1, coming from propositions 4.6.16 and 4.6.18 together.

Proposition 4.6.25. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\left[\frac{1}{|k|^{\delta}\sqrt{4\mu+1}}\,U_1\left(\tau\left(\frac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right)\right.$$
$$\left. -\tfrac{1}{\sqrt{\frac{1}{4}+\mu}}\,U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right)\right)\right]$$

is holomorphic on the half-plane

$$\mathrm{Re}\, s \quad > \quad 1-\tfrac{1}{2\delta} \quad ,$$

which contains 0 *if we have* $\delta \;<\; 1/2$.

Remark 4.6.26. There is no need here for any asymptotic expansion as μ goes to infinity, since there will be full compensations later for this term.

Proof of proposition 4.6.25. The proof is similar to that of the last two propositions, keeping in mind that the polynomial U_1 is given by

$$U_1(t) \quad = \quad -\tfrac{1}{24}\left(3t-5t^3\right) \quad = \quad -u_1(t) \quad ,$$

where u_1 is the polynomial appearing in the asymptotic expansions provided by Olver in [74, 75]. □

Seventh part. Having finished with the terms coming from proposition 4.6.16, we now turn to the ones associated with proposition 4.6.17, for which we need to assume that ω is not zero.

Proposition 4.6.27. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{3^s}\left(\frac{1}{4}+\mu\right)^{-s}\quad \log K_{2\sqrt{\frac{1}{4}+\mu}}(\omega a)$$

is holomorphic on $\mathbb{C}$, *and its derivative at* $s \;=\; 0$ *satisfies, as* μ *goes to infinity,*

$$\tfrac{\partial}{\partial s}{}_{|s=0}\left(\tfrac{\sin(\pi s)}{\pi}\;\tfrac{1}{3^s}\;\left(\tfrac{1}{4}+\mu\right)^{-s}\;\log K_{2\sqrt{\frac{1}{4}+\mu}}(\omega a)\right)$$
$$= \quad \sqrt{\mu}\,\log\mu + 2\left[2\log 2 - 1 - \log(\omega a)\right]\sqrt{\mu} - \tfrac{1}{4}\log\mu + \tfrac{1}{2}\log\tfrac{\pi}{4} + o(1)\,.$$

Proof. Since no series is involved, the result stems directly from the asymptotic expansions of the modified Bessel functions. □

Eighth part. With the exception of what we did in the second and sixth part above, we have yet to study the series associated to the terms appearing in proposition 4.6.18. The first one of them is taken care of in the next proposition.

Proposition 4.6.28. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\sqrt{\left(C_{\omega,k}a\right)^2+\frac{1}{4}+\mu}\ ,$$

which is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \quad > \quad \tfrac{1}{\delta} \ ,$$

has a holomorphic continuation to a neighborhood of 0*, whose derivative there satisfies*

$$\frac{\partial}{\partial s}_{|s=0} \left(\tfrac{\sin(\pi s)}{\pi} \ \tfrac{1}{4^s} \ \left(\tfrac{1}{4}+\mu\right)^{-s} \sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s} \ \sqrt{\left(C_{\omega,k}a\right)^2+\tfrac{1}{4}+\mu} \right)$$
$$= \ \frac{\partial}{\partial s}_{|s=0} \left(\tfrac{\sin(\pi s)}{\pi} \ \tfrac{1}{4^s} \ \tfrac{2\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}} \ \left(\tfrac{1}{4}+\mu\right)^{1-(1+\delta)s} \ (2\pi a)^{2\delta s-1} \ \tfrac{1}{(2\delta s-1)(2\delta s-2)} \right)$$
$$-\sqrt{\mu}+\tfrac{\omega^2}{2\pi}a+o\left(1\right)$$

as μ *goes to infinity.*

Remark 4.6.29. Even though, in this particular instance, the derivative at 0 left untouched in the proposition above can be computed precisely as μ goes to infinity, we have chosen not to do so, as this computation will become significantly simpler when this term is grouped with another one that will appear in the next section.

Proof of proposition 4.6.28. We begin by noting that the function

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s} \ \sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\tfrac{1}{4}+\mu} \ ,$$

is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$, which never contains the origin. In order to prove this proposition, we will use a method known as the *Ramanujan summation*, for which the reader is referred to [23]. This will allow us not only to extend the function above, which could be done using a Taylor expansion, but also to get a control of this continuation as μ goes to infinity. Let us first use the binomial formula on the half-plane $\operatorname{Re} s > 1/\delta$, which yields

$$\begin{aligned}
&\sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s} \ \sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\tfrac{1}{4}+\mu} \\
&\quad = \ \sum_{j\geqslant 0} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \sum_{|k|\geqslant 1} \tfrac{1}{|k|^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\tfrac{1}{4}+\mu} \\
&\quad = \ \sum_{j\geqslant 0} \tfrac{(s)_j}{j!}\cdot\tfrac{1}{4^j} \Bigg[\sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\tfrac{1}{4}+\mu} \\
&\qquad\qquad + \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k-\omega\right)a\right)^2+\tfrac{1}{4}+\mu}\Bigg].
\end{aligned}$$

The reason for splitting the sum into two parts is that there will be a slight variation in the argument we will use. We begin by dealing with the series

$$\sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\tfrac{1}{4}+\mu}$$

for any positive integer j. The first step towards using the Ramanujan summation process is to find a function, regular enough, that interpolates the terms appearing in the series. The only natural choice is to take, for s fixed in the half-plane mentioned above,

$$f_{s,j} \quad : \quad z \quad \longmapsto \quad \tfrac{1}{z^{2\delta(s+j)}}\sqrt{\left(\left(2\pi z+\omega\right)a\right)^2+\tfrac{1}{4}+\mu} \ ,$$

which is indeed well-defined and holomorphic on the half-plane $\operatorname{Re} z > 0$, since we have

$$\begin{aligned}\tfrac{1}{4} + \mu + (2\pi z + \omega)^2 a^2 \quad &= \quad \tfrac{1}{4} + \mu + 4\pi^2 a^2 \left((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 \right) + 4\pi\omega a^2 \operatorname{Re} z \\ &\qquad + \omega^2 a^2 + 4i\pi a^2 \left(2\pi \operatorname{Re} z + \omega \right) \operatorname{Im} z.\end{aligned}$$

It should be noted that the definition of the complex power and square root used above is relatively to the cut along the half-line of negative real numbers. The function $f_{s,j}$ is of *moderate growth*, in the sense of [23, Sec. 1.3.2], on the half-plane where it is defined. Furthermore, Carlson's theorem, presented in [23, App. B], states that $f_{s,j}$ is the only function of moderate growth which interpolates the terms considered in the series. There are two hypotheses we need to check if we want the Ramanujan summation process to behave nicely. The first one is that we should have

$$\lim_{k \to +\infty} f_{s,j}(k) \quad = \quad 0,$$

which holds, as we have chosen s to be in the half-plane where the associated series converges absolutely, which in particular implies that the terms comprising said series go to 0 as k goes to infinity. The second one is a bit more technical. We need to prove that we have

$$\lim_{k \to +\infty} \int_0^{+\infty} \tfrac{f_{s,j}(k+it) - f_{s,j}(k-it)}{e^{2\pi t} - 1} \, \mathrm{d}t \quad = \quad 0 \; .$$

This will be done by using both Lebesgue's dominated convergence theorem, and Taylor's formula. For any integer $k \geqslant 1$, and any positive real number t, we have

$$\begin{aligned} f_{s,j}(k+it) \quad &= \quad f_{s,j}(k) + i \int_0^t \, f'_{s,j}(k+ix) \, \mathrm{d}x, \\ f_{s,j}(k-it) \quad &= \quad f_{s,j}(k) - i \int_0^t \, f'_{s,j}(k-ix) \, \mathrm{d}x. \end{aligned}$$

This gives, after taking the difference,

$$f_{s,j}(k+it) - f_{s,j}(k-it) \quad = \quad i \int_0^t \, \left(f'_{s,j}(k+ix) + f'_{s,j}(k-ix) \right) \, \mathrm{d}x \; .$$

This manipulation allows us to keep the convergence at $t = 0$ in the integral we consider, as it cancels the singularity induced by $1/\left(e^{2\pi t} - 1\right)$. We further have, on the half-plane $\operatorname{Re} z > 0$,

$$f'_{s,j}(z) \quad = \quad \tfrac{1}{z^{2\delta(s+j)}} \cdot \tfrac{1}{2} \cdot \tfrac{4\pi a^2 (2\pi z + \omega)}{\sqrt{\frac{1}{4} + \mu + (2\pi z + \omega)^2 a^2}} - 2\delta(s+j) \cdot \tfrac{1}{z^{2\delta(s+j)+1}} \cdot \sqrt{\tfrac{1}{4} + \mu + (2\pi z + \omega)^2 a^2} \; .$$

We need to find precise estimates on the square roots appearing above. We have

$$\begin{aligned}\tfrac{1}{4} + \mu + (2\pi(k+ix) + \omega)^2 a^2 \quad &= \quad \tfrac{1}{4} + \mu + 4\pi^2 a^2 \left(k^2 - x^2\right) + 4\pi\omega a^2 k + \omega^2 a^2 \\ &\qquad + 4i\pi a^2 x \left(2k\pi + \omega\right),\end{aligned}$$

which gives

$$\begin{aligned}\left| \tfrac{1}{4} + \mu + (2\pi(k+ix) + \omega)^2 a^2 \right|^2 \quad &= \quad \left(\tfrac{1}{4} + \mu + 4\pi^2 a^2 \left(k^2 - x^2\right) + 4\pi\omega a^2 k + \omega^2 a^2 \right)^2 \\ &\qquad + 16\pi^2 a^4 x^2 \left(2k\pi + \omega\right)^2 \\ &= \quad \left(\tfrac{1}{4} + \mu - 4\pi^2 a^2 x^2 \right)^2 + 8\pi^2 a^4 x^2 \left(2k\pi + \omega\right)^2 + \left(2k\pi + \omega\right)^4 a^4 + 2\left(2k\pi + \omega\right)^2 \left(\tfrac{1}{4} + \mu\right) a^2 \\ &\geqslant \quad \left(2k\pi + \omega\right)^4 a^4.\end{aligned}$$

This bound then yields

$$\left|\sqrt{\tfrac{1}{4}+\mu+\left(2\pi\left(k+ix\right)+\omega\right)^{2}a^{2}}\right| \quad = \quad \sqrt{\left|\tfrac{1}{4}+\mu+\left(2\pi\left(k+ix\right)+\omega\right)^{2}a^{2}\right|} \quad \geqslant \quad \left(2k\pi+\omega\right)a.$$

Note that we also have

$$\left|\sqrt{\tfrac{1}{4}+\mu+\left(2\pi\left(k+ix\right)+\omega\right)^{2}a^{2}}\right| \quad \leqslant \quad \sqrt{\tfrac{1}{4}+\mu+\left(2\pi\sqrt{k^{2}+t^{2}}+\omega\right)^{2}a^{2}}.$$

Finally, we have

$$\begin{aligned}
\left(k+ix\right)^{-2\delta(s+j)} &= \left(k+ix\right)^{-2\delta j}\left(k+ix\right)^{-2\delta s} \\
&= \left(k+ix\right)^{-2\delta j}\exp\left(-2\delta\left(\operatorname{Re}s+i\operatorname{Im}s\right)\left(\tfrac{1}{2}\log\left(k^{2}+x^{2}\right)+i\arg\left(k+ix\right)\right)\right),
\end{aligned}$$

which allows us to bound this factor, as we have

$$\begin{aligned}
\left|\left(k+ix\right)^{-2\delta(s+j)}\right| &= \left(k^{2}+x^{2}\right)^{-\delta(j+\operatorname{Re}s)}\exp\left(2\delta\operatorname{Im}s\arg\left(k+ix\right)\right) \\
&\leqslant \tfrac{1}{\left(k^{2}+x^{2}\right)^{\delta(\operatorname{Re}s+j)}}\cdot\exp\left(\delta\left|\operatorname{Im}s\right|\pi\right).
\end{aligned}$$

We now have everything that we need to properly bound $f'_{s,j}\left(k+ix\right)+f'_{s,j}\left(k-ix\right)$ as we have estimates similar to those above when replacing $k+ix$ by $k-ix$. Indeed, we have

$$\begin{aligned}
&\left|f'_{s,j}\left(k+ix\right)+f'_{s,j}\left(k-ix\right)\right| \\
&\qquad\leqslant \tfrac{2\pi a^{2}\exp\left(\delta\left|\operatorname{Im}s\right|\pi\right)}{\left(k^{2}+x^{2}\right)^{\delta(\operatorname{Re}s+j)}}\cdot 2\cdot\tfrac{2\pi\left(k^{2}+x^{2}\right)^{1/2}+\omega}{\left(2\pi k+\omega\right)a}+2\delta\left(\left|s\right|+j\right)\cdot\tfrac{\exp\left(\delta\left|\operatorname{Im}s\right|\pi\right)}{\left(k^{2}+x^{2}\right)^{\delta(\operatorname{Re}s+j)+\frac{1}{2}}} \\
&\qquad\leqslant 4\exp\left(\delta\left|\operatorname{Im}s\right|\pi\right)\left[\pi a+\delta\left(\left|s\right|+j\right)\sqrt{\tfrac{1}{4}+\mu+\left(2\pi\sqrt{1+t^{2}}+\omega\right)^{2}a^{2}}\right].
\end{aligned}$$

This allows us to use the dominated convergence theorem, which yields

$$\lim_{k\to+\infty}\int_{0}^{+\infty}\tfrac{f_{s,j}(k+it)-f_{s,j}(k-it)}{e^{2\pi t}-1}\,\mathrm{d}t \quad = \quad 0\ .$$

Using theorem 2 from [23, Sec. 1.4.3], one then gets

$$\sum_{k=1}^{+\infty}f_{s,j}\left(k\right) \quad = \quad \sum_{k\geqslant 1}^{(\mathcal{R})}f_{s,j}\left(k\right)+\int_{1}^{+\infty}f_{s,j}\left(x\right)\,\mathrm{d}x\ ,$$

where the first symbol on the right-hand side stands for the Ramanujan sum. It should be noted, as is explained in [23], that this "sum" depends on the whole function f_j, and not simply its values at integers, even though, in our case, Carlson's theorem provides unicity for the interpolating function with moderate growth. The main difference between the usual sum and this Ramanujan sum is that, as is mentioned in theorem 9 of [23, Sec. 3.1.1], the function

$$s \quad \longmapsto \quad \sum_{k\geqslant 1}^{(\mathcal{R})}f_{s,j}\left(k\right)$$

is entire, *i.e.* holomorphic on the complex plane, as it is a Ramanujan sum of entire functions.

This allows us to say that the function

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{k\geqslant 1}^{(\mathcal{R})} f_{s,j}(k)$$

is entire, and that its derivative at $s=0$ is given by

$$\begin{aligned}
\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{k\geqslant 1}^{(\mathcal{R})} f_{s,j}(k)\right] &= \sum_{k\geqslant 1}^{(\mathcal{R})} f_{0,j}(k) \\
&= \frac{1}{2} f_{0,j}(1) + i\int_0^{+\infty} \frac{f_{0,j}(1+it)-f_{0,j}(1-it)}{e^{2\pi t}-1}\,\mathrm{d}t.
\end{aligned}$$

We will now prove that the integral above, whose dependence in μ is hidden within the definition of $f_{s,j}$, goes to 0 as μ goes to infinity. We have

$$\begin{aligned}
f_j(1+it) - f_j(1-it) &= \sqrt{\tfrac{1}{4}+\mu+(2\pi(1+it)+\omega)^2 a^2} - \sqrt{\tfrac{1}{4}+\mu+(2\pi(1-it)+\omega)^2 a^2} \\
&= \frac{a^2\left[(2\pi(1+it)+\omega)^2-(2\pi(1-it)+\omega)^2\right]}{\sqrt{\frac{1}{4}+\mu+(2\pi(1+it)+\omega)^2a^2}+\sqrt{\frac{1}{4}+\mu+(2\pi(1-it)+\omega)^2a^2}} \\
&= i\,\frac{8\pi a^2 t(2\pi+\omega)}{\sqrt{\frac{1}{4}+\mu+(2\pi(1+it)+\omega)^2a^2}+\sqrt{\frac{1}{4}+\mu+(2\pi(1-it)+\omega)^2a^2}}
\end{aligned}$$

which means we need only find a lower bound for the denominator. We have

$$\begin{aligned}
&\sqrt{\tfrac{1}{4}+\mu+(2\pi(1+it)+\omega)^2 a^2} + \sqrt{\tfrac{1}{4}+\mu+(2\pi(1-it)+\omega)^2 a^2} \\
&\quad = \sqrt{2}\Bigg[\left(\left(\tfrac{1}{4}+\mu-4\pi^2a^2t^2\right)^2 + (2\pi+\omega)^4 + 2\left(\tfrac{1}{4}+\mu-4\pi^2a^2t^2\right)(2\pi+\omega)^2a^2\right. \\
&\qquad\qquad \left. +16\pi^2a^4t^2(2\pi+\omega)^2\right)^{1/2} + \tfrac{1}{4}+\mu+4\pi^2a^2-4\pi^2a^2t^2+4\pi\omega a^2+\omega^2a^2\Bigg]^{1/2} \\
&\quad \geqslant \sqrt{2}\left[\underbrace{\left|\frac{1}{4}+\mu-4\pi^2a^2t^2\right| + \frac{1}{4}+\mu-4\pi^2a^2t^2}_{\geqslant 0} + \underbrace{4\pi^2a^2+4\pi\omega a^2+\omega^2a^2}_{(2\pi+\omega)^2a^2}\right]^{1/2} \\
&\quad \geqslant \sqrt{2}\,(2\pi+\omega)\,a.
\end{aligned}$$

which gives

$$\left|\sqrt{\tfrac{1}{4}+\mu+(2\pi(1+it)+\omega)^2 a^2} - \sqrt{\tfrac{1}{4}+\mu+(2\pi(1-it)+\omega)^2 a^2}\right| \quad \leqslant \quad 4\sqrt{2}\pi a t.$$

This allows us to use the dominated convergence theorem, and gives

$$\lim_{\mu\to+\infty} \int_0^{+\infty} \frac{\sqrt{\frac{1}{4}+\mu+(2\pi(1+it)+\omega)^2a^2}-\sqrt{\frac{1}{4}+\mu+(2\pi(1-it)+\omega)^2a^2}}{e^{2\pi t}-1} \quad = \quad 0 \ .$$

Hence, we have

$$\sum_{k\geqslant 1}^{(\mathcal{R})} f_{0,j}(k) \quad = \quad \frac{1}{2}\sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2a^2} + o(1) \quad = \quad \frac{1}{2}\sqrt{\mu}+o(1)$$

as μ goes to infinity. What remains to prove is that the function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \sum_{j \geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)}} \sqrt{\tfrac{1}{4}+\mu+(2\pi x+\omega)^2 a^2}\, \mathrm{d}x \ ,$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$, has a holomorphic continuation to an open neighborhood of the origin, and get an asymptotic control of its derivative there, as μ goes to infinity. For any integer $j \geqslant 0$, we have, on the appropriate half-plane,

$$\begin{aligned}
&\int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)}} \sqrt{\tfrac{1}{4}+\mu+(2\pi x+\omega)^2 a^2}\, \mathrm{d}x \\
&\quad = \left[-\tfrac{1}{2\delta(s+j)-1} \cdot \tfrac{1}{x^{2\delta(s+j)-1}} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi x+\omega)^2 a^2} \right]_1^{+\infty} \\
&\qquad\qquad + \tfrac{1}{2\delta(s+j)-1} \cdot \tfrac{1}{2} \cdot \int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)-1}} \cdot \tfrac{4\pi a^2(2\pi x+\omega)}{\sqrt{\frac{1}{4}+\mu+(2\pi x+\omega)^2 a^2}}\, \mathrm{d}x \\
&\quad = \tfrac{1}{2\delta(s+j)-1} \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2 a^2} + \tfrac{2\pi a}{2\delta(s+j)-1} \int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)-1}} \cdot \tfrac{1}{\sqrt{1+\frac{1/4+\mu}{(2\pi x+\omega)^2 a^2}}}\, \mathrm{d}x.
\end{aligned}$$

We will now compute the integral above using hypergeometric functions, for which the reader is referred to [74, 75]. We have

$$\begin{aligned}
&\int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)-1}} \cdot \left(1+\tfrac{\frac{1}{4}+\mu}{(2\pi x+\omega)^2 a^2}\right)^{-1/2} \mathrm{d}x \\
&\quad = \sqrt{\tfrac{1}{4}+\mu} \cdot \tfrac{1}{4\pi a} (2\pi)^{2\delta(s+j)-1} \int_0^{\frac{1/4+\mu}{(2\pi+\omega)^2 a^2}} \left(\tfrac{1}{a t^{1/2}} \sqrt{\tfrac{1}{4}+\mu} - \omega \right)^{-2\delta(s+j)+1} \cdot \tfrac{1}{t^{3/2}} \cdot \tfrac{\mathrm{d}t}{\sqrt{1+t}} \\
&\quad = \sqrt{\tfrac{1}{4}+\mu} \cdot \tfrac{1}{4\pi a} (2\pi a)^{2\delta(s+j)-1} \cdot \left(\tfrac{1}{4}+\mu\right)^{-\delta(s+j)+\frac{1}{2}} \\
&\qquad\qquad \cdot \int_0^{\frac{1/4+\mu}{(2\pi+\omega)^2 a^2}} t^{\delta(s+j)-2} \cdot \tfrac{1}{\sqrt{1+t}} \left(1 - \tfrac{a\omega}{\sqrt{1/4+\mu}} t^{1/2}\right)^{-2\delta(s+j)+1} \mathrm{d}t.
\end{aligned}$$

We now note that we have, on the interval of integration,

$$\tfrac{a\omega t^{1/2}}{\sqrt{1/4+\mu}} \quad \leqslant \quad \tfrac{a\omega}{\sqrt{1/4+\mu}} \cdot \tfrac{\sqrt{1/4+\mu}}{(2\pi+\omega)a} \quad = \quad \tfrac{\omega}{2\pi+\omega} \quad < \quad 1 \ ,$$

which allows us to use the binomial formula to expand the complex powers within the integral and interchange sums and integrals. We thus have

$$\begin{aligned}
&\int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)}} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi x+\omega)^2 a^2}\, \mathrm{d}x \\
&= \tfrac{1}{2\delta(s+j)-1} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2 a^2} + \tfrac{2\pi a}{2\delta(s+j)-1} \cdot \tfrac{1}{4\pi a} (2\pi a)^{2\delta(s+j)-1} \left(\tfrac{1}{4}+\mu\right)^{-\delta(s+j)+1} \\
&\qquad \cdot \sum_{n \geqslant 0} \tfrac{\Gamma(2\delta(s+j)+n-1)}{\Gamma(2\delta(s+j)-1)} \cdot \tfrac{1}{n!} \cdot \tfrac{a^n \omega^n}{(1/4+\mu)^{n/2}} \int_0^{\frac{1/4+\mu}{(2\pi+\omega)^2 a^2}} \tfrac{t^{(\delta(s+j)+n/2-1)-1}}{\sqrt{1+t}}\, \mathrm{d}t.
\end{aligned}$$

The integral remaining above can be computed, using hypergeometric functions and the formula provided in the proof of [47, Prop. 6.13]. We get

$$
\begin{aligned}
&\int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)}} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi x+\omega)^2\, a^2}\, \mathrm{d}x \\
&= \tfrac{1}{2\delta(s+j)-1} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2} + \tfrac{1}{2\delta(s+j)-1} \cdot (2\pi a)^{2\delta(s+j)-1} \cdot \tfrac{1}{2} \cdot \cancel{\left(\tfrac{1}{4}+\mu\right)^{-\delta(s+j)+1}} \\
&\qquad \cdot \sum_{n\geqslant 0} \Bigg[\tfrac{\Gamma(2\delta(s+j)+n-1)}{\Gamma(2\delta(s+j)-1)} \cdot \tfrac{1}{n!} \cdot \tfrac{a^n \omega^n}{\cancel{(1/4+\mu)^{n/2}}} \cdot \left(\tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2}\right)^{\delta(s+j)+n/2-1} \\
&\qquad\qquad \cdot \left(1+\tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2}\right)^{-1/2} \cdot \tfrac{2}{2\delta(s+j)+n-2}\, F\left(\tfrac{1}{2},\, 1,\, \delta\,(s+j)+n/2,\, \tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)\Bigg] \\
&= \tfrac{1}{2\delta(s+j)-1} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2} + \tfrac{1}{2}\,(2\pi a)^{2\delta(s+j)-1} \sum_{n\geqslant 0} \Bigg[\tfrac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))} \cdot \tfrac{2}{n!} \left(\tfrac{\omega}{2\pi+\omega}\right)^n \\
&\qquad \cdot ((2\pi+\omega)\, a)^{-2\delta(s+j)+2}\, \tfrac{(2\pi+\omega)a}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} F\left(\tfrac{1}{2},\, 1,\, \delta\,(s+j)+\tfrac{n}{2},\, \tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)\Bigg] \\
&= \tfrac{1}{2\delta(s+j)-1} \cdot \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2} + \left(\tfrac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2\, a^2 \sum_{n\geqslant 0} \Bigg[\tfrac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))} \\
&\qquad \cdot \tfrac{1}{n!} \cdot \left(\tfrac{\omega}{2\pi+\omega}\right)^n \cdot \tfrac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\tfrac{1}{2},\, 1,\, \delta\,(s+j)+\tfrac{n}{2},\, \tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)\Bigg].
\end{aligned}
$$

We will deal with both of these terms separately, assuming for simplicity, that $1/(2\delta)$ is not an integer, so that no positive integer j could satisfy $2\delta j = 1$. After summing over j and multiplying by the relevant factors,the first term yields

$$
\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \left(\sum_{j\geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \cdot \tfrac{1}{2\delta(s+j)-1}\right) \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2}\ ,
$$

which induces a holomorphic function around 0, whose derivative at $s=0$ is given by

$$
\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0} \left[\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \left(\sum_{j\geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \cdot \tfrac{1}{2\delta(s+j)-1}\right) \sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2} \right] \\
&\qquad = \quad -\sqrt{\tfrac{1}{4}+\mu+(2\pi+\omega)^2\, a^2} \quad = \quad -\sqrt{\mu}+o\,(1)\,,
\end{aligned}
$$

as μ goes to infinity. We can now study the following term, which is

$$
\begin{aligned}
&\sum_{j\geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \cdot \left(\tfrac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2\, a^2 \sum_{n\geqslant 0} \Bigg[\tfrac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))} \cdot \tfrac{1}{n!} \cdot \left(\tfrac{\omega}{2\pi+\omega}\right)^n \\
&\qquad \cdot \tfrac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\tfrac{1}{2},\, 1,\, \delta\,(s+j)+\tfrac{n}{2},\, \tfrac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)\Bigg].
\end{aligned}
$$

We will with this term by breaking apart the sum over n.

- For $n \geqslant 4$, we have

$$
\delta\,(\operatorname{Re} s+j)+\tfrac{n}{2}-\tfrac{3}{2} \quad \geqslant \quad \delta\,(\operatorname{Re} s+j)+\tfrac{1}{2}\ ,
$$

which means that the hypergeometric function

$$
t \quad \longmapsto \quad F\left(\tfrac{1}{2},\, 1,\, \delta\,(s+j)+\tfrac{n}{2},\, t\right)
$$

is bounded on $[0,1]$, as it is continuous on $[0,1[$ and has a finite limit at 1, a fact for which the reader is refered to [75, Sec. 15.4.ii], and this bound is furthermore uniform in j, as well as in s, assuming it stays in a neighborhood of the origin. This means that the term

$$\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2 a^2 \sum_{n\geqslant 4} \left[\frac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))} \right.$$

$$\left. \cdot \frac{1}{n!} \cdot \left(\frac{\omega}{2\pi+\omega}\right)^n \cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\frac{1}{2},\, 1,\, \delta\,(s+j)+\frac{n}{2},\, \frac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)\right]$$

induces a holomorphic function around 0, whose derivative at $s=0$ vanishes because of the Pochhammer symbol for $j \neq 0$, and because of $\Gamma\left(2\,(s+j)\right)$ for $j=0$.

- For $n=3$, we consider

$$\sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2 a^2 \frac{\Gamma(2\delta(s+j)+1)}{\Gamma(2\delta(s+j))} \cdot \frac{1}{6} \cdot \left(\frac{\omega}{2\pi+\omega}\right)^3$$

$$\cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\frac{1}{2},\, 1,\, \delta\,(s+j)+\frac{3}{2},\, \frac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right).$$

For $j \geqslant 1$, we can bound the hypergeometric function above as before, using the continuity on the interval $[0,1[$ and the finite limit at 1^- given by [75, Sec. 15.4.ii], uniformly in every parameter, for s in a neighborhood of 0, which means that the term

$$\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2 a^2 \frac{\Gamma(2\delta(s+j)+1)}{\Gamma(2\delta(s+j))}$$

$$\cdot \frac{1}{6} \cdot \left(\frac{\omega}{2\pi+\omega}\right)^3 \cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\frac{1}{2},\, 1,\, \delta\,(s+j)+\frac{3}{2},\, \frac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)$$

induces a holomorphic function around 0, whose derivative there vanishes because of $(s)_j$ for $j \neq 0$, and because of $\Gamma\left(2\delta\,(s+j)\right)$ for $j=0$.

- For $n=2$, we consider

$$\sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2 a^2 \cdot \frac{1}{2} \cdot \left(\frac{\omega}{2\pi+\omega}\right)^2$$

$$\cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\frac{1}{2},\, 1,\, \delta\,(s+j)+1,\, \frac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right).$$

When the integer j is large enough, we have

$$\delta\left(\mathrm{Re}\, s+j\right)-\frac{1}{2} \quad > \quad 0$$

for s in a neighborhood of the origin, and we can bound the hypergeometric function above uniformly in every parameter, for s close enough to 0. This means that the term

$$\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{j\geqslant 1} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1} (2\pi+\omega)^2 a^2 \cdot \frac{1}{2} \cdot \left(\frac{\omega}{2\pi+\omega}\right)^2$$

$$\cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2 a^2}} \cdot F\left(\frac{1}{2},\, 1,\, \delta\,(s+j)+1,\, \frac{1/4+\mu}{(2\pi+\omega)^2 a^2+1/4+\mu}\right)$$

induces a holomorphic function around 0, and that its derivative at this point vanishes, because of the Pochhammer symbol. Furthermore, the term associated to $j=0$, which is given by

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}(2\pi+\omega)^2 a^2\cdot\frac{1}{2}\cdot\left(\frac{\omega}{2\pi+\omega}\right)^2$$
$$\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,\delta s+1,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right)$$

also induces a holomorphic function around $s=0$, and its derivative there is

$$\frac{2\pi+\omega}{2\pi}(2\pi+\omega)^2 a^2\cdot\frac{1}{2}\cdot\left(\frac{\omega}{2\pi+\omega}\right)^2\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,1,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right)=\frac{a\omega^2}{4\pi}.$$

- For $n=1$, we consider

$$\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\cdot\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1}(2\pi+\omega)^2 a^2\,\frac{1}{2\delta(s+j)-1}\cdot\cdot\frac{\omega}{2\pi+\omega}$$
$$\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,\delta\,(s+j)+\frac{1}{2},\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

When the integer j is large enough, we have $\delta\,(\operatorname{Re}s+j)-1>0$ for s close to the origin, where we can bound the hypergeometric function uniformly in every parameter. The term

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 1}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\cdot\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1}(2\pi+\omega)^2 a^2\,\frac{1}{2\delta(s+j)-1}\cdot\frac{\omega}{2\pi+\omega}$$
$$\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,\delta\,(s+j)+\frac{1}{2},\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

thus induces a holomorphic function around 0, as we have assumed that $2\delta j$ never equals 1, whose derivative at $s=0$ vanishes, because of the Pochhammer symbol. The remaining term, corresponding to $j=0$, is given by

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}(2\pi+\omega)^2 a^2\,\frac{1}{2\delta s-1}\cdot\frac{\omega}{2\pi+\omega}$$
$$\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,\delta s+\frac{1}{2},\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

It also induces a holomorphic function around the origin, and its derivative there equals

$$-\frac{2\pi+\omega}{2\pi}\cdot(2\pi+\omega)^2 a^2\cdot\frac{\omega}{2\pi+\omega}\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\frac{1}{2},\,1,\,\delta s+\frac{1}{2},\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right)$$
$$=\ -\frac{\omega}{2\pi}\cdot\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}\ =\ -\frac{\omega}{2\pi}\sqrt{\mu}+o\,(1)$$

as μ goes to infinity.

- For $n=0$, we consider

$$\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\cdot\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1}\frac{(2\pi+\omega)^2a^2}{(2\delta(s+j)-1)(2\delta(s+j)-2)}\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}$$
$$\cdot F\left(\frac{1}{2},\,1,\,\delta\,(s+j)\,,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

For j large enough, we have

$$\delta\left(\operatorname{Re} s+j\right)-\tfrac{3}{2}\quad>\quad 0$$

for s close to the origin, and we can bound the hypergeometric function above uniformly in every parameter, still for s in a neighborhood of 0, which means that the term

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 1}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\cdot\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta(s+j)-1}\frac{(2\pi+\omega)^2a^2}{(2\delta(s+j)-1)(2\delta(s+j)-2)}$$
$$\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\cdot F\left(\tfrac{1}{2},\,1,\,\delta\left(s+j\right),\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

induces a holomorphic function around 0, whose derivative there vanishes because of the Pochhammer symbol. The only term left to be studied is the one corresponding to $j=0$, given by

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}\frac{(2\pi+\omega)^2a^2}{(2\delta s-1)(2\delta s-2)}\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}F\left(\tfrac{1}{2},\,1,\,\delta s,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right).$$

We now need to simplify the hypergeometric function above. We have

$$\begin{aligned}
&F\left(\tfrac{1}{2},\,1,\,\delta s,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right)\\
&=\frac{\Gamma(\delta s)\Gamma(\delta s-3/2)}{\Gamma(\delta s-1/2)\Gamma(\delta s-1)}\,F\left(\tfrac{1}{2},\,1,\,\tfrac{5}{2}-\delta s,\,\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\\
&\qquad+\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\Gamma(1/2)}\left(\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)^{\delta s-3/2}F\left(\delta s-\tfrac{1}{2},\delta s-1,\delta s-\tfrac{1}{2},\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\\
&=\frac{\delta s-1}{\delta s-3/2}\,F\left(\tfrac{1}{2},\,1,\,\tfrac{5}{2}-\delta s,\,\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\\
&\qquad+\frac{1}{\sqrt{\pi}}\Gamma\left(\delta s\right)\Gamma\left(3/2-\delta s\right)\left(\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)^{\delta s-3/2}\left(\frac{1/4+\mu}{1/4+\mu+(2\pi+\omega)^2a^2}\right)^{1-\delta s}\\
&=\frac{\delta s-1}{\delta s-3/2}\,F\left(\tfrac{1}{2},\,1,\,\tfrac{5}{2}-\delta s,\,\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\\
&\qquad+\frac{1}{\sqrt{\pi}}\Gamma\left(\delta s\right)\Gamma\left(3/2-\delta s\right)\sqrt{1/4+\mu+\left(2\pi+\omega\right)^2a^2}\left(\left(2\pi+\omega\right)a\right)^{2\delta s-3}\left(\tfrac{1}{4}+\mu\right)^{1-\delta s},
\end{aligned}$$

which then yields

$$\begin{aligned}
&\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}\frac{(2\pi+\omega)^2a^2}{(2\delta s-1)(2\delta s-2)}\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}F\left(\tfrac{1}{2},\,1,\,\delta s,\,\frac{1/4+\mu}{(2\pi+\omega)^2a^2+1/4+\mu}\right)\\
&\quad=\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}\frac{(2\pi+\omega)^2a^2}{(2\delta s-1)(2\delta s-2)}\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\\
&\qquad\cdot\left[\frac{\delta s-1}{\delta s-3/2}\,F\left(\tfrac{1}{2},\,1,\,\tfrac{5}{2}-\delta s,\,\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\right.\\
&\qquad\quad\left.+\frac{\Gamma(\delta s)}{\sqrt{\pi}}\Gamma\left(\tfrac{3}{2}-\delta s\right)\sqrt{\tfrac{1}{4}+\mu+\left(2\pi+\omega\right)^2a^2}\left(\left(2\pi+\omega\right)a\right)^{2\delta s-3}\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\right]\\
&\quad=\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\left(\frac{2\pi}{2\pi+\omega}\right)^{2\delta s-1}\frac{(2\pi+\omega)^2a^2}{(2\delta s-1)(2\delta s-2)}\cdot\frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}}\\
&\qquad\cdot\frac{\delta s-1}{\delta s-3/2}\,F\left(\tfrac{1}{2},\,1,\,\tfrac{5}{2}-\delta s,\,\frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)\\
&\qquad\quad+\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\,\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\,\left(\tfrac{1}{4}+\mu\right)^{1-(1+\delta)s}\,(2\pi a)^{2\delta s-1}\,\frac{1}{(2\delta s-1)(2\delta s-2)}.
\end{aligned}$$

We now note that the second term above, which is to be left untouched, as indicated by the statement of proposition we aim to prove, induces a holomorphic function around the origin, the pole induced by the factor $\Gamma(\delta s)$ being canceled by the factor $\sin(\pi s)$. The first term also induces a holomorphic function around 0, and its derivative there, which is given by

$$\frac{2\pi+\omega}{2\pi}\left(2\pi+\omega\right)^2 a^2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1/4+\mu+(2\pi+\omega)^2a^2}} \cdot \frac{2}{3}\, F\left(\frac{1}{2},\ 1,\ \frac{5}{2},\ \frac{(2\pi+\omega)^2a^2}{1/4+\mu+(2\pi+\omega)^2a^2}\right)$$

goes to 0 as μ goes to infinity. This completes the study of the first part of the considered series, that arose from the splitting of the sum over k into one bearing on strictly positive integers, and another on strictly negative ones. To sum, we have proved so far that the function

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\frac{1}{4}+\mu}$$

has a holomorphic continuation to a neighborhood of the origin. Its derivative there satisfies

$$\begin{aligned}
\frac{\partial}{\partial s}_{|s=0}&\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k+\omega\right)a\right)^2+\frac{1}{4}+\mu}\right]\\
&= \frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\,\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\,\left(\frac{1}{4}+\mu\right)^{1-(1+\delta)s}\left(2\pi a\right)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}\right]\\
&\qquad -\left(\frac{1}{2}+\frac{\omega}{2\pi}\right)\sqrt{\mu}+\frac{a\omega^2}{4\pi}+o\left(1\right)
\end{aligned}$$

as μ goes to infinity. We now move on to studying the term

$$\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k-\omega\right)a\right)^2+\frac{1}{4}+\mu}\ .$$

The method used here is extremely similar to the what is done above, if ω switches sign. There are, however, a few differences, which we will need to highlight in what follows. What can be done in the same way as before will not be detailed for clarity. The aim is here to use the Ramanujan summation process on each and every one of the series

$$\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{\left(\left(2\pi k-\omega\right)a\right)^2+\frac{1}{4}+\mu}\ .$$

We begin by considering the interpolating functions

$$g_{s,j} \quad : \quad z \quad \longmapsto \quad \frac{1}{z^{2\delta(s+j)}}\sqrt{\left(\left(2\pi z-\omega\right)a\right)^2+\frac{1}{4}+\mu}$$

which are well-defined and holomorphic on the half-plane $\operatorname{Re} z > \omega/\left(2\pi\right)$, since we have

$$\begin{aligned}
\frac{1}{4}+\mu+\left(\left(2\pi z-\omega\right)a\right)^2 \quad &= \quad \frac{1}{4}+\mu+4\pi^2a^2\left(\left(\operatorname{Re} z\right)^2-\left(\operatorname{Im} z\right)^2\right)-4\pi\omega a^2\operatorname{Re} z\\
&\qquad +\omega^2a^2+4i\pi a^2\left(2\pi\operatorname{Re} z-\omega\right)\operatorname{Im} z.
\end{aligned}$$

It is important to note here that the half-plane on which $g_{s,j}$ is defined contains 1, since ω is strictly below 2π. As before, we have

$$\lim_{k\to+\infty} g_{s,j}\left(k\right) \quad = \quad 0\ ,$$

since we have chosen s to be such that we have $\operatorname{Re} s > 1/\delta$. We further have

$$\lim_{k \to +\infty} \int_0^{+\infty} \tfrac{g_{s,j}(k+it) - g_{s,j}(k-it)}{e^{2\pi t}-1} \, \mathrm{d}t \quad = \quad 0 \ ,$$

still by using Taylor's formula and Lebesgue's dominated convergence theorem. We then have

$$\sum_{k=1}^{+\infty} g_{s,j}(k) \quad = \quad \sum_{k=1}^{(\mathcal{R})} g_{s,j}(k) + \int_1^{+\infty} g_{s,j}(x) \ \mathrm{d}x \ ,$$

and the analyticity theorem for Ramanujan sums implies that the function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4} + \mu\right)^{-s} \sum_{k=1}^{(\mathcal{R})} g_{s,j}(k)$$

is entire. Its derivative at $s = 0$ is given by

$$\begin{aligned}
\tfrac{\partial}{\partial s}_{|s=0} \left[\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4} + \mu\right)^{-s} \sum_{k=1}^{(\mathcal{R})} g_{s,j}(k) \right] \quad &= \quad \sum_{k=1}^{(\mathcal{R})} g_{0,j}(k) \\
&= \quad \tfrac{1}{2} g_{0,j}(1) + i \int_0^{+\infty} \tfrac{g_{0,j}(1+it) - g_{0,j}(1-it)}{e^{2\pi t}-1} \ \mathrm{d}t.
\end{aligned}$$

Similarly to what we did before, the dominated convergence theorem proves that we have

$$\lim_{\mu \to +\infty} \int_0^{+\infty} \tfrac{g_{0,j}(1+it) - g_{0,j}(1-it)}{e^{2\pi t}-1} \, \mathrm{d}t \quad = \quad 0 \ .$$

This yields the following asymptotic behavior of the Ramanujan sum

$$\sum_{k=1}^{(\mathcal{R})} g_{0,j}(k) \quad = \quad \tfrac{1}{2} \sqrt{\tfrac{1}{4} + \mu + (2\pi - \omega)^2 a^2} + o(1) \quad = \quad \tfrac{1}{2}\sqrt{\mu} + o(1)$$

as μ goes to infinity. We will now prove that the function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4} + \mu\right)^{-s} \sum_{j \geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \int_1^{+\infty} \tfrac{1}{x^{2\delta(s+j)}} \sqrt{\tfrac{1}{4} + \mu + (2\pi x - \omega)^2 a^2} \, \mathrm{d}x \ ,$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$, has a holomorphic continuation to an open neighborhood of 0, and get an asymptotic control on its derivative at this point, as μ goes to infinity. We will need to remove part of the integral before we can proceed. The function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4} + \mu\right)^{-s} \sum_{j \geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \int_1^{2} \tfrac{1}{x^{2\delta(s+j)}} \sqrt{\tfrac{1}{4} + \mu + (2\pi x - \omega)^2 a^2} \, \mathrm{d}x$$

is holomorphic around 0, and its derivative there is given by

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0} \left[\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{4^s} \left(\tfrac{1}{4} + \mu\right)^{-s} \sum_{j \geqslant 0} \tfrac{(s)_j}{j!} \cdot \tfrac{1}{4^j} \int_1^{2} \tfrac{1}{x^{2\delta(s+j)}} \sqrt{\tfrac{1}{4} + \mu + (2\pi x - \omega)^2 a^2} \, \mathrm{d}x \right] \\
&\qquad = \quad \sqrt{\mu} \ \left[1 + \int_1^2 \left(\sqrt{1 + \tfrac{1}{4\mu} + (2\pi x - \omega)^2 \tfrac{a^2}{\mu}} - 1 \right) \ \mathrm{d}x \ \right] \quad = \quad \sqrt{\mu} + o(1) \, .
\end{aligned}$$

Thus, the function we actually need to study is

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \int_2^{+\infty} \frac{1}{x^{2\delta(s+j)}} \sqrt{\frac{1}{4}+\mu+(2\pi x-\omega)^2 a^2}\, \mathrm{d}x \quad .$$

When s lies in the half-plane where this function is defined, we have

$$\begin{aligned}
&\int_2^{+\infty} \frac{1}{x^{2\delta(s+j)}} \sqrt{\frac{1}{4}+\mu+(2\pi x-\omega)^2 a^2}\, \mathrm{d}x \\
&\quad = \quad \frac{1}{2\delta(s+j)-1} \frac{1}{2^{2\delta(s+j)-1}} \sqrt{\frac{1}{4}+\mu+(4\pi-\omega)^2 a^2} + \frac{2\pi a}{2\delta(s+j)-1} \int_2^{+\infty} \frac{1}{x^{2\delta(s+j)-1}} \cdot \frac{1}{\sqrt{1+\frac{1/4+\mu}{(2\pi x-\omega)^2 a^2}}} \mathrm{d}x,
\end{aligned}$$

and we can once again compute this last integral using hypergeometric functions. We have

$$\begin{aligned}
\int_2^{+\infty} \frac{1}{x^{2\delta(s+j)-1}} \cdot \frac{1}{\sqrt{1+\frac{1/4+\mu}{(2\pi x-\omega)^2 a^2}}}\, \mathrm{d}x \quad = \quad & \frac{1}{4\pi a} \sqrt{\frac{1}{4}+\mu}\, (2\pi a)^{2\delta(s+j)-1} \cdot \left(\frac{1}{4}+\mu\right)^{-\delta(s+j)+1/2} \\
& \cdot \int_0^{\frac{1/4+\mu}{(4\pi-\omega)^2 a^2}} \frac{t^{\delta(s+j)-2}}{\sqrt{1+t}} \left(1+\frac{a\omega}{\sqrt{1/4+\mu}} t^{1/2}\right)^{-2\delta(s+j)+1} \mathrm{d}t.
\end{aligned}$$

The difference with the previous series appears here, as we have, on the interval of integration

$$\frac{a\omega}{\sqrt{1/4+\mu}} t^{1/2} \quad \leqslant \quad \frac{a\omega}{\sqrt{1/4+\mu}} \cdot \frac{\sqrt{1/4+\mu}}{(4\pi-\omega)a} \quad = \quad \frac{\omega}{4\pi-\omega} \quad < \quad 1.$$

Had not removed the integral ranging from 1 to 2, we would have gotten $\omega/(2\pi-\omega)$, which is not in general strictly smaller than 1. Fortunately, having done this manipulation allows us to obtain the required bound, which in turn enables the use of the binomial formula, and the interchange of sums and integrals. We thus have

$$\begin{aligned}
&\int_2^{+\infty} \frac{1}{x^{2\delta(s+j)}} \sqrt{\frac{1}{4}+\mu+(2\pi x-\omega)^2 a^2}\, \mathrm{d}x \\
&\quad = \quad \frac{1}{2\delta(s+j)-1} \cdot \frac{1}{2^{2\delta(s+j)-1}} \sqrt{\frac{1}{4}+\mu+(4\pi-\omega)^2 a^2} + \left(\frac{2\pi}{4\pi-\omega}\right)^{2\delta(s+j)-1} (4\pi-\omega)^2 a^2 \\
&\qquad \cdot \sum_{n \geqslant 0} \left[\frac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))} \cdot \frac{(-1)^n}{n!} \cdot \left(\frac{\omega}{4\pi-\omega}\right)^n \cdot \frac{1}{\sqrt{1/4+\mu+(4\pi-\omega)^2 a^2}} \right. \\
&\qquad\qquad \left. \cdot F\left(\frac{1}{2},\, 1,\, \delta(s+j)+\frac{n}{2},\, \frac{1/4+\mu}{1/4+\mu+(4\pi-\omega)^2 a^2}\right) \right].
\end{aligned}$$

Once again, we assume that $1/(2\delta)$ is not an integer, so as not to introduce artificial singularities. After summing over j and multiplying by the relevant factor, the first term above becomes

$$\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \left(\sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \frac{1}{2^{2\delta(s+j)-1}} \cdot \frac{1}{2\delta(s+j)-1} \right) \sqrt{\frac{1}{4}+\mu+(4\pi-\omega)^2 a^2}\ ,$$

and induces a holomorphic function around 0, whose derivative there is given by

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\frac{1}{4}+\mu\right)^{-s} \left(\sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{4^j} \cdot \frac{1}{2^{2\delta(s+j)-1}} \cdot \frac{1}{2\delta(s+j)-1} \right) \sqrt{\frac{1}{4}+\mu+(4\pi-\omega)^2 a^2} \right] \\
&\qquad = \quad -2\sqrt{\frac{1}{4}+\mu+(4\pi-\omega)^2 a^2} \quad = \quad -2\sqrt{\mu}+o(1)
\end{aligned}$$

as μ goes to infinity. We can now study the second term, given by

$$\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\left(\frac{2\pi}{4\pi-\omega}\right)^{2\delta(s+j)-1}(4\pi-\omega)^2 a^2$$
$$\cdot\sum_{n\geqslant 0}\left[\frac{\Gamma(2\delta(s+j)+n-2)}{\Gamma(2\delta(s+j))}\cdot\frac{(-1)^n}{n!}\cdot\left(\frac{\omega}{4\pi-\omega}\right)^n\cdot\frac{1}{\sqrt{1/4+\mu+(4\pi-\omega)^2a^2}}\right.$$
$$\left.\cdot F\left(\frac{1}{2},\,1,\,\delta\,(s+j)+\frac{n}{2},\,\frac{1/4+\mu}{1/4+\mu+(4\pi-\omega)^2a^2}\right)\right].$$

At this point, the study is very much the same as the one led above, and yields the holomorphic continuation around 0 of the term presented above, as well as the fact that its derivative at $s=0$ satisfies the following asymptotic expansion

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-(1+\delta)s}(2\pi a)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}\right]+\frac{a\omega^2}{4\pi}+\frac{\omega}{2\pi}\sqrt{\mu}+o\,(1)$$

as μ goes to infinity. This part of the computation thus proves that the function

$$s\quad\longmapsto\quad\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{((2\pi k-\omega)\,a)^2+\frac{1}{4}+\mu}$$

has a holomorphic continuation to a neighborhood of the origin, where its derivative satisfies

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{4^j}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)}}\sqrt{((2\pi k-\omega)\,a)^2+\frac{1}{4}+\mu}\right]$$
$$=\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-(1+\delta)s}(2\pi a)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}\right]$$
$$-\left(\frac{1}{2}-\frac{\omega}{2\pi}\right)\sqrt{\mu}+\frac{a\omega^2}{4\pi}+o\,(1)$$

as μ goes to infinity. Putting the two parts of the computation, we have proved that the function

$$s\quad\longmapsto\quad\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\sqrt{((2\pi k+\omega)\,a)^2+\frac{1}{4}+\mu}$$

has a holomorphic continuation to a neighborhood of 0, and that its derivative there satisfies

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\sqrt{((2\pi k+\omega)\,a)^2+\frac{1}{4}+\mu}\right]$$
$$=\;2\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-(1+\delta)s}(2\pi a)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}\right]$$
$$-\left(\frac{1}{2}+\frac{\omega}{2\pi}\right)\sqrt{\mu}+\frac{a\omega^2}{4\pi}-\left(\frac{1}{2}-\frac{\omega}{2\pi}\right)\sqrt{\mu}+\frac{a\omega^2}{4\pi}+o\,(1)$$
$$=\;2\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\frac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-(1+\delta)s}(2\pi a)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}\right]$$
$$-\sqrt{\mu}+\frac{\omega^2}{2\pi}a+o\,(1)$$

as μ goes to infinity. This completes the proof of this proposition.

$\square$

Ninth part. The next term coming from proposition 4.6.18 is dealt with below.

Proposition 4.6.30. *The function*

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s} \operatorname{Argsh}\left(\frac{\sqrt{1/4+\mu}}{C_{\omega,k}a}\right) ,$$

which is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \quad > \quad \tfrac{1}{2\delta} \ ,$$

has a holomorphic continuation to a neighborhood of 0*, whose derivative there satisfies*

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left(-\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^s}\,\left(\tfrac{1}{4}+\mu\right)^{-s+\frac{1}{2}}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\ \operatorname{Argsh}\left(\tfrac{\sqrt{\frac{1}{4}+\mu}}{C_{\omega,k}a}\right)\right)\\
&\quad = \ \tfrac{\partial}{\partial s}_{|s=0}\left(\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^s}\,\left(\tfrac{1}{4}+\mu\right)^{1-(1+\delta)s}(2\pi a)^{2\delta s-1}\,\tfrac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}}\,\tfrac{1}{2\delta s-1}\right)-\tfrac{1}{2}\sqrt{\mu}\log\mu\\
&\qquad + \left[2\int_0^{+\infty}\tfrac{1}{e^{2\pi t}-1}\left(\arctan\left(\tfrac{2\pi}{2\pi+\omega}t\right)+\arctan\left(\tfrac{2\pi}{2\pi-\omega}t\right)\right)\,dt-\log 2+2\right.\\
&\qquad\qquad \left.+\tfrac{\omega}{2\pi}\log\left(\tfrac{2\pi+\omega}{2\pi-\omega}\right)+\tfrac{1}{2}\log\left(\left(4\pi^2-\omega^2\right)a^2\right)\right]\sqrt{\mu}+o\left(1\right)
\end{aligned}$$

as μ *goes to infinity.*

Remark 4.6.31. Apart from the term left untouched, which will later be canceled, we note there is no contribution to the constant term in the asymptotic expansion as μ goes to infinity.

Proof of proposition 4.6.30. This result can be proved using, as for proposition 4.6.28, the Ramanujan summation process and hypergeometric functions. □

Tenth part. The next term from proposition 4.6.18 is the one associated to the logarithm.

Proposition 4.6.32. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\,\frac{1}{4^{s+1}}\left(\frac{1}{4}+\mu\right)^{-s} \sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s} \log\left(\left(C_{\omega,k}a\right)^2+\frac{1}{4}+\mu\right) ,$$

which is well-defined and holomorphic on the half-plane

$$\operatorname{Re} s \quad > \quad \tfrac{1}{2\delta} \ ,$$

has a holomorphic continuation to a neighborhood of 0*, whose derivative at* $s=0$ *satisfies*

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left(\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^{s+1}}\,\left(\tfrac{1}{4}+\mu\right)^{-s}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\ \log\left(\left(C_{\omega,k}a\right)^2+\tfrac{1}{4}+\mu\right)\right)\\
&\qquad\qquad = \ \tfrac{1}{2}\log\mu+\tfrac{1}{4a}\sqrt{\mu}+o\left(1\right),
\end{aligned}$$

as μ *goes to infinity.*

Proof. Once again, we can use the same arguments as in the proof of proposition 4.6.28, namely the Ramanujan summation process and hypergeometric functions.

□

Eleventh part. The only remaining difficult term to deal with is the polynomial term from proposition 4.6.18.

Proposition 4.6.33. *The function*

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s-\frac{1}{2}}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}U_1\left(\tau\left(\frac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right)\right),$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 0$ *has a holomorphic continuation to a neighborhood of* 0*, whose derivative at* $s = 0$ *satisfies*

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}&\left(-\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^s}\,\left(\frac{1}{4}+\mu\right)^{-s-\frac{1}{2}}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}U_1\left(\tau\left(\frac{C_{\omega,k}a}{\sqrt{\frac{1}{4}+\mu}}\right)\right)\right)\\ &= -\frac{1}{16\pi a}\left(1+\frac{1}{\delta}\right)\log\mu-\frac{1}{8\pi a\delta}\log 2+\frac{1}{8\pi a}\log(4\pi a)-\frac{5}{24\pi a}+o(1),\end{aligned}$$

as μ *goes to infinity.*

Proof. The proof is once more conducted as that of proposition 4.6.28, using the Ramanujan summation process and hypergeometric functions.

□

Twelth part. The last term we need to take care of is the one corresponding to $k = 0$ in proposition 4.6.18, for which we need to assume ω not to be zero.

Proposition 4.6.34. *The function*

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\,\frac{1}{3^s}\left(\frac{1}{4}+\mu\right)^{-s}\log K_{\sqrt{\frac{1}{4}+\mu}}(\omega a)$$

is holomorphic on $\mathbb{C}$*, and its derivative at* $s = 0$ *satisfies*

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}&\left(-\frac{\sin(\pi s)}{\pi}\,\frac{1}{3^s}\,\left(\frac{1}{4}+\mu\right)^{-s}\log K_{\sqrt{\frac{1}{4}+\mu}}(\omega a)\right)\\ &= -\frac{1}{2}\sqrt{\mu}\log\mu+(\log(\omega a)-\log 2+1)\sqrt{\mu}+\frac{1}{4}\log\mu-\frac{1}{2}\log\left(\frac{\pi}{2}\right)+o(1),\end{aligned}$$

as μ *goes to infinity.*

Proof. Since there is no series involved, this result is a direct consequence of the asymptotic expansion of the modified Bessel functions of the second kind.

□

Thirteenth part. However, as we have stated right before proposition 4.6.16, we have only dealt with the difference that appears in the expression of $F_{\mu,k}$. It is now time to say a few words about the rest, namely about the terms associated to

$$\begin{cases} -2\sqrt{\frac{1}{4}+\mu}\ \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\ \log K_t\left(C_{\omega,k}a\right) & \text{if} \quad k\neq 0 \\ -2\sqrt{\frac{1}{4}+\mu}\ \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\ \log K_t\left(\omega a\right) & \text{if} \quad k=0 \end{cases}$$

Proposition 4.6.35. *The function*

$$s \longmapsto \frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\Bigg[-2\sqrt{\frac{1}{4}+\mu}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s+1}\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right)$$
$$+2\,\frac{\Gamma(\delta s)\Gamma\left(\frac{3}{2}-\delta s\right)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-\delta s}(2\pi a)^{2\delta s-1}\frac{1}{(2\delta s-1)(2\delta s-2)}$$
$$+\frac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}}\left(\frac{1}{4}+\mu\right)^{1-\delta s}(2\pi a)^{2\delta s-1}\frac{1}{2\delta s-1}\Bigg],$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$*, has a holomorphic continuation to a neighborhood of* 0.

Proof. This result is a direct consequence of theorem 4.6.12 and of the several computations, stated in propositions 4.6.19, 4.6.20, 4.6.21, 4.6.23, 4.6.24, 4.6.25, 4.6.28, 4.6.30, 4.6.32, 4.6.33.

□

Remark 4.6.36. Note that we have omitted from the last proposition the factor $\sin(\pi s)$. The reason is that the cancellation we hinted at before proposition 4.6.16 will not be perfect, as explained in remark 4.7.4. However, since there are no series involved, the cancellation of the derivative at 0 for the term corresponding to $k = 0$ will be complete.

Corollary 4.6.37. *The function*

$$s \longmapsto \frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s} - 2\sqrt{\frac{1}{4}+\mu}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s}\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right),$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$*, has a holomorphic continuation to a neighborhood of* 0.

This concludes the study of the terms associated to $A_{\mu,k}$, and thus that of the integrals $L_{\mu,k}$, which was the point of this section.

4.7 Study of the integrals $M_{\mu,k}$

Recalling definition 4.5.2, we turn our attention to the integrals $M_{\mu,k}$, which were defined as

$$M_{\mu,k}(s) = \begin{cases} \dfrac{\sin(\pi s)}{\pi}\displaystyle\int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,k}(t)\ \mathrm{d}t & \text{if} \quad k\neq 0 \\ \dfrac{\sin(\pi s)}{\pi}\displaystyle\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} f_{\mu,0}(t)\ \mathrm{d}t & \text{if} \quad k=0 \text{ and } \omega\neq 0 \end{cases},$$

where, as indicated in definition 4.4.5, the function $f_{\mu,k}$ is given by

$$\begin{array}{rcccl} f_{\mu,k} & : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & & t & \longmapsto & \frac{\partial}{\partial t}\log K_t\left(C_{\omega,k}a\right) - \frac{t}{\sqrt{\frac{1}{4}+\mu}}\,\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right) \end{array}.$$

As always, we have denoted by $\mu \geqslant 0$ a positive real number, and by ω a real number in $[0, 2\pi[$. We can then split the integrals $M_{\mu,k}$ according to the two terms appearing in the definition of $f_{\mu,k}$, leading to the following definition.

Definition 4.7.1. On the strip $1 < \operatorname{Re} s < 2$, for any real numbers μ and ω as above, as well as any integer k, we define the integrals $\widetilde{M}_{\mu,k}$ and $R_{\mu,k}$ by

$$\widetilde{M}_{\mu,k}(s) = \begin{cases} \frac{\sin(\pi s)}{\pi}\int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}\frac{\partial}{\partial t}\log K_t\left(C_{\omega,k}a\right)\,\mathrm{d}t & \text{if } k \neq 0 \\ \frac{\sin(\pi s)}{\pi}\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}\frac{\partial}{\partial t}\log K_t\left(\omega a\right)\,\mathrm{d}t & \text{if } k = 0 \text{ and } \omega \neq 0 \end{cases},$$

for the first part of $f_{\mu,k}$, and

$$R_{\mu,k}(s) = \begin{cases} -\frac{\sin(\pi s)}{\pi}\frac{1}{\sqrt{\frac{1}{4}+\mu}}\int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}t\,\mathrm{d}t \\ \quad\cdot\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right) & \text{if } \quad k \neq 0 \\ -\frac{\sin(\pi s)}{\pi}\frac{1}{\sqrt{\frac{1}{4}+\mu}}\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}t\,\mathrm{d}t \\ \quad\cdot\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(\omega a\right) & \text{if } \quad k = 0 \text{ and } \omega \neq 0 \end{cases},$$

for the second part of $f_{\mu,k}$.

Remark 4.7.2. This splitting has been performed so as to have $M_{\mu,k}(s) = \widetilde{M}_{\mu,k}(s) + R_{\mu,k}(s)$. Studying $M_{\mu,k}$ will therefore be reduced to investigating the behavior of both terms above.

4.7.1 Study of the integrals $R_{\mu,k}$

We begin this phase of the study with the simplest term, which is the second one in the definition above. As the reader will notice, it can be computed quite explicitly.

Proposition 4.7.3. *For any complex number s with $1 < \operatorname{Re} s < 2$, any integer k, as well as ny μ and ω as above, we have*

$$R_{\mu,k}(s) = \begin{cases} \frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\frac{1}{4}+\mu\right)^{-s}\frac{1}{1-s}\cdot 2\sqrt{\frac{1}{4}+\mu}\left(|k|^{2\delta}-\frac{1}{4}\right)^{-s+1} \\ \quad\cdot\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right) & \textit{if} \quad k \neq 0 \\ \frac{\sin(\pi s)}{\pi}\frac{1}{4}\frac{1}{3^s}\left(\frac{1}{4}+\mu\right)^{-s}\frac{1}{1-s}\cdot 2\sqrt{\frac{1}{4}+\mu} \\ \quad\cdot\frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(\omega a\right) & \textit{if} \quad k = 0 \textit{ and } \omega \neq 0 \end{cases}.$$

Proof. This proposition can be summed up as the computation of the integral that appears in the definition of $R_{\mu,k}$, *i.e.* in definition 4.7.1.

□

Remark 4.7.4. The reason for the lack of compensation that was alluded to in remark 4.6.36 is the factor $1/(s-1)$ that appears in the computation of $R_{\mu,k}$. When the associated term is combined with that of proposition 4.6.35, one sees a factor

$$\frac{\sin(\pi s)}{\pi} \cdot \frac{s}{s-1}$$

appear, contributing to the cancellation of the derivative at 0. However, even these two factors together cannot give a cancellation when multiplied by a meromorphic function with a single pole at 0. This is the reason why, in proposition 4.6.35, we had to remove some problematic terms.

Proposition 4.7.5. *The function*

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} R_{\mu,k}(s) \quad ,$$

which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$*, has a holomorphic continuation to a neighborhood of* 0*, whose derivative at* $s=0$ *satisfies, as* μ *goes to infinity,*

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\sum_{|k|\geqslant 1} R_{\mu,k}(s)\right] \\
&= \quad -\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \left[-2\sqrt{\tfrac{1}{4}+\mu} \sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s+1} \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}} \log K_t\left(C_{\omega,k}a\right)\right.\right. \\
&\qquad \left.\left. +2\,\frac{\Gamma(\delta s)\Gamma\left(\frac{3}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)} + \frac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right]\right] \\
&\qquad +\frac{1}{4\pi a}\left(1+\tfrac{1}{\delta}\right)\mu\log\mu + \frac{1}{2\pi a}\left[-\tfrac{1}{2\delta}-\log(4\pi a)-\tfrac{3}{2}+\tfrac{1}{\delta}\log 2\right]\mu \\
&\qquad +\frac{1}{16\pi a}\left(1+\tfrac{1}{\delta}\right)\log\mu + \frac{1}{8\pi a\delta}\log 2 - \frac{1}{8\pi a}\log(4\pi a) - \frac{1}{8\pi a} + o(1)\,.
\end{aligned}$$

Proof. Using proposition 4.7.3, we note that the function to be considered here is, up to a holomorphic factor around 0, the same as the function studied in proposition 4.6.35. Hence, the function

$$s \quad \longmapsto \quad \sum_{|k|\geqslant 1} R_{\mu,k}(s)\,,$$

which is indeed well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1/\delta$, has a holomorphic continuation to a neighborhood of 0. Still using proposition 4.6.35, we see that we have

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\sum_{|k|\geqslant 1} R_{\mu,k}(s)\right] \\
&= \quad \frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \frac{(1/4+\mu)^{-s}}{s-1} \left[-2\sqrt{\tfrac{1}{4}+\mu} \sum_{|k|\geqslant 1} \left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s+1} \frac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}} \log K_t\left(C_{\omega,k}a\right)\right.\right. \\
&\qquad \left.\left. +2\,\frac{\Gamma(\delta s)\Gamma\left(\frac{3}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)} + \frac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right]\right] \\
&\quad -\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \frac{1}{s-1} \left(2\,\frac{\Gamma(\delta s)\Gamma\left(\frac{3}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)}\right)\right] \\
&\quad -\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \frac{1}{4^s} \left(\tfrac{1}{4}+\mu\right)^{-s} \frac{1}{s-1} \left(\frac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}} \left(\tfrac{1}{4}+\mu\right)^{1-\delta s} \frac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right)\right].
\end{aligned}$$

The first derivative above can be simplified, as the factor $1/(s-1)$ within only induces a change of sign. The result is precisely the derivative in the formula we wish to prove. Therefore, we only need to compute the last two derivative in this last equality. This will be achieved by using the Laurent expansions at 0 of the relevant terms. For simplicity, we will use the notation $O\left(s^k\right)$ to mean a holomorphic function around 0 with a zero of order at least k there. We have

$$\begin{aligned}
&-\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\tfrac{1}{s-1}\left(2\,\tfrac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\,\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\,\tfrac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)}\right)\\
&= \tfrac{1}{4\pi a}\left(\tfrac{1}{4}+\mu\right)\cdot\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-(1+\delta)s}\tfrac{1}{1-s}\Gamma(\delta s)\,\tfrac{\Gamma(3/2-\delta s)}{\frac{1}{2}\sqrt{\pi}}\,(2\pi a)^{2\delta s}\,\tfrac{1}{(2\delta s-1)(\delta s-1)}\\
&= \tfrac{1}{4\pi a}\left(\tfrac{1}{4}+\mu\right)\left(s+O\left(s^2\right)\right)\left(1-2\log(2)\,s+O\left(s^2\right)\right)\\
&\quad\cdot\left(1-(1+\delta)\log\left(\tfrac{1}{4}+\mu\right)s+O\left(s^2\right)\right)\left(1+s+O\left(s^2\right)\right)\left(\tfrac{1}{\delta s}-\gamma+O(s)\right)\\
&\quad\cdot\left(1+\delta\left(2\log 2+\gamma-2\right)s+O\left(s^2\right)\right)\left(1+2\delta\log(2\pi a)\,s+O\left(s^2\right)\right)\\
&\quad\cdot\left(1+3\delta s+O\left(s^2\right)\right).
\end{aligned}$$

Therefore, the derivative at $s=0$ of the considered term is given by

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left[-\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\tfrac{1}{s-1}\left(2\,\tfrac{\Gamma(\delta s)\Gamma(3/2-\delta s)}{\sqrt{\pi}}\,\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\,\tfrac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)}\right)\right]\\
&= \tfrac{1}{4\pi a\delta}\left(\tfrac{1}{4}+\mu\right)\left(-2\log 2-(1+\delta)\log\left(\tfrac{1}{4}+\mu\right)-\cancel{\gamma\delta}+2\delta\log 2+\cancel{\gamma\delta}\right.\\
&\quad\left.-2\delta+1+2\delta\log(2\pi a)+3\delta\right)\\
&= \tfrac{1}{4\pi a\delta}\left(\tfrac{1}{4}+\mu\right)\left(1+2\delta\log(4\pi a)+\delta-2\log 2\right)-\tfrac{1}{4\pi a}\left(1+\tfrac{1}{\delta}\right)\left(\tfrac{1}{4}+\mu\right)\log\left(\tfrac{1}{4}+\mu\right)\\
&= \tfrac{1}{4\pi a\delta}\left(\tfrac{1}{4}+\mu\right)\left(1+2\delta\log(4\pi a)+\delta-2\log 2\right)\\
&\quad-\tfrac{1}{4\pi a}\left(1+\tfrac{1}{\delta}\right)\left(\tfrac{1}{4}\log\mu+\mu\log\mu+\tfrac{1}{4}\right)+o(1)\\
&= -\tfrac{1}{4\pi a}\left(1+\tfrac{1}{\delta}\right)\mu\log\mu-\tfrac{1}{16\pi a}\left(1+\tfrac{1}{\delta}\right)\log\mu+\tfrac{1}{4\pi a\delta}\left(1+2\delta\log(4\pi a)+\delta-2\log 2\right)\mu\\
&\quad+\tfrac{1}{8\pi a}\log(4\pi a)-\tfrac{1}{8\pi a\delta}\log 2+o(1)
\end{aligned}$$

as μ goes to infinity. We can now move to determining the asymptotic behavior of the last derivative at 0 above as μ goes to infinity. We have

$$\begin{aligned}
&-\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\tfrac{1}{s-1}\left(\tfrac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}}\,\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\,\tfrac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right)\\
&= -\tfrac{1}{2\pi a}\left(\tfrac{1}{4}+\mu\right)\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\tfrac{1}{1-s}\left(\tfrac{1}{4}+\mu\right)^{-(1+\delta)s}\Gamma(\delta s)\,\tfrac{\Gamma(1/2-\delta s)}{\sqrt{\pi}}\,(2\pi a)^{2\delta s}\,\tfrac{1}{1-2\delta s}\\
&= -\tfrac{1}{2\pi a}\left(\tfrac{1}{4}+\mu\right)\left(s+O\left(s^3\right)\right)\left(1-2\log(2)\,s+O\left(s^2\right)\right)\left(1+s+O\left(s^2\right)\right)\\
&\quad\cdot\left(1-(1+\delta)\log\left(\tfrac{1}{4}+\mu\right)s+O\left(s^2\right)\right)\left(\tfrac{1}{\delta s}-\gamma+O(s)\right)\\
&\quad\cdot\left(1+\delta\left(2\log 2+\gamma\right)s+O\left(s^2\right)\right)\left(1+2\delta\log(2\pi a)\,s+O\left(s^2\right)\right)\\
&\quad\cdot\left(1+2\delta s+O\left(s^2\right)\right).
\end{aligned}$$

The derivative of this term at the origin is thus given by

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left[-\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s}\tfrac{1}{s-1}\left(\tfrac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}}\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\tfrac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right)\right] \\
&= -\tfrac{1}{2\pi a\delta}\left(\tfrac{1}{4}+\mu\right)\left(-2\log 2+1-(1+\delta)\log\left(\tfrac{1}{4}+\mu\right)-\cancel{\gamma\delta}\right. \\
&\qquad \left.+2\delta\log 2+\cancel{\gamma\delta}+2\delta\log(2\pi a)+2\delta\right) \\
&= \tfrac{1}{2\pi a\delta}(1+\delta)\,\mu\log\mu+\tfrac{1}{8\pi a}\left(1+\tfrac{1}{\delta}\right)\log\mu-\tfrac{1}{2\pi a\delta}\left(1-2\log 2+2\delta\log(4\pi a)+2\delta\right)\mu \\
&\qquad -\tfrac{1}{8\pi a\delta}\left(-2\log 2+1+2\delta\log 2+2\delta\log(2\pi a)+2\delta\right)+\tfrac{1}{8\pi a\delta}(1+\delta)+o(1) \\
&= \tfrac{1}{2\pi a\delta}(1+\delta)\,\mu\log\mu+\tfrac{1}{8\pi a}\left(1+\tfrac{1}{\delta}\right)\log\mu-\tfrac{1}{2\pi a\delta}\left(1-2\log 2+2\delta\log(4\pi a)+2\delta\right)\mu \\
&\qquad +\tfrac{1}{4\pi a\delta}\log 2-\tfrac{1}{8\pi a}-\tfrac{1}{4\pi a}\log(4\pi a)+o(1)
\end{aligned}$$

as μ goes to infinity. These last two computations then yield

$$\begin{aligned}
&\tfrac{\partial}{\partial s}_{|s=0}\left[\sum_{|k|\geqslant 1}R_{\mu,k}(s)\right] \\
&= \tfrac{\partial}{\partial s}_{|s=0}\left[\tfrac{\sin(\pi s)}{\pi}\tfrac{1}{4^s}\tfrac{(1/4+\mu)^{-s}}{s-1}\left[-2\sqrt{\tfrac{1}{4}+\mu}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s+1}\tfrac{\partial}{\partial t}_{|t=\sqrt{\frac{1}{4}+\mu}}\log K_t\left(C_{\omega,k}a\right)\right.\right. \\
&\qquad \left.\left.+2\,\tfrac{\Gamma(\delta s)\Gamma\left(\frac{3}{2}-\delta s\right)}{\sqrt{\pi}}\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\tfrac{(2\pi a)^{2\delta s-1}}{(2\delta s-1)(2\delta s-2)}+\tfrac{\Gamma(\delta s)\Gamma\left(\frac{1}{2}-\delta s\right)}{\sqrt{\pi}}\left(\tfrac{1}{4}+\mu\right)^{1-\delta s}\tfrac{(2\pi a)^{2\delta s-1}}{2\delta s-1}\right]\right] \\
&\qquad +\tfrac{1}{4\pi a}\left(1+\tfrac{1}{\delta}\right)\mu\log\mu+\tfrac{1}{2\pi a}\left[-\tfrac{1}{2\delta}-\log(4\pi a)-\tfrac{3}{2}+\tfrac{1}{\delta}\log 2\right]\mu \\
&\qquad +\tfrac{1}{16\pi a}\left(1+\tfrac{1}{\delta}\right)\log\mu+\tfrac{1}{8\pi a\delta}\log 2-\tfrac{1}{8\pi a}\log(4\pi a)-\tfrac{1}{8\pi a}+o(1)
\end{aligned}$$

as μ goes to infinity. This concludes the proof of the proposition.

□

This concludes the study of the integrals $R_{\mu,k}$ as μ goes to infinity. We now move on to investigating the behavior of $R_{0,k}$ as a goes to infinity. This will be done by taking advantage of the fact that the log-derivative of the modified Bessel function has an explicit expression in this case.

Proposition 4.7.6. *For every integer k, we have*

$$R_{0,k}(s) = \begin{cases} \tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{2}\,\tfrac{1}{1-s}\cdot\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s+1}\mathbb{E}_1\left(2C_{\omega,k}a\right)e^{2C_{\omega,k}a} & \textit{if} \quad k\neq 0 \\[2ex] \tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{2}\,\tfrac{1}{1-s}\cdot\left(\tfrac{3}{4}\right)^{-s+1}\mathbb{E}_1(2\omega a)\,e^{2\omega a} & \textit{if} \quad k=0 \textit{ and } \omega\neq 0 \end{cases},$$

where $\mathbb{E}_1$ denotes the exponential integral, for which the reader is referred to [75, Chap. 6]. The derivative at 0 of (the continuation of) function

$$s \longmapsto \sum_{|k|\geqslant 1}R_{0,k}(s),$$

then satisfies

$$\tfrac{\partial}{\partial s}_{|s=0}\left[\sum_{|k|\geqslant 1}R_{0,k}(s)\right] = o(1)$$

as a goes to infinity. Furthermore, assuming ω does not vanish, we have

$$\frac{\partial}{\partial s}_{|s=0} R_{0,0}(s) = o(1)$$

as a goes to infinity.

Proof. This result can be obtained by using the asymptotic expansion of the exponential integral, which the reader may find in [75, Sec. 6.12.2].

□

Remark 4.7.7. It should be noted that this sort of asymptotic evaluation would be significantly more difficult to obtain should we have not assumed μ to be zero.

4.7.2 Study of the integrals $\widetilde{M}_{\mu,k}(s)$

We can now move on to the core of this section, which is the study of the terms associated to

$$\widetilde{M}_{\mu,k}(s) = \begin{cases} \frac{\sin(\pi s)}{\pi} \int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k}a\right)\, \mathrm{d}t & \text{if } k \neq 0 \\ \frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t} \log K_t(\omega a)\, \mathrm{d}t & \text{if } k = 0 \\ & \text{and } \omega \neq 0 \end{cases}.$$

Proposition 4.7.8. *For any μ, ω as before, any integer k, and any real number t, we have*

$$\begin{aligned} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k}a\right) &= \operatorname{Argsh}\left(\frac{t}{C_{\omega,k}a}\right) - \frac{1}{2} \cdot \frac{t}{t^2 + \left(C_{\omega,k}a\right)^2} \\ &\quad + \frac{\partial}{\partial t}\left(\frac{1}{t}\, U_1\left(\tau\left(\frac{C_{\omega,k}a}{t}\right)\right)\right) + \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho}_2\left(t, \frac{C_{\omega,k}a}{t}\right)\right). \end{aligned}$$

This computation also holds for $k = 0$, assuming ω is non-zero.

Proof. Let $\mu \geqslant 0$ and ω be real numbers, with $\omega \in [0, 2\pi[$, and k be an integer. Recalling the asymptotic expansion of modified Bessel functions of the second kind, we have

$$\begin{aligned} \log K_t\left(C_{\omega,k}a\right) &= \frac{1}{2}\log\left(\frac{\pi}{2}\right) + t \operatorname{Argsh}\left(\frac{t}{C_{\omega,k}}\right) - \sqrt{t^2 + \left(C_{\omega,k}a\right)^2} - \frac{1}{4}\log\left(t^2 + \left(C_{\omega,k}a\right)^2\right) \\ &\quad - \frac{1}{8t}\left(1 + \frac{1}{t^2}\left(C_{\omega,k}a\right)^2\right)^{-1/2} + \frac{5}{24t}\left(1 + \frac{1}{t^2}\left(C_{\omega,k}a\right)^2\right)^{-3/2} \\ &\quad + \frac{1}{t^2}\widetilde{\rho}_2\left(t, \frac{C_{\omega,k}a}{t}\right) \end{aligned}$$

for any real number t. After differentiating, we get

$$\begin{aligned} \frac{\partial}{\partial t} \log K_t\left(C_{\omega,k}a\right) &= \operatorname{Argsh}\left(\frac{t}{C_{\omega,k}a}\right) + \cancel{\frac{t}{\sqrt{1+\frac{t^2}{\left(C_{\omega,k}a\right)^2}}} \cdot \frac{1}{C_{\omega,k}a}} - \cancel{\frac{t}{\sqrt{t^2+\left(C_{\omega,k}a\right)^2}}} \\ &\quad - \frac{1}{2} \cdot \frac{t}{t^2+\left(C_{\omega,k}a\right)^2} - \frac{1}{8}\frac{\partial}{\partial t}\left(\frac{1}{t}\left(1 + \frac{1}{t^2}\left(C_{\omega,k}a\right)^2\right)^{-1/2}\right) \\ &\quad + \frac{5}{24}\frac{\partial}{\partial t}\left(\frac{1}{t}\left(1 + \frac{1}{t^2}\left(C_{\omega,k}a\right)^2\right)^{-3/2}\right) + \frac{\partial}{\partial t}\left(\frac{1}{t^2}\widetilde{\rho}_2\left(t, \frac{C_{\omega,k}a}{t}\right)\right). \end{aligned}$$

This yields the proposition.

□

We will treat every term appearing in proposition 4.7.8, taking care of the μ-asymptotic expansion, and of the a-asymptotic expansion for $\mu = 0$ simultaneously, as computations are the same.

First part. We begin with the term that involves the remainder $\widetilde{\rho_2}$.

Proposition 4.7.9. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{C_{\omega,k}a}{t}\right)\right)\, dt$$

is well-defined and holomorphic on the half-plane $\operatorname{Re} s > -\frac{1}{4}$ *and its derivative at* $s = 0$ *satisfies*

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{C_{\omega,k}a}{t}\right)\right) dt\right] \quad = \quad o(1)$$

as μ goes to infinity. Furthermore, the same derivative, this time for $\mu = 0$, has the following asymptotic expansion, as a goes to infinity,

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{|k|^\delta}^{+\infty} \left(t^2 - \frac{1}{4}\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{C_{\omega,k}a}{t}\right)\right) dt\right] \quad = \quad o(1)\,.$$

Proof. We will begin by performing an integration by parts. We have

$$\begin{aligned}
&\int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s} \tfrac{\partial}{\partial t}\left(\tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right)\right) \mathrm{d}t \\
&\quad = \left[\left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s} \tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right)\right]_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \\
&\qquad\qquad +2s\int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} t\left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} \tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right) \mathrm{d}t \\
&\quad = -\tfrac{1}{4^{s+1}}\left(\tfrac{1}{4}+\mu\right)^{-s-1}\left(|k|^{2\delta} - \tfrac{1}{4}\right)^{-s} \cdot \tfrac{1}{|k|^{2\delta}} \cdot \widetilde{\rho_2}\left(2|k|^\delta\sqrt{\tfrac{1}{4}+\mu}, \tfrac{C_{\omega,k}a}{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}\right) \\
&\qquad\qquad +2s\int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} t\left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} \tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right) \mathrm{d}t
\end{aligned}$$

for s of real part large enough. We note that, the first term above has actually already been dealt with in proposition 4.6.19, in which a bound was proved on $\widetilde{\rho_2}$, thus yielding the asymptotic behaviors required. To take care of the second term, we will work in a similar fashion. For any integer $k \in \mathbb{Z}$ and any real number

$$t \quad \in \quad \left[2|k|^\delta\sqrt{\tfrac{1}{4}+\mu}, +\infty\right[,$$

we have

$$\begin{aligned}
\tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right) \quad &= \quad \sum_{n\geqslant 1} \tfrac{(-1)^{n+1}}{n}\left[\tfrac{1}{t^2}\ \rho_2\left(t, \tfrac{1}{t}(2\pi k+\omega)a\right) - \tfrac{1}{t}u_1\left(\tau\left(\tfrac{1}{t}(2\pi k+\omega)a\right)\right)\right]^n \\
&\qquad\qquad +\tfrac{1}{t}u_1\left(\tau\left(\tfrac{1}{t}(2\pi k+\omega)a\right)\right] \\
&= \quad \sum_{n\geqslant 2} \tfrac{(-1)^{n+1}}{n}\left[\tfrac{1}{t^2}\ \rho_2\left(t, \tfrac{1}{t}(2\pi k+\omega)a\right) - \tfrac{1}{t}u_1\left(\tau\left(\tfrac{1}{t}(2\pi k+\omega)a\right)\right)\right]^n \\
&\qquad\qquad +\tfrac{1}{t^2}\ \rho_2\left(t, \tfrac{1}{t}(2\pi k+\omega)a\right),
\end{aligned}$$

where the terms ρ_2 and u_1 are the ones used by Olver in [74]. We now have

$$\frac{1}{t^2}\left|\rho_2\left(t, \frac{(2\pi k+\omega)a}{t}\right)\right| \quad \leqslant \quad \frac{C}{t^2}\cdot\frac{1}{1+\frac{(2\pi k+\omega)^2a^2}{t^2}} \quad = \quad \frac{C}{t^2+(2\pi k+\omega)^2a^2} \quad \leqslant \quad \frac{C}{2t|2\pi k+\omega|a} \ ,$$

where $C>0$ is a strictly positive constant. We should note that this estimate on ρ_2 can be obtained in more details as in proposition 4.6.19. Furthermore, we have

$$\begin{aligned}
\frac{1}{t}\left|u_1\left(\tau\left(\tfrac{1}{t}\left(2\pi k+\omega\right)a\right)\right)\right| \quad &\leqslant \quad \frac{1}{24t}\left(3\tau\left(\frac{(2\pi k+\omega)a}{t}\right)+5\tau\left(\frac{(2\pi k+\omega)}{t}\right)^3\right)\\
&\leqslant \quad \frac{1}{3t}\cdot\frac{1}{\sqrt{1+\frac{(2\pi k+\omega)^2a^2}{t^2}}} \quad \leqslant \quad \frac{1}{3\sqrt{2}}\cdot\frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{|2\pi k+\omega|a}}.
\end{aligned}$$

This allows us to give explicit bounds for $\widetilde{\rho}_2$. We have

$$\begin{aligned}
\frac{1}{t^2}\left|\widetilde{\rho}_2\left(t,\frac{C_{\omega,k}a}{t}\right)\right| \quad &\leqslant \quad \frac{C}{2t|2\pi k+\omega|a}+\sum_{n\geqslant 2}\left[\frac{C}{2t|2\pi k+\omega|a}+\frac{1}{3\sqrt{2}}\cdot\frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{|2\pi k+\omega|a}}\right]^n\\
&\leqslant \quad \frac{C}{2t|2\pi k+\omega|a}+\sum_{n\geqslant 2}\frac{1}{t^{n/2}}\cdot\frac{1}{(|2\pi k+\omega|a)^{n/2}}\underbrace{\left[\frac{C}{2t\sqrt{t}\sqrt{|2\pi k+\omega|a}}+\frac{1}{3\sqrt{2}}\right]^2}_{<\frac{1}{2^n}\text{ for }|k|\text{ large enough}}\\
&\leqslant \quad \frac{C}{2t|2\pi k+\omega|a}+\frac{1}{t|2\pi k+\omega|a}\sum_{n\geqslant 2}\frac{1}{2^n}\\
&\leqslant \quad \frac{C'}{2t|2\pi k+\omega|a}
\end{aligned}$$

where C' is a strictly positive real constant. It is worth noting that the fact that this bound is only valid for $|k|$ large enough is of no consequence, as we have the required asymptotics for any of the individual terms appearing in the series. Thus, for $|k|$ large enough and any t in the interval mentioned above, we have

$$\begin{aligned}
\left|t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1}\frac{1}{t^2}\ \widetilde{\rho}_2\left(t,\frac{C_{\omega,k}a}{t}\right)\right| \quad &\leqslant \quad \cancel{t}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-\operatorname{Re}s-1}\cdot\frac{C'}{2\cancel{t}|2\pi k+\omega|a}\\
&\leqslant \quad \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-3/4}\cdot\frac{C'}{2|2\pi k+\omega|a},
\end{aligned}$$

the last inequality being valid for k large enough in absolute value, proving that the function

$$s \quad \longmapsto \quad \int_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1}\frac{1}{t^2}\ \widetilde{\rho}_2\left(t,\frac{C_{\omega,k}a}{t}\right)\ \mathrm{d}t$$

is holomorphic on the half-plane $\operatorname{Re}s>-1/4$, at least whenever $|k|$ is large enough, while the remaining terms can be deal with in a similar fashion. We then have, still for k far enough from 0,

$$\begin{aligned}
&\left|\int_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1}\frac{1}{t^2}\ \widetilde{\rho}_2\left(t,\frac{C_{\omega,k}a}{t}\right)\ \mathrm{d}t\right|\\
&\leqslant \quad \frac{C'}{2a|2\pi k+\omega|}\int_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty}\frac{\mathrm{d}t}{(t^2-(1/4+\mu))^{3/4}}\\
&\leqslant \quad \frac{C'}{2a|2\pi k+\omega|}\left(\left[\frac{2}{t}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{1/4}\right]_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty}+2\int_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty}\frac{1}{t^2}\underbrace{\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{1/4}}_{\leqslant t^{1/2}}\mathrm{d}t\right)\\
&\leqslant \quad \frac{C''}{|2\pi k+\omega|a}\cdot\left(\tfrac{1}{4}+\mu\right)^{-1/4}\cdot\frac{1}{|k|^{\delta/2}},
\end{aligned}$$

which means that this term, after summation over k, induces a holomorphic function around the origin. The derivative of the function

$$s \quad \longmapsto \quad 2s\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^\delta\sqrt{1/4+\mu}}^{+\infty} t\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s-1} \tfrac{1}{t^2}\ \widetilde{\rho_2}\left(t, \tfrac{C_{\omega,k}a}{t}\right)\ \mathrm{d}t$$

then vanishes, because of the presence of the factor $s\sin(\pi s)$. The proposition is thus proved.

□

Proposition 4.7.10. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{\omega a}{t}\right)\right)\ dt$$

is holomorphic on the half-plane $\operatorname{Re} s > -\frac{3}{2}$ *and its derivative at* $s=0$ *satisfies,*

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{\omega a}{t}\right)\right)\ dt\right] \quad = \quad o(1)$$

as μ *goes to infinity. Furthermore, the same derivative, taken with* $\mu=0$, *satisfies*

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi} \int_{1}^{+\infty} \left(t^2-\frac{1}{4}\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t^2}\ \widetilde{\rho_2}\left(t, \frac{\omega a}{t}\right)\right)\ dt\right] \quad = \quad o(1)$$

as a *goes to infinity.*

Proof. The proof can be conducted in a similar way as that of the last proposition, the argument being simpler, since there are no series involved.

□

Second part. We now move on to the Argsh term.

Proposition 4.7.11. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\frac{t}{C_{\omega,k}a}\right)\ dt\ ,$$

which is well-defined and holomorphic on a half-plane of complex numbers s *with large enough real part, has a holomorphic continuation to a neighborhood of* 0, *whose derivative there satisfies*

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\tfrac{t}{C_{\omega,k}a}\right)\ dt\right] \\
&= \quad -\frac{\partial}{\partial s}_{|s=0}\left(\frac{\sin(\pi s)}{\pi}\ \frac{1}{4^{s-1/2}}\ \left(\tfrac{1}{4}+\mu\right)^{-s+\frac{1}{2}} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s-\delta}}\ \operatorname{Argsh}\left(\frac{|k|^\delta\sqrt{4\mu+1}}{|2\pi k+\omega|\,a}\right)\right) \\
&\qquad + \frac{\partial}{\partial s}_{|s=0}\left(\frac{\sin(\pi s)}{\pi}\frac{1}{4^s}\left(\tfrac{1}{4}+\mu\right)^{-s} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}}\sqrt{(4\mu+1)\,|k|^{2\delta}+(2\pi k+\omega)^2 a^2}\right) \\
&\qquad\quad -\frac{1}{4\pi a\delta}\mu\log\mu + \frac{1}{2\pi a}\left[1+\log(4\pi a)-\tfrac{1}{\delta}\log 2\right]\mu - \frac{1}{16\pi a\delta}\log\mu \\
&\qquad\qquad -\frac{\omega^2}{2\pi}a + \frac{1}{8\pi a} + \frac{1}{8\pi a}\log(4\pi a) - \frac{1}{8\pi a\delta}\log 2 - \frac{1}{16\pi a\delta} + o(1)
\end{aligned}$$

as μ goes to infinity. Furthermore, the same derivative, this time taken for $\mu = 0$, satisfies

$$\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{|k|^\delta}^{+\infty} \left(t^2 - \tfrac{1}{4}\right)^{-s} \operatorname{Argsh}\left(\frac{t}{C_{\omega,k} a}\right) dt \right] = -\frac{\omega^2}{2\pi} a - \frac{\pi}{3} a + o(1)$$

as a goes to infinity.

Remark 4.7.12. It should be noted that two derivatives were left uncomputed in the proposition above. The first aims to cancel the one from proposition 4.6.23, while the second one cancels the derivative from proposition 4.6.21. Since the study of both asymptotic behaviors call for the same computations, they will be done simultaneously.

Proof of proposition 4.7.11. We begin by using the binomial formula, which holds on the interval of integration. We have

$$\begin{aligned}
&\sum_{|k|\geqslant 1} \int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\frac{t}{C_{\omega,k} a}\right) \mathrm{d}t \\
&\qquad = \sum_{j\geqslant 0} \frac{(s)_j}{j} \left(\tfrac{1}{4}+\mu\right)^j \sum_{|k|\geqslant 1} \int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \operatorname{Argsh}\left(\frac{t}{|2\pi k+\omega|a}\right) \mathrm{d}t,
\end{aligned}$$

as the interchanging of sums and integrals is also permitted. We will now compute the integrals above using hypergeometric functions, as well as an integration by parts. We have

$$\begin{aligned}
&\int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \operatorname{Argsh}\left(\frac{t}{|2\pi k+\omega|a}\right) \mathrm{d}t \\
&\qquad = \left[-\frac{1}{2s+2j-1} \cdot \frac{1}{t^{2s+2j-1}} \operatorname{Argsh}\left(\frac{t}{|2\pi k+\omega|a}\right)\right]_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \\
&\qquad\qquad + \frac{1}{2s+2j-1} \cdot \frac{1}{|2\pi k+\omega|a} \int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j-1}} \left(1 + \frac{t^2}{(2\pi k+\omega)^2 a^2}\right)^{-1/2} \mathrm{d}t \\
&\qquad = \frac{1}{2s+2j-1} \cdot \frac{1}{2^{2s+2j-1}} \left(\tfrac{1}{4}+\mu\right)^{-s-j+1/2} \cdot \frac{1}{|k|^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2|k|^\delta \sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right) \\
&\qquad\qquad + \frac{1}{2s+2j-1} \int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \cdot \left(1 + \frac{(2\pi k+\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t.
\end{aligned}$$

After summation over k and j, we get

$$\begin{aligned}
&\sum_{|k|\geqslant 1} \int_{2|k|^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\tfrac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\frac{t}{C_{\omega,k} a}\right) \mathrm{d}t \\
&= \left(\tfrac{1}{4}+\mu\right)^{-s+1/2} \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \cdot \frac{1}{2^{2s+2j-1}} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2|k|^\delta \sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right) \\
&\quad + \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \left(\tfrac{1}{4}+\mu\right)^j \left[\sum_{k\geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1 + \frac{(2\pi k+\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t \right. \\
&\qquad\qquad \left. + \sum_{k\geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1 + \frac{(2\pi k-\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t \right].
\end{aligned}$$

We will begin by studying the function associated to the first term, that is

$$s \longmapsto \frac{\sin(\pi s)}{\pi} \left(\tfrac{1}{4}+\mu\right)^{-s+1/2} \sum_{j\geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \cdot \frac{1}{2^{2s+2j-1}} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2|k|^\delta \sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right).$$

As we did in the proof of proposition 4.6.23, for any integer $j \geqslant 0$, we split the sum over k into the following two parts

$$\begin{aligned}
&\sqrt{4\mu+1} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2|k|^\delta\sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right) \\
&\quad = \sqrt{4\mu+1}\left[\sum_{k\geqslant 1} \frac{1}{k^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2k^\delta\sqrt{1/4+\mu}}{(2\pi k+\omega)a}\right) + \sum_{k\geqslant 1} \frac{1}{k^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2k^\delta\sqrt{1/4+\mu}}{(2\pi k-\omega)a}\right)\right]
\end{aligned}$$

and we note that the first of these two sums can be written as

$$\begin{aligned}
&\sqrt{4\mu+1} \sum_{k\geqslant 1} \frac{1}{k^{2\delta(s+j)-\delta}} \operatorname{Argsh}\left(\frac{2k^\delta\sqrt{1/4+\mu}}{(2\pi k+\omega)a}\right) \\
&\quad = \frac{4\mu+1}{2\pi a}\zeta\left(2\delta\left(s+j-1\right)+1\right) - \frac{4\mu+1}{2\pi a}\cdot\frac{\omega}{2\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j-1)+2}}\cdot\frac{1}{1+\frac{\omega}{2\pi k}} \\
&\qquad -\sqrt{4\mu+1}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s+j)-\delta}}\int_0^{\frac{k^\delta\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}}\frac{x}{(1+x^2)^{3/2}}\left(\frac{k^\delta\sqrt{4\mu+1}}{(2\pi k+\omega)\,a}-x\right)\,\mathrm{d}x.
\end{aligned}$$

We then realize that, after multiplication by the appropriate factor, the sum over $j \geqslant 2$ induces a holomorphic function around 0, whose derivative at this point vanishes because of the Pochhammer symbol. The term corresponding to $j = 1$ also induces a holomorphic function around the origin, and its derivative there is given by the term above involving the Riemann zeta function, which is

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\left(\frac{1}{4}+\mu\right)^{-s} s\cdot\frac{1}{2s+1}\cdot\frac{1}{2^{2s+2}}\cdot\frac{4\mu+1}{2\pi a}\zeta\left(2\delta s+1\right)\right] \;=\; \frac{4\mu+1}{8\pi a}\cdot\frac{1}{2\delta} \;=\; \frac{1}{4\pi a\delta}\mu+\frac{1}{16\pi a\delta}\;.$$

This derivative has been written that way so as to facilitate its consideration in the μ-asymptotic expansion. After having taken $\mu = 0$, the same derivative vanishes as a goes to infinity, which yields its contribution to the a-asymptotic expansion. The term associated to $j = 0$, *i.e.*

$$\frac{\sin(\pi s)}{\pi}\left(\frac{1}{4}+\mu\right)^{-s+1/2}\frac{1}{2s-1}\cdot\frac{1}{2^{2s-1}}\sum_{|k|\geqslant 1}\frac{1}{|k|^{2\delta s-\delta}}\operatorname{Argsh}\left(\frac{2|k|^\delta\sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right)$$

induces a holomorphic function around 0. A computation of its derivative there is not necessary for the asymptotic study as μ goes to infinity, as indicated in the statement of this proposition. However, it has to be done for the study as a goes to infinity, for which we take μ to be zero. Fortunately, the computation that was done above will yield the result quite directly. We have

$$\begin{aligned}
\sum_{k\geqslant 1}\frac{1}{k^{2\delta s-\delta}}\operatorname{Argsh}\left(\frac{k^\delta}{(2\pi k+\omega)a}\right) \;&=\; \frac{1}{2\pi a}\zeta\left(2\delta\left(s-1\right)+1\right)-\frac{1}{2\pi a}\cdot\frac{\omega}{2\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)+2}}\cdot\frac{1}{1+\frac{\omega}{2\pi k}} \\
&\quad -\sum_{k\geqslant 1}\frac{1}{k^{2\delta s-\delta}}\int_0^{\frac{k^\delta}{(2\pi k+\omega)\,a}}\frac{x}{(1+x^2)^{3/2}}\left(\frac{k^\delta}{(2\pi k+\omega)\,a}-x\right)\,\mathrm{d}x.
\end{aligned}$$

We note that the second sum over k can be dealt with by formally switching the sign of ω, yielding

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\left(\frac{1}{4}+\mu\right)^{-s+1/2}\frac{1}{2s-1}\cdot\frac{1}{2^{2s-1}}\sum_{|k|\geqslant 1}\frac{1}{|k|^{2\delta s-\delta}}\operatorname{Argsh}\left(\frac{2|k|^\delta\sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right)\right] \;=\; o\left(1\right)$$

as a goes to infinity. To sum up what we have proved so far, the function

$$s \;\longmapsto\; \frac{\sin(\pi s)}{\pi}\left(\frac{1}{4}+\mu\right)^{-s+1/2}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{2s+2j-1}\cdot\frac{1}{2^{2s+2j-1}}\sum_{|k|\geqslant 1}\frac{1}{|k|^{2\delta(s+j)-\delta}}\operatorname{Argsh}\left(\frac{2|k|^\delta\sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right)$$

has a holomorphic continuation to a neighborhood of 0, its derivative there is given by

$$-\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\left(\frac{1}{4}+\mu\right)^{-s+1/2}\frac{1}{2^{2s-1}}\sum_{|k|\geqslant 1}\frac{1}{|k|^{2\delta s-\delta}}\operatorname{Argsh}\left(\frac{2|k|^{\delta}\sqrt{1/4+\mu}}{|2\pi k+\omega|a}\right)\right]+\frac{1}{2\pi a\delta}\mu+\frac{1}{8\pi a\delta}$$

and vanishes as a goes to infinity, when μ equals zero. We move on to the next term, given by

$$s \quad\longmapsto\quad \frac{\sin(\pi s)}{\pi}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\cdot\frac{1}{2s+2j-1}\left(\frac{1}{4}+\mu\right)^j\sum_{k\geqslant 1}\int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\mathrm{d}t\ .$$

We will first prove that the sum over $j\geqslant 2$ plays no role in what we aim to prove. For any such integer j, any strictly positive integer k, and any t in the interval of integration, we have

$$\left|\frac{1}{t^{2s+2j}}\right| \quad\leqslant\quad \frac{1}{t^{2j-3}}\cdot\frac{1}{t^{2\operatorname{Re}s+3}} \quad\leqslant\quad \left(\frac{1}{4}+\mu\right)^{-j+3/2}\cdot\frac{1}{k^{\delta(2j-3)}}\cdot\frac{1}{t^{2\operatorname{Re}s+3}}\ ,$$

which then yields

$$\left|\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\right| \quad\leqslant\quad (4\mu+1)^{-j+3/2}\cdot\frac{1}{k^{\delta(2j-3)}}\cdot\frac{1}{(2\pi k+\omega)a}\cdot\frac{1}{t^{2\operatorname{Re}s+2}}\ .$$

This proves that, for any integer $j\geqslant 2$, the function

$$s \quad\longmapsto\quad \int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\mathrm{d}t$$

is holomorphic on a neighborhood of 0. For these integers, we further have

$$\begin{aligned}&\left|\int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\mathrm{d}t\right|\\ &\qquad\leqslant\quad (4\mu+1)^{-j+3/2}\cdot\frac{1}{k^{\delta(2j-3)}}\cdot\frac{1}{(2\pi k+\omega)a}\cdot\frac{1}{1+2\operatorname{Re}s}\left(2k^{\delta}\sqrt{\frac{1}{4}+\mu}\right)^{-2\operatorname{Re}s-1}.\end{aligned}$$

This proves that the following function is holomorphic on a neighborhood of the origin

$$s \quad\longmapsto\quad \frac{\sin(\pi s)}{\pi}\sum_{j\geqslant 2}\frac{(s)_j}{j!}\cdot\frac{1}{2s+2j-1}\left(\frac{1}{4}+\mu\right)^j\sum_{k\geqslant 1}\int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\mathrm{d}t.$$

Its derivative there vanishes because of the Pochhammer symbol. Only the terms corresponding to $j\in\{0,1\}$ remain. Let us compute the integrals above in the sum over k, using hypergeometric functions and a change of variables $x=(2\pi k+\omega)^2a^2t^{-2}$. For any integer $k\geqslant 1$, we have

$$\begin{aligned}&\int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j}}\left(1+\frac{(2\pi k+\omega)^2a^2}{t^2}\right)^{-1/2}\mathrm{d}t\\ &=\quad \frac{1}{2}\left((2\pi k+\omega)a\right)^{-2s-2j+1}\left[\frac{\Gamma(s+j-1/2)\Gamma(-s-j+1)}{\Gamma(1/2)}+\frac{1}{s+j-1}\cdot\frac{((2\pi k+\omega)a)^{2s+2j-2}}{(4\mu+1)^{s+j-1}k^{2\delta(s+j-1)}}\right.\\ &\qquad\qquad\left.\cdot F\left(\frac{1}{2},-s-j+1,-s-j+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\right]\\ &=\quad \frac{1}{2}\cdot\frac{1}{(2\pi k+\omega)a}\cdot\frac{1}{s+j-1}(4\mu+1)^{-s-j+1}\cdot\frac{1}{k^{2\delta(s+j-1)}}F\left(\frac{1}{2},-s-j+1,-s-j+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\qquad+\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s+2j-1}}\cdot\frac{1}{\sqrt{\pi}}\Gamma(s+j-1/2)\,\Gamma(-s-j+1)\,.\end{aligned}$$

Having this formula, we can now take care of both integers j that have yet to be studied.

- We begin by dealing with the case $j=1$. We consider

$$\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\sum_{k\geqslant 1}\left[\frac{1}{2}\cdot\frac{1}{(2\pi k+\omega)a}\cdot\frac{1}{s}\left(4\mu+1\right)^{-s}\cdot\frac{1}{k^{2\delta s}}F\left(\frac{1}{2},-s,-s+1,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\right.$$
$$\left.+\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s+1}}\cdot\frac{1}{\sqrt{\pi}}\Gamma\left(s+1/2\right)\Gamma\left(-s\right)\right].$$

We first consider the second term of the above, given by

$$\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\frac{1}{2\sqrt{\pi}}\Gamma\left(s+1/2\right)\Gamma\left(-s\right)\sum_{k\geqslant 1}\frac{1}{((2\pi k+\omega)a)^{2s+1}}\quad,$$

which is close to the Hurwitz zeta function ζ_H. Namely, we have

$$\begin{aligned}&\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\frac{1}{2\sqrt{\pi}}\Gamma\left(s+1/2\right)\Gamma\left(-s\right)\sum_{k\geqslant 1}\frac{1}{((2\pi k+\omega)a)^{2s+1}}\\&\qquad=\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\frac{1}{2\sqrt{\pi}}\Gamma\left(s+1/2\right)\Gamma\left(-s\right)\left(2\pi a\right)^{-2s-1}\zeta_H\left(2s+1,1+\frac{\omega}{2\pi}\right).\end{aligned}$$

Using the fact that ζ_H has a meromorphic continuation to $\mathbb{C}$, whose only pole is at 1 and is simple, we see that the above induces a holomorphic function around $s=0$. We will compute its derivative at the origin using Laurent expansions, as well as the previously used notation $O\left(s^k\right)$, which stands for a holomorphic function around 0 which has a zero of order at least k there. We have

$$\begin{aligned}&\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\frac{1}{2}\frac{\Gamma(s+1/2)}{\sqrt{\pi}}\Gamma\left(-s\right)\left(2\pi a\right)^{-2s-1}\zeta_H\left(2s+1,1+\frac{\omega}{2\pi}\right)\\&\quad=-\frac{1}{8\pi a}\left(\frac{1}{4}+\mu\right)s^2\left(1+O\left(s^2\right)\right)\left(1-2s+O\left(s^2\right)\right)\left(1-2\log\left(2\pi a\right)s+O\left(s^2\right)\right)\\&\qquad\cdot\left(\frac{1}{s}-2\psi\left(1+\frac{\omega}{2\pi}\right)+O\left(s\right)\right)\left(1-\left(2\log 2+\gamma\right)s+O\left(s^2\right)\right)\left(\frac{1}{s}+\gamma+O\left(s\right)\right)\\&\quad=-\frac{1}{8\pi a}\left(\frac{1}{4}+\mu\right)\left(1+O\left(s^2\right)\right)\left(1-2s+O\left(s^2\right)\right)\left(1-2\log\left(2\pi a\right)s+O\left(s^2\right)\right)\\&\qquad\cdot\left(1-2\psi\left(1+\frac{\omega}{2\pi}\right)s+O\left(s^2\right)\right)\left(1-\left(2\log 2+\gamma\right)s+O\left(s^2\right)\right)\left(1+\gamma s+O\left(s^2\right)\right),\end{aligned}$$

where ψ denotes the so-called *digamma function*, that is the log-derivative of the Gamma function. This means that we have

$$\begin{aligned}&\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{s}{2s+1}\left(\frac{1}{4}+\mu\right)\frac{1}{2}\frac{\Gamma(s+1/2)}{\sqrt{\pi}}\Gamma\left(-s\right)\left(2\pi a\right)^{-2s-1}\zeta_H\left(2s+1,1+\frac{\omega}{2\pi}\right)\right]\\&\qquad=-\frac{1}{8\pi a}\left(\frac{1}{4}+\mu\right)\left(-2-2\log\left(2\pi a\right)-2\psi\left(1+\frac{\omega}{2\pi}\right)-2\log 2-\cancel{\gamma}+\cancel{\gamma}\right)\\&\qquad=\frac{1}{4\pi a}\left(\frac{1}{4}+\mu\right)\left(1+\log\left(4\pi a\right)+\psi\left(1+\frac{\omega}{2\pi}\right)\right)\\&\qquad=\frac{1}{4\pi a}\left(1+\log\left(4\pi a\right)+\psi\left(1+\frac{\omega}{2\pi}\right)\right)\mu+\frac{1}{16\pi a}\left(1+\log\left(4\pi a\right)+\psi\left(1+\frac{\omega}{2\pi}\right)\right).\end{aligned}$$

We have elected to write this last line above to clearly see the contribution of this derivative to the μ-asymptotic expansion. Furthermore, this derivative vanishes as a goes to infinity, for $\mu=0$. This yields the full asymptotic study for this term. We now move on to the first term of the decomposition used to study the case $j=1$. We thus consider

$$\frac{\sin(\pi s)}{\pi}\frac{\cancel{s}}{2s+1}\frac{1}{8a}\left(4\mu+1\right)^{-s+1}\sum_{k\geqslant 1}\frac{1}{2\pi k+\omega}\cdot\frac{1}{\cancel{s}}\cdot\frac{1}{k^{2\delta s}}F\left(\frac{1}{2},-s,-s+1,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).$$

For any integer $k \geqslant 1$, we have

$$F\left(\tfrac{1}{2}, -s, -s+1, -\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)$$
$$= \left(1 + \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)^s F\left(-s+\tfrac{1}{2}, -s, -s+1, \tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2 a^2}\right).$$

It is worth noting that the point of using the last formula was to get a hypergeometric function whose last parameter lies in the interval $[0,1[$. The aim will now be to break down this hypergeometric function, in order to make factors s appear as much as possible, which will tend to simplify the computation of derivatives at 0. We begin by removing part of the hypergeometric function, which involves the use of *generalized hypergeometric functions*, defined in [75, Sec 16.2],

$$F\left(\tfrac{1}{2}, -s, -s+1, -\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)$$
$$= \left(1 + \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)^s \Big[1 - \tfrac{s(s-1/2)}{s-1} \cdot \tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2 a^2}$$
$$\cdot F\left(-s+\tfrac{3}{2}, -s+1, 1; -s+2, 2; \tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2 a^2}\right)\Big].$$

Using difference quotients, it can be seen that the generalized hypergeometric function above is bounded, uniformly in every parameter for s in a neighborhood of 0, since we have

$$\operatorname{Re}\left(-s+1-(-s)-\tfrac{1}{2}\right) = \tfrac{1}{2} > 0 .$$

After multiplication by the appropriate factor and summation over k, the term involving this generalized hypergeometric function induces a holomorphic function around 0, whose derivative there vanishes because of the factor s. Hence, we only have to deal with the term associated to

$$\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{8a} (4\mu+1)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{2\pi k+\omega} \cdot \tfrac{1}{k^{2\delta s}} \left(1 + \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)^s .$$

We can further simplify this term, by writting

$$\left(1 + \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}\right)^s = 1 + s\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}} (1+t)^{s-1}\,\mathrm{d}t \;\cdot$$

The term involving the integral remainder above behaves nicely, as we have

$$\left|\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}} (1+t)^{s-1}\,\mathrm{d}t\right| \leqslant \int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}} (1+t)^{\operatorname{Re} s-1}\,\mathrm{d}t \leqslant \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}$$

for instance on the strip $-1/2 \leqslant \operatorname{Re} s \leqslant 1/2$. Thus, the term

$$\tfrac{\sin(\pi s)}{\pi} \tfrac{s}{2s+1} \tfrac{1}{8a} (4\mu+1)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{2\pi k+\omega} \cdot \tfrac{1}{k^{2\delta s}} \int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2}} (1+t)^{s-1}\,\mathrm{d}t$$

induces a holomorphic function around 0, whose derivative there vanishes, because of the factor s. Thus, what remains to study is

$$\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{8a} (4\mu+1)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{2\pi k+\omega} \cdot \tfrac{1}{k^{2\delta s}} .$$

This term, which is in some sense is related to both the Riemann zeta function and the Hurwith zeta function, can be further broken apart by writting

$$\sum_{k\geqslant 1} \tfrac{1}{2\pi k+\omega} \cdot \tfrac{1}{k^{2\delta s}} \quad = \quad \tfrac{1}{2\pi} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta s+1}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \quad = \quad \tfrac{1}{2\pi} \left[\zeta\left(2\delta s+1\right) - \tfrac{\omega}{2\pi} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta s+2}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \right] .$$

The term associated to the series above is then given by

$$-\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{8a} \tfrac{\omega}{4\pi^2} \left(4\mu+1\right)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta s+2}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \ ,$$

induces a holomorphic function around $s = 0$, and its derivative there is

$$\tfrac{\partial}{\partial s}_{|s=0} \left[-\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{8a} \tfrac{\omega}{4\pi^2} \left(4\mu+1\right)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta s+2}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \right] \quad = \quad -\tfrac{\omega}{8\pi^2 a} \left(\tfrac{1}{4}+\mu\right) \sum_{k\geqslant 1} \tfrac{1}{k^2} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \ .$$

To complete this step, we therefore only need to evaluate the series above, which we can do using the fact that it ressembles both the Hurwitz and the Riemann zeta functions. We will use the notation Fp, which stands for the "finite part" of a meromorphic function at a point, *i.e.* the constant term in the Laurent expansion there. We have

$$\begin{aligned} -\tfrac{\omega}{2\pi} \sum_{k\geqslant 1} \tfrac{1}{k^2} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \quad &= \quad \sum_{k\geqslant 1} \left(\tfrac{1}{k+\frac{\omega}{2\pi}} - \tfrac{1}{k} \right) \quad = \quad Fp_{s=0} \left[\sum_{k\geqslant 1} \left(\tfrac{1}{\left(k+\frac{\omega}{2\pi}\right)^{s+1}} - \tfrac{1}{k^{s+1}} \right) \right] \\ &= \quad Fp_{s=0} \left[\zeta_H\left(1+s, 1+\tfrac{\omega}{2\pi}\right) - \zeta\left(1+s\right) \right] \quad = \quad -\gamma - \psi\left(1+\tfrac{\omega}{2\pi}\right) . \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\tfrac{\partial}{\partial s}_{|s=0} \left[-\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{8a} \tfrac{\omega}{4\pi^2} \left(4\mu+1\right)^{-s+1} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta s+2}} \cdot \tfrac{1}{1+\frac{\omega}{2\pi k}} \right] \\ &= \quad -\tfrac{1}{4\pi a} \left(\tfrac{1}{4}+\mu\right) \left(\gamma + \psi\left(1+\tfrac{\omega}{2\pi}\right)\right) \quad = \quad -\tfrac{1}{4\pi a} \left(\gamma + \psi\left(1+\tfrac{\omega}{2\pi}\right)\right)\mu - \tfrac{1}{16\pi a} \left(\gamma + \psi\left(1+\tfrac{\omega}{2\pi}\right)\right) . \end{aligned}$$

This provides the contribution of this term to the μ-asymptotic expansion. We also note that this derivative vanishes as a goes to infinity (when μ equals zero), which yields the required asymptotic expansion related to a. The last term related to the case $j = 1$ we must study is given by

$$\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{16a\pi} \left(4\mu+1\right)^{-s+1} \zeta\left(2\delta s+1\right) \ ,$$

which induces a holomorphic around 0, whose derivative there can be computed using Laurent expansions. We have

$$\begin{aligned} &\tfrac{\sin(\pi s)}{\pi} \tfrac{1}{2s+1} \tfrac{1}{16a\pi} \left(4\mu+1\right)^{-s+1} \zeta\left(2\delta s+1\right) \\ &\qquad = \quad \tfrac{1}{4\pi a} \left(\tfrac{1}{4}+\mu\right) \left(s+O\left(s^3\right)\right) \left(1-2\log\left(2\right)s+O\left(s^2\right)\right) \left(1-2s+O\left(s^2\right)\right) \\ &\qquad\qquad \cdot \left(1-\log\left(\tfrac{1}{4}+\mu\right)s+O\left(s^2\right)\right) \left(\tfrac{1}{2\delta s}+\gamma+O\left(s\right)\right) \\ &\qquad = \quad \tfrac{1}{4\pi a} \left(\tfrac{1}{4}+\mu\right) \left(1+O\left(s^2\right)\right) \left(1-2\log\left(2\right)s+O\left(s^2\right)\right) \left(1-2s+O\left(s^2\right)\right) \\ &\qquad\qquad \cdot \left(1-\log\left(\tfrac{1}{4}+\mu\right)s+O\left(s^2\right)\right) \left(\tfrac{1}{2\delta}+\gamma s+O\left(s^2\right)\right) . \end{aligned}$$

The derivative of this term at $s=0$ is thus given by

$$\frac{\partial}{\partial s}_{|s=0}\left[\frac{\sin(\pi s)}{\pi}\frac{1}{2s+1}\frac{1}{16a\pi}\left(4\mu+1\right)^{-s+1}\zeta\left(2\delta s+1\right)\right]$$

$$= \frac{1}{8\pi a\delta}\left(\frac{1}{4}+\mu\right)\left(-2\log 2-2-\log\left(\frac{1}{4}+\mu\right)+2\gamma\delta\right).$$

Before proceeding to the asymptotic study in μ, we note that the derivative above vanishes as a goes to infinity. Finding the contribution to the μ-asymptotic expansion is then simply a matter of performing a Taylor expansion in $1/\mu$. We have

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}&\left[\frac{\sin(\pi s)}{\pi}\frac{1}{2s+1}\frac{1}{16a\pi}\left(4\mu+1\right)^{-s+1}\zeta\left(2\delta s+1\right)\right]\\
&= -\frac{1}{8\pi a\delta}\mu\log\mu+\frac{1}{4\pi a\delta}\left(\gamma\delta-1-\log 2\right)\mu-\frac{1}{32\pi a\delta}\log\mu\\
&\qquad +\frac{1}{16\pi a\delta}\left(\gamma\delta-1-\log 2\right)-\frac{1}{32\pi a\delta}+o\left(1\right)\\
&= -\frac{1}{8\pi a\delta}\mu\log\mu+\frac{1}{4\pi a\delta}\left(\gamma\delta-1-\log 2\right)\mu-\frac{1}{32\pi a\delta}\log\mu\\
&\qquad +\frac{1}{16\pi a}\gamma-\frac{3}{32\pi a\delta}-\frac{1}{16\pi a\delta}\log 2+o\left(1\right).\end{aligned}$$

This concludes the study of the case $j=1$.

- We now turn to the last remaining case, which is $j=0$. We consider

$$\begin{aligned}\frac{\sin(\pi s)}{\pi}\frac{1}{2s-1}\sum_{k\geqslant 1}&\left[\frac{1}{2}\cdot\frac{1}{(2\pi k+\omega)a}\cdot\frac{1}{s-1}\left(4\mu+1\right)^{-s+1}\cdot\frac{1}{k^{2\delta(s-1)}}F\left(\frac{1}{2},-s+1,-s+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\right.\\
&\left.+\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s-1}}\cdot\frac{1}{\sqrt{\pi}}\Gamma\left(s-1/2\right)\Gamma\left(-s+1\right)\right].\end{aligned}$$

Once again, we begin by taking care of the second term above, namely

$$\frac{\sin(\pi s)}{\pi}\frac{1}{2s-1}\frac{1}{2}\frac{\Gamma(s-1/2)}{\sqrt{\pi}}\Gamma\left(-s+1\right)\sum_{k\geqslant 1}\frac{1}{((2\pi k+\omega)a)^{2s-1}}\quad .$$

This term induces a holomorphic function around 0, since we have

$$\begin{aligned}\frac{1}{2}\frac{\sin(\pi s)}{\pi}\frac{1}{2s-1}&\frac{\Gamma(s-1/2)}{\sqrt{\pi}}\Gamma\left(-s+1\right)\sum_{k\geqslant 1}\frac{1}{((2\pi k+\omega)a)^{2s-1}}\\
&= \frac{1}{2}\cdot\left(2\pi a\right)^{-2s+1}\cdot\frac{\sin(\pi s)}{\pi}\cdot\frac{1}{2s-1}\cdot\frac{\Gamma(s-1/2)}{\sqrt{\pi}}\Gamma\left(-s+1\right)\zeta_H\left(2s-1,1+\frac{\omega}{2\pi}\right),\end{aligned}$$

and its derivative there can be computed as follows

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}&\left[\frac{1}{2}\frac{\sin(\pi s)}{\pi}\frac{1}{2s-1}\frac{\Gamma(s-1/2)}{\sqrt{\pi}}\Gamma\left(-s+1\right)\sum_{k\geqslant 1}\frac{1}{((2\pi k+\omega)a)^{2s-1}}\right]\\
&= -\frac{1}{2}\cdot 2\pi a\cdot(-2)\,\zeta_H\left(-1,1+\frac{\omega}{2\pi}\right) = -\pi a\left[\left(1+\frac{\omega}{2\pi}\right)^2-\left(1+\frac{\omega}{2\pi}\right)+\frac{1}{6}\right]\\
&= -\frac{\omega^2}{4\pi}a-\frac{1}{2}\omega a-\frac{\pi}{6}a.\end{aligned}$$

It is worth noting that since this computation is exact, and not asymptotic in either μ or a, the associated contribution to either expansion is the same. We then move on to the first term of the above, given by

$$\frac{1}{2a}\frac{\sin(\pi s)}{\pi}\frac{1}{(2s-1)(s-1)}\sum_{k\geqslant 1}\frac{1}{2\pi k+\omega}\left(4\mu+1\right)^{-s+1}\frac{1}{k^{2\delta(s-1)}}F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).$$

Unlike what happened in the previous case, there is no s alone in the parameters of the hypergeometric function above that would help us to compute the derivative at 0. To remedy that problem, we will use one of the formulae related to "contiguous functions", for which the reader is referred to [75, Sec. 15.5.ii], which here yields

$$\begin{aligned}
-\left(s+\left(s-\tfrac{1}{2}\right)\cdot\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)&\\
+(s-1)\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(-s+1,\tfrac{1}{2},-s+1,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)&\\
+F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)&=0.
\end{aligned}$$

We note that the first hypergeometric function above is the one we wish to study, the second one can be computed precisely since its first and third argument are identical, and the third one has its argument equal to $-s$, which means that its derivative at 0 can be computed by successively breaking it down, as we did for the case $j=1$. First of all, we note that we have

$$\begin{aligned}
&\tfrac{1}{2}\cdot\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&=\; s\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)-F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&\qquad-(s-1)\cdot\tfrac{1}{(2\pi k+\omega)a}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}.
\end{aligned}$$

The point of this manipulation was to make factors s appear as much as possible. It should be noted that the hypergeometric function that appears on the left-hand side is also present on the right-hand side, with a crucial difference, as it has a factor s in front of it. Therefore, we consider

$$\begin{aligned}
&\tfrac{1}{2a}\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\tfrac{1}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{1}{2\pi k+\omega}\cdot\tfrac{1}{k^{2\delta(s-1)}}F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&=\;\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\tfrac{a}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}\cdot\tfrac{1}{2}\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&=\;\tfrac{\sin(\pi s)}{\pi}\tfrac{a(4\mu+1)^{-s}}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}\Bigg[s\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&\qquad-F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)-\tfrac{s-1}{(2\pi k+\omega)a}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\;\Bigg]\\
&=\;\tfrac{\sin(\pi s)}{\pi}\tfrac{as(4\mu+1)^{-s}}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&\qquad-\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\tfrac{1}{(2s-1)(s-1)}a\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\
&\qquad-\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\cdot\tfrac{1}{2s-1}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}.
\end{aligned}$$

We will begin by studying the third term of the above, given by

$$-\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\cdot\tfrac{1}{2s-1}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\ .$$

Using a Taylor expansion, as in the proof of proposition 4.6.21, we see that the function

$$s\quad\longmapsto\quad\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}$$

has a holomorphic continuation to a neighborhood of 0, whose derivative at this point is given by

$$\begin{aligned}\tfrac{\partial}{\partial s}_{|s=0}&\left[-\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\cdot\tfrac{1}{2s-1}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\right]\\ &=\tfrac{\partial}{\partial s}_{|s=0}\left[\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\right].\end{aligned}$$

As indicated in the statement of this proposition, this derivative is not to be evaluated as μ goes to infinity. However, it must be studied, for $\mu=0$, as a goes to infinity. We have

$$\tfrac{\partial}{\partial s}_{|s=0}\left[-\tfrac{\sin(\pi s)}{\pi}\cdot\tfrac{1}{2s-1}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{k^{2\delta}+(2\pi k+\omega)^2\,a^2}\right]\quad=\quad-\tfrac{\pi}{6}a-\tfrac{1}{2}\omega a+o\left(1\right)$$

as a goes to infinity. We can then move on to the next term, given by

$$\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\,\tfrac{as}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\ .$$

We will first need to work on the hypergeometric function. We have

$$\begin{aligned}&F\left(\tfrac{1}{2},-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\qquad=\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^{-1/2}F\left(\tfrac{1}{2},1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right).\end{aligned}$$

The term we wish to study therefore becomes

$$\begin{aligned}\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\,\tfrac{s}{(2s-1)(s-1)}\sum_{k\geqslant 1}&\left[\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\right.\\ &\left.\cdot F\left(\tfrac{1}{2},1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\right].\end{aligned}$$

The hypergeometric function we get in this last expression has something which is particularly nicer than the one that was originally to be considered: its second argument is 1. This allows us to expand the hypergeometric function as follows

$$\begin{aligned}&F\left(\tfrac{1}{2},1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\\ &\quad=1-\tfrac{1}{2(s-2)}\cdot\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\\ &\qquad+\tfrac{3}{4}\cdot\tfrac{1}{(s-2)(s-3)}\cdot\left(\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)^2F\left(\tfrac{5}{2},1,-s+4,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right).\end{aligned}$$

For s in a neighborhood of 0, we have the inequality

$$\operatorname{Re}\left(-s+4-1-\tfrac{5}{2}\right) \quad = \quad \operatorname{Re}\left(-s+\tfrac{1}{2}\right) \quad > \quad 0$$

meaning we can bound the last hypergeometric function above, uniformly in every parameter. Thus, the function

$$s \quad \longmapsto \quad \tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\,\tfrac{s}{(2s-1)(s-1)} \sum_{k\geqslant 1} \Bigg[\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}$$

$$\cdot\tfrac{3}{4}\cdot\tfrac{1}{(s-2)(s-3)}\cdot\left(\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)^2 F\left(\tfrac{5}{2},1,-s+4,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\Bigg]$$

is holomorphic in a neighborhood of 0, and its derivative there vanishes, due to the presence of the factor s. Thus, we only need to deal with

$$\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\,\tfrac{s}{(2s-1)(s-1)} \sum_{k\geqslant 1} \Bigg[\tfrac{1}{k^{2\delta s}}\sqrt{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}$$

$$\cdot\left(1-\tfrac{1}{2(s-2)}\cdot\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\Bigg].$$

The first step towards that is to study the second term of the above, namely

$$-\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\tfrac{s}{2(2s-1)(s-1)(s-2)} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s-1)}}\cdot\tfrac{1}{\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}}$$

For every integer $k\geqslant 1$, we have

$$\begin{aligned}\tfrac{1}{\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}} \quad &= \quad \tfrac{1}{(2\pi k+\omega)a}+\tfrac{1}{\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}}-\tfrac{1}{(2\pi k+\omega)a}\\ &= \quad \tfrac{1}{(2\pi k+\omega)a}-\tfrac{1}{\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}}\cdot\tfrac{1}{(2\pi k+\omega)a}\cdot\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)a+\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}}\ ,\end{aligned}$$

This proves that the following function is holomorphic around 0

$$s \quad \longmapsto \quad -\tfrac{\sin(\pi s)}{\pi}\,\tfrac{s(4\mu+1)^{-s+1}}{2(2s-1)(s-1)(s-2)} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s-1)}}\left(\tfrac{1}{\sqrt{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}}-\tfrac{1}{(2\pi k+\omega)a}\right),$$

and its derivative there vanishes, because of the factor s. Thus, we look at

$$-\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\tfrac{s}{2(2s-1)(s-1)(s-2)} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s-1)}}\cdot\tfrac{1}{(2\pi k+\omega)a},$$

and we can further write

$$\tfrac{1}{(2\pi k+\omega)a} \quad = \quad \tfrac{1}{2\pi ak}\cdot\tfrac{1}{1+\frac{\omega}{2\pi k}} \quad = \quad \tfrac{1}{2\pi ak}+\tfrac{1}{2\pi ak}\left(\tfrac{1}{1+\frac{\omega}{2\pi k}}-1\right) \quad = \quad \tfrac{1}{2\pi ak}-\tfrac{1}{2\pi ak}\cdot\tfrac{\omega}{2\pi k}\cdot\underbrace{\frac{1}{1+\frac{\omega}{2\pi k}}}_{\leqslant 1}.$$

Thus the function

$$s \quad \longmapsto \quad -\tfrac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\tfrac{s}{2(2s-1)(s-1)(s-2)} \sum_{k\geqslant 1} \tfrac{1}{k^{2\delta(s-1)}}\cdot\tfrac{1}{2\pi ak}\left(\tfrac{1}{1+\frac{\omega}{2\pi k}}-1\right)$$

is holomorphic around the origin, whose derivative there vanishes because of the factor s. We are then led to study the term

$$\begin{aligned}-\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\tfrac{s}{2(2s-1)(s-1)(s-2)}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta(s-1)}}\cdot\tfrac{1}{2\pi ak}\\ = -\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\tfrac{s}{2(2s-1)(s-1)(s-2)}\cdot\tfrac{1}{2\pi a}\zeta\left(1+2\delta\left(s-1\right)\right),\end{aligned}$$

which has a holomorphic continuation to a neighborhood of 0, whose derivative at this point vanishes. Therefore, we are left with studying

$$\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\tfrac{s}{(2s-1)(s-1)}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta s}}\sqrt{\left(4\mu+1\right)k^{2\delta}+\left(2\pi k+\omega\right)^2a^2}.$$

Using the same computations as those performed in the proof of proposition 4.6.21, we see that the above function has a holomorphic continuation to a neighborhood of 0, and unlike what happened in the referenced proposition, the term we are dealing with here has a vanishing derivative at 0, due to the presence of an extra factor s. The last term we need to take care of is

$$-\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\tfrac{1}{(2s-1)(s-1)}a\sum_{k\geqslant 1}\tfrac{2\pi k+\omega}{k^{2\delta s}}F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).$$

The first step in the study of this term is to slightly modify the hypergeometric function so that its last argument becomes trapped between 0 and 1. For every integer $k\geqslant 1$, we have

$$\begin{aligned}&F\left(-s,\tfrac{1}{2},-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\quad = \left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s F\left(-s,-s+\tfrac{3}{2},-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right),\end{aligned}$$

and we can expand this last hypergeometric function, by writting

$$\begin{aligned}&F\left(-s,-s+\tfrac{3}{2},-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\\ &\quad = 1-\tfrac{s(s-3/2)}{s-2}\cdot\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\\ &\qquad +2s\tfrac{(s-3/2)(s-5/2)(s-1)}{(s-2)(s-3)}\left(\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)^2\\ &\qquad\quad \cdot F\left(-s+\tfrac{7}{2},-s+2,1;-s+4,3;\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right),\end{aligned}$$

where a generalized hypergeometric function, as defined in [75, Sec. 16.2], appears. Since we have

$$\operatorname{Re}\left(-s+4+3-\left(-s+\tfrac{7}{2}-s+2+1\right)\right) = \operatorname{Re}s+\tfrac{1}{2} > 0$$

for s around 0, the generalized hypergeometric function above can be bounded uniformly in every parameter, which means that the function

$$\begin{aligned}s \longmapsto\ & -\tfrac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\tfrac{2}{(2s-1)(s-1)}as\tfrac{(s-3/2)(s-5/2)(s-1)}{(s-2)(s-3)}\\ &\sum_{k\geqslant 1}\left[\tfrac{2\pi k+\omega}{k^{2\delta s}}\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\left(\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)^2\right.\\ &\qquad\left.\cdot F\left(-s+\tfrac{7}{2},-s+2,1;-s+4,3;\tfrac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\right]\end{aligned}$$

is holomorphic on a neighborhood of 0, and its derivative there vanishes, because of the presence of a factor s. We will now take care of the term

$$\frac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s}\,\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}a\sum_{k\geqslant 1}\frac{2\pi k+\omega}{k^{2\delta s}}\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\cdot\frac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}.$$

To do that, we will successively break apart the last two factors that appear in the series above. For every integer $k\geqslant 1$, we have

$$\begin{aligned}\frac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2} &= \frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\cdot\frac{1}{1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}\\ &= \frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}-\frac{(4\mu+1)^2k^{4\delta}}{(2\pi k+\omega)^4a^4}\cdot\underbrace{\frac{1}{1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}}_{\leqslant 1}.\end{aligned}$$

This proves that the function

$$s\quad\longmapsto\quad\frac{\sin(\pi s)}{\pi}\,\frac{s(s-3/2)(4\mu+1)^{-s+1}}{a(2s-1)(s-1)(s-2)}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)}}\,\frac{1}{2\pi k+\omega}\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\left(\frac{1}{1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}-1\right)$$

is holomorphic on a neighborhood of 0, and its derivative at 0 vanishes. We are left with studying

$$\frac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}\,\frac{1}{a}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)}}\,\frac{1}{2\pi k+\omega}\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\quad,$$

and we start by breaking down the complex power above. For every integer $k\geqslant 1$, we have

$$\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\quad=\quad 1+s\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}(1+t)^{s-1}\,\mathrm{d}t\ ,$$

and the integral remainder satisfies

$$\left|\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}(1+t)^{s-1}\,\mathrm{d}t\right|\quad\leqslant\quad\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\ ,$$

for s in a neighborhood of 0. This means that the function

$$s\quad\longmapsto\quad\frac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\frac{s^2(s-3/2)}{(2s-1)(s-1)(s-2)}\,\frac{1}{a}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)}}\,\frac{1}{2\pi k+\omega}\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}(1+t)^{s-1}\,\mathrm{d}t$$

is holomorphic around 0, and its derivative at 0 vanishes. Hence we are led to study

$$\frac{\sin(\pi s)}{\pi}\,(4\mu+1)^{-s+1}\,\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}\,\frac{1}{a}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)}}\,\frac{1}{2\pi k+\omega}.$$

The point is now to link this term to one involving the Riemann zeta function, much in the spirit of what we have prevously done. For every integer $k\geqslant 1$, we have

$$\frac{1}{2\pi k+\omega}\quad=\quad\frac{1}{2\pi k}\cdot\frac{1}{1+\frac{\omega}{2\pi k}}\quad=\quad\frac{1}{2\pi k}+\frac{1}{2\pi k}\left(\frac{1}{1+\frac{\omega}{2\pi k}}-1\right)\quad=\quad\frac{1}{2\pi k}-\frac{\omega}{4\pi^2k^2}\cdot\frac{1}{1+\frac{\omega}{2\pi k}}.$$

The following function is therefore holomorphic around 0

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}\frac{1}{2\pi a}\sum_{k\geqslant 1}\frac{1}{k^{1+2\delta(s-1)}}\left(\frac{1}{1+\frac{\omega}{2\pi k}}-1\right),$$

and its derivative there vanishes. The term we need to study, which is

$$\begin{aligned}
&\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}\frac{1}{2\pi a}\sum_{k\geqslant 1}\frac{1}{k^{1+2\delta(s-1)}}\\
&\qquad\qquad = \quad \frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\frac{s(s-3/2)}{(2s-1)(s-1)(s-2)}\frac{1}{2\pi a}\zeta\left(1+2\delta\left(s-1\right)\right),
\end{aligned}$$

is holomorphic around 0, whose derivative there vanishes. Finally, we have to study

$$-\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\frac{1}{(2s-1)(s-1)}a\sum_{k\geqslant 1}\frac{2\pi k+\omega}{k^{2\delta s}}\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^{s}.$$

Once again, we need to work on the complex power above. For any integer $k\geqslant 1$, we have

$$\begin{aligned}
\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^{s} \quad &= \quad 1+s\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\\
&\qquad +\frac{1}{2}s\left(s-1\right)\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}\left(1-t\right)^{s-2}\left(\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}-t\right)\,\mathrm{d}t.
\end{aligned}$$

We now note that the integral remainder above satisfies

$$\left|\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}\left(1-t\right)^{s-2}\left(\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}-t\right)\,\mathrm{d}t\right| \quad \leqslant \quad \frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2},$$

for s close enough to 0. This proves that the function

$$s \quad \longmapsto \quad -\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\frac{s}{2s-1}\frac{a}{2}\sum_{k\geqslant 1}\frac{2\pi k+\omega}{k^{2\delta s}}\int_0^{\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}}\left(1-t\right)^{s-2}\left(\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}-t\right)\,\mathrm{d}t$$

is holomorphic around 0, and its derivative vanishes there. We move on to the next term, *i.e.*

$$-\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s+1}\frac{s}{(2s-1)(s-1)}\frac{1}{a}\sum_{k\geqslant 1}\frac{1}{k^{2\delta(s-1)}}\cdot\frac{1}{2\pi k+\omega}.$$

Using the same computations as those previously done in this proof, this term is holomorphic around 0, and its derivative at this point vanishes. The last term to be taken care of is therefore

$$\begin{aligned}
&-\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\frac{1}{(2s-1)(s-1)}a\sum_{k\geqslant 1}\frac{2\pi k+\omega}{k^{2\delta s}}\\
&\qquad\qquad = \quad -\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\frac{1}{(2s-1)(s-1)}a\left(2\pi\zeta\left(2\delta s-1\right)+\omega\zeta\left(2\delta s\right)\right).
\end{aligned}$$

Using the properties and special values of the Riemann zeta function, this last term induces a holomorphic function around 0, and its derivative there is given by

$$\begin{aligned}
\frac{\partial}{\partial s}_{|s=0}\left[-\frac{\sin(\pi s)}{\pi}\left(4\mu+1\right)^{-s}\frac{1}{(2s-1)(s-1)}a\sum_{k\geqslant 1}\frac{2\pi k+\omega}{k^{2\delta s}}\right] \quad &= \quad -a\left(2\pi\zeta\left(-1\right)+\omega\zeta\left(0\right)\right)\\
&= \quad a\left(\frac{\pi}{6}+\frac{1}{2}\omega\right).
\end{aligned}$$

To sum up what we have proved so far, the function

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \left(\frac{1}{4}+\mu\right)^j \sum_{k \geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k+\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t$$

has a holomorphic continuation to a neighborhood of 0, and its derivative there is given by

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \left(\frac{1}{4}+\mu\right)^j \sum_{k \geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k+\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t\right] \\
&\qquad = \frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} (4\mu+1)^{-s} \sum_{k \geqslant 1} \frac{1}{k^{2\delta s}} \sqrt{(4\mu+1)\, k^{2\delta} + (2\pi k+\omega)^2 a^2}\right] \\
&\qquad\quad - \frac{1}{8\pi a \delta} \mu \log \mu + \frac{1}{4\pi a} \left[1 + \log(4\pi a) - \frac{1}{\delta} - \frac{1}{\delta} \log 2\right] \mu - \frac{1}{32\pi a \delta} \log \mu \\
&\qquad\quad - \frac{\omega^2}{4\pi} a + \frac{1}{16\pi a} + \frac{1}{16\pi a} \log(4\pi a) - \frac{1}{16\pi a \delta} \log 2 + o(1)
\end{aligned}$$

as μ goes to infinity. Furthermore, the same derivative, taken at $\mu = 0$, satisfies

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \frac{1}{4^j} \sum_{k \geqslant 1} \int_{k^\delta}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k+\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t\right] \\
&\qquad = -\frac{\omega^2}{4\pi} a - \frac{\pi}{6} a - \frac{1}{2} \omega a + o(1)
\end{aligned}$$

as a goes to infinity. The same methods also prove that the function

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \left(\frac{1}{4}+\mu\right)^j \sum_{k \geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k-\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t$$

has a holomorphic continuation to a neighborhood of 0, and its derivative there is given by, as

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \left(\frac{1}{4}+\mu\right)^j \sum_{k \geqslant 1} \int_{2k^\delta \sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k-\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t\right] \\
&\qquad = \frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} (4\mu+1)^{-s} \sum_{k \geqslant 1} \frac{1}{k^{2\delta s}} \sqrt{(4\mu+1)\, k^{2\delta} + (2\pi k-\omega)^2 a^2}\right] \\
&\qquad\quad - \frac{1}{8\pi a \delta} \mu \log \mu + \frac{1}{4\pi a} \left[1 + \log(4\pi a) - \frac{1}{\delta} - \frac{1}{\delta} \log 2\right] \mu - \frac{1}{32\pi a \delta} \log \mu \\
&\qquad\quad - \frac{\omega^2}{4\pi} a + \frac{1}{16\pi a} + \frac{1}{16\pi a} \log(4\pi a) - \frac{1}{16\pi a \delta} \log 2 + o(1)
\end{aligned}$$

as μ goes to infinity. Furthermore, the same derivative, taken at $\mu = 0$, satisfies

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{j \geqslant 0} \frac{(s)_j}{j!} \cdot \frac{1}{2s+2j-1} \frac{1}{4^j} \sum_{k \geqslant 1} \int_{k^\delta}^{+\infty} \frac{1}{t^{2s+2j}} \left(1+\frac{(2\pi k-\omega)^2 a^2}{t^2}\right)^{-1/2} \mathrm{d}t\right] \\
&\qquad = -\frac{\omega^2}{4\pi} a - \frac{\pi}{6} a + \frac{1}{2} \omega a + o(1)
\end{aligned}$$

as a goes to infinity. It is worth noting that one may get these asymptotics expansions simply by changing the sign of ω in the study that was detailed. Put together, these last two results, as well as the computations made at the beginning of this proof, yield the full proposition.

□

We now take care of the remaining term corresponding to $k=0$ assuming that ω is not zero.

Proposition 4.7.13. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\frac{t}{\omega a}\right)\, dt \ ,$$

which is well-defined and holomorphic on a half-plane of complex numbers s with large enough real part, has a holomorphic continuation to a neighborhood of 0, whose derivative there satisfies

$$\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \operatorname{Argsh}\left(\frac{t}{\omega a}\right)\, dt\right]$$
$$= \quad -\sqrt{\mu}\log\mu - 2\left[2\log 2 - \log(\omega a)\right]\sqrt{\mu} + O\left(\frac{1}{\sqrt{\mu}}\right)$$

as μ goes to infinity. Furthermore, this derivative, taken for $\mu=0$, satisfies, as a goes to infinity

$$\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \int_{1}^{+\infty} \left(t^2-\frac{1}{4}\right)^{-s} \operatorname{Argsh}\left(\frac{t}{\omega a}\right)\, dt\right] \quad = \quad \omega a + o(1)\,.$$

Third part. We now deal with the term involving a rational fraction in t from proposition 4.7.8.

Proposition 4.7.14. *The function*

$$s \quad \longmapsto \quad -\frac{1}{2}\,\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{t}{t^2+(C_{\omega,k}a)^2}\, dt \ ,$$

which is well-defined and holomorphic on a half-plane of complex numbers s with large enough real part, has a holomorphic continuation to a neighborhood of 0, whose derivative there satisfies

$$\frac{\partial}{\partial s}_{|s=0} \left[-\frac{1}{2}\,\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{t}{t^2+(C_{\omega,k}a)^2}\, dt\right]$$
$$= \quad \frac{\partial}{\partial s}_{|s=0} \left(\frac{\sin(\pi s)}{\pi}\,\frac{1}{4^{s+1}} \left(\frac{1}{4}+\mu\right)^{-s} \sum_{|k|\geqslant 1} \frac{1}{|k|^{2\delta s}} \log\left(\left((2\pi k+\omega)\,a\right)^2 + (4\mu+1)\,|k|^{2\delta}\right)\right),$$

as μ goes to infinity. Furthermore, this derivative, taken for $\mu=0$, satisfies, as a goes to infinity

$$\frac{\partial}{\partial s}_{|s=0} \left[-\frac{1}{2}\,\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{|k|^{\delta}}^{+\infty} \left(t^2-\frac{1}{4}\right)^{-s} \frac{t}{t^2+(C_{\omega,k}a)^2}\, dt\right]$$
$$= \quad -\frac{1}{2}\log a - \frac{1}{2}\left[\log\Gamma\left(1-\frac{\omega}{2\pi}\right) + \log\Gamma\left(1+\frac{\omega}{2\pi}\right)\right] + o(1)\,.$$

Remark 4.7.15. The derivative above cancels a term that appeared in proposition 4.6.24.

Proof of proposition 4.7.14. Using the binomial formula, we have

$$\sum_{|k|\geqslant 1} \int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{t}{t^2+(C_{\omega,k}a)^2}\,\mathrm{d}t$$
$$= \sum_{j\geqslant 0} \frac{(s)_j}{j!} \left(\frac{1}{4}+\mu\right)^j \Bigg[\sum_{k\geqslant 1} \int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j-1}}\cdot\frac{\mathrm{d}t}{t^2+(2\pi k+\omega)^2a^2}$$
$$+ \sum_{k\geqslant 1} \int_{2k^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty} \frac{1}{t^{2s+2j-1}}\cdot\frac{\mathrm{d}t}{t^2+(2\pi k-\omega)^2a^2}\Bigg].$$

We will first study the term

$$-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\left(\frac{1}{4}+\mu\right)^j\sum_{k\geqslant 1}\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j-1}}\cdot\frac{t}{t^2+(2\pi k+\omega)^2a^2}\,\mathrm{d}t\ .$$

We begin by proving that the sum over $j\geqslant 2$ does not play any role. For any $j\geqslant 2$, we have

$$\begin{aligned}\left|\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j-1}}\cdot\frac{\mathrm{d}t}{t^2+(2\pi k+\omega)^2a^2}\right| &\leqslant \frac{1}{4^{j-2}k^{2\delta(j-2)}}\left(\frac{1}{4}+\mu\right)^{2-j}\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2\,\mathrm{Re}\,s+3}}\cdot\frac{\mathrm{d}t}{(2\pi k+\omega)^2a^2}\\ &\leqslant \frac{1}{4^{j-2}}\left(\frac{1}{4}+\mu\right)^{2-j}\cdot\frac{1}{k^{2\delta(j-2)}}\cdot\frac{1}{2(\mathrm{Re}\,s+1)}\cdot\frac{1}{(2\pi k+\omega)^2a^2}\cdot\frac{1}{4^{\mathrm{Re}\,s+1}k^{2\delta(\mathrm{Re}\,s+1)}}\left(\frac{1}{4}+\mu\right)^{-\mathrm{Re}\,s-1}\ .\end{aligned}$$

Hence, we see that the function

$$s\ \longmapsto\ -\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{j\geqslant 2}\frac{(s)_j}{j!}\left(\frac{1}{4}+\mu\right)^j\sum_{k\geqslant 1}\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j-1}}\cdot\frac{\mathrm{d}t}{t^2+(2\pi k+\omega)^2a^2}$$

is holomorphic around 0, and its derivative there vanishes because of the Pochhammer symbol. Thus, we only have to deal with the terms corresponding to $j=0$ and $j=1$. We have

$$\begin{aligned}\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j-1}}\cdot\frac{\mathrm{d}t}{t^2+(2\pi k+\omega)^2a^2} &= \int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j+1}}\cdot\frac{\mathrm{d}t}{1+\frac{(2\pi k+\omega)^2a^2}{t^2}}\\ &= \frac{1}{2}\left(2\pi k+\omega\right)a\cdot\frac{1}{((2\pi k+\omega)a)^{2s+2j+1}}\int_0^{\frac{(2\pi k+\omega)^2a^2}{(4\mu+1)k^{2\delta}}}\frac{x^{s+j+1/2}}{1+x}\cdot\frac{\mathrm{d}x}{x^{3/2}}\\ &= \frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s+2j}}\left[\frac{1}{s+j-1}\cdot\frac{((2\pi k+\omega)a)^{2s+2j-2}}{4^{s+j-1}(1/4+\mu)^{s+j-1}k^{2\delta(s+j-1)}}\right.\\ &\qquad\cdot F\left(1,-s-j+1,-s-j+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\qquad\left.+\Gamma\left(s+j\right)\Gamma\left(-s-j+1\right)\right]\\ &= \frac{1}{2}\cdot\frac{1}{s+j-1}\cdot\frac{1}{(2\pi k+\omega)^2a^2}\cdot\frac{1}{4^{s+j-1}}\left(\frac{1}{4}+\mu\right)^{-s-j+1}\cdot\frac{1}{k^{2\delta(s+j-1)}}\\ &\qquad\cdot F\left(1,-s-j+1,-s-j+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\qquad+\frac{(-1)^j}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s+2j}}\cdot\frac{\pi}{\sin(\pi s)}\ .\end{aligned}$$

- We begin by dealing with the case $j=1$. We consider

$$\begin{aligned}-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}s\left(\frac{1}{4}+\mu\right)\sum_{k\geqslant 1}&\left[\frac{1}{2}\cdot\frac{1}{s}\cdot\frac{1}{(2\pi k+\omega)^2a^2}\left(4\mu+1\right)^{-s}\cdot\frac{1}{k^{2\delta s}}F\left(1,-s,-s+1,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\right.\\ &\left.-\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2j+2}}\cdot\frac{\pi}{\sin(\pi s)}\right].\end{aligned}$$

We will first take care of the second term above, given by

$$\begin{aligned}-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}s\left(\frac{1}{4}+\mu\right)\sum_{k\geqslant 1}&\left[-\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2j+2}}\cdot\frac{\pi}{\sin(\pi s)}\right]\\ &= \frac{1}{4}s\left(\frac{1}{4}+\mu\right)\cdot\frac{1}{(2\pi a)^{2s+2}}\sum_{k\geqslant 1}\frac{1}{\left(k+\frac{\omega}{2\pi}\right)^{2s+2}}\\ &= \frac{1}{4}s\left(\frac{1}{4}+\mu\right)\cdot\frac{1}{(2\pi a)^{2s+2}}\zeta_H\left(2s+2,1+\frac{\omega}{2\pi}\right).\end{aligned}$$

This term induces a holomorphic function around 0, whose derivative there is given by

$$\begin{aligned}\frac{1}{4}\left(\frac{1}{4}+\mu\right)\cdot\frac{1}{(2\pi a)^2}\zeta_H\left(2,1+\frac{\omega}{2\pi}\right) &= \frac{1}{16\pi^2a^2}\left(\frac{1}{4}+\mu\right)\zeta_H\left(2,1+\frac{\omega}{2\pi}\right)\\ &= \frac{1}{16\pi^2a^2}\zeta_H\left(2,1+\frac{\omega}{2\pi}\right)\mu+\frac{1}{64\pi^2a^2}\zeta_H\left(2,1+\frac{\omega}{2\pi}\right).\end{aligned}$$

This computation yields the contribution to the μ-asymptotic expansion, and since this derivative vanishes as a goes to infinity, this also yields the contribution to the asymptotic study as a goes to infinity. We now move on to the first term above, given by

$$-\frac{1}{16}\cdot\frac{\sin(\pi s)}{\pi}(4\mu+1)^{-s+1}\cdot\frac{1}{a^2}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\cdot\frac{1}{(2\pi k+\omega)^2}F\left(1,-s,-s+1,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).$$

For any integer $k\geqslant 1$, we have

$$\begin{aligned}&F\left(1,-s,-s+1,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &\quad=\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\underbrace{F\left(-s,-s,-s+1,\frac{(4\mu+1)\,k^{2\delta}}{(4\mu+1)\,k^{2\delta}+(2\pi k+\omega)^2\,a^2}\right)}_{\text{bounded uniformly for } s \text{ around } 0},\end{aligned}$$

Noting that the convergence of the series above is provided by the factor $1/(2\pi k+\omega)^2$, the function

$$\begin{aligned}s\longmapsto\ &-\frac{1}{16}\cdot\frac{\sin(\pi s)}{\pi}(4\mu+1)^{-s+1}\cdot\frac{1}{a^2}\sum_{k\geqslant 1}\Bigg[\frac{1}{k^{2\delta s}}\cdot\frac{1}{(2\pi k+\omega)^2}\left(1+\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)^s\\ &\qquad\cdot F\left(-s,-s,-s+1,\frac{(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}+(2\pi k+\omega)^2a^2}\right)\Bigg]\end{aligned}$$

is holomorphic around 0, and its derivative there is given by

$$\begin{aligned}-\frac{1}{4}\left(\frac{1}{4}+\mu\right)\cdot\frac{1}{a^2}\cdot\frac{1}{4\pi^2}\sum_{k\geqslant 1}\frac{1}{\left(k+\frac{\omega}{2\pi}\right)^2} &= -\frac{1}{16\pi a^2}\left(\frac{1}{4}+\mu\right)\zeta_H\left(2,1+\frac{\omega}{2\pi}\right)\\ &= -\frac{1}{16\pi^2a^2}\zeta_H\left(2,1+\frac{\omega}{2\pi}\right)\mu-\frac{1}{64\pi^2a^2}\zeta_H\left(2,1+\frac{\omega}{2\pi}\right).\end{aligned}$$

In the same way as was noted above, this provides the relevant contribution to both asymptotic expansions, and these happen to cancel each other.

- We now move on to dealing with the case $j=0$. We consider

$$\begin{aligned}-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\Bigg[&\frac{1}{2}\cdot\frac{1}{s-1}\cdot\frac{1}{(2\pi k+\omega)^2a^2}(4\mu+1)^{-s+1}\frac{1}{k^{2\delta(s-1)}}F\left(1,-s+1,-s+2,-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\ &+\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s}}\cdot\frac{\pi}{\sin(\pi s)}\Bigg].\end{aligned}$$

We will first take care of the second term above, given by

$$-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\left(\frac{1}{2}\cdot\frac{1}{((2\pi k+\omega)a)^{2s}}\cdot\frac{\pi}{\sin(\pi s)}\right)=-\frac{1}{4}\cdot\frac{1}{(2\pi a)^{2s}}\zeta_H\left(2s,1+\frac{\omega}{2\pi}\right).$$

This term induces a holomorphic function around 0, and its derivative there is given by

$$\begin{aligned}&-\frac{1}{2}\left(-\zeta_H\left(0,1+\frac{\omega}{2\pi}\right)\log(2\pi a)+\zeta_H'\left(0,1+\frac{\omega}{2\pi}\right)\right)\\ &\quad=-\frac{1}{2}\left(\frac{1}{2}\log a+\frac{\omega}{2\pi}\log(2\pi a)+\log\Gamma\left(1+\frac{\omega}{2\pi}\right)\right).\end{aligned}$$

This computation, being exact, provides the contributions to both asymptotic studies required in this proof. We now move on to dealing with the first term of the above, namely

$$-\tfrac{1}{4}\cdot\tfrac{\sin(\pi s)}{\pi}\cdot\tfrac{1}{s-1}\,(4\mu+1)^{-s+1}\cdot\tfrac{1}{a^2}\sum_{k\geqslant 1}\tfrac{1}{k^{2\delta(s-1)}}\cdot\tfrac{1}{(2\pi k+\omega)^2}F\left(1,-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).$$

The point of what follows is to perform what could be summed up as a “partial evaluation” of the terms involved in the series above. To be more precise, for every integer $k\geqslant 1$, we have

$$\begin{aligned}&F\left(1,-s+1,-s+2,-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right)\\&\qquad=\tfrac{(2\pi k+\omega)^2a^2}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}F\left(1,1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}\right),\end{aligned}$$

and we will relate the last hypergeometric function above to its value at $s=0$, using Euler's integral formula, for which the reader is referred to [75, Sec. 15.6.1]. We have

$$\begin{aligned}&B(1,-s+1)\,F\left(1,1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}\right)\\&\qquad=\int_0^1(1-x)^{-s}\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x,\end{aligned}$$

where B denotes the beta function, defined in [75, Sec. 5.18.iii]. It can be computed here as

$$B(1,-s+1)=\tfrac{\Gamma(1)\Gamma(1-s)}{\Gamma(2-s)}=\tfrac{1}{1-s}.$$

These formulae can furthermore be evaluated at $s=0$, which then yields

$$\begin{aligned}&\tfrac{1}{1-s}F\left(1,1,-s+2,\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}\right)-F\left(1,1,2,\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}\right)\\&=\int_0^1(1-x)^{-s}\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x-\int_0^1\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\\&=\int_0^1\left[(1-x)^{-s}-1\right]\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\\&=s\int_0^1\left(\int_0^x(1-t)^{-s-1}\,\mathrm{d}t\right)\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\\&=s\int_0^1(1-t)^{-s-1}\left(\int_t^1\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\right)\mathrm{d}t,\end{aligned}$$

this last equality being a consequence of Fubini's theorem. Now, we note that we have

$$\begin{aligned}&\left|\int_0^1(1-t)^{-s-1}\left(\int_t^1\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\right)\mathrm{d}t\right|\\&\qquad\leqslant\int_0^1(1-t)^{-\operatorname{Re}s-1}\int_t^1\left(1-\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}x\right)^{-1}\mathrm{d}x\,\mathrm{d}t\\&\qquad\leqslant\left(\int_0^1(1-t)^{-\operatorname{Re}s}\,\mathrm{d}t\right)\cdot\tfrac{1}{1-\frac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2+(4\mu+1)k^{2\delta}}}\\&\qquad\leqslant\tfrac{1}{1-\operatorname{Re}s}\left(1+\tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2a^2}\right),\end{aligned}$$

at least whenever the real part of s stays strictly below 1. Therefore, the function

$$
\begin{aligned}
s \quad \longmapsto \quad & \tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s+1} \cdot \tfrac{1}{a^2} \sum_{k \geqslant 1} \Big[\tfrac{1}{k^{2\delta(s-1)}} \cdot \tfrac{1}{(2\pi k+\omega)^2} \tfrac{(2\pi k+\omega)^2 a^2}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}} \\
& \cdot \left(\tfrac{1}{1-s} F\left(1,1,-s+2, \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right) - F\left(1,1,2, \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right)\right)\Big]
\end{aligned}
$$

is holomorphic around 0, and its derivative there vanishes. Thus, we need to study

$$
\begin{aligned}
& \tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s+1} \cdot \cancel{\tfrac{1}{a^2}} \sum_{k \geqslant 1} \Big[\tfrac{1}{k^{2\delta(s-1)}} \cdot \cancel{\tfrac{1}{(2\pi k+\omega)^2}} \tfrac{\cancel{(2\pi k+\omega)^2 a^2}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}} \\
& \qquad \cdot F\left(1,1,2, \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right)\Big] .
\end{aligned}
$$

The advantage is that this hypergeometric function, unlike the previous one, can be computed much more explicitely. For every integer $k \geqslant 1$, we have

$$
F\left(1,1,2, \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right) \quad = \quad -\tfrac{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}{(4\mu+1)k^{2\delta}} \log\left(\tfrac{(2\pi k+\omega)^2 a^2}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right) \quad ,
$$

which then yields

$$
\begin{aligned}
& \tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s+1} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta(s-1)}} \cdot \tfrac{1}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}} \cdot F\left(1,1,2, \tfrac{(4\mu+1)k^{2\delta}}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right) \\
= \quad & -\tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s+1} \sum_{k \geqslant 1} \Big[\tfrac{1}{k^{2\delta(s-1)}} \cdot \cancel{\tfrac{1}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}} \\
& \qquad \cdot \tfrac{\cancel{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}}{(4\mu+1)k^{2\delta}} \log\left(\tfrac{(2\pi k+\omega)^2 a^2}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right)\Big] \\
= \quad & -\tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left(\tfrac{(2\pi k+\omega)^2 a^2}{(2\pi k+\omega)^2 a^2+(4\mu+1)k^{2\delta}}\right) \\
= \quad & -\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left((2\pi k+\omega)\, a\right) \\
& \qquad + \tfrac{1}{4} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left((2\pi k+\omega)^2 a^2 + (4\mu+1)\, k^{2\delta}\right) .
\end{aligned}
$$

One notes that, using the computations performed in the proof of proposition 4.6.24, the second term above induces a holomorphic function around 0, whose derivative there is not to be computed for the μ-asymptotic expansion. Before moving on to the asymptotic study in a, we need to complete the study in μ of the first term above, given by

$$
-\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left((2\pi k+\omega)\, a\right) \quad .
$$

In order to avoid any unnecessary computation, we will actually consider this term together with the other similar term otained by switching the sign of ω, which is also to be considered in the proof of the current proposition. Therefore, we consider

$$
\begin{aligned}
& -\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \left[\log\left((2\pi k+\omega)\, a\right) + \log\left((2\pi k-\omega)\, a\right)\right] \\
& \quad = \quad -\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left(\left(4\pi^2 k^2 - \omega^2\right) a^2\right) \\
& \quad = \quad -\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \left[2\log\left(2\pi a\right) + 2\log k + \log\left(1 - \tfrac{\omega^2}{4\pi^2 k^2}\right)\right] \\
& \quad = \quad -\tfrac{1}{2} \cdot \tfrac{\sin(\pi s)}{\pi} \cdot (4\mu+1)^{-s} \left[2\log\left(2\pi a\right)\zeta\left(2\delta s\right) - 2\zeta'\left(2\delta s\right) + \sum_{k \geqslant 1} \tfrac{1}{k^{2\delta s}} \log\left(1 - \tfrac{\omega^2}{4\pi^2 k^2}\right)\right].
\end{aligned}
$$

We see that this term induces a holomorphic function around 0, whose derivative there is given by

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0}\left[-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\cdot(4\mu+1)^{-s}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\left[\log\left((2\pi k+\omega)\,a\right)+\log\left((2\pi k-\omega)\,a\right)\right]\right]\\
&\qquad= -\log(2\pi a)\,\zeta(0)+\zeta'(0)-\frac{1}{2}\sum_{k\geqslant 1}\log\left(1-\frac{\omega^2}{4\pi^2k^2}\right)\\
&\qquad= -\frac{1}{2}\sum_{k\geqslant 1}\log\left(1-\frac{\omega^2}{4\pi^2k^2}\right).
\end{aligned}$$

We will now compute this last series. We have

$$\begin{aligned}
-\frac{1}{2}\sum_{k\geqslant 1}\log\left(1-\frac{\omega^2}{4\pi^2k^2}\right) &= \sum_{k\geqslant 1}\sum_{n\geqslant 1}\frac{1}{2n}\left(\frac{\omega}{2\pi}\right)^n\cdot\frac{1}{k^{2n}}\\
&= \sum_{n\geqslant 1}\frac{1}{2n}\left(\frac{\omega}{2\pi}\right)^n\sum_{k\geqslant 1}\frac{1}{k^{2n}} \quad \text{by Fubini's theorem}\\
&= \sum_{n\geqslant 1}\frac{1}{2n}\left(\frac{\omega}{2\pi}\right)^n\zeta(2n)\\
&= \log\left(\frac{\sin(\omega/2)}{\omega/2}\right),
\end{aligned}$$

this last computation being made using known formulae yielding sums whose terms involve the Riemann zeta function. Having finished the study in μ, we turn our attention to the part of the asymptotic study as a goes to infinity that was left out. We need to study the term

$$-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)\,a\right)+\frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)^2a^2+k^{2\delta}\right).$$

To do that, we will expand its second term. We have

$$\begin{aligned}
&\frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)^2a^2+k^{2\delta}\right)\\
&\quad= \frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)\,a\right)+\frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left(1+\frac{k^{2\delta}}{(2\pi k+\omega)^2a^2}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
&-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)\,a\right)+\frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left((2\pi k+\omega)^2a^2+k^{2\delta}\right)\\
&\quad= \frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\log\left(1+\frac{k^{2\delta}}{(2\pi k+\omega)^2a^2}\right) = \frac{1}{4}\cdot\frac{\sin(\pi s)}{\pi}\sum_{k\geqslant 1}\frac{1}{k^{2\delta s}}\cdot\int_0^{\frac{k^{2\delta}}{(2\pi k+\omega)^2a^2}}\frac{\mathrm{d}t}{1+t}.
\end{aligned}$$

We note that this term induces a holomorphic function around 0, whose derivative there vanishes as a goes to infinity. Finally, one notices that the study of the last remaining term, *i.e.* of

$$-\frac{1}{2}\cdot\frac{\sin(\pi s)}{\pi}\sum_{j\geqslant 0}\frac{(s)_j}{j!}\left(\frac{1}{4}+\mu\right)^j\sum_{k\geqslant 1}\int_{2k^\delta\sqrt{\frac{1}{4}+\mu}}^{+\infty}\frac{1}{t^{2s+2j-1}}\cdot\frac{t}{t^2+(2\pi k-\omega)^2a^2}\,\mathrm{d}t\ ,$$

is extremely similar to what we have done here, and part of it has actually been done in order to not compute unnecessary terms. Putting all these evaluations together yields the proposition.

□

We still need to talk about the case $k=0$, for which we assume that ω itself is not zero.

Proposition 4.7.16. *The function*

$$s \quad \longmapsto \quad -\frac{1}{2}\,\frac{\sin(\pi s)}{\pi}\,\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}\,\frac{t}{t^2+(\omega a)^2}\,dt\ ,$$

which is well-defined and holomorphic on a half-plane of complex numbers s with large enough real part, has a holomorphic continuation to a neighborhood of 0*, whose derivative there satisfies*

$$\frac{\partial}{\partial s}_{|s=0}\left[-\tfrac{1}{2}\,\tfrac{\sin(\pi s)}{\pi}\,\int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s}\,\tfrac{t}{t^2+(\omega a)^2}\,dt\right] \;=\; \tfrac{1}{4}\log\mu+\tfrac{1}{2}\log 2+o(1)\,,$$

as μ goes to infinity. Furthermore, this derivative, taken for $\mu=0$, satisfies, as a goes to infinity,

$$\frac{\partial}{\partial s}_{|s=0}\left[-\tfrac{1}{2}\,\tfrac{\sin(\pi s)}{\pi}\,\int_{1}^{+\infty}\left(t^2-\tfrac{1}{4}\right)^{-s}\,\tfrac{t}{t^2+(\omega a)^2}\,dt\right] \;=\; \tfrac{1}{2}\log(\omega a)+o(1)\,.$$

Fourth part. Finally, we can move to the last part of the study of $\widetilde{M}_{\mu,k}$.

Proposition 4.7.17. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi}\sum_{|k|\geqslant 1}\int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\frac{1}{4}+\mu\right)\right)^{-s}\frac{\partial}{\partial t}\left(\frac{1}{t}\,U_1\left(\tau\left(\frac{C_{\omega,k}a}{t}\right)\right)\right)\,dt\ ,$$

which is well-defined and holomorphic on a half-plane consisting on complex numbers s of real part large enough, has a holomorphic continuation to a neighborhood of 0*, whose derivative at $s=0$ satisfies*

$$\frac{\partial}{\partial s}_{|s=0}\left[\tfrac{\sin(\pi s)}{\pi}\sum_{|k|\geqslant 1}\int_{2|k|^{\delta}\sqrt{\frac{1}{4}+\mu}}^{+\infty}\left(t^2-\left(\tfrac{1}{4}+\mu\right)\right)^{-s}\tfrac{\partial}{\partial t}\left(\tfrac{1}{t}\,U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{t}\right)\right)\right)\,dt\right]$$

$$= \;-\frac{\partial}{\partial s}_{|s=0}\left(\tfrac{\sin(\pi s)}{\pi}\,\tfrac{1}{4^{s+\frac{1}{2}}}\,\left(\tfrac{1}{4}+\mu\right)^{-s-\frac{1}{2}}\sum_{|k|\geqslant 1}\left(|k|^{2\delta}-\tfrac{1}{4}\right)^{-s}\,\tfrac{1}{|k|^{\delta}p}\,U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{|k|^{\delta}\sqrt{4\mu+1}}\right)\right)\right)$$

$$+\tfrac{1}{16\pi a\delta}\log\mu-\tfrac{1}{8\pi a}\log(4\pi a)+\tfrac{1}{8\pi a\delta}\log 2+\tfrac{5}{24\pi a}+o(1)\,,$$

as μ goes to infinity. Furthermore, the same derivative, considered for $\mu=0$, satisfies

$$\frac{\partial}{\partial s}_{|s=0}\left[\tfrac{\sin(\pi s)}{\pi}\sum_{|k|\geqslant 1}\int_{|k|^{\delta}}^{+\infty}\left(t^2-\tfrac{1}{4}\right)^{-s}\tfrac{\partial}{\partial t}\left(\tfrac{1}{t}\,U_1\left(\tau\left(\tfrac{C_{\omega,k}a}{t}\right)\right)\right)\,dt\right] \;=\; o(1)$$

as a goes to infinity.

Remark 4.7.18. The derivative above that was left untouched is to be canceled by one the terms coming from proposition 4.6.25. It should be noted that the compensation does not only concern the derivative at 0, but the whole function.

Proof of proposition 4.7.17. The proof is similar to that of the last propositions.

□

We now end this study with the case $k = 0$, assuming ω is not zero.

Proposition 4.7.19. *The function*

$$s \quad \longmapsto \quad \frac{\sin(\pi s)}{\pi} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4} + \mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t}\, U_1\left(\tau\left(\frac{\omega a}{t}\right)\right)\right)\, dt \ ,$$

which is well-defined and holomorphic on a half-plane of complex numbers s with large enough real part, has a holomorphic continuation to a neighborhood of 0, whose derivative at $s = 0$ satisfies

$$\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \sum_{|k|\geqslant 1} \int_{2\sqrt{\frac{1}{4}+\mu}}^{+\infty} \left(t^2 - \left(\frac{1}{4}+\mu\right)\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t}\, U_1\left(\tau\left(\frac{\omega a}{t}\right)\right)\right)\, dt\right] \quad = \quad o\,(1)\,,$$

as μ goes to infinity. Furthermore, the same derivative, considered for $\mu = 0$, satisfies

$$\frac{\partial}{\partial s}_{|s=0} \left[\frac{\sin(\pi s)}{\pi} \int_{1}^{+\infty} \left(t^2 - \frac{1}{4}\right)^{-s} \frac{\partial}{\partial t}\left(\frac{1}{t}\, U_1\left(\tau\left(\frac{\omega a}{t}\right)\right)\right)\, dt\right] \quad = \quad o\,(1)$$

as a goes to infinity.

Proof. This study is similar to the one conducted in the last propositions.

□

4.8 Evaluation of the spectral zeta function around a cusp

After all these considerations, we can finally state the following theorem, which was the aim of this entire chapter. We will separate the case $\omega = 0$ from the rest, as the formulae take some space.

Theorem 4.8.1. *The spectral zeta function with parameters $\omega \in \,]0, 2\pi[$ and μ, which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1$, has a holomorphic continuation to an open neighborhood of 0, and its derivative at $s = 0$ satisfies*

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \zeta_\omega\,(\mu, s) \\
&= \quad \tfrac{1}{4\pi a}\mu\log\mu - \sqrt{\mu}\log\mu - \tfrac{1}{4\pi a}\mu + \Big[2\int_0^{+\infty} \tfrac{1}{e^{2\pi t}-1}\left(\arctan\left(\tfrac{2\pi}{2\pi+\omega}t\right) + \arctan\left(\tfrac{2\pi}{2\pi-\omega}t\right)\right)\, dt \\
&\qquad\quad + \log(\omega a) - 2\log 2 + \tfrac{\omega}{2\pi}\log\left(\tfrac{2\pi+\omega}{2\pi-\omega}\right) + \tfrac{1}{2}\log\left(\left(4\pi^2 - \omega^2\right)a^2\right) + \tfrac{1}{4a}\Big]\sqrt{\mu} \\
&\qquad\qquad + \tfrac{1}{4}\left(3 - \tfrac{1}{2\pi a}\right)\log\mu + o\,(1)
\end{aligned}$$

as μ goes to infinity. In particular, there is no constant term in this asymptotic expansion.

Theorem 4.8.2. *The spectral zeta function with parameters $\omega = 0$ and μ, which is well-defined and holomorphic on the half-plane $\operatorname{Re} s > 1$, has a holomorphic continuation to an open neighborhood of 0, and its derivative at $s = 0$ satisfies*

$$\begin{aligned}
&\frac{\partial}{\partial s}_{|s=0} \zeta_\omega\,(\mu, s) \\
&= \quad \tfrac{1}{4\pi a}\mu\log\mu - \tfrac{1}{2}\sqrt{\mu}\log\mu - \tfrac{1}{4\pi a}\mu + \Big[2\int_0^{+\infty} \tfrac{1}{e^{2\pi t}-1}\left(\arctan\left(\tfrac{2\pi}{2\pi+\omega}t\right) + \arctan\left(\tfrac{2\pi}{2\pi-\omega}t\right)\right)\, dt \\
&\qquad\quad - \log 2 + 1 + \log(2\pi a) + \tfrac{1}{4a}\Big]\sqrt{\mu} + \tfrac{1}{2}\left(1 - \tfrac{1}{4\pi a}\right)\log\mu + o\,(1)
\end{aligned}$$

as μ goes to infinity. In particular, there is no constant term in this asymptotic expansion.

We can now state the theorems pertaining to the study of the derivatives at $s = 0$ of the spectral zeta function with parameters as a goes to infinity, when μ equals zero.

Theorem 4.8.3. *The derivative at $s = 0$ of the spectral zeta function with parameters $\omega \in]0, 2\pi[$ and $\mu = 0$ satisfies, as a goes to infinity,*

$$\frac{\partial}{\partial s}_{|s=0} \zeta_\omega(0, s) \quad = \quad -\tfrac{\omega^2}{2\pi}a - \tfrac{\pi}{3}a + \omega a + \tfrac{1}{2}\log\left(\sin\tfrac{\omega}{2}\right) + \tfrac{1}{2}\log 2 + o(1) \ .$$

Theorem 4.8.4. *The derivative at $s = 0$ of the spectral zeta function with parameters ω and μ set at zero satisfies, as a goes to infinity,*

$$\frac{\partial}{\partial s}_{|s=0} \zeta_0(0, s) \quad = \quad -\tfrac{\pi}{3}a - \tfrac{1}{2}\log a + o(1) \ .$$

4.9 Evaluation of the relative determinant around cusps

The aim of this section is to provide reformulations of the four theorems given in the last section in the context of asymptotic evaluations for relative determinants. The notations used here are the same as the ones defined in the previous chapters. We briefly recall some of them for clarity. We consider a compactified modular curve X, arising from the quotient of $\mathbb{H}$ by a Fuchsian group of the first kind $\Gamma \subset PSL_2(\mathbb{R})$ without torsion, as well as a unitary representation

$$\rho \quad : \quad \Gamma \quad \longrightarrow \quad U_r(\mathbb{C}) \ .$$

For any cusp p, the image $\rho(\gamma_p)$ of a generator of the stabilizer Γ_p of p in Γ can be diagonalized. Its eigenvalues, being of modulus 1, can be written as $e^{2i\pi\alpha_{p,j}}$, for j between 1 and r, with $\alpha_{p,j}$ being 0 for j between 1 and k_p, and strictly positive for $j \geqslant k_p + 1$. We have also defined, for any strictly positive real number $\varepsilon > 0$, an open neighborhood $U_{p,\varepsilon}$ of the cusp p which can be identified to a product $S^1 \times]a(\varepsilon), +\infty[$, where we have set

$$a(\varepsilon) \quad = \quad \tfrac{1}{2\pi}\log\varepsilon^{-1} \ .$$

From ρ, we have built a flat unitary holomorphic vector bundle E of rank r, be written as

$$E \quad = \quad \bigoplus_{j=1}^{r} L_{p,j}$$

above $U_{p,\varepsilon}$, each $L_{p,j}$ being closely related to the eigenvalue $e^{2i\pi\alpha_{p,j}}$. If we denote, as in chapter 3, by Δ_ε the auxiliary Laplacian, we have, for any real number $\mu \geqslant 0$,

$$\begin{aligned} &\det(\Delta_{E,\mathrm{cusp},\varepsilon} + \mu, \Delta_\varepsilon + \mu) \\ &\quad = \quad \prod_{p \text{ cusp}} \left[\left(\prod_{j=1}^{k_p} \det\left(\Delta_{L_{p,j},p,\varepsilon} + \mu, \Delta_\varepsilon + \mu\right) \cdot \left(\prod_{j=k_p+1}^{r} \det\left(\Delta_{L_{p,j},p,\varepsilon} + \mu\right) \right) \right]. \end{aligned}$$

Each of the determinant appearing above has been computed in this chapter, as μ goes to infinity, and for $\mu = 0$ as ε goes to 0.

Theorem 4.9.1. *The constant coefficient in the asymptotic expansion as μ goes to infinity of the logarithm of the previous relative determinant is given by*

$$\mathrm{Fp}_{\mu=+\infty} \ \log\det(\Delta_{E,cusp,\varepsilon} + \mu, \Delta_{E,\varepsilon} + \mu) \quad = \quad 0 \ .$$

Proof. This is a direct consequence of the decomposition above, as well as of 4.8.1 and 4.8.2. □

Theorem 4.9.2. *As ε goes to 0^+, we have the following asymptotic expansion*

$$\begin{aligned}\log\det\left(\Delta_{E,cusp,\varepsilon},\Delta_{\varepsilon}\right) &= 2\pi a\left(\varepsilon\right)\sum_{p\ cusp}\ \sum_{j=k_p+1}^{r}\alpha_{p,j}^{2}-2\pi a\left(\varepsilon\right)\sum_{p\ cusp}\ \sum_{j=k_p+1}^{r}\alpha_{p,j}+\tfrac{1}{3}\pi hra\left(\varepsilon\right)\\ &\quad+\tfrac{1}{2}k\left(\Gamma,\rho\right)\log a\left(\varepsilon\right)-\tfrac{1}{2}\sum_{p\ cusp}\sum_{j=k_p+1}^{r}\log\sin\pi\alpha_{p,j}-\tfrac{1}{2}hr\log 2+o\left(1\right)\end{aligned}$$

where r is the rank of E, and h is the number of cusps.

Chapter 5

A Deligne–Riemann–Roch isometry

We now come to the core chapter of this text. Everything we have seen so far has been in preparation of proving an isometry of the same type as the one obtained by Deligne in [32]. We will first recall what the result we wish to emulate is in the situation considered by Deligne, and only then go back to the case of modular curves.

5.1 Deligne's theorem

For the purpose of this section, we will go back to a relatively abstract setting, which is the one examined in [32]. Let $\mathcal{X} \longrightarrow \mathcal{S}$ be a proper, smooth scheme morphism of pure relative dimension 1, and $\mathcal{E}$ be a vector bundle over $\mathcal{X}$ of rank r. The full result is the following.

Theorem 5.1.1. *Assuming $\mathcal{X}/\mathcal{S}$ has geometrically connected fibers, we have a functorial isomorphism of line bundles over the base $\mathcal{S}$*

$$\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})^{12} \quad \simeq \quad \left\langle \omega_{\mathcal{X}/\mathcal{S}}, \omega_{\mathcal{X}/\mathcal{S}} \right\rangle^{r} \left\langle \det \mathcal{E}, \det \mathcal{E} \otimes \omega_{\mathcal{X}/\mathcal{S}}^{-1} \right\rangle^{6} IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})^{-12} \ .$$

When $\mathcal{X}$ and $\mathcal{E}$ are both endowed with smooth metrics, this isomorphism becomes an isometry, up to an explicit factor $c(g,r)$ which only depends on the genus g of the fibers and on the rank r.

Remark 5.1.2. The unspecified constant $c(g,r)$ can be found in [89, Sec. 4.4], and is given by

$$c(g,r) \quad = \quad \exp\left[-r(1-g)\left(1-24\zeta'(-1)\right)\right] \ .$$

Remark 5.1.3. The theorem above is comprised of two parts: the functorial isomorphism, and the isometry. It is necessary to separate the two, as the first one will still hold for models of modular curves. The isomorphism being functorial, it is compatible with base change, so going from isomorphism to isometry is done by looking at what happens when the base is a point. Cursive letters, like $\mathcal{X}$, $\mathcal{S}$, and $\mathcal{E}$ will denote objects satisfying the hypotheses of theorem 5.1.1.

We will now recall the definitions of the various notions involved in this theorem, in the case of a base reduced to a point, with references being given for the relative versions. For clarity, we will denote by X a compact Riemann surface, and E a holomorphic vector bundle over X, corresponding to the data of Deligne's theorem when $\mathcal{S}$ is a point.

5.1.1 The determinant line bundle

In this section, we will recall the definition of the determinant line bundle on the compact Riemann surface X, related to the holomorphic vector bundle E. For more information on the notions of complex geometry we use here, the reader is referred to [99] and [100].

Definition 5.1.4. The *determinant line bundle* $\lambda_X(E)$ is the complex line

$$\lambda_X(E) \quad = \quad \det H^0(X,E) \otimes \det H^1(X,E)^{\vee} \ ,$$

where "det" of a finite dimensional vector space denotes its determinant, meaning its top exterior power, and the vector spaces involved are the Dolbeault cohomology spaces.

Remark 5.1.5. Let us say a few words about the relative version of the determinant line bundle over the base $\mathcal{S}$, denoted by $\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})$. Three approaches can be followed.

1. The most general definition is due to Knudsen and Mumford, and is presented in [64].
2. The second one, called the "algebraic approach" in [90, Sec. VI.1] is the one closest to the definition we saw before this remark. The idea is to replace the Dolbeault cohomology spaces by the right-derived functors of the pushforward to the base. The main problem is that these sheaves need not be locally free, which prevents us from taking their determinants. Assuming that $\mathcal{S}$ is smooth, however, one can take locally free resolutions and use them to define the required determinants. The resulting line bundle can be proved to be independant of these resolutions.
3. The third way to see the determinant line bundle is due to Bismut, Gillet, and Soulé. It can be found in [11], and is called the "analytic approach" in [90, Sec. VI.2].

Any two of these definitions, when they make sense, coincide, and provide a line bundle over $\mathcal{S}$ which is compatible with base change. In particular, for any complex point $s \in \mathcal{S}(\mathbb{C})$, we have

$$\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})_s \quad = \quad \lambda_{\mathcal{X}_s}(\mathcal{E}_s) \ ,$$

which coincide with definition 5.1.4.

We now assume that X is endowed with a smooth Riemannian metric g, and E with a smooth Hermitian metric h. The Dolbeault cohomology spaces are then endowed with the L^2-metric induced by integration of differential forms, through Hodge theory and the identification of H^0 and H^1 with the appropriate kernels of Laplacians.

Remark 5.1.6. It is possible to define the L^2-metric without mentionning Hodge theory, by using the *quotient metric*. In a more general setting, if $(V, \|\cdot\|)$ is a normed space and W is a subspace of V, the *quotient norm* is defined by

$$\|v+W\|_{V/W} \quad = \quad \inf_{w\in W} \|v+w\|_V \ .$$

In our setting, this quotient metric coincide with the one provided by Hodge theory, since harmonic representants are minimal for the L^2-norm.

Definition 5.1.7. The *L^2-metric* $\|\cdot\|_{L^2}$ on $\lambda_X(E)$ is defined as the metric induced by the L^2-metrics on the Dolbeault cohomology spaces by taking their determinant, dual, and tensor product.

Remark 5.1.8. This definition, coupled with the end of remark 5.1.5, yields an L^2-metric on the relative determinant line bundle $\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})$. However, this metric is not the one we will consider, because it does not vary smoothly with respect to the base point $s \in \mathcal{S}$. The main reason for that is the possible presence of jumps in dimension for the kernels of the Laplacians needed in the definition. To regularize this behavior, one needs to twist this L^2-metric by a factor, called the *holomorphic analytic torsion*. For more information on that, the reader is referred to the work of Ray, Singer, Bismut, Gillet, Soulé to name but a few, for instance in [9, 10, 11, 79]. When referring to this notion, the word "holomorphic" will be dropped, as no confusion can arise here. In our situation, this factor will be greatly simplified.

Definition 5.1.9. We define the *Quillen metric* $\|\cdot\|_Q$ on $\lambda_X(E)$ to be

$$\|\cdot\|_Q = \left(\det{}' \Delta^{g,h}_{\overline{\partial}_E}\right)^{-1/2} \|\cdot\|_{L^2} .$$

Remark 5.1.10. Looking at a more general definition of the analytic torsion, for instance given by Ray and Singer in [79], it is natural to wonder why the factor that appears in our case involves the Dolbeault Laplacian acting on sections of E, and not the one acting on 1-forms with values in E. The reason is that they are equal, and the one acting on sections is easier to work with.

Remark 5.1.11. As mentioned in remark 5.1.8, the idea behind the definition of the Quillen metric is to modify the L^2-metric to get a metric which would vary smoothly in family. The L^2-metric depending only on kernels of Laplacians, meaning only the eigenvalue 0, it is not surprising to regularize it by taking into account all the other eigenvalues of the Dolbeault Laplacian, which is done here by using the modified determinant.

5.1.2 Deligne pairing and first Chern classes

Having defined the left-hand side of theorem 5.1.1, we now turn our attention to the first two factors on the right-hand side, which are *Deligne pairings* of line bundles over $\mathcal{X}$. We will first give the intuition behind this pairing, before defining it properly when the base is a point. The definition of the relative version is hinted at by Deligne right before section 8.3 of [32], and is detailed by Elkik in [40].

Remark 5.1.12. What follows is a heuristics for Deligne pairings. We will not be too concerned with hypotheses for the purpose of this remark. Let $f : \mathcal{X} \longrightarrow \mathcal{S}$ be a family of schemes as in theorem 5.1.1. Let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles over $\mathcal{X}$. The Picard group $Pic(\mathcal{X})$, which is the group of line bundles over $\mathcal{X}$ modulo isomorphism, and the first Chow group $CH^1(\mathcal{X})$ are isomorphic by the first Chern class. Intersection theory then defines

$$c_1(\mathcal{L}) \cdot c_1(\mathcal{M}) \in CH^2(\mathcal{X}) ,$$

and we can pushforward this to the base, yielding

$$f_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{M})) \in CH^1(\mathcal{S}) .$$

Using the first Chern class on $\mathcal{S}$, we now know that this element of $CH^1(\mathcal{S})$ corresponds to a line bundle over $\mathcal{S}$, though we do not know much about it. The *Deligne pairing* provides a concrete line bundle over the base whose c_1 is precisely the pushforward of the intersection we consider.

We go back to the case where $\mathcal{S}$ is a point, meaning we consider a compact Riemann surface X of a genus g. The definition of the Deligne pairing we will see below is taken from [4, Sec. 13.5], though it can also be found in [32, Sec. 1.4].

Definition 5.1.13. Let L and M be two holomorphic line bundles over X. We denote by V the complex vector space freely generated by symbols $\langle l, m\rangle$, where l and m are meromorphic sections of L and M with disjoint divisors.

Proposition-Definition 5.1.14. *The binary relation $\sim$ on V induced by:*

$$\begin{aligned} \langle fl, m\rangle &\sim f(div\, m)\langle l, m\rangle , \\ \langle l, gm\rangle &\sim g(div\, l)\langle l, m\rangle \end{aligned}$$

for any meromorphic sections l and m of L and M respectively, as well as any meromorphic functions f and g, is an equivalence relation. The quotient space of V by $\sim$ is a complex line.

Proof. This proposition is a direct consequence of the Weil reciprocity law. □

Definition 5.1.15. The quotient space in the statement above is denoted by $\langle L, M\rangle$, and called the *Deligne pairing* of L and M.

Remark 5.1.16. As explained in [32, 40], there is a relative version of the Deligne pairing. For any line bundles $\mathcal{L}$ and $\mathcal{M}$ over the family $\mathcal{X}/\mathcal{S}$, we get a line bundle $\langle \mathcal{L}, \mathcal{M}\rangle$ over $\mathcal{S}$. It is compatible with base change, which implies that for any complex point $s \in \mathcal{S}(\mathbb{C})$, we have

$$\langle \mathcal{L}, \mathcal{M}\rangle_s \quad = \quad \langle \mathcal{L}_s, \mathcal{M}_s\rangle \ ,$$

where $\mathcal{L}_s$ and $\mathcal{M}_s$ are line bundles over the Riemann surface $\mathcal{X}_s$.

Proposition 5.1.17. *The Deligne pairing is bimultiplicative with respect to tensor product, meaning we have canonical isomorphisms, for any line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ on X,*

$$\begin{aligned}\langle \mathcal{L}_1 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_2, \mathcal{L}_3\rangle &\simeq \langle \mathcal{L}_1, \mathcal{L}_3\rangle \otimes_{\mathcal{O}_\mathcal{S}} \langle \mathcal{L}_2, \mathcal{L}_3\rangle \,, \\ \langle \mathcal{L}_1, \mathcal{L}_2 \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{L}_3\rangle &\simeq \langle \mathcal{L}_1, \mathcal{L}_2\rangle \otimes_{\mathcal{O}_\mathcal{S}} \langle \mathcal{L}_1, \mathcal{L}_3\rangle \,.\end{aligned}$$

of line bundles over $\mathcal{S}$. It is also symmetric, meaning we have a canonical isomorphism

$$\langle \mathcal{L}_1, \mathcal{L}_2\rangle \quad \simeq \quad \langle \mathcal{L}_2, \mathcal{L}_1\rangle$$

for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$.

We now turn to the definition of a canonical metric on Deligne pairings, a task for which we may consider the base to be a point. Let L_1 and L_2 be two holomorphic line bundles over X, respectively endowed with smooth hermitian metrics h_1 and h_2. We denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the associated norms. We will define a metric on the Deligne pairing $\langle L_1, L_2\rangle$, using the *star-product* of Green currents, for which the reader is referred to [90, Sec. II.3]

Definition 5.1.18. The *Deligne pairing metric* on $\langle L_1, L_2\rangle$ is defined by

$$\log \|\langle s_1, s_2\rangle\|^{-2} \quad = \quad \int g_1 * g_2 \ ,$$

for any two meromorphic sections s_1 and s_2 of L_1 and L_2 respectively, with disjoint divisors, where g_i is a Green current for the divisor of s_i. Here, the dd^c has been taken in the distributional sense and the integral is over X.

Remark 5.1.19. It can be noted that the function $\log \|s_j\|_j^{-2}$ is a Green current for $div\, s_j$, denoted by g_j. We then have

$$\log \|\langle s_1, s_2\rangle\|^{-2} \quad = \quad \int \left(\log \|s_2\|_2^{-2} \cdot dd^c \log \|s_1\|_1^{-2}\right) + \log \|s_1\|_1^{-2} [div\, s_2] + \log \|s_2\|_2^{-2} [div\, s_1] \ ,$$

where we have set

$$f\left[\sum_i \alpha_i P_i\right] \quad = \quad \sum_i \alpha_i f(P_i)$$

for any meromorphic function f. In case the metrics on L_1 and L_2 are conformally changed by

$$\|\cdot\|_{1,\psi_1} \quad = \quad e^{\psi_1} \|\cdot\|_1 \,, \qquad\qquad \|\cdot\|_{2,,\psi_2,} \quad = \quad e^{\psi_2} \|\cdot\|_2 \,,$$

for smooth functions ψ_1 and ψ_2, the Green currents for $div\, s_1$ and $div\, s_2$ respectively change by

$$g_{1,\psi_1} \quad = \quad g_1 - 2\psi_1, \qquad\qquad g_{2,\psi_2} \quad = \quad g_2 - 2\psi_2.$$

The integral of the star-product of Green currents then varies in the following way, where ω_i denotes the smooth differential form induced by the Poincaré-Lelong equation,

$$\int g_{1,\psi_1} * g_{2,\psi_2} \quad = \quad \int g_1 * g_2 - 2\int \psi_1\omega_2 - 2\int \psi_2\omega_1 + \tfrac{2}{i\pi}\int \partial\psi_1 \wedge \overline{\partial}\psi_2 \ .$$

This allows us, as was done in [14, Sec. 5.1], to give meaning to the integral of the star-product of Green currents, and thus to the Deligne pairing metric on $\langle L_1, L_2\rangle$ when the metrics on L_1 and L_2 only have an H^1-regularity, *i.e.* when they differ from smooth metrics by functions ψ_1 and ψ_2 which are only H^1. The Deligne pairing is therefore continuous with respect to the H^1 norm on the metrics attached to both line bundles considered.

Proposition 5.1.20. *The isomorphisms from proposition 5.1.17 become isometries, where every factor is endowed with the Deligne pairing metric.*

Proposition 5.1.21. *Let σ be a section of the structural morphism $f : \mathcal{X} \longrightarrow \mathcal{S}$, whose image is denoted by D, and $\mathcal{L}$ be a line bundle over $\mathcal{X}$. We have a canonical isomorphism*

$$\langle \mathcal{L}, \mathcal{O}_{\mathcal{X}}(D)\rangle \quad \simeq \quad \sigma^* L \ .$$

Remark 5.1.22. When specialized to the case where $\mathcal{S}$ is a point, such a section σ becomes a point P of X, and the isomorphism above becomes

$$\begin{array}{ccc} \langle L, \mathcal{O}_X(P)\rangle & \xrightarrow{\sim} & L_{|P} \\ \langle l, \mathbb{1}\rangle & \longmapsto & l_P \end{array} .$$

Remark 5.1.23. It is worth noting that taking $\mathcal{L} = \omega_{\mathcal{X}/\mathcal{S}}(D)$, where $\omega_{\mathcal{X}/\mathcal{S}}$ denotes the dualizing sheaf, in proposition 5.1.21 ans using the adjunction formula yield isomorphisms

$$\left\langle \omega_{\mathcal{X}/\mathcal{S}}(D), \mathcal{O}_{\mathcal{X}}(D)\right\rangle \quad \simeq \quad \sigma^*\omega_{\mathcal{X}/\mathcal{S}}(D) \quad \simeq \quad \mathcal{O}_{\mathcal{S}} \ .$$

It is not, however, necessarily an isometry when $\mathcal{O}_{\mathcal{S}}$ is endowed with the trivial metric.

5.1.3 Realization of the second Chern class

We finally come to the last factor of the right-hand side of theorem 5.1.1, the IC_2-bundle. As we did for the Deligne pairings, we will first see the motivation behind this bundle, and only then move on to recall some facts about it.

Remark 5.1.24. Let us see a heuristics for the definition of the IC_2-bundle. Consider a scheme morphism $f : \mathcal{X} \longrightarrow \mathcal{S}$ as in theorem 5.1.1, and $\mathcal{E}$ a vector bundle of rank r over $\mathcal{X}$. Its second Chern class is an element

$$c_2(\mathcal{E}) \quad \in \quad CH^2(\mathcal{X}) \ ,$$

which we can pushforward by f, yielding

$$\textstyle\int_{\mathcal{X}/\mathcal{S}} c_2(\mathcal{E}) \quad = \quad f_*(c_2(\mathcal{E})) \quad \in \quad CH^1(\mathcal{S}) \ .$$

The notation of the left-hand side above is commonly used for this pushforward, and called "integration along the fibers". This element of $CH^1(\mathcal{S})$ is then the c^1 of a line bundle on $\mathcal{S}$, but we do not have any information on it. The IC_2-bundle provides an explicit such line bundle on $\mathcal{S}$.

The following is entirely taken from [32, Sec. 9], to which the reader is referred for more details.

Definition 5.1.25. Let $\mathcal{X}/\mathcal{S}$ be a family as in theorem 5.1.1. The abelian Picard category $\mathfrak{A}_{\mathcal{X}/\mathcal{S}}$ is defined by the following data:

1. the objects are triplets $(n,\ \mathcal{L},\ \mathcal{M})$ where n is a locally constant integer on $\mathcal{X}$, and $\mathcal{L}$, respectively $\mathcal{M}$, is a line bundle over $\mathcal{X}$, respectively over $\mathcal{S}$;

2. the maps are the isomorphisms, given by equality of the locally constant integers, and isomorphisms of the respective line bundles;

3. the sum of two objects is defined by

$$(n_1,\ \mathcal{L}_1,\ \mathcal{M}_1) + (n_2,\ \mathcal{L}_2,\ \mathcal{M}_2) \quad = \quad (n_1 + n_2,\ \mathcal{L}_1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}_2,\ \mathcal{M}_1 \otimes_{\mathcal{O}_{\mathcal{S}}} \mathcal{M}_2 \otimes_{\mathcal{O}_{\mathcal{S}}} \langle \mathcal{L}_1, \mathcal{L}_2 \rangle)\ .$$

Proposition-Definition 5.1.26. *There exists, up to unique isomorphism, a unique functor going from the category of virtual bundles over $\mathcal{X}$ to that of line bundles over $\mathcal{S}$, called $IC_2(\mathcal{X}/\mathcal{S}, \cdot)$, which is compatible with base change, and satisfies the following properties.*

1. *The functor*

$$T \quad : \quad \mathcal{E} \quad \longmapsto \quad (\operatorname{rk}\mathcal{E},\ \det\mathcal{E},\ IC_2(\mathcal{X}/\mathcal{S},\ \mathcal{E}))$$

is endowed, for any short exact sequence of vector bundles over $\mathcal{X}$

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

with an isomorphism $T(\mathcal{E}) \simeq T(\mathcal{E}') + T(\mathcal{E}'')$ in the category $\mathfrak{A}_{\mathcal{X}/\mathcal{S}}$;

2. *For any line bundle $\mathcal{L}$ over $\mathcal{X}$, the line bundle $IC_2(\mathcal{X}/\mathcal{S}, \mathcal{L})$ is trivial;*

3. *For any σ be a section of $\mathcal{X} \longrightarrow \mathcal{S}$, whose image is denoted by D, and $\mathcal{M}$ be a line bundle over $\mathcal{S}$, we have an isomorphism which depends only on $\mathcal{M}$ and σ*

$$IC_2(\mathcal{X}/\mathcal{S}, \sigma_* \mathcal{M}) \quad \simeq \quad \mathcal{M}^{-1} \otimes_{\mathcal{O}_{\mathcal{S}}} \sigma^* \mathcal{O}_{\mathcal{X}}(-D)\ .$$

Remark 5.1.27. This construction being compatible with base change, we have, for any complex point $s \in \mathcal{S}(\mathbb{C})$,

$$IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})_s \quad \simeq \quad IC_2(\mathcal{X}_s, \mathcal{E}_s)\ ,$$

where the right-hand side is what we would get from the definition above if $\mathcal{S}$ was a point.

Remark 5.1.28. As hinted at in remark 5.1.24, we indeed have

$$c_1(IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})) \quad = \quad f_*(c_2(\mathcal{E})) \quad = \quad \int_{\mathcal{X}/\mathcal{S}} c_2(\mathcal{E})\ .$$

From now on, we will drop the subscript on the tensor product whenever no confusion can arise, and tensor product of line bundles will simply be denoted by a product.

Proposition 5.1.29. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector bundles over $\mathcal{X}$ of respective ranks r and r'. We have an isomorphism of line bundles over the base $\mathcal{S}$*

$$\begin{aligned} IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}\otimes\mathcal{E}'\right)^2 \quad \simeq \quad & IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}\right)^{2r'} IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}'\right)^{2r} \left\langle \det\mathcal{E},\det\mathcal{E}\right\rangle^{r'\left(r'-1\right)} \\ & \left\langle \det\mathcal{E}',\det\mathcal{E}'\right\rangle^{r(r-1)} \left\langle \det\mathcal{E},\det\mathcal{E}'\right\rangle^{2\left(rr'-1\right)}. \end{aligned}$$

Remark 5.1.30. We will only give the intuition behind the proof of this result, which may be done by using Elkik's formalism, presented in [40]. We have

$$ch\left(\mathcal{E}\otimes\mathcal{E}'\right) \quad = \quad \underbrace{\mathrm{rk}\left(\mathcal{E}\otimes\mathcal{E}'\right)}_{=rr'} + c_1\left(\mathcal{E}\otimes\mathcal{E}'\right) + \tfrac{1}{2}\Big(c_1\left(\mathcal{E}\otimes\mathcal{E}'\right)^2 - 2c_2\left(\mathcal{E}\otimes\mathcal{E}'\right)\Big) + \ \dots \ ,$$

the terms omitted on the right-hand side being in Chow groups $CH^n\left(\mathcal{X}\right)$ with $n \geqslant 3$, which means they play no role in the study of c_2. On the other hand, we have

$$\begin{aligned} ch\left(\mathcal{E}\otimes\mathcal{E}'\right) \quad &= \quad ch\left(\mathcal{E}\right)\ ch\left(\mathcal{E}'\right) \\ &= \quad \Big[r + c_1\left(\mathcal{E}\right) + \tfrac{1}{2}\Big(c_1\left(\mathcal{E}\right)^2 - 2c_2\left(\mathcal{E}\right)\Big) + \ \dots\Big] \cdot \Big[r' + c_1\left(\mathcal{E}'\right) + \tfrac{1}{2}\Big(c_1\left(\mathcal{E}'\right)^2 - 2c_2\left(\mathcal{E}'\right)\Big) + \ \dots\Big] \\ &= \quad rr' + \left[rc_1\left(\mathcal{E}'\right) + r'c_1\left(\mathcal{E}\right)\right] + \tfrac{1}{2}\Big[r\Big(c_1\left(\mathcal{E}'\right)^2 - 2c_2\left(\mathcal{E}'\right)\Big) + r'\Big(c_1\left(\mathcal{E}\right)^2 - 2c_2\left(\mathcal{E}\right)\Big)\Big] \\ &\qquad + c_1\left(\mathcal{E}\right)c_1\left(\mathcal{E}'\right) + \ \dots . \end{aligned}$$

Identifying terms which are in the same Chow groups, we get

$$c_1\left(\mathcal{E}\otimes\mathcal{E}'\right) \quad = \quad rc_1\left(\mathcal{E}'\right) + r'c_1\left(\mathcal{E}\right)\ ,$$

for terms in $CH^1\left(\mathcal{X}\right)$, and the terms in $CH^2\left(\mathcal{X}\right)$ yield

$$\begin{aligned} & 2c_2\left(\mathcal{E}\otimes\mathcal{E}'\right) \\ & \quad = \quad \left[rc_1\left(\mathcal{E}'\right) + r'c_1\left(\mathcal{E}\right)\right]^2 - rc_1\left(\mathcal{E}'\right)^2 + 2rc_2\left(\mathcal{E}'\right) - r'c_1\left(\mathcal{E}\right)^2 + r'c_2\left(\mathcal{E}\right) - 2c_1\left(\mathcal{E}\right)c_1\left(\mathcal{E}'\right) \\ & \quad = \quad r\left(r-1\right)c_1\left(\mathcal{E}'\right)^2 + r'\left(r'-1\right)c_1\left(\mathcal{E}\right)^2 + 2\left(rr'-1\right)c_1\left(\mathcal{E}\right)c_1\left(\mathcal{E}'\right) + 2rc_2\left(\mathcal{E}'\right) + 2r'c_2\left(\mathcal{E}\right). \end{aligned}$$

Remark 5.1.31. As a consequence of the definition of the IC_2 bundle, we have

$$IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}\oplus\mathcal{E}'\right) \quad = \quad IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}\right)\ IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}'\right)\ \left\langle \det\mathcal{E},\det\mathcal{E}'\right\rangle$$

for any vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on $\mathcal{X}$.

We now turn to the task of defining a smooth metric on a bundle $IC_2\left(\mathcal{X}/\mathcal{S},\mathcal{E}\right)$ when one is given on $\mathcal{E}$. What follows is taken from [32, Sec. 10]. Since defining a metric can be done by looking only at the case of a base $\mathcal{S}$ reduced to a point, we consider a compact Riemann surface X, endowed with a smooth Riemannian metric g. In this case, we note that $IC_2\left(X,E\right)$ is a complex line for any holomorphic vector bundle E over X.

Proposition 5.1.32. *For any holomorphic vector bundle E endowed with a smooth Hermitian metric h, there is one and only one way to define a metric on the complex line $IC_2\left(X,E\right)$ satisfying the following properties.*

1. *If E is of rank 1, the line $IC_2(X,E)$ is endowed with the trivial metric;*
2. *Consider a short exact sequence of holomorphic vector bundles*

$$\mathscr{E} \quad : \quad 0 \quad \longrightarrow \quad E' \quad \longrightarrow \quad E \quad \longrightarrow \quad E'' \quad \longrightarrow \quad 0$$

where each term is endowed with a smooth hermitian metric h', h, h'', respectively. We have a canonical isomorphism

$$IC_2(X,E) \quad \simeq \quad IC_2(X,E')\ IC_2(X,E')\ \langle \det E', \det E'' \rangle \ ,$$

the respective norms an element x on the left-hand side, and its corresponding element y on the right-hand side are related by

$$\log \|x\| \quad = \quad \log \|y\| + \int_X \widetilde{c_2}(\mathscr{E}) \ ,$$

where $\widetilde{c_2}$ denotes the secondary Bott-Chern class associated to c_2.

Remark 5.1.33. It is worth noting that, although a short exact sequence

$$\mathscr{E} \quad : \quad 0 \quad \longrightarrow \quad E' \quad \longrightarrow \quad E \quad \longrightarrow \quad E'' \quad \longrightarrow \quad 0$$

induces an isomorphism of vector bundles $E \simeq E' \oplus E''$, the isomorphism in the second point of the last proposition is not in general an isometry, since the metrics on E, E', and E'' are not required to be compatible. If they are, then the secondary Bott-Chern class $\widetilde{c_2}(\mathscr{E})$ vanishes.

Remark 5.1.34. We will later need to know that if we take a sequence $(h_k)_k$ of smooth hermitian metrics on E converging in the H^1-sense to an H^1-metric, the sequence of metrics on $IC_2(X,E)$ attached to $(h_k)_k$ converges to a well-defined metric on $IC_2(X,E)$, which does not depend on the sequence $(h_k)_k$. We will give a proof under certain hypotheses in a subsequent section.

Proposition 5.1.35. *Let E and E' be two holomorphic vector bundles over X, endowed with smooth Hermitian metrics h and h', respectively. When the tensor product $E \otimes E'$ is endowed with the smooth Hermitian metric $h \otimes h'$, the isomorphism*

$$\begin{aligned} IC_2(X, E \otimes E')^2 \quad \simeq \quad & IC_2(X,E)^{2r'}\ IC_2(X,E')^{2r}\ \langle \det E, \det E \rangle^{r'\left(r'-1\right)} \\ & \langle \det E', \det E' \rangle^{r(r-1)}\ \langle \det E, \det E' \rangle^{2\left(rr'-1\right)} \end{aligned}$$

induced by proposition 5.1.29 is an isometry, where r and r' are the ranks of E and E', respectively.

Proof. This result stems from the fact that the arithmetic first Chern class $\widehat{c_1}$ is a ring homomorphism, as indicated in [90, Sec. 4.1]. □

Remark 5.1.36. When applied to the case where $\mathcal{E}' = \mathcal{L}$ is a line bundle, we have an isometry

$$IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E} \otimes \mathcal{L})^2 \quad \simeq \quad IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})^2\ \langle \mathcal{L}, \mathcal{L} \rangle^{r(r-1)}\ \langle \det \mathcal{E}, \mathcal{L} \rangle^{2(r-1)} \ .$$

When applied by induction to the case of the m-th tensor power of $\mathcal{E}$, we get an isometry

$$IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E}^{\otimes m})^2 \quad \simeq \quad IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})^{2mr^{m-1}}\ \langle \det E, \det E \rangle^{mr^{m-1}\left(mr^{m-1}-1\right)-m(m-1)r^{m-2}} \ .$$

5.2 Arithmetic surfaces

We now go back to the situation we study in this document: modular curves, and families of such objects. We will begin by presenting the relative situation we consider, which is close to the one depicted in [47, Sec. 10.1], and then only consider a single modular curve, as going from isomorphism to isometry in theorem 5.1.1.

Definition 5.2.1. Let K be a number field, and $\mathcal{S} = \operatorname{Spec}\mathcal{O}_K$, where $\mathcal{O}_K$ denotes the ring of integers of K. An *arithmetic surface* $\mathcal{X}/\mathcal{S}$ is a regular, integral scheme $\mathcal{X}$ of dimension 2, with a structural morphism $f : \mathcal{X} \longrightarrow \mathcal{S}$ which is projective, flat, and with geometrically connected fibers.

Remark 5.2.2. The definition of this notion of arithmetic surface is not universal. For instance, Soulé gives two slightly different definitions, in [89, Sec. 0.1] and in [90, Sec. III.1.1].

Theorem 5.2.3. *We have a functorial isomorphism of line bundles over $\mathcal{S}$*

$$\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})^{12} \quad \simeq \quad \mathcal{O}(\delta)^r \left\langle \omega_{\mathcal{X}/\mathcal{S}}, \omega_{\mathcal{X}/\mathcal{S}} \right\rangle^r \left\langle \det\mathcal{E}, \det\mathcal{E} \otimes \omega_{\mathcal{X}/\mathcal{S}}^{-1} \right\rangle^6 IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})^{-12} \ ,$$

where δ is the discriminant of $\mathcal{X}/\mathcal{S}$.

Remark 5.2.4. The factor $\mathcal{O}(\delta)^r$, which is not present in theorem 5.1.1, has to be inserted here because $\mathcal{X}/\mathcal{S}$ is not smooth. However the regularity assumption means that it is generically smooth, as K is in particuler a perfect field. Deligne's theorem could only be applied as written here over $\operatorname{Spec} K$, which is not enough here. The reader is referred to [89, Sec. 4.2] for this version of Deligne's result, and to [70, 83] for more information on the notions of discriminant and Artin's conductor. To sum up, we have

$$\delta \quad = \quad \sum_{\mathfrak{p}} \delta_{\mathfrak{p}} \mathfrak{p} \ ,$$

where $\delta_{\mathfrak{p}}$ is the Artin conductor of $\mathcal{X}$ at $\mathfrak{p}$, the sum bearing over the maximal ideals of $\mathcal{O}_K$.

Remark 5.2.5. The definition of the line bundle $IC_2(\mathcal{X}/\mathcal{S}, \mathcal{E})$ requires some care, as the process explicited by Deligne, which was recalled in the last section, only works when the structural morphism $f : \mathcal{X} \longrightarrow \mathcal{S}$ is smooth, which it is not here. The reader is referred to [40, Sec. V.3] for this generalization.

From now on, we will restrict our attention to particular arithmetic surfaces. We assume the structural morphism $f : \mathcal{X} \longrightarrow \mathcal{S} = \operatorname{Spec}\mathcal{O}_K$ has sections $\sigma_1, \ldots, \sigma_h$ such that for every complex embedding $\tau : K \hookrightarrow \mathbb{C}$, the non-compact Riemann surface $\mathcal{X}_\tau(\mathbb{C}) \backslash \{\sigma_1(\tau), \ldots, \sigma_h(\tau)\}$ is a modular curve arising from a Fuchsian group of the first kind Γ_τ without torsion. Furthermore, we consider a vector bundle $\mathcal{E}$ of rank r over $\mathcal{X}$ such that every $\mathcal{E}_\tau$ is a flat unitary holomorphic vector bundle over the modular curve $\mathcal{X}_\tau(\mathbb{C})$. The aim of what follows is to get some isometry from this theorem, with the most natural metrics being chosen on $\omega_{\mathcal{X}/\mathcal{S}}$ and $\mathcal{E}$. Unfortunately, this cannot be done directly, as there are singularities at the cusps. Since metrics are defined above each complex point, we will work in the case of a single modular curve. In this setting, theorem 5.2.3 can be restated as the following, with the same notations as in previous chapters.

Theorem 5.2.6. *We have an isomorphism of complex lines*

$$\lambda_X(E)^{12} \quad \simeq \quad \langle \omega_X, \omega_X \rangle^r \left\langle \det E, \det E \otimes \omega_X^{-1} \right\rangle^6 IC_2(X, E)^{-12} \ .$$

Remark 5.2.7. In the arithmetic setting, the sheaf $\omega_{\mathcal{X}/\mathcal{S}}$ is the dualizing sheaf. When specialized to the case where $\mathcal{S}$ is a complex point, it is the holomorphic cotangent bundle.

Remark 5.2.8. Of course, having an isomorphism between two complex lines is not a deep result. One needs to remember here (and in what follows) that it is canonical, meaning it can be formulated in family.

5.3 Smooth and truncated isometries

Even though the whole point of what we are seeing is to obtain an isometry for the Poincaré metric on a modular curve and the canonical metric on the vector bundle, we can still apply Deligne's full result for smooth metrics on X and E. We denote by g the Poincaré metric on X, by g_ε its truncated version, by h the canonical metric on E, and by h_ε its truncated counterpart. We then choose smooth approximations

$$g_k \quad \longrightarrow \quad g_\varepsilon , \qquad\qquad h_k \quad \longrightarrow \quad h_\varepsilon$$

in the Sobolev H^1 sense, with g_k and h_k being equal to g and h respectively on X_ε and on an open neighborhood of Σ_ε. Applying Deligne's result, we get the following.

Theorem 5.3.1. *We have an isometry of complex lines*

$$\lambda_X (E)_{\varepsilon,k}^{12} \quad \simeq \quad \langle \omega_{X,\varepsilon,k}, \omega_{X,\varepsilon,k} \rangle^r \left\langle \det E_{\varepsilon,k}, \det E_{\varepsilon,k} \otimes \omega_{X,\varepsilon,k}^{-1} \right\rangle^6 IC_2 \left(X, E_{\varepsilon,k}\right)^{-12} \left(\mathbb{C}, c\left(g_\Gamma, r\right) |\cdot|\right) ,$$

where $c(g_\Gamma, r)$ is the constant given in remark 5.1.2, and g_Γ is the genus of X.

Remark 5.3.2. The subscripts in the theorem above have been put there so as to remember which metrics are considered on the tangent bundle and on the vector bundle E. The bundles themselves do not change.

Using the end of remark 5.1.19, we see that we can take the limit as k goes to infinity of the first two factors on the right-hand side of this isometry, as the Deligne pairing metric exists when the line bundles are endowed with Sobolev H^1-metrics. It remains to show that we can do the same for the IC_2 bundle. To that effect, we write

$$h_k \quad = \quad (h_0)_{\psi_k} \ ,$$

where ψ_k is a smooth section of the endomorphism bundle $\operatorname{End} E$. By definition of the convergence of metrics, the sequence $(\psi_k)_k$ converges in the H^1-sense to an H^1-section ψ of $\operatorname{End} E$. Recall that the metric h_k can then be expressed in terms of h_0 by

$$h_k (s,t) \quad = \quad h_0 \left(e^{\psi_k} s, e^{\psi_k} t\right)$$

for any smooth sections s and t of E. Furthermore, the assumption that h_k and h_0 coincide on X_ε and on an open neighborhood of Σ_ε means that the sections ψ_k all have compact support, with

$$\operatorname{Supp} \psi_k \quad \subset \quad \bigsqcup_{p \text{ cusp}} U_{p,\varepsilon} \ .$$

We recall once again that, above a cusp p, we have a decomposition

$$E_{|U_{p,\varepsilon}} \quad = \quad \bigoplus_{j=1}^{r} L_{p,j}$$

of E as an orthogonal sum of line bundles, and that the metrics h and h_ε are compatible with this decomposition. Their data over $U_{p,\varepsilon}$ is then equivalent to that of metrics $h_{p,j}$ and $h_{p,j,\varepsilon}$ on the line bundles $L_{p,j}$. We then further assume that all the metrics h_k are likewise compatible with this decomposition, and induced near a cusp p by metrics $h_{k,p,j}$ on $L_{p,j}$. This means that, still near a cusp p, every section ψ_k is naturally seen as a diagonal endomorphism of E, given by smooth functions $\psi_{k,p,j}$. The support of such a function is compact, included in the open subset $U_{p,\varepsilon}$.

Furthermore, we assume that all $\psi_{k,p,j}$ is identically zero should $L_{p,j}$ be metrically trivial. We can now state and prove the following proposition.

Proposition 5.3.3. *The hermitian lines $IC_2(X, E_{\varepsilon,k})$ converge to a hermitian line $IC_2(X, E_\varepsilon)$. In other words, we can define a sequence of metrics on the complex line $IC_2(X, E)$ attached to the sequence h_k, and this sequence converges to a metric on the same line as k goes to infinity.*

Proof. The key to proving this result is to compare the metric on $IC_2(X, E)$ attached to h_k to the one related to h_0 using the functions $\psi_{k,p,j}$ defined above. For that, we use the "trivial" exact sequence of vector bundles

$$\mathscr{E} \ : \ 0 \longrightarrow E \longrightarrow E \longrightarrow 0\,.$$

We now consider the metric h_0 on the first copy of E, and the metric h_k on the second one. We further denote by $\|\cdot\|_k$ the metric on $IC_2(X, E)$ related to h_k. Using the second point of proposition 5.1.32, we see that we have

$$\log\|\cdot\|_k = \log\|\cdot\|_0 + \int_X \widetilde{c_2}(\mathscr{E}) = \log\|\cdot\|_0 + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} \widetilde{c_2}(\mathscr{E})\,,$$

since all metrics coincide on the compact part X_ε, resulting in a vanishing secondary Bott-Chern class. To compute $\widetilde{c_2}(\mathscr{E})$ near each cusp, we will use the notations of arithmetic intersection theory adopted in [15, Sec. 2.1.1, 2.1.2]. Using the decomposition of E above $U_{p,\varepsilon}$, we have

$$\begin{aligned}
\log\|\cdot\|_k &= \log\|\cdot\|_0 + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} \widetilde{c_2}(\mathscr{E}) \\
&= \log\|\cdot\|_0 + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} \left[4\sum_{j<m} \psi_{k,p,j}\, \mathrm{dd}^c \psi_{k,p,m} - 2\sum_{j\neq m} \psi_{k,p,j}\,\omega\left(\widehat{c_1}\left(L_{p,m}, h_{0,p,m}\right)\right)\right] \\
&= \log\|\cdot\|_0 - \frac{2}{i\pi} \sum_{p \text{ cusp}} \sum_{j<m} \int_{U_{p,\varepsilon}} \partial\psi_{k,p,j} \wedge \overline{\partial}\psi_{k,p,m} \\
&\qquad -2 \sum_{p \text{ cusp}} \sum_{j\neq m} \int_{U_{p,\varepsilon}} \psi_{k,p,j}\,\omega\left(\widehat{c_1}\left(L_{p,m}, h_{0,p,m}\right)\right).
\end{aligned}$$

The limit as k goes to infinity can then be taken in this last expression, since we have assumed an H^1-convergence of metrics. The result does not depend on the particular choice of the smooth approximation of the truncated metric, and yields a metric on $IC_2(X, E)$ attached to h_ε.

□

Proposition 5.3.4. *For each integer k, denote by $\|\cdot\|_{Q,k}$ the Quillen metric on $\lambda_X(E)$. As k goes to infinity, this sequence of metrics converges to a metric on $\lambda_X(E)$ which does not depend on the particular approximations of the truncated metrics g_ε and h_ε.*

Proof. This result is a combination of the end of remark 5.1.19, in which we saw that the Deligne pairing metric makes sense for H^1-metrics, of proposition 5.3.3, which gives meaning to the metric on $IC_2(X, E)$ for an H^1-metric on E, and of the smooth Deligne-Riemann-Roch isometry presented in theorem 5.3.1.

□

Definition 5.3.5. The limit metric yielded by the prposition above is called the *ε-truncated Quillen metric*, and denoted by $\|\cdot\|_{Q,\varepsilon}$.

Taking the limit of theorem 5.3.1 as k goes to infinity, and using the fact that the result does not depend on the chosen approximations of the truncated metrics, we get the following *truncated Deligne-Riemann-Roch isometry.*

Theorem 5.3.6. *We have an isometry of complex lines*

$$\lambda_X(E)^{12}_{\varepsilon} \quad \simeq \quad \langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon}\rangle^r \left\langle \det E_\varepsilon, \det E_\varepsilon \otimes \omega_{X,\varepsilon}^{-1} \right\rangle^6 IC_2\left(X, E_\varepsilon\right)^{-12} \left(\mathbb{C}, c\left(g_\Gamma, r\right)|\cdot|\right) \ .$$

Remark 5.3.7. Recall that, for every integer k, the Quillen metric $\|\cdot\|_{Q,k}$ is defined by

$$\|\cdot\|_{Q,k} \quad = \quad \left(\det{}' \Delta^{g_k,h_k}_{\overline{\partial}_E}\right)^{-1/2} \|\cdot\|_{L^2,k} \ ,$$

where $\|\cdot\|_{L^2,k}$ is the L^2-metric provided, for instance, by Hodge theory. The smooth Deligne-Riemann-Roch isometry allows us to define the limit as k goes to infinity of these metrics, but we need to know that each of the factors involved in this Quillen metric has a well-defined limit, independant of any choice. This will be done by studying the L^2-metric.

Proposition 5.3.8. *As k goes to infinity, the L^2-metric $\|\cdot\|_{L^2,k}$ on $\lambda_X(E)$ attached to the smooth metrics g_k and h_k converges to a well-defined metric, still on the determinant line bundle $\lambda_X(E)$. This limit metric depends only on the truncated metrics, and not on their smooth approximations.*

Proof. We will need to study separately the L^2-metrics related to g_k and h_k on both Dolbeault cohomology spaces $H^0(X,E)$ and $H^1(X,E)$. There are two separate ways in which the metrics on X and E influence the L^2-metrics on these spaces: the first one relates to the inner-product on differential forms with values in E, while the second one has to do with the identification between the Dolbeault cohomology spaces and the spaces of harmonic forms. We begin by investigating how the kernels of the Dolbeault Laplacians acting on sections of E and on $(0,1)$-forms depend on the (smooth) metrics. For the purpose of this argument only, we denote by g and h two smooth metrics on X and E respectively, and we consider a smooth function φ on X, as well as a smooth section ψ of $\operatorname{End} E$. Let s be a smooth section of E. Using proposition 2.1.40, we have

$$\begin{aligned}
\Delta^{g_\varphi,h_\psi}_{\overline{\partial}_E} s \quad = \quad 0 \quad &\iff \quad e^{-2\varphi}e^{-\psi}\left(e^{-\psi}\right)^*\left(\overline{\partial}_E\right)^*_{g,h}\left(e^{\psi}\right)^* e^{\psi}\,\overline{\partial}_E s \quad = \quad 0 \\
&\iff \quad \left\|e^{\psi}\,\overline{\partial}_E s, s\right\|^2_{L^2(X,E)} \quad = \quad 0 \\
&\iff \quad \left\|\overline{\partial}_E s, s\right\|^2_{L^2(X,E)} \quad = \quad 0 \\
&\iff \quad \Delta^{g,h}_{\overline{\partial}_E} s \quad = \quad 0.
\end{aligned}$$

This proves that the identification of the Dolbeault cohomology space $H^0(X,E)$ with harmonic forms does not depend on the chosen smooth metrics, even though the Dolbeault Laplacian does. Unlike what we saw for the Laplacian on sections, the kernel of the Laplacian acting on $(0,1)$-forms depends on the metric considered on E. We now go back to previous notations, where g and h denote the Poincaré metric on X and the canonical metric on E, respectively. We denote by $\mathscr{H}_k(X,E)$ the space of Dolbeault harmonic sections of E for the metrics g_k and h_k. To sum up the results shown above, we have

$$\mathscr{H}_k(X,E) \quad = \quad \mathscr{H}_0(X,E)\,.$$

We can now give an expression for the L^2-metric on the determinant, meaning the top exterior power, of the 0-th Dolbeault cohomology space. Let s_1 ans s_2 be two elements of $H^0(X,E)$, which we identify to two smooth global harmonic sections $\widetilde{s_1}$ and $\widetilde{s_2}$ of E. We have

$$\begin{aligned}
\langle \widetilde{s_1}, \widetilde{s_2}\rangle^{g_k,h_k}_{L^2,E} \quad &= \quad \langle \widetilde{s_1}, \widetilde{s_2}\rangle^{g_k,h_k}_{L^2,E} \quad = \quad \int_X e^{2\varphi_k(z)}\, h_{\psi_k,z}\left(\widetilde{s_1}(z), \widetilde{s_2}(z)\right\rangle \,\mathrm{d}\mu_{g_0}(z) \\
&= \quad \int_{X_\varepsilon} h_{\psi_0,z}\left(\widetilde{s_1}(z), \widetilde{s_2}(z)\right\rangle \,\mathrm{d}\mu_{g_0}(z) + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} e^{2\varphi_k(z)}\, h_{\psi_k,z}\left(\widetilde{s_1}(z), \widetilde{s_2}(z)\right\rangle \,\mathrm{d}\mu_{g_0}(z)
\end{aligned}$$

using the fact that all φ_k and ψ_k vanish on the compact part of the modular curve. We now note that the sections $\widetilde{s_1}$ and $\widetilde{s_2}$, being harmonic, are constant, in the sense that they can be identified to constant vector-valued functions

$$\widetilde{s_i} \quad : \quad \mathbb{H} \quad \longrightarrow \quad \mathbb{C}^r$$

which are compatible with the representation ρ of Γ. It is of course not necessary that such non-zero sections exist. Restricted to $U_{p,\varepsilon}$, these sections can in particular be seen as functions

$$\widetilde{s_i} \quad : \quad \mathbb{R} \times \left]a\left(\varepsilon\right), +\infty\right[\quad \longrightarrow \quad \mathbb{C}^r$$

which are periodic in the first variable. By assumption, every $\psi_{k,p,j}$ corresponding to a metrically trivial line bundle $L_{p,j}$ vanish, which means that for every integer k and every point $z \in U_{p,\varepsilon}$, we have $e^{\psi_k(z)}\widetilde{s_i}\left(z\right) = \widetilde{s_i}\left(z\right)$. We then have

$$\begin{aligned}
\langle \widetilde{s_1}, \widetilde{s_2} \rangle_{L^2,E}^{g_k,h_k} \quad &= \quad \int_{X_\varepsilon} h_{\psi_0,z}\left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) \\
&\qquad + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} e^{2\varphi_k(z)} \, h_{\psi_0,z}\left(e^{\psi_k(z)}\widetilde{s_1}\left(z\right), e^{\psi_k(z)}\widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) \\
&= \int_{X_\varepsilon} h_{\psi_0,z}\left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} e^{2\varphi_k(z)} \, h_{\psi_0,z}\left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) \\
&= \int_X e^{2\varphi_k(z)} h_{\psi_0,z}\left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right).
\end{aligned}$$

Using the convergence of the sequence φ_k, we then see that the sequence of real numbers $\langle \widetilde{s_1}, \widetilde{s_2} \rangle_{L^2}^{g_k,h_k}$ converges to a real number which does not depend on the particular apparoximation of the truncated metrics. Considering a basis $s_1, \ldots, s_d$ of the cohomology space $H^0\left(X, E\right)$, the metric on the complex line $\det H^0\left(X, E\right)$ is determined by

$$\left\| s_1 \wedge \cdots \wedge s_d \right\|_{L^2}^{g_k,h_k} \quad = \quad \left(\det \left(\langle \widetilde{s_i}, \widetilde{s_m} \rangle_{L^2}^{g_k,h_k} \right)_{i,m} \right)^{1/2} .$$

Using the convergence of each term in the matrix whose determinant is considered, this norm converges to a real number which does not depend on the approximations of the truncated metrics. This strictly positive real number defines the L^2-metric on $\det H^0\left(X, E\right)$ associated to the truncated metrics g_ε and h_ε. We now move on to working on $\det H^1\left(X, E\right)$. For that, we use Serre duality, which gives an antilinear isometry

$$H^1\left(X, E\right) \quad \simeq \quad \left(H^0\left(X, \omega_X \otimes E^*\right)\right)^\vee$$

when both sides are endowed with the L^2-metrics attached to metrics on the tangent bundle and on the vector bundle, with the dual metrics being considered on ω and E^*. Let s_1 and s_2 be two elements of $H^0\left(X, \omega \otimes E^*\right)$, identified to their lifts $\widetilde{s_1}$ and $\widetilde{s_2}$ to the half-plane. We have

$$\begin{aligned}
\langle \widetilde{s_1}, \widetilde{s_2} \rangle_{L^2,\omega\otimes E}^{g_k,h_k} \quad &= \quad \int_X e^{2\varphi_k(z)} \left(g_{\phi_k,z}^* \otimes h_{\psi_k,z}^* \right) \left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) \\
&= \quad \int_{X_\varepsilon} \left(g_{\phi_0,z}^* \otimes h_{\psi_0,z}^* \right) \left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right) \\
&\qquad + \sum_{p \text{ cusp}} \int_{U_{p,\varepsilon}} \left(g_{\phi_0,z}^* \otimes h_{\psi_k,z}^* \right) \left(\widetilde{s_1}\left(z\right), \widetilde{s_2}\left(z\right)\right\rangle \, \mathrm{d}\mu_{g_0}\left(z\right).
\end{aligned}$$

Up to working on a smaller open subset included in $U_{p,\varepsilon}$, we may assume, without loss of generality, that $\widetilde{s_m}$ can be identified to an element $e^*_{p,j} \otimes \omega_m$, where $e^*_{p,1}, \ldots, e^*_{p,r}$ is the dual basis of the chosen basis diagonalizing $\rho(\gamma_p)$, where γ_p is a generator of the stabilizer Γ_p of γ_p in Γ, and ω_m is a holomorphic differential form on $U_{p,\varepsilon}$ such that we have

$$\gamma^* \omega_i = e^{2i\pi\alpha_{p,j}} \omega_m \quad .$$

Using this description, we now have

$$\begin{aligned} &\int_{U_{p,\varepsilon}} \left(g^*_{\phi_0,z} \otimes h^*_{\psi_k,z} \right) \langle \widetilde{s_1}(z), \widetilde{s_2}(z) \rangle \ \mathrm{d}\mu_{g_0}(z) \\ &\qquad = \int_{U_{p,\varepsilon}} e^{-2\psi_{k,p,j}(z)} h^*_{\psi_0,z} \left(e^*_{p,j}(z), e^*_{p,j}(z) \right) \omega_1 \wedge \overline{\omega_2} \end{aligned}$$

This last expression can be seen to converge as k goes to infinity. We can then use this result in turn to show that the L^2-metric on $\det H^0(X, \omega_X \otimes E^*)$, or alternatively on $\det H^1(X, E)$, attached to the metrics g_k and h_k converges to a strictly positive real number, defining the L^2-metric associated to the truncated metrics. This limit metric depends only on the truncated metrics, and not on their smooth approximations.

□

Definition 5.3.9. The limit metric obtained in this last proposition is denoted by $\|\cdot\|_{L^2,\varepsilon}$ and called the *ε-truncated L^2-metric.*

Proposition 5.3.10. *Assume that we have $\alpha_{p,j} < 1$ for all cusps p and all integers $j \in [\![1, r]\!]$. The L^2-metric on $\lambda_X(E)$ associated to the Poincaré metric on X and the canonical metric on E is then well-defined, and we have*

$$\lim_{\varepsilon \to 0^+} \|\cdot\|_{L^2,\varepsilon} = \|\cdot\|_{L^2} \quad .$$

Proof. We begin by proving that the L^2-metric on $H^0(X, E)$ associated to the singular metrics g and h is well-defined, and that we have

$$\langle \cdot, \cdot \rangle^{g_\varepsilon, h_\varepsilon}_{L^2, E} \underset{\varepsilon \to 0^+}{\longrightarrow} \langle \cdot, \cdot \rangle^{g,h}_{L^2, E} \quad .$$

The first of these points amounts to showing that, if s_1 and s_2 are constant global sections of E, then the function

$$z \longmapsto h_z(s_1(z), s_2(z))$$

is integrable on each $U_{p,\varepsilon}$. Using the definition of constant sections, we identify s_1 and s_2 to constant vector-valued functions defined on $\mathbb{H}$ and compatible with the representation ρ. The inner product h_z then becomes the canonical hermitian product on $\mathbb{C}^r$, and the function above is then constant. The result we would like to prove is then equivalent to saying that $U_{p,\varepsilon}$ has finite volume for the Poincaré metric. The convergence that follows can then be restated as saying that we have

$$\mathrm{Vol}(U_{p,\varepsilon}, g_\varepsilon) \underset{\varepsilon \to 0^+}{\longrightarrow} 0 \quad .$$

The volume for the truncated metric can be computed explicitely. We have

$$\mathrm{Vol}(U_{p,\varepsilon}, g_\varepsilon) = \frac{1}{\varepsilon^2 (\log \varepsilon)^2} \cdot \pi \varepsilon^2 = \frac{2\pi}{(\log \varepsilon)^2} \underset{\varepsilon \to 0^+}{\longrightarrow} 0 \quad .$$

This proves the result for $H^0(X,E)$. We now move to proving a similar result for $H^0(\omega\otimes E^*)$, which will involve the assumption we made on $\alpha_{p,j}$. Proving that the L^2-metric associated to g and h on $H^0(\omega\otimes E^*)$ is well-defiend can be reduced to showing that the L^2-product on $U_{p,\varepsilon}$ of functions

$$z \quad\longmapsto\quad \mathrm{d}z\otimes e_{p,j}^*$$

is well-defined. Since the basis $\left(e_{p,j}^*\right)_j$ is orthonormal, we only consider pairs of such functions involving the same integer j. We have

$$\left(g_z^*\otimes h_z^*\right)\left(\mathrm{d}z\otimes e_{p,j}^*,\mathrm{d}z\otimes e_{p,j}^*\right) \quad=\quad g_z^*\left(\mathrm{d}z,\mathrm{d}z\right)h_z^*\left(e_{p,j}^*,e_{p,j}^*\right) \quad=\quad |z|^2\left(\log|z|\right)^2\cdot|z|^{-2\alpha_{p,j}}\,.$$

This function is integrable on $U_{p,\varepsilon}$ for the Poincaré metric if the function

$$z \quad\longmapsto\quad |z|^{-2\alpha_{p,j}}$$

is integrable on the disk of radius ε for the trivial metric on the disk. We have

$$\int_{U_{p,\varepsilon}}\frac{1}{|z|^{2\alpha_{p,j}}}\,\mathrm{d}z \quad=\quad 2\pi\int_0^{\varepsilon}\frac{1}{r^{2\alpha_{p,j}-1}}\,\mathrm{d}r$$

by making a change of variables to polar coordinates. The function appearing above is then integrable if and only if we have $\alpha_{p,j}<1$. Furthermore, the convergence of the ε-truncated metric to this L^2-metric can be obtained by studying

$$\left(g_{\varepsilon,z}^*\otimes h_{\varepsilon,z}^*\right)\left(\mathrm{d}z\otimes e_{p,j}^*,\mathrm{d}z\otimes e_{p,j}^*\right) \quad=\quad \overline{f_2(z)}g_z^*\left(\mathrm{d}z,\mathrm{d}z\right)h_z^*\left(e_{p,j}^*,e_{p,j}^*\right) \quad=\quad \varepsilon^2\left(\log\varepsilon\right)^2\cdot\varepsilon^{-2\alpha_{p,j}}.$$

Integrating on $U_{p,\varepsilon}$ with respect to the truncated Poincaré metric yields

$$\int_{U_{p,\varepsilon}}\frac{1}{\varepsilon^{2\alpha_{p,j}}}\,\mathrm{d}z \quad=\quad \pi\varepsilon^2\cdot\frac{1}{\varepsilon^{2\alpha_{p,j}}} \quad=\quad \pi\varepsilon^{2-2\alpha_{p,j}} \quad\underset{\varepsilon\to0^+}{\longrightarrow}\quad 0$$

having assumed that we have $\alpha_{p,j}<1$. This completes the proof.

□

Proposition-Definition 5.3.11. *As k goes to infinity, the following limit exists*

$$\det{}'\,\Delta_{\overline{\partial}_E}^{g_\varepsilon,h_\varepsilon} \quad=\quad \lim_{k\to+\infty}\det{}'\,\Delta_{\overline{\partial}_E}^{g_k,h_k}\quad,$$

and only depends on the truncated metrics g_ε and h_ε, not on their smooth approximations. It is called the determinant of the Dolbeault Laplacian attached to the truncated metric.

Proof. This result is a direct consequence of the definition of the Quillen metric $\|\cdot\|_{Q,k}$, given for instance in remark 5.3.7, and of propositions 5.3.4 and 5.3.8.

□

Remark 5.3.12. It is important to note that this determinant is not actually the determinant of a Laplacian, as such operators cannot be defined for non-smooth metrics. Furthermore, the existence of this limit is entirely a consequence of the Deligne-Riemann-Roch isometry, and more precisely of a study of its right-hand side, which is arithmetic in nature. This quantity therefore lacks any spectral interpretation.

Remark 5.3.13. This particular result regarding the asymptotic behavior of the (modified) determinants of the Dolbeault Laplacian on the whole modular curve could also be obtained by means of a generalization of the Polyakov formula, for a vector bundle over a closed Riemann surface.

In order to get a Deligne-Riemann-Roch isometry for the singular metrics g and h, the most natural idea would be to let ε go to 0^+ in the truncated isometry presented in theorem 5.3.6. On the left-hand side, this would mean considering the limit of the ε-truncated Quillen metrics. Even though taking such a limit is possible for the L^2-metric on the determinant line bundle, the full Quillen metric diverges, which means a lot of care must be taken. There is also an issue on the right-hand side of the truncated isometry, because g does not have a Sobolev H^1-regularity, and that neither does h, unless it is smooth at every cusp. The aim of chapters 2, 3, and 4 is to study the left-hand side. We will now take care of the arithmetic part, before putting everything together.

5.4 Regularization of the isomorphism

This section will be devoted to the regularization of the right-hand side of theorem 5.3.6. The idea is to modify each factor, in a way that would make sense in family, so as to extract and set aside the singular behavior as ε goes to 0^+. Having done so will then allow us to take the required limit on the "regular" part. Let us recall some of the notations which will be important in this section. Let us recall that we have the following truncated isometry

$$\lambda_X(E)_\varepsilon^{12} \quad \simeq \quad \underbrace{\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon}\rangle^r}_{\text{1st Deligne pairing}} \underbrace{\left\langle \det E_\varepsilon, \det E_\varepsilon \otimes \omega_{X,\varepsilon}^{-1}\right\rangle^6}_{\text{2nd Deligne pairing}} IC_2(X, E_\varepsilon)^{-12}(\mathbb{C}, c(g_\Gamma, r)\,|\cdot|)\,.$$

5.4.1 Regularization of the Deligne pairings

We begin with the two Deligne pairings involved in the truncated isometry, for which we will use the same methods as those presented in [47, Sec. 4.2.3].

First Deligne pairing. The first factor we aim to regularize is the first Deligne pairing referenced above, which involves only the holomorphic cotangent bundle ω_X.

Definition 5.4.1. The *cusp-divisor* D is defined to be

$$D \quad = \quad \sum_{p \text{ cusp}} p \quad .$$

Definition 5.4.2. The line bundle $\mathcal{O}_X(-D)$ on X is defined to be the bundle of holomorphic functions which vanish at the cusps.

Remark 5.4.3. The typical metric considered on $\mathcal{O}_X(-D)$ is the trivial metric. On an open neighborhood $U_{p,\varepsilon}$ of a cusp p, which we see as an open disk of radius ε, this line bundle is generated by the section z given by the chosen local coordinate centered at p, and the trivial metric is given by the modulus. This metric degenerates at the cusps.

Definition 5.4.4. The *ε-truncated trivial metric* on $\mathcal{O}_X(-D)$ is given by the trivial metric on the compact part and on $U_{p,\varepsilon}$ by $|z|_\varepsilon = \varepsilon$.

Proposition 5.4.5. *The ε-truncated trivial metric is a Sobolev H^1-metric on $\mathcal{O}_X(-D)$.*

Remark 5.4.6. The dual of $\mathcal{O}_X(-D)$ is the line bundle, denoted by $\mathcal{O}_X(D)$ of meromorphic functions with a pole of order at most 1 at cusps. It can be endowed with the dual of the trivial metric, or that of its truncated version. We denote by $\mathcal{O}_X(D)_\varepsilon$ and $\mathcal{O}_X(-D)_\varepsilon$ these line bundles when endowed with the truncated metrics.

The main problem we faced with the holomorphic cotangent bundle ω_X was that the (dual of the) Poincaré metric on it around a cusp p was given by

$$g_z^*(\mathrm{d}z, \mathrm{d}z) = (|z| \log |z|)^2 \, |\mathrm{d}z|^2$$

with the usual local coordinated z on $U_{p,\varepsilon}$. This metric is not Sobolev H^1, as we have

$$(|z| \log |z|)^2 = e^{2 \log|z| + 2 \log \log|z|^{-1}}$$

and the function $z \mapsto \log |z|$ is not H^1 on a disk of radius ε. However, the function $z \mapsto \log \log |z|$ being H^1 on such an open disk, we only need to remove the factor $|z|^2$ in the metric ω_X to regularize it. This action of taking away some of the singularity of the metric can be done by transfering it to singularity of the objects we consider. Here, this means that, instead of taking the line bundle of holomorphic differential forms, we will work with the bundle of meromorphic differential forms with poles of order at most 1 at the cusps. More precisely, we consider

$$\omega_X(D) = \omega_X \otimes \mathcal{O}_X(D)$$

endowed with the tensor product metric of the (dual of the) Poincaré metric on ω_X together with the (dual of the) trivial metric on $\mathcal{O}_X(D)$.

Proposition 5.4.7. *The metric on $\omega_X(D)$ has a Sobolev H^1-regularity.*

Proof. Around a cusp, this line bundle is generated by the section $\mathrm{d}z/z$, whose norm is

$$\left\| \tfrac{\mathrm{d}z}{z} \right\|^2 = (\log |z|)^2 \, |\mathrm{d}z|^2 = e^{2 \log \log|z|^{-1}} \, |\mathrm{d}z|^2 \ .$$

As stated above, the function $z \mapsto \log \log |z|$ being H^1 on the punctured disk, the result is proved. $\square$

Remark 5.4.8. Using this proposition, we can define the Deligne pairing metric on the self-intersection $\langle \omega_X(D), \omega_X(D) \rangle$, for which the reader is referred to definition 5.1.18 and the end of remark 5.1.19.

Proposition 5.4.9. *The ε-truncated metric on $\omega_X(D)$ is given around cusps by*

$$\left\| \tfrac{\mathrm{d}z}{z} \right\|_\varepsilon^2 = 2 (\log \varepsilon)^2 \, |\mathrm{d}z|^2 \ .$$

Remark 5.4.10. In the following, we will write $\omega_X(D)_\varepsilon$ when using the ε-truncated metric is considered on the line bundle $\omega_X(D)$, and reserve the notation $\omega_X(D)$ for the singular H^1-metric. It is important to keep mind that the line bundle does not change for these two cases.

Proposition 5.4.11. *As ε goes to 0^+, we have the following convergence of complex lines*

$$\langle \omega_X(D)_\varepsilon, \omega_X(D)_\varepsilon \rangle \underset{\varepsilon \to 0^+}{\longrightarrow} \langle \omega_X(D), \omega_X(D) \rangle \ .$$

Proof. Since the Deligne pairing itself does not depend on ε, this limit is to be understood as a convergence of the relevant Deligne pairing metrics. The result is then a consequence of the end of remark 5.1.19. $\square$

We now want to replace the first factor on the right-hand side of the truncated isometry (see theorem 5.3.6) by $\langle \omega_X(D)_\varepsilon, \omega_X(D)_\varepsilon \rangle$, as this new factor has a well-defined limit as ε goes to 0^+.

Modifying the theorem in that way requires to add and remove some terms, which we then wish to express in simple terms. This is what we turn to.

Proposition 5.4.12. *We have a canonical isometry of complex lines*

$$\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \rangle \quad \simeq \quad \langle \omega_X (D)_\varepsilon , \omega_X (D)_\varepsilon \rangle \langle \omega_{X,\varepsilon}, \mathcal{O}_X (D)_\varepsilon \rangle^{-2} \langle \mathcal{O}_X (D)_\varepsilon , \mathcal{O}_X (D)_\varepsilon \rangle^{-1} \ .$$

Proof. This is a direct consequence of the (metric) bimultiplicativity of the Deligne pairing, recalled in propositions 5.1.17 and 5.1.20.

□

Remark 5.4.13. As always in this chapter, the word "canonical" here means that we would have such an isometry in families, the objects being in this case line bundles over the base instead of complex lines. When limiting ourselves to particular arithmetic surfaces whose set of complex points would yield modular curves, we had to assume the existence of sections $\sigma_1, \ldots, \sigma_h$ giving the cusps to make sense of the line bundle $\mathcal{O}_X (D)$ in family.

Looking at proposition 5.4.12, we now wish to compute more explicitely the last two factors of the right-hand side, in order to understand their behavior as ε goes to 0^+, which must be singular. To do that, we will proceed as in [47, Sec. 4.1.2], and use [14].

Definition 5.4.14. Let p be a cusp. The *current of integration againt the circle* $\partial U_{p,\varepsilon}$, denoted by $\delta_{\partial U_{p,\varepsilon}}$, is defined by

$$\delta_{\partial U_{p,\varepsilon}} f \quad = \quad \int_0^{2\pi} f \left(\varepsilon e^{i\vartheta} \right) \, \mathrm{d}\vartheta \ ,$$

for any smooth function f.

Proposition 5.4.15. *We have the following equality of currents*

$$c_1 (\omega_{X,\varepsilon}) \quad = \quad c_1 \left(\omega_{X \setminus \{p \ cusp\}} \right) \mathbb{1}_{X_\varepsilon} - \tfrac{1}{2\pi} \left(1 - \tfrac{1}{\log \varepsilon^{-1}} \right) \sum_{p \ cusp} \delta_{\partial U_{p,\varepsilon}} \ ,$$

where $\mathbb{1}_{X_\varepsilon}$ *is the characteristic function of the compact part* X_ε *of the modular curve* X.

Proof. Let χ be a smooth function, which vanishes on an open neighborhood of each $\overline{U_{p,\varepsilon}}$, and is constant equal to 1 on an open neighborhood of $X_{2\varepsilon}$. We further consider a global meromorphic differential form ω on X, which has neither zeros nor poles on the open subsets $U_{p,2\varepsilon}$. The first Chern current $c_1 (\omega_{X,\varepsilon})$ is then represented by

$$c_1 (\omega_{X,\varepsilon}) \quad = \quad -\tfrac{1}{2i\pi} \partial\overline{\partial} \log \|\omega\|_\varepsilon^{-2} + \delta_{\operatorname{div} \omega} \ ,$$

where $\|\cdot\|_\varepsilon$ denotes the truncated metric on ω_X. Let f be a smooth function. After applying the first Chern current to f, we get

$$\begin{aligned} \langle c_1 (\omega_{X,\varepsilon}) \, , \, f \rangle \quad &= \quad \langle c_1 (\omega_{X,\varepsilon}) \, , \, \chi f \rangle + \langle c_1 (\omega_{X,\varepsilon}) \, , \, (1-\chi) \, f \rangle \\ &= \quad \langle c_1 (\omega_{X,\varepsilon}) \, , \, \chi f \rangle - \tfrac{1}{2i\pi} \sum_{p \text{ cusp}} \int_{U_{p,2\varepsilon}} \log \|\omega\|_\varepsilon^{-2} \, \partial\overline{\partial} \, (1-\chi) \, f, \end{aligned}$$

using the vanishing of $1-\chi$ on $X_{2\varepsilon}$. We will compute each of the integrals appearing above. For any cusp p, we have

$$\begin{aligned} \int_{U_{p,2\varepsilon}} \log \|\omega\|_\varepsilon^{-2} \, \partial\overline{\partial} \, (1-\chi) \, f \quad = \quad & \int_{U_{p,2\varepsilon} \setminus U_{p,\varepsilon}} \log \|\omega\|_\varepsilon^{-2} \, \partial\overline{\partial} \, (1-\chi) \, f \\ & + \int_{U_{p,\varepsilon}} \log \|\omega\|_\varepsilon^{-2} \, \partial\overline{\partial} \, (1-\chi) \, f. \end{aligned}$$

The integral over $U_{p,\varepsilon}$ can be computed more easily than the other. For that, we note that the differential form ω can be written as

$$\omega(z) \quad = \quad u(z)\,\mathrm{d}z$$

over $U_{p,2\varepsilon}$, where u is a holomorphic function on $U_{p,2\varepsilon}$. Up to reducing ε, we may assume that u does not vanish on $U_{p,2\varepsilon}$, except possibly at the cusp itself. We have

$$\begin{aligned}\int_{U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\partial\overline{\partial}(1-\chi)f \quad &= \quad \int_{U_{p,\varepsilon}} \log|u(z)|^{-2}\,\partial\overline{\partial}(1-\chi)f \\ &\qquad + \int_{U_{p,\varepsilon}} \log\|\mathrm{d}z\|_\varepsilon^{-2}\,\partial\overline{\partial}(1-\chi)f.\end{aligned}$$

As we will now see, the first of these integrals does not play any role, while the second one can be computed quite explicitely. We have

$$\int_{U_{p,\varepsilon}} \log\|\mathrm{d}z\|_\varepsilon^{-2}\,\partial\overline{\partial}(1-\chi)f \quad = \quad -\left(\log 2 + 2\log\varepsilon + 2\log\log\varepsilon^{-1}\right)\int_{\partial U_{p,\varepsilon}} \overline{\partial}f,$$

using Stokes' formula to get the last equality, as well as the vanishing of χ. This term will be later canceled. We now have

$$\begin{aligned}\int_{U_{p,\varepsilon}} \log|u(z)|^{-2}\,\partial\overline{\partial}(1-\chi)f \quad &= \quad \int_{U_{p,\varepsilon}} \log|u(z)|^{-2}\,\partial\overline{\partial}f \\ &= \int_{U_{p,\varepsilon}} \partial\left(\log|u(z)|^{-2}\,\overline{\partial}f\right) - \int_{U_{p,\varepsilon}} \partial\log|u(z)|^{-2}\wedge\overline{\partial}f \\ &= \int_{\partial U_{p,\varepsilon}} \log|u(z)|^{-2}\,\overline{\partial}f + \int_{U_{p,\varepsilon}} \overline{\partial}\left(f\partial\log|u(z)|^{-2}\right) - \cancel{\int_{U_{p,\varepsilon}} f\overline{\partial}\partial\log|u(z)|^{-2}} \\ &= \int_{\partial U_{p,\varepsilon}} \log|u(z)|^{-2}\,\overline{\partial}f + \int_{\partial U_{p,\varepsilon}} f\partial\log|u(z)|^{-2},\end{aligned}$$

using once again Stoke's formula, and the fact that u is holomorphic, and non-vanishing (except possibly at the cusp) to cancel the $\overline{\partial}\partial$ term. Now, we have

$$\begin{aligned}&\int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\partial\overline{\partial}(1-\chi)f \\ &\quad= \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} \partial\left(\log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}(1-\chi)f\right) - \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} \partial\log\|\omega\|_\varepsilon^{-2}\wedge\overline{\partial}(1-\chi)f \\ &\quad= \cancel{\int_{\partial U_{p,2\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}(1-\chi)f} - \int_{\partial U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}\underbrace{(1-\chi)}_{=1}f \\ &\qquad\quad + \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} \overline{\partial}\left((1-\chi)f\partial\log\|\omega\|_\varepsilon^{-2}\right) - \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi)f\overline{\partial}\partial\log\|\omega\|_\varepsilon^{-2} \\ &\quad= -\int_{\partial U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}f + \int_{\partial U_{p,2\varepsilon}} \underbrace{(1-\chi)}_{=0}f\partial\log\|\omega\|_\varepsilon^{-2} \\ &\qquad\qquad - \int_{\partial U_{p,\varepsilon}} \underbrace{(1-\chi)}_{=1}f\partial\log\|\omega\|_\varepsilon^{-2} - \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi)f\overline{\partial}\partial\log\|\omega\|_\varepsilon^{-2} \\ &\quad= -\int_{\partial U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}f - \int_{\partial U_{p,\varepsilon}} f\partial\log\|\omega\|_\varepsilon^{-2} - \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi)f\overline{\partial}\partial\log\|\omega\|_\varepsilon^{-2}.\end{aligned}$$

We can now put these last few computations together, which yields

$$\begin{aligned}&\int_{U_{p,2\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\partial\overline{\partial}(1-\chi)f \\ &\quad= -\left(\log 2 + 2\log\varepsilon + 2\log\log\varepsilon^{-1}\right)\int_{\partial U_{p,\varepsilon}} \overline{\partial}f + \int_{\partial U_{p,\varepsilon}} \log|u(z)|^{-2}\,\overline{\partial}f \\ &\qquad + \int_{\partial U_{p,\varepsilon}} f\partial\log|u(z)|^{-2} - \int_{\partial U_{p,\varepsilon}} \log\|\omega\|_\varepsilon^{-2}\,\overline{\partial}f - \int_{\partial U_{p,\varepsilon}} f\partial\log\|\omega\|_\varepsilon^{-2} \\ &\qquad - \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi)f\overline{\partial}\partial\log\|\omega\|_\varepsilon^{-2}.\end{aligned}$$

We now recall that we have $\overline{\partial}\partial = -\partial\overline{\partial}$, as well as

$$\log \|\omega\|_{\varepsilon}^{-2} \quad = \quad -\log 2 - 2\log|z| - 2\log\log|z|^{-1} + \log|u(z)|^{-2}$$

on $U_{p,2\varepsilon} \setminus U_{p,\varepsilon}$, which gives

$$\partial \log \|\omega\|_{\varepsilon}^{-2} \quad = \quad -\left(\tfrac{1}{z} + \tfrac{1}{z\log|z|}\right) \mathrm{d}z + \partial \log|u(z)|^{-2} \ .$$

This gives

$$\begin{aligned}
&\int_{U_{p,2\varepsilon}} \log\|\omega\|_{\varepsilon}^{-2}\ \partial\overline{\partial}(1-\chi) f \\
&\qquad = \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi) f \partial\overline{\partial} \log \|\omega\|_{\varepsilon}^{-2} + \left(1 + \tfrac{1}{\log\varepsilon}\right) \int_{\partial U_{p,\varepsilon}} f \tfrac{\mathrm{d}z}{z},
\end{aligned}$$

and we finally have

$$\begin{aligned}
&\langle c_1(\omega_{X,\varepsilon}), f\rangle \\
&= \langle c_1(\omega_{X,\varepsilon}), \chi f\rangle - \tfrac{1}{2i\pi} \sum_{p \text{ cusp}} \int_{U_{p,2\varepsilon}\setminus U_{p,\varepsilon}} (1-\chi) f \partial\overline{\partial} \log\|\omega\|_{\varepsilon}^{-2} \\
&\qquad - \tfrac{1}{2i\pi}\left(1 + \tfrac{1}{\log\varepsilon}\right) \sum_{p \text{ cusp}} \int_{\partial U_{p,\varepsilon}} f \tfrac{\mathrm{d}z}{z} \\
&= \langle c_1(\omega_{X,\varepsilon}) \mathbb{1}_{X_\varepsilon}, \chi f\rangle + \langle c_1(\omega_{X,\varepsilon}) \mathbb{1}_{X_\varepsilon}, (1-\chi) f\rangle - \tfrac{1}{2i\pi}\left(1 + \tfrac{1}{\log\varepsilon}\right) \sum_{p \text{ cusp}} \int_{\partial U_{p,\varepsilon}} f \tfrac{\mathrm{d}z}{z} \\
&= \langle c_1(\omega_{X,\varepsilon}) \mathbb{1}_{X_\varepsilon}, f\rangle - \tfrac{1}{2\pi}\left(1 + \tfrac{1}{\log\varepsilon}\right) \sum_{p \text{ cusp}} \int_0^{2\pi} f\left(\varepsilon e^{i\vartheta}\right) \mathrm{d}\vartheta \\
&= \left\langle c_1(\omega_{X,\varepsilon}) \mathbb{1}_{X_\varepsilon} - \tfrac{1}{2\pi}\left(1 - \tfrac{1}{\log\varepsilon^{-1}}\right) \sum_{p \text{ cusp}} \delta_{\partial U_{p,\varepsilon}}, f \right\rangle.
\end{aligned}$$

This completes the proof of the proposition.

□

Remark 5.4.16. The important part of this last proposition is that the first Chern current for the truncated metric is supported on X_ε, which by convention includes the boundary of the open neighborhoods of the cusps.

Proposition 5.4.17. *For any cusp p, we have a canonical isometry of complex lines*

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X(D)_\varepsilon \rangle \quad \simeq \quad \bigotimes_{p \text{ cusp}} \left(\omega_{X,p}, \sqrt{2}\,\varepsilon \log\varepsilon^{-1} |\cdot|_p\right) \ ,$$

where the norm $|\cdot|_p$ on $\omega_{X,p}$ is defined by $|\mathrm{d}z_p|_p = 1$ around a cusp p.

Proof. Using the metric bimultiplicativity of the Deligne pairing, we have

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X(D)_\varepsilon \rangle \quad \simeq \quad \bigotimes_{p \text{ cusp}} \langle \omega_{X,\varepsilon}, \mathcal{O}_X(p)_\varepsilon \rangle \ ,$$

which we can treat each cusp separately. We will thus prove that we have a canonical isometry

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X(p)_\varepsilon \rangle \quad \simeq \quad \left(\omega_{X,p}, \sqrt{2}\,\varepsilon \log\varepsilon^{-1} |\cdot|_p\right) \ ,$$

with $|\cdot|_p$ is defined as in the statement of the proposition. The canonical isomorphism (*i.e.* coming from an isomorphism in family) mentioned here results from the version of proposition 5.1.21 described in remark 5.1.22. Proving the proposition is then simply a matter of comparing the metrics. To do that, we need to consider a particular global meromorphic section of $\mathcal{O}_X(p)$, for instance the one given by the constant function which equals 1, denoted by $\mathbb{1}_p$. We have

$$\operatorname{div} \mathbb{1}_p \quad = \quad p \ ,$$

as we can see from writing the constant function 1 in the canonical trivialization of $\mathcal{O}_X(p)$ near p. Denoting by $\|\cdot\|_\varepsilon$ the truncated metric on $\mathcal{O}_X(p)$, we have

$$\log \|\mathbb{1}_p\|_\varepsilon \quad = \quad \begin{cases} 0 & \text{on} \quad X \setminus U_{p,\varepsilon} \\ \log \frac{|z|}{\varepsilon} & \text{on} \quad U_{p,\varepsilon} \end{cases} \ ,$$

where z denotes the usual coordinate on $U_{p,\varepsilon}$ resulting from its description as an open disk of radius ε. Since the first Chern current $c_1(\omega_{X,\varepsilon})$ is supported in X_ε and $\log \|\mathbb{1}_p\|_\varepsilon$ is supported in its complement, we have

$$\int_X \log \|\mathbb{1}_p\|_\varepsilon^{-2} c_1(\omega_{X,\varepsilon}) \quad = \quad 0 \ .$$

We now consider a global meromorphic differential form ω which can be written as

$$\omega(z) \quad = \quad u(z)\,\mathrm{d}z$$

above $U_{p,\varepsilon}$, with u holomorphic and satisfying $u(0) = 1$. We have the isomorphism

$$\begin{array}{ccc} \langle \omega_{X,\varepsilon}, \mathcal{O}_X(p)_\varepsilon \rangle & \simeq & \omega_{X,p} \\ \langle \omega, \mathbb{1}_p \rangle & \longmapsto & \mathrm{d}z_p \end{array} \ ,$$

and we thus only need to compute the Deligne pairing norm of $\langle \omega, \mathbb{1}_p \rangle$. Using the beginning of remark 5.1.19, we have

$$\log \|\langle \omega, \mathbb{1}_p \rangle\|^{-2} \quad = \quad \cancel{\int_X \log \|\mathbb{1}_p\|_\varepsilon^{-2} c_1(\omega_{X,\varepsilon})} + \log \|\omega\|_\varepsilon^{-2}(p) \quad = \quad -\log\left(2(\varepsilon \log \varepsilon)^2\right) \ ,$$

which completes the proof of the proposition, as it gives

$$\|\langle \omega, \mathbb{1}_p \rangle\| \quad = \quad \sqrt{2}\varepsilon \log \varepsilon^{-1} \ .$$

□

Proposition 5.4.18. *Let p be a cusp. We have the following equality of currents*

$$c_1(\mathcal{O}_X(p)) \quad = \quad \tfrac{1}{2\pi} \delta_{\partial U_{p,\varepsilon}} \ .$$

Proof. The method we use here is similar to the one we used in the proof of proposition 5.4.15. We consider a smooth function χ which equals 1 on an open neighborhood of $X \setminus U_{p,2\varepsilon}$, and vanishes on an open neighborhood of $\overline{U_{p,\varepsilon}}$. The first Chern current $c_1(\mathcal{O}_X(p)_\varepsilon)$ is represented by

$$c_1(\mathcal{O}_X(p)_\varepsilon) \quad = \quad -\tfrac{1}{2i\pi} \partial\overline{\partial} \log \|\mathbb{1}_p\|_\varepsilon^{-2} + \delta_p \ .$$

Let f be a smooth function on X. Applying the appropriate first Chern current to f yields

$$\langle c_1 (\mathcal{O}_X (p)_\varepsilon), f\rangle \quad = \quad \tfrac{1}{i\pi} \left\langle \partial\overline{\partial} \log \|\mathbb{1}_p\|_\varepsilon , f \right\rangle + f(p) \quad = \quad \tfrac{1}{i\pi} \int_X \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f + f(p) \, .$$

We can now use the fact that $\log \|\mathbb{1}_p\|_\varepsilon$ vanishes everywhere except on $U_{p,\varepsilon}$ to simplify the computation of the integral. We thus have

$$\langle c_1 (\mathcal{O}_X (p)_\varepsilon), f\rangle \quad = \quad \tfrac{1}{i\pi} \int_{U_{p,\varepsilon}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f + f(p) \ \ .$$

It is worth noting that the logarithmic singularity of $\log \|\mathbb{1}_p\|_\varepsilon$ at the cusp p does not prevent integrability here. However, since we wish to integrate by parts, we need to write

$$\tfrac{1}{i\pi} \int_{U_{p,\varepsilon}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f \quad = \quad \tfrac{1}{i\pi} \lim_{\eta \to 0^+} \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f \ \ ,$$

and work instead with the integral appearing within the limit. We have

$$\begin{aligned}
\int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f \quad &= \quad \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \partial \left(\log \|\mathbb{1}_p\|_\varepsilon \, \overline{\partial} f \right) - \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \partial \log \|\mathbb{1}_p\|_\varepsilon \wedge \overline{\partial} f \\
&= \quad \int_{\partial U_{p,\varepsilon}} \underbrace{\log \|\mathbb{1}_p\|_\varepsilon}_{=0} \overline{\partial} f - \int_{\partial U_{p,\eta}} \underbrace{\log \|\mathbb{1}_p\|_\varepsilon}_{=\log \frac{\eta}{\varepsilon}} \overline{\partial} f + \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \overline{\partial} \left(f \partial \log \|\mathbb{1}_p\|_\varepsilon \right) \\
&\qquad - \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} f \overline{\partial} \partial \log \|\mathbb{1}_p\|_\varepsilon \\
&= \quad - \log \tfrac{\eta}{\varepsilon} \int_{\partial U_{p,\eta}} \overline{\partial} f + \int_{\partial U_{p,\varepsilon}} f \partial \log \|\mathbb{1}_p\|_\varepsilon - \int_{\partial U_{p,\eta}} f \partial \log \|\mathbb{1}_p\|_\varepsilon - \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} f \overline{\partial} \partial \log \|\mathbb{1}_p\|_\varepsilon \\
&= \quad - \log \tfrac{\eta}{\varepsilon} \int_{\partial U_{p,\eta}} \overline{\partial} f + \int_{\partial U_{p,\varepsilon}} f \partial \log \|\mathbb{1}_p\|_\varepsilon - \int_{\partial U_{p,\eta}} f \partial \log \|\mathbb{1}_p\|_\varepsilon + \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} f \partial \overline{\partial} \log \|\mathbb{1}_p\|_\varepsilon \, .
\end{aligned}$$

We now note that, on $U_{p,\varepsilon} \setminus U_{p,\eta}$, we have

$$\partial \log \|\mathbb{1}_p\|_\varepsilon \quad = \quad \tfrac{\mathrm{d}z}{2z} \ \ ,$$

which then gives

$$\int_{\partial U_{p,\varepsilon}} f \partial \log \|\mathbb{1}_p\|_\varepsilon - \int_{\partial U_{p,\eta}} f \partial \log \|\mathbb{1}_p\|_\varepsilon \quad = \quad \tfrac{i}{2} \left[\int_0^{2\pi} f \left(\varepsilon e^{i\vartheta} \right) \mathrm{d}\vartheta - \int_0^{2\pi} f \left(\eta e^{i\vartheta} \right) \mathrm{d}\vartheta \right] \ .$$

Using this, we get

$$\begin{aligned}
\int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f \quad &= \quad - \log \tfrac{\eta}{\varepsilon} \int_{\partial U_{p,\eta}} \overline{\partial} f + \tfrac{i}{2} \int_0^{2\pi} f \left(\varepsilon e^{i\vartheta} \right) \mathrm{d}\vartheta - \tfrac{i}{2} \int_0^{2\pi} f \left(\eta e^{i\vartheta} \right) \mathrm{d}\vartheta \\
&\qquad + \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} f \partial \overline{\partial} \log \|\mathbb{1}_p\|_\varepsilon \, .
\end{aligned}$$

Noting that we have $\partial\overline{\partial} \log \|\mathbb{1}_p\|_\varepsilon = 0$ on the open subset $U_{p,\varepsilon} \setminus U_{p,\eta}$, and taking the limit as η goes to 0^+, we have

$$\tfrac{1}{i\pi} \lim_{\eta \to 0^+} \int_{U_{p,\varepsilon} \setminus U_{p,\eta}} \log \|\mathbb{1}_p\|_\varepsilon \, \partial\overline{\partial} f \quad = \quad \tfrac{1}{2\pi} \int_0^{2\pi} f \left(\varepsilon e^{i\vartheta} \right) \mathrm{d}\vartheta - f(0) \, .$$

Here, we need to recall that the local coordinate z on $U_{p,\varepsilon}$ is centered at the cusp p, so that $f(0)$ is to be understood as $f(p)$. We thus have

$$\langle c_1 (\mathcal{O}_X (p)_\varepsilon), f\rangle \quad = \quad \tfrac{1}{2\pi} \int_0^{2\pi} f \left(\varepsilon e^{i\vartheta} \right) \mathrm{d}\vartheta - f(p) + f(p) \quad = \quad \tfrac{1}{2\pi} \left\langle \delta_{\partial U_{p,\varepsilon}}, f \right\rangle \ \ ,$$

thereby proving the proposition.

□

Remark 5.4.19. Here again, the important point is that the first Chern current $c_1\left(\mathcal{O}_X(p)_\varepsilon\right)$ is supported on the circle $\partial U_{p,\varepsilon}$, and thus outside the open subset $U_{p,\varepsilon}$.

Proposition 5.4.20. *We have a canonical isometry of complex lines*

$$\left\langle \mathcal{O}_X(p)_\varepsilon, \mathcal{O}_X(q)_\varepsilon \right\rangle \quad \simeq \quad (\mathbb{C}, |\cdot|) \ ,$$

where $|\cdot|$ denotes the usual complex modulus, for any two distinct cusps p and q.

Proof. We first note that the two canonical sections $\mathbb{1}_p$ and $\mathbb{1}_q$ of $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$, respectively, have disjoint divisors, since the cusps p and q are themselves distinct. We now use the canonical isomorphism provided in proposition 5.1.21

$$\begin{array}{ccc} \left\langle \mathcal{O}_X(p), \mathcal{O}_X(q)\right\rangle & \simeq & \mathbb{C} \\ \left\langle \mathbb{1}_p, \mathbb{1}_q \right\rangle & \longmapsto & 1 \end{array} .$$

Proving the result is then only a matter of computing the Deligne pairing norm of $\left\|\left\langle \mathbb{1}_p, \mathbb{1}_q\right\rangle\right\|_\varepsilon$ associated to the truncated metrics. The first Chern current $c_1\left(\mathcal{O}_X(p)_\varepsilon\right)$ being supported on the boundary of $U_{p,\varepsilon}$, we have

$$\int_X \log \left\|\mathbb{1}_q\right\|_\varepsilon^{-2} c_1\left(\mathcal{O}_X(p)_\varepsilon\right) \quad = \quad 0 \ ,$$

since we assumed ε to be small enough so that $U_{p,\varepsilon}$ and $U_{q,\varepsilon}$ are far apart from one another. The Deligne pairing norm we wish to compute is then given by

$$\log \left\|\left\langle \mathbb{1}_p, \mathbb{1}_q\right\rangle\right\|_\varepsilon^{-2} \quad = \quad \log \left\|\mathbb{1}_p\right\|_\varepsilon^{-2}(q) \quad = \quad 0 \ .$$

This completes the proof.

□

Proposition 5.4.21. *For any cusp p, we have a canonical isometry of complex lines*

$$\left\langle \mathcal{O}_X(p)_\varepsilon, \mathcal{O}_X(p)_\varepsilon \right\rangle \quad \simeq \quad \left(\omega_{X,p}, \varepsilon\, |\cdot|_p\right)^{-1} \ ,$$

where the norm $|\cdot|_p$ on $\omega_{X,p}$ is defined by $\left|\mathrm{d}z_p\right|_p = 1$ around a cusp p.

Proof. The idea here is to use remark 5.1.23, which gives a canonical isomorphism

$$\begin{array}{ccc} \left\langle \omega_X(p), \mathcal{O}_X(p)\right\rangle & \simeq & \mathbb{C} \\ \left\langle \omega_p, \mathbb{1}_p \right\rangle & \longmapsto & 1 \end{array} ,$$

where ω_p is a global meromorphic form on X with a single pole and residue 1 at the cusp p. Up to reducing ε, we assume that ω_p has no other pole or zero in $U_{p,\varepsilon}$. The divisor of ω_p, seen as a section of $\omega_X(p)$, does not involve p, and is thus disjoint from that of $\mathbb{1}_p$. Using propositions 5.4.15 and 5.4.18, we have the equality of currents

$$c_1\left(\omega_X(p)_\varepsilon\right) \quad = \quad c_1\left(\omega_{X\setminus\{p \text{ cusp}\}}\right) \mathbb{1}_{X_\varepsilon} - \tfrac{1}{2\pi}\left(1 - \tfrac{1}{\log \varepsilon^{-1}}\right) \sum_{q \neq p \text{ cusp}} \delta_{\partial U_{q,\varepsilon}} + \tfrac{1}{2\pi \log \varepsilon^{-1}} \delta_{\partial U_{p,\varepsilon}} \ ,$$

which means in particular that the first Chern current of $\omega_X(p)_\varepsilon$ is supported outside $U_{p,\varepsilon}$. This means that we have

$$\int_X \log \|\mathbb{1}_p\|_\varepsilon^{-2} c_1(\omega_X(p)_\varepsilon) \quad = \quad 0 \ .$$

The Deligne pairing norm of $\langle \omega_p, \mathbb{1}_p \rangle$ is then given by

$$\log \|\langle \omega_p, \mathbb{1}_p \rangle\|^{-2} \quad = \quad \log \|\omega\|_\varepsilon^{-2}(p) \quad = \quad -\log 2 - 2\log\log\varepsilon^{-1} \ .$$

This gives a canonical isometry

$$\langle \omega_X(p)_\varepsilon, \mathcal{O}_X(p)_\varepsilon \rangle \quad \simeq \quad \left(\mathbb{C}, \tfrac{1}{\sqrt{2}\log\varepsilon^{-1}} |\cdot|\right) \ ,$$

with $|\cdot|$ being the usual complex modulus. Combining this with the isometry

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X(p)_\varepsilon \rangle \quad \simeq \quad \left(\omega_{X,p}, \sqrt{2}\varepsilon\log\varepsilon^{-1} |\cdot|_p\right)$$

proved in proposition 5.4.17, and using the metric bimultiplicativity of the Deligne pairing, we get the required result.

□

These last two results then immediatly yield the following.

Proposition 5.4.22. *We have a canonical isometry of complex lines*

$$\langle \mathcal{O}_X(D)_\varepsilon, \mathcal{O}_X(D)_\varepsilon \rangle \quad \simeq \quad \bigotimes_{p \ cusp} \left(\omega_{X,p}, \varepsilon |\cdot|_p\right)^{-1} \ .$$

Proof. Recalling that the cusp-divisor D is defined by

$$D \quad = \quad \sum_{p \text{ cusp}} p \ ,$$

we can use the metric bimultiplicativity of the Deligne pairing to get a canonical isometry

$$\langle \mathcal{O}_X(D)_\varepsilon, \mathcal{O}_X(D)_\varepsilon \rangle \quad \simeq \quad \left(\bigotimes_{p \text{ cusp}} \langle \mathcal{O}_X(p)_\varepsilon, \mathcal{O}_X(p)_\varepsilon \rangle\right) \otimes \left(\bigotimes_{p \neq q \text{ cusps}} \langle \mathcal{O}_X(p)_\varepsilon, \mathcal{O}_X(q)_\varepsilon \rangle\right) \ .$$

Propositions 5.4.20 and 5.4.21 then give a canonical isometry

$$\langle \mathcal{O}_X(D)_\varepsilon, \mathcal{O}_X(D)_\varepsilon \rangle \quad \simeq \quad \bigotimes_{p \text{ cusp}} \left(\omega_{X,p}, \varepsilon |\cdot|_p\right)^{-1} \ ,$$

as was required in the statement of this proposition.

□

We now have all the necessary tools to regularize the first Deligne pairing appearing in the truncated Deligne-Riemann-Roch isometry. This is summed up in the following theorem.

Theorem 5.4.23. *We have a canonical isometry of complex lines*

$$\langle \omega_{X,\varepsilon}, \omega_{X,\varepsilon} \rangle \quad \simeq \quad \langle \omega_X(D)_\varepsilon, \omega_X(D)_\varepsilon \rangle \otimes \bigotimes_{p \ cusp} \left(\omega_{X,p}, |\cdot|_p\right)^{-1} \otimes \left(\mathbb{C}, R_\omega(\varepsilon) |\cdot|\right) \ ,$$

where $|\cdot|$ denotes the usual modulus on $\mathbb{C}$, and $R_\omega(\varepsilon)$ is defined by

$$\log R_\omega(\varepsilon) \quad = \quad -h\log 2 - h\log\varepsilon - 2h\log\log\varepsilon^{-1} \quad ,$$

where h is the number of cusps.

Proof. This result is a direct combination of propositions 5.4.12, as well as 5.4.17 and 5.4.22. □

Second Deligne pairing. We now turn to the second Deligne pairing appearing in the right-hand side of the truncated Deligne-Riemann-Roch isometry, which involves the determinant of the vector bundle E. Let us first recall a few facts about E. Around a cuso p, we have a decomposition

$$E_{|U_{p,\varepsilon}} \quad = \quad \bigoplus_{j=1}^{r} L_{p,j} \quad ,$$

each $L_{p,j}$ being a line bundle over $U_{p,\varepsilon}$, trivialized by a section s_j with

$$\|s_{p,j}\|_z^2 \quad = \quad |z|^{2m_{p,j}} \quad ,$$

with $\alpha_{p,j}$ begin a rational number in $[0,1[$. We write

$$\alpha_{p,j} \quad = \quad \tfrac{m_{p,j}}{m} \quad ,$$

where m is a strictly positive integer independant of either p or j, and $k_{p,j} \in [\![0, m-1]\!]$ is an integer. Around the cusp p, the determinant of E is triviliazed by the section $s_{p,1}\wedge\cdots\wedge s_{p,r}$, whose norm is given by

$$\|s_{p,1}\wedge\cdots\wedge s_{p,r}\|_z^2 \quad = \quad |z|^{2\sum\limits_{j=1}^{r}\alpha_{p,j}} \quad .$$

The presence of denominators will force us to elevate the truncated Deligne-Riemann-Roch isometry to the power m^2 in order to regularize the second Deligne pairing.

Definition 5.4.24. We define the *vector bundle divisor* D' to be

$$D' \quad = \quad \sum_{p\text{ cusp}} \left(\sum_{j=1}^{r} m_{p,j}\right) p \quad .$$

Definition 5.4.25. The line bundle $\mathcal{O}_X(-D')$ on X is defined to be the bundle of holomorphic functions with a zero at each cusp p of order at least

$$m_p \quad = \quad \sum_{j=1}^{r} m_{p,j} \quad .$$

Remark 5.4.26. The most natural metric on $\mathcal{O}_X(-D')$ is the trivial metric, which degenerates at the cusps, all integers $k_{p,j}$ vanish.

Definition 5.4.27. The *ε-truncated trivial metric* on $\mathcal{O}_X(-D')$ is given by the trivial metric on the compact part X_ε and on $U_{p,\varepsilon}$ by

$$|z^{m_p}|_\varepsilon \quad = \quad \varepsilon^{m_p} \quad .$$

Proposition 5.4.28. *The ε-truncated trivial metric is a Sobolev H^1-metric on $\mathcal{O}_X(-D')$.*

Remark 5.4.29. The dual of $\mathcal{O}_X(-D')$ is the line bundle $\mathcal{O}_X(D')$ of meromorphic functions with a pole of order at most m_p at the cusp p. It can be endowed with the dual of the trivial metric, or that of its truncated version. We denote these line bundles by $\mathcal{O}_X(-D')_\varepsilon$ and $\mathcal{O}_X(-D')_\varepsilon$ when endowed with the truncated metrics.

Similarly to what happened with the holomorphic cotangent bundle ω_X, the main problem we face with the determinant of the vector bundle E is that the determinant of the canonical metric on it degenerates (in general) at the cusps. To remedy that problem, we will consider the vector bundle

$$(\det E)^m(D') \quad = \quad (\det E)^m \otimes \mathcal{O}_X(D') \ ,$$

endowed with the m-th power of the determinant of the canonical metric on E tensored by the trivial metric on $\mathcal{O}_X(D')$.

Proposition 5.4.30. *The metric on the line bundle $(\det E)^m(D')$ is smooth.*

Proof. Around a cusp, the line bundle is generated by the section $(s_{p,1}\wedge\cdots\wedge s_{p,r})^{\otimes m}\otimes\frac{1}{z^{m_p}}$, whose norm is given by

$$\left\|(s_{p,1}\wedge\cdots\wedge s_{p,r})^{\otimes m}\otimes\tfrac{1}{z^{m_p}}\right\|_z^2 \quad = \quad \|s_{p,1}\wedge\cdots\wedge s_{p,r}\|_z^{2m}\cdot\tfrac{1}{|z|^{2m_p}} \quad = \quad 1 \ .$$

□

Remark 5.4.31. Using this last proposition, and the Sobolev H^1-regularity of the tensor product metric on $\omega_X(D)$, we can define the Deligne pairing metric on

$$\langle(\det E)^m(D'),(\det E)^m(D')\rangle\left\langle(\det E)^m(D'),\omega_X(D)^{-1}\right\rangle^m \ .$$

Remark 5.4.32. The truncation process we perform on the different metrics we consider commutes with tensor products, and with taking the determinant, meaning the top exterior power. It should be noted that the divisors D and D' are not the same, and that both are involved here.

Proposition 5.4.33. *As ε goes to 0^+, we have the following convergence of complex lines*

$$\langle(\det E)^m(D')_\varepsilon,(\det E)^m(D')_\varepsilon\rangle\left\langle(\det E)^m(D')_\varepsilon,\omega_X(D)_\varepsilon^{-1}\right\rangle^m$$
$$\underset{\varepsilon\to0^+}{\longrightarrow} \quad \langle(\det E)^m(D'),(\det E)^m(D')\rangle\left\langle(\det E)^m(D'),\omega_X(D)^{-1}\right\rangle^m .$$

Proof. Once again, the line bundles themselves do not depend on ε, only the metrics do. This result is then a convergence of metrics, which is a direct consequence of the end of remark 5.1.19. □

We now want to replace the second Deligne pairing appearing in the m^2-th power of the truncated Deligne-Riemann-Roch isometry (see theorem 5.3.6) by

$$\langle(\det E)^m(D')_\varepsilon,(\det E)^m(D')_\varepsilon\rangle\left\langle(\det E)^m(D')_\varepsilon,\omega_X(D)_\varepsilon^{-1}\right\rangle^m \ ,$$

since this new factor converges as ε goes to 0^+. Making this modification requires some care, as we need to carefully remove the terms added this way.

Proposition 5.4.34. *We have a canonical isometry of complex lines*

$$\begin{aligned}\left\langle \det E_\varepsilon, \det E_\varepsilon \otimes \omega_{X,\varepsilon}^{-1} \right\rangle^{m^2} &\simeq \left\langle (\det E)^m (D')_\varepsilon, (\det E)^m (D')_\varepsilon \right\rangle \left\langle (\det E)^m (D')_\varepsilon, \omega_X (D)_\varepsilon \right\rangle^{-m} \\ &\quad \left\langle \det E_\varepsilon, \mathcal{O}_X (D')_\varepsilon \right\rangle^{-2m} \left\langle \mathcal{O}_X (D')_\varepsilon, \mathcal{O}_X (D')_\varepsilon \right\rangle^{-1} \\ &\quad \left\langle \det E_\varepsilon, \mathcal{O}_X (D)_\varepsilon \right\rangle^{m^2} \left\langle \omega_{X,\varepsilon}, \mathcal{O}_X (D')_\varepsilon \right\rangle^{m} \\ &\quad \left\langle \mathcal{O}_X (D')_\varepsilon, \mathcal{O}_X (D)_\varepsilon \right\rangle^{m}\end{aligned}$$

Proof. This is a consequence of the metric bimultiplicativity of the Deligne pairing. □

Proposition 5.4.35. *We have the following equality of currents*

$$c_1 (\det E_\varepsilon) \quad = \quad c_1 \left((\det E)_{|X \setminus \{p \ cusp\}} \right) \mathbb{1}_{X_\varepsilon} - \frac{1}{2\pi} \sum_{p \ cusp} \left(\sum_{j=1}^{r} \alpha_{p,j} \right) \delta_{\partial U_{p,\varepsilon}} \ ,$$

where $\mathbb{1}_{X_\varepsilon}$ is the characteristic function of the compact part X_ε.

Proof. This proposition can be obtained with the same type of arguments of those used in proposition 5.4.18. □

Compared to the canonical isometries obtained in the regularization of the first Deligne pairing, only one more is required here. It is the object of the next proposition.

Proposition 5.4.36. *Let p be a cusp. We have a canonical isometry of complex lines*

$$\langle \det E_\varepsilon, \mathcal{O}_X (p)_\varepsilon \rangle \quad \simeq \quad \left(\det E_p, \ \varepsilon^{m_p/m} \left\| \cdot \right\|_p \right) \ ,$$

where the norm $\|\cdot\|_p$ on E_p is characterized by $\|s_{p,1} \wedge \cdots \wedge s_{p,r}\|_p = 1$.

Proof. The method used here follows closely the proof of propositions 5.4.17, 5.4.20, and 5.4.21. Using proposition 5.4.35, the first Chern current $c_1 (\det E_\varepsilon)$ is supported outside $U_{p,\varepsilon}$, which gives

$$\int_X \log \|\mathbb{1}_p\|_\varepsilon^{-2} c_1 (\det E_\varepsilon) \quad = \quad 0 \ .$$

Furthermore, properties of the Deligne pairing give a canonical isomorphism of complex lines

$$\begin{aligned} \langle \det E, \mathcal{O}_X (p) \rangle \quad &\simeq \quad \det E_p \\ \langle s, \mathbb{1}_p \rangle \quad &\longmapsto \quad s_p \end{aligned}$$

where s is a meromorphic section of $\det E$ which equals $s_{p,1} \wedge \cdots \wedge s_{p,r}$ on $U_{p,\varepsilon}$. Proving the required isometry is now only a matter of computing the Deligne norm of $\langle s, \mathbb{1}_p \rangle$. We have

$$\log \|\langle s, \mathbb{1}_p \rangle\|_\varepsilon^{-2} \quad = \quad \log \|s_{p,1} \wedge \cdots \wedge s_{p,r}\|_\varepsilon^{-2} (p) \quad = \quad -2 \frac{m_p}{m} \log \varepsilon \ ,$$

which completes the proof of the proposition. □

We can now rewrite the last five factors on the right-hand side of proposition 5.4.34.

Proposition 5.4.37. *We have a canonical isometry of complex lines*

$$\langle \det E_\varepsilon, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle^{-2m} \quad \simeq \quad \bigotimes_{p \ cusp} \left(\det E_p, \varepsilon^{m_p/m} \left\| \cdot \right\|_p \right)^{-2mm_p} \quad .$$

Proof. We first recall that the vector bundle divisor D' is defined by

$$D' \quad = \quad \sum_{p \text{ cusp}} m_p \, p.$$

This means that we have a canonical isometry of complex lines

$$\langle \det E_\varepsilon, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle^{-2m} \quad \simeq \quad \bigotimes_{p \text{ cusp}} \langle \det E_\varepsilon, \mathcal{O}_X\left(p\right) \rangle^{-2mm_p} \quad .$$

Using proposition 5.4.36 completes the proof.

□

Proposition 5.4.38. *We have a canonical isometry of complex lines*

$$\langle \mathcal{O}_X\left(D'\right)_\varepsilon, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle^{-1} \quad \simeq \quad \bigotimes_{p \ cusp} \left(\omega_{X,p}, \varepsilon \left| \cdot \right|_p \right)^{m_p^2} \quad .$$

Proof. Using the definition of D' and the metric bimultiplicativity of the Deligne pairing, we have a canonical isometry of complex lines

$$\langle \mathcal{O}_X\left(D'\right)_\varepsilon, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle^{-1} \quad \simeq \quad \left(\bigotimes_{\substack{p \\ \text{cusp}}} \langle \mathcal{O}_X\left(p\right), \mathcal{O}_X\left(p\right) \rangle^{-m_p^2} \right) \left(\bigotimes_{\substack{p \neq q \\ \text{cusps}}} \langle \mathcal{O}_X\left(p\right), \mathcal{O}_X\left(q\right) \rangle^{-m_p m_q} \right).$$

Using propositions 5.4.20 and 5.4.21, we get the result.

□

Proposition 5.4.39. *We have a canonical isometry of complex lines*

$$\langle \det E_\varepsilon, \mathcal{O}_X\left(D\right)_\varepsilon \rangle^{m^2} \quad \simeq \quad \bigotimes_{p \ cusp} \left(\det E_p, \varepsilon^{m_p/m} \left\| \cdot \right\|_p \right)^{m^2} \quad .$$

Proof. Proving this result is a matter of using the definition of D, the metric bimultiplicativity of the Deligne pairing, and proposition 5.4.36.

□

Proposition 5.4.40. *We have a canonical isometry of complex lines*

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle^{m} \quad \simeq \quad \bigotimes_{p \ cusp} \left(\omega_{X,p}, \sqrt{2}\varepsilon \log \varepsilon^{-1} \left| \cdot \right|_p \right)^{mm_p} \quad .$$

Proof. Using the definition of the vector bundle divisor D', and the metric bimultiplicativity of the Deligne pairing, we have a canonical isometry of complex lines

$$\langle \omega_{X,\varepsilon}, \mathcal{O}_X\left(D'\right)_\varepsilon \rangle \quad \simeq \quad \bigotimes_{p \text{ cusp}} \langle \omega_{X,\varepsilon}, \mathcal{O}_X\left(p\right)_\varepsilon \rangle^{m_p} \quad ,$$

and proposition 5.4.17 completes the proof.

□

Proposition 5.4.41. *We have a canonical isometry of complex lines*

$$\left\langle \mathcal{O}_X\left(D'\right)_\varepsilon, \mathcal{O}_X\left(D\right)_\varepsilon \right\rangle^m \simeq \bigotimes_{p\ cusp} \left(\omega_{X,p}, \varepsilon\left|\cdot\right|_p\right)^{-mm_p}.$$

Proof. Using the definitions of the cusp divisor D and the vector bundle divisor D', as well as the metric bimultiplicativity of the Deligne pairing, we have a canonical isometry of complex lines

$$\left\langle \mathcal{O}_X\left(D'\right)_\varepsilon, \mathcal{O}_X\left(D\right)_\varepsilon \right\rangle \simeq \left(\bigotimes_{p\ \mathrm{cusp}} \left\langle \mathcal{O}_X\left(p\right), \mathcal{O}_X\left(p\right)\right\rangle^{m_p}\right)\left(\bigotimes_{\substack{p\neq q \\ \mathrm{cusps}}} \left\langle \mathcal{O}_X\left(p\right), \mathcal{O}_X\left(q\right)\right\rangle^{m_p}\right).$$

Propositions 5.4.20 and 5.4.21 then complete the proof.

□

We can now fully state the regularization of the second Deligne pairing appearing on the right-hand side of the m^2-th power of the truncated Deligne-Riemann-Roch isometry.

Theorem 5.4.42. *We have a canonical isometry of complex lines*

$$\begin{aligned}&\left\langle \det E_\varepsilon, \det E_\varepsilon \otimes \omega_{X,\varepsilon}^{-1}\right\rangle^{m^2} \\ &\quad\simeq \left\langle (\det E)^m \left(D'\right)_\varepsilon, (\det E)^m \left(D'\right)_\varepsilon\right\rangle \left\langle (\det E)^m \left(D'\right)_\varepsilon, \omega_X\left(D\right)_\varepsilon\right\rangle^{-m} \\ &\qquad\otimes \bigotimes_{p\ cusp}\left(\det E_p, \left\|\cdot\right\|_p\right)^{m(m-2m_p)} \otimes \bigotimes_{p\ cusp}\left(\omega_{X,p}, \left|\cdot\right|_p\right)^{m_p^2} \otimes \left(\mathbb{C}, R_E\left(\varepsilon\right)\left|\cdot\right|\right)\end{aligned}$$

where $R_E(\varepsilon)$ is defined by

$$\log R_E\left(\varepsilon\right) = \frac{m}{2}\left(\sum_{p\ cusp} m_p\right)\log 2 + \left(\sum_{p\ cusp} m_p\left(m-m_p\right)\right)\log\varepsilon + m\left(\sum_{p\ cusp} m_p\right)\log\log\varepsilon^{-1}.$$

Remark 5.4.43. Note that if all the integers m_p vanish, meaning if E is trivial at the cusps, the term $R_E(\varepsilon)$ vanishes identically, as no regularization was actually needed. The complex lines involving E_p are canonically trivial, thus not contributing, and no factor $\omega_{X,p}$ is to be considered.

5.4.2 Regularization of the IC_2 bundle

We now turn our attention to regularizing the last factor on the right-hand side of the truncated Deligne-Riemann-Roch, meaning the factor $IC_2(X, E_\varepsilon)$. The assumption made in the last section that all $\alpha_{p,j}$ be rational is still made here and in the rest of this text. As we will see, however, this regularization cannot be made unless one of two unrelated hypotheses is made. The reason behind this need is that we need to perform the regularization on the vector bundle E, which is not of rank 1. If we were to try the methods used in the last section, we would have to tensor E so as to allow poles of different order in different directions around the cusps. This cannot be done simply.

The monodromy is scalar at the cusps. The first hypothesis under which the regularization of $IC_2(X, E_\varepsilon)$ can be performed is that the representation

$$\rho : \Gamma \longrightarrow U_r(\mathbb{C})$$

should be scalar at the cusps, which we will assume for the remainder of this paragraph. Let us define that precisely.

Definition 5.4.44. The representation ρ of Γ is said to be *scalar at the cusps* if we have

$$\alpha_{p,1} \quad = \quad \dots \quad = \quad \alpha_{p,r}$$

for every cusp, which means that the unitary matrix $\rho(\gamma_p)$ is scalar, where γ_p is a generator of the stabilizer Γ_p of the cusp p in Γ. In this case, we denote by α_p this common rational number.

Remark 5.4.45. Similarly to what we did in the last section, we write

$$\alpha_p \quad = \quad \frac{\widetilde{m_p}}{m} \quad ,$$

where m is a strictly positive integer and $\widetilde{m_p}$ is an integer between 1 and $m-1$.

Remark 5.4.46. This hypothesis on ρ is interesting, as we can hope to regularize E by allowing a pole of a single order in all directions for every cusp. Of the two hypotheses we will study, this one is the most natural.

Definition 5.4.47. We define the *scalar cusp divisor* D'' to be given by

$$D'' \quad = \quad \sum_{p \text{ cusp}} \widetilde{m_p}\, p$$

Proposition 5.4.48. *The metric on $E^{\otimes m}(D'')$ arising from the canonical metric on E and the trivial metric on $\mathcal{O}_X(D'')$ is smooth on the compactified modular curve X.*

Proof. It is enough to prove smoothness around a cusp p. Over the open subset $U_{p,\varepsilon}$, the vector bundle E^m is generated by the sections

$$s_{p,I} \quad = \quad s_{p,i_1} \otimes \cdots \otimes s_{p,i_m} \quad ,$$

where I is a collection of integers between 1 and r, more specifically

$$I \quad = \quad \{i_1, \dots, i_m\} \quad \in \quad [\![1, r]\!]^m \quad .$$

For the tensor product metric on E^m, we note that we have

$$h_z^{\otimes m}(s_{p,I}, s_{p,J}) \quad = \quad \prod_{k=1}^{m} h_z(s_{p,i_k}, s_{p,j_k}) \quad ,$$

which is non-zero if and only if we have $I = J$. In this case, we have

$$h_z^{\otimes m}(s_{p,I}, s_{p,I}) \quad = \quad \prod_{k=1}^{m} h_z(s_{p,i_k}, s_{p,i_k}) \quad = \quad |z|^{2m\alpha_p} \quad = \quad |z|^{2\widetilde{m_p}} \quad .$$

Over $U_{p,\varepsilon}$, the vector bundle $E^m(D'')$ being generated by the sections $s_{p,I}/z^{\widetilde{m_p}}$, with I being as above, the proposition is proved.

□

Proposition 5.4.49. *We have a canonical isometry of complex lines*

$$\begin{aligned} IC_2(X, E_\varepsilon)^{2m^2 r^{m-1}} \quad \simeq \quad & IC_2\left(X, E^{\otimes m}(D'')_\varepsilon\right)^{2m} \left\langle (\det E_\varepsilon)^m, (\det E_\varepsilon)^m \right\rangle^{r^{m-2}(m(1-r^m)+r-1)} \\ & \left[\left\langle (\det E_\varepsilon)^m, \mathcal{O}_X(D'')_\varepsilon\right\rangle^2 \left\langle \mathcal{O}_X(D'')_\varepsilon, \mathcal{O}_X(D'')_\varepsilon\right\rangle^r\right]^{mr^{m-1}(1-r^m)} . \end{aligned}$$

Proof. Using the first part of remark 5.1.36, we have a canonical isometry of complex lines

$$IC_2\left(X, E^{\otimes m}\left(D''\right)_\varepsilon\right)^2 \simeq IC_2\left(X, E_\varepsilon^{\otimes m}\right)^2 \left\langle \mathcal{O}_X\left(D''\right)_\varepsilon, \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{r^m\left(r^m-1\right)} \left\langle \det\left(E_\varepsilon^{\otimes m}\right), \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{2\left(r^m-1\right)} .$$

We can now combine this with the canonical isometry

$$\det\left(E_\varepsilon^{\otimes m}\right) \simeq \left(\det E_\varepsilon\right)^{mr^{m-1}} ,$$

and the second part of remark 5.1.36, which yields a canonical isometry of complex lines

$$IC_2\left(E_\varepsilon^{\otimes m}\right)^2 \simeq IC_2\left(E_\varepsilon\right)^{2mr^{m-1}} \left\langle \det E_\varepsilon, \det E_\varepsilon\right\rangle^{mr^{m-1}\left(mr^{m-1}-1\right)-m(m-1)r^{m-2}} .$$

This completes the proof of the proposition.

□

Remark 5.4.50. Since ρ is scalar at the cusps, the vector bundle divisor D', introduced in definition 5.4.24, and the scalar cusp divisor D'', in definition 5.4.47, are related by

$$D' = rD'' .$$

The first one is used to regularize $\det E_\varepsilon$, while the second one concerns E itself.

Proposition 5.4.51. *We have a canonical isometry of complex lines*

$$\left\langle \left(\det E_\varepsilon\right)^m, \left(\det E_\varepsilon\right)^m\right\rangle \simeq \left\langle \left(\det E\right)^m\left(rD''\right)_\varepsilon, \left(\det E\right)^m\left(rD''\right)_\varepsilon\right\rangle \left\langle \left(\det E_\varepsilon\right)^m, \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{-2r} \left\langle \mathcal{O}_X\left(D''\right)_\varepsilon, \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{-r^2} .$$

Proof. This result is a direct consequence of the metric bimultiplicativity of the Deligne pairing.

□

Proposition 5.4.52. *We have a canonical isometry of complex lines*

$$\begin{aligned} & IC_2\left(X, E_\varepsilon\right)^{2m^2r^{m-1}} \\ & \simeq IC_2\left(X, E^{\otimes m}\left(D''\right)_\varepsilon\right)^{2m} \left\langle \left(\det E\right)^m\left(D'\right)_\varepsilon, \left(\det E\right)^m\left(D'\right)_\varepsilon\right\rangle^{r^{m-2}\left(m\left(1-r^m\right)+r-1\right)} \\ & \qquad \left\langle \left(\det E_\varepsilon\right)^m, \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{2r^{m-1}(1-r)} \left\langle \mathcal{O}_X\left(D''\right)_\varepsilon, \mathcal{O}_X\left(D''\right)_\varepsilon\right\rangle^{r^m(1-r)} . \end{aligned}$$

Proof. This a combination of propositions 5.4.49 and 5.4.51.

□

Remark 5.4.53. As was already mentioned, the divisors D' and D'' are closely related. In the last proposition, one or the other can be used in the second and third factor on the right-hand side, with the appropriate modification of the exponent while the fourth term should involve D'' and not D', as the exponent must remain integral.

Proposition 5.4.54. *As ε goes to 0^+, we have the following convergence of complex lines*

$$IC_2\left(X, E^{\otimes m}\left(D''\right)_\varepsilon\right) \underset{\varepsilon\to 0^+}{\longrightarrow} IC_2\left(X, E^{\otimes m}\left(D''\right)\right) .$$

Proof. It can be noted that the ε-truncated metric on $E^{\otimes m}(D'')$ equals the metric induced by the canonical metric on E and the trivial metric on $\mathcal{O}_X(D'')$. The sequence of complex lines being constant, we have the required result. $\square$

Proposition 5.4.55. *As ε goes to 0^+, we have the following convergence of complex lines*

$$\langle (\det E)^m (D')_\varepsilon , (\det E)^m (D')_\varepsilon \rangle \underset{\varepsilon \to 0^+}{\longrightarrow} \langle (\det E)^m (D') , (\det E)^m (D') \rangle \ .$$

Proof. As in the last proposition, we note that the ε-truncated metric on $(\det E)^m (D')$ equals the metric induced by the canonical metric on E and the trivial metric on $\mathcal{O}_X(D')$. The proposition is thus proved. $\square$

Proposition 5.4.56. *We have a canonical isometry of complex lines*

$$\langle (\det E_\varepsilon)^m , \mathcal{O}_X(D'')_\varepsilon \rangle^{2r^{m-1}(1-r)} \simeq \bigotimes_{p \ cusp} \left(\det E_p, \varepsilon^{m_p/m} \left|\cdot\right|_p \right)^{2m\widetilde{m_p}r^{m-1}(1-r)} .$$

Proof. This is a consequence of the definition of D'', of the metric bimultiplicativity of the Deligne pairing, and of proposition 5.4.36. $\square$

Proposition 5.4.57. *We have a canonical isometry of complex lines*

$$\langle \mathcal{O}_X(D'')_\varepsilon , \mathcal{O}_X(D'')_\varepsilon \rangle^{r^m(1-r)} \simeq \bigotimes_{p \ cusp} \left(\omega_{X,p}, \varepsilon \left|\cdot\right| \right)^{-\widetilde{m_p}^2 r^m (1-r)} .$$

Proof. This is a consequence of the definition of D'', of the metric bimultiplicativity of the Deligne pairing, and of propositions 5.4.20 and 5.4.21. $\square$

Theorem 5.4.58. *We have a canonical isometry of complex lines*

$$\begin{aligned} & IC_2(X, E_\varepsilon)^{2m^2 r^{m-1}} \\ & \quad \simeq IC_2\left(X, E^{\otimes m}(D'')_\varepsilon\right)^{2m} \langle (\det E)^m (D')_\varepsilon , (\det E)^m (D')_\varepsilon \rangle^{r^{m-2}(m(1-r^m)+r-1)} \\ & \qquad \otimes \bigotimes_{p \ cusp} \left(\det E_p, \left\|\cdot\right\|_p \right)^{m\widetilde{m_p}r^{m-1}(m(r^m-1)+2(1-r))} \otimes \bigotimes_{p \ cusp} \left(\omega_{X,p}, \left|\cdot\right|_p \right)^{-\widetilde{m_p}^2 r^m (1-r)} \\ & \qquad \otimes \left(\mathbb{C}, R^s_{IC_2}(\varepsilon) \left|\cdot\right| \right) , \end{aligned}$$

where $R^s_{IC_2}(\varepsilon)$ is defined by

$$\log R^s_{IC_2}(\varepsilon) = \left(\sum_{p \ cusp} \widetilde{m_p}^2 \right) r^m (1-r) \log \varepsilon \ .$$

Proof. This theorem stems from propositions 5.4.52, 5.4.56, and 5.4.57. $\square$

The monodromy is finite. This paragraph is intended as a remark to present an idea which can be envisionned to regularize the IC_2 bundle. The precise regularization cannot yet be stated, because of the way Deligne extensions of unitary flat vector bundles behave with respect to tensor product of representations. This is based on the following classical result.

Theorem 5.4.59 (Fell's absorption principle)**.** *Let G be a discrete group, and*

$$\chi \quad : \quad G \quad \longrightarrow \quad U_r(\mathbb{C})$$

be a unitary representation of G of rank r. Denote by λ the left regular representation of G, and by $\mathbb{1}_r$ the trivial representation of G of rank r. We have a unitary equivalence of representations

$$\lambda \otimes \mathbb{1}_r \quad \simeq \quad \lambda \otimes \chi \ .$$

Proof. The first point to note is that the left regular representation is induced by the action of G on itself by left-translation. More explicitely, it is given by

$$\begin{array}{rccccc} \lambda & : & G & \longrightarrow & \mathrm{End}\left(L^2(G)\right) & \\ & & g & \longmapsto & [\delta_h \quad \longmapsto \quad \delta_{gh}] & . \end{array}$$

The tensor product representation $\lambda \otimes \chi$ can then be seen as

$$\begin{array}{rcccc} \lambda \otimes \chi & : & G & \longrightarrow & \mathrm{End}\left(L^2(G, \mathbb{C}^r)\right) \\ & & g & \longmapsto & \left[\sum\limits_{h \in G} \delta_h \otimes v_h \quad \longmapsto \quad \sum\limits_{h \in G} \delta_{gh} \otimes \chi(g) v_h \right] . \end{array}$$

Similarly, we can view the representation $\lambda \otimes \mathbb{1}_r$ as

$$\begin{array}{rcccc} \lambda \otimes \mathbb{1}_r & : & G & \longrightarrow & \mathrm{End}\left(L^2(G, \mathbb{C}^r)\right) \\ & & g & \longmapsto & \left[\sum\limits_{h \in G} \delta_h \otimes v_h \quad \longmapsto \quad \sum\limits_{h \in G} \delta_{gh} \otimes v_h \right] . \end{array}$$

We can then consider the endomorphism

$$\begin{array}{rcccc} U_r & : & L^2(G, \mathbb{C}^r) & \longrightarrow & L^2(G, \mathbb{C}^r) \\ & & \sum\limits_{h \in G} \delta_h \otimes v_h & \longmapsto & \sum\limits_{h \in G} \delta_h \otimes \chi(h) v_h \end{array}$$

of $L^2(G, \mathbb{C}^r)$. This application will be realize the unitary equivalence we require. It can be first be seen that it is bijective, since we can define its inverse using $\chi(h)^{-1}$ instead of $\chi(h)$. It only remains to show that it is compatible with the representations, since U_r is unitary. We have

$$\begin{aligned} U_r\left((\lambda \otimes \mathbb{1}_r)(g) \left(\sum_{h \in G} \delta_h \otimes v_h \right) \right) &= U_r\left(\sum_{h \in G} \delta_{gh} \otimes v_h \right) \\ &= \sum_{h \in G} \delta_{gh} \otimes \chi(gh) v_h \\ &= (\lambda \otimes \chi)(g) \left(U_r \left(\sum_{h \in G} \delta_h \otimes v_h \right) \right) . \end{aligned}$$

□

This result is a classical one, for instance presented in [16, Thm 2.5.5]. We want to apply it to the case of a Fuchsian group of the first kind Γ, endowed with a unitary representation ρ of rank r. In this setting, Fell's absorption principle is not satisfactory, because all Fuchsian groups of the first kind are infinite, yielding left regular representations of infinite rank. Looking closely at the proof above, we see that the important part is for

$$U_r\left(\sum_{h\in G}\delta_h\otimes v_h\right) \quad = \quad \sum_{h\in G}\delta_h\otimes\chi(h)\,v_h$$

to be well-defined. Summing over the full group G is unnecessary to achieve that, as we could restrict ourselves to $G/\ker\chi$. We get the following result.

Theorem 5.4.60 (Modified Fell's absorption principle)**.** *Let G be a discrete group, and*

$$\chi \;:\; G \longrightarrow U_r(\mathbb{C})$$

be a unitary representation of G of rank r. Denote by λ_χ the representation

$$\begin{array}{rcccl} \lambda_\chi & : & G & \longrightarrow & \operatorname{End}\left(L^2\left(G/\ker\chi\right)\right) \\ & & g & \longmapsto & \left[\delta_{[h]} \longmapsto \delta_{[gh]}\right] \end{array}$$

induced by the action of G on $G/\ker\chi$ by left translation, and by $\mathbb{1}_r$ the trivial representation of the group G of rank r. We have a unitary equivalence of representations

$$\lambda_\chi\otimes\mathbb{1}_r \quad\simeq\quad \lambda_\chi\otimes\chi\ .$$

Proof. The proof of this result is completely similar to that of theorem 5.4.59, the only difference being that, instead of U_r, we need to define and use the operator

$$V_r\left(\sum_{[h]\in G/\ker\chi}\delta_{[h]}\otimes v_{[h]}\right) \quad = \quad \sum_{[h]\in G/\ker\chi}\delta_{[h]}\otimes\chi(h)\,v_{[h]}\ .$$

□

5.4.3 Singular behavior

We can now sum up the results produced by the regularization of all three factors appearing on the right-hand side of the truncated Deligne-Riemann-Roch isometry. We will get two slightly different results, depending on which hypothesis we make to regularize the IC_2 bundle.

The monodromy is scalar at the cusps. We begin with the assumption that ρ be scalar at the cusps, presented in definition 5.4.44.

Theorem 5.4.61. *Assume the representation ρ is scalar at the cusps. We then have a canonical isometry of complex lines*

$$\begin{aligned} \lambda_X(E)_\varepsilon^{12m^2r^{m-1}} & \\ \simeq\ & \langle\omega_X(D)_\varepsilon,\omega_X(D)_\varepsilon\rangle^{m^2r^m}\langle(\det E)^m(D')_\varepsilon,(\det E)^m(D')_\varepsilon\rangle^{6r^{m-2}(1-m(1-r^m))} \\ & \langle(\det E)^m(D')_\varepsilon,\omega_X(D)_\varepsilon\rangle^{-6mr^{m-1}}IC_2\left(X,E^{\otimes m}(D'')_\varepsilon\right)^{-12m} \\ & \otimes\psi_{\omega,s}\ \otimes\ \psi_{E,s}\ \otimes\ (\mathbb{C},R_s(\varepsilon)\,|\cdot|)\otimes(\mathbb{C},c(g_\Gamma,r)\,|\cdot|)\,, \end{aligned}$$

where the complex lines $\psi_{\omega,s}$ and $\psi_{E,s}$ are defined by

$$\begin{aligned}\psi_{\omega,s} &= \bigotimes_{p\ cusp}\left(\omega_{X,p},|\cdot|_p\right)^{\left(6\widetilde{m_p}-m^2\right)r^m},\\ \psi_{E,s} &= \bigotimes_{p\ cusp}\left(\det E_p,\|\cdot\|_p\right)^{6mr^{m-1}\left(m-2\widetilde{m_p}\right)},\end{aligned}$$

and the remainder $R_s(\varepsilon)$ under the "scalar at the cusps" hypothesis is given by

$$\begin{aligned}\log R_s(\varepsilon) &= mr^m\left[3\sum\widetilde{m_p}-mh\right]\log 2+6mr^m\left[\sum_{p\ cusp}\widetilde{m_p}-\tfrac{1}{3}mh\right]\log\log\varepsilon^{-1}\\ &\quad +r^m\left[m^2h-6m\sum_{p\ cusp}\widetilde{m_p}+6\sum_{p\ cusp}\widetilde{m_p}^2\right]\log\varepsilon^{-1}.\end{aligned}$$

Proof. This theorem is a concatenation of the truncated Deligne-Riemann-Roch isometry, presented in theorem 5.3.6, and of the three canonical isometries obtained in theorems 5.4.23, 5.4.42, and 5.4.58.

□

Remark 5.4.62. The purpose of this last theorem was to isolate the singular behavior as ε goes to 0^+ on the right-hand side, which manifests itself as the term $R_s(\varepsilon)$. This information will allow us to define a Quillen metric under the "scalar at the cusps" hypothesis, which we will do in the next section.

5.5 The Quillen metric on a modular curve

Similarly to the presentation we followed in the last section, we will now define a Quillen metric on modular curves by treating separately the two types of hypotheses we made in order to regularize the IC_2 bundle. The corresponding Deligne-Riemann-Roch isometry will also depend on which assumption we make.

The monodromy is scalar at the cusps. The aim of this paragraph is to attach a natural Quillen metric to the Poincaré metric on X and the canonical metric on E, by looking at the bahavior as ε goes to 0^+ of the ε-truncated Quillen metric, introduced in definition 5.3.5. However, this truncated metric diverges, which means we cannot simply take its limit. Instead, we rely on the following proposition. We assume that the representation ρ is scalar at the cusps.

Proposition-Definition 5.5.1. *The singular determinant $\det'\Delta_{\overline{\partial}_E}$ associated to the singular metrics g and h is defined as*

$$\det'\Delta_{\overline{\partial}_E} = \lim_{\varepsilon\to 0^+} R_s(\varepsilon)^{\frac{1}{6m^2r^{m-1}}}\det'\Delta_{\overline{\partial}_E}^{g_\varepsilon,h_\varepsilon},$$

this limit being well-defined and yielding a strictly positive real number.

Proof. The convergence as ε goes to 0^+ stems from the truncated Deligne-Riemann-Roch isometry obtained under the "scalar at the cusps" hypothesis, which constitutes theorem 5.4.61.

□

Remark 5.5.2. The function $R_s(\varepsilon)$ being arithmetic in nature, insofar at it is only defined using the right-hand side of the truncated Deligne-Riemann-Roch isometry, this *naive determinant* loses its spectral interpretation. The purpose of chapters 2, 3, and 4 is to restore it.

Proposition 5.5.3. *As ε goes to 0^{+}, we have the following asymptotic expansion*

$$\begin{aligned}
&\log\det{}'\,\Delta_{\overline{\partial}_E}^{g_\varepsilon,h_\varepsilon} \\
&= -\tfrac{1}{6m^2r^{m-1}}\log R_s(\varepsilon)+\Big(\log\tfrac{C(\varepsilon)}{\det' N_{E,\varepsilon}}+\Big(\Big(\sum_{p\ cusp}\sum_{j=1}^{r}\alpha_{p,j}\Big)-\tfrac{1}{3}rh\Big)\log\log\varepsilon^{-1}\Big) \\
&\quad +\tfrac{1}{2}k(\Gamma,\rho)\log 2+\log Z^{(d)}(1)+\log(d!)+\tfrac{r}{3}\left[g_\Gamma-1\right]\log 2 \\
&\quad +\tfrac{1}{2\pi}rV\left[2\zeta'(-1)+\tfrac{1}{2}\log 2\pi-\tfrac{1}{4}\right]+\tfrac{1}{2}\Big(\sum_{p\ cusp}\sum_{j=1}^{r}\alpha_{p,j}\Big)\log 2+o(1)\,.
\end{aligned}$$

Proof. This is a combination of corollary 2.5.14 and theorem 5.4.61. □

Remark 5.5.4. The fact that the limit

$$\lim_{\varepsilon\to 0^+}\left(\log\tfrac{C(\varepsilon)}{\det' N_{E,\varepsilon}}+\left(\left(\sum_{p\text{ cusp}}\sum_{j=1}^{r}\alpha_{p,j}\right)-\tfrac{1}{3}rh\right)\log\log\varepsilon^{-1}\right)$$

exists and is finite should be viewed here as a consequence of the truncated Deligne-Riemann-Roch isometry, as presented in theorem 5.4.61. When the vector bundle E is smooth at the cusps, *i.e.* when all weights $\alpha_{p,j}$ vanish, it can be proved independantly, and the limit can be computed exactly, using spectral geometry tools such as Polyakov formulae, and exact computation of determinants of Laplacians for trivial metrics, as well as the conformal invariance of the jump operator.

Definition 5.5.5. The *truncation constant* C_E is defined as

$$C_E = \lim_{\varepsilon\to 0^+}\left(\log\tfrac{C(\varepsilon)}{\det' N_{E,\varepsilon}}+\left(\left(\sum_{p\text{ cusp}}\sum_{j=1}^{r}\alpha_{p,j}\right)-\tfrac{1}{3}rh\right)\log\log\varepsilon^{-1}\right).$$

Corollary 5.5.6. *The singular determinant is given by*

$$\begin{aligned}
\log\det{}'\,\Delta_{\overline{\partial}_E} &= C_E+\tfrac{1}{2}k(\Gamma,\rho)\log 2+\log Z^{(d)}(1)+\log(d!)+\tfrac{r}{3}\left[g_\Gamma-1\right]\log 2 \\
&\quad +\tfrac{1}{2\pi}rV\left[2\zeta'(-1)+\tfrac{1}{2}\log 2\pi-\tfrac{1}{4}\right]+\tfrac{1}{2}\Big(\sum_{p\ cusp}\sum_{j=1}^{r}\alpha_{p,j}\Big)\log 2.
\end{aligned}$$

Definition 5.5.7. The Quillen metric $\|\cdot\|_Q$ associated to the singular metrics g and h is given by

$$\|\cdot\|_Q = \left(\det{}'\,\Delta_{\overline{\partial}_E}^{g,h}\right)^{-1/2}\|\cdot\|_{L^2}\;.$$

Theorem 5.5.8. *Assume the representation ρ is scalar at the cusps. We have a canonical isometry of complex lines*

$$\begin{aligned}
&\lambda_X(E)_Q^{12m^2r^{m-1}} \\
&\simeq \langle\omega_X(D),\omega_X(D)\rangle^{m^2r^m}\langle(\det E)^m(D'),(\det E)^m(D')\rangle^{6r^{m-2}(1-m(1-r^m))} \\
&\quad \langle(\det E)^m(D'),\omega_X(D)\rangle^{-6mr^{m-1}}\,IC_2\left(X,E^{\otimes m}(D'')\right)^{-12m}\otimes\psi_{\omega,s}\otimes\psi_{E,s} \\
&\quad \otimes(\mathbb{C},c(g_\Gamma,r)\,|\cdot|)\,,
\end{aligned}$$

where the complex lines ψ_ω and ψ_E are as in theorem 5.4.61, and the metric on $\lambda_X(E)$ is the Quillen metric $\|\cdot\|_Q$ defined above.

5.6 An arithmetic Riemann-Roch formula

In what we have seen so far, the word "canonical" was used when talking about isometries of complex lines to mean an "isometry induced by one that works in family". We will now state the version of the Deligne-Riemann-Roch isometry in family. Let us first recall the setting. We consider an arithmetic surface over the ring of integers $\mathcal{O}_K$ of a number field

$$f \quad : \quad \mathcal{X} \quad \longrightarrow \quad \mathcal{S} \; = \; \operatorname{Spec} \mathcal{O}_K$$

We assume that f has disjoint sections $\sigma_1, \ldots, \sigma_h$ such that for every complex embedding of the number field $\tau : K \hookrightarrow \mathbb{C}$, the non-compact Riemann surface

$$\mathcal{Z}_\tau \quad = \quad \mathcal{X}_\tau(\mathbb{C}) \setminus \{\sigma_1(\tau), \ldots, \sigma_h(\tau)\}$$

is a modular curve, arising from a Fuchsian group of the first kind Γ_τ without torsion. We also consider a vector bundle $\mathcal{E}$ of rank r over $\mathcal{X}$ such that for every complex embedding τ, the vector bundle $\mathcal{E}_\tau$ is a vector bundle over $\mathcal{X}_\tau(\mathbb{C})$, which is the extension of a flat unitary vector bundle over $\mathcal{Z}_\tau$ associated to a unitary representation ρ_τ of Γ_τ. We further assume that the weights of $\mathcal{E}_\tau$ at the cusp $\sigma_i(\tau)$ are rational and independant of τ. At the cusp $\sigma_i(\tau)$, the weights are denoted by $\alpha_{i,j}$, and further written as

$$\alpha_{i,j} \quad = \quad \frac{m_{i,j}}{m} \quad .$$

We then denote by m_i the sum of the integers $m_{i,j}$. Here, we will work under the "scalar at the cusps", which means that $m_{i,j}$ does not depend on j, and is denoted by $\widetilde{m_i}$. We will now define the relative versions of the three divisors D, D', and D'' used in the regularizations.

Definition 5.6.1. The *cusp divisor in family D_σ* is defined as

$$\mathcal{D} \quad = \quad \sum_{i=1}^{h} \sigma_i \quad ,$$

where the sections $\sigma_1, \ldots, \sigma_h$ of the structural morphism of $\mathcal{X}/\mathcal{S}$ define the cusps for each complex embedding. The *vector bundle divisor in family D'_σ* is defined as

$$\mathcal{D} \quad = \quad \sum_{i=1}^{h} \left(\sum_{j=1}^{r} m_{i,j} \right) \sigma_i \quad .$$

The *scalar cusp divisor in family D''_σ* is defined as

$$\mathcal{D}' \quad = \quad \sum_{i=1}^{h} \widetilde{m_i} \sigma_i \quad .$$

Definition 5.6.2. We define the *relative deficiency line bundles* $\psi^{\mathrm{rel}}_{\omega,s}$ and $\psi^{\mathrm{rel}}_{E,s}$ by

$$\psi^{\mathrm{rel}}_{\omega,s} \quad = \quad \bigotimes_{i=1}^{h} \left(\sigma_i^* \omega_{\mathcal{X}/\mathcal{S}}, |\cdot|_i\right)^{\left(6\widetilde{m_i} - m^2\right) r^m} , \qquad \psi^{\mathrm{rel}}_{E,s} \quad = \quad \bigotimes_{i=1}^{h} \left(\sigma_i^* \det \mathcal{E}, \|\cdot\|_i\right)^{6mr^{m-1}(m - 2\widetilde{m_i})} .$$

Theorem 5.6.3. *We have an isometry of line bundles over $\mathcal{S}$*

$$\begin{aligned}
&\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})_Q^{12m^2r^{m-1}} \\
&\simeq \left\langle \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}) \right\rangle^{m^2r^m} \left\langle (\det \mathcal{E})^m (\mathcal{D}'), (\det \mathcal{E})^m (\mathcal{D}') \right\rangle^{6r^{m-2}(1-m(1-r^m))} \\
&\quad \left\langle (\det \mathcal{E})^m (\mathcal{D}'), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}) \right\rangle^{-6mr^{m-1}} IC_2\left(\mathcal{X}, \mathcal{E}^{\otimes m}(\mathcal{D}'')\right)^{-12m} \otimes \psi_{\omega,s}^{rel} \otimes \psi_{E,s}^{rel} \\
&\quad \otimes \left(\mathcal{O}_{\mathcal{S}}, c(g,r)\right)^{m^2r^{m-1}} \otimes \mathcal{O}(\delta)^{m^2r^m},
\end{aligned}$$

where g denotes the genus of any modular curve $X_\tau(\mathbb{C})$, and the metric on $\mathcal{O}_{\mathcal{S}}$ is $c(g,r)$ times the trivial metric.

Corollary 5.6.4 (Arithmetic Riemann-Roch theorem)**.** *Assuming the sections σ_i are pairwise disjoint, we have an equality of real numbers*

$$\begin{aligned}
&12m^2r^{m-1}\widehat{\deg}\lambda_{\mathcal{X}/\mathcal{S}}(\mathcal{E})_Q \\
&= m^2r^m \left(\omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D}), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D})\right) + 6r^{m-2}\left(1 - m\left(1 - r^m\right)\right)\left((\det \mathcal{E})^m (\mathcal{D}'), (\det \mathcal{E})^m (\mathcal{D}')\right) \\
&\quad -6mr^{m-1}\left((\det \mathcal{E})^m (\mathcal{D}'), \omega_{\mathcal{X}/\mathcal{S}}(\mathcal{D})\right) - 12m\,\widehat{\deg}\, IC_2\left(\mathcal{X}, \mathcal{E}^{\otimes m}(\mathcal{D}'')\right) \\
&\quad +\widehat{\deg}\,\psi_{\omega,s}^{rel} + \widehat{\deg}\,\psi_{E,s}^{rel} + m^2r^m\delta + m^2r^{m-1}\log\left(r(1-g)\left(1 - 24\zeta'(-1)\right)\right),
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the arithmetic intersection pairing.

Proof. This formula is obtained by applying the arithmetic degree $\widehat{\deg}$ to the isometry presented in theorem 5.6.3.

$\square$

Remark 5.6.5. We had to assume that the sections σ_i are disjoint, so that for every $i \neq j$, the Deligne pairings $\langle \mathcal{O}(\sigma_i), \mathcal{O}(\sigma_j) \rangle$ are metrically trivial.

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