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Introduction

A l'origine, l'unification était parfaite, et les onze dimensions de l'univers identiques. La superbe symétrie rendait impossible la distinction entre particules d'interaction et particules de matière, qui avaient d'ailleurs toutes la même masse nulle. Un jour, la merveilleuse symétrie se brisa spontanément, et sept des onze dimensions s'enroulèrent sur une sphère minuscule, se dérobant à notre perception. Il devint également possible de séparer les bosons porteurs d'interactions de leurs super-partenaires, les fermions matériels. Dans les $3 + 1$ dimensions d'espace-temps restantes, apparut alors une particule portant le nom du physicien écossais Peter Higgs. Ultime survivant de la géométrie des dimensions cachées, le boson de Higgs se chargea d'attribuer leurs différentes masses à toutes les particules vivant dans les quatres dimensions visibles de notre monde...

Dans cette thèse, nous discutons quelques aspects mathématiques relatifs au semi-rêve ci-dessus, construit par les physiciens depuis les années 1970. Du point de vue de la théorie des champs, nous n'aborderons que le côté classique, c'est-à-dire non-quantique, des différentes théories que nous allons considérer.

Signalons cependant que les théories de Higgs, avec leur origine dans les univers multidimensionnels et les symétries qu'elles mettent en jeu, sont directement liées aux théories de supercordes, qui constituent une tentative sérieuse d'unification de la gravité avec les principes quantiques. D'un autre côté, la recherche du boson de Higgs est l'un des problèmes prioritaires de la physique des particules, auquel le nouvel accélérateur LHC tant attendu devra fournir une réponse à partir de 2008.

Indépendamment des résultats nouveaux qui peuvent ou non avoir un intérêt pour la physique, notre travail s'inscrit surtout dans une tentative de compréhension et de clarification mathématique de théories, que les physiciens théoriciens ont élaborées dans leur quête d'une description unifiée de l'univers dans sa structure fondamentale. Les théories supersymétriques, et particulièrement la supergravité et ses compactifications, recèlent des trésors mathématiques. Elles mettent en jeu les fondements de notre conception d'espace et de géométrie, à qui elles demandent la plus grande généralité, celle qu'Elie Cartan nous a dévoilée. Et même plus, puisque pour inclure la matière spinorielle dans l'unification, il nous faut passer à la supergéométrie (de Cartan), où la notion de point perd un peu de sa substance, et c'est la perspective des faisceaux structuraux de Grothendieck qui prend la relève. En toile de fond, les (super-)groupes de Lie et leurs représentations qui structurent la géométrie et les champs, ainsi que les invariances de la théorie, principes à l'origine des équations de la physique.

D'un point de vue moderne, la géométrie différentielle, souvent enrichie de structures additionnelles, se décrit bien dans le formalisme général des espaces fibrés, des connexions

d’Ehresmann et de Cartan avec leur courbure et leur torsion, des G -structures, etc... C’est dans ce langage que nous nous proposons de reprendre certaines théories de champs classiques, supersymétriques et gravitationnelles, en espérant que cette approche rendra ces théories un peu plus transparentes auprès des mathématiciens intéressés par la physique.

Commençons par décrire rapidement le contenu de la thèse, divisée en cinq parties.

Dans la première partie, nous nous intéressons à la réduction de Kaluza-Klein d’une théorie de gravitation d’Einstein sur un univers multidimensionnel, fibré en espaces homogènes compacts G/H au-dessus d’un espace-temps ordinaire. Partant du point de vue Coquereaux et Jadczyk, qui ont étudié dans [CJ1] la réduction d’une métrique G -invariante sur un tel espace-temps étendu, nous définissons pour la théorie réduite un espace affine de champs scalaires, correspondant à une partie hyperbolique de la métrique dans l’espace interne, et naturellement associé à la restriction à H de la représentation adjointe de G . Pour cela, nous nous appuyons sur les caractéristiques des exemples développés par Manton et Witten. Nous calculons ensuite le potentiel de ces champs scalaires, et nous montrons que c’est une fonction polynomiale bornée inférieurement, de degré plus petit ou égal à 6. Ce potentiel, ainsi que les couplages avec le secteur gravitationnel et le secteur Yang-Mills de la théorie réduite, possèdent les propriétés d’une théorie de Yang-Mills-Higgs standard, et pourraient donner naissance à de nouveaux types de monopoles. Avec $G = \mathrm{SU}(5)$ et $H = \mathrm{U}(1)$, nous construisons un exemple où le potentiel est exactement de degré 6.

Il est en fait artificiel de découpler le champs de Higgs du champ de gravitation. Beaucoup de physiciens sont persuadés que c’est la gravité qui assure la stabilité du champ de Higgs, et peut-être de l’ensemble des champs. D’où notre volonté de décrire plus à fond l’ensemble formé par la gravitation, les champs de Higgs et les champs de Yang-Mills ; le bon cadre semble être celui des géométries de Cartan. Il doit nous permettre aussi d’étendre les constructions du chapitre 1 aux théories de supergravité.

La seconde partie présente les théories de gravité selon le point de vue géométrique de Cartan. Nous présentons en premier lieu la notion de géométrie de Klein, qui fournit le réceptacle pour des théories “à background fixé” (c’est-à-dire sans gravité dynamique). En vue de la dernière partie de la thèse, un petit rappel sur les σ -modèles non-linéaires à valeurs dans un espace homogène est ensuite donné, ainsi qu’une réalisation de l’algèbre de Lie du groupe exceptionnel E_7 dans sa représentation fondamentale. En second lieu, nous présentons la notion de géométrie de Cartan, qui constitue à notre avis l’objet adéquat pour décrire la gravité. Une théorie de gravité est ainsi vue comme géométrie de Cartan modelée sur un espace-temps de Minkowski, puis Anti-de-Sitter.

Dans la troisième partie, nous passons à la supersymétrie. Nous revenons dans un premier temps sur les fondements de la théorie des supervariétés, en insistant sur les différences entre deux classes de différentiabilité possibles pour les super-fonctions, appelées G^∞ et H^∞ respectivement. La seconde, donnant une catégorie de supervariétés équivalente à celle de Berezin et Kostant dans l’approche des faisceaux, permettrait d’éviter l’exigence d’anticommutation pour les spineurs qui constituent la partie impaire des superchamps. Nous montrons que le choix de la catégorie H^∞ avec des spineurs ordinaires permet d’obtenir de nouvelles équations de champs classiques, dans le cadre des lagrangiens de σ -modèles non-linéaires supersymétriques en dimension 1 (les super-géodésiques),

et en dimension 2 (les super-nappes extrémales).

Dans la quatrième partie, nous abordons les théories de supergravité dans leur formulation en super-espace, en adoptant une présentation géométrique intrinsèque. Nous nous basons sur le travail de John Lott, qui a trouvé une interprétation géométrique unifiée des contraintes de torsion pour plusieurs théories de supergravité, en termes de G -structures (cf. [Lot]). Dans cette présentation, les contraintes s'expliquent par le désir de maintenir une platitude au premier ordre pour des G -structures correspondant à des supergroupes spécifiques. Enfin, nous proposons de regarder la supergravité d'un point de vue affine, en tant que théorie de jauge pour le supergroupe de Poincaré. En considérant l'extension affine du groupe de jauge, nous montrons que les contraintes de torsion traduisent l'existence d'une jauge où l'action sur le supervielbein des super-difféomorphismes coincide avec l'action des supertranslations de jauge.

Dans la cinquième et dernière partie, nous nous concentrons sur le cas de la supergravité à 11 dimensions, construite par Cremmer, Scherk et Julia dans [CrJS], puis largement développée par de Wit et Nicolai, Duff, etc... Nous formulons dans un langage géométrique intrinsèque les travaux de DeWit et Nicolai, qui ont montré dans [dWN2] que le secteur de masse nulle de la théorie réduite par compactification de la supergravité à 11 dimensions sur une sphère de dimension 7 coïncidait avec la supergravité jaugée à 8 supersymétries en dimension 4. Nous insistons enfin sur le secteur scalaire obtenu par cette réduction, qui correspond à un sigma-modèle non-linéaire à valeurs dans l'espace homogène $E_7/SU(8)$. Signalons que les champs scalaires à valeurs dans $E_7/SU(8)$ (ou des champs analogues pour d'autres compactifications) sont à l'origine des dualités T, S, U, \dots si importantes dans les théories des cordes et les théories quantiques des champs. Il est possible de relier le potentiel de ces champs scalaires à des réductions du type envisagé dans la première partie, pour des métriques obtenues par projection de métriques invariantes par $SO(8) \times SO(8)$.

Détaillons à présent le contenu de chacune des cinq parties.

0.1 Réduction dimensionnelle de la gravité et champs de Higgs

Dans les théories de jauge usuelles, les champs de Higgs apparaissent au niveau classique comme des champs scalaires ϕ , dont la dynamique est prescrite par un potentiel polynomial de degré 4

$$V(\phi) = \lambda (|\phi|^2 - a^2)^2 \quad (\lambda > 0)$$

provoquant un mécanisme de brisure spontanée de la symétrie, à l'issue duquel sont générées des masses pour les bosons de jauge. Ces champs de Higgs, s'ils étaient découverts, formeraient la dernière brique du "modèle standard".

A première vue, de tels potentiels de Higgs semblent être mis à la main, alors qu'il serait intéressant de disposer d'une compréhension géométrique de leur origine. L'approche considérée ici est celle des théories de réduction dimensionnelle, dont l'idée centrale est l'existence d'une théorie de champs simple et invariante par un groupe de symétrie, formulée dans un univers multidimensionnel, et conduisant à travers un processus de réduction dimensionnelle aux théories de champs que nous percevons dans notre espace-temps

quadri-dimensionnel. Signalons qu'il y a aussi une approche différente par la géométrie non-commutative due à Alain Connes (mais qui n'est peut-être pas si éloignée).

Un premier pas dans cette direction est de chercher des théories multidimensionnelles qui donneraient, par réduction dimensionnelle, des théories de champs contenant des champs scalaires de type Higgs. Une solution bien connue à ce problème consiste à prendre une théorie de Yang-Mills sur l'univers multidimensionnel. Sous des conditions appropriées, la réduction dimensionnelle d'une telle théorie, étudiée en premier lieu par Manton et Witten(cf. [Man], [CM]), donne une théorie de Yang-Mills avec des champs de Higgs possédant un potentiel qui brise la symétrie, du type mentionné ci-dessus.

En fait, des études faites par A. Jadczyk, R. Coquereaux et K. Pilch (cf. [JP], [Jad] and [CJ2]) montrèrent que si l'on veut effectuer la réduction dimensionnelle d'un champ de Yang-Mills symétrique dans un cadre suffisamment général, alors l'objet naturel à réduire n'est pas un champ de Yang-Mills symétrique isolé, mais une combinaison convenable de deux champs : un champ de Yang-Mills symétrique + un champ de gravitation symétrique.

Ils ont montré aussi que la meilleure façon d'obtenir une telle combinaison est de considérer que l'univers multidimensionnel dans lequel vivent ces deux champs symétriques est lui-même le résultat de la réduction dimensionnelle d'un autre univers avec encore plus de dimensions, et qu'en ajoutant un champ constant au couple symétrique, on peut obtenir tous les champs à partir d'un unique champ gravitationnel très symétrique sur l'univers de dimension maximale. Dans ce secteur, on est en présence des théories de Kaluza-Klein, dans leur extension, due à Richard Kerner (cf. [Ker1]), au cas où les dimensions supplémentaires compactifiées sont celles d'un groupe de Lie non-nécessairement abélien.

Ainsi, la théorie quadri-dimensionnelle contenant des champs scalaires de type Higgs semble provenir en fait de la réduction d'un champ de *gravitation* symétrique. Mais ces champs ϕ qui présentent un potentiel polynomial de degré 4 ne sont pas les seuls champs que l'on obtient par réduction de la théorie gravitationnelle très symétrique évoquée ci-dessus. Il y a d'autres champs également, en particulier des champs scalaires ayant des dynamiques différentes.

En fait, partant du "grand" univers particulier ci-dessus, muni du champ gravitationnel particulier très symétrique, on a le choix entre deux manières de réduire (cf. [CJ2]) : soit on fait une première réduction partielle pour avoir le couple moins symétrique, et ensuite on réduit ce dernier pour obtenir notre théorie quadri-dimensionnelle avec champs de Higgs, soit on effectue directement une réduction dimensionnelle complète du champ de gravitation symétrique dans le grand univers, ce qui donne en particulier des champs de Higgs en dimension 4, que nous aurons à distinguer des autres champs scalaires obtenus.

Ces deux méthodes semblent équivalentes, ce qui suggère qu'on pourrait omettre l'étape intermédiaire. En d'autres termes, si l'on cherche une compréhension plus fondamentale et géométrique de la nature des champs de Higgs, il faut voir du côté de la réduction des théories de gravitation.

Cette idée est aussi appuyée par le travail d'O. Maspfuhl, qui a montré dans la première partie de [Mas] que par réduction de Poisson en présence d'une contrainte coisotrope, on pouvait obtenir la dynamique de l'espace des phases de particules évoluant dans un champ

gravitationnel couplé à un secteur de Yang-Mills-Higgs, en partant d'une classe générique de Hamiltoniens définis sur une variété symplectique.

Dans le chapitre 1, nous commençons par un rappel sur la réduction dimensionnelle de la gravitation symétrique. En résumé, l'univers multidimensionnel est vu comme une variété U fibrée en espaces homogènes au-dessus d'une variété M . Au-dessus de chaque point $x \in M$, la fibre U_x est une copie d'un espace homogène G/H , où G agit à droite sur U . Coquereaux et Jadzcyk montrent dans [CJ1] que la réduction dimensionnelle à M d'une métrique G -invariante g sur U fournit un triplet (γ, α, f) où γ est une métrique sur M , α est une connexion sur un certain fibré principal de groupe structural $N(H)|H$, et f est un champ scalaire, directement relié à la famille de métriques obtenue en considérant, pour chaque $x \in M$, la restriction de g à la fibre U_x .

Nous portons ensuite une attention particulière à ce champ scalaire f , dit de Thiry, qui prend ses valeurs dans le cône $S_2^{H++}(\mathfrak{m})$ des produits scalaires $\text{Ad}(H)$ -invariants sur un supplémentaire réductif \mathfrak{m} de \mathfrak{h} dans \mathfrak{g} . Le potentiel du champ scalaire de Thiry a été calculé dans [CJ1] : il n'est pas borné inférieurement, ce qui est plutôt gênant pour la physique.

Par conséquent, nous proposons une définition nouvelle de champs scalaires de type Higgs, en distinguant des composantes particulières du champ scalaire général, sur lesquels nous récupérons un potentiel polynomial positif. Plus précisément, pour une décomposition $\text{Ad}(N(H)|H)$ -invariante $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, nous définissons les champs de Higgs comme prenant leurs valeurs dans l'espace vectoriel $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ des applications linéaires $\text{Ad}(H)$ -équivariantes de \mathfrak{m}_1 dans \mathfrak{m}_2 . Nous calculons alors le potentiel de ces nouveaux champs, obtenant le théorème suivant, qui constitue le résultat principal de cette première partie :

Theorem 0.1.1 *1. Le potentiel pour le champ scalaire ϕ est un polynôme de degré 6 au plus en ϕ , dont le terme de degré 6 est positif et est donné par :*

$$V^{(6)}(\phi) = \frac{1}{4} \| \phi ad_{12}(\phi) \phi \|_{hhk}^2$$

De plus, $V^{(6)} = 0$ si et seulement si la condition de 3ème degré suivante est satisfaite pour tout ϕ : $\{\phi \circ ad_{12}(\phi(X)) \circ \phi = 0 \quad \forall X \in \mathfrak{m}_1\}$.

2. Si $V^{(6)} = 0$, le potentiel $V(\phi)$ devient un polynôme de degré 4 au plus en ϕ , dont le terme de degré 4 est positif et est donné par :

$$\begin{aligned} V^{(4)}(\phi) &= \frac{1}{4} \| \phi ad_{11}(\phi) - ad_{22}(\phi) \phi + \phi ad_{12} \phi \|_{hhk}^2 + \frac{1}{4} \| ad_{12}(\phi) \phi \|_{hhk}^2 \\ &\quad + \frac{1}{4} \| \phi ad_{12}(\phi) \|_{hk}^2 + \frac{1}{4} \| \phi ad_{12} \phi \|_{khk}^2 \end{aligned}$$

De plus, $V^{(4)} = 0$ si et seulement si les conditions quadratiques suivantes sont satisfaites pour tout ϕ :

$$(C) \left\{ \begin{array}{ll} \phi ad_{11}(\phi(X)) - ad_{22}(\phi(X)) \phi + \phi ad_{12} \phi &= 0 & \forall X \in \mathfrak{m}_1 \\ ad_{12}(\phi(X)) \phi &= 0 & \forall X \in \mathfrak{m}_1 \\ \phi ad_{12}(\phi(X)) &= 0 & \forall X \in \mathfrak{m}_1 \\ \phi ad_{12} \phi &= 0 & \forall X \in \mathfrak{m}_2 \end{array} \right.$$

Nous calculons ensuite le terme de Yang-Mills et le terme cinétique de la théorie réduite, obtenant des résultats compatibles avec les théories de Yang-Mills-Higgs standards.

Finalement, nous donnons pour exemple un modèle avec $SU(5)/U(1)$ comme espace interne, conduisant à une théorie de jauge abélienne admettant un tore comme groupe de jauge.

Le chapitre 1 a donné lieu à un article paru dans la revue "Journal of Geometry and Physics", Volume 57, Issue 4, March 2007, Pages 1215-1237.

L'ansatz adopté pour les réductions dimensionnelles envisagées ci-dessus consiste à retenir exclusivement les métriques G -invariantes. Coquereaux et Jadczyk ont démontré que cet ansatz est *cohérent*, c'est-à-dire que toute solution de la théorie réduite peut être réinterprétée comme solution de la théorie multidimensionnelle. Or l'ansatz le plus souvent considéré dans les années 80, en particulier pour les réductions de supergravité, *n'est pas* l'ansatz G -invariant. C'est un ansatz moins contraignant, appelé *l'ansatz populaire*. Bien que génériquement incohérent, l'ansatz populaire se trouve être cohérent dans quelques cas exceptionnels, qui se révèlent être justement les cas les plus intéressants, comme par exemple la réduction sur la sphère S^7 de la supergravité à 11 dimensions, que nous allons aborder dans la dernière partie. Coquereaux et Jadczyk ont proposé une interprétation géométrique de l'ansatz populaire, que nous présentons dans le chapitre 2. Les métriques non-invariantes de l'ansatz populaire apparaissent alors comme des métriques projetées de métriques $(G \times G)$ -invariantes sur un fibré tautologique obtenu en élargissant trivialement l'univers multidimensionnel par le groupe G . On est ainsi ramené au même schéma de réduction considéré dans le chapitre précédent pour des métriques invariantes, mais cette fois-ci par $G \times G$ au lieu de G . Une conséquence est qu'une telle réduction basée sur l'ansatz populaire donne naturellement un espace de champs scalaires du type que nous avons défini dans le premier chapitre, et fournit ainsi un nouvel exemple entrant dans le cadre de notre construction.

Enfin, le chapitre 3 constitue un appendice pour la première partie. Il peut parfaitement être omis dans une première lecture. En un premier temps, nous y généralisons à $SU(n)/U(1)$ l'exemple construit dans le chapitre 1 avec $n = 5$. Les paragraphes qui suivent sont en français, et ont fait l'objet de notre mémoire de DEA (appelé Master 2 maintenant). Ils constituent un travail préliminaire à la thèse, qui nous a introduit au sujet de cette première partie. Elles peuvent intéresser le lecteur désirant avoir plus de détails sur les réductions dimensionnelles de théories de gravitation et de Yang-Mills invariantes par un groupe de symétrie, ainsi que sur la structure des champs scalaires et des lagrangiens qu'elles font intervenir. Elles peuvent également servir de motivation à la première partie, mais n'ont par ailleurs aucune incidence sur la thèse.

0.2 La gravitation vue comme géométrie de Cartan

Le 19ème siècle a vu apparaître plusieurs géométries ne satisfaisant pas les axiomes de la géométrie euclidienne. Les géométries hyperbolique, elliptique, affine, projective, sphérique, de Moebius, de Laguerre, etc... se constituèrent chacune de son côté. Ce fut Félix Klein qui, dans son *Erlanger Programm*, mit en évidence le principe unificateur pour toutes ces géométries, en constatant que l'espace de chaque géométrie est une variété connexe munie de l'action transitive d'un groupe G . Il mit l'accent sur le fait que toutes les propriétés des figures étudiées dans une géométrie donnée étaient invariantes par les

transformations du groupe. Ainsi, ce qui constitue une géométrie est entièrement contenu dans son groupe de symétrie G , que Klein appelle *le groupe principal de la géométrie*. Dans la perspective de Klein, les objets d'études en géométrie sont les *espaces homogènes*.

Dans son beau livre [Sha], R.W. Sharpe définit une *géométrie de Klein* comme étant un couple (G, H) où G est un groupe de Lie et H est un sous-groupe fermé de G , tel que l'espace homogène G/H soit connexe.

Ainsi, d'une certaine façon, une géométrie de Klein n'est rien d'autre qu'un espace homogène (pour un groupe de Lie donné). Toutefois, la portée de cette notion par rapport aux fondements de la géométrie (et de la physique !), surtout à travers sa généralisation par Cartan, justifie qu'elle soit mise en évidence de cette manière.

Etant donné une géométrie de Klein (G, H) , on peut y associer naturellement un fibré principal, que nous appelons *le fibré canonique de la géométrie de Klein* : c'est le fibré d'espace total G , de base G/H , et de groupe structural H (la fibration étant simplement la projection canonique $\pi : G \longrightarrow G/H$). Dans le cas réductif (*i.e.* lorsqu'il existe une décomposition $\text{Ad}(H)$ -invariante $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$), la forme invariante (mettons à gauche) de Maurer-Cartan θ sur G induit naturellement une connexion ω à valeurs dans \mathfrak{h} sur le fibré canonique, ainsi qu'une 1-forme tensorielle e à valeurs dans l'espace vectoriel \mathfrak{m} . Cette dernière est appelée *forme de soudure*; c'est la version intrinsèque du *vielbein* (dans le background) des physiciens. Notons qu'ici, e et ω sont définis canoniquement, une fois qu'on s'est donné une géométrie de Klein (G, H) .

Passons maintenant à la géométrie de Cartan. D'abord, il faut noter qu'une autre généralisation de la géométrie euclidienne, "orthogonale" à celle de Klein, fut établie au 19ème siècle par Bernhard Riemann, et donna ce qu'on appelle aujourd'hui la géométrie riemannienne. Alors que la généralisation de Klein consistait à remplacer le groupe euclidien par un groupe quelconque, la généralisation de Riemann maintenait le groupe orthogonal mais permettait la courbure. Chacune des deux généralisations précédentes peut être généralisée à son tour ! D'un côté, on peut "mettre de la courbure" dans une géométrie de Klein, et d'un autre côté on peut remplacer le groupe orthogonal de la géométrie riemannienne par un groupe quelconque. En fait les deux côtés se rejoignent, et donnent la même conception générale de géométrie qui englobe toutes les précédentes : celle des *espaces généralisés* d'Elie Cartan.

Ainsi, de même que la géométrie riemannienne peut être vue comme localement modélisée sur la géométrie euclidienne, une géométrie de Cartan peut être vue comme localement modélisée sur une géométrie de Klein, autrement dit sur un espace homogène quelconque.

En suivant toujours la présentation moderne effectuée par Sharpe des idées fondatrices d'E. Cartan, une *géométrie de Cartan modelée sur* (G, H) est définie comme étant un couple (P, θ) où P est un H -fibré principal au-dessus d'une variété M , et θ est une 1-forme différentielle sur P vérifiant certaines propriétés. θ est appelée une *connexion de Cartan*; ce n'est pas une connexion principale au sens usuel (d'Ehresmann). Il faut voir le fibré principal P au-dessus de M comme généralisant le groupe principal G (qui est fibré au-dessus de G/H), et la connexion de Cartan θ comme généralisant la forme de Maurer-Cartan θ . Dans le cas réductif, une connexion de Cartan se décompose en une connexion principale ω à valeurs dans \mathfrak{h} et une forme de soudure e à valeurs dans \mathfrak{m} .

La courbure de Cartan d'une connexion de Cartan est définie par $\Theta = d\theta + \frac{1}{2}[\theta, \theta]$. Elle mesure l'obstruction locale de P à être un groupe, autrement dit l'obstruction à trouver un isomorphisme local entre le fibré principal $P \rightarrow M$ et son modèle homogène $G \rightarrow G/H$. Dans le cas de la géométrie de Klein elle-même, c'est-à-dire lorsque $P = G$, $M = G/H$, et θ est la forme de Maurer-Cartan $\dot{\theta}$ de G , on a alors naturellement $\dot{\Theta} = 0$ (c'est l'équation de structure de Maurer-Cartan), et c'est en ce sens que l'on considère les espaces homogènes comme "plats". Dans le cas réductif, on peut définir la courbure $\Omega = \mathcal{D}^\omega \omega$ et la torsion $T = \mathcal{D}^\omega e$ (qui n'ont aucune raison d'être nulles dans le cas général, même pour le modèle homogène).

La pertinence de ce point de vue pour les théories de champs en physique n'est pas difficile à dégager. En un premier temps, une géométrie de Klein constitue la "géométrie background", c'est-à-dire le réceptacle fixe, dans lequel évoluent les champs. On demande souvent à ce réceptacle d'être "symétrique", voire "maximamente symétrique", d'où l'intérêt de considérer des espaces homogènes (avec un groupe de symétrie G). Ce réceptacle a le plus souvent été choisi plat au sens où $\dot{\Omega} = 0$, et sans torsion : $\dot{T} = 0$. Tel est le cas par exemple de la géométrie de Klein ($\mathbb{R}^4 \rtimes SO^\dagger(3, 1)$, $SO^\dagger(3, 1)$) dont l'espace n'est autre que l'espace-temps de Minkowski. Mais l'on peut très bien choisir d'autres réceptacles, comme l'espace-temps Anti-de Sitter AdS₄, associé à la géométrie de Klein ($SO^\dagger(3, 2)$, $SO^\dagger(3, 1)$). Dans ce cas, $\dot{\Omega} \neq 0$ (mais $\dot{T} = 0$, car c'est un *espace symétrique*...). Il y a aussi des cas où $\dot{T} \neq 0$; un exemple important -mais qui exige qu'on étende toute cette histoire au monde de la supergéométrie- est celui du super-espace-temps de Minkowski, associé à la supergéométrie de Klein ($\mathbb{R}^{4|4} \rtimes SO^\dagger(3, 1)$, $SO^\dagger(3, 1)$). Ce dernier exemple confirme pleinement l'intuition de Cartan quand il ne voyait aucune raison de négliger la torsion, en présence de la matière spinorielle...

La gravitation nous oblige à "sortir" du réceptacle fixe de la géométrie de Klein. En déformant l'homogénéité du modèle, elle introduit la courbure Θ , qui dit dans quelle mesure on s'écarte du modèle. La gravité se conçoit alors comme une géométrie de Cartan. Plus exactement, la gravité devient une théorie où la variable dynamique est une géométrie de Cartan. Cette théorie doit admettre comme solution particulière l'espace homogène "maximamente symétrique" de la géométrie de Klein, plat au sens de Cartan, qui lui sert de modèle. Le modèle Kleinien apparaît alors comme *état de vide*, valeur moyenne autour de laquelle fluctue la géométrie de Cartan. Les divers vides fournissant différents états asymptotiques sont vus par les physiciens comme des différentes phases de la théorie.

Signalons que notre idée de formuler ainsi la gravitation à l'aide des géométries de Cartan se retrouve également chez Derek K. Wise dans [Wis], qui la relie à la version de la (super-)gravité de MacDowell et Mansouri (cf. [MM]).

Le chapitre 4 commence par la définition d'une géométrie de Klein et des notions qui s'y rattachent. S'ensuit une discussion plus technique sur la manière dont s'exprime l'invariance des objets canoniquement définis par une géométrie de Klein, lorsqu'on fixe un choix de jauge. Ce paragraphe, dans lequel nous reprenons intégralement la discussion de Bernard de Wit dans [dW], mais sans utiliser de coordonnées locales, montre l'origine des compensateurs qu'introduisent les physiciens pour maintenir un choix de jauge. Dans le paragraphe qui suit, nous présentons tout simplement l'espace-temps de Minkowski habituel, mais d'une manière complètement intrinsèque, et nous terminons en

fixant les notations pour d'autres géométries de Klein qui interviendront dans la suite. L'avant-dernier paragraphe, également inspiré de [dW], rappelle quelques notions de base relatives aux σ -modèles non-linéaires à valeurs dans un espace homogène. Ceci nous sera utile dans la dernière partie, étant donné que la réduction dimensionnelle de la supergravité à 11 dimensions donne naissance à un σ -modèle à valeurs dans $E_7/SU(8)$. C'est pour cette raison d'ailleurs que nous terminons ce chapitre en donnant une réalisation, directement inspirée de F. Adams (cf. [Ad2]), de l'algèbre de Lie du groupe exceptionnel E_7 dans sa représentation fondamentale de dimension 56.

Le chapitre 5 commence par la définition d'une géométrie de Cartan et des notions qui s'y rattachent. Puis, nous évoquons brièvement le lien entre les connexions de Cartan et celles d'Ehresmann, à la suite de Sharpe (cf. [Sha]). Nous définissons ensuite la notion générale de symétrie de jauge pour une géométrie de Cartan, et nous exprimons cette transformation de jauge en terme de la connexion principale ω et la forme de soudure e . C'est là que se trouve, à notre avis, l'origine naturelle de toutes les invariances de jauge, notamment l'invariance par transformation de jauge de Lorentz, et l'invariance par translation de jauge pour une théorie de gravitation modelée sur l'espace-temps de Minkowski (cette dernière invariance par translation se traduit comme étant l'invariance par difféomorphismes, lorsqu'on passe à la formulation métrique habituelle de la relativité générale).

De même, c'est dans cette notion de symétrie de jauge pour une *supergéométrie de Cartan* que nous voyons l'origine naturelle de l'invariance par transformations de jauge de supersymétrie, principe qui fonde les théories de supergravité, qu'elles soient modelées sur le super-espace-temps de Minkowski, sur l'espace super-Anti-de Sitter, où sur n'importe quel super-espace homogène.

Dans le paragraphe qui suit (dans le chapitre 5), nous montrons que les géométries de Cartan offrent également le cadre adéquat pour définir les *spineurs de Killing*, qui apparaissent alors comme des spineurs parallèles pour une connexion de Cartan. Nous terminons ce chapitre par une brève présentation des spineurs de Killing sur la sphère de dimension 7, en raison du rôle qu'ils vont jouer dans la dernière partie.

Dans le chapitre 6, nous formulons deux théories de gravité, en partant de [dW], mais en les exprimant dans le langage des géométries de Cartan. La première, dite de Poincaré, est modelée sur l'espace-temps de Minkowski. La seconde, dite Anti-de-Sitter, est modelée sur l'espace Anti-de Sitter. En ajoutant dans chacun des cas, un champ de spineurs de Rarita-Schwinger, on devient très proche de la supergravité, dans sa formulation dite *en champs composants*. Les paramètres des transformations de supersymétries (ici jaugées puisqu'il s'agit de la supergravité) ne peuvent cependant pas être des nombres ordinaires, mais doivent être des "nombres" anticommutants, appartenant à la partie impaire d'une algèbre de Grassmann.

A ce stade, nous sommes obligés de quitter le monde "rassurant" des mathématiques non-supersymétriques, pour plonger (dans la 3ème partie) dans un super-monde où tous les objets mathématiques vont subir une graduation en parties paire et impaire, de sorte à pouvoir répondre au désir des physiciens de mettre à pied d'égalité les champs bosoniques et fermioniques.

0.3 Supervariétés et σ -modèles supersymétriques en dimensions 1 et 2

Depuis Galilée, l'idée de symétrie en physique s'est révélée être un principe fondamental, au cœur des lois de la physique. Par exemple, le "principe de relativité" de Galilée correspond à l'invariance des équations du mouvement de la mécanique classique par ce qu'on appelle aujourd'hui le *groupe de Galilée*.

En physique des particules élémentaires, on réalisa qu'on pouvait penser les états de particules comme correspondant à des représentations spécifiques du *groupe de Poincaré* (qui peut être vu comme l'extension relativiste du groupe de Galilée). Chaque représentation irréductible unitaire (d'énergie positive) du groupe de Poincaré est classifiée par un nombre réel m (la masse) et un (demi-)entier $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ (le spin). Ainsi, les bosons (resp. les fermions) correspondant à une valeur entière (resp. demi-entière) de s sont permutés entre eux par la symétrie de Poincaré.

Pour des raisons qu'il serait un peu long d'exposer ici, les physiciens ont cherché une symétrie qui pourrait échanger les bosons et les fermions, c'est-à-dire, les faire apparaître dans la même représentation irréductible d'un certain groupe de Lie. Ils découvrirent qu'une telle symétrie ne peut pas être réalisée au moyen d'une algèbre de Lie ordinaire, mais nécessitait une structure de *super-algèbre de Lie*, avec une partie paire correspondant à une algèbre de Lie ordinaire, et une partie impaire sur laquelle les crochets étaient réalisés par des anticommutateurs. Nous renvoyons aux articles fondateurs [CoM] et [HLS] pour plus de détails.

Des théories supersymétriques de champs furent alors construites. Elles correspondent à des théories de champs invariantes par une super-algèbre de Lie spécifique, appelée la *super-algèbre de Poincaré*. Celle-ci contient l'algèbre de Poincaré ordinaire, ainsi que des translations par des paramètres impairs, anticommutants. Ces translations impaires sont appelées *transformations de supersymétrie*).

La formulation initiale des théories supersymétriques, dite *en champs composants*, fait intervenir, comme dans la plupart des théories de champs usuelles, des champs bosoniques et fermioniques assujettis à des équations de champs. Le caractère supersymétrique de la théorie est relié au fait que ces champs portent des représentations de la super-algèbre de Poincaré, et que les équations des champs sont invariantes sous ces transformations "de super-Poincaré".

L'introduction du concept de super-espace ([SS], [FWZ]) permit une formulation élégante des théories supersymétriques, utilisant la notion de *superchamp*. Dans cette approche, on retrouve un véritable groupe (agissant sur le super-espace et sur les superchamps) : le groupe des supertranslations. Les transformations de supersymétrie acquièrent alors un sens géométrique naturel : elles correspondent à des translations dans les directions impaires du super-espace (la partie spinorielle des supertranslations). De plus, l'invariance par supersymétrie d'une théorie donnée est alors manifeste, et n'a pas besoin d'être explicitement vérifiée comme dans la formulation en champs composants. Pour une présentation moderne dans le langage des mathématiciens des théories de champs supersymétriques en super-espace, voir par exemple [DF].

Le chapitre 7 commence par les définitions de base nécessaires pour introduire les supervariétés. Nous suivons une approche de géométrie différentielle, dans l'esprit de Bryce DeWitt. Comme A. Bahraini l'a démontré dans la première partie de sa thèse (cf. [Bah]), cette approche est en définitive équivalente à celle des faisceaux de Berezin et Kostant (plus proche de la géométrie algébrique). Pour avoir une notion de supervariété, il faut se donner un modèle local, et y définir une notion de superfonction. Mais définissons d'abord les super-nombres. Ce sont les éléments d'une algèbre extérieure $L = \Lambda(\mathbb{R}^l)$, où l est un entier assez grand. L possède une graduation naturelle : $L = L^{ev} \oplus L^{od}$, où L^{ev} est formé des éléments de degré pair et L^{od} des éléments de degré impair. D'autre part, L se décompose de la manière suivante : $L = \mathbb{R} \oplus L^*$, ce qui permet d'associer à tout super-nombre $x \in L$ son *corps* $\beta(x) \in \mathbb{R}$ et son *âme* $\sigma(x) \in L^*$. Enfin, L^{ev} se décompose lui aussi : $L^{ev} = \mathbb{R} \oplus L^{ev,*}$. Le modèle local est alors :

$$U^{m|n} = U \times (\mathbb{R}^m \otimes L^{ev,*}) \times (\mathbb{R}^n \otimes L^{od})$$

où U est un ouvert de \mathbb{R}^m . Pour définir une notion de différentiabilité pour les *superfonctions* $F : U^{m|n} \rightarrow L$, on commence par définir la *super-extension* d'une fonction ordinaire $f \in C^\infty(U, \mathbb{R})$: c'est une application $\tilde{f} : U \times (\mathbb{R}^m \otimes L^{ev,*}) \rightarrow L^{ev}$, canoniquement définie par la série de Taylor :

$$\tilde{f}(x^1, \dots, x^m) = f(\beta(x^1), \dots, \beta(x^m)) + \sum_{k=1}^{+\infty} \frac{1}{k!} \frac{\partial^k f}{\partial x^{\mu_1} \dots \partial x^{\mu_k}}(\beta(x^1), \dots, \beta(x^m)) \sigma(x^1)^{\mu_1} \dots \sigma(x^k)^{\mu_k}$$

où $\beta(x) = (\beta(x^1), \dots, \beta(x^m)) \in U$ pour tout $x = (x^1, \dots, x^m) \in U \times (\mathbb{R}^m \otimes L^{ev,*}) \subset (L^{ev})^m$.

Une définition naturelle de *superfonction indéfiniment différentiable* $F : U^{m|n} \rightarrow L$ est alors la suivante : F est une application de $U^{m|n}$ dans L pour laquelle il existe des fonctions ordinaires $f_0, f_{\alpha_1 \dots \alpha_k} \in C^\infty(U, \mathbb{R})$ tel que :

$$F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = \tilde{f}_0(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} \tilde{f}_{\alpha_1 \dots \alpha_k}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k}$$

Avec cette définition, F sera dite de classe H^∞ . Cette définition est simple et naturelle dans le sens où les fonctions composantes de F sont des super-extensions de fonctions ordinaires de U dans \mathbb{R} . De plus, elle est naturellement reliée à la catégorie des supervariétés définie en terme de faisceaux par Berezin et Kostant, d'après le théorème d'équivalence démontré par A. Bahraini dans [Bah]. Or, ce n'est pas la définition adoptée par la majorité des physiciens, qui considèrent une classe beaucoup plus large, définie de la manière suivante : F est une application de $U^{m|n}$ dans L de la forme suivante :

$$F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = F_0(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} F_{\alpha_1 \dots \alpha_k}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k}$$

où les $F_0, F_{\alpha_1 \dots \alpha_k}$ sont chacune une application $F_A : U \times (\mathbb{R}^m \otimes L^{ev,*}) \rightarrow L$ de la forme :

$$F_A(x^1, \dots, x^m) = \tilde{f}_0(x^1, \dots, x^m) + \sum_{k=1}^{+\infty} \tilde{f}_{i_1 \dots i_k}(x^1, \dots, x^m) \zeta^{i_1} \dots \zeta^{i_k}$$

les ζ^i étant les générateurs de L . Avec cette définition, F sera dite de classe G^∞ .

On voit que la différentiabilité G^∞ fait intervenir un nombre énormément plus élevé de fonctions composantes. Les deux notions de différentiabilité donnent lieu à deux catégories de supervariétés aux propriétés assez différentes. Nous poursuivons le chapitre avec les définitions de *super-champ de vecteurs*, *super-applications*, *supervariétés*, *super-espace tangent*,... dans chacune des deux catégories, en essayant de voir les différences. Dans le dernier paragraphe, une attention particulière est portée au super-espace tangent dans la catégorie H^∞ .

Dans la suite, nous prenons le parti de la catégorie H^∞ . Nous examinons dans le chapitre 8 les conséquences de ce choix pour les σ -modèles non-linéaires supersymétriques en dimensions 1 puis 2. Commençons par la dimension 1. Après avoir défini la supervariété plate $\mathbb{R}^{1|1}$, nous détaillons l'action des supertranslations sur $\mathbb{R}^{1|1}$ et sur les superfonctions définies dessus. Ensuite, nous considérons une supervariété quelconque $M^{m|n}$, associée à un fibré vectoriel E de rang n au-dessus d'une variété ordinaire M de dimension m . La donnée d'une métrique g sur M , d'une connexion ∇ sur E , et d'une 2-forme antisymétrique B sur E compatible avec ∇ , se réinterprète en termes d'une *supermétrique* G sur la supervariété $M^{m|n}$. De même, la donnée d'un chemin $x : I \longrightarrow M$ et d'une section ψ de E le long de x se réinterprète en termes d'un *superchemin* $X : \mathbb{R}^{1|1} \longrightarrow M^{m|n}$. Il est alors possible d'écrire une action manifestement supersymétrique portant le superchemin X :

$$\mathcal{A}(X) = \frac{1}{2} \int_{\mathbb{R}^{1|1}} G_{X(t,\tau)}(\mathcal{H}X(t,\tau), \mathcal{D}X(t,\tau)) d\tau dt$$

Les superchemins qui extrémisent cette action sont appelés des *supergéodésiques*. Nous montrons alors que :

En restant dans la catégorie H^∞ , et grâce à l'introduction de la 2-forme antisymétrique B , on obtient de nouvelles équations, classiques mais supersymétriques, portant sur le chemin x et la section ψ le long de x :

$$\left\{ \begin{array}{l} \frac{D\dot{x}}{dt} + \frac{1}{2} \mathbf{R}_{\dot{x},\psi,\psi} = 0 \\ \frac{\nabla\psi}{dt} = 0 \end{array} \right.$$

Ces équations ne semblent pas avoir été considérées dans la littérature ; elles peuvent être comparées à des équations de supergéodésiques différentes obtenues dans [DE] par Deligne et Freed (qui travaillent plutôt avec la catégorie G^∞).

Des exemples particuliers peuvent être donnés. Si l'on considère par exemple le cas où M est une sphère, alors les équations ci-dessus admettent *tous* les cercles de cette sphère comme solutions. On pourrait considérer également le cas où M est un espace projectif.

Enfin, nous reprenons l'étude en dimension 2. Après un petit rappel sur les propriétés des spineurs en signature $(1, 1)$ (notamment l'existence de spineurs *Majorana-Weyl*), nous sommes en mesure de définir la supervariété plate $\mathring{\Sigma}^{2|(1,1)}$ associée à un espace-temps de Minkowski de signature $(1, 1)$ et à un fibré \mathbb{S} de spineurs naturellement défini dessus. Nous détaillons ensuite l'action des supertranslations sur $\mathring{\Sigma}^{2|(1,1)}$ et sur les superfonctions définies dessus. Puis, comme nous l'avons fait en dimension 1, nous considérons une supervariété quelconque $M^{m|n}$, associée à un fibré vectoriel E de rang n au-dessus d'une

variété ordinaire M de dimension m . La donnée d'une métrique g sur M , d'une connexion ∇ sur E , et d'une 2-forme antisymétrique B sur E compatible avec ∇ , se réinterprète en termes d'une *supermétrique* G sur la supervariété $M^{m|n}$. De même, la donnée d'une nappe $X : \mathring{\Sigma} \longrightarrow M$, d'une section ψ de $E \otimes \mathbb{S}$ le long de X et d'un champ de vecteurs F le long de X , se réinterprète en termes d'une *super-nappe* $Y : \mathring{\Sigma}^{2|(1,1)} \longrightarrow M^{m|n}$. Il est alors possible d'écrire une action manifestement supersymétrique portant la super-nappe Y :

$$\mathcal{A}(Y) = \frac{1}{2\pi} \int_{\mathring{\Sigma}^{2|(1,1)}} G_{Y(\sigma,\tau)}(\mathcal{D}_+ Y(\sigma,\tau), \mathcal{D}_- Y(\sigma,\tau)) d^2\tau d^2\sigma$$

Les super-nappes qui extrémisent cette action seront simplement appelées des *super-nappes extrémales*. Nous montrons alors que :

En restant dans la catégorie H^∞ , et grâce à l'introduction de la 2-forme antisymétrique B , on obtient de nouvelles équations, classiques mais supersymétriques, portant sur la nappe X , le champ de spineurs ψ le long de X , et le champ de vecteurs F le long de X :

$$\left\{ \begin{array}{l} \square^\nabla X + \frac{1}{2} \mathbf{R}_{\partial X, \psi, \psi} = 0 \\ \nabla_+ \psi^+ = 0 \\ \nabla_- \psi^- = 0 \\ F = 0 \end{array} \right.$$

Ces équations ne semblent pas avoir été considérées dans la littérature ; elle peuvent être vues comme une généralisation au cas où M est une variété quelconque (courbe) et E un fibré vectoriel quelconque (pas nécessairement le fibré tangent) d'équations obtenues dans [GSW] avec $M = \mathbb{R}^m$ et $E = T\mathbb{R}^m$. Notons que Green, Schwarz et Witten se placent plutôt dans la catégorie G^∞ .

Enfin, nous commençons dans le chapitre 9 par décrire la théorie de Wess-Zumino, connue pour être la première théorie de champs supersymétrique sur l'espace-temps plat de Minkowski. Nous présentons sa formulation en super-espace (sur le super-espace-temps plat -mais avec supertorsion !- de Minkowski). Cette théorie constitue un exemple de σ -modèle linéaire supersymétrique en dimension 4. Nous y présentons des calculs explicites en coordonnées. Ensuite, nous passons à une brève présentation des théories de jauge supersymétriques, en suivant directement la belle présentation de [Gie].

0.4 Géométrie de la supergravité en super-espace

Pour construire une théorie supersymétrique de la gravité, les physiciens réalisèrent qu'ils avaient besoin de promouvoir les transformations de supersymétries à des "transformations de supersymétrie jaugées" (c'est-à-dire des transformations qui dépendent de chaque point de l'espace-temps, comme dans les théories de Yang-Mills). Ceci donna ce qu'on appelle : *les théories de supergravité*.

Les théories de supergravité furent d'abord données dans la formulation en champs composants. Ensuite la version en super-espace fut construite pour la supergravité simple à quatre dimensions (la version complète, off-shell, de la supergravité $N = 1$ en $d = 4$ se

trouve par exemple dans [WB]) (N désigne *le nombre de supersymétries*, c'est à dire le nombre de copies de la représentation spinorielle irréductible intervenant dans la superalgèbre de Poincaré de la théorie). Des théories de supergravité dans d'autres dimensions furent aussi formulées en super-espace (les supergravités $d = 10$ et $d = 11$ ne sont connues que sur la couche de masse (on-shell), voir [HW] et [BH]), et sans lagrangien (quoiqu'un progrès a été fait dans cette direction [DLT]).

Un aspect particulier de la supergravité en super-espace est la nécessité des "contraintes de torsion". Les champs de base dans la formulation en super-espace de la supergravité sont : le *supervielbein* (repère mobile), et la *superconnexion*. Ces deux objets induisent deux tenseurs covariants : la *supertorsion* et la *supercourbure*. Pour chaque théorie, il faut trouver un ensemble convenable de contraintes sur le tenseur de supertorsion, qui permettent, en utilisant les identités de Bianchi, d'éliminer la plupart des superfonctions composantes en les exprimant en termes d'un petit nombre de superchamps indépendants.

Trouver les bonnes contraintes de torsion n'est pas aisés. On exige d'elles d'être covariantes de Lorentz et supersymétriques, et en général, on fait appel à des arguments d'analyse dimensionnelle pour trouver les contraintes appropriées. De plus, leur origine géométrique n'est pas claire.

Nous commençons par une présentation de la géométrie -plus précisément de la supergéométrie- sous-jacente aux théories de supergravité. Nous utilisons un langage intrinsèque de géométrie différentielle (en essayant d'éviter les indices). Après avoir mis en place un cadre général pour l'étude de la gravité, avec la notion de G -structure, nous passons rapidement au cas supersymétrique, en considérant un super-fibré vectoriel $\mathbb{V}^{d|n}$ (de rang $(d|n)$) au-dessus d'une supervariété $M^{d|n}$ (de dimension $(d|n)$) qui joue le rôle de super-espace-temps.

G étant un supergroupe de Lie, sous-groupe de $\mathrm{GL}(d|n)$, nous définissons alors une G -structure sur $\mathbb{V}^{d|n}$ comme étant une réduction P à G du fibré des repères de $\mathbb{V}^{d|n}$. Ensuite nous fixons une fois pour toutes une G -structure, dite *auxiliaire*, sur $\mathbb{V}^{d|n}$. Pour définir une G -structure sur la supervariété $M^{d|n}$, autrement sur son super-fibré tangent, il suffit de se donner une *super-forme de soudure*, c'est-à-dire un isomorphisme $e : TM^{d|n} \longrightarrow \mathbb{V}^{d|n}$ (version intrinsèque du *supervielbein* des physiciens), avec lequel on rappelle la G -structure auxiliaire. La super-forme de soudure e est alors prise comme variable dynamique de la théorie ; en la faisant varier, on fait varier avec elle la G -structure sur $M^{d|n}$.

En s'inspirant de J. Lott (cf. [Lot]), on peut alors définir la *fonction de structure du premier ordre* de la G -structure sur $M^{d|n}$ définie par e (où plus brièvement : la *fonction de structure du premier ordre* de e). C'est une fonction sur P à valeurs dans le groupe de cohomologie de Spencer $H_{\mathfrak{g}}^{0,2}$, définie à partir de la supertorsion d'un couple (e, ω) , mais dépendant uniquement de e et pas de la superconnexion spin ω . Elle mesure l'obstruction locale de la G -structure définie par e à être *plate au premier ordre*, autrement dit à pouvoir trouver une superconnexion spin ω tel que la supertorsion du couple (e, ω) soit égale à la supertorsion plate canonique (plus généralement à la supertorsion canonique du modèle de Klein).

Dans ces conditions, les contraintes sur la supertorsion se ramènent à demander une platitude au premier ordre de G -structures, pour un supergroupe spécifique G . Il se trouve

qu'en prenant pour G la réalisation naturelle du groupe de Lorentz sur la fibre type de $V^{d|n}$, les contraintes obtenues sont trop restrictives et forcent le super-espace-temps à être plat. Lott propose alors une certaine extension du supergroupe G , et exige la platitude au premier ordre pour la structure étendue (la structure initiale cessant alors d'être plate au premier ordre). De cette façon, il retrouve les contraintes de supertorsion pour plusieurs théories de supergravité, en particulier pour la supergravité $N = 1$ en dimension 4.

Dans le dernier paragraphe du chapitre 10, nous proposons de regarder la supergravité d'un autre point de vue, en tant que théorie de jauge pour le supergroupe de Poincaré. La variable dynamique est alors initialement une superconnexion $\check{\omega}$ à valeurs dans la super-algèbre de Lie de Poincaré. Elle se décompose en une superconnexion spin ω et une super-forme de soudure e . Partant de la transformation de jauge naturelle sur $\check{\omega}$, nous en déduisons des transformations de jauge sur e et ω . En particulier, nous obtenons très naturellement de cette manière l'action des supertranslations de jauge $\xi \in \Gamma(\mathbb{V}^{d|n})$ sur le supervielbein e :

$$\delta_\xi e = [\xi, e] + \mathcal{D}^\omega \xi$$

Par ailleurs, nous définissons une autre action naturelle sur le supervielbein, celle des champs de vecteurs $X \in \Gamma(TM^{d|n})$ agissant par *dérivée de Lie covariante* : $\forall Y \in \Gamma(TM^{d|n})$,

$$(\mathcal{L}_X^\omega e)(Y) = \nabla_X^\omega(e(Y)) - e([X, Y])$$

Il reste à faire le lien entre ces deux façons d'agir sur le supervielbein. Il s'avère alors que les contraintes sur la supertorsion entraînent l'existence d'une jauge où ces deux transformations sur le supervielbein deviennent équivalentes. Nous avons ainsi obtenu le théorème suivant, qui peut être vu comme le résultat principal de cette partie de la thèse :

Theorem 0.4.1 *Soit $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ une super-forme de soudure, $\omega \in \mathcal{C}(P)$ une superconnexion, et T la supertorsion du couple (e, ω) . Il y a équivalence entre les supertranslations de jauge infinitésimales agissant sur e et les super-diffeomorphismes infinitésimaux agissant sur e , si et seulement si la contrainte $T(X, Y) = [e(X), e(Y)]$ est satisfaite (pour tous $X, Y \in \Gamma(TM^{d|n})$).*

0.5 Compactification sphérique de la supergravité à onze dimensions

La théorie de la supergravité à onze dimensions, construite en 1978 par Cremmer, Julia et Scherk dans [CrJS], a suscité un immense intérêt parmi les physiciens, apparaissant à l'époque comme le premier candidat potentiel pour une *théorie du tout*. Parmi les raisons à l'origine de cet intérêt considérable, le fait que 11 est le nombre maximal de dimensions où il est possible d'éviter des particules de spin strictement supérieur à 2 (en maintenant une seule dimension temporelle), et en même temps, 11 paraissait comme le nombre minimal de dimensions nécessaires pour produire un groupe de jauge qui contienne ceux du "modèle standard", c'est-à-dire U(1) pour l'interaction électromagnétique, SU(2) pour l'interaction nucléaire faible et SU(3) pour l'interaction nucléaire forte. La supergravité $N = 1$ en $d = 11$ de Cremmer, Julia et Scherk est la seule théorie *classique* connue en

dimension 11 à être invariante par les transformations de supersymétrie jaugées (sans faire intervenir des champs de spin > 2).

Notons qu'à priori, le plus grand groupe de jauge qu'on obtient ainsi pour agir sur les fermions est $\text{SO}(8)$. Cependant, nous verrons apparaître des symétries pour $\text{SO}(8) \times \text{SO}(8)$, et surtout, grâce aux dualités en dimension 10, les groupes $\text{SO}(32)$ et $E_8 \times E_8$ apparaissent naturellement.

D'un autre côté, Freund et Rubin ont montré dans [FR] que la supergravité à onze dimensions donnait lieu naturellement à une compactification spontanée préférentielle de 7 dimensions, générant 4 dimensions d'espace-temps (en fait, la compactification de Freund et Rubin permet aussi 7 dimensions non-compactes d'espace-temps + 4 dimensions compactes, sans qu'il soit évident d'exclure mathématiquement ce scénario). Il existe d'autres compactifications bien sûr, mais celle de Freund-Rubin est capable de préserver toutes les supersymétries, auquel cas les quatre dimensions non-compactes forment l'espace-temps Anti-de-Sitter AdS_4 et les sept dimensions compactifiées forment la sphère ronde S^7 .

Parallèlement, en effectuant une compactification de la supergravité $d = 11$ sur un tore T^7 , Cremmer et Julia (cf. [CrJ]) ont construit la supergravité étendue $N = 8$ en dimension 4 ($N = 8$ est le nombre maximal de supersymétries en dimension 4). En étudiant les champs scalaires obtenus par leur compactification toroïdale, Cremmer et Julia s'aperçoivent que ceux-ci constituent un σ -modèle non-linéaire à valeurs dans l'espace homogène $E_7/SU(8)$. En plus de l'invariance de jauge $SU(8)$ (due au caractère étendu de la théorie et à l'action des supersymétries sur les champs scalaires), le groupe exceptionnel E_7 apparaît comme une symétrie globale de la théorie $N = 8$ en dimension 4. Cremmer et Julia relient cette symétrie à des considérations de dualité. Signalons enfin que les champs scalaires obtenus n'ont pas un potentiel dans cette théorie, qui possède par ailleurs une invariance globale par un groupe $\text{SO}(8)$.

Ensuite, dans [dWN], de Wit et Nicolai retrouvent cette même théorie $N = 8$, mais en la construisant directement dans la dimension 4. Cependant, ils trouvent aussi qu'en modifiant convenablement le lagrangien et les transformations de supersymétrie, on obtient une théorie de supergravité $N = 8$ en dimension 4 où l'invariance globale $\text{SO}(8)$ est devenue une invariance de jauge. Dans cette supergravité $N = 8$ jaugée de de Wit et Nicolai, les champs scalaires forment toujours un σ -modèle non-linéaire à valeurs dans $E_7/SU(8)$, mais acquièrent en plus un potentiel scalaire.

Donnons au passage le spectre de la supergravité $N = 8$ jaugée en dimension 4 :

spin	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	0
champ	e	ψ	A	χ	u	v
Rep. de $\text{SU}(8)$	\mathbb{R}	\mathbb{C}^8	\mathbb{R}	$\Lambda^3(\mathbb{C}^8)$	$\Lambda^2(\mathbb{C}^8)$	$\Lambda^2(\bar{\mathbb{C}}^8)$
Rep. de $\text{SO}(8)$	\mathbb{R}	\mathbb{R}	$\Lambda^2(\mathbb{R}^8)$	\mathbb{R}	$\Lambda^2(\mathbb{R}^8)$	$\Lambda^2(\mathbb{R}^8)$

où e est la 1-forme de soudure, ψ est la 1-forme de Rarita-Schwinger, A est la connexion pour $\text{SO}(8)$, χ est un "trispineur", et u et v forment le champ scalaire à valeurs dans

$E_7/SU(8)$.

Cette théorie $N = 8$ jaugée (mais aussi d'autres supergravités pour lesquelles un groupe de symétrie globale était jaugé par un mécanisme analogue) a particulièrement retenu l'attention des physiciens. Dans le cas $N = 8$, cela était surtout lié à la conjecture suivante, qui s'était imposée naturellement : *la supergravité $N = 8$ jaugée de Cremmer et Julia devait correspondre à la troncature au secteur sans masse de la théorie obtenue en compactifiant la supergravité à 11 dimensions sur la sphère S^7* . La démonstration complète de ce résultat se révéla un problème ardu, que de Wit et Nicolai résolvèrent quelques années plus tard dans [dWN1] et [dWN2].

La solution élaborée par de Wit et Nicolai repose sur une reformulation de la supergravité à 11 dimensions de [CrJS], où l'invariance de Lorentz explicite est perdue (pour le groupe de Lorentz en signature $(10, 1)$), mais qui exhibe une symétrie de jauge pour le groupe $\text{Spin}^\dagger(3, 1) \times SU(8)$. Cette reformulation s'avère très adaptée pour effectuer la réduction dimensionnelle jusqu'à $d = 4$.

D'un point de vue géométrique, les constructions très poussées de de Wit et Nicolai demeurent cependant assez obscures. Ces derniers se basent essentiellement sur les transformations de supersymétrie en champs composants pour dégager (avec une virtuosité technique admirable) toutes les redéfinitions judicieuses qui conduisent à des champs possédant une covariance $SU(8)$. Cependant, les expressions des transformations de supersymétrie en champs composants cachent parfois le sens géométrique (ce qui est d'ailleurs le cas dans toutes les théories supersymétriques). La signification géométrique transparente de la supersymétrie ne se révèle qu'avec l'approche en super-espace. Or, celle-ci ne semble pas aisée à formuler (en dehors de la couche de masse) pour les théories en dimension 11 (cf. [BH]) et 10 (cf. [HW]), ni pour les théories avec supersymétries étendues ($N > 1$) en dimension 4.

Des structures mathématiques très intéressantes sont sous-jacentes à ces constructions des physiciens : la théorie des groupes bien sûr (notamment les groupes exceptionnels), la théorie des représentations (omni-présente), mais de Wit et Nicolai voient déjà certaines de leurs équations comme des équations de structure (de type Maurer-Cartan), et ils pressentent une géométrie qui ressemble à la géométrie riemannienne, mais plus générale.

A nouveau, nous avons la conviction que le cadre naturel de ces théories de supergravité et de leurs compactifications est celui des géométries de Cartan. Si l'on s'en tient aux champs composants, la clarification restera toutefois partielle. Il nous semble qu'il faille développer une théorie des supergéométries de Cartan, en tenant compte aussi de l'approche par les G -structures (utilisée par Lott pour comprendre géométriquement les contraintes de supertorsion, cf. la troisième partie).

Signalons que la réduction des fermions pose des problèmes difficiles qui ne seront pas abordés ici ; en particulier parce qu'en dimension 4, les fermions connus font jouer un rôle différent aux diverses représentations spinorielles (on dit qu'ils sont chiraux). Une solution qui a été envisagée est de pratiquer des projections au niveau de la théorie quantique (théorie GSO), ou de réduire par des orbifolds plutôt que par des variétés sans bord.

Dans ce dernier chapitre (11) de la thèse, nous poursuivons essentiellement deux buts :

- Reprendre d'abord la formulation de la supergravité de [CrJS], toujours en champs composants, mais en termes plus intrinsèques, et en appliquant le cadre des géométries de Cartan dans lequel nous nous sommes déjà placés dans la deuxième partie (pour les théories de gravité). Ensuite, nous essayons de décrire la compactification dans ce langage. Le point de départ est une réduction du groupe structural $\text{Spin}^\dagger(10, 1)$ au groupe $\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)$. Or ceci revient à changer la géométrie modèle et son groupe principal (Poincaré en dimension $(10, 1)$). Un choix possible naturel est alors $\text{Spin}^\dagger(3, 2) \times \text{Spin}(8)$, et correspond au background compactifié $\text{AdS}_4 \times S^7$. Nous remarquons en passant que l'expression dans ce background d'une certaine dérivée covariante généralisée, qui apparaît dans les équations de la supergravité de [CrJS], et qui dépend de la 3-forme \hat{A} que cette théorie fait intervenir, peut être vue comme une dérivée covariante associée à une connexion de Cartan. Cela confirme le souhait de voir la 3-forme apparaître comme la partie spin-spin du superelfbein, et non comme une composante d'une super-3-forme. Nous esquissons d'ailleurs très rapidement le point de vue en super-espace, que nous espérons par la suite développer davantage. Une théorie naturelle pour la compactification devrait être obtenue en prenant comme variable dynamique une supergéométrie de Cartan modelée sur la supergéométrie de Klein ($\text{Osp}(4|8)$, $\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)$).
- Les champs de Higgs, et faire le lien avec la première partie de la thèse. Pour cela, nous devons d'abord effectuer la réduction des champs de la supergravité à 11 dimensions autour du vide $\text{AdS}_4 \times S^7$. En traduisant toujours [dWN1] et [dWN2], on trouve que la partie interne de l'elfbein fournit un secteur scalaire correspondant à un σ -modèle non-linéaire à valeurs dans $E_7/SU(8)$. L'invariance par supersymétrie de jauge de l'elfbein se traduit alors par l'action de E_7 sur le champ scalaire Φ obtenu. Par ailleurs, le champ spinoriel de Rarita-Schwinger dans les 11 dimensions se réduit quant à lui en un champ analogue en dimension 4, et en une partie interne qui correspond à un *trispineur* en dimension 4. L'invariance par supersymétrie de jauge de ces deux spineurs réduits fait apparaître deux fonctions A_1 et A_2 à valeurs dans des représentations irréductibles de $SU(8)$. Ces deux fonctions dépendent directement du champ scalaire Φ ; elles sont polynômales et de degré 3 dans les coordonnées naturelles sur la représentation de dimension 56 de E_7 . Dans [dWN2], de Wit et Nicolai montrent que la compactification sphérique de la supergravité en $d = 11$ fournit un potentiel quadratique en A_1 et A_2 et donc de degré 6 en les coordonnées de Φ :

$$\mathcal{V}(\Phi) = -g^2 \left\{ \frac{3}{4}|A_1|^2 - \frac{1}{24}|A_2|^2 \right\}$$

Ce potentiel coincide avec celui qu'ils avaient calculé dans [dWN] pour la supergravité $N = 8$ jaugée en $d = 4$. Il est tentant d'essayer de le rapprocher du potentiel que nous avons calculé dans la première partie. Notons que l'espace homogène non-compact $E_7/SU(8)$ qui intervient ici se réalise naturellement comme une sous-variété de dimension 70 dans l'espace riemannien symétrique $\text{Sp}(56, \mathbb{R})/\text{SU}(56)$ (lui-même sous-variété de $\text{SL}(56, \mathbb{R})/\text{SO}(56)$), qui s'interprète comme l'espace des métriques $(\text{SO}(8) \times \text{SO}(8))$ -invariantes sur $\text{SO}(8) \times \text{SO}(8)$.

Bien que ces espaces symétriques possèdent des structures affines naturelles, le plongement dans $\text{SL}(56, \mathbb{R})$ n'est pas affine; même si on est tombé sur des potentiels polynomiaux de degré 6, on est donc à priori dans une situation différente que

celle qu'on a vue au chapitre 1. Pourtant, nous verrons qu'il existe une infinité de sous-variétés affines naturelles de dimension 7, correspondant à des sous-groupes paraboliques de $\mathrm{SL}(8, \mathbb{R})$, via l'identification de \mathfrak{e}_7 avec $\mathfrak{sl}(8, \mathbb{R}) \oplus \bigwedge^4(\mathbb{R}^8)$. Sur celles-ci, les coordonnées affines globales correspondent à celles qui viennent de la dimension 56, et on a bien des polynômes de degré 6. Les familles de métriques qui leur correspondent viennent des facteurs invariants de l'espace des métriques projetées (comme au chapitre 2), mais ne semblent pas rentrer dans le cadre envisagé au chapitre 1.

Finalement, mentionnons que le potentiel \mathcal{V} n'est pas borné inférieurement sur $E_7/SU(8)$. Cependant G.W. Gibbons, C.M. Hull and N.P. Warner ont démontré un théorème de masse positive dans cette situation (asymptotiquement AdS_4 , avec $N = 8$ supersymétries), qui montre que la gravitation stabilise la solution (cf. [GHW]).

Chapitre 1

Canonical Higgs fields from higher-dimensional gravity

1.1 Introduction

In the usual gauge theories, Higgs fields appear at the classical level as scalar fields ϕ , whose dynamics are prescribed by a fourth-degree polynomial potential

$$V(\phi) = \lambda (|\phi|^2 - a^2)^2 \quad (\lambda > 0)$$

at least when we are in front of a spontaneous symmetry breaking mechanism that will generate mass for the gauge particles.

At first sight, such Higgs potentials seem to be put in by hand, while it would be interesting to have a geometrical understanding of their origin. The approach considered here is that of dimensional reduction theories, behind which the central idea is the existence of a simple and symmetric field theory in a higher-dimensional universe, leading through a process of dimensional reduction, to the field theories we see in our four-dimensional space-time.

A first step in this direction is to search for multidimensional theories that would give, by dimensional reduction, field theories containing Higgs-like scalar fields.

A well-known solution for this problem is to take a symmetric Yang-Mills theory on a multidimensional universe. Under appropriate conditions, the dimensional reduction of such a theory, studied first by Manton (cf. [Man], [CM]), gives : a Yang-Mills theory with Higgs fields exhibiting a symmetry-breaking potential like the one above.

Actually, studies made by A. Jadczyk, R. Coquereaux and K. Pilch (cf. [JP], [Jad] and [CJ2]) showed the following : if one wants to perform the dimensional reduction of a symmetric Yang-Mills field in a sufficiently general setting, then the natural object to reduce is not a symmetric Yang-Mills field alone, but an appropriate combination of two fields : a symmetric Yang-Mills field + a symmetric gravity field.

They also showed that the best way to get such an appropriate combination, is to consider that the multidimensional universe in which these two symmetric fields live is itself the result of the dimensional reduction of another universe with even more dimensions,

and by adding a constant field to the symmetric couple, we can get all the fields from a single and very symmetric gravitational field on the universe with the highest dimension. In this sector, we are in front of Kaluza-Klein theories, as they have been extended by Richard Kerner (cf. [Ker1]) to the case where the compactified extra-dimensions are those of a non-necessarily abelian Lie group.

Thus, the four-dimensional theory containing Higgs-like scalar fields seems to come in fact from the dimensional reduction of a symmetric *gravity* field. But those ϕ fields which exhibit a fourth-degree polynomial potential are not the only fields one obtains from the reduction of the very symmetric gravitational theory above. There are other fields also, in particular scalar fields with different dynamics.

In fact, starting from the above particular "big" universe, with the particular very symmetric gravitational field, we have the choice between two ways of reducing (cf. [CJ2]) : either we make a first partial reduction to get the less symmetric couple, and then reduce this last one to get our four-dimensional theory with Higgs fields, either we make directly a complete dimensional reduction of the symmetric gravity field in the big universe, which gives in particular Higgs fields on space-time, that we'll have to distinguish from other scalar fields obtained.

These two methods seem equivalent, which implies that the intermediate level could be omitted ! In other words, if one is seeking for a more fundamental and geometrical understanding of the nature of the Higgs fields, this is to be found in the dimensional reduction of a gravitational theory.

This idea is supported also by the work of O. Maspfuhl, who has showed in the first part of [Mas] that by Poisson reduction in the presence of a coisotropic constraint, one can get the phase space dynamics of particles in gravitational, Yang-Mills and Higgs fields, by starting from a generic class of Hamiltonians defined on a symplectic manifold.

In section 1.2, we will make a short recap on dimensional reduction of symmetric gravity, with a focus on the scalar fields obtained from such a reduction that are interpreted as collections of invariant metrics on homogenous spaces, and are known to have a rather different behavior than the usual Higgs fields. Therefore, we will propose in section 1.3 a definition of Higgs-like scalar fields, by distinguishing components of the general scalar fields on which we recover a positive polynomial potential. We study then the Yang-Mills term and the kinetic term of the reduced theory. Finally, we study in section 1.4 a model with $SU(5)/U(1)$ as compactified space, leading to an abelian gauge theory having a torus as gauge group.

1.2 Dimensional reduction of symmetric gravitational fields

1.2.1 The dimensional reduction theorem

Coquereaux and Jadzcyk performed the dimensional reduction of symmetric gravitational fields in the general context of fiber bundles with homogenous fibers (cf. [CJ1]), and the short recap we present in this first part is directly inspired from their work.

One starts with a manifold U , on which a compact Lie group G is acting on the right. Let M the quotient manifold of U by the action of G . One chooses then some point u_0 in U , and let H be the stabilizer of u_0 under G . The action of G is supposed to be **regular** (or **simple**) in the following sense : every stabilizer under G is conjugated to H . By the orbit theorem, which states the elementary fact that each orbit uG is diffeomorphic to $G/\text{stab}(u)$, we deduce that all the orbits are in fact diffeomorphic to the same homogeneous space which is G/H . Therefore, U can be realized as a fiber bundle over M , the fibers being homogeneous spaces for the group G , copies of the typical fiber G/H .

We would like to see the fiber bundle $(U, M, G/H)$ as associated to some principal fiber bundle. For this, we introduce the normalizer $N(H)$ of H in G . Notice that since we started with a right action of G on U , G/H is the space of right cosets Ha , and thus G/H is equipped with a right action of G . So G does not act naturally on G/H on the left. But if we take the normalizer $N(H)$, then $N(H)$ acts of course on G/H on the right, but it also acts naturally on G/H on the left ! Indeed, all one has to do is to set $n(Ha) = Hna$ and then one gets a left action of $N(H)$ on G/H , and since $(nh)(Ha) = n(Ha)$, we see that $N(H)|H$ acts also on G/H , on the left. On the other hand, the right action of G on U induces a right action of $N(H)$ on U . Let Q be the submanifold of U whose points have exactly H as stabilizer. It's easy to see then that Q is invariant under the right action of $N(H)$: indeed, for $q \in Q$ and $n \in N(H)$, $\text{stab}(qn) = n^{-1}\text{stab}(q)n = n^{-1}Hn = H$, so $qn \in Q$. Notice also that $q(nh) = qh'n = qn$, so here also, we get an action of $N(H)|H$, but this time on Q and on the right. The difference between the right-action of $N(H)$ on Q , and that of $N(H)|H$, is that the latter is free. Therefore, Q is realized as a principal fiber bundle on M , with $N(H)|H$ as structure group. Now, using the left action of $N(H)|H$ on G/H , we may construct the associated fiber bundle $Q \times_{N(H)|H} G/H$, and it's not hard to see that $(q, Ha)N(H)|H \mapsto qa$ is an isomorphism from $Q \times_{N(H)|H} G/H$ to U .

Now we get to the infinitesimal level. Let \mathfrak{g} denote the Lie algebra of G , and \mathfrak{h} that of H . The group G being compact, it is possible to choose a scalar product \langle , \rangle on \mathfrak{g} that is invariant by the adjoint representation of G . Let us call \mathfrak{m} the orthogonal complement of \mathfrak{h} in \mathfrak{g} . Since H acts on \mathfrak{h} and the scalar product is in particular $\text{Ad}(H)$ -invariant, it is clear that we get then a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ (reductive simply means that the subspace \mathfrak{m} is invariant under the restriction to H of the adjoint representation). Thus, we have a representation $\text{Ad}_{|H}^{\mathfrak{m}} : H \longrightarrow GL(\mathfrak{m})$ that will be important in the following. We also introduce a vector space that is going to play a significant role :

Let $S_2^H(\mathfrak{m})$ denote the vector space of all $\text{Ad}(H)$ -invariant symmetric bilinear forms on \mathfrak{m} . In other terms, $S_2^H(\mathfrak{m}) = \{f \in \text{Sym}^2(\mathfrak{m}^*) / {}^t\text{Ad}_h(f) = f \quad \forall h \in H\}$.

We shall endow this vector space with a representation of the group $N(H)|H$: first, notice that \mathfrak{m} , being the orthogonal complement of \mathfrak{h} in \mathfrak{g} for our $\text{Ad}(G)$ -invariant scalar product on \mathfrak{g} , is not only $\text{Ad}(H)$ -invariant, but also $\text{Ad}(N(H))$ -invariant (since $N(H)$ acts on \mathfrak{h} and the scalar product is in particular $\text{Ad}(N(H))$ -invariant). Thus, $N(H)$ acts on \mathfrak{m} , and therefore, it acts on the space of symmetric bilinear form on \mathfrak{m} . Actually, it's easy to check that the subspace $S_2^H(\mathfrak{m})$ is invariant under $N(H)$, and that this representation induces a representation ρ of $N(H)|H$ on $S_2^H(\mathfrak{m})$, given by : $\rho_{[n]}(f) = {}^t\text{Ad}_{n^{-1}}(f)$.

Now, starting from a G -invariant metric g on U , we can construct a triple of fields (γ, α, f) , that can be considered locally as defined on M .

- . At each point u in U , $g(u)$ is a scalar product on the tangent space $T_u U$. If we call Z_u the orthogonal complement of the vertical subspace $V_u(U)$ at u , we get a horizontal distribution on the fiber bundle $(U, M, G/H)$, that is, a connection α on the principal fiber bundle $(Q, M, N(H)|H)$.
- . Next, the connection α provides an isomorphism at each point u between the horizontal space Z_u and the tangent space $T_x M$ of M at $x = \pi(u)$ ($\pi : U \rightarrow M$ being the bundle's projection). This isomorphism allows us to project the scalar product $g(u)|_{Z_u \times Z_u}$ to a scalar product $\gamma(x)$ on $T_x M$. Hence, we get a metric γ on M .
- . Finally, for $x \in M$, let f_x denote the pull-back of g by the canonical injection of the fiber U_x in U . Then, for each $x \in M$, f_x is a G -invariant metric on the fiber U_x . Therefore, for each $q \in Q_x$, it defines on $T_q U_x$ a scalar product $f_x(q)$ which is invariant by the isotropy representation of H . If we pull-back $f_x(q)$ by the H -equivariant isomorphism from \mathfrak{m} to $T_q U_x$ defined by the choice of q , we get an $\text{Ad}(H)$ -invariant scalar product $f(q)$ on \mathfrak{m} . Hence, we have an $N(H)|H$ -equivariant map $f : Q \rightarrow S_2^H(\mathfrak{m})$, that we shall call a **Thiry scalar field**.

Thus, we have the following dimensional reduction theorem, proven in [CJ1] :

Theorem 1.2.1 (*Coquereaux & Jadczyk*) *The preceding construction defines a one-to-one correspondence between the set of G -invariant metrics on U , and the set of triples (γ, α, f) , where :*

- γ is a metric on the base manifold M ,
- α is a connection form on the principal fiber bundle $(Q, M, N(H)|H)$,
- f is a scalar field on Q taking values in $S_2^H(\mathfrak{m})$ (equivariant for the group $N(H)|H$).

1.2.2 The potential for the Thiry scalar field

It is possible to perform now the dimensional reduction of the action density. On the multidimensional universe, we have a pure gravity theory, therefore we can write the Einstein-Hilbert action. The constraint of G -invariance of the metric field implies that the Lagrangian of the multidimensional theory does not depend on the internal variables, but only the space-time variables. Therefore, one can integrate over the internal space to get the reduced action density on the base manifold. What we obtain is of course an Einstein-Yang-Mills theory interacting with scalar field f . In general, we have a kinetic term for the Thiry scalar field f , when it interacts with the gauge field. We also get a potential term, that we are going to study carefully here, leaving the Yang-Mills term and the kinetic term to section 1.3.4.

The potential appearing in the reduced action is a real-valued function V defined on the open set $S_2^{H++}(\mathfrak{m})$ made of positive definite elements of $S_2^H(\mathfrak{m})$. In fact, V can be more generally defined on the open subset of all non-degenerate elements of $S_2^H(\mathfrak{m})$, but we shall restrict ourselves to the positive definite case.

It is well known (cf. [CE] for example) that $S_2^{H++}(\mathfrak{m})$ is in one-to-one correspondence with the set of all G -invariant metrics on the homogeneous space G/H .

As one might expect, the potential obtained by dimensional reduction is then the opposite of the scalar curvature functional of G/H restricted to the set of G -invariant

metrics.

There are several ways to compute the scalar curvature of a riemannian homogeneous space (cf. for example [CE], [KN2], [Ber], [WZ]) but most of them give the expression after having chosen a fixed G -invariant metric and an orthonormal basis on the tangent space at the origin. Here, we are rather interested in the scalar curvature functional, so we would like the dependance on the metric to be explicit. The formula we present here is an intrinsic version of that of [CJ1].

Let us introduce some notations first. We denote by $\widetilde{\text{ad}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of \mathfrak{g} . For any $X \in \mathfrak{m}$, let $\text{ad}(X)$ be the endomorphism of \mathfrak{m} defined by : $\text{ad}(X) = \pi_{\mathfrak{m}} \circ \widetilde{\text{ad}}(X) \circ \iota_{\mathfrak{m}}$, where $\iota_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{g}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \longrightarrow \mathfrak{m}$ are the canonical injection and projection respectively. Now for any bilinear form b on \mathfrak{m} , we denote by $\hat{b} : \mathfrak{m} \longrightarrow \mathfrak{m}^*$ the linear map canonically associated to b (that is : $\hat{b}(X)(Y) = b(X, Y)$ for any $X, Y \in \mathfrak{m}$). With these notations, the potential $V : S_2^{H++}(\mathfrak{m}) \longrightarrow \mathbb{R}$ is given by :

$$V(f) = \frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D}) + \frac{1}{2} \text{tr}(\hat{f}^{-1} \circ \hat{B}) + \text{tr}(\hat{f}^{-1} \circ \hat{D}^{\mathfrak{h}})$$

where D , B and $D^{\mathfrak{h}}$ are the bilinear forms on \mathfrak{m} defined by :

$$\begin{aligned} D(X, Y) &= \text{tr}(\hat{f}^{-1} \circ {}^t \text{ad}(X) \circ \hat{f} \circ \text{ad}(Y)) \\ B(X, Y) &= \text{tr}(\text{ad}(X) \circ \text{ad}(Y)) \\ D^{\mathfrak{h}}(X, Y) &= \text{tr}(\pi_{\mathfrak{m}} \circ \widetilde{\text{ad}}(Y) \circ \iota_{\mathfrak{h}} \circ \pi_{\mathfrak{h}} \circ \widetilde{\text{ad}}(X) \circ \iota_{\mathfrak{m}}) \end{aligned}$$

The first term of V will be of interest to us, so we will begin by expressing it in a different way. Let $\mathcal{L}(\mathfrak{m})$ denote the space of endomorphisms of \mathfrak{m} . Each $f \in S_2^{H++}(\mathfrak{m})$ defines a scalar product on $\mathcal{L}(\mathfrak{m})$ given by : $\langle u, v \rangle_{ff} = \text{tr}(\hat{f}^{-1} \circ {}^t u \circ \hat{f} \circ v)$. If $\widehat{ff} : \mathcal{L}(\mathfrak{m}) \longrightarrow \mathcal{L}(\mathfrak{m})^*$ denotes the isomorphism associated to $\langle \cdot, \cdot \rangle_{ff}$, then we get a scalar product on the space $\mathcal{L}(\mathfrak{m}, \mathcal{L}(\mathfrak{m}))$ of linear maps from \mathfrak{m} to $\mathcal{L}(\mathfrak{m})$ by setting : $\langle \alpha, \beta \rangle_{fff} = \text{tr}(\hat{f}^{-1} \circ {}^t \alpha \circ \widehat{ff} \circ \beta)$. (That $\langle \cdot, \cdot \rangle_{ff}$ and $\langle \cdot, \cdot \rangle_{fff}$ are positive definite symmetric bilinear forms is not difficult to prove). Using the preceding definitions, we can write :

$$\begin{aligned} D(X, Y) &= \text{tr}(\hat{f}^{-1} \circ {}^t \text{ad}(X) \circ \hat{f} \circ \text{ad}(Y)) \\ &= \langle \text{ad}(X), \text{ad}(Y) \rangle_{ff} \\ &= {}^t \text{ad}(\langle \cdot, \cdot \rangle_{ff})(X, Y) \end{aligned}$$

so $\hat{D} = {}^t \text{ad} \circ \widehat{ff} \circ \text{ad}$, therefore

$$\begin{aligned} \text{tr}(\hat{f}^{-1} \circ \hat{D}) &= \text{tr}(\hat{f}^{-1} \circ {}^t \text{ad} \circ \widehat{ff} \circ \text{ad}) \\ &= \langle \text{ad}, \text{ad} \rangle_{fff} \end{aligned}$$

Thus, we have

$$\frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D}) = \frac{1}{4} \| \text{ad} \|_{fff}^2$$

which proves the positivity of this term.

It is convenient to write the representation ρ of the gauge group $N(H)|H$ on the vector space $S_2^H(\mathfrak{m})$ in terms of the linear maps $\hat{f} : \mathfrak{m} \longrightarrow \mathfrak{m}^* : \rho_{[n]}(\hat{f}) = {}^t \text{Ad}_{n-1} \circ \hat{f} \circ \text{Ad}_{n-1}$. When we will study the kinetic term, we will need the expression of the covariant derivative corresponding to the connexion α and acting on the $S_2^H(\mathfrak{m})$ -valued Thiry scalar fields.

Therefore, we need to write also the infinitesimal version of ρ , that is, the representation ρ' of the Lie algebra $\mathfrak{k} = \mathfrak{n}(\mathfrak{h})|\mathfrak{h}$ on $S_2^H(\mathfrak{m})$. It is easy to check that this last one is given by : $\rho'_{[\tilde{A}]}(\hat{f}) = -({}^t\text{ad}_{\tilde{A}} \circ \hat{f}) - (\hat{f} \circ \text{ad}_{\tilde{A}})$ for every $\tilde{A} \in \mathfrak{n}(\mathfrak{h})$.

Proposition 1.2.2 *The potential V is invariant under the representation of the gauge group $N(H)|H$.*

Proof : We begin with the first term of V : $\text{tr}(\hat{f}^{-1} \circ \hat{D}_f)$

$$\begin{aligned} D_{\rho_{[n]}(\hat{f})}(X, Y) &= \text{tr}(\rho_{[n]}(\hat{f})^{-1} \circ {}^t\text{ad}(X) \circ \rho_{[n]}(\hat{f}) \circ \text{ad}(Y)) \\ &= \text{tr}(({}^t\text{Ad}_{n-1} \circ \hat{f} \circ \text{Ad}_{n-1})^{-1} \circ {}^t\text{ad}(X) \circ {}^t\text{Ad}_{n-1} \circ \hat{f} \circ \text{Ad}_{n-1} \circ \text{ad}(Y)) \\ &= \text{tr}(\text{Ad}_n \circ \hat{f}^{-1} \circ {}^t\text{Ad}_n \circ {}^t\text{ad}(X) \circ {}^t\text{Ad}_{n-1} \circ \hat{f} \circ \text{Ad}_{n-1} \circ \text{ad}(Y)) \\ &= \text{tr}(\hat{f}^{-1} \circ {}^t(\text{Ad}_{n-1} \circ \text{ad}(X) \circ \text{Ad}_n) \circ \hat{f} \circ (\text{Ad}_{n-1} \circ \text{ad}(Y) \circ \text{Ad}_n)) \\ &= \text{tr}(\hat{f}^{-1} \circ {}^t\alpha_n(X) \circ \hat{f} \circ \alpha_n(Y)) \end{aligned}$$

if we set $\alpha_n(X) = \text{Ad}_{n-1} \circ \text{ad}(X) \circ \text{Ad}_n$ for all $X \in \mathfrak{m}$.

So $\hat{D}_{\rho_{[n]}(\hat{f})} = {}^t\alpha_n \circ \widehat{ff} \circ \alpha_n$. Therefore,

$$\begin{aligned} \text{tr}(\rho_{[n]}(\hat{f})^{-1} \circ \hat{D}_{\rho_{[n]}(\hat{f})}) &= \text{tr}(\text{Ad}_n \circ \hat{f}^{-1} \circ {}^t\text{Ad}_n \circ {}^t\alpha_n \circ \widehat{ff} \circ \alpha_n) \\ &= \text{tr}(\hat{f}^{-1} \circ {}^t(\alpha_n \circ \text{Ad}_n) \circ \widehat{ff} \circ (\alpha_n \circ \text{Ad}_n)) \end{aligned}$$

But for all $X, Y \in \mathfrak{m}$,

$$\begin{aligned} \alpha_n \circ \text{Ad}_n(X)(Y) &= \alpha_n(\text{Ad}_n(X))(Y) \\ &= \text{Ad}_{n-1} \circ \text{ad}(\text{Ad}_n(X)) \circ \text{Ad}_n(Y) \\ &= \text{Ad}_{n-1}(\text{ad}(\text{Ad}_n(X))(\text{Ad}_n(Y))) \\ &= \text{Ad}_{n-1}([\text{Ad}_n(X), \text{Ad}_n(Y)]_{\mathfrak{m}}) \\ &= \text{Ad}_{n-1} \circ \text{Ad}_n([X, Y]_{\mathfrak{m}}) \\ &= [X, Y]_{\mathfrak{m}} \\ &= \text{ad}(X)(Y) \end{aligned}$$

Therefore,

$$\begin{aligned} \text{tr}(\rho_{[n]}(\hat{f})^{-1} \circ \hat{D}_{\rho_{[n]}(\hat{f})}) &= \text{tr}(\hat{f}^{-1} \circ {}^t\text{ad} \circ \widehat{ff} \circ \text{ad}) \\ &= \text{tr}(\hat{f}^{-1} \circ \hat{D}_f) \end{aligned}$$

The invariance of the two other terms follows from the invariance of the Killing form. \square

We saw that V is, up to a sign, the scalar curvature functional of a homogenous space. This function has no reason to have a fixed sign. There are many examples (cf. [WZ], [CJ1]) in which this potential is not bounded from below. At this level, one may wonder if it is possible to construct a physical theory out of such a potential.

In order to recover a more realistic pattern for the potential, we shall try to distinguish certain directions in the space $S_2^H(\mathfrak{m})$ of the scalar fields, on which we might have a positive polynomial potential, closer to that of the Higgs fields.

1.3 Recovering scalar fields with polynomial potential

1.3.1 Introduction

Let us consider the vector space $S_2^H(\mathfrak{m})$. There are at least two interesting ways to decompose it. First, we may start by finding the decomposition of \mathfrak{m} into irreducible representations of the group H :

$$\mathfrak{m} = \bigoplus \mathfrak{m}_i$$

with : $\text{Ad}(H)\mathfrak{m}_i \subset \mathfrak{m}_i$, and the \mathfrak{m}_i 's are irreducible.

Associated to this decomposition, one then writes a decomposition of $S_2^H(\mathfrak{m})$, which allows for example the calculation of the dimension of $S_2^H(\mathfrak{m})$.

Another natural decomposition of $S_2^H(\mathfrak{m})$ is the one in terms of irreducible representations of $N(H)|H$. It would seem interesting to look at some $N(H)|H$ -invariant subspace of $S_2^H(\mathfrak{m})$, constructed out of $\text{Ad}(H)$ -invariant factors in the decomposition $\mathfrak{m} = \bigoplus \mathfrak{m}_i$. In particular models of symmetric gauge fields dimensional reduction, we find Higgs fields as intertwining operators between two representative spaces, so this suggest we make look at fields of the following type : $\text{Ad}(H)$ -equivariant maps $\phi : \mathfrak{m}_i \longrightarrow \mathfrak{m}_j$ coming from the fields $f \in S_2^H(\mathfrak{m})$.

1.3.2 Decomposition of $S_2^H(\mathfrak{m})$

Let us study the following situation : we fix two $\text{Ad}(N(H))$ -invariant vector subspaces \mathfrak{m}_1 and \mathfrak{m}_2 in \mathfrak{m} , getting what we shall call an **Ad($N(H)$)-invariant splitting** : $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We denote by $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ the space of $\text{Ad}(H)$ -equivariant linear maps from \mathfrak{m}_1 to \mathfrak{m}_2 . Then, we can state the lemma on which all our following results will lie :

Lemma 1.3.1 *There is a one-to-one correspondence between the space $S_2^H(\mathfrak{m})$ and the direct product $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) \times S_2^H(\mathfrak{m}_1) \times S_2^H(\mathfrak{m}_2)$.*

It is given in matrix form by :

$$(\phi, h, k) \longmapsto f = \begin{pmatrix} h + {}^t\phi k \phi & -{}^t\phi k \\ -k \phi & k \end{pmatrix}$$

Proof :

1) Taking $f \in S_2^H(\mathfrak{m})$, we set :

- . $\phi = -\text{pr}_{|\mathfrak{m}_1}^{\mathfrak{m}_2}$, where $\text{pr}^{\mathfrak{m}_2}$ is the orthogonal projector on \mathfrak{m}_2 for the euclidian structure defined by f .
- . $\forall X_1, Y_1 \in \mathfrak{m}_1$, $h(X_1, Y_1) = f(X_1 + \phi(X_1), Y_1 + \phi(Y_1))$
- . $k = f|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$

For all $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ ($X_1, Y_1 \in \mathfrak{m}_1$, $X_2, Y_2 \in \mathfrak{m}_2$), letting $X = (X_1 + \phi(X_1)) + (-\phi(X_1) + X_2)$ and $Y = (Y_1 + \phi(Y_1)) + (-\phi(Y_1) + Y_2)$, we compute $f(X, Y)$ and obtain :

$$\begin{aligned} f(X, Y) &= h(X_1, Y_1) + k(\phi(X_1), \phi(Y_1)) - k(\phi((X_1), Y_2) \\ &\quad - k(\phi(Y_1), X_2) + k(X_2, Y_2) \end{aligned}$$

- 2) Conversely, let $(\phi, h, k) \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) \times S_2^H(\mathfrak{m}_1) \times S_2^H(\mathfrak{m}_2)$. It is not difficult to see that if we define $f \in S_2^H(\mathfrak{m})$ by the preceeding formula, then :
- . $\phi = -\text{pr}_{|\mathfrak{m}_1}^{\mathfrak{m}_2}$, where $\text{pr}^{\mathfrak{m}_2}$ is the orthogonal projector on \mathfrak{m}_2 for the euclidian structure defined by f .
 - . $\forall X_1, Y_1 \in \mathfrak{m}_1, h(X_1, Y_1) = f(X_1 + \phi(X_1), Y_1 + \phi(Y_1))$
 - . $k = f|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$

□

Remark 1.3.2 *The space of Higgs fields that we obtained depends on the choice of the decomposition of \mathfrak{m} into \mathfrak{m}_1 and \mathfrak{m}_2 . In certains cases, there exists natural decompositions, as we are going to see in the next chapter.*

It is easy to check the following :

- 1) f is positive definite if and only if h and k are positive definite.
- 2) The assumption of $\text{Ad}(N(H))$ -invariance for \mathfrak{m}_1 and \mathfrak{m}_2 implies the existence of a representation of $N(H)|H$ on $S_2^H(\mathfrak{m}_1)$, on $S_2^H(\mathfrak{m}_2)$, and also on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. Denoting by $\text{Ad}^{(1)}$ (resp. $\text{Ad}^{(2)}$) the representation of $N(H)$ on \mathfrak{m}_1 (resp. on \mathfrak{m}_2), and by $\text{ad}^{(1)}$ (resp. $\text{ad}^{(2)}$) the representation of $\mathfrak{n}(\mathfrak{h})$ on \mathfrak{m}_1 (resp. on \mathfrak{m}_2), the representations on the fields spaces are given by :

$$\begin{aligned}\rho_{[n]}^1(\hat{h}) &= {}^t\text{Ad}_{n^{-1}}^{(1)} \circ \hat{h} \circ \text{Ad}_{n^{-1}}^{(1)} \\ \rho_{[n]}^2(\hat{k}) &= {}^t\text{Ad}_{n^{-1}}^{(2)} \circ \hat{k} \circ \text{Ad}_{n^{-1}}^{(2)} \\ \rho_{[n]}^0(\phi) &= \text{Ad}_n^{(2)} \circ \phi \circ \text{Ad}_{n^{-1}}^{(1)}\end{aligned}$$

We define the corresponding representations $\eta^i = (\rho^i)'$ of the Lie algebra $\mathfrak{k} = \mathfrak{n}(\mathfrak{h})|\mathfrak{h}$:

For every $\tilde{A} \in \mathfrak{n}(\mathfrak{h})$,

$$\begin{aligned}\eta_{[\tilde{A}]}^1(\hat{h}) &= -({}^t\text{ad}_{\tilde{A}}^{(1)} \circ \hat{h}) - (\hat{h} \circ \text{ad}_{\tilde{A}}^{(1)}) \\ \eta_{[\tilde{A}]}^2(\hat{k}) &= -({}^t\text{ad}_{\tilde{A}}^{(2)} \circ \hat{k}) - (\hat{k} \circ \text{ad}_{\tilde{A}}^{(2)}) \\ \eta_{[\tilde{A}]}^0(\phi) &= (\text{ad}_{\tilde{A}}^{(2)} \circ \phi) - (\phi \circ \text{ad}_{\tilde{A}}^{(1)})\end{aligned}$$

1.3.3 The potential for the scalar field ϕ

For h and k fixed in $S_2^{H++}(\mathfrak{m}_1)$ and $S_2^{H++}(\mathfrak{m}_2)$ respectively, let us compute the potential V in terms of ϕ . Here we denote by V the real-valued function defined on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ by :

$$V(\phi) = \frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D}) + \frac{1}{2} \text{tr}(\hat{f}^{-1} \circ \hat{B}) + \text{tr}(\hat{f}^{-1} \circ \hat{D}^{\mathfrak{h}})$$

where f is given by :

$$\hat{f} = {}^t\pi_1 \circ \hat{h} \circ \pi_1 + {}^t(\phi \circ \pi_1 - \pi_2) \circ \hat{k} \circ (\phi \circ \pi_1 - \pi_2)$$

We will need to split $\text{ad}(X)$ in terms of \mathfrak{m}_1 and \mathfrak{m}_2 , so we define :

$$\begin{aligned}\text{ad}_{11}(X) &= \pi_1 \circ \text{ad}(X) \circ \iota_1 \in \mathcal{L}(\mathfrak{m}_1) \\ \text{ad}_{12}(X) &= \pi_1 \circ \text{ad}(X) \circ \iota_2 \in \mathcal{L}(\mathfrak{m}_2, \mathfrak{m}_1) \\ \text{ad}_{21}(X) &= \pi_2 \circ \text{ad}(X) \circ \iota_1 \in \mathcal{L}(\mathfrak{m}_2, \mathfrak{m}_1) \\ \text{ad}_{22}(X) &= \pi_2 \circ \text{ad}(X) \circ \iota_2 \in \mathcal{L}(\mathfrak{m}_2)\end{aligned}$$

With the notations that we will to introduce below, we are going to prove the following theorem :

Theorem 1.3.3 1. The potential for the scalar field ϕ is a polynomial of degree at most 6 in ϕ , whose sixth-degree term is positive and is given by :

$$V^{(6)}(\phi) = \frac{1}{4} \| \phi ad_{12}(\phi) \phi \|_{hhk}^2$$

Besides, $V^{(6)} = 0$ if and only if the third-degree condition is satisfied for every $\phi : \{\phi \circ ad_{12}(\phi(X)) \circ \phi = 0 \quad \forall X \in \mathfrak{m}_1\}$.

2. If $V^{(6)} = 0$, the potential $V(\phi)$ becomes a polynomial of degree at most 4 in ϕ , whose fourth-degree term is positive and is given by :

$$\begin{aligned} V^{(4)}(\phi) &= \frac{1}{4} \| \phi ad_{11}(\phi) - ad_{22}(\phi) \phi + \phi ad_{12} \phi \|_{hhk}^2 + \frac{1}{4} \| ad_{12}(\phi) \phi \|_{hhk}^2 \\ &\quad + \frac{1}{4} \| \phi ad_{12}(\phi) \|_{hhk}^2 + \frac{1}{4} \| \phi ad_{12} \phi \|_{hhk}^2 \end{aligned}$$

Besides, $V^{(4)} = 0$ if and only if the quadratic conditions are satisfied for every $\phi :$

$$(C) \left\{ \begin{array}{ll} \phi ad_{11}(\phi(X)) - ad_{22}(\phi(X)) \phi + \phi ad_{12} \phi &= 0 & \forall X \in \mathfrak{m}_1 \\ ad_{12}(\phi(X)) \phi &= 0 & \forall X \in \mathfrak{m}_1 \\ \phi ad_{12}(\phi(X)) &= 0 & \forall X \in \mathfrak{m}_1 \\ \phi ad_{12} \phi &= 0 & \forall X \in \mathfrak{m}_2 \end{array} \right.$$

We begin by computing the first term $\frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D})$. For all $X, Y \in \mathfrak{m}$, we have :

$$\begin{aligned} \hat{f}^{-1} \circ {}^t \text{ad}(X) \circ \hat{f} \circ \text{ad}(Y) &= [(\iota_1 + \iota_2 \circ \phi) \circ \hat{h}^{-1} \circ {}^t(\iota_1 + \iota_2 \circ \phi) + \iota_2 \circ \hat{k}^{-1} \circ {}^t \iota_2] \circ {}^t \text{ad}(X) \\ &\quad \circ [{}^t \pi_1 \circ \hat{h} \circ \pi_1 + {}^t(\phi \circ \pi_1 - \pi_2) \circ \hat{k} \circ (\phi \circ \pi_1 - \pi_2)] \circ \text{ad}(Y) \end{aligned}$$

We can already see from this expression that for each $X, Y \in \mathfrak{m}$, $D(X, Y)$, which is the trace of the above expression, is a polynomial of degree at most 4 in ϕ . The fourth-degree term in $D(X, Y)$ is :

$$\begin{aligned} D^{(4)}(X, Y) &= \text{tr}(\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2 \circ {}^t \text{ad}(X) \circ {}^t \pi_1 \circ {}^t \phi \circ \hat{k} \circ \phi \circ \pi_1 \circ \text{ad}(Y)) \\ &= \text{tr}(\phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2 \circ {}^t \text{ad}(X) \circ {}^t \pi_1 \circ {}^t \phi \circ \hat{k} \circ \phi \circ \pi_1 \circ \text{ad}(Y) \circ \iota_2) \\ &= \text{tr}(\phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \text{ad}_{12}(X) \circ {}^t \phi \circ \hat{k} \circ \phi \circ \text{ad}_{12}(Y)) \\ &= \text{tr}(\hat{h}^{-1} \circ {}^t \phi \circ {}^t \text{ad}_{12}(X) \circ {}^t \phi \circ \hat{k} \circ \phi \circ \text{ad}_{12}(Y) \circ \phi) \end{aligned}$$

On the other hand, \hat{f}^{-1} is a polynomial of degree 2 in ϕ , whose second-degree term is : $\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2$.

We deduce therefore that the potential V is a polynomial of degree at most 6 in ϕ , whose sixth-degree term is :

$$\begin{aligned} V^{(6)}(\phi) &= \frac{1}{4} \text{tr}(\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2 \circ \hat{D}^{(4)}) \\ &= \frac{1}{4} \text{tr}(\phi \circ \hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2 \circ \hat{D}^{(4)} \circ \iota_2) \\ &= \frac{1}{4} \text{tr}(\hat{h}^{-1} \circ {}^t \phi \circ {}^t \iota_2 \circ \hat{D}^{(4)} \circ \iota_2 \circ \phi) \\ &= \frac{1}{4} \text{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi)) \end{aligned}$$

where we have set $\hat{P}^{(6)}(\phi) = {}^t \phi \circ {}^t \iota_2 \circ \hat{D}^{(4)} \circ \iota_2 \circ \phi$.

Thus, we have :

$$V^{(6)}(\phi) = \frac{1}{4} \operatorname{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi))$$

Here also, it is possible to write $V^{(6)}(\phi)$ in a more synthetic way. For this, we make use of the scalar products h and k to define the following scalar products :

$$\begin{aligned} \text{On } \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2) & : \langle u, v \rangle_{hk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t u \circ \hat{k} \circ v) = \widehat{hk}(u)(v) \\ \text{On } \mathcal{L}(\mathfrak{m}_1) & : \langle u, v \rangle_{hh} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t u \circ \hat{h} \circ v) = \widehat{hh}(u)(v) \\ \text{On } \mathcal{L}(\mathfrak{m}_2) & : \langle u, v \rangle_{kk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t u \circ \hat{k} \circ v) = \widehat{kk}(u)(v) \end{aligned}$$

which we use to define again the following scalar products :

$$\begin{aligned} \text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2)) & : \langle \alpha, \beta \rangle_{hhk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t \alpha \circ \widehat{hk} \circ \beta) \\ \text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1)) & : \langle \alpha, \beta \rangle_{hhh} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t \alpha \circ \widehat{hh} \circ \beta) \\ \text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_2)) & : \langle \alpha, \beta \rangle_{hkk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t \alpha \circ \widehat{kk} \circ \beta) \\ \text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2)) & : \langle \alpha, \beta \rangle_{khk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t \alpha \circ \widehat{hk} \circ \beta) \\ \text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_1)) & : \langle \alpha, \beta \rangle_{khh} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t \alpha \circ \widehat{hh} \circ \beta) \\ \text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_2)) & : \langle \alpha, \beta \rangle_{kkk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t \alpha \circ \widehat{kk} \circ \beta) \end{aligned}$$

Using the preceding definitions, we can write :

$$\begin{aligned} P^{(6)}(\phi)(X, Y) & = \operatorname{tr}(\hat{h}^{-1} \circ {}^t \phi \circ {}^t \operatorname{ad}_{12}(\phi(X)) \circ {}^t \phi \circ \hat{k} \circ \phi \circ \operatorname{ad}_{12}(\phi(Y)) \circ \phi) \\ & = \langle \phi \circ \operatorname{ad}_{12}(\phi(X)) \circ \phi, \phi \circ \operatorname{ad}_{12}(\phi(Y)) \circ \phi \rangle_{hk} \\ & = {}^t(\phi \operatorname{ad}_{12}(\phi) \phi)(\langle, \rangle_{hk})(X, Y) \end{aligned}$$

where $\phi \operatorname{ad}_{12}(\phi) \phi$ denotes the element of $\mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2))$ which associates $\phi \circ \operatorname{ad}_{12}(\phi(X)) \circ \phi$ to each $X \in \mathfrak{m}_1$.

Thus, $\hat{P}^{(6)}(\phi) = {}^t(\phi \operatorname{ad}_{12}(\phi) \phi) \circ \widehat{hk} \circ (\phi \operatorname{ad}_{12}(\phi) \phi)$, therefore

$$\begin{aligned} \operatorname{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi)) & = \operatorname{tr}(\hat{h}^{-1} \circ {}^t(\phi \operatorname{ad}_{12}(\phi) \phi) \circ \widehat{hk} \circ (\phi \operatorname{ad}_{12}(\phi) \phi)) \\ & = \langle \phi \operatorname{ad}_{12}(\phi) \phi, \phi \operatorname{ad}_{12}(\phi) \phi \rangle_{hhk} \end{aligned}$$

Thus, we have

$$V^{(6)}(\phi) = \frac{1}{4} \| \phi \operatorname{ad}_{12}(\phi) \phi \|_{hhk}^2$$

which proves the positivity of this term.

Moreover, if we denote by $\Gamma^{(6)}$ the set of zeroes of $V^{(6)}$, we see immediatly that :

$$\Gamma^{(6)} = \{ \phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) / \phi \circ \operatorname{ad}_{12}(\phi(X)) \circ \phi = 0 \quad \forall X \in \mathfrak{m}_1 \}$$

Thus, we have a condition of the third-degree in ϕ , and if this condition is verified for every ϕ that is, if $\Gamma^{(6)} = \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$, then $V^{(6)} = 0$.

Because of the positivity of $\frac{1}{4} \operatorname{tr}(\hat{f}^{-1} \circ \hat{D})$, V cannot be of degree 5, therefore it becomes a polynomial of degree 4 at most, whose quartic terms can be computed in a similar manner to give :

$$\begin{aligned} V^{(4)}(\phi) & = \frac{1}{4} \| \phi \operatorname{ad}_{11}(\phi) - \operatorname{ad}_{22}(\phi) \phi + \phi \operatorname{ad}_{12} \phi \|_{hhk}^2 + \frac{1}{4} \| \operatorname{ad}_{12}(\phi) \phi \|_{hhk}^2 \\ & \quad + \frac{1}{4} \| \phi \operatorname{ad}_{12}(\phi) \|_{hk}^2 + \frac{1}{4} \| \phi \operatorname{ad}_{12} \phi \|_{hk}^2 \end{aligned}$$

which proves the positivity of $V^{(4)}$.

Moreover, if we denote by $\Gamma^{(4)}$ the set of zeroes of $V^{(4)}$, we see immediatly that $\Gamma^{(4)}$ is the set of the elements ϕ in $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ that satisfy the quadratic conditions (C) .

If these conditions is verified for every ϕ that is, if $\Gamma^{(4)} = \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$, then $V^{(4)} = 0$.

Because of the positivity of $\frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D})$, V cannot be of degree 3, therefore it becomes a polynomial of degree 2 at most. It is also possible to compute the quadratic terms that appear in $\frac{1}{4} \text{tr}(\hat{f}^{-1} \circ \hat{D})$, which are necessarily positive, but to get all the quadratic contributions in V , one has to include also the quadratic terms coming from $\frac{1}{2} \text{tr}(\hat{f}^{-1} \circ \hat{B})$ and $\text{tr}(\hat{f}^{-1} \circ \hat{D}^\mathfrak{h})$, and these have no reason to be of fixed sign.

Proposition 1.3.4 *If the elements h and k , fixed respectively in $S_2^{H++}(\mathfrak{m}_1)$ and $S_2^{H++}(\mathfrak{m}_2)$, are chosen $\text{Ad}(N(H))$ -invariant, then the potential $V(\phi)$ is invariant under the action of the group $N(H)|H$.*

Proof : The proof is similar to that of Proposition 1.2.2. □

Let us take the special case in which \mathfrak{m}_2 is a Lie algebra. This case covers the known situations in which we have Higgs fields as equivariant maps between two representation spaces. If \mathfrak{m}_2 is a Lie algebra, then ad_{12} vanishes identically, and therefore $V^{(6)}(\phi)$ is zero. But in this case, we can verify that $V^{(5)}(\phi)$ will also be zero. Thus, we are left with a quartic potential which is exactly what we have in the known situations.

1.3.4 The Yang-Mills term and the kinetic term

As outlined in the first paragraph of section 1.2.2, the Lagrangian of the multidimensional G -invariant gravity theory reduces, after integration on the internal variables, to the Lagrangian of an Einstein-Yang-Mills theory coupled to a scalar field f . We will begin by writing an intrinsic expression of this reduced Lagrangian, and then explain the notations used while investigating each term.

$$\mathcal{L}(\gamma, \alpha, f)(x) = -V(\gamma)(x) - V(\Omega(q)) + K(f(q)) - V(f(q)) \quad \text{where } q \in Q_x$$

where V stands for "potential" and K for "kinetic". Explicitly,

$$\mathcal{L}(\gamma, \alpha, f)(x) = \rho_M(\gamma)(x) - \frac{1}{2} \| \bar{\Omega}(q) \|_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}^2 + \frac{1}{2} \| D\hat{f}(q) \|_{\tilde{\gamma}(q)f(q)f^*(q)}^2 + \rho_{G/H}(f(q))$$

ρ means "scalar curvature functional". The term $\rho_{G/H}(f(q))$ is the opposite of the potential for the Thiry scalar field f , and has already been discussed in 1.2.2 and 1.3.3. $\rho_M(\gamma)(x)$ is the Lagrangian of the gravity sector in M , and we'll have nothing more to say about it. The remainder of this section will be devoted to the two middle terms.

- The Yang-Mills term $V(\Omega(q)) = \frac{1}{2} \| \bar{\Omega}(q) \|_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}^2$

Ω denotes the curvature two-form $D\alpha$ of the connection α . Let \mathfrak{k} be the Lie algebra of the gauge group $N(H)|H$. Then $\Omega(q)$ is a \mathfrak{k} -valued antisymmetric bilinear form on $T_q Q$, and by horizontality of Ω , we can consider only its restriction to the horizontal subspace $Z_q = \text{Ker } \alpha(q)$. We denote by $\bar{\Omega}(q) : Z_q \longrightarrow \mathcal{L}(Z_q, \mathfrak{k})$ the linear map canonically associated to $\Omega(q)|_{Z_q \times Z_q}$. Let $\tilde{\gamma}(q)$ be the scalar product on Z_q obtained by

horizontal lift of $\gamma(x)$, and $\bar{\gamma}(q) : Z_q \longrightarrow Z_q^*$ the linear map canonically associated to $\tilde{\gamma}(q)$. Finally, let $\tilde{f}(q)$ be the restriction to \mathfrak{k} of $f(q)$, and $\bar{f}(q) : \mathfrak{k} \longrightarrow \mathfrak{k}^*$ the linear map canonically associated to $\tilde{f}(q)$.

Similarly to what we have done when expressing the potential of the scalar fields, we introduce the scalar product :

$$\text{On } \mathcal{L}(Z_q, \mathfrak{k}) : \langle u, v \rangle_{\tilde{\gamma}(q)\tilde{f}(q)} = \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t u \circ \bar{f}(q) \circ v) = \overline{\gamma(q)f(q)}(u)(v)$$

which we use to define again the following scalar product :

$$\text{On } \mathcal{L}(Z_q, \mathcal{L}(Z_q, \mathfrak{k})) : \langle \alpha, \beta \rangle_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)} = \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t \alpha \circ \overline{\gamma(q)f(q)} \circ \beta)$$

Now we have introduced all the notations that give sense to the expression :

$$V(\Omega(q)) = \frac{1}{2} \| \bar{\Omega}(q) \|_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}^2$$

but let us write it also in the expanded way :

$$\begin{aligned} V(\Omega(q)) &= \frac{1}{2} \langle \bar{\Omega}(q), \bar{\Omega}(q) \rangle_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)} \\ &= \frac{1}{2} \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t \bar{\Omega}(q) \circ \overline{\gamma(q)f(q)} \circ \bar{\Omega}(q)) \\ &= \frac{1}{2} \text{tr}(\bar{\gamma}(q)^{-1} \circ \bar{D}) \end{aligned}$$

where $\bar{D} = {}^t \bar{\Omega}(q) \circ \overline{\gamma(q)f(q)} \circ \bar{\Omega}(q)$ is a linear map from Z_q to Z_q^* , to which is associated the following bilinear form on Z_q :

$$\begin{aligned} D(X, Y) &= {}^t \bar{\Omega}(q)(\langle, \rangle_{\tilde{\gamma}(q)\tilde{f}(q)})(X, Y) \\ &= \langle {}^t \bar{\Omega}(q)(X), {}^t \bar{\Omega}(q)(Y) \rangle_{\tilde{\gamma}(q)\tilde{f}(q)} \\ &= \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t \bar{\Omega}(q)(X) \circ \bar{f}(q) \circ \bar{\Omega}(q)(Y)) \end{aligned}$$

for every $X, Y \in Z_q$.

Our next purpose is to expand this Yang-Mills term with respect to our decomposition of f in terms of h, k, ϕ . For this, we set : $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{m}_1$ and $\mathfrak{k}_2 = \mathfrak{k} \cap \mathfrak{m}_2$, so that : $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. Let $\bar{\pi}_1 : \mathfrak{k} \longrightarrow \mathfrak{k}_1$ and $\bar{\pi}_2 : \mathfrak{k} \longrightarrow \mathfrak{k}_2$ be the corresponding projectors. We define :

- . $\bar{\phi}(q) = -\text{pr}_{|\mathfrak{k}_1}^{\mathfrak{k}_2}$, where $\text{pr}^{\mathfrak{k}_2}$ is the orthogonal projector on \mathfrak{k}_2 for the euclidian structure defined by $\tilde{f}(q)$.
- . $\forall X_1, Y_1 \in \mathfrak{k}_1, \bar{h}(q)(X_1, Y_1) = \tilde{f}(q)(X_1 + \bar{\phi}(q)(X_1), Y_1 + \bar{\phi}(q)(Y_1))$
- . $\tilde{k} = \tilde{f}|_{\mathfrak{k}_2 \times \mathfrak{k}_2}$
so that $\bar{h}(q) \in S_2(\mathfrak{k}_1)$, $\tilde{k}(q) \in S_2(\mathfrak{k}_2)$, $\bar{\phi}(q) \in \mathcal{L}(\mathfrak{k}_1, \mathfrak{k}_2)$, and :

$$\bar{f}(q) = {}^t \bar{\pi}_1 \circ \bar{h}(q) \circ \bar{\pi}_1 + {}^t (\bar{\phi}(q) \circ \bar{\pi}_1 - \bar{\pi}_2) \circ \bar{k}(q) \circ (\bar{\phi}(q) \circ \bar{\pi}_1 - \bar{\pi}_2)$$

Finally, we set :

$$\begin{aligned} \bar{\Omega}_1(q)(X) &= \bar{\pi}_1 \circ \bar{\Omega}(q)(X) \in \mathcal{L}(Z_q, \mathfrak{k}_1) \\ \bar{\Omega}_2(q)(X) &= \bar{\pi}_2 \circ \bar{\Omega}(q)(X) \in \mathcal{L}(Z_q, \mathfrak{k}_2) \end{aligned}$$

With all these definitions, we have :

$$\begin{aligned}
D(X, Y) &= \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ \bar{h}(q) \circ \bar{\Omega}_1(q)(Y)) \\
&= + \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_2(q)(X) \circ \bar{k}(q) \circ \bar{\Omega}_2(q)(Y)) \\
&= + \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ {}^t\phi(q) \circ \bar{k}(q) \circ \bar{\phi}(q) \circ \bar{\Omega}_1(q)(Y)) \\
&= - \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ {}^t\bar{\phi}(q) \circ \bar{k}(q) \circ \bar{\Omega}_2(q)(Y)) \\
&= - \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_2(q)(X) \circ \bar{k}(q) \circ \bar{\phi}(q) \circ \bar{\Omega}_1(q)(Y))
\end{aligned}$$

We can state now the following theorem :

Theorem 1.3.5 *If the trivial representation of H lies completely either in \mathfrak{m}_1 or in \mathfrak{m}_2 , then there is no direct coupling between the scalar fields ϕ and the Yang-Mills strength Ω , and we have :*

$$V(\Omega(q)) = \frac{1}{2} \| \bar{\Omega}_1(q) \|_{\bar{\gamma}(q)\bar{\gamma}(q)\bar{h}(q)}^2 + \frac{1}{2} \| \bar{\Omega}_2(q) \|_{\bar{\gamma}(q)\bar{\gamma}(q)\bar{k}(q)}^2$$

Proof : It is not difficult to see that \mathfrak{k} carries the trivial representation of H (cf. [CJ1] for example). The assumption of the theorem implies then that at least one of the two subspaces \mathfrak{k}_1 and \mathfrak{k}_2 is zero. Therefore, we have $\mathcal{L}(\mathfrak{k}_1, \mathfrak{k}_2) = 0$. We deduce then that in the last expression of $D(X, Y)$, we have $\bar{\phi} = 0$, and therefore we are left only with the first two terms. \square

- The kinetic term $K(f(q)) = \frac{1}{2} \| \mathcal{D}\hat{f}(q) \|_{\bar{\gamma}(q)f(q)f^*(q)}^2$

We shall follow the same steps that for the Yang-Mills term. $\mathcal{D}f$ denotes the covariant derivative of f , therefore $\mathcal{D}f(q)$ is a linear map from $T_q Q$ to $S_H^2(\mathfrak{m})$. By horizontality of $\mathcal{D}f$, we can consider only the restriction of $\mathcal{D}f(q)$ to the horizontal subspace Z_q , and in fact we shall work with $\mathcal{D}\hat{f}(q) : Z_q \longrightarrow \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)$. For every $X \in Z_q$, we have :

$$\begin{aligned}
\mathcal{D}\hat{f}(q)(X) &= d\hat{f}(q)(X) + \rho'_{\alpha(q)(X)}(\hat{f}(q)) \\
&= d\hat{f}(q)(X) - ({}^t\text{ad}_{\alpha(q)(X)} \circ \hat{f}(q)) - (\hat{f}(q) \circ \text{ad}_{\alpha(q)(X)})
\end{aligned}$$

(see the remark before Proposition 1.2.2). $\bar{\gamma}(q)$ has been defined in the previous paragraph.

In the same spirit, we introduce the following scalar product :

$$\text{On } \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*) : \langle u, v \rangle_{f(q)f^*(q)} = \text{tr}(\hat{f}(q)^{-1} \circ {}^t u \circ {}^t \hat{f}(q)^{-1} \circ v) = \widehat{f(q)f^*(q)}(u)(v)$$

which we use to define again the following scalar product :

$$\text{On } \mathcal{L}(Z_q, \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)) : \langle \alpha, \beta \rangle_{\bar{\gamma}(q)f(q)f^*(q)} = \text{tr}(\bar{\gamma}(q)^{-1} \circ {}^t \alpha \circ \widehat{f(q)f^*(q)} \circ \beta)$$

$(\widehat{f(q)f^*(q)} : \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*) \longrightarrow \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)^*$ is the isomorphism canonically associated to $\langle \cdot, \cdot \rangle_{f(q)f^*(q)}$).

Now we have introduced all the notations that give sense to the expression :

$$K(f(q)) = \frac{1}{2} \| \mathcal{D}\hat{f}(q) \|_{\bar{\gamma}(q)f(q)f^*(q)}^2$$

but let us write it also in the expanded way :

$$\begin{aligned}
K(f(q)) &= \frac{1}{2} \langle \mathcal{D}\hat{f}(q), \mathcal{D}\hat{f}(q) \rangle_{\tilde{\gamma}(q)f(q)f^*(q)} \\
&= \frac{1}{2} \operatorname{tr}(\tilde{\gamma}(q)^{-1} \circ {}^t \mathcal{D}\hat{f}(q) \circ \widehat{f(q)f^*(q)} \circ \mathcal{D}\hat{f}(q)) \\
&= \frac{1}{2} \operatorname{tr}(\tilde{\gamma}(q)^{-1} \circ \bar{E})
\end{aligned}$$

where $\bar{E} = {}^t \mathcal{D}\hat{f}(q) \circ \widehat{f(q)f^*(q)} \circ \mathcal{D}\hat{f}(q)$ is a linear map from Z_q to Z_q^* , to which is associated the following bilinear form on Z_q :

$$\begin{aligned}
E(X, Y) &= {}^t \mathcal{D}\hat{f}(q)(\langle, \rangle_{f(q)f^*(q)})(X, Y) \\
&= \langle {}^t \mathcal{D}\hat{f}(q)(X), {}^t \mathcal{D}\hat{f}(q)(Y) \rangle_{f(q)f^*(q)} \\
&= \operatorname{tr}(\hat{f}(q)^{-1} \circ {}^t \mathcal{D}\hat{f}(q)(X) \circ {}^t \hat{f}(q)^{-1} \circ \mathcal{D}\hat{f}(q)(Y))
\end{aligned}$$

for every $X, Y \in Z_q$.

Again, our next purpose is to expand this kinetic term with respect to our decomposition of f in terms of h, k, ϕ . For this, we recall that :

$$\hat{f}(q) = {}^t \pi_1 \circ \hat{h}(q) \circ \pi_1 + {}^t (\phi(q) \circ \pi_1 - \pi_2) \circ \hat{k}(q) \circ (\phi(q) \circ \pi_1 - \pi_2)$$

and

$$\hat{f}(q)^{-1} = (\iota_1 + \iota_2 \circ \phi(q)) \circ \hat{h}(q)^{-1} \circ {}^t (\iota_1 + \iota_2 \circ \phi(q)) + \iota_2 \circ \hat{k}(q)^{-1} \circ {}^t \iota_2$$

and we write ${}^t \hat{f}(q)^{-1}$. Then, we compute the covariant derivatives $\mathcal{D}\hat{f}(q)(Y)$ and ${}^t \mathcal{D}\hat{f}(q)(X)$ with the usual Leibnitz rule for products, getting the sum of nine terms. Notice for example that :

$$\begin{aligned}
&\mathcal{D}({}^t \pi_1 \circ \hat{h}(q) \circ \pi_1)(X) \\
&= {}^t \pi_1 \circ d\hat{h}(q)(X) \circ \pi_1 - ({}^t \operatorname{ad}_{\alpha(q)(X)} \circ {}^t \pi_1 \circ \hat{h}(q) \circ \pi_1) - ({}^t \pi_1 \circ \hat{h}(q) \circ \pi_1 \circ \operatorname{ad}_{\alpha(q)(X)}) \\
&= {}^t \pi_1 \circ d\hat{h}(q)(X) \circ \pi_1 - ({}^t \pi_1 \circ {}^t \operatorname{ad}_{\alpha(q)(X)}^{(1)} \circ \hat{h}(q) \circ \pi_1) - ({}^t \pi_1 \circ \hat{h}(q) \circ \operatorname{ad}_{\alpha(q)(X)}^{(1)} \circ \pi_1) \\
&= {}^t \pi_1 \circ [d\hat{h}(q)(X) - ({}^t \operatorname{ad}_{\alpha(q)(X)}^{(1)} \circ \hat{h}(q)) - (\hat{h}(q) \circ \operatorname{ad}_{\alpha(q)(X)}^{(1)})] \circ \pi_1 \\
&= {}^t \pi_1 \circ \mathcal{D}\hat{h}(q)(X) \circ \pi_1
\end{aligned}$$

where $\mathcal{D}\hat{h}(q)$ is naturally defined by :

$$\begin{aligned}
\mathcal{D}\hat{h}(q)(X) &= d\hat{h}(q)(X) - ({}^t \operatorname{ad}_{\alpha(q)(X)}^{(1)} \circ \hat{h}(q)) - (\hat{h}(q) \circ \operatorname{ad}_{\alpha(q)(X)}^{(1)}) \\
&= d\hat{h}(q)(X) + \eta_{\alpha(q)(X)}^1(\hat{h}(q))
\end{aligned}$$

Similarly,

$$\mathcal{D}({}^t \pi_2 \circ \hat{k}(q) \circ \pi_2)(X) = {}^t \pi_2 \circ \mathcal{D}\hat{k}(q)(X) \circ \pi_2$$

where $\mathcal{D}\hat{k}(q)$ is naturally defined by :

$$\begin{aligned}\mathcal{D}\hat{k}(q)(X) &= d\hat{k}(q)(X) - ({}^t \text{ad}_{\alpha(q)(X)}^{(2)} \circ \hat{k}(q)) - (\hat{k}(q) \circ \text{ad}_{\alpha(q)(X)}^{(2)}) \\ &= d\hat{k}(q)(X) + \eta_{\alpha(q)(X)}^2(\hat{k}(q))\end{aligned}$$

As for $\mathcal{D}\phi(q)$, it is naturally defined by :

$$\begin{aligned}\mathcal{D}\phi(q)(X) &= d\phi(q)(X) + \eta_{\alpha(q)(X)}^0(\phi(q)) \\ &= d\phi(q)(X) + (\text{ad}_{\alpha(q)(X)}^{(2)} \circ \phi(q)) - (\phi(q) \circ \text{ad}_{\alpha(q)(X)}^{(1)})\end{aligned}$$

Finally, we replace in the expression of $E(X, Y)$ and expand ! The long calculation gives a large number of terms, between which we discover happily an equally large number of cancellations, and we are left finally with the terms :

$$\| \mathcal{D}\phi(q) \|_{h(q)h(q)k(q)}^2 + \text{four terms depending on } \mathcal{D}k$$

Thus, we can state the following result :

Theorem 1.3.6 *If, in the decomposition of f in terms of h , k , ϕ , the matter field k is taken constant, then the kinetic term of the scalar field ϕ is proportional to the standard term : $\frac{1}{2} \| \mathcal{D}\phi(q) \|_{h(q)h(q)k(q)}^2$.*

This result agrees with that of [CJ2], where the field k was taken constant : the same kinetic term is obtained for ϕ .

We conclude this section by checking the gauge invariance of the theory :

Proposition 1.3.7 *If we freeze the scalar fields $h : Q \longrightarrow S_2^H(\mathfrak{m}_1)$ and $k : Q \longrightarrow S_2^H(\mathfrak{m}_2)$ by taking them constant, the value of each being $\text{Ad}(N(H))$ -invariant and positive definite, and if the trivial representation of H lies completely either in \mathfrak{m}_1 or in \mathfrak{m}_2 , then the Lagrangian $\mathcal{L}(\gamma, \alpha, \phi)$ is invariant under the action of the group $\mathcal{G} = C_{N(H)|H}^\infty(Q, N(H)|H)$ of gauge transformations.*

Proof : We already mentioned the invariance of the potential $V(\phi)$ in Proposition 1.3.4. The gravity sector in M brings no problem. We are left with the Yang-Mills term of Theorem 1.3.5 and the standard kinetic term of Theorem 1.3.6. And these are easily checked to be invariant when a gauge transformation $n : Q \longrightarrow N(H)|H$ acts on all the fields. \square

1.4 Example with $G = \text{SU}(5)$, $H \simeq \text{U}(1)$ and $N(H)|H \simeq \mathbb{T}^3$

We begin by setting some notations that we are going to use in the following.

- For each $n \in \mathbb{N}^*$, \mathbb{T}^n denotes the n -dimensional torus $\text{U}(1) \times \dots \times \text{U}(1)$.
- For each $p \in \mathbb{Z}$, \mathbb{C}_p denotes the vector space \mathbb{C} endowed with the irreducible representation of $\text{U}(1)$ labeled by the integer p .

$U(1)$ is embedded in $SU(5)$ by the following injective homomorphism of Lie groups :
 $\lambda : U(1) \longrightarrow SU(5)$ defined by :

$$\lambda(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{i2\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{i3\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{i4\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-i10\theta} \end{pmatrix}$$

We set $H = \lambda(U(1))$. The normalizer of H in G is then :

$$N(H) = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(\theta_1+\theta_2+\theta_3+\theta_4)} \end{pmatrix} ; \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \right\} \simeq \mathbb{T}^4$$

First, we study the adjoint action of H on $\mathfrak{g} = \mathfrak{su}(5)$. Let $h = \lambda(e^{i\theta}) \in H$, and

$$A = \begin{pmatrix} iy_{11} & -\bar{z}_{21} & z_{13} & z_{14} & z_{15} \\ z_{21} & iy_{22} & -\bar{z}_{32} & z_{24} & z_{25} \\ -\bar{z}_{13} & z_{32} & iy_{33} & -\bar{z}_{43} & z_{35} \\ -\bar{z}_{14} & -\bar{z}_{24} & z_{43} & iy_{44} & z_{45} \\ -\bar{z}_{15} & -\bar{z}_{25} & -\bar{z}_{35} & -\bar{z}_{45} & -i(y_{11} + y_{22} + y_{33} + y_{44}) \end{pmatrix} \in \mathfrak{su}(5)$$

$$\text{Then : } hAh^{-1} = \begin{pmatrix} iy_{11} & -e^{-i\theta}\bar{z}_{21} & e^{-i2\theta}z_{13} & e^{-i3\theta}z_{14} & e^{i11\theta}z_{15} \\ e^{i\theta}z_{21} & iy_{22} & -e^{-i\theta}\bar{z}_{32} & e^{-i2\theta}z_{24} & e^{i12\theta}z_{25} \\ -e^{i2\theta}\bar{z}_{13} & e^{i\theta}z_{32} & iy_{33} & -e^{-i\theta}\bar{z}_{43} & e^{i13\theta}z_{35} \\ -e^{i3\theta}\bar{z}_{14} & -e^{i2\theta}\bar{z}_{24} & e^{i\theta}z_{43} & iy_{44} & e^{i14\theta}z_{45} \\ -e^{-i11\theta}\bar{z}_{15} & -e^{-i12\theta}\bar{z}_{25} & -e^{-i13\theta}\bar{z}_{35} & -e^{-i14\theta}\bar{z}_{45} & -i(y_{11} + y_{22} + y_{33} + y_{44}) \end{pmatrix}$$

We deduce the decomposition of $\mathfrak{g} = \mathfrak{su}(5)$ under the adjoint action of H :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{l}_{21} \oplus \mathfrak{l}_{32} \oplus \mathfrak{l}_{43} \oplus \mathfrak{l}_{13} \oplus \mathfrak{l}_{24} \oplus \mathfrak{l}_{14} \oplus \mathfrak{l}_{15} \oplus \mathfrak{l}_{25} \oplus \mathfrak{l}_{35} \oplus \mathfrak{l}_{45}$$

where :

$$\mathfrak{h} = \left\{ \begin{pmatrix} i\theta & 0 & 0 & 0 & 0 \\ 0 & i2\theta & 0 & 0 & 0 \\ 0 & 0 & i3\theta & 0 & 0 \\ 0 & 0 & 0 & i4\theta & 0 \\ 0 & 0 & 0 & 0 & -i10\theta \end{pmatrix} ; \theta \in \mathbb{R} \right\} \simeq \text{Lie}(H) \simeq \mathfrak{u}(1) \simeq i\mathbb{R}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} ix_1 & 0 & 0 & 0 & 0 \\ 0 & ix_2 & 0 & 0 & 0 \\ 0 & 0 & ix_3 & 0 & 0 \\ 0 & 0 & 0 & -i(x_1 + x_2 + x_3) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; x_j \in \mathbb{R} \right\} \simeq \text{Lie}(N(H)|H) \simeq \text{Lie}(\mathbb{T}^3) \simeq \mathbb{R}^3$$

—

$$\text{For } 1 \leq j, k \leq 5, \quad \mathfrak{l}_{jk} = \left\{ \begin{pmatrix} & & & & z \\ & & & & -\bar{z} \end{pmatrix}; z \in \mathbb{C} \right\} \simeq \mathbb{C}$$

(in the last matrix, z is the (j, k) -entry and $-\bar{z}$ is the (k, j) -entry).

\mathfrak{h} and \mathfrak{k} carry the trivial representation of H , and for $1 \leq j < k \leq 5$, $\overset{p}{\mathfrak{l}_{jk}} \simeq \mathbb{C}_p$

Second, we study the adjoint action of $N(H)$ on $\mathfrak{g} = \mathfrak{su}(5)$. Noticing that $N(H) \simeq \mathbb{T}^4$ is nothing but the natural maximal torus of $\mathfrak{su}(5)$, we see immediately that the decomposition of \mathfrak{g} under $N(H)$ reads :

$$\begin{aligned} \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus & \overset{(-1,1,0,0)}{\mathfrak{l}_{21}} \oplus \overset{(0,-1,1,0)}{\mathfrak{l}_{32}} \oplus \overset{(0,0,-1,1)}{\mathfrak{l}_{43}} \oplus \overset{(1,0,-1,0)}{\mathfrak{l}_{13}} \oplus \overset{(0,1,0,-1)}{\mathfrak{l}_{24}} \oplus \overset{(1,0,0,-1)}{\mathfrak{l}_{14}} \\ & \oplus \overset{(2,1,1,1)}{\mathfrak{l}_{15}} \oplus \overset{(1,2,1,1)}{\mathfrak{l}_{25}} \oplus \overset{(1,1,2,1)}{\mathfrak{l}_{35}} \oplus \overset{(1,1,1,2)}{\mathfrak{l}_{45}} \end{aligned}$$

where $\overset{(p_1,p_2,p_3,p_4)}{\mathfrak{l}_{jk}} \simeq \mathbb{C}$ carrying the irreducible representation of \mathbb{T}^4 labeled by the element $(p_1, p_2, p_3, p_4) \in \mathbb{Z}^4$.

Let us define the two following subspaces of \mathfrak{g} :

$$\begin{aligned} \mathfrak{m}_1 &= \mathfrak{k} \oplus \overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}} \oplus \overset{-2}{\mathfrak{l}_{13}} \\ \mathfrak{m}_2 &= \overset{-3}{\mathfrak{l}_{14}} \oplus \overset{1}{\mathfrak{l}_{43}} \oplus \overset{-2}{\mathfrak{l}_{24}} \oplus \overset{11}{\mathfrak{l}_{15}} \oplus \overset{12}{\mathfrak{l}_{25}} \oplus \overset{13}{\mathfrak{l}_{35}} \oplus \overset{14}{\mathfrak{l}_{45}} \end{aligned}$$

It is clear that \mathfrak{m}_1 and \mathfrak{m}_2 are $\text{Ad}(N(H))$ -invariant complementary subspaces of \mathfrak{m} , and thus we have an $\text{Ad}(N(H))$ -invariant splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We deduce a space of Higgs fields :

$$\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = \mathcal{L}(\overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}}, \overset{1}{\mathfrak{l}_{43}}) \oplus \mathcal{L}(\overset{-2}{\mathfrak{l}_{13}}, \overset{-2}{\mathfrak{l}_{24}})$$

It is not difficult to see that : $\dim \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = 6$ (on \mathbb{R}).

A natural basis for the vector space \mathfrak{m} is given by the following matrices :

$$\begin{aligned} \left(\begin{array}{ccccc} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) & \text{(basis for } \mathfrak{k} \text{)}; \\ \text{for } 1 \leq j, k \leq 5, \quad a_{jk} = \left(\begin{array}{c} 1 \\ -1 \end{array} \right), \quad b_{jk} = \left(\begin{array}{c} i \\ i \end{array} \right) & \text{(basis for } \mathfrak{l}_{jk} \text{)}. \end{aligned}$$

(In a_{jk} , 1 is the (j, k) -entry and -1 is the (k, j) -entry, etc...)

Let $\phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. In fact, ϕ is completely determined by its restriction (also denoted by ϕ) :

$$\phi : \mathfrak{l}_{21}^1 \oplus \mathfrak{l}_{32}^1 \oplus \mathfrak{l}_{13}^{-2} \longrightarrow \mathfrak{l}_{43}^1 \oplus \mathfrak{l}_{24}^{-2}$$

We set $\mathfrak{l}_1 = \mathfrak{l}_{21}^1 \oplus \mathfrak{l}_{32}^1 \oplus \mathfrak{l}_{13}^{-2}$, $\mathfrak{l}_2 = \mathfrak{l}_{43}^1 \oplus \mathfrak{l}_{24}^{-2}$, and define the six real numbers $\varphi_{21}, \psi_{21}, \varphi_{32}, \psi_{32}, \varphi_{13}, \psi_{13}$ by :

$$\begin{aligned} \phi(a_{21}) &= \varphi_{21} a_{43} + \psi_{21} b_{43} & (\text{and } \phi(b_{21}) &= -\psi_{21} a_{43} + \varphi_{21} b_{43}) \\ \phi(a_{32}) &= \varphi_{32} a_{43} + \psi_{32} b_{43} & (\text{and } \phi(b_{32}) &= -\psi_{32} a_{43} + \varphi_{32} b_{43}) \\ \phi(a_{13}) &= \varphi_{13} a_{24} + \psi_{13} b_{24} & (\text{and } \phi(b_{13}) &= -\psi_{13} a_{24} + \varphi_{13} b_{24}) \end{aligned}$$

The real numbers $\varphi_{21}, \psi_{21}, \varphi_{32}, \psi_{32}, \varphi_{13}, \psi_{13}$ constitute the six components of the Higgs field ϕ in the bases of \mathfrak{m}_1 and \mathfrak{m}_2 defined above.

We shall need the three complex components of ϕ , defined by :

$$\phi_{21} = \varphi_{21} + i \psi_{21}, \quad \phi_{32} = \varphi_{32} + i \psi_{32}, \quad \phi_{13} = \varphi_{13} + i \psi_{13}$$

$$\text{If } A = \begin{pmatrix} 0 & -\bar{z}_{21} & z_{13} & 0 & 0 \\ z_{21} & 0 & -\bar{z}_{32} & 0 & 0 \\ -\bar{z}_{13} & z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_1,$$

$$\text{then } \phi(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{13} z_{13} & 0 \\ 0 & 0 & 0 & -\bar{\phi}_{21} \bar{z}_{21} - \bar{\phi}_{32} \bar{z}_{32} & 0 \\ 0 & -\bar{\phi}_{13} \bar{z}_{13} & \phi_{21} z_{21} + \phi_{32} z_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_2.$$

Let us study the representation of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2)$. For this, we first look at the representation of $N(H)$ on $\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2)$:

$$\text{Given } P = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(\theta_1+\theta_2+\theta_3+\theta_4)} \end{pmatrix} \in N(H),$$

$$\text{we have } \text{Ad}_{P^{-1}}(A) = P^{-1}AP = \begin{pmatrix} 0 & -e^{i(-\theta_1+\theta_2)} \bar{z}_{21} & e^{i(-\theta_1+\theta_3)} z_{13} & 0 & 0 \\ e^{-i(-\theta_1+\theta_2)} z_{21} & 0 & -e^{i(-\theta_2+\theta_3)} \bar{z}_{32} & 0 & 0 \\ -e^{-i(-\theta_1+\theta_3)} \bar{z}_{13} & e^{-i(-\theta_2+\theta_3)} z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \text{Ad}_P \circ \phi \circ \text{Ad}_{P^{-1}}(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\bar{\beta} & 0 \\ 0 & -\bar{\alpha} & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\alpha = e^{i(\theta_2-\theta_4-\theta_1+\theta_3)}\phi_{13} z_{13}$ and $\beta = e^{-i(\theta_3-\theta_4-\theta_1+\theta_2)}\phi_{21} z_{21} + e^{-i(\theta_3-\theta_4-\theta_2+\theta_3)}\phi_{32} z_{32}$. This shows that ϕ_{13} , ϕ_{21} and ϕ_{32} transform respectively according to the representations $(-1, 1, 1, -1)$, $(1, -1, -1, 1)$ and $(0, 1, -2, 1)$ of $\mathbb{T}^4 \simeq N(H)$.

Now we turn to the action of $N(H)|H$. First, we define an homomorphism of Lie groups $f : N(H) \longrightarrow G$ by setting :

$$f(P) = \begin{pmatrix} e^{i(4\theta_1-\theta_4)} & 0 & 0 & 0 & 0 \\ 0 & e^{i(3\theta_1-\theta_3)} & 0 & 0 & 0 \\ 0 & 0 & e^{i(2\theta_1-\theta_2)} & 0 & 0 \\ 0 & 0 & 0 & e^{-i(9\theta_1-\theta_2-\theta_3-\theta_4)} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Next, we set :

$$K = \left\{ \begin{pmatrix} e^{ix_1} & 0 & 0 & 0 & 0 \\ 0 & e^{ix_2} & 0 & 0 & 0 \\ 0 & 0 & e^{ix_3} & 0 & 0 \\ 0 & 0 & 0 & e^{-i(x_1+x_2+x_3)} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

It is not difficult to see that $\text{Ker } f = H$ and $\text{Im } f = K$. Therefore, we have a surjective homomorphism that we also denote by f , and this $f : N(H) \longrightarrow K$ induces an isomorphism $\bar{f} : N(H)|H \longrightarrow K$. Thus, $N(H)|H \simeq K \simeq \mathbb{T}^3$.

Now if $\rho : N(H) \longrightarrow GL(\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2))$ is the representation previously computed ($\rho(P)(\phi) = \text{Ad}_P \circ \phi \circ \text{Ad}_{P^{-1}}$), we define $\bar{\rho} : K \longrightarrow GL(\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2))$ the following way : for each $Q \in K$, let $\bar{\rho}(Q) = \rho(P)$, where P is any element of $N(H)$ such that $f(P) = Q$. For example, one can take :

$$P = \begin{pmatrix} e^{ix_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i(2x_1-x_3)} & 0 & 0 & 0 \\ 0 & 0 & e^{i(3x_1-x_2)} & 0 & 0 \\ 0 & 0 & 0 & e^{i3x_1} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(9x_1-x_2-x_3)} \end{pmatrix}$$

We have previously shown that the Higgs field ϕ transforms under the action of $P \in N(H)$ according to :

$$\begin{cases} \phi_{13} \longrightarrow e^{i(-\theta_1+\theta_2+\theta_3-\theta_4)}\phi_{13} \\ \phi_{21} \longrightarrow e^{i(\theta_1-\theta_2-\theta_3+\theta_4)}\phi_{21} \\ \phi_{32} \longrightarrow e^{i(\theta_2-2\theta_3+\theta_4)}\phi_{32} \end{cases}$$

We deduce that ϕ transforms under the action of $Q \in K$ according to :

$$\begin{cases} \phi_{13} \longrightarrow e^{i(x_1-x_2-x_3)}\phi_{13} \\ \phi_{21} \longrightarrow e^{i(-x_1+x_2+x_3)}\phi_{21} \\ \phi_{32} \longrightarrow e^{i(-x_1+2x_2-x_3)}\phi_{32} \end{cases}$$

Thus, ϕ_{13} , ϕ_{21} and ϕ_{32} belong respectively to the representations $(1, -1, -1)$, $(-1, 1, 1)$ and $(-1, 2, -1)$ of $\mathbb{T}^3 \simeq N(H)|H$.

The sixth degree term of the potential of the scalar field ϕ is proportional to the squared norm of the tensor $\phi \text{ ad}_{12}(\phi) \phi$, which can be identified with the bilinear map $B_\phi : \mathfrak{l}_1 \times \mathfrak{l}_1 \longrightarrow \mathfrak{l}_2$ defined by : $B_\phi(X, X') = \phi(\pi_1[\phi(X), \phi(X')])$.

$$\text{Let } X = \begin{pmatrix} 0 & -\bar{z}_{21} & z_{13} & 0 & 0 \\ z_{21} & 0 & -\bar{z}_{32} & 0 & 0 \\ -\bar{z}_{13} & z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and } X' = \begin{pmatrix} 0 & -\bar{z}'_{21} & z'_{13} & 0 & 0 \\ z'_{21} & 0 & -\bar{z}'_{32} & 0 & 0 \\ -\bar{z}'_{13} & z'_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_1.$$

Then :

$$\pi_1[\phi(X), \phi(X')] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & -\bar{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{where } \gamma = \phi_{13} z_{13} (\phi_{21} z'_{21} + \phi_{32} z'_{32}) - \phi_{13} z'_{13} (\phi_{21} z_{21} + \phi_{32} z_{32}).$$

We deduce that :

$$B_\phi(X, X') = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\phi}_{32} \gamma & 0 \\ 0 & 0 & -\phi_{32} \bar{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$\begin{aligned} \bar{\phi}_{32} \gamma &= \bar{\phi}_{32} \phi_{13} z_{13} (\phi_{21} z'_{21} + \phi_{32} z'_{32}) - \bar{\phi}_{32} \phi_{13} z'_{13} (\phi_{21} z_{21} + \phi_{32} z_{32}) \\ &= \bar{\phi}_{32} \phi_{13} \phi_{21} z_{13} z'_{21} + \bar{\phi}_{32} \phi_{13} \phi_{32} z_{13} z'_{32} - \bar{\phi}_{32} \phi_{13} \phi_{21} z'_{13} z_{21} - \bar{\phi}_{32} \phi_{13} \phi_{32} z'_{13} z_{32} \\ &= \bar{\phi}_{32} \phi_{13} \phi_{21} (z_{13} z'_{21} - z'_{13} z_{21}) + \bar{\phi}_{32} \phi_{13} \phi_{32} (z_{13} z'_{32} - z'_{13} z_{32}) \end{aligned}$$

Thus, B_ϕ can be identified with the antisymmetric complex bilinear form (also denoted by B_ϕ) :

$$B_\phi : \mathfrak{l}_1 \times \mathfrak{l}_1 \longrightarrow \mathbb{C}$$

defined by : $B_\phi(X, X') = \bar{\phi}_{32} \phi_{13} \phi_{21} (z_{13} z'_{21} - z'_{13} z_{21}) + \bar{\phi}_{32} \phi_{13} \phi_{32} (z_{13} z'_{32} - z'_{13} z_{32})$.

\mathfrak{l}_1 is isomorphic to the three-dimensional complex vector space \mathbb{C}^3 . If we set $(e_1, e_2, e_3) = (a_{21}, a_{32}, a_{13})$, then $(e_i)_{1 \leq i \leq 3}$ is a basis of \mathfrak{l}_1 , and B_ϕ is completely determined by the 3 complex numbers $B_\phi(e_i, e_j)$, $1 \leq i < j \leq 3$. It is easy to check that :

$$\begin{aligned} B_\phi(a_{21}, a_{32}) &= 0 \\ B_\phi(a_{21}, a_{13}) &= \bar{\phi}_{32} \phi_{13} \phi_{21} \\ B_\phi(a_{32}, a_{13}) &= \bar{\phi}_{32} \phi_{13} \phi_{32} \end{aligned}$$

Finally, we need to define the $\text{Ad}(N(H))$ -invariant scalar products h and k on \mathfrak{l}_1 and \mathfrak{l}_2 respectively, and use them to compute the squared norm of the tensor $\phi \text{ ad}_{12}(\phi) \phi$. We consider the natural $\text{Ad}(\text{SU}(5))$ -invariant scalar product on $\mathfrak{su}(5)$ defined by : $\langle A, B \rangle = -\frac{1}{2}\text{tr}(AB)$ for every $A, B \in \mathfrak{su}(5)$. Then, h and k are defined to be the restrictions of \langle , \rangle to $\mathfrak{l}_1 \times \mathfrak{l}_1$ and $\mathfrak{l}_2 \times \mathfrak{l}_2$ respectively. Then we take the restriction of k to \mathfrak{l}_{43} and obtain the canonical scalar product on $\mathbb{C} \simeq \mathfrak{l}_{43}$.

Thus, we have in this model : $V^6(\phi) \propto |\bar{\phi}_{32} \phi_{13} \phi_{21}|^2 + |\bar{\phi}_{32} \phi_{13} \phi_{32}|^2$.

1.5 Conclusion

In the context of Kaluza-Klein theory with, as internal spaces, copies of a homogeneous space G/H , we defined a new type of scalar fields as intertwining operators between two $\text{Ad}(N(H))$ -invariant complementary subspaces of the orthogonal complement of \mathfrak{h} in \mathfrak{g} for a fixed bi-invariant metric on G .

We studied the potential of those scalar fields and found it to be a polynomial of the sixth degree at most, bounded from below when it was exactly of degree 6 or 4. And we found the necessary and sufficient conditions in which the degree is 6 or 4.

We investigated an eventual coupling of the scalar fields with the Yang-Mills strength, and found that there is no such direct coupling if we choose the two complementary subspaces in a such way that all the trivial representation is contained in only one of them. We also computed the kinetic term of the scalar fields and found a standard result. An example leading to an abelian gauge theory was given.

Our scalar fields exhibit therefore properties that are very close to those of the Higgs fields ordinarily expected in the physical models, and they have been constructed in a natural way, by looking at hyperbolic directions in the space of Thiry scalar fields, this last one appearing whenever one takes a gravitational theory with some symmetry.

Although sixth-degree polynomial potentials give rise to a non-renormalizable quantum theory in four-dimensional spacetime, they can be very interesting in three-dimensional spacetime, where they lead to a renormalizable theory. Quantum field theories in three dimensions are related to Chern-Simons theories, which have both physical and mathematical applications. In particular, sixth-degree polynomial potentials lead in these theories to vortices, which are solutions of the soliton type.

It would seem interesting to carry further the study of the potential of such "canonical Higgs fields", looking for example at the corresponding symmetry breaking schemes, and seeking what happens at the quantum level. Soliton-like solutions may also appear in the BPS limit, and the introduction of supersymmetry could also be relevant to make contact with certain types of dualities.

Let us mention that among the principal models that retain the attention of the physicists, the *little Higgs* models are somehow close to those treated in this chapter. But there are lots of other schemes that have been considered (cf. [?]), especially models related to supersymmetry, and in particular the extended supersymmetries coming from compactified supergravity theories, which we are going to study in the last part of this thesis.

Finally, let us not forget that Quantum Field Theory is not the last word ; it is possible that well-understood gravitation would allow the quantization of sixth-degree polynomial potentials (even in four dimensions), and therefore one should maybe go to superstring theories (and in particular include supergravity).

Chapitre 2

Dimensional reduction of projected metrics

To perform a Kaluza-Klein reduction of a gravitational theory, one generally starts with a $(d+n)$ -dimensional general relativity theory (which amounts to write an equation for which solutions are particular couples (U, g) , where U is a $(d+n)$ -dimensional manifold, and g is a metric on U). Then, one seeks for a "ground-state compactifying solution", that is, a solution $(\mathring{U}, \mathring{g})$ of the general relativity equation such that :

- $\mathring{U} = \mathring{M} \times \mathring{K}$, where \mathring{M} is a d -dimensional non-compact manifold "with symmetries" (that is for example a homogeneous space for some non-compact Lie group), and \mathring{K} is an n -dimensional compact manifold "with symmetries" (typically a compact Lie group G or a compact homogeneous space G/H).
- $\mathring{g} = \mathring{\eta} \times \mathring{\delta}$ is a product metric on $\mathring{U} = \mathring{M} \times \mathring{K}$, where $\mathring{\eta}$ is a "maximally symmetric" Lorentzian metric on \mathring{M} , and $\mathring{\delta}$ is a "maximally symmetric" Euclidian metric on \mathring{K} .

In the case where $\mathring{K} = G$, a compact Lie group, the requirement of maximal symmetry for $\mathring{\delta}$ could mean for example that it has to be a *bi-invariant* metric. The isometry group of $\mathring{\delta}$ contains then the group $G \times G$, which acts on $\mathring{K} = G$.

In the case where $\mathring{K} = G/H$, a compact homogeneous space, the requirement of maximal symmetry for $\mathring{\delta}$ could mean for example that it has to be a *normal* metric. The isometry group of $\mathring{\delta}$ contains then the group $(N(H)|H) \times G$, which acts on $\mathring{K} = G/H$.

The next step, in order to obtain an effective d -dimensional theory, is to impose an *ansatz* on the $(d+n)$ -dimensional metric g on U . This means the following. First, as a differentiable manifold, U is now fixed to be $\mathring{M} \times \mathring{K}$ (which carries a right action of the subgroup G), or more generally a fiber bundle carrying a right action of G , whose fibers are copies of \mathring{K} . Then, instead of considering any metric g on U (which would correspond to infinitely many fields on $M = U/G$), one selects only a subset of "dimensionally reducible" metrics on U (which correspond to a finite number of fields on M). To search for solutions in this subset, one writes the $(d+n)$ -dimensional Einstein Lagrangian in terms of the finite number of fields on M , and then integrates it over G/H , obtaining thus a Langrangian on M . If the ansatz is *consistent*, any solution of the reduced d -dimensional theory corresponds to an element of the selected subset of metrics on U that is a solution of the $(d+n)$ -dimensional theory.

The chosen ansatz for the dimensional reductions considered in the first chapter amounts to retain only the G -invariant metrics. Coquereaux and Jadczyk have shown that this ansatz is *consistent*, which means that every solution of the reduced theory can be expressed as a solution of the higher-dimensional theory. However, the most considered ansatz in the 80's, in particular for supergravity reductions, is *not* the G -invariant ansatz. It is a less restrictive ansatz, called *the popular ansatz*. Although generically inconsistent, the popular ansatz is shown to be consistent in a few exceptionnal cases, which are precisely the most interesting cases, like for example the reduction on the sphere S^7 of eleven-dimensional supergravity, that we are going to discuss in the last part. Coquereaux and Jadczyk have proposed a geometrical interpretation of the popular ansatz, that we present in this chapter. The non-invariant metrics appear then as projected metrics from $(G \times G)$ -invariant metrics on a tautological bundle obtained by extending trivially the higher-dimensional universe with the group G . We are thus brought back to the same scheme of reduction as the one considered in the preceding chapter for invariant metrics, but this time with $G \times G$ instead of G . A consequence is that such a reduction based on the popular ansatz gives naturally a space of scalar fields of the type that we defined in the first chapter, and provides thus a new example belonging to the framework of our construction.

2.1 The geometrical interpretation of the popular ansatz

We recall here the geometrical construction proposed by Coquereaux and Jadczyk (cf. [CJ3]) that gives the so-called "popular ansatz", which is dimensionally reducible, but not G -invariant.

In the preceding chapter, we considered a smooth manifold U , and a Lie group G operating on U on the right, in such a way that all the stabilizers are conjugated between themselves. For $u_0 \in U$ fixed, let $H = \text{Stab}(u_0)$, $M = U/G$, and $\pi : U \longrightarrow M$ the canonical surjection. We know that U is a locally trivializable fibre bundle above M (π being the fibration), with typical fiber G/H . Let $Q = \{u \in U / \text{Stab}(u) = H\}$. Then $N(H)|H$ acts on Q on the right, and this action is free. Moreover $M \simeq Q/(N(H)|H)$. Thus, $(Q, M, N(H)|H)$ is a principal fibre bundle, and $(U, M, G/H)$ is the fiber bundle associated to it by the natural left action of $N(H)|H$ on G/H .

We saw in the preceding chapter that the dimensional reduction of a G -invariant metric on U leads to : a metric γ on M , a connection α on $(Q, M, N(H)|H)$ (which corresponds to a Yang-Mills gauge potential valued in the Lie algebra of the structure group $N(H)|H$), and an element h of $C_{N(H)|H}^\infty(Q, \text{Sym}_H^2 \mathfrak{m}^*)$ (which corresponds to a scalar field valued in the vector space of $\text{Ad}(H)$ -invariant symmetric bilinear forms on \mathfrak{m} , the orthogonal complement of \mathfrak{h} in \mathfrak{g} for an $\text{Ad}(G)$ -invariant scalar product on \mathfrak{g}).

A more general Kaluza-Klein ansatz is expected to yield to a Yang-Mills gauge potential valued in the Lie algebra of the isometry group of $\overset{\circ}{K}$ (with its normal metric $\overset{\circ}{\delta}$), which contains $(N(H)|H) \times G$. Thus, it would be interesting to enlarge the structure group $N(H)|H$ (that naturally emerges from the above construction) to $(N(H)|H) \times G$. This is the purpose of the following construction.

We set $\bar{U} = U \times G$, equipped with the right action $\bar{R} : G \times G \longrightarrow \text{Diff}(\bar{U})$, defined by : $\bar{R}_{(a,b)}(u, c) = (ua, b^{-1}c)$. For $c \in G$, let us first determine the stabilizer of the point $(u_0, c) \in \bar{U}$ (we already have $\text{Stab}(u_0) = H$).

$$\begin{aligned} (a, b) \in \text{Stab}(u_0, c) &\iff \bar{R}_{(a,b)}(u_0, c) = (u_0, c) \\ &\iff (u_0a, b^{-1}c) = (u_0, c) \\ &\iff u_0a = u_0 \text{ and } b^{-1}c = c \\ &\iff a \in H \text{ and } b = e \\ &\iff (a, b) \in H \times \{e\} \end{aligned}$$

Thus, $\text{Stab}(u_0, c) = H \times \{e\}$. Moreover, it is easy to show that all stabilizers are conjugated between themselves for the action of $G \times G$.

Proposition 2.1.1 $\bar{U}/(G \times G) \simeq M$

Proof : Let $\bar{\pi} : \bar{U} \longrightarrow M$ the map defined by : $\bar{\pi}(u, c) = \pi(u)$. It is clear that $\bar{\pi}$ is surjective because $\forall x \in M, \exists u \in U$ such that $\pi(u) = x$; then, $\bar{\pi}(u, e) = x$. Next, $\bar{\pi}$ is $(G \times G)$ -invariant : $\bar{\pi}(\bar{R}_{(a,b)}(u, c)) = \bar{\pi}(ua, b^{-1}c) = \pi(ua) = \pi(u) = \bar{\pi}(u, c)$. Finally, if $\bar{\pi}(u, c) = \bar{\pi}(u', c')$, then $\pi(u) = \pi(u')$ and therefore it exists $a \in G$ such as $u' = ua$. Then, setting $b = cc'^{-1}$, we have : $u' = ua$ and $c' = b^{-1}c$, which implies : $(u', c') = (ua, b^{-1}c) = \bar{R}_{(a,b)}(u, c)$. Thus, the fibres of $\bar{\pi}$ are identified to the orbits of $G \times G$, and we have a diffeomorphism $\bar{U}/(G \times G) \longrightarrow M$. \square

Thus, \bar{U} is a locally trivializable fibre bundle above M , with typical fibre $(G \times G)/(H \times \{e\})$, exactly like U is a locally trivializable fibre bundle above M , with typical fibre G/H . Also, in the same way that we realized $(U, M, G/H)$ as the associated bundle to a principal bundle $(Q, M, N(H)|H)$, we may realize $(\bar{U}, M, (G \times G)/(H \times \{e\}))$ as the associated bundle to a principal bundle $(\bar{Q}, M, (N(H)|H) \times G)$ that we construct below.

Set $\bar{Q} = \{(u, c) \in \bar{U} / \text{Stab}(u, c) = H \times \{e\}\}$. We have :

$$\begin{aligned} (u, c) \in \bar{Q} &\iff \bar{R}_{(a,e)}(u, c) = (u, c) \quad \forall a \in H \\ &\iff (ua, e^{-1}c) = (u, c) \quad \forall a \in H \\ &\iff (ua, c) = (u, c) \quad \forall a \in H \\ &\iff ua = u \quad \forall a \in H \\ &\iff \text{Stab}(u) = H \\ &\iff u \in Q \end{aligned}$$

Thus, $\bar{Q} = Q \times G$.

Let $(a, b) \in N(H \times \{e\})$, so that $\forall c \in H, (a, b)(c, e)(a^{-1}, b^{-1}) \in H \times \{e\}$. This implies $aca^{-1} \in H, \forall c \in H$, that is : $a \in N(H)$. So $N(H \times \{e\}) = N(H) \times G$. Also, $N(H \times \{e\})/(H \times \{e\}) = (N(H) \times G)/(H \times \{e\}) = (N(H)|H) \times G$.

Thus, $(\bar{Q}, M, (N(H)|H) \times G)$ is a principal fibre bundle, and $(\bar{U}, M, (G \times G)/(H \times \{e\}))$ is an associated fibre bundle to this principal fibre bundle, by the natural action of $(N(H)|H) \times G$ on $(G \times G)/(H \times \{e\})$.

Now we consider another fibration of \bar{U} , this time above U .

Let Δ be the diagonal embedding of G into $G \times G$. We can then define a right action $R : G \longrightarrow \text{Diff}(\bar{U})$ by setting : $R_a(u, c) = \bar{R}_{(a,a)}(u, c) = (ua, a^{-1}c)$, that is, $R = \bar{R} \circ \Delta$. Being free, this action defines on \bar{U} a principal G -bundle structure, and we have $\bar{U}/G \simeq U$. In fact, the surjection $s : \bar{U} \longrightarrow U$ defined by $s(u, c) = uc$ is $R(G)$ -invariant, and defines therefore a principal fibration of \bar{U} above U , with G as structure group.

We fix an $\text{Ad}(G)$ -invariant scalar product k_0 on \mathfrak{g} . We may consider then the constant map $k : \bar{U} \longrightarrow \text{Sym}^2 \mathfrak{g}^*$ defined by : $k(u, c) = k_0$, $\forall (u, c) \in \bar{U}$. For every $a \in G$, we have : $k(R_a(u, c)) = k(ua, a^{-1}c) = k_0 = {}^t\text{Ad}_a(k_0) = {}^t\text{Ad}_a(k(u, c))$. We deduce that : $k \in C_G^\infty(\bar{U}, \text{Sym}^2 \mathfrak{g}^*)$.

Let g be a metric on U , and ω a connection on the principal bundle (\bar{U}, U, G) . We know that we can associate to the triple (g, ω, k) a $R(G)$ -invariant metric \bar{g} on \bar{U} .

In what follows, we consider only the couples (g, ω) such that the G -invariant metric \bar{g} associated to the triple (g, ω, k) is $(G \times G)$ -invariant.

On $\mathfrak{g} \times \mathfrak{g}$, we consider the $\text{Ad}(G \times G)$ -invariant scalar product $k_0 \times k_0$; if \mathfrak{m} is the orthogonal complement of \mathfrak{h} in \mathfrak{g} (with respect to k_0), then $\bar{\mathfrak{m}} = \mathfrak{m} \times \mathfrak{g}$ is the orthogonal complement of $\mathfrak{h} \times \{0\}$ in $\mathfrak{g} \times \mathfrak{g}$ (with respect to $k_0 \times k_0$).

The dimensional reduction of \bar{g} by $G \times G$ gives :

- a metric γ on M ,
- a connection $\bar{\alpha}$ on the principal bundle $(\bar{Q}, M, (N(H)|H) \times G)$,
- a map $\bar{h} \in C_{(N(H)|H) \times G}^\infty(\bar{Q}, \text{Sym}_{H \times \{e\}}^2 \bar{\mathfrak{m}}^*)$

Thus, we obtained a Yang-Mills-Higgs theory on M with structure group $(N(H)|H) \times G$, as expected.

We conclude this section by recovering an explicit expression of the popular ansatz.

Consider a local chart (W, φ) on M , such that the open set W is trivializing for the principal bundle \bar{Q} . Let $\sigma : W \longrightarrow \bar{Q}$ be a local cross-section of \bar{Q} above W . For each $x \in W$, we denote by $(\partial_{\mu|x})$ the basis of $T_x M$ associated with the chart (W, φ) . For $(q, c) \in \bar{Q}_x$, we denote by $e_{\mu|(q,c)} \in T_{(q,c)} \bar{Q}$ the horizontal lift of $\partial_{\mu|x}$ for the connection $\bar{\alpha}$. On the other hand, let $(\bar{e}_i) = (\bar{e}_{\hat{a}}, \bar{e}_{\hat{a}}, \bar{e}_a)$ be an adapted basis of the Lie algebra $\mathfrak{g} \times \mathfrak{g} = (\mathfrak{h} \times \{0\}) \oplus (\mathfrak{k} \times \mathfrak{g}) \oplus (\mathfrak{l} \times \{0\})$. We denote by $\tilde{\bar{e}}_{\hat{a}}$ the fundamental vector field on \bar{Q} associated to $\bar{e}_{\hat{a}}$ by the right action of $(N(H)|H) \times G$ on \bar{Q} .

It is not difficult to see that $(e_\mu, \tilde{\bar{e}}_{\hat{a}})$ is a local moving frame on \bar{Q} .

If $\bar{\pi} : \bar{Q} \longrightarrow M$ is the projection of the principal bundle $(\bar{Q}, M, (N(H)|H) \times G)$, then any point $(q, c) \in \bar{\pi}^{-1}(W)$ may be written : $(q, c) = \sigma(x)([n], c)$ with $x \in W$ and $([n], c) \in (N(H)|H) \times G$. Set ${}^\sigma \bar{\alpha} = \sigma^* \bar{\alpha}$ and ${}^\sigma \bar{\alpha}_\mu = {}^\sigma \bar{\alpha}_{\mu}^{\hat{a}} \bar{e}_{\hat{a}}$.

Finally, making use of the local cross-section σ , we may define another local moving frame on \bar{Q} by setting : ${}^\sigma \partial_{\mu|(q,c)} = T_x \sigma(\partial_{\mu|x})$ and ${}^\sigma \bar{e}_{\hat{a}|(q,c)} = \text{Ad}_{([n], c)^{-1}}(\tilde{\bar{e}}_{\hat{a}}|_{(q,c)})$. The trivialization-

dependent local moving frame on \bar{Q} is then : $(\sigma \partial_\mu, \sigma \bar{e}_{\hat{a}})$. This last frame has the nice property to be made of right-invariant vector fields.

The $(G \times G)$ -invariant ansatz on the inverse metric of \bar{g} is then :

$$\begin{aligned} \bar{g}_{|(q,c)}^{-1} &= \gamma^{\mu\nu}(x)(\sigma \partial_{\mu|(q,c)} \otimes \sigma \partial_{\nu|(q,c)}) + \gamma^{\mu\nu}(x) \sigma \bar{\alpha}_\nu^{\hat{b}}(x)(\sigma \partial_{\mu|(q,c)} \otimes \sigma \bar{e}_{\hat{b}|(q,c)}) \\ &+ \gamma^{\mu\nu}(x) \sigma \bar{\alpha}_\mu^{\hat{a}}(x)(\sigma \bar{e}_{\hat{a}|(q,c)} \otimes \sigma \partial_{\nu|(q,c)}) + (\gamma^{\mu\nu}(x) \sigma \bar{\alpha}_\mu^{\hat{a}}(x) \sigma \bar{\alpha}_\nu^{\hat{b}}(x) + \bar{h}^{\hat{a}\hat{b}}(x))(\sigma \bar{e}_{\hat{a}|(q,c)} \otimes \sigma \bar{e}_{\hat{b}|(q,c)}) \end{aligned}$$

2.2 The scalar sector resulting from a projected metric

Let us now investigate the scalar sector resulting from the above reduction scheme. The matter field obtained is a $((N(H)|H) \times G)$ -equivariant map $\bar{h} : \bar{Q} \longrightarrow \text{Sym}_{H \times \{e\}}^2 \bar{\mathfrak{m}}^*$.

For $x \in M$, let $U_x = \pi^{-1}(\{x\})$ and $\bar{U}_x = \bar{\pi}^{-1}(U_x)$. Consider $(u, c) \in \bar{U}_x$ so that $\bar{\pi}(u, c) = x$.

Proposition 2.2.1 (\bar{U}_x, U_x, G) is a principal fibre bundle.

Proof : We have $\pi(u) = \bar{\pi}(u, c) = x$ so $u \in U_x$, and $s(u, c) = uc \in U_x$. Therefore, the projection $s : \bar{U} \longrightarrow U$ can be restricted to a map $s : \bar{U}_x \longrightarrow U_x$. Moreover, for every $u \in U_x$, we have $s(u, e) = u$, with $(u, e) \in \bar{U}_x$ since $\bar{\pi}(u, e) = \pi(u) = x$. This shows that s is surjective. Note that $\bar{\pi}(R_a(u, c)) = \bar{\pi}(ua, a^{-1}c) = \pi(ua) = \pi(u) = x$, for every $(u, c) \in \bar{U}_x$. Thus, $\forall (u, c) \in \bar{U}_x$, $R_a(u, c) \in \bar{U}_x$, which shows that G acts on \bar{U}_x : we have therefore $R : G \longrightarrow \text{Diff}(\bar{U}_x)$. The surjection s is $R(G)$ -invariant since $s(R_a(u, c)) = s(ua, a^{-1}c) = uaa^{-1}c = uc = s(u, c)$. And if $s(u, c) = s(u', c')$, we have $uc = u'c'$. Therefore $\pi(uc) = \pi(u'c')$, i.e. $\pi(u) = \pi(u')$, and therefore $\exists a \in G$ such as $u' = ua$. Thus, we have $uc = uac'$, which implies $c = ac'$ that is $c' = a^{-1}c$. Therefore $(u', c') = (ua, a^{-1}c) = R_a(u, c)$. Thus, the fibres of s are identified to the orbits of $R(G)$, and we have a diffeomorphism $\bar{U}_x/R(G) \longrightarrow U_x$. We deduce that (\bar{U}_x, U_x, G) is a principal fibre bundle. \square

Let $\iota_x : \bar{Q}_x \hookrightarrow \bar{Q}$ be the canonical inclusion. Set $\bar{h}_x = \bar{h} \circ \iota_x$.

Then $\bar{h}_x \in C_{(N(H)|H) \times G}^\infty(\bar{Q}_x, \text{Sym}_{H \times \{e\}}^2 \bar{\mathfrak{m}}^*)$. But

$$C_{(N(H)|H) \times G}^\infty(\bar{Q}_x, \text{Sym}_{H \times \{e\}}^2 \bar{\mathfrak{m}}) \simeq \mathcal{M}et_{G \times G}(\bar{U}_x) \quad (\simeq \text{Sym}_{H \times \{e\}}^2 \bar{\mathfrak{m}}^*)$$

Thus, \bar{h}_x can be seen as a $(G \times G)$ -invariant metric on the homogeneous space \bar{U}_x . In particular, \bar{h}_x is a $R(G)$ -invariant metric on \bar{U}_x . The dimensional reduction of \bar{h}_x by G gives :

- a metric h_x on U_x
- a connection φ_x on the principal bundle (\bar{U}_x, U_x, G)
- a map $k_x \in C_G^\infty(\bar{U}_x, \text{Sym}^2 \mathfrak{g}^*)$.

Let us characterize each of these fields.

For k_x , it is immediate that $k_x = k \circ \iota_x$, where $\iota_x : \bar{U}_x \hookrightarrow \bar{U}$ is the canonical inclusion and $k \in C_G^\infty(\bar{U}, \text{Sym}^2 \mathfrak{g}^*)$ is the constant map of value k_0 .

Next we turn to the connection φ_x . First, notice that :

$$\begin{aligned}\mathfrak{g} \times \mathfrak{g} &= (\mathfrak{g} \times \{0\}) \oplus (\{0\} \times \mathfrak{g}) \\ &= ((\mathfrak{h} \oplus \mathfrak{m}) \times \{0\}) \oplus (\{0\} \times \mathfrak{g}) \\ &= (\mathfrak{h} \times \{0\}) \oplus (\mathfrak{m} \times \{0\}) \oplus (\{0\} \times \mathfrak{g})\end{aligned}$$

Thus, $\bar{\mathfrak{m}} = (\mathfrak{m} \times \{0\}) \oplus (\{0\} \times \mathfrak{g}) (= \mathfrak{m} \times \mathfrak{g})$.

We are going to see that the connection φ_x is defined by a map $\phi_x : \bar{Q}_x \longrightarrow \text{Hom}_H(\mathfrak{m}, \mathfrak{g})$.

Let $(q, c) \in \bar{Q}_x$, and denote by $\mathfrak{g}_{(q,c)}^\perp$ the orthogonal complement of $(\{0\} \times \mathfrak{g})$ in $\bar{\mathfrak{m}}$ for the scalar product $\bar{h}_x(q, c)$ on $\bar{\mathfrak{m}}$ (which is $\text{Ad}(H \times \{e\})$ -invariant). Let $p_x(q, c) : \bar{\mathfrak{m}} \longrightarrow (\{0\} \times \mathfrak{g})$ be the orthogonal projector on $(\{0\} \times \mathfrak{g})$ for the scalar product $\bar{h}_x(q, c)$ (that is, the projector on $(\{0\} \times \mathfrak{g})$ parallel to $\mathfrak{g}_{(q,c)}^\perp$). We denote by $j_1 : \mathfrak{m} \longrightarrow \mathfrak{m} \times \{0\}$ the canonical injection ($j_1(X) = (X, 0)$) and by $\pi_2 : (\{0\} \times \mathfrak{g}) \longrightarrow \mathfrak{g}$ the canonical projection ($\pi_2(0, X) = X$).

Then $\phi_x(q, c) \in \text{Hom}(\mathfrak{m}, \mathfrak{g})$ is defined by : $\phi_x(q, c) = -\pi_2 \circ p_x(q, c)|_{\mathfrak{m} \times \{0\}} \circ j_1$.

Proposition 2.2.2 .

1. $\phi_x(q, c) \in \text{Hom}_H(\mathfrak{m}, \mathfrak{g})$
2. $\mathfrak{g}_{(q,c)}^\perp = \{(X, \phi_x(q, c)(X)) ; X \in \mathfrak{m}\}$

Proof :

1. We have a trivial adjoint action of H on $\{0\} \times \mathfrak{g}$. Since $\bar{h}_x(q, c)$ is an $\text{Ad}(H)$ -invariant scalar product on $\bar{\mathfrak{m}}$, the orthogonal complement $\mathfrak{g}_{(q,c)}^\perp$ of $\{0\} \times \mathfrak{g}$ in $\bar{\mathfrak{m}}$ for $\bar{h}_x(q, c)$ is also $\text{Ad}(H)$ -invariant. We therefore have an $\text{Ad}(H)$ -invariant decomposition $\bar{\mathfrak{m}} = \mathfrak{g}_{(q,c)}^\perp \oplus (\{0\} \times \mathfrak{g})$, and the associated projector $p_x(q, c) : \bar{\mathfrak{m}} \longrightarrow \{0\} \times \mathfrak{g}$ is then necessarily $\text{Ad}(H)$ -equivariant. Since on the other hand j_1 and π_2 are $\text{Ad}(H)$ -equivariants, we deduce that : $\phi_x(q, c) \in \text{Hom}_H(\mathfrak{m}, \mathfrak{g})$.

2. We have : $(X, Y) = ((X, 0) - p_x(q, c)|_{\mathfrak{m} \times \{0\}}(X, 0)) + p_x(q, c)(X, 0) + (0, Y)$. Therefore,

$$\begin{aligned}\mathfrak{g}_{(q,c)}^\perp &= \{(X, 0) - p_x(q, c)|_{\mathfrak{m} \times \{0\}} \circ j_1(X) ; X \in \mathfrak{m}\} \\ &= \{(X, 0) + (0, \phi_x(q, c)(X)) ; X \in \mathfrak{m}\} \\ &= \{(X, \phi_x(q, c)(X)) ; X \in \mathfrak{m}\}\end{aligned}\quad \square$$

We set $\varphi_x(q, c) = \pi_2 \circ p_x(q, c)$. For every $(X, Y) \in \bar{\mathfrak{m}}$,

$$\begin{aligned}\varphi_x(q, c)(X, Y) &= \pi_2 \circ p_x(q, c)(X, Y) \\ &= \pi_2 \circ p_x(q, c)|_{\mathfrak{m} \times \{0\}}(X, 0) + \pi_2 \circ p_x(q, c)(0, Y) \\ &= \pi_2 \circ p_x(q, c)|_{\mathfrak{m} \times \{0\}} \circ j_1(X) + \pi_2(0, Y)\end{aligned}$$

So $\varphi_x(q, c)(X, Y) = -\phi_x(q, c)(X) + Y$.

Finally, we deal with h_x . Let $r \in Q_x$, and $(q, c) \in \bar{Q}_x$ such as $qc = r$ (which requires $c \in N(H)$). $h_x(r)$ is defined by the requirement that the natural isomorphism

$$(\mathfrak{m}, h_x(r)) \longrightarrow (\mathfrak{g}_{(q,c)}^\perp, \bar{h}_x(q, c)|_{\mathfrak{g}_{(q,c)}^\perp \times \mathfrak{g}_{(q,c)}^\perp})$$

must be an isometry. Therefore, we set : $\forall r \in Q_x, \forall X, Y \in \mathfrak{m}$,

$$h_x(r)(X, Y) = \bar{h}_x(q, c)((X, \phi_x(q, c)(X)), (Y, \phi_x(q, c)(Y)))$$

(where (q, c) is any element of \bar{Q}_x such that $qc = r$).

In conclusion, we have obtained the decomposition of $\bar{h} \in C_{(N(H)|H) \times G}^\infty(\bar{Q}, \text{Sym}_{H \times \{e\}}^2 \bar{m}^*)$ into :

- $h \in C^\infty(\bar{Q}, \text{Sym}^2 \mathfrak{m}^*)$ (defined by $h(q, c) = h_{\pi(q)}(qc)$)
- $\phi \in C_G^\infty(\bar{Q}, \text{Hom}_H(\mathfrak{m}, \mathfrak{g}))$
- $k \in C_G^\infty(\bar{U}, \text{Sym}^2 \mathfrak{g}^*)$.

The preceding computation is a particular case of lemma 1.3.1. The space of Higgs fields considered in the first chapter coincides here with the linear space of H -equivariant homomorphisms from $\mathfrak{m} \times \{0\}$ into $\{0\} \times \mathfrak{g}$. As the action of H on $\{0\} \times \mathfrak{g}$ is trivial and $\mathfrak{k} = \text{Lie}(N(H)|H)$ is the subspace of \mathfrak{m} carrying the trivial representation of H , this space of Higgs fields reduces to $\text{Hom}(\mathfrak{k}, \mathfrak{g})$.

Chapitre 3

Appendix to the first part

3.1 Example with $G = \mathrm{SU}(n)$, $H \simeq \mathrm{U}(1)$ and $N(H)|H \simeq \mathbb{T}^{n-2}$

We begin by setting some notations that we are going to use in the following.

For $1 \leq j, k \leq n$, E_{jk} denotes the $n \times n$ matrix whose all entries are zero except the (j, k) -entry which is equal to 1.

$\mathrm{U}(1)$ is embedded in $\mathrm{SU}(n)$ by the following injective homomorphism of Lie groups : $\lambda_{m,l} : \mathrm{U}(1) \longrightarrow \mathrm{SU}(n)$ defined by :

$$\lambda_{m,l}(e^{i\theta}) = \mathrm{diag}(e^{im\theta}, e^{i(m-l)\theta}, e^{i(m-2l)\theta}, \dots, e^{i(m-(n-2)l)\theta}, e^{-i(n-1)[m-(n-2)\frac{l}{2}]\theta})$$

where m and l are integers. We set $H = \lambda_{m,l}(\mathrm{U}(1))$.

We shall focus on the interesting case $l \neq 0$. Here,

$$N(H) = \{\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\dots+\theta_{n-1})}) ; \theta_1, \dots, \theta_{n-1} \in \mathbb{R}\}$$

Therefore, $N(H) \simeq \mathbb{T}^{n-1}$.

First, we study the adjoint action of H on $\mathfrak{g} = \mathfrak{su}(n)$. Let $h = \lambda_{m,l}(e^{i\theta}) \in H$, and $A = [z_{jk}] \in \mathfrak{su}(n)$. We have to compute $hAh^{-1} = [z_{jk}^h]$, which is facilitated by the fact that h is a diagonal matrix :

- For $1 \leq j \leq n-1$, $z_{jj}^h = e^{i(m-(j-1)l)\theta} z_{jj} e^{-i(m-(j-1)l)\theta}$. Therefore $z_{jj}^h = z_{jj}$ (which is true also for $j = n$), so the diagonal elements of A are not changed.
 - For $1 \leq j < k \leq n-1$, $z_{jk}^h = e^{i(m-(j-1)l)\theta} z_{jk} e^{-i(m-(k-1)l)\theta} = e^{i(k-j)l\theta} z_{jk}$
 - For $1 \leq j \leq n-1$ and $k = n$, $z_{jn}^h = e^{i(m-(j-1)l)\theta} z_{jn} e^{-i(n-1)[m-(n-2)\frac{l}{2}]\theta} = e^{i[nm-(n^2-3n+2j)\frac{l}{2}]\theta} z_{jn}$
- We deduce the decomposition of $\mathfrak{g} = \mathfrak{su}(n)$ under the adjoint action of H :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \bigoplus_{r=1}^{n-2} \bigoplus_{j=1}^{n-(r+1)} \mathfrak{l}_{j,j+r} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}$$

where :

$$\begin{aligned}\mathfrak{h} &= \{\text{diag}(im\theta, i(m-l)\theta, i(m-2l)\theta, \dots, i(m-(n-2)l)\theta, -i(n-1)[m-(n-2)\frac{l}{2}]\theta) ; \theta \in \mathbb{R}\} \\ &\simeq \text{Lie}(H) \simeq \mathfrak{u}(1) \simeq i\mathbb{R}\end{aligned}$$

$$\mathfrak{k} = \{\text{diag}(ix_1, \dots, ix_{n-2}, -i(x_1 + \dots + x_{n-2}), 0) ; x_1, \dots, x_{n-2} \in \mathbb{R}\} \simeq \mathbb{R}^{n-2}$$

$$\mathfrak{l}_{jk} = \{zE_{jk} - \bar{z}E_{kj} ; z \in \mathbb{C}\} \simeq \mathbb{C} \quad (j < k)$$

$$(\text{and } \mathfrak{m} = \mathfrak{k} \oplus \bigoplus_{r=1}^{n-2} \bigoplus_{j=1}^{n-(r+1)} \mathfrak{l}_{j,j+r} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn})$$

\mathfrak{h} and \mathfrak{k} carry the trivial representation of H , and :

- for $2 \leq r+1 \leq n-1$ and $1 \leq j \leq n-(r+1)$, $\mathfrak{l}_{j,j+r} \simeq \mathbb{C}_{rl}$
- for $1 \leq s \leq n-1$, $\mathfrak{l}_{sn} \simeq \mathbb{C}_{nm-(n^2-3n+2s)\frac{l}{2}}$

Second, we study the adjoint action of $N(H)$ on $\mathfrak{g} = \mathfrak{su}(n)$. Noticing that $N(H) \simeq \mathbb{T}^{n-1}$ is nothing but the natural maximal torus of $\mathfrak{su}(n)$, we see immediately that the decomposition of \mathfrak{g} under $N(H)$ reads :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \bigoplus_{1 \leq j < k \leq n-1} \mathfrak{l}_{jk} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}$$

where :

- $\mathfrak{l}_{jk} \simeq \mathbb{C}$ carrying the irreducible representation of \mathbb{T}^{n-1} labeled by the element $(0, \dots, 1, \dots, -1, \dots, 0) \in \mathbb{Z}^{n-1}$, 1 being in the j^{th} place, -1 in the k^{th} place, and 0 in the $n-3$ remaining places.
- $\mathfrak{l}_{sn} \simeq \mathbb{C}$ carrying the irreducible representation of \mathbb{T}^{n-1} labeled by the element $(1, \dots, 2, \dots, 1, \dots, 1) \in \mathbb{Z}^{n-1}$, 2 being in the s^{th} place, and 1 in the $n-2$ remaining places.

Let us define the two following subspaces of \mathfrak{g} :

$$\begin{aligned}\mathfrak{m}_1 &= \mathfrak{k} \oplus \bigoplus_{r=1}^{n-3} \bigoplus_{j=1}^{n-(r+2)} \mathfrak{l}_{j,j+r} \\ \mathfrak{m}_2 &= \mathfrak{l}_{1,n-1} \oplus \bigoplus_{r=1}^{n-3} \mathfrak{l}_{n-1-r,n-1} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}\end{aligned}$$

It is clear that \mathfrak{m}_1 and \mathfrak{m}_2 are $\text{Ad}(N(H))$ -invariant complementary subspaces of \mathfrak{m} , and thus we have an $\text{Ad}(N(H))$ -invariant splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We deduce a space of Higgs fields :

$$\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = \bigoplus_{r=1}^{n-3} \mathcal{L}\left(\bigoplus_{j=1}^{n-(r+2)} \mathfrak{l}_{j,j+r}, \mathfrak{l}_{n-1-r,n-1}\right)$$

The dimension of $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ can be easily computed :

$$\dim \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = 2 \sum_{r=1}^{n-3} [n - (r+2)].1 = (n-2)(n-3)$$

A natural basis for the vector space \mathfrak{m} is given by the following matrices :
 $i(E_{jj} - E_{n-1,n-1})$, $1 \leq j \leq n-2$ (basis for \mathfrak{k})

$a_{jk} = E_{jk} - E_{kj}$ and $b_{jk} = i(E_{jk} + E_{kj})$, $1 \leq j < k \leq n$ (basis for \mathfrak{l}_{jk})

This basis is adapted to our decompositions, since the vectors :

$i(E_{jj} - E_{n-1,n-1})$, $1 \leq j \leq n-2$, $a_{j,j+r}$ and $b_{j,j+r}$ for $1 \leq r \leq n-3$ and $1 \leq j \leq n-(r+2)$ form a basis of the subspace \mathfrak{m}_1 , and the vectors :

$a_{1,n-1}$, $b_{1,n-1}$, $a_{n-1-r,n-1}$ and $b_{n-1-r,n-1}$ for $1 \leq r \leq n-3$, a_{sn} and b_{sn} for $1 \leq s \leq n-1$ form a basis of the subspace \mathfrak{m}_2 .

Let $\phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. We have : $\phi(i(E_{jj} - E_{n-1,n-1})) = 0$.

We define the real numbers $\varphi_{j,j+r}$, $\psi_{j,j+r}$ for $1 \leq r \leq n-3$ and $1 \leq j \leq n-(r+2)$ by :

$$\begin{aligned}\phi(a_{j,j+r}) &= \varphi_{j,j+r} a_{n-1-r,n-1} + \psi_{j,j+r} b_{n-1-r,n-1} \\ \phi(b_{j,j+r}) &= -\psi_{j,j+r} a_{n-1-r,n-1} + \varphi_{j,j+r} b_{n-1-r,n-1}\end{aligned}$$

The $\varphi_{j,j+r}$ and $\psi_{j,j+r}$ constitute the $(n-2)(n-3)$ components of the Higgs field ϕ in the bases of \mathfrak{m}_1 and \mathfrak{m}_2 defined above.

Third, we study the representation of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. For this, we first look at the representation of $N(H)$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$:

Given $P = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\dots+\theta_{n-1})}) \in N(H)$, and $A = [z_{jk}] \in \mathfrak{m}_1$, we begin by computing $\text{Ad}_{P^{-1}}(A) = P^{-1}AP = [z_{jk}^{P^{-1}}]$, which again is facilitated because P is diagonal.

- For $1 \leq j \leq n-1$, $z_{jj}^{P^{-1}} = e^{-i\theta_j} z_{jj} e^{i\theta_j}$. Therefore $z_{jj}^{P^{-1}} = z_{jj}$ (which is true also for $j = n$), so the diagonal elements of A are not changed.
- For $1 \leq j < k \leq n-2$, $z_{jk}^{P^{-1}} = e^{-i\theta_j} z_{jk} e^{i\theta_k} = e^{i(-\theta_j+\theta_k)} z_{jk}$

Then, we compute $\text{Ad}_P \circ \phi \circ \text{Ad}_{P^{-1}}(A)$. Writing

$$A = \sum_{j=1}^{n-2} x_j i(E_{jj} - E_{n-1,n-1}) + \sum_{r=1}^{n-3} \sum_{j=1}^{n-(r+2)} (x_{j,j+r} a_{j,j+r} + y_{j,j+r} b_{j,j+r})$$

and noting that $x_{jk} a_{jk} + y_{jk} b_{jk} = z_{jk} E_{jk} - \bar{z}_{jk} E_{kj}$ if $z_{jk} = x_{jk} + i y_{jk}$, we see that :

$$\text{Ad}_{P^{-1}}(A) = \sum_{j=1}^{n-2} x_j i(E_{jj} - E_{n-1,n-1}) + \sum_{r=1}^{n-3} \sum_{j=1}^{n-(r+2)} (e^{i(-\theta_j+\theta_{j+r})} z_{j,j+r} E_{j,j+r} - e^{-i(-\theta_j+\theta_{j+r})} \bar{z}_{j,j+r} E_{j+r,j})$$

and we deduce, letting $\phi_{jk} = \varphi_{jk} + i \psi_{jk}$, that :

$$\begin{aligned}\phi(\text{Ad}_{P^{-1}}(A)) &= \sum_{r=1}^{n-3} \left(\sum_{j=1}^{n-(r+2)} e^{i(-\theta_j+\theta_{j+r})} \phi_{j,j+r} z_{j,j+r} E_{n-1-r,n-1} \right. \\ &\quad \left. - \sum_{j=1}^{n-(r+2)} e^{-i(-\theta_j+\theta_{j+r})} \bar{\phi}_{j,j+r} \bar{z}_{j,j+r} E_{n-1,n-1-r} \right)\end{aligned}$$

Finally,

$$\begin{aligned}\text{Ad}_P(\phi(\text{Ad}_{P^{-1}}(A))) &= \sum_{r=1}^{n-3} \left(\sum_{j=1}^{n-(r+2)} e^{i(\theta_{n-1-r,n-1}-\theta_{n-1}-\theta_j+\theta_{j+r})} \phi_{j,j+r} z_{j,j+r} E_{n-1-r,n-1} \right. \\ &\quad \left. - \sum_{j=1}^{n-(r+2)} e^{-i(\theta_{n-1-r,n-1}-\theta_{n-1}-\theta_j+\theta_{j+r})} \bar{\phi}_{j,j+r} \bar{z}_{j,j+r} E_{n-1,n-1-r} \right)\end{aligned}$$

We define the homomorphism of Lie groups $f : N(H) \longrightarrow G$ by :

$$f(P) = \text{diag}(e^{i[(1-(n-2)\frac{l}{m})\theta_1-\theta_{n-1}]}, \dots, e^{i[(1-\frac{l}{m})\theta_1-\theta_2]}, e^{-i[(n-2)(1-(n-1)\frac{l}{2m})\theta_1-(\theta_2+\dots+\theta_{n-1})]}, 1)$$

for each $P = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\dots+\theta_{n-1})}) \in N(H)$.

We see that $\text{Ker } f = H$, and if we set

$K = \{\text{diag}(e^{ix_1}, \dots, e^{ix_{n-2}}, e^{-i(x_1+\dots+x_{n-2})}, 1) ; x_1, \dots, x_{n-2} \in \mathbb{R}\}$, then $K = \text{Im } f$, and we obtain a surjective homomorphism that we also denote by f . Thus, $f : N(H) \longrightarrow K$ induces an isomorphism $\bar{f} : N(H)|H \longrightarrow K$. It is easy now to explicit the action of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$: if $\rho : N(H) \longrightarrow GL(\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2))$ is the representation previously computed ($\rho(P)(\phi) = \text{Ad}_P \circ \phi \circ \text{Ad}_{P^{-1}}$), we define $\bar{\rho} : K \longrightarrow GL(\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2))$ the following way : for each $Q = \text{diag}(e^{ix_1}, \dots, e^{ix_{n-2}}, e^{-i(x_1+\dots+x_{n-2})}, 1) \in K$, let $\bar{\rho}(Q) = \rho(P)$, where P is any element of $N(H)$ such that $f(P) = Q$. For example, one can take

$$P = \text{diag}(e^{ix_1}, e^{i[(1-\frac{l}{m})x_1-x_{n-2}]}, \dots, e^{i[(1-(n-3)\frac{l}{m})x_1-x_2]}, e^{-i(n-2)\frac{l}{m}x_1}, e^{-i[(n-2)(1-(n-1)\frac{l}{2m})x_1-(x_2+\dots+x_{n-2})]})$$

We have shown that the Higgs field ϕ transform under the action of $P = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\dots+\theta_{n-1})}) \in N(H)$ according to :

$$\phi_{j,j+r} \longrightarrow e^{i(\theta_{n-1-r,n-1}-\theta_{n-1}-\theta_j+\theta_{j+r})} \phi_{j,j+r}$$

Using the relations :

$$\left\{ \begin{array}{lcl} \theta_j & = & (1 - (j-1)\frac{l}{m})x_1 - x_{n-j} \quad (\text{for } j \geq 2) \\ \theta_{j+r} & = & (1 - (j+r-1)\frac{l}{m})x_1 - x_{n-j-r} \\ \dots & & \\ \theta_{n-1-r} & = & (1 - (n-r-2)\frac{l}{m})x_1 - x_{r+1} \\ \theta_{n-1} & = & (1 - (n-2)\frac{l}{m})x_1 - x_1 \quad (= -(n-2)\frac{l}{m}x_1) \end{array} \right.$$

We derive the transformation rules of ϕ under the action of $Q = \text{diag}(e^{ix_1}, \dots, e^{ix_{n-2}}, e^{-i(x_1+\dots+x_{n-2})}, 1)$:

$$\phi_{j,j+r} \longrightarrow e^{i(x_1-x_{r+1}+x_{n-j}-x_{n-j-r})} \phi_{j,j+r} \quad (\text{for } j \geq 2)$$

Our final task is the computation of the Higgs potential of this model. For this, we shall need first some structure constants which are given by the following lemma :

- Lemma 3.1.1**
- 0) $[E_{jk}, E_{j'k'}] = \delta_{kj'}E_{jk'} - \delta_{k'j}E_{j'k}$
 - 1) $[a_{jk}, a_{j'k'}] = \delta_{kj'}a_{jk'} - \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'}$
 - 2) $[a_{jk}, b_{j'k'}] = \delta_{kj'}b_{jk'} - \delta_{k'j}b_{j'k} + \delta_{kk'}b_{jj'} - \delta_{jj'}b_{kk'}$
 - 3) $[b_{jk}, a_{j'k'}] = \delta_{kj'}b_{jk'} - \delta_{k'j}b_{j'k} - \delta_{kk'}b_{jj'} + \delta_{jj'}b_{kk'}$
 - 4) $[b_{jk}, b_{j'k'}] = -\delta_{kj'}a_{jk'} + \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'}$

Proof : 0) follows immediately by expressing the matrix product of E_{jk} and $E_{j'k'}$ in terms of Kronecker symbols. For 1),

$$\begin{aligned} [a_{jk}, a_{j'k'}] &= [E_{jk} - E_{kj}, E_{j'k'} - E_{k'j'}] \\ &= [E_{jk}, E_{j'k'}] - [E_{jk}, E_{k'j'}] - [E_{kj}, E_{j'k'}] + [E_{kj}, E_{k'j'}] \\ &= \delta_{kj'}E_{jk'} - \delta_{k'j}E_{j'k} - \delta_{kk'}E_{jj'} + \delta_{j'j}E_{k'k} - \delta_{jj'}E_{kk'} + \delta_{k'k}E_{j'j} + \delta_{jk'}E_{kj'} - \delta_{j'k}E_{k'j} \\ &= \delta_{kj'}a_{jk'} - \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'} \end{aligned}$$

The computations are similar for 2), 3) and 4). \square

Next, we have to find the components of the tensor $\phi \text{ ad}_{12}(\phi) \phi$, which can be viewed as the bilinear map $B_\phi : \mathfrak{m}_1 \times \mathfrak{m}_1 \longrightarrow \mathfrak{m}_2$ defined by : $B_\phi(X, Y) = \phi(\pi_1[\phi(X), \phi(Y)])$. Thus, we need to express the components of $B_\phi(a_{j,j+r}, a_{j',j'+r'})$, $B_\phi(a_{j,j+r}, b_{j',j'+r'})$, etc... in the corresponding basis of \mathfrak{m}_2 .

$$\begin{aligned}
& [\phi(a_{j,j+r}), \phi(a_{j',j'+r'})] = \\
& [\varphi_{j,j+r} a_{n-1-r,n-1} + \psi_{j,j+r} b_{n-1-r,n-1}, \varphi_{j',j'+r'} a_{n-1-r',n-1} + \psi_{j',j'+r'} b_{n-1-r',n-1}] \\
= & \varphi_{j,j+r} \varphi_{j',j'+r'} (\delta_{r'0} a_{n-1-r,n-1} - \delta_{r0} a_{n-1-r',n-1} - a_{n-1-r,n-1-r'} - \delta_{rr'} a_{n-1,n-1}) \\
& + \varphi_{j,j+r} \psi_{j',j'+r'} (\delta_{r'0} b_{n-1-r,n-1} - \delta_{r0} b_{n-1-r',n-1} + b_{n-1-r,n-1-r'} - \delta_{rr'} b_{n-1,n-1}) \\
& + \psi_{j,j+r} \varphi_{j',j'+r'} (\delta_{r'0} b_{n-1-r,n-1} - \delta_{r0} b_{n-1-r',n-1} - b_{n-1-r,n-1-r'} + \delta_{rr'} b_{n-1,n-1}) \\
& + \psi_{j,j+r} \psi_{j',j'+r'} (-\delta_{r'0} a_{n-1-r,n-1} + \delta_{r0} a_{n-1-r',n-1} - a_{n-1-r,n-1-r'} - \delta_{rr'} a_{n-1,n-1})
\end{aligned}$$

In particular, noticing that $a_{jk} = -a_{kj}$ (which implies $a_{jj} = 0$) and that $b_{jk} = b_{kj}$, we have :

$$\begin{aligned}
& [\phi(a_{j,j+r}), \phi(a_{j',j'})] = 2 \varphi_{j,j+r} \psi_{j',j'} b_{n-1-r,n-1} - 2 \psi_{j,j+r} \psi_{j',j'} a_{n-1-r,n-1} \text{ if } r \neq 0 \\
& [\phi(a_{j,j}), \phi(a_{j',j'+r'})] = -2 \psi_{j,j} \varphi_{j',j'+r'} b_{n-1-r',n-1} + 2 \psi_{j,j} \psi_{j',j'+r'} a_{n-1-r',n-1} \text{ if } r' \neq 0 \\
& [\phi(a_{j,j}), \phi(a_{j',j'})] = 0 \\
& [\phi(a_{j,j+r}), \phi(a_{j',j'+r'})] = (\varphi_{j,j+r} \psi_{j',j'+r'} - \psi_{j,j+r} \varphi_{j',j'+r'}) (b_{n-1-r,n-1-r} - b_{n-1,n-1}) \text{ if } r \neq 0 \\
& [\phi(a_{j,j+r}), \phi(a_{j',j'+r'})] = -\varphi_{j,j+r} \varphi_{j',j'+r'} a_{n-1-r,n-1-r'} + \varphi_{j,j+r} \psi_{j',j'+r'} b_{n-1-r,n-1-r'} - \\
& \psi_{j,j+r} \varphi_{j',j'+r'} b_{n-1-r,n-1-r'} - \psi_{j,j+r} \psi_{j',j'+r'} a_{n-1-r,n-1-r'} \text{ if } r \neq 0, r' \neq 0 \text{ and } r \neq r'
\end{aligned}$$

If $1 \leq r - r' \leq n - 3$ and $1 \leq n - 1 - r \leq n - (r - r' + 2)$ then $a_{n-1-r,n-1-r'} \in \mathfrak{m}_1$ and $b_{n-1-r,n-1-r'} \in \mathfrak{m}_1$. Therefore,

$$\pi_1([\phi(a_{j,j+r}), \phi(a_{j',j'+r'})]) = (-\varphi_{j,j+r} \varphi_{j',j'+r'} - \psi_{j,j+r} \psi_{j',j'+r'}) a_{n-1-r,n-1-r'} + (\varphi_{j,j+r} \psi_{j',j'+r'} - \psi_{j,j+r} \varphi_{j',j'+r'}) b_{n-1-r,n-1-r'}$$

Using the fact that :

$$\begin{aligned}
\phi(a_{n-1-r,n-1-r'}) &= \varphi_{n-1-r,n-1-r'} a_{n-1-(r-r'),n-1} + \psi_{n-1-r,n-1-r'} b_{n-1-(r-r'),n-1} \\
\phi(b_{n-1-r,n-1-r'}) &= -\psi_{n-1-r,n-1-r'} a_{n-1-(r-r'),n-1} + \varphi_{n-1-r,n-1-r'} b_{n-1-(r-r'),n-1}
\end{aligned}$$

we deduce :

$$\begin{aligned}
& B_\phi(a_{j,j+r}, a_{j',j'+r'}) \\
= & (-\varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r} - \psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} \\
& - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} + \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) a_{n-1-(r-r'),n-1} \\
& + (-\varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) \\
& + \varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) b_{n-1-(r-r'),n-1}
\end{aligned}$$

Completely analogous calculations lead to :

$$\begin{aligned}
& B_\phi(a_{j,j+r}, b_{j',j'+r'}) \\
&= (\varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} \\
&\quad - \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) a_{n-1-(r-r'),n-1} \\
&\quad + (\varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} \\
&\quad + \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} + \psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) b_{n-1-(r-r'),n-1} \\
& B_\phi(b_{j,j+r}, a_{j',j'+r'}) \\
&= (\psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} \\
&\quad + \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} + \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) a_{n-1-(r-r'),n-1} \\
&\quad + (\psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} \\
&\quad - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) b_{n-1-(r-r'),n-1} \\
& B_\phi(b_{j,j+r}, b_{j',j'+r'}) \\
&= (\psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} \\
&\quad + \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) a_{n-1-(r-r'),n-1} \\
&\quad + (-\psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} \\
&\quad - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} + \varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) b_{n-1-(r-r'),n-1}
\end{aligned}$$

Finally, we need to define the $\text{Ad}(N(H))$ -invariant scalar products h and k on \mathfrak{m}_1 and \mathfrak{m}_2 respectively, and use them to compute the squared norm of the tensor $\phi \text{ ad}_{12}(\phi)$. We consider the natural $\text{Ad}(\text{SU}(n))$ -invariant scalar product on $\mathfrak{su}(n)$ defined by : $\langle A, B \rangle = -\frac{1}{2}\text{tr}(AB)$ for every $A, B \in \mathfrak{su}(n)$. Then, h and k are defined to be the restrictions of \langle , \rangle to $\mathfrak{m}_1 \times \mathfrak{m}_1$ and $\mathfrak{m}_2 \times \mathfrak{m}_2$ respectively.

Lemma 3.1.2 *The a_{jk} and b_{jk} constitute an orthonormal family in $\mathfrak{su}(n)$, i.e.*

- 1) $\langle a_{jk}, a_{j'k'} \rangle = \delta_{jj'} \delta_{kk'}$
- 2) $\langle a_{jk}, b_{j'k'} \rangle = 0$
- 3) $\langle b_{jk}, a_{j'k'} \rangle = 0$
- 4) $\langle b_{jk}, b_{j'k'} \rangle = \delta_{jj'} \delta_{kk'}$

Proof : First, notice that $\text{tr}(E_{jk}) = \delta_{jk}$ and that the a_{jk}, b_{jk} are only defined for $j < k$. We prove only 1), the calculations being similar for 2), 3) and 4).

$$\begin{aligned}
a_{jk} a_{j'k'} &= (E_{jk} - E_{kj})(E_{j'k'} - E_{k'j'}) \\
&= E_{jk}E_{j'k'} - E_{jk}E_{k'j'} - E_{kj}E_{j'k'} + E_{kj}E_{k'j'} \\
&= \delta_{kj'} E_{jk} - \delta_{kk'} E_{jj'} - \delta_{jj'} E_{kk'} + \delta_{jk} E_{kj'} \\
\langle a_{jk}, a_{j'k'} \rangle &= -\frac{1}{2}\text{tr}(a_{jk} a_{j'k'}) \\
&= -\frac{1}{2}(\delta_{kj'} \delta_{jk} - \delta_{kk'} \delta_{jj'} - \delta_{jj'} \delta_{kk'} + \delta_{jk} \delta_{kj'}) \\
&= \delta_{jj'} \delta_{kk'}
\end{aligned}$$

□

h and k are thus represented by the identity matrices in the corresponding bases of \mathfrak{m}_1 and \mathfrak{m}_2 , and the sixth-degree term of the Higgs potential $V^{(6)}(\phi) = \frac{1}{4} \|\phi \text{ ad}_{12}(\phi)\|_{hhk}^2$ reads in this model :

$$\begin{aligned}
V^{(6)}(\phi) = & \frac{1}{4} \sum \{ \quad (-\varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r} - \psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) \\
& - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} + \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& + (-\varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& + \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& (\varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) \\
& - \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& + (\varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) \\
& + \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} + \psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'})^2 \\
& (\psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) \\
& + \psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} + \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& + (\psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) \\
& - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'})^2 \\
& (\psi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'}) \\
& + \psi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'})^2 \\
& + (-\psi_{j,j+r} \psi_{j',j'+r'} \psi_{n-1-r,n-1-r'} - \varphi_{j,j+r} \varphi_{j',j'+r'} \psi_{n-1-r,n-1-r'}) \\
& - \psi_{j,j+r} \varphi_{j',j'+r'} \varphi_{n-1-r,n-1-r'} + \varphi_{j,j+r} \psi_{j',j'+r'} \varphi_{n-1-r,n-1-r'})^2 \quad \}
\end{aligned}$$

the sum being made on the positive integers j, r, j', r' satisfying the following constraints :

$$\left\{ \begin{array}{l} 1 \leq r \leq n-3 \\ 1 \leq j \leq n-(r+2) \\ 1 \leq r' \leq n-3 \\ 1 \leq j' \leq n-(r'+2) \\ 1 \leq r-r' \leq n-3 \\ 1 \leq n-1-r \leq n-(r-r'+2) \end{array} \right\}$$

3.2 Réduction dimensionnelle sur un espace à fibres homogènes

L'idée selon laquelle notre univers aurait des dimensions supplémentaires qui nous sont cachées remonterait à Platon. Plus récemment, dans les années 1921 à 1926, Kaluza et Klein ont suggéré qu'un pas était possible dans les tentatives d'unification des interactions fondamentales (en particulier la gravitation avec l'électromagnétisme), si l'on supposait que l'univers possédait une dimension de plus par rapport aux quatre dimensions d'espace-temps de la relativité générale. Plus précisément, il existerait un petit cercle (espace interne) au-dessus de chaque point de l'espace-temps, et l'univers (espace-temps étendu) aurait la topologie de $\mathbb{R}^4 \times S^1$.

Cette notion d'univers multidimensionnel s'est généralisée et on s'est rendu compte que les objets mathématiques les plus naturels qui permettent de décrire ces espaces-

temps étendus étaient les espaces fibrés.

Au départ, l'intérêt fut porté presque exclusivement au cas où l'espace interne au-dessus de chaque point de l'espace-temps est une copie d'un groupe de Lie (ce qui revient à considérer les fibrés principaux). Par la suite, il s'est avéré qu'on avait besoin d'une classe plus générale de fibrés : ceux dont les espaces internes sont des copies d'un espace homogène G/H . Ces fibrés à fibres homogènes apparaissent en fait dans un cadre assez général : à chaque fois qu'un groupe de Lie compact G opère sur une variété E de telle manière que les orbites soient "de même type", E devient un espace fibré sur l'espace des orbites M , dont les fibres sont des copies de G/H , où H est le stabilisateur d'un point de E . En nous basant sur les travaux de Coquereaux et Jadczyk [CJ1], nous précisons cette structure géométrique dans le paragraphe 3.2.1.

On s'intéresse ensuite à la réduction dimensionnelle d'une théorie des champs donnée sur E . Dans le cas d'un champ de gravitation sur E invariant par G , on a alors le résultat essentiel [CJ1] : "Il y a équivalence entre : la donnée d'une métrique G -invariante sur E , et la donnée d'un triplet (γ, A, h) de champs définis sur M , où : γ est une métrique sur M , A est un champ de jauge sur M à valeurs dans l'algèbre de Lie de $N|H$ (N étant le normalisateur de H dans G), et h est une famille de champs scalaires (pour chaque $x \in M$, h_x est une métrique G -invariante sur la fibre au-dessus de x)".

Nous nous intéresserons particulièrement par la suite aux champs scalaires qui résultent de cette réduction dimensionnelle. Puisque pour chaque $x \in M$, h_x est une métrique G -invariante sur la fibre au-dessus de x , et qu'une telle fibre est une copie de G/H , on est amené à regarder les métriques G -invariantes sur G/H , autrement dit les produits scalaires $Ad(H)$ -invariants sur l'espace tangent à G/H en l'origine. Pour cela, nous introduirons au paragraphe 3.2.2 des décompositions convenables de l'algèbre de Lie de G , qui nous seront utiles dans toute la suite.

Enfin, on démontre au paragraphe 3.2.3 le théorème de réduction énoncé ci-haut.

3.2.1 Géométrie des espaces-temps étendus

Theorem 3.2.1 *Soit E une variété sur laquelle opère à droite un groupe de Lie compact G . On suppose qu'il existe $y_0 \in E$ tel que tous les stabilisateurs soient conjugués au stabilisateur H de y_0 . Notons M l'espace des orbites, N le normalisateur de H dans G , et $G/H = \{Ha ; a \in G\}$. Alors M est une variété, et $(E, M, G/H)$ est un espace fibré de groupe structural $N|H$.*

Nous donnerons uniquement les grandes lignes de la démonstrations. Pour plus de détails, le lecteur est invité à regarder [CJ1]. On commence par établir le lemme suivant :

Lemma 3.2.2 *Le groupe $N|H$ opère à gauche naturellement sur G/H , par permutations G -équivariantes. Mieux : les groupes $N|H$ et $Aut_G(G/H)$ sont isomorphes, $Aut_G(G/H)$ étant le groupe des permutations G -équivariantes de G/H .*

Démonstration du lemme : On commence par définir une action de N sur G/H : pour tout $n \in N$, soit $\sigma_n : G/H \longrightarrow G/H$ l'application définie par : $\sigma_n(Ha) = Hna$. On vérifie que σ_n est bien définie, bijective, et qu'elle respecte l'action à droite de G sur G/H . Par suite, $\sigma_n \in Aut_G(G/H)$. Soit alors $\varphi : N \longrightarrow Aut_G(G/H)$ l'application définie

par : $\varphi(n) = \sigma_n$. On vérifie que φ est un morphisme surjectif de groupes, de noyau H . En passant au quotient, on obtient une action de $N|H$ sur G/H , et un isomorphisme entre les groupes $N|H$ et $Aut_G(G/H)$. \square

Démonstration du théorème : Remarquons d'abord le fait suivant : on peut considérer au départ une variété M , un groupe de Lie compact G , un sous-groupe fermé H de G , et noter N le normalisateur de H dans G . Dans ces conditions, à chaque fibré principal P sur M de groupe $N|H$, on peut associer un fibré $(E, M, G/H)$ par l'action à gauche de $N|H$ sur G/H . Il n'est pas difficile de montrer que G opère alors sur E par une action telle que tous les stabilisateurs soient conjugués à H . De plus, P s'identifie à la sous-variété de E formée des éléments dont le stabilisateur est précisément H . Pour démontrer le théorème, on raisonne en sens inverse : partant des hypothèses de l'énoncé, on note $P = \{y \in E / H_y = H\}$ et on vérifie que $(P, M, N|H)$ est un fibré principal, et que E s'identifie au fibré associé à P par l'action naturelle de $N|H$ sur G/H . \square

3.2.2 Décomposition des algèbres de Lie

Dans toute la suite, G désigne un groupe de Lie, et H un sous-groupe fermé de G . On pose : $\mathfrak{g} = Lie(G)$, $\mathfrak{h} = Lie(H)$ et $\mathfrak{n} = Lie(N)$.

Pour simplifier, nous supposerons que G et H sont compacts et connexes. La compacité de G assure l'existence d'une métrique riemannienne bi-invariante sur G , i.e. d'un produit scalaire $Ad(G)$ -invariant sur \mathfrak{g} . Rappelons au passage que si G est simple (ce qui revient à dire que la représentation adjointe de G est irréductible), il n'existe, à un facteur multiplicatif près, qu'un seul produit scalaire $Ad(G)$ -invariant sur \mathfrak{g} . Sans faire l'hypothèse de simplicité pour G , nous fixons une fois pour toutes un tel produit scalaire \langle , \rangle sur \mathfrak{g} .

Soit maintenant \mathfrak{m} le supplémentaire orthogonal de \mathfrak{h} dans \mathfrak{g} (pour \langle , \rangle). Il est facile de voir que \mathfrak{h} est un sous-espace $Ad(H)$ -invariant de \mathfrak{g} . Par suite, et grâce à l'orthogonalité pour \langle , \rangle de la représentation adjointe, \mathfrak{m} est aussi un sous-espace $Ad(H)$ -invariant.

\mathfrak{g} admet donc une décomposition réductive $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, c'est-à-dire qu'il existe un supplémentaire \mathfrak{m} de \mathfrak{h} dans \mathfrak{g} tel que : $Ad(H)\mathfrak{m} \subset \mathfrak{m}$ (où encore $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, par connexité de H).

Soit \mathfrak{k} le supplémentaire orthogonal de \mathfrak{h} dans \mathfrak{n} . Alors de même, $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{k}$ est une décomposition réductive. De plus, on voit tout de suite que $\mathfrak{k} = \mathfrak{n} \cap \mathfrak{m}$.

Proposition 3.2.3 $[\mathfrak{h}, \mathfrak{k}] = \{0\}$

Démonstration : Comme H est distingué dans N , on a $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{h}$, et donc $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h}$. Par ailleurs, on a $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$, par réductivité de $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{k}$. Il en résulte que $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h} \cap \mathfrak{k}$, d'où : $[\mathfrak{h}, \mathfrak{k}] = \{0\}$. \square

Soit \mathfrak{z} le centralisateur de \mathfrak{h} dans \mathfrak{g} , i.e.

$$\mathfrak{z} = \{X \in \mathfrak{g} / [X, \mathfrak{h}] = \{0\}\}$$

Alors d'une part, on a $\mathfrak{c} \subset \mathfrak{z}$, où $\mathfrak{c} = \mathfrak{z} \cap \mathfrak{h}$ est le centre de \mathfrak{h} . D'autre part, on a $\mathfrak{k} \subset \mathfrak{z} \cap \mathfrak{m}$, puisque $\mathfrak{k} \subset \mathfrak{m}$ et $[\mathfrak{h}, \mathfrak{k}] = \{0\}$. En fait, on peut montrer que $\mathfrak{k} = \mathfrak{z} \cap \mathfrak{m}$. Ainsi,

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{m} / [X, \mathfrak{h}] = \{0\}\} \\ &= \{X \in \mathfrak{m} / Ad_h(X) = X, \forall h \in H\}\end{aligned}$$

Proposition 3.2.4 \mathfrak{k} est une sous-algèbre de Lie de \mathfrak{n} .

Démonstration : Puisque \mathfrak{h} est un idéal de \mathfrak{n} , on a $[\mathfrak{n}, \mathfrak{h}] \subset \mathfrak{h}$, d'où $Ad(N)\mathfrak{h} \subset \mathfrak{h}$. Donc \mathfrak{h} est un sous-espace $Ad(N)$ -invariant de \mathfrak{n} . Par suite, et grâce à l'orthogonalité pour $<, >$ de la représentation adjointe, \mathfrak{k} est aussi un sous-espace $Ad(N)$ -invariant, c'est-à-dire $Ad(N)\mathfrak{k} \subset \mathfrak{k}$, ou encore $[\mathfrak{n}, \mathfrak{k}] \subset \mathfrak{k}$. Ainsi, \mathfrak{k} est aussi un idéal de \mathfrak{n} , et à fortiori une sous-algèbre de Lie de \mathfrak{n} ($[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$). \square

\mathfrak{k} s'identifie donc à l'algèbre de Lie de $N|H$.

Proposition 3.2.5

$$\mathfrak{z} = \mathfrak{c} \oplus \mathfrak{k}$$

Démonstration : Il est clair que $\mathfrak{c} \cap \mathfrak{k} = \{0\}$ et que $\mathfrak{c} \oplus \mathfrak{k} \subset \mathfrak{z}$. Soit $X \in \mathfrak{z}$. D'après $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, on peut écrire : $X = X_{\mathfrak{h}} + X_{\mathfrak{m}}$ avec $X_{\mathfrak{h}} \in \mathfrak{h}$ et $X_{\mathfrak{m}} \in \mathfrak{m}$. Dans ces conditions, pour tout $Y \in \mathfrak{h}$, $0 = [X, Y] = [X_{\mathfrak{h}}, Y] + [X_{\mathfrak{m}}, Y]$, d'où $[X_{\mathfrak{h}}, Y] = -[X_{\mathfrak{m}}, Y]$. Or $[X_{\mathfrak{h}}, Y] \in \mathfrak{h}$ et $[X_{\mathfrak{m}}, Y] \in \mathfrak{m}$ par réductivité de $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Par suite, $[X_{\mathfrak{h}}, Y] \in \mathfrak{h} \cap \mathfrak{m}$ et donc $[X_{\mathfrak{h}}, Y] = 0$, ce qui entraîne : $X_{\mathfrak{h}} \in \mathfrak{c}$. Par ailleurs, $X_{\mathfrak{m}} = X - X_{\mathfrak{h}}$ avec $X_{\mathfrak{m}} \in \mathfrak{m}$ et $X - X_{\mathfrak{h}} \in \mathfrak{z}$. Par suite, $X_{\mathfrak{m}} \in \mathfrak{z} \cap \mathfrak{m}$, c'est-à-dire $X_{\mathfrak{m}} \in \mathfrak{k}$. \square

Enfin, on note \mathfrak{l} le supplémentaire orthogonal de \mathfrak{n} dans \mathfrak{g} . Alors, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{l}$ est une décomposition réductive. De plus, on a la décomposition orthogonale $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{l}$, et $Ad(H)\mathfrak{l} \subset \mathfrak{l}$. La décomposition $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{l}$ est orthogonale pour $<, >$, mais en fait, comme \mathfrak{k} et \mathfrak{l} portent des représentations disjointes de H , ils sont orthogonaux pour n'importe quel produit scalaire $Ad(H)$ -invariant sur \mathfrak{m} .

3.2.3 Réduction dimensionnelle d'un champ de gravitation

Enonçons d'abord, sans démonstration, le théorème de réduction dans le cas particulier des fibrés principaux. Nous aurons l'occasion de l'appliquer dans les paragraphes qui suivent.

Theorem 3.2.6 *On considère un fibré principal (P, M, G) . Il y a équivalence entre : la donnée d'une métrique G -invariante sur P , et la donnée d'un triplet (γ, β, h) de champs, où : γ est une métrique sur M , β est une connexion (P, M, G) , et h est une famille de champs scalaires (pour chaque $x \in M$, h_x est une métrique G -invariante sur la fibre au-dessus de x).*

La différence essentielle avec la version généralisée aux espaces à fibres homogènes (que nous démontrons ci-dessous) réside dans le fait que la connexion β obtenue va être à valeurs dans l'algèbre de Lie de $N|H$ au lieu de G .

Theorem 3.2.7 *On considère la structure géométrique définie dans le paragraphe précédent. Il y a équivalence entre : la donnée d'une métrique G -invariante sur E , et la donnée d'un triplet (γ, β, h) de champs, où : γ est une métrique sur M , β est une connexion sur le fibré principal $(P, M, N|H)$, et h est une famille de champs scalaires (pour chaque $x \in M$, h_x est une métrique G -invariante sur la fibre au-dessus de x).*

Démonstration :

1. Soit g une métrique G -invariante sur E .
 - Soit $x \in M$, et notons E_x la fibre au-dessus de x . Pour tout $y \in E_x$, posons $h_x(y) = g(y)|_{T_y E_x \times T_y E_x}$. Alors h_x est une métrique G -invariante sur E_x .
 - Pour tout $y \in E$, on note \mathcal{H}_y le supplémentaire orthogonal du sous-espace vertical $\mathcal{V}_y(E)$, pour le produit scalaire $g(y)$ dans $T_y E$. On a alors $\mathcal{H}_{ya} = \mathcal{H}_y a$ pour tout $a \in G$. Il reste à montrer que pour tout $y \in P$, $\mathcal{H}_y \subset T_y P$. Pour cela, nous allons utiliser la décomposition orthogonale $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{l}$ vue au paragraphe précédent. Soit $y \in P$, et posons $\mathfrak{l}_y = T_e \sigma_y(\mathfrak{l})$, où $\sigma_y : G \longrightarrow E$ est l'application définie par $\sigma_y(g) = yg$. Dans ces conditions, pour tous $a \in H$ et $\xi \in T_y P$, on a $\xi a = \xi$. Autrement dit, l'action de H sur $T_y P$ est triviale, ce qui est naturel puisque P est l'ensemble des points qui ont H pour stabilisateur. Ainsi, pour $y \in P$, $T_y P$ et \mathfrak{k} se correspondent par l'application injective $(T_e \sigma_y)|_{\mathfrak{m}} : \mathfrak{m} \longrightarrow T_y E$. Par ailleurs, le produit scalaire $g(y)$ sur $T_y E$ est tiré en arrière par l'injection précédente pour donner un produit scalaire f $Ad(H)$ -invariant sur \mathfrak{m} . De la décomposition orthogonale (pour f) $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{l}$, on déduit la décomposition orthogonale (pour $g(y)$) suivante : $T_y E = T_y P \oplus \mathfrak{l}_y$. Dans ces conditions, puisque $\mathfrak{l}_y \subset \mathcal{V}_y(E)$, on déduit que $\mathcal{H}_y \subset T_y P$. Finalement, $(\mathcal{H}_y)_{y \in P}$ détermine une connexion principale sur $(P, M, N|H)$.
 - Soient $x \in M$, et $\xi, \eta \in T_x M$. On pose $\gamma(x)(\xi, \eta) = g(y)(\tilde{\xi}, \tilde{\eta})$, où y est un point quelconque de E_x , et $\tilde{\xi}$ et $\tilde{\eta}$ sont les relèvements horizontaux respectifs de ξ et η . Alors γ est une métrique sur M .
2. Inversement, étant donné un triplet (γ, β, h) , notons π la projection de E sur M , et soit $y \in E$, et $u, v \in T_y E$. Posons :

$$g(y)(u, v) = \gamma(\pi(y))(T_y \pi(u), T_y \pi(v)) + h_{\pi(y)}(ver_y(u), ver_y(v))$$

Alors g est une métrique G -invariante sur E . □

3.3 Réduction des champs d'Einstein-Yang-Mills symétriques

Nous avons étudié dans le paragraphe précédent la réduction dimensionnelle d'un champ de gravitation g défini sur un espace à fibres homogènes $(E, M, G/H)$, avec la condition que la métrique g soit invariante par le groupe G . On change maintenant de notations en se donnant un espace à fibres homogènes $(E, M, S/I)$, où S est encore un groupe de Lie compact, et I un sous-groupe fermé de S . Et on s'intéresse à la réduction dimensionnelle d'un champ de Yang-Mills défini sur E , invariant par S . Puisqu'un champ de Yang-Mills sur E correspond à une connexion ω sur un fibré principal (U, E, R) , le problème se ramène à étudier les connexions sur un fibré principal (U, E, R) qui sont invariantes sous une action de S sur U qui relève l'action de S sur E . Ce problème a été largement étudié, et il fut abordé dans différents cas de figures.

Tout d'abord, on a considéré le cas particulier où le fibré $(E, M, S/I)$ est trivial (cf. [Man], [CM] et [CS], puis [HSV] et [HST]). Il a été démontré que dans ce cas, cette réduction dimensionnelle aboutit à :

- (i) un champ de Yang-Mills sur M à valeurs dans l'algèbre de Lie du centralisateur Z d'un sous-groupe de R isomorphe à I ,

(ii) un champ scalaire ϕ sur M qui peut être interprété comme un champ de Higgs.

Les choses se compliquent lorsque la trivialité du fibré $(E, M, S/I)$ est seulement supposée locale. Dans ce cas, les études de Jadzcyk, Coquereaux et Pilch (cf. [JP], [Jad] et [CJ2]) montrent qu'il est nécessaire de fixer au préalable une connexion β sur le fibré principal $(P, M, N(I)|I)$, où $P = \{y \in E / I_y = I\}$ (cf. chap 1).

D'une manière plus précise, on commence par associer à l'action de S un morphisme $\lambda : I \longrightarrow R$ et on considère le sous-groupe H de $G = S \times R$ défini comme étant le graphe de λ . En faisant agir G sur U , on constate que H n'est autre que le stabilisateur d'un point de U . L'action de G sur U étant supposée simple, cela détermine alors sur U une structure d'espace à fibres homogènes $(U, M, G/H)$. En posant $Q = \{u \in U / H_u = H\}$, on obtient un fibré principal $(Q, M, N(H)|H)$.

Dans ces conditions, si l'on fixe une connexion β sur P , la réduction dimensionnelle de la connexion S -invariante ω sur U conduit à :

- (i) une connexion α sur Q à valeurs dans l'algèbre de Lie de $N(H)|H$,
- (ii) un champ scalaire ϕ , section d'un fibré vectoriel associé à Q .

En fait, on verra que $\mathfrak{n}(\mathfrak{h})|\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{z}$, où \mathfrak{z} est l'algèbre de Lie du centralisateur de $\lambda(I)$ dans R , et $\mathfrak{k} = \mathfrak{n}(\mathfrak{i})|\mathfrak{i}$. Par suite, on a $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{z}}$, les formes $\alpha_{\mathfrak{k}}$ et $\alpha_{\mathfrak{z}}$ étant à valeurs dans \mathfrak{k} et \mathfrak{z} respectivement. La forme $\alpha_{\mathfrak{k}}$ provient de la connexion fixée β , tandis que $\alpha_{\mathfrak{z}}$ est fournie par la réduction dimensionnelle de ω .

Si l'on veut avoir un "bon" Lagrangien pour α , ainsi que des interactions scalaires, il faut ajouter un terme cinétique à β . Or ceci est automatiquement réalisé lorsque la connexion β provient de la réduction dimensionnelle d'une métrique S -invariante sur E . Rappelons que la réduction d'une telle métrique donne une métrique γ sur M , "une" métrique h sur G/H , et, lorsque E est non trivial, une connexion β pour "coller" γ et h .

Ceci dit, si E était trivial, β n'aurait pas apparu, et α serait réduite à sa composante $\alpha_{\mathfrak{z}}$. On aurait abouti à une connexion sur le fibré principal (Q, M, Z) , obtenu par réduction de $(Q, M, N(H)|H)$, une telle réduction ne pouvant d'ailleurs exister que si P est trivial. On retrouve ainsi, dans le cas où E est trivial, les résultats de Manton.

Revenons au cas général. Ce qui précède montre que l'objet naturel à réduire est en fait un système de deux champs sur E : un champ de gravitation, et un champ de Yang-Mills, tous deux S -invariants. La réduction d'un tel système (g, ω) donne alors :

- (i) une métrique γ sur M (provenant de g),
- (ii) une métrique S -invariante h_x sur chaque copie de S/I (provenant de g),
- (iii) une connexion α sur Q (provenant de g et de ω),
- (iv) un champ scalaire ϕ (provenant de ω), section d'un fibré vectoriel associé à Q , et interprété comme champ de Higgs.

Pour parvenir à ce dernier résultat, nous allons ramener le problème de la réduction du couple (g, ω) défini sur E , à celui de la réduction d'une métrique \tilde{g} définie sur U . En effet, fixons une métrique riemannienne bi-invariante k sur R . k détermine alors une métrique R -invariante k_y sur chaque fibre U_y du fibré principal (U, E, R) . Le triplet (g, ω, k) apparaît alors naturellement comme le résultat de la réduction dimensionnelle à E d'une

métrique \tilde{g} , R -invariante sur U . Comme g et ω étaient au départ S -invariantes, \tilde{g} sera G -invariante.

Dans ces conditions, on est ramené à réduire la métrique G -invariante \tilde{g} définie sur le fibré $(U, M, G/H)$. Les résultats du chapitre 1 conduisent alors à :

- (i) une métrique γ sur M ,
- (ii) une métrique G -invariante \tilde{h}_x sur chaque copie de G/H ,
- (iii) une connexion α sur Q .

Il ne reste plus qu'à faire le lien entre \tilde{h}_x d'une part, et (h_x, ϕ) d'autre part. Là encore, le théorème de réduction entre en jeu. \tilde{h}_x peut être vue comme une métrique R -invariante sur le fibré principal (U_x, E_x, R) . Sa réduction fournit :

- (i) une métrique h_x sur E_x ,
- (ii) "une" métrique R -invariante k sur chaque copie de R ,
- (iii) un champ de Yang-Mills $\phi(x)$ sur E_x , à valeurs dans l'algèbre de Lie de R .

Pour tout $y \in E_x$, on a donc une application linéaire $\phi(x)(y) : T_y E_x \longrightarrow \mathfrak{r}$. Si \mathfrak{s} admet une décomposition réductive $\mathfrak{s} = \mathfrak{i} \oplus \mathfrak{p}$, alors $T_y E_x$ s'identifie à \mathfrak{p} , si bien que le champ $\phi(x)$ est représenté algébriquement par des applications linéaires $\phi : \mathfrak{p} \longrightarrow \mathfrak{r}$. Comme \tilde{h}_x était au départ G -invariante, la métrique réduite h_x va être S -invariante, et le champ ϕ va porter une équivariance : il va entrelacer les représentations de I sur \mathfrak{p} et sur \mathfrak{r} (via λ).

Dans ce qui suit, nous allons essentiellement préciser les notions que nous venons d'évoquer, et démontrer une partie des résultats mentionnés. Dans le paragraphe 3.3.1, on se base principalement sur [KN1] pour rappeler de manière précise les notions d'invariance par un groupe de symétries pour les fibrés et les connexions. Notons que le théorème de Wang, les connexions invariantes sont justement caractérisées par des applications linéaires $\phi : \mathfrak{p} \longrightarrow \mathfrak{r}$, sauf que ce résultat est démontré dans le cas particulier où S opère transitivement sur E , de sorte que l'espace-temps M est réduit à un point ! Il existe toutefois des généralisations du théorème de Wang (cf. [HSV]).

Dans le paragraphe 3.3.2, nous précisons toute la structure géométrique déjà évoquée pour la réduction des champs d'Einstein-Yang-Mills.

3.3.1 Symétries des fibrés et des connexions

Soit (U, E, R) un fibré principal, le groupe de Lie R étant compact. On notera $Aut(U)$ le groupe des **automorphismes** de U , c'est-à-dire le groupe des difféomorphismes $f : U \longrightarrow U$ qui vérifient : $f(ur) = f(u)r$ pour tous $u \in U$ et $r \in R$. On dira qu'une connexion ω sur U est **invariante** par l'automorphisme f de U si $f^*\omega = \omega$, autrement dit si pour tous $u \in U$ et $\xi \in T_u U$, $\omega_{f(u)}(T_{uf}f(\xi)) = \omega_u(\xi)$.

Proposition 3.3.1 Soit (f_t) un groupe à un paramètre d'automorphismes du fibré principal U , et X le générateur infinitésimal de (f_t) . Soit u_0 un point fixé de U , $t \longrightarrow u(t)$ la courbe de U définie par : $u(t) = f_t(u_0)$, et $t \longrightarrow x(t)$ la projection de cette courbe sur M . Enfin, soit ω une connexion invariante par (f_t) , et $t \longrightarrow v(t)$ le relèvement horizontal de $t \longrightarrow x(t)$ à U (par rapport à ω). Dans ces conditions, l'élément $A = \omega_{u_0}(X_{u_0})$ de

$\mathfrak{r} = \text{Lie}(R)$ est le générateur infinitésimal du sous-groupe à un paramètre $t \rightarrow r(t)$ de R défini par : $u(t) = v(t)r(t)$.

Démonstration : En différentiant la relation $u(t) = v(t)r(t)$, on peut écrire : $u'(t) = v'(t)r(t) + v(t)r'(t)$. Par suite,

$$\begin{aligned}\omega_{u(t)}(u'(t)) &= \omega_{u(t)}(v'(t)r(t)) + \omega_{u(t)}(v(t)r'(t)) \\ &= \text{Ad}_{r(t)^{-1}}(\omega_{v(t)}(v'(t))) + \omega_{u(t)}(u(t)r(t)^{-1}r'(t)) \\ &= \text{Ad}_{r(t)^{-1}}(\omega_{v(t)}(v'(t))) + r(t)^{-1}r'(t)\end{aligned}$$

Par horizontalité du relèvement $t \rightarrow v(t)$, on a $\omega_{v(t)}(v'(t)) = 0$, d'où :

$$\omega_{u(t)}(u'(t)) = r(t)^{-1}r'(t)$$

Par ailleurs, la relation $u(t) = f_t(u_0)$ entraîne $u'(t) = T_{u_0}f_t(u'(0)) = T_{u_0}f_t(X_{u_0})$. Il en résulte :

$$\begin{aligned}r(t)^{-1}r'(t) &= \omega_{u(t)}(u'(t)) \\ &= \omega_{f_t(u_0)}(T_{u_0}f_t(X_{u_0})) \\ &= \omega_{u_0}(X_{u_0}) \\ &= A\end{aligned}$$

En particulier, $A = r(0)^{-1}r'(0)$ d'où $A = r'(0)$ et donc $r(t) = \exp(tA)$. \square

Remarquons que la condition " ω est invariante par (f_t) " équivaut à " $L_X\omega = 0$ ".

Soit S un groupe de Lie compact, opérant sur U par automorphismes. On a donc $s(ur) = (su)r$ pour tous $s \in S$, $u \in U$ et $r \in R$. Par suite, l'action de S sur U induit une action de S sur E (il suffit de poser pour tous $s \in S$ et $y \in E$: $sy = \pi(su)$, $\pi : U \rightarrow E$ étant la projection du fibré, et u un point quelconque de la fibre au-dessus de y). Dans la suite, nous fixerons une fois pour toutes un point u_0 de U , et nous noterons I le stabilisateur de $y_0 = \pi(u_0)$. Il est facile de voir que pour tout $s \in I$, les points u_0 et su_0 sont dans la même fibre. Par suite, il existe un unique élément $r \in R$ vérifiant : $su_0 = u_0r$. Soit alors $\lambda : I \rightarrow R$ l'application qui à chaque s fait correspondre l'unique r tel que $su_0 = u_0r$.

Proposition 3.3.2 *L'application $\lambda : I \rightarrow R$ est un morphisme de groupes de Lie.*

Démonstration : Par définition de λ , on a $su_0 = u_0\lambda(s)$ pour tout $s \in I$. Par conséquent, pour tous $s, t \in I$, $u_0\lambda(st) = (st)u_0 = s(tu_0) = s(u_0\lambda(t)) = (su_0)\lambda(t) = (u_0\lambda(s))\lambda(t) = u_0\lambda(s)\lambda(t)$, d'où : $\lambda(st) = \lambda(s)\lambda(t)$. \square

Corollary 3.3.3 *L'application $\lambda' : \mathfrak{i} \rightarrow \mathfrak{r}$ est un morphisme d'algèbres de Lie.*

L'action de S sur U permet d'associer à chaque $X \in \mathfrak{s}$ un champ de vecteurs \tilde{X} sur U , appelé champ fondamental, et défini en notation abrégée par $\tilde{X}_u = Xu$. D'une manière plus précise, $\tilde{X}_u = T_e\sigma_u(X)$, $\sigma_u : S \rightarrow U$ étant l'application définie par $\sigma_u(s) = su$.

Maintenant à chaque connexion ω sur U , on peut associer une application linéaire $\Lambda : \mathfrak{s} \rightarrow \mathfrak{r}$ en posant : $\Lambda(X) = \omega_{u_0}(\tilde{X}_{u_0})$.

Proposition 3.3.4 *L'application linéaire $\Lambda : \mathfrak{s} \rightarrow \mathfrak{r}$ vérifie la condition suivante :*

$$(1) \quad \Lambda|_{\mathfrak{i}} = \lambda'$$

Démonstration : Soit $X \in \mathfrak{i}$, de sorte que $\exp(tX) \in I$. Par suite, on a : $\exp(tX)u_0 = u_0\lambda(\exp(tX))$. La dérivation des deux membres de cette égalité en $t = 0$ donne $\tilde{X}_{u_0} = u_0\lambda'(X)$, d'où : $\omega_{u_0}(\tilde{X}_{u_0}) = \omega_{u_0}(u_0\lambda'(X))$, c'est-à-dire : $\Lambda(X) = \lambda'(X)$. \square

Proposition 3.3.5 Si la connexion ω est invariante par S , alors $\Lambda : \mathfrak{s} \longrightarrow \mathfrak{r}$ vérifie la condition suivante :

$$(2) \quad \Lambda \circ Ad_s = Ad_{\lambda(s)} \circ \Lambda, \text{ pour tout } s \in I$$

Démonstration : Remarquons d'abord que l'invariance de la connexion ω par le groupe S se traduit de la manière suivante : pour tous $s \in S$, $u \in U$ et $\xi \in T_u U$, $\omega_{su}(s\xi) = \omega_u(\xi)$. Soient $s \in I$, $X \in \mathfrak{s}$ et posons $Y = Ad_s(X)$. Soit $u(t) = \exp(tY)u_0$. Alors $u(t) = s[\exp(tX)]s^{-1}u_0 = s[\exp(tX)u_0]\lambda(s^{-1})$. On en déduit que $\tilde{Y}_{u_0} = u'(0) = s\tilde{X}_{u_0}\lambda(s^{-1})$. Dans ces conditions,

$$\begin{aligned} \Lambda(Ad_s(X)) &= \Lambda(Y) \\ &= \omega_{u_0}(\tilde{Y}_{u_0}) \\ &= \omega_{u_0}(s\tilde{X}_{u_0}\lambda(s^{-1})) \\ &= \omega_{ss^{-1}u_0}(s\tilde{X}_{u_0}\lambda(s^{-1})) \\ &= \omega_{s^{-1}u_0}(\tilde{X}_{u_0}\lambda(s^{-1})) \\ &= \omega_{u_0\lambda(s^{-1})}(\tilde{X}_{u_0}\lambda(s^{-1})) \\ &= Ad_{\lambda(s)}(\omega_{u_0}(\tilde{X}_{u_0})) \\ &= Ad_{\lambda(s)}(\Lambda(X)) \end{aligned}$$

\square

Nous pouvons à présent énoncer le théorème de Wang :

Theorem 3.3.6 Si l'action de S sur E est transitive, alors il existe une bijection naturelle entre l'ensemble des connexions S -invariantes sur U , et l'ensemble des applications linéaires $\Lambda : \mathfrak{s} \longrightarrow \mathfrak{r}$ qui vérifient les deux conditions :

- (1) $\Lambda|_{\mathfrak{i}} = \lambda'$
- (2) $\Lambda \circ Ad_s = Ad_{\lambda(s)} \circ \Lambda, \text{ pour tout } s \in I$

Démonstration : cf. [KN1] par exemple. \square

Corollary 3.3.7 On suppose que \mathfrak{s} admet une décomposition réductive $\mathfrak{s} = \mathfrak{i} \oplus \mathfrak{p}$. Dans ce cas, si l'action de S sur E est transitive, alors il existe une bijection naturelle entre l'ensemble des connexions S -invariantes sur U , et l'ensemble des applications linéaires $\phi : \mathfrak{p} \longrightarrow \mathfrak{r}$ qui vérifient la condition :

$$(2) \quad \phi \circ Ad_s = Ad_{\lambda(s)} \circ \phi, \text{ pour tout } s \in I$$

Démonstration : Soit $\Lambda : \mathfrak{s} \longrightarrow \mathfrak{r}$ vérifiant (1) et (2), et posons $\phi = \Lambda|_{\mathfrak{p}}$. Il est facile de voir que la correspondance $\Lambda \longrightarrow \phi$ nous donne la bijection désirée. \square

3.3.2 Structure géométrique adaptée

On se donne une fibré principal (U, E, R) , R étant compact. On considère ensuite un groupe de Lie compact S opérant sur U par automorphismes. L'action de S sur U induit une action de S sur E , comme nous l'avons vu au paragraphe précédent. Nous fixons de même un point u_0 de U , et nous notons I le stabilisateur de $y_0 = \pi(u_0)$, $\pi : U \longrightarrow E$

étant la projection du fibré. Enfin, on considère l'application $\lambda : I \longrightarrow R$ qui à chaque s fait correspondre l'unique r tel que $su_0 = u_0r$.

Notons $N(I)$ le normalisateur de I dans S , et posons $K = N(I)|I$. Soit alors $P = \{y \in E / I_y = I\}$. L'action de S sur E étant supposée simple, on sait d'après le chapitre 1 que (P, M, K) est un fibré principal, et que $(E, M, S/I)$ est un fibré associé à (P, M, K) par l'action naturelle de K sur S/I .

Les actions de S et R sur U donnent naissance à une action de $G = S \times R$ sur U ; il suffit de poser $u(s, r) = s^{-1}ur$. On vérifie aisément que stabilisateur H de u_0 pour cette action n'est autre que le graphe du morphisme $\lambda : I \longrightarrow R$, c'est-à-dire : $H = \{(i, \lambda(i)); i \in I\}$. Il n'est pas plus difficile de caractériser le normalisateur $N(H)$ de H dans G :

$$(s, r) \in N(H) \iff s \in N(I) \text{ et } r\lambda(i)r^{-1} = \lambda(sis^{-1}) \text{ pour tout } i \in I$$

Introduisons le centralisateur Z de $\lambda(I)$ dans R . Par définition,

$$Z = \{r \in R / r\lambda(i)r^{-1} = \lambda(i), \forall i \in I\}$$

Proposition 3.3.8 Z s'identifie à un sous-groupe distingué de $N(H)|H$.

Démonstration : D'abord, on a $\{e\} \times Z \subset N(H)$. En effet, soit $(s, r) \in \{e\} \times Z$, alors $s = e \in N(I)$ et pour tout $i \in I$, $r\lambda(i)r^{-1} = \lambda(i) = \lambda(eie^{-1}) = \lambda(sis^{-1})$, donc $(s, r) \in N(H)$. Ensuite, il est facile de vérifier que l'application $j : Z \longrightarrow N(H)|H$ définie par $j(r) = (e, r)H$ est un morphisme injectif de groupes, qui induit donc un isomorphisme de Z sur le sous-groupe $j(Z)$ de $N(H)|H$. Il reste à montrer que $j(Z)$ est distingué dans $N(H)|H$. Posons $[(s, r)] = (s, r)H$ pour tout $(s, r) \in N(H)$, et soit $[(e, z)] \in j(Z)$. Pour tout $[(s, r)] \in N(H)|H$, $[(s, r)][(e, z)][(s, r)]^{-1} = [(ses^{-1}, rzs^{-1})] = [(e, rzs^{-1})]$ et il suffit de vérifier que $rzs^{-1} \in Z$ pour conclure que $[(e, rzs^{-1})] \in j(Z)$. Pour tout $i \in I$, $(rzs^{-1})\lambda(i)(rzs^{-1})^{-1} = rz(r^{-1}\lambda(i)r)z^{-1}r^{-1} = rz\lambda(s^{-1}is)z^{-1}r^{-1} = r\lambda(s^{-1}is)r^{-1} = r(r^{-1}\lambda(i)r)r^{-1} = \lambda(i)$. \square

Dans la suite, \mathfrak{k} désignera l'algèbre de Lie de K .

Proposition 3.3.9 Localement, on a $N(H)|H = K \times Z$.

Démonstration : Comme $\mathfrak{n}(\mathfrak{i}) = \mathfrak{i} \oplus \mathfrak{k}$, avec $[\mathfrak{i}, \mathfrak{k}] = \{0\}$, on a localement $N(I) = IK$. Soit $(s, r) \in N(H)$, de sorte que $s \in N(I)$ et $r\lambda(i)r^{-1} = \lambda(sis^{-1})$ pour tout $i \in I$. Soit $s = \sigma k$, avec $\sigma \in I$ et $k \in K$. Alors pour tout $i \in I$,

$$\lambda(sis^{-1}) = \lambda(\sigma kik^{-1}\sigma^{-1}) = \lambda(\sigma i\sigma^{-1}) = \lambda(\sigma)\lambda(i)\lambda(\sigma)^{-1}$$

et donc $\lambda(\sigma)^{-1}r \in Z$, c'est-à-dire $r = \lambda(\sigma)z$ avec $z \in Z$. Donc localement, on a $N(H) = \{(\sigma k, \lambda(\sigma)z)\}$ et par suite $N(H)|H = K \times Z$. \square

Corollary 3.3.10

$$\mathfrak{n}(\mathfrak{h})|\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{z}$$

Soit maintenant $Q = \{u \in E / H_u = H\}$. L'action de G sur U étant supposée simple, on déduit que $(Q, M, N(H)|H)$ est un fibré principal, et que $(U, M, G/H)$ est un fibré associé à $(Q, M, N(H)|H)$ par l'action naturelle de $N(H)|H$ sur G/H .

Remarquons que pour tout $s \in I$ et $u \in Q$, on a $u = u(s, \lambda(s))$ (car $u \in Q$ et $(s, \lambda(s)) \in H$) d'où $u = s^{-1}u\lambda(s)$ et donc $su = u\lambda(s)$. Par suite, $\pi(su) = \pi(u)$ et donc $s\pi(u) = \pi(u)$. On en déduit que $\pi(u) \in P$. Ainsi, la projection $\pi : U \longrightarrow E$ induit une projection $\pi : Q \longrightarrow P$.

3.4 Les champs scalaires et leurs représentations

Nous avons vu au paragraphe précédent que la réduction d'un champ d'Einstein-Yang-Mills (g, ω) invariant par un groupe S est équivalente à la réduction d'une métrique \tilde{g} invariante par un groupe non simple G avec plongement diagonal du stabilisateur.

Cette dernière réduction fournissait en particulier un champ scalaire \tilde{h} , constitué d'une métrique G -invariante \tilde{h}_x sur chaque fibre U_x . Nous avons vu également qu'en réduisant la métrique \tilde{h}_x (qui est en particulier R -invariante) sur le fibré principal (U_x, E_x, R) , on aboutissait à un triplet de champs $(h_x, \phi(x), k)$, où :

- (i) h_x est une métrique S -invariante sur E_x ,
- (ii) k est une métrique riemannienne bi-invariante sur R ,
- (iii) $\phi(x)$ est un champ de Yang-Mills sur S/I à valeurs dans l'algèbre de Lie de R .

Ces trois champs peuvent être vus comme des champs scalaires, dans le sens où ils se réalisent naturellement comme sections de fibrés vectoriels associés à des fibrés principaux. Notre premier objectif dans ce chapitre est justement de préciser la nature mathématique de ces champs. Nous verrons ainsi au paragraphe 3.2 que :

\tilde{h} s'identifie à une section du fibré vectoriel $Q \times_{\rho} S_2^H(\mathfrak{m})$, où ρ est une représentation naturelle de $N(H)|H$ sur l'espace vectoriel $S_2^H(\mathfrak{m})$ des formes bilinéaires symétriques $Ad(H)$ -invariantes sur \mathfrak{m} .

On aura alors de même :

h s'identifie à une section du fibré vectoriel $P \times_{\rho} S_2^I(\mathfrak{p})$, où ρ est une représentation naturelle de $N(I)|I$ sur l'espace vectoriel $S_2^I(\mathfrak{p})$ des formes bilinéaires symétriques $Ad(I)$ -invariantes sur \mathfrak{p} .

Regardons à présent le champ ϕ . Nous verrons au paragraphe 3.3 que :

ϕ s'identifie à une section du fibré vectoriel $Q \times_{\rho} \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$, où ρ est une représentation naturelle de $N(H)|H$ sur l'espace vectoriel $\mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$ des applications linéaires $\phi : \mathfrak{p} \longrightarrow \mathfrak{r}$ qui entrelacent les représentations de I sur \mathfrak{p} et sur \mathfrak{r} (via λ).

Nous pouvons maintenant formuler la situation au niveau algébrique :

D'une part, on a $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

D'autre part, on a $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r} = \mathfrak{i} \oplus \mathfrak{p} \oplus \mathfrak{r}$.

Or $H = \{(i, \lambda(i)) ; i \in I\}$ entraîne que $\mathfrak{h} = \{X + \lambda'(X) ; X \in \mathfrak{i}\}$, et on voit que \mathfrak{h} , comme graphe de λ' , est isomorphe à \mathfrak{i} . Si bien qu'on a l'identification :

$$\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{r}$$

Par ailleurs, rappelons que localement, on a $N(H)|H = K \times Z$ avec $K = N(I)|I$.

En étudiant alors les représentations de $N(H)|H = K \times Z$ sur $S_2^H(\mathfrak{m})$, on déduit qu'un produit scalaire $Ad(H)$ -invariant sur \mathfrak{m} va se décomposer en :

- (i) un produit scalaire $Ad(I)$ -invariant sur \mathfrak{p} ,
- (ii) un produit scalaire $Ad(R)$ -invariant sur \mathfrak{r} ,
- (iii) une application linéaire $\phi : \mathfrak{p} \longrightarrow \mathfrak{r}$ qui entrelace les représentations de I sur \mathfrak{p} et sur \mathfrak{r} (via λ).

On aurait donc quelque chose comme :

$$S_2^H(\mathfrak{m}) = S_2^I(\mathfrak{p}) \oplus S_2^R(\mathfrak{r}) \oplus \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$$

En examinant la forme du potentiel de ces champs scalaires, on constate que ceux qui sont représentés par $\mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$ ont une allure semblable aux potentiels de Higgs qui apparaissent dans le modèle standard de la physique des particules. Ils se couplent de la même manière aux champs de jauge et ont une interaction quartique. C'est pourquoi ils sont interprétés comme des champs de Higgs.

3.4.1 Représentation associée aux métriques G -invariantes sur G/H

On commence par envisager le cas particulier d'un fibré principal (P, M, G) .

On note $S_2(\mathfrak{g})$ l'espace vectoriel des formes bilinéaires symétriques sur \mathfrak{g} . La représentation adjointe de G induit une représentation naturelle de G sur $S_2(\mathfrak{g})$. En effet, il suffit de considérer l'application $\rho : G \longrightarrow GL(S_2(\mathfrak{g}))$ définie par $\rho_a(f) = {}^t Ad_{a^{-1}}(f)$ pour tous $a \in G$ et $f \in S_2(\mathfrak{g})$. Cette représentation ρ permet alors de définir le fibré vectoriel associé :

$$\mathcal{S} = P \times_\rho S_2(\mathfrak{g})$$

Notons que $\mathcal{S} = \{(y, f)G ; (y, f) \in P \times S_2(\mathfrak{g})\}$ où $(y, f)G = \{(ya, {}^t Ad_a(f)) ; a \in G\}$.

Proposition 3.4.1 *Il existe une bijection entre l'ensemble $\Gamma(\mathcal{S}^{++})$ des sections du fibré $\mathcal{S}^{++} = P \times_\rho S_2^{++}(\mathfrak{g})$ (où $S_2^{++}(\mathfrak{g})$ est l'ensemble des produits scalaires sur \mathfrak{g}), et l'ensemble des familles $(h_x)_{x \in M}$ telles que pour tout $x \in M$, h_x est une métrique riemannienne G -invariante sur P_x .*

Démonstration : Remarquons d'abord que $S_2^{++}(\mathfrak{g})$ est un ouvert de l'espace vectoriel $S_2(\mathfrak{g})$. Soit $s \in \Gamma(\mathcal{S})$, et $\varphi : P \longrightarrow S_2(\mathfrak{g})$ l'application équivariante associée à la section s . Pour tous $x \in M$ et $u \in P_x$, posons $h_x(u) = {}^t(T_e\sigma_u)^{-1}(\varphi(u))$, $\sigma_u : G \longrightarrow P$ étant l'application définie par $\sigma_u(g) = ug$. Dans ces conditions, si $f \in S_2^{++}(\mathfrak{g})$, alors $h_x(u)$ est un produit scalaire sur $T_u P_x$, et donc h_x est une métrique riemannienne sur P_x . Il ne reste plus qu'à montrer que h_x est G -invariante. Pour chaque $X \in \mathfrak{g}$, notons \mathbf{X} le champ fondamental sur P associé à X : pour tout $u \in P$, $\mathbf{X}_u = T_e\sigma_u(X)$. Dans ces conditions, pour tous $a \in G$, et $X, Y \in \mathfrak{g}$, on a :

$$\begin{aligned}
h_x(ua)(\mathbf{X}_u a, \mathbf{Y}_u a) &= {}^t(T_e \sigma_{ua})^{-1}(\varphi(ua))(\mathbf{X}_u a, \mathbf{Y}_u a) \\
&= \varphi(ua)((T_e \sigma_{ua})^{-1}(\mathbf{X}_u a), (T_e \sigma_{ua})^{-1}(\mathbf{Y}_u a)) \\
&= {}^t Ad_a(\varphi(u))((T_e \sigma_{ua})^{-1}(\mathbf{Ad}_{a^{-1}}(\mathbf{X})_{ua}), (T_e \sigma_{ua})^{-1}(\mathbf{Ad}_{a^{-1}}(\mathbf{Y})_{ua})) \\
&= {}^t Ad_a(\varphi(u))(Ad_{a^{-1}}(X), Ad_{a^{-1}}(Y)) \\
&= \varphi(u)(X, Y) \\
&= \varphi(u)((T_e \sigma_u)^{-1}(\mathbf{X}_u), (T_e \sigma_u)^{-1}(\mathbf{Y}_u)) \\
&= {}^t(T_e \sigma_u)^{-1}(\varphi(u))(\mathbf{X}_u, \mathbf{Y}_u) \\
&= h_x(u)(\mathbf{X}_u, \mathbf{Y}_u)
\end{aligned}$$

d'où la G -invariance de h_x . \square

Revenons maintenant au cas général d'un espace à fibres homogènes $(E, M, G/H)$.

Pour simplifier, nous supposons que G et H sont compacts et connexes. La compacité de G assure l'existence d'une métrique riemannienne bi-invariante sur G , i.e. d'un produit scalaire $Ad(G)$ -invariant sur \mathfrak{g} . Nous fixons une fois pour toutes un tel produit scalaire \langle , \rangle sur \mathfrak{g} .

Soit maintenant \mathfrak{m} le supplémentaire orthogonal de \mathfrak{h} dans \mathfrak{g} (pour \langle , \rangle). On sait que $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ est une décomposition réductive.

Notons alors $S_2^H(\mathfrak{m})$ l'ensemble des formes bilinéaires symétriques $Ad(H)$ -invariantes sur \mathfrak{m} . On vérifie facilement que $S_2^H(\mathfrak{m})$ est un espace vectoriel.

Rappelons le résultat essentiel suivant :

Theorem 3.4.2 *Il existe une bijection entre l'ensemble des métriques G -invariantes sur G/H , et l'ensemble $(S_2^H)^{++}(\mathfrak{m})$ des produits scalaires $Ad(H)$ -invariants sur \mathfrak{m} .*

Démonstration : cf. [CE] par exemple. \square

Nous allons munir l'espace vectoriel $S_2^H(\mathfrak{m})$ d'une représentation linéaire du groupe $N(H)|H$. Pour cela, on commence par vérifier que pour tout $n \in N(H)$, l'automorphisme $\tilde{\rho}_n : f \longrightarrow {}^t Ad_{n^{-1}}(f)$ de $S_2^H(\mathfrak{m})$ est bien défini.

Soit $f \in S_2^H(\mathfrak{m})$. Alors pour tout $a \in H$, il existe $b \in H$ tel que $n^{-1}a = bn^{-1}$ (par définition du normalisateur). Dans ces conditions,

$$\begin{aligned}
{}^t Ad_a(\tilde{\rho}_n(f)) &= {}^t Ad_a({}^t Ad_{n^{-1}}(f)) \\
&= {}^t(Ad_{n^{-1}} \circ Ad_a)(f) \\
&= {}^t Ad_{n^{-1}a}(f) \\
&= {}^t Ad_{bn^{-1}}(f) \\
&= {}^t(Ad_b \circ Ad_{n^{-1}})(f) \\
&= {}^t Ad_{n^{-1}}({}^t Ad_b(f)) \\
&= {}^t Ad_{n^{-1}}(f) \\
&= \tilde{\rho}_n(f)
\end{aligned}$$

Donc $\tilde{\rho}_n(f) \in S_2^H(\mathfrak{m})$, et par suite, on a une représentation $\tilde{\rho} : N(H) \longrightarrow GL(S_2^H(\mathfrak{m}))$. Maintenant pour tout pour tout $a \in H$,

$$\begin{aligned}
\tilde{\rho}_{na}(f) &= {}^t Ad_{a^{-1}n^{-1}}(f) \\
&= {}^t(Ad_{a^{-1}} \circ Ad_{n^{-1}})(f) \\
&= {}^t Ad_{n^{-1}}({}^t Ad_{a^{-1}}(f)) \\
&= {}^t Ad_{n^{-1}}(f) \\
&= \tilde{\rho}_n(f)
\end{aligned}$$

On peut donc définir la représentation quotient $\rho : N(H)|H \longrightarrow GL(S_2^H(\mathfrak{m}))$, en posant $\rho_{[n]} = \tilde{\rho}_n$.

Cette représentation ρ permet alors de définir le fibré vectoriel suivant :

$$\mathcal{S} = P \times_\rho S_2^H(\mathfrak{m})$$

où $P = \{y \in E / H_y = H\}$.

Notons que $\mathcal{S} = \{(y, f)N(H)|H ; (y, f) \in P \times S_2^H(\mathfrak{m})\}$
où $(y, f)N(H)|H = \{(y[n], \rho_{[n]^{-1}}(f)) ; [n] \in N(H)|H\}$.

Proposition 3.4.3 *Il existe une bijection entre l'ensemble $\Gamma(\mathcal{S}^{++})$ des sections du fibré $\mathcal{S}^{++} = P \times_\rho (S_2^H)^{++}(\mathfrak{m})$ (où $(S_2^H)^{++}(\mathfrak{m})$ est l'ensemble des produits scalaires $Ad(H)$ -invariants sur \mathfrak{m}), et l'ensemble des familles $(h_x)_{x \in M}$ telles que pour tout $x \in M$, h_x est une métrique riemannienne G -invariante sur E_x .*

3.4.2 Représentation associée aux champs de Yang-Mills sur S/I

On suppose que \mathfrak{s} admet une décomposition réductive $\mathfrak{s} = \mathfrak{n}(\mathfrak{i}) \oplus \mathfrak{l}$, et on pose $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{l}$, de sorte que $\mathfrak{s} = \mathfrak{i} \oplus \mathfrak{p}$ soit une décomposition réductive.

On définit alors les espaces vectoriels suivants :

$$\mathcal{L}_0(\mathfrak{p}, \mathfrak{r}) = \{\phi \in \mathcal{L}(\mathfrak{p}, \mathfrak{r}) / \phi \circ Ad_s = Ad_{\lambda(s)} \circ \phi, \forall s \in I\}$$

$$\mathcal{L}_0(\mathfrak{k}, \mathfrak{r}) = \{\phi_{\mathfrak{k}} \in \mathcal{L}(\mathfrak{k}, \mathfrak{r}) / \phi_{\mathfrak{k}} \circ Ad_s = Ad_{\lambda(s)} \circ \phi_{\mathfrak{k}}, \forall s \in I\}$$

$$\mathcal{L}_0(\mathfrak{l}, \mathfrak{r}) = \{\phi_{\mathfrak{l}} \in \mathcal{L}(\mathfrak{l}, \mathfrak{r}) / \phi_{\mathfrak{l}} \circ Ad_s = Ad_{\lambda(s)} \circ \phi_{\mathfrak{l}}, \forall s \in I\}$$

Chacun de ces espaces vectoriels peut être muni d'une représentation linéaire du groupe $N(H)|H$. En effet, nous allons d'abord vérifier que pour tout $(s, r) \in N(H)$, l'automorphisme $\tilde{\rho}_{(s,r)} : \phi \longrightarrow Ad_r \circ \phi \circ Ad_{s^{-1}}$ de $\mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$ est bien défini.

Soit $\phi \in \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$. Alors pour tout $i \in I$,

$$\begin{aligned}
\tilde{\rho}_{(s,r)}(\phi) \circ Ad_i &= Ad_r \circ \phi \circ Ad_{s^{-1}} \circ Ad_i \\
&= Ad_r \circ \phi \circ Ad_{s^{-1}is} \circ Ad_s \circ Ad_{s^{-1}} \\
&= Ad_r \circ \phi \circ Ad_{s^{-1}is} \circ Ad_{s^{-1}} \\
&= Ad_r \circ Ad_{\lambda(s^{-1}is)} \circ \phi \circ Ad_{s^{-1}} \\
&= Ad_r \circ Ad_{r^{-1}\lambda(i)r} \circ \phi \circ Ad_{s^{-1}} \\
&= Ad_r \circ Ad_{r^{-1}} \circ Ad_{\lambda(i)r} \circ \phi \circ Ad_{s^{-1}} \\
&= Ad_{\lambda(i)} \circ Ad_r \circ \phi \circ Ad_{s^{-1}} \\
&= Ad_{\lambda(i)} \circ \tilde{\rho}_{(s,r)}(\phi)
\end{aligned}$$

Donc $\tilde{\rho}_{(s,r)}(\phi) \in \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$, et par suite, on a une représentation $\tilde{\rho} : N(H) \longrightarrow GL(\mathcal{L}_0(\mathfrak{p}, \mathfrak{r}))$. Maintenant pour tout $h = (i, \lambda(i)) \in H$, $(s, r)h = (si, r\lambda(i))$ et donc

$$\begin{aligned}\tilde{\rho}_{(s,r)h}(\phi) &= Ad_{r\lambda(i)} \circ \phi \circ Ad_{i^{-1}s^{-1}} \\ &= Ad_r \circ Ad_{\lambda(i)} \circ \phi \circ Ad_{i^{-1}} \circ Ad_{s^{-1}} \\ &= Ad_r \circ Ad_{\lambda(i)} \circ Ad_{\lambda(i^{-1})} \circ \phi \circ Ad_{s^{-1}} \\ &= Ad_r \circ \phi \circ Ad_{s^{-1}} \\ &= \tilde{\rho}_{(s,r)}(\phi)\end{aligned}$$

On peut alors définir la représentation quotient $\rho : N(H)|H \longrightarrow GL(\mathcal{L}_0(\mathfrak{k}, \mathfrak{r}))$, en posant $\rho_{[(s,r)]} = \tilde{\rho}_{(s,r)}$.

On définit de la même manière les représentations :

$$\begin{aligned}\rho^{\mathfrak{k}} : N(H)|H &\longrightarrow GL(\mathcal{L}_0(\mathfrak{k}, \mathfrak{r})) \\ \rho^{\mathfrak{l}} : N(H)|H &\longrightarrow GL(\mathcal{L}_0(\mathfrak{l}, \mathfrak{r}))\end{aligned}$$

Ce qui précède nous permet alors de considérer les fibrés vectoriels suivants, associés à $(Q, M, N(H)|H)$:

$$\begin{aligned}\mathcal{F} &= Q \times_{\rho} \mathcal{L}_0(\mathfrak{p}, \mathfrak{r}) \\ \mathcal{F}_{\mathfrak{k}} &= Q \times_{\rho^{\mathfrak{k}}} \mathcal{L}_0(\mathfrak{k}, \mathfrak{r}) \\ \mathcal{F}_{\mathfrak{l}} &= Q \times_{\rho^{\mathfrak{l}}} \mathcal{L}_0(\mathfrak{l}, \mathfrak{r})\end{aligned}$$

Proposition 3.4.4 Pour tout $\phi \in \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$, on a $\phi(\mathfrak{k}) \subset \mathfrak{z}$.

Démonstration : Comme $\phi \in \mathcal{L}_0(\mathfrak{p}, \mathfrak{r})$, on a $\phi \circ Ad_s = Ad_{\lambda(s)} \circ \phi$ pour tout $s \in I$, donc $\phi(Ad_s(X)) = Ad_{\lambda(s)}(\phi(X))$ pour tous $s \in I$ et $X \in \mathfrak{p}$. Au niveau infinitésimal, cette condition s'écrit : $\phi([X, Y]) = [\phi(X), \lambda'(Y)]$ pour tous $Y \in \mathfrak{i}$ et $X \in \mathfrak{p}$. Si $X \in \mathfrak{k}$, alors $[X, Y] = 0$ pour tout $Y \in \mathfrak{i}$, et donc $[\phi(X), \lambda'(Y)] = 0$ pour tout $Y \in \mathfrak{i}$, c'est-à-dire $\phi(X) \in \mathfrak{z}$. \square

3.4.3 Décomposition de $S_2^H(\mathfrak{m})$

Nous allons maintenant examiner de plus près l'espace vectoriel $S_2^H(\mathfrak{m})$ des formes bilinéaires symétriques $Ad(H)$ -invariantes sur \mathfrak{m} . On voudrait par exemple calculer la dimension de cet espace, ce qui fournira le "nombre" de champs scalaires.

Pour cela, il faut décomposer $S_2^H(\mathfrak{m})$ en somme directe de sous-espaces dont on connaît les dimensions. A cette fin, on commence par décomposer la représentation isotrope $Ad : H \longrightarrow GL(\mathfrak{m})$ en représentations réelles irréductibles.

Rappelons d'abord quelques notations : si V , V_1 et V_2 sont trois G -modules sur \mathbb{K} (où $\mathbb{K} = \mathbb{R}$, \mathbb{C} ou \mathbb{H}), alors :

- $\mathcal{L}_{\mathbb{K}}^G(V_1, V_2)$ désigne l'espace vectoriel des applications linéaires G -équivariantes de V_1 dans V_2 ,

- $\mathcal{L}_2^G(V)$ désigne l'espace vectoriel des formes bilinéaires G -invariantes sur V ,
- $S_2^G(V)$ désigne l'espace vectoriel des formes bilinéaires symétriques G -invariantes sur V ,
- $\mathcal{L}_{\frac{3}{2}}^G(V)$ désigne l'espace vectoriel des formes sesquilinearaires G -invariantes sur V (qui se confondent avec les formes bilinéaires lorsque $\mathbb{K} = \mathbb{R}$),
- $S_{\frac{3}{2}}^G(V)$ désigne l'espace vectoriel des formes hermitiennes G -invariantes sur V (qui se confondent avec les formes bilinéaires symétriques lorsque $\mathbb{K} = \mathbb{R}$),
- $MS_n(\mathbb{K})$ désigne l'espace vectoriel (de dimension $\frac{n(n+1)}{2}$ sur \mathbb{K}) des matrices symétriques d'éléments de \mathbb{K} ,
- $MH_n(\mathbb{K})$ désigne l'espace vectoriel des matrices hermitiennes d'éléments de \mathbb{K} ($MH_n(\mathbb{R})$ n'est autre que $MS_n(\mathbb{R})$, $MH_n(\mathbb{C})$ est de dimension n^2 sur \mathbb{R} et $MH_n(\mathbb{H})$ est de dimension $n(2n - 1)$ sur \mathbb{R}).

Les sous-modules réels irréductibles de \mathfrak{m} (sous l'action adjointe de H) sont alors classés en trois types :

- Ceux qui sont **de type réel** : notés par la lettre D , ils sont caractérisés par le fait que leurs complexifiés $V = e_{\mathbb{R}}^{\mathbb{C}}(D) = \mathbb{C} \otimes_{\mathbb{R}} D$ sont des H -modules complexes irréductibles. Les V sont nécessairement auto-conjugués (isomorphes à leurs conjugués).
- Ceux qui sont **de type complexe** : notés par la lettre E , ce sont les H -modules réels sous-jacents à des H -modules complexes irréductibles V **non auto-conjugués**. On écrit $E = r_{\mathbb{R}}^{\mathbb{C}}(V)$, en gardant à l'esprit que E et V désignent le même ensemble sous-jacent.
- Ceux qui sont **de type quaternionique** : notés par la lettre F , ce sont les H -modules réels sous-jacents à des H -modules complexes irréductibles V qui admettent une structure quaternionique. On écrit $F = r_{\mathbb{R}}^{\mathbb{C}}(V)$ avec $V = r_{\mathbb{C}}^{\mathbb{H}}(W)$, où W est un H -module quaternionique irréductible. Il faut garder à l'esprit que F , V et W désignent les mêmes ensembles sous-jacents. Ici, comme dans le cas du type réel, les V sont auto-conjugués.

La décomposition de \mathfrak{m} s'écrit donc de manière générale :

$$\mathfrak{m} = (D_1)^{n_1} \oplus \dots \oplus (D_i)^{n_i} \oplus (E_1)^{m_1} \oplus \dots \oplus (E_j)^{m_j} \oplus (F_1)^{p_1} \oplus \dots \oplus (F_k)^{p_k}$$

les $n_1 \dots n_i, m_1 \dots m_j, p_1 \dots p_k$ désignant les multiplicités respectives des représentations réelles irréductibles $D_1 \dots D_i, E_1 \dots E_j, F_1 \dots F_k$.

La détermination de $S_2^H(\mathfrak{m})$ s'effectue alors en trois temps : d'abord, on détermine $S_2^H(D^n)$, où D est une représentation réelle irréductible de H de type réel.

$$\begin{aligned}
e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_2^H(D^n) &= e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{R}}^H(D^n, (D^n)^*) \\
&= \mathcal{L}_{\mathbb{C}}^H(V^n, (V^n)') \\
&= \mathcal{L}_{\mathbb{C}}^H(V \otimes \mathbb{C}^n, V' \otimes (\mathbb{C}^n)') \\
&= \mathcal{L}_{\mathbb{C}}^H(V, V') \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, (\mathbb{C}^n)') \\
&= \mathbb{C} \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, (\mathbb{C}^n)') \\
&= \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, (\mathbb{C}^n)') \\
&= \mathcal{L}_2(\mathbb{C}^n) \\
&= e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_2(\mathbb{R}^n)
\end{aligned}$$

Par suite,

$$\mathcal{L}_2^H(D^n) = \mathcal{L}_2(\mathbb{R}^n) = M_n(\mathbb{R})$$

et on en déduit que :

$$S_2^H(D^n) = S_2(\mathbb{R}^n) = M S_n(\mathbb{R})$$

Ensuite, on détermine $S_2^H(E^m)$, où E est une représentation réelle irréductible de H de type complexe.

$$\begin{aligned}
e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_2^H(E^m) &= e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{R}}^H(E^m, (E^m)^*) \\
&= \mathcal{L}_{\mathbb{C}}^H(V^m \oplus \bar{V}^m, (V^m)' \oplus (\bar{V}^m)') \\
&= \mathcal{L}_{\mathbb{C}}^H(V^m, (V^m)') \oplus \mathcal{L}_{\mathbb{C}}^H(V^m, (\bar{V}^m)') \oplus \mathcal{L}_{\mathbb{C}}^H(\bar{V}^m, (V^m)') \oplus \mathcal{L}_{\mathbb{C}}^H(\bar{V}^m, (\bar{V}^m)') \\
&= e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{C}}^H(V^m, (V^m)') \oplus e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{C}}^H(V^m, (\bar{V}^m)') \\
&= e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} (\mathcal{L}_{\mathbb{C}}^H(V, V') \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^m, (\mathbb{C}^m)')) \oplus e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} (\mathcal{L}_{\mathbb{C}}^H(V, \bar{V}') \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^m, (\bar{\mathbb{C}}^m)')) \\
&= e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} (0 \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^m, (\mathbb{C}^m)')) \oplus e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} (\mathbb{C} \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^m, (\bar{\mathbb{C}}^m)')) \\
&= e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{C}}(\mathbb{C}^m, (\bar{\mathbb{C}}^m)') \\
&= e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\frac{3}{2}}(\mathbb{C}^m)
\end{aligned}$$

Par suite,

$$\mathcal{L}_2^H(E^m) = r_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\frac{3}{2}}(\mathbb{C}^m) = r_{\mathbb{R}}^{\mathbb{C}} M_m(\mathbb{C})$$

et on en déduit que :

$$S_2^H(E^m) = S_{\frac{3}{2}}(\mathbb{C}^m) = M H_m(\mathbb{C})$$

Enfin, on détermine $S_2^H(F^p)$, où F est une représentation réelle irréductible de H de type quaternionique.

$$\begin{aligned}
e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_2^H(F^p) &= e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_{\mathbb{R}}^H(F^p, (F^p)^*) \\
&= \mathcal{L}_{\mathbb{C}}^H(V^p \oplus \bar{V}^p, (V^p)' \oplus (\bar{V}^p)') \\
&= \mathcal{L}_{\mathbb{C}}^H(V^{2p}, (V^{2p})') \\
&= \mathcal{L}_{\mathbb{C}}^H(V, V') \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^{2p}, (\mathbb{C}^{2p})') \\
&= \mathbb{C} \otimes \mathcal{L}_{\mathbb{C}}(\mathbb{C}^{2p}, (\mathbb{C}^{2p})') \\
&= \mathcal{L}_{\mathbb{C}}(\mathbb{C}^{2p}, (\mathbb{C}^{2p})') \\
&= \mathcal{L}_2(\mathbb{C}^{2p}) \\
&= e_{\mathbb{R}}^{\mathbb{C}} \mathcal{L}_2(\mathbb{R}^{2p})
\end{aligned}$$

Par suite,

$$\mathcal{L}_2^H(F^p) = \mathcal{L}_2(\mathbb{R}^{2p}) = M_{2p}(\mathbb{R}) = M_p(\mathbb{H})$$

et on en déduit que :

$$S_2^H(F^p) = MH_p(\mathbb{H})$$

Détaillons ce dernier point : il est facile de voir que pour tout $C = [c_{ij}] \in M_p(\mathbb{H})$, l'application $f : \mathbb{H}^p \times \mathbb{H}^p \rightarrow \mathbb{R}$ définie par :

$$f(X, Y) = \operatorname{Re}({}^t(CX)\bar{Y})$$

où

$$X = \begin{bmatrix} q_1 \\ \vdots \\ q_p \end{bmatrix}, \quad Y = \begin{bmatrix} q'_1 \\ \vdots \\ q'_p \end{bmatrix}$$

est une forme bilinéaire $Ad(H)$ -invariante sur \mathbb{H}^p . De plus, ce qui précède montre que toutes les formes bilinéaires $Ad(H)$ -invariantes sur \mathbb{H}^p sont ainsi obtenues.

Il reste à vérifier que f est symétrique si et seulement si C est hermitienne quaternionique. D'abord, remarquons f ne peut pas s'écrire de la manière suivante : $f(X, Y) = \operatorname{Re}({}^t X {}^t C \bar{Y})$, car lorsque A et B sont des matrices quaternioniques, on n'a pas : ${}^t(AB) = {}^t B {}^t A$, et on n'a pas non plus : $\bar{A}\bar{B} = \bar{B}\bar{A}$. Par contre, on a toujours $(AB)^* = B^*A^*$, où $A^* = {}^t \bar{A}$ est la matrice adjointe de A .

Revenons à notre condition de symétrie. On a :

$$f(X, Y) = \operatorname{Re}(\sum_{i=1}^p \sum_{j=1}^p c_{ij} q_j \bar{q}'_i)$$

et

$$\begin{aligned} f(Y, X) &= \operatorname{Re}(\sum_{i=1}^p \sum_{j=1}^p c_{ij} q'_j \bar{q}_i) \\ &= \operatorname{Re}(\sum_{i=1}^p \sum_{j=1}^p q_i \bar{q}'_j \bar{c}_{ij}) \\ &= \operatorname{Re}(\sum_{i=1}^p \sum_{j=1}^p q_j \bar{q}'_i \bar{c}_{ji}) \\ &= \sum_{i=1}^p \sum_{j=1}^p \operatorname{Re}((q_j \bar{q}'_i) \bar{c}_{ji}) \\ &= \sum_{i=1}^p \sum_{j=1}^p \operatorname{Re}(\bar{c}_{ji} (q_j \bar{q}'_i)) \\ &= \operatorname{Re}(\sum_{i=1}^p \sum_{j=1}^p \bar{c}_{ji} q_j \bar{q}'_i) \end{aligned}$$

Par conséquent,

$$f(X, Y) = f(Y, X) \iff c_{ij} = \bar{c}_{ji} \iff C \in MH_p(\mathbb{H})$$

On a finalement démontré le théorème suivant :

Theorem 3.4.5 Soit $\mathfrak{m} = (D_1)^{n_1} \oplus \dots \oplus (D_i)^{n_i} \oplus (E_1)^{m_1} \oplus \dots \oplus (E_j)^{m_j} \oplus (F_1)^{p_1} \oplus \dots \oplus (F_k)^{p_k}$ une décomposition de \mathfrak{m} en représentations irréductibles sous l'action adjointe de H . Alors :

$$S_2^H(\mathfrak{m}) = MS_{n_1}(\mathbb{R}) \oplus \dots \oplus MS_{n_i}(\mathbb{R}) \oplus MH_{m_1}(\mathbb{C}) \oplus \dots \oplus MH_{m_j}(\mathbb{C}) \oplus MH_{p_1}(\mathbb{H}) \oplus \dots \oplus MH_{p_k}(\mathbb{H})$$

Corollary 3.4.6

$$\dim S_2^H(\mathfrak{m}) = \frac{n_1(n_1+1)}{2} + \dots + \frac{n_i(n_i+1)}{2} + m_1^2 + \dots + m_j^2 + p_1(2p_1-1) + \dots + p_k(2p_k-1)$$

Plusieurs modèles ont été construits selon le schéma présenté dans les pages précédentes (cf. [Man] et [CM]). Nous en citerons deux sans toutefois développer le second en détail. Le lecteur est prié de consulter [CM] pour les détails du second exemple.

3.4.4 Exemple de Manton avec $R = SU(3)$

Ce premier modèle consiste à prendre : $R = SU(3)$, $S = SO(3)$ et $I = SO(2)$.

Les fibres de E sont alors des copies de la sphère $\mathbb{S}^2 = SO(3)/SO(2)$.

$SO(2)$ est plongé de manière habituelle dans $SO(3)$, c'est-à-dire on définit I comme étant le sous-groupe de S isomorphe à $SO(2)$ suivant :

$$I = \{i_a = \begin{bmatrix} \cos(a) & -\sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{bmatrix} ; a \in \mathbb{R}\}$$

On pose ensuite $G = S \times R$.

Le morphisme $\lambda : I \longrightarrow R$ est défini par :

$$\lambda(i_a) = \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & e^{-2ia} \end{bmatrix}$$

$\lambda(I)$ est alors un sous-groupe de R isomorphe à $U(1)$.

On définit ensuite $H = \{(i_a, \lambda(i_a)); i_a \in I\}$, qui est un sous-groupe de G isomorphe à $U(1)$. On voit qu'on peut écrire (du moins localement) :

$$H = \{h_a = (\begin{bmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{bmatrix}, \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & e^{-2ia} \end{bmatrix}) ; a \in \mathbb{R}\}$$

Passons maintenant aux algèbres de Lie. On a $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathfrak{su}(3)$. Par suite :

$$\mathfrak{g} = \{(\begin{bmatrix} ia & \beta \\ -\bar{\beta} & -ia \end{bmatrix}, \begin{bmatrix} i(g+f) & \gamma & \delta \\ -\bar{\gamma} & i(g-f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2ig \end{bmatrix}) ; a, f, g \in \mathbb{R} \text{ et } \beta, \gamma, \delta, \varepsilon \in \mathbb{C}\}$$

$$\mathfrak{h} = \left\{ \left(\begin{bmatrix} ia & 0 \\ 0 & -ia \end{bmatrix}, \begin{bmatrix} ia & 0 & 0 \\ 0 & ia & 0 \\ 0 & 0 & -2ia \end{bmatrix} \right) ; a \in \mathbb{R} \right\}$$

On choisit ensuite $\mathfrak{m} \subset \mathfrak{g}$ de façon à avoir une décomposition réductive $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

$$\mathfrak{m} = \left\{ \left(\begin{bmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} i(g' + f) & \gamma & \delta \\ -\bar{\gamma} & i(g' - f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2ig' \end{bmatrix} \right) ; f, g' \in \mathbb{R} \text{ et } \beta, \gamma, \delta, \varepsilon \in \mathbb{C} \right\}$$

Il s'agit d'étudier la représentation isotrope $Ad : H \longrightarrow GL(\mathfrak{m})$. Pour cela, commençons par déterminer la représentation $Ad : H \longrightarrow GL(\mathfrak{g})$. Pour tout $X \in \mathfrak{g}$,

$$\begin{aligned} Ad_{h_a}(X) &= h_a X h_a^{-1} = \left(\begin{bmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{bmatrix}, \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & e^{-2ia} \end{bmatrix} \right) \\ &\left(\begin{bmatrix} ia & \beta \\ -\bar{\beta} & -ia \end{bmatrix}, \begin{bmatrix} i(g + f) & \gamma & \delta \\ -\bar{\gamma} & i(g - f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2ig \end{bmatrix} \right) \left(\begin{bmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{bmatrix}, \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & e^{-2ia} \end{bmatrix} \right)^{-1} \\ &= \left(\begin{bmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{bmatrix} \begin{bmatrix} ia & \beta \\ -\bar{\beta} & -ia \end{bmatrix} \begin{bmatrix} e^{-ia/2} & 0 \\ 0 & e^{ia/2} \end{bmatrix}, \right. \\ &\quad \left. \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & e^{-2ia} \end{bmatrix} \begin{bmatrix} i(g + f) & \gamma & \delta \\ -\bar{\gamma} & i(g - f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2ig \end{bmatrix} \begin{bmatrix} e^{-ia} & 0 & 0 \\ 0 & e^{-ia} & 0 \\ 0 & 0 & e^{2ia} \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} ia & e^{ia}\beta \\ -e^{-ia}\bar{\beta} & -ia \end{bmatrix}, \begin{bmatrix} i(g + f) & \gamma & e^{3ia}\delta \\ -\bar{\gamma} & i(g - f) & e^{3ia}\varepsilon \\ -e^{-3ia}\bar{\delta} & -e^{-3ia}\bar{\varepsilon} & -2ig \end{bmatrix} \right) \end{aligned}$$

Maintenant si

$$X = \left(\begin{bmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} i(g - a + f) & \gamma & \delta \\ -\bar{\gamma} & i(g - a - f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2i(g - a) \end{bmatrix} \right) \in \mathfrak{m}$$

alors

$$Ad_{h_a}(X) = \left(\begin{bmatrix} 0 & e^{ia}\beta \\ -e^{-ia}\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} i(g - a + f) & \gamma & e^{3ia}\delta \\ -\bar{\gamma} & i(g - a - f) & e^{3ia}\varepsilon \\ -e^{-3ia}\bar{\delta} & -e^{-3ia}\bar{\varepsilon} & -2i(g - a) \end{bmatrix} \right) \in \mathfrak{m}$$

ce qui montre au passage que \mathfrak{m} est bien un sous-espace $Ad(H)$ -invariant.

On pose :

$$D = \left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i(g - a + f) & \gamma & 0 \\ -\bar{\gamma} & i(g - a - f) & 0 \\ 0 & 0 & -2i(g - a) \end{bmatrix} \right) ; f, g - a, \gamma_1, \gamma_2 \in \mathbb{R} \right\}$$

$$E_1 = \{ \left(\begin{bmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) ; \beta \in \mathbb{C} \}$$

$$E_2 = \{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \delta \\ 0 & 0 & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & 0 \end{bmatrix} \right) ; \delta, \varepsilon \in \mathbb{C} \}$$

de sorte que $\mathfrak{m} = D \oplus E_1 \oplus E_2$.

Alors :

$$\forall X \in D, Ad_{h_a}(X) = X \in D$$

$$\forall X \in E_1, Ad_{h_a}(X) = \left(\begin{bmatrix} 0 & e^{ia}\beta \\ -e^{-ia}\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \in E_1$$

$$\forall X \in E_2, Ad_{h_a}(X) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & e^{3ia}\delta \\ 0 & 0 & e^{3ia}\varepsilon \\ -e^{-3ia}\bar{\delta} & -e^{-3ia}\bar{\varepsilon} & 0 \end{bmatrix} \right) \in E_2$$

(ce qui montre au passage que les sous-espaces D , E_1 et E_2 sont $Ad(H)$ -invariants).

Or :

$D = (\mathbb{R})^4$, où \mathbb{R} porte la représentation triviale de H ,

$E_1 = \mathbb{C}$, où \mathbb{C} porte la représentation irréductible $z \mapsto e^{ia}z$ de $H = U(1)$,

$E_2 = (\mathbb{C})^2$, où \mathbb{C} porte la représentation irréductible $z \mapsto e^{3ia}z$ de $H = U(1)$.

Finalement, la décomposition de la représentation isotrope $Ad : H \rightarrow GL(\mathfrak{m})$ en représentations réelles irréductibles s'écrit :

$$\mathfrak{m} = (\mathbb{R})^4 \oplus \mathbb{C} \oplus (\mathbb{C})^2$$

Cette décomposition correspond ainsi aux représentations irréductibles de $U(1)$ suivantes : $n = 0$, $n = 1$ et $n = 3$. La première est de type réel, et les deux autres sont de type complexe.

Dans ces conditions, le théorème établi au paragraphe précédent fournit la décomposition de $S_2^H(\mathfrak{m})$ associée :

$$S_2^H(\mathfrak{m}) = MS_4(\mathbb{R}) \oplus MH_1(\mathbb{C}) \oplus MH_2(\mathbb{C})$$

d'où :

$$\dim S_2^H(\mathfrak{m}) = 15$$

Pour chaque

$$X = \left(\begin{bmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{bmatrix}, \begin{bmatrix} i(g-a+f) & \gamma & \delta \\ -\bar{\gamma} & i(g-a-f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2i(g-a) \end{bmatrix} \right) \in \mathfrak{m}$$

on pose :

$$X_0 = [f \ g-a \ \gamma_1 \ \gamma_2] \in \mathbb{R}^4$$

$$X_1 = \beta \in \mathbb{C}$$

$$X_3 = [\delta \ \varepsilon] \in \mathbb{C}^2$$

Dans ces conditions, $f \in S_2^H(\mathfrak{m})$ équivaut à l'existence d'un triplet (A, λ, B) , où $A \in MS_4(\mathbb{R})$, $\lambda \in \mathbb{R}$ et $B \in MH_2(\mathbb{C})$, tel que :

$$f(X, Y) = X_0 A^t Y_0 + \lambda \operatorname{Re}(X_1 \bar{Y}_1) + \operatorname{Re}(X_3 B Y_3^*)$$

Vérifions "à la main" que la forme bilinéaire symétrique $f : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathbb{R}$ définie ci-dessus est bien $Ad(H)$ -invariante. En effet, pour tout $h_a \in H$,

$$\begin{aligned} f(Ad_{h_a}(X), Ad_{h_a}(Y)) &= X_0 A^t Y_0 + \lambda \operatorname{Re}(e^{ia} X_1 e^{ia} \bar{Y}_1) + \operatorname{Re}(e^{3ia} X_3 B (e^{3ia} Y_3)^*) \\ &= X_0 A^t Y_0 + \lambda \operatorname{Re}(e^{ia} X_1 e^{-ia} \bar{Y}_1) + \operatorname{Re}(e^{3ia} X_3 B e^{-3ia} Y_3^*) \\ &= X_0 A^t Y_0 + \lambda \operatorname{Re}(X_1 \bar{Y}_1) + \operatorname{Re}(X_3 B Y_3^*) \\ &= f(X, Y) \end{aligned}$$

Remarquons que la forme quadratique q associée à f s'écrit :

$$q(X) = X_0 A^t X_0 + \lambda |X_1|^2 + X_3 B X_3^*$$

A présent, nous allons nous intéresser à l'action de $N(H)|H$ sur \mathfrak{m} et sur $S_2^H(\mathfrak{m})$.

On vérifie sans difficulté que :

$$\begin{aligned} N(I) &= \left\{ \begin{bmatrix} P & 0 \\ 0 & (\det P)^{-1} \end{bmatrix} ; P \in O(2) \right\} \\ Z(\lambda(I)) &= \left\{ \begin{bmatrix} u & 0 \\ 0 & (\det u)^{-1} \end{bmatrix} ; u \in U(2) \right\} \end{aligned}$$

Par conséquent,

$$N(I)|I = O(2)|SO(2) = \mathbb{Z}_2$$

et localement, on a

$$N(H)|H = N(I)|I \times Z(\lambda(I)) = \mathbb{Z}_2 \times U(2)$$

On dispose d'un morphisme surjectif de noyau \mathbb{Z}_2 :

$$(\begin{bmatrix} \mu & \nu \\ -\bar{\nu} & \bar{\mu} \end{bmatrix}, \begin{bmatrix} e^{ig} & 0 \\ 0 & e^{ig} \end{bmatrix}) \longrightarrow \begin{bmatrix} U(2) \\ e^{ig}\mu & e^{ig}\nu \\ -e^{ig}\bar{\nu} & e^{ig}\bar{\mu} \end{bmatrix} (= \begin{bmatrix} \mu & \nu \\ -\bar{\nu} & \bar{\mu} \end{bmatrix} \begin{bmatrix} e^{ig} & 0 \\ 0 & e^{ig} \end{bmatrix})$$

de sorte que $U(2)$ s'identifie au quotient du produit direct $SU(2) \times U(1)$. Tout $u \in U(2)$ s'écrit $u = s\omega$ avec $s \in SU(2)$ et $\omega \in U(1)$.

Au niveau des algèbres de Lie, on déduit alors un isomorphisme :

$$(\begin{bmatrix} if & \gamma \\ -\bar{\gamma} & -if \end{bmatrix}, \begin{bmatrix} ig & 0 \\ 0 & ig \end{bmatrix}) \longrightarrow \begin{bmatrix} i(g+f) & \gamma \\ -\bar{\gamma} & i(g-f) \end{bmatrix} (= \begin{bmatrix} if & \gamma \\ -\bar{\gamma} & -if \end{bmatrix} + \begin{bmatrix} ig & 0 \\ 0 & ig \end{bmatrix})$$

de sorte que $L \in \mathfrak{u}(2)$ s'écrit de manière unique $L = Y + ig$ avec $Y \in \mathfrak{su}(2)$ et $ig \in \mathfrak{u}(1)$.

En fait, l'écriture $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ correspond à la décomposition de la représentation adjointe $Ad : U(2) \longrightarrow GL(\mathfrak{u}(2))$ en représentations irréductibles : en effet, pour tout $u = s\omega \in U(2)$ et pour tout $L = Y + ig \in \mathfrak{u}(2)$, $Ad_u(L) = uLu^{-1} = (s\omega)(Y + ig)(s\omega)^{-1} = s\omega Y \omega^{-1} s^{-1} + s\omega i g \omega^{-1} s^{-1} = sYs^{-1} + ig = Ad_s(Y) + ig = Ad_u(Y) + ig$.

Puisque $Ad_u(Y) = Ad_s(Y)$, alors $Ad_u(Y) \in \mathfrak{su}(2)$ et donc $\mathfrak{su}(2)$ est $Ad(U(2))$ -invariant. De même, puisque $Ad_u(ig) = ig$, alors $Ad_u(ig) \in \mathfrak{u}(1)$ et donc $\mathfrak{u}(1)$ est $Ad(U(2))$ -invariant.

Etudions la décomposition de \mathfrak{m} sous l'action de $N(H)|H$. On a vu que sous l'action de H , $\mathfrak{m} = D \oplus E_1 \oplus E_2$. Or $D \oplus E_2 = \mathfrak{su}(3)$, donc $\mathfrak{m} = \mathfrak{su}(3) \oplus E_1$.

L'action de \mathbb{Z}_2 étant triviale, il suffit d'étudier l'action de $U(2)$. Or $U(2)$ agit trivialement sur $E_1 = \mathbb{C}$. Finalement, on est ramené à l'étude de l'action adjointe de $U(2) \subset SU(3)$ sur $\mathfrak{su}(3)$. On a :

$$\begin{aligned} U(2) &= \left\{ \begin{bmatrix} u & 0 \\ 0 & (det u)^{-1} \end{bmatrix} ; u \in U(2) \right\} = \left\{ \begin{bmatrix} s\omega & 0 \\ 0 & \omega^{-2} \end{bmatrix} ; s \in SU(2) \text{ et } \omega \in U(1) \right\} \subset SU(3) \\ \mathfrak{su}(3) &= \left\{ \begin{bmatrix} i(g+f) & \gamma & \delta \\ -\bar{\gamma} & i(g-f) & \varepsilon \\ -\bar{\delta} & -\bar{\varepsilon} & -2ig \end{bmatrix} ; f, g \in \mathbb{R} \text{ et } \gamma, \delta, \varepsilon \in \mathbb{C} \right\} \\ &= \left\{ \begin{bmatrix} L = Y + ig & v \\ -{}^t \bar{v} & -tr(L) \end{bmatrix} ; L \in \mathfrak{u}(2) \text{ et } v \in \mathbb{C}^2 \right\} \quad (v = \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix}) \end{aligned}$$

Pour tout $u \in U(2)$ et $X \in \mathfrak{su}(3)$,

$$\begin{aligned} Ad_u(X) &= uXu^{-1} = \begin{bmatrix} s\omega & 0 \\ 0 & \omega^{-2} \end{bmatrix} \begin{bmatrix} Y + ig & v \\ -{}^t \bar{v} & -2ig \end{bmatrix} \begin{bmatrix} \omega^{-1}s^{-1} & 0 \\ 0 & \omega^2 \end{bmatrix} \\ &= \begin{bmatrix} s\omega Y + s\omega i g & s\omega v \\ -\omega^{-2} {}^t \bar{v} & -2\omega^{-2} i g \end{bmatrix} \begin{bmatrix} \omega^{-1}s^{-1} & 0 \\ 0 & \omega^2 \end{bmatrix} \\ &= \begin{bmatrix} sYs^{-1} + ig & s\omega^3 v \\ -\omega^{-3}s^{-1} {}^t \bar{v} & -2ig \end{bmatrix} \end{aligned}$$

On pose

$$U_{f,\gamma_1,\gamma_2} = \left\{ \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} ; Y \in \mathfrak{su}(2) \right\}$$

$$U_{\delta,\varepsilon} = \left\{ \begin{bmatrix} 0 & v \\ -{}^t \bar{v} & 0 \end{bmatrix} ; v \in \mathbb{C}^2 \right\}$$

$$U_g = \left\{ \begin{bmatrix} ig & 0 & 0 \\ 0 & ig & 0 \\ 0 & 0 & -2ig \end{bmatrix} ; g \in \mathbb{R} \right\}$$

de sorte que $\mathfrak{su}(3) = U_{f,\gamma_1,\gamma_2} \oplus U_{\delta,\varepsilon} \oplus U_g$ et donc $\mathfrak{m} = U_{f,\gamma_1,\gamma_2} \oplus U_{\delta,\varepsilon} \oplus U_g \oplus U_\beta$, où $U_\beta = E_1 = \mathbb{C}$.

Alors :

$$\forall X \in U_{f,\gamma_1,\gamma_2}, \quad Ad_u(X) = \begin{bmatrix} sYs^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in U_{f,\gamma_1,\gamma_2}$$

$$\forall X \in U_{\delta,\varepsilon}, \quad Ad_u(X) = \begin{bmatrix} 0 & s\omega^3 v \\ -\omega^{-3}s^{-1}{}^t \bar{v} & 0 \end{bmatrix} \in U_{\delta,\varepsilon}$$

$$\forall X \in U_g, \quad Ad_u(X) = X \in U_g$$

$$\forall X \in U_\beta, \quad Ad_u(X) = X \in U_\beta$$

(ce qui montre au passage que les sous-espaces $U_{f,\gamma_1,\gamma_2}, U_{\delta,\varepsilon}, U_g$ et U_β sont $Ad(U(2))$ -invariants).

Or :

$U_{f,\gamma_1,\gamma_2} = \mathfrak{su}(2) = \mathbb{R}^3$, où $\mathfrak{su}(2)$ porte la représentation adjointe de $SU(2)$ (qui est une sous-représentation irréductible de la représentation adjointe de $U(2)$),

$U_{\delta,\varepsilon} = \mathbb{C}^2$, où \mathbb{C}^2 porte la représentation $\rho^{\frac{1}{2}} \otimes (\rho_3 \oplus \rho_3)$, $\rho^{\frac{1}{2}}$ étant la représentation irréductible standard de $SU(2)$ sur \mathbb{C}^2 , et ρ_3 la représentation irréductible $z \mapsto e^{3ia}z$ de $U(1)$ sur \mathbb{C} ,

$U_{g,\beta} = (\mathbb{R})^3$, où \mathbb{R} porte la représentation triviale de $U(2)$.

Finalement, la décomposition de \mathfrak{m} en représentations réelles irréductibles sous l'action de $N(H)|H$ s'écrit :

$$\mathfrak{m} = \mathbb{R}^3 \oplus \mathbb{C}^2 \oplus (\mathbb{R})^3$$

Etudions maintenant la décomposition de $S_2^H(\mathfrak{m})$ sous l'action de $N(H)|H$. On a vu que :

$$S_2^H(\mathfrak{m}) = MS_4(\mathbb{R}) \oplus MH_1(\mathbb{C}) \oplus MH_2(\mathbb{C})$$

L'action de \mathbb{Z}_2 étant triviale, il suffit d'étudier l'action du centralisateur $U(2)$. Soit :

$$u = \begin{bmatrix} s\omega & 0 \\ 0 & \omega^{-2} \end{bmatrix} \in U(2)$$

Comme $s \in SU(2)$, $Ad_s \in GL(\mathfrak{su}(2))$. En fait, $Ad_s \in SO(\mathfrak{su}(2)) = SO(\mathbb{R}^3) = SO(3)$.

Dans ces conditions, si :

$$A = \begin{bmatrix} A' & w \\ {}^t_w & d \end{bmatrix} \in MS_4(\mathbb{R}) \text{ avec } A' \in MS_4(\mathbb{R}), w \in \mathbb{R}^3 \text{ et } d \in \mathbb{R}$$

- $\lambda \in \mathbb{R}$
- $B \in MH_2(\mathbb{C})$

alors :

$$\begin{aligned} Ad_u(A) &= \begin{bmatrix} Ad_s A' Ad_s^{-1} & Ad_s w \\ {}^t_w Ad_s^{-1} & d \end{bmatrix} \\ - Ad_u(\lambda) &= \lambda \\ - Ad_u(B) &= sBs^{-1} \end{aligned}$$

Grâce à l'identification :

$$\begin{array}{ccc} i\mathfrak{su}(2) & \longrightarrow & \mathfrak{su}(2) \\ X & \longrightarrow & iX \end{array}$$

on a : $MS_3(\mathbb{R}) = (MS_3(\mathbb{R}) \cap \mathfrak{sl}_3(\mathbb{R})) \oplus \mathbb{R}$ et $MH_2(\mathbb{C}) = i\mathfrak{su}_2 \oplus \mathbb{R}$,

et donc la décomposition $S_2^H(\mathfrak{m}) = (MS_3(\mathbb{R}) \oplus \mathbb{R}^3 \oplus \mathbb{R}) \oplus \mathbb{R} \oplus MH_2(\mathbb{C})$ entraîne :

$$\begin{aligned} S_2^H(\mathfrak{m}) &= ((MS_3(\mathbb{R}) \cap \mathfrak{sl}_3(\mathbb{R})) \oplus \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}) \oplus \mathbb{R} \oplus (i\mathfrak{su}_2 \oplus \mathbb{R}) \\ &= \mathbb{R}^5 \oplus (\mathbb{R}^4 \oplus (\mathbb{R}^3)^2) \end{aligned}$$

Le terme \mathbb{R}^3 figurant dans l'avant-dernière expression est la représentation adjointe de $SU(2)$, et les quatre \mathbb{R} correspondent respectivement à $tr(A')$, d , λ et $tr(B)$.

3.4.5 Exemple de Manton avec $R = SU(5)$

Ce second modèle consiste à prendre : $R = SU(5)$, $S = U(2, \mathbb{H}) = Sp(2)$ et $I = SU(2) = Sp(1)$.

Les fibres de E sont alors des copies de la sphère $\mathbb{S}^7 = Sp(2)/Sp(1)$.

On a alors $G = Sp(2) \times SU(5)$ et $H = \{(s, \lambda(s)) ; s \in SU(2)\}$.

On peut choisir $\mathfrak{m} \subset \mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{su}(5)$ de façon à avoir une décomposition réductive $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. On a ici $\dim \mathfrak{g} = 10 + 24 = 34$, $\dim \mathfrak{h} = 3$ et $\dim \mathfrak{m} = 31$.

On vérifie que $N(I) = SU(2) \times SU(2)$ et $Z = SU(3) \times U(1)$, de sorte que $K = N(I)|I = SU(2)$.

Localement, on a : $N(H)|H = K \times Z = SU(3) \times SU(2) \times U(1)$. Par suite, $\dim \mathfrak{n}(\mathfrak{h})|\mathfrak{h} = 8 + 3 + 1 = 12$.

On peut alors calculer le "nombre" de champs de Higgs. Tous calculs faits, on trouve [JP] :

$$\dim \mathcal{L}_0(\mathfrak{k}, \mathfrak{r}) = 30$$

$$\dim \mathcal{L}_0(\mathfrak{l}, \mathfrak{r}) = 12$$

3.5 Calcul des Lagrangiens

Nous présentons dans ce paragraphe les expressions des Lagrangiens qui interviennent dans les processus de réduction dimensionnelle étudiés dans les paragraphes précédents. Les symboles C_{ij}^k désignent les constantes de structure de l'algèbre de Lie \mathfrak{g} relativement à une base (T_i) précisée ci-dessous.

Nous écrirons au paragraphe 3.5.1 le Lagrangien d'Einstein d'une métrique G -invariante sur un espace à fibres homogènes $(E, M, G/H)$, en fonction des champs réduits sur la base.

Nous donnerons ensuite au paragraphe 3.5.2 l'expression du Lagrangien d'un champ d'Einstein-Yang-Mills S -invariant sur $(E, M, S/I)$, toujours en fonction des quantités obtenues sur la base.

Enfin, une interprétation des potentiels obtenus en lien avec les champs de Higgs est esquissée au dernier paragraphe.

3.5.1 Lagrangien d'Einstein

Lemma 3.5.1 *Avec les notations du chapitre 1, on considère une décomposition réductive $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ de l'algèbre de Lie du groupe compact G . Soit $(T_{\tilde{\alpha}}, T_\alpha)$ une base de \mathfrak{g} adaptée à cette décomposition, et posons $g_{\alpha\beta} = f(T_\alpha, T_\beta)$, où f est un produit scalaire $Ad(H)$ -invariant sur \mathfrak{m} . Alors la courbure scalaire $\rho(G/H)$ pour la métrique G -invariante associée à f est constante et vaut :*

$$\rho(G/H) = -g^{\alpha\alpha'} \left(\frac{1}{2} C_{\alpha\beta}^\gamma C_{\alpha'\gamma}^\beta + \frac{1}{4} g^{\beta\beta'} g_{\gamma\gamma'} C_{\alpha\beta}^\gamma C_{\alpha'\beta'}^{\gamma'} + C_{\alpha\beta}^{\tilde{\gamma}} C_{\alpha'\tilde{\gamma}}^\beta \right)$$

Theorem 3.5.2 *En reprenant toutes les notations du chapitre 1, soit $(T_{\tilde{\alpha}}, T_{\tilde{a}}, T_a)$ une base de \mathfrak{g} adaptée à la décomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{l}$. Soit W un ouvert trivialisant domaine d'une carte (x^μ) de M , $\sigma : W \rightarrow P$ une section locale trivialisante de P , $A = \sigma^*\omega$, $F = \sigma^*D\omega$, avec $A = A_\mu dx^\mu$, $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, $A_\mu = A_{\mu}^{\tilde{a}} T_{\tilde{a}}$, $F_{\mu\nu} = F_{\mu\nu}^{\tilde{a}} T_{\tilde{a}}$. On pose $h_{\alpha\beta}(x) = h_x(\sigma(x))(e_\alpha(\sigma(x)), e_\beta(\sigma(x))) \quad \forall x \in W$, e_α étant le champ fondamental associé à T_α par l'action de G . Enfin, $D_\mu h_{\alpha\beta} = \partial_\mu h_{\alpha\beta} + C_{\alpha\tilde{a}}^\delta A_{\mu}^{\tilde{a}} h_{\delta\beta} + C_{\beta\tilde{a}}^\delta A_{\mu}^{\tilde{a}} h_{\delta\alpha}$ est la dérivée covariante de $h_{\alpha\beta}$ par rapport à ω , et ∇_μ est la dérivée covariante par rapport à la connexion de Levi-Civita sur M . Dans ces conditions, on a :*

$$\rho(E) = \rho(M) + \rho(G/H) + YM(M, N|H) + EC(M, G/H)$$

où :

$$YM(M, N|H) = -\frac{1}{4} F_{\mu\nu}^{\tilde{a}} F_{\tilde{a}}^{\mu\nu}$$

$$EC(M, G/H) = -\frac{1}{4}h^{\alpha\beta}h^{\gamma\delta}(D_\mu h_{\alpha\gamma}D^\mu h_{\beta\delta} + D_\mu h_{\alpha\beta}D^\mu h_{\gamma\delta}) - \nabla_\mu(h^{\alpha\beta}D^\mu h_{\alpha\beta})$$

$$\rho(G/H) = -h^{\alpha\alpha'}(\frac{1}{2}C_{\alpha\beta}^\gamma C_{\alpha'\gamma}^\beta + \frac{1}{4}h^{\beta\beta'}h_{\gamma\gamma'}C_{\alpha\beta}^\gamma C_{\alpha'\beta'}^{\gamma'} + C_{\alpha\beta}^{\tilde{\gamma}}C_{\alpha'\tilde{\gamma}}^\beta)$$

Dans le cas particulier où la réduction dimensionnelle s'effectue sur un fibré principal (P, M, G) (ce qui revient à prendre $H = \{e\}$ dans ce qui précède), on a :

$$\rho(P) = \rho(M) + \rho(G) + YM(M, G) + EC(M, G)$$

Si de plus, chaque h_x est la copie d'une métrique riemannienne bi-invariante sur G , alors $D_\mu h_{\alpha\beta} = 0$, et par suite $EC(M, G) = 0$. Dans ce cas, $\rho(G)$ est une constante.

3.5.2 Lagrangien d'Einstein-Yang-Mills

On reprend toutes les notations du chapitre 2. On considère alors des bases adaptées aux décompositions des algèbres de Lie qu'on a vues, ainsi que des systèmes de coordonnées et jauge locales appropriées. Les formules que nous allons énoncer peuvent être établies dans ces systèmes, cependant nous passons sur les détails pour lesquels nous renvoyons le lecteur à [CJ2].

On a vu que le couple S -invariant (g, ω) accompagné d'une métrique riemannienne bi-invariante sur R , pouvait être vu comme le résultat de la réduction dimensionnelle d'une métrique R -invariante (en fait G -invariante) sur le fibré principal (U, E, R) . D'après le paragraphe précédent, on a alors :

$$\rho(U) = \rho(E) + \rho(R) + YM(E, R)$$

puisque l'on est alors dans le cas très particulier d'une métrique bi-invariante sur la fibre (d'où $EC(E, R) = 0$).

Ecrivons à présent le Lagrangien d'Einstein de g lorsqu'on la réduit de E à M :

$$\rho(E) = \rho(M) + \rho(S/I) + YM(M, K) + EC(M, S/I)$$

Ecrivons de même le Lagrangien de \tilde{g} lorsqu'on la réduit de U à M :

$$\rho(U) = \rho(M) + \rho(G/H) + YM(M, N(H)|H) + EC(M, G/H)$$

Ecrivons enfin le Lagrangien de \tilde{h}_x lorsqu'on la réduit de G/H à S/I :

$$\rho(G/H) = \rho(S/I) + \rho(R) + YM(S/I, R)$$

On a également $EC(M, G/H) = EC(M, S/I) + EC(\phi)$, avec :

$$EC(\phi) = -\frac{1}{2}h^{\alpha\beta}\gamma^{\mu\nu}k_{ij}D_\mu\phi_\alpha^iD_\nu\phi_\beta^j$$

et $YM(M, N(H)|H) = YM(M, K) + YM(M, Z) + \Delta$, avec :

$$\Delta = -\frac{1}{4}\gamma^{\mu\mu'}\gamma^{\nu\nu'}k_{\tilde{i}\tilde{j}}\phi_{\tilde{\alpha}}^{\tilde{i}}F_{\mu\nu}^{\tilde{\alpha}}(\phi_{\tilde{\beta}}^{\tilde{i}}F_{\mu'\nu'}^{\tilde{\beta}} - 2F_{\mu'\nu'}^{\tilde{j}})$$

De ces relations, on peut d'abord déduire le Lagrangien du couple (g, ω) lorsqu'on le réduit de E à M :

$$\begin{aligned}
EYM &= \rho(E) + YM(E, R) \\
&= \rho(U) - \rho(R) \\
&= (\rho(M) + \rho(G/H) + YM(M, N(H)|H) + EC(M, G/H)) - \rho(R) \\
&= \rho(M) + (\rho(S/I) + \rho(R) + YM(S/I, R)) + YM(M, N(H)|H) + EC(M, G/H) - \rho(R) \\
&= \rho(M) + \rho(S/I) + YM(S/I, R) + YM(M, N(H)|H) + EC(M, G/H) \\
&= \rho(M) + \rho(S/I) + YM(S/I, R) + YM(M, N(H)|H) + EC(M, S/I) + EC(\phi) \\
&= \rho(M) + YM(M, N(H)|H) + EC(h) - V(h) + EC(\phi) - V(\phi)
\end{aligned}$$

où :

$$\begin{aligned}
EC(h) &= EC(M, S/I) \\
V(h) &= -\rho(S/I) \\
V(\phi) &= -YM(S/I, R)
\end{aligned}$$

On peut déduire ensuite le Lagrangien de Yang-Mills de ω seule lorsqu'on la réduit de E à M :

$$\begin{aligned}
YM &= EYM(E) - \rho(E) \\
&= \rho(M) + \rho(S/I) + YM(S/I, R) + YM(M, N(H)|H) + EC(M, S/I) + EC(\phi) - \rho(E) \\
&= YM(M, Z) + \Delta + YM(S/I, R) + EC(\phi) \\
&= YM(M, Z) + EC(\phi) - V(\phi) + \Delta
\end{aligned}$$

Notons que :

$$V(\phi) = \frac{1}{4} h^{\alpha\alpha'} h^{\beta\beta'} k_{ij} F_{\alpha\beta}^i F_{\alpha'\beta'}^j$$

3.5.3 Lien avec les champs de Higgs et conclusion

On a vu au chapitre 3.3 qu'il y avait essentiellement une équivalence entre les champs scalaires obtenus en réduisant un champ d'Einstein-Yang-Mills (g, ω) invariant par un groupe S , et ceux obtenus en réduisant une métrique \tilde{g} invariante par un groupe G contenant S .

En pratique, et avec les notations des chapitres précédents,

- soit on réduit d'abord \tilde{g} de U à E , obtenant ainsi le triplet (g, ω, k) , puis on réduit (g, ω) de E à M , obtenant le quadruplet $(\gamma, h_x, \alpha, \phi(x))$ (et donc finalement les champs scalaires $h_x, \phi(x)$ et k),
- soit on réduit directement \tilde{g} de U à M , obtenant le triplet $(\gamma, \alpha, \tilde{h}_x)$ et nous avons vu qu'alors, les champs scalaires \tilde{h}_x correspondent aux champs scalaires $h_x, \phi(x)$ et k .

On en déduit que la réduction dimensionnelle d'une métrique G -invariante (avec éventuellement un groupe G non simple) est en définitive suffisante, pour déterminer à elle seule, tous les champs scalaires susceptibles d'être obtenus.

Il s'avère alors intéressant de classifier ces champs scalaires qui apparaissent (par l'intermédiaire des représentations qui leurs sont associées), de pouvoir déterminer ainsi ceux qui sont physiquement acceptables, et de comprendre les mécanismes mis en jeu par ces champs, notamment les brisures de symétries.

En particulier, on aimerait identifier dans l'ensemble de ces champs, ceux qui correspondent aux champs de Higgs du modèle standard, responsables de la brisure spontanée de la symétrie électrofaible.

Pour cela, on essaie de calculer les potentiels des champs scalaires, en vue de savoir s'ils ont ou non "la bonne forme", celle présentée par les champs de Higgs qui brisent la symétrie.

Plus généralement, on envisage l'existence d'un lien explicite entre les représentations obtenues pour les champs, et le potentiel qu'ils déterminent. Ce lien doit permettre de ramener la prédiction de la dynamique des champs scalaires, à des critères purement algébriques de la théorie des groupes.

Ceci ouvre la voie à une étude plus approfondie, incluant aussi le secteur fermionique et la supersymétrie, étude que nous souhaitons poursuivre en thèse...

Chapitre 4

Klein geometries

4.1 Klein geometries

Definition 4.1.1 A Klein geometry is a pair (G, H) , where G is a Lie group and H is a closed subgroup of G such that the homogenous space G/H is connected.

G/H denotes the space of orbits under the action of H on G by right translations, that is, $G/H = \{aH ; a \in G\}$. We set $\overset{\circ}{M} = G/H$ so that $\overset{\circ}{M}$ is a smooth manifold equipped with a transitive left action of G . We set $x_0 = eH = H$, which is viewed as the origin of the homogenous space $\overset{\circ}{M}$. The canonical projection is the surjective map $\pi : G \longrightarrow \overset{\circ}{M}$ defined by : $\pi(a) = ax_0$. Finally H is nothing but the stabilizer of x_0 under the action of G , that is, $H = \{a \in G / ax_0 = x_0\}$.

The group G is called the **principal group of the Klein geometry** (G, H) , and the homogenous space $\overset{\circ}{M}$ is called the **space of the Klein geometry** (G, H) .

Remark 4.1.2 If one starts with a connected smooth manifold $\overset{\circ}{M}$ and considers a group G acting transitively (say by the left) on $\overset{\circ}{M}$, then by choosing a point $x_0 \in \overset{\circ}{M}$, one obtains a closed subgroup H of G by taking the stabilizer of x_0 under the action of G . Thus, one obtains a Klein geometry (G, H) whose space G/H is naturally identified to $\overset{\circ}{M}$ via the surjective map $\pi : G \longrightarrow M$ defined by : $\pi(a) = ax_0$.

Definition 4.1.3

1. The **kernel of a Klein geometry** (G, H) is the largest normal subgroup of G contained in H .
2. A Klein geometry (G, H) is said to be **effective** (resp. **locally effective**) if its kernel is $\{1\}$ (resp. is discrete).
3. A Klein geometry (G, H) is said to be **reductive** if there exists a vector subspace \mathfrak{m} of the Lie algebra \mathfrak{g} such that : $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad(H)\mathfrak{m} \subset \mathfrak{m}$ (reductive decomposition).

To each Klein geometry (G, H) , we may associate a principal fibre bundle. The total space of this bundle is the group G itself, which is realized as a principal bundle over $\overset{\circ}{M}$ with structural group H , the fibration being the projection $\pi : G \longrightarrow \overset{\circ}{M}$. The principal fibre bundle $(G, \overset{\circ}{M}, H)$ will be called **the canonical bundle of the Klein geometry**.

Let (G, H) be a reductive Klein geometry, and choose a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We may consider the vector bundle $\overset{\circ}{V}$ associated to the canonical bundle of (G, H) by the restricted adjoint representation $Ad : H \longrightarrow GL(\mathfrak{m})$. That is, we set :

$$\mathring{\mathbb{V}} = G \times_{Ad} \mathfrak{m}.$$

Denote by $\mathcal{F}(\mathring{\mathbb{V}})$ the linear frame bundle of $\mathring{\mathbb{V}}$. $\mathcal{F}(\mathring{\mathbb{V}})$ is a principal fibre bundle over \mathring{M} with structural group $GL(\mathfrak{m})$, and by "frame at a point $x \in \mathring{M}$ " we mean an isomorphism from \mathfrak{m} to $\mathring{\mathbb{V}}_x$. There is a natural map $\lambda : G \longrightarrow \mathcal{F}(\mathring{\mathbb{V}})$ which associates to each $a \in G$ the frame $\lambda(a) : \mathfrak{m} \longrightarrow \mathring{\mathbb{V}}_{ax_0}$ defined by $\lambda(a)(X) = (a, X)H$.

Remark 4.1.4 *If the restricted adjoint representation $Ad : H \longrightarrow GL(\mathfrak{m})$ is faithful, then one can check that λ is in fact an embedding of G in $\mathcal{F}(\mathring{\mathbb{V}})$.*

It is easy to see that $(\lambda(G), \mathring{M}, Ad(H))$ is a principal subbundle of the linear frame bundle $(\mathcal{F}(\mathring{\mathbb{V}}), \mathring{M}, GL(\mathfrak{m}))$, that is, $\lambda(G)$ defines an $Ad(H)$ -structure on the vector bundle $\mathring{\mathbb{V}}$.

Suppose now that we are given an $Ad(H)$ -invariant scalar product on \mathfrak{m} . Then $Ad(H) \subset O(\mathfrak{m})$, and the $Ad(H)$ -structure $\lambda(G)$ induces naturally an $O(\mathfrak{m})$ -structure on the vector bundle $\mathring{\mathbb{V}}$, that is, a *bundle metric* on $\mathring{\mathbb{V}}$. More generally, consider any $Ad(H)$ -invariant structure on the vector space \mathfrak{m} . Then the group of automorphisms of this structure will contain $Ad(H)$, and this allows to transport to the structure to the vector bundle $\mathring{\mathbb{V}}$ (and then to the tangent bundle $T\mathring{M}$, as we are going to see below).

On the group G , the left translation $l : G \longrightarrow G$ induces a natural \mathfrak{g} -valued one-form : the **left Maurer-Cartan form** ${}^l\dot{\theta}$, defined by : ${}^l\dot{\theta}_a = T_a l_{a^{-1}}$.

Similarly, the right translation $r : G \longrightarrow G$ induces a natural \mathfrak{g} -valued one-form on G : the **right Maurer-Cartan form** ${}^r\dot{\theta}$, defined by : ${}^r\dot{\theta}_a = T_a r_{a^{-1}}$.

In the remainder of this chapter, we shall be concerned only with the left Maurer-Cartan form, which we denote simply by $\dot{\theta}$.

Let (G, H) be a Klein geometry. The left Maurer-Cartan form $\dot{\theta}$ on G satisfies the following properties :

1. For each $a \in G$, the linear map $\dot{\theta}_a : T_a G \longrightarrow \mathfrak{g}$ is an isomorphism.
2. $((r_h)^*\dot{\theta})_a = Ad_{h^{-1}} \circ \dot{\theta}_a$ for any $a \in G$ and $h \in H$.
3. $\dot{\theta}(\tilde{X}) = X$ for any $X \in \mathfrak{h}$, where \tilde{X} denotes the right fundamental vector field associated to X by the action of H on G by right translations.

Remark 4.1.5 .

- In 2., *Ad* refers to the unrestricted adjoint representation $(Ad : G \longrightarrow GL(\mathfrak{g}))$, and the property holds in fact for any $a \in G$ and $h \in G$. Also, 3. remains clearly true for any $X \in \mathfrak{g}$, for \tilde{X} being the right fundamental vector field associated to X by the action of G on itself by right translations (which is also the left-invariant vector field associated to X). However, as they are stated, properties 1. 2. and 3. will suggest the definition of an object which generalizes the Maurer-Cartan form, namely the Cartan connection that we will present in the next chapter.
- The left Maurer-Cartan form has also the important property to be **left-invariant**, that is $(l_a)^*\dot{\theta} = \dot{\theta}$ for all $a \in G$.

Let (G, H) be a reductive Klein geometry, and choose a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\pi_{\mathfrak{h}} : \mathfrak{g} \longrightarrow \mathfrak{h}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \longrightarrow \mathfrak{m}$ the corresponding projectors, and let $\mathring{\mathbb{V}} = G \times_{Ad} \mathfrak{m}$. For any $a \in G$, we set $\mathring{\omega}_a = \pi_{\mathfrak{h}} \circ \mathring{\theta}_a$ and $\mathring{e}_a = \pi_{\mathfrak{m}} \circ \mathring{\theta}_a$.

Proposition 4.1.6 .

1. $\mathring{\omega}$ is a connection one-form on the canonical bundle (G, \mathring{M}, H) .
2. \mathring{e} is a tensorial (i.e. an H -equivariant and horizontal) \mathfrak{m} -valued one-form on the canonical bundle (G, \mathring{M}, H) (i.e. $\mathring{e} \in \bar{\Lambda}_H^1(G, \mathfrak{m})$).

$\mathring{\omega}$ is called the **canonical connection** of the reductive Klein geometry (G, H) .

\mathring{e} is called the **canonical soldering form** of the reductive Klein geometry (G, H) .

Remark 4.1.7 .

- It is well-known that $\bar{\Lambda}_H^1(G, \mathfrak{m}) \simeq \Lambda^1(\mathring{M}, \mathring{\mathbb{V}})$. So we may view \mathring{e} as a one-form on \mathring{M} with values in the vector bundle $\mathring{\mathbb{V}}$. It is easy then to show that $\mathring{e} : T\mathring{M} \longrightarrow \mathring{\mathbb{V}}$ is in fact an isomorphism of vector bundles.
- It is well-known that $Ad(H)$ -invariant scalar products on \mathfrak{m} are in one-to-one correspondence with G -invariant metrics on \mathring{M} . In fact, this has a natural interpretation in our setting. We have already said that any $Ad(H)$ -invariant scalar product on \mathfrak{m} naturally gives rise to a bundle metric on $\mathring{\mathbb{V}}$. Using the isomorphism $\mathring{e} : T\mathring{M} \longrightarrow \mathring{\mathbb{V}}$, we deduce a metric \mathring{g} on \mathring{M} which is easily shown to be G -invariant.
- Since the Maurer-Cartan form $\mathring{\theta}$ is left-invariant, the same can be said for the canonical connection and soldering form. So we have : $(l_a)^* \mathring{\omega} = \mathring{\omega}$ and $(l_a)^* \mathring{e} = \mathring{e}$ for all $a \in G$.
- Since the canonical connection $\mathring{\omega}$ is left-invariant, it follows from Wang's theorem that $\mathring{\omega}$ is entirely characterized by an $Ad(H)$ -equivariant linear map $\varphi_{can} : \mathfrak{m} \longrightarrow \mathfrak{h}$. Here, φ_{can} is in fact the restriction to \mathfrak{m} of the projector $\pi_{\mathfrak{h}}$.

Definition 4.1.8 .

1. The **canonical curvature** of the reductive Klein geometry (G, H) is the curvature of the canonical connection $\mathring{\omega}$, that is the two-form $\mathring{\Omega} \in \bar{\Lambda}_H^2(G, \mathfrak{h})$ defined by : $\mathring{\Omega} = \mathcal{D}^{\mathring{\omega}} \mathring{\omega}$.
2. The **canonical torsion** of the reductive Klein geometry (G, H) is the two-form $\mathring{T} \in \bar{\Lambda}_H^2(G, \mathfrak{m})$ defined by : $\mathring{T} = \mathcal{D}^{\mathring{\omega}} \mathring{e}$.

Theorem 4.1.9 The canonical curvature and torsion satisfy the following equations, called the **structure equations** of the reductive Klein geometry :

$$\left\{ \begin{array}{lcl} \mathring{\Omega} & = & d\mathring{\omega} + \frac{1}{2}[\mathring{\omega}, \mathring{\omega}] = -\frac{1}{2}[\mathring{e}, \mathring{e}]_{\mathfrak{h}} \\ \mathring{T} & = & d\mathring{e} + [\mathring{\omega}, \mathring{e}] = -\frac{1}{2}[\mathring{e}, \mathring{e}]_{\mathfrak{m}} \end{array} \right.$$

Proof : To obtain the above equations, it is sufficient to project the Maurer-Cartan structure equation on \mathfrak{h} and \mathfrak{m} :

$$d\mathring{\theta} + \frac{1}{2}[\mathring{\theta}, \mathring{\theta}] = 0$$

$$\begin{aligned}
&\iff d\dot{\omega} + d\dot{e} + \frac{1}{2}[\dot{\omega} + \dot{e}, \dot{\omega} + \dot{e}] = 0 \\
&\iff d\dot{\omega} + \frac{1}{2}[\dot{\omega}, \dot{\omega}] + d\dot{e} + [\dot{\omega}, \dot{e}] + \frac{1}{2}[\dot{e}, \dot{e}] = 0 \\
&\iff \ddot{\Omega} + \ddot{T} + \frac{1}{2}[\ddot{e}, \ddot{e}]_{\mathfrak{h}} + \frac{1}{2}[\ddot{e}, \ddot{e}]_{\mathfrak{m}} = 0 \\
&\iff (\ddot{\Omega} + \frac{1}{2}[\ddot{e}, \ddot{e}]_{\mathfrak{h}}) + (\ddot{T} + \frac{1}{2}[\ddot{e}, \ddot{e}]_{\mathfrak{m}}) = 0
\end{aligned}$$

The first (resp. second) term in brackets takes its values in \mathfrak{h} (resp. in \mathfrak{m}). Therefore both terms vanish. \square

Remark 4.1.10 .

- If \mathring{M} is a symmetric space, we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. In this case, the canonical torsion vanishes.
- The canonical curvature is characterized by the $Ad(H)$ -equivariant bilinear map $\mathcal{R}_{can} : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{h}$ defined by : $\mathcal{R}_{can}(X, Y) = -[X, Y]_{\mathfrak{h}}$.
- The canonical torsion is characterized by the $Ad(H)$ -equivariant bilinear map $\mathcal{T}_{can} : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ defined by : $\mathcal{T}_{can}(X, Y) = -[X, Y]_{\mathfrak{m}}$.

Now we define the infinitesimal version of Klein geometries.

Definition 4.1.11 A Klein pair is a pair $(\mathfrak{g}, \mathfrak{h})$, where \mathfrak{g} is a Lie algebra and \mathfrak{h} is a Lie subalgebra of \mathfrak{g} .

Definition 4.1.12 .

1. The kernel of a Klein pair $(\mathfrak{g}, \mathfrak{h})$ is the largest ideal of \mathfrak{g} contained in \mathfrak{h} .
2. A Klein pair $(\mathfrak{g}, \mathfrak{h})$ is said to be **effective** if its kernel is $\{0\}$.
3. A Klein pair $(\mathfrak{g}, \mathfrak{h})$ is said to be **reductive** if there exists a vector subspace \mathfrak{m} of the Lie algebra \mathfrak{g} such that : $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ (reductive decomposition).

For any Klein geometry (G, H) , if \mathfrak{g} and \mathfrak{h} are the Lie algebras of G and H respectively, then $(\mathfrak{g}, \mathfrak{h})$ is a Klein pair called the **Klein pair associated to the Klein geometry** (G, H) . It is easy to see that the Lie algebra of the kernel of a Klein geometry (G, H) is the kernel of the Klein pair $(\mathfrak{g}, \mathfrak{h})$ associated to (G, H) . In particular, if (G, H) is locally effective, then $(\mathfrak{g}, \mathfrak{h})$ is effective.

4.2 Fixing a gauge

Let W be a G -invariant open set in \mathring{M} , and $s : W \longrightarrow G$ a local cross-section trivializing the canonical bundle (G, \mathring{M}, H) . We say that the gauge s is *invariant* or *preserved* by the transformations of G if $s(W) \subset G$ is invariant under the action of G on itself by left translations (or, equivalently, if s is G -equivariant).

In general, a gauge s has no reason to be invariant under the action of G : if $a \in G$ and $x \in W$, we have generally $as(x) \neq s(ax)$ (and even : $as(x) \notin s(W)$).

However, for any $a \in G$ and $x \in W$, we have : $\pi(as(x)) = a \pi(s(x)) = ax = \pi(s(ax))$. Therefore, for any $a \in G$, there exists a unique function ${}^s h_a : W \longrightarrow H$ (called a *compensating gauge transformation*) such that : $as(x) = s(ax) {}^s h_a(x)$. The role of ${}^s h_a$ is somehow to "bring back" to the gauge s . It is not difficult to check that for each $x \in W$, the map ${}^s \Lambda_x : G \longrightarrow H$ defined by : ${}^s \Lambda_x(a) = {}^s h_a(x)$ is a group homomorphism.

At the infinitesimal level, the action of \mathfrak{g} on $\overset{\circ}{M}$ is given by the fundamental vector fields. For each $X \in \mathfrak{g}$, the **fundamental vector field associated to X** by the action of G on $\overset{\circ}{M}$ is the vector field $\tilde{X} \in \Gamma(T\overset{\circ}{M})$ defined by : $\tilde{X}_x = T_e \pi_x$, where $\pi_x : G \longrightarrow \overset{\circ}{M}$ is the map defined by : $\pi_x(a) = ax$. Equivalently, \tilde{X} is defined by : $\tilde{X}_x = \frac{d}{dt}(\exp(tX)x)|_{t=0}$.

Remark 4.2.1 Let \dot{g} be the G -invariant metric on $\overset{\circ}{M}$ associated to an $Ad(H)$ -invariant scalar product on \mathfrak{m} . At the infinitesimal level, the G -invariance of the metric \dot{g} is equivalent to the following condition :

"For any $X \in \mathfrak{g}$, the fundamental vector field \tilde{X} associated to X is a **Killing vector field** for \dot{g} ".

By definition of Killing vector fields, this means that $L_{\tilde{X}} \dot{g} = 0$, that is :

$$\tilde{X} \dot{g}(Y, Z) = \dot{g}([\tilde{X}, Y], Z) + \dot{g}(Y, [\tilde{X}, Z]) \quad \text{for any } Y, Z \in \Gamma(T\overset{\circ}{M}) \quad (\text{or equivalently} :$$

$$\dot{g}(\overset{\circ}{\nabla}_Y \tilde{X}, Z) + \dot{g}(Y, \overset{\circ}{\nabla}_Z \tilde{X}) = 0 \quad \text{for any } Y, Z \in \Gamma(T\overset{\circ}{M}), \text{ where } \overset{\circ}{\nabla} \text{ is the Levi-Civita connection of } \dot{g}.$$

Now we can give the infinitesimal version of the compensating gauge transformation ${}^s h_a$. For any $X \in \mathfrak{g}$, the *compensating infinitesimal gauge transformation* is the function ${}^s Y_X : W \longrightarrow \mathfrak{h}$ defined by : ${}^s Y_X(x) = \frac{d}{dt}({}^s h_{\exp(tX)}(x))|_{t=0}$. Notice that $\varphi : \mathbb{R} \longrightarrow H$ defined by : $\varphi(t) = {}^s h_{\exp(tX)}(x)$ is a one-parameter subgroup of H ($\varphi(t) = {}^s \Lambda_x(\exp(tX))$). The infinitesimal generator of this one-parameter subgroup is of course ${}^s Y_X(x)$, so ${}^s h_{\exp(tX)}(x) = \exp(t {}^s Y_X(x))$.

Later, it will be useful to have an expression of the canonical connection and soldering form of (G, H) in the gauge s , especially when applying these one-forms to a fundamental vector field. This is done in the following proposition, for the proof of which we need first the following lemma :

Lemma 4.2.2 For any $X \in \mathfrak{g}$ and $x \in W$, we have : $T_x s(\tilde{X}_x) = X s(x) - s(x) {}^s Y_X(x)$.

Proof : We know that $T_x s : T_x \overset{\circ}{M} \longrightarrow T_{s(x)} G$, and the above relation means precisely : $T_x s(\tilde{X}_x) = T_1 r_{s(x)}(X) - T_1 l_{s(x)}({}^s Y_X(x))$. We have : $\exp(tX)s(x) = s(\exp(tX)x) {}^s h_{\exp(tX)}(x)$, that is : $s(\exp(tX)x) = \exp(tX)s(x) {}^s h_{\exp(tX)}(x)^{-1}$. By taking the derivative of both sides with respect to t at $t = 0$, we obtain the desired relation. \square

We define ${}^s \overset{\circ}{\theta} = s^* \overset{\circ}{\theta} \in \Lambda^1(W, \mathfrak{g})$, ${}^s \overset{\circ}{\omega} = s^* \overset{\circ}{\omega} \in \Lambda^1(W, \mathfrak{h})$ and ${}^s \overset{\circ}{e} = s^* \overset{\circ}{e} \in \Lambda^1(W, \mathfrak{m})$.

Proposition 4.2.3 For any $X \in \mathfrak{g}$ and $x \in W$, we have :

$${}^s\dot{\omega}_x(\tilde{X}_x) = [Ad_{s(x)^{-1}}(X)]_{\mathfrak{h}} - {}^sY_X(x)$$

$${}^s\dot{e}_x(\tilde{X}_x) = [Ad_{s(x)^{-1}}(X)]_{\mathfrak{m}}$$

$$\begin{aligned} \text{Proof : } & \overset{\circ}{\theta} = \overset{\circ}{\omega} + \overset{\circ}{e} \text{ implies } {}^s\overset{\circ}{\theta} = {}^s\overset{\circ}{\omega} + {}^s\overset{\circ}{e}. \text{ In particular, } {}^s\dot{\omega}_x(\tilde{X}_x) + {}^s\dot{e}_x(\tilde{X}_x) = {}^s\overset{\circ}{\theta}_x(\tilde{X}_x) \\ & = (\overset{\circ}{\theta})_{s(x)}(\mathrm{T}_x s(\tilde{X}_x)) \\ & = (\overset{\circ}{\theta})_{s(x)}(\mathrm{T}_1 r_{s(x)}(X) - \mathrm{T}_1 l_{s(x)}({}^sY_X(x))) \\ & = \mathrm{T}_{s(x)}l_{s(x)^{-1}}(\mathrm{T}_1 r_{s(x)}(X) - \mathrm{T}_1 l_{s(x)}({}^sY_X(x))) \\ & = \mathrm{T}_{s(x)}l_{s(x)^{-1}} \circ \mathrm{T}_1 r_{s(x)}(X) - \mathrm{T}_{s(x)}l_{s(x)^{-1}} \circ \mathrm{T}_1 l_{s(x)}({}^sY_X(x)) \\ & = \mathrm{Ad}_{s(x)^{-1}}(X) - {}^sY_X(x) \\ & = - {}^sY_X(x) + [\mathrm{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{h}} + [\mathrm{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{m}} \end{aligned}$$

So we have the equality :

$({}^s\dot{\omega}_x(\tilde{X}_x)) + ({}^s\dot{e}_x(\tilde{X}_x)) = (- {}^sY_X(x) + [\mathrm{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{h}}) + ([\mathrm{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{m}})$, where in both sides, the first term in brackets is in \mathfrak{h} while the second is in \mathfrak{m} . We deduce the claimed expressions for ${}^s\dot{\omega}_x(\tilde{X}_x)$ and ${}^s\dot{e}_x(\tilde{X}_x)$. \square

Next, we want to express in the gauge s the G -invariance of the canonical connection and soldering form of (G, H) .

Proposition 4.2.4 In the gauge s ,

- the condition $(l_a)^*\overset{\circ}{\omega} = \overset{\circ}{\omega}$ becomes : $(l_a)^*{}^s\overset{\circ}{\omega} = {}^s h_a {}^s\overset{\circ}{\omega} {}^s h_a^{-1} + {}^s h_a d({}^s h_a)$,
- the condition $(l_a)^*\overset{\circ}{e} = \overset{\circ}{e}$ becomes : $(l_a)^*{}^s\overset{\circ}{e} = {}^s h_a {}^s\overset{\circ}{e} {}^s h_a^{-1}$.

Proof : First, notice that $s(ax) = as(x) {}^s h_a(x)^{-1}$ is equivalent to : $s \circ l_a(x) = l_a \circ (s {}^s h_a)(x)$.

$$\begin{aligned} - (l_a)^*{}^s\overset{\circ}{\omega} &= (l_a)^*s^*\overset{\circ}{\omega} = (s \circ l_a)^*\overset{\circ}{\omega} = (l_a \circ (s {}^s h_a))^*\overset{\circ}{\omega} = (s {}^s h_a)^*l_a^*\overset{\circ}{\omega} = (s {}^s h_a)^*\overset{\circ}{\omega} = (s {}^s h_a)\overset{\circ}{\omega} \\ &= {}^s h_a {}^s\overset{\circ}{\omega} {}^s h_a^{-1} + {}^s h_a d({}^s h_a) \\ - (l_a)^*{}^s\overset{\circ}{e} &= (l_a)^*s^*\overset{\circ}{e} = (s \circ l_a)^*\overset{\circ}{e} = (l_a \circ (s {}^s h_a))^*\overset{\circ}{e} = (s {}^s h_a)^*l_a^*\overset{\circ}{e} = (s {}^s h_a)^*\overset{\circ}{e} = (s {}^s h_a)\overset{\circ}{e} \\ &= {}^s h_a {}^s\overset{\circ}{e} {}^s h_a^{-1} \end{aligned} \quad \square$$

At the infinitesimal level, the G -invariance of the canonical connection and soldering form is expressed through the following conditions : $L_{\tilde{X}}\overset{\circ}{\omega} = 0$ and $L_{\tilde{X}}\overset{\circ}{e} = 0$ for every $X \in \mathfrak{g}$, where $\tilde{X} \in \Gamma(TG)$ is the left fundamental (and right-invariant) vector field associated to X by the action of G on itself by left translations. Let us write these infinitesimal conditions in the gauge s .

Proposition 4.2.5 In the gauge s ,

- the G -invariance of the canonical connection is expressed by :

$$L_{\tilde{X}}\overset{\circ}{\omega} = -[{}^s\overset{\circ}{\omega}, {}^sY_X] - d({}^sY_X) \quad (= -\mathcal{D}^{\overset{\circ}{\omega}}({}^sY_X)) \quad \text{for every } X \in \mathfrak{g},$$

- the G -invariance of the canonical soldering form is expressed by :

$$L_{\tilde{X}}\overset{\circ}{e} = -[{}^s\overset{\circ}{e}, {}^sY_X] \quad \text{for every } X \in \mathfrak{g}.$$

It will be useful to have a "covariantized" version of these conditions. For this, we introduce the following "local infinitesimal gauge transformations" associated to an element $X \in \mathfrak{g}$:

$${}^s\hat{Y}_X : W \longrightarrow \mathfrak{h} \text{ defined by : } {}^s\hat{Y}_X(x) = [\mathrm{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{h}}$$

${}^s\hat{X} : W \longrightarrow \mathfrak{m}$ defined by : ${}^s\hat{X}(x) = [\text{Ad}_{s(x)^{-1}}(X)]_{\mathfrak{m}}$

Then, Proposition 4.2.3 implies :

$$\begin{aligned} {}^s\hat{Y}_X &= {}^sY_X + {}^s\dot{\omega}(\tilde{X}) \\ {}^s\hat{X} &= {}^s\dot{e}(\tilde{X}) \end{aligned}$$

The first relation shows that ${}^s\hat{Y}_X$ is a covariantized version of the compensating infinitesimal gauge transformation sY_X .

The second relation shows that ${}^s\hat{X}$ gives another way to realize the infinitesimal action of the symmetry group G : the first way was through the fundamental vector field (infinitesimal diffeomorphism) \tilde{X} associated to $X \in \mathfrak{g}$; now we see that, owing to the soldering form, the infinitesimal action of G can be thought as an \mathfrak{m} -valued infinitesimal gauge transformation ${}^s\hat{X}$ associated to $X \in \mathfrak{g}$.

We would like to express the invariance conditions of the proposition 4.2.5 in terms of ${}^s\hat{Y}_X$ and ${}^s\hat{X}$, and this makes the curvature and torsion appear. In fact, we have the following proposition :

Proposition 4.2.6 .

1. The condition $L_{\tilde{X}}\dot{\omega} = -\mathcal{D}^{\dot{\omega}}({}^sY_X)$ is equivalent to :

$$i_{\tilde{X}} {}^s\dot{\Omega} = -\mathcal{D}^{\dot{\omega}}({}^s\hat{Y}_X)$$

2. The condition $L_{\tilde{X}}\dot{e} = -[{}^s\dot{e}, {}^sY_X]$ is equivalent to :

$$i_{\tilde{X}} {}^s\dot{T} = -[{}^s\dot{e}, {}^s\hat{Y}_X] - \mathcal{D}^{\dot{\omega}}({}^s\hat{X})$$

Proof :

1. We have : ${}^s\dot{\Omega} = {}^s*\dot{\Omega} = \mathcal{D}^{\dot{\omega}}({}^s\dot{\omega})$. For any $Z \in \Gamma(T\mathring{M})$,

$$\begin{aligned} i_{\tilde{X}} {}^s\dot{\Omega}(Z) &= {}^s\dot{\Omega}(\tilde{X}, Z) = d {}^s\dot{\omega}(\tilde{X}, Z) + [{}^s\dot{\omega}(\tilde{X}), {}^s\dot{\omega}(Z)] \\ &= \tilde{X} {}^s\dot{\omega}(Z) - Z {}^s\dot{\omega}(\tilde{X}) - {}^s\dot{\omega}([\tilde{X}, Z]) + [{}^s\dot{\omega}(\tilde{X}), {}^s\dot{\omega}(Z)] \\ &= (L_{\tilde{X}} {}^s\dot{\omega})(Z) - Z {}^s\dot{\omega}(\tilde{X}) - [{}^s\dot{\omega}(Z), {}^s\dot{\omega}(\tilde{X})] \\ &= -\mathcal{D}^{\dot{\omega}}({}^sY_X)(Z) - \nabla_Z^{\dot{\omega}} {}^s\dot{\omega}(\tilde{X}) = -\nabla_Z^{\dot{\omega}} {}^sY_X - \nabla_Z^{\dot{\omega}} {}^s\dot{\omega}(\tilde{X}) \\ &= -\nabla_Z^{\dot{\omega}} ({}^sY_X + {}^s\dot{\omega}(\tilde{X})) = -\nabla_Z^{\dot{\omega}} {}^s\hat{Y}_X = -\mathcal{D}^{\dot{\omega}}({}^s\hat{Y}_X)(Z) \end{aligned}$$

2. We have : ${}^s\dot{T} = {}^s*\dot{T} = \mathcal{D}^{\dot{\omega}}({}^s\dot{e})$. For any $Z \in \Gamma(T\mathring{M})$,

$$\begin{aligned} i_{\tilde{X}} {}^s\dot{T}(Z) &= {}^s\dot{T}(\tilde{X}, Z) = d {}^s\dot{e}(\tilde{X}, Z) + [{}^s\dot{e}(\tilde{X}), {}^s\dot{e}(Z)] - [{}^s\dot{e}(Z), {}^s\dot{e}(\tilde{X})] \\ &= \tilde{X} {}^s\dot{e}(Z) - Z {}^s\dot{e}(\tilde{X}) - {}^s\dot{e}([\tilde{X}, Z]) + [{}^s\dot{e}(\tilde{X}), {}^s\dot{e}(Z)] - [{}^s\dot{e}(Z), {}^s\dot{e}(\tilde{X})] \\ &= (L_{\tilde{X}} {}^s\dot{e})(Z) - Z {}^s\dot{e}(\tilde{X}) + [{}^s\dot{e}(\tilde{X}), {}^s\dot{e}(Z)] - [{}^s\dot{e}(Z), {}^s\dot{e}(\tilde{X})] \\ &= -[{}^s\dot{e}(Z), {}^sY_X] - Z {}^s\dot{e}(\tilde{X}) + [{}^s\dot{e}(\tilde{X}), {}^s\dot{e}(Z)] - [{}^s\dot{e}(Z), {}^s\dot{e}(\tilde{X})] \\ &= -[{}^s\dot{e}(Z), {}^sY_X] - [{}^s\dot{e}(Z), {}^s\dot{\omega}(\tilde{X})] - Z {}^s\dot{e}(\tilde{X}) - [{}^s\dot{e}(Z), {}^s\dot{e}(\tilde{X})] \\ &= -[{}^s\dot{e}(Z), {}^s\hat{Y}_X] - \nabla_Z^{\dot{\omega}} {}^s\dot{e}(\tilde{X}) = -[{}^s\dot{e}(Z), {}^s\hat{Y}_X] - \mathcal{D}^{\dot{\omega}}({}^s\hat{X})(Z) \end{aligned}$$

□

There is a natural way to express the invariance of the canonical soldering form \mathring{e} , if we introduce the notion of *covariant Lie derivative*.

Definition 4.2.7 *The covariant Lie derivative of the canonical soldering form \mathring{e} with respect to the fundamental vector field \tilde{X} is defined by : $\forall Z \in \Gamma(\mathring{T}M)$,*

$$(L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e})(Z) = \nabla_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e}(Z) - {}^s\mathring{e}([\tilde{X}, Z])$$

From the definition, we have : $(L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e})(Z) = \tilde{X} {}^s\mathring{e}(Z) + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{e}(Z)] - {}^s\mathring{e}([\tilde{X}, Z])$. So we see that : $L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e} = L_{\tilde{X}} {}^s\mathring{e} + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{e}]$.

Consequently, $L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e} = -[{}^s\mathring{e}, {}^sY_X] + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{e}] = -[{}^s\mathring{e}, {}^sY_X] - [{}^s\mathring{e}, {}^s\mathring{\omega}(\tilde{X})]$.

Therefore, the condition $L_{\tilde{X}}\mathring{e} = -[{}^s\mathring{e}, {}^sY_X]$ is equivalent to :

$$L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e} = -[{}^s\mathring{e}, {}^s\hat{Y}_X]$$

We deduce then the following relation :

$$L_{\tilde{X}}^{\mathring{\omega}} {}^s\mathring{e} = i_{\tilde{X}} {}^s\mathring{T} + \mathcal{D}^{\mathring{\omega}}({}^s\hat{X})$$

Remark 4.2.8 *This last relation could also have been obtained directly from the formula : $L_{\tilde{X}}^{\mathring{\omega}} = i_{\tilde{X}} \mathcal{D}^{\mathring{\omega}} + \mathcal{D}^{\mathring{\omega}} i_{\tilde{X}}$.*

From the structure equations, we have : $\forall Z \in \Gamma(\mathring{T}M)$, ${}^s\mathring{T}(\tilde{X}, Z) = -[{}^s\mathring{e}(\tilde{X}), {}^s\mathring{e}(Z)]_m$. We deduce that $(i_{\tilde{X}} {}^s\mathring{T})(Z) = -[{}^s\hat{X}, {}^s\mathring{e}(Z)]_m$ for every $Z \in \Gamma(\mathring{T}M)$. Thus,

$$i_{\tilde{X}} {}^s\mathring{T} + [{}^s\hat{X}, {}^s\mathring{e}]_m = 0$$

Finally, we compute the commutator of two covariant compensating infinitesimal gauge transformations. For this, we first note that for any $X, X' \in \mathfrak{g}$, $[\tilde{X}, \tilde{X}'] = -\widetilde{[X, X']}$ (since we are dealing with a *left* action), and $[{}^sY_X, {}^sY_{X'}] = {}^sY_{[X, X']}$ (since the map $X \mapsto {}^sY_X$ is a Lie algebra homomorphism from \mathfrak{g} to \mathfrak{h}).

Proposition 4.2.9 *For any $X, X' \in \mathfrak{g}$, we have :*

$$[{}^s\hat{Y}_X, {}^s\hat{Y}_{X'}] = {}^s\hat{Y}_{[X, X']} + {}^s\Omega(\tilde{X}, \tilde{X}')$$

Proof : $[{}^s\hat{Y}_X, {}^s\hat{Y}_{X'}] = [{}^sY_X + {}^s\mathring{\omega}(\tilde{X}), {}^sY_{X'} + {}^s\mathring{\omega}(\tilde{X}')] = [{}^sY_X, {}^sY_{X'}] + [{}^sY_X, {}^s\mathring{\omega}(\tilde{X}')] + [{}^s\mathring{\omega}(\tilde{X}), {}^sY_{X'}] + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} - [{}^s\mathring{\omega}(\tilde{X}'), {}^sY_X] + [{}^s\mathring{\omega}(\tilde{X}), {}^sY_{X'}] + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} - \tilde{X}' {}^sY_X - [{}^s\mathring{\omega}(\tilde{X}'), {}^sY_X] + [{}^s\mathring{\omega}(\tilde{X}), {}^sY_{X'}] + \tilde{X} {}^sY_{X'} + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} - \nabla_{\tilde{X}'}^{\mathring{\omega}} {}^sY_X + \nabla_{\tilde{X}}^{\mathring{\omega}} {}^sY_{X'} + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} - \mathcal{D}^{\mathring{\omega}}({}^sY_X)(\tilde{X}') + \mathcal{D}^{\mathring{\omega}}({}^sY_{X'})(\tilde{X}) + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} + (L_{\tilde{X}} {}^s\mathring{\omega})(\tilde{X}') - (L_{\tilde{X}}, {}^s\mathring{\omega})(\tilde{X}) + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} - {}^s\mathring{\omega}([\tilde{X}, \tilde{X}']) + \tilde{X} {}^s\mathring{\omega}(\tilde{X}') - \tilde{X}' {}^s\mathring{\omega}(\tilde{X}) + {}^s\mathring{\omega}([\tilde{X}', \tilde{X}]) + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^sY_{[X, X']} + {}^s\mathring{\omega}([\tilde{X}, \tilde{X}']) + \tilde{X} {}^s\mathring{\omega}(\tilde{X}') - \tilde{X}' {}^s\mathring{\omega}(\tilde{X}) - {}^s\mathring{\omega}([\tilde{X}, \tilde{X}']) + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^s\hat{Y}_{[X, X']} + d {}^s\mathring{\omega}(\tilde{X}, \tilde{X}') + [{}^s\mathring{\omega}(\tilde{X}), {}^s\mathring{\omega}(\tilde{X}')] = {}^s\hat{Y}_{[X, X']} + {}^s\Omega(\tilde{X}, \tilde{X}')$

□

4.3 Minkowski spacetime and other Klein geometries

- Definition 4.3.1**
1. A **Lorentzian vector space of signature $(d - 1, 1)$** is a real vector space V of dimension d , equipped with a non-degenerate symmetric bilinear form η of signature $(d - 1, 1)$.
 2. A **Minkowski spacetime of signature $(d - 1, 1)$** is a Lorentzian affine space directed by V , that is, a set $\overset{\circ}{M}$ equipped with a free and transitive right action of the underlying additive group of V .

One can easily show that a Minkowski spacetime $\overset{\circ}{M}$ has a natural structure of smooth manifold. Let $t : V \longrightarrow \text{Diff}(\overset{\circ}{M})$ be the injective group anti-homomorphism defining the right action of V on $\overset{\circ}{M}$. The $+$ symbol is defined by $: x + v := t_v(x)$ for any point $x \in \overset{\circ}{M}$ and vector $v \in V$. Since V (as additive group) is abelian, t is also a group homomorphism. Therefore, $\text{Im } t$ is an abelian subgroup of $\text{Diff}(\overset{\circ}{M})$ isomorphic to V , denoted by $\mathcal{T}(\overset{\circ}{M})$ and called **the group of translations of $\overset{\circ}{M}$** .

We denote by $\text{SO}^\dagger(V)$ the *Lorentz group* of V (which is the connected component of the identity in $\text{O}(V)$).

- Definition 4.3.2** A **Poincaré transformation** is a direct affine isometry of $\overset{\circ}{M}$, that is a diffeomorphism $u : \overset{\circ}{M} \longrightarrow \overset{\circ}{M}$ for which there exists a Lorentz transformation $\Lambda \in \text{SO}^\dagger(V)$ such that $u(x + v) = u(x) + \Lambda(v)$ for any point $x \in \overset{\circ}{M}$ and vector $v \in V$.

It is easy to see that if there exists $\Lambda \in \text{SO}^\dagger(V)$ such that $u(x + v) = u(x) + \Lambda(v)$ for any $x \in \overset{\circ}{M}$ and $v \in V$, then Λ is unique. It is called the *linear part* of the Poincaré transformation u . The set of Poincaré transformations of $\overset{\circ}{M}$ is a subgroup of $\text{Diff}(\overset{\circ}{M})$, denoted by $\Pi(\overset{\circ}{M})$ and called the **Poincaré group**. One can easily show that $\Pi(\overset{\circ}{M})$ has a natural structure of Lie group. We denote by $L : \Pi(\overset{\circ}{M}) \longrightarrow \text{SO}^\dagger(V)$ the map which associates to each Poincaré transformation its linear part. It is easy to check that L is a surjective group homomorphism whose kernel is nothing but $\mathcal{T}(\overset{\circ}{M})$. In particular, $\mathcal{T}(\overset{\circ}{M})$ is a normal abelian subgroup of $\Pi(\overset{\circ}{M})$, and $\Pi(\overset{\circ}{M})/\mathcal{T}(\overset{\circ}{M})$ is canonically isomorphic to $\text{SO}^\dagger(V)$. Thus, we have an exact sequence of Lie groups :

$$0 \longrightarrow V \longrightarrow \Pi(\overset{\circ}{M}) \longrightarrow \text{SO}^\dagger(V) \longrightarrow 1$$

the injection being t , and the surjection being L .

The Poincaré group $\Pi(\overset{\circ}{M})$ acts transitively on $\overset{\circ}{M}$, since it contains the subgroup $\mathcal{T}(\overset{\circ}{M})$ which clearly acts transitively on $\overset{\circ}{M}$.

Let us describe now the Lie algebra of the Poincaré group.

- Definition 4.3.3** An **infinitesimal Poincaré transformation** is an affine map $X : \overset{\circ}{M} \longrightarrow V$ whose linear part $L(X)$ is an antisymmetric endomorphism of the Lorentzian vector space V . We denote by $\pi(\overset{\circ}{M}, V)$ the set of infinitesimal Poincaré transformations.

- Proposition 4.3.4** $\pi(\overset{\circ}{M}, V)$ has a natural Lie algebra structure. It is called the **Poincaré Lie algebra**.

Proof : $\pi(\overset{\circ}{M}, V)$ has a natural vector space structure coming from V .

For $X, Y \in \pi(\overset{\circ}{M}, V)$, set :

$$[X, Y] = L(X) \circ Y - L(Y) \circ X$$

It is not difficult to check that this defines a Lie bracket on $\pi(\overset{\circ}{M}, V)$. \square

Let $\iota : V \longrightarrow \pi(\overset{\circ}{M}, V)$ be the canonical injection which sends a vector $v \in V$ to the constant map $\overset{\circ}{M} \longrightarrow V$ with value v . It is clear $\text{Im } \iota$ is an abelian Lie subalgebra of $\pi(\overset{\circ}{M}, V)$ isomorphic to the trivial Lie algebra V . We denote it by \mathfrak{m} and call it **the Lie algebra of infinitesimal translations**.

On the other hand, the map $L : \pi(\overset{\circ}{M}, V) \longrightarrow \mathfrak{so}(V)$ which associates to each infinitesimal Poincaré transformation its linear part is easily seen to be a surjective Lie algebras homomorphism whose kernel is nothing but \mathfrak{m} . In particular, \mathfrak{m} is an abelian ideal of $\pi(\overset{\circ}{M}, V)$, and $\pi(\overset{\circ}{M}, V)|\mathfrak{m}$ is canonically isomorphic to $\mathfrak{so}(V)$. Thus, we have an exact sequence of Lie algebras :

$$0 \longrightarrow V \longrightarrow \pi(\overset{\circ}{M}, V) \longrightarrow \mathfrak{so}(V) \longrightarrow 0$$

the injection being ι , and the surjection being L .

Notice that until now, our discussion was purely affine : *we have not made any choice of origin yet*. Let us now choose an origin $x_0 \in \overset{\circ}{M}$. This amounts to the choice of a splitting for the above exact sequences.

Indeed, *once an origin x_0 has been chosen*, we may define an injective group homomorphism $\sigma : \text{SO}^\dagger(V) \longrightarrow \Pi(\overset{\circ}{M})$ by associating to every $\Lambda \in \text{SO}^\dagger(V)$ the Poincaré transformation σ_Λ defined by : $\forall x \in \overset{\circ}{M}, \sigma_\Lambda(x) = x_0 + \Lambda(\overrightarrow{x_0 x})$ where $\overrightarrow{x_0 x}$ is the unique vector of V such that $x_0 + \overrightarrow{x_0 x} = x$. It is easy to check that $L(\sigma_\Lambda) = \Lambda$ which shows that σ is a splitting of the above exact sequence of Lie groups. Besides, we have $\sigma_\Lambda(x_0) = x_0$.

Set $H = \text{Im } \sigma$. It is immediate that H is a subgroup of $\Pi(\overset{\circ}{M})$ isomorphic to $\text{SO}^\dagger(V)$. On the other hand, it is not difficult to show that H is nothing but the stabilizer of x_0 under the action of $G := \Pi(\overset{\circ}{M})$. We have thus obtained a Klein geometry $(G, H) = (\Pi(\overset{\circ}{M}), \text{Im } \sigma)$. We insist again that σ , H and therefore the Klein geometry (G, H) depend on the choice of the origin $x_0 \in \overset{\circ}{M}$. However, another choice of origin would lead to an isomorphic Klein geometry, since the corresponding stabilizer would be conjugated to H . The space of the Klein geometry (G, H) is of course $\Pi(\overset{\circ}{M})/\text{Im } \sigma$ which is identified to $\overset{\circ}{M}$ via the surjective map $\pi : \Pi(\overset{\circ}{M}) \longrightarrow \overset{\circ}{M}$ defined by : $\pi(u) = u(x_0)$.

At the infinitesimal level, we may define an injective Lie algebra homomorphism $\sigma' : \mathfrak{so}(V) \longrightarrow \pi(\overset{\circ}{M}, V)$ by associating to every $\lambda \in \mathfrak{so}(V)$ the infinitesimal Poincaré transformation σ'_λ defined by : $\sigma'_\lambda(x) = \lambda(\overrightarrow{x_0 x})$. It is easy to check that $L(\sigma'_\lambda) = \lambda$ which shows that σ' is a splitting of the above exact sequence of Lie algebras. Besides, we have $\sigma'_\lambda(x_0) = 0$.

Set $\mathfrak{h} = \text{Im } \sigma'$. It is immediate that \mathfrak{h} is a Lie subalgebra of $\pi(\overset{\circ}{M}, V)$ isomorphic to $\mathfrak{so}(V)$. We have thus obtained the Klein pair $(\mathfrak{g}, \mathfrak{h}) = (\pi(\overset{\circ}{M}, V), \sigma')$ associated to the Klein geometry $(G, H) = (\Pi(\overset{\circ}{M}), \text{Im } \sigma)$. Besides, the Lie algebras \mathfrak{h} and \mathfrak{m} define a natural reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In the following, we shall make no distinction between a vector $v \in V$, and the constant map $\overset{\circ}{M} \longrightarrow V$ with value v ; thus we identify the abelian ideal \mathfrak{m} of infinitesimal translations with V . The reductive decomposition becomes : $\mathfrak{g} = \mathfrak{h} \oplus V$. Notice that the restricted adjoint representation $\text{Ad} : H \longrightarrow \text{GL}(\mathfrak{m})$ is equivalent here to the standard representation of $\text{SO}^\dagger(V)$ on V . We set $\overset{\circ}{V} = \Pi(\overset{\circ}{M}) \times_{\text{Ad}} V$.

The choice of an origin in $x_0 \in \overset{\circ}{M}$ has other consequences also. For example, it induces a bijection $V \longrightarrow \overset{\circ}{M}$ which associates to each vector $v \in V$ the point $x_0 + v \in \overset{\circ}{M}$. And there is more : consider the semi-direct product $V \rtimes \text{SO}^\dagger(V)$. This is the product $V \times \text{SO}^\dagger(V)$ equipped with the following group law : $(v, \Lambda)(v', \Lambda') = (v + \Lambda(v'), \Lambda \circ \Lambda')$. The choice of an origin $x_0 \in \overset{\circ}{M}$ provides us with a group isomorphism $V \rtimes \text{SO}^\dagger(V) \longrightarrow \Pi(\overset{\circ}{M})$. Indeed, to a pair $(v, \Lambda) \in V \rtimes \text{SO}^\dagger(V)$, it is sufficient to associate the Poincaré transformation u defined by : $u(x) = \sigma_\Lambda(x) + v$ for every $x \in \overset{\circ}{M}$.

At the infinitesimal level, consider the semi-direct sum $V \oplus' \mathfrak{so}(V)$. This is the product $V \times \mathfrak{so}(V)$ equipped with the following Lie bracket : $[(v, \lambda), (w, \mu)] = (\lambda(w) - \mu(v), [\lambda, \mu])$. The choice of an origin $x_0 \in \overset{\circ}{M}$ provides us with a Lie algebras isomorphism $V \oplus' \mathfrak{so}(V) \longrightarrow \pi(\overset{\circ}{M}, V)$. Indeed, to a pair $(v, \lambda) \in V \oplus' \mathfrak{so}(V)$, it is sufficient to associate the infinitesimal Poincaré transformation X defined by : $X(x) = \sigma'_\lambda(x) + v$ for every $x \in \overset{\circ}{M}$.

Moreover, with the choice of an origin $x_0 \in \overset{\circ}{M}$, $\Pi(\overset{\circ}{M})$ is realized as a principal fibre bundle over $\overset{\circ}{M}$, with structural group H (the fibration being the projection $\pi : \Pi(\overset{\circ}{M}) \longrightarrow \overset{\circ}{M}$). This is nothing but the canonical bundle of the Klein geometry $(\Pi(\overset{\circ}{M}), \text{Im } \sigma)$.

An interesting feature of the Klein geometry $(\Pi(\overset{\circ}{M}), \text{Im } \sigma)$ is the fact that it is *parallelizable*, which means that its canonical bundle is trivializable. In fact, it admits a canonical parallelism, given by the canonical global cross-section $\dot{s} : \overset{\circ}{M} \longrightarrow \Pi(\overset{\circ}{M})$ defined by : $\dot{s}_x = t_{\overrightarrow{x_0 x}}$.

If we consider the canonical soldering form \dot{e} of the reductive Klein geometry $(\Pi(\overset{\circ}{M}), \text{Im } \sigma)$, we may define its pull-back by \dot{s} . We obtain thus a canonical one-form $\dot{s}^* \dot{e} \in \Lambda^1(\overset{\circ}{M}, V)$, such that the corresponding linear map $\Gamma(T\overset{\circ}{M}) \longrightarrow C^\infty(\overset{\circ}{M}, V)$ is an isomorphism.

On the other hand, the pull-back by \dot{s} of the canonical connection $\dot{\omega}$ of $(\Pi(\overset{\circ}{M}), \text{Im } \sigma)$ is zero.

The Klein geometry $(\Pi(\overset{\circ}{M}), \text{Im } \sigma)$ is effective, but has the disadvantage of not allowing the definition of spinor fields on $\overset{\circ}{M}$. In order to deal with spinors, we must turn to a locally effective but not effective Klein geometry, obtained by replacing the groups G and H by their universal covering.

The universal covering of $\text{SO}^\dagger(V)$ is the group $\text{Spin}^\dagger(V)$. The covering map $\text{Spin}^\dagger(V) \longrightarrow \text{SO}^\dagger(V) \subset \text{GL}(V)$ is denoted by ρ_1 and is called the *vector representation*.

Definition 4.3.5 *The simply connected Poincaré group is the universal covering of the Poincaré group $V \rtimes \text{SO}^\dagger(V)$, that is : $\tilde{\Pi} = V \rtimes_{\rho_1} \text{Spin}^\dagger(V)$ (the group law being given by : $(v, \tilde{\Lambda})(v', \tilde{\Lambda}') = (v + \rho_1(\tilde{\Lambda})(v'), \tilde{\Lambda}\tilde{\Lambda}')$).*

With the choice of the point $x_0 \in \overset{\circ}{M}$, the group $\tilde{\Pi}$ acts transitively on $\overset{\circ}{M}$ by : $(v, \tilde{\Lambda})x = x_0 + \rho_1(\tilde{\Lambda})(\overrightarrow{x_0 x}) + v$. The stabilizer of x_0 under this action is the $\text{Spin}^\dagger(V)$ subgroup of $\tilde{\Pi}$. Thus, we obtain a locally effective (but non-effective) Klein geometry $(\tilde{G}, \tilde{H}) = (\tilde{\Pi}, \text{Spin}^\dagger(V))$. The space of the Klein geometry (\tilde{G}, \tilde{H}) is of course $\tilde{\Pi}/\text{Spin}^\dagger(V)$ which is identified to $\overset{\circ}{M}$ via the surjective map $\pi : \tilde{\Pi} \longrightarrow \overset{\circ}{M}$ defined by : $\pi(v, \tilde{\Lambda}) = (v, \tilde{\Lambda})x_0$ ($= x_0 + v$). This map is nothing but the projection of the canonical

bundle associated to $(\tilde{\Pi}, \text{Spin}^\dagger(V))$.

The map $\dot{\lambda} : \tilde{\Pi} \longrightarrow \Pi(\mathring{M})$ which associates to each element $(v, \tilde{\Lambda}) \in \tilde{\Pi}$ the Poincaré transformation u defined by : $u(x) = x_0 + \rho_1(\tilde{\Lambda})(\overrightarrow{x_0x}) + v$ is a bundle homomorphism over $\text{Id}_{\mathring{M}}$ satisfying : $\dot{\lambda}((v, \tilde{\Lambda})\tilde{\Lambda}') = \dot{\lambda}((v, \tilde{\Lambda})) \circ \rho_1(\tilde{\Lambda}')$ for all $\tilde{\Lambda}' \in \text{Spin}^\dagger(V)$. Therefore, $(\tilde{\Pi}, \dot{\lambda})$ defines a *canonical spinorial structure* on the vector bundle \mathring{V} (and therefore on M , owing to the canonical soldering form \mathring{e}).

Consider a faithful representation $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(V) \longrightarrow \text{GL}(S)$, called the *spinor representation*. The **spinor bundle** is the associated vector bundle : $\mathring{S} = \tilde{\Pi} \times \rho_{\frac{1}{2}} S$. There is a natural embedding $\iota : \tilde{\Pi} \longrightarrow \mathcal{F}(\mathring{S})$ (where $\mathcal{F}(\mathring{S})$ is the linear frame bundle of \mathring{S} , with structural group $\text{GL}(S)$). ι associates to each $(v, \tilde{\Lambda}) \in \tilde{\Pi}$ the frame $\iota(v, \tilde{\Lambda}) : S \longrightarrow \mathring{S}_{x_0+v}$ defined by : $\iota(v, \tilde{\Lambda})(\Psi) = ((v, \tilde{\Lambda}), \Psi)\text{Spin}^\dagger(V)$.

Remark 4.3.6 $\iota(\tilde{\Pi})$ is a $\rho_{\frac{1}{2}}(\text{Spin}^\dagger(V))$ -structure on \mathring{S} (with $\text{Spin}^\dagger(V) \simeq \rho_{\frac{1}{2}}(\text{Spin}^\dagger(V)) \subset \text{GL}(S)$).

The Klein geometry $(\tilde{\Pi}, \text{Spin}^\dagger(V))$ is also parallelizable. It admits a canonical parallelism, given by the canonical global cross-section $\dot{s} : \mathring{M} \longrightarrow \tilde{\Pi}$ defined by : $\dot{s}_x = (\overrightarrow{x_0x}, 0)$. As a consequence, there exists a canonical isomorphism $\Gamma(\mathring{S}) \longrightarrow C^\infty(M, S)$. Indeed, for any $x \in \mathring{M}$, $\iota(\dot{s}_x)$ is the frame $S \longrightarrow \mathring{S}_x$ defined by : $\iota(\dot{s}_x)(\Psi) = ((\overrightarrow{x_0x}, 0), \Psi)\text{Spin}^\dagger(V)$. Now for any spinor field $\psi \in \Gamma(\mathring{S})$, set $(\dot{s}^*\psi)(x) = \iota(\dot{s}_x)^{-1}(\psi_x)$. Then $\dot{s}^*\psi \in C^\infty(\mathring{M}, S)$.

Consider the canonical connection $\dot{\omega} \in \Lambda^1_{\text{Spin}^\dagger(V)}(\tilde{\Pi}, \mathfrak{spin}(V))$ of the Klein geometry $(\tilde{\Pi}, \text{Spin}^\dagger(V))$. The covariant derivative $\mathring{\nabla}$ acting on spinor fields is given in the canonical global trivialization by : $\mathring{\nabla}(\dot{s}^*\psi) = d(\dot{s}^*\psi) + \rho'_{\frac{1}{2}}(\dot{s}^*\dot{\omega})(\dot{s}^*\psi)$, where $\rho'_{\frac{1}{2}} : \mathfrak{spin}(V) \longrightarrow \mathfrak{gl}(S)$ is the derivative of $\rho_{\frac{1}{2}}$. Since $\dot{s}^*\dot{\omega} = 0$, we have $\mathring{\nabla}(\dot{s}^*\psi) = d(\dot{s}^*\psi)$.

We end this section by fixing the notations for other Klein geometries that will play a role in the next chapters of the thesis.

In what follows \mathbb{R}^{n+1} is equipped with the non-degenerate symmetric bilinear form of signature $(p, q+1)$ canonically associated to the following matrix : $\eta = \text{diag}(-1, -1, \dots, -1, +1, \dots, +1)$ where the number of (-1) is $q+1$, and the number of $(+1)$ is p . Set : $O(p, q+1) = \{P \in \text{GL}_{n+1}(\mathbb{R}) / {}^t\Lambda\eta\Lambda = \eta\}$, and let $\text{SO}^\dagger(p, q+1)$ be the connected component of the identity in $O(p, q+1)$. Finally, set : $\text{SO}(p+1) = \text{SO}^\dagger(p+1, 0)$.

Replacing in the above paragraph $n+1$ by n , and $q+1$ by q , we obtain a group $\text{SO}^\dagger(p, q)$. We denote by H the following subgroup of $G := \text{SO}^\dagger(p, q+1)$:

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} ; \Lambda \in \text{SO}^\dagger(p, q) \right\}$$

Then (G, H) is a Klein geometry, denoted by $(\text{SO}^\dagger(p, q+1), \text{SO}^\dagger(p, q))$. We set :

$$\mathring{M}_{p,q} = \text{SO}^\dagger(p, q+1)/\text{SO}^\dagger(p, q)$$

Important examples are :

- $\mathring{M}_{0,n} = \mathrm{SO}^\dagger(0, n+1)/\mathrm{SO}^\dagger(0, n)$. This is the *round sphere* S^n .
- $\mathring{M}_{n,0} = \mathrm{SO}^\dagger(n, 1)/\mathrm{SO}(n)$. This is the *hyperbolic space* H^n .
- $\mathring{M}_{1,n-1} = \mathrm{SO}^\dagger(1, n)/\mathrm{SO}^\dagger(1, n-1)$. This is the *de Sitter space* dS_n .
- $\mathring{M}_{n-1,1} = \mathrm{SO}^\dagger(n-1, 2)/\mathrm{SO}^\dagger(n-1, 1)$. This is the *Anti-de Sitter space* AdS_n .

4.4 Sigma-models with homogenous space as target

Let (G, H) be a reductive Klein geometry, and choose a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\pi_{\mathfrak{h}} : \mathfrak{g} \longrightarrow \mathfrak{h}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \longrightarrow \mathfrak{m}$ the corresponding projectors. Let $\mathring{\theta}$ be the left-invariant Maurer-Cartan form on G .

Definition 4.4.1 *Let M be a spacetime. A (non-linear) **σ -model on M valued in G/H** is a couple (P, Φ) where P is a principal H -bundle over M and $\Phi : P \longrightarrow G$ is an H -equivariant map.*

Let us specify what is meant here by H -equivariance for Φ . We consider the following left action of H on $G : \forall h \in H, \forall a \in G, \hat{l}_h(a) = ah^{-1}$. Then Φ is H -equivariant means that $\forall h \in H, \forall p \in P, \Phi(ph) = \hat{l}_{h^{-1}}(\Phi(p)) (= \Phi(p)h)$. In other terms, $\Phi \circ \check{r}_h = r_h \circ \Phi \quad \forall h \in H$, where \check{r} denotes the right action of H on P , and r denotes the action of H on G by right translations ($r_h = \hat{l}_{h^{-1}}$).

Let (P, Φ) be a σ -model on M valued in G/H . Then we may associate to this σ -model an H -equivariant \mathfrak{g} -valued one-form on P : simply take $\Phi^*\mathring{\theta} \in \Lambda_H^1(P, \mathfrak{g})$. Moreover, since (G, H) is reductive, we can set for any $p \in P : \mathcal{Q}_p = \pi_{\mathfrak{h}} \circ (\Phi^*\mathring{\theta})_p$ and $\mathcal{P}_p = \pi_{\mathfrak{m}} \circ (\Phi^*\mathring{\theta})_p$.

Then $\mathcal{Q} = \Phi^*\mathring{\omega} \in \Lambda_H^1(P, \mathfrak{h})$ and $\mathcal{P} = \Phi^*\mathring{e} \in \bar{\Lambda}_H^1(P, \mathfrak{m})$. Therefore, \mathcal{Q} is a connection one-form on P and \mathcal{P} is a tensorial \mathfrak{m} -valued one-form on P .

Remark 4.4.2 *The H -equivariance of the one-forms $\Phi^*\mathring{\theta}$, \mathcal{Q} and \mathcal{P} results from the H -equivariance of Φ and from the (right) H -equivariance of the Maurer-Cartan form $\mathring{\theta}$. Indeed, $\forall h \in H, \forall p \in P, (\check{r}_h^*(\Phi^*\mathring{\theta}))_p = ((\Phi \circ \check{r}_h)^*\mathring{\theta})_p = ((r_h \circ \Phi)^*\mathring{\theta})_p = (\Phi^*r_h^*\mathring{\theta})_p = (r_h^*\mathring{\theta})_{\Phi(p)} \circ T_p \Phi = Ad_{h^{-1}} \circ \mathring{\theta}_{\Phi(p)} \circ T_p \Phi = Ad_{h^{-1}} \circ (\Phi^*\mathring{\theta})_p$ which proves the H -equivariance of $\Phi^*\mathring{\theta}$, and therefore, that of \mathcal{Q} and \mathcal{P} .*

Let $C_H^\infty(P, G)$ be the set of H -equivariant maps from P to G . We have a natural left action of the (infinite-dimensional) Lie group $C^\infty(P, H)$ on $C_H^\infty(P, G)$: for any $h \in C^\infty(P, H)$ and $\Phi \in C_H^\infty(P, G)$, set : $(h \cdot \Phi)(p) = \hat{l}_{h(p)}(\Phi(p)) = \Phi(p)h(p)^{-1}, \forall p \in P$. We say that we have an action by *gauge transformations* (for the group H) on Φ .

The preceding action induces a natural action of $C^\infty(P, H)$ on \mathcal{Q} and \mathcal{P} ; for any $h \in C^\infty(P, H)$, we define $h \cdot \mathcal{Q}$ and $h \cdot \mathcal{P}$ by :

$$(h \cdot \mathcal{Q})_p = \pi_{\mathfrak{h}} \circ ((h \cdot \Phi)^* \dot{\theta})_p$$

$$(h \cdot \mathcal{P})_p = \pi_{\mathfrak{m}} \circ ((h \cdot \Phi)^* \dot{\theta})_p$$

for any $p \in P$.

Proposition 4.4.3 *The above action of $C^\infty(P, H)$ on \mathcal{Q} and \mathcal{P} is nothing but the action by gauge transformations (for the group H) on the connection \mathcal{Q} and the tensorial one-form \mathcal{P} in the usual sense, that is :*

$$(h \cdot \mathcal{Q})_p = Ad_{h(p)} \circ \mathcal{Q}_p + ((h^{-1})^* \dot{\theta}_H)_p$$

$$(h \cdot \mathcal{P})_p = Ad_{h(p)} \circ \mathcal{P}_p$$

for any $h \in C^\infty(P, H)$ and $p \in P$ (here $\dot{\theta}_H$ is the Maurer-Cartan form of H).

Proof : It is sufficient to prove that

$$((h \cdot \Phi)^* \dot{\theta})_p = Ad_{h(p)} \circ (\Phi^* \dot{\theta})_p + ((h^{-1})^* \dot{\theta}_H)_p.$$

We can see quickly why this is true by reverting to physicists condensed notations (in which $\Phi^* \dot{\theta} = \Phi^{-1} d\Phi$) :

Φ transforms by : $\Phi \longrightarrow \Phi h^{-1}$. Consequently, $\Phi^{-1} d\Phi$ transforms by :

$$\Phi^{-1} d\Phi \longrightarrow (\Phi h^{-1})^{-1} d(\Phi h^{-1}) = h\Phi^{-1}((d\Phi)h^{-1} + \Phi(dh^{-1})).$$

Therefore, we have $(\Phi h^{-1})^{-1} d(\Phi h^{-1}) = h(\Phi^{-1} d\Phi)h^{-1} + h dh^{-1}$.

Projecting this relation to \mathfrak{h} and \mathfrak{m} , we get :

$$h \cdot \mathcal{Q} = h \mathcal{Q} h^{-1} + h dh^{-1}$$

$$h \cdot \mathcal{P} = h \mathcal{P} h^{-1}$$

□

Now we turn to the action of G . We have a natural left action of G on $C_H^\infty(P, G)$: for any $a \in G$ and $\Phi \in C_H^\infty(P, G)$, set : $l_a(\Phi) := l_a \circ \Phi$ (where l denotes the action of G on itself by left translations).

The preceding action induces a natural action on \mathcal{Q} and \mathcal{P} ; for any $a \in G$, we define $l_a(\mathcal{Q})$ and $l_a(\mathcal{P})$ by :

$$l_a(\mathcal{Q})_p = \pi_{\mathfrak{h}} \circ (l_a(\Phi)^* \dot{\theta})_p$$

$$l_a(\mathcal{P})_p = \pi_{\mathfrak{m}} \circ (l_a(\Phi)^* \dot{\theta})_p$$

for any $p \in P$.

Proposition 4.4.4 *\mathcal{Q} and \mathcal{P} are invariant under the above action.*

Proof : It is sufficient to prove that : $l_a(\Phi)^* \dot{\theta} = \Phi^* \dot{\theta}$. This is a direct consequence of the left-invariance of the Maurer-Cartan form $\dot{\theta}$. Indeed,

$l_a(\Phi)^* \dot{\theta} = (l_a \circ \Phi)^* \dot{\theta} = \Phi^* l_a^* \dot{\theta} = \Phi^* \dot{\theta}$. Therefore, we have $l_a(\mathcal{Q}) = \mathcal{Q}$ and $l_a(\mathcal{P}) = \mathcal{P}$ for every $a \in G$. □

It is possible to define the covariant differential of Φ with respect to the connection \mathcal{Q} . Denote by $\iota_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{g}$ and $\iota_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{g}$ the canonical injections.

Definition 4.4.5 *The covariant differential of Φ with respect to the connection \mathcal{Q} is the element $\mathcal{D}^\mathcal{Q}\Phi$ of $\Lambda^1(P, \Phi^{-1}TG)$ defined by :*

$$(\mathcal{D}^\mathcal{Q}\Phi)_p = (d\Phi)_p - T_e l_{\Phi(p)} \circ \iota_{\mathfrak{h}} \circ \mathcal{Q}_p$$

(where $(d\Phi)_p = T_p\Phi \in \text{Hom}(T_p P, T_{\Phi(p)} G)$).

Remark 4.4.6 In condensed notations, the above definition may be written : $\mathcal{D}^Q\Phi = d\Phi - \Phi Q$.

We know that for any $p \in P$, $(\Phi^*\dot{\theta})_p = \dot{\theta}_{\Phi(p)} \circ T_p\Phi = T_{\Phi(p)}l_{\Phi(p)^{-1}} \circ (d\Phi)_p$ (this is why $\Phi^*\dot{\theta}$ is denoted by $\Phi^{-1}d\Phi$ in physicists notations). Now using the covariant differential, we may consider another H -equivariant \mathfrak{g} -valued one-form on P , denoted by $\Phi^{-1}\mathcal{D}^Q\Phi$ and defined by : $(\Phi^{-1}\mathcal{D}^Q\Phi)_p = T_{\Phi(p)}l_{\Phi(p)^{-1}} \circ (\mathcal{D}^Q\Phi)_p$ for any $p \in P$.

Proposition 4.4.7 We have : $(\Phi^{-1}\mathcal{D}^Q\Phi)_p = \iota_{\mathfrak{m}} \circ \mathcal{P}_p$ for every $p \in P$.

Proof : $(\Phi^{-1}\mathcal{D}^Q\Phi)_p = T_{\Phi(p)}l_{\Phi(p)^{-1}} \circ (d\Phi)_p - T_{\Phi(p)}l_{\Phi(p)^{-1}} \circ T_e l_{\Phi(p)} \circ \iota_{\mathfrak{h}} \circ Q_p = (\Phi^*\dot{\theta})_p - \iota_{\mathfrak{h}} \circ Q_p = \iota_{\mathfrak{h}} \circ Q_p + \iota_{\mathfrak{m}} \circ \mathcal{P}_p - \iota_{\mathfrak{h}} \circ Q_p = \iota_{\mathfrak{m}} \circ \mathcal{P}_p$. \square

To write a Lagrangian for the σ -model, we first notice that since \mathcal{P} is horizontal, \mathcal{P}_p is in fact entirely determined for every p by its restriction to $\mathcal{H}_p = \text{Ker } Q_p$. We also denote by \mathcal{P}_p the linear map $\mathcal{H}_p \longrightarrow \mathfrak{m}$. Then we need a scalar product on $\text{Hom}(\mathcal{H}_p, \mathfrak{m})$. If k denotes an $\text{Ad}(H)$ -invariant scalar product on \mathfrak{m} , and \bar{g}_p the scalar product on \mathcal{H}_p deduced from a given Lorentzian metric g on M , then we can define a scalar product $\langle \cdot, \cdot \rangle_{\bar{g}_p k}$ on $\text{Hom}(\mathcal{H}_p, \mathfrak{m})$ by setting :

$$\forall u, v \in \text{Hom}(\mathcal{H}_p, \mathfrak{m}), \quad \langle u, v \rangle_{\bar{g}_p k} = \text{tr}(\bar{g}_p^{-1} \circ {}^t u \circ k \circ v)$$

($\bar{g}_p : \mathcal{H}_p \longrightarrow \mathcal{H}_p^*$ and $k : \mathfrak{m} \longrightarrow \mathfrak{m}^*$ being the isomorphisms canonically associated to the corresponding scalar products). It is not difficult to check that $\|\mathcal{P}_{ph}\|_{\bar{g}_p k}^2 = \|\mathcal{P}_p\|_{\bar{g}_p k}^2$ for any $h \in H$.

The action density of Φ is then the function $\mathcal{L}(\Phi) \in C^\infty(M, \mathbb{R})$ given by :

$$\mathcal{L}(\Phi)(x) = \frac{1}{2} \|\mathcal{P}_p\|_{\bar{g}_p k}^2$$

where p is any element of the fiber P_x of P above x .

Theorem 4.4.8 .

1. For any $a \in G$, $\mathcal{L}(l_a(\Phi)) = \mathcal{L}(\Phi)$.
2. For any $h \in C^\infty(P, H)$, $\mathcal{L}(h \cdot \Phi) = \mathcal{L}(\Phi)$.

In other terms, the action density \mathcal{L} is invariant under the action of G and is gauge-invariant for the group H .

Proof :

1. For any $a \in G$, $x \in M$ and $p \in P_x$, we have : $\mathcal{L}(l_a(\Phi))(x) = \frac{1}{2} \|l_a(\mathcal{P})_p\|_{\bar{g}_p k}^2 = \frac{1}{2} \|\mathcal{P}_p\|_{\bar{g}_p k}^2 = \mathcal{L}(\Phi)(x)$ (since $l_a(\mathcal{P}) = \mathcal{P}$).
2. For any $h \in C^\infty(P, H)$, $x \in M$ and $p \in P_x$, we have : $\mathcal{L}(h \cdot \Phi)(x) = \frac{1}{2} \|(h \cdot \mathcal{P})_p\|_{\bar{g}_p k}^2 = \frac{1}{2} \|\text{Ad}_{h(p)} \circ \mathcal{P}_p\|_{\bar{g}_p k}^2 = \frac{1}{2} \text{tr}(\bar{g}_p^{-1} \circ {}^t \mathcal{P}_p \circ {}^t \text{Ad}_{h(p)} \circ h \circ \text{Ad}_{h(p)} \circ \mathcal{P}_p) = \frac{1}{2} \text{tr}(\bar{g}_p^{-1} \circ {}^t \mathcal{P}_p \circ h \circ \mathcal{P}_p) = \mathcal{L}(\Phi)(x)$

$$= \frac{1}{2} \|\mathcal{P}_p\|_{\tilde{g}_p k}^2 = \mathcal{L}(\Phi)(x) \quad (\text{since } k \text{ is } \text{Ad}(H)\text{-invariant}).$$

□

Set $\mathcal{F} = \Phi^* \dot{\Omega} = \mathcal{D}^Q \mathcal{Q}$ and $\Phi^* \dot{T} = \mathcal{D}^Q \mathcal{P}$.

Theorem 4.4.9 \mathcal{Q} and \mathcal{P} satisfy the following **structure equations** :

$$\left\{ \begin{array}{lcl} \mathcal{F} & = & d\mathcal{Q} + \frac{1}{2}[\mathcal{Q}, \mathcal{Q}] = -\frac{1}{2}[\mathcal{P}, \mathcal{P}]_{\mathfrak{h}} \\ \mathcal{D}^Q \mathcal{P} & = & d\mathcal{P} + [\mathcal{Q}, \mathcal{P}] = -\frac{1}{2}[\mathcal{P}, \mathcal{P}]_{\mathfrak{m}} \end{array} \right.$$

4.5 $E_7/\mathbf{SU}(8)$

Let W be an n -dimensional complex vector space. Recall that for every $k \in \{1, \dots, n\}$, there exists a canonical non-degenerate bilinear form $\bigwedge^k(W^*) \times \bigwedge^k(W) \rightarrow \mathbb{C}$, which associates to each $(f_1 \wedge \dots \wedge f_k, \xi_1 \wedge \dots \wedge \xi_k)$ the complex number $\sum_{\sigma \in S_k} \varepsilon(\sigma) f_1(\xi_{\sigma(1)}) \dots f_k(\xi_{\sigma(k)})$.

Therefore, $\bigwedge^k(W)^* \simeq \bigwedge^k(W^*)$.

Recall also that a *volume element* (or, equivalently, an $SL(W)$ -*structure*) on W is an element of $\bigwedge^n(W) \setminus \{0\}$. Since $\dim \bigwedge^n(W) = 1$, choosing a volume element on W amounts to choosing a basis of $\bigwedge^n(W)$, that is, an isomorphism $\bigwedge^n(W) \rightarrow \mathbb{C}$.

Proposition 4.5.1 *The choice of a volume element on W gives a natural non-degenerate bilinear form $\bigwedge^k(W) \times \bigwedge^{n-k}(W) \rightarrow \mathbb{C}$.*

Proof : It is sufficient to compose the natural bilinear map $\bigwedge^k(W) \times \bigwedge^{n-k}(W) \rightarrow \bigwedge^n(W)$, $(\xi, \eta) \mapsto \xi \wedge \eta$ with the isomorphism $\bigwedge^n(W) \rightarrow \mathbb{C}$ provided by the volume element. □

From the preceding proposition, we have $\bigwedge^k(W) \simeq \bigwedge^{n-k}(W)^*$ when W is equipped with a volume element. Since $\bigwedge^{n-k}(W)^* \simeq \bigwedge^{n-k}(W^*)$, we deduce the existence of a natural isomorphism between $\bigwedge^k(W)$ and $\bigwedge^{n-k}(W^*)$.

Definition 4.5.2 *Let W be an n -dimensional complex vector space endowed with a volume element α . The **generalized Hodge star** is the natural isomorphism*

$$\hat{*} : \bigwedge^k(W) \rightarrow \bigwedge^{n-k}(W^*)$$

Explicitely, to $\xi \in \bigwedge^k(W)$, we associate $\hat{*}\xi \in \bigwedge^{n-k}(W^*)$ in the following way :

we consider the linear form $\varphi_\xi \in \bigwedge^{n-k}(W)^*$ which sends every $\eta \in \bigwedge^{n-k}(W)$ to the unique complex number $\varphi_\xi(\eta)$ such that : $\xi \wedge \eta = \varphi_\xi(\eta)\alpha$. Then $\hat{*}\xi$ is defined as the unique element of $\bigwedge^{n-k}(W^*)$ whose image is φ_ξ under the canonical isomorphism $\bigwedge^{n-k}(W^*) \rightarrow \bigwedge^{n-k}(W)^*$.

We denote also by $\hat{*}$ the isomorphism $\bigwedge^{n-k}(W^*) \rightarrow \bigwedge^{n-(n-k)}(W^{**}) \simeq \bigwedge^k(W)$. If $\hat{*} \circ \hat{*}$ denotes the composition $\bigwedge^k(W) \rightarrow \bigwedge^{n-k}(W^*) \rightarrow \bigwedge^k(W)$, one can show that

$$\hat{*} \circ \hat{*} = (-1)^{k(n-k)} \text{Id}.$$

We shall need the natural representations of the Lie algebra $\mathfrak{sl}(W)$ on $\Lambda^2(W)$ and $\Lambda^2(W^*)$, respectively defined by :

$$\lambda \cdot (\xi_1 \wedge \xi_2) = \lambda(\xi_1) \wedge \xi_2 + \xi_1 \wedge \lambda(\xi_2) \quad \text{and} \quad \lambda \cdot (f_1 \wedge f_2) = {}^t\lambda(f_1) \wedge f_2 + f_1 \wedge {}^t\lambda(f_2).$$

From now, W is an 8-dimensional complex vector space endowed with a volume element α .

We define the vector spaces : $\mathfrak{e}_7^{\mathbb{C}} = \mathfrak{sl}(W) \oplus \Lambda^4(W)$ and $F^{\mathbb{C}} = \Lambda^2(W) \oplus \Lambda^2(W^*)$,

and the injective linear map : $\rho^{\mathbb{C}} : \mathfrak{e}_7^{\mathbb{C}} \longrightarrow \mathfrak{gl}(F^{\mathbb{C}})$ given by :

$$\rho^{\mathbb{C}}(\lambda, \sigma)(\xi, f) = (\lambda \cdot \xi + \hat{*}(\hat{*}\sigma \wedge f), \lambda \cdot f + \hat{*}(\sigma \wedge \xi))$$

for every $(\lambda, \sigma) \in \mathfrak{e}_7^{\mathbb{C}}$ and $(\xi, f) \in F^{\mathbb{C}}$.

Notice that since $\sigma \in \Lambda^4(W)$ and $f \in \Lambda^2(W^*)$, we have $\hat{*}\sigma \in \Lambda^4(W^*)$ and $\hat{*}\sigma \wedge f \in \Lambda^6(W^*)$. Consequently, $\hat{*}(\hat{*}\sigma \wedge f) \in \Lambda^2(W)$.

Similarly, since $\sigma \in \Lambda^4(W)$ and $\xi \in \Lambda^2(W)$, we have $\sigma \wedge \xi \in \Lambda^6(W)$. Consequently, $\hat{*}(\sigma \wedge \xi) \in \Lambda^2(W^*)$.

We have : $\dim_{\mathbb{C}} \mathfrak{e}_7^{\mathbb{C}} = \dim_{\mathbb{C}} \mathfrak{sl}(W) + \dim_{\mathbb{C}} \Lambda^4(W) = 63 + 70 = 133$,

and $\dim_{\mathbb{C}} F^{\mathbb{C}} = \dim_{\mathbb{C}} \Lambda^2(W) + \dim_{\mathbb{C}} \Lambda^2(W^*) = 28 + 28 = 56$.

Theorem 4.5.3 $\rho^{\mathbb{C}}(\mathfrak{e}_7^{\mathbb{C}})$ is a complex Lie subalgebra of $\mathfrak{gl}(F^{\mathbb{C}})$.

We refer to [Ad2] for a proof.

Thus, there is a complex Lie algebra structure on $\mathfrak{e}_7^{\mathbb{C}}$ such that the above linear map $\rho^{\mathbb{C}}$ is a faithful Lie algebra representation.

Now we assume that W is also endowed with an hermitian structure compatible with the volume element α (so that if (e_i) is an orthonormal basis of W , then $e_1 \wedge \dots \wedge e_8 = \pm \alpha$). Let $\theta : W \longrightarrow W^*$ be the conjugate-linear isomorphism defined by the hermitian structure on W . Then θ induces a conjugate-linear isomorphism $\hat{\theta} : \Lambda^2(W) \longrightarrow \Lambda^2(W^*)$ defined by : $\hat{\theta}(\xi_1 \wedge \xi_2) = \theta(\xi_1) \wedge \theta(\xi_2)$.

It is not difficult to show that the map $c : F^{\mathbb{C}} \longrightarrow F^{\mathbb{C}}$ defined by : $c(\xi, f) = (\hat{\theta}^{-1}(f), \hat{\theta}(\xi))$ is a conjugation on $F^{\mathbb{C}}$. Thus, we get a real structure by setting $F = \text{Ker}(c - \text{Id}_{F^{\mathbb{C}}})$ (so that : $F^{\mathbb{C}} = F \oplus iF$).

On the other hand, the map $\mathfrak{sl}(W) \longrightarrow \mathfrak{sl}(W)$, $\lambda \mapsto \lambda^\dagger$ is a conjugation on $\mathfrak{sl}(W)$ (λ^\dagger being the adjoint of λ for the hermitian structure defined on W). The eigenspace corresponding to $+1$ is the real vector space $i\mathfrak{su}(W)$ of traceless hermitian endomorphisms of W , while the eigenspace corresponding to -1 is the Lie algebra $\mathfrak{su}(W)$ of traceless antihermitian endomorphisms of W , and we have : $\mathfrak{sl}(W) = i\mathfrak{su}(W) \oplus \mathfrak{su}(W)$.

Finally, θ induces a conjugate-linear isomorphism $\hat{\theta} : \Lambda^4(W) \longrightarrow \Lambda^4(W^*)$ defined by :
 $\hat{\theta}(\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4) = \theta(\xi_1) \wedge \theta(\xi_2) \wedge \theta(\xi_3) \wedge \theta(\xi_4)$.

Composing $\hat{\theta}^{-1}$ with generalized Hodge star $\hat{*} : \Lambda^4(W) \longrightarrow \Lambda^4(W^*)$, we obtain a conjugate-linear involution $\hat{\theta}^{-1} \circ \hat{*} : \Lambda^4(W) \longrightarrow \Lambda^4(W)$, that is, a conjugation on $\Lambda^4(W)$.

Thus, we get a real structure by setting : $\Lambda_+^4(W) = \text{Ker}(\hat{\theta}^{-1} \circ \hat{*} - \text{Id})$ and

$$\Lambda_-^4(W) = \text{Ker}(\hat{\theta}^{-1} \circ \hat{*} + \text{Id}) = i \Lambda_+^4(W) \quad (\text{so that} : \Lambda^4(W) = \Lambda_+^4(W) \oplus \Lambda_-^4(W)).$$

In fact, $\hat{\theta}^{-1} \circ \hat{*}$ is nothing but the usual Hodge star operator $*$ on the hermitian vector space W , and therefore $\Lambda_+^4(W)$ (resp. $\Lambda_-^4(W)$) is the real vector space of *self-dual* (resp. *anti-self-dual*) elements of $\Lambda^4(W)$.

We define the vector space : $\mathfrak{e}_7 = \mathfrak{su}(W) \oplus \Lambda_+^4(W)$,

and the injective linear map : $\rho : \mathfrak{e}_7 \longrightarrow \mathfrak{gl}(F)$ given by :

$$\rho(\lambda, \sigma)(\xi, \hat{\theta}(\xi)) = (\lambda \cdot \xi + \hat{*}(\hat{*}\sigma \wedge \hat{\theta}(\xi)), \lambda \cdot \hat{\theta}(\xi) + \hat{*}(\sigma \wedge \xi))$$

for every $(\lambda, \sigma) \in \mathfrak{e}_7$ and $(\xi, \hat{\theta}(\xi)) \in F$,
with $F = \text{Ker}(c - \text{Id}_{F^\mathbb{C}}) = \{(\xi, \hat{\theta}(\xi)) ; \xi \in \Lambda^2(W)\}$.

We have : $\dim_{\mathbb{R}} \mathfrak{e}_7 = \dim_{\mathbb{R}} \mathfrak{su}(W) + \dim_{\mathbb{R}} \Lambda_+^4(W) = 63 + 70 = 133$,

and $\dim_{\mathbb{R}} F = 56$.

Proposition 4.5.4 $\rho(\mathfrak{e}_7)$ is a real Lie subalgebra of $\mathfrak{gl}(F)$.

Thus, there is a real Lie algebra structure on \mathfrak{e}_7 such that the above linear map ρ is a faithful Lie algebra representation.

Let J be the unique complex structure on F such that the linear map
 $\Phi : \Lambda^2(W) \longrightarrow F$ defined by $\Phi(\xi) = (\xi, \hat{\theta}(\xi))$ is an isomorphism of complex vector spaces.
We have : $J(\xi, \hat{\theta}(\xi)) = (i\xi, -i\hat{\theta}(\xi))$.

On F we consider also the unique hermitian structure such that the linear map
 $\Phi : \Lambda^2(W) \longrightarrow F$ defined by $\Phi(\xi) = (\xi, \hat{\theta}(\xi))$ is an isometry of hermitian vector spaces
($\Lambda^2(W)$ has a natural hermitian structure induced by that of W ; it is defined by the conjugate-linear isomorphism $\hat{\theta} : \Lambda^2(W) \longrightarrow \Lambda^2(W^*) \simeq \Lambda^2(W)^*$).

Let a be the imaginary part of the hermitian product on F . Then a is a symplectic form on F . We denote by $\text{Sp}_{\mathbb{R}}(F)$ the symplectic group of F with respect to a , and by $\mathfrak{sp}_{\mathbb{R}}(F)$ the Lie algebra of $\text{Sp}_{\mathbb{R}}(F)$.

Proposition 4.5.5 The symplectic form a on F is \mathfrak{e}_7 -invariant, that is, $\rho(\mathfrak{e}_7) \subset \mathfrak{sp}_{\mathbb{R}}(F)$.

Let $\mathfrak{gl}_{\mathbb{C}}(F) = \{u \in \mathfrak{gl}_{\mathbb{R}}(F) / u \circ J = J \circ u\}$. Notice that the complex structure J on F is not \mathfrak{e}_7 -invariant, that is, $\rho(\mathfrak{e}_7) \not\subset \mathfrak{gl}_{\mathbb{C}}(F)$. Indeed, $\rho(\lambda, \sigma)(J(\xi, \hat{\theta}(\xi))) = \rho(\lambda, \sigma)(i\xi, -i\hat{\theta}(\xi))$

$$\begin{aligned}
&= (\lambda \cdot i\xi + \hat{*}(\hat{\sigma} \wedge \hat{\theta}(i\xi)) , \lambda \cdot \hat{\theta}(i\xi) + \hat{*}(\sigma \wedge i\xi)) \\
&= (i\lambda \cdot \xi - i \hat{*}(\hat{\sigma} \wedge \hat{\theta}(\xi)) , -i\lambda \cdot \hat{\theta}(\xi) + i \hat{*}(\sigma \wedge \xi)) \\
&\neq (i\lambda \cdot \xi + i \hat{*}(\hat{\sigma} \wedge \hat{\theta}(\xi)) , -i\lambda \cdot \hat{\theta}(\xi) - i \hat{*}(\sigma \wedge \xi)) \\
&= (i(\lambda \cdot \xi + \hat{*}(\hat{\sigma} \wedge \hat{\theta}(\xi))) , -i \hat{\theta}(\lambda \cdot \xi + \hat{*}(\hat{\sigma} \wedge \hat{\theta}(\xi)))) = J(\rho(\lambda, \sigma)(\xi, \hat{\theta}(\xi)))
\end{aligned}$$

Consequently, the hermitian structure on F is not \mathfrak{e}_7 -invariant, that is, $\rho(\mathfrak{e}_7) \not\subset \mathfrak{u}(F)$ (if $\rho(\mathfrak{e}_7) \subset \mathfrak{u}(F)$, we would have $\rho(\mathfrak{e}_7) \subset \mathfrak{gl}_{\mathbb{C}}(F)$ since $\mathfrak{u}(F) \subset \mathfrak{gl}_{\mathbb{C}}(F)$).

Let s be the real part of the hermitian product. Then s is a scalar product on F . This scalar product is not \mathfrak{e}_7 -invariant, otherwise the hermitian product would be \mathfrak{e}_7 -invariant since the symplectic form a is \mathfrak{e}_7 -invariant (and $\mathfrak{u}(F) = \mathfrak{o}_{\mathbb{R}}(F) \cap \mathfrak{sp}_{\mathbb{R}}(F)$).

Chapitre 5

Cartan geometries

5.1 Cartan geometries

Definition 5.1.1 A model geometry is a triple $((\mathfrak{g}, \mathfrak{h}), H, Ad)$ where :

1. $(\mathfrak{g}, \mathfrak{h})$ is an effective Klein pair,
2. H is a Lie group admitting \mathfrak{h} as Lie algebra,
3. $Ad : H \longrightarrow GL(\mathfrak{g})$ is a representation extending the adjoint representation of H .

Definition 5.1.2 .

1. The kernel of a model geometry is the kernel of the representation $Ad : H \longrightarrow GL(\mathfrak{g})$.
2. A model geometry is said to be **effective** if its kernel is $\{0\}$.
3. A model geometry is said to be **reductive** if there exists a vector subspace \mathfrak{m} of the Lie algebra \mathfrak{g} such that : $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ (reductive decomposition).

We note that the effectiveness assumption in the definition of a model geometry implies that the kernel is a discrete subgroup of H . We deduce that any locally effective Klein geometry (G, H) determines naturally a model geometry $((\mathfrak{g}, \mathfrak{h}), H, Ad)$.

Definition 5.1.3 Let $((\mathfrak{g}, \mathfrak{h}), H, Ad)$ be a model geometry and M a smooth manifold. A **Cartan geometry on M modeled on $((\mathfrak{g}, \mathfrak{h}), H, Ad)$** is a pair (P, θ) where :

1. P is a principal bundle over M with structural group H ,
2. θ is a \mathfrak{g} -valued one-form on P satisfying the following properties :
 - (a) For each $p \in P$, the linear map $\theta_p : T_p P \longrightarrow \mathfrak{g}$ is an isomorphism.
 - (b) $((r_h)^* \theta)_p = Ad_{h^{-1}} \circ \theta_p$ for any $p \in P$ and $h \in H$.
 - (c) $\theta(\tilde{X}) = X$ for any $X \in \mathfrak{h}$, where \tilde{X} denotes the right fundamental vector field associated to X by the right action of H on P .

$\theta \in \Lambda^1_H(P, \mathfrak{g})$ is called a **Cartan connection**. It is not a connection in the usual sense (*i.e.* an Ehresmann connection), but it can be related to an Ehresmann connection, as we will see later.

It results from the definition of a Cartan geometry that $\dim M = \dim \mathfrak{g} - \dim \mathfrak{h}$.

Definition 5.1.4 A Cartan geometry is said to be **effective** (*resp.* **reductive**) if its model geometry is effective (*resp.* reductive).

Definition 5.1.5 *The Cartan curvature of a Cartan geometry (P, θ) is the two-form $\Theta \in \bar{\Lambda}_H^2(P, \mathfrak{g})$ defined by :*

$$\Theta = d\theta + \frac{1}{2}[\theta, \theta]$$

Let $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$ be a reductive model geometry, and choose a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\pi_{\mathfrak{h}} : \mathfrak{g} \longrightarrow \mathfrak{h}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \longrightarrow \mathfrak{m}$ the corresponding projectors. Let M be a smooth manifold. (P, θ) being a Cartan geometry on M modeled on $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$, let $\mathbb{V} = P \times_{Ad} \mathfrak{m}$. For any $p \in P$, we set $\omega_p = \pi_{\mathfrak{h}} \circ \theta_p$ and $e_p = \pi_{\mathfrak{m}} \circ \theta_p$.

Proposition 5.1.6 .

1. ω is a connection one-form (in the usual sense of Ehresmann) on the principal bundle (P, M, H) .
2. e is a tensorial (i.e. an H -equivariant and horizontal) \mathfrak{m} -valued one-form on the principal bundle (P, M, H) (i.e. $e \in \bar{\Lambda}_H^1(P, \mathfrak{m})$).

ω is called the **connection** of the reductive Cartan geometry (P, θ) on M .

e is called the **soldering form** of the reductive Cartan geometry (P, θ) on M .

Remark 5.1.7 *It is well-known that $\bar{\Lambda}_H^1(P, \mathfrak{m}) \simeq \Lambda^1(M, \mathbb{V})$. So we may view e as a one-form on M with values in the vector bundle \mathbb{V} . It is easy then to show that $e : TM \longrightarrow \mathbb{V}$ is in fact an isomorphism of vector bundles.*

Definition 5.1.8 .

1. *The curvature of the reductive Cartan geometry (P, θ) on M is the curvature of the connection ω , that is the two-form $\Omega \in \bar{\Lambda}_H^2(P, \mathfrak{h})$ defined by : $\Omega = \mathcal{D}^\omega \omega$.*
2. *The torsion of the reductive Cartan geometry (P, θ) on M is the two-form $T \in \bar{\Lambda}_H^2(P, \mathfrak{m})$ defined by : $T = \mathcal{D}^\omega e$.*

Theorem 5.1.9 *The curvature and torsion satisfy the following equations, called the structure equations of the reductive Cartan geometry :*

$$\begin{cases} \Omega = d\omega + \frac{1}{2}[\omega, \omega] = \Theta_{\mathfrak{h}} - \frac{1}{2}[e, e]_{\mathfrak{h}} \\ T = de + [\omega, e] = \Theta_{\mathfrak{m}} - \frac{1}{2}[e, e]_{\mathfrak{m}} \end{cases}$$

Proof : To obtain the above equations, it is sufficient to project the definition of the Cartan curvature on \mathfrak{h} and \mathfrak{m} :

$$\begin{aligned} d\theta + \frac{1}{2}[\theta, \theta] &= \Theta \\ \iff d\omega + de + \frac{1}{2}[\omega + e, \omega + e] &= \Theta \\ \iff d\omega + \frac{1}{2}[\omega, \omega] + de + [\omega, e] + \frac{1}{2}[e, e] &= \Theta \\ \iff \Omega + T + \frac{1}{2}[e, e]_{\mathfrak{h}} + \frac{1}{2}[e, e]_{\mathfrak{m}} &= \Theta_{\mathfrak{h}} + \Theta_{\mathfrak{m}} \end{aligned}$$

$$\iff (\Omega + \frac{1}{2}[e, e]_{\mathfrak{h}} - \Theta_{\mathfrak{h}}) + (T + \frac{1}{2}[e, e]_{\mathfrak{m}} - \Theta_{\mathfrak{m}}) = 0$$

The first (resp. second) term in brackets takes its values in \mathfrak{h} (resp. in \mathfrak{m}). Therefore both terms vanish. \square

As we have said, every locally effective Klein geometry (G, H) determines naturally a model geometry $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$. In fact, every locally effective Klein geometry (G, H) defines canonically a Cartan geometry $(G, \dot{\theta})$ on $\dot{M} = G/H$, modeled on $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$. This canonical Cartan geometry is thus defined by : $P = G$ and $\theta = \dot{\theta}$. Therefore, the Cartan connection θ of this canonical Cartan geometry is nothing but the Maurer-Cartan form $\dot{\theta}$ on G . Moreover, the Cartan curvature $\dot{\Theta}$ of this canonical Cartan geometry vanishes, since $\dot{\Theta} = 0$ is exactly the Maurer-Cartan structure equation of the group G .

Finally, we define the Ricci and scalar curvatures of a Cartan geometry (P, θ) , if \mathfrak{m} is oriented and endowed with an $\text{Ad}(H)$ -invariant scalar product (giving rise to an orientation and a bundle metric η on $\mathbb{V} = P \times_{\text{Ad}} \mathfrak{m}$).

Composing $\text{Ad} : H \longrightarrow \text{SO}^\dagger(\mathfrak{m})$ with $\text{Ad} : \text{SO}^\dagger(\mathfrak{m}) \longrightarrow \text{GL}(\mathfrak{so}(\mathfrak{m}))$, we obtain a representation $\rho : H \longrightarrow \text{GL}(\mathfrak{so}(\mathfrak{m}))$, so we may consider the associated fiber bundle $\mathfrak{so}(\mathbb{V}) = P \times_\rho \mathfrak{so}(\mathfrak{m})$ (we have denoted this adjoint vector bundle by $\mathfrak{so}(\mathbb{V})$ because for each $x \in M$, the fiber above x is the Lie algebra $\mathfrak{so}(\mathbb{V}_x)$ of antisymmetric endomorphisms of \mathbb{V}_x). Now since $\bar{\Lambda}_H^2(P, \mathfrak{h}) \simeq \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$, we may view the curvature Ω as an element $R^\theta \in \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$.

Definition 5.1.10 .

1. *The Ricci curvature of the Cartan geometry (P, θ) is the element $Ric^\theta \in \Gamma(Sym^2 \mathbb{V}^*)$ defined by : $\forall x \in M, \forall u, v \in \mathbb{V}_x, Ric_x^\theta(u, v) = \text{tr}(r_x^\theta(u, v))$, where $r_x^\theta(u, v)$ is the endomorphism of \mathbb{V}_x defined by : $\forall w \in \mathbb{V}_x, r_x^\theta(u, v)(w) = R_x^\theta(e_x^{-1}(w), e_x^{-1}(u))(v)$.*
2. *The scalar curvature of the Cartan geometry (P, θ) is the function $R_{scal}^\theta : M \longrightarrow \mathbb{R}$ defined by : $\forall x \in M, R_{scal}^\theta(x) = \text{tr}(\eta_x^{-1} \circ Ric_x^\theta)$ (where the scalar product η_x and the symmetric bilinear form Ric_x^θ are viewed as isomorphisms from \mathbb{V}_x to \mathbb{V}_x^*).*

5.2 Gauge symmetries and Cartan geometries

Let $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$ be the model geometry deduced from a Klein geometry (G, H) , and M a smooth manifold of dimension $\dim \mathfrak{g} - \dim \mathfrak{h}$. We fix a principal bundle P over M with structural group H . We are going to show that the infinite-dimensional Lie group $C^\infty(P, G)$ acts naturally on the space of Cartan connections on P (modeled on $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad})$).

For this and for other purposes, it will be useful to rely on the following theorem, which relates Cartan connections and Ehresmann connections.

First, notice that whenever we have a Klein geometry (G, H) , and a principal bundle P over M with structural group H , we may extend P to a principal bundle Q over M with

structural group G . Indeed, it is sufficient to take the associate bundle $Q = P \times_H G$, with H acting on G by left translations. We have then a canonical injection $\iota : P \longrightarrow Q$ defined by : $\iota(p) = (p, 1_G)H$. Note that ι is H -equivariant.

Theorem 5.2.1 *Let \mathcal{C}_C be the space of Cartan connections on P modeled on $((\mathfrak{g}, \mathfrak{h}), H, Ad)$, and \mathcal{C}_E the space of Ehresmann connections $\check{\omega}$ on Q satisfying the following condition : $\forall p \in P, \text{Ker } \check{\omega}_{\iota(p)} \cap T_p \mu(T_p P) = \{0\}$. Then the map $\iota^* : \mathcal{C}_E \longrightarrow \mathcal{C}_C$ is a bijection.*

We know that there is a natural representation of the infinite-dimensional Lie algebra $C_G^\infty(Q, \mathfrak{g})$ on the space of Ehresmann connections on Q ; it is given by : $\delta_{\check{\Lambda}} \check{\omega} = \mathcal{D}^{\check{\omega}} \check{\Lambda} = d\check{\Lambda} + [\check{\omega}, \check{\Lambda}]$.

Let $\check{\omega} \in \mathcal{C}_E$ and $\theta = \iota^* \check{\omega}$ the associated Cartan connection on P . We still denote by $\check{\Lambda}$ the restriction $\iota^* \check{\Lambda} = \check{\Lambda} \circ \iota \in C_H^\infty(P, \mathfrak{g})$ of $\check{\Lambda} \in C_G^\infty(Q, \mathfrak{g})$.

Then $\iota^* \delta_{\check{\Lambda}} \check{\omega} = d\iota^* \check{\Lambda} + [\iota^* \check{\omega}, \iota^* \check{\Lambda}]$, that is :

$$\delta_{\check{\Lambda}} \theta = d\check{\Lambda} + [\theta, \check{\Lambda}] =: \mathcal{D}^\theta \check{\Lambda}$$

If the model geometry is reductive and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition, we may define $\Lambda \in C_H^\infty(P, \mathfrak{h})$ and $\xi \in C_H^\infty(P, \mathfrak{m})$ by $\Lambda = \pi_{\mathfrak{h}} \circ \check{\Lambda}$ and $\xi = \pi_{\mathfrak{m}} \circ \check{\Lambda}$ respectively. Then the above representation of $C_H^\infty(P, \mathfrak{g})$ on the space of Cartan connections splits :

$$\begin{aligned} \delta_{\check{\Lambda}} \omega + \delta_{\check{\Lambda}} e &= d\Lambda + d\xi + [\omega + e, \Lambda + \xi] = d\Lambda + d\xi + [\omega, \Lambda] + [\omega, \xi] + [e, \Lambda] + [e, \xi] \\ &= (d\Lambda + [\omega, \Lambda] + [e, \xi]_{\mathfrak{h}}) + (d\xi + [\omega, \xi] + [e, \xi]_{\mathfrak{m}} + [e, \Lambda]). \end{aligned}$$

We deduce that :

$$\begin{cases} \delta_{\check{\Lambda}} \omega &= \mathcal{D}^\omega \Lambda + [e, \xi]_{\mathfrak{h}} \\ \delta_{\check{\Lambda}} e &= \mathcal{D}^\omega \xi + [e, \xi]_{\mathfrak{m}} + [e, \Lambda] \end{cases}$$

Taking $\Lambda = 0$, we may restrict our attention to the action of the "gauge translation" $\xi \in C_H^\infty(P, \mathfrak{m})$:

$$\begin{cases} \delta_\xi \omega &= [e, \xi]_{\mathfrak{h}} \\ \delta_\xi e &= \mathcal{D}^\omega \xi + [e, \xi]_{\mathfrak{m}} \end{cases}$$

Let us consider again the general representation of $C_G^\infty(Q, \mathfrak{g})$ on the space of Ehresmann connections on Q : $\delta_{\check{\Lambda}} \check{\omega} = \mathcal{D}^{\check{\omega}} \check{\Lambda} = d\check{\Lambda} + [\check{\omega}, \check{\Lambda}]$.

A Cartan geometry is said to be *invariant* under the infinitesimal gauge transformation $\check{\Lambda}$ if : $\delta_{\check{\Lambda}} \theta = 0$ (which is equivalent to $\delta_{\check{\Lambda}} \check{\omega} = 0$, if $\theta = \iota^* \check{\omega}$).

Suppose θ invariant under $\check{\Lambda}$, so that : $\mathcal{D}^{\check{\omega}} \check{\Lambda} = 0$. For any $Y \in \Gamma(TQ)$, we have : $\mathcal{D}_Y^{\check{\omega}} \check{\Lambda} = 0$. We deduce that for any $X, Y \in \Gamma(TQ)$, we have : $\mathcal{D}_X^{\check{\omega}} \mathcal{D}_Y^{\check{\omega}} \check{\Lambda} = 0$ and $\mathcal{D}_Y^{\check{\omega}} \mathcal{D}_X^{\check{\omega}} \check{\Lambda} = 0$. Thus, $\mathcal{D}_X^{\check{\omega}} \mathcal{D}_Y^{\check{\omega}} \check{\Lambda} - \mathcal{D}_Y^{\check{\omega}} \mathcal{D}_X^{\check{\omega}} \check{\Lambda} = 0$, and therefore : $\check{\Omega}(X, Y) \check{\Lambda} = 0$ for all $X, Y \in \Gamma(TQ)$.

From $\theta = \iota^*\check{\omega}$, we deduce that : $\Theta = \iota^*\check{\Omega}$. Therefore, the above condition becomes :

$$\Theta(X, Y)\check{\Lambda} = 0$$

for all $X, Y \in \Gamma(TQ)$. Thus, the existence of a family of solutions $\check{\Lambda}$ for the equation $\mathcal{D}^\theta\check{\Lambda} = 0$ requires the integrability condition :

$$\Theta = 0$$

But $\Theta = 0$ is equivalent to the Maurer-Cartan structure equations :

$$\begin{cases} \Omega &= -\frac{1}{2}[e, e]_{\mathfrak{h}} \\ T &= -\frac{1}{2}[e, e]_{\mathfrak{m}} \end{cases}$$

Discarding global topological considerations, the above equation admits a unique solution : it is the canonical Cartan geometry associated to the Klein geometry (G, H) with space $\overset{\circ}{M} = G/H$.

Suppose now \mathfrak{m} is oriented and endowed with an $\text{Ad}(H)$ -invariant scalar product. We also assume the existence of a group homomorphism $\widetilde{\text{Ad}} : H \longrightarrow \text{Spin}^\dagger(\mathfrak{m})$ which lifts the adjoint representation $\text{Ad} : H \longrightarrow \text{SO}^\dagger(\mathfrak{m})$.

Consider a faithful representation $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(\mathfrak{m}) \longrightarrow \text{GL}(S)$, called the *spinor representation*.

Composing $\widetilde{\text{Ad}} : H \longrightarrow \text{Spin}^\dagger(\mathfrak{m})$ with $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(\mathfrak{m}) \longrightarrow \text{GL}(S)$, we obtain a representation $\rho : H \longrightarrow \text{GL}(S)$, so we may consider the associated vector bundle $\mathbb{S} = P \times_\rho S$. This is the **spinor bundle**.

In some cases, and this will be true for the interesting cases we are going to consider, the action of the group H on S extends to G , so that we may consider \mathbb{S} as the vector bundle associated to Q by the action $\check{\rho}$ of G on S , that is : $P \times_\rho S =: \mathbb{S} = Q \times_{\check{\rho}} S$. The derivative of the representation $\check{\rho} : G \longrightarrow \text{GL}(S)$ is the Lie algebra representation $\check{\rho}' : \mathfrak{g} \longrightarrow \mathfrak{gl}(S)$. It restricts to the representation $\rho' : \mathfrak{h} \longrightarrow \mathfrak{gl}(S)$, and \mathfrak{m} acts on S by Clifford multiplication : $\gamma : Cl(\mathfrak{m}) \longrightarrow \text{End}(S)$.

For each $\alpha \in \mathbb{R} \cup i\mathbb{R}$, there is a covariant derivative for the connection $\check{\omega}$ that acts on spinor fields : if $\epsilon \in C_G^\infty(Q, S) \simeq \Gamma(\mathbb{S})$, we set :

$$\mathcal{D}^{\check{\omega}, \alpha}\epsilon = \mathcal{D}^\omega\epsilon - \alpha \gamma(e)\epsilon$$

Here, $\gamma(e)$ denotes the element of $\bar{\Lambda}_H^1(P, \text{End}(S))$ defined by : $\gamma(e)_p = \gamma \circ e_p$ for every $p \in P$.

Definition 5.2.2 Let $\alpha \in \mathbb{R} \cup i\mathbb{R}$. A **Killing spinor with Killing constant α** is a spinor field $\epsilon \in \Gamma(\mathbb{S})$ such that : $\mathcal{D}^\omega\epsilon = \alpha \gamma(e)\epsilon$.

The above condition is equivalent to : $\mathcal{D}^{\check{\omega}, \alpha} \epsilon = 0$. Thus, Killing spinors with Killing constant α are in fact *parallel spinors* for the connection $\mathcal{D}^{\check{\omega}, \alpha}$.

In terms of the Cartan geometry (P, θ) , Killing spinors with Killing constant α are the solutions of the equation :

$$\mathcal{D}^{\theta, \alpha} \epsilon = 0$$

with $\mathcal{D}^{\theta, \alpha} = \mathcal{D}^\omega - \alpha \gamma(e)$ acting on spinor fields $\epsilon \in C_H^\infty(P, S) \simeq \Gamma(\mathbb{S})$.

Suppose ϵ is a Killing spinor, so that : $\mathcal{D}^{\check{\omega}, \alpha} \epsilon = 0$. For any $Y \in \Gamma(TQ)$, we have : $\mathcal{D}_Y^{\check{\omega}, \alpha} \epsilon = 0$. We deduce that for any $X, Y \in \Gamma(TQ)$, we have : $\mathcal{D}_X^{\check{\omega}, \alpha} \mathcal{D}_Y^{\check{\omega}, \alpha} \epsilon = 0$ and $\mathcal{D}_Y^{\check{\omega}, \alpha} \mathcal{D}_X^{\check{\omega}, \alpha} \epsilon = 0$. Thus, $\mathcal{D}_X^{\check{\omega}, \alpha} \mathcal{D}_Y^{\check{\omega}, \alpha} \epsilon - \mathcal{D}_Y^{\check{\omega}, \alpha} \mathcal{D}_X^{\check{\omega}, \alpha} \epsilon = 0$, and therefore : $\check{\Omega}(X, Y)\epsilon = 0$ for all $X, Y \in \Gamma(TQ)$.

In terms of the Cartan curvature Θ , the above condition becomes :

$$\Theta(X, Y)\epsilon = 0$$

for all $X, Y \in \Gamma(TQ)$.

Proposition 5.2.3 *If M admits a Killing spinor with Killing constant α , then :*

$$Ric^\theta = 4\alpha^2(d-1) \eta$$

5.3 Killing spinors on the seven-sphere

When we will consider the spontaneous compactification of eleven-dimensional supergravity, the Killing spinors on the seven-sphere will be of fundamental importance. Let us describe them.

We consider the canonical Cartan geometry associated to the Klein geometry $(\text{Spin}(8), \text{Spin}(7))$ with space the round seven-sphere of radius m^{-1} , denoted by $S^7(m^{-1})$. It is known that $\text{Spin}(7)$ admits a faithful irreducible eight-dimensional complex representation $\rho_{\frac{1}{2}} : \text{Spin}(7) \longrightarrow \text{GL}(W)$. We may consider then the spinor bundle $\mathbb{W} = \text{Spin}(7) \times_{\rho_{\frac{1}{2}}} W$.

For each $m > 0$, there exists Killing spinor fields $\tilde{\eta} \in \Gamma(\mathbb{W})$ on $S^7(m^{-1})$ with Killing constant $\alpha = \frac{1}{2}m$. We are going to show how they are constructed. In fact, when a group G acts on a manifold M (in particular when we have an homogeneous space $M = G/H$), one may associate to each $X \in \mathfrak{g}$ a **fundamental vector field** $\tilde{X} \in \Gamma(TM)$. If M is equipped with a G -invariant metric, one can show then that these fundamental vector fields are Killing vector fields for this metric. We proceed in an analogous way for spinors. We construct first **fundamental spinor fields** $\tilde{\eta} \in \Gamma(\mathbb{W})$ associated to the (constant) elements η of W . Then it is possible to show that these fundamental spinor fields $\tilde{\eta}$ satisfy the Killing equation : $\mathcal{D}^{\theta, \alpha} \tilde{\eta} = 0$

with $\mathcal{D}^{\theta, \alpha} = \mathcal{D}^\omega - \frac{1}{2}m \gamma(e)$ acting on spinor fields $\epsilon \in C_{\text{Spin}(7)}^\infty(\text{Spin}(8), W) \simeq \Gamma(\mathbb{W})$.

It is well-known that the Klein geometry $(\text{Spin}(8), \text{Spin}(7))$ is parallelizable.

Let $\sigma : S^7(m^{-1}) \longrightarrow \text{Spin}(8)$ be a global cross-section of the canonical bundle $(\text{Spin}(8), S^7(m^{-1}), \text{Spin}(7), \pi)$, and $\pi_2 : \text{Spin}(8) \longrightarrow \text{Spin}(7)$ the map which associates to

each $g \in \text{Spin}(8)$ the unique element $h \in \text{Spin}(7)$ such that : $g = \sigma(\pi(g))h$. For each $\eta \in W$, the fundamental spinor field is then defined by : $\tilde{\eta}(g) = \pi_2(g)^{-1}\eta$. It is easy to check that $\tilde{\eta} \in C_{\text{Spin}(7)}^\infty(\text{Spin}(8), W) \simeq \Gamma(\mathbb{W})$. One then proves that $\mathcal{D}^\omega \tilde{\eta} = \frac{1}{2}m \gamma(e)\tilde{\eta}$.

Chapitre 6

Gravities and supergravities in component formulation

6.1 Poincaré gravity and supergravity in component formulation

Let :

- V be a Lorentzian vector space of signature $(d - 1, 1)$,
- $\mathfrak{g} = \tilde{\pi}(V) = V \oplus' \mathfrak{spin}(V)$ (this is the product $V \times \mathfrak{spin}(V)$ equipped with the following Lie bracket : $[(v, \tilde{\lambda}), (w, \tilde{\mu})] = (\tilde{\lambda}(w) - \tilde{\mu}(v), [\tilde{\lambda}, \tilde{\mu}])$),
- $\mathfrak{h} = \mathfrak{spin}(V)$ (the $\mathfrak{spin}(V)$ Lie subalgebra of \mathfrak{g}),
- $H = \text{Spin}^\dagger(V)$,
- $\text{Ad} : H \longrightarrow \text{GL}(\mathfrak{g})$ defined by : $\text{Ad}_{\tilde{\Lambda}}(v, \tilde{\lambda}) = (\rho_1(\tilde{\Lambda})(v), \text{Ad}_{\tilde{\Lambda}}(\tilde{\lambda}))$.

Then $((\mathfrak{g}, \mathfrak{h}), H, \text{Ad}) = ((\tilde{\pi}(V), \mathfrak{spin}(V)), \text{Spin}^\dagger(V), \text{Ad})$ is a reductive model geometry, the natural reductive decomposition being : $\mathfrak{g} = \mathfrak{h} \oplus V$.

The above model geometry is the model geometry for Poincaré Einstein gravity. Note that we could have replaced $\mathfrak{spin}(V)$ and $\text{Spin}^\dagger(V)$ by $\mathfrak{so}(V)$ and $\text{SO}^\dagger(V)$ respectively, obtaining in this way an *effective* model geometry. This would be sufficient for a pure gravity theory, but since we are interested with coupling gravity to spinorial matter fields, we take the universal coverings in the definition of the model geometry.

The dynamical variable in a pure Einstein gravity theory is a Cartan geometry (P, θ) on a smooth d -dimensional manifold M , modeled on $((\tilde{\pi}(V), \mathfrak{spin}(V)), \text{Spin}^\dagger(V), \text{Ad})$.

Let ω (resp. e) be the connection (resp. the soldering form) of the reductive Cartan geometry (P, θ) . Then $\omega \in \Lambda^1_{\text{Spin}^\dagger(V)}(P, \mathfrak{spin}(V))$ is called the **spin connection** and $e \in \bar{\Lambda}^1_{\text{Spin}^\dagger(V)}(P, V)$ is called the **vielbein** by the physicists.

Let $\mathbb{V} = P \times_{\text{Ad}} V$. Since $\bar{\Lambda}^1_{\text{Spin}^\dagger(V)}(P, V) \simeq \Lambda^1(M, \mathbb{V})$, we may view e as a one-form on M with values in the vector bundle \mathbb{V} , and $e : TM \longrightarrow \mathbb{V}$ is an isomorphism of vector

bundles. We recall that there is a natural bundle metric η and orientation on \mathbb{V} .

We draw the reader's attention that the symbol η refers here to the bundle metric on \mathbb{V} , and not to the Minkowski scalar product on V .

Consider the curvature $\Omega \in \bar{\Lambda}_{Spin^\dagger(V)}^2(P, \mathfrak{spin}(V))$ and the torsion $T \in \bar{\Lambda}_{Spin^\dagger(V)}^2(P, V)$ of the reductive Cartan geometry (P, θ) .

Since V is an abelian Lie algebra, Maurer-Cartan structure equations implicate the torsion of the canonical Cartan geometry defined by the Klein geometry $(\tilde{\Pi}(V), Spin^\dagger(V))$ is $\dot{T} = 0$. This is also related to the fact that the model geometry $((\tilde{\pi}(V), \mathfrak{spin}(V)), Spin^\dagger(V), Ad)$ is *first-order flat*, as we are going to see in the fourth part of the thesis (cf. Remark 10.5.3).

We deduce that for any soldering form $e \in \bar{\Lambda}_{Spin^\dagger(V)}^1(P, V)$, there exists a unique spin connection $\omega \in \Lambda_{Spin^\dagger(V)}^1(P, \mathfrak{spin}(V))$ such that the torsion T of the Cartan geometry defined by e and ω vanishes.

Let g be the metric on M defined by $g = e^* \eta$. As a consequence of the Theorem 10.6.14 that we are going to establish in the fourth part of the thesis (and which remains true in the particular case of gravity), there is an equivalence between the action of $C^\infty(P, V)$ on e by infinitesimal gauge translations, and the action of the infinitesimal diffeomorphisms of M on g .

If we compose $\rho_1 : Spin^\dagger(V) \longrightarrow SO^\dagger(V)$ with $Ad : SO^\dagger(V) \longrightarrow GL(\mathfrak{so}(V))$, we obtain a representation $Ad \circ \rho_1 : Spin^\dagger(V) \longrightarrow GL(\mathfrak{so}(V))$, so we may consider the associated vector bundle $\mathfrak{so}(\mathbb{V}) = P \times_{Ad \circ \rho_1} \mathfrak{so}(V)$. Now since $\bar{\Lambda}_{Spin^\dagger(V)}^2(P, \mathfrak{spin}(V)) \simeq \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$, we may view the curvature Ω as an element $R^\theta \in \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$.

Denoting by vol_η the volume form on \mathbb{V} defined by the bundle metric and orientation of \mathbb{V} , the action functional of pure Einstein gravity is then :

$$\mathcal{A}(\theta) = \int_M -\frac{1}{2} R_{scal}^\theta e^* vol_\eta$$

This action is invariant under gauge translations and gauge Lorentz transformations.

The corresponding field equation is the Einstein equation :

$$Ric^\theta - \frac{1}{2} R_{scal}^\theta \eta = 0$$

which is equivalent to : $Ric^\theta = 0$.

Its solutions can be related to *Ricci-flat metrics* (that is, *Einstein metrics* with Einstein constant $c = 0$). If (P, θ) is a solution, then : $R_{scal}^\theta = 0$.

Consider the following equation :

$$R^\theta = 0$$

It is not immediate that any solution of $R^\theta = 0$ is also a solution of the field equation $Ric^\theta = 0$.

Solutions of $R^\theta = 0$ are called Cartan geometries *with zero curvature*.

In fact, discarding global topological considerations, the above equation admits a unique solution : it is the canonical Cartan geometry associated to the Klein geometry $(\tilde{\Pi}(V), \text{SO}^\dagger(V))$ whose space is Minkowski space.

Consider a faithful representation $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(V) \longrightarrow \text{GL}(S)$, called the *spinor representation*. The **spinor bundle** is the associated vector bundle : $\mathbb{S} = P \times \rho_{\frac{1}{2}} S$.

There exists a $\text{Spin}^\dagger(V)$ -invariant non-degenerate bilinear form

$\varepsilon : S \times S \longrightarrow \mathbb{R}$. For any $\Psi \in S$, we denote by Ψ^* the element of S^* defined by : $\Psi^* = \varepsilon(\Psi, .)$, so that $\Psi_1^* \Psi_2 = \varepsilon(\Psi_1, \Psi_2)$.

A **Rarita-Schwinger spinor field** is an element of $\Lambda^1(M, \mathbb{S})$. The spin connection ω induces a covariant derivative \mathcal{D}^ω which sends an element $\psi \in \Lambda^1(M, \mathbb{S})$ to $\mathcal{D}^\omega \psi \in \Lambda^2(M, \mathbb{S})$.

We define : $\gamma^*(e) = \gamma \circ \eta^{-1} \circ {}^t e^{-1} : T^*M \longrightarrow \mathbb{V}^* \longrightarrow \mathbb{V} \longrightarrow \text{End}(\mathbb{S})$, and we set : $\gamma_{(3)}^*(e) = \gamma^*(e) \wedge \gamma^*(e) \wedge \gamma^*(e)$.

We concentrate here on the soldering form and the Rarita-Schwinger spinor field, which are the basic fields in Poincaré supergravity. But one should be aware that supergravity theories involve in general other fields also (for example a 3-form in eleven dimensions, but even in four dimensions one has to include other auxiliary fields when dealing with the off-shell formulation).

The action functional of Poincaré supergravity is :

$$\mathcal{A}(\theta, \psi) = \int_M -\frac{1}{2} \{ R_{scal}^\theta + \text{Tr}(\psi^* \wedge \gamma_{(3)}^*(e) \wedge \mathcal{D}^\omega \psi) + \dots \} e^* \text{vol}_\eta$$

where $\text{Tr}(\psi^* \wedge \gamma_{(3)}^*(e) \wedge \mathcal{D}^\omega \psi)$ is the function obtained by exterior product, spinor contractions and tensor contractions of the following maps :

$$\psi^* : TM \longrightarrow \mathbb{S}^*$$

$$\gamma_{(3)}^*(e) : T^*M \wedge \bigwedge^2 T^*M \longrightarrow \text{End}(\mathbb{S})$$

$$\mathcal{D}^\omega \psi : \bigwedge^2 TM \longrightarrow \mathbb{S}$$

If we take the canonical Cartan geometry associated to the Klein geometry $(\tilde{\Pi}(V), \text{Spin}^\dagger(V))$, together with the Rarita-Schwinger spinor field $\dot{\psi} = 0$, then we have clearly a solution of the field equations associated to the above action.

Remark 6.1.1 *The above action is still invariant under gauge translations and gauge Lorentz transformations, but it is also invariant (up to terms that are cubic in the Rarita-Schwinger field) under a new kind of symmetry discovered by the physicists : the supersymmetry transformations, capable of exchanging bosonic fields with fermionic fields (cf. [HLS] for example). These transformations (that are gauged here, i.e. depend on the spacetime point) may be written :*

$$\delta e = \frac{1}{2} \Gamma(\epsilon, \psi) \quad \text{and} \quad \delta \psi = \mathcal{D}^\omega \epsilon$$

where the parameter $\epsilon \in \Gamma(\mathbb{S} \otimes L^{od})$, L^{od} being the odd part of a Grassmann algebra, so that ϵ is an anticommuting spinor. At this stage, and in order to explore further these supersymmetries, we have to jump to the world of supersymmetric mathematics, where all the mathematical objects will be graded into even and odd part, in order to give a transparent geometric understanding of supersymmetry transformations like the one written above. This will be the subject of the next parts of the thesis.

6.2 Anti-de Sitter gravity and supergravity in component formulation

Now we turn to Anti-de Sitter gravity and supergravity, which correspond to a different model geometry.

\mathbb{R}^{d+1} being equipped with the non-degenerate symmetric bilinear form of signature $(d-1, 2)$ associated to the matrix $\tilde{\eta} = \text{diag}(-1, \eta) = \text{diag}(-1, -1, +1, \dots, +1)$ in the canonical basis $(e_m)_{-1 \leq m \leq d-1}$, and μ being a non-negative real number, we define **the Anti-de Sitter space of radius μ^{-1}** as :

$$\text{AdS}_d(\mu^{-1}) = \{ Y \in \mathbb{R}^{d+1} / {}^t Y \tilde{\eta} Y = -\mu^{-2} \}$$

Let $G := \text{SO}^\uparrow(d-1, 2)$ be the connected component of the identity in $\text{O}(d-1, 2)$, and H the following subgroup of G :

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} ; \Lambda \in \text{SO}^\uparrow(d-1, 1) \right\}$$

Then $\text{AdS}_d(\mu^{-1})$ is clearly invariant under the standard representation of $G = \text{SO}^\uparrow(d-1, 2)$ on \mathbb{R}^{d+1} . In fact, $\text{SO}^\uparrow(d-1, 2)$ acts transitively on $\text{AdS}_d(\mu^{-1})$, and the stabilizer of $\mu^{-1}e_{-1}$ is nothing but the subgroup H .

We deduce that $(G, H) = (\text{SO}^\uparrow(d-1, 2), \text{SO}^\uparrow(d-1, 1))$ is an effective Klein geometry, whose space $\text{SO}^\uparrow(d-1, 2)/\text{SO}^\uparrow(d-1, 1)$ is naturally identified to $\text{AdS}_d(\mu^{-1})$ through the surjective map $\pi : \text{SO}^\uparrow(d-1, 2) \longrightarrow \text{AdS}_d(\mu^{-1})$ defined by : $\pi(a) = a \mu^{-1}e_{-1}$.

Let $\mathfrak{g} = \mathfrak{so}(d-1, 2)$ and $\mathfrak{h} = \mathfrak{so}(d-1, 1)$ be the Lie algebras of G and H respectively. Writing :

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & {}^t v \eta \\ v & \lambda \end{pmatrix} ; v \in \mathbb{R}^d \text{ and } \lambda \in \mathfrak{so}(d-1, 1) \right\}$$

we see that we have a natural reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where :

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} ; \lambda \in \mathfrak{so}(d-1, 1) \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & {}^t v \eta \\ v & 0 \end{pmatrix} ; v \in \mathbb{R}^d \right\}$$

Here, η is the Lorentzian scalar product obtained by restricting $\tilde{\eta}$ to the d -dimensional vector subspace $V = (\mathbb{R}e_{-1})^\perp$. In the basis $(e_m)_{0 \leq m \leq d-1}$ of V , we have $\eta = \text{diag}(-1, +1, \dots, +1)$.

It is easy to check that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Therefore, $\text{AdS}_d(\mu^{-1})$ is a symmetric space, and the canonical Cartan geometry associated to the Klein geometry $(\text{SO}^\dagger(d-1, 2), \text{SO}^\dagger(d-1, 1))$ has vanishing torsion.

The adjoint representation $\text{Ad} : H \longrightarrow \text{GL}(\mathfrak{g})$ can be explicitly computed :

$$\text{Ad}_h(X) = hXh^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 0 & {}^t v\eta \\ v & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & {}^t v\eta \Lambda^{-1} \\ \Lambda v & \Lambda \lambda \Lambda^{-1} \end{pmatrix}$$

Notice that the adjoint representation of H on \mathfrak{m} is equivalent to the standard representation of $\text{SO}^\dagger(d-1, 1)$ on \mathbb{R}^d .

We have defined an effective model geometry :

$$((\mathfrak{g}, \mathfrak{h}), H, \text{Ad}) = ((\mathfrak{so}(d-1, 2), \mathfrak{so}(d-1, 1)), \text{SO}^\dagger(d-1, 1), \text{Ad}).$$

Since we are interested with a supergravity theory (including spinorial matter fields), we define another model geometry by taking the universal coverings :

$$((\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}), \tilde{H}, \text{Ad}) = ((\mathfrak{spin}(d-1, 2), \mathfrak{spin}(d-1, 1)), \text{Spin}^\dagger(d-1, 1), \text{Ad}). \text{ This model geometry is not effective, but it is reductive.}$$

The above model geometry is the model geometry for Anti-de Sitter gravity.

The dynamical variable in a pure Anti-de Sitter gravity theory is a Cartan geometry (P, θ) on a smooth d -dimensional manifold M , modeled on

$$((\mathfrak{so}(d-1, 2), \mathfrak{so}(d-1, 1)), \text{SO}^\dagger(d-1, 1), \text{Ad}).$$

Let ω (resp. e) be the connection (resp. the soldering form) of the reductive Cartan geometry (P, θ) . Then $\omega \in \Lambda_{\text{Spin}^\dagger(d-1, 1)}^1(P, \mathfrak{spin}(d-1, 1))$ is called the **spin connection** and $e \in \bar{\Lambda}_{\text{Spin}^\dagger(d-1, 1)}^1(P, \mathfrak{m})$ is called the **vielbein** by the physicists.

Let $\mathbb{V} = P \times_{Ad} \mathfrak{m}$. Since $\bar{\Lambda}_{\text{Spin}^\dagger(d-1, 1)}^1(P, \mathfrak{m}) \simeq \Lambda^1(M, \mathbb{V})$, we may view e as a one-form on M with values in the vector bundle \mathbb{V} , and $e : TM \longrightarrow \mathbb{V}$ is an isomorphism of vector bundles.

Consider the curvature $\Omega \in \bar{\Lambda}_{\text{Spin}^\dagger(d-1, 1)}^2(P, \mathfrak{spin}(d-1, 1))$ and the torsion $T \in \bar{\Lambda}_{\text{Spin}^\dagger(d-1, 1)}^2(P, \mathfrak{m})$ of the reductive Cartan geometry (P, θ) . Composing $\rho_1 : \text{Spin}^\dagger(d-1, 1) \longrightarrow \text{SO}^\dagger(d-1, 1)$ with $\text{Ad} : \text{SO}^\dagger(d-1, 1) \longrightarrow \text{GL}(\mathfrak{so}(d-1, 1))$, we obtain a representation $\text{Ad} \circ \rho_1 : \text{Spin}^\dagger(d-1, 1) \longrightarrow \text{GL}(\mathfrak{so}(d-1, 1))$, so we may consider the associated vector bundle $\mathfrak{so}(\mathbb{V}) = P \times_{Ad \circ \rho_1} \mathfrak{so}(d-1, 1)$. Now since $\bar{\Lambda}_{\text{Spin}^\dagger(d-1, 1)}^2(P, \mathfrak{spin}(d-1, 1)) \simeq \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$, we may view the curvature Ω as an element $R^\theta \in \Lambda^2(M, \mathfrak{so}(\mathbb{V}))$.

Denoting by vol_η the volume form on \mathbb{V} defined by the bundle metric and orientation of \mathbb{V} , the action functional of pure Anti-de Sitter gravity is then :

$$\mathcal{A}(\theta) = \int_M -\frac{1}{2} \{ R_{scal}^\theta + \mu^2(d-1)(d-2) \} e^* vol_\eta$$

This action is invariant under $\text{Spin}^\dagger(d-1, 2)$ gauge transformations.

The corresponding field equation is the Einstein equation with cosmological constant $\Lambda = -\frac{1}{2}\mu^2(d-1)(d-2)$:

$$Ric^\theta - \frac{1}{2}R_{scal}^\theta \eta - \frac{1}{2}\mu^2(d-1)(d-2)\eta = 0$$

Proposition 6.2.1 .

1. Assuming $d > 2$, the two following statements are equivalent for any $\Lambda \in \mathbb{R}$:
 - (a) The Cartan geometry (P, θ) satisfies the Einstein equation with cosmological constant Λ :
- $$Ric^\theta - \frac{1}{2}R_{scal}^\theta \eta + \Lambda \eta = 0$$
- (b) The Cartan geometry (P, θ) satisfies the following equation : $Ric^\theta = c \eta$
with $c = \frac{2}{d-2}\Lambda$.
2. When one of the above statements is satisfied, $R_{scal}^\theta = c d = \frac{2d}{d-2}\Lambda$.

Thus, the field equation may be written : $Ric^\theta = -\mu^2(d-1)\eta$.

Its solutions can be related to *Einstein metrics* with Einstein constant $c = -\mu^2(d-1)$. If (P, θ) is a solution, then : $R_{scal}^\theta = -\mu^2 d(d-1)$.

Consider the following equation :

$$R^\theta = -2\mu^2 \eta(e) \wedge e$$

where $\eta(e) \wedge e$ is the element of $\Lambda^2(M, \mathfrak{so}(\mathbb{V}))$ defined by :

$$(\eta(e) \wedge e)(X, Y) = \eta(e(X)) \otimes e(Y) - \eta(e(Y)) \otimes e(X) \text{ for every } X, Y \in \Gamma(TM).$$

It is not difficult to see that any solution of $R^\theta = -2\mu^2 \eta(e) \wedge e$ is also a solution of the field equation $Ric^\theta = -\mu^2(d-1)\eta$.

Solutions of $R^\theta = -2\mu^2 \eta(e) \wedge e$ are called Cartan geometries *with constant curvature* μ .

In fact, discarding global topological considerations, the above equation admits a unique solution : it is the canonical Cartan geometry associated to the Klein geometry $(SO^\dagger(d-1, 2), SO^\dagger(d-1, 1))$ with space $AdS_d(\mu^{-1})$.

Consider a faithful representation $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(d-1, 1) \longrightarrow \text{GL}(S)$, called the *spinor representation*. The **spinor bundle** is the associated vector bundle : $\mathbb{S} = P \times \rho_{\frac{1}{2}} S$.

There exists a $\text{Spin}^\dagger(V)$ -invariant non-degenerate bilinear form

$\varepsilon : S \times S \longrightarrow \mathbb{R}$. For any $\Psi \in S$, we denote by Ψ^* the element of S^* defined by : $\Psi^* = \varepsilon(\Psi, .)$, so that $\Psi_1^* \Psi_2 = \varepsilon(\Psi_1, \Psi_2)$.

We recall that $\gamma(e) = \gamma \circ e : TM \longrightarrow \mathbb{V} \longrightarrow \text{End}(\mathbb{S})$. Moreover, we define :

$\gamma^*(e) = \gamma \circ \eta^{-1} \circ {}^t e^{-1} : T^*M \longrightarrow \mathbb{V}^* \longrightarrow \mathbb{V} \longrightarrow \text{End}(\mathbb{S})$, and we set :

$$\gamma_{(3)}^*(e) = \gamma^*(e) \wedge \gamma^*(e) \wedge \gamma^*(e).$$

A **Rarita-Schwinger spinor field** is an element of $\Lambda^1(M, \mathbb{S})$. The spin connection ω induces a covariant derivative \mathcal{D}^ω which sends an element $\psi \in \Lambda^1(M, \mathbb{S})$ to $\mathcal{D}^\omega \psi \in \Lambda^2(M, \mathbb{S})$.

The action functional of Anti-de Sitter supergravity is then :

$$\mathcal{A}(\theta, \psi) = \int_M -\frac{1}{2} \{ R_{scal}^\theta + \mu^2(d-1)(d-2) + \text{Tr}(\psi^* \wedge \gamma_{(3)}^*(e) \wedge \mathcal{D}^{\theta,\alpha}\psi) + \dots \} e^* vol_\eta$$

$$\text{where } \mathcal{D}^{\theta,\alpha}\psi = \mathcal{D}^\omega\psi + \frac{1}{2} i\mu \gamma(e) \wedge \psi,$$

and $\text{Tr}(\psi^* \wedge \gamma_{(3)}^*(e) \wedge \mathcal{D}^{\theta,\alpha}\psi)$ is the function obtained by exterior product, spinor contractions and tensor contractions of the following maps :

$$\begin{aligned} \psi^* : TM &\longrightarrow \mathbb{S}^* \\ \gamma_{(3)}^*(e) : T^*M \wedge \bigwedge^2 T^*M &\longrightarrow \text{End}(\mathbb{S}) \\ \mathcal{D}^{\theta,\alpha}\psi : \bigwedge^2 TM &\longrightarrow \mathbb{S} \end{aligned}$$

Remark 6.2.2 This action is still invariant under $\text{Spin}^\dagger(d-1, 2)$ gauge transformations, but it is also invariant (up to terms that are cubic in the Rarita-Schwinger field) under "gauge supersymmetry transformations", with parameter $\epsilon \in \Gamma(\mathbb{S} \otimes L^{od})$ (cf. Remark 6.1.1) :

$$\delta_\epsilon e = \frac{1}{2} \Gamma(\epsilon, \psi) \quad \text{and} \quad \delta_\epsilon \psi = \mathcal{D}^{\theta,\alpha}\epsilon$$

If we take the canonical Cartan geometry associated to the Klein geometry $(SO^\dagger(d-1, 2), SO^\dagger(d-1, 1))$ with space $\text{AdS}_d(\mu^{-1})$, together with the Rarita-Schwinger spinor field $\dot{\psi} = 0$, then we have a solution of the field equations associated to the above action.

Consider now any Cartan geometry (P, θ) , together with the Rarita-Schwinger spinor field $\dot{\psi} = 0$, such that $(\theta, \dot{\psi})$ is a solution of supergravity field equation. Such a solution is said to be **supersymmetric** if there exists $\epsilon \in \Gamma(\mathbb{S})$ such that : $\delta_\epsilon e = 0$ and $\delta_\epsilon \dot{\psi} = 0$. It is said to be **fully supersymmetric** if there exists a family of spinors $\epsilon \in \Gamma(\mathbb{S})$ such that : $\delta_\epsilon e = 0$ and $\delta_\epsilon \dot{\psi} = 0$. Since $\dot{\psi} = 0$, we have $\Gamma(\epsilon, \dot{\psi}) = 0$, therefore the condition $\delta_\epsilon e = 0$ is always satisfied. We are left with the condition :

$$\mathcal{D}^{\theta,\alpha}\epsilon = 0$$

Solutions of this equation are the Killing spinors with Killing constant $\alpha = -\frac{1}{2} i\mu$.

Proposition 6.2.3 .

1. A necessary condition for the existence of Killing spinors is the following integrability condition : "the Cartan geometry (P, θ) must be a solution of the matter-free field equation : $\text{Ric}^\theta = -\mu^2(d-1)\eta$ ",
2. A necessary and sufficient condition for a solution to be fully supersymmetric is that $R^\theta = -2\mu^2 \eta(e) \wedge e$, so that (P, θ) is the canonical Cartan geometry associated to the Klein geometry $(SO^\dagger(d-1, 2), SO^\dagger(d-1, 1))$ with space $\text{AdS}_d(\mu^{-1})$.

Chapitre 7

On the definition of supermanifolds

7.1 H^∞ and G^∞ superfunctions

Definition 7.1.1 A complex (resp. real) **super-vector space** is a \mathbb{Z}_2 -graded complex (resp. real) vector space, that is a vector space V together with a decomposition $V = V_0 \oplus V_1$, where V_0 and V_1 are complementary vector subspaces of V . An element x of V is said to be **even** (resp. **odd**) if $x \in V_0$ (resp. $x \in V_1$). An element x of V is said to be **pure** if it is even or odd. For any pure element x of a super-vector space, we define **the partity of x** by : $|x| = 0$ (mod 2) if x is even, and $|x| = 1$ (mod 2) if x is odd.

Definition 7.1.2 A complex (resp. real) **Lie superalgebra** is a complex (resp. real) super-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, endowed with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the **Lie superbracket**, satisfying the following conditions :

1. $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$, and $(\mathfrak{g}_0, [,])$ is a Lie algebra.
2. $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, and the map $\mathfrak{g}_0 \longrightarrow \mathfrak{gl}(\mathfrak{g}_1)$ which associates to every $b \in \mathfrak{g}_0$ the endomorphism $f \mapsto [b, f]$ of \mathfrak{g}_1 defines a representation of the Lie algebra \mathfrak{g}_0 .
3. $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$, and the bilinear map $[,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is symmetric and \mathfrak{g}_0 -equivariant.
4. For any pure elements $x, y, z \in \mathfrak{g}$, the following identity holds (**super-Jacobi identity**) :

$$[x, [y, z]] + (-1)^{|x||y|+|x||z|}[y, [z, x]] + (-1)^{|x||z|+|y||z|}[z, [x, y]] = 0$$

Remark 7.1.3 \mathfrak{g}_0 (resp. \mathfrak{g}_1) is called the **bosonic** (resp. the **fermionic**) part of \mathfrak{g} . The super-Jacobi identity is equivalent to the following identities, for any $b_1, b_2, b_3 \in \mathfrak{g}_0$, $f_1, f_2, f_3 \in \mathfrak{g}_1$:

$$\begin{aligned} [b_1, [b_2, b_3]] - [[b_1, b_2], b_3] - [b_2, [b_1, b_3]] &= 0 \\ [[b_1, b_2], f_3] - [b_1, [b_2, f_3]] + [b_2, [b_1, f_3]] &= 0 \\ [b_1, [f_2, f_3]] - [[b_1, f_2], f_3] - [f_2, [b_1, f_3]] &= 0 \\ [f_1, [f_2, f_3]] - [[f_1, f_2], f_3] + [f_2, [f_1, f_3]] &= 0 \end{aligned}$$

The first identity in 4. is nothing but the usual Jacobi identity for the Lie algebra \mathfrak{g}_0 . The second one expresses that \mathfrak{g}_1 is a representation of the Lie algebra \mathfrak{g}_0 . The third one expresses the \mathfrak{g}_0 -equivariance of the symmetric bilinear map $[,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$. Thus, the first three conditions in 4. are just consequences of 1., 2. and 3. The fourth one is

additional and expresses a \mathbb{Z}_2 -graded Leibniz rule.

Definition 7.1.4 *The Grassmann algebra L is the exterior algebra $\Lambda(\mathbb{R}^l)$, where l is a sufficiently large integer. We set : $L^k = \Lambda^k(\mathbb{R}^l)$. The Grassmann algebra L has a natural \mathbb{Z}_2 -grading : $L = L^{ev} \oplus L^{od}$, where :*

$$\begin{aligned} L^{ev} &= \mathbb{R} \oplus L^2 \oplus L^4 \oplus \dots \\ L^{od} &= L^1 \oplus L^3 \oplus L^5 \oplus \dots \end{aligned}$$

We also set : $L^{ev,*} = L^2 \oplus L^4 \oplus \dots$ and $L^* = L^{ev,*} \oplus L^{od}$, so that $L^{ev} = \mathbb{R} \oplus L^{ev,*}$ and $L = \mathbb{R} \oplus L^*$. Under this last decomposition, any element $a \in L$ may be written : $a = \beta(a) + \sigma(a)$, where $\beta(a) \in \mathbb{R}$ is called the **body** of a , and $\sigma(a) \in L^*$ is called the **soul** of a .

In the Grassmann algebra L , we omit the wedge symbol in the product of two elements. Besides, we fix a basis $(\zeta^i)_{1 \leq i \leq l}$ in $L^1 \simeq \mathbb{R}^l$. The ζ^i are called the **generators of L** (in fact, as an algebra, L is generated by $1, \zeta^1, \dots, \zeta^l$). Then, a basis of L^k is given by : $(\zeta^{i_1} \dots \zeta^{i_k})_{i_1 < \dots < i_k}$.

Definition 7.1.5 *A supermodule is a \mathbb{Z}_2 -graded left L^{ev} -module, that is a left L^{ev} -module \mathcal{M} together with a decomposition $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, where \mathcal{M}_0 and \mathcal{M}_1 are complementary submodules of \mathcal{M} .*

Remark 7.1.6 *Since $\mathbb{R} \subset L^{ev}$, a supermodule has an underlying structure of super-vector space. In particular, it is also an ordinary vector space.*

Definition 7.1.7 *A Lie supermodule is a supermodule $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that the underlying super-vector space has also a Lie superalgebra structure, the Lie superbracket being compatible with the left multiplication by elements of L^{ev} : $[aX_1, X_2] = a[X_1, X_2]$ for every $a \in L^{ev}$ and $X_1, X_2 \in \mathcal{A}$.*

Definition 7.1.8 *A graded L -module is a \mathbb{Z}_2 -graded left L -module, that is a left L -module \mathcal{M} together with a decomposition $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$, where \mathcal{M}_0 and \mathcal{M}_1 are complementary sub- L^{ev} -modules of \mathcal{M} , such that the left multiplication by elements of L^{od} exchanges \mathcal{M}_0 and \mathcal{M}_1 .*

Remark 7.1.9 *Since $\mathbb{R} \subset L$, a graded L -module has an underlying structure of super-vector space. In particular, it is also an ordinary vector space.*

Definition 7.1.10 *A graded Lie L -module is a graded L -module $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that the underlying super-vector space has also a Lie superalgebra structure, the Lie superbracket being compatible with the left multiplication by elements of L : $[aX_1, X_2] = a[X_1, X_2]$ for every $a \in L$ and $X_1, X_2 \in \mathcal{A}$.*

Let us now give the local model of a supermanifold : for any open set U in \mathbb{R}^m , we define :

$$U^{m|n} = U \times (\mathbb{R}^m \otimes L^{ev,*}) \times (\mathbb{R}^n \otimes L^{od})$$

In particular, if $U = \mathbb{R}^m$, we set : $\mathbb{R}^{m|n} = U^{m|n}$, so that : $\mathbb{R}^{m|n} = (L^{ev})^m \times (L^{od})^n$.

Definition 7.1.11 *Let $V = V_0 \oplus V_1$ be a super-vector space.*

1. *We say that V is of dimension $(m|n)$ if $\dim V_0 = m$ and $\dim V_1 = n$.*

2. The supermodule associated with V is the set $V_0^{m|n} = (V_0 \otimes L^{ev}) \oplus (V_1 \otimes L^{od})$.

Remark 7.1.12 The supermodule $V_0^{m|n}$ is also said to be of dimension $(m|n)$. Notice that $V_0^{m|n}$ is also a super-vector space. In particular, it is also an ordinary vector space, and as such, $\dim V_0^{m|n} = (m+n)2^{l-1}$. As a standard example, take the super-vector space $\mathbb{R}^m \times \mathbb{R}^n$. The supermodule associated with $\mathbb{R}^m \times \mathbb{R}^n$ is $\mathbb{R}^{m|n}$.

Definition 7.1.13 Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra of dimension $(m|n)$. The **Lie supermodule associated with \mathfrak{g}** is the supermodule associated with the underlying super-vector space \mathfrak{g} , that is : $\mathfrak{g}_0^{m|n} = (\mathfrak{g}_0 \otimes L^{ev}) \oplus (\mathfrak{g}_1 \otimes L^{od})$, together with a natural extension of the Lie superbracket from \mathfrak{g} to $\mathfrak{g}_0^{m|n}$.

Next, we need to define the concept of differentiable superfunction on $U^{m|n}$. We view a superfunction on $U^{m|n}$ as a map $F : U^{m|n} \rightarrow L$. There is at least two ways to define differentiability for a superfunction. They lead to two distinct differentiability classes, called H^∞ and G^∞ respectively. In the following definitions, the indices μ_j run between and 1 and m , and Einstein's summation convention is assumed on them.

Definition 7.1.14 Let $f : U \rightarrow \mathbb{R}$ a C^∞ function. The **super-extension** of f is the map $\tilde{f} : U \times (\mathbb{R}^m \otimes L^{ev,*}) \rightarrow L^{ev}$ defined by :

$$\tilde{f}(x^1, \dots, x^m) = f(\beta(x^1), \dots, \beta(x^m)) + \sum_{k=1}^{+\infty} \frac{1}{k!} \frac{\partial^k f}{\partial x^{\mu_1} \dots \partial x^{\mu_k}}(\beta(x^1), \dots, \beta(x^m)) \sigma(x^1)^{\mu_1} \dots \sigma(x^k)^{\mu_k}$$

where $\beta(x) = (\beta(x^1), \dots, \beta(x^m)) \in U$ for any $x = (x^1, \dots, x^m) \in U \times (\mathbb{R}^m \otimes L^{ev,*}) \subset (L^{ev})^m$.

In the following definitions, the indices α_j run between and 1 and n , and Einstein's summation convention is assumed on them. Moreover, the $f_{\alpha_1 \dots \alpha_k}$, $\tilde{f}_{\alpha_1 \dots \alpha_k}$ and $F_{\alpha_1 \dots \alpha_k}$ are totally antisymmetric in their indices.

Definition 7.1.15 A **superfunction of class H^∞ on $U^{m|n}$** is a map $F : U^{m|n} \rightarrow L$ for which there exists functions $f_0, f_{\alpha_1 \dots \alpha_k} \in C^\infty(U, \mathbb{R})$ such that :

$$F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = \tilde{f}_0(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} \tilde{f}_{\alpha_1 \dots \alpha_k}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k}$$

Definition 7.1.16 A **superfunction of class G^∞ on $U \times (\mathbb{R}^m \otimes L^{ev,*})$** is a map $F : U \times (\mathbb{R}^m \otimes L^{ev,*}) \rightarrow L$ for which there exists functions $f_0, f_{i_1 \dots i_k} \in C^\infty(U, \mathbb{R})$ such that :

$$F(x^1, \dots, x^m) = \tilde{f}_0(x^1, \dots, x^m) + \sum_{k=1}^{+\infty} \tilde{f}_{i_1 \dots i_k}(x^1, \dots, x^m) \zeta^{i_1} \dots \zeta^{i_k}$$

Definition 7.1.17 A **superfunction of class G^∞ on $U^{m|n}$** is a map $F : U^{m|n} \rightarrow L$ for which there exists G^∞ superfunctions $F_0, F_{\alpha_1 \dots \alpha_k} : U \times (\mathbb{R}^m \otimes L^{ev,*}) \rightarrow L$ such that :

$$F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = F_0(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} F_{\alpha_1 \dots \alpha_k}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k}$$

Definition 7.1.18 A superfunction $F : U^{m|n} \rightarrow L$ is said to be **even** (resp. **odd**) if $F(U^{m|n}) \subset L^{ev}$ (resp. $F(U^{m|n}) \subset L^{od}$). A superfunction is said to be **pure** if it is even or odd.

According to whether it is H^∞ or G^∞ , the parity of a pure superfunction F will have two completely different meanings.

In the H^∞ case, an even superfunction $F : U^{m|n} \rightarrow L$ will be of the form :

$$\begin{aligned} F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) &= \tilde{f}_0(x^1, \dots, x^m) + \frac{1}{2!} \tilde{f}_{\alpha_1 \alpha_2}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \\ &= + \frac{1}{4!} \tilde{f}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \theta^{\alpha_3} \theta^{\alpha_4} + \dots \end{aligned}$$

while an odd superfunction will be of the form :

$$\begin{aligned} F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) &= \tilde{f}_{\alpha_1}(x^1, \dots, x^m) \theta^{\alpha_1} + \frac{1}{3!} \tilde{f}_{\alpha_1 \alpha_2 \alpha_3}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \theta^{\alpha_3} \\ &= + \frac{1}{5!} \tilde{f}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \theta^{\alpha_3} \theta^{\alpha_4} \theta^{\alpha_5} + \dots \end{aligned}$$

In the G^∞ case, both even and odd superfunctions $F : U^{m|n} \rightarrow L$ will be of the form :

$$\begin{aligned} F(x^1, \dots, x^m, \theta^1, \dots, \theta^n) &= F_0(x^1, \dots, x^m) + F_{\alpha_1}(x^1, \dots, x^m) \theta^{\alpha_1} + \frac{1}{2!} F_{\alpha_1 \alpha_2}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \\ &= + \frac{1}{3!} F_{\alpha_1 \alpha_2 \alpha_3}(x^1, \dots, x^m) \theta^{\alpha_1} \theta^{\alpha_2} \theta^{\alpha_3} + \dots \end{aligned}$$

but with $F_0, F_{\alpha_1 \alpha_2}, \dots$ even (resp. odd) and $F_{\alpha_1}, F_{\alpha_1 \alpha_2 \alpha_3}, \dots$ odd (resp. even) if F is even (resp. odd).

Thus, we see that in the H^∞ case, it would be too restrictive to work with pure superfunctions. The contrary is true in the G^∞ case : the class of G^∞ superfunctions is too large, so most physicists prefer to work only with pure superfunctions.

We define the partial derivatives of the super-extension of a C^∞ function $f : U \rightarrow \mathbb{R}$:

$$\frac{\partial \tilde{f}}{\partial x^\mu}(x) = \frac{\partial f}{\partial x^\mu}(\beta(x^1), \dots, \beta(x^m)) + \sum_{k=1}^{+\infty} \frac{1}{k!} \frac{\partial^{k+1} f}{\partial x^\mu \partial x^{\mu_1} \dots \partial x^{\mu_k}}(\beta(x^1), \dots, \beta(x^m)) \sigma(x^1)^{\mu_1} \dots \sigma(x^k)^{\mu_k}$$

This allows us to define the partial derivatives of a superfunction F of class H^∞ on $U^{m|n}$:

$$\begin{aligned} \frac{\partial F}{\partial x^\mu}(x^1, \dots, x^m, \theta^1, \dots, \theta^n) &= \frac{\partial \tilde{f}_0}{\partial x^\mu}(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} \frac{\partial \tilde{f}_{\alpha_1 \dots \alpha_k}}{\partial x^\mu}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k} \\ \frac{\partial F}{\partial \theta^\alpha}(x^1, \dots, x^m, \theta^1, \dots, \theta^n) &= \sum_{k=1}^n \frac{1}{(k-1)!} \tilde{f}_{\alpha \alpha_1 \dots \alpha_{k-1}}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_{k-1}} \end{aligned}$$

We may also define the partial derivatives of a pure superfunction F of class G^∞ on $U^{m|n}$:

$$\frac{\partial F}{\partial x^\mu}(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = \frac{\partial F_0}{\partial x^\mu}(x^1, \dots, x^m) + \sum_{k=1}^n \frac{1}{k!} \frac{\partial F_{\alpha_1 \dots \alpha_k}}{\partial x^\mu}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_k}$$

$$\frac{\partial F}{\partial \theta^\alpha}(x^1, \dots, x^m, \theta^1, \dots, \theta^n) = (-1)^{|F|} \sum_{k=1}^n (-1)^k \frac{1}{(k-1)!} F_{\alpha \alpha_1 \dots \alpha_{k-1}}(x^1, \dots, x^m) \theta^{\alpha_1} \dots \theta^{\alpha_{k-1}}$$

where $|F|$ is the parity of F .

It is easy to see that $\frac{\partial}{\partial x^\mu}$ (resp. $\frac{\partial}{\partial \theta^\alpha}$) preserves (resp. reverses) the parity of a pure superfunction.

Denoting by $H^\infty(U^{m|n}, L^{ev})$ (resp. $H^\infty(U^{m|n}, L^{od})$) the space of even (resp. odd) H^∞ superfunctions on $U^{m|n}$, it is not difficult to see that

$$H^\infty(U^{m|n}, L) = H^\infty(U^{m|n}, L^{ev}) \oplus H^\infty(U^{m|n}, L^{od})$$

has a structure of supermodule.

Let $\text{End}(H^\infty(U^{m|n}, L))_0$ (resp. $\text{End}(H^\infty(U^{m|n}, L))_1$) be the space of endomorphisms of $H^\infty(U^{m|n}, L)$ which preserve (resp. reverse) the parity of the pure superfunctions. It is not difficult to see that

$$\text{End}(H^\infty(U^{m|n}, L)) = \text{End}(H^\infty(U^{m|n}, L))_0 \oplus \text{End}(H^\infty(U^{m|n}, L))_1$$

has a structure of supermodule. Moreover, it has the additional structure of Lie supermodule, the Lie superbracket being defined by : $[X_1, X_2] = X_1 X_2 - (-1)^{|X_1| \cdot |X_2|} X_2 X_1$.

Remark 7.1.19 *End($H^\infty(U^{m|n}, L)$) is also a $H^\infty(U^{m|n}, L^{ev})$ -module.*

Definition 7.1.20 *An H^∞ super-vector field on $U^{m|n}$ is an element $X \in \text{End}(H^\infty(U^{m|n}, L))$ satisfying the following condition : $X(fg) = (Xf)g + (-1)^{|f| \cdot |X|} f(Xg)$ for any $f, g \in H^\infty(U^{m|n}, L)$*

We denote by $\chi^{H^\infty}(U^{m|n})$ the space of H^∞ super-vector fields on $U^{m|n}$, and we set :
 $\chi^{H^\infty}(U^{m|n})_0 = \chi^{H^\infty}(U^{m|n}) \cap \text{End}(H^\infty(U^{m|n}, L))_0$
 $\chi^{H^\infty}(U^{m|n})_1 = \chi^{H^\infty}(U^{m|n}) \cap \text{End}(H^\infty(U^{m|n}, L))_1$

Remark 7.1.21 *We have : $\frac{\partial}{\partial x^\mu} \in \chi^{H^\infty}(U^{m|n})_0$ and $\frac{\partial}{\partial \theta^\alpha} \in \chi^{H^\infty}(U^{m|n})_1$.*

It is not difficult to see that

$$\chi^{H^\infty}(U^{m|n}) = \chi^{H^\infty}(U^{m|n})_0 \oplus \chi^{H^\infty}(U^{m|n})_1$$

is a sub-supermodule of $\text{End}(H^\infty(U^{m|n}, L))$. Moreover, one can prove that $\chi^{H^\infty}(U^{m|n})$ is stable under the Lie superbracket of $\text{End}(H^\infty(U^{m|n}, L))$. Therefore, $\chi^{H^\infty}(U^{m|n})$ has the additional structure of Lie supermodule.

Remark 7.1.22 *$\chi^{H^\infty}(U^{m|n})$ is also a $H^\infty(U^{m|n}, L^{ev})$ -module.*

For the G^∞ superfunctions, the above definitions are essentially the same, the difference being that we obtain a graded L -module structure instead of a supermodule structure :

Denoting by $G^\infty(U^{m|n}, L^{ev})$ (resp. $G^\infty(U^{m|n}, L^{od})$) the space of even (resp. odd) G^∞ superfunctions on $U^{m|n}$, it is not difficult to see that

$$G^\infty(U^{m|n}, L) = G^\infty(U^{m|n}, L^{ev}) \oplus G^\infty(U^{m|n}, L^{od})$$

has a structure of graded L -module.

Let $\text{End}(G^\infty(U^{m|n}, L))_0$ (resp. $\text{End}(G^\infty(U^{m|n}, L))_1$) be the space of endomorphisms of $G^\infty(U^{m|n}, L)$ which preserve (resp. reverse) the parity of the pure superfunctions. It is not difficult to see that

$$\text{End}(G^\infty(U^{m|n}, L)) = \text{End}(G^\infty(U^{m|n}, L))_0 \oplus \text{End}(G^\infty(U^{m|n}, L))_1$$

has a structure of graded L -module. Moreover, it has the additional structure of graded Lie L -module, the Lie superbracket being defined by : $[X_1, X_2] = X_1 X_2 - (-1)^{|X_1| \cdot |X_2|} X_2 X_1$.

Remark 7.1.23 $\text{End}(G^\infty(U^{m|n}, L))$ is also a $G^\infty(U^{m|n}, L)$ -module.

Definition 7.1.24 A G^∞ super-vector field on $U^{m|n}$ is an element $X \in \text{End}(G^\infty(U^{m|n}, L))$ satisfying the following condition : $X(fg) = (Xf)g + (-1)^{|f| \cdot |X|} f(Xg)$ for any $f, g \in G^\infty(U^{m|n}, L)$

We denote by $\chi^{G^\infty}(U^{m|n})$ the space of G^∞ super-vector fields on $U^{m|n}$, and we set :

$$\chi^{G^\infty}(U^{m|n})_0 = \chi^{G^\infty}(U^{m|n}) \cap \text{End}(G^\infty(U^{m|n}, L))_0$$

$$\chi^{G^\infty}(U^{m|n})_1 = \chi^{G^\infty}(U^{m|n}) \cap \text{End}(G^\infty(U^{m|n}, L))_1$$

Remark 7.1.25 We have : $\frac{\partial}{\partial x^\mu} \in \chi^{G^\infty}(U^{m|n})_0$ and $\frac{\partial}{\partial \theta^\alpha} \in \chi^{G^\infty}(U^{m|n})_1$.

It is not difficult to see that

$$\chi^{G^\infty}(U^{m|n}) = \chi^{G^\infty}(U^{m|n})_0 \oplus \chi^{G^\infty}(U^{m|n})_1$$

is a submodule of $\text{End}(G^\infty(U^{m|n}, L))$. Moreover, one can prove that $\chi^{G^\infty}(U^{m|n})$ is stable under the Lie superbracket of $\text{End}(G^\infty(U^{m|n}, L))$. Therefore, $\chi^{G^\infty}(U^{m|n})$ has the additional structure of graded Lie L -module.

Remark 7.1.26 $\chi^{G^\infty}(U^{m|n})$ is also a $G^\infty(U^{m|n}, L)$ -module.

Definition 7.1.27 U_1 (resp. U_2) being an open set in \mathbb{R}^m (resp. in \mathbb{R}^p), a supermap is a map $\Phi : U_1^{m|n} \rightarrow U_2^{p|q}$ whose p first components are even superfunctions, and q last components are odd superfunctions. Φ is said to be of class H^∞ (resp. G^∞) if all its components are of class H^∞ (resp. G^∞).

Remark 7.1.28 From the definition of H^∞ superfunctions, we have $\Phi(U_1^{m|0}) \subset U_2^{p|0}$ for any H^∞ supermap Φ . The same is not true for G^∞ supermaps.

We are now able to give the definition of a supermanifold.

Definition 7.1.29 Let \mathcal{M} be a Hausdorff topological space.

1. An $(m|n)$ -chart of \mathcal{M} is a couple (\mathcal{W}, Φ) where \mathcal{W} is an open subset of \mathcal{M} , and $\varphi : \mathcal{W} \rightarrow U^{m|n}$ is an homeomorphism.
2. An $(m|n)$ -atlas of class H^∞ (resp. G^∞) on \mathcal{M} is a family of $(m|n)$ -charts whose domains cover \mathcal{M} , and such that the transition maps are supermaps of class H^∞ (resp. G^∞).
3. We say that \mathcal{M} is a supermanifold of dimension $(m|n)$ and of class H^∞ (resp. G^∞) if \mathcal{M} is equipped with a maximal $(m|n)$ -atlas of class H^∞ (resp. G^∞).

It is straightforward to extend the definitions of H^∞ (resp. G^∞) superfunctions, supermaps, and super-vector fields to a supermanifold.

Let $M^{m|n}$ be a supermanifold of dimension $(m|n)$ and of class H^∞ , and (\mathcal{W}, Φ) a chart of $M^{m|n}$. Consider the subset of \mathcal{W} which is sent by Φ to $U^{m|0}$. From the preceding remark, any other chart will also send this subset to some $U^{m|0}$. The union of all points sent to a $U^{m|0}$ by some chart is therefore a well-defined subset of $M^{m|n}$, called the **body of** $M^{m|n}$ and denoted by $M^{m|0}$. For $p \in \mathcal{W}$, we set : $\Phi(p) = (x^1, \dots, x^m, \theta^1, \dots, \theta^n)$. We define :

$$\frac{\partial}{\partial x^\mu} : H^\infty(\mathcal{W}, L) \longrightarrow H^\infty(\mathcal{W}, L) \text{ by } : \frac{\partial F}{\partial x^\mu} = \frac{\partial(F \circ \Phi^{-1})}{\partial x^\mu} \circ \Phi$$

$$\frac{\partial}{\partial \theta^\alpha} : H^\infty(\mathcal{W}, L) \longrightarrow H^\infty(\mathcal{W}, L) \text{ by } : \frac{\partial F}{\partial \theta^\alpha} = \frac{\partial(F \circ \Phi^{-1})}{\partial \theta^\alpha} \circ \Phi$$

$$\text{Then } \frac{\partial}{\partial x^\mu} \in \chi^{H^\infty}(\mathcal{W})_0 \text{ and } \frac{\partial}{\partial \theta^\alpha} \in \chi^{H^\infty}(\mathcal{W})_1.$$

Let $M^{m|n}$ be a supermanifold of dimension $(m|n)$ and of class G^∞ , and (\mathcal{W}, Φ) a chart of $M^{m|n}$. It is not possible to define the body of $M^{m|n}$ when $M^{m|n}$ is G^∞ . For $p \in \mathcal{W}$, we set : $\Phi(p) = (x^1, \dots, x^m, \theta^1, \dots, \theta^n)$. We define :

$$\frac{\partial}{\partial x^\mu} : G^\infty(\mathcal{W}, L) \longrightarrow G^\infty(\mathcal{W}, L) \text{ by } : \frac{\partial F}{\partial x^\mu} = \frac{\partial(F \circ \Phi^{-1})}{\partial x^\mu} \circ \Phi$$

$$\frac{\partial}{\partial \theta^\alpha} : G^\infty(\mathcal{W}, L) \longrightarrow G^\infty(\mathcal{W}, L) \text{ by } : \frac{\partial F}{\partial \theta^\alpha} = \frac{\partial(F \circ \Phi^{-1})}{\partial \theta^\alpha} \circ \Phi$$

$$\text{Then } \frac{\partial}{\partial x^\mu} \in \chi^{G^\infty}(\mathcal{W})_0 \text{ and } \frac{\partial}{\partial \theta^\alpha} \in \chi^{G^\infty}(\mathcal{W})_1.$$

For $p \in M^{m|n}$ and $X \in \chi^{G^\infty}(M^{m|n})$, we define $X_p : G^\infty(M^{m|n}, L) \longrightarrow L$ by : $X_p(F) = X(F)(p)$. The **tangent graded L-module of** $M^{m|n}$ at p is then : $T_p M^{m|n} = \{X_p ; X \in \chi^{G^\infty}(M^{m|n})\}$.

Setting $(T_p M^{m|n})_0 = \{X_p ; X \in \chi^{G^\infty}(M^{m|n})_0\}$ and $(T_p M^{m|n})_1 = \{X_p ; X \in \chi^{G^\infty}(M^{m|n})_1\}$, we have : $(T_p M^{m|n}) = (T_p M^{m|n})_0 \oplus (T_p M^{m|n})_1$.

Notice that is not possible to define an analogous notion of super-tangent space at a point for H^∞ supermanifolds.

Definition 7.1.30 An H^∞ (resp. G^∞) **Lie supergroup** is a supermanifold G of class H^∞ (resp. G^∞) equipped with a group structure such that the composition and the inverse maps are H^∞ (resp. G^∞).

7.2 The super-tangent bundle in the H^∞ category

Let E (resp. S) be an m -dimensional (resp. n -dimensional) vector space, and U an open set in E . Let $U^{(m|n)} = U \times (E \otimes L^{ev,*}) \times (S \otimes L^{od})$ be the corresponding cylindrical open set in the Lie supermodule $(E \otimes L^{ev}) \times (S \otimes L^{od})$. The model of the associated H^∞ super-tangent bundle is :

$${}^s T U^{(2m+n|2n+m)} = U \times E \times S \times [(E \times E \times S) \otimes L^{ev,*} \times (S \times E \times S) \otimes L^{od}]. \quad (7.1)$$

Let (e_1, \dots, e_m) be a basis of E and (s_1, \dots, s_n) a basis of S ; the canonical super-extensions by Taylor series of the real-valued coordinate functions associated to these bases provide

(even) coordinates (x^1, \dots, x^m) on $E \otimes L^{ev}$ taking values in L , as well as (odd) coordinates $(\theta^1, \dots, \theta^n)$ on $S \otimes L^{od}$, also taking values in L . But we can as well deduce from the same bases (even) coordinates $(x^1, \dots, x^m, \xi^1, \dots, \xi^m, \varphi^1, \dots, \varphi^n)$ on $E \times E \times S$, and (odd) coordinates $(\theta^1, \dots, \theta^n, \Xi^1, \dots, \Xi^m, \Theta^1, \dots, \Theta^n)$ on $S \times E \times S$; it is sufficient for that to identify the different E factors between them, and same for S .

In this way, the superfunctions $x^g, \xi^h, \varphi^i, \theta^j, \Xi^k, \Theta^l$ constitute an H^∞ system of coordinates on ${}^sTU^{(2m+n|2n+m)}$; this system is *even* in the sense that even coordinates are even and odd ones are odd. In what follows, we will exclusively consider these kinds of coordinates.

For each superfunction $\Phi : U^{(m|n)} \rightarrow L$ in the H^∞ (resp. G^∞) category, Φ being even or odd, we canonically associate a superfunction ${}^s d\Phi : ({}^sTU)^{(2m+n|2n+m)} \rightarrow L$ in the same category, through the following formula :

$${}^s d\Phi(x, \xi, \varphi, \theta, \Xi, \Theta) = \nabla_x \Phi(x, \theta) \cdot (\xi + \Xi) + \nabla_\theta \Phi(x, \theta) \cdot (\varphi + \Theta) \quad (7.2)$$

If the decomposition of Φ in (super-)Taylor series, ordered following the subsets A of the set $\{1, \dots, n\}$, is written

$$\Phi(x_1, \dots, x_m, \theta_1, \dots, \theta_n) = \sum_A \widetilde{\Phi}_A(x_1, \dots, x_m) \theta^A \quad (7.3)$$

then ${}^s d\Phi$ is defined by the series

$$\begin{aligned} {}^s d\Phi(x_1, \dots, x_m, \xi_1, \dots, \xi_m, \varphi_1, \dots, \varphi_n, \theta_1, \dots, \theta_n, \chi_1, \dots, \chi_m) &= \\ \sum_A \sum_i \partial_{x_i}(\widetilde{\Phi}_A(x_1, \dots, x_m) \theta^A) (\xi_i + \Xi_i) + \sum_A \sum_j \partial_{\theta_j}(\widetilde{\Phi}_A(x_1, \dots, x_m) \theta^A) (\varphi_j + \Theta_j) &\quad (7.4) \end{aligned}$$

We notice easily that ${}^s d\Phi$ is the super-extension of an H^∞ function when Φ is itself H^∞ , it is sufficient for that to introduce, besides the ordinary partial derivatives of the $\Phi_A \theta^A$ in the directions x^i, θ^j multiplied by ξ^i or φ^j , the new components $\partial_{x_i} \Phi_A \theta^A \cdot \Xi_i$ and $\partial_{\theta_j} \Phi_A \theta^A \cdot \Theta_j$.

Lemma 7.2.1 *For each even H^∞ -morphism $F : U^{(m|n)} \rightarrow V^{(p|q)}$, there exists a unique even H^∞ -morphism ${}^s TF : ({}^s TU^{(m|n)}) \rightarrow ({}^s TV^{(p|q)})$ such that for each superfunction $\Psi : V^{(p|q)} \rightarrow L$, the superfunction $\Phi = \Psi \circ F$ verifies ${}^s d\Phi = ({}^s d\Psi(F)) \circ ({}^s TF)$.*

Let us introduce the coordinates (x, θ) on $U^{(m|n)}$ and (y, λ) on $V^{(p|q)}$, from which we deduce canonically the coordinates $(x, \xi, \varphi, \theta, \Xi, \Theta)$ on $({}^s TU^{(m|n)})$ and $(y, \eta, \psi, \lambda, H, \Lambda)$ on $({}^s TV^{(p|q)})$. Then, let us define the components of ${}^s TF$:

$$\begin{aligned} \eta_i({}^s TV^{(p|q)}) &= \sum_a \partial/\partial x_a(y_i \circ F) \cdot \xi_a, & H_i({}^s TV^{(p|q)}) &= \sum_a \partial/\partial x_a(y_i \circ F) \cdot \Xi_a, \\ \psi_j({}^s TV^{(p|q)}) &= \sum_b \partial/\partial \theta_b(\lambda_j \circ F) \cdot \varphi_b, & \Lambda_j({}^s TV^{(p|q)}) &= \sum_b \partial/\partial \theta_b(\lambda_j \circ F) \cdot \Theta_b. \end{aligned}$$

We may write :

$$\begin{aligned} {}^s d\Phi(x, \xi, \varphi, \theta, \Xi, \Theta) &= \nabla_x \Phi \cdot (\xi + \Xi) + \nabla_\theta \Phi \cdot (\varphi + \Theta) \\ &= \nabla_y \Psi(F(x, \theta)) \circ D_x(F)(\xi + \Xi) + \nabla_\lambda \Psi(F(x, \theta)) \circ D_\theta(F)(\varphi + \Theta) \\ &= \nabla_y \Psi(F(x, \theta)) \cdot (\eta + H)({}^s TV^{(p|q)}) + \nabla_\lambda \Psi(F(x, \theta)) \cdot (\psi + \Lambda)({}^s TV^{(p|q)}) \\ &= ({}^s d\Psi(F)) \circ ({}^s TF)(\xi + \Xi, \varphi + \Theta). \end{aligned}$$

Lemma 7.2.2 *The construction (^sT) constitute a functor from the category of the cylindrical models with the even G^∞ morphisms in itself, which respects the sub-category H^∞ .*

Proof : The demonstration is immediate from lemma 7.2.1. \square

Consequently, for each supermanifold $M^{(m|n)}$ of one or the other category (provided that it is even), there is a naturally associated super-tangent supermanifold ${}^sTM^{(m|n)}$. Moreover, the canonical projections

$${}^s\pi_U : {}^sTU^{(2m+n|2n+m)} \rightarrow U^{(m|n)}$$

constitute a natural transformation of the functor sT towards the identity functor . They are glued in the charts of the supermanifolds in natural projections

$${}^s\pi_M : {}^sTM^{(m|n)} \longrightarrow M^{(m|n)}.$$

We have already defined the super-vector fields of G^∞ and H^∞ classes : they are the endomorphisms of the supermodules $G^\infty(U^{(m|n)}, L)$ and $H^\infty(U^{(m|n)}, L)$ respectively, verifying the conditions of natural ε -derivations (and the corresponding conditions of ε -linearities) ; among these fields some are even and other odd, depending whether they preserve or reverse the parity of superfunctions.

Let $M^{(m|n)}$ a supermanifold and σ a section of ${}^s\pi_M : {}^sTM^{(m|n)} \longrightarrow M^{(m|n)}$, that is a morphism of the supermanifold ${}^sTM^{(m|n)}$ in the supermanifold $M^{(m|n)}$ such that ${}^s\pi_M \circ \sigma = \text{Id}_{M^{(m|n)}}$; let us consider an element Φ of $G^\infty(M^{(m|n)}, L)$ (resp. $H^\infty(M^{(m|n)}, L)$), and let us denote by $\sigma * \Phi$ the composed map ${}^s d\Phi \circ \sigma$, it is an element of $G^\infty(M^{(m|n)}, L)$ (resp. $H^\infty(M^{(m|n)}, L)$), and it is easy to verify that the endomorphism

$$\Phi \mapsto \sigma * \Phi$$

of $G^\infty(U^{(m|n)}, L)$ (resp. $H^\infty(U^{(m|n)}, L)$) defines a vector field on $M^{(m|n)}$. We will refer to this field by $\chi(\sigma)$, so that for any superfunction Φ we have $\chi(\sigma)(\Phi) = \sigma * \Phi$.

Proposition 7.2.3 *The map χ defines a natural bijection between the set of even G^∞ (resp. H^∞) cross-sections of ${}^s\pi_M$ and the set $\chi^{G^\infty}(M^{(m|n)})$ (resp. $\chi^{H^\infty}(M^{(m|n)})$) of all the vectors fields on $M^{(m|n)}$.*

Proof :

Above each chart domain $U^{(m|n)}$, provided with coordinates (x^i, θ^j) , $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, a basis of the module $\chi^{G^\infty}(U^{(m|n)})$ (resp. $\chi^{H^\infty}(U^{(m|n)})$) on the ring $G^\infty(U^{(m|n)}, L)$ (resp. $H^\infty(U^{(m|n)}, L)$) is constituted by the vectors fields $\partial/\partial x_i$ and $\partial/\partial \theta_j$.

Let X a super-vector field of class G^∞ . There exists (unique) elements $\Phi^i, i = 1, \dots, m$, $\Psi^j, j = 1, \dots, n$ of $G^\infty(U^{(m|n)}, L)$ such that $X = \Phi^i \frac{\partial}{\partial x^i} + \Psi^j \frac{\partial}{\partial \theta^j}$. Then, for each superfunction $\Phi \in G^\infty(U^{(m|n)}, L)$, we have :

$$X(\Phi) = \sum_i \Phi_i E_i * \Phi + \sum_j \Psi_j S_j * \Phi.$$

Let us decompose the superfunctions Φ_i, Ψ_j in even and odd parts :

$$\Phi_i = \Phi_i^{ev} + \Phi_i^{od}, \quad \Psi_j = \Psi_j^{ev} + \Psi_j^{od};$$

and let us define a cross-section σ_X of ${}^s\pi_U$ by setting

$$\begin{aligned} \sigma_X(x_1, \dots, x_m, \theta_1, \dots, \theta_n) = \\ (x_1, \dots, x_m, \Phi_1^{ev}, \dots, \Phi_m^{ev}, \Psi_1^{ev}, \dots, \Psi_n^{ev}, \theta_1, \dots, \theta_n, \Phi_1^{od}, \dots, \Phi_m^{od}, \Psi_1^{od}, \dots, \Psi_n^{od}). \end{aligned} \quad (7.5)$$

It is immediate that σ_X is even and that we have $\chi(\sigma_X) = X$.

When X is of class H^∞ , the superfunctions Φ_i et Ψ_j belong to $H^\infty(U^{(m|n)}, L)$, therefore their even and odd parts too, and consequently (7.5) defines an even morphism in the category H^∞ .

Finally the formula (7.5) shows clearly that the map $X \mapsto \sigma_X$ is a bijection from the set of super-vectors fields above $U^{(m|n)}$ on the set of even sections of ${}^s\pi_U$.

The proposition is deduced by naturality and glueing. \square

The super-vector fields motivated by supersymmetric theories are included in this category.

For each $i \in \{1, \dots, m\}$ (resp. $j \in \{1, \dots, n\}$), let us define the section E_i (resp. S_j) of ${}^s\pi_M$ by requiring the coordinate ξ_i (resp. φ_j) to be 1 and all other coordinates $\xi, \varphi, \Xi, \Theta$ to be zero.

For each superfunction $\Phi \in G^\infty(U^{(m|n)}, L)$ (resp. $\Phi \in H^\infty(U^{(m|n)}, L)$), we have, for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$,

$$\begin{aligned} E_i * \Phi &= \partial\Phi/\partial x_i, \\ S_j * \Phi &= \partial\Phi/\partial\theta_j. \end{aligned}$$

Notice that we cannot define E_i^{od} (resp. S_j^{od}) by requiring the coordinate Ξ_i (resp. Θ_j) to be 1 and all other coordinates $\xi, \varphi, \Xi, \Theta$ to be zero, because Ξ_i as well as Θ_j must be odd.

On the numerical space $\mathbb{R}^{1|1}$, supplied with the coordinates (x, θ) , the odd field $Q = \partial/\partial_\theta - \theta \partial/\partial_x$ corresponds to the following cross-section in the even category

$$\sigma_Q(x, \theta) = (x, 0, 1, \theta, -\theta, 0).$$

More generally, the odd fields Q_j on a space $\mathbb{R}^{m|n}$ (supplied with the coordinates $(x_g, \xi_h, \varphi_i, \theta_j, \Xi_k, \Theta_l)$) that span the supersymmetries correspond to formulas of the following type :

$$\begin{aligned} \forall h, \quad \xi_h &= 0, \\ \forall i \neq j, \quad \varphi_i &= 0, \\ \varphi_j &= 1, \\ \forall k, \quad \Xi_k &= - \sum_i \Gamma_k^{i,j} \theta_i, \\ \forall l, \quad \Theta_l &= 0, \end{aligned}$$

for suitable numerical constants $\Gamma_k^{i,j}$.

Let $M^{(m|n)}$ an H^∞ supermanifold. For each point $p \approx \mathbb{R}^{0|0}$ of the body of $M^{(m|n)}$, is attached a vector super-tangent space to $M^{(m|n)}$, it is $T_p(M) = (T_p(M))_0 + (T_p(M))_1$; but we must be aware that $(T_p(M))_0$, constituted by the values of even fields, is itself a super-space, isomorphic to $E \otimes L^{ev} \oplus S \otimes L^{od}$ and that $(T_p(M))_1$, constituted of the values of odd fields, is also a super-space, isomorphic to $S \otimes L^{ev} \oplus E \otimes L^{od}$. If we have chosen local coordinates (x_i, θ_j) , $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$, on $M^{(m|n)}$, we have on $(T_p(M))_0$ the coordinates (ξ_i, Θ_j) and on $(T_p(M))_1$ the coordinates (φ_j, Ξ_i) .

Chapitre 8

Supersymmetric sigma-models

8.1 The superspace $\mathbb{R}^{1|1}$

Consider the super-vector space $\mathbb{R}^2 = (\mathbb{R} \times \{0\}) \oplus (\{0\} \times \mathbb{R})$, and let (e_1, e_2) be its canonical (adapted) basis. We are going to endow \mathbb{R}^2 with a Lie superalgebra structure such that $\mathbb{R} \times \{0\}$ is the bosonic part and $\{0\} \times \mathbb{R}$ is the fermionic part. For this, it is sufficient to define a Lie superbracket on the basis vectors. It is easy to see that setting $[e_1, e_1] = 0$, $[e_1, e_2] = 0$ and $[e_2, e_2] = -2 e_1$ defines a Lie superalgebra structure on \mathbb{R}^2 .

Next, we consider the Lie supermodule associated to the Lie superalgebra \mathbb{R}^2 . This is nothing but $\mathbb{R}^{1|1}$, with a natural extension of the Lie superbracket, so that for any $(t, \tau), (t', \tau') \in \mathbb{R}^{1|1}$, we have : $[(t, \tau), (t', \tau')] = (2\tau t', 0)$.

There is a Lie supergroup naturally associated to the Lie supermodule $\mathbb{R}^{1|1}$. The underlying supermanifold is $\mathbb{R}^{1|1}$ itself (here we consider it as an H^∞ supermanifold), and the group law is obtained by using the Campbell-Baker-Hausdorff formula, remembering that in this case, the exponential map is the identity, and triple brackets are zero : so for any $z, z' \in \mathbb{R}^{1|1}$,

$$z * z' = z + z' + \frac{1}{2}[z, z']$$

Writing $z = (t, \tau)$, $z' = (t', \tau')$, we get : $z * z' = (t + t' + \tau t', \tau + \tau')$.

Notice that the group obtained is not abelian. This is an important difference with the ordinary (non-supersymmetric) case.

Since $\mathbb{R}^{1|1}$ is a group, it acts on itself by left multiplication ($l_u(z) = u * z$), which we call here **left supertranslation**, and also by right multiplication ($r_u(z) = z * u$), which we call here **right supertranslation**. Contrary to ordinary translation in time, left and right supertranslations are distinct ($l \neq r$), due to the fact that $\mathbb{R}^{1|1}$ is not abelian.

Here, we pause a little bit to discuss a subtle point concerning the super-differentiability of the objects we are dealing with. Our initial wish was to describe different supersymmetric models (in particular the non-linear supersymmetric sigma-models in dimensions 1 and 2) by restricting ourselves to the H^∞ class. However, while for example the supermap $(z, z') \mapsto z * z'$ from $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$ into $\mathbb{R}^{1|1}$ is easily seen to be H^∞ , the same is not true for the partial supermaps l_u and r_u introduced above. This is related to the fact that a point outside the body of an H^∞ supermanifold is not an H^∞ sub-supermanifold. In particular,

it is not possible to isolate a specific superpoint in an H^∞ -consistent way ! Therefore, if one wants to work completely within the H^∞ category, one should avoid any point-fixing. Such a point of view would nevertheless obscure the geometrical intuition, especially when we want to consider supersymmetries as transformations associated to elements of a Lie supergroup or supermodule. Consequently, we shall allow ourselves to leave sometimes the H^∞ category, especially in the proofs, and work in the larger G^∞ category. We will try then to check that the final results that we obtain have a meaning within the H^∞ category.

When viewed as a Lie supergroup, $\mathbb{R}^{1|1}$ will be denoted by $\overset{\circ}{M}{}^{1|1}$. It admits $\mathbb{R}^{1|1}$ as tangent Lie supermodule at $(0, 0)$. We recall that the left-invariant Maurer-Cartan form on $\overset{\circ}{M}{}^{1|1}$ may be viewed as the isomorphism ${}^l e : \chi^{H^\infty}(\overset{\circ}{M}{}^{1|1}) \longrightarrow H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1})$ defined by : ${}^l e(X)(z) = T_z l_{z^{-1}}(X_z)$. The inverse isomorphism ${}^l e^{-1} : H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1}) \longrightarrow \chi^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ is defined by : ${}^l e^{-1}(w)_z = T_0 l_z(w(z))$.

If we identify $\mathbb{R}^{1|1}$ with the subset of constant superfunctions in $H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1})$, and if we denote by $\chi_l^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ the image of this subset by ${}^l e^{-1}$, it is possible to restrict ${}^l e^{-1}$ to an isomorphism ${}^l e^{-1} : \mathbb{R}^{1|1} \longrightarrow \chi_l^{H^\infty}(\overset{\circ}{M}{}^{1|1})$. From now on we reserve the notation ${}^l e^{-1}$ for the restricted isomorphism. The elements of $\chi_l^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ are called the **left-invariant super-vector fields** on $\overset{\circ}{M}{}^{1|1}$.

We want to define a moving frame on $\overset{\circ}{M}{}^{1|1}$, made of left-invariant super-vector fields. Of course, the vector e_2 of the canonical basis of \mathbb{R}^2 does not belong to $\mathbb{R}^{1|1}$. In order to have a basis of $\mathbb{R}^{1|1}$, we redefine e_1 and e_2 by setting : $e_1 = \frac{\partial}{\partial t}_{|(0,0)}$ and $e_2 = \frac{\partial}{\partial \tau}_{|(0,0)}$.

Now set $H = {}^l e^{-1}(e_1)$, and $Q = {}^l e^{-1}(e_2)$. H and Q are left-invariant super-vector fields on $\overset{\circ}{M}{}^{1|1}$, and (H, Q) is a *global moving frame* on $\overset{\circ}{M}{}^{1|1}$. Notice that we have on $\overset{\circ}{M}{}^{1|1}$ another global moving frame, namely the one induced by the canonical global chart of $\overset{\circ}{M}{}^{1|1}$: $(\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau})$, with : $\frac{\partial}{\partial t}_{|z} = e_1$ and $\frac{\partial}{\partial \tau}_{|z} = e_2$ for any $z \in \overset{\circ}{M}{}^{1|1}$.

Proposition 8.1.1 *We have :*

$$\begin{cases} H &= \frac{\partial}{\partial t} \\ Q &= \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial t} \end{cases}$$

Proof : Let $w = a e_1 + \epsilon e_2 \in \mathbb{R}^{1|1}$. Then, for any $z = (t, \tau) \in \overset{\circ}{M}{}^{1|1}$, we have : ${}^l e^{-1}(w)_z = T_0 l_z(w) = (a + \tau \epsilon) e_1 + \epsilon e_2 = (a - \epsilon \tau) \frac{\partial}{\partial t}_{|z} + \epsilon \frac{\partial}{\partial \tau}_{|z} = a \frac{\partial}{\partial t}_{|z} + \epsilon (\frac{\partial}{\partial \tau}_{|z} - \tau \frac{\partial}{\partial t}_{|z})$.

In particular, taking $w = e_1$, we have : $H = {}^l e^{-1}(e_1) = \frac{\partial}{\partial t}$.

Similarly, taking $w = e_2$, we have : $Q = {}^l e^{-1}(e_2) = \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial t}$. \square

Proposition 8.1.2 *H and Q satisfy the supercommutation relations of the Lie superalgebra \mathbb{R}^2 , that is :*

$$[H, H] = 0 \quad , \quad [H, Q] = 0 \quad \text{and} \quad [Q, Q] = -2H$$

Proof : The left-invariant super-vector fields H and Q are also the *right fundamental* super-vector fields that generate the right action of $\overset{\circ}{M}{}^{1|1}$ on itself. Consequently, they satisfy exactly the same supercommutation relations as the Lie superalgebra \mathbb{R}^2 . \square

We may consider also the right-invariant Maurer-Cartan form on $\overset{\circ}{M}{}^{1|1}$, which may be viewed as the isomorphism ${}^r e : \chi^{H^\infty}(\overset{\circ}{M}{}^{1|1}) \longrightarrow H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1})$ defined by : ${}^r e(X)(z) = T_z r_{z^{-1}}(X_z)$. The inverse isomorphism ${}^r e^{-1} : H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1}) \longrightarrow \chi^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ is defined by : ${}^r e^{-1}(w)_z = T_0 r_z(w(z))$.

If we identify $\mathbb{R}^{1|1}$ with the subset of constant superfunctions in $H^\infty(\overset{\circ}{M}{}^{1|1}, \mathbb{R}^{1|1})$, and if we denote by $\chi_r^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ the image of this subset by ${}^r e^{-1}$, it is possible to restrict ${}^r e^{-1}$ to an isomorphism ${}^r e^{-1} : \mathbb{R}^{1|1} \longrightarrow \chi_r^{H^\infty}(\overset{\circ}{M}{}^{1|1})$. From now on we reserve the notation ${}^r e^{-1}$ for the restricted isomorphism. The elements of $\chi_r^{H^\infty}(\overset{\circ}{M}{}^{1|1})$ are called the **right-invariant super-vector fields** on $\overset{\circ}{M}{}^{1|1}$.

We set $H = {}^r e^{-1}(e_1)$, and $D = {}^r e^{-1}(e_2)$. H and D are right-invariant super-vector fields on $\overset{\circ}{M}{}^{1|1}$, and (H, D) is a *global moving frame* on $\overset{\circ}{M}{}^{1|1}$.

Proposition 8.1.3 *We have :*

$$\begin{cases} H &= \frac{\partial}{\partial t} \\ D &= \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial t} \end{cases}$$

Proof : Let $w = a e_1 + \epsilon e_2 \in \mathbb{R}^{1|1}$. Then, for any $z = (t, \tau) \in \overset{\circ}{M}{}^{1|1}$, we have : ${}^r e^{-1}(w)_z = T_0 r_z(w) = (a + \epsilon \tau) e_1 + \epsilon e_2 = (a + \epsilon \tau) \frac{\partial}{\partial t|z} + \epsilon \frac{\partial}{\partial \tau|z} = a \frac{\partial}{\partial t|z} + \epsilon \left(\frac{\partial}{\partial \tau|z} + \tau \frac{\partial}{\partial t|z} \right)$.

In particular, taking $w = e_1$, we have : $H = {}^l e^{-1}(e_1) = \frac{\partial}{\partial t}$.

Similarly, taking $w = e_2$, we have : $D = {}^l e^{-1}(e_2) = \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial t}$. \square

Proposition 8.1.4 *H and D satisfy the following supercommutation relations :*

$$[H, H] = 0 \quad , \quad [H, D] = 0 \quad \text{and} \quad [D, D] = 2H$$

Proof : The right-invariant super-vector fields H and D are also the *left fundamental* super-vector fields that generate the left action of $\overset{\circ}{M}{}^{1|1}$ on itself. Consequently, they satisfy the supercommutation relations of the Lie superalgebra \mathbb{R}^2 up to a sign. \square

If we start with the action of $\overset{\circ}{M}{}^{1|1}$ on itself by **right** supertranslations, then we may associate two natural representations of the Lie supergroup $\overset{\circ}{M}{}^{1|1}$ on the supermodule $H^\infty(\overset{\circ}{M}{}^{1|1}, L)$: for $u \in \overset{\circ}{M}{}^{1|1}$ and $F \in H^\infty(\overset{\circ}{M}{}^{1|1}, L)$, we set $(uF)(t, \tau) = F((t, \tau) * u)$ (which gives a left action) and $(Fu)(t, \tau) = F((t, \tau) * u^{-1})$ (which gives a right action).

Infinitesimally, we have two corresponding representations of the Lie supermodule $\mathbb{R}^{1|1}$ on $H^\infty(\overset{\circ}{M}{}^{1|1}, L)$, which are naturally expressed in terms of the **right** fundamental (and **left-invariant**) super-vector fields H and Q : for $u = (a, \epsilon) \in \overset{\circ}{M}{}^{1|1}$, the left action reads :

$$\begin{aligned}
(uF)(t, \tau) &= F((t, \tau) * (a, \epsilon)) = F(t + a + \tau\epsilon, \tau + \epsilon) = F(t + a - \epsilon\tau, \tau + \epsilon) \\
&= F(t, \tau) + (a - \epsilon\tau) \frac{\partial F}{\partial t}(t, \tau) + \epsilon \frac{\partial F}{\partial \tau}(t, \tau) = F(t, \tau) + a \frac{\partial F}{\partial t}(t, \tau) + \epsilon \left(\frac{\partial F}{\partial \tau}(t, \tau) - \tau \frac{\partial F}{\partial t}(t, \tau) \right) \\
&= F(t, \tau) + a HF(t, \tau) + \epsilon QF(t, \tau) . \text{ So } \delta_a F = aHF \text{ and } \delta_\epsilon F = \epsilon QF .
\end{aligned}$$

and the right action reads :

$$\begin{aligned}
(Fu)(t, \tau) &= F((t, \tau) * (-a, -\epsilon)) = F(t - a - \tau\epsilon, \tau - \epsilon) = F(t - a + \epsilon\tau, \tau - \epsilon) \\
&= F(t, \tau) - (a - \epsilon\tau) \frac{\partial F}{\partial t}(t, \tau) - \epsilon \frac{\partial F}{\partial \tau}(t, \tau) = F(t, \tau) - a \frac{\partial F}{\partial t}(t, \tau) - \epsilon \left(\frac{\partial F}{\partial \tau}(t, \tau) - \tau \frac{\partial F}{\partial t}(t, \tau) \right) \\
&= F(t, \tau) - a HF(t, \tau) - \epsilon QF(t, \tau) . \text{ So } \delta_a F = -aHF \text{ and } \delta_\epsilon F = -\epsilon QF .
\end{aligned}$$

We could have started with the action of $\overset{\circ}{M}{}^{1|1}$ on itself by **left** supertranslations. In this case, we may associate two natural representations of the Lie supergroup $\overset{\circ}{M}{}^{1|1}$ on the supermodule $H^\infty(\overset{\circ}{M}{}^{1|1}, L)$: for $u \in \overset{\circ}{M}{}^{1|1}$ and $F \in H^\infty(\overset{\circ}{M}{}^{1|1}, L)$, we set $(uF)(t, \tau) = F(u^{-1} * (t, \tau))$ (which gives a left action) and $(Fu)(t, \tau) = F(u * (t, \tau))$ (which gives a right action).

Infinitesimally, we have two corresponding representations of the Lie supermodule $\mathbb{R}^{1|1}$ on $H^\infty(\overset{\circ}{M}{}^{1|1}, L)$, which are naturally expressed in terms of the **left** fundamental (and *right-invariant*) super-vector fields H and D : for $u = (a, \epsilon) \in \overset{\circ}{M}{}^{1|1}$, the left action reads :

$$\begin{aligned}
(uF)(t, \tau) &= F((-a, -\epsilon) * (t, \tau)) = F(-a + t - \epsilon\tau, -\epsilon + \tau) = F(t - a - \epsilon\tau, \tau - \epsilon) \\
&= F(t, \tau) - (a + \epsilon\tau) \frac{\partial F}{\partial t}(t, \tau) - \epsilon \frac{\partial F}{\partial \tau}(t, \tau) = F(t, \tau) - a \frac{\partial F}{\partial t}(t, \tau) - \epsilon \left(\frac{\partial F}{\partial \tau}(t, \tau) + \tau \frac{\partial F}{\partial t}(t, \tau) \right) \\
&= F(t, \tau) - a HF(t, \tau) - \epsilon DF(t, \tau) . \text{ So } \delta_a F = -aHF \text{ and } \delta_\epsilon F = -\epsilon DF .
\end{aligned}$$

and the right action reads :

$$\begin{aligned}
(Fu)(t, \tau) &= F((a, \epsilon) * (t, \tau)) = F(a + t + \epsilon\tau, \epsilon + \tau) = F(t + a + \epsilon\tau, \tau + \epsilon) \\
&= F(t, \tau) + (a + \epsilon\tau) \frac{\partial F}{\partial t}(t, \tau) + \epsilon \frac{\partial F}{\partial \tau}(t, \tau) = F(t, \tau) + a \frac{\partial F}{\partial t}(t, \tau) + \epsilon \left(\frac{\partial F}{\partial \tau}(t, \tau) + \tau \frac{\partial F}{\partial t}(t, \tau) \right) \\
&= F(t, \tau) + a HF(t, \tau) + \epsilon DF(t, \tau) . \text{ So } \delta_a F = aHF \text{ and } \delta_\epsilon F = \epsilon DF .
\end{aligned}$$

8.2 Supersymmetric sigma-model in one dimension

Let M be a smooth manifold of dimension m , and E a vector bundle of rank n over M . Denote by π be the projection $E \rightarrow M$, and let $(W_i, \varphi_i)_{i \in I}$ be an atlas of M , such that the open sets W_i trivialize the vector bundle E (so we have trivializations $\pi^{-1}(W_i) \rightarrow W_i \times \mathbb{R}^n$ which associate to each $u \in \pi^{-1}(W_i)$ a couple $(\pi(u), \lambda_i(u))$). Let $\zeta_i : \pi^{-1}(W_i) \rightarrow \varphi_i(W_i) \times \mathbb{R}^n$ be the map defined by $\zeta_i(u) = (\varphi_i(\pi(u)), \lambda_i(u))$, then $(\pi^{-1}(W_i), \zeta_i)_{i \in I}$ is an atlas of E .

Definition 8.2.1 *The supermanifold associated with the vector bundle E is the H^∞ supermanifold $M^{m|n}$ admitting an atlas (\mathcal{W}_i, Φ_i) such that $\Phi_i(\mathcal{W}_i) = \varphi_i(W_i)^{m|n}$ for each $i \in I$.*

On E we consider the following data :

1. a Lorentzian metric g on M ,
2. a connection ∇ on E ,
3. an antisymmetric two-form B on E , compatible with ∇ (i.e. $\nabla B = 0$).

Finally, let us specify the dynamical content of the theory.

1. We consider a path $x : I \longrightarrow M$, I being an open interval of \mathbb{R} . Locally, in a chart (W, φ) of M , we have $x(t) = (x^1(t), \dots, x^m(t))$.
2. We consider a **cross-section of E along x** , that is, a lift ψ of x to the vector bundle E . In other words, $\psi : I \longrightarrow E$ is a smooth map such that $\pi \circ \psi = x$.

The velocity of x is a lift of x to the tangent bundle TM ; in the chart (W, φ) , we may write : $\dot{x}(t) = (x^1(t), \dots, x^m(t), \dot{x}^1(t), \dots, \dot{x}^m(t))$.

Let $\frac{\nabla\psi}{dt}$ be the covariant derivative of ψ along x . Then $\frac{\nabla\psi}{dt}$ is also a lift of x to the vector bundle E , so that we have : $\frac{\nabla\psi}{dt}(t) \in E_{x(t)}$ for every $t \in I$. We consider a local trivialization of E over a trivializing chart domain W of M . The local trivialization of E ensures the existence of a family of local cross-sections $(e_\alpha)_{1 \leq \alpha \leq n}$ such that for every $x \in W$, $(e_{\alpha|x})_{1 \leq \alpha \leq n}$ is a basis of E_x . Then, we may write : $\psi(t) = \psi^\alpha(t) e_{\alpha|x(t)}$ and $\frac{\nabla\psi}{dt}(t) = (\frac{\nabla\psi}{dt})^\alpha(t) e_{\alpha|x(t)}$. On the other hand, we have : $\dot{x}(t) = \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu}|_{x(t)}$. It is not difficult to find an expression for $(\frac{\nabla\psi}{dt})^\alpha(t)$.

Proposition 8.2.2 *Let $(\Gamma_{\mu\beta}^\alpha)$ be the Christoffel symbols of the connection ∇ is the local basis $(e_\alpha)_{1 \leq \alpha \leq n}$. Then*

$$(\frac{\nabla\psi}{dt})^\alpha(t) = \dot{\psi}^\alpha(t) + \Gamma_{\mu\beta}^\alpha(x(t)) \dot{x}^\mu(t) \psi^\beta(t)$$

Proof : First, notice that if a cross-section φ along x comes from a cross-section s of E (that is : $\varphi(t) = s_{x(t)}$ for every $t \in I$), then $\frac{\nabla\varphi}{dt}(t) = \nabla_{\dot{x}(t)}s$. Applying this for $s = e_\beta$, we get :

$$\begin{aligned} \frac{\nabla\psi}{dt}(t) &= \frac{\nabla}{dt}(\psi^\beta(t) e_{\beta|x(t)}) \\ &= \dot{\psi}^\beta(t) e_{\beta|x(t)} + \psi^\beta(t) \frac{\nabla(e_{\beta|x(t)})}{dt}(t) \\ &= \dot{\psi}^\beta(t) e_{\beta|x(t)} + \psi^\beta(t) \nabla_{\dot{x}(t)}e_\beta \\ &= \dot{\psi}^\beta(t) e_{\beta|x(t)} + \dot{x}^\mu(t) \psi^\beta(t) \nabla_{\frac{\partial}{\partial x^\mu}|_{x(t)}}e_\beta \\ &= \dot{\psi}^\beta(t) e_{\beta|x(t)} + \dot{x}^\mu(t) \psi^\beta(t) \Gamma_{\mu\beta}^\alpha(x(t)) e_{\alpha|x(t)} \\ &= \dot{\psi}^\alpha(t) e_{\alpha|x(t)} + \dot{x}^\mu(t) \psi^\beta(t) \Gamma_{\mu\beta}^\alpha(x(t)) e_{\alpha|x(t)} \\ &= (\dot{\psi}^\alpha(t) + \dot{x}^\mu(t) \psi^\beta(t) \Gamma_{\mu\beta}^\alpha(x(t))) e_{\alpha|x(t)} \end{aligned}$$

□

Remark 8.2.3 In the particular case where $E = TM$ and $\psi = \dot{x}$, we have :

$$\left(\frac{\nabla \dot{x}}{dt}\right)^\mu(t) = \ddot{x}^\mu(t) + \Gamma_{\nu\rho}^\mu(x(t)) \dot{x}^\nu(t) \dot{x}^\rho(t)$$

and x is a **geodesic** for the connection ∇ if and only if $\frac{\nabla \dot{x}}{dt} = 0$. We recover thus the well-known equation for geodesics.

Definition 8.2.4 The superpath associated with the cross-section ψ is the H^∞ supermap $X : \overset{\circ}{M}{}^{1|1} \longrightarrow M^{m|n}$ which reads in the chart (\mathcal{W}, Φ) of $M^{m|n}$ (associated to the chart $(\pi^{-1}(W), \psi)$ of E) :

$$X(t, \tau) = (\tilde{x}^1(t), \dots, \tilde{x}^m(t), \tilde{\psi}^1(t)\tau, \dots, \tilde{\psi}^n(t)\tau)$$

where $\tilde{x}^\mu, \tilde{\psi}^\alpha : L^{ev} \longrightarrow L^{ev}$ are the super-extensions of $x^\mu, \psi^\alpha : \mathbb{R} \longrightarrow \mathbb{R}$. In the following, we omit the tilda superscripts.

If we start with the action of $\overset{\circ}{M}{}^{1|1}$ on itself by **right** supertranslations, then we may associate two natural actions of $\overset{\circ}{M}{}^{1|1}$ on the space of superpaths $H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$: for $u \in \overset{\circ}{M}{}^{1|1}$ and $X \in H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$, we set $(uX)(t, \tau) = X((t, \tau) * u)$ (which gives a left action) and $(Xu)(t, \tau) = X((t, \tau) * u^{-1})$ (which gives a right action).

In order to describe these two actions at the infinitesimal level, we first notice that the space $H^\infty(\overset{\circ}{M}{}^{1|1}, TM^{m|n})$ may be realized as an infinite-dimensional fibre bundle over $H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$, by setting $\hat{\pi}(\Psi) = \pi \circ \Psi$ for any $\Psi \in H^\infty(\overset{\circ}{M}{}^{1|1}, TM^{m|n})$, π denoting here the projection $TM^{m|n} \longrightarrow M^{m|n}$.

For any superpath $X \in H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$, let $X^{-1}TM^{m|n}$ be the *induced bundle* over $\overset{\circ}{M}{}^{1|1}$. The cross-sections of this induced bundle are nothing but the *super-vector fields along X* , that is the lifts of X to the super-tangent bundle $TM^{m|n}$. It is not difficult to see that $\Gamma(X^{-1}TM^{m|n})$ is also $\hat{\pi}^{-1}(\{X\})$, the fiber of $H^\infty(\overset{\circ}{M}{}^{1|1}, TM^{m|n})$ above X .

Proposition 8.2.5 The infinitesimal action of the supermodule $\mathbb{R}^{1|1}$ on the space of superpaths $H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$ is given by a map $\mathbb{R}^{1|1} \longrightarrow \Gamma(H^\infty(\overset{\circ}{M}{}^{1|1}, TM^{m|n}))$. In other terms, we associate respectively to e_1 and e_2 two cross-sections \mathcal{H} and \mathcal{Q} of the fibre bundle $H^\infty(\overset{\circ}{M}{}^{1|1}, TM^{m|n})$. For any $X \in H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$, $\mathcal{H}X$ and $\mathcal{Q}X$ are two super-vector fields along X (that is, $\mathcal{H}X, \mathcal{Q}X \in \Gamma(X^{-1}TM^{m|n}) = \hat{\pi}^{-1}(\{X\})$). In the chart (\mathcal{W}, Φ) , we have :

$$\begin{aligned} \mathcal{H}X(t, \tau) &= \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu}|_{X(t, \tau)} + \tau \left(\frac{\nabla \psi}{dt} \right)^\alpha(t) \frac{\partial}{\partial \theta^\alpha}|_{X(t, \tau)} \in (T_{X(t, \tau)} M^{m|n})_0 \\ \mathcal{Q}X(t, \tau) &= -\tau \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu}|_{X(t, \tau)} + \psi^\alpha(t) \frac{\partial}{\partial \theta^\alpha}|_{X(t, \tau)} \in (T_{X(t, \tau)} M^{m|n})_1 \end{aligned}$$

It is not difficult now to deduce that the left action of $\overset{\circ}{M}{}^{1|1}$ on the space of superpaths $H^\infty(\overset{\circ}{M}{}^{1|1}, M^{m|n})$ is expressed infinitesimally by the following "representation" of $\mathbb{R}^{1|1}$: for $(a, \epsilon) \in \mathbb{R}^{1|1}$, $\delta_a X = a\mathcal{H}X$ and $\delta_\epsilon X = \epsilon\mathcal{Q}X$.

Similarly, the right action of $\mathring{M}^{1|1}$ on $H^\infty(\mathring{M}^{1|1}, M^{m|n})$ is expressed infinitesimally by the following "representation" of $\mathbb{R}^{1|1}$: for $(a, \epsilon) \in \mathbb{R}^{1|1}$, $\delta_a X = -a\mathcal{H}X$ and $\delta_\epsilon X = -\epsilon\mathcal{Q}X$.

Remark 8.2.6 We could have started with the action of $\mathring{M}^{1|1}$ on itself by **left** super-translations. In this case, we may associate two natural actions of $\mathring{M}^{1|1}$ on the space of superpaths $H^\infty(\mathring{M}^{1|1}, M^{m|n})$: for $u \in \mathring{M}^{1|1}$ and $X \in H^\infty(\mathring{M}^{1|1}, M^{m|n})$, we set $(uX)(t, \tau) = X(u^{-1} * (t, \tau))$ (which gives a left action) and $(Xu)(t, \tau) = X(u * (t, \tau))$ (which gives a right action). In this case, instead of the cross-section \mathcal{Q} , we would have taken the cross-section \mathcal{D} given in the chart (\mathcal{W}, Φ) by :

$$\mathcal{D}X(t, \tau) = \tau \dot{x}^\mu(t) \frac{\partial}{\partial x^\mu|_{X(t, \tau)}} + \psi^\alpha(t) \frac{\partial}{\partial \theta^\alpha|_{X(t, \tau)}} \in (T_{X(t, \tau)} M^{m|n})_1$$

To write an action functional for the superpath X , we make use of the data introduced on E , namely the metric g on M and the antisymmetric two-form B on E .

Set $g_{\mu\nu}(x) = g(\frac{\partial}{\partial x^\mu|_x}, \frac{\partial}{\partial x^\nu|_x})$ and $B_{\alpha\beta}(x) = B(e_\alpha|_x, e_\beta|_x)$. Then, consider the super-extensions of $g_{\mu\nu}$ and $B_{\alpha\beta}$. We can now define a supermetric G on $M^{m|n}$, that is for any $p \in M^{m|n}$, a super-bilinear form $G_p : T_p M^{m|n} \times T_p M^{m|n} \rightarrow L$ such that :

$G_p(\frac{\partial}{\partial x^\mu|_p}, \frac{\partial}{\partial x^\nu|_p}) = g_{\mu\nu}(x)$ and $G_p(e_\alpha|_p, e_\beta|_p) = B_{\alpha\beta}(x)$ for any $p \in \mathcal{W}$. In particular, if we denote by $(T_p M^{m|n})_+$ (resp. by $(T_p M^{m|n})_-$) the subspace of $T_p M^{m|n}$ spanned by the $\frac{\partial}{\partial x^\mu|_p}$ (resp. by the $\frac{\partial}{\partial \theta^\alpha|_p}$), the restriction of G_p to $(T_p M^{m|n})_- \times (T_p M^{m|n})_-$ is antisymmetric, while its restriction on $(T_p M^{m|n})_+ \times (T_p M^{m|n})_+$ is symmetric.

We are now able to write an action functional for X :

$$\mathcal{A}(X) = \frac{1}{2} \int_{\mathring{M}^{1|1}} G_{X(t, \tau)}(\mathcal{H}X(t, \tau), \mathcal{D}X(t, \tau)) d\tau dt$$

Proposition 8.2.7 The above action is equivalent to the following component action :

$$\mathcal{A}(x, \psi) = \frac{1}{2} \int_{\mathbb{R}} \{ g_{x(t)}(\dot{x}(t), \dot{x}(t)) + B_{x(t)}(\frac{\nabla \psi}{dt}(t), \psi(t)) \} dt$$

Proof :

$$G_{X(t, \tau)}(\mathcal{H}X(t, \tau), \mathcal{D}X(t, \tau))$$

$$\begin{aligned} &= G_{X(t, \tau)}(\dot{x}^\mu(t) \frac{\partial}{\partial x^\mu|_{X(t, \tau)}}, \tau(\frac{\nabla \psi}{dt})^\alpha(t) \frac{\partial}{\partial \theta^\alpha|_{X(t, \tau)}}), \tau \dot{x}^\nu(t) \frac{\partial}{\partial x^\nu|_{X(t, \tau)}} + \psi^\beta(t) \frac{\partial}{\partial \theta^\beta|_{X(t, \tau)}} \\ &= G_{X(t, \tau)}(\dot{x}^\mu(t) \frac{\partial}{\partial x^\mu|_{X(t, \tau)}}, \tau \dot{x}^\nu(t) \frac{\partial}{\partial x^\nu|_{X(t, \tau)}}) + G_{X(t, \tau)}(\tau(\frac{\nabla \psi}{dt})^\alpha(t) \frac{\partial}{\partial \theta^\alpha|_{X(t, \tau)}}, \psi^\beta(t) \frac{\partial}{\partial \theta^\beta|_{X(t, \tau)}}) \\ &= \tau \{ G_{X(t, \tau)}(\frac{\partial}{\partial x^\mu|_{X(t, \tau)}}, \frac{\partial}{\partial x^\nu|_{X(t, \tau)}}) \dot{x}^\mu(t) \dot{x}^\nu(t) + G_{X(t, \tau)}(\frac{\partial}{\partial \theta^\alpha|_{X(t, \tau)}}, \frac{\partial}{\partial \theta^\beta|_{X(t, \tau)}}) (\frac{\nabla \psi}{dt})^\alpha(t) \psi^\beta(t) \} \\ &= \tau \{ g_{\mu\nu}(x(t)) \dot{x}^\mu(t) \dot{x}^\nu(t) + B_{\alpha\beta}(x(t)) (\frac{\nabla \psi}{dt})^\alpha(t) \psi^\beta(t) \} \\ &= \tau \{ g_{x(t)}(\dot{x}(t), \dot{x}(t)) + B_{x(t)}(\frac{\nabla \psi}{dt}(t), \psi(t)) \} \end{aligned}$$

$$\begin{aligned}
\mathcal{A}(X) &= \frac{1}{2} \int_{\mathring{M}^{1|1}} G_{X(t,\tau)}(\mathcal{H}X(t,\tau), \mathcal{D}X(t,\tau)) d\tau dt \\
&= \frac{1}{2} \int_{\mathring{M}^{1|1}} \tau \{ g_{x(t)}(\dot{x}(t), \dot{x}(t)) + B_{x(t)}\left(\frac{\nabla\psi}{dt}(t), \psi(t)\right) \} d\tau dt \\
&= \frac{1}{2} \int_{\mathbb{R}} \{ g_{x(t)}(\dot{x}(t), \dot{x}(t)) + B_{x(t)}\left(\frac{\nabla\psi}{dt}(t), \psi(t)\right) \} dt
\end{aligned}$$

□

Remark 8.2.8 We didn't say anything about the restriction of G_p to $(T_p M^{m|n})_+ \times (T_p M^{m|n})_-$ and to $(T_p M^{m|n})_- \times (T_p M^{m|n})_+$, and we proved the preceding proposition as if they were zero. In fact, we can leave the restriction to $(T_p M^{m|n})_+ \times (T_p M^{m|n})_-$ arbitrary (and take its (super-)transpose on $(T_p M^{m|n})_- \times (T_p M^{m|n})_+$). This would change nothing to the preceding result, as the term $G_{X(t,\tau)}(\tau(\frac{\nabla\psi}{dt})^\alpha(t) \frac{\partial}{\partial\theta^\alpha}|_{X(t,\tau)}, \tau \dot{x}^\nu(t) \frac{\partial}{\partial x^\nu}|_{X(t,\tau)})$ vanishes since it is proportional to τ^2 , and the term $G_{X(t,\tau)}(\dot{x}^\mu(t) \frac{\partial}{\partial x^\mu}|_{X(t,\tau)}, \psi^\beta(t) \frac{\partial}{\partial\theta^\beta}|_{X(t,\tau)})$ will vanish when integrated with respect to τ .

Let us now derive the Euler-Lagrange equations associated to the above action.

Consider a variation of x and ψ , that is a one-parameter family of paths $x_s : I \longrightarrow M$ and maps $\psi_s : I \longrightarrow E$ for s in a neighbourhood $]-\varepsilon, \varepsilon[$ of 0 in \mathbb{R} , such that ψ_s is a cross-section of E along x_s for each s , and $(x_0, \psi_0) = (x, \psi)$.

In order to derive the equations of motion, we compute the variation of the Lagrangian density : $\mathcal{L}(x_s, \psi_s) = \frac{1}{2} \{ g_{x_s}(\dot{x}_s, \dot{x}_s) + B_{x_s}\left(\frac{\nabla\psi_s}{dt}, \psi_s\right) \}$. In what follows, D denotes the Levi-Civita connection of the metric g , and R the curvature of the connection ∇ on E .

$$\begin{aligned}
2 \frac{\partial \mathcal{L}}{\partial s}(x_s, \psi_s) &= \frac{\partial}{\partial s} g_{x_s}(\dot{x}_s, \dot{x}_s) + \frac{\partial}{\partial s} B_{x_s}\left(\frac{\nabla\psi_s}{dt}, \psi_s\right) \\
&= 2 g_{x_s}\left(\frac{D}{\partial s}\dot{x}_s, \dot{x}_s\right) + B_{x_s}\left(\frac{\nabla}{\partial s}\frac{\nabla\psi_s}{\partial t}, \psi_s\right) + B_{x_s}\left(\frac{\nabla\psi_s}{\partial t}, \frac{\nabla}{\partial s}\psi_s\right) \\
&= 2 g_{x_s}\left(\frac{D}{\partial s}\frac{\partial}{\partial t}x_s, \dot{x}_s\right) + B_{x_s}\left(\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\psi_s, \psi_s\right) + B_{x_s}\left(\frac{\nabla\psi_s}{\partial t}, \frac{\nabla\psi_s}{\partial s}\right) \\
&= 2 g_{x_s}\left(\frac{D}{\partial t}\frac{\partial}{\partial s}x_s, \dot{x}_s\right) + B_{x_s}\left(\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\psi_s, \psi_s\right) + B_{x_s}\left(\frac{\nabla\psi_s}{\partial t}, \frac{\nabla\psi_s}{\partial s}\right)
\end{aligned}$$

because $\frac{D}{\partial s}\frac{\partial}{\partial t} - \frac{D}{\partial t}\frac{\partial}{\partial s} = [\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]$ since D is torsionless, and $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$.

Up to a boundary term resulting from an integration by parts, $2 \frac{\partial \mathcal{L}}{\partial s}(x_s, \psi_s)$ is equal to :

$$\begin{aligned}
&-2 g_{x_s}\left(\frac{\partial}{\partial s}x_s, \frac{D}{\partial t}\dot{x}_s\right) + B_{x_s}\left(\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\psi_s, \psi_s\right) + B_{x_s}\left(\frac{\nabla\psi_s}{\partial t}, \frac{\nabla\psi_s}{\partial s}\right) \\
&= -2 g_{x_s}\left(\frac{\partial x_s}{\partial s}, \frac{D\dot{x}_s}{\partial t}\right) + B_{x_s}\left(\frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\psi_s, \psi_s\right) + B_{x_s}\left(R_{x_s}\left(\frac{\partial x_s}{\partial s}, \frac{\partial x_s}{\partial t}\right)\psi_s, \psi_s\right) + B_{x_s}\left(\frac{\nabla\psi_s}{\partial t}, \frac{\nabla\psi_s}{\partial s}\right) \\
&\text{because } \frac{\nabla}{\partial s}\frac{\nabla}{\partial t} - \frac{\nabla}{\partial t}\frac{\nabla}{\partial s} = R_{x_s}\left(\frac{\partial x_s}{\partial s}, \frac{\partial x_s}{\partial t}\right).
\end{aligned}$$

Up to a boundary term resulting from an integration by parts, $2 \frac{\partial \mathcal{L}}{\partial s}(x_s, \psi_s)$ is equal to :

$$-2 g_{x_s} \left(\frac{\partial x_s}{\partial s}, \frac{D\dot{x}_s}{\partial t} \right) - B_{x_s} \left(\frac{\nabla}{\partial s} \psi_s, \frac{\nabla}{\partial t} \psi_s \right) - B_{x_s} \left(R_{x_s} \left(\frac{\partial x_s}{\partial t}, \frac{\partial x_s}{\partial s} \right) \psi_s, \psi_s \right) - B_{x_s} \left(\frac{\nabla \psi_s}{\partial s}, \frac{\nabla \psi_s}{\partial t} \right)$$

Defining the vector $\mathbf{R}_{\dot{x}, \psi, \psi}$ by : $g_{x_s}(\mathbf{R}_{\dot{x}, \psi, \psi}, \frac{\partial x_s}{\partial s}) = B_{x_s}(\mathbf{R}_{x_s}(\frac{\partial x_s}{\partial t}, \frac{\partial x_s}{\partial s}) \psi_s, \psi_s)$, the last expression reads :

$$\begin{aligned} & -2 g_{x_s} \left(\frac{\partial x_s}{\partial s}, \frac{D\dot{x}_s}{\partial t} \right) - B_{x_s} \left(\frac{\nabla \psi_s}{\partial s}, \frac{\nabla \psi_s}{\partial t} \right) - g_{x_s} \left(\frac{\partial x_s}{\partial s}, \mathbf{R}_{\dot{x}, \psi, \psi} \right) - B_{x_s} \left(\frac{\nabla \psi_s}{\partial s}, \frac{\nabla \psi_s}{\partial t} \right) \\ & = -g_{x_s} \left(\frac{\partial x_s}{\partial s}, 2 \frac{D\dot{x}_s}{\partial t} + \mathbf{R}_{\dot{x}, \psi, \psi} \right) - B_{x_s} \left(\frac{\nabla \psi_s}{\partial s}, 2 \frac{\nabla \psi_s}{\partial t} \right) \end{aligned}$$

At a stationnary configuration, this last expression vanishes for any variation of x and ψ , which gives the supergeodesics equations :

$$\left\{ \begin{array}{l} \frac{D\dot{x}}{dt} + \frac{1}{2} \mathbf{R}_{\dot{x}, \psi, \psi} = 0 \\ \frac{\nabla \psi}{dt} = 0 \end{array} \right.$$

The invariance of these equations by supersymmetry transformations is a direct consequence of the manifest supercovariant form of the action. One could also check the supersymmetry invariance at the component level.

Particular examples can be given. If we consider for instance the case where M is a sphere, then the above equations admit *all* the circles of this sphere as solutions. One could also consider the case where M is a projective space.

8.3 The superspace $\overset{\circ}{\Sigma}{}^{2|(1,1)}$

Let $\overset{\circ}{\Sigma}$ be a two-dimensional Minkowski spacetime directed by a Lorentzian vector plane (V, η) of signature $(1, 1)$. We fix an origin $o \in \overset{\circ}{\Sigma}$, and choose in V an orthonormal basis $(e_a)_{a \in \{0,1\}}$ in which the matrix of η is $\text{diag}(-1, 1)$. The canonical global chart of $\overset{\circ}{\Sigma}$ is the diffeomorphism $\overset{\circ}{\Sigma} \longrightarrow \mathbb{R}^2$ which associates to each $\sigma \in \overset{\circ}{\Sigma}$ the couple of coordinates $(\sigma^\alpha)_{\alpha \in \{0,1\}}$ defined by : $\sigma = o + \sigma^0 e_0 + \sigma^1 e_1$.

Let $S_{\mathbb{C}}$ be an irreducible complex representation of the Clifford algebra $Cl(V)$. It is well-known that $\dim S_{\mathbb{C}} = 2^{[\frac{2}{2}]} = 2$. Since we are in an even dimension, $S_{\mathbb{C}}$ constitutes a reducible representation of $Cl^+(V)$ (and of $\text{Spin}^\dagger(V)$). In fact we have :

$S_{\mathbb{C}} = (S_{\mathbb{C}})_+ \oplus (S_{\mathbb{C}})_-$, where $(S_{\mathbb{C}})_+$ and $(S_{\mathbb{C}})_-$ are inequivalent irreducible one-dimensional complex representations of $\text{Spin}^\dagger(V)$.

On the other hand, each of $(S_{\mathbb{C}})_+$ and $(S_{\mathbb{C}})_-$ is an irreducible complex representation of *real type*, in other terms there exists on each of $(S_{\mathbb{C}})_+$ and $(S_{\mathbb{C}})_-$ a $\text{Spin}^\dagger(V)$ -equivariant conjugation. Choosing a $\text{Spin}^\dagger(V)$ -equivariant conjugation σ_+ (resp. σ_-) on $(S_{\mathbb{C}})_+$ (resp. on $(S_{\mathbb{C}})_-$), we see that $S_+ = \text{Inv}(\sigma_+)$ and $S_- = \text{Inv}(\sigma_-)$ are two inequivalent irreducible one-dimensional real representations of $\text{Spin}^\dagger(V)$.

Finally, putting $S = S_+ \oplus S_-$, we obtain a reducible two-dimensional real representation of $\text{Spin}^\dagger(V)$.

Physicists call the elements of $S_{\mathbb{C}}$ (resp. $(S_{\mathbb{C}})_+$ and $(S_{\mathbb{C}})_-$, S , S_+ and S_-) **Dirac spinors** (resp. **Weyl spinors**, **Majorana spinors**, **Majorana-Weyl spinors**).

Notice that S is an irreducible real representation of $Cl(V)$, and that

$\gamma : Cl(V) \longrightarrow \text{End}(S)$ is a real algebras isomorphism. Moreover, for every $v \in V$, $\gamma(v)(S_+) \subset S_-$ and $\gamma(v)(S_-) \subset S_+$. We denote by $\gamma_{(+)}(v) : S_+ \longrightarrow S_-$ and $\gamma_{(-)}(v) : S_- \longrightarrow S_+$ the restrictions of $\gamma(v)$.

There exists a $\text{Spin}^\dagger(V)$ -invariant non-degenerate bilinear form $\varepsilon : S_+ \times S_- \longrightarrow \mathbb{R}$; it induces $\text{Spin}^\dagger(V)$ -equivariant isomorphisms $\hat{\varepsilon} : S_+ \longrightarrow S_-^*$ and ${}^t\hat{\varepsilon} : S_- \longrightarrow S_+^*$.

It is not difficult to check that for every $v \in V$, the map $\gamma_{(+)}(v) : S_+ \longrightarrow S_+^*$ given by : $\gamma_{(+)}(v) = {}^t\hat{\varepsilon} \circ \gamma_{(-)}(v)$ and the map $\gamma_{(-)}(v) : S_- \longrightarrow S_-^*$ given by : $\gamma_{(-)}(v) = \hat{\varepsilon} \circ \gamma_{(+)}(v)$ are symmetric. Therefore, for any $s, t \in S_+$, if we define $\Gamma_{(+)}(s, t)$ to be the unique vector of V such that $\eta(\Gamma_{(+)}(s, t), v) = \gamma_{(+)}(v)(s)(t)$ for every $v \in V$, we obtain a $\text{Spin}^\dagger(V)$ -equivariant symmetric bilinear map $\Gamma_{(+)} : S_+ \times S_+ \longrightarrow V$. Similarly, for any $s, t \in S_-$, if we define $\Gamma_{(-)}(s, t)$ to be the unique vector of V such that $\eta(\Gamma_{(-)}(s, t), v) = \gamma_{(-)}(v)(s)(t)$ for every $v \in V$, we obtain a $\text{Spin}^\dagger(V)$ -equivariant symmetric bilinear map $\Gamma_{(-)} : S_- \times S_- \longrightarrow V$.

Finally, $\Gamma_{(+)}$ and $\Gamma_{(-)}$ extend naturally to a $\text{Spin}^\dagger(V)$ -equivariant symmetric bilinear map $\Gamma : S \times S \longrightarrow V$ (by setting $\Gamma|_{S_+ \times S_-} = \Gamma|_{S_- \times S_+} = 0$).

There exists on S a $\text{Spin}^\dagger(V)$ -invariant symplectic form a , defined by :
 $a|_{S_+ \times S_+} = a|_{S_- \times S_-} = 0$, $a|_{S_+ \times S_-} = \varepsilon$ and $a|_{S_- \times S_+} = -{}^t\varepsilon$.

We may find a basis $(f_A) = (f_+, f_-)$ of $S = S_+ \oplus S_-$ in which the endomorphisms $\gamma(e_0)$ and $\gamma(e_1)$ are represented respectively by the matrices :

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and such that $\varepsilon(f_+, f_-) = 1$.

We have then : $\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}I_2$, and the symplectic form a is represented in (f_A) by the matrix γ_0 .

Let us express the symmetric bilinear map $\Gamma : S \times S \longrightarrow V$ in the basis (f_A) . First, we notice that for any $v \in V$ and $s, t \in S_+$, we have :

$$\eta(\Gamma(s, t), v) = \gamma_{(+)}(v)(s)(t) = {}^t\hat{\varepsilon}(\gamma(v)(s))(t) = \varepsilon(t, \gamma(v)(s)).$$

In particular, for any $s, t \in S_+$, we have $\Gamma^a(s, t) = \varepsilon(t, \gamma(e_a)(s))$, and therefore :

$$\Gamma_{++}^0 := \Gamma^0(f_+, f_+) = \varepsilon(f_+, \gamma(e_0)(f_+)) = \varepsilon(f_+, -f_-) = -1$$

$$\Gamma_{++}^1 := \Gamma^1(f_+, f_+) = \varepsilon(f_+, \gamma(e_1)(f_+)) = \varepsilon(f_+, f_-) = 1$$

Similarly, for any $v \in V$ and $s, t \in S_-$, we have :

$$\eta(\Gamma(s, t), v) = \gamma_{(-)}(v)(s)(t) = \hat{\varepsilon}(\gamma(v)(s))(t) = \varepsilon(\gamma(v)(s), t).$$

In particular, for any $s, t \in S_-$, we have $\Gamma^a(s, t) = \varepsilon(\gamma(e_a)(s), t)$, and therefore :

$$\Gamma_{--}^0 := \Gamma^0(f_-, f_-) = \varepsilon(\gamma(e_0)(f_-), f_-) = \varepsilon(f_+, f_-) = 1$$

$$\Gamma_{--}^1 := \Gamma^1(f_-, f_-) = \varepsilon(\gamma(e_1)(f_-), f_-) = \varepsilon(f_+, f_-) = 1$$

Of course, $\Gamma_{+-}^0 = \Gamma_{-+}^0 = \Gamma_{+-}^1 = \Gamma_{-+}^1 = 0$ by definition of Γ .

An element of S is denoted by :

$$\Psi = (\Psi^A) = \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix}$$

For $\Psi \in S$, we set :

$$\tilde{\Psi} = \Psi^\dagger \gamma^0 = {}^t \Psi \gamma^0 = \begin{pmatrix} \Psi^+ & \Psi^- \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \Psi^- & -\Psi^+ \end{pmatrix}$$

Finally, let $\tilde{S} = \tilde{\Pi} \times \rho_{\frac{1}{2}} S$ be the associated spinor bundle, $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(V) \longrightarrow \text{GL}(S)$ being the reducible two-dimensional real representation of $\text{Spin}^\dagger(V)$.

Proposition 8.3.1 *The symmetric bilinear map Γ induces a natural Lie superalgebra structure on $V \times S$.*

We get a super-bracket by setting : $[V, V] = [V, S] = [S, V] = 0$ and for any $s, t \in S$, $[s, t] = -2 \Gamma(s, t)$.

We set : $\Gamma(f_A, f_B) = \Gamma_{AB}^a e_a$.

Then we have : $[f_A, f_B] = -2 \Gamma_{AB}^a e_a$.

Proposition 8.3.2 *The symmetric bilinear map Γ induces a natural Lie supermodule structure on $V^{2|(1,1)} = (V \otimes L^{ev}) \times (S_+ \otimes L^{od}) \oplus (S_- \otimes L^{od})$.*

Let us expand the super-bracket of two elements $u = v^a \otimes e_a + s^A \otimes f_A$ and $u' = v'^a \otimes e_a + s'^A \otimes f_A$ of $V^{2|(1,1)}$.

$$\begin{aligned} [u, u'] &= [s^A \otimes f_A, s'^B \otimes f_B] \\ &= -s^A s'^B [f_A, f_B] \\ &= -s^A s'^B (-2 \Gamma_{AB}^a e_a) \\ &= 2 \Gamma_{AB}^a s^A s'^B e_a \end{aligned}$$

Proposition 8.3.3 *There exists a natural structure of Lie supergroup on $V^{2|(1,1)}$, the group law being given by : $u * u' = u + u' + \frac{1}{2}[u, u']$ for any $u, u' \in V^{2|(1,1)}$.*

In the chosen bases of V and S , we have :

$$u * u' = (v^a + v'^a + \Gamma_{AB}^a s^A s'^B) \otimes e_a + (s^A + s'^A) \otimes f_A$$

Definition 8.3.4 *The super-Poincaré group is the Lie supergroup defined by :*

$$S\Pi(V^{2|(1,1)}) = V^{2|(1,1)} \rtimes_\rho \text{Spin}^\dagger(V)$$

Proposition 8.3.5 *The group law in $S\Pi(V^{2|(1,1)})$ is given by :*

$$(u, \sigma)(u', \sigma') = (u * \rho(\sigma)(u'), \sigma\sigma')$$

Let $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ be the underlying supermanifold of the supermodule $V^{2|(1,1)}$. Then $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ is a superspacetime associated to $\overset{\circ}{\Sigma}$ and S , called the **flat superspacetime** (or the **Minkowski superspacetime**). We will generally view $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ as a group supermanifold, endowed with the left and right actions (by supertranslation) of $V^{2|(1,1)}$, considered as a Lie supergroup. On the other hand, writing $T_0 \overset{\circ}{\Sigma}{}^{2|(1,1)} = V^{2|(1,1)}$, we can view $V^{2|(1,1)}$ as the tangent Lie supermodule of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$.

Remark :

We have a natural affine action $\tau : S\Pi(V^{2|(1,1)}) \longrightarrow \text{GA}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ given by : $\tau_{(u,\sigma)}(z) = u * \rho(\sigma)(z)$ for any $(u,\sigma) \in S\Pi(V^{2|(1,1)})$ and $z \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$. By restricting to the Lie supergroup $V^{2|(1,1)}$, we obtain the action by left supertranslations $l : V^{2|(1,1)} \longrightarrow \text{GA}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ defined by : $l_u(z) = u * z$. We also have the action by right supertranslations $r : V^{2|(1,1)} \longrightarrow \text{GA}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ defined by : $r_u(z) = z * u$. Since $V^{2|(1,1)}$ is not abelian, we have $l \neq r$.

We recall that the left-invariant Maurer-Cartan form on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ may be viewed as the isomorphism ${}^l e : \chi^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)}) \longrightarrow H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)})$ defined by : ${}^l e(X)(z) = T_z l_{z^{-1}}(X_z)$. The inverse isomorphism ${}^l e^{-1} : H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)}) \longrightarrow \chi^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ is defined by : ${}^l e^{-1}(w)_z = T_0 l_z(w(z))$.

If we identify $V^{2|(1,1)}$ with the subset of constant superfunctions in $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)})$, and if we denote by $\chi_l^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ the image of this subset by ${}^l e^{-1}$, it is possible to restrict ${}^l e^{-1}$ to an isomorphism ${}^l e^{-1} : V^{2|(1,1)} \longrightarrow \chi_l^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$. From now on we reserve the notation ${}^l e^{-1}$ for the restricted isomorphism. The elements of $\chi_l^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ are called the **left-invariant super-vector fields** on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$.

We want to define a moving frame on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$, made of left-invariant super-vector fields. Of course, the vectors f_A do not belong to $V^{2|(1,1)}$. In order to have a basis of $V^{2|(1,1)}$, we redefine e_a and f_A by setting : $e_\alpha = \frac{\partial}{\partial \sigma^\alpha}|_{(0,0)}$ and $f_\Lambda = \frac{\partial}{\partial \tau^\Lambda}|_{(0,0)}$.

Now set $P_a = {}^l e^{-1}(e_a)$, and $Q_A = {}^l e^{-1}(f_A)$. P_a and Q_A are left-invariant super-vector fields on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$, and (P_a, Q_A) is a *global moving frame* on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$. Notice that we have on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ another global moving frame, namely the one induced by the canonical global chart of $\overset{\circ}{\Sigma}{}^{2|(1,1)} : (\frac{\partial}{\partial \sigma^\alpha}, \frac{\partial}{\partial \tau^\Lambda})$, with : $\frac{\partial}{\partial \sigma^\alpha}|_z = e_\alpha$ and $\frac{\partial}{\partial \tau^\Lambda}|_z = f_\Lambda$ for any $z \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$.

Proposition 8.3.6 *We have :*

$$\begin{cases} P_a &= \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha} \\ Q_A &= \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda} - \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha} \end{cases}$$

Proof : Let $w = v^a e_a + \epsilon^A f_A \in V^{2|(1,1)}$. Then, for any $z = (\sigma^\alpha, \tau^\Lambda) \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$, we have : ${}^l e^{-1}(w)_z = T_0 l_z(w) = (v^a + \tau^\Pi \epsilon^A \Gamma_{AB}^a \delta_\Pi^B) e_a + \epsilon^A f_A = (v^a - \epsilon^A \tau^\Pi \Gamma_{AB}^a \delta_\Pi^B) \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}|_z + \epsilon^A \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda}|_z = v^a (\delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}|_z) + \epsilon^A (\delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda}|_z - \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha}|_z)$.

In particular, taking $w = e_a$, we have : $P_a = {}^l e^{-1}(e_a) = \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}$.

Similarly, taking $w = f_A$, we have : $Q_A = {}^l e^{-1}(f_A) = \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda} - \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha}$. \square

Proposition 8.3.7 P_a and Q_A satisfy the supercommutation relations of the Lie superalgebra $V \times S$, that is :

$$[P_a, P_b] = 0 \quad , \quad [P_a, Q_A] = 0 \quad \text{and} \quad [Q_A, Q_B] = -2 \Gamma_{AB}^a P_a$$

Proof : The left-invariant super-vector fields P_a and Q_A are also the *right fundamental* super-vector fields that generate the right action of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on itself. Consequently, they satisfy exactly the same supercommutation relations as the Lie superalgebra $V \times S$. \square

We may consider also the right-invariant Maurer-Cartan form on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$, which may be viewed as the isomorphism ${}^r e : \chi^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)}) \longrightarrow H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)})$ defined by :

${}^r e(X)(z) = T_z r_{z^{-1}}(X_z)$. The inverse isomorphism ${}^r e^{-1} : H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)}) \longrightarrow \chi^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ is defined by : ${}^r e^{-1}(w)_z = T_0 r_z(w(z))$.

If we identify $V^{2|(1,1)}$ with the subset of constant superfunctions in $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, V^{2|(1,1)})$, and if we denote by $\chi_r^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ the image of this subset by ${}^r e^{-1}$, it is possible to restrict ${}^r e^{-1}$ to an isomorphism ${}^r e^{-1} : V^{2|(1,1)} \longrightarrow \chi_r^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$. From now on we reserve the notation ${}^r e^{-1}$ for the restricted isomorphism. The elements of $\chi_r^{H^\infty}(\overset{\circ}{\Sigma}{}^{2|(1,1)})$ are called the **right-invariant super-vector fields** on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$.

We set $\partial_a = {}^r e^{-1}(e_a)$, and $D_A = {}^r e^{-1}(f_A)$. ∂_a and D_A are right-invariant super-vector fields on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$, and (∂_a, D_A) is a *global moving frame* on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$.

Proposition 8.3.8 We have :

$$\begin{cases} \partial_a = \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha} \\ D_A = \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda} + \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha} \end{cases}$$

Proof : Let $w = v^a e_a + \epsilon^A f_A \in V^{2|(1,1)}$. Then, for any $z = (\sigma^\alpha, \tau^\Lambda) \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$, we have : ${}^r e^{-1}(w)_z = T_0 r_z(w) = (v^a + \epsilon^A \tau^\Pi \Gamma_{AB}^a \delta_\Pi^B) e_a + \epsilon^A f_A = (v^a + \epsilon^A \tau^\Pi \Gamma_{AB}^a \delta_\Pi^B) \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}|_z + \epsilon^A \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda}|_z = v^a (\delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}|_z) + \epsilon^A (\delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda}|_z + \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha}|_z)$.

In particular, taking $w = e_a$, we have : $\partial_a = {}^l e^{-1}(e_a) = \delta_a^\alpha \frac{\partial}{\partial \sigma^\alpha}$.

Similarly, taking $w = f_A$, we have : $D_A = {}^l e^{-1}(f_A) = \delta_A^\Lambda \frac{\partial}{\partial \tau^\Lambda} + \tau^\Pi(\delta_\Pi^B \Gamma_{AB}^a \delta_a^\alpha) \frac{\partial}{\partial \sigma^\alpha}$. \square

Proposition 8.3.9 ∂_a and D_A satisfy the following supercommutation relations :

$$[\partial_a, \partial_b] = 0 \quad , \quad [\partial_a, D_A] = 0 \quad \text{and} \quad [D_A, D_B] = 2 \Gamma_{AB}^a \partial_a$$

Proof : The right-invariant super-vector fields ∂_a and D_A are also the *left fundamental* super-vector fields that generate the left action of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on itself. Consequently, they satisfy the supercommutation relations of the Lie superalgebra $V \times S$ up to a sign. \square

Remark 8.3.10 Up to now, we were making a distinction between :

1. objects coming from a chart on $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ (the canonical global chart) and carrying greek indices, like for example the global moving frame : $(\frac{\partial}{\partial\sigma^\alpha}, \frac{\partial}{\partial\tau^\Lambda})$
2. objects coming from the supermodule $V^{2|(1,1)}$ through Maurer-Cartan forms and carrying latin indices, like for example the left- and right-invariant super-vector fields $P_\alpha, Q_\Lambda, \partial_a, D_A$.

While this distinction is absolutely necessary in curved superspacetimes (as in supergravity theories), it is possible in a flat background to convert all the indices to greek indices, owing to the canonical identification of the tangent bundle with the auxiliary bundle (via the canonical soldering form). Consequently, the expressions of the invariant super-vector fields take a simpler form :

$$\begin{cases} P_\alpha = \frac{\partial}{\partial\sigma^\alpha} \\ Q_\Lambda = \frac{\partial}{\partial\tau^\Lambda} - \Gamma_{\Lambda\Pi}^\alpha \tau^\Pi \frac{\partial}{\partial\sigma^\alpha} \end{cases}$$

$$\begin{cases} \partial_\alpha = \frac{\partial}{\partial\sigma^\alpha} \\ D_\Lambda = \frac{\partial}{\partial\tau^\Lambda} + \Gamma_{\Lambda\Pi}^\alpha \tau^\Pi \frac{\partial}{\partial\sigma^\alpha} \end{cases}$$

In particular, $Q_+ = \frac{\partial}{\partial\tau^+} - \Gamma_{++}^\alpha \tau^+ \frac{\partial}{\partial\sigma^\alpha}$ and $Q_- = \frac{\partial}{\partial\tau^-} - \Gamma_{--}^\alpha \tau^- \frac{\partial}{\partial\sigma^\alpha}$.

Similarly, $D_+ = \frac{\partial}{\partial\tau^+} + \Gamma_{++}^\alpha \tau^+ \frac{\partial}{\partial\sigma^\alpha}$ and $D_- = \frac{\partial}{\partial\tau^-} + \Gamma_{--}^\alpha \tau^- \frac{\partial}{\partial\sigma^\alpha}$.

If we start with the action of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on itself by **right** supertranslations, then we may associate two natural representations of the Lie supergroup $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on the supermodule $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, L)$: for $u \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$ and $F \in H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, L)$, we set $(uF)(\sigma^\alpha, \tau^\Lambda) = F((\sigma^\alpha, \tau^\Lambda) * u)$ (which gives a left action) and $(Fu)(\sigma^\alpha, \tau^\Lambda) = F((\sigma^\alpha, \tau^\Lambda) * u^{-1})$ (which gives a right action).

Infinitesimally, we have two corresponding representations of the Lie supermodule $V^{2|(1,1)}$ on $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, L)$, which are naturally expressed in terms of the **right** fundamental (and *left-invariant*) super-vector fields P_α and Q_Λ : for $u = (v^\alpha, \epsilon^\Lambda) \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$, the left action reads :

$$(uF)(\sigma^\alpha, \tau^\Lambda) = F((\sigma^\alpha, \tau^\Lambda) * (v^\alpha, \epsilon^\Lambda)) = F(\sigma^\alpha + v^\alpha + \tau^\Pi \epsilon^\Lambda \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda + \epsilon^\Lambda)$$

$$= F(\sigma^\alpha + v^\alpha - \epsilon^\Lambda \tau^\Pi \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda + \epsilon^\Lambda)$$

$$= F(\sigma^\alpha, \tau^\Lambda) + (v^\alpha - \epsilon^\Lambda \tau^\Pi \Gamma_{\Pi\Lambda}^\alpha) \frac{\partial F}{\partial\sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda \frac{\partial F}{\partial\tau^\Lambda}(\sigma^\alpha, \tau^\Lambda)$$

$$\begin{aligned}
&= F(\sigma^\alpha, \tau^\Lambda) + v^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda \left(\frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) - \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) \right) \\
&= F(\sigma^\alpha, \tau^\Lambda) + v^\alpha P_\alpha F(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda Q_\Lambda F(\sigma^\alpha, \tau^\Lambda). \text{ So } \delta_v F = v^\alpha P_\alpha F \text{ and } \delta_\epsilon F = \epsilon^\Lambda Q_\Lambda F.
\end{aligned}$$

and the right action reads :

$$\begin{aligned}
(Fu)(\sigma^\alpha, \tau^\Lambda) &= F((\sigma^\alpha, \tau^\Lambda) * (-v^\alpha, -\epsilon^\Lambda)) = F(\sigma^\alpha - v^\alpha - \tau^{\Pi\Lambda} \epsilon^\Lambda \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda - \epsilon^\Lambda) \\
&= F(\sigma^\alpha - v^\alpha + \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda - \epsilon^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) - (v^\alpha - \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha) \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda \frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) - v^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda \left(\frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) - \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) \right) \\
&= F(\sigma^\alpha, \tau^\Lambda) - v^\alpha P_\alpha F(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda Q_\Lambda F(\sigma^\alpha, \tau^\Lambda). \text{ So } \delta_v F = -v^\alpha P_\alpha F \text{ and } \delta_\epsilon F = -\epsilon^\Lambda Q_\Lambda F.
\end{aligned}$$

We could have started with the action of $\dot{\Sigma}^{2|(1,1)}$ on itself by **left** supertranslations. In this case, we may associate two natural representations of the Lie supergroup $\dot{\Sigma}^{2|(1,1)}$ on the supermodule $H^\infty(\dot{\Sigma}^{2|(1,1)}, L)$: for $u \in \dot{\Sigma}^{2|(1,1)}$ and $F \in H^\infty(\dot{\Sigma}^{2|(1,1)}, L)$, we set

$$\begin{aligned}
(uF)(\sigma^\alpha, \tau^\Lambda) &= F(u^{-1} * (\sigma^\alpha, \tau^\Lambda)) \text{ (which gives a left action) and} \\
(Fu)(\sigma^\alpha, \tau^\Lambda) &= F(u * (\sigma^\alpha, \tau^\Lambda)) \text{ (which gives a right action).}
\end{aligned}$$

Infinitesimally, we have two corresponding representations of the Lie supermodule $V^{2|(1,1)}$ on $H^\infty(\dot{\Sigma}^{2|(1,1)}, L)$, which are naturally expressed in terms of the **left** fundamental (and *right-invariant*) super-vector fields ∂_α and D_Λ : for $u = (v^\alpha, \epsilon^\Lambda) \in \dot{\Sigma}^{2|(1,1)}$, the left action reads :

$$\begin{aligned}
(uF)(\sigma^\alpha, \tau^\Lambda) &= F((-v^\alpha, -\epsilon^\Lambda) * (\sigma^\alpha, \tau^\Lambda)) = F(-v^\alpha + \sigma^\alpha - \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha, -\epsilon^\Lambda + \tau^\Lambda) \\
&= F(\sigma^\alpha - v^\alpha - \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda - \epsilon^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) - (v^\alpha + \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha) \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda \frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) - v^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda \left(\frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) + \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) \right) \\
&= F(\sigma^\alpha, \tau^\Lambda) - v^\alpha \partial_\alpha F(\sigma^\alpha, \tau^\Lambda) - \epsilon^\Lambda D_\Lambda F(\sigma^\alpha, \tau^\Lambda). \text{ So } \delta_v F = -v^\alpha \partial_\alpha F \text{ and } \delta_\epsilon F = -\epsilon^\Lambda D_\Lambda F.
\end{aligned}$$

and the right action reads :

$$\begin{aligned}
(Fu)(\sigma^\alpha, \tau^\Lambda) &= F((v^\alpha, \epsilon^\Lambda) * (\sigma^\alpha, \tau^\Lambda)) = F(v^\alpha + \sigma^\alpha + \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha, \epsilon^\Lambda + \tau^\Lambda) \\
&= F(\sigma^\alpha + v^\alpha + \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha, \tau^\Lambda + \epsilon^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) + (v^\alpha + \epsilon^\Lambda \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha) \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda \frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) \\
&= F(\sigma^\alpha, \tau^\Lambda) + v^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda \left(\frac{\partial F}{\partial \tau^\Lambda}(\sigma^\alpha, \tau^\Lambda) + \tau^{\Pi\Lambda} \Gamma_{\Pi\Lambda}^\alpha \frac{\partial F}{\partial \sigma^\alpha}(\sigma^\alpha, \tau^\Lambda) \right)
\end{aligned}$$

$$= F(\sigma^\alpha, \tau^\Lambda) + v^\alpha \partial_\alpha F(\sigma^\alpha, \tau^\Lambda) + \epsilon^\Lambda D_\Lambda F(\sigma^\alpha, \tau^\Lambda). \text{ So } \delta_v F = v^\alpha \partial_\alpha F \text{ and } \delta_\epsilon F = \epsilon^\Lambda D_\Lambda F.$$

8.4 Supersymmetric sigma-model in two dimensions

Let M be a smooth manifold of dimension m , and E a vector bundle of rank n over M . Denote by π be the projection $E \rightarrow M$, and let $(W_i, \varphi_i)_{i \in I}$ be an atlas of M , such that the open sets W_i trivialize the vector bundle E (so we have trivializations $\pi^{-1}(W_i) \rightarrow W_i \times \mathbb{R}^n$ which associate to each $u \in \pi^{-1}(W_i)$ a couple $(\pi(u), \lambda_i(u))$). Let $\zeta_i : \pi^{-1}(W_i) \rightarrow \varphi_i(W_i) \times \mathbb{R}^n$ be the map defined by $\zeta_i(u) = (\varphi_i(\pi(u)), \lambda_i(u))$, then $(\pi^{-1}(W_i), \zeta_i)_{i \in I}$ is an atlas of E .

Definition 8.4.1 *The supermanifold associated with the vector bundle E is the H^∞ supermanifold $M^{m|n}$ admitting an atlas (\mathcal{W}_i, Φ_i) such that $\Phi_i(\mathcal{W}_i) = \varphi_i(W_i)^{m|n}$ for each $i \in I$.*

On E we consider the following data :

1. a Lorentzian metric g on M ,
2. a connection ∇ on E ,
3. an antisymmetric two-form B on E , compatible with ∇ (i.e. $\nabla B = 0$).

Finally, let us specify the dynamical content of the theory.

1. We consider a surface $X : \dot{\Sigma} \rightarrow M$. Locally, in a chart (W, φ) of M , we have $X(\sigma) = (X^1(\sigma), \dots, X^m(\sigma))$.
2. We consider a cross-section ψ of the vector bundle $(X^{-1}E \otimes \dot{\mathbb{S}}, \dot{\Sigma}, \mathbb{R}^n \otimes S)$. In other words, $\psi : \dot{\Sigma} \rightarrow X^{-1}E \otimes \dot{\mathbb{S}}$ is a smooth map such that $\psi(\sigma) \in E_{X(\sigma)} \otimes \dot{\mathbb{S}}_\sigma$ for any $\sigma \in \dot{\Sigma}$.
3. A vector field along X , that is a lift F of X to the tangent bundle TM .

For each $\alpha \in \{0, 1\}$, $\partial_\alpha X$ is a lift of X to the tangent bundle TM ; in the chart (W, φ) , we may write : $\partial_\alpha X(\sigma) = (X^1(\sigma), \dots, X^m(\sigma), \partial_\alpha X^1(\sigma), \dots, \partial_\alpha X^m(\sigma))$.

The connection ∇ on E induces a connection $X^*\nabla$ on the induced bundle $X^{-1}E$. For any *cross-section* λ of E along X , the covariant derivatives of λ along X are defined by : $\frac{\nabla \lambda}{\partial \sigma^\alpha}(\sigma) = ((X^*\nabla)_{\partial_\alpha} \lambda)_\sigma$. For each $\alpha \in \{0, 1\}$, $\frac{\nabla \lambda}{\partial \sigma^\alpha}$ is a lift of X to the vector bundle E , so that we have : $\frac{\nabla \lambda}{\partial \sigma^\alpha}(\sigma) \in E_{X(\sigma)}$ for any $\sigma \in \dot{\Sigma}$.

We consider a local trivialization of E over a trivializing chart domain W of M . The local trivialization of E ensures the existence of a family of local cross-sections $(a_i)_{1 \leq i \leq n}$ such that for every $x \in W$, $(a_i|_x)_{1 \leq i \leq n}$ is a basis of E_x . Then, we may write :

$\lambda(\sigma) = \lambda^i(\sigma) a_i|_{X(\sigma)}$ and $\frac{\nabla \lambda}{\partial \sigma^\alpha}(\sigma) = \left(\frac{\nabla \lambda}{\partial \sigma^\alpha}\right)^i(\sigma) a_i|_{X(\sigma)}$. On the other hand, we have : $\partial_\alpha X(\sigma) = \partial_\alpha X^\mu(\sigma) \frac{\partial}{\partial x^\mu|_{X(\sigma)}}$. It is not difficult to find an expression for $\left(\frac{\nabla \lambda}{\partial \sigma^\alpha}\right)^i(\sigma)$.

Proposition 8.4.2 Let $(\Gamma_{\mu j}^i)$ be the Christoffel symbols of the connection ∇ is the local basis $(a_i)_{1 \leq i \leq n}$. Then

$$\left(\frac{\nabla \lambda}{\partial \sigma^\alpha}\right)^i(\sigma) = \partial_\alpha \lambda^i(\sigma) + \Gamma_{\mu j}^i(X(\sigma)) \partial_\alpha X^\mu(\sigma) \lambda^j(\sigma)$$

Proof : First, notice that if a cross-section φ along X comes from a cross-section s of E (that is : $\varphi(\sigma) = s_{X(\sigma)}$ for every $\sigma \in \dot{\Sigma}$), then $\frac{\nabla \varphi}{\partial \sigma^\alpha}(\sigma) = \nabla_{\partial_\alpha X(\sigma)} s$. Applying this for $s = a_j$, we get :

$$\begin{aligned} \frac{\nabla \lambda}{\partial \sigma^\alpha}(\sigma) &= \frac{\nabla}{\partial \sigma^\alpha}(\lambda^j(\sigma) a_j|_{X(\sigma)}) \\ &= \partial_\alpha \lambda^j(\sigma) a_j|_{X(\sigma)} + \lambda^j(\sigma) \frac{\nabla(a_j \circ X)}{\partial \sigma^\alpha}(\sigma) \\ &= \partial_\alpha \lambda^j(\sigma) a_j|_{X(\sigma)} + \lambda^j(\sigma) \nabla_{\partial_\alpha X(\sigma)} a_j \\ &= \partial_\alpha \lambda^j(\sigma) a_j|_{X(\sigma)} + \partial_\alpha X^\mu(\sigma) \lambda^j(\sigma) \nabla_{\frac{\partial}{\partial X^\mu}|_{X(\sigma)}} a_j \\ &= \partial_\alpha \lambda^j(\sigma) a_j|_{X(\sigma)} + \partial_\alpha X^\mu(\sigma) \lambda^j(\sigma) \Gamma_{\mu j}^i(X(\sigma)) a_i|_{X(\sigma)} \\ &= \partial_\alpha \lambda^i(\sigma) a_i|_{X(\sigma)} + \partial_\alpha X^\mu(\sigma) \lambda^j(\sigma) \Gamma_{\mu j}^i(X(\sigma)) a_i|_{X(\sigma)} \\ &= (\partial_\alpha \lambda^i(\sigma) + \partial_\alpha X^\mu(\sigma) \lambda^j(\sigma) \Gamma_{\mu j}^i(X(\sigma))) a_i|_{X(\sigma)} \end{aligned}$$

□

On the other hand, we may consider the covariant derivative $\hat{\nabla}$ acting on spinor fields, and associated to the canonical connection $\hat{\omega}$ of the Klein geometry $(\tilde{\Pi}, \text{Spin}^\dagger(V))$. A connection on $X^{-1}E \otimes \dot{\mathbb{S}}$ is now obtained by setting : $\hat{\nabla} = (X^* \nabla) \otimes \text{Id} \oplus \text{Id} \otimes \nabla$. In particular, $\frac{\hat{\nabla}}{\partial \sigma^\alpha} = \frac{\nabla}{\partial \sigma^\alpha} \otimes \text{Id} \oplus \text{Id} \otimes \hat{\nabla}_{\partial_\alpha}$.

From now on, we work in the canonical global trivialization \dot{s} . Consequently, spinor fields on $\dot{\Sigma}$ are identified with elements of $C^\infty(\dot{\Sigma}, S)$, and we know that $\hat{\nabla} = d$. In particular, a cross-section ψ of $X^{-1}E \otimes \dot{\mathbb{S}}$ is identified with a cross-section of the vector bundle $(X^{-1}E \otimes S, \dot{\Sigma}, \mathbb{R}^n \otimes S)$, and we have :

$$\frac{\hat{\nabla} \psi}{\partial \sigma^\alpha}(\sigma) = \begin{pmatrix} \frac{\nabla \psi_+}{\partial \sigma^\alpha}(\sigma) \\ \frac{\nabla \psi_-}{\partial \sigma^\alpha}(\sigma) \end{pmatrix} \in E_{X(\sigma)} \otimes S$$

Recalling that we have a symplectic form a on S , we set $(\tau)^2 = a(\tau, \tau)$, which is non-zero precisely because $\tau \in S \otimes L^{od}$.

Definition 8.4.3 The supersurface associated with the cross-section ψ is the H^∞ supermap $Y : \dot{\Sigma}^{2|(1,1)} \longrightarrow M^{m|n}$ which reads in the chart (\mathcal{W}, Φ) of $M^{m|n}$ (associated to the chart $(\pi^{-1}(W), \psi)$ of E) :

$$Y(\sigma, \tau) = (\tilde{X}^1(\sigma) + \frac{1}{2}(\tau)^2 \tilde{F}^1(\sigma), \dots, \tilde{X}^m(\sigma) + \frac{1}{2}(\tau)^2 \tilde{F}^m(\sigma), a(\tau, \tilde{\psi}^1(\sigma)), \dots, a(\tau, \tilde{\psi}^n(\sigma)))$$

where $\tilde{X}^\mu, \tilde{F}^\mu : (L^{ev})^2 \longrightarrow L^{ev}$ and $\tilde{\psi}^i : (L^{ev})^2 \longrightarrow L^{ev} \otimes S$ are the super-extensions of $X^\mu, F^\mu : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\psi^i : \mathbb{R}^2 \longrightarrow S$. In the following, we omit the tilda superscripts on X^μ, F^μ, ψ^i .

If we start with the action of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on itself by **right** supertranslations, then we may associate two natural actions of $\overset{\circ}{\Sigma}{}^{2|(1,1)}$ on the space of supersurfaces $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$: for $u \in \overset{\circ}{\Sigma}{}^{2|(1,1)}$ and $Y \in H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$, we set $(uY)(\sigma, \tau) = Y((\sigma, \tau) * u)$ (which gives a left action) and $(Yu)(\sigma, \tau) = Y((\sigma, \tau) * u^{-1})$ (which gives a right action).

In order to describe these two actions at the infinitesimal level, we first notice that the space $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, TM^{m|n})$ may be realized as an infinite-dimensional fibre bundle over $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$, by setting $\hat{\pi}(\Psi) = \pi \circ \Psi$ for any $\Psi \in H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, TM^{m|n})$, π denoting here the projection $TM^{m|n} \longrightarrow M^{m|n}$.

For any supersurface $Y \in H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$, let $Y^{-1}TM^{m|n}$ be the *induced bundle* over $\overset{\circ}{\Sigma}{}^{2|(1,1)}$. The cross-sections of this induced bundle are nothing but the *super-vector fields along* Y , that is the lifts of Y to the super-tangent bundle $TM^{m|n}$. It is not difficult to see that $\Gamma(Y^{-1}TM^{m|n})$ is also $\hat{\pi}^{-1}(\{Y\})$, the fiber of $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, TM^{m|n})$ above Y .

Proposition 8.4.4 *The infinitesimal action of the supermodule $V^{2|(1,1)}$ on the space of supersurfaces $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$ is given by a map $V^{2|(1,1)} \longrightarrow \Gamma(H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, TM^{m|n}))$. In other terms, we associate respectively to e_a and f_A two cross-sections \mathcal{P}_α and \mathcal{Q}_Λ of the fibre bundle $H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, TM^{m|n})$. For any $Y \in H^\infty(\overset{\circ}{\Sigma}{}^{2|(1,1)}, M^{m|n})$, $\mathcal{P}_\alpha Y$ and $\mathcal{Q}_\Lambda Y$ are two super-vector fields along Y (that is, $\mathcal{P}_\alpha Y, \mathcal{Q}_\Lambda Y \in \Gamma(Y^{-1}TM^{m|n}) = \hat{\pi}^{-1}(\{Y\})$). In the chart (\mathcal{W}, Φ) , we have :*

$$\begin{aligned}\mathcal{P}_\alpha Y(\sigma, \tau) &= \{\partial_\alpha X^\mu(\sigma) + \frac{1}{2}(\tau)^2(\frac{DF}{\partial\sigma^\alpha})^\mu(\sigma)\} \frac{\partial}{\partial x^\mu}|_{Y(\sigma, \tau)} + a(\tau, (\frac{\nabla\psi}{\partial\sigma^\alpha})^i(\sigma)) \frac{\partial}{\partial\theta^i}|_{Y(\sigma, \tau)} \\ \mathcal{Q}_+ Y(\sigma, \tau) &= \{-\Gamma_{++}^\alpha \tau^+ \partial_\alpha X^\mu(\sigma) + \tau^- F^\mu(\sigma)\} \frac{\partial}{\partial x^\mu}|_{Y(\sigma, \tau)} + \{\psi^{i-}(\sigma) + \frac{1}{2}(\tau)^2 \Gamma_{++}^\alpha (\frac{\nabla\psi^+}{\partial\sigma^\alpha})^i(\sigma)\} \frac{\partial}{\partial\theta^i}|_{Y(\sigma, \tau)} \\ \mathcal{Q}_- Y(\sigma, \tau) &= \{-\Gamma_{--}^\alpha \tau^- \partial_\alpha X^\mu(\sigma) - \tau^+ F^\mu(\sigma)\} \frac{\partial}{\partial x^\mu}|_{Y(\sigma, \tau)} + \{-\psi^{i+}(\sigma) + \frac{1}{2}(\tau)^2 \Gamma_{--}^\alpha (\frac{\nabla\psi^-}{\partial\sigma^\alpha})^i(\sigma)\} \frac{\partial}{\partial\theta^i}|_{Y(\sigma, \tau)}\end{aligned}$$

with $\mathcal{P}_\alpha Y(\sigma, \tau) \in (T_{Y(\sigma, \tau)} M^{m|n})_0$, $\mathcal{Q}_+ Y(\sigma, \tau), \mathcal{Q}_- Y(\sigma, \tau) \in (T_{Y(\sigma, \tau)} M^{m|n})_1$ and where $\frac{DF}{\partial\sigma^\alpha}$ is the covariant derivative of F along X for the Levi-Civita connection associated to the metric g on M .

Proof :

$$\text{First, } (\tau)^2 = a(\tau, \tau) = {}^t\tau\gamma_0\tau = \begin{pmatrix} \tau^+ & \tau^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tau^+ \\ \tau^- \end{pmatrix} = \tau^+\tau^- - \tau^-\tau^+$$

so we have :

$$(\tau)^2 = 2\tau^+\tau^-$$

On the other hand, we have :

$$a(\tau, \psi^i(\sigma)) = {}^t\tau\gamma_0\psi^i(\sigma) = \begin{pmatrix} \tau^+ & \tau^- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi^{i+}(\sigma) \\ \psi^{i-}(\sigma) \end{pmatrix} = \tau^+\psi^{i-}(\sigma) - \tau^-\psi^{i+}(\sigma)$$

We deduce that :

$$\begin{aligned}Y(\sigma, \tau) &= (X^\mu(\sigma) + \frac{1}{2}(\tau)^2 F^\mu(\sigma), a(\tau, \psi^i(\sigma))) \\ &= (X^\mu(\sigma) + \tau^+\tau^- F^\mu(\sigma), \tau^+\psi^{i-}(\sigma) - \tau^-\psi^{i+}(\sigma))\end{aligned}$$

Now we apply P_α , Q_+ and Q_- to this expression, but with taking care to replace the ordinary derivatives by covariant derivatives each time it is necessary. Thus, we obtain

the desired expressions. \square

It is not difficult now to deduce that the left action of $\mathring{\Sigma}^{2|(1,1)}$ on the space of supersurfaces $H^\infty(\mathring{\Sigma}^{2|(1,1)}, M^{m|n})$ is expressed infinitesimally by the following "representation" of $V^{2|(1,1)}$: for $(v^\alpha, \epsilon^\Lambda) \in V^{2|(1,1)}$, $\delta_v Y = v^\alpha \mathcal{P}_\alpha Y$ and $\delta_\epsilon Y = \epsilon^\Lambda \mathcal{Q}_\Lambda Y$.

Similarly, the right action of $\mathring{\Sigma}^{2|(1,1)}$ on $H^\infty(\mathring{\Sigma}^{2|(1,1)}, M^{m|n})$ is expressed infinitesimally by the following "representation" of $V^{2|(1,1)}$: for $(v^\alpha, \epsilon^\Lambda) \in V^{2|(1,1)}$, $\delta_v Y = -v^\alpha \mathcal{P}_\alpha Y$ and $\delta_\epsilon Y = -\epsilon^\Lambda \mathcal{Q}_\Lambda Y$.

Remark 8.4.5 We could have started with the action of $\mathring{\Sigma}^{2|(1,1)}$ on itself by **left** supertranslations. In this case, we may associate two natural actions of $\mathring{\Sigma}^{2|(1,1)}$ on the space of supersurfaces $H^\infty(\mathring{\Sigma}^{2|(1,1)}, M^{m|n})$: for $u \in \mathring{\Sigma}^{2|(1,1)}$ and $Y \in H^\infty(\mathring{\Sigma}^{2|(1,1)}, M^{m|n})$, we set $(uY)(\sigma, \tau) = Y(u^{-1} * (\sigma, \tau))$ (which gives a left action) and $(Yu)(\sigma, \tau) = Y(u * (\sigma, \tau))$ (which gives a right action). In this case, instead of the cross-section \mathcal{Q}_Λ , we would have taken the cross-section \mathcal{D}_Λ given in the chart (\mathcal{W}, Φ) by :

$$\begin{aligned}\mathcal{D}_+ Y(\sigma, \tau) &= \{\Gamma_{++}^\alpha \tau^+ \partial_\alpha X^\mu(\sigma) + \tau^- F^\mu(\sigma)\} \frac{\partial}{\partial x^\mu|_{Y(\sigma, \tau)}} + \{\psi^{i-}(\sigma) - \frac{1}{2}(\tau)^2 \Gamma_{++}^\alpha (\frac{\nabla \psi^+}{\partial \sigma^\alpha})^i(\sigma)\} \frac{\partial}{\partial \theta^i|_{Y(\sigma, \tau)}} \\ \mathcal{D}_- Y(\sigma, \tau) &= \{\Gamma_{--}^\alpha \tau^- \partial_\alpha X^\mu(\sigma) - \tau^+ F^\mu(\sigma)\} \frac{\partial}{\partial x^\mu|_{Y(\sigma, \tau)}} + \{-\psi^{i+}(\sigma) - \frac{1}{2}(\tau)^2 \Gamma_{--}^\alpha (\frac{\nabla \psi^-}{\partial \sigma^\alpha})^i(\sigma)\} \frac{\partial}{\partial \theta^i|_{Y(\sigma, \tau)}}\end{aligned}$$

To write an action functional for the supersurface Y , we make use of the data introduced on E , namely the metric g on M and the antisymmetric two-form B on E .

Set $g_{\mu\nu}(x) = g(\frac{\partial}{\partial x^\mu|_x}, \frac{\partial}{\partial x^\nu|_x})$ and $B_{ij}(x) = B(e_i|_x, e_j|_x)$. Then, consider the superextensions of $g_{\mu\nu}$ and B_{ij} . We can now define a supermetric G on $M^{m|n}$, that is for any $p \in M^{m|n}$, a super-bilinear form $G_p : T_p M^{m|n} \times T_p M^{m|n} \rightarrow L$ such that :

$G_p(\frac{\partial}{\partial x^\mu|_p}, \frac{\partial}{\partial x^\nu|_p}) = g_{\mu\nu}(x)$ and $G_p(e_i|_p, e_j|_p) = B_{ij}(x)$ for any $p \in \mathcal{W}$. In particular, if we denote by $(T_p M^{m|n})_+$ (resp. by $(T_p M^{m|n})_-$) the subspace of $T_p M^{m|n}$ spanned by the $\frac{\partial}{\partial x^\mu|_p}$ (resp. by the $\frac{\partial}{\partial \theta^i|_p}$), the restriction of G_p to $(T_p M^{m|n})_- \times (T_p M^{m|n})_-$ is antisymmetric, while its restriction on $(T_p M^{m|n})_+ \times (T_p M^{m|n})_+$ is symmetric.

We are now able to write an action functional for Y :

$$\mathcal{A}(Y) = \frac{1}{2\pi} \int_{\mathring{\Sigma}^{2|(1,1)}} G_{Y(\sigma, \tau)}(\mathcal{D}_+ Y(\sigma, \tau), \mathcal{D}_- Y(\sigma, \tau)) d^2\tau d^2\sigma$$

Proposition 8.4.6 The above action is equivalent to the following component action :

$$\mathcal{A}(X, \psi, F) = \frac{1}{2\pi} \int_{\mathring{\Sigma}} \{ \|dX_\sigma\|_{\eta g_X(\sigma)}^2 + \|F_\sigma\|_{g_X(\sigma)}^2 - \langle \psi(\sigma), \nabla \psi(\sigma) \rangle_{B_X(\sigma)} \} d^2\sigma$$

Proof :

We expand $G_{Y(\sigma, \tau)}(\mathcal{D}_+ Y(\sigma, \tau), \mathcal{D}_- Y(\sigma, \tau))$, retaining only the terms that will not vanish when integrating with respect to τ^+ and τ^- , that is the terms proportional to $(\tau)^2$.

The terms proportional to $(\tau)^2$ in $G_{Y(\sigma, \tau)}(\mathcal{D}_+ Y(\sigma, \tau), \mathcal{D}_- Y(\sigma, \tau))$ are :

$$\begin{aligned}
& G_{Y(\sigma,\tau)}(\Gamma_{++}^\alpha \tau^+ \partial_\alpha X^\mu(\sigma) \frac{\partial}{\partial x^\mu}_{|Y(\sigma,\tau)}, \Gamma_{--}^\beta \tau^- \partial_\beta X^\nu(\sigma) \frac{\partial}{\partial x^\nu}_{|Y(\sigma,\tau)}) \\
& - G_{Y(\sigma,\tau)}(\tau^- F^\mu(\sigma) \frac{\partial}{\partial x^\mu}_{|Y(\sigma,\tau)}, \tau^+ F^\nu(\sigma) \frac{\partial}{\partial x^\nu}_{|Y(\sigma,\tau)}) \\
& - G_{Y(\sigma,\tau)}(\psi^{i-}(\sigma) \frac{\partial}{\partial \theta^i}_{|Y(\sigma,\tau)}, \frac{1}{2}(\tau)^2 \Gamma_{--}^\alpha (\frac{\nabla \psi^-}{\partial \sigma^\alpha})^j(\sigma) \frac{\partial}{\partial \theta^j}_{|Y(\sigma,\tau)}) \\
& + G_{Y(\sigma,\tau)}(\frac{1}{2}(\tau)^2 \Gamma_{++}^\alpha (\frac{\nabla \psi^+}{\partial \sigma^\alpha})^i(\sigma) \frac{\partial}{\partial \theta^i}_{|Y(\sigma,\tau)}, \psi^{j+}(\sigma) \frac{\partial}{\partial \theta^j}_{|Y(\sigma,\tau)}) \\
\\
& = \frac{1}{2}(\tau)^2 \{ G_{Y(\sigma,\tau)}(\frac{\partial}{\partial x^\mu}_{|Y(\sigma,\tau)}, \frac{\partial}{\partial x^\nu}_{|Y(\sigma,\tau)}) \Gamma_{++}^\alpha \Gamma_{--}^\beta \partial_\alpha X^\mu(\sigma) \partial_\beta X^\nu(\sigma) \\
& + G_{Y(\sigma,\tau)}(\frac{\partial}{\partial x^\mu}_{|Y(\sigma,\tau)}, \frac{\partial}{\partial x^\nu}_{|Y(\sigma,\tau)}) F^\mu(\sigma) F^\nu(\sigma) \\
& - G_{Y(\sigma,\tau)}(\frac{\partial}{\partial \theta^i}_{|Y(\sigma,\tau)}, \frac{\partial}{\partial \theta^j}_{|Y(\sigma,\tau)}) \Gamma_{--}^\alpha \psi^{i-}(\sigma) (\frac{\nabla \psi^-}{\partial \sigma^\alpha})^j(\sigma) \\
& + G_{Y(\sigma,\tau)}(\frac{\partial}{\partial \theta^i}_{|Y(\sigma,\tau)}, \frac{\partial}{\partial \theta^j}_{|Y(\sigma,\tau)}) \Gamma_{++}^\alpha (\frac{\nabla \psi^+}{\partial \sigma^\alpha})^i(\sigma) \psi^{j+}(\sigma) \} \\
\\
& = \frac{1}{2}(\tau)^2 \{ g_{\mu\nu}(X(\sigma)) \Gamma_{++}^\alpha \Gamma_{--}^\beta \partial_\alpha X^\mu(\sigma) \partial_\beta X^\nu(\sigma) + g_{\mu\nu}(X(\sigma)) F^\mu(\sigma) F^\nu(\sigma) \\
& - B_{ij}(X(\sigma)) \Gamma_{--}^\alpha \psi^{i-}(\sigma) (\frac{\nabla \psi^-}{\partial \sigma^\alpha})^j(\sigma) + B_{ij}(X(\sigma)) \Gamma_{++}^\alpha (\frac{\nabla \psi^+}{\partial \sigma^\alpha})^i(\sigma) \psi^{j+}(\sigma) \} \\
\\
& = \frac{1}{2}(\tau)^2 \{ g_{\mu\nu}(X(\sigma)) (\Gamma_{++}^\alpha \Gamma_{--}^\beta + \Gamma_{++}^\beta \Gamma_{--}^\alpha) \partial_\alpha X^\mu(\sigma) \partial_\beta X^\nu(\sigma) + g_{\mu\nu}(X(\sigma)) F^\mu(\sigma) F^\nu(\sigma) \\
& - B_{ij}(X(\sigma)) \Gamma_{--}^\alpha \psi^{i-}(\sigma) (\frac{\nabla \psi^-}{\partial \sigma^\alpha})^j(\sigma) - B_{ij}(X(\sigma)) \Gamma_{++}^\alpha \psi^{i+}(\sigma) (\frac{\nabla \psi^+}{\partial \sigma^\alpha})^j(\sigma) \} \\
\\
& = \frac{1}{2}(\tau)^2 \{ g_{\mu\nu}(X(\sigma)) \eta^{\alpha\beta} \partial_\alpha X^\mu(\sigma) \partial_\beta X^\nu(\sigma) + g_{X(\sigma)}(F_\sigma, F_\sigma) \\
& - B_{X(\sigma)}(\psi^-(\sigma), \Gamma_{--}^\alpha \frac{\nabla \psi^-}{\partial \sigma^\alpha}(\sigma)) - B_{X(\sigma)}(\psi^+(\sigma), \Gamma_{++}^\alpha \frac{\nabla \psi^+}{\partial \sigma^\alpha}(\sigma)) \} \\
\\
& = \frac{1}{2}(\tau)^2 \{ \|dX_\sigma\|_{\eta g_{X(\sigma)}}^2 + \|F_\sigma\|_{g_{X(\sigma)}}^2 - B_{X(\sigma)}(\psi^-(\sigma), (\nabla \psi^-)(\sigma)) - B_{X(\sigma)}(\psi^+(\sigma), (\nabla \psi^+)(\sigma)) \} \\
\\
& = \frac{1}{2}(\tau)^2 \{ \|dX_\sigma\|_{\eta g_{X(\sigma)}}^2 + \|F_\sigma\|_{g_{X(\sigma)}}^2 - \langle \psi(\sigma), \nabla \psi(\sigma) \rangle_{B_{X(\sigma)}} \}
\end{aligned}$$

$$\mathcal{A}(Y) = \frac{1}{2\pi} \int_{\dot{\Sigma}^2(1,1)} G_{Y(\sigma,\tau)}(\mathcal{D}_+ Y(\sigma,\tau), \mathcal{D}_- Y(\sigma,\tau)) d^2\tau d^2\sigma$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{\dot{\Sigma}^2 \setminus (1,1)} \frac{1}{2}(\tau)^2 \left\{ \|dX_\sigma\|_{\eta g_X(\sigma)}^2 + \|F_\sigma\|_{g_X(\sigma)}^2 - \langle \psi(\sigma), \nabla \psi(\sigma) \rangle_{B_X(\sigma)} \right\} d^2\tau d^2\sigma \\
&= \frac{1}{2\pi} \int_{\dot{\Sigma}} \left\{ \|dX_\sigma\|_{\eta g_X(\sigma)}^2 + \|F_\sigma\|_{g_X(\sigma)}^2 - \langle \psi(\sigma), \nabla \psi(\sigma) \rangle_{B_X(\sigma)} \right\} d^2\sigma \quad \square
\end{aligned}$$

Remark 8.4.7 Instead of taking the symplectic form a in the definition of $Y(\sigma, \tau)$, we could have taken the $\text{Spin}^\dagger(V)$ -invariant symmetric bilinear form c on S defined by : $c|_{S_+ \times S_+} = c|_{S_- \times S_-} = 0$, $c|_{S_+ \times S_-} = \varepsilon$ and $c|_{S_- \times S_+} = {}^t\varepsilon$. This would result in changing the sign of the ψ^+ terms in \mathcal{D}_+ and \mathcal{D}_- , but these changes cancel when computing the action at the component level. Thus, whatever choice of pairing we make on S , the coupling between ψ and $\nabla \psi$ in the final action is antisymmetric, as for one-dimensional supersymmetric sigma-models.

Let us now derive the Euler-Lagrange equations associated to the above action.

Consider a variation of X , ψ and F , that is, for each s in a neighbourhood $]-\varepsilon, \varepsilon[$ of 0 in \mathbb{R} , a surface $X_s : \dot{\Sigma} \longrightarrow M$, a cross-section ψ_s of the vector bundle $(X_s^{-1}E \otimes \dot{\mathbb{S}}, \dot{\Sigma}, \mathbb{R}^n \otimes S)$, and a vector field F_s along X_s , such that : (X_0, ψ_0, F_0) .

In order to derive the equations of motion, we compute the variation of the Lagrangian density :

$$\mathcal{L}(X_s, \psi_s, F_s) = \eta^{\alpha\beta} g_{X_s}(\partial_\alpha X_s, \partial_\beta X_s) + g_{X_s}(F_s, F_s) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+)$$

In what follows, D denotes the Levi-Civita connection of the metric g , and R the curvature of the connection ∇ on E .

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial s}(X_s, \psi_s, F_s) &= \eta^{\alpha\beta} \frac{\partial}{\partial s} g_{X_s}(\partial_\alpha X_s, \partial_\beta X_s) + \frac{\partial}{\partial s} g_{X_s}(F_s, F_s) \\
&\quad - \frac{\partial}{\partial s} B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) - \frac{\partial}{\partial s} B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) \\
&= \eta^{\alpha\beta} g_{X_s}\left(\frac{D}{\partial s} \partial_\alpha X_s, \partial_\beta X_s\right) + \eta^{\alpha\beta} g_{X_s}(\partial_\alpha X_s, \frac{D}{\partial s} \partial_\beta X_s) + 2 g_{X_s}\left(\frac{D}{\partial s} F_s, F_s\right) - B_{X_s}\left(\frac{\nabla}{\partial s} \psi_s^-, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-\right) \\
&\quad - B_{X_s}\left(\psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \nabla_\alpha \psi_s^-\right) - B_{X_s}\left(\frac{\nabla}{\partial s} \psi_s^+, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+\right) - B_{X_s}\left(\psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial s} \nabla_\alpha \psi_s^+\right) \\
&= \eta^{\alpha\beta} g_{X_s}\left(\frac{D}{\partial s} \frac{\partial}{\partial \sigma^\alpha} X_s, \partial_\beta X_s\right) + \eta^{\alpha\beta} g_{X_s}(\partial_\alpha X_s, \frac{D}{\partial s} \frac{\partial}{\partial \sigma^\beta} X_s) + 2 g_{X_s}\left(\frac{D}{\partial s} F_s, F_s\right) - B_{X_s}\left(\frac{\nabla \psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-\right) \\
&\quad - B_{X_s}\left(\frac{\nabla \psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+\right) - B_{X_s}\left(\psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \nabla_\alpha \psi_s^-\right) - B_{X_s}\left(\psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial s} \nabla_\alpha \psi_s^+\right) \\
&= \eta^{\alpha\beta} g_{X_s}\left(\frac{D}{\partial \sigma^\alpha} \frac{\partial}{\partial s} X_s, \partial_\beta X_s\right) + \eta^{\alpha\beta} g_{X_s}(\partial_\alpha X_s, \frac{D}{\partial \sigma^\beta} \frac{\partial}{\partial s} X_s) + 2 g_{X_s}\left(\frac{D}{\partial s} F_s, F_s\right) - B_{X_s}\left(\frac{\nabla \psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-\right)
\end{aligned}$$

$$-B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \frac{\nabla}{\partial \sigma^\alpha} \psi_s^-) - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial s} \frac{\nabla}{\partial \sigma^\alpha} \psi_s^+)$$

because $\frac{D}{\partial s} \frac{\partial}{\partial \sigma^\gamma} - \frac{D}{\partial \sigma^\gamma} \frac{\partial}{\partial s} = [\frac{\partial}{\partial s}, \frac{\partial}{\partial \sigma^\gamma}]$ since D is torsionless, and $[\frac{\partial}{\partial s}, \frac{\partial}{\partial \sigma^\gamma}] = 0$.

Up to a boundary term resulting from an integration by parts, $\frac{\partial \mathcal{L}}{\partial s}(X_s, \psi_s, F_s)$ is equal to :

$$\begin{aligned} & -\eta^{\alpha\beta} g_{X_s}(\frac{\partial}{\partial s} X_s, \frac{D}{\partial \sigma^\alpha} \partial_\beta X_s) - \eta^{\alpha\beta} g_{X_s}(\frac{D}{\partial \sigma^\beta} \partial_\alpha X_s, \frac{\partial}{\partial s} X_s) + 2 g_{X_s}(\frac{D}{\partial s} F_s, F_s) - B_{X_s}(\frac{\nabla\psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) \\ & - B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \frac{\nabla}{\partial \sigma^\alpha} \psi_s^-) - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial s} \frac{\nabla}{\partial \sigma^\alpha} \psi_s^+) \\ & = -\eta^{\alpha\beta} g_{X_s}(\frac{\partial X_s}{\partial s}, \frac{D}{\partial \sigma^\alpha} \partial_\beta X_s) - \eta^{\alpha\beta} g_{X_s}(\frac{D}{\partial \sigma^\beta} \partial_\alpha X_s, \frac{\partial X_s}{\partial s}) + 2 g_{X_s}(\frac{D}{\partial s} F_s, F_s) - B_{X_s}(\frac{\nabla\psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) \\ & - B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial \sigma^\alpha} \frac{\nabla}{\partial s} \psi_s^-) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^-) \\ & - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial \sigma^\alpha} \frac{\nabla}{\partial s} \psi_s^+) - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^+) \\ & \text{because } \frac{\nabla}{\partial s} \frac{\nabla}{\partial \sigma^\alpha} - \frac{\nabla}{\partial \sigma^\alpha} \frac{\nabla}{\partial s} = R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}). \end{aligned}$$

Up to a boundary term resulting from an integration by parts, $\frac{\partial \mathcal{L}}{\partial s}(X_s, \psi_s, F_s)$ is equal to :

$$\begin{aligned} & -\eta^{\alpha\beta} g_{X_s}(\frac{\partial X_s}{\partial s}, \frac{D}{\partial \sigma^\alpha} \partial_\beta X_s) - \eta^{\alpha\beta} g_{X_s}(\frac{D}{\partial \sigma^\beta} \partial_\alpha X_s, \frac{\partial X_s}{\partial s}) + 2 g_{X_s}(\frac{D}{\partial s} F_s, F_s) - B_{X_s}(\frac{\nabla\psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) \\ & - B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) + B_{X_s}(\frac{\nabla}{\partial \sigma^\alpha} \psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \psi_s^-) - B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^-) \\ & + B_{X_s}(\frac{\nabla}{\partial \sigma^\alpha} \psi_s^+, \Gamma_{++}^\alpha \frac{\nabla}{\partial s} \psi_s^+) - B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^+) \end{aligned}$$

Defining the vectors $\mathbf{R}_{\partial X, \psi, \psi}^+$ and $\mathbf{R}_{\partial X, \psi, \psi}^-$ by :

$$\begin{aligned} g_{X_s}(\mathbf{R}_{\partial X, \psi, \psi}^+, \frac{\partial X_s}{\partial s}) &= B_{X_s}(\psi_s^+, \Gamma_{++}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^+) \\ g_{X_s}(\mathbf{R}_{\partial X, \psi, \psi}^-, \frac{\partial X_s}{\partial s}) &= B_{X_s}(\psi_s^-, \Gamma_{--}^\alpha R_{X_s}(\frac{\partial X_s}{\partial s}, \frac{\partial X_s}{\partial \sigma^\alpha}) \psi_s^-) \end{aligned}$$

the last expression reads :

$$\begin{aligned} & -\eta^{\alpha\beta} g_{X_s}(\frac{\partial X_s}{\partial s}, \frac{D}{\partial \sigma^\alpha} \partial_\beta X_s) - \eta^{\alpha\beta} g_{X_s}(\frac{D}{\partial \sigma^\beta} \partial_\alpha X_s, \frac{\partial X_s}{\partial s}) + 2 g_{X_s}(\frac{D}{\partial s} F_s, F_s) - B_{X_s}(\frac{\nabla\psi_s^-}{\partial s}, \Gamma_{--}^\alpha \nabla_\alpha \psi_s^-) \\ & - B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, \Gamma_{++}^\alpha \nabla_\alpha \psi_s^+) + B_{X_s}(\frac{\nabla}{\partial \sigma^\alpha} \psi_s^-, \Gamma_{--}^\alpha \frac{\nabla}{\partial s} \psi_s^-) - g_{X_s}(\mathbf{R}_{\partial X, \psi, \psi}^-, \frac{\partial X_s}{\partial s}) \end{aligned}$$

$$+B_{X_s}(\frac{\nabla}{\partial\sigma^\alpha}\psi_s^+, \Gamma_{++}^\alpha\frac{\nabla}{\partial s}\psi_s^+) - g_{X_s}(\mathbf{R}_{\partial X, \psi, \psi}^+, \frac{\partial X_s}{\partial s})$$

$$\begin{aligned} &= -g_{X_s}(\frac{\partial X_s}{\partial s}, \eta^{\alpha\beta}(\frac{D}{\partial\sigma^\alpha}\partial_\beta X_s + \frac{D}{\partial\sigma^\beta}\partial_\alpha X_s) + \mathbf{R}_{\partial X, \psi, \psi}^+ + \mathbf{R}_{\partial X, \psi, \psi}^-) + 2 g_{X_s}(\frac{D}{\partial s}F_s, F_s) \\ &\quad - B_{X_s}(\frac{\nabla\psi_s^-}{\partial s}, 2\Gamma_{--}^\alpha\frac{\nabla\psi_s^-}{\partial\sigma^\alpha}) - B_{X_s}(\frac{\nabla\psi_s^+}{\partial s}, 2\Gamma_{++}^\alpha\frac{\nabla\psi_s^+}{\partial\sigma^\alpha}) \end{aligned}$$

At a stationnary configuration, this last expression vanishes for any variation of X , ψ and F , which gives the following equations :

$$\eta^{\alpha\beta}\frac{D}{\partial\sigma^\alpha}\partial_\beta X + \frac{1}{2}\mathbf{R}_{\partial X, \psi, \psi}^+ + \frac{1}{2}\mathbf{R}_{\partial X, \psi, \psi}^- = 0 \quad , \quad \Gamma_{++}^\alpha\frac{\nabla\psi^+}{\partial\sigma^\alpha} = 0 \quad , \quad \Gamma_{--}^\alpha\frac{\nabla\psi^-}{\partial\sigma^\alpha} = 0 \quad , \quad F = 0.$$

These equations may be written in an intrinsic form. For this, we define a covariantized version \square^∇ of the d'Alembertian operator. \square^∇ will associate to a surface

$X : \overset{\circ}{\Sigma} \longrightarrow M$ a vector field $\square^\nabla X$ along X . We know that $\partial_\alpha X$ is a vector field along X . Setting $dX = \partial_\alpha X \, d\sigma^\alpha$, we obtain a vector-bundle-valued one-form dX on $\overset{\circ}{\Sigma}$; we have : $dX \in \Lambda^1(\overset{\circ}{\Sigma}, X^{-1}TM)$. Let $*$ be the Hodge star operator on $\overset{\circ}{\Sigma}$. Then we have $*dX \in \Lambda^1(\overset{\circ}{\Sigma}, X^{-1}TM)$ as well. Denoting by d^∇ the covariant differential associated with the connection $X^*\nabla$ on $\overset{\circ}{\Sigma}$, we see that $d^\nabla(*dX) \in \Lambda^2(\overset{\circ}{\Sigma}, X^{-1}TM)$. Taking again the Hodge dual, we finally obtain a zero-form $*d^\nabla(*dX) \in \Lambda^0(\overset{\circ}{\Sigma}, X^{-1}TM) = \Gamma(X^{-1}TM)$. Thus,

$$\square^\nabla X = *d^\nabla(*dX)$$

is vector field along X , and we have : $\square^\nabla X = \eta^{\alpha\beta}\frac{D}{\partial\sigma^\alpha}\partial_\beta X$. Setting besides $\mathbf{R}_{\partial X, \psi, \psi} = \mathbf{R}_{\partial X, \psi, \psi}^+ + \mathbf{R}_{\partial X, \psi, \psi}^-$, the equations become :

$$\left\{ \begin{array}{l} \square^\nabla X + \frac{1}{2}\mathbf{R}_{\partial X, \psi, \psi} = 0 \\ \nabla_+\psi^+ = 0 \\ \nabla_-\psi^- = 0 \\ F = 0 \end{array} \right.$$

The invariance of these equations by supersymmetry transformations is a direct consequence of the manifest supercovariant form of the action. One could also check the supersymmetry invariance at the component level.

Chapitre 9

Supersymmetric field theories in Minkowski spacetime

9.1 The Wess-Zumino model

In this section, we consider the case $D = 4$, and we identify $\text{Spin}^\dagger(3, 1)$ with $\text{SL}_2(\mathbb{C})$. We define S_+ (resp. S_-) to be the subspace $\mathbb{C}^2 \times \{0\}$ (resp. $\{0\} \times \mathbb{C}^2$) of \mathbb{C}^4 carrying the representation $(\psi, 0) \mapsto (g\psi, 0)$ (resp. $(0, \chi) \mapsto (0, \bar{g}\chi)$) (with $g \in \text{Spin}^\dagger(3, 1)$). We set $S_{\mathbb{C}} = S_+ \oplus S_-$ ($S_{\mathbb{C}} = \mathbb{C}^4$ carrying the direct sum representation). The elements of $S_{\mathbb{C}}$ (resp. S_+, S_-) are called **Dirac spinors** (resp. **left Weyl spinors, right Weyl spinors**).

Proposition 9.1.1 *The map $\tau : S_{\mathbb{C}} \longrightarrow S_{\mathbb{C}}$ defined by : $\tau(\psi, \chi) = (\bar{\chi}, \bar{\psi})$ is a $\text{Spin}^\dagger(3, 1)$ -equivariant conjugation of $S_{\mathbb{C}}$. The subspaces S_+ and S_- are exchanged by τ .*

We set $S = \text{Inv}(\tau) = \ker(\tau - \text{Id}_{S_{\mathbb{C}}})$. Then S is a four-dimensional real faithful representation of $\text{Spin}^\dagger(3, 1)$, irreducible and of complex type. By definition, we have :

$$S = \{(\psi, \bar{\psi}) \in S_{\mathbb{C}} ; \psi \in \mathbb{C}^2\}$$

The complex structure of S is naturally induced by the complex structure of \mathbb{C}^2 : for any $z \in \mathbb{C}$, we set $z(\psi, \bar{\psi}) = (z\psi, \bar{z}\bar{\psi})$. The elements of S are called **Majorana spinors**.

Let $\varepsilon_+ : S_+ \times S_+ \longrightarrow \mathbb{C}$ be the $\text{Spin}^\dagger(3, 1)$ -invariant antisymmetric bilinear form defined by : $\varepsilon_+((\psi, 0), (\psi', 0)) = \psi^1\psi'^2 - \psi^2\psi'^1$ (where $\psi = (\psi^1, \psi^2)$ and $\psi' = (\psi'^1, \psi'^2)$).

Let $\varepsilon_- : S_- \times S_- \longrightarrow \mathbb{C}$ be the $\text{Spin}^\dagger(3, 1)$ -invariant antisymmetric bilinear form defined by : $\varepsilon_-((0, \chi), (0, \chi')) = \chi^{\dot{1}}\chi'^{\dot{2}} - \chi^{\dot{2}}\chi'^{\dot{1}}$ (where $\chi = (\chi^{\dot{1}}, \chi^{\dot{2}})$ and $\chi' = (\chi'^{\dot{1}}, \chi'^{\dot{2}})$).

We deduce a $\text{Spin}^\dagger(3, 1)$ -invariant antisymmetric bilinear form $\varepsilon_{\mathbb{C}} : S_{\mathbb{C}} \times S_{\mathbb{C}} \longrightarrow \mathbb{C}$ defined by : $\varepsilon_{\mathbb{C}}((\psi, \chi), (\psi', \chi')) = \varepsilon_+((\psi, 0), (\psi', 0)) + \varepsilon_-((0, \chi), (0, \chi'))$.

Let $(\psi, \bar{\psi}), (\psi', \bar{\psi}') \in S$. Then $\varepsilon_{\mathbb{C}}((\psi, \bar{\psi}), (\psi', \bar{\psi}')) = 2 \operatorname{Re}(\psi^1\psi'^2 - \psi^2\psi'^1)$. We deduce that $\varepsilon_{\mathbb{C}}(S \times S) \subset \mathbb{R}$. Let us denote by $\varepsilon : S \times S \longrightarrow \mathbb{R}$ the restriction of $\varepsilon_{\mathbb{C}}$ to $S \times S$. Then ε is a $\text{Spin}^\dagger(3, 1)$ -invariant antisymmetric bilinear form on S .

We define $V_{\mathbb{C}} = S_+ \otimes S_-$. Then $V_{\mathbb{C}}$ is a complex vector space of dimension 4.

Proposition 9.1.2 *The map $\nu : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ defined by : $\nu((\psi, 0) \otimes (0, \chi)) = (\bar{\chi}, 0) \otimes (0, \bar{\psi})$ is a $\text{Spin}^{\dagger}(3, 1)$ -equivariant conjugation of $V_{\mathbb{C}}$.*

We set $V = \text{Inv}(\nu) = \ker(\nu - \text{Id}_{V_{\mathbb{C}}})$. Then V is a real vector space of dimension 4, carrying an irreducible representation of $\text{Spin}^{\dagger}(3, 1)$. This representation is not faithful and descends into a representation of $\text{SO}^{\dagger}(3, 1)$.

We have a $\text{Spin}^{\dagger}(3, 1)$ -invariant non-degenerate symmetric bilinear form $\eta_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}$ defined by : $\eta_{\mathbb{C}}((\psi, 0) \otimes (0, \chi), (\psi', 0) \otimes (0, \chi')) = -2 \varepsilon_+((\psi, 0), (\psi', 0)) \varepsilon_-((0, \chi), (0, \chi'))$.

Proposition 9.1.3 *We have $\eta_{\mathbb{C}}(V \times V) \subset \mathbb{R}$, and denoting by $\eta : V \times V \longrightarrow \mathbb{R}$ the restriction of $\eta_{\mathbb{C}}$ to $V \times V$, we have that η is a $\text{Spin}^{\dagger}(3, 1)$ -invariant symmetric bilinear form of signature $(3, 1)$ on V .*

Let $\Gamma_{\mathbb{C}} : S_{\mathbb{C}} \times S_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ be the $\text{Spin}^{\dagger}(3, 1)$ -equivariant symmetric bilinear map defined by :

$$\Gamma_{\mathbb{C}}((\psi, \chi), (\psi', \chi')) = (\psi, 0) \otimes (0, \chi') + (\psi', 0) \otimes (0, \chi)$$

Notice that : $\Gamma_{\mathbb{C}}(S_+ \times S_+) = \Gamma_{\mathbb{C}}(S_- \times S_-) = 0$.

Proposition 9.1.4 *The symmetric bilinear map $\Gamma_{\mathbb{C}}$ induces a natural complex Lie superalgebra structure on $V_{\mathbb{C}} \times S_{\mathbb{C}}$.*

We get a super-bracket by setting : $[V_{\mathbb{C}}, V_{\mathbb{C}}] = [V_{\mathbb{C}}, S_{\mathbb{C}}] = [S_{\mathbb{C}}, V_{\mathbb{C}}] = 0$ and for any $s, t \in S_{\mathbb{C}}$, $[s, t]_+ = -2 \Gamma_{\mathbb{C}}(s, t)$.

Let $(\psi, \bar{\psi}), (\psi', \bar{\psi}') \in S$. Then $\Gamma_{\mathbb{C}}((\psi, \bar{\psi}), (\psi', \bar{\psi}')) = (\psi, 0) \otimes (0, \bar{\psi}') + (\psi', 0) \otimes (0, \bar{\psi})$ which is clearly invariant by ν . We deduce that $\Gamma_{\mathbb{C}}(S \times S) \subset V$. Let us denote by $\Gamma : S \times S \longrightarrow V$ the restriction of $\Gamma_{\mathbb{C}}$ to $S \times S$. Then Γ is a $\text{Spin}^{\dagger}(3, 1)$ -equivariant positive definite symmetric bilinear map, such that for any $(\psi, \bar{\psi}) \in S$, $\eta(\Gamma((\psi, \bar{\psi}), (\psi, \bar{\psi})), \Gamma((\psi, \bar{\psi}), (\psi, \bar{\psi}))) = 0$.

We define $V_+^{4|2} = (V_{\mathbb{C}} \otimes L_{\mathbb{C}}^{ev}) \times (S_+ \otimes L_{\mathbb{C}}^{od})$ and $V_-^{4|2} = (V_{\mathbb{C}} \otimes L_{\mathbb{C}}^{ev}) \times (S_- \otimes L_{\mathbb{C}}^{od})$.

Also, we define : $V_{\mathbb{C}}^{4|4} = (V_{\mathbb{C}} \otimes L_{\mathbb{C}}^{ev}) \times (S_{\mathbb{C}} \otimes L_{\mathbb{C}}^{od})$ and $V^{4|4} = (V \otimes L^{ev}) \times (S \otimes L^{od})$.

Proposition 9.1.5 *The symmetric bilinear map $\Gamma_{\mathbb{C}}$ (resp. Γ) induces a natural complex (resp. real) Lie supermodule structure on $V_{\mathbb{C}}^{4|4}$ (resp. $V^{4|4}$).*

Proposition 9.1.6 *There exists a natural structure of complex (resp. real) Lie supergroup on $V_{\mathbb{C}}^{4|4}$ (resp. $V^{4|4}$), the group law being given by : $u * u' = u + u' + \frac{1}{2}[u, u']$.*

Recall that we have an isomorphism : ${}^l e^{-1} : V_{\mathbb{C}}^{4|4} \longrightarrow \Gamma_l(T\mathring{M}_{\mathbb{C}}^{4|4})$, which associates to every $u \in V_{\mathbb{C}}^{4|4}$ a left-invariant super-vector field : ${}^l e^{-1}(u)_z = T_0 l_z(u)$. We set :

$$\begin{cases} P_m &= {}^l e^{-1}(e_m) \\ Q_a &= {}^l e^{-1}(f_a) \\ Q_b &= {}^l e^{-1}(f_b) \end{cases}$$

Recall that we have an isomorphism : ${}^r e^{-1} : V_{\mathbb{C}}^{4|4} \longrightarrow \Gamma_r(\mathbf{T}\mathring{M}_{\mathbb{C}}^{4|4})$, which associates to every $u \in V_{\mathbb{C}}^{4|4}$ a right-invariant super-vector field : ${}^r e^{-1}(u)_z = \mathbf{T}_0 r_z(u)$. We set :

$$\begin{cases} \partial_m &= {}^r e^{-1}(e_m) \\ D_a &= {}^r e^{-1}(f_a) \\ D_{\dot{a}} &= {}^r e^{-1}(f_{\dot{a}}) \end{cases}$$

Let $\Phi \in H^\omega(\mathring{M}^{4|4}, L_{\mathbb{C}})$. Then Φ can be continued to $\Phi_{\mathbb{C}} \in H^\omega(\mathring{M}_{\mathbb{C}}^{4|4}, L_{\mathbb{C}})$.

Definition 9.1.7 *We say that $\Phi \in H^\omega(\mathring{M}^{4|4}, L_{\mathbb{C}})$ is a **chiral scalar superfield** if :*

$$D_{\dot{a}}\Phi = 0$$

We may write an action functional for the chiral scalar superfield :

$$\mathcal{S}(\Phi) = \int_{\mathring{M}^{4|4}} \bar{\Phi}(x, \theta, \bar{\theta}) \Phi(x, \theta, \bar{\theta}) d^4x d^2\theta d^2\bar{\theta}$$

Theorem 9.1.8 *The preceding action is invariant under the infinitesimal supersymmetry transformations acting on Φ :*

$$\delta_\xi \Phi = (\xi^a Q_a + \bar{\xi}^{\dot{a}} Q_{\dot{a}}) \Phi$$

Let us express the above model in coordinates.

Let $f_1 = (1, 0, 0, 0)$, $f_2 = (0, 1, 0, 0)$, $f_{\dot{1}} = (0, 0, 1, 0)$, $f_{\dot{2}} = (0, 0, 0, 1)$, so that $(f_a)_{1 \leq a \leq 2}$ is a basis of S_+ and $(f_{\dot{a}})_{1 \leq \dot{a} \leq \dot{2}}$ is a basis of S_- .

The representative matrix (ε_{ab}) of ε_+ (resp. $(\varepsilon_{\dot{a}\dot{b}})$ of ε_-) in the basis $(f_a)_{1 \leq a \leq 2}$ (resp. $(f_{\dot{a}})_{1 \leq \dot{a} \leq \dot{2}}$) is :

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that :

$$\begin{aligned} \varepsilon_+((\psi, 0), (\psi', 0)) &= {}^t \psi \epsilon \psi' = \varepsilon_{ab} \psi^a \psi'^b = \psi^a \psi'_a = -\psi_b \psi'^b \\ \varepsilon_-((0, \chi), (0, \chi')) &= {}^t \chi \epsilon \chi' = \varepsilon_{\dot{a}\dot{b}} \chi^{\dot{a}} \chi'^{\dot{b}} = \chi^{\dot{a}} \chi'_{\dot{a}} = -\chi_{\dot{b}} \chi'^{\dot{b}} \end{aligned}$$

For $\theta = (\theta^1, \theta^2) \in \mathbb{C}^2 \otimes L_{\mathbb{C}}^{od}$, we set :

$$(\theta)^2 = {}^t \theta \epsilon \theta = \theta^a \theta_a = \varepsilon_{ab} \theta^a \theta^b = \theta^1 \theta^2 - \theta^2 \theta^1 = 2 \theta^1 \theta^2$$

Defining $e_{ab} = f_a \otimes f_b$, we obtain a basis (e_{ab}) of $V_{\mathbb{C}}$. We have :

$$\nu(e_{ab}) = e_{b\dot{a}} \quad , \quad \Gamma_{\mathbb{C}}(f_a, f_b) = \Gamma_{\mathbb{C}}(f_{\dot{b}}, f_a) = e_{ab} \quad , \quad [f_a, f_b]_+ = -2 \Gamma_{\mathbb{C}}(f_a, f_b)$$

Each $u \in V_{\mathbb{C}}^{4|4}$ may be written : $u = v^{ab} \otimes e_{ab} + s^a \otimes f_a + t^{\dot{b}} \otimes f_{\dot{b}}$, with $v^{ab} \in L_{\mathbb{C}}^{ev}$, and $s^a, t^{\dot{b}} \in L_{\mathbb{C}}^{od}$.

Let us expand the super-bracket of two elements $u = v^{ab} \otimes e_{ab} + s^a \otimes f_a + t^{\dot{b}} \otimes f_{\dot{b}}$ and $u' = v'^{ab} \otimes e_{ab} + s'^a \otimes f_a + t'^{\dot{b}} \otimes f_{\dot{b}}$ of $V_{\mathbb{C}}^{4|4}$.

$$[u, u'] = [s^a f_a, t'^{\dot{b}} f_{\dot{b}}] + [t^{\dot{b}} f_{\dot{b}}, s'^a f_a]$$

$$\begin{aligned}
[s^a f_a, t'^b f_b] &= s^a f_a t'^b f_b - t'^b f_b s^a f_a \\
&= -s^a f_a f_b t'^b - s^a f_b f_a t'^b \\
&= -s^a [f_a, f_b]_+ t'^b \\
&= 2 s^a t'^b e_{ab}
\end{aligned}
\quad
\begin{aligned}
[t^b f_b, s'^a f_a] &= t^b f_b s'^a f_a - s'^a f_a t^b f_b \\
&= -t^b f_b f_a s'^a - t^b f_a f_b s'^a \\
&= -t^b [f_b, f_a]_+ s'^a \\
&= 2 t^b s'^a e_{ab} \\
&= -2 s'^a t^b e_{ab}
\end{aligned}$$

Finally,

$$[u, u'] = 2 (s^a t'^b - s'^a t^b) e_{ab}$$

Let us expand the group product of two elements $u = v^{ab} \otimes e_{ab} + s^a \otimes f_a + t^b \otimes f_b$ and $u' = v'^{ab} \otimes e_{ab} + s'^a \otimes f_a + t'^b \otimes f_b$ of $V_{\mathbb{C}}^{4|4}$.

$$u * u' = (v^{ab} + v'^{ab} + s^a t'^b - s'^a t^b) \otimes e_{ab} + (s^a + s'^a) \otimes f_a + (t^b + t'^b) \otimes f_b$$

Set :

$$\left\{
\begin{array}{lcl}
e_0 & = & -\frac{1}{2}(e_{1i} + e_{2\dot{i}}) \\
e_1 & = & \frac{1}{2}(e_{1\dot{i}} + e_{2i}) \\
e_2 & = & -\frac{1}{2i}(e_{1\dot{i}} - e_{2i}) \\
e_3 & = & \frac{1}{2}(e_{1i} - e_{2\dot{i}})
\end{array}
\right.
\text{ so that : }
\left\{
\begin{array}{lcl}
e_{1i} & = & -e_0 + e_3 \\
e_{2\dot{i}} & = & -e_0 - e_3 \\
e_{1\dot{i}} & = & e_1 - ie_2 \\
e_{2i} & = & e_1 + ie_2
\end{array}
\right.$$

which is equivalent to : $e_{ab} = \sigma_{ab}^m e_m$ (where the σ^m are the Pauli matrices).

Then $(e_m)_{0 \leq m \leq 3}$ is a basis of $V_{\mathbb{C}}$. We check directly that for each $m \in \{0, 1, 2, 3\}$, $\nu(e_m) = e_m$ (which implies $e_m \in V$). We deduce that over \mathbb{R} , $(e_m)_{0 \leq m \leq 3}$ is a basis of V , and the representative matrix of η in this basis is $\text{diag}(-1, +1, +1, +1)$.

Each $u \in V^{4|4}$ may be written : $u = v^m \otimes e_m + s^a \otimes f_a + \bar{s}^{\dot{a}} \otimes f_{\dot{a}}$, with $v^m \in L^{ev}$, and $s^a, \bar{s}^{\dot{a}} \in L_{\mathbb{C}}^{od}$.

Let $u = v^m \otimes e_m + s^a \otimes f_a + \bar{s}^{\dot{a}} \otimes f_{\dot{a}}$ and $u' = v'^m \otimes e_m + s'^a \otimes f_a + \bar{s}'^{\dot{a}} \otimes f_{\dot{a}}$ in $V^{4|4}$.

$$[u, u'] = 2 \sigma_{ab}^m (s^a \bar{s}^{\dot{b}} - s'^a \bar{s}'^{\dot{b}}) e_m$$

$$u * u' = (v^m + v'^m + \sigma_{ab}^m (s^a \bar{s}^{\dot{b}} - s'^a \bar{s}'^{\dot{b}})) \otimes e_m + (s^a + s'^a) \otimes f_a + (\bar{s}^{\dot{b}} + \bar{s}'^{\dot{b}}) \otimes f_{\dot{b}}$$

On the complex supermanifold $\mathring{M}_{\mathbb{C}}^{4|4}$, we have a global chart $\varphi^{-1} : \mathbb{C}^{4|4} \longrightarrow \mathring{M}_{\mathbb{C}}^{4|4}$ defined by :

$$\varphi^{-1}(x^\mu, \theta^\alpha, \tau^{\dot{\beta}}) = x^\mu \otimes e_\mu + \theta^\alpha \otimes f_\alpha + \tau^{\dot{\beta}} \otimes f_{\dot{\beta}}$$

where $\mathbb{C}^{4|4} = (L_{\mathbb{C}}^{ev})^4 \times (L_{\mathbb{C}}^{od})^4$. This chart will be called **the canonical chart** on $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$. It induces a global moving frame $(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \tau^\beta})$, with

$$\frac{\partial}{\partial x^\mu}|_z = e_\mu \quad , \quad \frac{\partial}{\partial \theta^\alpha}|_z = f_\alpha \quad , \quad \frac{\partial}{\partial \tau^\beta}|_z = f_\beta$$

for each point $z = x^\mu \otimes e_\mu + \theta^\alpha \otimes f_\alpha + \tau^\beta \otimes f_\beta \in \overset{\circ}{M}_{\mathbb{C}}^{4|4}$. The above three vectors belong of course to the tangent supermodule $T_z \overset{\circ}{M}_{\mathbb{C}}^{4|4} \simeq \overset{\circ}{M}_{\mathbb{C}}^{4|4}$.

Finally, let us write the action of an element $u = \xi^\alpha \otimes f_\alpha + \bar{\xi}^\dot{\alpha} \otimes f_{\dot{\alpha}} \in V^{4|4}$ on a point $z \in \overset{\circ}{M}^{4|4}$ by supertranslation on the right :

$$r_u(z) = z * u = (x^\mu + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \bar{\xi}^\dot{\beta} - \xi^\alpha \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^\dot{\beta}) \otimes e_\mu + (\theta^\alpha + \xi^\alpha) \otimes f_\alpha + (\bar{\theta}^\dot{\alpha} + \bar{\xi}^\dot{\alpha}) \otimes f_{\dot{\alpha}}$$

Thus, we obtain what the physicists call the **supersymmetry transformations on superspace** :

$$\begin{cases} x'^\mu &= x^\mu + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \bar{\xi}^\dot{\beta} - \xi^\alpha \sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^\dot{\beta} \\ \theta'^\alpha &= \theta^\alpha + \xi^\alpha \\ \bar{\theta}'^{\dot{\alpha}} &= \bar{\theta}^\dot{\alpha} + \bar{\xi}^\dot{\alpha} \end{cases}$$

Recall that we have an isomorphism ${}^l e^{-1} : V_{\mathbb{C}}^{4|4} \longrightarrow \Gamma_l(T\overset{\circ}{M}_{\mathbb{C}}^{4|4})$, which associates to every $u = v^m \otimes e_m + s^a \otimes f_a + t^b \otimes f_b \in V_{\mathbb{C}}^{4|4}$ a left-invariant super-vector field : ${}^l e^{-1}(u)_z = T_0 l_z(u)$. Let γ be a path in $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$ satisfying : $\gamma(0) = 0$ and $\dot{\gamma}(0) = u$. We set :

$\gamma(t) = \lambda^\mu(t) \otimes e_\mu + \xi^\alpha(t) \otimes f_\alpha + \zeta^\dot{\beta}(t) \otimes f_{\dot{\beta}}$, so that :

$$\begin{aligned} {}^l e^{-1}(u)_z &= T_0 l_z(u) = \frac{d}{dt}[l_z(\gamma(t))]|_{t=0} = \frac{d}{dt}[z * \gamma(t)]|_{t=0} \\ &= \frac{d}{dt}[(x^\mu + \lambda^\mu(t) + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \zeta^\dot{\beta}(t) - \xi^\alpha(t) \sigma_{\alpha\dot{\beta}}^\mu \tau^\dot{\beta}) \otimes e_\mu + (\theta^\alpha + \xi^\alpha(t)) \otimes f_\alpha + (\tau^\dot{\beta} + \zeta^\dot{\beta}(t)) \otimes f_{\dot{\beta}}]|_{t=0} \\ &= (v^\mu - \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu t^\dot{\beta} - s^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^\dot{\beta}) \otimes \frac{\partial}{\partial x^\mu}|_z + s^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}|_z + t^\dot{\beta} \otimes \frac{\partial}{\partial \tau^\dot{\beta}}|_z \end{aligned}$$

In particular, taking $u = e_\nu = \delta_\nu^\mu \otimes e_\mu$, we have : ${}^l e^{-1}(e_\nu)_z = \delta_\nu^\mu \otimes \frac{\partial}{\partial x^\mu}|_z = \frac{\partial}{\partial x^\nu}|_z$. Then :

$$P_m = {}^l e^{-1}(e_m) = \delta_m^\mu \frac{\partial}{\partial x^\mu}.$$

Similarly, taking $u = f_\beta = \delta_\beta^\alpha \otimes f_\alpha$, we have : ${}^l e^{-1}(f_\beta)_z = \delta_\beta^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}|_z - \delta_\beta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^\dot{\beta} \otimes \frac{\partial}{\partial x^\mu}|_z$
 $= \frac{\partial}{\partial \theta^\beta}|_z - \sigma_{\beta\dot{\beta}}^\mu \tau^\dot{\beta} \otimes \frac{\partial}{\partial x^\mu}|_z$. Then : $Q_a = {}^l e^{-1}(f_a) = \delta_a^\alpha \frac{\partial}{\partial \theta^\alpha} - \tau^\dot{\beta} (\delta_\beta^\dot{b} \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu}$.

Similarly, taking $u = f_{\dot{\alpha}} = \delta_{\dot{\alpha}}^\dot{\beta} \otimes f_{\dot{\beta}}$, we have : ${}^l e^{-1}(f_{\dot{\alpha}})_z = \delta_{\dot{\alpha}}^\dot{\beta} \otimes \frac{\partial}{\partial \tau^\dot{\beta}}|_z - \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \delta_{\dot{\alpha}}^\dot{\beta} \otimes \frac{\partial}{\partial x^\mu}|_z$
 $= \frac{\partial}{\partial \tau^{\dot{\alpha}}}|_z - \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \otimes \frac{\partial}{\partial x^\mu}|_z$. Then : $Q_b = {}^l e^{-1}(f_b) = \delta_b^\dot{\beta} \frac{\partial}{\partial \tau^\dot{\beta}} - \theta^\alpha (\delta_\alpha^a \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu}$.

We have obtained the expressions of P_m , Q_a and Q_b in the canonical chart of $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$:

$$\left\{ \begin{array}{lcl} P_m & = & \delta_m^\mu \frac{\partial}{\partial x^\mu} \\ Q_a & = & \delta_a^\alpha \frac{\partial}{\partial \theta^\alpha} - \tau^{\dot{\beta}} (\delta_{\dot{\beta}}^b \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu} \\ Q_b & = & \delta_b^{\dot{\beta}} \frac{\partial}{\partial \tau^{\dot{\beta}}} - \theta^\alpha (\delta_\alpha^a \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu} \end{array} \right.$$

Proposition 9.1.9

$$[Q_a, Q_b] = -2\sigma_{ab}^m P_m$$

$$[Q_a, Q_b] = [{}^l e^{-1}(f_a), {}^l e^{-1}(f_b)] = {}^l e^{-1}([f_a, f_b]_+) = {}^l e^{-1}(-2\sigma_{ab}^m e_m) = -2\sigma_{ab}^m P_m$$

Recall that we have an isomorphism ${}^r e^{-1} : V_{\mathbb{C}}^{4|4} \longrightarrow \Gamma_r(T\overset{\circ}{M}_{\mathbb{C}}^{4|4})$, which associates to every $u = v^m \otimes e_m + s^a \otimes f_a + t^b \otimes f_b \in V_{\mathbb{C}}^{4|4}$ a right-invariant super-vector field : ${}^r e^{-1}(u)_z = T_0 r_z(u)$.

Let γ be a path in $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$ satisfying : $\gamma(0) = 0$ and $\dot{\gamma}(0) = u$. We set :

$$\gamma(t) = \lambda^\mu(t) \otimes e_\mu + \xi^\alpha(t) \otimes f_\alpha + \zeta^{\dot{\beta}}(t) \otimes f_{\dot{\beta}}, \text{ so that :}$$

$$\begin{aligned} {}^r e^{-1}(u)_z &= T_0 r_z(u) = \frac{d}{dt} [r_z(\gamma(t))]_{|t=0} = \frac{d}{dt} [\gamma(t) * z]_{|t=0} \\ &= \frac{d}{dt} [(\lambda^\mu(t) + x^\mu + \xi^\alpha(t) \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}} - \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \zeta^{\dot{\beta}}(t)) \otimes e_\mu + (\xi^\alpha(t) + \theta^\alpha) \otimes f_\alpha + (\zeta^{\dot{\beta}}(t) + \tau^{\dot{\beta}}) \otimes f_{\dot{\beta}}]_{|t=0} \\ &= (v^\mu + s^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}} + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu t^{\dot{\beta}}) \otimes \frac{\partial}{\partial x^\mu}_{|z} + s^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}_{|z} + t^{\dot{\beta}} \otimes \frac{\partial}{\partial \tau^{\dot{\beta}}}_{|z} \end{aligned}$$

In particular, taking $u = e_\nu = \delta_\nu^\mu \otimes e_\mu$, we have : ${}^r e^{-1}(e_\nu)_z = \delta_\nu^\mu \otimes \frac{\partial}{\partial x^\mu}_{|z} = \frac{\partial}{\partial x^\nu}_{|z}$. Then :

$$\partial_m = {}^r e^{-1}(e_m) = \delta_m^\mu \frac{\partial}{\partial x^\mu}.$$

Similarly, taking $u = f_\beta = \delta_\beta^\alpha \otimes f_\alpha$, we have : ${}^r e^{-1}(f_\beta)_z = \delta_\beta^\alpha \otimes \frac{\partial}{\partial \theta^\alpha}_{|z} + \delta_\beta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}} \otimes \frac{\partial}{\partial x^\mu}_{|z}$
 $= \frac{\partial}{\partial \theta^\beta}_{|z} + \sigma_{\beta\dot{\beta}}^\mu \tau^{\dot{\beta}} \otimes \frac{\partial}{\partial x^\mu}_{|z}$. Then : $D_a = {}^r e^{-1}(f_a) = \delta_a^\alpha \frac{\partial}{\partial \theta^\alpha} + \tau^{\dot{\beta}} (\delta_{\dot{\beta}}^b \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu}$.

Similarly, taking $u = f_{\dot{\alpha}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \otimes f_{\dot{\beta}}$, we have : ${}^r e^{-1}(f_{\dot{\alpha}})_z = \delta_{\dot{\alpha}}^{\dot{\beta}} \otimes \frac{\partial}{\partial \tau^{\dot{\beta}}}_{|z} + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \delta_{\dot{\alpha}}^{\dot{\beta}} \otimes \frac{\partial}{\partial x^\mu}_{|z}$
 $= \frac{\partial}{\partial \tau^{\dot{\alpha}}}_{|z} + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \otimes \frac{\partial}{\partial x^\mu}_{|z}$. Then : $D_b = {}^r e^{-1}(f_b) = \delta_b^{\dot{\beta}} \frac{\partial}{\partial \tau^{\dot{\beta}}} + \theta^\alpha (\delta_\alpha^a \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu}$.

We have obtained the expressions of ∂_m , D_a and D_b in the canonical chart of $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$:

$$\left\{ \begin{array}{lcl} \partial_m & = & \delta_m^\mu \frac{\partial}{\partial x^\mu} \\ D_a & = & \delta_a^\alpha \frac{\partial}{\partial \theta^\alpha} + \tau^{\dot{\beta}} (\delta_{\dot{\beta}}^b \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu} \\ D_b & = & \delta_b^{\dot{\beta}} \frac{\partial}{\partial \tau^{\dot{\beta}}} + \theta^\alpha (\delta_\alpha^a \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x^\mu} \end{array} \right.$$

Proposition 9.1.10

$$[D_a, D_b] = 2\sigma_{ab}^m \partial_m$$

$$[D_a, D_b] = [{}^r e^{-1}(f_a), {}^r e^{-1}(f_b)] = - {}^r e^{-1}([f_a, f_b]_+) = - {}^r e^{-1}(-2\sigma_{ab}^m e_m) = 2\sigma_{ab}^m \partial_m$$

The representative matrix of the isomorphism ${}^r e^{-1}(z) : V_{\mathbb{C}}^{4|4} \longrightarrow T_z \mathring{M}_{\mathbb{C}}^{4|4}$ in the bases (e_m, f_a, f_b) of $V_{\mathbb{C}}^{4|4}$ and $(\frac{\partial}{\partial x^\mu}|_z, \frac{\partial}{\partial \theta^\alpha}|_z, \frac{\partial}{\partial \tau^\beta}|_z)$ of $T_z \mathring{M}_{\mathbb{C}}^{4|4}$ is :

$$(e_A^M(z)) = \begin{pmatrix} \delta_m^\mu & \tau^{\dot{\beta}}(\delta_{\dot{\beta}}^b \sigma_{ab}^m \delta_m^\mu) & \theta^\alpha(\delta_\alpha^a \sigma_{ab}^m \delta_m^\mu) \\ 0 & \delta_a^\alpha & 0 \\ 0 & 0 & \delta_b^{\dot{\beta}} \end{pmatrix}$$

In e_A^M , the latin super-index A refers to (m, a, b) while the greek super-index M refers to (μ, α, β) . The above matrix represents **the canonical supervierbein** in the canonical chart of $\mathring{M}_{\mathbb{C}}^{4|4}$.

Theorem 9.1.11 *There exists a global chart $(x_+^\mu, \theta_+^\alpha, \tau_+^{\dot{\beta}})$ on $\mathring{M}_{\mathbb{C}}^{4|4}$ such that :*

$$\left\{ \begin{array}{lcl} \partial_m & = & \delta_m^\mu \frac{\partial}{\partial x_+^\mu} \\ D_a & = & \delta_a^\alpha \frac{\partial}{\partial \theta_+^\alpha} + 2 \tau^{\dot{\alpha}}(\delta_{\dot{\alpha}}^b \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x_+^\mu} \\ D_b & = & \delta_b^{\dot{\beta}} \frac{\partial}{\partial \tau_+^{\dot{\beta}}} \end{array} \right.$$

(∂_m, D_b) is a family of independent vector fields that commute between themselves (but not with the D_a 's). By Frobenius' theorem, we deduce the existence of coordinates such that : $\partial_m = \delta_m^\mu \frac{\partial}{\partial x_+^\mu}$ and $D_b = \delta_b^{\dot{\beta}} \frac{\partial}{\partial \tau_+^{\dot{\beta}}}$.

Let us define explicitly a chart on $\mathring{M}_{\mathbb{C}}^{4|4}$, called **the chiral chart**, which satisfies the above properties, with the given expression for D_a .

The chiral chart $\varphi_+^{-1} : \mathbb{C}^{4|4} \longrightarrow \mathring{M}_{\mathbb{C}}^{4|4}$ is defined by :

$$\varphi_+^{-1}(x_+^\mu, \theta_+^\alpha, \tau_+^{\dot{\beta}}) = (x_+^\mu - \theta_+^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau_+^{\dot{\beta}}) \otimes e_\mu + \theta_+^\alpha \otimes f_\alpha + \tau_+^{\dot{\beta}} \otimes f_{\dot{\beta}}$$

Thus, we have performed a change of coordinates by setting :

$$\left\{ \begin{array}{lcl} x_+^\mu & = & x^\mu + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}} \\ \theta_+^\alpha & = & \theta^\alpha \\ \tau_+^{\dot{\beta}} & = & \tau^{\dot{\beta}} \end{array} \right. \quad \text{which is equivalent to :} \quad \left\{ \begin{array}{lcl} x^\mu & = & x_+^\mu - \theta_+^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau_+^{\dot{\beta}} \\ \theta^\alpha & = & \theta_+^\alpha \\ \tau^{\dot{\beta}} & = & \tau_+^{\dot{\beta}} \end{array} \right.$$

The change of basis matrix from $(\frac{\partial}{\partial x^\nu}|_z, \frac{\partial}{\partial \theta^\beta}|_z, \frac{\partial}{\partial \tau^\dot{\alpha}}|_z)$ to $(\frac{\partial}{\partial x^\mu}_+|_z, \frac{\partial}{\partial \theta^\alpha}_+|_z, \frac{\partial}{\partial \tau^\dot{\beta}}|_z)$ is :

$$J(z) = \begin{pmatrix} \frac{\partial x^\nu}{\partial x^\mu}_+|_z & \frac{\partial x^\nu}{\partial \theta^\alpha}_+|_z & \frac{\partial x^\nu}{\partial \tau^\dot{\beta}}|_z \\ \frac{\partial \theta^\beta}{\partial x^\mu}_+|_z & \frac{\partial \theta^\beta}{\partial \theta^\alpha}_+|_z & \frac{\partial \theta^\beta}{\partial \tau^\dot{\beta}}|_z \\ \frac{\partial \tau^\dot{\alpha}}{\partial x^\mu}_+|_z & \frac{\partial \tau^\dot{\alpha}}{\partial \theta^\alpha}_+|_z & \frac{\partial \tau^\dot{\alpha}}{\partial \tau^\dot{\beta}}|_z \end{pmatrix} = \begin{pmatrix} \delta_\mu^\nu & -\sigma_{\alpha\dot{\beta}}^\nu \tau^\dot{\beta} & \theta^\alpha_+ \sigma_{\alpha\dot{\beta}}^\nu \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & 0 & \delta_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix}$$

The change of basis matrix from $(\frac{\partial}{\partial x^\mu}_+|_z, \frac{\partial}{\partial \theta^\alpha}_+|_z, \frac{\partial}{\partial \tau^\dot{\beta}}|_z)$ to $(\frac{\partial}{\partial x^\nu}|_z, \frac{\partial}{\partial \theta^\beta}|_z, \frac{\partial}{\partial \tau^\dot{\alpha}}|_z)$ is :

$$J(z)^{-1} = \begin{pmatrix} \frac{\partial x^\mu_+}{\partial x^\nu}|_z & \frac{\partial x^\mu_+}{\partial \theta^\beta}|_z & \frac{\partial x^\mu_+}{\partial \tau^\dot{\alpha}}|_z \\ \frac{\partial \theta^\alpha_+}{\partial x^\nu}|_z & \frac{\partial \theta^\alpha_+}{\partial \theta^\beta}|_z & \frac{\partial \theta^\alpha_+}{\partial \tau^\dot{\alpha}}|_z \\ \frac{\partial \tau^\dot{\beta}}{\partial x^\nu}|_z & \frac{\partial \tau^\dot{\beta}}{\partial \theta^\beta}|_z & \frac{\partial \tau^\dot{\beta}}{\partial \tau^\dot{\alpha}}|_z \end{pmatrix} = \begin{pmatrix} \delta_\nu^\mu & \sigma_{\beta\dot{\beta}}^\mu \tau^\dot{\beta} & -\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & 0 & \delta_{\dot{\alpha}}^{\dot{\beta}} \end{pmatrix}$$

$$\text{That is : } \begin{cases} \frac{\partial}{\partial x^\nu}|_z = \delta_\nu^\mu \frac{\partial}{\partial x^\mu}_+|_z \\ \frac{\partial}{\partial \theta^\beta}|_z = \sigma_{\beta\dot{\beta}}^\mu \tau^\dot{\beta} \frac{\partial}{\partial x^\mu}_+|_z + \delta_\beta^\alpha \frac{\partial}{\partial \theta^\alpha}_+|_z \\ \frac{\partial}{\partial \tau^\dot{\alpha}}|_z = -\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \frac{\partial}{\partial x^\mu}_+|_z + \delta_{\dot{\alpha}}^{\dot{\beta}} \frac{\partial}{\partial \tau^\dot{\beta}}|_z \end{cases}$$

We deduce the expressions of ∂_m , D_a and D_b in the chiral chart of $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$:

$$\left\{ \begin{array}{lcl} \partial_m & = & \delta_m^\nu \frac{\partial}{\partial x^\nu} \\ & = & \delta_m^\mu \frac{\partial}{\partial x_+^\mu} \\ \\ D_a & = & \delta_a^\beta \frac{\partial}{\partial \theta^\beta} + \tau^{\dot{\alpha}} (\delta_{\dot{\alpha}}^b \sigma_{ab}^m \delta_m^\nu) \frac{\partial}{\partial x^\nu} \\ & = & \delta_a^\alpha \frac{\partial}{\partial \theta_+^\alpha} + 2 \tau^{\dot{\alpha}} (\delta_{\dot{\alpha}}^b \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x_+^\mu} \\ \\ D_b & = & \delta_b^{\dot{\alpha}} \frac{\partial}{\partial \tau^{\dot{\alpha}}} + \theta^\beta (\delta_\beta^a \sigma_{ab}^m \delta_m^\nu) \frac{\partial}{\partial x^\nu} \\ & = & \delta_b^{\dot{\alpha}} \frac{\partial}{\partial \tau_+^{\dot{\alpha}}} + \theta^\beta (\delta_\beta^a \sigma_{ab}^m \delta_m^\mu) \frac{\partial}{\partial x_+^\mu} \end{array} \right.$$

The representative matrix of the isomorphism ${}^r e^{-1}(z) : V_{\mathbb{C}}^{4|4} \longrightarrow T_z \overset{\circ}{M}_{\mathbb{C}}^{4|4}$ in the bases (e_m, f_a, f_b) of $V_{\mathbb{C}}^{4|4}$ and $(\frac{\partial}{\partial x_+^\mu}|_z, \frac{\partial}{\partial \theta_+^\alpha}|_z, \frac{\partial}{\partial \tau_+^{\dot{\beta}}}|_z)$ of $T_z \overset{\circ}{M}_{\mathbb{C}}^{4|4}$ is :

$$(e_A^M(z)) = \begin{pmatrix} \delta_m^\mu & 2\tau^{\dot{\alpha}} (\delta_{\dot{\alpha}}^b \sigma_{ab}^m \delta_m^\mu) & 0 \\ 0 & \delta_a^\alpha & 0 \\ 0 & 0 & \delta_b^{\dot{\beta}} \end{pmatrix}$$

The above matrix represents **the canonical supervierbein** in the chiral chart of $\overset{\circ}{M}_{\mathbb{C}}^{4|4}$.

Let $\Phi \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, L_{\mathbb{C}})$.

In the canonical chart $(x^\mu, \theta^\alpha, \tau^{\dot{\beta}})$, Φ is represented by the superfunction $\Phi_{can} : \mathbb{C}^{4|4} \longrightarrow L_{\mathbb{C}}$ defined by : $\Phi_{can}(x^\mu, \theta^\alpha, \tau^{\dot{\beta}}) = \Phi(z)$, where $z = x^\mu \otimes e_\mu + \theta^\alpha \otimes f_\alpha + \tau^{\dot{\beta}} \otimes f_{\dot{\beta}}$.

In the chiral chart $(x_+^\mu, \theta_+^\alpha, \tau_+^{\dot{\beta}})$, Φ is represented by the superfunction $\Phi_+ : \mathbb{C}^{4|4} \longrightarrow L_{\mathbb{C}}$ defined by : $\Phi_+(x_+^\mu, \theta_+^\alpha, \tau_+^{\dot{\beta}}) = \Phi(z)$, where
 $z = (x_+^\mu - \theta_+^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau_+^{\dot{\beta}}) \otimes e_\mu + \theta_+^\alpha \otimes f_\alpha + \tau_+^{\dot{\beta}} \otimes f_{\dot{\beta}}$. The chirality condition $D_a \Phi = 0$ takes now the very simple form :

$$\frac{\partial \Phi_+}{\partial \tau_+^{\dot{\beta}}} = 0$$

Thus, if Φ is chiral, Φ_+ does not depend on τ_+ and can be written :

$$\Phi_+(x_+, \theta_+, \tau_+) = \phi(x_+) + \theta_+^\alpha \psi_\alpha(x_+) + (\theta_+)^2 F(x_+)$$

We deduce that if Φ is chiral, Φ_{can} can be written :

$$\Phi_{can}(x^\mu, \theta^\alpha, \tau^{\dot{\beta}}) = \Phi_+(x_+^\mu, \theta_+^\alpha, \tau_+^{\dot{\beta}}) = \Phi_+(x^\mu + \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}}, \theta^\alpha, \tau^{\dot{\beta}}) = e^{\theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \tau^{\dot{\beta}} \partial_\mu} \Phi_+(x^\mu, \theta^\alpha, \tau^{\dot{\beta}})$$

9.2 Super-Yang-Mills theories

We consider a compact Lie group G , and let \mathfrak{g} be the Lie algebra of G . We assume that G is a subgroup of $SU(n)$, so that \mathfrak{g} is a Lie subalgebra of $\mathfrak{su}(n)$. We denote by $(iT_k)_{1 \leq k \leq \dim G}$ a basis of \mathfrak{g} (for every $k \in \{1, \dots, \dim G\}$, $T_k \in \text{MH}_n(\mathbb{C})$). The basis (iT_k) is chosen such that $\text{tr}(T_k T_l) = \delta_{kl}$.

Let $\mathcal{A} \in \Lambda^1(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, \mathfrak{g} \otimes L_{\mathbb{C}})$ be a superconnection (in the canonical trivialization of $\overset{\circ}{M}_{\mathbb{C}}^{4|4} \times G$). Under a gauge transformation $g = \exp(iT_k \Lambda^{(k)})$ (with $\Lambda^{(k)} \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, L_{\mathbb{C}})$), we have the following transformation rule : $\mathcal{A} \longrightarrow g^{-1}\mathcal{A} g - g^{-1}dg$.

The supercurvature $\mathcal{F} \in \Lambda^2(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, \mathfrak{g} \otimes L_{\mathbb{C}})$ of \mathcal{A} is defined by : $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. Under a gauge transformation $g = \exp(iT_k \Lambda^{(k)})$, we have the following transformation rule : $\mathcal{F} \longrightarrow g^{-1}\mathcal{F} g$.

Let $\rho : G \longrightarrow \text{GL}(E)$ be a representation of G , and $\Phi \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, E \otimes L_{\mathbb{C}})$ a superfield.

We define the covariant derivative of Φ by : $\mathcal{D}^{\mathcal{A}}\Phi = d\Phi + \Phi \wedge \mathcal{A}$. In particular,

- if ρ is the fundamental representation of G on \mathbb{C}^n , $\mathcal{D}^{\mathcal{A}}\Phi = d\Phi + \Phi\mathcal{A} = d\Phi - \mathcal{A}\Phi$
- if ρ is the adjoint representation of G on \mathfrak{g} , $\mathcal{D}^{\mathcal{A}}\Phi = d\Phi + [\Phi, \mathcal{A}] = d\Phi - [\mathcal{A}, \Phi]$

We have the Bianchi identity : $\mathcal{D}^{\mathcal{A}}\mathcal{F} = 0$

which may be written $d\mathcal{F} - [\mathcal{A}, \mathcal{F}] = 0$ or $d\mathcal{F} - \mathcal{A}\mathcal{F} + \mathcal{F}\mathcal{A} = 0$.

Under a gauge transformation $g = \exp(iT_k \Lambda^{(k)})$, we have the following transformation rules : $\Phi \longrightarrow g^{-1}\Phi$, $\mathcal{D}^{\mathcal{A}}\Phi \longrightarrow g^{-1}\mathcal{D}^{\mathcal{A}}\Phi = \mathcal{D}^{\mathcal{A}'}\Phi'$ (where $\mathcal{A}' = g^{-1}\mathcal{A} g - g^{-1}dg$, and $\Phi' = g^{-1}\Phi$).

Let $(e^A) = (e^m, e^a, e^b)$ be the dual basis of $(D_A) = (\partial_m, D_a, D_b)$. We set : $\mathcal{A} = iT_k \mathcal{A}^{(k)}$, with $\mathcal{A}^{(k)} = e^m \mathcal{A}_m^{(k)} + e^a \mathcal{A}_a^{(k)} + e^b \mathcal{A}_b^{(k)} = e^A \mathcal{A}_A^{(k)}$.

Similarly, we set : $\mathcal{F} = iT_k \mathcal{F}^{(k)}$, with $\mathcal{F}^{(k)} = \frac{1}{2} e^A \wedge e^B \mathcal{F}_{BA}^{(k)}$.

Let (τ_{AB}^C) be the structure functions of the moving frame (D_A) , that is : $[D_A, D_B] = \tau_{AB}^C D_C$. We determine an expression for $\mathcal{F}_{BA} = iT_k \mathcal{F}_{BA}^{(k)}$:

$$\begin{aligned} \mathcal{F}_{BA} &= (d\mathcal{A})_{BA} + (\mathcal{A} \wedge \mathcal{A})_{BA} \\ &= d\mathcal{A}(D_B, D_A) + (\mathcal{A} \wedge \mathcal{A})_{BA} \\ &= D_B \mathcal{A}(D_A) - (-1)^{|A||B|} D_A \mathcal{A}(D_B) - \mathcal{A}([D_B, D_A]) + (\mathcal{A} \wedge \mathcal{A})_{BA} \\ &= D_B \mathcal{A}_A - (-1)^{|A||B|} D_A \mathcal{A}_B - \mathcal{A}(\tau_{BA}^C D_C) + [\mathcal{A}_B, \mathcal{A}_A] \end{aligned}$$

Finally,

$$\mathcal{F}_{BA} = D_B \mathcal{A}_A - (-1)^{|A||B|} D_A \mathcal{A}_B - \tau_{BA}^C \mathcal{A}_C + [\mathcal{A}_B, \mathcal{A}_A]$$

with $[\mathcal{A}_B, \mathcal{A}_A] = \mathcal{A}_B \mathcal{A}_A - (-1)^{|A||B|} \mathcal{A}_A \mathcal{A}_B$.

Notice that from $T(D_B, D_A) = [{}^r e(D_B), {}^r e(D_A)]$, we have $T_{BA}^C \tilde{e}_C = -{}^r e([D_B, D_A])$, that is $T_{BA}^C \tilde{e}_C = -{}^r e(\tau_{BA}^C D_C)$. We deduce that $T_{BA}^C \tilde{e}_C = -\tau_{BA}^C \tilde{e}_C$, so finally :

$$T_{BA}^C = -\tau_{BA}^C$$

We define $\mathcal{D}_A^A \Phi := \mathcal{D}^A \Phi(D_A) = d\Phi(D_A) + \Phi \wedge \mathcal{A}(D_A) = D_A \Phi + \Phi \wedge \mathcal{A}_A$.

Noticing that $\mathcal{D}_A^A \Phi = \nabla_{D_A}^A \Phi$, we may write :

$$\begin{aligned} [\mathcal{D}_A^A, \mathcal{D}_B^A] \Phi &= [\nabla_{D_A}^A, \nabla_{D_B}^A] \Phi = \nabla_{[D_A, D_B]}^A \Phi + \mathcal{F}_{AB} \Phi = \nabla_{\tau_{AB}^C D_C}^A \Phi + \mathcal{F}_{AB} \Phi \\ &= \tau_{AB}^C \nabla_{D_C}^A \Phi + \mathcal{F}_{AB} \Phi = \tau_{AB}^C \mathcal{D}_C^A \Phi + \mathcal{F}_{AB} \Phi \end{aligned}$$

Finally,

$$[\mathcal{D}_A^A, \mathcal{D}_B^A] = \tau_{AB}^C \mathcal{D}_C^A + \mathcal{F}_{AB}$$

Definition 9.2.1 We say that $\Phi \in H^\omega(\overset{\circ}{M}{}^{4|4}, E \otimes L_{\mathbb{C}})$ is a **covariantly chiral** (resp. **covariantly antichiral**) superfield if :

$$\mathcal{D}_{\dot{a}}^A \Phi = 0 \quad (\text{resp. } \mathcal{D}_a^A \Phi = 0)$$

We have $[\mathcal{D}_{\dot{a}}^A, \mathcal{D}_{\dot{b}}^A] = \tau_{\dot{a}\dot{b}}^C \mathcal{D}_C^A + \mathcal{F}_{\dot{a}\dot{b}}$ so $\mathcal{D}_{\dot{a}}^A \mathcal{D}_{\dot{b}}^A \Phi + \mathcal{D}_{\dot{b}}^A \mathcal{D}_{\dot{a}}^A \Phi = \tau_{\dot{a}\dot{b}}^C \mathcal{D}_C^A \Phi + \mathcal{F}_{\dot{a}\dot{b}} \Phi$.

Since $\tau_{\dot{a}\dot{b}}^C = 0$, if Φ is covariantly chiral, then $\mathcal{F}_{\dot{a}\dot{b}} \Phi = 0$. Thus, we must have :

$$\mathcal{F}_{\dot{a}\dot{b}} = 0$$

We have $[\mathcal{D}_a^A, \mathcal{D}_b^A] = \tau_{ab}^C \mathcal{D}_C^A + \mathcal{F}_{ab}$ so $\mathcal{D}_a^A \mathcal{D}_b^A \Phi + \mathcal{D}_b^A \mathcal{D}_a^A \Phi = \tau_{ab}^C \mathcal{D}_C^A \Phi + \mathcal{F}_{ab} \Phi$.

Since $\tau_{ab}^C = 0$, if Φ is covariantly antichiral, then $\mathcal{F}_{ab} \Phi = 0$. Thus, we must have :

$$\mathcal{F}_{ab} = 0$$

On the other hand, we impose the conventional constraint :

$$\mathcal{F}_{\dot{a}\dot{b}} = 0$$

Since $\mathcal{F}_{ab} = D_a \mathcal{A}_b + D_b \mathcal{A}_a - \tau_{ab}^m \mathcal{A}_m + [\mathcal{A}_a, \mathcal{A}_b]$, the preceding constraint allows us to redefine \mathcal{A}_m in terms of \mathcal{A}_a , \mathcal{A}_b and their spinorial derivatives.

The solution of the Bianchi identity $\mathcal{D}^A \mathcal{F} = 0$ subject to the constraints $\mathcal{F}_{ab} = 0$, $\mathcal{F}_{\dot{a}\dot{b}} = 0$ and $\mathcal{F}_{ab} = 0$ can be written :

$$\left\{ \begin{array}{lcl} \mathcal{F}_{ma} & = & \sigma_{mab} W_1^b \\ \mathcal{F}_{m\dot{a}} & = & W_2^b \sigma_{mb\dot{a}} \\ \mathcal{F}_{mn} & = & \frac{1}{2} (\mathcal{D}^A \sigma_{mn} W_2 - \bar{\mathcal{D}}^A \bar{\sigma}_{mn} W_1) \end{array} \right.$$

where $W_1 \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, S_- \otimes \mathfrak{g} \otimes L_{\mathbb{C}})$ and $W_2 \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, S_+ \otimes \mathfrak{g} \otimes L_{\mathbb{C}})$ are such that W_2^b is covariantly chiral, and W_1^b is covariantly antichiral. So we have :

$$\mathcal{D}_{\dot{a}}^A W_2^b = 0 \quad , \quad \mathcal{D}_a^A W_1^{\dot{b}} = 0$$

$$\text{We also have : } \mathcal{D}_a^A W_2^a = \mathcal{D}_{\dot{a}}^A W_1^{\dot{a}}$$

Under a gauge transformation $g = \exp(iT_k \Lambda^{(k)})$, we have the following transformation rules : $W_2^b \longrightarrow g^{-1} W_2^b g$ and $W_1^{\dot{b}} \longrightarrow g^{-1} W_1^{\dot{b}} g$.

Let us find the explicit solution of the constraints.

$\mathcal{F}_{ab} = 0 \iff D_a \mathcal{A}_b + D_b \mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_b] = 0$. This is an equation of Maurer-Cartan type. It admits a "pure gauge solution" :

$$\mathcal{A}_a = -\mathcal{V}^{-1} D_a \mathcal{V}$$

$$\text{where } \mathcal{V} = \exp(iT_k \nu^{(k)}).$$

$\mathcal{F}_{\dot{a}\dot{b}} = 0 \iff D_{\dot{a}} \mathcal{A}_{\dot{b}} + D_{\dot{b}} \mathcal{A}_{\dot{a}} + [\mathcal{A}_{\dot{a}}, \mathcal{A}_{\dot{b}}] = 0$. This is an equation of Maurer-Cartan type. It admits a "pure gauge solution" :

$$\mathcal{A}_{\dot{b}} = -\mathcal{U}^{-1} D_{\dot{b}} \mathcal{U}$$

$$\text{where } \mathcal{U} = \exp(iT_k \nu^{(k)}).$$

Finally, $\mathcal{F}_{ab} = 0$ allows to define \mathcal{A}_m in terms of \mathcal{A}_a and \mathcal{A}_b , and therefore in terms of \mathcal{U} and \mathcal{V} , which are called **prepotentials**. Under a gauge transformation $g = \exp(iT_k \Lambda^{(k)})$, we have the following transformation rules : $\mathcal{U} \longrightarrow \mathcal{U} g$ and $\mathcal{V} \longrightarrow \mathcal{V} g$, which induce on \mathcal{A}_a and \mathcal{A}_b the following transformation rules : $\mathcal{A}_a \longrightarrow g^{-1} \mathcal{A}_a g - g^{-1} D_a g$ and $\mathcal{A}_b \longrightarrow g^{-1} \mathcal{A}_b g - g^{-1} D_b g$.

Let $g_+ = \exp(iT_k \Lambda_+^{(k)})$ (resp. $g_- = \exp(iT_k \Lambda_-^{(k)})$) a transformation such that $\Lambda_+ \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, \mathfrak{g} \otimes L_{\mathbb{C}})$ is chiral (resp. $\Lambda_- \in H^\omega(\overset{\circ}{M}_{\mathbb{C}}^{4|4}, \mathfrak{g} \otimes L_{\mathbb{C}})$ is antichiral). We consider on the prepotentials \mathcal{U} and \mathcal{V} the following transformation rules : $\mathcal{U} \longrightarrow g_+^{-1} \mathcal{U} g$ and $\mathcal{V} \longrightarrow g_-^{-1} \mathcal{V} g$. These transformations are called **pregauge transformations**. They leave \mathcal{A}_a and \mathcal{A}_b invariant.

Thus, the total transformation laws for \mathcal{U} and \mathcal{V} are :

$$\begin{aligned} \mathcal{U} &\longrightarrow g_+^{-1} \mathcal{U} g \\ \mathcal{V} &\longrightarrow g_-^{-1} \mathcal{V} g \end{aligned}$$

Finally, we set :

$$\mathcal{W} = \mathcal{V} \mathcal{U}^{-1}$$

The preceding transformation laws for \mathcal{U} and \mathcal{V} induce on \mathcal{W} the following transformation law :

$$\mathcal{W} \longrightarrow g_-^{-1} \mathcal{W} g_+$$

The real gauge realization

We define a conjugation on $\Lambda^1(\overset{\circ}{M}_\mathbb{C}^{4|4}, \mathfrak{g} \otimes L_\mathbb{C})$ by setting :

$$\mathcal{A}^\dagger := \mathcal{A}_m^\dagger \nu(e^m) + \mathcal{A}_a^\dagger \tau(e^a) + \mathcal{A}_{\dot{b}}^\dagger \tau(e^{\dot{b}}) = \mathcal{A}_m^\dagger e^m + \mathcal{A}_{\dot{b}}^\dagger e^{\dot{b}} + \mathcal{A}_a^\dagger e^a$$

The reality condition on \mathcal{A} is :

$$\mathcal{A}^\dagger = -\mathcal{A}$$

This is equivalent to :

$$\begin{cases} \mathcal{A}_m^\dagger = -\mathcal{A}_m \\ \mathcal{A}_a^\dagger = -\mathcal{A}_{\dot{a}} \\ \mathcal{A}_{\dot{a}}^\dagger = -\mathcal{A}_a \end{cases}$$

For the supercurvature, we use the formula

$$\mathcal{F}_{BA} = D_B \mathcal{A}_A - (-1)^{|A||B|} D_A \mathcal{A}_B - \tau_{BA}^C \mathcal{A}_C + [\mathcal{A}_B, \mathcal{A}_A] \quad \text{to obtain :}$$

$$\begin{cases} \mathcal{F}_{mn}^\dagger = -\mathcal{F}_{mn} , \quad \mathcal{F}_{ma}^\dagger = -\mathcal{F}_{m\dot{a}} \\ \mathcal{F}_{ab}^\dagger = \mathcal{F}_{\dot{a}\dot{b}} , \quad \mathcal{F}_{\dot{a}\dot{b}}^\dagger = \mathcal{F}_{ab} \end{cases}$$

which is equivalent to :

$$\mathcal{F}^\dagger = \mathcal{F}$$

From $\mathcal{F}_{ma}^\dagger = -\mathcal{F}_{m\dot{a}}$, we deduce that : $(W_2^a)^\dagger = W_1^{\dot{a}}$

Therefore, it is sufficient to consider $W^a := W_2^a$, which is covariantly chiral ($\mathcal{D}_{\dot{a}}^A \Phi = 0$).

The gauge invariance of the condition $\mathcal{A}^\dagger = -\mathcal{A}$ requires : $\Lambda^\dagger = -\Lambda$.

Finally, we may write an action functional for the covariantly chiral superfield W^a :

$$\mathcal{S}(W) = \int_{\overset{\circ}{M}^{4|4}} \text{tr}(W^a W_a) d^4x d^2\theta$$

Let $\Phi \in H^\omega(\overset{\circ}{M}^{4|4}, E \otimes L_\mathbb{C})$ a covariantly chiral scalar superfield (so that Φ^\dagger is a covariantly antichiral scalar superfield). We may write an action functional for Φ :

$$\mathcal{S}(\Phi) = \int_{\overset{\circ}{M}^{4|4}} \Phi^\dagger \Phi d^4x d^2\theta$$

The condition $\mathcal{A}_a^\dagger = -\mathcal{A}_{\dot{a}}$ is equivalent to $-\mathcal{U}^{-1} D_{\dot{a}} \mathcal{U} = -(-\mathcal{V}^{-1} D_a \mathcal{V})^\dagger$, that is $-\mathcal{U}^{-1} D_{\dot{a}} \mathcal{U} = -\mathcal{V}^\dagger D_{\dot{a}} (\mathcal{V}^{-1})^\dagger$, which requires :

$$\mathcal{U}^\dagger = \mathcal{V}^{-1}$$

This last condition is invariant under pre-gauge transformations if and only if : $\Lambda_- = \Lambda_+^\dagger$.

We have $\mathcal{W} = \mathcal{V} \mathcal{V}^\dagger$. If we set $\Lambda := \Lambda_+$, the transformation rule of \mathcal{W} reads :

$$\mathcal{W} \longrightarrow e^{-i\Lambda^\dagger} \mathcal{W} e^{i\Lambda}$$

Notice that : $\mathcal{W}^\dagger = \mathcal{W}$. Therefore, there exists a superfield $V \in H^\omega(\mathring{M}_\mathbb{C}^{4|4}, \mathfrak{g} \otimes L_\mathbb{C})$ such that : $\mathcal{W} = e^V$, with $V = V^\dagger$.

Finally, \mathcal{U} and \mathcal{V} (with $\mathcal{V}^\dagger = \mathcal{U}^{-1}$) have the following transformation law :

$$\begin{aligned}\mathcal{U} &\longrightarrow e^{-i\Lambda} \mathcal{U} e^\chi \\ \mathcal{V} &\longrightarrow e^{-i\Lambda^\dagger} \mathcal{V} e^\chi\end{aligned}$$

where $D_{\dot{a}}\Lambda = 0$, $D_a\Lambda^\dagger = 0$ and $\chi^\dagger = -\chi$.

The chiral gauge realization

We set :

$$\begin{aligned}\mathcal{A}_+ &:= \mathcal{U} \mathcal{A} \mathcal{U}^{-1} - \mathcal{U}^{-1} d\mathcal{U} \\ \mathcal{A}_- &:= \mathcal{V} \mathcal{A} \mathcal{V}^{-1} - \mathcal{V}^{-1} d\mathcal{V}\end{aligned}$$

\mathcal{A}_+ and \mathcal{A}_- have the following transformation rules :

$$\begin{aligned}\mathcal{A}_+ &\longrightarrow e^{-i\Lambda_+} \mathcal{A}_+ e^{i\Lambda_+} - e^{-i\Lambda_+} d(e^{i\Lambda_+}) \\ \mathcal{A}_- &\longrightarrow e^{-i\Lambda_-} \mathcal{A}_- e^{i\Lambda_-} - e^{-i\Lambda_-} d(e^{i\Lambda_-})\end{aligned}$$

where $D_{\dot{a}}\Lambda_+ = 0$ and $D_a\Lambda_- = 0$.

We set :

$$\mathcal{F}_+ = \frac{1}{2} e^A \wedge e^B (\mathcal{F}_+)_{BA} = d\mathcal{A}_+ + \mathcal{A}_+ \wedge \mathcal{A}_+$$

$$\mathcal{F}_- = \frac{1}{2} e^A \wedge e^B (\mathcal{F}_-)_{BA} = d\mathcal{A}_- + \mathcal{A}_- \wedge \mathcal{A}_-$$

Then :

$$\mathcal{F}_+ = \mathcal{U} \mathcal{F}_+ \mathcal{U}^{-1}$$

$$\mathcal{F}_- = \mathcal{V} \mathcal{F}_- \mathcal{V}^{-1}$$

Proposition 9.2.2 *We have :*

$$\mathcal{A}_- = \mathcal{W} \mathcal{A}_+ \mathcal{W}^{-1} - \mathcal{W} d\mathcal{W}^{-1}$$

Proposition 9.2.3 *We have :*

$$\mathcal{F}_- = \mathcal{W} \mathcal{F}_+ \mathcal{W}^{-1}$$

Chapitre 10

On the supergeometrical formulation of supergravity theories

10.1 Introduction

Since Galileo, the idea of symmetry in physics revealed to be a truly fundamental principle, which is at the core of the fundamental laws of physics. For instance, the "relativity principle" of Galileo amounts to the invariance of the equations of motion of classical mechanics by what we call today the *Galilean group*.

In the physics of elementary particles, it was realized that one can think of particle states as corresponding to specific representations of the *Poincaré group* (which may be viewed as the relativistic extension of the Galilean group). Each unitary irreducible representation (with positive energy) of the Poincaré group is classified by a real number m (the mass) and a (half-)integer number $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ (the spin). Thus, bosons (resp. fermions) corresponding to an integer (resp. half-integer) value of s are mixed together by Poincaré symmetry.

For some reasons that would be too long to discuss here, physicists searched for a symmetry that could mix bosons with fermions (that is, make them appear in the same irreducible representation of some Lie group). It was discovered that such a symmetry could not be realized by means of an ordinary Lie algebra, but requested the structure of a *Lie superalgebra*, with an even part corresponding to an ordinary Lie algebra, and an odd part on which the brackets are realized as anticommutators. We refer to the pioneer papers [CoM] and [HLS] for further details.

Supersymmetric field theories were constructed. They correspond to field theories invariant under a specific Lie superalgebra, called the super-Poincaré algebra. The super-Poincaré algebra contains the ordinary Poincaré algebra, as well as shift transformations by odd, anticommuting parameters (the so-called supersymmetry transformations).

The initial formulation of supersymmetric theories -the component formulation- involved, as in most of the usual field theories, bosonic and fermionic fields subjected to field equations. The supersymmetric character of the theory was related to the fact that these fields carried representations of the super-Poincaré algebra, and the field equations were invariant under these super-Poincaré transformations.

The introduction of the concept of superspace ([SS], [FWZ]) allowed an elegant formulation of supersymmetric theories, using the notion of superfield. In this approach, one recovers a genuine group (acting on superspace and superfields) : the group of supertranslations. Supersymmetry transformations acquired then a natural geometrical meaning : they corresponded to translations in the odd directions of superspace (the spinorial part of supertranslations). Besides the supersymmetry invariance of a given theory was then manifest, and did not need to be explicitly checked as in the component formulation. For a modern, mathematical presentation of supersymmetric field theories in the superspace approach, see for example [DF].

In order to construct a supersymmetric theory of gravity, physicists realized that they needed to promote supersymmetry transformations to "gauge supersymmetry transformations" (that is, transformations which depend on each point of spacetime like in Yang-Mills theory). This gave what is called : supergravity theories.

Supergravity theories were first given in component formulation. Then the superspace version was constructed for the simple supergravity in four dimensions (the complete, off-shell $N = 1$ supergravity in $d = 4$ may be found in [WB] for example). Supergravity theories in other dimensions were also formulated in superspace ($d = 10$ and $d = 11$ superspace supergravities are only known on-shell, cf. [HW] and [BH]), and without a lagrangian formulation (although some progress has been made in this direction [DLT]).

One particular aspect of superspace supergravity is the need for "torsion constraints". The basic fields in a superspace formulation of supergravity are the supervielbein (moving frame), and the superconnection. From these are constructed two covariant tensors : the supertorsion and the supercurvature. For each theory, one has to find a suitable set of constraints on the supertorsion tensor, which allow, using the Bianchi identities, to eliminate most of the component superfunctions by expressing them in terms of a small number of independent superfields.

Finding the right torsion constraints is not easy. One requires them to be Lorentz covariant and supersymmetric, and in general, one make use of dimensional analysis arguments to find the proper constraints. Besides, their geometric origin is not clear.

In the following, we give a presentation of the geometry -more precisely the supergeometry- underlying supergravity theories. We use an intrinsic differential-geometric language (trying to avoid indices), which would be closer to the mathematically-minded readers. We rely on the work of John Lott (cf. [Lot]), who has found a unified geometrical interpretation of the torsion constraints for many supergravity theories, based on the use of G -structures. In this language, the constraints amount to requiring first-order flatness of G -structures, for a specific supergroup G . Finally, we propose to look at supergravity from an affine viewpoint (as a gauge theory for the super-Poincaré group). Then we prove that requiring first-order flatness amounts to requiring the equivalence, up to gauge transformations, between infinitesimal gauge supertranslations acting on the supervielbein and infinitesimal super-diffeomorphisms acting on the supervielbein.

10.2 General setting

We denote by V a vector space of dimension d equipped with the non-degenerate symmetric bilinear form η associated to the matrix $\text{diag}(-1, +1, \dots, +1)$ in a given basis $(e_m)_{0 \leq m \leq d-1}$ of V . We denote by $\text{SO}^\dagger(V)$ the *Lorentz group* of V (which is the connected component of the identity in $\text{O}(V)$), and by $\text{Spin}^\dagger(V)$ the universal covering of $\text{SO}^\dagger(V)$. The covering map $\text{Spin}^\dagger(V) \rightarrow \text{SO}^\dagger(V) \subset \text{GL}(V)$ is denoted by ρ_1 and is called the *vector representation*. We fix a faithful real representation $\rho_{\frac{1}{2}} : \text{Spin}^\dagger(V) \rightarrow \text{GL}(S)$, called the *spinor representation*, and we set $n = \dim S$. Finally, let : $\overset{\circ}{C}_+$ be the interior of the *future timelike cone* $C_+ = \{v \in V / \eta(v, v) \leq 0 \text{ and } v^0 \geq 0\}$, $\overset{\circ}{C}_-$ be the interior of the *past timelike cone* $C_- = \{v \in V / \eta(v, v) \leq 0 \text{ and } v^0 \leq 0\}$.

Definition 10.2.1 *The simply connected Poincaré group is the Lie group defined by :*

$$\Pi(V) = V \rtimes_{\rho_1} \text{Spin}^\dagger(V)$$

Remark 10.2.2 *The group law in $\Pi(V)$ is given by : $(v, \sigma)(v', \sigma') = (v + \rho_1(\sigma)(v'), \sigma\sigma')$ for any $(v, \sigma), (v', \sigma') \in \Pi(V)$.*

Definition 10.2.3 *We shall call a spacetime of dimension d a smooth connected non-compact spinorial manifold M of dimension d .*

Example 10.2.4 *Let $\overset{\circ}{M}$ be the underlying manifold of the vector space V . Then $\overset{\circ}{M}$ is a spacetime of dimension d , called the **flat spacetime** (or the **Minkowski spacetime**). We will generally view $\overset{\circ}{M}$ as a group manifold, endowed with the left and right actions (by translation) of V , considered as a Lie group. On the other hand, writing $T_0\overset{\circ}{M} = V$, we can view V as the tangent Lie algebra of $\overset{\circ}{M}$.*

Remark 10.2.5 *We have a natural affine action $\tau : \Pi(V) \rightarrow GA(\overset{\circ}{M})$ given by : $\tau_{(v, \sigma)}(x) = v + \rho_1(\sigma)(x)$ for any $(v, \sigma) \in \Pi(V)$ and $x \in \overset{\circ}{M}$. By restricting to the abelian Lie group V , we obtain the action by left translations $l : V \rightarrow GA(\overset{\circ}{M})$ defined by : $l_v(x) = v + x$. We also have the action by right translations $r : V \rightarrow GA(\overset{\circ}{M})$ defined by : $r_v(x) = x + v$. Since V is abelian, we have $l = r$.*

We fix an auxiliary vector bundle \mathbb{V} over M with typical fiber V , isomorphic to the tangent bundle TM , and we denote by $\mathcal{F}(\mathbb{V})$ the linear frame bundle of \mathbb{V} (with structural group $\text{GL}(V)$).

Definition 10.2.6 *For any Lie subgroup G of $\text{GL}(V)$, a **G -structure on the vector bundle \mathbb{V}** is a reduction to G of the frame bundle $\mathcal{F}(\mathbb{V})$, that is, a principal G -subbundle of $\mathcal{F}(\mathbb{V})$.*

Example 10.2.7 *The choice of an $\text{SO}^\dagger(V)$ -structure on \mathbb{V} is equivalent to the choice of a bundle metric on \mathbb{V} , together with a space and time orientation of \mathbb{V} .*

Definition 10.2.8 *Let \mathcal{L} be an $\text{SO}^\dagger(V)$ -structure on \mathbb{V} . A **spinorial structure on the vector bundle \mathbb{V}** (equipped with \mathcal{L}) is an extension to $\text{Spin}^\dagger(V)$ of the principal $\text{SO}^\dagger(V)$ -bundle \mathcal{L} , that is, a couple (\mathcal{S}, λ) where \mathcal{S} is a principal $\text{Spin}^\dagger(V)$ -bundle over M , and $\lambda : \mathcal{S} \rightarrow \mathcal{L}$ is a bundle homomorphism over Id_M satisfying $\lambda(s\sigma) = \lambda(s)\rho_1(\sigma)$ for any $s \in \mathcal{S}$ and $\sigma \in \text{Spin}^\dagger(V)$.*

We fix an $\mathrm{SO}^\dagger(V)$ -structure \mathcal{L} on \mathbb{V} , and a spinorial structure \mathcal{S} on $(\mathbb{V}, \mathcal{L})$. These structures always exist since \mathbb{V} has been taken isomorphic to TM , and M is assumed spinorial. We fix them once for all as auxiliary structures.

\mathbb{V} is naturally identified with the associated vector bundle $\mathcal{S} \times_{\rho_1} V$. We denote by \mathbb{S} the associated vector bundle $\mathcal{S} \times_{\rho_{\frac{1}{2}}} S$. There is a natural embedding $\iota : \mathcal{S} \longrightarrow \mathcal{F}(\mathbb{S})$ (where $\mathcal{F}(\mathbb{S})$ is the linear frame bundle of \mathbb{S} , with structural group $\mathrm{GL}(S)$). Then $\iota(\mathcal{S})$ is a $\rho_{\frac{1}{2}}(\mathrm{Spin}^\dagger(V))$ -structure on \mathbb{S} (with $\mathrm{Spin}^\dagger(V) \simeq \rho_{\frac{1}{2}}(\mathrm{Spin}^\dagger(V)) \subset \mathrm{GL}(S)$).

We shall not carry any further the study of G -structures, since we are going to define and study them thoroughly at the supersymmetric level in paragraph 4.

10.3 The supermodule $V^{d|n}$ and superspacetime

Definition 10.3.1 *The Grassmann algebra L is the exterior algebra $\Lambda(\mathbb{R}^l)$, where l is a sufficiently large integer. We set : $L^k = \Lambda^k(\mathbb{R}^l)$, $L^{ev} = \mathbb{R} \oplus L^2 \oplus L^4 \oplus \dots$, $L^{od} = L^1 \oplus L^3 \oplus L^5 \oplus \dots$, so that $L = L^{ev} \oplus L^{od}$. We also set : $L^{ev,*} = L^2 \oplus L^4 \oplus \dots$ and $L^* = L^{ev,*} \oplus L^{od}$, so that $L^{ev} = \mathbb{R} \oplus L^{ev,*}$ and $L = \mathbb{R} \oplus L^*$.*

Definition 10.3.2 *We define the supermodule $V^{d|n} = (V \otimes L^{ev}) \times (S \otimes L^{od})$.*

Remark : We have

$$V^{d|n} = V \oplus (V \otimes L^{ev,*}) \times (S \otimes L^{od})$$

Definition 10.3.3 *1. A symmetric bilinear map $\Gamma : S \times S \longrightarrow V$ is said to be **positive** (resp. **negative**) if for any $s \in S$, we have $\Gamma(s, s) \in C_+$ (resp. $\Gamma(s, s) \in C_-$).
2. A symmetric bilinear map $\Gamma : S \times S \longrightarrow V$ is said to be **positive definite** (resp. **negative definite**) if for any $s \in S \setminus \{0\}$, we have $\Gamma(s, s) \in \mathring{C}_+$ (resp. $\Gamma(s, s) \in \mathring{C}_-$).*

Theorem 10.3.4 *Let S be a faithful real representation of $\mathrm{Spin}^\dagger(V)$. There exists a non-zero $\mathrm{Spin}^\dagger(V)$ -equivariant symmetric bilinear map $\Gamma : S \times S \longrightarrow V$. Besides, if S is irreducible, Γ is projectively unique, and is either positive definite or negative definite.*

For a proof, see [DF] and references therein. We choose a positive definite Γ .

Proposition 10.3.5 *The symmetric bilinear map Γ induces a natural Lie superalgebra structure on $V \times S$.*

Proof : We get a super-bracket by setting : $[V, V] = [V, S] = [S, V] = 0$ and for any $s, t \in S$, $[s, t] = -2 \Gamma(s, t)$. \square

We denoted by $(e_m)_{0 \leq m \leq d-1}$ a basis of V . We introduce a basis $(f_a)_{1 \leq a \leq n}$ of S . Setting $\tilde{e}_A = e_A$ for $A \in \{0, \dots, d-1\}$ and $\tilde{e}_A = f_{A-d+1}$ for $A \in \{d, \dots, d+n-1\}$, we obtain a basis $(\tilde{e}_A)_{0 \leq A \leq d+n-1}$ of $V \times S$.

We set : $\Gamma(f_a, f_b) = \Gamma_{ab}^m e_m$.

Then we have : $[f_a, f_b] = -2\Gamma_{ab}^m e_m$.

Proposition 10.3.6 *The symmetric bilinear map Γ induces a natural Lie supermodule structure on $V^{d|n}$.*

Let us expand the super-bracket of two elements $u = u^A \otimes \tilde{e}_A = v^m \otimes e_m + s^a \otimes f_a$ and $u' = u'^A \otimes \tilde{e}_A = v'^m \otimes e_m + s'^a \otimes f_a$ of $V^{d|n}$.

$$\begin{aligned} [u, u'] &= [s^a \otimes f_a, s'^b \otimes f_b] \\ &= -s^a s'^b [f_a, f_b] \\ &= -s^a s'^b (-2\Gamma_{ab}^m e_m) \\ &= 2\Gamma_{ab}^m s^a s'^b e_m \end{aligned}$$

Proposition 10.3.7 *There exists a natural structure of Lie supergroup on $V^{d|n}$, the group law being given by : $u * u' = u + u' + \frac{1}{2}[u, u']$ for any $u, u' \in V^{d|n}$.*

In the chosen bases of V and S , we have :

$$u * u' = (v^m + v'^m + \Gamma_{ab}^m s^a s'^b) \otimes e_m + (s^a + s'^a) \otimes f_a$$

Let π be the projection $\mathbb{S} \longrightarrow M$, and let $(W_i, \varphi_i)_{i \in I}$ be an atlas of M , such that the open sets W_i trivialize the vector bundle \mathbb{S} (so we have trivialisations $\pi^{-1}(W_i) \longrightarrow W_i \times S$ which associate to each $\Psi \in \pi^{-1}(W_i)$ a couple $(\pi(\Psi), \lambda_i(\Psi))$). Let $\psi_i : \pi^{-1}(W_i) \longrightarrow \varphi_i(W_i) \times S$ be the map defined by $\psi_i(\Psi) = (\varphi_i(\pi(\Psi)), \lambda_i(\Psi))$, then $(\pi^{-1}(W_i), \psi_i)_{i \in I}$ is an atlas of \mathbb{S} .

Definition 10.3.8 *Let M be a spacetime of dimension d . We shall call a **superspace-time associated to M and S** an H^∞ -supermanifold $\mathring{M}^{d|n}$ admitting an atlas (\mathcal{W}_i, Φ_i) such that $\Phi_i(\mathcal{W}_i) = \varphi_i(W_i) \oplus (V \otimes L^{ev,*}) \times (S \otimes L^{od})$ for each $i \in I$.*

Example 10.3.9 *Let $\mathring{M}^{d|n}$ be the underlying supermanifold of the supermodule $V^{d|n}$. Then $\mathring{M}^{d|n}$ is a superspacetime associated to M and S , called the **flat superspacetime (or the Minkowski superspacetime)**. We will generally view $\mathring{M}^{d|n}$ as a group supermanifold, endowed with the left and right actions (by supertranslation) of $V^{d|n}$, considered as a Lie supergroup. On the other hand, writing $T_0 \mathring{M}^{d|n} = V^{d|n}$, we can view $V^{d|n}$ as the tangent Lie supermodule of $\mathring{M}^{d|n}$.*

Remark 10.3.10 *We have a natural affine action $\tau : S\Pi(V^{d|n}) \longrightarrow GA(\mathring{M}^{d|n})$ given by : $\tau_{(u,\sigma)}(z) = u * \rho(\sigma)(z)$ for any $(u, \sigma) \in S\Pi(V^{d|n})$ and $z \in \mathring{M}^{d|n}$. By restricting to the Lie supergroup $V^{d|n}$, we obtain the action by left supertranslations $l : V^{d|n} \longrightarrow GA(\mathring{M}^{d|n})$ defined by : $l_u(z) = u * z$. We also have the action by right supertranslations $r : V^{d|n} \longrightarrow GA(\mathring{M}^{d|n})$ defined by : $r_u(z) = z * u$. Since $V^{d|n}$ is not abelian, we have $l \neq r$.*

10.4 G-structures on superspace

We fix an auxiliary super-vector bundle $\mathbb{V}^{d|n}$ over $M^{d|n}$ with typical fiber $V^{d|n}$, isomorphic to $TM^{d|n}$, and we denote by $\mathcal{F}(\mathbb{V}^{d|n})$ the super-linear frame bundle of $\mathbb{V}^{d|n}$ (with structural supergroup $GL(V^{d|n})$).

Definition 10.4.1 *For any Lie sub-supergroup G of $GL(V^{d|n})$, a **G-structure on the super-vector bundle $\mathbb{V}^{d|n}$** is a reduction to G of the frame bundle $\mathcal{F}(\mathbb{V}^{d|n})$, that is, a principal G -subbundle of $\mathcal{F}(\mathbb{V}^{d|n})$.*

Remark 10.4.2 *A choice of a Lie supergroup $G \subset GL(V^{d|n})$ may be thought as a choice of a supergeometry. Thus, the word supergeometry will be for us a synonym of the word supergroup. Once a supergeometry G is chosen, one can consider the collection of all the corresponding structures (G-structures) that can be defined on the super-vector bundle $\mathbb{V}^{d|n}$ (or on the supermanifold $M^{d|n}$, as we are going to see).*

Let us give an important example of supergeometry. From the vector representation ρ_1 and the spinor representation $\rho_{\frac{1}{2}}$, we construct a representation $\rho : \text{Spin}^\dagger(V) \longrightarrow \text{GL}(V^{d|n})$, by setting : $\rho(\sigma)(v, s) = (\rho_1(\sigma)(v), \rho_{\frac{1}{2}}(\sigma)(s))$.

Definition 10.4.3 *The super-Lorentz group, denoted by $SO^\dagger(V^{d|n})$, is the Lie supergroup of $GL(V^{d|n})$ defined by the above representation. Thus, $SO^\dagger(V^{d|n}) := \rho(\text{Spin}^\dagger(V))$.*

Remark 10.4.4 *The super-Lorentz group $SO^\dagger(V^{d|n})$ should not be confused with the orthosymplectic group $Osp(V^{d|n})$. As it is well known, the supergeometry corresponding to Poincaré supergravity theories is the super-Lorentzian geometry, and not the orthosymplectic geometry (but one can also consider other supergravities than Poincaré supergravity ; in that case, as we are going to see in the last chapter of the thesis, the orthosymplectic geometry may be of great interest). In the next section, we are already going to be interested with more general supergeometries than the the super-Lorentzian geometry. For this reason, we shall consider in the remainder of this section a fixed but unspecified Lie supergroup $G \subset GL(V^{d|n})$.*

We consider the homomorphism $\delta : \text{Hom}(V^{d|n}, \mathfrak{g}) \longrightarrow \text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$ defined by : $\delta(f) = \text{Id} \wedge f$, i.e. $\delta(f)(u_1, u_2) = f(u_2)(u_1) - f(u_1)(u_2)$.

Definition 10.4.5 *The Spencer cohomology group $H_g^{0,2}$ is defined to be quotient super-vector space $\text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n}) / \text{Im } \delta$.*

Remark 10.4.6 *We have a natural representation of G on $\text{Hom}(V^{d|n}, \mathfrak{g})$ (resp. on $\text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$), given by : $(g \cdot f)(u) = g \circ f(g^{-1}u) \circ g^{-1}$ (resp. by : $(g \cdot \alpha)(u_1, u_2) = g \circ \alpha(g^{-1}u_1, g^{-1}u_2)$). Furthermore, it is not difficult to show that the homomorphism δ defined above is G -equivariant. Therefore, $\text{Im } \delta$ is a G -invariant subspace of $\text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$, and we get a representation of G on the quotient $H_g^{0,2}$ (by setting : $g \cdot [\alpha] = [g \cdot \alpha]$).*

We consider a G -structure P on $\mathbb{V}^{d|n}$, fixed once for all as an auxiliary structure.

Definition 10.4.7 *A soldering superform is a one-form $e \in \Lambda^1(M^{d|n}, \mathbb{V}^{d|n})$ such that for any $z \in M^{d|n}$, the superlinear map $e_z : T_z M^{d|n} \longrightarrow \mathbb{V}_z^{d|n}$ is an isomorphism.*

Remark 10.4.8 *In supergravity theories, the soldering superform plays a central role. It is, in a certain sense, the dynamical variable of the theory. Each soldering superform e defines a G -structure on $M^{d|n}$ (that is, on $TM^{d|n}$), obtained by pulling-back with e the fixed auxiliary G -structure P on $\mathbb{V}^{d|n}$.*

Remark 10.4.9 *For any $e \in \Lambda^1(M^{d|n}, \mathbb{V}^{d|n})$, we denote also by e the associated element in $\bar{\Lambda}_G^1(P, V^{d|n})$.*

Definition 10.4.10 *Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform, and $\omega \in \Lambda_G^1(P, \mathfrak{g})$ a superconnection.*

- *The supercurvature of the superconnection ω is the two-form $\Omega \in \bar{\Lambda}_G^2(P, \mathfrak{g})$ defined by : $\Omega = \mathcal{D}^\omega \omega$.*
- *The supertorsion of the couple (e, ω) is the two-form $T \in \bar{\Lambda}_G^2(P, V^{d|n})$ defined by : $T = \mathcal{D}^\omega e$.*

Remark 10.4.11 *We may view the supertorsion of the couple (e, ω) as a G -equivariant map*

$\mathcal{T} : P \longrightarrow \text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$. Indeed, if $H_p = \text{Ker } \omega_p$, then e_p restricts to an isomorphism between H_p and $V^{d|n}$, so by setting $\mathcal{T}(p) = (e_p^{-1})^*(T_p)$, we are associating to the antisymmetric bilinear map $T_p : H_p \times H_p \longrightarrow V^{d|n}$ an antisymmetric bilinear map $\mathcal{T}(p) : V^{d|n} \times V^{d|n} \longrightarrow V^{d|n}$.

Proposition 10.4.12 Let $\mathcal{T} : P \longrightarrow \text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$ be the supertorsion map associated with a couple (e, ω) . For each $p \in P$, the Spencer cohomology class $[\mathcal{T}(p)]$ is independent of the choice of the superconnection ω . It depends only on the G -structure defined by the soldering superform e .

Proof : Let ω' be another superconnection, and \mathcal{T}' the supertorsion map associated with the couple (e, ω') . For any $p \in P$, we have :

$$\begin{aligned} \mathcal{T}(p) - \mathcal{T}'(p) &= (e_p^{-1})^*(T_p) - (e_p^{-1})^*(T'_p) = (e_p^{-1})^*(T_p - T'_p) \\ &= (e_p^{-1})^*(de_p + e_p \wedge \omega_p - de_p - e_p \wedge \omega'_p) = (e_p^{-1})^*(e_p \wedge (\omega_p - \omega'_p)) = \text{Id} \wedge (e_p^{-1})^*(\omega_p - \omega'_p) \end{aligned}$$

Therefore, $\mathcal{T}(p) - \mathcal{T}'(p) \in \text{Im } \delta$, that is : $[\mathcal{T}(p)] = [\mathcal{T}'(p)]$. \square

Definition 10.4.13 Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform.

The **first-order structure function** of the G -structure defined on $M^{d|n}$ by e is the map $C(e) : P \longrightarrow H_{\mathfrak{g}}^{0,2}$ given by : $C(e)(p) = [\mathcal{T}(p)]$, where \mathcal{T} is the supertorsion map associated with the couple (e, ω) for any superconnection $\omega \in \Lambda_G^1(P, \mathfrak{g})$.

Example 10.4.14 Consider the Minkowski superspacetime $\overset{\circ}{M}{}^{d|n}$, over which we take the trivial super-vector bundle $\overset{\circ}{V}{}^{d|n} = \overset{\circ}{M}{}^{d|n} \times V^{d|n}$ (which is isomorphic to $T\overset{\circ}{M}{}^{d|n}$). Then $\mathcal{F}(\overset{\circ}{V}{}^{d|n}) = \overset{\circ}{M}{}^{d|n} \times GL(V^{d|n})$. We consider the canonical auxiliary G -structure $\overset{\circ}{P} = \overset{\circ}{M}{}^{d|n} \times G$. Let $s : \overset{\circ}{M}{}^{d|n} \longrightarrow \overset{\circ}{P}$ be the canonical trivialisation of $\overset{\circ}{P}$ (given by : $s(z) = (z, Id)$), and $\pi_2 : \overset{\circ}{P} \longrightarrow G$ the canonical projection (given by : $\pi_2(z, g) = g$). We consider the canonical soldering superform $\dot{e} \in \bar{\Lambda}_G^1(\overset{\circ}{P}, V^{d|n})$, defined by : $s^*\dot{e} = {}^r e$, where ${}^r e$ is the right-invariant Maurer-Cartan form on $\overset{\circ}{M}{}^{d|n}$. Finally, let $\tau = [,]$ be the Lie superbracket of $V^{d|n}$. Notice that $\tau \in \text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$. In the following proposition, we compute the first-order structure function of the G -structure defined on $\overset{\circ}{M}{}^{d|n}$ by \dot{e} , called the **canonical G -structure** on $\overset{\circ}{M}{}^{d|n}$.

Proposition 10.4.15 $\forall p \in \overset{\circ}{P}, C(\dot{e})(p) = \pi_2(p)^{-1} \cdot [\tau]$

Proof : Let $\dot{\omega}$ be the canonical flat connection on the trivial bundle $\overset{\circ}{P}$ (that is, $\dot{\omega} = \pi_2^* \theta_G$, where θ_G is the Maurer-Cartan form of G). Notice that $\dot{\omega}$ is also defined by the property : $s^*\dot{\omega} = 0$. First, we compute the supertorsion $\overset{\circ}{T}$ of the couple $(\dot{e}, \dot{\omega})$, in the canonical trivialisation s . From the definition, $\overset{\circ}{T} = d\dot{\omega}\dot{e} = d\dot{e} + \dot{e} \wedge \dot{\omega}$. Therefore, $s^*\overset{\circ}{T} = d(s^*\dot{e}) + (s^*\dot{e}) \wedge (s^*\dot{\omega}) = d{}^r e$. Next, we evaluate the associated supertorsion map $\overset{\circ}{\mathcal{T}}$, in the canonical trivialisation s . We have : $(s^*\overset{\circ}{\mathcal{T}})(z) = \overset{\circ}{\mathcal{T}}(s(z)) = (\dot{e}_{s(z)}^{-1})^*(\overset{\circ}{T}_{s(z)}) = ({}^r e_z^{-1})^* \circ ({}^r e_z)^* \circ (\dot{e}_{s(z)}^{-1})^*(\overset{\circ}{T}_{s(z)})$
 $= ({}^r e_z^{-1})^* \circ (\dot{e}_{s(z)}^{-1} \circ {}^r e_z)^*(\overset{\circ}{T}_{s(z)}) = ({}^r e_z^{-1})^* \circ (\dot{e}_{s(z)}^{-1} \circ \dot{e}_{s(z)} \circ T_z s)^*(\overset{\circ}{T}_{s(z)})$
 $= ({}^r e_z^{-1})^* \circ (T_z s)^*(\overset{\circ}{T}_{s(z)}) = ({}^r e_z^{-1})^*((s^*\overset{\circ}{T})_z)$.

Evaluating on the basis of $V^{d|n}$, we find : $(s^*\overset{\circ}{\mathcal{T}})(z)(e_A, e_B) = ({}^r e_z^{-1})^*((s^*\overset{\circ}{T})_z)(e_A, e_B)$
 $= ({}^r e_z^{-1})^*((d{}^r e)_z)(e_A, e_B) = (d{}^r e)_z({}^r e_z^{-1}(e_A), {}^r e_z^{-1}(e_B)) = (d{}^r e)_z((D_A)_{|z}, (D_B)_{|z})$
 $= -{}^r e_z([D_A, D_B]_{|z}) = [e_A, e_B] = \tau(e_A, e_B)$. Thus, for each $z \in \overset{\circ}{M}{}^{d|n}$,
 $\overset{\circ}{\mathcal{T}}(s(z)) = (s^*\overset{\circ}{\mathcal{T}})(z) = \tau$. We deduce that for any $p = s(z)\pi_2(p) \in \overset{\circ}{P}$,
 $\overset{\circ}{\mathcal{T}}(p) = \overset{\circ}{\mathcal{T}}(s(z)\pi_2(p)) = \pi_2(p)^{-1} \cdot \overset{\circ}{\mathcal{T}}(s(z)) = \pi_2(p)^{-1} \cdot \tau$. Going to the cohomology classes, we get : $C(\dot{e})(p) = [\overset{\circ}{\mathcal{T}}(p)] = [\pi_2(p)^{-1} \cdot \tau] = \pi_2(p)^{-1} \cdot [\tau]$. \square

Definition 10.4.16 Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform.

The G -structure defined on $M^{d|n}$ by e is said to be **first-order flat** if there exists a smooth function $g : P \longrightarrow G$ such that for any $p \in P$, $C(e)(p) = g(p) \cdot [\tau]$.

Remark 10.4.17 Obviously, the canonical G -structure on $\overset{\circ}{M}{}^{d|n}$ is first-order flat.

10.5 First-order flat super-Lorentzian and generalized structures

Let us consider the super-Lorentzian geometry, which corresponds to the choice of $G = \text{SO}^\dagger(V^{d|n})$. Notice that since the symmetric bilinear map $\Gamma : S \times S \longrightarrow V$ is $\text{Spin}^\dagger(V)$ -equivariant, the Lie superbracket $\tau \in \text{Hom}(V^{d|n} \wedge V^{d|n}, V^{d|n})$ is $\text{SO}^\dagger(V^{d|n})$ -invariant.

Proposition 10.5.1 *For $G = \text{SO}^\dagger(V^{d|n})$, we have $\text{Ker } \delta = \{0\}$.*

Proposition 10.5.2 *Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform. The following two statements are equivalent :*

1. *The super-Lorentzian structure defined on $M^{d|n}$ by e is first-order flat.*
2. *There exists a superconnection ω such that the supertorsion of (e, ω) is given by : $T(X, Y) = [e(X), e(Y)]$ for any $X, Y \in \Gamma(TM^{d|n})$.*

Moreover, when one of these statements is true, the superconnection ω is unique.

Proof : Suppose 1. is true. Let ω' be any superconnection, and \mathcal{T}' the supertorsion map associated with (e, ω') . Then for any $p \in P$, $C(e)(p) = [\mathcal{T}'(p)]$. The first-order flatness of the $\text{SO}^\dagger(V^{d|n})$ -structure defined by e implies the existence of $g : P \longrightarrow \text{SO}^\dagger(V^{d|n})$ such that for any $p \in P$, $C(e)(p) = g(p) \cdot [\tau]$. Since τ is $\text{SO}^\dagger(V^{d|n})$ -invariant, this is equivalent to $C(e)(p) = [\tau]$ for any $p \in P$, that is $[\mathcal{T}'(p)] = [\tau]$ for any $p \in P$. We deduce the existence of a $\text{SO}^\dagger(V^{d|n})$ -equivariant map $\beta : P \longrightarrow \text{Hom}(V^{d|n}, \mathfrak{g})$ such that : $\mathcal{T}'(p) - \tau = \text{Id} \wedge \beta(p)$. Let $\omega \in \Lambda_G^1(P, \mathfrak{g})$ defined by : $\omega_p = \omega'_p - (e_p)^*(\beta(p))$. It is easy to check that ω is a connection one-form on P . Let \mathcal{T} the supertorsion map associated with (e, ω) . Then

$$\begin{aligned} \mathcal{T}(p) &= (e_p^{-1})^*(T_p) = (e_p^{-1})^*(de_p + e_p \wedge \omega_p) = (e_p^{-1})^*(de_p + e_p \wedge \omega'_p - e_p \wedge (e_p)^*(\beta(p))) \\ &= (e_p^{-1})^*(de_p + e_p \wedge \omega'_p) - (e_p^{-1})^*(e_p \wedge (e_p)^*(\beta(p))) = (e_p^{-1})^*(T'_p) - \text{Id} \wedge \beta(p) = \mathcal{T}'(p) - \text{Id} \wedge \beta(p) \\ &= \tau + \text{Id} \wedge \beta(p) - \text{Id} \wedge \beta(p) = \tau. \end{aligned}$$

So we have $\mathcal{T}(p) = \tau$ for each $p \in P$, which is equivalent to $T_p = (e_p)^*(\tau)$ for each $p \in P$, that is : $T(X, Y) = [e(X), e(Y)]$ for any $X, Y \in \Gamma(TM^{d|n})$. Conversely, if 2. is true, it is now easy to see that 1. will also be true. The unicity of ω follows from $\text{Ker } \delta = \{0\}$. \square

Remark 10.5.3 *We could have carried all the preceding study at the ordinary (non-supersymmetric) level, obtaining a formulation of gravity by taking the Lorentzian geometry, which corresponds to the choice : $G = \text{SO}^\dagger(V)$. In this case, we have $\text{Ker } \delta = \{0\}$, but also $\tau = 0$ (translations do commute) and $H_{\mathfrak{g}}^{0,2} = \{0\}$ so there is no obstruction for first-order flatness. We deduce that for any soldering form $e \in \bar{\Lambda}_G^1(P, V)$, there exists a unique connection ω on P such that the torsion of the couple (e, ω) vanishes.*

Theorem 10.5.4 (Lott) *Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform such that the super-Lorentzian structure defined on $M^{d|n}$ by e is first-order flat. Let ω be the unique superconnection such that the supertorsion of (e, ω) is given by : $T(X, Y) = [e(X), e(Y)]$ for any $X, Y \in \Gamma(TM^{d|n})$. In addition, we suppose the existence of a $\text{Spin}^\dagger(V)$ -equivariant isomorphism $D : S^* \longrightarrow S$ such that for each $X \in \mathfrak{spin}(V)$, $\rho'_{\frac{1}{2}}(X) \circ D$ is symmetric. Then the supercurvature Ω of ω vanishes.*

The last theorem (see [Lot] for the proof) shows that imposing the constraint of first-order flatness on a super-Lorentzian structure is too restrictive, since it leads only to flat solutions. Following Lott, we consider a natural generalization of the super-Lorentzian geometry, by choosing a larger supergroup G . Namely, if \mathfrak{m} is a $\text{Spin}^\dagger(V)$ -invariant subspace of $\text{Hom}(V, S)$, we define for each $\sigma \in \text{Spin}^\dagger(V)$ and each $\mu \in \mathfrak{m}$ the automorphism

$\rho(\sigma, \mu)$ of $V^{d|n}$ given by : $\rho(\sigma, \mu)(v, s) = (\rho_1(\sigma)(v), \mu(v) + \rho_{\frac{1}{2}}(\sigma)(s))$. Then, we take : $G = \{\rho(\sigma, \mu) ; (\sigma, \mu) \in \text{Spin}^\dagger(V) \times \mathfrak{m}\}$. Setting $H = \text{SO}^\dagger(V^{d|n})$, we see that $H \subset G$.

Proposition 10.5.5 *The stabilizer of the Lie superbracket τ (and of $[\tau]$) is H (thus, $[\tau]$ is not G -invariant).*

Proof : For any $a \in G$, we have : $a \in \text{Stab}(\tau) \iff a \cdot \tau = \tau \iff \forall u_1, u_2 \in V^{d|n}, a [u_1, u_2] = [a u_1, a u_2]$.

Set $a = \rho(\sigma, \mu)$, $u_1 = (v_1, s_1)$, $u_2 = (v_2, s_2)$. Then : $a [u_1, u_2] = [a u_1, a u_2]$

$$\iff \rho(\sigma, \mu)((v_1, s_1), (v_2, s_2)) = [\rho(\sigma, \mu)(v_1, s_1), \rho(\sigma, \mu)(v_2, s_2)]$$

$$\iff \rho(\sigma, \mu)([s_1, s_2], 0) = [(\rho_1(\sigma)(v_1), \mu(v_1) + \rho_{\frac{1}{2}}(\sigma)(s_1)), (\rho_1(\sigma)(v_2), \mu(v_2) + \rho_{\frac{1}{2}}(\sigma)(s_2))]$$

$$\iff (\rho_1(\sigma)([s_1, s_2]), \mu([s_1, s_2])) = ([\mu(v_1) + \rho_{\frac{1}{2}}(\sigma)(s_1), \mu(v_2) + \rho_{\frac{1}{2}}(\sigma)(s_2)], 0)$$

$$\iff \mu = 0 \text{ and } \rho_1(\sigma)([s_1, s_2]) = [\rho_{\frac{1}{2}}(\sigma)(s_1), \rho_{\frac{1}{2}}(\sigma)(s_2)]$$

$$\iff a = \rho(\sigma, 0) \in H.$$

□

Proposition 10.5.6 *Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform. The following three statements are equivalent :*

1. *The G -structure defined on $M^{d|n}$ by e is first-order flat.*
2. *For each $z_0 \in M^{d|n}$, there exists an open neighborhood W of z_0 , and a local cross-section $s : W \rightarrow P$ such that for any $z \in W$, $C(e)(s(z)) = [\tau]$.*
3. *For each $p \in P$, $C(e)(p)$ and $[\tau]$ are in the same G -orbit.*

Proof :

1. \implies 2. From the definition of first-order flatness, there exists a smooth function $g : P \rightarrow G$ such that for any $p \in P$, we have : $C(e)(p) = g(p) \cdot [\tau]$. Let $z_0 \in M^{d|n}$, and W an open neighborhood of z_0 that trivializes P . Let $s' : W \rightarrow P$ a local cross-section. Then for any $z \in W$, we have : $C(e)(s'(z)) = g(s'(z)) \cdot [\tau]$. Define a new local cross-section $s : W \rightarrow P$ by setting : $s(z) = s'(z) g(s'(z))$. Then, for each $z \in W$, we have : $C(e)(s(z)) = g(s'(z))^{-1} \cdot C(e)(s'(z)) = g(s'(z))^{-1} g(s'(z)) \cdot [\tau] = [\tau]$.
2. \implies 3. Let $p \in P$, and $z_0 = \pi(p)$. Let W be an open neighborhood of z_0 , and $s : W \rightarrow P$ a local cross-section such that for any $z \in W$, $C(e)(s(z)) = [\tau]$. There exists $a \in G$ such that $p = s(z_0)a$. We deduce that $C(e)(p) = a^{-1} \cdot C(e)(s(z_0)) = a^{-1} \cdot [\tau]$, that is : $C(e)(p)$ and $[\tau]$ are in the same G -orbit.
3. \implies 1. First, we notice that $G|H$ is isomorphic to the subgroup $K = \{\rho(1, \mu) ; \mu \in \mathfrak{m}\}$ of G (where $\rho(1, \mu)(v, s) = (v, \mu(v) + s)$ for every $(v, s) \in V^{d|n}$). The subgroup K acts of course on $H_{\mathfrak{g}}^{0,2}$. Denote by $K \cdot [\tau]$ the K -orbit of $[\tau]$. Then the map $\varphi : K \rightarrow K \cdot [\tau]$ defined by : $\varphi(k) = k \cdot [\tau]$ is a diffeomorphism. On the other hand, for any $p \in P$, since $C(e)(p)$ and $[\tau]$ are assumed in the same G -orbit, there exists $a \in G$ such that $C(e)(p) = a \cdot [\tau]$. But then, there exists a unique couple $(h, k) \in H \times K$ such that $a = kh$, and therefore we have $C(e)(p) = k \cdot [\tau]$. So the target of $C(e)$ is in fact $K \cdot [\tau]$. Now put $g = \varphi^{-1} \circ C(e)$. Then $g : P \rightarrow K \hookrightarrow G$ is smooth, and we have $C(e)(p) = g(p) \cdot [\tau]$ for any $p \in P$. Thus, the G -structure defined on $M^{d|n}$ by e is first-order flat. □

Proposition 10.5.7 *Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform. The following two statements are equivalent :*

1. *The G -structure defined on $M^{d|n}$ by e is first-order flat.*
2. *There exists a reduction Q of P to the super-Lorentz subgroup H , a superconnection $\omega \in \Lambda_H^1(Q, \mathfrak{h})$, and a tensorial one-form $\alpha \in \Lambda_H^1(Q, \mathfrak{m})$ such that the supertorsion of the couple (e, ω) is given by : $T(X, Y) = [e(X), e(Y)] - \alpha(Y)e(X) + \alpha(X)e(Y)$ for any $X, Y \in \Gamma(TM^{d|n})$.*

Moreover, when one of these statements is true, then for each choice of α the superconnection ω is unique.

Proof : Suppose 1. is true. Let ω' be any superconnection on P , and \mathcal{T}' the supertorsion map associated with (e, ω') . Then for any $p \in P$, $C(e)(p) = [\mathcal{T}'(p)]$. The first-order flatness of the G -structure defined by e implies the existence of $g : P \longrightarrow G$ such that for any $p \in P$, $C(e)(p) = g(p) \cdot [\tau]$. We deduce the existence of a G -equivariant map $\beta : P \longrightarrow \text{Hom}(V^{d|n}, \mathfrak{g})$ such that $\mathcal{T}'(p) - g(p) \cdot \tau = \text{Id} \wedge \beta(p)$. Let $\tilde{\omega} \in \Lambda_G^1(P, \mathfrak{g})$ defined by $\tilde{\omega}_p = \omega'_p - (e_p)^*(\beta(p))$. It is easy to check that $\tilde{\omega}$ is a connection one-form on P . Let $\tilde{\mathcal{T}}$ the supertorsion map associated with $(e, \tilde{\omega})$. Then

$$\begin{aligned}\tilde{\mathcal{T}}(p) &= (e_p^{-1})^*(\tilde{\mathcal{T}}_p) = (e_p^{-1})^*(de_p + e_p \wedge \tilde{\omega}_p) = (e_p^{-1})^*(de_p + e_p \wedge \omega'_p - e_p \wedge (e_p)^*(\beta(p))) \\ &= (e_p^{-1})^*(de_p + e_p \wedge \omega'_p) - (e_p^{-1})^*(e_p \wedge (e_p)^*(\beta(p))) = (e_p^{-1})^*(T'_p) - \text{Id} \wedge \beta(p) = \mathcal{T}'(p) - \text{Id} \wedge \beta(p) \\ &= g(p) \cdot \tau + \text{Id} \wedge \beta(p) - \text{Id} \wedge \beta(p) = g(p) \cdot \tau.\end{aligned}$$

So we have $\tilde{\mathcal{T}}(p) = g(p) \cdot \tau$ for each $p \in P$. Define $Q = \{p \in P / \tilde{\mathcal{T}}(p) = \tau\}$. It is not difficult to check that Q is principal H -bundle over $M^{d|n}$. Let ι be the canonical injection of Q in P . There exists a unique superconnection $\omega \in \Lambda_H^1(Q, \mathfrak{h})$, and a unique tensorial one-form $\alpha \in \Lambda_H^1(Q, \mathfrak{m})$ such that $\iota^*\tilde{\omega} = \omega + \alpha$. Then we have $\tilde{\mathcal{T}}(p) = \tau$ for each $p \in Q$, which is equivalent to $(e_p^{-1})^*(de_p + e_p \wedge \omega_p + e_p \wedge \alpha_p) = \tau$ for each $p \in Q$. Since $T_p = de_p + e_p \wedge \omega_p$, we get $T_p = (e_p)^*(\tau) - e_p \wedge \alpha_p$. We deduce that $T(X, Y) = [e(X), e(Y)] - \alpha(Y)e(X) + \alpha(X)e(Y)$ for any $X, Y \in \Gamma(TM^{d|n})$. Conversely, if 2. is true, it is now easy to see that 1. will also be true. The unicity of ω for a given α follows from $\text{Ker } \delta = \{0\}$ (for $H = \text{SO}^\dagger(V^{d|n})$). \square

10.6 The affine viewpoint

In this section, we consider supergravity as a gauge theory for the generalized super-Poincaré group (which is essentially the semi-direct product of G with the supertranslations group $V^{d|n}$). We define then a sort of affine spin frame bundle, and we look at affine superconnections defined on it. It is easy to see that having an affine superconnection is equivalent to having a couple $(e, \tilde{\omega})$ where e is a soldering superform and $\tilde{\omega}$ is a superconnection for the group G . For an introduction to affine frame bundles and affine connections, we refer to [KN1]. The soldering superform e may be subjected to two different types of transformations : infinitesimal super-diffeomorphisms (i.e. supervector fields) act on e through the *covariant Lie derivative* (which is $\tilde{\omega}$ -dependent). On the other hand, gauge supertranslations act on e (also in an $\tilde{\omega}$ -dependent way). In the last theorem, we prove that these transformations on e are equivalent up to gauge transformations if and only if the G -structure defined by e is first-order flat (which is the constraint we impose).

Definition 10.6.1 *The generalized super-Poincaré group is the Lie supergroup defined by :*

$$S\Pi_G(V^{d|n}) = V^{d|n} \rtimes_\rho (\text{Spin}^\dagger(V) \times \mathfrak{m})$$

Remark 10.6.2 *The group law in $S\Pi_G(V^{d|n})$ is given by :*

$$(u, (\sigma, \mu))(u', (\sigma', \mu')) = (u * \rho(\sigma, \mu)(u'), (\sigma, \mu)(\sigma', \mu'))$$

for any $(u, (\sigma, \mu)), (u', (\sigma', \mu')) \in S\Pi_G(V^{d|n})$.

Set : $S\mathcal{P} = \mathbb{V}^{d|n} \times_{M^{d|n}} P$ (fiber product over $M^{d|n}$).

Then $S\mathcal{P}$ is a $S\Pi_G(V^{d|n})$ -principal bundle over $M^{d|n}$. Denote by $\mathcal{C}(P)$ (resp. $\mathcal{C}(S\mathcal{P})$) the (infinite-dimensional) affine space of superconnections on P (resp. on $S\mathcal{P}$).

We have a short sequence :

$$0 \longrightarrow V^{d|n} \longrightarrow S\Pi_G(V^{d|n}) \longrightarrow G \longrightarrow 1$$

which is exact and split, and the canonical injection $\iota : G \longrightarrow S\Pi_G(V^{d|n})$ induces an injective homomorphism of principal fiber bundles $\iota : P \longrightarrow S\mathcal{P}$.

At the Lie algebra level, we have the short exact sequence :

$$0 \longrightarrow V^{d|n} \longrightarrow \text{Lie}(S\Pi_G(V^{d|n})) \longrightarrow \mathfrak{g} \longrightarrow 0$$

and $\text{Lie}(S\Pi_G(V^{d|n})) = V^{d|n} \oplus_{\rho'} \mathfrak{g}$ (semi-direct sum).

Proposition 10.6.3 *There exists a one-to-one correspondence between the space $\mathcal{C}(S\mathcal{P})$ of superconnections on $S\mathcal{P}$, and the space $\bar{\Lambda}_G^1(P, V^{d|n}) \times \mathcal{C}(P)$.*

Proof : Using the splitting $\text{Lie}(S\Pi_G(V^{d|n})) = V^{d|n} \oplus \mathfrak{g}$, we set for any $\check{\omega} \in \mathcal{C}(S\mathcal{P})$: $\iota^*\check{\omega} = e + \tilde{\omega}$, with $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ and $\tilde{\omega} \in \mathcal{C}(P)$. \square

Definition 10.6.4 *The supergauge group associated to P (resp. $S\mathcal{P}$) is the (infinite-dimensional) supergroup $\text{Gau}(P) = H_G^\infty(P, G)$ (resp. $\text{Gau}(S\mathcal{P}) = H_{S\Pi_G(V^{d|n})}^\infty(S\mathcal{P}, S\Pi_G(V^{d|n}))$).*

Definition 10.6.5 *The supergauge algebra associated to P (resp. $S\mathcal{P}$) is the (infinite-dimensional) supermodule $\mathfrak{gau}(P) = H_G^\infty(P, \mathfrak{g})$ (resp. $\mathfrak{gau}(S\mathcal{P}) = H_{S\Pi_G(V^{d|n})}^\infty(S\mathcal{P}, \text{Lie}(S\Pi_G(V^{d|n})))$).*

Proposition 10.6.6 *There is a one-to-one correspondence between the super-Poincaré gauge algebra $\mathfrak{gau}(S\mathcal{P})$, and the product $H_G^\infty(P, V^{d|n}) \times \mathfrak{gau}(P)$, where $H_G^\infty(P, V^{d|n})$ is the supertranslation gauge algebra, and $\mathfrak{gau}(P)$ the Lorentz gauge algebra.*

We have a natural affine infinitesimal action of $\mathfrak{gau}(P)$ (resp. $\mathfrak{gau}(S\mathcal{P})$) on $\mathcal{C}(P)$ (resp. $\mathcal{C}(S\mathcal{P})$). Let us write these actions explicitly.

Let $\Lambda \in \mathfrak{gau}(P)$ and $\tilde{\omega} \in \mathcal{C}(P)$. Then $\delta_\Lambda \tilde{\omega} = \mathcal{D}^{\tilde{\omega}} \Lambda = d\Lambda + [\Lambda, \tilde{\omega}]$.

Let $\check{\Lambda} = (\xi, \Lambda) \in H_G^\infty(P, V) \times \mathfrak{gau}(P) \simeq \mathfrak{gau}(S\mathcal{P})$, and $\check{\omega} = (e, \tilde{\omega}) \in \bar{\Lambda}_G^1(P, V^{d|n}) \times \mathcal{C}(P) \simeq \mathcal{C}(S\mathcal{P})$. Then $\delta_{\check{\Lambda}} \check{\omega} = \mathcal{D}^{\check{\omega}} \check{\Lambda} = d\check{\Lambda} + [\check{\Lambda}, \check{\omega}]$.

$$\begin{aligned} \delta_{(\xi, \Lambda)}(e, \tilde{\omega}) &= (d\xi, d\Lambda) + [(\xi, \Lambda), (e, \tilde{\omega})] \\ &= (d\xi, d\Lambda) + ([\xi, e] + [\xi, \tilde{\omega}] + [\Lambda, e], [\Lambda, \tilde{\omega}]) \\ &= ([\xi, e] + d\xi + [\xi, \tilde{\omega}] + [\Lambda, e], d\Lambda + [\Lambda, \tilde{\omega}]) \\ &= ([\xi, e] + \mathcal{D}^{\tilde{\omega}} \xi + [\Lambda, e], \mathcal{D}^{\tilde{\omega}} \Lambda) \end{aligned}$$

Finally,

$$\begin{cases} \delta_{(\xi, \Lambda)} e &= [\xi, e] + \mathcal{D}^{\tilde{\omega}} \xi + [\Lambda, e] \\ \delta_{(\xi, \Lambda)} \tilde{\omega} &= \mathcal{D}^{\tilde{\omega}} \Lambda \end{cases}$$

Let us consider two particular cases :

- We have a natural action of the supertranslation gauge algebra $H_G^\infty(P, V^{d|n})$ on the space $\mathcal{C}(S\mathcal{P}) \simeq \bar{\Lambda}_G^1(P, V^{d|n}) \times \mathcal{C}(P) : \delta_\xi(e, \tilde{\omega}) = ([\xi, e] + \mathcal{D}^{\tilde{\omega}}\xi, 0)$. That is,

$$\begin{cases} \delta_\xi e = [\xi, e] + \mathcal{D}^{\tilde{\omega}}\xi \\ \delta_\xi \tilde{\omega} = 0 \end{cases}$$

- We have a natural action of the Lorentz gauge algebra $\mathfrak{gau}(P)$ on the space $\mathcal{C}(S\mathcal{P}) \simeq \bar{\Lambda}_G^1(P, V^{d|n}) \times \mathcal{C}(P) : \delta_\Lambda(e, \tilde{\omega}) = ([\Lambda, e], \mathcal{D}^{\tilde{\omega}}\Lambda)$. That is,

$$\begin{cases} \delta_\Lambda e = [\Lambda, e] \\ \delta_\Lambda \tilde{\omega} = \mathcal{D}^{\tilde{\omega}}\Lambda \end{cases}$$

Remark 10.6.7 Since the Lie superbracket $\tau = [,]$ appearing in the action of the supertranslation gauge algebra on e is not G -invariant, it is clear that this action is not covariant under G . In other terms, the formula $\delta_\xi e = [\xi, e] + \mathcal{D}^{\tilde{\omega}}\xi$ is not invariant under gauge transformations for the group G (it is invariant only under gauge transformations for the group H).

Definition 10.6.8 To each superconnection $\tilde{\omega} \in \mathcal{C}(P)$, we may associate a Lie derivative acting on the sections of $\mathbb{V}^{d|n}$, and on the elements of $\Lambda^k(M^{d|n}, \mathbb{V}^{d|n})$. It is the **covariant Lie derivative** $\mathcal{L}^{\tilde{\omega}}$, defined for each $X \in \Gamma(TM^{d|n})$ by :

$$\begin{aligned} \mathcal{L}_X^{\tilde{\omega}}\Phi &= \nabla_X^{\tilde{\omega}}\Phi, \text{ for any } \Phi \in \Gamma(\mathbb{V}^{d|n}) \\ (\mathcal{L}_X^{\tilde{\omega}}a)(Y) &= \nabla_X^{\tilde{\omega}}(a(Y)) - a([X, Y]), \text{ for any } a \in \Lambda^1(M^{d|n}, \mathbb{V}^{d|n}) \text{ (and } Y \in \Gamma(TM^{d|n})) \end{aligned}$$

Let $\tilde{\omega} \in \mathcal{C}(P)$ be a fixed superconnection. One can show the following two propositions :

Proposition 10.6.9 Let $a \in \Lambda^1(M^{d|n}, \mathbb{V}^{d|n})$. Then $\mathcal{D}^{\tilde{\omega}}a \in \Lambda^2(M^{d|n}, \mathbb{V}^{d|n})$, and for every $X, Y \in \Gamma(TM^{d|n})$, $\mathcal{D}^{\tilde{\omega}}a(X, Y) = \nabla_X^{\tilde{\omega}}(a(Y)) - \nabla_Y^{\tilde{\omega}}(a(X)) - a([X, Y])$.

Proposition 10.6.10 For every $X \in \Gamma(TM^{d|n})$, we have : $\mathcal{L}_X^{\tilde{\omega}} = i_X \mathcal{D}^{\tilde{\omega}} + \mathcal{D}^{\tilde{\omega}} i_X$.

Remark 10.6.11 Let $a \in \Lambda^1(M^{d|n}, \mathbb{V}^{d|n})$, and $X, Y \in \Gamma(TM^{d|n})$.

We have : $\mathcal{L}_X^{\tilde{\omega}}a = i_X \mathcal{D}^{\tilde{\omega}}a + \mathcal{D}^{\tilde{\omega}}i_Xa$,
therefore $(\mathcal{L}_X^{\tilde{\omega}}a)(Y) = (i_X \mathcal{D}^{\tilde{\omega}}a)(Y) + \mathcal{D}^{\tilde{\omega}}(a(X))(Y) = \mathcal{D}^{\tilde{\omega}}a(X, Y) + \nabla_Y^{\tilde{\omega}}(a(X))$
 $= \nabla_X^{\tilde{\omega}}(a(Y)) - \nabla_Y^{\tilde{\omega}}(a(X)) - a([X, Y]) + \nabla_Y^{\tilde{\omega}}(a(X)) = \nabla_X^{\tilde{\omega}}(a(Y)) - a([X, Y])$.
Thus, we recover the formula defining $\mathcal{L}_X^{\tilde{\omega}}a$.

Proposition 10.6.12 Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform, and \tilde{T} the supertorsion of the couple $(e, \tilde{\omega})$. We have : $\mathcal{L}_X^{\tilde{\omega}}e = i_X \tilde{T} + \mathcal{D}^{\tilde{\omega}}(e(X))$.

Proof : $\mathcal{L}_X^{\tilde{\omega}}e = i_X \mathcal{D}^{\tilde{\omega}}e + \mathcal{D}^{\tilde{\omega}}i_Xe = i_X \tilde{T} + \mathcal{D}^{\tilde{\omega}}(e(X))$. □

Remark 10.6.13 Unlike the action of the supertranslation gauge algebra on e , the action of infinitesimal super-diffeomorphisms on e through the covariant Lie derivative is fully covariant under G . In other terms, the formula $\mathcal{L}_X^{\tilde{\omega}}e = i_X \tilde{T} + \mathcal{D}^{\tilde{\omega}}(e(X))$ is invariant under gauge transformations for the whole group G .

Theorem 10.6.14 Let $e \in \bar{\Lambda}_G^1(P, V^{d|n})$ be a soldering superform, $\tilde{\omega} \in \mathcal{C}(P)$ a superconnection, and \tilde{T} the supertorsion of the couple $(e, \tilde{\omega})$. There is an equivalence between infinitesimal gauge supertranslations acting on e and infinitesimal super-diffeomorphisms acting on e , if and only if the constraint $\tilde{T}(X, Y) = [e(X), e(Y)]$ is satisfied (for all $X, Y \in \Gamma(TM^{d|n})$).

Proof : Let $\xi \in \Gamma(\mathbb{V}^{d|n})$. We set $X = e^{-1}(\xi) \in \Gamma(TM^{d|n})$. Then :
 $\delta_\xi e = \mathcal{L}_X^{\tilde{\omega}} e \iff [\xi, e] + \mathcal{D}^{\tilde{\omega}}\xi = i_X \tilde{T} + \mathcal{D}^{\tilde{\omega}}(e(X)) \iff [\xi, e] + \mathcal{D}^{\tilde{\omega}}\xi = i_X \tilde{T} + \mathcal{D}^{\tilde{\omega}}(\xi)$
 $\iff [\xi, e] = i_X \tilde{T} \iff [\xi, e(Y)] = (i_X \tilde{T})(Y) \iff [e(X), e(Y)] = \tilde{T}(X, Y).$ \square

From the proof of Proposition 10.5.7, the first-order flatness of the G -structure defined on $M^{d|n}$ by e is characterized by the existence of a superconnection $\tilde{\omega}$ and a smooth function $g : P \rightarrow G$ such that the supertorsion map \tilde{T} associated with $(e, \tilde{\omega})$ satisfies $\tilde{T}(p) = g(p) \cdot \tau$. But the assertion 2. of Proposition 10.5.6 implies the existence of a gauge in which the preceding condition is given by : $\tilde{T}(X, Y) = [e(X), e(Y)]$. We deduce that the equivalence between infinitesimal gauge supertranslations acting on e and infinitesimal super-diffeomorphisms acting on e holds up to gauge transformations for the group G , provided that the G -structure defined on $M^{d|n}$ by e is first-order flat. The gauge-dependence of this result is natural since the action of infinitesimal gauge supertranslations is not covariant under G while the action of infinitesimal super-diffeomorphisms is fully covariant (cf. Remarks 10.6.7 and 10.6.13).

Chapitre 11

Eleven-dimensional supergravity and its spherical compactification

11.1 Eleven-dimensional supergravity in component formulation

The conventional $N = 1$ supergravity in dimension $d = 11$ is a Poincaré supergravity, so it is built on the model geometry $((\hat{\mathfrak{g}}, \hat{\mathfrak{h}}), \hat{H}, \text{Ad}) = ((\tilde{\pi}(\hat{V}), \mathfrak{spin}(\hat{V})), \text{Spin}^\dagger(\hat{V}), \text{Ad})$, where \hat{V} is a Lorentzian vector space of signature $(10, 1)$.

Let $\hat{S}_{\mathbb{C}}$ be an irreducible complex representation of $Cl(\hat{V})$. It is well-known that $\dim_{\mathbb{C}} \hat{S}_{\mathbb{C}} = 2^{\lceil \frac{11}{2} \rceil} = 32$. Since $\hat{S}_{\mathbb{C}}$ is an irreducible complex representation of *real type*, it admits a $\text{Spin}^\dagger(\hat{V})$ -invariant conjugation σ . Then $\hat{S} = \text{Inv}(\hat{S}_{\mathbb{C}})$ is an irreducible real representation of $\text{Spin}^\dagger(\hat{V})$, and $\dim_{\mathbb{R}} \hat{S} = 32$.

There exists a $\text{Spin}^\dagger(\hat{V})$ -invariant non-degenerate antisymmetric bilinear form $\hat{\varepsilon} : \hat{S} \times \hat{S} \longrightarrow \mathbb{R}$. For any $\Psi \in \hat{S}$, we denote by Ψ^* the element of \hat{S}^* defined by : $\Psi^* = \hat{\varepsilon}(\Psi, .)$, so that $\Psi_1^* \Psi_2 = \hat{\varepsilon}(\Psi_1, \Psi_2)$.

The dynamical variables in $N = 1$ supergravity in dimension $d = 11$ are :

- A Cartan geometry $(\hat{P}, \hat{\theta})$ on a smooth eleven-dimensional manifold \hat{M} , modeled on $((\tilde{\pi}(\hat{V}), \mathfrak{spin}(\hat{V})), \text{Spin}^\dagger(\hat{V}), \text{Ad})$,
- A Rarita-Schwinger spinor field $\hat{\psi} \in \Lambda^1(\hat{M}, \hat{\mathbb{S}})$, where $\hat{\mathbb{S}}$ is the associated spinor bundle : $\hat{\mathbb{S}} = \hat{P} \times_{\rho_{\frac{1}{2}}} \hat{S}$,
- A 3-form $\hat{A} \in \Lambda^3(\hat{M}, \mathbb{R})$.

Let $\hat{\omega}$ (resp. \hat{e}) be the connection (resp. the soldering form) of the reductive Cartan geometry $(\hat{P}, \hat{\theta})$. Then $\hat{\omega} \in \Lambda^1_{\text{Spin}^\dagger(\hat{V})}(\hat{P}, \mathfrak{spin}(\hat{V}))$ is the spin connection and $\hat{e} \in \bar{\Lambda}^1_{\text{Spin}^\dagger(\hat{V})}(\hat{P}, \hat{V})$ is the elfbein.

Let $\hat{\mathbb{V}} = \hat{P} \times_{\text{Ad}} \hat{V}$. Since $\bar{\Lambda}^1_{\text{Spin}^\dagger(\hat{V})}(\hat{P}, \hat{V}) \simeq \Lambda^1(\hat{M}, \hat{\mathbb{V}})$, we may view \hat{e} as a one-form on \hat{M} with values in the vector bundle $\hat{\mathbb{V}}$, and $\hat{e} : T\hat{M} \longrightarrow \hat{\mathbb{V}}$ is an isomorphism of vector bundles. We recall that there is a natural bundle metric $\hat{\eta}$ and orientation on $\hat{\mathbb{V}}$.

Consider the curvature $\hat{\Omega} \in \bar{\Lambda}_{Spin^{\dagger}(\hat{V})}^2(\hat{P}, \mathfrak{spin}(\hat{V}))$ and the torsion $\hat{T} \in \bar{\Lambda}_{Spin^{\dagger}(\hat{V})}^2(\hat{P}, \hat{V})$ of the reductive Cartan geometry $(\hat{P}, \hat{\theta})$.

Since \hat{V} is an abelian Lie algebra, Maurer-Cartan structure equations impile the torsion of the canonical Cartan geometry defined by the Klein geometry $(\tilde{\Pi}(\hat{V}), Spin^{\dagger}(\hat{V}))$ is $\hat{T} = 0$. This is also related to the fact that the model geometry $((\tilde{\pi}(\hat{V}), \mathfrak{spin}(\hat{V})), Spin^{\dagger}(\hat{V}), \text{Ad})$ is *first-order flat*, as we have seen in the fourth part of the thesis (cf. Remark 10.5.3).

We deduce that for any soldering form $\hat{e} \in \bar{\Lambda}_{Spin^{\dagger}(\hat{V})}^1(\hat{P}, \hat{V})$, there exists a unique spin connection $\hat{\omega}_{LC} \in \Lambda_{Spin^{\dagger}(\hat{V})}^1(\hat{P}, \mathfrak{spin}(\hat{V}))$ such that the torsion \hat{T} of the Cartan geometry defined by \hat{e} and $\hat{\omega}_{LC}$ vanishes.

Set $\hat{\gamma}(\hat{e}) := \hat{\gamma} \circ \hat{e} : T\hat{M} \longrightarrow \hat{V} \longrightarrow \text{End}(\hat{\mathbb{S}})$ and $\hat{\gamma}_{(n)}(\hat{e}) = \hat{\gamma}(\hat{e}) \wedge \dots \wedge \hat{\gamma}(\hat{e})$ (n times), so that $\hat{\gamma}_{(n)}(\hat{e}) : \bigwedge^n T\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$. On the other hand, if we define :

$\hat{\gamma}^*(\hat{e}) = \hat{\gamma} \circ \hat{\eta}^{-1} \circ {}^t \hat{e}^{-1} : T^*\hat{M} \longrightarrow \hat{V}^* \longrightarrow \hat{V} \longrightarrow \text{End}(\hat{\mathbb{S}})$, we may also set :

$\hat{\gamma}_{(n)}^*(\hat{e}) = \hat{\gamma}^*(\hat{e}) \wedge \dots \wedge \hat{\gamma}^*(\hat{e})$ (n times), so that $\hat{\gamma}_{(n)}^*(\hat{e}) : \bigwedge^n T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$.

The connection $\hat{\omega}_{LC}$ is not supercovariant. To obtain a supercovariant connection $\hat{\omega}_{susy}$, we first define a connection $\hat{\omega}$ by adding to $\hat{\omega}_{LC}$ the following elements of $\bar{\Lambda}_{Spin^{\dagger}(\hat{V})}^1(\hat{P}, \mathfrak{spin}(\hat{V}))$:

- . $\frac{i}{2} \hat{\psi}^* \otimes (\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1})$. This element is obtained in the following way : performing the exterior product and spinor contraction of $\hat{\gamma} \circ \hat{\eta}^{-1} : \hat{V}^* \longrightarrow \hat{V} \longrightarrow \text{End}(\hat{\mathbb{S}})$ with $\hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} : \hat{V}^* \longrightarrow \hat{V} \longrightarrow T\hat{M} \longrightarrow \hat{\mathbb{S}}$ gives : $\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} : \bigwedge^2 \hat{V}^* \longrightarrow \hat{\mathbb{S}}$. It remains to make the tensor product and spinor contraction of $\hat{\psi}^* : T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$ with $\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1}$ to get : $\hat{\psi}^* \otimes (\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1}) : T\hat{M} \otimes \bigwedge^2 \hat{V}^* \longrightarrow \mathbb{R}$. Finally, $\text{Hom}(T\hat{M} \otimes \bigwedge^2 \hat{V}^*, \mathbb{R}) \simeq \text{Hom}(T\hat{M}, \bigwedge^2 \hat{V}) \simeq \Lambda^1(\hat{M}, \bigwedge^2 \hat{V}) \simeq \bar{\Lambda}_{Spin^{\dagger}(\hat{V})}^1(\hat{P}, \bigwedge^2 \hat{V})$ and $\bigwedge^2 \hat{V} \simeq \mathfrak{spin}(\hat{V})$.
- . $\frac{i}{2} (\hat{\psi}^* \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} \otimes \hat{\gamma}(\hat{e}) \otimes \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1})$. This element is obtained by performing the tensor product and spinor contractions of the three maps :
 $\hat{\psi}^* \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} : \hat{V}^* \longrightarrow \hat{V} \longrightarrow T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$
 $\hat{\gamma}(\hat{e}) : T\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$
 $\hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} : \hat{V}^* \longrightarrow \hat{V} \longrightarrow T\hat{M} \longrightarrow \hat{\mathbb{S}}$
- . $K = -\frac{i}{4} \hat{\psi}^* \otimes \hat{\gamma}_{(5)}^*(\hat{e}) \circ (\hat{g} \otimes {}^t \hat{e} \otimes {}^t \hat{e} \otimes \text{Id} \otimes \text{Id}) \otimes \hat{\psi}$. This element is obtained by tensor product, spinor contractions and tensor contractions of the following maps :
 $\hat{\psi}^* : T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$
 $\hat{\gamma}_{(5)}^*(\hat{e}) \circ (\hat{g} \otimes {}^t \hat{e} \otimes {}^t \hat{e} \otimes \text{Id} \otimes \text{Id}) : T\hat{M} \otimes \hat{V}^* \otimes \hat{V}^* \otimes T^*\hat{M} \otimes T^*\hat{M}$
 $\longrightarrow T^*\hat{M} \otimes T^*\hat{M} \otimes T^*\hat{M} \otimes T^*\hat{M} \otimes T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$
 $\hat{\psi} : T\hat{M} \longrightarrow \hat{\mathbb{S}}$

Thus, the connection $\hat{\omega}$ is defined by :

$$\hat{\omega} = \hat{\omega}_{LC} + \frac{i}{2} \hat{\psi}^* \otimes (\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1}) + \frac{i}{2} (\hat{\psi}^* \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} \otimes \hat{\gamma} \circ \hat{e} \otimes \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1}) + K$$

and the supercovariant connection $\hat{\omega}_{susy}$ is defined by : $\hat{\omega}_{susy} = \hat{\omega} - K$, so that :

$$\hat{\omega}_{susy} = \hat{\omega}_{LC} + \frac{i}{2} \hat{\psi}^* \otimes (\hat{\gamma} \circ \hat{\eta}^{-1} \wedge \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1}) + \frac{i}{2} (\hat{\psi}^* \circ \hat{e}^{-1} \circ \hat{\eta}^{-1} \otimes \hat{\gamma} \circ \hat{e} \otimes \hat{\psi} \circ \hat{e}^{-1} \circ \hat{\eta}^{-1})$$

We define the field strength $\hat{F} \in \Lambda^4(\hat{M}, \mathbb{R})$ of the 3-form \hat{A} by : $\hat{F} = d\hat{A}$.

\hat{F} is not supercovariant, we define a supercovariant field strength \hat{F}_{susy} by setting :

$$\hat{F}_{susy} = \hat{F} - 3 \hat{\psi}^* \wedge \hat{\gamma}_{(2)}(\hat{e}) \wedge \hat{\psi}$$

where spinor contractions have been performed with the exterior products so that : $\hat{\psi}^* \wedge \hat{\gamma}_{(2)}(\hat{e}) \wedge \hat{\psi} \in \Lambda^4(\hat{M}, \mathbb{R})$.

The action functional of eleven-dimensional supergravity is then :

$$\begin{aligned} \mathcal{A}(\hat{e}, \hat{\psi}, \hat{A}) = & \int_{\hat{M}} \{ c_1 R_{scal}^{\hat{e}, \hat{\omega}} + c_2 \text{Tr}(i \hat{\psi}^* \wedge \hat{\gamma}_{(3)}^*(\hat{e}) \wedge \mathcal{D}^{\frac{1}{2}(\hat{\omega} + \hat{\omega}_{susy})} \hat{\psi}) + c_3 *(\hat{F} \wedge * \hat{F}) \\ & + c_4 *(\hat{A} \wedge \hat{F} \wedge \hat{F}) + c_5 \text{Tr}(\hat{\psi}^* \wedge \hat{\gamma}_{(6)}^*(\hat{e}) \wedge \hat{\psi} \wedge (\hat{F} + \hat{F}_{susy})) \\ & + c_6 \text{Tr}((\hat{\psi}^* \circ \hat{g}^{-1}) \wedge \hat{\gamma}_{(2)}^*(\hat{e}) \wedge (\hat{\psi} \circ \hat{g}^{-1}) \wedge (\hat{F} + \hat{F}_{susy})) \} \hat{e}^* vol_{\hat{\eta}} \end{aligned}$$

where :

- . c_1, \dots, c_6 are numerical constants that can be precisely computed.
- . $\text{Tr}(i \hat{\psi}^* \wedge \hat{\gamma}_{(3)}^*(\hat{e}) \wedge \mathcal{D}^{\frac{1}{2}(\hat{\omega} + \hat{\omega}_{susy})} \hat{\psi})$ is the function obtained by exterior product, spinor contractions and tensor contractions of the following maps :
 - $\hat{\psi}^* : T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$
 - $i\hat{\gamma}_{(3)}^*(\hat{e}) : T^*\hat{M} \wedge \bigwedge^2 T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$
 - $\mathcal{D}^{\frac{1}{2}(\hat{\omega} + \hat{\omega}_{susy})} \hat{\psi} : \bigwedge^2 T\hat{M} \longrightarrow \hat{\mathbb{S}}$
- . $\text{Tr}(\hat{\psi}^* \wedge \hat{\gamma}_{(6)}^*(\hat{e}) \wedge \hat{\psi} \wedge (\hat{F} + \hat{F}_{susy}))$ is the function obtained by exterior product, spinor contractions and tensor contractions of the following maps :
 - $\hat{\psi}^* : T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$
 - $\hat{\gamma}_{(6)}^*(\hat{e}) : T^*\hat{M} \wedge T^*\hat{M} \wedge \bigwedge^4 T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$
 - $\hat{\psi} : T\hat{M} \longrightarrow \hat{\mathbb{S}}$
 - $\hat{F} + \hat{F}_{susy} : \bigwedge^4 T\hat{M} \longrightarrow \mathbb{R}$
- . $\text{Tr}((\hat{\psi}^* \circ \hat{g}^{-1}) \wedge \hat{\gamma}_{(2)}^*(\hat{e}) \wedge (\hat{\psi} \circ \hat{g}^{-1}) \wedge (\hat{F} + \hat{F}_{susy}))$ is the function obtained by exterior product, spinor contractions and tensor contractions of the following maps :
 - $\hat{\psi}^* \circ \hat{g}^{-1} : T^*\hat{M} \longrightarrow T\hat{M} \longrightarrow \hat{\mathbb{S}}^*$
 - $\hat{\gamma}_{(2)}^*(\hat{e}) : \bigwedge^2 T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$
 - $\hat{\psi} \circ \hat{g}^{-1} : T^*\hat{M} \longrightarrow T\hat{M} \longrightarrow \hat{\mathbb{S}}$
 - $\hat{F} + \hat{F}_{susy} : \bigwedge^4 T\hat{M} \longrightarrow \mathbb{R}$

The field equations of eleven-dimensional supergravity are :

$$\left\{ \begin{array}{l} Ric^{\hat{e}, \hat{\omega}_{susy}} - \frac{1}{2} R_{scal}^{\hat{e}, \hat{\omega}_{susy}} \hat{\eta} = \frac{1}{3} (\hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \text{Id}) \otimes \hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \hat{g}^{-1}) \\ \quad - \frac{1}{8} * (\hat{F}_{susy} \wedge * \hat{F}_{susy}) \hat{\eta}) \\ \hat{\gamma}_{(3)}^*(\hat{e}) \wedge \mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\psi} = 0 \\ d * \hat{F}_{susy} = - \hat{F}_{susy} \wedge \hat{F}_{susy} \end{array} \right.$$

where :

$\hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \text{Id}) \otimes \hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \hat{g}^{-1})$ is the element of $\Gamma(\text{Sym}^2 \hat{\mathbb{V}}^*)$ obtained by tensor product and tensor contraction of the two following maps :

$$\begin{aligned} \hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \text{Id}) : \hat{\mathbb{V}} \otimes T\hat{M} \otimes T\hat{M} \otimes T\hat{M} &\longrightarrow T\hat{M} \otimes T\hat{M} \otimes T\hat{M} \otimes T\hat{M} \longrightarrow \mathbb{R} \\ \hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \hat{g}^{-1}) : \hat{\mathbb{V}} \otimes T^*\hat{M} \otimes T^*\hat{M} \otimes T^*\hat{M} &\longrightarrow T\hat{M} \otimes T\hat{M} \otimes T\hat{M} \otimes T\hat{M} \longrightarrow \mathbb{R} \end{aligned}$$

$$\mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\psi} = \mathcal{D}^{\hat{\omega}_{susy}} \hat{\psi} - \frac{i}{144} (\hat{\gamma}_{(5)}^*(\hat{e}) - 8(\hat{\gamma}_{(3)}^*(\hat{e}) \otimes \hat{g}^{-1})) \circ (\hat{g} \otimes \bigwedge^4 \text{Id}) \wedge \hat{F}_{susy} \wedge \hat{\psi}$$

and $(\hat{\gamma}_{(5)}^*(\hat{e}) - 8(\hat{\gamma}_{(3)}^*(\hat{e}) \otimes \hat{g}^{-1})) \circ (\hat{g} \otimes \bigwedge^4 \text{Id}) \wedge \hat{F}_{susy} \wedge \hat{\psi}$ is the element of $\Lambda^2(\hat{M}, \hat{\mathbb{S}})$ obtained by exterior product, spinor contractions and tensor contractions of the following maps :

$$(\hat{\gamma}_{(5)}^*(\hat{e}) - 8(\hat{\gamma}_{(3)}^*(\hat{e}) \otimes \hat{g}^{-1})) \circ (\hat{g} \otimes \bigwedge^4 \text{Id}) : T\hat{M} \otimes \bigwedge^4 T^*\hat{M} \longrightarrow T^*\hat{M} \otimes \bigwedge^4 T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$$

$$\hat{F}_{susy} : \bigwedge^4 T\hat{M} \longrightarrow \mathbb{R}$$

$$\hat{\psi} : T\hat{M} \longrightarrow \hat{\mathbb{S}}.$$

$\hat{\gamma}_{(3)}^*(\hat{e}) \wedge \mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\psi}$ is the element of $\Gamma(T\hat{M} \otimes \hat{\mathbb{S}})$ obtained by exterior product, spinor contractions and tensor contractions of the two following maps :

$$\hat{\gamma}_{(3)}^*(\hat{e}) : T^*\hat{M} \wedge \bigwedge^2 T^*\hat{M} \longrightarrow \text{End}(\hat{\mathbb{S}})$$

$$\mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\psi} : \bigwedge^2 T\hat{M} \longrightarrow \hat{\mathbb{S}}$$

The above equations are in particular invariant under Poincaré gauge transformations, abelian gauge transformations, and the following gauge supersymmetry transformations, with parameter $\hat{\epsilon} \in \hat{\mathbb{S}}$:

$$\delta_{\hat{\epsilon}} \hat{e} = i \hat{\Gamma}(\hat{\epsilon}, \hat{\psi}) , \quad \delta_{\hat{\epsilon}} \hat{\psi} = \mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\epsilon} , \quad \delta_{\hat{\epsilon}} \hat{A} = \hat{\epsilon}^* \hat{\gamma}_{(2)}(e) \wedge \hat{\psi}$$

11.2 The spherical compactification of eleven-dimensional supergravity

In order to perform a dimensional reduction to four dimensions, we make a reduction of the structural group from $\hat{H} = \text{Spin}^\dagger(\hat{V})$ to $H = \text{Spin}^\dagger(3, 1) \times \text{Spin}(7)$, this choice being motivated by our desire to obtain the preferential compactifying solution of Freund and Rubin (cf. [FR]), that is, 4 non-compact spacetime dimensions + 7 compact internal dimensions (in fact, their ansatz allows also 7 non-compact spacetime dimensions + 4 compact internal dimensions, and it is not clear how to rule out mathematically this possibility).

Thus, we are changing the model geometry, but we still want it to be the model of a Cartan geometry on an eleven-dimensional manifold. Therefore we must also reduce the principal Lie algebra $\hat{\mathfrak{g}} = \tilde{\pi}(\hat{V})$ to a Lie algebra \mathfrak{g} containing $\mathfrak{h} = \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)$, and such that $\dim \mathfrak{g} = \dim \mathfrak{h} + 11 = 6 + 21 + 11 = 38$. A natural choice is : $\mathfrak{g} = \mathfrak{spin}(3, 2) \times \mathfrak{spin}(8)$.

Thus, we can take :

$$((\mathfrak{g}, \mathfrak{h}), H, \text{Ad}) = ((\mathfrak{spin}(3, 2) \times \mathfrak{spin}(8), \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7), \text{Ad})$$

as a model geometry to describe the compactification of eleven-dimensional supergravity to four dimensions.

The above model geometry is reductive : from reductive decompositions $\mathfrak{spin}(3, 2) = \mathfrak{spin}(3, 1) \oplus \mathfrak{m}^4$ and $\mathfrak{spin}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}^7$, by setting $\mathfrak{m}^{11} = \mathfrak{m}^4 \times \mathfrak{m}^7$ we obtain a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}^{11}$:

$$\mathfrak{spin}(3, 2) \times \mathfrak{spin}(8) = (\mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)) \oplus (\mathfrak{m}^4 \times \mathfrak{m}^7)$$

The gravitational part of the spontaneously compactified eleven-dimensional supergravity theory is then described by a Cartan geometry $(\hat{P}, \hat{\theta})$ on a smooth eleven-dimensional manifold \hat{M} , modeled on

$$((\mathfrak{spin}(3, 2) \times \mathfrak{spin}(8), \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7), \text{Ad})$$

Let $\hat{\omega}$ (resp. \hat{e}) be the connection (resp. the soldering form) of the reductive Cartan geometry $(\hat{P}, \hat{\theta})$. Then $\hat{\omega} \in \Lambda_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}^1(\hat{P}, \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7))$ is the spin connection and $\hat{e} \in \bar{\Lambda}_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}^1(\hat{P}, \mathfrak{m}^4 \times \mathfrak{m}^7)$ is the elfbein.

Let $\hat{\mathbb{V}} = \hat{P} \times_{\text{Ad}} (\mathfrak{m}^4 \times \mathfrak{m}^7)$. Since $\bar{\Lambda}_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}^1(\hat{P}, \mathfrak{m}^4 \times \mathfrak{m}^7) \simeq \Lambda^1(\hat{M}, \hat{\mathbb{V}})$, we may view \hat{e} as a one-form on \hat{M} with values in the vector bundle $\hat{\mathbb{V}}$, and $\hat{e} : T\hat{M} \longrightarrow \hat{\mathbb{V}}$ is an isomorphism of vector bundles. We recall that there is a natural bundle metric $\hat{\eta}$ and orientation on $\hat{\mathbb{V}}$. Besides, the reduction of the structure group from $\text{Spin}^\dagger(10, 1)$ to $\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)$ endows $\hat{\mathbb{V}}$ with a natural product structure : $\hat{\mathbb{V}} = \hat{\mathbb{V}}^4 \oplus \hat{\mathbb{V}}^7$. In other terms, we have a $(\text{Spin}^\dagger(3, 1) \times \text{Spin}(7))$ -structure on the vector bundle $\hat{\mathbb{V}}$.

Let $S_{\mathbb{C}}^{(4)}$ be an irreducible complex representation of $Cl(\mathfrak{m}^4)$. It is well-known that $\dim_{\mathbb{C}} S_{\mathbb{C}}^{(4)} = 2^{[\frac{4}{2}]} = 4$. Since we are in an even dimension, $S_{\mathbb{C}}^{(4)}$ constitutes a reducible representation of $Cl^+(\mathfrak{m}^4)$ (and of $\text{Spin}^\dagger(\mathfrak{m}^4)$). In fact, the volume element of \mathfrak{m}^4 (for a given orientation) induces an involutive automorphism χ of $S_{\mathbb{C}}^{(4)}$ (the *chirality operator*). Setting $(S_{\mathbb{C}}^{(4)})_+ = \text{Ker}(\chi - \text{Id})$ and $(S_{\mathbb{C}}^{(4)})_- = \text{Ker}(\chi + \text{Id})$, we have : $S_{\mathbb{C}}^{(4)} = (S_{\mathbb{C}}^{(4)})_+ \oplus (S_{\mathbb{C}}^{(4)})_-$, and $(S_{\mathbb{C}}^{(4)})_+$, $(S_{\mathbb{C}}^{(4)})_-$ are inequivalent irreducible two-dimensional complex representations of $\text{Spin}^\dagger(\mathfrak{m}^4)$. Changing the orientation of \mathfrak{m}^4 switches $(S_{\mathbb{C}}^{(4)})_+$ and $(S_{\mathbb{C}}^{(4)})_-$.

On the other hand, $S_{\mathbb{C}}^{(4)}$ is a representation of real type, in other terms it admits a $\text{Spin}^\dagger(\mathfrak{m}^4)$ -invariant conjugation $\sigma^{(4)}$. Then, $S^{(4)} = \text{Inv}(\sigma^{(4)})$ is an irreducible real representation of $\text{Spin}^\dagger(\mathfrak{m}^4)$, and $\dim_{\mathbb{R}} S^{(4)} = 4$.

Note that $S^{(4)}$ is an irreducible real representation of *complex type* : it can be endowed either with the complex structure of $(S_{\mathbb{C}}^{(4)})_+$ (via the isomorphism

$\frac{1}{2}(\text{Id} + \chi) : S^{(4)} \longrightarrow (S_{\mathbb{C}}^{(4)})_+$, or with the complex structure of $(S_{\mathbb{C}}^{(4)})_-$ (via the isomorphism $\frac{1}{2}(\text{Id} - \chi) : S^{(4)} \longrightarrow (S_{\mathbb{C}}^{(4)})_-$). Of course, the chiral projectors $\frac{1}{2}(\text{Id} \pm \chi)$ are $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant, and the complex representations obtained on $S^{(4)}$ are inequivalent.

Finally, we denote by C the algebra of $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant endomorphisms of $S^{(4)}$. It is known (cf. [BD] for example) that C is a field, isomorphic to \mathbb{R} , \mathbb{C} or \mathbb{H} . Here, since $S^{(4)}$ is an irreducible real representation of complex type, C is isomorphic to the field \mathbb{C} of complex numbers. Each of the two complex structures on $S^{(4)}$ induces an isomorphism from C to \mathbb{C} . Independently of this choice, $S^{(4)}$ is clearly a left C -module, in a canonical way (via evaluation).

Let $S^{(32)}$ be an irreducible real representation of $Cl(\mathfrak{m}^{11})$, so that $\dim_{\mathbb{R}} S^{(32)} = 32$. As a representation of $\text{Spin}^\dagger(\mathfrak{m}^{11})$, we know that $S^{(32)}$ is an *irreducible* real representation of *real type*. Besides, we have a positive-definite $\text{Spin}^\dagger(\mathfrak{m}^{11})$ -equivariant symmetric bilinear map $\Gamma^{(11)} : S^{(32)} \times S^{(32)} \longrightarrow \mathfrak{m}^{11}$.

We want to consider $S^{(32)}$ as a representation of $\text{Spin}^\dagger(\mathfrak{m}^4) \subset \text{Spin}^\dagger(\mathfrak{m}^{11})$. Then, as a representation of $\text{Spin}^\dagger(\mathfrak{m}^4)$, we see that $S^{(32)}$ is a *reducible* real representation.

Let $W = \text{Hom}_{\mathbb{R}}^{\text{Spin}^\dagger(\mathfrak{m}^4)}(S^{(4)}, S^{(32)})$.

Proposition 11.2.1 .

1. *W has a canonical structure of right C-module (via composition).*
2. *The map $W \otimes_C S^{(4)} \longrightarrow S^{(32)}$ which sends $w \otimes s$ to $w(s)$ is a canonical isomorphism.*
3. *For each of the two choices of complex structure on $S^{(4)}$, we have a corresponding structure of complex vector space on W, and $\dim_{\mathbb{C}} W = 8$.*

Proof :

1. Immediate.
2. We know that $S^{(32)}$ is reducible as a representation of $\text{Spin}^\dagger(\mathfrak{m}^4)$, and we may write :

$$S^{(32)} = \bigoplus_{i=1}^8 S_i$$
where each S_i is a real irreducible representation of $\text{Spin}^\dagger(\mathfrak{m}^4)$, equivalent to $S^{(4)}$. For each $i \in \{1, \dots, 8\}$, let $w_i : S^{(4)} \longrightarrow S^{(32)}$ inducing a $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant isomorphism from $S^{(4)}$ to S_i . On the other hand, let $p_i : S^{(32)} \longrightarrow S^{(32)}$ the projector on S_i . Then, for every $w \in W$, the linear map $p_i \circ w : S^{(4)} \longrightarrow S^{(32)} \longrightarrow S^{(32)}$ induces a $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant linear map from $S^{(4)}$ to S_i . By Schur's lemma, there exists a unique $c_i \in C$ such that $p_i \circ w = w_i c_i$. Therefore, $w = \text{Id} \circ w = (\sum_{i=1}^8 p_i) \circ w = \sum_{i=1}^8 (p_i \circ w) = \sum_{i=1}^8 w_i c_i$. Thus, $(w_i)_{1 \leq i \leq 8}$ is a basis of the C -module W and $\dim_C W = 8$. Now since $W \otimes_C S^{(4)} = \bigoplus_{i=1}^8 (w_i C) \otimes_C S^{(4)}$, we have $\dim_{\mathbb{R}}(W \otimes_C S^{(4)}) = 8 \dim_{\mathbb{R}}(S^{(4)}) = \dim_{\mathbb{R}}(S^{(32)})$. On the other hand, each element of $S^{(32)}$ is the sum of elements $w_i(s_i) \in S_i$, and each $w_i(s_i)$ is the image of $w_i \otimes s_i$. Therefore the map in question is surjective. Hence it is an isomorphism.

3. Let J_+ (resp. J_-) be the complex structure on $S^{(4)}$ obtained when taking the pull-back by $\frac{1}{2}(\text{Id}+\chi)$ (resp. $\frac{1}{2}(\text{Id}-\chi)$) of the complex structure of $(S_{\mathbb{C}}^{(4)})_+$ (resp. $(S_{\mathbb{C}}^{(4)})_-$). Then $J_- = -J_+$. Setting $\tilde{J}_+(w) = w \circ J_+$ and $\tilde{J}_-(w) = w \circ J_-$ for every $w \in W$, we obtain two conjugated (*i.e.* $\tilde{J}_- = -\tilde{J}_+$) complex structures on W . On the other hand, setting $\hat{J}_+(c) = J_+ \circ c$ and $\hat{J}_-(c) = J_- \circ c$ for every $c \in C$, we obtain two conjugate complex structures on C , and hence two isomorphisms from \mathbb{C} to C , which proves $\dim_{\mathbb{C}} W = 8$ when W is equipped with either \tilde{J}_+ or \tilde{J}_- . \square

Thus, we may choose any of the two complex structures on W , which will be denoted by W_+ (resp. W_-) if we made the choice corresponding to $(S_{\mathbb{C}}^{(4)})_+$ (resp. $(S_{\mathbb{C}}^{(4)})_-$). When physicists speak about "chiral SU(8)", they have in mind this possibility of choosing two complex structures on W , depending on the two possible chiralities of four-dimensional spinors. When the orientation of \mathfrak{m}^4 is reversed, the chiralities are exchanged, and so are the complex structures on W .

We set $\Gamma^{(4)} = p^{(4)} \circ \Gamma^{(11)}$, where $p^{(4)}$ is the canonical projection $\mathfrak{m}^{11} \longrightarrow \mathfrak{m}^4$.

Proposition 11.2.2 .

1. *There is a natural irreducible complex representation of $\text{Spin}(\mathfrak{m}^7)$ on W_+ (resp. W_-).*
2. *The irreducible complex representations W_+ and W_- of $\text{Spin}(\mathfrak{m}^7)$ are of real type, and we have : $W_+ \simeq W_-$ (as $\text{Spin}(\mathfrak{m}^7)$ -modules).*
3. $S^{(32)} \simeq W_+ \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_+ \simeq W_- \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_-$
4. *There is bijection between the set of positive-definite $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant symmetric bilinear maps $\Gamma^{(4)} : S^{(32)} \times S^{(32)} \longrightarrow \mathfrak{m}^4$, and the set of hermitian products on W_+ (resp. W_-).*

Thus, the positive-definite $\text{Spin}^\dagger(\mathfrak{m}^4)$ -equivariant symmetric bilinear map $\Gamma^{(4)} : S^{(32)} \times S^{(32)} \longrightarrow \mathfrak{m}^4$ provides W_+ and W_- with a natural hermitian structure.

Notice that since the irreducible complex representations W_+ and W_- of $\text{Spin}(\mathfrak{m}^7)$ are of real type, choosing a $\text{Spin}(\mathfrak{m}^7)$ -equivariant conjugation $\sigma_+^{(7)}$ on W_+ (resp. $\sigma_-^{(7)}$ on W_-) allows us to define $(W_{\mathbb{R}})_+ = \text{Inv}(\sigma_+^{(7)})$ and $(W_{\mathbb{R}})_- = \text{Inv}(\sigma_-^{(7)})$. The irreducible real representations $(W_{\mathbb{R}})_+$ and $(W_{\mathbb{R}})_-$ of $\text{Spin}(\mathfrak{m}^7)$ are of real type, and we have : $(W_{\mathbb{R}})_+ \simeq (W_{\mathbb{R}})_-$ (as $\text{Spin}(\mathfrak{m}^7)$ -modules).

We may also consider $S^{(32)}$ as a representation of $\text{Spin}(\mathfrak{m}^7) \subset \text{Spin}^\dagger(\mathfrak{m}^{11})$. Then, as a representation of $\text{Spin}(\mathfrak{m}^7)$, we see that $S^{(32)}$ is a *reducible* real representation. We set $\Gamma^{(7)} = p^{(7)} \circ \Gamma^{(11)}$, where $p^{(7)}$ is the canonical projection $\mathfrak{m}^{11} \longrightarrow \mathfrak{m}^7$.

Let \mathfrak{m}^8 be a Euclidian vector space, and $S^7(m^{-1})$ the sphere of center 0 and radius m^{-1} in \mathfrak{m}^8 . Let $y_0 \in S^7(m^{-1})$, and H the stabilizer of y_0 under the action of $\text{Spin}(\mathfrak{m}^8)$ on $S^7(m^{-1})$. Then we have a natural representation of H on the hyperplane $\mathfrak{m}^7 := (\mathbb{R}y_0)^\perp$ of \mathfrak{m}^8 , and $H \simeq \text{Spin}(\mathfrak{m}^7)$.

Consider now the irreducible complex representations W_+ and W_- of $\text{Spin}(\mathfrak{m}^7)$ defined above. Then W_+ and W_- constitute naturally two irreducible complex representations of

$\text{Spin}(\mathfrak{m}^8)$. However, as $\text{Spin}(\mathfrak{m}^8)$ -modules, W_+ and W_- are inequivalent.

When considered as representations of $\text{Spin}(\mathfrak{m}^8)$, the irreducible complex representations W_+ and W_- are also of real type. Therefore, choosing a $\text{Spin}(\mathfrak{m}^8)$ -equivariant conjugation $\sigma_+^{(8)}$ on W_+ (resp. $\sigma_-^{(8)}$ on W_-) allows us to define $(W_{\mathbb{R}})_+ = \text{Inv}(\sigma_+^{(8)})$ and $(W_{\mathbb{R}})_- = \text{Inv}(\sigma_-^{(8)})$. The irreducible real representations $(W_{\mathbb{R}})_+$ and $(W_{\mathbb{R}})_-$ of $\text{Spin}(\mathfrak{m}^8)$ are of real type, and they are inequivalent (as $\text{Spin}(\mathfrak{m}^8)$ -modules).

Let us recall a remarkable fact about the following three inequivalent representations of $\text{Spin}(\mathfrak{m}^8)$: $(W_{\mathbb{R}})_+$, $(W_{\mathbb{R}})_-$ and \mathfrak{m}^8 . We already know that we have a natural isomorphism : $Cl(\mathfrak{m}^8) \longrightarrow \text{End}((W_{\mathbb{R}})_+ \oplus (W_{\mathbb{R}})_-)$. One can show the existence of a natural Euclidian structure on $(W_{\mathbb{R}})_+$ (resp. on $(W_{\mathbb{R}})_-$) such that we have a natural isomorphism : $Cl(W_{\mathbb{R}})_- \longrightarrow \text{End}(\mathfrak{m}^8 \oplus (W_{\mathbb{R}})_+)$, and a natural isomorphism : $Cl((W_{\mathbb{R}})_+) \longrightarrow \text{End}((W_{\mathbb{R}})_- \oplus \mathfrak{m}^8)$. Thus, the roles of $(W_{\mathbb{R}})_+$, $(W_{\mathbb{R}})_-$ and \mathfrak{m}^8 can be interchanged. This result from Elie Cartan is known as *triality*. It will play an important role when we will identify the scalar fields resulting from the compactification of eleven-dimensional supergravity.

Consider the representations of $\text{Spin}^\dagger(\mathfrak{m}^4) \times \text{Spin}(\mathfrak{m}^7)$ on $S^{(32)}$, on $(S_{\mathbb{C}}^{(4)})_+$ and $(S_{\mathbb{C}}^{(4)})_-$ ($\text{Spin}(\mathfrak{m}^7)$ acting trivially), on W_+ and W_- ($\text{Spin}^\dagger(\mathfrak{m}^4)$ acting trivially), and finally on $(W_{\mathbb{R}})_+$ and $(W_{\mathbb{R}})_-$ ($\text{Spin}^\dagger(\mathfrak{m}^4)$ acting trivially).

All these representations give rise to spinor bundles over \hat{M} , associated to the principal bundle \hat{P} . In particular, we have :

$$\hat{\mathbb{S}}^{(32)} \simeq \hat{\mathbb{W}}_+ \otimes_{\mathbb{C}} (\hat{\mathbb{S}}_{\mathbb{C}}^{(4)})_+ \simeq \hat{\mathbb{W}}_- \otimes_{\mathbb{C}} (\hat{\mathbb{S}}_{\mathbb{C}}^{(4)})_-$$

A supergravity modeled on

$$((\mathfrak{spin}(3,2) \times \mathfrak{spin}(8), \mathfrak{spin}(3,1) \times \mathfrak{spin}(7)), \text{Spin}^\dagger(3,1) \times \text{Spin}(7), \text{Ad})$$

should clearly admit as solution the canonical Cartan geometry associated to the Klein geometry $(\text{Spin}^\dagger(3,2) \times \text{Spin}(8), \text{Spin}^\dagger(3,1) \times \text{Spin}(7))$ with space $\text{AdS}_4(\mu^{-1}) \times S^7(m^{-1})$ (for some μ and m), together with the Rarita-Schwinger spinor field $\dot{\psi} = 0$, and specific value(s) for the additional field(s).

It is well-known that Poincaré eleven-dimensional supergravity admits such a solution. Indeed, removing first the spinorial matter by setting $\dot{\psi} = 0$, we see that $\hat{\omega}_{susy} = \hat{\omega} = \hat{\omega}_{LC}$ and $\hat{F}_{susy} = \hat{F}$, so that the field equations become :

$$\left\{ \begin{array}{l} Ric^{\hat{e}, \hat{\omega}_{susy}} - \frac{1}{2} R_{scal}^{\hat{e}, \hat{\omega}_{susy}} \hat{\eta} = \frac{1}{3} (\hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \text{Id}) \otimes \hat{F}_{susy} \circ (\hat{e}^{-1} \otimes \bigotimes^3 \hat{g}^{-1}) \\ \quad - \frac{1}{8} * (\hat{F}_{susy} \wedge * \hat{F}_{susy}) \hat{\eta}) \\ d * \hat{F} = - \hat{F} \wedge \hat{F} \end{array} \right.$$

Imposing next the following ansatz (called the *Freund-Rubin ansatz*)

$$\left\{ \begin{array}{lcl} \hat{M} & = & \mathring{M}^4 \times \mathring{M}^7 \\ \hat{g}(x, y) & = & \mathring{g}^{(4)}(x) \times \mathring{g}^{(7)}(y) \\ \hat{\psi}(x, y) & = & 0 \\ \hat{F}(x, y) & = & \mathring{F}^{(4)}(x) \times \mathring{F}^{(7)}(y) \\ \mathring{F}^{(4)}(x) & = & 3m \operatorname{vol}_{\mathring{g}^{(4)}} \\ \mathring{F}^{(7)}(y) & = & 0 \end{array} \right.$$

the field equations become :

$$\left\{ \begin{array}{lcl} Ric^{\hat{e}^{(4)}, \hat{\omega}_{LC}^{(4)}} & = & -12m^2 \eta^{(4)} \\ Ric^{\hat{e}^{(7)}, \hat{\omega}_{LC}^{(7)}} & = & 6m^2 \eta^{(7)} \end{array} \right.$$

It is clear now that the canonical Cartan geometry associated to the Klein geometry $(\operatorname{Spin}^\dagger(3, 2) \times \operatorname{Spin}(8), \operatorname{Spin}^\dagger(3, 1) \times \operatorname{Spin}(7))$ with space $\operatorname{AdS}_4((2m)^{-1}) \times S^7(m^{-1})$ is a solution of the above equations. From now on, the notation AdS_4 (resp. S^7) will refer to $\operatorname{AdS}_4((2m)^{-1})$ (resp. $S^7(m^{-1})$).

It is interesting to evaluate the operator

$$\mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\psi} = \mathcal{D}^{\hat{\omega}_{susy}} \hat{\psi} - \frac{i}{144} (\hat{\gamma}_{(5)}^*(\hat{e}) - 8(\hat{\gamma}_{(3)}^*(\hat{e}) \otimes \hat{g}^{-1})) \circ (\hat{g} \otimes \wedge^4 \operatorname{Id}) \wedge \hat{F}_{susy} \wedge \hat{\psi}$$

in the background given by the above Klein geometry. One can check that $\mathcal{D}^{\hat{\omega}_{susy}, \hat{A}}$ splits into the two following four-dimensional and seven-dimensional parts (which are nothing but the covariant derivatives generalized in the Cartan sense!) :

$$\mathcal{D}^{\hat{\theta}^{(4)}, \alpha^{(4)}} = \mathcal{D}^{\hat{\omega}^{(4)}} + im \gamma^{(4)}(\hat{e}^{(4)})$$

$$\mathcal{D}^{\hat{\theta}^{(7)}, \alpha^{(7)}} = \mathcal{D}^{\hat{\omega}^{(7)}} - \frac{1}{2}m \gamma^{(7)}(\hat{e}^{(7)})$$

where : $\alpha^{(4)} = -im$, $\alpha^{(7)} = \frac{1}{2}m$, and :

$$\gamma^{(4)}(\hat{e}^{(4)}) = \gamma^{(4)} \circ \hat{e}^{(4)} : T\operatorname{AdS}_4 \longrightarrow \mathring{\mathbb{V}}^4 \longrightarrow \operatorname{End}(\mathring{\mathbb{S}}^{(4)})$$

$$\gamma^{(7)}(\hat{e}^{(7)}) = \gamma^{(7)} \circ \hat{e}^{(7)} : T\operatorname{S}^7 \longrightarrow \mathring{\mathbb{V}}^7 \longrightarrow \operatorname{End}(\mathring{\mathbb{W}}_{\mathbb{R}})$$

This shows that the 3-form \hat{A} is intimately related to the endomorphisms of spinors. This comforts our wish to consider \hat{A} as the spin-spin part of the superelfbein. In the next section, we mention quickly a possible starting point for superspace compactification of eleven-dimensional supergravity.

11.3 On superspace compactification of eleven-dimensional supergravity

It is possible to develop a theory of Cartan supergeometries by proceeding in analogy with the theory of Cartan geometries. A possible formulation of eleven-dimensional superspace

supergravity, suitable to describe its compactification to four dimensions, is the following one :

Start with the model supergeometry :

$$((\mathfrak{g}, \mathfrak{h}), H, \text{Ad}) = ((\mathfrak{osp}(4|8), \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7), \text{Ad})$$

where $\mathfrak{osp}(4|8)$ is the following Lie superalgebra :

$$\mathfrak{osp}(4|8) = (\mathfrak{spin}(3, 2) \times \mathfrak{spin}(8)) \times (W_+ \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_+)$$

The above model geometry is reductive : from reductive decompositions

$$\mathfrak{spin}(3, 2) = \mathfrak{spin}(3, 1) \oplus \mathfrak{m}^4 \quad \text{and} \quad \mathfrak{spin}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}^7,$$

by setting $\mathfrak{m}^{11|32} = (\mathfrak{m}^4 \times \mathfrak{m}^7) \times (W_+ \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_+)$ we obtain a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}^{11|32}$:

$$\mathfrak{osp}(4|8) = (\mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)) \oplus ((\mathfrak{m}^4 \times \mathfrak{m}^7) \times (W_+ \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_+))$$

We denote by $\mathfrak{osp}_L(4|8)$ the Lie supermodule associated to $\mathfrak{osp}(4|8)$, that is :

$$\mathfrak{osp}_L(4|8) = ((\mathfrak{spin}(3, 2) \times \mathfrak{spin}(8)) \otimes L^{ev}) \times (W_+ \otimes_{\mathbb{C}} (S_{\mathbb{C}}^{(4)})_+ \otimes L^{od})$$

The exponential map sends $\mathfrak{osp}_L(4|8)$ to the Lie supergroup $\text{Osp}(4|8)$.

The dynamical variable for eleven-dimensional superspace supergravity is a Cartan supergeometry $(\hat{\mathbf{P}}, \hat{\theta})$ on a supermanifold $M^{11|32}$, modeled on

$$((\mathfrak{osp}(4|8), \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7)), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7), \text{Ad}).$$

Let $\hat{\omega}$ (resp. $\hat{\mathbf{e}}$) be the superconnection (resp. the soldering superform) of the reductive Cartan supergeometry $(\hat{\mathbf{P}}, \hat{\theta})$. Then $\hat{\omega} \in \Lambda^1_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}(\hat{\mathbf{P}}, \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7))$ is the **spin superconnection** and $\hat{\mathbf{e}} \in \bar{\Lambda}^1_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}(\hat{\mathbf{P}}, \mathfrak{m}^{11|32})$ is the **superelfbein**.

Let $\mathbb{V}^{11|32} = \hat{\mathbf{P}} \times_{Ad} \mathfrak{m}^{11|32}$. Since $\bar{\Lambda}^1_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}(\hat{\mathbf{P}}, \mathfrak{m}^{11|32}) \simeq \Lambda^1(M^{11|32}, \mathbb{V}^{11|32})$, we may view $\hat{\mathbf{e}}$ as a one-form on $M^{11|32}$ with values in the super-vector bundle $\mathbb{V}^{11|32}$, and $\hat{\mathbf{e}} : TM^{11|32} \longrightarrow \mathbb{V}^{11|32}$ is an isomorphism of super-vector bundles.

Then one can define the **supercurvature** $\hat{\Omega} \in \bar{\Lambda}^2_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}(\hat{\mathbf{P}}, \mathfrak{spin}(3, 1) \times \mathfrak{spin}(7))$ and the **supertorsion** $\hat{\mathbf{T}} \in \bar{\Lambda}^2_{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}(\hat{\mathbf{P}}, \mathfrak{m}^{11|32})$, and write the appropriate constraints for the supertorsion (by requiring first-order flatness, eventually in a suitable extension of the structural group). Finally after using the constraints together with Bianchi identities, one should be able to write superfield equations.

The superfield equations should clearly admit as solution the canonical Cartan supergeometry associated to the following Klein supergeometry : $(\text{Osp}(4|8), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7))$ with superspace :

$$\overset{\circ}{M}{}^{11|32} = \frac{\text{Osp}(4|8)}{\text{Spin}^\dagger(3, 1) \times \text{Spin}(7)}$$

This supermanifold is nothing but the supermanifold associated to the vector bundle $\hat{\mathbb{W}}_+ \otimes_{\mathbb{C}} (\hat{\mathbb{S}}_{\mathbb{C}}^{(4)})_+$ over $\text{AdS}_4 \times S^7$ (in the sense of definition 8.2.1 in the third part of the thesis).

$\overset{\circ}{M}{}^{11|32}$ can serve as a background supergeometry for superspace compactification.

11.4 Dimensional reduction of the elfbein

We deal now with the dimensional reduction of the elfbein. We fix the eleven-dimensional manifold to be $\hat{M} = \text{AdS}_4 \times S^7$. Let $\pi^{(4)} : \hat{M} \rightarrow \text{AdS}_4$ and $\pi^{(7)} : \hat{M} \rightarrow S^7$ be the canonical projections. Starting with the canonical vector bundles $\mathring{\mathbb{V}}^4$ over AdS_4 and $\mathring{\mathbb{V}}^7$ over S^7 , we may define the two induced vector bundles $\hat{\mathbb{V}}^4 = (\pi^{(4)})^{-1}\mathring{\mathbb{V}}^4$ and $\hat{\mathbb{V}}^7 = (\pi^{(7)})^{-1}\mathring{\mathbb{V}}^7$ over \hat{M} . If we set $\hat{\mathbb{V}} = \hat{\mathbb{V}}^4 \oplus \hat{\mathbb{V}}^7$, then $\hat{\mathbb{V}}$ is nothing but the canonical vector bundle of the Klein geometry $(\text{Spin}^\dagger(3, 2) \times \text{Spin}(8), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7))$. In fact, the splitting $\hat{\mathbb{V}} = \hat{\mathbb{V}}^4 \oplus \hat{\mathbb{V}}^7$ is exactly the one provided by the $(\text{Spin}^\dagger(3, 1) \times \text{Spin}(7))$ -structure on $\hat{\mathbb{V}}$.

We consider an elfbein $\hat{e} \in \Lambda^1(\text{AdS}_4 \times S^7, \hat{\mathbb{V}}^4 \oplus \hat{\mathbb{V}}^7)$, viewed as a deviation from the canonical elfbein \mathring{e} of the Klein geometry $(\text{Spin}^\dagger(3, 2) \times \text{Spin}(8), \text{Spin}^\dagger(3, 1) \times \text{Spin}(7))$.

For each $(x, y) \in \text{AdS}_4 \times S^7$, we have thus an isomorphism :

$$\hat{e}_{(x,y)} : T_x \text{AdS}_4 \times T_y S^7 \longrightarrow \hat{\mathbb{V}}_{(x,y)}^4 \oplus \hat{\mathbb{V}}_{(x,y)}^7$$

We may assume that $\hat{e}_{(x,y)}(\{0_x\} \times T_y S^7) \subset \hat{\mathbb{V}}_{(x,y)}^7$.

For each $x \in \text{AdS}_4$, let $\iota_x : S^7 \longrightarrow \hat{M}$ be the injection defined by $\iota_x(y) = (x, y)$. Then $T_y \iota_x : T_y S^7 \longrightarrow T_x \text{AdS}_4 \times T_y S^7$ is defined by $T_y \iota_x(\xi) = (0_x, \xi)$. On the other hand, let $\hat{\pi}^{(7)} : \hat{\mathbb{V}}^7 \longrightarrow \mathring{\mathbb{V}}^7$ be the canonical projection (that lifts $\pi^{(7)} : \hat{M} \rightarrow S^7$).

We are now ready to define, for each $x \in \text{AdS}_4$, a *siebenbein* $e^{(7)}(x) \in \Lambda^1(S^7, \mathring{\mathbb{V}}^7)$. For each $y \in S^7$, set :

$$e^{(7)}(x)_y = \hat{\pi}^{(7)} \circ \hat{e}_{(x,y)} \circ T_y \iota_x$$

so that $e^{(7)}(x)_y : T_y S^7 \longrightarrow T_x \text{AdS}_4 \times T_y S^7 \longrightarrow \hat{\mathbb{V}}_{(x,y)}^7 \longrightarrow \mathring{\mathbb{V}}_y^7$.

Recall that the irreducible real representations $(W_{\mathbb{R}})_+$ and $(W_{\mathbb{R}})_-$ of $\text{Spin}(\mathfrak{m}^7)$ are equivalent. We denote by $W_{\mathbb{R}}$ one of them, and by $\mathring{W}_{\mathbb{R}}$ the associated canonical spinor bundle over S^7 .

It is well-known that there exists a $\text{Spin}(\mathfrak{m}^7)$ -equivariant morphism $\Gamma : \bigwedge^2(W_{\mathbb{R}}) \longrightarrow \mathfrak{m}^7$. We can therefore associate to the siebenbein $e^{(7)}(x)$ a cross-section $e_{\text{spin}}^{(7)}(x) \in \Gamma(T^*S^7 \otimes \bigwedge^2(\mathring{W}_{\mathbb{R}}^*))$ in the following way : for each $y \in S^7$,

$$e_{\text{spin}}^{(7)}(x)_y = {}^t \Gamma_y \circ \delta_y^{(7)} \circ e^{(7)}(x)_y : T_y S^7 \longrightarrow \mathring{\mathbb{V}}_y^7 \longrightarrow (\mathring{\mathbb{V}}_y^7)^* \longrightarrow \bigwedge^2(\mathring{W}_{\mathbb{R}}^*)_y$$

Notice that the field $e_{\text{spin}}^{(7)}$ on AdS_4 takes its values in the space $\Gamma(T^*S^7 \otimes \bigwedge^2(\mathring{W}_{\mathbb{R}}^*))$ (the space of cross-sections of the vector bundle $T^*S^7 \otimes \bigwedge^2(\mathring{W}_{\mathbb{R}}^*)$ over S^7), which is an infinite-dimensional vector space.

If we want to describe the low-energy sector, we do not need this infinite number of degrees of freedom. It is sufficient to retain only a finite number of degrees of freedom, that is, it is sufficient to retain only the projection of $e_{\text{spin}}^{(7)}(x)$ to a finite-dimensional vector subspace of $\Gamma(T^*S^7 \otimes \bigwedge^2(\mathring{W}_{\mathbb{R}}^*))$.

The harmonic analysis of cross-sections of vector bundles over compact homogenous spaces (namely the seven-spheres) provides us with a finite-dimensional vector subspace

of $\Gamma(T^*S^7 \otimes \Lambda^2(\dot{\mathbb{W}}_{\mathbb{R}}^*))$, that we shall call *the fundamental subspace* and denote by $\Gamma_{fond}(T^*S^7 \otimes \Lambda^2(\dot{\mathbb{W}}_{\mathbb{R}}^*))$. Its elements are the *fundamental cross-sections*; in this case, these are the cross-sections of the form $\tilde{K}^* \otimes (\tilde{\eta}_1^* \wedge \tilde{\eta}_2^*)$, where \tilde{K}^* is associated to a fundamental (Killing) vector field on S^7 and $\tilde{\eta}_1^*, \tilde{\eta}_2^*$ are fundamental (Killing) spinor fields on S^7 .

In other terms, using the map $\mathfrak{spin}(\mathfrak{m}^8) \longrightarrow \Gamma(TS^7)$ which associates to each $K \in \mathfrak{spin}(\mathfrak{m}^8)$ the fundamental vector field $\tilde{K} \in \Gamma(TS^7)$, as well as the map $W_{\mathbb{R}} \longrightarrow \Gamma(\dot{\mathbb{W}}_{\mathbb{R}})$ -provided by a parallelism on S^7 - which associates to each $\eta \in W_{\mathbb{R}}$ the fundamental spinor field $\tilde{\eta} \in \Gamma(\dot{\mathbb{W}}_{\mathbb{R}})$,

we obtain the isomorphism : $\Gamma_{fond}(T^*S^7 \otimes \Lambda^2(\dot{\mathbb{W}}_{\mathbb{R}}^*)) \simeq \Lambda^2(\mathfrak{m}^8)^* \otimes \Lambda^2(W_{\mathbb{R}}^*)$ (since $\mathfrak{spin}(\mathfrak{m}^8) \simeq \Lambda^2(\mathfrak{m}^8)$).

From Fourier analysis, we have therefore a projector

$$\text{pr} : \Gamma(T^*S^7 \otimes \Lambda^2(\dot{\mathbb{W}}_{\mathbb{R}}^*)) \longrightarrow \text{Hom}(\Lambda^2(\mathfrak{m}^8), \Lambda^2(W_{\mathbb{R}}^*))$$

so we end up with a scalar field $w \in C^\infty(\text{AdS}_4, \text{Hom}(\Lambda^2(\mathfrak{m}^8), \Lambda^2(W_{\mathbb{R}}^*)))$, defined by :

$$w(x) = \text{pr}(e_{\text{spin}}^{(7)}(x))$$

Owing to the triality property of the real irreducible representations of $\text{Spin}(8)$, we have a natural isomorphism $t : \Lambda^2(W_{\mathbb{R}}^*) \longrightarrow \Lambda^2(\mathfrak{m}^8)$. Therefore, $w(x) \circ t \in \text{End}(\Lambda^2(W_{\mathbb{R}}^*))$. In fact, one can show that $w(x) \circ t \in \text{GL}(\Lambda^2(W_{\mathbb{R}}^*))$. Thus, $w(x) \circ t$ is an automorphism of a 28-dimensional real vector space.

For reasons related to duality and supersymmetry invariance, we would like at this point to have the double of the number of degrees of freedom we actually have. In other terms, we would like to have for each $x \in \text{AdS}_4$ an automorphism of a 56-dimensional real vector space. In the toroidal compactification considered by Cremmer and Julia in [CrJ], the missing degrees of freedom for the scalar fields arise from the 3-form \hat{A} , which did not play a role here. In fact, de Wit and Nicolai have the same problem in [dWN1] : their e_{AB}^m are real and therefore they do not carry a representation of $\text{SU}(8)$. They solve this dilemma by postulating extra gauge degrees of freedom, so that the reality of the fields is only true for a particular gauge choice. Therefore, we may replace $W_{\mathbb{R}}^*$ by W^* , so that the field $w(x) \circ t$ is replaced by an automorphism $\Phi(x) \in \text{GL}(F)$, where $F = \{(\xi, \hat{\theta}(\xi)) \in \Lambda^2(W) \oplus \Lambda^2(W^*) ; \xi \in \Lambda^2(W)\}$ (here $\hat{\theta}$ denotes here the conjugate-linear isomorphism $\Lambda^2(W) \longrightarrow \Lambda^2(W^*)$ provided by the hermitian structure on W).

In fact, Cremmer and Julia, followed by de Wit and Nicolai have shown that we obtain a complete and coherent theory by imposing $\Phi(x) \in E_7 \subset \text{Sp}(F) \subset \text{GL}(F)$, and then Φ leads to a map $\phi \in C^\infty(\text{AdS}_4, E_7/\text{SU}(W))$.

11.5 Dimensional reduction of the Rarita-Schwinger spinor field

Now we turn our attention to the dimensional reduction of the Rarita-Schwinger spinor field. Starting with the canonical spinor bundles $\mathring{\mathbb{S}}^{(4)}$ over AdS_4 and $\mathring{\mathbb{W}}_{\mathbb{R}}$ over S^7 , we may define the two induced spinor bundles $\hat{\mathbb{S}}^{(4)} = (\pi^{(4)})^{-1}\mathring{\mathbb{S}}^{(4)}$ and $\hat{\mathbb{W}}_{\mathbb{R}} = (\pi^{(7)})^{-1}\mathring{\mathbb{W}}_{\mathbb{R}}$ over $\hat{M} = \text{AdS}_4 \times S^7$. We denote by $\hat{\pi}^{(4)} : \hat{\mathbb{S}}^{(4)} \longrightarrow \mathring{\mathbb{S}}^{(4)}$ (resp. $\hat{\pi}^{(7)} : \hat{\mathbb{W}}_{\mathbb{R}} \longrightarrow \mathring{\mathbb{W}}_{\mathbb{R}}$) the canonical projection that lifts $\pi^{(4)} : \hat{M} \longrightarrow \text{AdS}^4$ (resp. $\pi^{(7)} : \hat{M} \longrightarrow S^7$).

We consider a Rarita-Schwinger spinor field

$$\hat{\psi} \in \Lambda^1(\text{AdS}_4 \times S^7, \hat{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} \hat{\mathbb{S}}^{(4)})$$

For each $(x, y) \in \text{AdS}_4 \times S^7$, we have thus a linear map :

$$\hat{\psi}_{(x,y)} : T_x \text{AdS}_4 \times T_y S^7 \longrightarrow (\hat{\mathbb{W}}_{\mathbb{R}})_{(x,y)} \otimes_{\mathbb{R}} (\hat{\mathbb{S}}^{(4)})_{(x,y)}$$

Set :

$$(\psi_x^{(7)})_y = (\hat{\pi}^{(7)} \otimes \hat{\pi}^{(4)}) \circ \hat{\psi}_{(x,y)} \circ T_y \iota_x \circ e^{(7)}(x)_y^{(-1)}$$

so that : $(\psi_x^{(7)})_y : \mathring{\mathbb{V}}_y^7 \longrightarrow T_y S^7 \longrightarrow T_x \text{AdS}_4 \times T_y S^7 \longrightarrow (\hat{\mathbb{W}}_{\mathbb{R}})_{(x,y)} \otimes_{\mathbb{R}} (\hat{\mathbb{S}}^{(4)})_{(x,y)}$
 $\longrightarrow (\mathring{\mathbb{W}}_{\mathbb{R}})_y \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x$

Thus, for each $x \in \text{AdS}_4$, we have a cross-section $\psi_x^{(7)}$ of the vector bundle $(\mathring{\mathbb{V}}^7)^* \otimes \mathring{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x$ over S^7 . On the other hand, there exists a $\text{Spin}(\mathfrak{m}^7)$ -equivariant morphism $\Gamma : \Lambda^2(W_{\mathbb{R}}^*) \longrightarrow \mathfrak{m}^7$. We can therefore associate to $\psi_x^{(7)}$ a cross-section $(\psi_{\text{spin}}^{(7)})_x$ (called the *trispinor*) of the vector bundle $\Lambda^3(\mathring{\mathbb{W}}_{\mathbb{R}}) \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x$ over S^7 , in the following way : set

$$(\psi_{\text{spin}}^{(7)})_x = {}^t\Gamma \wedge \psi_x^{(7)}$$

obtained by contraction and exterior product of the two following maps :

$${}^t\Gamma : (\mathring{\mathbb{V}}^7)^* \longrightarrow \Lambda^2(\mathring{\mathbb{W}}_{\mathbb{R}}) \quad \text{and} \quad \psi_x^{(7)} : \mathring{\mathbb{V}}^7 \longrightarrow \mathring{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x.$$

Let $p_+ : \Lambda^3(\mathring{\mathbb{W}}_{\mathbb{R}}) \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x \longrightarrow \Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$ be the chiral projector (associated to $\frac{1}{2}(\text{Id} + \chi)$).

Thus, we have $p_+ \circ (\psi_{\text{spin}}^{(7)})_x \in \Gamma(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x)$ (the space of cross-sections of the vector bundle $\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$ over S^7), which is an infinite-dimensional vector space.

If we want to describe the low-energy sector, we do not need this infinite number of degrees of freedom. It is sufficient to retain only a finite number of degrees of freedom, that is, it is sufficient to retain only the projection of $p_+ \circ (\psi_{\text{spin}}^{(7)})_x$ to a finite-dimensional vector subspace of $\Gamma(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x)$.

The harmonic analysis of cross-sections of vector bundles over compact homogenous spaces (namely the seven-spheres) provides us with a finite-dimensional vector subspace of $\Gamma(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x)$, that we shall call *the fundamental subspace* and denote by $\Gamma_{\text{fond}}(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x)$. Its elements are the *fundamental trispinors*; in this case, these are the trispinors of the form $(\tilde{\eta}_1 \wedge \tilde{\eta}_2 \wedge \tilde{\eta}_3) \otimes s_x$, where $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$ are fundamental

(Killing) spinor fields on S^7 , and $s_x \in ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$.

In other terms, using the map $W_+ \longrightarrow \Gamma(\mathring{\mathbb{W}}_+)$ -provided by a parallelism on S^7 - which associates to each $\eta \in W_+$ the fundamental spinor field $\tilde{\eta} \in \Gamma(\mathring{\mathbb{W}}_+)$,

we obtain the isomorphism : $\Gamma_{fond}(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x) \simeq \Lambda^3(W_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$

From Fourier analysis, we have therefore a projector

$$\text{pr} : \Gamma(\Lambda^3(\mathring{\mathbb{W}}_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x) \longrightarrow \Lambda^3(W_+) \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$$

so we end up with the four-dimensional trispinor $\chi \in \Gamma(\Lambda^3(W_+) \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$ (the space of cross-sections of the vector bundle $\Lambda^3(W_+) \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+$ over AdS_4), defined by :

$$\chi_x = \text{pr}(p_+ \circ (\psi_{spin}^{(7)})_x)$$

The dimensional reduction of the supersymmetry parameter $\hat{\epsilon} \in \Gamma(\hat{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} \hat{\mathbb{S}}^{(4)})$ is similar and simpler. For each $(x, y) \in \text{AdS}_4 \times S^7$, we have : $\hat{\epsilon}_{(x,y)} \in (\hat{\mathbb{W}}_{\mathbb{R}})_{(x,y)} \otimes_{\mathbb{R}} (\hat{\mathbb{S}}^{(4)})_{(x,y)}$. Set : $(\epsilon_x^{(7)})_y = (\hat{\pi}^{(7)} \otimes \hat{\pi}^{(4)})(\hat{\epsilon}_{(x,y)})$, so that : $(\epsilon_x^{(7)})_y \in (\mathring{\mathbb{W}}_{\mathbb{R}})_y \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x$. Thus, for each $x \in \text{AdS}_4$, $(\epsilon^{(7)})_x$ is a cross-section of the vector bundle $\mathring{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x$ over S^7 . Let $p_+ : \mathring{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} (\mathring{\mathbb{S}}^{(4)})_x \longrightarrow \mathring{\mathbb{W}}_+ \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$ be the chiral projector (associated to $\frac{1}{2}(\text{Id} + \chi)$). Then $p_+ \circ (\epsilon^{(7)})_x$ belongs to the infinite-dimensional vector space $\Gamma(\mathring{\mathbb{W}}_+ \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x)$ (the space of cross-sections of the vector bundle $\mathring{\mathbb{W}}_+ \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$ over S^7). Projecting to the finite-dimensional vector space $\Gamma_{fond}(\mathring{\mathbb{W}}_+ \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x) \simeq W_+ \otimes_{\mathbb{C}} ((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)_x$, we obtain the gauge supersymmetry parameter of the four-dimensional theory : $\epsilon \in \Gamma(W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$ (the space of cross-sections of the vector bundle $W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+$ over AdS^4).

We have a symplectic form $\varepsilon_+ : (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+ \times (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+ \longrightarrow C^\infty(\text{AdS}_4, \mathbb{C})$ coming from the linear symplectic form $\varepsilon_+ : (S_{\mathbb{C}}^{(4)})_+ \times (S_{\mathbb{C}}^{(4)})_+ \longrightarrow \mathbb{C}$. We set : $\epsilon^* = \varepsilon_+(\epsilon, .) \in \Gamma(W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+^*)$. We are now able to define : $\sigma = \epsilon^* \wedge \chi + *(\epsilon^* \wedge \chi) \in C^\infty(\text{AdS}_4, \Lambda_+^4(W_+))$, where $\epsilon^* \wedge \chi \in C^\infty(\text{AdS}_4, \Lambda^4(W_+))$ is the element obtained by performing the exterior product and spinor contraction of $\epsilon^* \in \Gamma(W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+^*)$ with $\chi \in \Gamma(\Lambda^3(W_+) \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$.

We have seen that the dimensional reduction of the elfbein \hat{e} leads to a scalar field $\Phi \in C^\infty(\text{AdS}_4, E_7)$. One can prove that the dimensional reduction of the gauge supersymmetry transformation on \hat{e} (given by $\delta_\epsilon \hat{e} = i \hat{\Gamma}(\hat{\epsilon}, \hat{\psi})$) leads to the following infinitesimal action of $\sigma = \epsilon^* \wedge \chi + *(\epsilon^* \wedge \chi)$ on Φ :

$$\delta\Phi(x) = \Phi(x) \circ \rho(0, \sigma(x))$$

where $\rho : \mathfrak{e}_7 \longrightarrow \mathfrak{gl}(F)$ is the fundamental representation of $\mathfrak{e}_7 = \mathfrak{su}(W_+) \oplus \Lambda_+^4(W_+)$ (with $\rho(0, \sigma(x))(\xi, \hat{\theta}(\xi)) = (\hat{*}(\hat{*}\sigma(x) \wedge \hat{\theta}(\xi)), \hat{*}(\sigma(x) \wedge \xi))$ for every $(\xi, \hat{\theta}(\xi)) \in F$).

Starting from the Rarita-Schwinger spinor field $\hat{\psi} \in \Lambda^1(\text{AdS}_4 \times S^7, \hat{\mathbb{W}}_{\mathbb{R}} \otimes_{\mathbb{R}} \hat{\mathbb{S}}^{(4)})$, we performed a dimensional reduction and obtained the trispinor $\chi \in \Gamma(\Lambda^3(W_+) \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$. Of course this is not the only field that one obtains from $\hat{\psi}$; there is also a Rarita-Schwinger spinor field $\psi \in \Lambda^1(\text{AdS}_4, W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$.

11.6 The potential of de Wit and Nicolai

Performing the dimensional reduction of the gauge supersymmetry transformation on $\hat{\psi}$ (given by $\delta_\epsilon \hat{\psi} = \mathcal{D}^{\hat{\omega}_{susy}, \hat{A}} \hat{\epsilon}$), de Wit and Nicolai obtained the gauge supersymmetry transformation on ψ and χ . They showed the existence of two tensor-valued functions : $A_1 \in C^\infty(\text{AdS}_4, \text{Sym}^2(W_+))$ and $A_2 \in C^\infty(\text{AdS}_4, (\bigwedge^3(W_+) \otimes (W_+)^*)_0)$ (where $(\bigwedge^3(W_+) \otimes (W_+)^*)_0$ is the traceless subspace of $\bigwedge^3(W_+) \otimes (W_+)^*$), such that :

$$\delta_\epsilon \psi = \dots + \frac{\sqrt{2}}{2} g A_1 \otimes \gamma(e)(\epsilon^*)$$

$$\delta_\epsilon \chi = \dots - g A_2 \otimes \epsilon$$

Here, $A_1 \otimes \gamma(e)(\epsilon^*)$ is the element of $\Lambda^1(\text{AdS}_4, W_+ \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$ defined by spinor and tensor contraction of the following maps :

$$\begin{aligned} A_1 &: \text{AdS}_4 \longrightarrow \text{Sym}^2(W_+) \\ \gamma(e) &: T\text{AdS}_4 \longrightarrow \text{End}((\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+) \\ \epsilon^* &\in \Gamma((W_+)^* \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+) \end{aligned}$$

and $A_2 \otimes \epsilon$ is the element of $\Gamma(\bigwedge^3(W_+) \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+)$ defined by tensor contraction of the following maps :

$$\begin{aligned} A_2 &: \text{AdS}_4 \longrightarrow (\bigwedge^3(W_+) \otimes (W_+)^*)_0 \\ \epsilon &\in \Gamma((W_+)^* \otimes_{\mathbb{C}} (\mathring{\mathbb{S}}_{\mathbb{C}}^{(4)})_+) \end{aligned}$$

g is a coupling constant.

The two tensor-valued functions A_1 and A_2 appear to be polynomials of the third-order in u and v , the components of the scalar field Φ ; de Wit and Nicolai give the following expressions (in index notations) for A_1 and A_2 (cf. [dWN]) :

$$\begin{aligned} A_1^{ij} &= \frac{4}{21} (u_{IJ}^{kj} + v^{kjIJ}) (u_{km}^{JK} u_{KI}^{im} - v_{kmJK} v^{imKI}) \\ A_{2i}^{jkl} &= -\frac{4}{3} (u_{IJ}^{[kl} + v^{[klIJ}) (u_{im}^{JK} u_{KI}^{j]m} - v_{imJK} v^{j]mKI}) \end{aligned}$$

where :

$$\Phi(x) = \begin{pmatrix} u_{ij}^{IJ}(x) & v_{ijKL}(x) \\ v^{klIJ}(x) & u_{KL}^{kl}(x) \end{pmatrix}$$

(Here, raising and lowering indices corresponds to complex conjugation.)

Finally, they find the following potential for the scalar field Φ :

$$\mathcal{V}(\Phi) = -g^2 \left\{ \frac{3}{4} |A_1|^2 - \frac{1}{24} |A_2|^2 \right\}$$

In terms of u and v , we see that this potential is a polynomial of the sixth-degree, defined on the space of scalar fields, which is in fact $E_7/\text{SU}(W_+)$.

Let us give some remarks about the relationship between the scalar field Φ , and the construction introduced in the second chapter.

- On any vector space, there exists a flat parabolic family of metrics for an embedding of the affine group in n dimensions into the linear group in $n + 1$ dimensions; this family corresponds to translations, and the exponential map on these matrices is an affine map. When we go to the riemannian symmetric space $\mathrm{SL}(n+1, \mathbb{R})/\mathrm{SO}(n)$ through the map $M \mapsto {}^t M M$, we keep affine coordinates, and the image is a non-totally geodesic flat submanifold $E_7/SU(8)$ which contains $\mathrm{SL}(8, \mathbb{R})/\mathrm{SO}(8)$, and therefore contains also a seven-dimensional flat parabolic family.
- In the approach of Cremmer and Julia of $N = 8$ supergravity, such a seven-dimensional family corresponds exactly to the seven scalar fields obtained by Hodge duality from the $A_{\mu\nu i}$ components of the 3-form in eleven dimensions (cf. formula 4.10 in [CrJ]). Let us stress on the fact that it is through these fields that Cremmer and Julia are able to transform the local $\mathrm{SO}(7)$ symmetry into an $\mathrm{SO}(8)$ symmetry.
- In the gauged theory of de Wit and Nicolai, the parabolic family of fields comes from metrics on $\Lambda^2(\mathbb{R}^8)$ that are induced from metrics on \mathbb{R}^8 , and therefore of $(\mathrm{SO}(8) \times \mathrm{SO}(8))$ -invariant metrics on $\mathrm{SO}(8) \times \mathrm{SO}(8)$ (popular ansatz and projected metrics). Therefore it seems that there exists other remarkable affine subspaces in the $S_2^H(\mathfrak{m})$ space of chapter 1, that we find here owing to supersymmetry, and that lead to six-degree polynomial potentials.
- Finally, we mention that the potential \mathcal{V} is unbounded from below on $E_7/SU(8)$. However, G.W. Gibbons, C.M. Hull and N.P. Warner have proven a positive mass theorem in this situation (asymptotically AdS_4 , with $N = 8$ supersymmetries), which shows that the gravitation stabilizes the solution (cf. [GHW]).

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