Université de Paris

École Doctorale de Sciences Mathématiques de Paris Centre (ED 386)
Institut de Mathématiques de Jussieu - Paris Rive Gauche (UMR 7586)

Moduli spaces of Willmore immersions
Espaces de modules des immersions de Willmore

Par Nicolas MARQUE

Thèse de doctorat de mathématiques

Dirigée par Paul LAURAIN

Présentée et soutenue publiquement le 9 décembre 2019

Devant un jury composé de :

M. Yann BERNARD	SL	Monash University	Examinateur
M. Olivier DRUET	DR	Université Claude Bernard	Examinateur
M. Laurent HAUSWIRTH	PR	Université Paris-Est Créteil	Examinateur
M. Frédéric HÉLEIN	PR	Université Paris Diderot	Examinateur
M. Paul LAURAIN	MCF	Université Paris Diderot	Directeur
M. Jan METZGER	PR	Universität Potsdam	Rapporteur
Mme Ilaria MONDELLO	MCF	Université Paris-Est Créteil	Examinatrice
M. Andrea MONDINO	PR	University of Warwick	Rapporteur

Institut de Mathématiques de Jussieu-Paris Rive Gauche
(CNRS – UMR 7586)
Université Paris Diderot (Paris 7)
Bâtiment Sophie Germain - Boîte Courrier 7012
8, place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

École Doctorale de Sciences Mathématiques de Paris Centre
Boîte Courrier 290
4, place Jussieu
75252 Paris Cedex 05
FRANCE

Abstract

In this doctoral work we start by exposing a synthesis of the weak Willmore immersions formalism. To that end, we introduce conservation laws and exploit them to recover the ε-regularity theorems, as well as an innovative weak regularity result. We then present a study of the conformal Gauss map and its links with the Willmore surface notion. From this, we deduce an exchange law for residues as well as an original characterization of surfaces that are conformal transforms of constant mean curvature surfaces. We then apply these tools to sequences of Willmore immersions. We first show that they are not compact with a first instance of concentration for Willmore surfaces. However, relying upon an ε-regularity result based on a small control on the mean curvature, we show compactness below a given threshold.

Keywords : minimal surfaces, Willmore surfaces, conformal geometry, compactness, blow-up analysis, Lorentz spaces, De Sitter spaces, conservation laws.

Résumé

Dans ce travail doctoral, nous commençons par présenter une synthèse du formalisme des immersions faibles de Willmore. A cet effet, nous introduisons les lois de conservation et les exploitons pour retrouver les résultats d'ε-régularité, ainsi qu'un résultat de régularité faible inédit. Nous présentons ensuite une étude de l'application de Gauss conforme et de ses liens avec la notion de surface de Willmore. Nous en déduisons une loi d'échange de résidus ainsi que d'une caractérisation originale des surfaces étant transformations de surfaces à courbure moyenne constante. Nous appliquons ensuite ces outils aux suites d'immersions de Willmore. Nous montrons tout d'abord qu'elles ne sont pas compactes avec un premier exemple de concentration pour les surfaces de Willmore. Cependant, en se basant sur un résultat d'ε-régularité demandant un contrôle sur la courbure moyenne, nous montrons une compacité sous un certain plafond d'énergie.

Mots clefs : surfaces minimales, surfaces de Willmore, géométrie conforme, compacité, concentration, espaces de Lorentz, espace de De Sitter, lois de conservation.

Remerciements

Avant de commencer à exposer mon travail doctoral, il me faut remercier tous ceux sans qui ce travail n'aurait pas été possible.

Tout d'abord, tous mes remerciements vont à mon directeur de thèse, Paul Laurain. Je le remercie tout d'abord de m'avoir accepté en stage de master dès 2014 et présenté ce qui allait devenir mon sujet de recherche, d'avoir continué de s'intéresser à moi lors de mon année de préparation à l'agrégation, et enfin accepté de diriger ma thèse en 2016. Sous sa tutelle j'ai été formé à mon sujet de recherche, à la vie académique : j'ai appris à être un chercheur; les contributions que j'expose sont le résultat de son pilotage. Je voudrais également lui confier toute ma gratitude pour sa patience et sa gentillesse face à mes fréquents moments de doute et de panique, qu'il a su atténuer et dissiper. Merci pour cette thèse.

I would then give my sincere gratitude to Jan Metzger and Andrea Mondino for having accepted to review this doctoral work. I hope the hours spent on it did not feel like wasted time.

Je voudrais également remercier tous les membres du jury. Que mon travail suscite l'intérêt de chercheurs confirmés et établis reste pour moi une source d'émerveillement. Je remercie donc encore Andrea Mondino qui a accepté de se déplacer. Je remercie aussi Yann Bernard pour cela, mais également pour de très fructueuses discussions à l'été 2018 qui ont grandement contribué à cristalliser mes idées en ce qui deviendra le chapitre 4 de cette thèse. Mes sincères remerciements vont également vers Olivier Druet, dont les cours de géométrie différentielle et Riemannienne ainsi que sur les surfaces minimales à l'ENS de Lyon sont à la source de mon intérêt pour l'analyse géométrique, Laurent Hauswirth, Frédéric Hélein, que je remercie aussi pour ses discussions portant sur l'application de Gauss conforme, et Ilaria Mondello, dont l'intérêt porté à mes travaux a été particulièrement motivant.

I would also thank Jorge Lira and all of the Mathematics department of the UFC Fortaleza (notably, but without being exhaustive, Fernanda Roing, Rodrigo Avalos and Vincent Grandjean) for having welcomed me in June 2019. Thanks to them, I learned much during this stay, tasted *feijoada* and had the occasion to give my first "real" talk outside of PhD students seminars.

Je voudrais d'ailleurs remercier tous ceux qui m'ont offert mes premières occasions de parler de mes résultats et, ainsi, de rencontrer d'autres mathématiciens partageant mes centres d'intérêts et d'apprendre à leur contact. Outre Jorge Lira, je remercie le comité organisateur du 1st Joint Meeting Brazil-France in Mathematics pour l'invitation à l'IMPA, Eric Toubiana pour la possibilité de parler au séminaire d'équipe, Bruno Premoselli pour l'invitation à Bruxelles (et l'intéressante discussion ayant suivi) et Benoît Daniel pour son invitation à Nancy. I would also thank Jan Metzger for his invitation to Potsdam.

Pour conclure les remerciements mathématiques, je remercie tous ceux m'ayant aidé dans l'enseignement et appris cette partie du métier. En particulier, je remercie Samy Abbès et Mathilde Herblot auprès de qui j'ai commencé l'enseignement, Romain Pétrides pour son cours de théorie de la mesure (et pour une discussion et une balade à Ripo), Zoé Mesnil, Patrick Simonetta, Bertrand Gentou et bien d'autres. Je remercie également Frédérique Petit, et encore une fois Mathilde Herblot, pour leur organisation de la fête de la science, à laquelle participer a toujours été un vrai plaisir.

Je tiens aussi à offrir mes remerciements et ma gratitude à tout le personnel administratif et technique de l'IMJ-PRG et l'UFR de mathématiques, qui a su me guider avec bienveillance et gentillesse dans toutes les formalités, procédures et programmes et ce malgré mon incurie administrative et informatique. Je remercie ainsi Emilie Jacquemot, Claire Lavollay, Nadine Fournaiseau, Evariste Ciret, Sakina Madel-Kawami, Amina Hariti et Nawel Aissani.

J'en arrive maintenant à remercier mes collègues en thèse à l'IMJ-PRG (ou affiliés). Sans eux et sans elles, ces années auraient été bien moins agréables, et si on me voyait si souvent à Sophie Germain, c'était en grande partie grâce à leur compagnie. Merci donc. Merci à ceux qui sont partis au début de mon séjour : Aurélien, Baptiste (qui m'a accueilli à mon arrivée), David, Kévin D., Marco (et son enthousiasme contagieux), Martin, Victoria. Merci à ceux que j'ai pu cotoyer plus longtemps et dont les péripéties thèsiennes servaient d'exemples quand venait notre tour : Alexandre V. (qui connaît tous les jeux, et toutes les énigmes), Antoine J. (dont la bonne humeur apparemment permanente, le goût de la discussion et la tarte à l'oignon de sa soutenance resteront gravés dans ma mémoire), Esther, Juan-Pablo, Kévin B., Noé, Pooneh et Siarhei. Merci à ceux dont l'évolution parallèle était un réconfort et une source de joie, ceux avec qui j'ai partagé mon bureau : Andreas (viking au trône phtonogène), Grégoire, Mario (les livres se succédant sur son bureau me fascinent autant que les peluches y restant m'inquiètent), Yi, Hao et Claudia, et ceux avec qui j'ai partagé le labo : Elie G. (merci pour les conseils de conduite en Crète), Léa (grand soutien, particulièrement en troisième année, et force infatigable au service du labo), Macarena, Omar (et les moments de travail musicaux en salle commune), Parisa, Radko, Rahman, Reda, Sacha ("On a commencé ensemble, on soutient ensemble, on est hyper synchros!"), Soledad, Théophile (Vosgien sylviculteur exilé helvète et rare camarade analyste dans un océan d'algèbre). Merci à ceux arrivés après moi : Alex (le bourbaketteur qui comprend quand ça parle d'algèbre), Alexandre L., Amandine (merci pour les génoises framboise), Andrei, Charles V. (le meilleur Père Noël : il n'a pas besoin de lutins), Colin (copain!), Corentin M. (le didacticien de Notre Dame), Daniele, Elie C., Emmanuel, Fatna, Farid, Gabriel, Helmy, Jérémie, Kévin M. (grand gourou prosélyte des échecs), Léonardo, Macarena, Marie (malgré le *high five de la trahison*), Mathieu B., Maud (merci pour les résumés!), Oussama, Pierre Ge, Pierre Gi (meta-thesard), Romain, Tancrède, Yann, Wille, Xavier. Merci également à l'indéboulonnable Jérome, grand détenteur de la tradition doctorante, et contre qui je défendrai toujours la bande dessinée d'outre Atlantique. Merci à tous de m'avoir soutenu, fait rire et aidé en de très nombreuses occasions. Et rappelez-vous qu'on a toujours besoin d'orateurs pour les Bourbakettes.

Je voudrais aussi remercier des camarades extra-Germain, Alexis (avec qui c'est un plaisir de discuter), Amélie, Anna (repasse à Paris quand tu veux), Mathieu D. (algébriste mais sympathique), et Sophie (qui a eu la gentillesse de relire mon introduction), mais aussi tous ceux qui ont été mes colocataires ou mes camarades, à Lyon ou en prépa (Lucas, Victor, Simon, Alejandro, Jonathan, Félix...).

Je remercie également tous mes professeurs, TDmen, TDwomen, de l'ENS ou d'ailleurs, de mathématiques ou autres, qui m'ont accompagné, encouragé et canalisé tout au long de ma scolarité.

Ich will auch der Familie Mohr danken, die mich aufgenommen hat und immer sehr freundlich ist. Dankeschön von ganzem Herzen.

Merci aux amis de la Garonne qui sont là même quand je suis loin. Merci à Alain, Antonin et Mike, à Alice, Jolan, Laure et Marvin, à Jean, Maryne, Sylvain, Pascale.

Merci enfin à ma famille, à mon père, à ma mère (à Michel aussi), à mon frère et à ma soeur. Merci à toute la grande famille et ce qu'elle comporte de grands-parents, oncles et tantes, cousins et cousines. Je vous dois plus que je sais dire.

[illegible] Manu, [illegible].

"Mais sans technique, un don n'est rien
Qu'un' sal' manie..."
Georges Brassens, Le mauvais sujet repenti

A Manu,

Contents

Introduction

Introduction (English)

In order to introduce the reader to the topics adressed in this doctoral work, we first detail an overall chronological state of the art, before detailing its content on a chapter by chapter basis. The original contributions of the author will be highlighted, but we will only give the core results, in a sometimes simplified form, to avoid detailing technicalities in what is wished as a discussion of ideas. All of them will be properly written and proven in their specific chapters.

Mean curvature and elastic energy :

In 1680, R. Hooke, an English natural philosopher, rubbed a violin archet on a thin metallic plate covered by a slim stratum of sand. He noticed that the sand then organized in peculiar geometric shapes. At the onset of the XIXth century, E. Chladni systemized this experiment and highlighted the dependance of the sand patterns on the shape of plate. These interesting *Chladni patterns* entered the field of geometry.

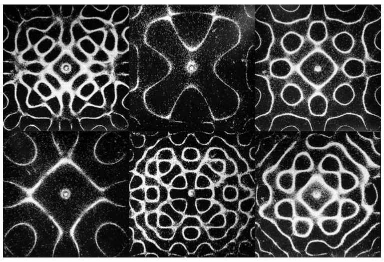

Figure 1 – Chladni patterns

This intrigued Napoleon Bonaparte enough to organize a scientific competition in order to explain this phenomenon. Among the competitors were S. Germain and S. Poisson. The former won the competition with her explanation describing the plate as a vibrating elastic surface. The sand naturally accumulates in its non-oscillating zones, leading to the Chladni patterns. In her memoir [Ger31], S. Germain then linked the elastic behavior of a surface in $\mathbb{R}^3$ at a given point to its *mean curvature* H at this point. Following from the study of the elastic curves led by D. Bernoulli and L. Euler, one can extrapolate from the 1-dimensional elastic energy of a given curve γ, $E = \int_\gamma \kappa^2$, where κ is the curvature of γ, and find the 2-dimensional energy of an immersed surface Σ :

$$W = \int_\Sigma H^2.$$

This quantity around which the present work revolves is called the Willmore energy.

Conformal invariance :

In the XXth century, G. Thomsen, see [Tho23], and W. Blaschke, see [Bla55], pioneered the angle of study of the Willmore energy which we will adopt : the conformal geometry approach. At the inception are the *conformal invariance* properties of W, meaning that W remains unchanged under the action of isometries of $\mathbb{R}^3$, dilations, and inversions which do not change the topology of the surface. Together, these transformations generate the *conformal group* of $\mathbb{R}^3$, i.e. the group of ambient angle-preserving maps. Although the Willmore energy is not invariant under all inversions (and is thus merely a contextual conformal invariant) a closely linked quantity is. The tracefree total curvature

$$\mathcal{E} = \int_\Sigma |\mathring{A}|^2,$$

where $\mathring{A}$ is the tracefree part of the second fundamental form of the immersion, is indeed invariant under all the diffeomorphisms generating the conformal group. Interestingly, it only differs from W by a topological constant, hence the peculiar properties of W when subjected to inversions.

Willmore immersions :

Given the elastic nature of the Willmore energy, it is natural to try to find minimizers. The first examples come from minimal surfaces, for whom the mean curvature is null. However such surfaces can never be compact, and thus do not provide satisfying answers. In the 1960's, T. Willmore, in [Wil65], gave the absolute minimum of the Willmore energy, 4π, reached by the round sphere, and conjectured that for tori the minimum was $2\pi^2$ and reached by the Clifford torus. This conjecture took his name and stood until 2015 when F. Marques and A. Neves solved it (in [MN16]) using geometric measure theory tools developped from F. Almgren and J. Pitts min-max theory, see [Pit81]. Broadening the scope of the study we will consider not only minimizers, but also critical points of the Willmore energy. They are called Willmore surfaces, or Willmore immersions depending on whether we consider the object in $\mathbb{R}^3$ or its parametrization. Given the properties of W, it is a notion invariant by conformal transformations. Inversions of minimal surfaces then offer a broad spectrum of examples with varied properties while the Clifford torus shows that Willmore immersions are not reduced to inversions of minimal surfaces in $\mathbb{R}^3$. The corresponding Euler-Lagrange equation is the Willmore equation :

$$\Delta H + H|\mathring{A}|^2 = 0.$$

The study of Willmore surfaces dates as far back as G. Thomsen and W. Blashcke's works, with significant contributions by T. Willmore ([Wil93]), R. Bryant ([Bry84]), E. Kuwert and R. Schätzle (who proved the first ε-regularity result in [KS01a]). In the following we will adopt T. Rivière's formalism developed throughout several of his works. Starting with the minimal assumptions to define the Willmore energy, he introduced the notion of weak immersions ([Riv08]). Even though in that context the Willmore equation does not have a correct distributional sense, conservation laws (that Y. Bernard later showed in [Ber16] were a consequence of the conformal invariance) allow for the introduction of *weak Willmore* immersions (in [Riv08] still). Further exploitation of the conservation laws and the Jacobian-like equations they induce leads to another ε-regularity theorem, thanks to integrability by compensation, which insures that weak Willmore immersions are smooth.

Using the same methods and an analysis on punctured disks, Y. Bernard and T. Rivière then extended the domain of study to branched Willmore immersions (see [BR13]) and fully described the behavior of branched Willmore immersions around a branch point by two quantites called residues. The *first residue* is truly a residue in the mathematical sense, as it comes from the aforementioned conservation laws. The *second residue* however does not, and is merely a way to sharply describe the behavior of the mean curvature at the branch point.

Sequences of Willmore immersions :

The framework of weak immersions is highly useful when considering sequences of uniformly bounded Willmore energy. The hypotheses they satisfy yield just enough regularity to imply a weak convergence up to extraction, away from a finite number of concentration points (see [Riv16]). At these points the limit immersion may degenerate and lead to branch points. These phenomena of concentration-compactness, as coined by W. Sacks and K. Uhlenbeck ([SU81]), have been studied with success, for instance in the case of constant mean curvature surfaces (see H. Brezis and J.-M. Coron's [BC85] for the setting of the problem, and P. Laurain's study of concentration in [Lau12b]). For their part, sequences of weak Willmore immersions converge smoothly away from the concentration points, thanks to the aforementioned ε-regularity. Blow-ups performed on the concentration points reveal a bubble tree made of possibly branched, possibly non compact Willmore spheres, glued on the concentration points thanks to neck domains. Y. Bernard and T. Rivière in [BR14], and P. Laurain and T. Rivière with other hypotheses in [LR18a], showed that sequences of weak Willmore immersions satisfy an *energy quantization* result : the Willmore energy of the sequence tends toward the sum of the Willmore energy of the limit surface and the Willmore energy of all the bubbles. This is equivalent to the *no-neck energy* principle : the neck domains do not carry any energy at the limit. Concentration phenomena represent a loss of compactness for weak Willmore immersions since they may degenerate into branched Willmore immersions. However these Willmore bubble trees cannot be arbitrary and are in fact constrained. Indeed, in [LR18a] P. Laurain and T. Rivière have eliminated enough bubbling configurations to ensure compactness of Willmore immersions with an energy strictly below 12π. The extension of this theorem is a major result of the following memoir.

Conformal Gauss map :

Another way to approach Willmore surfaces was pioneered in [Bry84] by R. Bryant. In it he used the notion of conformal Gauss maps to study Willmore immersions. The conformal Gauss map may be thought of as a generalization of the osculating circles of a curve in $\mathbb{R}^2$: it associates to $p \in \Sigma$ the tangent sphere of radius $[H(p)]^{-1}$. Seen as a map with values in the space of spheres, represented as the de Sitter space $\mathbb{S}^{4,1} \subset \mathbb{R}^{4,1}$, this yields a map Y which happens to be minimal in $\mathbb{S}^{4,1}$ if and only if the starting immersion is Willmore. Several surveys and studies have been conducted on this map, for instance J.-H. Eschenburg's [Esc88] or B. Palmer's [Pal91], and many properties on Willmore surfaces have been translated in the conformal Gauss map language. In fact in [MR17] A. Michelat and T. Rivière have established a deep parallel between the role of the conformal Gauss map for Willmore surfaces and the one played by the Gauss map in the constant mean curvature case. Using loop groups methods on the conformal Gauss map, F. Hélein even showed in [Hél98] that Willmore immersions could be described by a non-explicit Weierstrass representation.

Conformal Gauss maps have also proven to be pivotal in determining whether a Willmore immersion is the conformal mapping of a minimal immersion (in other words is conformally minimal). Indeed, in [Bry84], R. Bryant introduced the Bryant's quartic $\mathcal{Q}$ defined as

$$\mathcal{Q} = \left\langle \partial^2 Y, \partial^2 Y \right\rangle .$$

He then showed that a given immersion is conformally minimal if and only if $\mathcal{Q} = 0$. Using more complex techniques invented by J. Dorfmeister, F. Pedit and H. Wu (the DPW method, see [DPW98]), these results have been extended to conformal transformations of minimal immersions in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, and even to conformal transformations of surfaces of constant mean curvature. Results on this subject can be found in [Eji88], [Ric97], [Boh12] and [DW19].

Outline :

The first chapter will focus on properly introducing the basic notions surrounding the Willmore immersions. We will go through the definitions and present the computations of the conservation laws when one considers both W *and* $\mathcal{E}$ as Lagrangians. From this, we will find the Willmore equations as first exposed by T. Rivière. Further, we will modify two of them into an original and less algebraically remarkable form (we will lose the Jacobian shape), but one that involves only mean curvature terms. This new form will prove useful in chapter 4. We will then expose the weak Willmore immersion formalism and offer improvements of already existing estimates. Namely, we will obtain a sharper control on the quantity $\vec{L}$ derived from the first conservation law depending only on the mean curvature, instead of the whole Gauss map. As an illustration of what can be expected when following through these reasonings, we will achieve a weak regularity result for Willmore surfaces, namely :

Theorem A. Let Φ be a conformal weak Willmore immersion. Then for any $r < 1$ there exists a constant $C \in \mathbb{R}$ such that

$$\|H\nabla\Phi\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D}_\rho)},$$

and

$$\|\nabla\vec{n}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C\,\|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)} .$$

Theorem A contrasts with the more classical ε-regularity results, which we will recall.

In the second chapter, we will study the conformal Gauss map both in its relation to Willmore surfaces, and as a tool for determining if a surface is the conformal mapping of a constant mean curvature surface (we will call such surfaces : of conformally constant mean curvature, or conformally CMC). We will introduce the conformal Gauss map in the general case, highlight its properties and how its geometry sheds light on the immersion itself. at the core will be its behavior when the immersion is Willmore or conformally

CMC. While we will mostly reprove results found using the DPW method, the originality will lie in using only basic differential geometry in Lorentz spaces. Concluding this chapter, two new characterizations of conformally CMC immersions will have been achieved. First is the following :

Theorem B. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$). We assume that X has no umbilic point. One of the representation of X is conformally *CMC* in its ambient space if and only if its Bryant's quartic $\mathcal{Q}$ is holomorphic and X is isothermic. More precisely, if $\mathcal{W}$ is the Willmore operator, $\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}$ is necessarily real and

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

A slight variation on theorem B gives the equivalent :

Theorem C. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$). We assume X has no umbilic point. One of the representation of X is conformally *CMC* in its ambient space if and only if $\mathcal{Q}$ is holomorphic and $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$, where $\omega \in \mathbb{C}$ is the tracefree curvature of X. More precisely

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

The originality of theorems B and C lies both in the caracterization of the ambient space, unknown till now, and, for the latter, in replacing the isothermic condition by $\overline{\omega^2 \mathcal{Q}} \in \mathbb{R}$. It then shows that $\mathcal{Q}$ has deep ties to the isothermic nature of the surface.

The third chapter will deal with the compactness of sequences of weak immersions. It will mostly detail the state of the art and the energy quantization result, but also present a strong correspondance between bubbles and the surface they are glued on. We will show that the branching order of the bubble is heavily constrained by the branching order of the concentration point it is glued on :

Theorem D. One can only glue a Willmore branch point on a Willmore branched end of same multiplicity, and vice versa.

Moreover we will offer the first explicit example of Willmore bubbling.

Theorem E. There exists $\Phi_k : \mathbb{S}^2 \to \mathbb{R}^3$ a sequence of Willmore immersions such that

$$W(\Phi_k) = 16\pi,$$

and

$$\Phi_k \to \Phi_\infty,$$

smoothly on $\mathbb{S}^2 \backslash \{0\}$, where Φ_∞ is the inversion of a López surface. Further

$$\lim_{k \to \infty} E(\Phi_k) = E(\Phi_\infty) + E(\Psi_\infty),$$

where $\Psi_\infty : \mathbb{C} \to \mathbb{R}^3$ is the immersion of an Enneper surface.

Theorem E highlights a lack of compactness for Willmore immersions of high energy.

Finally, the fourth chapter will study the configuration of one simple minimal bubble glued onto a branch point, in order to extend P. Laurain and T. Rivière's compactness result. We will first show a new ε-regularity result with only a small control on H :

Theorem F. Let Φ be a conformal weak Willmore immersion. Then there exists ε_0' such that if

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0'$$

then for any $r < 1$ there exists a constant $C \in \mathbb{R}$ such that

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

and

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \leq C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

Then, through successive expansions, we will prove a control of the second residue of the limit surface.

Theorem G. Let Φ_k be a sequence of Willmore immersions of a closed surface Σ of uniformly bounded Willmore energy, and whose induced conformal classes are in a compact subset of the moduli space. Then at each concentration point $p \in \Sigma$ of multiplicity $\theta_p + 1$ on which a simple minimal bubble is blown, the second residue α_p of the limit immersion Φ_∞ satisfies

$$\alpha_p \leq \theta_p - 1.$$

Since, *a priori* the second residue only satisfies $\alpha_p \leq \theta_p$, this represents a real gain of regularity. One must note that while the example offered by theorem E does satisfy this new estimate, it will successfully eliminate several bubbling configurations. We will notably show that inverted Enneper surfaces, and more broadly speaking, inverted Chen-Gackstatter surfaces of any genus cannot be the recipient of simple minimal bubbling. From these results, we deduce an improvement of compactness for Willmore immersions of low energy :

Theorem H. Let Σ be a closed surface of genus 1 and $\Phi_k : \Sigma \to \mathbb{R}^3$ a sequence of Willmore immersions such that the induced metric remains in a compact set of the moduli space and

$$\limsup_{k\to\infty} W\left(\Phi_k\right) \leq 12\pi.$$

Then there exists a diffeomorphism ψ_k of Σ and a conformal transformation Θ_k of $\mathbb{R}^3 \cup \{\infty\}$, such that $\Theta_k \circ \Phi_k \circ \psi_k$ converges up to a subsequence toward a smooth Willmore immersion $\Phi_\infty : \Sigma \to \mathbb{R}^3$ in $C^\infty(\Sigma)$.

We will finally detail a stronger control under an additional assumption to highlight how the lack of compactness of Willmore immersions evidenced by theorem E can be seen as a consequence of the lackluster properties of the conformal group.

Theorems B and C were part of the preprint [Mar19a], theorems A and F can be found in [Mar19c], and theorems D, E, G and H in [Mar19b].

Introduction (Français)

Pour familiariser le lecteur aux sujets abordés dans ce travail doctoral, nous présentons un état de l'art chronologique, avant de détailler son contenu chapitre par chapitre. Les innovations de l'auteur y sont soulignées, mais nous ne donnerons que les résultats centraux, et parfois dans une forme simplifiée pour ne pas avoir à s'encombrer de détails techniques dans une partie conçue avant tout comme une discussion d'idées. Toutes les notions, tous les résultats, tous les théorèmes seront introduits, écrits et détaillés dans leurs chapitres respectifs.

Courbure moyenne et énergie élastique :

En 1680, R. Hooke, un philosophe anglais, frotta un archet sur une fine plaque métallique couverte d'une mince couche de sable. Il remarqua que le sable s'organisait selon d'étonnants motifs géométriques. Au début du XIXème siècle, E. Chladni systématisa cette expérience et mit en évidence la dépendance du motif formé à l'égard de la forme de la plaque. Il légua son nom à ces intéressantes *figures de Chladni*.

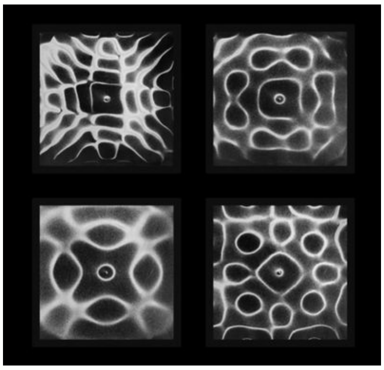

Figure 2 – Figures de Chladni

Cela intrigua suffisamment Napoléon Bonaparte pour qu'il organisât un concours scientifique afin d'expliquer ce phénomène. Parmi les concurrents étaient S. Germain et S. Poisson. Cette dernière gagna le concours grâce à son explication décrivant la plaque comme une surface élastique en vibration. Le sable s'accumulait alors dans les zones n'oscillant pas, menant aux figures de Chladni. Dans son mémoire [Ger31], S. Germain relia le comportement élastique d'une surface de $\mathbb{R}^3$ en un point donné à sa *courbure moyenne* H en ce point. En s'inspirant de l'étude des courbes élastiques menées par D. Bernoulli et L. Euler, on peut extrapoler à partir de l'énergie élastique d'une courbe γ donnée, $E = \int_\gamma \kappa^2$, où κ est la courbure de γ, et trouver l'énergie élastique bidimensionnelle d'une surface immergée Σ :

$$W = \int_\Sigma H^2.$$

Cette quantité, sur laquelle est basée ce travail, est appelée l'énergie de Willmore.

Invariance conforme :

Au XXème siècle, G. Thomsen, dans [Tho23], et W. Blaschke, dans [Bla55], développèrent l'angle d'approche de l'énergie de Willmore que nous adopterons : l'étude par la géométrie conforme. A la base sont les propriétés d'*invariance conforme* de W, c'est-à-dire que W reste inchangée sous l'action des isométries de $\mathbb{R}^3$, des dilatations, et des inversions qui ne changent pas la topologie de la surface. Ensemble, ces transformations génèrent le *groupe conforme* de $\mathbb{R}^3$, i.e. le groupe des tranformations ambientes préservant les angles. Bien que l'énergie de Willmore ne soit pas invariante sous toutes les inversions (et ne soit donc qu'un invariant conforme contextuel) une quantité très proche l'est. La courbure sans trace totale

$$\mathcal{E} = \int_\Sigma |\mathring{A}|^2,$$

où $\mathring{A}$ est la partie sans trace de la seconde forme fondamentale de l'immersion est, elle, bel et bien invariante sous tous les difféomorphismes engendrant le groupe conforme. Il est intéressant de remarquer qu'elle ne diffère de W que par une constante topologique, d'où les propriétés remarquables de W sous l'action des inversions.

Immersions de Willmore :

Etant donnée la nature élastique de l'énergie de Willmore, il est naturel d'essayer d'en trouver les minimiseurs. Les premiers exemples viennent des surfaces minimales, pour qui la courbure moyenne est nulle. Cependant, de telles surfaces ne peuvent être compactes, et donc ne répondent pas pleinement à nos interrogations. Dans les années 1960, T. Willmore, dans [Wil65], donna le minimum absolu de l'énergie de Willmore, 4π, atteint par la sphère ronde, et conjectura que le minimum pour les tores était à $2\pi^2$, atteint par le tore de Clifford. Cette conjecture garda son nom, et est restée non démontrée jusqu'en 2015 quand F. Marques et A. Neves la resolvèrent (dans [MN16]) en utilisant des outils de théorie géométrique de la mesure développés à partir de la théorie min-max de F. Almgren et J. Pitts ([Pit81]). Nous élargirons notre champ d'étude et nous ne considérerons pas seulement les minimiseurs, mais les points critiques de l'énergie de Willmore. Ils portent le nom de surfaces de Willmore, ou d'immersions de Willmore, selon que nous considérons l'objet dans $\mathbb{R}^3$ ou son paramétrage. Les inversions de surfaces minimales offrent alors un large spectrum d'exemples aux propriétés variées, alors que le tore de Clifford montre que les immersions de Willmore ne sont pas limitées aux inversions de surfaces minimales de $\mathbb{R}^3$. L'équation d'Euler-Lagrange correspondante porte le nom d'équation de Willmore :

$$\Delta H + H|\mathring{A}|^2 = 0.$$

L'étude des surfaces de Willmore date des travaux de G. Thomsen and W. Blaschke, avec d'importantes contributions de T. Willmore ([Wil93]), R. Bryant ([Bry84]), E. Kuwert et R. Schätzle (qui prouva le premier résultat d'ε-régularité dans [KS01a]). Dans la suite, nous adopterons le formalisme de T. Rivière développé à travers plusieurs de ses publications. Partant des hypothèses nécessaires pour définir l'énergie de Willmore, il introduisit la notion d'immersion faible ([Riv08]). Même si, dans ce contexte, l'équation de Willmore n'a pas de sens rigoureux, des lois de conservation (qui, comme Y. Bernard le montra dans[Ber16], sont une conséquence de l'invariance conforme) permirent d'introduire la notion d'immersions *faibles de Willmore* (toujours fait dans [Riv08]). Une exploitation plus poussée des lois de conservation et des équations de type Jacobiennes qu'elles impliquent, menèrent à un autre résultat d'ε-régularité, grâce à l'intégrabilité par compensation, qui assurèrent la régularité des immersions faibles de Willmore.

En utilisant les mêmes méthodes et une analyse sur des disques épointés, Y. Bernard et T. Rivière étendirent le domaine d'étude aux immersions de Willmore ramifiées (voir [BR13]) et décrirent complètement le comportement d'immersions de Willmore ramifiées autour d'un point de ramification par deux quantités appelées résidus. Le *premier résidu* est vraiment un résidu, au sens mathématique du terme, puisqu'il vient des lois de conservation suscitées. Le *second résidu*, cependant n'en est pas un, et est simplement une manière de décrire précisément le comportement de la courbure moyenne au point de ramification.

Suites d'immersions de Willmore :

Le cadre des immersions faibles est particulièrement utile quand on considère des suites dont l'énergie de Willmore est uniformément bornée. Les hypothèses qu'elles satisfont donnent juste assez de régularité et de contrôle pour impliquer une convergence faible à extraction près, loin d'un nombre fini de points de concentration (voir [Riv16]). En ces points, l'immersion limite peut potentiellement dégénérer et donner un point de ramification. Ces phénomènes de concentration-compacité, selon les termes de W. Sacks et K. Uhlenbeck ([SU81]), ont été étudiés avec succès, par exemple dans le cas des surfaces à courbure moyenne constante (voir H. Brézis et J.-M. Coron [BC85] pour le problème en lui-même, et l'étude de la concentration faite par P. Laurain dans [Lau12b]). Pour leur part, les suites d'immersions faibles de Willmore convergent régulièrement loin des points de concentration, grâce à l'ε-régularité. Des "blow-up" effectués sur les points de concentration révèlent un arbre de bulles constitué de sphères de Willmore, potentiellement ramifiées, potentiellement non compactes, collées sur les points de concentration à l'aide de domaines de type cou. Y. Bernard et T. Rivière, dans [BR14], et P. Laurain et T. Rivière avec d'autres hypothèses dans [LR18a], montrèrent que les suites d'immersions faibles de Willmore satisfont un résultat de *quantification de l'énergie* : l'énergie de Willmore de la suite tend vers la somme de l'énergie de Willmore de la surface limite et de l'énergie de Willmore de toutes les bulles. Ceci est équivalent avec le principe d'*absence d'énergie dans le cou* : les domaines de type cou n'ont pas d'énergie à la limite. Les phénomènes de concentration représentent une perte de compacité pour les immersions faibles de Willmore puisqu'elle peuvent dégénérer en immersions ramifiées. Cependant, ces arbres de bulles de Willmore ne peuvent être arbitraires et sont en fait contraints. En effet, dans [LR18a] P. Laurain et T. Rivière éliminèrent assez de configurations pour assurer la compacité des immersions de Willmore avec une énergie strictement inférieure à 12π. L'extension de ce théorème est un des résultats majeurs de ce mémoire.

Application de Gauss conforme :

Un autre moyen d'étudier les surfaces de Willmore fut introduit dans [Bry84] par R. Bryant. Dans cet article, il utilisa la notion d'application de Gauss conforme pour étudier les immersions de Willmore. L'application de Gauss conforme peut être comprise comme une généralisation des cercles osculateurs d'une courbe dans $\mathbb{R}^2$: à un point $p \in \Sigma$, elle associe la sphère tangente de rayon $[H(p)]^{-1}$. Vue comme une application à valeurs dans l'espace des sphères, représenté par l'espace de de Sitter $\mathbb{S}^{4,1} \subset \mathbb{R}^{4,1}$, elle donne une application Y qui est minimale dans $\mathbb{S}^{4,1}$ si et seulement si l'immersion de départ est Willmore. Plusieurs études furent conduites sur cette application, par exemple [Esc88] par J.-H. Eschenburg ou [Pal91] par B. Palmer, et beaucoup des propriétés des surfaces de Willmore transcrites dans le langage de la Gauss conforme. En fait, dans [MR17], A. Michelat et T. Rivière établirent un parallèle profond entre le rôle de la Gauss conforme pour les surfaces de Willmore et celui joué par l'application de Gauss dans le cas des surfaces à courbure moyenne constante. En utilisant des méthodes de groupes cycliques sur l'application de Gauss conforme, F. Hélein obtint même, dans [Hél98], une représentation de Weierstrass, malheureusement non explicite.

Les applications de Gauss conformes se sont également révélées cruciales s'agissant de déterminer si une immersion de Willmore est la transformation conforme d'une immersion minimale (autrement dit, si elle conformément minimale). En effet, dans [Bry84], R. Bryant introduisit la quartique de Bryant $\mathcal{Q}$, définie comme :

$$\mathcal{Q} = \langle \partial^2 Y, \partial^2 Y \rangle.$$

Il montra ensuite qu'une immersion donnée est conformément minimale si et seulement si $\mathcal{Q} = 0$. Grace à des techniques plus avancées inventées par J. Dorfmeister, F. Pedit et H.Wu (la méthode DPW, voir [DPW98]), ces résultats furent étendus aux transformations conformes d'immersions minimales de $\mathbb{R}^3$, $\mathbb{S}^3$ et $\mathbb{H}^3$, et même aux transformations conformes de surfaces à courbure moyenne constante. Des résultats à ce sujet peuvent être trouvés dans [Eji88], [Ric97], [Boh12] et [DW19].

Description du contenu :

Le premier chapitre se concentrera sur une introduction correcte des notions de base tournant autour des immersions de Willmore. Nous parcourrons les définitions et présenterons les calculs des lois de conservation en considérant W *et* $\mathcal{E}$ comme Lagrangiens. A partir de cela, nous trouverons les équations de Willmore telles qu'elles furent exposées par T. Rivière. De plus, nous modifierons deux d'entre elles pour leur donner une forme originale, bien que moins remarquable algébriquement (nous perdrons l'aspect Jacobien), mais qui ne fait appel qu'à la courbure moyenne. Cette nouvelle forme sera utile au chapitre 4. Nous exposerons ensuite le formalisme des immersions faibles de Willmore, en offrant des améliorations marginales sur des estimées déjà existantes. Par exemple, nous obtiendrons un contrôle plus précis, car ne dépendant que de la courbure moyenne au lieu de l'application de Gauss, sur la quantité $\vec{L}$ qui dérive de la première loi de conservation. Pour illustrer ce que nous pouvons attendre en suivant ces raisonnements plus précis, nous obtiendrons un résultat de régularité faible pour les surfaces de Willmore :

Théorème A. Soit Φ une immersion conforme faible de Willmore. Alors pour tout $r < 1$ il existe une constante $C \in \mathbb{R}$ telle que

$$\|H \nabla \Phi\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)},$$

et

$$\|\nabla \vec{n}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \, \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)} \, .$$

Le théorème A contraste avec le classique résultat d'ε-régularité, que nous rappellerons également.

Dans le deuxième chapitre, nous étudierons l'application de Gauss conforme, à la fois pour son lien avec les surfaces de Willmore, et en tant qu'outil pour déterminer si une surface est l'image par une application conforme d'une surface à courbure moyenne constante (nous appellerons de telles surfaces : conformément à courbure moyenne constante ou conformément CMC). Nous introduirons l'application de Gauss conforme dans le cas général, nous mettrons en avant ses propriétés et comment sa géométrie éclaire l'immersion. Au coeur de ce chapitre, nous étudierons son comportement quand l'immersion est Willmore ou conformément CMC. Si la plupart des résultats ont déjà été obtenus avec la méthode DPW, l'originalité de ces travaux est dans les preuves, qui ne reposent que sur de la géométrie différentielle de base dans les espaces de Lorentz. En conclusion du chapitre, nous obtiendrons deux caractérisations des immersions conformément CMC. Tout d'abord :

Théorème B. Soit X une immersion conforme sur $\mathbb{D}$ dans $\mathbb{S}^3$, et Φ (respectivement Z) sa représentation dans $\mathbb{R}^3$ (respectivement dans $\mathbb{H}^3$). On suppose que X n'a pas de point ombilic. Une des représentations de X est conformément *CMC* dans son espace ambient si et seulement si sa quartique de Bryant $\mathcal{Q}$ est holomorphe et X est isothermique. Plus précisément si $\mathcal{W}$ est l'opérateur de Willmore, $\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}$ est réel et

— Φ est conformément CMC (respectivement minimal) dans $\mathbb{R}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X est conformément CMC (respectivement minimal) dans $\mathbb{S}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z est conformément CMC (respectivement minimal) dans $\mathbb{H}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Les immersions conformément minimales vérifient $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

Une variation du théorème B donne le résultat équivalent :

Théorème C. Soit X une immersion conforme sur $\mathbb{D}$ dans $\mathbb{S}^3$, et Φ (respectivement Z) sa représentation dans $\mathbb{R}^3$ (respectivement dans $\mathbb{H}^3$). On suppose que X n'a pas de point ombilic. Une des représentations de X est conformément *CMC* dans son espace ambient si et seulement si $\mathcal{Q}$ est holomorphe et $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$, où $\omega \in \mathbb{C}$ est la courbure sans trace de X. More precisely

— Φ est conformément CMC (respectivement minimal) dans $\mathbb{R}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X est conformément CMC (respectivement minimal) dans $\mathbb{S}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z est conformément CMC (respectivement minimal) dans $\mathbb{H}^3$ si et seulement si

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Les immersions conformément minimales vérifient $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

L'originalité des théorèmes B et C repose dans la caractérisation explicite de l'espace ambient, inconnue jusque là, ainsi que dans la substitution de la condition isothermique par $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$. Ceci met donc en exergue un lien profond entre $\mathcal{Q}$ et le caractère isothermique de la surface.

Le troisième chapitre traitera de la compacité des suites d'immersions faibles. Il détaillera principalement l'état de la recherche et les théorèmes de quantification de l'énergie, mais présentera également une forte correspondance entre les bulles et la surface sur laquelle on les colle. En effet nous montrerons que l'ordre de ramification de la bulle est fortement contraint par l'ordre de ramification du point de concentration sur laquelle on le colle :

Théorème D. On ne peut coller un point de ramification de Willmore que sur un bout de Willmore de même multiplicité, et vice versa.

De plus, nous offrirons le premier exemple explicie de concentration pour les surfaces de Willmore.

Théorème E. Il existe $\Phi_k : \mathbb{S}^2 \to \mathbb{R}^3$, une suite d'immersions de Willmore telles que

$$W(\Phi_k) = 16\pi,$$

et

$$\Phi_k \to \Phi_\infty,$$

régulièrement sur $\mathbb{S}^2 \backslash \{0\}$, où Φ_∞ est l'inversion d'une surface de López. De plus

$$\lim_{k\to\infty} E(\Phi_k) = E(\Phi_\infty) + E(\Psi_\infty),$$

où $\Psi_\infty : \mathbb{C} \to \mathbb{R}^3$ est l'immersion d'une surface d'Enneper.

Le théorème E met en exergue la non compacité des immersions de Willmore à haute énergie.

Enfin, le quatrième chapitre étudiera la configuration composée d'une bulle minimale simple collée sur un point de ramification, afin d'étendre le résultat de compacité de P. Laurain et T. Rivière. Nous montrerons tout d'abord un nouveau résultat d'ε-régularité avec seulement un contrôle sur H :

Théorème F. Soit Φ une immersion faible conforme de Willmore. Alors il existe ε_0' tel que si

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0'$$

alors pour tout $r < 1$ il exite une constante $C \in \mathbb{R}$ telle que

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

et

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \leq C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

pour tout $p < \infty$.

Puis, grâce à des développements successifs, nous prouverons un contrôle sur le second résidu de la surface limite.

Théorème G. Soit Φ_k une suite d'immersions de Willmore d'une surface compacte Σ dont l'énergie de Willmore est uniformément bornée, et dont les classes conformes induites sont dans un compact de l'espace de module. Alors, à chaque point de concentration $p \in \Sigma$ de multiplicité $\theta_p + 1$ sur lesquels une bulle minimale simple se développe, le second résidu α_p de l'immersion limite Φ_∞ vérifie

$$\alpha_p \leq \theta_p - 1.$$

Puisque le second résidu ne vérifie *à priori* que $\alpha_p \leq \theta_p$, ceci représente un gain réel de régularité. Il faut également remarquer que, si l'exemple offert par le théorème E vérifie bien cette nouvelle estimée, elle suffit pour éliminer plusieurs configurations d'arbres de bulles. Notamment, nous montrerons que les surfaces d'Enneper inversées, et plus généralement, toutes les Chen-Gackstatter inversées, quel que soit le genre, ne peuvent accueillir une bulle minimale simple. A partir de ceci, nous déduirons une amélioration de la compacité pour les immersions de Willmore à basse énergie.

Théorème H. Soit Σ une surface fermée de genre 1 et $\Phi_k : \Sigma \to \mathbb{R}^3$ une suite d'immersions de Willmore telle que la métrique induite reste dans un compact de l'espace des modules et telle que

$$\limsup_{k\to} W(\Phi_k) \leq 12\pi.$$

Alors il existe une suite de difféomorphismes ψ_k de Σ et une suite de transformations conformes Θ_k de $\mathbb{R}^3 \cup \{\infty\}$, telle que $\Theta_k \circ \Phi_k \circ \psi_k$ converge à extraction près vers une immersion de Willmore lisse $\Phi_\infty \Sigma \to \mathbb{R}^3$.

Nous détaillerons enfin un contrôle plus fort sous une hypothèse additionnelle pour mettre en évidence la manière dont l'absence de compacité suscitée peut être vue comme une conséquence des propriétés du groupe conforme.

Les théorèmes B et C font partie de la prépublication [Mar19a], les théorèmes A et F peuvent être trouvés dans [Mar19c], et les théorèmes D, E, G et H dans [Mar19b].

Part I

Willmore surfaces

Table of contents

1

Analysis of Willmore surfaces

Abstract.

1.1 Introduction

This introduction is to be seen as a quick walkthrough of the present chapter in which the original contributions of the author will be highlighted. While we will only give broad ideas and concepts to contextualize the important results, they will be reframed more rigorously in the core (notably all the involved quantities will be precisely defined). In that way the introduction is to be seen as an independant and heuristic explanation of the content of this chapter.

In it we will mostly establish notations and the state of the art. As a result some proofs will be glossed over. However we will offer some marginal improvements over a few of preexisting results, in which case the demonstration will be detailed. Those results are showcased in this introduction.

Given an immersion Φ of a Riemann surface (Σ, g) into $\mathbb{R}^3$, its Willmore energy is a measure of its mean curvature over all the surface :

$$W(\Phi) = \int_\Sigma H^2 d\mathrm{vol}_g.$$

It was first introduced as a tool for the study of 2 dimensional elasticas by S. Germain and S. Poisson. We will however consider it in the conformal setting, first conceived by W. Blashcke and then furthered by T. Willmore. Indeed the Willmore energy is a *contingent* conformal invariant, meaning that it is invariant under the conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ that do not change the topology of the surface. More precisely the total tracefree curvature, defined by

$$\mathcal{E}(\Phi) = \int_\Sigma |\mathring{A}|^2 d\mathrm{vol}_g,$$

is the true conformal invariant. It only differs from W by a topological quantity, thanks to Gauss-Bonnet theorem. An interesting question revolving around the Willmore energy is the search for minimizers, first globals (found by T. Willmore), and then for a given genus. The genus 1 case has been solved recently by F. Marques and A. Neves using geometric measure theory tools (see [MN16]).

Broadening our scope (in section 1.2.2), we then introduce the notion of Willmore immersion (critical points of the Willmore energy) and conformal Willmore immersion (critical points in a conformal class). The corresponding Euler-Lagrange equation can then be computed : it is called the Willmore equation. Since its analytical properties are lackluster, we wish to find a set of more interesting equations. Exploiting the conformal invariance thanks to Noether's theorem then yields four conservation laws (section 1.2.3).

Theorem 1.1.1. Let Σ be a Riemann surface and $\Phi \in C^\infty\left(\Sigma, \mathbb{R}^3\right)$ a Willmore immersion. Then Φ satisfies the following conservation laws :

$$\mathrm{div}\left(V_{\mathrm{tra}}\right) = \mathrm{div}\left(V_{\mathrm{dil}}\right) = \mathrm{div}\left(V_{\mathrm{rot}}\right) = \mathrm{div}\left(\tilde{V}_{\mathrm{rot}}\right) = \mathrm{div}\left(V_{\mathrm{inv}}\right) = 0,$$

where

$$\begin{cases} V_{\mathrm{tra}} = -2\left(\nabla H \vec{n} + H\mathring{A}\nabla\Phi\right) \\ V_{\mathrm{dil}} = \langle \Phi, V_{\mathrm{tra}} \rangle \\ V_{\mathrm{rot}} = \Phi \times V_{\mathrm{tra}} + 2H\nabla\Phi \times \vec{n} \\ \tilde{V}_{\mathrm{rot}} = \Phi \times V_{\mathrm{tra}} + 2\left(\mathring{A}\nabla\Phi\right) \times \vec{n} \\ V_{\mathrm{inv}} = -|\Phi|^2 V_{\mathrm{tra}} + 2\langle \Phi, V_{\mathrm{tra}} \rangle \Phi - 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right). \end{cases}$$

If Φ is branched, these stand away from the branch points.

The result itself is not new, the conservation laws were already found by T. Rivière (see theorem I.4 [Riv08]). In [Ber16], Y. Bernard showed that the first three resulted from conservation laws, and conjectured the origin of the fourth. We will use his formalism in section 1.2.3 to detail this. We must however point out that these were already computed in section 3.1 of [MR17] by A. Michelat and T. Rivière. We will use a different formalism, closer to the one in [Ber16].

Using these conservation laws, one can introduce auxiliary quantities $\vec{L}$, S and $\vec{R}$ which satisfy a classical Jacobian-like system (section 1.2.4) :

$$\begin{cases} \Delta S = -\left\langle \nabla \vec{n}, \nabla^\perp \vec{R} \right\rangle \\ \Delta \vec{R} = \nabla \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \nabla \vec{n} \\ \Delta \Phi = \frac{1}{2}\left(\nabla^\perp S . \nabla \Phi + \nabla^\perp \vec{R} \times \nabla \Phi \right). \end{cases}$$

Going into the details and the nature of the quantities introduced, this sytem can be modified into an original form which can be exploited later.

Theorem 1.1.2. Let $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ satisfy the hypotheses of theorem 1.2.16. Then

$$\begin{cases} \Delta S = \left\langle H \nabla \Phi, \nabla^\perp \vec{R} \right\rangle \\ \Delta \vec{R} = -H \nabla \Phi \times \nabla^\perp \vec{R} - \nabla^\perp S H \nabla \Phi \\ \Delta \Phi = \frac{1}{2}\left(\nabla^\perp S . \nabla \Phi + \nabla^\perp \vec{R} \times \nabla \Phi \right). \end{cases}$$

We can then introduce the notion of weak immersion as devised by T. Rivière to enjoy a weak framework for Willmore immersions (this will be the subject of section 1.3). First, with weak hypotheses in local conformal charts, we can prove a Harnack inequality on the conformal factor in domains of small energies. With an added parameter r_0 measuring the number of small energy disks required to cover the domain of a conformal chart, we can extend this result to domains of merely bounded energy. That parameter is defined as

$$r_0 = \frac{1}{\rho} \inf \left\{ s \,\middle|\, \int_{B_s(p)} |\nabla \vec{n}|^2 = \frac{8\pi}{6},\ \forall p \in \mathbb{D}_\rho \text{ s.t. } B_s(p) \subset \mathbb{D}_\rho \right\}.$$

Corollary 1.1.1. Let $\Phi \in \mathcal{E}\left(\mathbb{D}_\rho\right)$ conformal, $\vec{n}$ be its Gauss map and λ its conformal factor. We assume that

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0.$$

Then for any $r < 1$ there exists $c_{\rho,r} \in \mathbb{R}$ and $C \in \mathbb{R}$ depending on r, C_0 and r_0 (defined by (1.3.67)) such that

$$\|\lambda - c_{\rho,r}\|_{L^\infty(\mathbb{D}_{r\rho})} \leq C.$$

This Harnack inequality gives meaning to the first conservation law in the weak framework, and thus gives sense to the notion of *weak Willmore immersion*. From this, one can prove low-regularity results for weak Willmore immersions in Lorentz spaces. First is an improvement on the controls on the quantity $\vec{L}$:

Theorem 1.1.3. Let $\Phi \in \mathcal{E}\left(\mathbb{D}_\rho\right)$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and λ its conformal factor.

We assume

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0.$$

Then for any $r < 1$ there exists a constant $\vec{\mathcal{L}}_{\rho,r} \in \mathbb{R}^3$ and a constant $C \in \mathbb{R}$ depending on r, C_0 and r_0 (defined in (1.3.67)) such that

$$\left\| e^\lambda \left(\vec{L} - \vec{\mathcal{L}}_{\rho,r} \right) \right\|_{L^{2,\infty}(\mathbb{D}_{r\rho})} \leq C \, \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)}.$$

This is an improvement over theorem 7.4 of [Riv16], in that the control depends only on the mean curvature instead of on the whole second fundamental form. Estimate (A.2.11) ensures that one can find theorem 7.4 back from our result. With this base, one can find an overall control in Lorentz spaces, without a small energy hypothesis. The following regularity result is thus remarkable in that it differs from the ε-regularity result.

Theorem A. Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$ be a weak conformal weak Willmore immersion. We assume

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0.$$

Then for any $r < 1$ there exists a constant $C \in \mathbb{R}$ depending on r, C_0 and r_0 such that

$$\|H\nabla\Phi\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D}_\rho)},$$

and

$$\|\nabla\vec{n}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C\,\|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)}.$$

The last four results were part of the preprint [Mar19c]. Due to the critical nature of the equation systems for Willmore immersions, it is unreasonable to expect better controls without further assumptions.

However, if we consider disks of small energy, then classical ε-regularity results have been found, like theorem I.5 in [Riv08] and theorem I.1 in [BR14]. These results control the immersion, its Gauss map and all its derivatives by $\|\nabla\vec{n}\|_{L^2}$, and thus ensure the smoothness of weak Willmore immersions.

The present chapter will then conclude in section 1.5 with a study of branched Willmore immersions near the branch points based on Y. Bernard and T. Rivière work in [BR13] with an emphasis on describing the behavior of what we will later study as Bryant's quartic, around the branch point.

1.2 Willmore surfaces

1.2.1 The Willmore energy

Consider Φ an immersion from a closed Riemann surface Σ into $\mathbb{R}^3$. We denote by $g := \Phi^*\xi$ the pullback by Φ of the euclidean metric ξ of $\mathbb{R}^3$, also called the first fundamental form of Φ or the induced metric. Let $d\mathrm{vol}_g$ be the volume form associated with g. The Gauss map $\vec{n}$ of Φ is the normal to the surface. In local coordinates (x, y) :

$$\vec{n} := \frac{\Phi_x \times \Phi_y}{|\Phi_x \times \Phi_y|},$$

where $\Phi_x = \partial_x\Phi$, $\Phi_y = \partial_y\Phi$ and $\times$ is the usual vectorial product in $\mathbb{R}^3$. Denoting $\pi_{\vec{n}}$ the orthonormal projection on the normal (meaning $\pi_{\vec{n}}(v) = \langle\vec{n}, v\rangle\vec{n}$), the second fundamental form of Φ at the point $p \in \Sigma$ is defined as follows.

$$\vec{A}_p(X, Y) := A_p(X, Y)\vec{n} := \pi_{\vec{n}}\left(d^2\Phi\,(X, Y)\right) \text{ for all } X, Y \in T_p\Sigma.$$

The mean curvature of the immersion at p is then

$$\vec{H}(p) = H(p)\vec{n} = \frac{1}{2}Tr_g\,(A)\,\vec{n},$$

while its tracefree second fundamental form is

$$\mathring{A}_p(X, Y) = A_p(X, Y) - H(p)g_p(X, Y).$$

Definition 1.2.1. The Willmore energy W of Φ is defined as

$$W(\Phi) := \int_\Sigma H^2 d\mathrm{vol}_g.$$

This quantity, which is at the core of the present work, arose naturally in the study of elasticity in the first third of the XIXth century in the works of S. Poisson [Poi14] and S. Germain [Ger31]. It is indeed an elastic energy that can be thought to measure how extrinsiquely curved an immersion of a Riemann surface is. G. Thomsen and W. Blaschke then studied it in the framework of conformal geometry due to its conformal invariance properties. To explore these we briefly recall basic notions.

Definition 1.2.2. Let (X, g) and (Y, h) be two Riemannian manifolds. A diffeomorphism $\varphi : X \to Y$ is *conformal* if and only if g and the induced metric on X by φ are proportional. In other words, if and only if there exists $\mu : X \to \mathbb{R}$ such that

$$\varphi^* h = e^{2\mu} g.$$

The conformal group of (X, g), denoted $\mathrm{Conf}(X, g)$ is the set of conformal diffeomorphim $(X, g) \to (X, g)$.

Since, in this section, we will mostly work with immersions in $\mathbb{R}^3$ with the classic euclidean product, the corresponding conformal group $\mathrm{Conf}(\mathbb{R}^3 \cup \{\infty\})$ will be of crucial interest to us. We will often perform a slight abuse of notations and write $\mathrm{Conf}(\mathbb{R}^3)$ instead of the more formally correct $\mathrm{Conf}(\mathbb{R}^3 \cup \{\infty\})$. This group is fully described by Liouville theorem (see theorem 1.1.1 of [AG96] for a proof).

Theorem 1.2.1. Any $\varphi \in \mathrm{Conf}(\mathbb{R}^3)$ satisfies

$$\varphi = T_{\vec{a}} \circ R_\Theta \circ D_\lambda$$

if $\varphi(\infty) = \infty$,

$$\varphi = T_{\vec{b}} \circ R_\Theta \circ D_\lambda \circ \iota \circ T_{\vec{a}}$$

otherwise. Here $T_{\vec{a}}$ and $T_{\vec{b}}$ denote translations, D_λ a dilation, R_Θ a rotation and $\iota : x \mapsto \frac{x}{|x|^2}$ the inversion at the origin. Such decompositions are unique.

Proposition 1.2.2. The Willmore energy is invariant under the action of translations, rotations and dilations. It is not left invariant by the inversions.

Proof. Since H is an extrinsic metric invariant it is left unchanged under the action of ambient isometries. A straightforward computations shows that a dilation of factor f changes H into $f^{-1}H$, while $d$vol turns into $f^2 d$vol. Then $H^2 d$vol is pointwise invariant under the action of dilations. The Willmore energy is thus as stated unchanged by translations, rotations and dilations.

A round sphere has a Willmore energy of 4π, while a plane has a Willmore energy of 0. Since inverting a round sphere at one of its point yields a plane, the Willmore energy is not an inversion invariant. □

Proposition 1.2.2 does not allow us to consider W as a full conformal invariant due to the action of some inversions. However the tracefree fundamental form yields a real conformal invariant quantity.

Proposition 1.2.3. The quantity $\left|\mathring{A}\right|_g^2 d\mathrm{vol}_g$ is a pointwise conformal invariant.

Proof. We refer the reader to theorem 7.3.1 of [Wil93] for a detailed proof. $\square$

The "real" conformal invariant energy is then the total tracefree curvature

$$\mathcal{E}(\Phi) = \int_\Sigma |\mathring{A}|_g d\mathrm{vol}_g.$$

Besides, this quantity is deeply correlated with the Willmore energy, which will allow us to recover conformal invariance properties for W.

Lemma 1.2.1. If we denote $K := \det\left(g^{-1}A\right)$ the Gauss curvature of Φ, we have

$$|\mathring{A}|_g^2 = \frac{1}{2}|A|_g^2 - K = 2H^2 - 2K. \tag{1.2.1}$$

Proof. At a given point, we write

$$g^{-1}A = \begin{pmatrix} \epsilon & \phi \\ \phi & \gamma \end{pmatrix}.$$

Then

$$H = \frac{\epsilon + \gamma}{2}$$

and

$$g^{-1}\mathring{A} = \begin{pmatrix} \frac{\epsilon-\gamma}{2} & \phi \\ \phi & -\frac{\epsilon-\gamma}{2} \end{pmatrix}.$$

Hence

$$\begin{aligned} |\mathring{A}|_g^2 &= 2\left(\left(\frac{\epsilon-\gamma}{2}\right)^2 + \phi^2\right) \\ &= \frac{\epsilon^2}{2} + \frac{\gamma^2}{2} + \phi^2 - \left(\epsilon\gamma - \phi^2\right) = \frac{1}{2}|A|_g^2 - K \\ &= 2\left(\frac{\epsilon+\gamma}{2}\right)^2 - 2\left(\epsilon\gamma - \phi^2\right) = 2H^2 - 2K. \end{aligned}$$

This concludes the proof. $\square$

Using the Gauss-Bonnet formula, we can conclude that the Willmore energy and the total tracefree curvature differ by a topological quantity :

Proposition 1.2.4. Let $\chi(\Sigma)$ denote the Euler characteristic of Σ, and

$$E(\Phi) = \int_\Sigma |A|_g^2 \, d\mathrm{vol}_g = \int_\Sigma |\nabla_g \vec{n}|^2 \, d\mathrm{vol}_g.$$

Then

$$\begin{aligned} \mathcal{E}(\Phi) &= 2W(\Phi) - 2\chi(\Sigma) \\ &= \frac{1}{2}E(\Phi) - \chi(\Sigma). \end{aligned} \tag{1.2.2}$$

Equalities (1.2.2) show that W (and E) are *contingent* invariant under the action of ambient conformal diffeomorphisms *as long as the topology is not changed* (meaning as long as you do not center an inversion on the surface). These conformal invariance properties introduced by W. Blaschke ([Bla55]) and rediscovered by T. Willmore ([Wil93]) make the interest of these energies in conformal geometry clear.

It is worth mentioning that these properties are not generic for second fundamental form based functionals. Indeed, in [MN18], A. Mondino and H. Nguyen have shown that among all the possible curvature functionals depending on the second fundamental form, the Willmore functional is the only conformal invariant, up to topological terms. This property helps explain the interest the Willmore functional has garnered through the years, and its importance in conformal geometry.

Since many prominent examples of minimal surfaces with branched ends (namely the Enneper surface, the Chen-Gackstatter surfaces and López surfaces) will be pivotal in what follows, we extend our studies to non compact branched immersions.

Definition 1.2.3. Let Σ be a compact Riemann surface. An application $\Phi : \Sigma \to \mathbb{R}^3$ is a branched immersion if and only if it is an immersion away from a finite number of points $p_1, \dots, p_n$, around which $|\nabla \Phi| \sim_{p_i} C r^{l_i}$ with $l_i \in \mathbb{N}^*$. The p_i are *branch points* of multiplicity $l_i + 1$.
It is a non compact branched immersion if and only if it is a branched immersion away from a finite number of points $q_1, \dots, q_m$, around which $|\nabla \Phi| \sim_{q_i} C r^{-l_i}$ with $l_i \in \mathbb{N}^*$. The point q_i is then an *end* (possibly branched) of multiplicity $l_i - 1$.

There exists a phrasing of Gauss-Bonnet formula for branched (non necessarily compact) immersions :

Theorem 1.2.5. Let $\Phi : \Sigma \to \mathbb{R}^3$ be a branched immersion. Let $(p_i)_{i=1,..n}$ be its branch points of multiplicity n_i, and $(q_j)_{j=1,..m}$ be its ends of multiplicity m_j. Then

$$\int_\Sigma K d\mathrm{vol}_g = 2\pi \left(\chi(\Sigma) + \sum_{i=1}^{n} (n_i - 1) - \sum_{j=1}^{m} (m_j + 1) \right). \tag{1.2.3}$$

Proof. We only give the ideas of the proof and refer the reader to theorem 2.6 of [LN15] for details.

One can apply Gauss-Bonnet formula with boundary to

$$\Sigma_r = \Sigma \backslash \left(\bigcup_{i=1}^{n} B_r(p_i) \cup \bigcup_{j=1}^{m} B_r(q_j) \right).$$

Denoting k_g the geodesic curvature obtain :

$$\int_{\Sigma_r} K d\mathrm{vol}_g + \sum_{i=1}^{n} \int_{\partial B_r(p_i)} k_g ds + \sum_{j=1}^{m} \int_{\partial B_r(q_j)} k_g ds = 2\pi \chi(\Sigma_r).$$

Letting $r \to 0$ yields the desired result, given the behavior of $\nabla \Phi$ around the branch points and ends. □

Equality (1.2.3) gives a very efficient way to compute the transformations of W under the action of inversions.

Theorem 1.2.6. Let $\Phi : \Sigma \to \mathbb{R}^3$ be a branched immersion, and $\Theta \in \mathrm{Conf}(\mathbb{R}^3)$. Let $\Psi = \Theta \circ \Phi$. We denote $p_1, \dots p_a$ the branch points of Φ *that become ends* of Ψ and $q_1, \dots q_b$ the ends of Φ *that become branch points* of Ψ. We denote $n_1 \dots n_a$ and $m_1 \dots m_b$ their respective multiplicity. Then

$$W(\Phi) = W(\Psi) + 4\pi \left(\sum_{i=1}^{a} n_i - \sum_{j=1}^{b} m_j \right).$$

Proof. We denote $p_1, \dots p_a, p_{a+1}, \dots p_n$ the branch points and $q_1, \dots q_b, q_{b+1}, \dots q_m$ the ends of Φ and denote n_i (respectively m_j) their multiplicity. Applying (1.2.3) to Φ yields :

$$\int_\Sigma K_\Phi d\mathrm{vol}_{g_\Phi} = 2\pi \left(\chi(\Sigma) + \sum_{i=1}^{a} (n_i - 1) - \sum_{j=1}^{b} (m_j + 1) + \sum_{i=a+1}^{n} (n_i - 1) - \sum_{j=b+1}^{m} (m_j + 1) \right). \tag{1.2.4}$$

The branch points of Ψ are $q_1, \dots, q_b, p_{a+1}, \dots, p_n$ of multiplicity $m_1, \dots, m_b, n_{a+1}, \dots, n_n$. Its ends are $p_1, \dots, p_a, q_{b+1}, \dots q_m$. of mutliplicity $n_1, \dots, n_a, m_{b+1}, \dots m_m$. Then applying (1.2.3) to Φ gives

$$\int_\Sigma K_\Psi d\mathrm{vol}_{g_\Psi} = 2\pi \left(\chi(\Sigma) - \sum_{i=1}^{a} (n_i + 1) + \sum_{j=1}^{b} (m_j - 1) + \sum_{i=a+1}^{n} (n_i - 1) - \sum_{j=b+1}^{m} (m_j + 1) \right). \tag{1.2.5}$$

Integrating equality (1.2.1) applied to Φ and Ψ states, since $\mathcal{E}$ is a conformal invariant :

$$2W(\Phi) - 2\int_\Sigma K_\Phi d\mathrm{vol}_{g_\Phi} = \mathcal{E}(\Phi) = \mathcal{E}(\Psi) = 2W(\Psi) - 2\int_\Sigma K_\Psi d\mathrm{vol}_{g_\Psi}.$$

Consequently

$$W(\Phi) = W(\Psi) + \int_\Sigma K_\Phi d\mathrm{vol}_{g_\Phi} - \int_\Sigma K_\Psi d\mathrm{vol}_{g_\Psi}. \tag{1.2.6}$$

Injecting (1.2.4) and (1.2.5) into (1.2.6) yields

$$W(\Phi) = W(\Psi) + 4\pi \left(\sum_{i=1}^{a} n_i - \sum_{j=1}^{b} m_j \right),$$

which concludes the proof. □

An immediate consequence is a characterization of the conformal transformations of a minimal surface (that we call *conformally minimal*) :

Corollary 1.2.1. A branched immersion $\Phi : \Sigma \to \mathbb{R}^3$ is the conformal transform of a branched minimal immersion in $\mathbb{R}^3$ if and only if

$$W(\Phi) = 4\pi\theta(p, \Phi),$$

where p is the point of $\mathbb{R}^3$ of highest density for Φ.

Trying to bound the Willmore energy from below for compact surfaces (minimal surfaces offer a trivial bound for non-compact ones) has proven an interesting and difficult question. While it is not the main subject of the present work, it is worth going over, if only briefly.

The first bound came from T. Willmore (in [Wil65], see also theorem 7.2.2 of [Wil93]) :

Theorem 1.2.7. Let Σ be a closed orientable surface and $\Phi : \Sigma \to \mathbb{R}^3$ be a smooth immersion of Σ into $\mathbb{R}^3$. Then

$$W(\Phi) \geq 4\pi.$$

Moreover $W(\Phi) = 4\pi$ if and only if $\Phi(\Sigma)$ is a round sphere.

T. Willmore formulated in [Wil65] its eponym conjecture, which was proved in 2015 by F. Marques and A. Neves (see [MN16]).

Theorem 1.2.8. The minimum of the Willmore energy among surfaces of genus 1 is $2\pi^2$. It is reached by the Clifford torus.

The Clifford torus can be seen equivalently as the surface parametrized by

$$\begin{cases} x = \left(1 + \frac{\sqrt{2}}{2}\cos u\right)\cos v \\ y = \left(1 + \frac{\sqrt{2}}{2}\cos u\right)\sin v \\ z = \frac{\sqrt{2}}{2}\sin u, \end{cases}$$

or as the image by a stereographic projection of the natural embedding of $\frac{\mathbb{S}^1}{\sqrt{2}} \times \frac{\mathbb{S}^1}{\sqrt{2}}$ in $\mathbb{S}^3$.

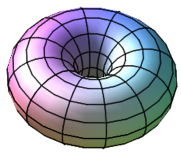

Figure 1.1 – The Clifford Torus

Little is known concerning the minima of the Willmore energy for higher genus beyond the conjectured shape of the minimizer for the genus 2 (see [Kus89] by R. Kusner). The following inequality is however worth mentioning :

Theorem 1.2.9. Let $\Phi : \Sigma \to \mathbb{R}^3$. Then for all $p \in \mathbb{R}^3$

$$W(\Phi) \geq 4\pi\theta\left(\Phi, p\right), \tag{1.2.7}$$

where $\theta\left(\Phi, p\right)$ is the density of Φ at p.

This result known as Li-Yau inequality (see theorem 6 in [LY82]), combined with the existence of examples of energy strictly below 8π for any genus (the Lawson surfaces, see [Law70]), proves that the minimizers of the Willmore energy are embedded.

1.2.2 Willmore surfaces

Definition 1.2.4. A Willmore immersion $\Phi \in C^\infty\left(\Sigma, \mathbb{R}^3\right)$ is a smooth immersion which is a critical point of W (or equivalently, given (1.2.2), of E and $\mathcal{E}$).

The immersed surface $\Phi(\Sigma)$ is then a Willmore surface. We will sometimes refer to $\Phi(\Sigma)$ as a *Willmore sphere* or a *Willmore torus* depending on its genus.

While $\mathcal{E}$ has the best invariance properties of all three energies, W is not only the historically studied quantity, it more organically leads to exploitable equations as we will see in subsections 1.2.3 and 1.2.4. Further since W is non-invariant by only a Lebesgue neglectible subset of the inversions of $\mathbb{R}^3$ (those with center on the surface), it will have little consequences on the analysis of Willmore immersions.

Since $\mathcal{E}$ is a conformal invariant, the notion of Willmore immersion (respectively of Willmore surface) is invariant by conformal transformations.

Proposition 1.2.10. Let $\Phi \in C^{\infty}\left(\Sigma, \mathbb{R}^3\right)$ and $\Theta \in \mathrm{Conf}(\mathbb{R}^3)$. Then Φ is a Willmore immersion if and only if $\Theta \circ \Phi$ is a Willmore immersion.

Corollary 1.2.2. Minimal immersions are non compact Willmore immersions. Inversions of minimal immersions are Willmore immersions. When the inversions are centered away from the surface, they are compact Willmore immersions.

In fact, Willmore surfaces were partly conceived as a generalization of the conformal transformations of minimal surfaces, and W. Blaschke used the terminology "conformal minimal surfaces" in [Bla55]. From this we can build a zoology of Willmore immersions and branched Willmore immersions having a wide specter of properties.

Example 1.2.1. The round sphere is a Willmore surface of Willmore energy

$$W\left(\mathbb{S}^2\right) = 4\pi.$$

Example 1.2.2. The Bryant's surface is a four ended immersed minimal surface. An exemple of parametrization over $\mathbb{C}$ is given by the following (see theorem E [Bry84] for details or below in section 3.3 where we play with the relative position of the ends) :

$$\Phi = 2\Re\left(v_0 z - \frac{v_1}{z-1} - \frac{v_2}{z-j} - \frac{v_3}{z-j^2}\right),$$

where $j^3 = 1$ and v_0 v_1, v_2, $v_3 \in \mathbb{C}^3$ satisfy

$$\begin{aligned}
&\langle v_i, v_j \rangle = \lambda \neq 0, \quad 1 \leq i < j \leq 3\\
&\langle v_i, v_i \rangle = 0, \quad 1 \leq i \leq 3,\\
&v_0 = \frac{1}{3}\left(v_1 + j v_2 + j^2 v_3\right).
\end{aligned}$$

Its inverse forms a compact immersed Willmore sphere of Willmore energy 16π. Inverting it at a point of density 1 yields a non-compact non minimal Willmore sphere of Willmore energy 12π.

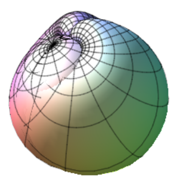

Figure 1.2 – An inverted Bryant's surface

Example 1.2.3. The López surface is a two ended minimal sphere, with one end of multiplicity 1 and one of multiplicity 3. A parametrization is given by :

$$\Phi = 2\Re\left(\frac{3}{2z^3}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{z}{8}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + \frac{3}{2z}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right).$$

Its inverse is a branched Willmore sphere, with a point of density 4, split into a branch point of multiplicity 3 and one of multiplicity 1. They have Willmore energy 16π.

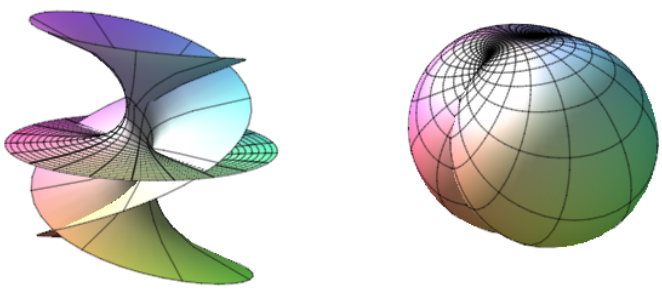

Figure 1.3 – The López surface and an inversion

Example 1.2.4. The Clifford torus is a Willmore torus of Willmore energy $W = 2\pi^2$. Given corollary 1.2.1 it is thus not the conformal transform of a minimal surface in $\mathbb{R}^3$.

We can give a few additionnal examples of branched Willmore surfaces.

Example 1.2.5. The Enneper surface, parametrized on $\mathbb{C}$ by

$$E(z) = 2\Re\left(\int \frac{1}{2}\begin{pmatrix} 1-z^2 \\ i(1+z^2) \\ 2z \end{pmatrix}\right),$$

is a minimal surface of total curvature

$$\int_{\mathbb{C}} K d\mathrm{vol} = -4\pi.$$

Its inverse is a Willmore sphere of Willmore energy $W(E) = 12\pi$.

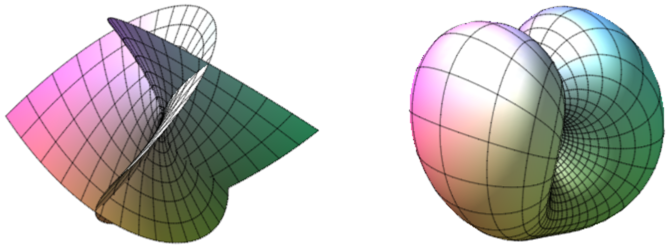

Figure 1.4 – The Enneper surface and an inversion

Example 1.2.6. The Chen-Gackstatter torus is a minimal torus of Enneper-Weierstrass data on $\mathbb{C}^2/\mathbb{Z}^2$: $(f, g) = \left(2\mathfrak{p}(z), A\frac{\mathfrak{p}_z}{\mathfrak{p}}(z)\right)$ (see [CG82]) where $\mathfrak{p}$ is the Weierstrass elliptic function, of elliptic invariants (see [Apo90])

$$\begin{aligned} g_2 &= 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+ni)^4} > 0, \\ g_3 &= 0, \end{aligned}$$

and

$$A = \sqrt{\frac{3\pi}{2g_2}} \in \mathbb{R}_+.$$

It has a branched end of multiplicity 3 asymptotic to the Enneper surface and is thus of total curvature

$$\int K d\mathrm{vol} = -8\pi.$$

Its inverse is a Willmore torus of Willmore energy $W = 12\pi$.

Example 1.2.7. One can define Chen-Gackstatter surfaces of arbitrarily high genus g. They have a single branched end of multiplicity 3 asymptotic to the Enneper surface and have a total curvature of $-4\pi\,(g+1)$. Their inverses are branched Willmore surfaces of genus g and of Willmore energy $W = 12\pi$.

The example of the Clifford torus ensures that not all Willmore surfaces are conformal transformations of minimal surfaces in $\mathbb{R}^3$. There are however classification results for Willmore spheres :

Theorem 1.2.11. Every immersed Willmore sphere in $\mathbb{R}^3$ is the conformal transform of a minimal sphere in $\mathbb{R}^3$.

Theorem 1.2.12. Every branched Willmore sphere in $\mathbb{R}^3$ with less than 3 branch points is the conformal transform of a branched minimal sphere in $\mathbb{R}^3$.

Theorem 1.2.11 (due to R. Bryant, theorem E in [Bry84]) and theorem 1.2.12 (partly due to T. Lamm and H. Nguyen [LN15] and A. Michelat and T. Rivière [MR17]) will be proved below (section 2.6) with the conformal Gauss map tools.

A corollary to theorems 1.2.11 and 1.2.12 allows one to enumerate low energy Willmore spheres :

Corollary 1.2.3. The round sphere and the catenoïd are the only Willmore spheres of energy strictly lower than 12π.

One must point out that while the Clifford torus is not the conformal transform of a minimal surface in $\mathbb{R}^3$, it is the stereographic projection (and thus a conformal transform) of a minimal surface in $\mathbb{S}^3$. We will detail below necessary and sufficient conditions for a surface to be conformally minimal in $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ (see section 2.5). Beyond that, one can conjecture that Willmore surfaces are conformal transformations of minimal surfaces in a subset of $\mathbb{R}^3$ imbued with a conformally flat metric, although this remains somewhat speculative.

1.2.3 Conservation laws

The aim of this subsection is to draw the Lagrange equation satisfied by Willmore surfaces, and to exploit the conformal invariance of W to highlight four conservation laws, in accordance with Noether's theorem. Indeed the latter states in spirit

"Each infinitesimal symmetry induces a conservation law."

While we refer the reader to theorem 1.3.1 of [Hél02] for a proper wording of the theorem, this mere idea will satisfy our current needs. Since the Willmore energy is a conformal invariant, one can expect a conservation law for each of the four fundamental conformal transformations (translation, dilation, rotation, inversion). The ideas and much of the computations are taken from Y. Bernard's [Ber16] which thoroughly studies W. We will however slightly extend our scope to $\mathcal{E}$ and E.

We consider Σ a Riemann surface (that may have a boundary), an (eventually branched) immersion $\Phi \in C^\infty\left(\Sigma, \mathbb{R}^3\right)$, Ω an open subset of Σ (away from the possible branch points and branched ends) and $X \in C^\infty_c\left(\Omega, \mathbb{R}^3\right)$. We use the notations introduced in subsection 1.2.1. We will study the following perturbation of the immersion Φ :

$$\Phi_t := \Phi + tX.$$

We will use the tensorial language in a local map and denote D the Levi-Civita connection. Then if $V = V^{i_1,\dots,i_{k_1}}_{j_1,\dots,j_{k_2}} \partial_{i_1} \dots \partial_{i_{k_1}} \partial^{j_1} \dots \partial^{j_{k_2}}$, one has $DV = \nabla_p V^{i_1,\dots,i_{k_1}}_{j_1,\dots,j_{k_2}} \partial_{i_1} \dots \partial_{i_{k_1}} \partial^{j_1} \dots \partial^{j_{k_2}} \partial^p$. We will denote $\delta := \frac{d}{dt}\big|_{t=0}$. For computational convenience we will decompose

$$X := \vec{N} + \vec{T} := N\vec{n} + T^p \nabla_p \Phi.$$

The endgame here is to compute $\delta\left(H^2 d\mathrm{vol}_g\right)$ and $\delta\left(\left|\mathring{A}\right|^2_g d\mathrm{vol}_g\right)$, or equivalently in any local chart $\delta\left(H^2 |g|^{\frac{1}{2}}\right)$ and $\delta\left(\left|\mathring{A}\right|^2_g |g|^{\frac{1}{2}}\right)$. Then

$$\delta\left(\nabla_i \Phi_t\right) = \nabla_i \vec{N} + \nabla_i T^p \nabla_p \Phi + T^p \nabla_{ip} \Phi. \tag{1.2.8}$$

However by definition of the Levi-Civita connection, in a local chart :

$$\nabla_{ip}\Phi = \partial_{ip}\Phi - \Gamma^k_{ip}\partial_k \Phi,$$

where $\Gamma^k_{ip} := \frac{1}{2} g^{kl}\left(\partial_i g_{pl} + \partial_p g_{il} - \partial_l g_{ip}\right)$ are the Christoffel's symbols. Then, since $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$, one can compute

$$\begin{aligned}
\Gamma^k_{ip} &= \frac{1}{2} g^{kl}\left(\langle \partial_{ip}\Phi, \partial_l \Phi\rangle + \langle \partial_p \Phi, \partial_{il}\Phi\rangle + \langle \partial_{ip}\Phi, \partial_l\Phi\rangle + \langle \partial_i\Phi, \partial_{pl}\Phi\rangle - \langle \partial_{il}\Phi, \partial_p\Phi\rangle - \langle \partial_i\Phi, \partial_{pl}\Phi\rangle\right)\\
&= g^{kl}\langle \partial_{ip}\Phi, \partial_l \Phi\rangle.
\end{aligned}$$

Then $\Gamma^k_{ip}\partial_k\Phi = g^{kl}\langle \partial_{ip}\Phi, \partial_l\Phi\rangle \partial_k \Phi$ is exactly the tangent part of $\partial_{ip}\Phi$. Consequently $\nabla_{ip}\Phi = \partial_{ip}\Phi - \Gamma^k_{ip}\partial_k\Phi$ is the leftover part, that is the normal part. By definition of the second fundamental form introduced in subsection 1.2.1,

$$\nabla_{ip}\Phi = \pi_{\vec{n}}\left(\partial_{ip}\Phi\right) = \vec{A}_{ip}. \tag{1.2.9}$$

Then, injecting (1.2.9) into (1.2.8) yields

$$\delta\left(\nabla_i \Phi_t\right) = \nabla_i \vec{N} + \nabla_i T^p \nabla_p \Phi + T^p \vec{A}_{ip}. \tag{1.2.10}$$

From this, we find

$$\begin{aligned}\delta\left(g_{tij}\right) &= \langle\delta\left(\nabla_i\Phi_t\right),\nabla_j\Phi\rangle + \langle\delta\left(\nabla_j\Phi_t\right),\nabla_i\Phi\rangle \\ &= \langle\nabla_i N\vec{n} + A_{ip}T^p\vec{n} - A_{ip}\nabla^p\Phi + \nabla_i T^p\nabla_p\Phi, \nabla_j\Phi\rangle \\ &+ \langle\nabla_j N\vec{n} + A_{jp}T^p\vec{n} - A_{jp}\nabla^p\Phi + \nabla_j T^p\nabla_p\Phi, \nabla_i\Phi\rangle \\ &= \nabla_i T_j + \nabla_j T_i - 2NA_{ij}.\end{aligned} \tag{1.2.11}$$

Then

$$\delta\left(g_t{}^{ij}\right) = -g^{ip}\delta\left(g_{tpq}\right)g^{qj} = 2NA^{ij} - \nabla^i T^j - \nabla^j T^i. \tag{1.2.12}$$

Consequently,

$$\begin{aligned}\delta\left(|g_t|\right) &= |g|\mathrm{Tr}\left(g^{-1}\delta g\right) = |g|g^{ij}\left(\nabla_i T_j + \nabla_j T_i - 2NA_{ij}\right) \\ &= 2|g|\left(\nabla_p T^p - 2NH\right),\end{aligned} \tag{1.2.13}$$

and

$$\delta\left(|g_t|^{\frac{1}{2}}\right) = |g|^{\frac{1}{2}}\left(\nabla_p T^p - 2NH\right). \tag{1.2.14}$$

Using (1.2.9), we can compute the perturbation of the second fundamental form in the following manner (note that since the goal is to compute the norm of the relevant quantities, only the normal terms are incidental) :

$$\begin{aligned}\delta\left(\vec{A}_{t\,ij}\right) &= \delta\left(\nabla_i\nabla_j\Phi_t\right) \\ &= \nabla_i\left(\nabla_j\vec{N}\right) + \nabla_i\nabla_j T^p\nabla_p\Phi + \nabla_j T^p\vec{A}_{ip} + \nabla_i T^p\vec{A}_{jp} + T^p\nabla_i\vec{A}_{jp} \\ &+ \text{ tangent terms} \\ &= \nabla_i\left(\nabla_j\vec{N}\right) + \nabla_i\nabla_j T^p\nabla_p\Phi + \nabla_j T^p\vec{A}_{ip} + \nabla_i T^p\vec{A}_{jp} + T^p\nabla_i A_{jp}\vec{n} \\ &+ \text{ tangent terms} \\ &= \nabla_i\left(\nabla_j\vec{N}\right) + \nabla_i\nabla_j T^p\nabla_p\Phi + \nabla_j T^p\vec{A}_{ip} + \nabla_i T^p\vec{A}_{jp} + T^p\nabla_p A_{ij}\vec{n} \\ &+ \text{ tangent terms.}\end{aligned} \tag{1.2.15}$$

To obtain this last expression, we have used the Gauss-Codazzi equation (see [Wil65], chapter 3) :

$$\nabla_i A_{jp} = \nabla_j A_{ip} = \nabla_p A_{ij} \quad \forall i,j,p. \tag{1.2.16}$$

From (1.2.15), we deduce

$$\begin{aligned}\delta\left(\vec{A}^i_{t\,j}\right) &= \delta\left(g_t^{ip}\right)\vec{A}_{pj} + g^{ip}\delta\left(\vec{A}_{t\,pj}\right) \\ &= \left(2NA^{ip} - \nabla^i T^p - \nabla^p T^i\right)\vec{A}_{pj} + \nabla^i\left(\nabla_j\vec{N}\right) + \nabla_j T^p\vec{A}^i_p + \nabla^i T^p\vec{A}_{jp} \\ &+ T^p\nabla_p A^i_j\vec{n} + \text{ tangent terms} \\ &= \nabla^i\left(\nabla_j\vec{N}\right) + 2\vec{N}A^i_pA^p_j + \nabla_j T^p\vec{A}^i_p - \nabla^p T^i\vec{A}_{pj} + T^p\nabla_p A^i_j\vec{n} \\ &+ \text{ tangent terms.}\end{aligned} \tag{1.2.17}$$

Hence,

$$\delta\left(\vec{H}_t\right) = \delta\left(\frac{\vec{A}^i_{t\,i}}{2}\right) = \frac{1}{2}\Delta_g\vec{N} + |A|^2_g\vec{N} + T^p\nabla_p H\vec{n} + \text{ tangent terms.} \tag{1.2.18}$$

This yields

$$
\begin{aligned}
\delta\left(H_t^2\right) &= \left\langle \Delta_g \vec{N}, \vec{H}\right\rangle + 2\,|A|_g^2\, NH + T_p \nabla_p\left(H^2\right)\\
&= \nabla_i\left(\left\langle \nabla^i \vec{N}, \vec{H}\right\rangle\right) - \nabla^i N \nabla_i H - NH\left\langle \nabla^i \vec{n}, \nabla_i \vec{n}\right\rangle + 2NH\,|A|^2\\
&+ T^p \nabla_p\left(H^2\right)\\
&= \nabla_i\left(\left\langle \nabla^i \vec{N}, \vec{H}\right\rangle - \left\langle \vec{N}, \nabla^i \vec{H}\right\rangle\right) + N\Delta_g H + NH\,|A|_g^2 + T^p\nabla_p\left(H^2\right).
\end{aligned}
\tag{1.2.19}
$$

One must point out that to obtain the last equality we have computed

$$
\left\langle \nabla^i \vec{n}, \nabla_i \vec{n}\right\rangle = \left\langle -A^{ip}\nabla_p \Phi, -A_i^q \nabla_q \Phi\right\rangle = A_q^i A_i^q = |A|_g^2\,. \tag{1.2.20}
$$

From (1.2.14) and (1.2.19) (and using (1.2.1) for the different formulations) we finally reach

$$
\begin{aligned}
\delta\left(H_t^2\,|g_t|^{\frac{1}{2}}\right) &= |g|^{\frac{1}{2}}\,\nabla_p\left(\left\langle \nabla^p \vec{N}, \vec{H}\right\rangle - \left\langle \vec{N}, \nabla\vec{H}\right\rangle + T^p H^2\right)\\
&+ |g|^{\frac{1}{2}}\,N\left(\Delta_g H + H\left(|A|_g^2 - 2H^2\right)\right)\\
&= |g|^{\frac{1}{2}}\,\nabla_p\left(\left\langle \nabla^p \vec{N}, \vec{H}\right\rangle - \left\langle \vec{N}, \nabla\vec{H}\right\rangle + T^p H^2\right)\\
&+ |g|^{\frac{1}{2}}\,N\left(\Delta_g H + 2H\left(H^2 - K\right)\right)\\
&= |g|^{\frac{1}{2}}\,\nabla_p\left(\left\langle \nabla^p \vec{N}, \vec{H}\right\rangle - \left\langle \vec{N}, \nabla\vec{H}\right\rangle + T^p H^2\right)\\
&+ |g|^{\frac{1}{2}}\,N\left(\Delta_g H + H\big|\mathring{A}\big|_g^2\right).
\end{aligned}
\tag{1.2.21}
$$

To simplify notations we denote $\mathcal{W}(\Phi) = \Delta_g H + H\big|\mathring{A}\big|_g^2$. We now compute with Gauss-Codazzi

$$
\begin{aligned}
\delta\left(\frac{|A_t|_g^2}{2}\right) &= \left\langle \delta\left(\vec{A}^i_{t\,j}\right), \vec{A}^i_j\right\rangle\\
&= \left\langle \nabla^i\left(\nabla_j \vec{N}\right), \vec{A}_i^j\right\rangle + 2NA_p^iA_j^pA_i^j + T^p\nabla_p\left(\frac{|A|_g^2}{2}\right)\\
&= \nabla_i\left(\left\langle \nabla_j\vec{N}, \vec{A}^{ji}\right\rangle\right) - \left\langle \nabla_j \vec{N}, \nabla^i \vec{A}_i^j\right\rangle + 2NA_p^iA_j^pA_i^j + T^p\nabla_p\left(\frac{|A|_g^2}{2}\right)\\
&= \nabla_i\left(\left\langle \nabla_j\vec{N}, \vec{A}^{ji}\right\rangle\right) - \nabla_j N \nabla^i A_i^j + T^p\nabla_p\left(\frac{|A|_g^2}{2}\right)\\
&= \nabla_i\left(\left\langle \nabla_j\vec{N}, \vec{A}^{ji}\right\rangle\right) - 2\nabla_j N \nabla^j H + T^p\nabla_p\left(\frac{|A|_g^2}{2}\right)\\
&= \nabla_i\left(\left\langle \nabla_j\vec{N}, \vec{A}^{ji}\right\rangle - 2\left\langle \vec{N}, \nabla^i \vec{H}\right\rangle\right)\\
&+ 2N\Delta_g H + N\mathrm{Tr}\left(\left(g^{-1}A\right)^3\right) + T^p\nabla_p\left(\frac{|A|_g^2}{2}\right).
\end{aligned}
\tag{1.2.22}
$$

This yields :

$$\begin{aligned}\delta\left(\frac{|A_t|_g^2\,|g_t|^{\frac{1}{2}}}{2}\right)&=|g|^{\frac{1}{2}}\nabla_p\left(\left\langle\nabla_q\vec{N},\vec{A}^{pq}\right\rangle-2\left\langle\vec{N},\nabla^p\vec{H}\right\rangle+T^p\frac{|A|_g^2}{2}\right)\\&\quad+2|g|^{\frac{1}{2}}N\left(\Delta_g H+\frac{\mathrm{Tr}\left(\left(g^{-1}A\right)^3\right)}{2}-H\frac{|A|_g^2}{2}\right).\end{aligned}\tag{1.2.23}$$

Reusing the notations of the proof of lemma 1.2.1, we find

$$\begin{aligned}\mathrm{Tr}\left(\left(g^{-1}A\right)^3\right)&=\epsilon\left(\epsilon^2+\phi^2\right)+2\left(\epsilon+\gamma\right)\phi^2+\gamma\left(\gamma^2+\phi^2\right)\\&=8H^3-6HK.\end{aligned}$$

And with (1.2.1), we conclude

$$\frac{\mathrm{Tr}\left(\left(g^{-1}A\right)^3\right)}{2}-H\frac{|A|_g^2}{2}=2H(H^2-K).\tag{1.2.24}$$

Injecting (1.2.24) into (1.2.23) then yields

$$\begin{aligned}\delta\left(\frac{|A_t|_g^2\,|g_t|^{\frac{1}{2}}}{2}\right)&=|g|^{\frac{1}{2}}\nabla_p\left(\left\langle\nabla_q\vec{N},\vec{A}^{pq}\right\rangle-2\left\langle\vec{N},\nabla^p\vec{H}\right\rangle+T^p\frac{|A|_g^2}{2}\right)\\&\quad+2|g|^{\frac{1}{2}}N\mathcal{W}\left(\Phi\right).\end{aligned}\tag{1.2.25}$$

Since $\left|\mathring{A}\right|_g^2=|A|^2-2H^2$, we conclude with

$$\begin{aligned}\delta\left(\frac{\left|\mathring{A}_t\right|_g^2|g_t|^{\frac{1}{2}}}{2}\right)&=|g|^{\frac{1}{2}}\nabla_p\left(\left\langle\nabla_q\vec{N},\vec{\mathring{A}}^{pq}\right\rangle-\left\langle\vec{N},\nabla^p\vec{H}\right\rangle+T^p\frac{|\mathring{A}|_g^2}{2}\right)\\&\quad+|g|^{\frac{1}{2}}N\mathcal{W}\left(\Phi\right).\end{aligned}\tag{1.2.26}$$

We will slightly modify (1.2.21) and (1.2.26).

$$\begin{aligned}\left\langle\nabla^p\vec{N},\vec{H}\right\rangle-\left\langle\vec{N},\nabla^p\vec{H}\right\rangle+T^pH^2&=\left\langle\nabla^p\left(\langle X,\vec{n}\rangle\vec{n}\right),\vec{H}\right\rangle-\langle X,\nabla^pH\vec{n}\rangle+H^2\left\langle X,\nabla^p\Phi\right\rangle\\&=\left\langle\nabla^pX,\vec{H}\right\rangle+\langle X,H\nabla^p\vec{n}\rangle-\langle X,\nabla^pH\vec{n}\rangle+H^2\left\langle X,\nabla^p\Phi\right\rangle\\&=\left\langle\nabla^pX,\vec{H}\right\rangle+\left\langle X,H\nabla^p\vec{n}-\nabla^pH\vec{n}+H^2\nabla^p\Phi\right\rangle.\end{aligned}$$

Similarly if we let $\mu_1=\left\langle\nabla_q\vec{N},\vec{\mathring{A}}^{pq}\right\rangle-\left\langle\vec{N},\nabla^p\vec{H}\right\rangle+T^p\frac{|\mathring{A}|_g^2}{2}$, then

$$\begin{aligned}\mu_1&=\left\langle\nabla_q\left(\langle X,\vec{n}\rangle\vec{n}\right),\vec{\mathring{A}}^{pq}\right\rangle-\langle X,\vec{n}\rangle\nabla^pH+\left\langle X,\nabla^p\Phi\right\rangle\frac{|\mathring{A}|_g^2}{2}\\&=\left\langle\nabla_qX,\vec{\mathring{A}}^{pq}\right\rangle+\left\langle X,\nabla_q\vec{n}\right\rangle\mathring{A}^{pq}-\langle X,\vec{n}\rangle\nabla^pH+\left\langle X,\nabla^p\Phi\right\rangle\frac{|\mathring{A}|_g^2}{2}\\&=\left\langle\vec{n},\mathring{A}^{pq}\nabla_qX\right\rangle+\left\langle X,\mathring{A}^{pq}\nabla_q\vec{n}-\nabla^pH\vec{n}+\nabla^p\Phi\frac{|\mathring{A}|_g^2}{2}\right\rangle.\end{aligned}$$

This can be simplified yet further. Let $\mu_2 = \mathring{A}^{pq}\nabla_q\vec{n} - \nabla^p H\vec{n} + \nabla^p\Phi\frac{|\mathring{A}|_g^2}{2}$, and

$$\begin{aligned}\mu_2 &= -H\mathring{A}^{pq}\nabla_q\Phi - \mathring{A}^{pq}\mathring{A}_{ql}\nabla^l\Phi - \nabla^p H\vec{n} + \frac{|\mathring{A}|_g^2}{2}\nabla^p\Phi \\ &= -\left(\nabla^p H\vec{n} + H\mathring{A}^{pq}\nabla\Phi\right) + \frac{|\mathring{A}|_g^2}{2}\nabla^p\Phi - \mathring{A}^p_q\mathring{A}^q_l\nabla^l\Phi.\end{aligned} \tag{1.2.27}$$

If we notice that $\mathring{A}^p_q\mathring{A}^q_l = \left(\mathring{A}^2\right)^p_l$, and using once more the notations of the proof of lemma 1.2.1, we compute

$$\begin{aligned}\mathring{A}^2 &= \begin{pmatrix} \frac{\epsilon-\gamma}{2} & \phi \\ \phi & -\frac{\epsilon-\gamma}{2}\end{pmatrix}^2 \\ &= \begin{pmatrix} \left(\frac{\epsilon-\gamma}{2}\right)^2 + \phi^2 & 0 \\ 0 & \left(\frac{\epsilon-\gamma}{2}\right)^2 + \phi^2\end{pmatrix} \\ &= \frac{\left|\mathring{A}\right|_g^2}{2} Id.\end{aligned} \tag{1.2.28}$$

Injecting the latter into (1.2.27) then means :

$$\mathring{A}^{pq}\nabla_q\vec{n} - \nabla^p H\vec{n} + \nabla^p\Phi\frac{|\mathring{A}|_g^2}{2} = -\left(\nabla^p H\vec{n} + H\mathring{A}^{pq}\nabla_q\Phi\right).$$

Similarly

$$H\nabla^p\vec{n} - \nabla^p H\vec{n} + H^2\nabla^p\Phi = -\left(\nabla^p H\vec{n} + H\mathring{A}^{pq}\nabla_q\Phi\right).$$

Canonically, the expression $\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H} = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)$, where $\times$ is the vectorial product of $\mathbb{R}^3$ and $\nabla^\perp = \begin{pmatrix} -\partial_y \\ \partial_x\end{pmatrix}$, is often used in local conformal charts. We can then conclude these computations

$$\begin{aligned}\delta\left(H_t^2\,|g_t|^{\frac{1}{2}}\right) &= \frac{|g|^{\frac{1}{2}}}{2}\nabla.\left(2\left\langle\nabla X, \vec{H}\right\rangle + \left\langle X, -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)\right\rangle\right) \\ &\quad + |g|^{\frac{1}{2}}\,N\mathcal{W}(\Phi), \\ \delta\left(\frac{\left|\mathring{A}_t\right|_g^2|g_t|^{\frac{1}{2}}}{2}\right) &= \frac{|g|^{\frac{1}{2}}}{2}\nabla.\left(2\left\langle\mathring{A}\nabla X, \vec{n}\right\rangle + \left\langle X, -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)\right\rangle\right) \\ &\quad + |g|^{\frac{1}{2}}N\mathcal{W}(\Phi),\end{aligned} \tag{1.2.29}$$

where $\nabla.V = \nabla_p V^p = \mathrm{div}_g V$.

From this we deduce the Euler-Lagrange equation for Willmore surfaces :

Theorem 1.2.13. Let Σ be a Riemann surface and $\Phi\in C^\infty\left(\Sigma,\mathbb{R}^3\right)$ an immersion. Then Φ is a Willmore immersion if and only if it satisfies the *Willmore equation*

$$\mathcal{W}(\Phi) := \Delta_g H + H\left|\mathring{A}\right|_g^2 = 0. \tag{1.2.30}$$

If Φ is branched, it is Willmore if and only if (1.2.30) stands away from the branch points.

Proof. The immersion Φ is Willmore if and only if it is a critical point of W (equivalently of $\mathcal{E}$), meaning if and only if, using the introduced notations, for all perturbations X :

$$\delta\left(\int_\Sigma H_t^2 \mathrm{dvol}_{g_t}\right) = 0.$$

This condition is equivalent to

$$\int_\Sigma \delta\left(H_t^2 \mathrm{dvol}_{g_t}\right) = \int_\Sigma \operatorname{div}\left(\dots\right) + |g|^{\frac{1}{2}} N\mathcal{W}\left(\Phi\right) = \int_\Sigma |g|^{\frac{1}{2}} N\mathcal{W} = 0.$$

Since this is true for all compactly supported N, this concludes the proof. The computations with $\mathcal{E}$ yield the same result. □

This equation was known since the works of W. Blaschke. There exists a similar equation in higher codimension (see T. Rivière [Riv08]), but it is outside the frame of the present work. One must notice that the divergence terms have not (classically) impacted the Euler-Lagrange equation. They will however be pivotal in establishing conservation laws. Let (φ_t) be a sequence of maps of $\mathbb{R}^3$ leaving $H^2|g|^{\frac{1}{2}}$ invariant, with $\varphi_0 = Id$. Let Φ be a Willmore surface and $\Phi_t = \varphi_t \circ \Phi = \Phi + tX + o(t)$. The vector X is the infinitesimal symmetry associated to φ_t alluded to in the incipit of this subsection. Then by definition of φ_t :

$$H_t^2 |g_t|^{\frac{1}{2}} = H^2 |g|^{\frac{1}{2}} + t\delta\left(H_t^2 |g_t|^{\frac{1}{2}}\right) + o(t) = H^2 |g|^{\frac{1}{2}}.$$

Then, given that Φ is a Willmore surface,

$$\delta\left(H_t^2 |g_t|^{\frac{1}{2}}\right) = \frac{|g|^{\frac{1}{2}}}{2}\nabla.\left(2\left\langle \nabla X, \vec{H}\right\rangle + \left\langle X, \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right\rangle\right) = 0.$$

This is the conservation law associated to the infinitesimal symmetry X. We then only have to plug in the infinitesimal symetries corresponding to the generators of the conformal group to conclude.

Translations :

Here $\varphi_t(x) = x + t\vec{a}$ with $\vec{a} \in \mathbb{R}^3$, then $X = \vec{a}$ and the corresponding conservation law is

$$\forall \vec{a} \in \mathbb{R}^3 \quad \operatorname{div}\left(\left\langle \vec{a}, -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)\right\rangle\right) = 0.$$

Consequently one can introduce the first conserved quantity :

$$V_{\text{tra}} = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right). \tag{1.2.31}$$

This quantity satisfies $\operatorname{div}\left(V_{\text{tra}}\right) = 0$.

Dilations :

Here $\varphi_t(x) = (1+t)x$, then $X = \Phi$ and the corresponding conservation law is

$$\operatorname{div}\left(\langle \Phi, V_{\text{tra}}\rangle\right) = 0.$$

The second conserved quantity is then

$$V_{\text{dil}} = \left\langle \Phi, -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right)\right\rangle = \langle \Phi, V_{\text{tra}}\rangle. \tag{1.2.32}$$

Rotations :

Here $\varphi_t(x) = M_{t,\vec{a}}x$ where $M_{t,\vec{a}}$ is the rotation of angle t and axis $\vec{a} \in \mathbb{R}^3$. Then $X = \vec{a} \times \Phi$. The induced conservation law is then

$$\forall \vec{a} \in \mathbb{R}^3 \quad \mathrm{div}\left(\langle \vec{a}, \Phi \times V_{\mathrm{tra}} + 2H\nabla\Phi \times \vec{n}\rangle\right) = 0.$$

The third conserved quantity is then

$$V_{\mathrm{rot}} = \Phi \times V_{\mathrm{tra}} + 2H\nabla\Phi \times \vec{n}. \tag{1.2.33}$$

Interestingly, working with $\left|\mathring{A}\right|^2 |g|^{\frac{1}{2}}$ yields a somewhat different conservation law :

$$\forall \vec{a} \in \mathbb{R}^3 \quad \mathrm{div}\left(\left\langle \vec{a}, \Phi \times V_{\mathrm{tra}} + 2(\mathring{A}\nabla\Phi) \times \vec{n}\right\rangle\right) = 0,$$

which yields a divergence free variation on V_{rot} :

$$\tilde{V}_{\mathrm{rot}} = \Phi \times V_{\mathrm{tra}} + 2\left(\mathring{A}\nabla\Phi\right) \times \vec{n}. \tag{1.2.34}$$

It can be shown through (straightforward in a conformal chart) computations (see (A.2.5) in the appendix) that

$$\tilde{V}_{\mathrm{rot}} = V_{\mathrm{rot}} + 2\nabla^{\perp}\vec{n}. \tag{1.2.35}$$

Inversions :

As has been noticed in propositions 1.2.2 and 1.2.1, $H^2|g|^{\frac{1}{2}}$ is not invariant under the action of inversions, but $\left|\mathring{A}\right|^2|g|^{\frac{1}{2}}$ is. We will then work with the latter. Let

$$\varphi_t(x) = \frac{\frac{x}{|x|^2} - t\vec{a}}{\left|\frac{x}{|x|^2} - t\vec{a}\right|^2},$$

with $\vec{a} \in \mathbb{R}^3$. Hence

$$\Phi_t := \varphi_t \circ \Phi = \Phi - t\left(|\Phi|^2\,\vec{a} - 2\,\langle\Phi, \vec{a}\rangle\,\Phi\right) + o(t).$$

One can then inject $X = |\Phi|^2\,\vec{a} - 2\,\langle\Phi, \vec{a}\rangle\,\Phi$ in (1.2.29) to conclude. To that end we compute

$$\begin{aligned}
\left\langle \mathring{A}\nabla X, \vec{n}\right\rangle &= 2\left\langle \mathring{A}\nabla\Phi, \Phi\right\rangle \langle\vec{n}, \vec{a}\rangle - 2\left\langle \mathring{A}\nabla\Phi, \vec{a}\right\rangle \langle\Phi, \vec{n}\rangle \\
&= 2\left\langle \vec{a}, \left\langle \mathring{A}\nabla\Phi, \Phi\right\rangle \vec{n} - \langle\Phi, \vec{n}\rangle\,\mathring{A}\nabla\Phi\right\rangle \\
&= 2\left\langle \vec{a}, \Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right)\right\rangle.
\end{aligned}$$

Moreover

$$\langle X, V_{\mathrm{tra}}\rangle = \left\langle \vec{a}, |\Phi|^2 V_{\mathrm{tra}} - 2\,\langle\Phi, V_{\mathrm{tra}}\rangle\,\Phi\right\rangle.$$

We deduce :

$$\forall \vec{a} \in \mathbb{R}^3 \quad \mathrm{div}\left(\left\langle \vec{a}, |\Phi|^2 V_{\mathrm{tra}} - 2\,\langle\Phi, V_{\mathrm{tra}}\rangle\,\Phi + 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right)\right\rangle\right) = 0.$$

From this, we find the fourth conserved quantity :

$$\begin{aligned}
V_{\mathrm{inv}} &= -|\Phi|^2 V_{\mathrm{tra}} + 2\,\langle\Phi, V_{\mathrm{tra}}\rangle\,\Phi - 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right) \\
&= -|\Phi|^2 V_{\mathrm{tra}} + 2V_{\mathrm{dil}}\Phi - 4\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right).
\end{aligned} \tag{1.2.36}$$

We have thus proven the following result :

Theorem 1.2.14. Let Σ be a Riemann surface and $\Phi \in C^\infty\left(\Sigma, \mathbb{R}^3\right)$ a Willmore immersion. Then Φ satisfies the following conservation laws :

$$\operatorname{div}\left(V_{\text{tra}}\right) = \operatorname{div}\left(V_{\text{dil}}\right) = \operatorname{div}\left(V_{\text{rot}}\right) = \operatorname{div}\left(\tilde{V}_{\text{rot}}\right) = \operatorname{div}\left(V_{\text{inv}}\right) = 0, \tag{1.2.37}$$

where

$$\begin{cases} V_{\text{tra}} = -2\left(\nabla H \vec{n} + H \mathring{A} \nabla \Phi\right) \\ V_{\text{dil}} = \langle \Phi, V_{\text{tra}} \rangle \\ V_{\text{rot}} = \Phi \times V_{\text{tra}} + 2H \nabla \Phi \times \vec{n} \\ \tilde{V}_{\text{rot}} = \Phi \times V_{\text{tra}} + 2\left(\mathring{A} \nabla \Phi\right) \times \vec{n} \\ V_{\text{inv}} = -|\Phi|^2 V_{\text{tra}} + 2\langle \Phi, V_{\text{tra}} \rangle \Phi - 4\Phi \times \left(\vec{n} \times \mathring{A} \nabla \Phi\right). \end{cases} \tag{1.2.38}$$

If Φ is branched, these stand away from the branch points.

These conserved quantities were first found by T. Rivière through purely computational means (see theorem I.4 [Riv08]). Y. Bernard then showed in [Ber16] that they stemmed from the conformal invariance of the Willmore energy, did the computations pertaining to the mean curvature and corresponding to the first three conserved quantities. As far as we know the computations corresponding to $\tilde{V}_{\text{rot}}$ and V_{inv} have not been done (although Y. Bernard conjectured them) in that way (see section 3.1 in [MR17] by A. Michelat and T. Rivière for a mildly different formalism).

Remark 1.2.1. The conservation law proceeding from the invariance by translations is in fact the Willmore equation put in divergence form (see [Riv08] for this process). It requires much weaker assumptions for it to have a distributional meaning, compared to (1.2.30) and will thus be central in defining a notion of "weak Willmore immersion" (see section 1.4).

In the following, we will often work in local conformal charts on the Riemann surface (this can be done without losing any generality see the discussion in section 1.3) . The immersions can then be seen as immersions Φ from the unit disk $\mathbb{D}$ into $\mathbb{R}^3$ that are conformal (when the two spaces are imbued with their canonical euclidean metric). Denoting $\partial_x \Phi = \Phi_x$ and $\partial_y \Phi = \Phi_y$ this translates as

$$\begin{aligned} \langle \Phi_x, \Phi_x \rangle = \langle \Phi_y, \Phi_y \rangle &:= e^{2\lambda}, \\ \langle \Phi_x, \Phi_y \rangle &= 0. \end{aligned} \tag{1.2.39}$$

The function λ is called the conformal factor of Φ. Working in that framework simplifies computations greatly. For instance, since $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$ and $\nabla^\perp = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}$, one has

$$\nabla \Phi \times \vec{n} = \nabla^\perp \Phi. \tag{1.2.40}$$

In the appendix, we give a family of similar formulas for conformal immersions, in real (section A.2.1) and complex (section A.2.2) notations (for instance (1.2.40) is recalled as (A.2.4) in real notations and A.2.15 in complex notations). These simplify the Willmore equation and the conserved quantities.

Theorem 1.2.15. Let $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ be a conformal immersion, let λ be its conformal factor. Then Φ is Willmore if and only if

$$\Delta H + 2He^{2\lambda}\left(H^2 - K\right) = 0. \tag{1.2.41}$$

The conserved quantities are then

$$\begin{cases} V_{\mathrm{tra}} = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right) = \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^{\perp}\vec{n}\times\vec{H} \\ V_{\mathrm{dil}} = \langle\Phi, V_{\mathrm{tra}}\rangle \\ V_{\mathrm{rot}} = \Phi\times V_{\mathrm{tra}} + 2H\nabla^{\perp}\Phi \\ \tilde{V}_{\mathrm{rot}} = \Phi\times V_{\mathrm{tra}} + 2\mathring{A}\nabla^{\perp}\Phi \\ V_{\mathrm{inv}} = -|\Phi|^2 V_{\mathrm{tra}} + 2\langle\Phi, V_{\mathrm{tra}}\rangle\Phi + 4\Phi\times\left(\mathring{A}\nabla^{\perp}\Phi\right). \end{cases} \tag{1.2.42}$$

Using the conserved quantities in local conformal charts allows one to find a set of equations satisfied by Willmore immersions with better analytical properties than the Willmore equation. Indeed the latter has an overall $\Delta f = f^3$ shape which is unadapted to a weak formulation (it is super critical for the Calderón-Zygmund theory) which we will require to study compactness (see section 3.2.1). The conserved quantities will then prove pivotal for what is to follow.

1.2.4 Willmore equations

Following T. Rivière's original computations in [Riv08], we can modify the conserved quantities into an analytically handier form :

Theorem 1.2.16. Let $\Phi\in C^{\infty}\left(\mathbb{D},\mathbb{R}^3\right)$ be a Willmore conformal immersion of conformal factor $\lambda = \ln|\Phi_x|$, of Gauss map $\vec{n}$, of mean curvature H and of tracefree second fundamental form $\mathring{A}$. Then there exists $\vec{L}\in C^{\infty}\left(\mathbb{D},\mathbb{R}^3\right)$, $S\in C^{\infty}(\mathbb{D},\mathbb{R})$, $\vec{R}\in C^{\infty}\left(\mathbb{D},\mathbb{R}^3\right)$, $\tilde{R}\in C^{\infty}\left(\mathbb{D},\mathbb{R}^3\right)$ such that :

$$\begin{cases} \nabla^{\perp}\vec{L} = -2\left(\nabla H\vec{n} + H\mathring{A}\nabla\Phi\right) = \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^{\perp}\vec{n}\times\vec{H} \\ \nabla^{\perp}S = \left\langle\nabla^{\perp}\Phi, \vec{L}\right\rangle \\ \nabla^{\perp}\vec{R} = \vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi \\ \nabla^{\perp}\tilde{R} = \vec{L}\times\nabla^{\perp}\Phi + 2\mathring{A}\nabla^{\perp}\Phi \end{cases} \tag{1.2.43}$$

Proof. Since $\operatorname{div}(V_{\mathrm{tra}}) = 0$ on $\mathbb{D}$ (which is simply connected), there exists $\vec{L}$ as required. Further

$$\begin{aligned} V_{\mathrm{dil}} = \langle\Phi, V_{\mathrm{tra}}\rangle &= \left\langle\Phi, \nabla^{\perp}\vec{L}\right\rangle \\ &= \nabla^{\perp}\left(\left\langle\Phi,\vec{L}\right\rangle\right) - \left\langle\nabla^{\perp}\Phi,\vec{L}\right\rangle. \end{aligned}$$

Since

$$\operatorname{div}\left(V_{\mathrm{dil}} - \nabla^{\perp}\left(\left\langle\Phi,\vec{L}\right\rangle\right)\right) = 0,$$

there exists S as desired. We proceed similarly for $\vec{R}$ and $\tilde{R}$ and conclude the proof. □

At the inception, $\vec{R}$ and S were obtained through astute computations in arbitrary codimensions, trying to find other divergence free quantities from $\vec{L}$. While Noether's theorem clarifies the fundamental origin of the conservation laws, how to derive S and $\vec{R}$ from them is less obvious. In our case, the applied modifications can be seen as striving to work with invariant quantities. Indeed, while $\left\langle\Phi,\nabla^{\perp}\vec{L}\right\rangle$ changes under the action of translations, moving the $\nabla^{\perp}$ onto Φ leads to a translational invariant, and thus a potentially more interesting quantity.

Remark 1.2.2. All four $\vec{L}$, S, $\vec{R}$, $\tilde{R}$ are defined up to a constant.

Remark 1.2.3. Identity (1.2.35) implies that $\nabla^\perp \tilde{R} = \nabla^\perp \vec{R} + 2\nabla^\perp \vec{n}$.

Remark 1.2.4. While S, $\vec{R}$, $\tilde{R}$ are dilation invariants, $\vec{L}$ is not. Indeed for $\mu \in \mathbb{R}$, $\Phi_\mu = \mu\Phi$ is a Willmore conformal immersion of the disk. It can be shown through a straightforward computation that its quantities $\vec{L}_\mu$, S_μ, $\vec{R}_\mu$ and $\tilde{R}_\mu$ defined by (1.2.43) satisfy

$$\begin{cases} \vec{L}_\mu = \dfrac{1}{\mu}\vec{L} \\ S_\mu = S \\ \vec{R}_\mu = \vec{R} \\ \tilde{R}_\mu = \tilde{R}. \end{cases}$$

Remark 1.2.5. In complex notations one has

$$\begin{cases} \vec{L}_z = -2i\left(H_z\vec{n} + H\Omega e^{-2\lambda}\Phi_{\bar{z}}\right) \\ S_z = \left\langle \Phi_z, \vec{L}\right\rangle \\ \vec{R}_z = \vec{L} \times \Phi_z + 2H\Phi_z, \end{cases} \tag{1.2.44}$$

where $\Omega = 2\langle \Phi_{zz}, \vec{n}\rangle$ is the tracefree curvature of Φ (in essence the complex analogue of $\mathring{A}$, see section A.2.2 in the appendix).

The following results, taken from theorem I.4 [Riv08], link $\vec{R}$ and S.

Lemma 1.2.2. Let $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ satisfy the hypotheses of theorem 1.2.16. Then

$$\begin{cases} \nabla S = -\left\langle \vec{n}, \nabla^\perp \vec{R}\right\rangle \\ \nabla \vec{R} = \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \vec{n}. \end{cases} \tag{1.2.45}$$

Proof. We simply use (A.2.4) and compute :

$$\begin{aligned} \left\langle \vec{n}, \nabla^\perp \vec{R}\right\rangle &= \left\langle \vec{n}, \vec{L} \times \nabla^\perp\Phi + 2H\nabla^\perp\Phi\right\rangle = \left\langle \vec{L}, \nabla^\perp\Phi \times \vec{n}\right\rangle \\ &= -\left\langle \vec{L}, \nabla\Phi\right\rangle = -\nabla S. \end{aligned}$$

Similarly

$$\begin{aligned} \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S\vec{n} &= \vec{n} \times \left(\vec{L} \times \nabla^\perp\Phi + 2H\nabla^\perp\Phi\right) + \left\langle \vec{L}, \nabla^\perp\Phi\right\rangle \vec{n} \\ &= \left\langle \vec{L}, \nabla^\perp\Phi\right\rangle \vec{n} - \left\langle \vec{L}, \vec{n}\right\rangle \nabla^\perp\Phi + 2H\nabla\Phi \\ &= \vec{L} \times \left(\vec{n} \times \nabla^\perp\Phi\right) + 2H\nabla\Phi = \nabla\vec{R}. \end{aligned}$$

This concludes the proof. $\square$

From lemma 1.2.2, one deduces :

Theorem 1.2.17. Let $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ satisfy the hypotheses of theorem 1.2.16. Then

$$\begin{cases} \Delta S = -\left\langle \nabla\vec{n}, \nabla^\perp \vec{R}\right\rangle \\ \Delta\vec{R} = \nabla\vec{n} \times \nabla^\perp\vec{R} + \nabla^\perp S \nabla\vec{n} \\ \Delta\Phi = \dfrac{1}{2}\left(\nabla^\perp S.\nabla\Phi + \nabla^\perp\vec{R} \times \nabla\Phi\right). \end{cases} \tag{1.2.46}$$

Proof. As in [Riv08], the first two equalities are obtained by taking the divergence of (1.2.45), and the third one can be found through direct computations. We will however frame it as a consequence of the fourth conservation laws, as was suspected in [Ber16].

Indeed one has

$$\begin{aligned} V_{\text{inv}} &= -|\Phi|^2 V_{\text{tra}} + 2\langle \Phi, V_{\text{tra}}\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp \Phi\right) \\ &= -|\Phi|^2 \nabla^\perp \vec{L} + 2\left\langle \Phi, \nabla^\perp \vec{L}\right\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp\Phi\right) \\ &= \nabla^\perp\left(-|\Phi|^2 \vec{L} + 2\left\langle \Phi, \vec{L}\right\rangle \Phi\right) + 2\left\langle \nabla^\perp\Phi, \Phi\right\rangle \vec{L} - 2\left\langle \Phi, \vec{L}\right\rangle \nabla^\perp \Phi \\ &- 2\left\langle \nabla^\perp \Phi, \vec{L}\right\rangle \Phi + 4\Phi \times \left(\mathring{A}\nabla^\perp\Phi\right) \\ &= \nabla^\perp(\dots) + 2\Phi \times \left(\vec{L}\times \nabla^\perp\Phi + 2\mathring{A}\nabla^\perp\Phi\right) - 2\nabla^\perp S\Phi \\ &= \nabla^\perp(\dots) + 2\Phi\times\nabla^\perp\tilde{R} - 2\nabla^\perp S\Phi. \end{aligned} \tag{1.2.47}$$

Thanks to remark 1.2.3, one has

$$\begin{aligned} V_{\text{inv}} &= \nabla^\perp(\dots) + 2\Phi\times\nabla^\perp\vec{R} + 4\Phi\times\nabla^\perp\vec{n} - 2\nabla^\perp S\Phi \\ &= \nabla^\perp(\dots) + 2\Phi\times\nabla^\perp\vec{R} - 2\nabla^\perp S\Phi - 4\nabla^\perp\Phi\times\vec{n} \\ &= \nabla^\perp(\dots) + +2\Phi\times\nabla^\perp\vec{R} - 2\nabla^\perp S\Phi + 4\nabla\Phi. \end{aligned} \tag{1.2.48}$$

Taking the divergence of (1.2.48) yields :

$$\Delta\Phi = \frac{1}{2}\nabla^\perp S\nabla\Phi - \frac{1}{2}\nabla\Phi\times\nabla^\perp\vec{R} = \frac{1}{2}\nabla^\perp S\nabla\Phi + \frac{1}{2}\nabla^\perp\vec{R}\times\nabla\Phi.$$

Which concludes the proof. □

Although critical for the Calderón-Zygmund theory, system (1.2.46) is remarkable in its Jacobian-like form which allows for the use of Wente's lemmas (see section A.3.2 for a panel of integrability by compensation results). This algebraic structure will thus be key in the weak framework.

Remark 1.2.6. One can simply take the divergence of (1.2.47) and find

$$\nabla\Phi\times\nabla^\perp\tilde{R} - 2\nabla^\perp S\nabla\Phi = 0.$$

This may help explain the relative lack of interest revolving around $\tilde{R}$ since while $\vec{R}$ allows one to recover controls on $\Delta\Phi$, $\tilde{R}$ does not. Further injecting the content of remark 1.2.3 into (1.2.46) shows that $\tilde{R}$, like $\vec{n}$, does not satisfy a remarkable enough equation.

The system (1.2.46) has been modified in [Mar19c] to only involve mean curvature terms.

Theorem 1.2.18. Let $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ satisfy the hypotheses of theorem 1.2.16. Then

$$\begin{cases} \Delta S = \left\langle H\nabla\Phi, \nabla^\perp\vec{R}\right\rangle \\ \Delta\vec{R} = -H\nabla\Phi\times\nabla^\perp\vec{R} - \nabla^\perp S H\nabla\Phi \\ \Delta\Phi = \frac{1}{2}\left(\nabla^\perp S.\nabla\Phi + \nabla^\perp\vec{R}\times\nabla\Phi\right). \end{cases} \tag{1.2.49}$$

Proof. We reproduce the proof given in [Mar19c], which consists essentially in rephrasing (1.2.46). We use the notations of section A.2.1.

To that end we compute

$$\begin{aligned}
\left\langle \mathring{A}\nabla\Phi, \nabla^{\perp}\vec{R}\right\rangle &= \left\langle \mathring{A}\nabla\Phi, \vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi\right\rangle \\
&= -e^{-2\lambda}\left\langle \frac{e-g}{2}\Phi_x + f\Phi_y, \vec{L}\times\Phi_y + 2H\Phi_y\right\rangle \\
&+ e^{-2\lambda}\left\langle f\Phi_x + \frac{g-e}{2}\Phi_y, \vec{L}\times\Phi_x + 2H\Phi_x\right\rangle \\
&= \frac{g-e}{2}e^{-2\lambda}\left(\left\langle \Phi_x, \vec{L}\times\Phi_y\right\rangle + \left\langle \Phi_y, \vec{L}\times\Phi_x\right\rangle\right) - 2Hf + 2Hf \\
&= \frac{g-e}{2}e^{-2\lambda}\left(\left\langle \vec{L}, \Phi_y\times\Phi_x\right\rangle + \left\langle \vec{L}, \Phi_x\times\Phi_y\right\rangle\right) \\
&= 0.
\end{aligned} \tag{1.2.50}$$

Further

$$\begin{aligned}
\mathring{A}\nabla\Phi\times\nabla^{\perp}\vec{R} &= \mathring{A}\nabla\Phi\times\left(\vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi\right) \\
&= \left\langle \mathring{A}\nabla\Phi . \nabla^{\perp}\Phi\right\rangle\vec{L} - \left\langle \mathring{A}\nabla\Phi, \vec{L}\right\rangle\nabla^{\perp}\Phi + 2H\mathring{A}\nabla\Phi\times\nabla^{\perp}\Phi \\
&= -e^{-2\lambda}\left\langle \frac{e-g}{2}\Phi_x + f\Phi_y, \Phi_y\right\rangle\vec{L} \\
&+ e^{-2\lambda}\left\langle \frac{e-g}{2}\Phi_x + f\Phi_y, \vec{L}\right\rangle\Phi_y \\
&- e^{-2\lambda}2H\left(\frac{e-g}{2}\Phi_x + f\Phi_y\right)\times\Phi_y \\
&+ e^{-2\lambda}\ \left\langle f\Phi_x + \frac{g-e}{2}\Phi_y, \Phi_x\right\rangle\vec{L} \\
&- e^{-2\lambda}\left\langle f\Phi_x + \frac{g-e}{2}\Phi_y, \vec{L}\right\rangle\Phi_x \\
&+ 2H\left(f\Phi_x + \frac{g-e}{2}\Phi_y\right)\times\Phi_x \\
&= \frac{e-g}{2}\left\langle \Phi_x, \vec{L}\right\rangle\Phi_y + f\left\langle \Phi_y, \vec{L}\right\rangle\Phi_y \\
&- f\left\langle \Phi_x, \vec{L}\right\rangle\Phi_x - \frac{g-e}{2}\left\langle \Phi_y, \vec{L}\right\rangle\Phi_x \\
&+ 2H\left(-\frac{e-g}{2}\vec{n} + \frac{g-e}{2}(-\vec{n})\right) \\
&= \left(\frac{e-g}{2}S_x\Phi_y + fS_y\Phi_y - fS_x\Phi_x - \frac{g-e}{2}S_y\Phi_x\right) \\
&= -\nabla^{\perp}S\mathring{A}\nabla\Phi.
\end{aligned} \tag{1.2.51}$$

We have used (1.2.43) to obtain the second to last equality. The decomposition (A.2.3) then yields

$$\left\langle \nabla\vec{n}, \nabla^{\perp}\vec{R}\right\rangle = -\left\langle H\nabla\Phi + \mathring{A}\nabla\Phi, \nabla^{\perp}\vec{R}\right\rangle = -\left\langle H\nabla\Phi, \nabla^{\perp}\vec{R}\right\rangle,$$

with (1.2.50). Similarly, with (1.2.51) we compute

$$\begin{aligned}
\nabla\vec{n}\times\nabla^{\perp}\vec{R}+\nabla^{\perp}S\nabla\vec{n} &= -H\nabla\Phi\times\nabla^{\perp}\vec{R}-H\nabla\Phi\nabla^{\perp}S-\mathring{A}\nabla\Phi\nabla^{\perp}\vec{R}-\mathring{A}\nabla\Phi\nabla^{\perp}S\\
&= -H\nabla\Phi\times\nabla^{\perp}\vec{R}-H\nabla\Phi\nabla^{\perp}S+\nabla^{\perp}S\mathring{A}\nabla\Phi-\nabla^{\perp}S\mathring{A}\nabla\Phi\\
&= -H\nabla\Phi\times\nabla^{\perp}\vec{R}-H\nabla\Phi\nabla^{\perp}S.
\end{aligned}$$

Injecting these last two equalities in (1.2.46), we can conclude that $\vec{R}$, S and Φ satisfy the desired system. □

Remark 1.2.7. We could have done the proof of theorem 1.2.18 using complex notations. From (1.2.44), it is obvious that

$$\begin{aligned}
&\left\langle\vec{R}_z,\Phi_z\right\rangle=0,\\
&\vec{R}_z\times\Phi_z+S_z\Phi_z=0.
\end{aligned}$$

From this, the complexified version of (1.2.50) and (1.2.51) stands :

$$\begin{aligned}
&\left\langle\vec{R}_z,\overline{\Omega}e^{-2\lambda}\Phi_z\right\rangle=0,\\
&\vec{R}_z\times\left(\overline{\Omega}e^{-2\lambda}\Phi_z\right)+S_z\left(\overline{\Omega}e^{-2\lambda}\Phi_z\right)=0.
\end{aligned}$$

Taking the imaginary part of the last two equalities yields precisely (1.2.50) and (1.2.51), which concludes the complex proof.

System (1.2.49) is critical and does not have the Jacobian-like structure necessary for Wente's lemma. However, once criticality is broken it offers a way to bound $\nabla\vec{R}$ and ∇S by $H\nabla\Phi$. Besides, one can deduce from (1.2.43) that thanks to the properties of the vectorial product

$$\left|\nabla\vec{R}\right|^2=\left|\vec{L}\times\nabla^{\perp}\Phi\right|^2+4\left|H\nabla\Phi\right|^2.$$

This yields an interesting estimate :

$$|H\nabla\Phi|\leq\frac{1}{2}\left|\nabla\vec{R}\right|. \tag{1.2.52}$$

Consequently (1.2.49) and (1.2.52) offer a closed bootstraping loop which we will use in section 4.2.1.

In fact building from remark 1.2.7, one can express $\vec{R}_z$ in the frame $(\Phi_z,\Phi_{\bar{z}},\vec{n})$. Indeed from $\vec{R}_z=\vec{L}\times\Phi_z+2H\Phi_z$, we can compute

$$\begin{aligned}
\left\langle\vec{R}_z,\Phi_{\bar{z}}\right\rangle&=He^{2\lambda}+i\frac{e^{2\lambda}}{2}\left\langle\vec{L},\vec{n}\right\rangle\\
\left\langle\vec{R}_z,\vec{n}\right\rangle&=\left\langle\vec{L},\Phi_z\times\vec{n}\right\rangle=-i\left\langle\vec{L},\Phi_z\right\rangle=-iS_z.
\end{aligned}$$

Hence if we denote $V=\frac{i}{2}\left\langle\vec{L},\vec{n}\right\rangle$,

$$\vec{R}_z=2\left(H+iV\right)\Phi_z-iS_z\vec{n}. \tag{1.2.53}$$

1.2.5 Conformal Willmore immersions

Since the first conservation law is the Willmore equation in divergence form (see remark 1.2.1), satisfying it is tantamount to being a Willmore immersion. Since S and $\vec{R}$ are solutions of a self-sufficient system, it is natural to wonder if satisfying the two corresponding conservation laws yields the same result.

We then consider $\Phi \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ a conformal immersion. We assume there exists $\vec{L} \in C^\infty\left(\mathbb{D}, \mathbb{R}^3\right)$ such that

$$\begin{cases} \operatorname{div}\left(\left\langle \vec{L}, \nabla^\perp \Phi \right\rangle\right) = 0 \\ \operatorname{div}\left(\vec{L} \times \nabla^\perp \Phi + 2H\nabla^\perp \Phi\right) = 0. \end{cases} \tag{1.2.54}$$

We do not assume anything on the nature of $\vec{L}$. In particular we do not assume that $\vec{L}$ satisfies the first conservation law of (1.2.43). System (1.2.54) is enough to go from (1.2.43) to (1.2.46) and find the corresponding quantities S and $\vec{R}$, which satisfy (1.2.46). We will work in complex coordinates and write

$$\begin{cases} S_z = \left\langle \vec{L}, \Phi_z \right\rangle \\ \vec{R}_z = \vec{L} \times \Phi_z + 2H\Phi_z. \end{cases} \tag{1.2.55}$$

We decompose $\vec{L}_z = a\Phi_z + b\Phi_{\bar{z}} + c\vec{n}$, with $a, b, c \in C^\infty(\mathbb{D}, \mathbb{C})$. The system (1.2.54) implies

$$\begin{cases} \left\langle \nabla \vec{L}, \nabla^\perp \Phi \right\rangle = 0, \\ \nabla \vec{L} \times \nabla^\perp \Phi + 2\nabla H \nabla^\perp \Phi = 0. \end{cases} \tag{1.2.56}$$

In complex coordinates, this translates as

$$\begin{cases} \Im\left(\left\langle \vec{L}_z, \Phi_{\bar{z}} \right\rangle\right) = \Im\left(\dfrac{ae^{2\lambda}}{2}\right) = 0, \\ \Im\left(\vec{L}_z \times \Phi_{\bar{z}} + 2H_z\Phi_{\bar{z}}\right) = \Im\left(i\dfrac{ae^{2\lambda}}{2}\vec{n} + \left(2H_z - ic\right)\Phi_{\bar{z}}\right) = 0. \end{cases} \tag{1.2.57}$$

Consequently we find

$$\begin{cases} a = 0 \\ c = -2iH_z. \end{cases}$$

Hence $\vec{L}_z = b\Phi_{\bar{z}} - 2iH_z\vec{n}$. If we compute $\vec{L}_{z\bar{z}}$ we find

$$\begin{aligned} \vec{L}_{z\bar{z}} &= b_{\bar{z}}\Phi_{\bar{z}} + 2\lambda_{\bar{z}} b\Phi_{\bar{z}} + \frac{\overline{\Omega} b}{2}\vec{n} - 2iH_{z\bar{z}}\vec{n} + 2iH_z H\Phi_{\bar{z}} + 2iH_z\overline{\Omega}e^{-2\lambda}\Phi_z \\ &= 2iH_z\overline{\Omega}e^{-2\lambda}\Phi_z + \left(b_{\bar{z}} + 2\lambda_{\bar{z}} b + 2iH_z H\right)\Phi_{\bar{z}} + \left(\frac{\overline{\Omega} b}{2} - 2iH_{z\bar{z}}\right)\vec{n}. \end{aligned}$$

Given that $\vec{L} \in \mathbb{R}^3$, $\vec{L}_{z\bar{z}} \in \mathbb{R}^3$ and thus :

$$\begin{cases} b_{\bar{z}} + 2\lambda_{\bar{z}} b + 2iH_z H = -2iH_{\bar{z}}\Omega e^{-2\lambda} \\ -4iH_{z\bar{z}} + \dfrac{\Omega\bar{b}}{2} - \dfrac{\overline{\Omega} b}{2} = 0. \end{cases} \tag{1.2.58}$$

We can work on the expression :

$$\begin{aligned} b_{\bar z} + 2\lambda_{\bar z} b + 2iH_z H + 2iH_{\bar z}\Omega e^{-2\lambda} &= \left(be^{2\lambda} + 2iH\Omega\right)_{\bar z} e^{-2\lambda} - 2iH\Omega_{\bar z} e^{-2\lambda} + 2iHH_z \\ &= \left(be^{2\lambda} + 2iH\Omega\right)_{\bar z} e^{-2\lambda}. \end{aligned}$$

To obtain the last equality we have used the Gauss-Codazzi identity in complex notations (see (A.2.20)). We can then find a holomorphic function F such that

$$b = e^{-2\lambda} F - 2iH\Omega e^{-2\lambda}. \tag{1.2.59}$$

Injecting (1.2.59) into the second equality of (1.2.58) yields :

$$\mathcal{W}(\Phi) = \Im\left(F\overline{\Omega} e^{-2\lambda}\right). \tag{1.2.60}$$

This is a notion closely linked to Willmore immersions which will be useful later, called conformal Willmore immersions.

Definition 1.2.5. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal immersion. Then Φ is said to be a conformal Willmore immersion if there exists an holomorphic function F such that $\mathcal{W}(\Phi) = \Re\left(\overline{F}\Omega e^{-2\lambda}\right)$.

This notion is yet again linked with a behavior of the Willmore energy.

Proposition 1.2.19. Conformal Willmore immersions are critical points of the Willmore functional *in a conformal class* and F acts as a Lagrange multiplier.

Remark 1.2.8. The notion of "conformal Willmore immersion" is then invariant by conformal mappings of the ambient space.

We do not give details for brevity and refer the reader to the subsection X.7.4 in [Riv12]. Conformal Willmore immersions can be apprehended as an extension of constant mean curvature surfaces the same way Willmore immersions are an extension of minimal surfaces. We will give in section 2.5 a necessary and sufficient condition for a conformal Willmore immersion to be the conformal transform of a constant mean curvature immersion.

Interestingly enough, F can be expressed in function of S and $\vec{R}$. Following from (1.2.59), we find

$$\begin{aligned} \frac{F}{2} - iH\Omega = \frac{e^{2\lambda} b}{2} &= \left\langle \vec{L}_z, \Phi_z \right\rangle = \left(\left\langle \vec{L}, \Phi_z \right\rangle\right)_z - \left\langle \vec{L}, \Phi_{zz} \right\rangle \\ &= S_{zz} - 2\lambda_z \left\langle \vec{L}, \Phi_z \right\rangle - \frac{\Omega}{2}\left\langle \vec{L}, \vec{n} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z - \frac{\Omega}{2}\left\langle \vec{L}, \vec{n} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda}\left\langle \vec{L}, \Phi_z \times \Phi_{\bar z} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda}\left\langle \vec{L}, \Phi_z \times \Phi_{\bar z} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda}\left\langle \vec{L} \times \Phi_z, \Phi_{\bar z} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda}\left\langle \vec{R}_z - 2H\Phi_z, \Phi_{\bar z} \right\rangle \\ &= S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda}\left\langle \vec{R}_z, \Phi_{\bar z} \right\rangle - iH\Omega. \end{aligned}$$

Hence
$$\frac{F}{2} = S_{zz} - 2\lambda_z S_z + i\Omega e^{-2\lambda} \left\langle \vec{R}_z, \Phi_{\bar{z}} \right\rangle. \tag{1.2.61}$$

We gave the definition of conformal Willmore immersion for conformal immersions of the disk. It can of course be extended to conformal Willmore immersions of a surface.

Definition 1.2.6. Let Σ be a Riemann surface. Then $\Phi : \Sigma \to \mathbb{R}^3$ is a conformal Willmore immersion if for all $x \in \Sigma$, (1.2.60) stands in any local conformal chart centered at x.

Actually (1.2.60) can be given meaning on the whole surface, and not merely in conformal charts. To that end we can consider the Weingarten tensor $h_0 = \langle \partial^2 \Phi, \vec{n} \rangle$, which is a $(2,0)$ form on the Riemann surface. In a local complex chart it is written

$$h_0 = \Omega dz^2.$$

Here ∂ is the complex differentiation operator on the Riemann surface. In a local complex chart : $\partial = \partial_z dz$. Then $\Phi : \Sigma \to \mathbb{R}^3$ is a conformal Willmore immersion if there exists a holomorphic 2-form f such that

$$\left(\partial \overline{\partial} H + \frac{1}{2} g^{-1} \otimes h_0 \otimes \overline{h_0} H \right) g = \Im \left(h_0 \otimes \overline{f} \right).$$

This is clear since in local conformal charts

$$\begin{aligned}
\partial \overline{\partial} H + \frac{1}{2} g^{-1} \otimes h_0 \otimes \overline{h_0} H &= \left(H_{z\bar{z}} + \frac{|\Omega|^2}{2} e^{-2\lambda} H \right) dz d\overline{z} \\
&= \frac{\mathcal{W}(\Phi)}{4} dz d\overline{z}, \\
g &= e^{2\lambda} dz d\overline{z}, \\
\text{and } f &= F dz^2.
\end{aligned}$$

The formalism of differential forms may thus allow one to work globally. However,mostly for simplicity, our proof are written locally, and we only translate the results globally when needed.

1.3 Weak immersions with L^2 second fundamental form

1.3.1 Definition

Let Σ be an arbitrary closed compact two-dimensional manifold. Let g_0 be a smooth "reference" metric on Σ. The Sobolev spaces $W^{k,p}\left(\Sigma, \mathbb{R}^3\right)$ of measurable maps from Σ into $\mathbb{R}^3$ is defined as

$$W^{k,p}\left(\Sigma, \mathbb{R}^3\right) := \left\{ f \text{ measurable} : \Sigma \to \mathbb{R}^3 \text{ s.t } \sum_{l=0}^{k} \int_\Sigma \left| \nabla^l_{g_0} f \right|^p_{g_0} d\text{vol}_{g_0} < \infty \right\}.$$

Since Σ is assumed to be compact this definition does not depend on g_0.

We will work with the concept of weak immersions introduced by T. Rivière, which represents the correct starting framework for studying Willmore immersions. One might notice that the presentation of this notion has changed through the years (compare definition I.1 in [Riv08] with its equivalent in subsection 1.2 in [LR18a]). While we use the latter, which is sufficient for our needs, one could take slightly less demanding (albeit more complex) starting hypotheses.

Definition 1.3.1. Let $\Phi : \Sigma \to \mathbb{R}^3$. Let $g_\Phi = \Phi^*\xi$ be the first fundamental form of Φ and $\vec{n}$ its Gauss map. Then Φ is a weak immersion with locally L^2-bounded second fundamental form if $\Phi \in W^{1,\infty}(\Sigma)$, if there exists a constant C_Φ such that

$$\frac{1}{C_\Phi} g_0 \leq g_\Phi \leq C_\Phi g_0,$$

and if

$$\int_\Sigma |d\vec{n}|^2_{g_\Phi} \, d\mathrm{vol}_\Phi < \infty.$$

The set of weak immersions with L^2-bounded second fundamental form on Σ will be denoted $\mathcal{E}(\Sigma)$.

One of the advantages of such weak immersions is that they allow us to work with conformal maps as shown by theorem 5.1.1 of [Hél02].

Theorem 1.3.1. Let Φ be a weak immersion from Σ into $\mathbb{R}^3$ with L^2-bounded second fundamental form. Then for every $x \in \Sigma$, there exists an open disk D in Σ containing x and a homeomorphism $\Psi : \mathbb{D} \to D$ such that $\Phi \circ \Psi$ is a conformal bilipschitz immersion. The induced metric $g = (\Phi \circ \Psi)^* \xi$ is continuous. Moreover, the Gauss map $\vec{n}$ of this immersion is in $W^{1,2}(\mathbb{D}, \mathbb{S}^2)$.

Further, proving estimates on the Green function of Σ, P. Laurain and T. Rivière have shown in theorem 3.1 of [LR18b] that up to choosing a specific atlas, one could have further control on the conformal factor.

Theorem 1.3.2. Let (Σ, g) be a closed Riemann surface of fixed genus greater than one. Let h denote the metric with constant curvature (and volume equal to one in the torus case) in the conformal class of g and $\Phi \in \mathcal{E}(\Sigma)$ conformal, that is :

$$\Phi^*\xi = e^{2u} h.$$

Then there exists a finite conformal atlas (U_i, Ψ_i) and a positive constant C depending only on the genus of Σ, such that

$$\|\nabla \lambda_i\|_{L^{2,\infty}(V_i)} \leq C \, \|\nabla_{\Phi^*\xi} \vec{n}\|^2_{L^2(\Sigma)},$$

with $\lambda_i = \frac{1}{2} \log \frac{|\nabla \Phi|^2}{2}$ the conformal factor of $\Phi \circ \Psi_i^{-1}$ in $V_i = \Psi_i(U_i)$.

Thus given $\tilde{\Phi} \in \mathcal{E}(\Sigma)$ we can choose a conformal atlas such that, in a local chart on $\mathbb{D}$ of this atlas, $\tilde{\Phi}$ yields $\Phi \in \mathcal{E}(\mathbb{D})$ satisfying

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D})} \leq C_0. \tag{1.3.62}$$

One can then systematically study any $\tilde{\Phi} \in \mathcal{E}(\Sigma)$ in such local conformal charts, as a conformal bilipschitz map $\Phi \in \mathcal{E}(\mathbb{D})$ satisfying (1.3.62).

We can now introduce the notion of weak Willmore immersions (definition I.2 in [Riv08]).

Definition 1.3.2. Let $\Phi \in \mathcal{E}(\Sigma)$, Φ is a weak Willmore immersion if

$$\operatorname{div}\left(\nabla \vec{H} - 3\pi_{\vec{n}}\left(\nabla \vec{H}\right) + \nabla^\perp \vec{n} \times \vec{H}\right) = 0 \tag{1.3.63}$$

holds in a distributional sense in every conformal parametrization $\Psi : \mathbb{D} \to D$ on every neighborhood D of x , for all $x \in \Sigma$. Here, the operators div, ∇ and $\nabla^\perp$ are to be understood with respect to the flat metric on $\mathbb{D}$.

Of course, the *weak Willmore equation* (1.3.63) is merely the first conservation law for Willmore immersions (see theorem 1.2.16), which is, as was said in remark 1.2.1, the Willmore equation put in divergence form. A smooth weak Willmore immersion is then a Willmore immersion, since it satisfies the Willmore equation. All the stakes then revolve around using the system (1.2.46) (which remains valid in this weak framework since the conservation laws have been reached through purely computational means by T. Rivière in [Riv08]) to regain this regularity.

1.3.2 Harnack inequalities on the conformal factor

Works by F. Hélein (see [Hél02]) ensured that in disks of small energy, and that up to a reasonable (see (1.3.62)) assumption on $\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D})}$, the conformal factor could be controlled pointwise. We here give a version from [Riv16] (theorem 5.5).

Theorem 1.3.3. Let $\Phi \in \mathcal{E}(\mathbb{D})$, conformal. Let $\vec{n}$ be its Gauss map and λ its conformal factor. We assume

$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 < \frac{8\pi}{3},$$

and

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D})} \leq C_0. \tag{1.3.64}$$

Then for any $r < 1$ there exists $c \in \mathbb{R}$ and $C \in \mathbb{R}$ depending on r and C_0 such that

$$\|\lambda - c\|_{L^{\infty}(\mathbb{D}_r)} \leq C.$$

While we will make use of this result throughout this work, the proof itself is outside its scope. The idea is to rely on the Liouville equation (see (A.2.8)) and to write it in a Jacobian form thanks to a moving Coulomb frame. The details are in the referenced texts.

This theorem can be adapted to disks of arbitrary radii without losing control on the constant.

Corollary 1.3.1. Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$, conformal. Let $\vec{n}$ be its Gauss map and λ its conformal factor. We assume

$$\int_{\mathbb{D}_\rho} |\nabla \vec{n}|^2 < \frac{8\pi}{3}$$

and

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} \leq C_0.$$

Then for any $r < 1$ there exists $c_{\rho,r} \in \mathbb{R}$ and $C \in \mathbb{R}$ depending on r and C_0 such that

$$\|\lambda - c_{\rho,r}\|_{L^{\infty}(\mathbb{D}_{r\rho})} \leq C.$$

Proof. Let $\Phi_\rho = \Phi(\rho .)$, let $\vec{n}_\rho$ be its Gauss map and λ_ρ its conformal factor. Straightforward computations yield

$$e^{\lambda_\rho} = \rho e^{\lambda}(\rho .) \tag{1.3.65}$$

and

$$\vec{n}_\rho = \vec{n}(\rho .). \tag{1.3.66}$$

Then

$$\int_{\mathbb{D}} |\nabla \vec{n}_\rho|^2 \, dz = \int_{\mathbb{D}_\rho} |\nabla \vec{n}|^2 \, dz < \frac{8\pi}{3}$$

and, thanks to (1.3.65),

$$\|\nabla\lambda_\rho\|_{L^{2,\infty}(\mathbb{D})} = \|\nabla\left(\lambda(\rho.)+\ln\rho\right)\|_{L^{2,\infty}(\mathbb{D})} = \|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} \leq C_0$$

owing to the scaling-invariance properties of the L^2 and $L^{2,\infty}$ norms. Applying theorem 1.3.3, one finds there exists $c \in \mathbb{R}$ and $C \in \mathbb{R}$ depending on r and C_0 such that

$$\|\lambda_\rho - c_r\|_{L^\infty(\mathbb{D}_r)} \leq C.$$

However, using (1.3.65),

$$\|\lambda - c_{\rho,r}\|_{L^\infty(\mathbb{D}_{r\rho})} \leq C$$

with $c_{\rho,r} = c_r - \ln\rho$ and the same C. $\square$

We can extend the control to domains with merely $\int_{\mathbb{D}_\rho}|\nabla\vec{n}|^2 < \infty$ up to adding an additionnal parameter r_0 to the constant :

$$r_0 = \frac{1}{\rho}\inf\left\{ s \,\middle|\, \int_{B_s(p)} |\nabla\vec{n}|^2 = \frac{8\pi}{6},\ \forall p \in \mathbb{D}_\rho \text{ s.t. } B_s(p) \subset \mathbb{D}_\rho \right\}. \tag{1.3.67}$$

This parameter marks how relatively small a ball has to be to ensure that it does not contain too much energy, and its inverse will bound the number of balls with small energy covering the disk. Alternatively, anticipating on the vocabulary of concentration-compactness (see section 3.2), it measures how concentrated $\nabla\vec{n}$ is on a disk.

Corollary 1.3.2. Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$ conformal, $\vec{n}$ be its Gauss map and λ its conformal factor. We assume that

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0.$$

Then for any $r < 1$ there exists $c_{\rho,r} \in \mathbb{R}$ and $C \in \mathbb{R}$ depending on r, C_0 and r_0 (defined by (1.3.67)) such that

$$\|\lambda - c_{\rho,r}\|_{L^\infty(\mathbb{D}_{r\rho})} \leq C.$$

Proof. We reproduce the proof given in [Mar19c] and prove the result on $\mathbb{D}$. Then working as in the proof of corollary 1.3.1 we can extend the result to $\mathbb{D}_\rho$.

If $\int_{\mathbb{D}}|\nabla\vec{n}|^2 < \frac{8\pi}{6}$, then one can simply apply theorem 1.3.3. Else let $r < 1$, and

$$r_1 = \min\left(\frac{1-r}{2}, r_0\right).$$

We cover $\mathbb{D}_r$ with a finite number of open disks $\left(B_{\frac{r_1}{10}}(p_i)\right)_{i\in I}$. Using Vitali's covering theorem (see for instance theorem 1.24 p 36 of [EG15]) one can extract N disjoint disks $\left(B_{\frac{r_1}{10}}(p_{i_j})\right)_{j=1..N}$ of this covering such that

$$\bigcup_{i\in I} B_{\frac{r_1}{10}}(p_i) \subset \bigcup_{j=1}^{N} B_{\frac{r_1}{2}}(p_{i_j}).$$

As a consequence

$$\bigsqcup_{j=1}^{N} B_{\frac{r_1}{10}}(p_{i_j}) \subset \bigcup_{j=1}^{N} B_{\frac{r_1}{2}}(p_{i_j}) \subset \mathbb{D}$$

which implies

$$\sum_{j=1}^{N} \lambda\left(B_{\frac{r_1}{10}}(p_{i_j})\right) \leq \lambda(\mathbb{D}),$$

with λ the Lebesgue measure. Thus

$$N \leq \frac{100}{r_1^2}. \tag{1.3.68}$$

For simplicity's sake we will renumber the (p_i) such that $(p_{i_j}) = (p_i)_{i=..N}$.

One can then apply corollary 1.3.1 on each $B_{r_1}(p_i)$ and find $c_i \in \mathbb{R}$ such that

$$\|\lambda - c_i\|_{L^\infty\left(B_{\frac{r_1}{2}}(p_i)\right)} \leq C. \tag{1.3.69}$$

Here C is a constant depending only on C_0. Let $i, j \in I$ such that $B_{\frac{r_1}{2}}(p_i) \cap B_{\frac{r_1}{2}}(p_j) \neq \emptyset$. Then

$$\begin{aligned} |c_i - c_j| &\leq |c_i - \lambda(x)| + |c_j - \lambda(x)| \\ &\leq \|\lambda - c_i\|_{L^\infty\left(B_{\frac{r_1}{2}}(p_i)\right)} + \|\lambda - c_j\|_{L^\infty\left(B_{\frac{r_1}{2}}(p_j)\right)} \\ &\leq 2C. \end{aligned} \tag{1.3.70}$$

Taking any $i, j \in I$, let γ_{ij} be a straight line linking any fixed $x_i \in B_{\frac{r_1}{2}}(p_i)$ to any fixed $x_j \in B_{\frac{r_1}{2}}(p_j)$. γ_{ij} goes through the disks $\left(B_{\frac{r_1}{2}}(p_{q_l})\right)_{q_l \in J \subset I}$, ordered such that

$$B_{\frac{r_1}{2}}(p_{q_l}) \cap B_{\frac{r_1}{2}}\left(p_{q_{l+1}}\right) \neq \emptyset.$$

Then, thanks to (1.3.70),

$$\begin{aligned} |c_i - c_j| &\leq \sum_l \left|c_{q_l} - c_{q_{l+1}}\right| \\ &\leq \sum_l 2C \\ &\leq 2NC, \end{aligned}$$

since γ_{ij} goes through at most N disks.

Setting $c_{\rho,r} = c_1$, one deduces

$$|c_{\rho,r} - c_i| \leq 2NC \quad \forall i \in I. \tag{1.3.71}$$

Then given any $x \in \mathbb{D}_r$ we find a $i \in I$ such that $x \in B_{\frac{r_1}{2}}(p_i)$ and have, using (1.3.69) and (1.3.71),

$$|\lambda(x) - c_{\rho,r}| \leq |\lambda(x) - c_i| + |c_{\rho,r} - c_i| \leq (2N+1)\, C.$$

Taking the supremum over x we conclude with

$$\|\lambda - c_{\rho,r}\|_{L^\infty(\mathbb{D}_r)} \leq (2N+1)\, C$$

which is as announced given that N depends only on r and r_0. □

This control ensures that (1.3.63) has a distributional meaning in conformal charts. Indeed if we consider $\Phi \in \mathcal{E}(\mathbb{D})$ satisfying hypothesis (1.3.64), $\nabla \vec{n} \in L^2(\mathbb{D})$ and its respective tracefull and tracefree part $H \nabla \Phi$ and $\mathring{A} \nabla \Phi$ are properly defined as $L^2(\mathbb{D})$ functions (see

(A.2.6) for details). Then corollary 1.3.2 ensures that for any $r<1$, there exists $\Lambda\in\mathbb{R}$ such that on $\mathbb{D}_r$ we have the following Harnack inequality :

$$\frac{e^{\Lambda}}{C}\leq e^{\lambda}\leq Ce^{\Lambda}. \tag{1.3.72}$$

Hence, since $|H|=\frac{1}{\sqrt{2}}e^{-\lambda}\,|H\nabla\Phi|$ we have on $\mathbb{D}_r$

$$\begin{aligned}\|H\|_{L^2(\mathbb{D}_r)}&\leq e^{-\Lambda}C\|H\nabla\Phi\|_{L^2(\mathbb{D})}\\&\leq e^{-\Lambda}C\|\nabla\vec{n}\|_{L^2(\mathbb{D})}<+\infty.\end{aligned}\tag{1.3.73}$$

As a result (1.3.63) is well-defined in the distributional sense, which allows us to introduce $\vec{R}$ and S, and thus the regularizing system (1.2.46). The following definition and proposition sum up these considerations.

Definition 1.3.3. Let $\Phi\in\mathcal{E}(\mathbb{D})$ be a weak Willmore immersion. Then there exists $\vec{L}\in\mathcal{D}'(\mathbb{D})$ such that

$$\nabla^{\perp}\vec{L}=\nabla\vec{H}-3\pi_{\vec{n}}\left(\nabla\vec{H}\right)+\nabla^{\perp}\vec{n}\times\vec{H}. \tag{1.3.74}$$

Proposition 1.3.4. Let $\Phi\in\mathcal{E}(\mathbb{D})$ be a weak Willmore immersion. Then for any $\vec{L}\in\mathcal{D}'(\mathbb{D})$ satisfying (1.3.74) we have

$$\begin{aligned}&\operatorname{div}\left(\langle\vec{L},\nabla^{\perp}\Phi\rangle\right)=0\\&\operatorname{div}\left(\vec{L}\times\nabla^{\perp}\Phi+2H\nabla^{\perp}\Phi\right)=0.\end{aligned}$$

Consequently, there exists S and $\vec{R}\in\mathcal{D}'(\mathbb{D})$ such that

$$\begin{aligned}\nabla^{\perp}S&=\langle\vec{L},\nabla^{\perp}\Phi\rangle\\\nabla^{\perp}\vec{R}&=\vec{L}\times\nabla^{\perp}\Phi+2H\nabla^{\perp}\Phi.\end{aligned}\tag{1.3.75}$$

1.4 Regularity for weak Willmore immersions

1.4.1 Controls on $\vec{L}$

This section is devoted to the following result which is an improvement (given estimate (A.2.11)) over theorem 7.4 of [Riv16], with a control by $H\nabla\Phi$ replacing one by $\nabla\vec{n}$. We will however follow *mutatis mutandis* the previous proof. This theorem appeared in the prepublication [Mar19c].

Theorem 1.4.1. Let $\Phi\in\mathcal{E}(\mathbb{D}_\rho)$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and λ its conformal factor.

We assume

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)}+\|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)}\leq C_0.$$

Then, for any $r<1$, there exists a constant $\vec{\mathcal{L}}_{\rho,r}\in\mathbb{R}^3$ and a constant $C\in\mathbb{R}$ depending on r, C_0 and r_0 (defined in (1.3.67)) such that

$$\left\|e^{\lambda}\left(\vec{L}-\vec{\mathcal{L}}_{\rho,r}\right)\right\|_{L^{2,\infty}(\mathbb{D}_{r\rho})}\leq C\,\|H\nabla\Phi\|_{L^2(\mathbb{D}_\rho)},$$

where $\vec{L}$ is given by (1.3.74).

Proof. As before, we will prove the theorem on $\mathbb{D}$. The proof on $\mathbb{D}_\rho$ follows as in corollary 1.3.1. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion, $\vec{n}$ its Gauss map, H its mean curvature and λ its conformal factor. We assume that

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D})} + \|\nabla\vec{n}\|_{L^2(\mathbb{D})} \leq C_0.$$

Let $r < 1$ and $\vec{L} \in \mathcal{D}'(\mathbb{D})$ satisfying (1.3.74).

Step 1 : Control of the conformal factor

Applying corollary 1.3.2 we find $\Lambda \in \mathbb{R}$ and C depending on r, C_0 and r_0 such that

$$\|\lambda - \Lambda\|_{L^\infty\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq C.$$

Consequently λ satisfies (1.3.72),

$$\forall x \in \mathbb{D}_{\frac{r+1}{2}} \quad \frac{e^\Lambda}{C} \leq e^{\lambda(x)} \leq Ce^\Lambda.$$

Step 2 : Control on $\nabla\vec{L}$

Estimates (1.3.73) then stands :

$$\|H\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \leq Ce^{-\Lambda}\|H\nabla\Phi\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}.$$

We can exploit it to control the right-hand side of (1.3.74). First, using the fact that the tangent part of $\nabla\vec{H}$, $\pi_T\left(\nabla\vec{H}\right)$, satisfies $\pi_T\left(\nabla\vec{H}\right) = H\nabla\vec{n}$, we recast (1.3.74) as

$$\begin{aligned}\nabla^\perp\vec{L} &= \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H} \\ &= \nabla\vec{H} - 3\nabla\vec{H} + 3\pi_T\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H} \\ &= -2\nabla\vec{H} + 3H\nabla\vec{n} + \nabla^\perp\vec{n}\times\vec{H}.\end{aligned} \tag{1.4.76}$$

Then we control each term of the right-hand side as follows. With theorem 1, section 5.9.1 in [EG15] we find

$$\begin{aligned}\left\|\nabla\vec{H}\right\|_{H^{-1}\left(\mathbb{D}_{\frac{r+1}{2}}\right)} &\leq \left\|\vec{H}\right\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \\ &\leq Ce^{-\Lambda}\|H\nabla\Phi\|_{L^2(\mathbb{D})}.\end{aligned}$$

Moreover

$$\begin{aligned}\|\nabla^\perp\vec{n}\times\vec{H}\|_{L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right)} &\leq \|\nabla\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}\|\vec{H}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \\ &\leq Ce^{-\Lambda}\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\|H\nabla\Phi\|_{L^2(\mathbb{D})},\end{aligned}$$

while

$$\begin{aligned}\|H\nabla\vec{n}\|_{L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right)} &\leq \|\nabla\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)}\|\vec{H}\|_{L^2\left(\mathbb{D}_{\frac{r+1}{2}}\right)} \\ &\leq Ce^{-\Lambda}\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\|H\nabla\Phi\|_{L^2(\mathbb{D})}.\end{aligned}$$

The last three estimates combined give

$$\nabla\vec{L} \in H^{-1}\left(\mathbb{D}_{\frac{r+1}{2}}\right) \oplus L^1\left(\mathbb{D}_{\frac{r+1}{2}}\right).$$

Step 3 : Conclusion
Thanks to Step 2 and theorem A.3.2 (in the appendix)

$$\exists \vec{\mathcal{L}}_r \in \mathbb{R}^3 \quad \left\| \vec{L} - \vec{\mathcal{L}}_r \right\|_{L^{2,\infty}(\mathbb{D}_r)} \leq C e^{-\Lambda} \|H \nabla \Phi\|_{L^2(\mathbb{D})}$$

with C a real constant that depends on r, C_0 and r_0. Hence, with (1.3.72) :

$$\begin{aligned} \left\| \left(\vec{L} - \vec{\mathcal{L}}_r \right) e^{\lambda} \right\|_{L^{2,\infty}(\mathbb{D}_r)} &\leq e^{\Lambda} \left\| \vec{L} - \vec{\mathcal{L}}_r \right\|_{L^{2,\infty}(\mathbb{D}_r)} \\ &\leq C \|H \nabla \Phi\|_{L^2(\mathbb{D})}, \end{aligned}$$

with C as desired. This concludes the proof on $\mathbb{D}$. $\square$

1.4.2 Low regularity controls : proof of theorem A

Without any small control on H or $\vec{n}$, some results can be achieved in term of Lorentz spaces estimates as shown by the following.

Theorem A. Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$ be a conformal weak Willmore immersion satisfying the hypotheses of theorem 1.4.1. Then, for any $r < 1$, there exists a constant $C \in \mathbb{R}$ depending on r, C_0 and r_0 (defined in (1.3.67)) such that

$$\|H \nabla \Phi\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)},$$

and

$$\|\nabla \vec{n}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)}.$$

We first prove a more flexible result than theorem A (in that it does not reference r_0) controlling the $L^{2,1}$ norm of $\nabla \vec{n}$ under $L^{2,\infty}$ assumptions on $\vec{L}$.

Theorem 1.4.2. Let $\Phi \in \mathcal{E}(\mathbb{D}_\rho)$ be a conformal weak Willmore immersion, $\vec{n}$ its Gauss map, H its mean curvature, λ its conformal factor and $\vec{L}$ its first Willmore quantity. We assume

$$\|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0,$$

and that there exists $r' < 1$ and and $C_1 > 0$ such that

$$\left\| \vec{L} e^{\lambda} \right\|_{L^{2,\infty}\left(\mathbb{D}_{r'\rho}\right)} \leq C_1 \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)}.$$

Then for any $r < r'$ there exists a constant C depending on r, r', C_0 and C_1 such that

$$\|H \nabla \Phi\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)},$$

and

$$\|\nabla \vec{n}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|\nabla \vec{n}\|_{L^2(\mathbb{D}_\rho)}.$$

Furthermore the associated second and third Willmore quantities also satisfy

$$\|\nabla S\|_{L^{2,1}(\mathbb{D}_{r\rho})} + \|\nabla \vec{R}\|_{L^{2,1}(\mathbb{D}_{r\rho})} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_\rho)}.$$

Proof. As before it is enough to work on the unit disk and conclude with a dilation to obtain the result on disks of arbitrari radii.

Step 1 : $L^{2,1}$ control of ∇S and $\nabla\vec{R}$

Let $r' < 1$ and $\vec{L}$ (defined in (1.2.43)) such that

$$\left\|\vec{L}e^{\lambda}\right\|_{L^{2,\infty}(\mathbb{D}_{r'})} \leq C_1 \left\|H\nabla\Phi\right\|_{L^2(\mathbb{D})}.$$

Then S and $\vec{R}$ defined as

$$\nabla^{\perp} S = \langle\vec{L},\nabla\Phi\rangle$$
$$\nabla^{\perp}\vec{R} = \vec{L}\times\nabla^{\perp}\Phi + 2H\nabla^{\perp}\Phi,$$

satisfy :

$$\begin{aligned}\|\nabla S\|_{L^{2,\infty}(\mathbb{D}_{r'})} + \|\nabla\vec{R}\|_{L^{2,\infty}(\mathbb{D}_{r'})} &\leq \left\|\vec{L}e^{\lambda}\right\|_{L^{2,\infty}(\mathbb{D}_{r'})} + \|H\nabla\Phi\|_{L^2(\mathbb{D}_{r'})} \\ &\leq (C_1+1)\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D})}.\end{aligned} \tag{1.4.77}$$

Noticing that S and $\vec{R}$ are defined up to an additive constant, we can choose S and $\vec{R}$ to be of null average value on $\mathbb{D}_{r'}$.

The classic Poincaré–Wirtinger's inequality (see theorem 2, section 5.8.1 in [Eva10]) yields for any $1<p<\infty$ and any u such that $\nabla u \in L^p(\mathbb{D}_{r'})$:

$$\|u-\bar{u}\|_{L^p(\mathbb{D}_{r'})} \leq C_{p,r'}\,\|\nabla u\|_{L^p(\mathbb{D}_{r'})}$$

with $C_{p,r'}\in\mathbb{R}_+$ and $\bar{u}$ the mean value of u on $\mathbb{D}_{r'}$. These inequalities can be extended using Marcinkiewitz interpolation theorem (see for example theorem 3.3.3 of [Hél02], recalled as theorem A.1.2 in the appendix) to $L^{2,\infty}$: there exists $C_{r'}$ such that for any u with $\nabla u\in L^{2,\infty}(\mathbb{D})$

$$\|u-\bar{u}\|_{L^{2,\infty}(\mathbb{D}_{r'})} \leq C_{r'}\|\nabla u\|_{L^{2,\infty}(\mathbb{D}_{r'})}.$$

Applied to S and $\vec{R}$ (which are of null mean value), this yields :

$$\|S\|_{W^{1,(2,\infty)}(\mathbb{D}_{r'})} + \|\vec{R}\|_{W^{1,(2,\infty)}(\mathbb{D}_{r'})} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

where C depends on r'. Since, thanks to (1.2.46)

$$\Delta S = \langle\nabla\vec{R},\nabla^{\perp}\vec{n}\rangle,$$

one can decompose $S=\sigma+s$ where s is harmonic and σ is a solution of

$$\begin{cases}\Delta\sigma = \nabla\vec{R}.\nabla^{\perp}\vec{n} \text{ in } \mathbb{D}_{r'} \\ \sigma = 0 \text{ on } \partial\mathbb{D}_{r'}.\end{cases}$$

Using Wente's lemma (theorem A.3.5, in appendix), one finds :

$$\begin{aligned}\|\nabla\sigma\|_{L^2(\mathbb{D}_{r'})} &\leq C\|\nabla\vec{R}\|_{L^{2,\infty}(\mathbb{D}_{r'})}\|\nabla\vec{n}\|_{L^2(\mathbb{D}_{r'})} \\ &\leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},\end{aligned} \tag{1.4.78}$$

where C depends on C_0 and C_1. Meanwhile Poisson's formula yields for s :

$$\|\nabla s\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C\|S\|_{L^1(\partial\mathbb{D}_{r'})} \tag{1.4.79}$$

where C depends on r, and r'.

Using Marcinkiewitz interpolation theorem on trace operators yields

$$\|S\|_{L^1(\partial\mathbb{D}_{r'})} \leq C\,\|\nabla S\|_{L^{2,\infty}(\mathbb{D}_{r'})} \tag{1.4.80}$$

with C depending on r'. Combining (1.4.77), (1.4.79) and (1.4.80) yields :

$$\|\nabla s\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}, \tag{1.4.81}$$

where C depends on r, r', C_1 and C_0. Together (1.4.78) and (1.4.81) yield :

$$\|\nabla S\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}.$$

Working similarly on $\vec{R}$, one finds

$$\|\nabla S\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \|\nabla\vec{R}\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \tag{1.4.82}$$

This estimate can still be improved : let $S = \sigma' + s'$ with s' harmonic and σ'

$$\begin{cases} \Delta\sigma' = \nabla\vec{R}.\nabla^{\perp}\vec{n} \text{ in } \mathbb{D}_{\frac{r+r'}{2}} \\ \sigma' = 0 \text{ on } \partial\mathbb{D}_{\frac{r+r'}{2}}. \end{cases}$$

Using theorem A.3.6 (in appendix) and (1.4.82) ensures

$$\begin{aligned} \left\|\nabla\sigma'\right\|_{L^{2,1}\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} &\leq C\|\nabla\vec{R}\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)}\|\nabla\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \\ &\leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \end{aligned} \tag{1.4.83}$$

Using Poisson's formula allows one to control s':

$$\left\|\nabla s'\right\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C\|S\|_{L^1\left(\partial\mathbb{D}_{\frac{r+r'}{2}}\right)}. \tag{1.4.84}$$

As before, Marcinkiewitz interpolation on trace theorems yields

$$\left\|\nabla s'\right\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \tag{1.4.85}$$

Together (1.4.83) and (1.4.85) ensure

$$\|\nabla S\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}.$$

Working analogously on $\vec{R}$, one finds

$$\|\nabla S\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} + \|\nabla\vec{R}\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \tag{1.4.86}$$

Once more, C depends on r, r', C_0 and C_1 which concludes Step 1.

Step 2 : $L^{2,1}$ control of $H\nabla\Phi$

We simply use inequality (1.2.52) :

$$|H\nabla\Phi| \leq \frac{1}{2}\left|\nabla\vec{R}\right|.$$

Combining it with (1.4.86), we find

$$\|H\nabla\Phi\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \le C\|H\nabla\Phi\|_{L^2(\mathbb{D})}, \tag{1.4.87}$$

which gives us the desired control on $H\nabla\Phi$.

Step 3 : $L^{2,1}$ control of $\nabla\vec{n}$

To expand these estimates to $\nabla\vec{n}$, we will use equation (A.2.23) (see section A.2.2 in the appendix)

$$\Delta\vec{n} + \nabla\vec{n}\times\nabla^{\perp}\vec{n} + 2\mathrm{div}\,(H\nabla\Phi) = 0.$$

Using corollary A.3.2 and (1.4.87) there exists $\alpha \in W^{1,(2,1)}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)$ such that

$$\Delta\alpha = \mathrm{div}\,(H\nabla\Phi) \tag{1.4.88}$$

and

$$\begin{aligned} \|\alpha\|_{W^{1,(2,1)}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} &\le \|H\nabla\Phi\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \\ &\le C\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \end{aligned} \tag{1.4.89}$$

Setting $\nu = \vec{n} - 2\alpha$ and using (1.4.89) yields

$$\begin{aligned} \|\nabla\nu\|_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} &\le \|\nabla(\vec{n}-2\alpha)\|_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \\ &\le \|\nabla\vec{n}\|_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} + 2\|\nabla\alpha\|_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \\ &\le \|\nabla\vec{n}\|_{L^2(\mathbb{D})} + 2C\|\nabla\alpha\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \\ &\le \|\nabla\vec{n}\|_{L^2(\mathbb{D})} + C\|H\nabla\Phi\|_{L^2(\mathbb{D})} \\ &\le C\|\nabla\vec{n}\|_{L^2(\mathbb{D})}. \end{aligned} \tag{1.4.90}$$

Besides, ν satisfies

$$\Delta\nu + \nabla\vec{n}\times\nabla^{\perp}\vec{n} = 0.$$

We split $\nu = \nu_1 + \nu_2$, with ν_2 harmonic and ν_1 solution of

$$\begin{cases} \Delta\nu_1 + \nabla\vec{n}\times\nabla^{\perp}\vec{n} = 0 \text{ in } \mathbb{D}_{\frac{3r+r'}{4}} \\ \nu_1 = 0 \text{ on } \partial\mathbb{D}_{\frac{3r+r'}{4}}. \end{cases}$$

Using theorem A.3.6, we bound

$$\|\nabla\nu_1\|_{L^{2,1}\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \le C\|\nabla\vec{n}\|^2_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)}. \tag{1.4.91}$$

Using the same method as for the estimates on s' (see (1.4.84) and (1.4.85)), and applying (1.4.90) we find

$$\|\nabla\nu_2\|_{L^{2,1}(\mathbb{D}_r)} \le C\|\nabla\nu\|_{L^2\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \le C\|\nabla\vec{n}\|_{L^2(\mathbb{D})}. \tag{1.4.92}$$

Combining (1.4.91) and (1.4.92) yields

$$\|\nabla\nu\|_{L^{2,1}(\mathbb{D}_r)} \le C\|\nabla\vec{n}\|_{L^2(\mathbb{D})}. \tag{1.4.93}$$

Since $\vec{n} = \nu + 2\alpha$, (1.4.89) and (1.4.93) ensure that

$$\begin{aligned}\|\nabla\vec{n}\|_{L^{2,1}(\mathbb{D}_r)} &\le \|\nabla\nu\|_{L^{2,1}(\mathbb{D}_r)} + 2\|\nabla\alpha\|_{L^{2,1}(\mathbb{D}_r)} \\ &\le C\,\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\,,\end{aligned}$$

which concludes the proof. □

Theorem A follows from combining theorems 1.4.1 and 1.4.2.

1.4.3 ε-regularity results

In this section, we briefly recall T. Rivière's ε-regularity results. We will not detail the proofs (since we do not improve on them) but give the overall ideas to contextualize them. Further, the proof of theorem 4.2.1 will flow in a very similar way and offer enough illustration for these methods.

Following is a combination of theorem I.5 in [Riv08] and theorem I.1 in [BR14].

Theorem 1.4.3. Let $\Phi \in \mathcal{E}(\mathbb{D})$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and $\lambda = \frac{1}{2}\log\left(\frac{|\nabla\Phi|^2}{2}\right)$ its conformal factor. We assume

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D})} \le C_0.$$

Then there exists $\varepsilon_0 > 0$ such that if

$$\int_{\mathbb{D}} |\nabla\vec{n}|^2 < \varepsilon_0, \tag{1.4.94}$$

then for any $r < 1$ and for any $k \in \mathbb{N}$

$$\begin{aligned}&\|\nabla^k\vec{n}\|^2_{L^\infty(\mathbb{D}_r)} \le C\int_{\mathbb{D}}|\nabla\vec{n}|^2\,,\\ &\|e^{-\lambda}\nabla^k\Phi\|^2_{L^\infty(\mathbb{D}_r)} \le C\left(\int_{\mathbb{D}}|\nabla\vec{n}|^2 + 1\right),\end{aligned}$$

with C a real constant depending on r, C_0 and k.

Proof. The original result by T. Rivière was in fact formulated in any codimension. The idea behind the proof is to apply theorem A.3.4 to system (1.2.46) on balls $B_t(p)$ to find the following Morrey-type inequalities for ε_0 small enough :

$$\|\nabla S\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} + \|\nabla\vec{R}\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} \le \frac{3}{4}\left(\|\nabla S\|^2_{L^2(B_t(p))} + \|\nabla\vec{R}\|^2_{L^2(B_t(p))}\right). \tag{1.4.95}$$

Through classical estimates on Riesz potentials, see for instance theorem 3.1 in [Ada75] , it entails

$$\exists q > 2 \text{ s.t. } \|\nabla S\|_{L^q\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} + \left\|\nabla\vec{R}\right\|_{L^q\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \le C_q\left(\|\nabla S\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \left\|\nabla\vec{R}\right\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)}\right). \tag{1.4.96}$$

Since $q > 2$, the criticality of (1.2.46) is broken. Its third equation, as well as (A.2.23) allows one to pass the regularity and the controls on to $\vec{n}$ and $\nabla\Phi$. Bootstrapping yields the desired result. □

One of the consequences of this ε-regularity is to vindicate the weak immersion formalism. Indeed as defined weak Willmore immersions are smooth, and thus Willmore immersions in the sense of definition 1.2.4. Besides the controls displayed can be readily obtained uniformly away from *concentration points* and thus exploited for the compactness results.

Theorem 1.4.3 in fact followed a preexisting result by E. Kuwert and R. Schatzle (theorem 2.10 of [KS01b],) :

Theorem 1.4.4. There exists $\varepsilon_0 > 0$ such that if $\Phi : \Sigma \to \mathbb{R}^3$ is an immersed surface, $\Sigma_l = \Phi^{-1}(B_l(x_0)) \subset\subset \Sigma$ such that

$$\int_{\Sigma_l} |A|^2 \, d\mu < \varepsilon_0,$$

then

$$\|A\|^2_{L^\infty\left(\Sigma_{\frac{l}{2}}\right)} \leq C\left(W(\Phi) + \frac{1}{l^2}\|A\|_{L^2(\Sigma_l)}\right)\|A\|_{L^2(\Sigma_l)}.$$

While similar in appearance they differ fundamentally. Indeed theorem 1.4.4 does not start with a weak immersion and considers extrinsic balls (meaning balls of the ambient space), while theorem 1.4.3 deals with intrinsic ones. Thus while these two results intersect they do not *a priori* overlap. We refer the reader to the discussion in [BWV18] (between estimate (I.11) and the end of the introduction) for more details.

1.5 Branched Willmore immersions

1.5.1 Behavior around the branch point

So far we have either considered Willmore immersions, or branched Willmore immersions away from the branch point. Studying the behavior around a branch point requires a specific analysis. Carried out by Y. Bernard and T. Rivière (below is theorem 1.8 of [BR13]) it lead to a description through an expansion aroung the point.

Theorem 1.5.1. Let $\Phi \in C^\infty(\mathbb{D}\backslash\{0\}) \cap \left(W^{2,2} \cap W^{1,\infty}\right)(\mathbb{D})$ be a Willmore conformal branched immersion whose Gauss map $\vec{n}$ lies in $W^{1,2}(\mathbb{D})$ and with a branch point at 0 of multiplicity $\theta + 1$. Let λ be its conformal factor, $\vec{\gamma}_0$ the *first residue* defined as

$$\vec{\gamma}_0 := \frac{1}{4\pi}\int_{\partial\mathbb{D}} \vec{\nu}.\left(\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n} \times \vec{H}\right).$$

Then there exists $\alpha \in \mathbb{Z}$ such that $\alpha \leq \theta$ and locally around the origin, Φ has the following asymptotic expansion :

$$\Phi(z) = \Re\left(\vec{A}z^{\theta+1} + \sum_{j=1}^{\theta+1-\alpha}\vec{B}_j z^{\theta+1+j} + \vec{C}_\alpha|z|^{2(\theta+1)}z^{-\alpha}\right) - C\vec{\gamma}_0\left(\ln|z|^{2(m+1)} - 4\right) + \xi(z), \tag{1.5.97}$$

where $\vec{B}_j$, $\vec{C}_\alpha \in \mathbb{C}^3$ are constant vectors, $\vec{A} \in \mathbb{C}\backslash\{0\}$, and $C \in \mathbb{R}$. Furthermore ξ satisfies the estimates

$$\nabla^j\xi(z) = O\left(|z|^{2(\theta+1)-\alpha-j+1-\upsilon}\right) \text{ for all } \upsilon > 0 \text{ and } j \leq \theta + 2 - \alpha,$$
$$|z|^{-\theta}\,\nabla^{\theta-\alpha+3}\xi \in L^p \text{ for all } p < \infty.$$

In particular :

$$\vec{H}(z) = \Re\left(\vec{E}_\alpha \bar{z}^{-\alpha}\right) - \vec{\gamma}_0 \ln|z| + \eta(z), \tag{1.5.98}$$

where $\vec{E}_\alpha \in \mathbb{C}^3 \backslash \{0\}$. The function η satisfies

$$\nabla^j \eta(z) = O\left(|z|^{1-j-\alpha-\upsilon}\right) \text{ for all } \upsilon > 0 \text{ and } j \leq \theta - \alpha,$$
$$|z|^\theta \nabla^{\theta+1-\alpha}\eta \in L^p \text{ for all } p < \infty.$$

Once more we will not detail the proof and merely explain the relevant ideas. Here the difficulty is that the Willmore equation and the conservation laws are only true on $\mathbb{D}\backslash\{0\}$. To extend them to the whole disk requires to introduce a term $\gamma_i \delta_0$ where γ_i is the corresponding residue and δ_0 a Dirac at 0. More explicitely around the branch point, the conservation laws induce the following five residues :

$$\begin{cases} \vec{\gamma}_0 := \dfrac{1}{4\pi}\displaystyle\int_{\partial\mathbb{D}} \vec{\nu}.\left(\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right) \\ \gamma_1 = \dfrac{1}{4\pi}\displaystyle\int_{\partial\mathbb{D}} \vec{\nu}.\left\langle \Phi, \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right\rangle \\ \vec{\gamma}_2 = \dfrac{1}{4\pi}\displaystyle\int_{\partial\mathbb{D}} \vec{\nu}.\left(\Phi\times\left(\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right) + 2H\nabla^\perp\Phi\right) \\ \tilde{\gamma}_2 = \dfrac{1}{4\pi}\displaystyle\int_{\partial\mathbb{D}} \vec{\nu}.\left(\Phi\times\left(\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right) - 2\mathring{A}\nabla^\perp\Phi\right) \\ \vec{\gamma}_3 = \dfrac{1}{4\pi}\displaystyle\int_{\partial\mathbb{D}} \vec{\nu}.\left(-|\Phi|^2\left(\nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right)\right. \\ \quad \left. +2\left\langle \Phi, \nabla\vec{H} - 3\pi_{\vec{n}}\left(\nabla\vec{H}\right) + \nabla^\perp\vec{n}\times\vec{H}\right\rangle\Phi + 4\Phi\times\left(\mathring{A}\nabla^\perp\Phi\right)\right). \end{cases} \tag{1.5.99}$$

These diracs reverberate throughout the process and in the expansions, creating the logarithmic terms. However only the residue corresponding to the invariance by translations has an impact on the expansion of Φ and H. It is called the *first residue*. The quantity $\alpha \in \mathbb{Z}$ is called the *second residue* (see definition 1.7 in [BR13]), although it is not actually a residue. It describes precisely how H behaves at the branch point. Together $\vec{\gamma}_0$ and α control the regularity of Φ across the branch point. The expansions themselves are obtained by going through the Willmore equations using weighted Calderón-Zygmund theorems (detailed in the appendix, section A.3.4).

A particularly noteworthy case occurs when all the residues $\vec{\gamma}_0, \ldots, \vec{\gamma}_3$ cancel out. This configuration which occurs naturally when considering sequences of Willmore immersions (see section 3.2 for context) offers greater regularity and fairer expansions. A Willmore surface whose residues are all null will be called *a true Willmore surface* (this terminology stems from [MR17]).

It is worth mentioning that, using the formalism and the methods introduced in [KS01b], E. Kuwert and R. Schatzle have found corresponding results for branched Willmore surfaces (see [KS07]).

1.5.2 Expansions for True Willmore surfaces

Following is a work in preparation for the analysis of the Bryant's quartic (section 2.6 below), centered around finding expansions for more terms than just Φ and H in continuation of Y. Bernard and T. Rivière works in [BR13] (some of the expansions below were actually already found in this paper).

We then consider $\Phi : \mathbb{D} \to \mathbb{R}^3$ a true Willmore conformal branched immersion with a single branch point at 0, of multiplicity $\theta+1$. Then applying theorem 1.5.1 we can expand Φ around 0 in the following way :

$$\begin{aligned}\Phi(z) = 2\Re\Bigg(& \frac{1}{2(\theta+1)}\begin{pmatrix}1\\-i\\0\end{pmatrix} z^{\theta+1} + \sum_{j=1}^{\theta+1-\alpha}\frac{\vec{A}_j}{\theta+1+j} z^{\theta+1+j} \\ & + \left(\frac{C z^{\theta+1-\alpha}\bar{z}^{\theta+1} + \overline{C} z^{\theta+1}\bar{z}^{\theta+1-\alpha}}{(\theta+1-\alpha)(\theta+1)}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\Bigg) + \xi,\end{aligned} \tag{1.5.100}$$

where ξ satisfies

$$\nabla^j \xi = O\left(|z|^{2(\theta+1)-\alpha-j+1-\nu}\right) \text{ for all } v>0 \text{ and } j \leq \theta+2-\alpha.$$

Further if we do the conformal change of variables $Z^{\theta+1} = z^{\theta+1} + Az^{\theta+2}$, (1.5.100) becomes

$$\begin{aligned}\Phi(Z) = 2\Re\Bigg(& \frac{1}{2(\theta+1)}\begin{pmatrix}1\\-i\\0\end{pmatrix} Z^{\theta+1} + \left(\frac{\vec{A}_1}{\theta+2} + A\begin{pmatrix}1\\-i\\0\end{pmatrix}\right) Z^{\theta+2} \\ & + \left(\frac{C Z^{\theta+1-\alpha}\overline{Z}^{\theta+1} + \overline{C} Z^{\theta+1}\overline{Z}^{\theta+1-\alpha}}{(\theta+1-\alpha)(\theta+1)}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\Bigg) + O\left(|Z|^{\theta+3}\right).\end{aligned}$$

Thus up to doing a conformal change of charts we can assume that $\vec{A}_1$ has no component along $\begin{pmatrix}1\\-i\\0\end{pmatrix}$, meaning :

$$\vec{A}_1 = \frac{U}{2}\begin{pmatrix}1\\i\\0\end{pmatrix} + V\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{1.5.101}$$

Using Φ conformal, we can expand $\langle \Phi_z, \Phi_z \rangle$ to the order $z^{2\theta+1}$ and conclude that

$$U = \left\langle \vec{A}_1, \begin{pmatrix}1\\-i\\0\end{pmatrix}\right\rangle = 0.$$

We deduce the following expansion for Φ_z :

$$\begin{aligned}\Phi_z = & \frac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} z^{\theta} + \sum_{j=2}^{\theta+1-\alpha}\vec{A}_j z^{\theta+j} \\ & + \left(V z^{\theta+1} + \frac{C}{\theta+1} z^{\theta-\alpha}\bar{z}^{\theta+1} + \frac{\overline{C}}{\theta+1-\alpha} z^{\theta}\bar{z}^{\theta+1-\alpha}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + \xi_z.\end{aligned} \tag{1.5.102}$$

Then, wishing to expand the Gauss map $\vec{n}$ we compute :

$$|\Phi_z|^2 = \frac{r^{2\theta}}{2} + \tilde{\xi}, \tag{1.5.103}$$

with

$$\nabla^j \tilde{\xi} = O\left(|z|^{2\theta+2-j}\right) \text{ for all } v>0 \text{ and } j \leq \theta+3-\alpha.$$

$$\begin{aligned}\Phi_z \times \Phi_{\bar{z}} &= \left(\frac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} + V z^{\theta+1}\begin{pmatrix}0\\0\\1\end{pmatrix} + O(r^{\theta+2})\right) \times \left(\frac{1}{2}\begin{pmatrix}1\\i\\0\end{pmatrix} + \overline{V}\bar{z}^{\theta+1}\begin{pmatrix}0\\0\\1\end{pmatrix} + O(r^{\theta+2})\right)\\ &= \frac{ir^{2\theta}}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{iVz^{\theta+1}\bar{z}^{\theta}}{2}\begin{pmatrix}1\\i\\0\end{pmatrix} - \frac{i\overline{V}z^{\theta}\bar{z}^{\theta+1}}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} + O(r^{2\theta+2}).\end{aligned}$$

Hence we can write

$$\vec{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} - Vz\begin{pmatrix}1\\i\\0\end{pmatrix} - \overline{V}\bar{z}\begin{pmatrix}1\\-i\\0\end{pmatrix} + \nu. \tag{1.5.104}$$

Here

$$\begin{gathered}\frac{|\nu|}{r^2} + \frac{|\nabla \nu|}{r} \leq C,\\ \nabla^2 \nu \in L^p \quad \forall p < \infty.\end{gathered}$$

Since we are studying a fourth order problem on Φ (or second order on H) it is natural to desire expansions valid for the fourth derivatives of Φ and the second of H. With theorem 1.5.1 this only seems possible for $\alpha \leq \theta - 1$. However using the same techniques as in [BR13], fourth order expansions can be drawn even when $\alpha = \theta$.

Proposition 1.5.2. Let Φ satisfy the hypothesis of theorem 1.5.1. We further assume that Φ is a true Willmore immersion. Then in the expansion (1.5.97) ξ satisfies :

$$|z|^{\alpha+1-2\theta}\nabla^4 \xi \in L^p(\mathbb{D}) \quad \forall p \in \mathbb{N}.$$

Similarly η in the expansion (1.5.98) satisfies

$$|z|^{\alpha+1}\nabla^2 \eta \in L^p(\mathbb{D}) \quad \forall p \in \mathbb{N}.$$

Proof. Since this theorem is a slight extension of the previous result, since it corresponds to theorem 2.3 in [LN15] (although with a different formalism) and since we will do this procedure in details and in a more general case (section 4.4), we will only give the outline of the proof.

Starting with the expansions (1.5.97) and (1.5.98), and injecting them into (1.2.43) one finds :

$$\begin{aligned}\vec{L}e^{\lambda} &\in L^\infty,\\ \nabla S, \nabla \vec{R} &\in L^\infty,\\ H\nabla\Phi &\in L^\infty.\end{aligned} \tag{1.5.105}$$

These estimates in fact correspond to (2.30) in [BR13]. Now since Φ is assumed to be a *true* Willmore immersion, system (1.2.46) (which is a priori only defined on the punctured disk) extends on the whole disk. One can then inject (1.5.105) into the first two equations of (1.2.46) and apply theorem A.3.9 to find expansions on ∇S and $\nabla \vec{R}$, which are valid for $\nabla^2 S$ and $\nabla^2 \vec{R}$. Since in fact, thanks to (1.5.98), we can expand $H\nabla\Phi$, and have a control on its first order derivatives, we can apply once more theorem A.3.9 to the first two equations of (1.2.46), and extend the expansions to the third derivatives of S and $\vec{R}$. Theorem A.3.9 applied to the third equation of system (1.2.46) extend (1.5.97) to the fourth order derivatives in the desired way. Similarly, one can extend (1.5.98) as wanted. □

Proposition 1.5.2 propagates to the subsequent expansions, for instance in (1.5.104) $r\nabla^3\nu \in L^p$ for all $p<\infty$. One can differentiate (1.5.102) into :

$$\Phi_{zz} = \frac{\theta}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} z^{\theta-1} + \sum_{j=1}^{\theta+1-\alpha}(\theta+j)\,\vec{A}_j z^{\theta-1+j}$$
$$+\left(\frac{(\theta-\alpha)C}{\theta+1}z^{\theta-1-\alpha}\bar{z}^{\theta+1} + \frac{\theta\overline{C}}{\theta+1-\alpha}z^{\theta-1}\bar{z}^{\theta+1-\alpha}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + \xi_{zz}.$$

Taking the scalar product with (1.5.104) we find :

$$\frac{\Omega}{2} = (\theta+1)Vz^{\theta} - \theta Vz^{\theta} + \frac{T}{2} = Vz^{\theta} + \frac{T}{2}. \tag{1.5.106}$$

On the other hand

$$\Phi_{z\bar{z}} = \left(Cz^{\theta-\alpha} + \overline{C}z^{\theta}\bar{z}^{\theta-\alpha}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + \xi_{z\bar{z}}.$$

Taking the scalar product with (1.5.104) we find :

$$\frac{He^{2\lambda}}{2} = Cz^{\theta-\alpha}\bar{z}^{\theta} + \overline{C}z^{\theta}\bar{z}^{\theta-\alpha} + \eta_0.$$

Combined with (1.5.103) we find :

$$H = 2\overline{C}z^{-\alpha} + 2C\bar{z}^{-\alpha} + \eta_1. \tag{1.5.107}$$

If we compare (1.5.98) and (1.5.107) we can deduce $\vec{E_\alpha} = 2\overline{C}$.

From (1.5.106) we find

$$\begin{aligned}\Omega &= O(r^{\theta})\\ \Omega_z &= O(r^{\theta-1}).\end{aligned} \tag{1.5.108}$$

Thanks to Gauss-Codazzi equation (A.2.20) we can evaluate $\Omega_{\bar{z}}$ through the derivatives of H. Then with (1.5.107), one finds

$$\begin{aligned}\Omega_{\bar{z}} &= H_z e^{2\lambda} = -2\alpha C z^{\theta-\alpha-1}\bar{z}^{\theta} + O\left(r^{2\theta-\alpha}\right) = O(r^{2\theta-\alpha-1})\\ \Omega_{z\bar{z}} &= H_{zz}e^{2\lambda} + H\left(2\left|\Phi_z\right|^2\right)_z = -2\alpha(\theta-\alpha-1)Cz^{\theta-\alpha-2}\bar{z}^{\theta} + O(r^{2\theta-\alpha-1}) = O(r^{2\theta-\alpha-2}).\end{aligned} \tag{1.5.109}$$

As a conclusion, combining (1.5.103), (1.5.108) and (1.5.109) ensures that :

$$\begin{aligned}e^{-2\lambda}\left(\Omega_{z\bar{z}}\Omega - \Omega_z\Omega_{\bar{z}}\right) &= 2\alpha(\alpha+1)CVz^{\theta-\alpha-2} + O\left(r^{\theta-\alpha-1}\right)\\ &= O\left(r^{\theta-\alpha-2}\right) = O\left(r^{-2}\right).\end{aligned} \tag{1.5.110}$$

This term will be pivotal in section 2.4.2 since it is the leading order term of the Bryant's quartic (see (2.4.38)), which will thus have at most a pole of order 2 at branch points of the immersion.

2

Conformal Gauss map approaches

Abstract.

2.1 Introduction

Once more, this introduction will give a quick and stand-alone look at the notions developed in this chapter, with an emphasis on the author's contributions.

Following we mostly study concepts and ideas revolving around the conformal Gauss map and its deep link with the notion of Willmore surfaces. While most of the results presented here were already known and obtained through the Dorfmeister-Pedit-Wu (DPW) method (see for instance [DPW98], [DW19], [Eji88] or [Ric97]), the originality of the present work lies in our approach. Indeed, we will merely employ basic differential geometry. Consequently, most of our result will present clear geometric interpretations, often absent when obtained through DPW methods. Most of this chapter was part of the preprint [Mar19a].

We will work with three models : the euclidean space $\mathbb{R}^3$, the 3-dimensional round sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$. Relying on the local conformal equivalences induced by the stereographic projections, we can jointly describe the space of spheres of the three models as the de Sitter space $\mathbb{S}^{4,1}$ of $\mathbb{R}^{4,1}$. From the characterization of conformal maps as those who preserve the set of spheres, we can then find *explicit* correspondances between the conformal groups and $SO(4,1)$, described in the following result.

Corollary 2.1.1. $SO(4,1)$ acts transitively through conformal diffeomorphisms on

— $X \in \mathbb{S}^3$:

$$M.X = \frac{V_\circ}{V_5}$$

where

$$V = M\begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} V_\circ \\ V_5 \end{pmatrix}.$$

— $x \in \mathbb{R}^3$:

$$M.x = \frac{y_\diamond}{y_5 - y_4}$$

where

$$y = M\begin{pmatrix} x \\ \frac{|x|^2-1}{2} \\ \frac{|x|^2+1}{2} \end{pmatrix} = \begin{pmatrix} y_\diamond \\ y_4 \\ y_5 \end{pmatrix}.$$

While the fact that $\mathrm{Conf}(\mathbb{S}^3)$ and $SO(4,1)$ are isomorphic is well-known, the explicit correspondance is rather uncommon.

From this, in section 2.3, we will consider the Conformal Gauss map of an immersion. This notion will be to Willmore surfaces what the Gauss map is to CMC surfaces. It can be apprehended as the map that, to a point x on the surface associates the tangent sphere of radius the inverse of the mean curvature. It is thus an application $Y : \Sigma \to \mathbb{R}^{4,1}$. Studying its geometry allows one to build a natural normal frame based on the immersion. Computing the corresponding mean and trace free curvatures then highlights that Y is the conformal Gauss map of a Willmore immersion if and only if it is a minimal immersion in $\mathbb{S}^{4,1}$, with the minimal equation for the conformal Gauss map being equivalent to the Willmore equation for the immersion. This will be detailed in section 2.4.

Further the conservation laws of the Willmore surfaces can be read thanks to the conformal Gauss map.

Theorem 2.1.1. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Let

$$\mu = \left(\nabla Y_i Y_j - Y_i \nabla Y_j\right) = \nabla Y Y^T - Y \nabla Y^T.$$

Then $\mathrm{div}_g(\mu) = 0$, and

$$2\mu = \begin{pmatrix} U & -\frac{V_{\mathrm{tra}} - V_{\mathrm{inv}}}{2} & \frac{V_{\mathrm{tra}} + V_{\mathrm{inv}}}{2} \\ \left(\frac{V_{\mathrm{tra}} - V_{\mathrm{inv}}}{2}\right)^T & 0 & V_{\mathrm{dil}} \\ -\left(\frac{V_{\mathrm{inv}} + V_{\mathrm{tra}}}{2}\right)^T & -V_{\mathrm{dil}} & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 0 & -\tilde{V}_{\mathrm{rot}\,3} & \tilde{V}_{\mathrm{rot}\,2} \\ \tilde{V}_{\mathrm{rot}\,3} & 0 & -\tilde{V}_{\mathrm{rot}\,1} \\ -\tilde{V}_{\mathrm{rot}\,2} & \tilde{V}_{\mathrm{rot}\,1} & 0 \end{pmatrix}$$

with $V_{\mathrm{tra}}, V_{\mathrm{dil}}, V_{\mathrm{rot}}, \tilde{V}_{\mathrm{rot}}$ and V_{inv} defined in theorem 1.2.14.

This observation, coupled with the behavior of the conformal Gauss map under the action of the conformal group tells us how the conserved quantities change under the action of the conformal tranforms. Considering the specific case of the inversions yields an elementary proof of theorem 3.9 of [MR17].

Corollary 2.1.2. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Let $\iota : x \mapsto \frac{x}{|x|^2}$ be the inversion at the origin. Let $V_{*,\iota}$ be the conserved quantity corresponding to the transformation $*$ for $\iota \circ \Phi$. Then

$$\begin{aligned} V_{\mathrm{tra},\,\iota} &= V_{\mathrm{inv}} \\ V_{\mathrm{inv},\,\iota} &= V_{\mathrm{tra}} \\ V_{\mathrm{dil},\,\iota} &= -V_{\mathrm{dil}} \\ \tilde{V}_{\mathrm{rot},\,\iota} &= \tilde{V}_{\mathrm{rot}}. \end{aligned}$$

The usefullness of the conformal Gauss map will also be seen in the study of branched Willmore surfaces, where its behavior at a branch point will allow us to compute the second residue. Applied to inversions of minimal immersions, it will make computing both residues at branch points obtained easier.

We will finally study another relevant quantity : Bryant's quartic. This object, which can be seen as the product of the tracefree curvatures of the conformal Gauss map takes center stage in the problem of conformally CMC surfaces, subject of the section 2.4.3. This issue revolves around finding equivalent conditions for a conformal Willmore surface to be the conformal transform of a CMC surface. We say that Φ (respectively X, Z) is conformally CMC (respectively minimal) if and only if there exists a conformal diffeomorphism φ of $\mathbb{R}^3 \cup \{\infty\}$ (respectively $\mathbb{S}^3$, $\mathbb{H}^3$) such that $\varphi \circ \Phi$ (respectively $\varphi \circ X$, $\varphi \circ Z$) has constant mean curvature (respectively is minimal) in $\mathbb{R}^3$ (respectively $\mathbb{S}^3$, $\mathbb{H}^3$). One such geometric condition on the conformal Gauss map arises naturally.

Theorem 2.1.2. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a smooth conformal immersion, and X (respectively Z) its representation in $\mathbb{S}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). Let Y be its conformal Gauss map. We assume the set of umbilic points of Φ (or equivalently, see (A.2.29) and (A.2.42), X or Z) to be nowhere dense.

Then

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with lightlike normal.

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with timelike normal.

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with spacelike normal.

From this, with a careful study of the geometry of conformal Gauss maps, two similar characterizations follow. The first one is based on the notion of isothermic immersion, which we will briefly go over in section 2.4.2. It can easily be compared to theorem 4.4 from [Boh12], reached with other means.

Theorem B. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). We assume that X (or equivalently, see (A.2.29) and (A.2.42), Φ or Z) has no umbilic point. One of the representation of X is conformally *CMC* in its ambient space if and only if $\mathcal{Q}$ is holomorphic and X is isothermic. More precisely $\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}$ is then necessarily real and

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

This last theorem is in accordance with previous results obtained through DPW methods in the previously mentioned works. Its originality lies in the numeric determination of the space in which an immersion is potentially CMC. The same analysis that led to this theorem can be extended to give it a more outstanding shape.

Theorem C. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). We assume X (or equivalently, see (A.2.29) and (A.2.42), Φ or Z) has no umbilic point. One of the representation of X is conformally *CMC* in its ambient space if and only if $\mathcal{Q}$ is holomorphic and $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$. More precisely

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

While theorem C is a modification of theorem B, it is one we think fruitful. Indeed the replacing of "X isothermic" by "$\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$ seems to suggest that $\mathcal{Q}$ could be seen as a measure of isothermicity, and at least when it is holomorphic, $\sqrt{\mathcal{Q}}$ is necessarily the isothermic 1-form.

These type of theorems can be applied to classify Willmore immersions of a sphere, in the fashion of theorem E of [Bry84], or even to some branched immersions of the sphere (following from theorem F in [MR17]), which we will prove in section 2.6 using analytical methods.

We will conclude this section by an exposition of a study by A. Michelat and T. Rivière in section 4 of [MR17] of the case when the second residue is better than expected. This study will prove highly relevant when framed with theorem G.

2.2 Conformal Geometry in three model spaces

2.2.1 Local conformal equivalences

In the following $\langle ., . \rangle$ will denote the standard product on the relevant contextual space. For instance if $u, v \in \mathbb{R}^m$ with $m \in \mathbb{N}$, $\langle u, v \rangle$ denotes the euclidean product of u and v in $\mathbb{R}^m$. If u and v are stated to be in $\mathbb{R}^{m,1}$ then $\langle u, v \rangle = \sum_{i=1}^{m} u_i v_i - u_{m+1} v_{m+1}$ denotes the $(m, 1)$ Lorentzian product of u and v in $\mathbb{R}^{m+1}$.

We will focus on immersions into the Euclidean space $\mathbb{R}^3$, into the round sphere $\mathbb{S}^3$ and into the hyperbolic space $\mathbb{H}^3$.

These three spaces are locally conformally equivalent and thus their respective conformal geometry can be linked. Namely the stereographic projection from the north pole N

$$\pi : \begin{cases} \mathbb{S}^3 \backslash \{N\} \to \mathbb{R}^3 \\ (x, y, z, t) \mapsto \dfrac{1}{1-t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{cases}$$

is a conformal diffeomorphism whose inverse is

$$\pi^{-1} : \begin{cases} \mathbb{R}^3 \to \mathbb{S}^3 \backslash \{N\} \\ (x, y, z) \mapsto \dfrac{1}{1+r^2} \begin{pmatrix} 2x \\ 2y \\ 2z \\ r^2 - 1 \end{pmatrix} \end{cases}$$

which extends to a conformal diffeomorphism $\mathbb{R}^3 \cup \{\infty\} \to \mathbb{S}^3$. Consequently one can link $\mathrm{Conf}\left(\mathbb{R}^3\right)$ and $\mathrm{Conf}\left(\mathbb{S}^3\right)$.

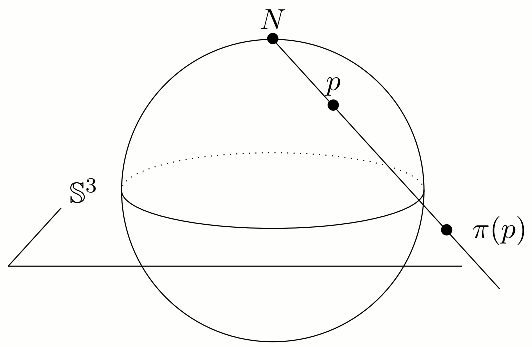

Figure 2.1 – Stereographic projection.

Proposition 2.2.1. π realises an isomorphism between $\mathrm{Conf}\left(\mathbb{R}^3\right)$ and $\mathrm{Conf}\left(\mathbb{S}^3\right)$, with $\mathrm{Conf}(X)$ being the group of conformal diffeomorphisms of X. We remind the reader that we make a slight abuse of notations and use $\mathrm{Conf}\left(\mathbb{R}^3\right)$ for the conformal group of $\mathbb{R}^3 \cup \{\infty\}$.

Combining both Liouville theorem (see theorem 1.2.1) and proposition 2.2.1 ensures a description of conformal diffeomorphisms of $\mathbb{S}^3$.

Corollary 2.2.1. Any conformal mapping $\varphi \in \mathrm{Conf}(\mathbb{S}^3)$ satisfies either

$$\varphi = \pi^{-1} \circ T_{\vec{b}} \circ R_\Theta \circ D_\lambda \circ T_{\vec{a}} \circ \pi$$

if $\varphi(N) = N$,

$$\varphi = \pi^{-1} \circ T_{\vec{b}} \circ R_{\Theta} \circ D_{\lambda} \circ \iota \circ T_{\vec{a}} \circ \pi$$

otherwise.

Using the Poincaré disk model of the hyperbolic space one finds an isometry $\tilde{\pi}_0 \, : \, \mathbb{H}^3 \to \left(B_1(0), \frac{\langle .,. \rangle}{(1-|p|^2)^2} \right)$ and thus a conformal diffeomorphism between $\mathbb{H}^3$ and the unit ball of $\mathbb{R}^3$. It will be convenient in the following to consider $\mathbb{H}^3$ as the upper part of the quadric $\left\{ v \in \mathbb{R}^{3,1} \, | \langle v, v \rangle = -1 \right\}$ in $\mathbb{R}^{3,1}$:

$$\mathbb{H}^3 = \left\{ (x,y,z,t) \,\middle|\, x^2 + y^2 + z^2 - t^2 + 1 = 0 \text{ and } t \geq 0 \right\} \subset \mathbb{R}^{3,1}.$$

Then, the following projection yields an explicit conformal diffeomorphism

$$\tilde{\pi} : \begin{cases} \mathbb{H}^3 \to B_1(0) \\ (x,y,z,t) \mapsto \dfrac{1}{1+t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{cases}$$

of inverse

$$\tilde{\pi}^{-1} : \begin{cases} B_1(0) \to \mathbb{H}^3 \\ (x,y,z) \mapsto \dfrac{1}{1-r^2} \begin{pmatrix} 2x \\ 2y \\ 2z \\ r^2+1 \end{pmatrix} \end{cases}.$$

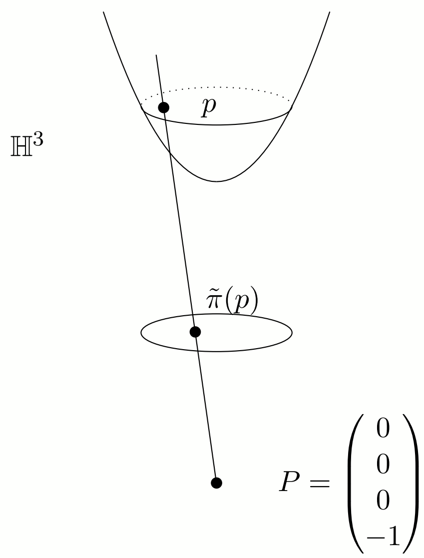

Figure 2.2 – Hyperbolic projection.

In conclusion, our three model spaces are locally conformally equivalent, and their conformal geometries will be intertwined.

2.2.2 Spaces of spheres

In the present subsection, we wish to properly represent the geometry of geodesic spheres of $\mathbb{S}^3$. Our motivation comes from the following result, drawn from chapter 1 in [AG96].

Theorem 2.2.2. Let (M, g) and (N, h) be two Riemann manifolds and $\varphi : M \to N$. φ is conformal if and only if it sends a geodesic sphere of M into a geodesic sphere of N.

Thanks to theorem 2.2.2, one would then expect to be able to detail conformal diffeomorphisms of $\mathbb{S}^3$. Moreover since $\mathbb{H}^3 \hookrightarrow \mathbb{R}^3 \hookrightarrow \mathbb{S}^3$ conformally, we would subsequently be able to represent geodesic spheres in $\mathbb{H}^3$ and $\mathbb{R}^3$.

The stereographic projection ensures that $\mathbb{R}^3 \cup \{\infty\} \simeq \mathbb{S}^3$ conformally, and thus geodesic spheres in $\mathbb{S}^3$ are images by π^{-1} of euclidean spheres and planes ("spheres" going through ∞) of $\mathbb{R}^3$. They will be called spheres in $\mathbb{S}^3$. More precisely :

Definition 2.2.1. A sphere in $\mathbb{S}^3$ is equivalently defined as follows :

— The inverse of the stereographic projection of a sphere or a plane in $\mathbb{R}^3$.

— $\{x \in \mathbb{S}^3 \quad d(q, x) = r\}$ for a given q in $\mathbb{S}^3$. q is then the center of the sphere, of radius $r \leq \frac{\pi}{2}$.

An equator of $\mathbb{S}^3$ is a sphere of maximum radius $r = \frac{\pi}{2}$.

One can easily check that spheres in $\mathbb{S}^3$ are orientable.

Definition 2.2.2. Let

$$\begin{aligned} \mathbb{M}_0 &= \{\text{non-oriented spheres in } \mathbb{S}^3\}, \\ \mathbb{E}_0 &= \{\text{non-oriented equatorial spheres in } \mathbb{S}^3\}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{M} &= \{\text{oriented spheres in } \mathbb{S}^3\}, \\ \mathbb{E} &= \{\text{oriented spheres in } \mathbb{S}^3\}. \end{aligned}$$

Let σ be a non-oriented sphere of radius $r < \frac{\pi}{2}$. Let $X_\sigma \in \sigma$ be any point on the sphere and $\vec{N_\sigma}$ the inward pointing (relative to σ) normal to σ at X_σ. Then $p_\sigma = X_\sigma + \tan r \vec{N_\sigma}$ is the summit of the tangent cone to $\mathbb{S}^3$ along σ. Since a sphere in $\mathbb{S}^3$ has constant mean curvature $h = \frac{1}{\tan r}$ (see (A.2.36) in the appendix A.2.4), $p_\sigma = X_\sigma + \frac{1}{h}\vec{N_\sigma}$. This gives us a representation of $\mathbb{M}_0 \backslash \mathbb{E}_0$:

$$P_0 : \begin{cases} \mathbb{M}_0 \backslash \mathbb{E}_0 \to \mathbb{R}^4 \backslash B_1(0) \\ \sigma \mapsto p_\sigma, \end{cases}$$

as shown in figure 1.

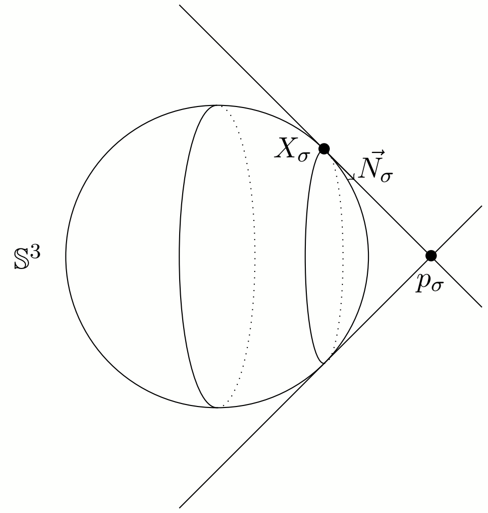

Figure 2.3 – Construction of p_σ.

Conversely given any $p \in \mathbb{R}^4 \backslash B_1(0)$ there exists a unit cone of summit p tangent to $\mathbb{S}^3$, along a sphere of $\mathbb{S}^3$. P_0 is then a bijection. As σ becomes equatorial, $h \to 0$, meaning $p \to \infty$ and $\vec{N}_\sigma \to \vec{\nu}$ with $\vec{\nu} \in \mathbb{R}^4$ independant on the chosen X_σ. To properly represent all of $\mathbb{M}_0$ we define $\underline{p_\sigma} = \begin{pmatrix} p_\sigma \\ 1 \end{pmatrix}$. Then

$$\frac{\underline{p_\sigma}}{|p_\sigma|} = \begin{pmatrix} \frac{p_\sigma}{|p_\sigma|} \\ \frac{1}{|p_\sigma|} \end{pmatrix} \to \begin{pmatrix} \vec{\nu} \\ 0 \end{pmatrix}$$

as σ tends toward an equatorial sphere of constant normal $\vec{\nu}$. Then one can represent $\mathbb{M}_0$ in $\mathbb{RP}^4$, with equatorial spheres being sent to $\left[\begin{pmatrix} \vec{\nu} \\ 0 \end{pmatrix}\right]$ typed directions (where $[d]$ denotes the direction of $d \in \mathbb{R}^5$).

$$P_1 : \begin{cases} \mathbb{M}_0 \to \mathbb{RP}^4 \\ \sigma \mapsto \left[\underline{p_\sigma}\right]. \end{cases}$$

Since any equatorial sphere is fully determined by its normal, P_1 remains injective. However for non equatorial spheres p_σ is necessarily outside $B_1(0)$, and thus P_1 cannot be surjective. Going further will require some basic notions in semi-Riemannian geometry.

Definition 2.2.3. Let $m \in \mathbb{N}$ and $v \in \mathbb{R}^{m,1}$. Then v is said to be

— *spacelike* if $\langle v, v \rangle > 0$,

— *lightlike* if $\langle v, v \rangle = 0$,

— *timelike* if $\langle v, v \rangle < 0$.

Accordingly a direction $d \in \mathbb{RP}^{m+1}$ is called

— *spacelike* if there exists $v \in \mathbb{R}^{m,1}$ such that $\langle v, v \rangle > 0$ and $[v] = d$,

— *lightlike* if there exists $v \in \mathbb{R}^{m,1}$ such that $\langle v, v \rangle = 0$ and $[v] = d$,

— *timelike* if there exists $v \in \mathbb{R}^{m,1}$ such that $\langle v, v \rangle < 0$ and $[v] = d$.

We also define

— the *De Sitter* space of $\mathbb{R}^{m,1}$ as the set of unit spacelike vectors.
It will be denoted $\mathbb{S}^{m,1} := \left\{ v \in \mathbb{R}^{m,1} \quad \langle v, v \rangle = 1 \right\}$,

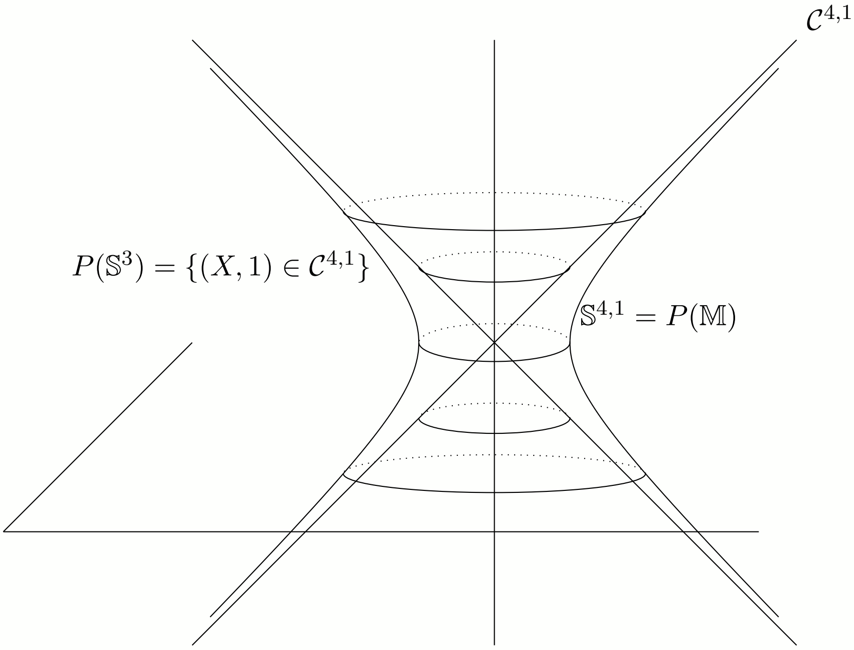

Figure 2.4 – De Sitter and the isotropic cone.

— the *isotropic cone* of $\mathbb{R}^{m,1}$ as the set of lightlike vectors.
It will be denoted $\mathcal{C}^{m,1} := \{v \in \mathbb{R}^{m,1} \quad \langle v, v\rangle = 0\}$.

One can realize that the image of P_1 is the set of all the space-like directions of $\mathbb{R}^{4,1}$ which is isomorphic to $\mathbb{S}^{4,1}/\{\pm Id\}$. We finally obtain our representation of non-oriented spheres :

$$P : \begin{cases} \mathbb{M}_0 \to \mathbb{S}^{4,1}/\{\pm Id\} \\ \sigma \mapsto \dfrac{\underline{p}}{\|\underline{p}\|} \end{cases}$$

where $\|\underline{p}\| = \sqrt{\langle \underline{p_\sigma}, \underline{p_\sigma}\rangle}$. P is easily extended to $\mathbb{M}$ by taking the natural two covering of $\mathbb{S}^{4,1}/\{\pm Id\}$. Two opposite points in the De Sitter space then represent the same sphere with opposite orientations.

$$P : \begin{cases} \mathbb{M} \to \mathbb{S}^{4,1} \\ \sigma \mapsto \dfrac{\underline{p}}{\|\underline{p}\|} = h \begin{pmatrix} X_\sigma \\ 1 \end{pmatrix} + \begin{pmatrix} \vec{N}_\sigma \\ 0 \end{pmatrix} \end{cases} \tag{2.2.1}$$

for any $X_\sigma \in \sigma$.

As $h \to \infty$ (that is the radius of the sphere goes to 0 and thus the sphere collapses on a point $X \in \mathbb{S}^3$), $\frac{P(\sigma)}{h} \to (X, 1)$, meaning that $P(\sigma)$ tends to ∞ in an isotropic direction of $\mathbb{R}^{4,1}$ bijectively and smoothly linked with the point of collapse X. One can then continuously extend P

$$P : \begin{cases} \mathbb{M} \cup \mathbb{S}^3 \to \mathbb{S}^{4,1} \cup \mathcal{C}^{4,1} \\ \sigma \in \mathbb{M} \mapsto \dfrac{\underline{p}}{\|\underline{p}\|} = h \begin{pmatrix} X_\sigma \\ 1 \end{pmatrix} + \begin{pmatrix} \vec{N}_\sigma \\ 0 \end{pmatrix} \in \mathbb{S}^{4,1} \\ X \in \mathbb{S}^3 \mapsto \begin{pmatrix} X \\ 1 \end{pmatrix} \in \mathcal{C}^{4,1}. \end{cases} \tag{2.2.2}$$

Since the stereographic projection is a conformal diffeomorphism, the set of non-oriented (respectively oriented) spheres and planes or $\mathbb{R}^3$ is in bijection with $\mathbb{M}_0$ (respectively $\mathbb{M}$) and can be represented using P. Using formula (A.2.30) (see appendix A.2.3) one finds

$$P : \begin{cases} \left(\mathbb{R}^3 \cup \{\infty\}\right) \cup \mathbb{M} \to \mathbb{S}^{4,1} \cup \mathcal{C}^{4,1} \\ \sigma \in \mathbb{M} \mapsto H_\sigma \begin{pmatrix} \Phi_\sigma \\ \frac{|\Phi_\sigma|^2 - 1}{2} \\ \frac{|\Phi_\sigma|^2+1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n}_\sigma \\ \langle \vec{n}_\sigma, \Phi_\sigma \rangle \\ \langle \vec{n}_\sigma, \Phi_\sigma \rangle \end{pmatrix} \text{ for any } \Phi_\sigma \in \sigma \\ \Phi \in \mathbb{R}^3 \mapsto \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} \in \mathcal{C}^{4,1} \\ \infty \mapsto \left(0,1,1\right) \in \mathcal{C}^{4,1}. \end{cases} \tag{2.2.3}$$

Similarly consider $\mathbb{M}_{\mathbb{H}^3}$ the set of oriented geodesic spheres in $\mathbb{H}^3$. The function $\pi^{-1} \circ \tilde{\pi}$ sends $\mathbb{H}^3$ injectively into $\mathbb{S}^3$ and thus maps $\mathbb{M}_{\mathbb{H}^3}$ injectively into $\mathbb{M}$. $\mathbb{M}_{\mathbb{H}^3}$ can then be represented using P (see formula (A.2.43) in appendix A.2.5) one finds

$$P : \begin{cases} \mathbb{H}^3 \cup \mathbb{M}_{\mathbb{H}^3} \to \mathbb{S}^{4,1} \cup \mathcal{C}^{4,1} \\ \sigma \in \mathbb{M}_{\mathbb{H}^3} \mapsto H^Z_\sigma \begin{pmatrix} Z_{h\,\sigma} \\ -1 \\ Z_{4\,\sigma} \end{pmatrix} + \begin{pmatrix} \vec{n}^Z_{h\,\sigma} \\ 0 \\ \vec{n}^Z_{4\,\sigma} \end{pmatrix} \text{ for any } \begin{pmatrix} Z_{h\,\sigma} \\ Z_{4\,\sigma} \end{pmatrix} \in \sigma \\ Z = \begin{pmatrix} Z_h \\ Z_4 \end{pmatrix} \in \mathbb{H}^3 \mapsto \begin{pmatrix} Z_h \\ -1 \\ Z_4 \end{pmatrix} \in \mathcal{C}^{4,1}. \end{cases} \tag{2.2.4}$$

2.2.3 $\mathrm{Conf}(\mathbb{S}^3) \simeq SO(4,1)$

As foreshadowed in subsection 2.2.2, we can use P to study conformal diffeomorphisms of $\mathbb{S}^3$.

Theorem 2.2.3. *P realises an isomorphism between $\mathrm{Conf}(\mathbb{S}^3)$ and $SO(4,1)$.*

Proof. According to proposition 2.2.1, showing $\mathrm{Conf}\left(\mathbb{R}^3\right) \simeq SO(4,1)$ is enough. We proceed in three steps : we define the correspondance, show that it represents a morphism and conclude by proving it is bijective.

Step 1 : Defining the correspondance $M \to \varphi_M$

The core idea here is that isotropic directions in $\mathbb{R}^{4,1}$ are in bijection with $\mathbb{R}^3 \cup \{\infty\}$, and that any $M \in SO(4,1)$ shuffles them. Thus M yields a transformation of $\mathbb{R}^3 \cup \{\infty\}$. Its conformality is all one needs to prove.

Let $p(x) := \begin{pmatrix} x \\ \frac{|x|^2-1}{2} \\ \frac{|x|^2+1}{2} \end{pmatrix} = P_{|_{\mathbb{R}^3 \cup \{\infty\}}}(x)$. One easily shows that for all i, j :

$$\langle \partial_i p, \partial_j p \rangle = \delta_{ij},$$

that is $p : \mathbb{R}^3 \to P\left(\mathbb{R}^3 \cup \{\infty\}\right)$ is an isometry. As $x \to \infty$, $\frac{p(x)}{|p(x)|} \to \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Noticing that $P\left(\mathbb{R}^3 \cup \{\infty\}\right) = \left\{p \in \mathcal{C}^{4,1} \text{ s.t. } p_5 - p_4 = 1\right\} \cup \{(0,0,0,1,1)\}$, one can conversely associate to any $p \in \mathcal{C}^{4,1}$ a point $x = \frac{(p_1,p_2,p_3)}{p_5 - p_4} \in \mathbb{R}^3 \cup \{\infty\}$ depending only on the direction of p.

Given $M \in SO(4,1)$ let $y = Mp(x) = \begin{pmatrix} y_\diamond \\ y_4 \\ y_5 \end{pmatrix}$. Then

$$\begin{aligned}
\langle y_\diamond, y_\diamond \rangle &= y_5^2 - y_4^2 \\
\langle \partial_i y_\diamond, y_\diamond \rangle &= \partial_i y_5 y_5 - \partial_i y_4 y_4 \\
\langle \partial_i y_\diamond, \partial_j y_\diamond \rangle &= \delta_{ij} + \partial_i y_5 \partial_j y_5 - \partial_i y_4 \partial_j y_4.
\end{aligned}$$

Renormalizing as suggested, let $\varphi_M(x) = \frac{y_\diamond}{y_5 - y_4} = p^{-1}\left(\frac{Mp(x)}{(Mp(x))_5 - (Mp(x))_4}\right)$. φ_M is a transformation of $\mathbb{R}^3 \cup \{\infty\}$. Let us show it is conformal :

$$\begin{aligned}
\langle \partial_i \varphi_M, \partial_j \varphi_M \rangle &= \left\langle \frac{\partial_i y_\diamond}{y_5 - y_4} - \frac{(\partial_i y_5 - \partial_i y_4)\, y_\diamond}{(y_5 - y_4)^2}, \frac{\partial_j y_\diamond}{y_5 - y_4} - \frac{(\partial_j y_5 - \partial_j y_4)\, y_\diamond}{(y_5 - y_4)^2} \right\rangle \\
&= \frac{1}{(y_5 - y_4)^2} \langle \partial_i y_\diamond, \partial_j y_\diamond \rangle + \frac{(\partial_i y_5 - \partial_i y_4)(\partial_j y_5 - \partial_j y_4)}{(y_5 - y_4)^3} \langle y_\diamond, y_\diamond \rangle \\
&- \frac{1}{(y_5 - y_4)^3} \left((\partial_i y_5 - \partial_i y_4) \langle \partial_j y_\diamond, y_\diamond \rangle - (\partial_j y_5 - \partial_j y_4) \langle \partial_i y_\diamond, y_\diamond \rangle \right) \\
&= \frac{\delta_{ij}}{(y_5 - y_4)^2} + \frac{\partial_i y_5 \partial_j y_5 - \partial_i y_4 \partial_j y_4}{(y_5 - y_4)^2} \\
&+ \frac{1}{(y_5 - y_4)^3} (\partial_i y_5 - \partial_i y_4)(\partial_j y_5 - \partial_j y_4)(\partial_i y_5 y_5 - \partial_i y_4 y_4) \\
&- \frac{(\partial_i y_5 - \partial_i y_4)(\partial_j y_5 y_5 - \partial_j y_4 y_4)}{(y_5 - y_4)^3} \\
&+ \frac{(\partial_j y_5 - \partial_j y_4)(\partial_i y_5 y_5 - \partial_i y_4 y_4)}{(y_5 - y_4)^3} \\
&= \frac{\delta_{ij}}{(y_5 - y_4)^2}.
\end{aligned}$$

Then $\varphi_M \in \mathrm{Conf}(\mathbb{R}^3)$.

Step 2 : $M \to \varphi_M$ **is a morphism**

Given M_1 and $M_2 \in SO(4,1)$, we compute

$$\begin{aligned}
\varphi_{M_1} \circ \varphi_{M_2}(x) &= p^{-1} \left(\frac{M_1 \frac{M_2 p(x)}{(M_2 p(x))_5 - (M_2 p(x))_4}}{\left(M_1 \frac{M_2 p(x)}{(M_2 p(x))_5 - (M_2 p(x))_4} \right)_5 - \left(M_1 \frac{M_2 p(x)}{(M_2 p(x))_5 - (M_2 p(x))_4} \right)_4} \right) \\
&= p^{-1} \circ \left(\frac{M_1 M_2 p(x)}{(M_1 M_2 p(x))_5 - (M_1 M_2 p(x))_4} \right) \\
&= \varphi_{M_1 M_2}(x).
\end{aligned}$$

Thus $M \mapsto \varphi_M$ is a morphism between $SO(4,1)$ and $\mathrm{Conf}\left(\mathbb{R}^3 \cup \{\infty\}\right)$.

Step 3 : $M \to \varphi_M$ **is an isomorphism**

Bijectivity is the only property left to show. According to theorem 1.2.1, exhibiting $M \in SO(4,1)$ for dilations, translations, rotations and the inversion is enough to ensure surjectivity. Computing we find

Dilations :

For $D_\lambda(x) = e^\lambda x$,

$$M_{D_\lambda} = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{ch}\lambda & \mathrm{sh}\lambda \\ 0 & \mathrm{sh}\lambda & \mathrm{ch}\lambda \end{pmatrix} \in SO(4,1). \tag{2.2.5}$$

Rotations :
For $R_\Theta(x) = \Theta x$, with $\Theta \in O(3)$,

$$M_{R_\Theta} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(4,1). \tag{2.2.6}$$

Inversion :
For $\iota(x) = \frac{x}{|x|^2}$,

$$M_\iota = \begin{pmatrix} -Id & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(4,1). \tag{2.2.7}$$

Translations :
For $T_{\vec{a}}(x) = x + \vec{a}$, with $\vec{a} \in \mathbb{R}^3$,

$$M_{T_{\vec{a}}} = \begin{pmatrix} Id & -\vec{a} & \vec{a} \\ \vec{a}^T & 1 - \frac{|\vec{a}|^2}{2} & \frac{|\vec{a}|^2}{2} \\ \vec{a}^T & -\frac{|\vec{a}|^2}{2} & 1 + \frac{|\vec{a}|^2}{2} \end{pmatrix} \in SO(4,1). \tag{2.2.8}$$

$M \to \varphi_M$ is then surjective. With injectivity stemming from the uniqueness of the decomposition in theorem 1.2.1, $M \to \varphi_M$ is bijective, which concludes the proof. $\square$

A direct consequence of the proof is the explicit formula for the conformal actions of $SO(4,1)$ on $\mathbb{S}^3$ and $\mathbb{R}^3$.

Corollary 2.2.2. $SO(4,1)$ acts transitively through conformal diffeomorphisms on

— $\mathbb{S}^3$:

$$M.X = \frac{V_\circ}{V_5}$$

where

$$V = M \begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} V_\circ \\ V_5 \end{pmatrix}.$$

— $\mathbb{R}^3$:

$$M.x = \frac{y_\diamond}{y_5 - y_4}$$

where

$$y = M \begin{pmatrix} x \\ \frac{|x|^2-1}{2} \\ \frac{|x|^2+1}{2} \end{pmatrix} = \begin{pmatrix} y_\diamond \\ y_4 \\ y_5 \end{pmatrix}.$$

While $\mathrm{Conf}(\mathbb{S}^3) \simeq SO(4,1)$ is well known, the explicit action of $SO(4,1)$ on elements of $\mathbb{S}^3$ is less commonly found.

We will work in the three models and frequently switch from one to the other. For simplicity, we define notations once and for all. Given Σ a Riemman surface and $\Phi : \Sigma \to \mathbb{R}^3$, we refer to $X = \pi^{-1} \circ \Phi$ as the representation of Φ in $\mathbb{S}^3$ and $Z = \tilde{\pi}^{-1} \circ \Phi$ as the representation of Φ in $\mathbb{H}^3$ (whenever $\Phi(\Sigma) \subset B_1(0)$). We will often decompose $Z = (Z_h, Z_4)$ with $Z_h = (Z_1, Z_2, Z_3)$.

2.3 The Conformal Gauss map

The previous considerations on the representation of spheres in the de Sitter space can be applied to the study of the geometry of immersed surface through the conformal Gauss map. To lighten notations, we will denote

$$\begin{aligned} p(\Phi) &= \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} \text{ for } \Phi \in \mathbb{R}^3, \\ p(X) &= \begin{pmatrix} X \\ 1 \end{pmatrix} \text{ for } X \in \mathbb{S}^3, \\ p(Z) &= \begin{pmatrix} Z_h \\ -1 \\ Z_4 \end{pmatrix} \text{ for } Z = \begin{pmatrix} Z_h \\ Z_4 \end{pmatrix} \in \mathbb{H}^3. \end{aligned}$$

2.3.1 Enveloping spherical congruences

We first introduce the notion of enveloping spherical congruences.

Definition 2.3.1. Let Σ be a Riemann surface. A spherical congruence on Σ is a smooth application $Y : \Sigma \to \mathbb{S}^{4,1}$, that is, a family of oriented spheres parametrized on Σ. Given $\Phi : \Sigma \to \mathbb{R}^3$, or equivalently X its representation in $\mathbb{S}^3$, or $Z = (Z_h, Z_4)$ in $\mathbb{H}^3$, Y envelopes Φ, or equivalently X or Z, if and only if

$$\langle Y, p(\Phi)\rangle = 0 \tag{2.3.9}$$

and

$$\langle Y, \nabla p(\Phi)\rangle = 0, \tag{2.3.10}$$

or equivalently

$$\langle Y, p(X)\rangle = 0 \tag{2.3.11}$$

and

$$\langle Y, \nabla p(X)\rangle = 0, \tag{2.3.12}$$

or

$$\langle Y, p(Z)\rangle = 0 \tag{2.3.13}$$

and

$$\langle Y, \nabla p(Z)\rangle = 0. \tag{2.3.14}$$

Geometrically speaking Y envelopes Φ at the point $p \in \Sigma$ if the generalized sphere $Y(p)$ is tangent to $\Phi(\Sigma)$ at the point $\Phi(p)$.

Proof. We here show the equivalence of the three definitions. Since $p(\Phi)$, $p(X)$ and $p(Z)$ are pairwise colinear, one finds (2.3.9), (2.3.11) and (2.3.13) to be equivalent.

Moreover, assuming (2.3.9), (2.3.11), and (2.3.13), one deduces

$$\langle Y, \nabla p(\Phi)\rangle = \nabla\left(\langle Y, p(\Phi)\rangle\right) - \langle \nabla Y, p(\Phi)\rangle = -\langle \nabla Y, p(\Phi)\rangle,$$

$$\langle Y, \nabla p(X)\rangle = \nabla\left(\langle Y, p(X)\rangle\right) - \langle \nabla Y, p(X)\rangle = -\langle \nabla Y, p(X)\rangle$$

and

$$\langle Y, \nabla p(Z)\rangle = \nabla \left(\langle Y, p(Z)\rangle\right) - \langle \nabla Y, p(Z)\rangle = -\langle \nabla Y, p(Z)\rangle ,$$

which ensures that (2.3.10), (2.3.12) and (2.3.14) are equivalent. $\square$

Example 2.3.1. The conformal Gauss map :
Let Σ be a Riemann surface and $\Phi : \Sigma \to \mathbb{R}^3$. The conformal Gauss map Y, which to a point $z \in \Sigma$ associates the tangent sphere to the surface at $\Phi(z)$ of center $\Phi(z) + \frac{\vec{n}(z)}{|H(z)|}$ if $H(z) \neq 0$, and the tangent plane if $H(z) = 0$, is a spherical congruence enveloping Φ.

Y can be written as :

$$Y = H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi\rangle \\ \langle \vec{n}, \Phi\rangle \end{pmatrix}. \tag{2.3.15}$$

One can notice :

$$\begin{aligned} \langle p(\Phi), p(\Phi)\rangle &= 0 \\ \langle p(\Phi), \nabla p(\Phi)\rangle &= 0 \\ \left\langle p(\Phi), \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi\rangle \\ \langle \vec{n}, \Phi\rangle \end{pmatrix} \right\rangle &= 0 \end{aligned}$$

and in local coordinates

$$\begin{aligned} \nabla Y &= \nabla H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + H \nabla \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + \nabla \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi\rangle \\ \langle \vec{n}, \Phi\rangle \end{pmatrix} \\ &= \nabla H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + H \begin{pmatrix} \nabla\Phi \\ \langle \nabla\Phi, \Phi\rangle \\ \langle \nabla\Phi, \Phi\rangle \end{pmatrix} + \begin{pmatrix} \nabla\vec{n} \\ \langle \nabla\vec{n}, \Phi\rangle \\ \langle \nabla\vec{n}, \Phi\rangle \end{pmatrix} \\ &= \nabla H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} - \mathring{A} \begin{pmatrix} \nabla\Phi \\ \langle \nabla\Phi, \Phi\rangle \\ \langle \nabla\Phi, \Phi\rangle \end{pmatrix}. \end{aligned} \tag{2.3.16}$$

Hence :

$$\begin{aligned} \langle \partial_i Y, \partial_j Y\rangle &= \langle \mathring{A}_i^p \partial_p \Phi, \mathring{A}_j^q \partial_q \Phi\rangle \\ &= \mathring{A}_i^p \mathring{A}_{pj} = \mathring{A}_i^p \mathring{A}_p^l g_{jl} = \left(\mathring{A}^T \mathring{A} g\right)_{ij} \\ &= \frac{1}{2}|\mathring{A}|^2 g_{ij} \end{aligned} \tag{2.3.17}$$

since $\mathring{A}$ is symetric tracefree (see (1.2.28)). We then deduce that $Y : (\Sigma, g) \to \mathbb{S}^{4,1}$ is conformal. One may notice that the umbilic points of Φ are critical points of Y.

As an enveloping spherical congruence, the conformal Gauss map carries many informations on the geometry of the immersion. Its key role is further emphasized by the fact it is the only conformal enveloping spherical congruence, up to orientation.

Theorem 2.3.1. Let Σ be a Riemann surface and $\Phi : \Sigma \to \mathbb{R}^3$ an immersion. We denote g its first fundamental form and Y its conformal Gauss map. If the set of umbilic points of Φ is nowhere dense then Y and $-Y$ are the only smooth conformal $(\Sigma, g) \to \mathbb{R}^3$ spherical congruences enveloping Φ.

Proof. As stated when we introduced it, the conformal Gauss map is a spherical congruence enveloping Φ which happens to be conformal.

Conversely we consider a spherical congruence G enveloping Φ.
Let $E = \mathrm{Vect}\left(p(\Phi), \partial_x p(\Phi), \partial_y p(\Phi)\right)$. Equations (2.3.9) and (2.3.10) force G to lie in $(E)^\perp$. Since Φ is an immersion, E is of dimension 3, and its orthogonal is then of dimension 2. Y envelopes Φ and $p(\Phi)$is isotropic, hence $(Y, p(\Phi))$ is a basis of $(E)^\perp$. G can then be written as

$$G = \mu Y + \lambda p(\Phi)$$

with μ, $\lambda \in \mathbb{R}$.

Since $\langle G, G \rangle = \mu^2 = 1$ one finds $\mu = \pm 1$ and deduce $\nabla \mu = 0$. We then only need to compute the first fundamental form of G :

$$\begin{aligned}
\langle \partial_i G, \partial_j G \rangle &= \left\langle \mu \partial_i Y + \partial_i \lambda p(\Phi) + \lambda \begin{pmatrix} \partial_i \Phi \\ \langle \partial_i \Phi, \Phi \rangle \\ \langle \partial_i \Phi, \Phi \rangle \end{pmatrix}, \mu \partial_j Y + \partial_j \lambda p(\Phi) + \lambda \begin{pmatrix} \partial_j \Phi \\ \langle \partial_j \Phi, \Phi \rangle \\ \langle \partial_j \Phi, \Phi \rangle \end{pmatrix} \right\rangle \\
&= \left\langle \lambda \begin{pmatrix} \partial_i \Phi \\ \langle \partial_i \Phi, \Phi \rangle \\ \langle \partial_i \Phi, \Phi \rangle \end{pmatrix} - \mathring{A}_i^p \begin{pmatrix} \partial_p \Phi \\ \langle \partial_p \Phi, \Phi \rangle \\ \langle \partial_\Phi, \Phi \rangle \end{pmatrix}, \lambda \begin{pmatrix} \partial_j \Phi \\ \langle \partial_j \Phi, \Phi \rangle \\ \langle \partial_j \Phi, \Phi \rangle \end{pmatrix} - \mathring{A}_j^p \begin{pmatrix} \partial_p \Phi \\ \langle \partial_p \Phi, \Phi \rangle \\ \langle \partial_\Phi, \Phi \rangle \end{pmatrix} \right\rangle
\end{aligned}$$

using expression (2.3.16) of ∇Y and the fact that $p(\Phi) \in E^\perp$. Then

$$\begin{aligned}
\langle \partial_i G, \partial_j G \rangle &= \langle \lambda \partial_i \Phi - \mathring{A}_i^p \partial_p \Phi, \lambda \partial_j \Phi - \mathring{A}_j^q \partial_q \Phi \rangle \\
&= \lambda^2 g_{ij} + \mathring{A}_i^p \mathring{A}_{pj} - 2\lambda \mathring{A}_{ij} \\
&= \left(\lambda^2 + \frac{|\mathring{A}|^2}{2} \right) g_{ij} - 2\lambda \mathring{A}_{ij}
\end{aligned}$$

where we have used (1.2.28). By hypothesis the set of umbilic points is nowhere dense, G is then conformal if and only if $\lambda = 0$. We then have $G = \pm Y$ which concludes the proof. $\square$

Taking $-Y$ instead of Y is tantamount to changing the orientation of the surface (taking $-\vec{n}$ instead of $\vec{n}$ as a Gauss map).

Geometrically speaking Y can be seen as the 2-dimensional generalization of the osculating circles for curves in euclidian spaces, and it will be of major importance in the study of Willmore surfaces, playing much of the same role as the Gauss map in the case of constant mean curvature surfaces.

Since Y conserves the conformal structure on Σ, it is convenient, and will not induce any loss of generality, to work in complex coordinates in local conformal charts (see subsection A.2.2 in the appendix for more details). In the following we will then consider $\Phi : \mathbb{D} \to \mathbb{R}^3$ a smooth conformal immersion, that is satisfying $\langle \Phi_z, \Phi_z \rangle = 0$. Let $\vec{n} = \frac{\Phi_z \times \Phi_{\bar{z}}}{i |\Phi_z|^2}$ denote its Gauss map with $\times$ the classic vectorial product in $\mathbb{R}^3$, $\lambda = \frac{1}{2} \log\left(2 |\Phi_z|^2\right)$ its conformal factor and $H = \left\langle \frac{\Phi_{z\bar{z}}}{|\Phi_z|^2}, \vec{n} \right\rangle$ its mean curvature. Its tracefree curvature is defined as follows

$$\Omega := 2 \langle \Phi_{zz}, \vec{n} \rangle .$$

Its representation in $\mathbb{S}^3$, $X = \pi^{-1} \circ \Phi = \frac{1}{1+|\Phi|^2} \begin{pmatrix} 2\Phi \\ |\Phi|^2 - 1 \end{pmatrix}$ is conformal. Let $\Lambda := \frac{1}{2} \log\left(2 |X_z|^2\right)$ be its conformal factor, $\vec{N}$ such that $\left(X, e^{-\Lambda} X_x, e^{-\Lambda} X_y, \vec{N}\right)$ is a direct orthonormal basis of $\mathbb{R}^4$ its Gauss map , $h = \left\langle \frac{X_{z\bar{z}}}{|X_z|^2}, \vec{N} \right\rangle$ its mean curvature and $\omega :=$ $2\left\langle X_{zz}, \vec{N} \right\rangle$ its tracefree curvature. Similarly its representation in $\mathbb{H}^3$, $Z = \tilde{\pi}^{-1} \circ \Phi$ is conformal. Let $\lambda^Z := \frac{1}{2}\log\left(2\langle Z_z, Z_{\bar{z}}\rangle\right)$ be its conformal factor,$\vec{n}^Z$ such that $(Z, e^{-\lambda^Z} Z_x, e^{-\lambda^Z} Z_y, \vec{n}^Z)$ is a direct orthonormal basis of $\mathbb{R}^{3,1}$ its Gauss map , $H^Z = \left\langle \frac{Z_{z\bar{z}}}{|Z_z|^2}, \vec{n}^Z \right\rangle$ its mean curvature and $\Omega^Z := 2\left\langle Z_{zz}, \vec{n}^Z \right\rangle$ its tracefree curvature. One can then express Y as the conformal Gauss map of an immersion in $\mathbb{S}^3$ or in $\mathbb{H}^3$.

Proposition 2.3.2. Let Φ be a smooth conformal immersion on $\mathbb{D}$, and X (respectively Z) its representation in $\mathbb{S}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). Let Y be its conformal Gauss map. Then

$$\begin{aligned} Y &= h\begin{pmatrix} X \\ 1 \end{pmatrix} + \begin{pmatrix} \vec{N} \\ 0 \end{pmatrix} \\ &= H^Z \begin{pmatrix} Z_h \\ -1 \\ Z_4 \end{pmatrix} + \begin{pmatrix} \vec{n}^Z_h \\ 0 \\ \vec{n}^Z_4 \end{pmatrix} \end{aligned}$$

where $Z = \begin{pmatrix} Z_h \\ Z_4 \end{pmatrix}$ and $\vec{n}^Z = \begin{pmatrix} \vec{n}^Z_h \\ \vec{n}^Z_4 \end{pmatrix}$, while h and H^Z are the respective mean curvatures.

Proof. The computations are done in the appendix, respectively in subsections A.2.1, A.2.3 and A.2.5. □

It is interesting to study how Y changes under the action of conformal diffeomorphisms.

Proposition 2.3.3. Let $\varphi \in \mathrm{Conf}(\mathbb{S}^3)$ corresponding to $M \in SO(4,1)$. Let $X : \Sigma \to \mathbb{S}^3$ be a smooth conformal immersion of conformal Gauss map Y. We assume the set of umbilic points of X to be nowhere dense. Let Y_φ be the conformal Gauss map of $\varphi \circ X$. Then

$$Y_\varphi = MY.$$

Proof. We work in a conformal chart on a disk. Thanks to theorem 2.3.1 one just needs to prove that MY is conformal, envelopes $\varphi \circ X$ and has the same orientation as Y_φ.

We first show that MY is conformal. Since $(MY)_z = MY_z$ and $M \in SO(4,1)$,

$$\langle (MY)_z, (MY)_z \rangle = \langle Y_z, Y_z \rangle.$$

Given that Y is conformal, one finds $\langle (MY)_z, (MY)_z \rangle = 0$, that is MY is conformal. We then justify that MY envelopes $\varphi \circ X$. To that aim, let $V = M\begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} V_\circ \\ V_5 \end{pmatrix}$. In accordance with corollary 2.2.2, $\varphi(X) = \frac{V_\circ}{V_5}$, which translates to

$$p(\varphi(X)) = \frac{1}{V_5} M p(X). \tag{2.3.18}$$

Then

$$\begin{aligned} \langle MY, p(\varphi(X)) \rangle &= \frac{1}{V_5} \langle MY, Mp(X) \rangle \\ &= \frac{1}{V_5} \langle Y, p(X) \rangle \\ &= 0, \end{aligned} \tag{2.3.19}$$

which proves (2.3.11), and

$$\begin{aligned}\langle MY, \nabla p(\varphi(X))\rangle &= \nabla\left(\langle MY, p(\varphi(X))\rangle\right) - \langle M\nabla Y, p(\varphi(X))\rangle \\ &= -\frac{1}{V_5}\langle M\nabla Y, Mp(X)\rangle \\ &= -\frac{1}{V_5}\langle \nabla Y, p(X)\rangle \\ &= 0,\end{aligned}$$

which shows (2.3.12) and that MY envelopes $\varphi(X)$.

Finally one need only adress the orientation of $\varphi(X)$ to conclude. Let N_φ be the Gauss map of $\varphi \circ X$ induced by the Gauss map N of X, namely $N_\varphi = \frac{d\varphi(N)}{|d\varphi(N)|}$. Given the expression (A.2.44) of the conformal Gauss map, $MY = Y_\varphi$ if and only if $\left\langle MY, \begin{pmatrix} N_\varphi \\ 0 \end{pmatrix}\right\rangle = 1$, $MY = -Y_\varphi$ otherwise. Let $W = M\begin{pmatrix} N \\ 0\end{pmatrix} = \begin{pmatrix} W_\circ \\ W_5 \end{pmatrix}$. With a straightforward computation one finds

$$d\varphi(N) = \frac{W_\circ}{V_5} - \frac{W_5}{V_5^2}V_\circ,$$

which yields

$$N_\varphi = W_\circ - \frac{W_5}{V_5}V_\circ.$$

Then

$$\begin{aligned}\left\langle MY, \begin{pmatrix} N_\varphi \\ 0\end{pmatrix}\right\rangle &= \left\langle MY, \begin{pmatrix} W_\circ - \frac{W_5}{V_5}V_\circ \\ 0 \end{pmatrix}\right\rangle \\ &= \left\langle MY, \begin{pmatrix} W_\circ \\ W_5 \end{pmatrix} - \begin{pmatrix} \frac{W_5}{V_5}V_\circ \\ W_5\end{pmatrix}\right\rangle \\ &= \left\langle MY, M\begin{pmatrix} N \\ 0\end{pmatrix} - \frac{W_5}{V_5}\begin{pmatrix} V_\circ \\ V_5 \end{pmatrix}\right\rangle\end{aligned}$$

thanks to the definition of W. Then since $\left\langle MY, M\begin{pmatrix} N \\ 0\end{pmatrix}\right\rangle = \left\langle Y, \begin{pmatrix} N \\ 0\end{pmatrix}\right\rangle = 1$, one finds

$$\begin{aligned}\left\langle MY, \begin{pmatrix} N_\varphi \\ 0\end{pmatrix}\right\rangle &= 1 - \left\langle MY, \frac{W_5}{V_5}\begin{pmatrix} V_\circ \\ V_5\end{pmatrix}\right\rangle \\ &= 1 - \frac{W_5}{V_5}\langle MY, p(X)\rangle,\end{aligned}$$

by definition of V. The equality (2.3.19) gives the expected result.

Then $MY = Y_\varphi$ which is the desired result.

□

One has similar results in the $\mathbb{R}^3$ and $\mathbb{H}^3$ settings.

Proposition 2.3.4. Let $\varphi \in \mathrm{Conf}(\mathbb{R}^3)$ corresponding to $M \in SO(4,1)$. Let $\Phi \in C^\infty(\Sigma, \mathbb{R}^3)$ be a smooth immersion and Y its conformal Gauss map. We assume the set of umbilic points of Φ to be nowhere dense. Let Y_φ be the conformal Gauss map of $\varphi \circ \Phi$. Then

$$Y_\varphi = MY.$$

Proposition 2.3.5. Let $\varphi \in \mathrm{Conf}(\mathbb{H}^3)$ corresponding to $M \in SO(4,1)$. Let $Z \in C^\infty(\Sigma, \mathbb{H}^3)$ be a smooth conformal immersion and Y its conformal Gauss map. We assume the set of umbilic points of Z to be nowhere dense. Let Y_φ be the conformal Gauss map of $\varphi \circ Z$. Then

$$Y_\varphi = MY.$$

2.3.2 Geometry of Conformal Gauss maps

Enveloping conditions (2.3.9) and (2.3.10) (or equivalently (2.3.11) and (2.3.12) or (2.3.13) and (2.3.14)) ensure that $p(\Phi)$ (or equivalently $p(X)$ or $p(Z)$) is an isotropic vector field normal to Y in $\mathbb{R}^{4,1}$.

We wish to complete $(Y, Y_z, Y_{\bar{z}}, p(\Phi))$ into a moving frame of $\mathbb{R}^{4,1}$ compatible with the decomposition $\mathbb{R}^{4,1} = TY \bigoplus NY$, in order to introduce the mean and tracefree curvatures of Y as an immersion in $\mathbb{R}^{4,1}$. As we pointed out prior, finding another immersion enveloped by Y is enough to complete the moving frame. We will use the notations introduced in subsection A.2.7 in the appendix.

Theorem 2.3.6. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a smooth conformal immersion with no umbilic points. Then there exists

$$\Phi^* = \Phi - \frac{4H_z\overline{\Omega}e^{-2\lambda}}{T(\Phi)}\Phi_z - \frac{4H_{\bar{z}}\Omega e^{-2\lambda}}{T(\Phi)}\Phi_{\bar{z}} + \frac{2H\,|\Omega|^2\,e^{-2\lambda}}{T(\Phi)}\vec{n}$$

where $T(\Phi) = |\nabla H|^2 + H^2\,|\Omega|^2\,e^{-2\lambda}$, such that

$$\langle Y, p(\Phi^*)\rangle = 0 \tag{2.3.20}$$

and

$$\langle \nabla Y, p(\Phi^*)\rangle = 0. \tag{2.3.21}$$

Proof. We search for Φ^* under the form

$$\Phi^* = \Phi + u\Phi_z + \overline{u}\Phi_{\bar{z}} + v\vec{n}.$$

Applying first (2.3.20) then (2.3.21) yields

$$\begin{aligned} v &= \frac{|u|^2e^{2\lambda} + v^2}{2}H \\ \Omega u &= -H_z\left(|u|^2e^{2\lambda} + v^2\right). \end{aligned}$$

Solving the resulting system gives us the desired values for u and v. □

One can work similarly with immersions in $\mathbb{S}^3$.

Theorem 2.3.7. Let $X : \mathbb{D} \to \mathbb{S}^3$ be a smooth conformal immersion with no umbilic points. Then there exists

$$X^* = \frac{h^2\,|\omega|^2 + 4\,|h_z|^2\,e^{2\Lambda} - |\omega|^2}{T(X)}X - \frac{4h_z\overline{\omega}}{T(X)}X_z - \frac{4h_{\bar{z}}\omega}{T(X)}X_{\bar{z}} + \frac{2\,|\omega|^2\,h}{T(X)}N$$

where $T(X) = |\omega|^2\left(1 + h^2\right) + 4\,|h_z|^2\,e^{2\Lambda}$, such that

$$\langle Y, p(X^*)\rangle = 0 \tag{2.3.22}$$

and

$$\langle \nabla Y, p(X^*)\rangle = 0. \tag{2.3.23}$$

Proof. We search for X^* under the form

$$X^* = \alpha X + \beta X_z + \beta X_{\bar{z}} + \gamma N.$$

Applying first (2.3.22), then (2.3.23) yields

$$\begin{aligned}\gamma &= (1-\alpha)h,\\ 2h_z(\alpha-1) &= \omega\beta.\end{aligned}$$

Further $\langle X^*, X^*\rangle = 1$ ensures

$$\alpha^2 + |\beta|^2 e^{2\Lambda} + \gamma^2 = 1.$$

Solving the resulting system gives the desired result. $\square$

Let $e_\Phi := (Y, Y_z, Y_{\bar{z}}, p(\Phi), p(\Phi^*))$ and $e_X := (Y, Y_z, Y_{\bar{z}}, p(X), p(X^*))$ denote our two frames. Since $p(\Phi)$ and $p(X)$ are colinear, necessarily $p(\Phi^*)$ and $p(X^*)$ are too, meaning $X^* = \pi^{-1} \circ \Phi^*$, that is X^* is the representation of Φ^* in $\mathbb{S}^3$.

Since Y is conformal, (2.3.20) and (2.3.21) (respectively (2.3.22) and (2.3.23)), (2.3.9) and (2.3.10) (respectively (2.3.11) and (2.3.12)) ensure e_Φ (respectively e_X) is orthogonal. For convenience's sake, we will mainly work with e_X. Indeed while Φ is not necessarily contained in a compact, and thus neither is $p(\Phi)$, $X \in \mathbb{S}^3$ makes for easier computations. Each result has its counterpart in $\mathbb{R}^3$.

Let

$$\nu = p(X) = \begin{pmatrix} X \\ 1 \end{pmatrix}, \tag{2.3.24}$$

$$l = \langle p(X), p(X^*)\rangle = \frac{-2|\omega|^2}{|\omega|^2(h^2+1) + |\nabla h|^2 e^{2\Lambda}},$$

and

$$\begin{aligned}\nu^* &= -\frac{1}{l} p(X^*)\\ &= \frac{|\omega|^2\left(h^2+1\right) + |\nabla h|^2 e^{2\Lambda}}{2|\omega|^2} p(X^*)\\ &= \begin{pmatrix} \left(\frac{h^2-1}{2} + \frac{|\nabla h|^2 e^{2\Lambda}}{2|\omega|^2}\right) X - \frac{2h_z}{\omega} X_z - \frac{2h_{\bar{z}}}{\overline{\omega}} X_{\bar{z}} + hN \\ \frac{h^2+1}{2} + \frac{|\nabla h|^2 e^{2\Lambda}}{2|\omega|^2}\end{pmatrix}.\end{aligned} \tag{2.3.25}$$

By design, we have $\langle \nu, \nu^*\rangle = -1$. Thus defined $|\nu^*| < \infty$ away from umbilic points.

One computes easily, with Gauss-Codazzi (see (A.2.35) in appendix) to obtain the second equality,

$$\langle \nu_z, \nu^*\rangle = -\frac{h_{\bar{z}} e^{2\Lambda}}{\overline{\omega}} = -\frac{\overline{\omega}_z}{\overline{\omega}}. \tag{2.3.26}$$

Using computations done for conformal immersions in $\mathbb{R}^{4,1}$ in a nice frame, (see (A.2.51) in the appendix), one finds

$$Y_{z\bar{z}} = \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\begin{pmatrix} X \\ 1\end{pmatrix} - \frac{|\omega|^2 e^{-2\Lambda}}{2} Y = \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\nu - \frac{|\omega|^2 e^{-2\Lambda}}{2} Y \tag{2.3.27}$$

where

$$\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4} = h_{z\bar{z}} + \frac{|\omega|^2 e^{-2\Lambda}}{2} h \in \mathbb{R}$$

as defined in (A.2.52). With the notations of section A.2.7, see (A.2.61), this yields

$$H_\nu = 0, \tag{2.3.28}$$

$$e^{2\mathcal{L}} = |\omega|^2 e^{-2\Lambda}, \tag{2.3.29}$$

and

$$H_{\nu^*} = \frac{-\mathcal{W}_{\mathbb{S}^3}(X)}{2|\omega|^2 e^{-2\Lambda}}. \tag{2.3.30}$$

Similarly, applying (A.2.56) to (A.2.48) we find

$$\Omega_\nu = 2\langle Y_{zz}, \nu\rangle = \omega, \tag{2.3.31}$$

and

$$\begin{aligned}
\Omega_{\nu^*} &= 2\langle Y_{zz}, \nu^*\rangle \\
&= 2\left\langle h_{zz}\begin{pmatrix} X \\ 1\end{pmatrix} + h_z \begin{pmatrix} X_z \\ 0\end{pmatrix} - \left(\omega e^{-2\Lambda}\right)_z \begin{pmatrix} X_{\bar{z}} \\ 0\end{pmatrix} - \omega\left(\frac{h}{2}\begin{pmatrix} \vec{N} \\ 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix} X \\ 0\end{pmatrix}\right), \nu^*\right\rangle \\
&= 2\left(h_{zz} + \frac{\omega}{2}\right)\left(\frac{h^2-1}{2} + \frac{|\nabla h|^2 e^{2\Lambda}}{2|\omega|^2}\right) + \frac{h_z}{\omega}\left(\omega e^{-2\Lambda}\right)_z e^{2\Lambda} - \frac{h_z h_{\bar{z}}}{\overline{\omega}} e^{2\Lambda} - \frac{\omega h^2}{2} \\
&\quad - h_{zz}\left(\frac{h^2+1}{2} + \frac{|\nabla h|^2 e^{2\Lambda}}{2|\omega|^2}\right) \\
&= -\omega\frac{h^2+1}{2} + 2\frac{|\omega_{\bar{z}}|^2 e^{-2\Lambda}}{\overline{\omega}} + 2\frac{\omega_{\bar{z}}\left(\omega e^{-2\Lambda}\right)_z}{\omega} - 2\frac{\omega_{\bar{z}}\overline{\omega}_z e^{-2\Lambda}}{\overline{\omega}} - 2\left(\omega_{\bar{z}} e^{-2\Lambda}\right)_z \\
&= -\omega\frac{h^2+1}{2} + 2\frac{\omega_{\bar{z}}\left(\omega e^{-2\Lambda}\right)_z}{\omega} - 2\left(\omega_{\bar{z}} e^{-2\Lambda}\right)_z \\
&= 2\left(\frac{\omega_{\bar{z}}\omega_z}{\omega} - \omega_{z\bar{z}}\right)e^{-2\Lambda} - \omega\frac{h^2+1}{2},
\end{aligned}$$

where we have used (A.2.35) for the fourth equality. This yields

$$\begin{aligned}
\Omega_{\nu^*} &= -2\omega e^{-2\Lambda}\left(\left(\frac{\omega_{\bar{z}}}{\omega}\right)_z + \frac{h^2+1}{4}e^{2\Lambda}\right) \\
&= -2\omega e^{-2\Lambda}\left(\left(\frac{\omega_z}{\omega}\right)_{\bar{z}} + \frac{h^2+1}{4}e^{2\Lambda}\right).
\end{aligned} \tag{2.3.32}$$

A consequence of these computations is that the conformal Gauss map of an immersion X is necessarily of vanishing mean curvature in the direction $p(X)$. This is in fact an equivalence.

Theorem 2.3.8. Let $Y : \mathbb{D} \to \mathbb{S}^{4,1}$ be a spacelike (that is $\langle Y_z, Y_{\bar{z}}\rangle > 0$) conformal immersion. Then Y is the conformal Gauss map of $X : \mathbb{D} \to \mathbb{S}^3$ if and only if there exists an isotropic normal direction ν such that $H_\nu = 0$, where H_ν is the mean curvature in the ν direction defined in (A.2.56). Moreover, ν is parallel to $p(X)$.

Proof. We have shown in (2.3.28) that if Y is the conformal Gauss map of X then Y is of null mean curvature in the isotropic $p(X)$ direction.

Reciprocally consider Y of null mean curvature in the isotropic direction ν Let us build $X : \mathbb{D} \to \mathbb{S}^3$ such that Y is the conformal Gauss map of X. Since $\langle \nu, \nu\rangle = 0$ and $\nu \neq 0$, the last coordinate ν_5 of ν is necessarily non null. One can then renormalize ν to $\frac{\nu}{\nu_5} = p(X)$. There then exists $X : \mathbb{D} \to \mathbb{S}^3$ such that

$$\begin{aligned}
\langle Y, p(X) \rangle &= 0, \\
\langle Y_z, p(X) \rangle &= 0, \\
\langle Y_{z\bar{z}}, p(X) \rangle &= 0.
\end{aligned}$$

One checks that hypotheses (2.3.11) and (2.3.12) are satisfied and that Y envelopes X. We now just have to prove that X is conformal, and apply 2.3.1 to conclude.

Since $\langle X_z, X_z \rangle = \langle p(X)_z, p(X)_z \rangle$ and according to (A.2.72)

$$\langle p(X)_z, p(X)_z \rangle = H_{p(X)} \Omega_{p(X)} = 0,$$

X is shown to be conformal, which concludes the proof. □

We must draw the reader's attention to the fact that Y is *not* a priori the conformal Gauss map of X^*. Indeed, while Y envelopes X^*, X^* is not necessarily conformal :

$$\begin{aligned}
\langle X^*_z, X^*_z \rangle &= \langle p(X^*)_z, p(X^*)_z \rangle \\
&= \langle (l\nu^*)_z, (l\nu^*)_z \rangle \\
&= l^2 \langle \nu^*_z, \nu^*_z \rangle
\end{aligned}$$

since (A.2.70) stands and ν^* is isotropic. Then using (A.2.72)

$$\begin{aligned}
\langle X^*_z, X^*_z \rangle &= l^2 H_{\nu^*} \Omega_{\nu^*} \\
&= l^2 \omega e^{-2\Lambda} \left(\left(\frac{\omega_z}{\omega} \right)_{\bar{z}} + \frac{h^2+1}{4} e^{2\Lambda} \right) \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2 |\omega|^2 e^{-2\Lambda}},
\end{aligned}$$

with (2.3.32) and (2.3.30).

Then

$$\langle X^*_z, X^*_z \rangle = \frac{\omega |\omega|^2}{\left(|\omega|^2 (h^2+1) + |\nabla h|^2 e^{2\Lambda} \right)^2} \mathcal{W}_{\mathbb{S}^3}(X) \left(\left(\frac{\omega_z}{\omega} \right)_{\bar{z}} + \frac{h^2+1}{4} e^{2\Lambda} \right). \tag{2.3.33}$$

One can notice that a simple condition to ensure that X^* is conformal is $\mathcal{W}_{\mathbb{S}^3}(X) = 0$, that is X is a Willmore immersion. The computations for an immersion Φ in $\mathbb{R}^3$ (see (A.2.45)-(A.2.49)) bring to the forefront the quantity

$$\mathcal{W}(\Phi) = 4H_{z\bar{z}} + 2 |\Omega|^2 e^{-2\lambda} H \in \mathbb{R}.$$

We refer the reader to (A.2.53) for the proof that

$$\mathcal{W}_{\mathbb{S}^3}(X) = \frac{|\Phi|^2+1}{2} \mathcal{W}(\Phi).$$

Given how the left-hand term of the Willmore equation appears organically as a geometric term for the conformal Gauss map it becomes natural and interesting to consider the conformal Gauss map of Willmore immersions.

2.4 Conformal Gauss map of Willmore immersions

2.4.1 Another look at the conservation laws

Equality (A.2.46) (or equivalently (A.2.51)) yields the following well known theorem (found in [BR14] for instance).

Theorem 2.4.1. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal immersion of representation X in $\mathbb{S}^3$ and Z in $\mathbb{H}^3$. Then Φ is Willmore if and only if its conformal Gauss map Y is minimal, that is if it is conformal and satisfies

$$Y_{z\bar{z}} + \langle Y_z, Y_{\bar{z}} \rangle Y = 0$$

which in real notations is tantamount to

$$\Delta Y + \langle \nabla Y . \nabla Y \rangle Y = 0. \tag{2.4.34}$$

Remark 2.4.1. One could then define a notion of Willmore immersion in $\mathbb{S}^3$ of $\mathbb{H}^3$ by using their representation in $\mathbb{R}^3$. Actually, the whole process of defining Willmore immersions can be followed through in a general Riemannian setting. Any smooth immersion of a surface into a Riemannian manifold defines a mean curvature, which allows us to introduce a Willmore energy, whose critical points are Willmore immersions. This notion is invariant under conformal diffeomorphisms from one Riemannian setting into the other. The two ways to define Willmore immersions on $\mathbb{S}^3$ or $\mathbb{H}^3$ naturally coincide.

Assuming (2.4.34), for all $i, j \in \{1 \dots 5\}$ one has

$$\operatorname{div}\left(\nabla Y_i Y_j - Y_i \nabla Y_j\right) = \Delta Y_i Y_j - \Delta Y_j Y_i = 0.$$

Y then satisfies the following conservation laws (that can actually be thought to follow from the invariance group $SO(4,1)$ of the energy $E(Y) = \int_{\mathbb{D}} \langle \nabla Y . \nabla Y \rangle \, dz$) :

$$\operatorname{div}\left(\nabla Y Y^T - Y \nabla Y^T\right) = 0. \tag{2.4.35}$$

These conservation laws stem from the seminal works of F. Hélein on harmonic maps in the euclidean spheres (see [Hél02] for an extensive study) and the generalization of M. Zhu to harmonic maps in de Sitter spaces in [Zhu13].

Theorem 2.4.2. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Let

$$\mu = \left(\nabla Y_i Y_j - Y_i \nabla Y_j\right) = \nabla Y Y^T - Y \nabla Y^T.$$

Then $\operatorname{div}(\mu) = 0$ and

$$2\mu = \begin{pmatrix} U & -\frac{V_{\mathrm{tra}} - V_{\mathrm{inv}}}{2} & \frac{V_{\mathrm{tra}} + V_{\mathrm{inv}}}{2} \\ \left(\frac{V_{\mathrm{tra}} - V_{\mathrm{inv}}}{2}\right)^T & 0 & V_{\mathrm{dil}} \\ -\left(\frac{V_{\mathrm{inv}} + V_{\mathrm{tra}}}{2}\right)^T & -V_{\mathrm{dil}} & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 0 & -\tilde{V}_{\mathrm{rot}\,3} & \tilde{V}_{\mathrm{rot}\,2} \\ \tilde{V}_{\mathrm{rot}\,3} & 0 & -\tilde{V}_{\mathrm{rot}\,1} \\ -\tilde{V}_{\mathrm{rot}\,2} & \tilde{V}_{\mathrm{rot}\,1} & 0 \end{pmatrix}$$

with $V_{\mathrm{tra}}, V_{\mathrm{dil}}, V_{\mathrm{rot}}, \tilde{V}_{\mathrm{rot}}$ and V_{inv} defined in theorem 1.2.14.

Proof. We decompose μ in blocks :

$$\mu = \begin{pmatrix} P & a & b \\ -a^T & 0 & \omega \\ -b^T & -\omega & 0 \end{pmatrix},$$

with $P \in M_3(\mathbb{R})$ antisymetric, $a, b \in \mathbb{R}^3$ and $\omega \in \mathbb{R}$. Let $\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. Then given any $a, b \in \mathbb{R}^5$,

$$a^T \epsilon b = \langle a, b \rangle,$$

where $\langle .,. \rangle$ is the Lorentzian product in $\mathbb{R}^{4,1}$.

For any $w \in \mathbb{R}^3$,

$$\begin{aligned} \mu\epsilon \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} &= \left\langle Y, \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} \right\rangle \nabla Y - \left\langle \nabla Y, \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} \right\rangle Y \\ &= \langle H\Phi + \vec{n}, w \rangle \left[\nabla H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2 - 1}{2} \\ \frac{|\Phi|^2 + 1}{2} \end{pmatrix} - \mathring{A} \begin{pmatrix} \nabla\Phi \\ \langle \nabla\Phi, \Phi \rangle \\ \langle \nabla\Phi, \Phi \rangle \end{pmatrix} \right] \\ &\quad - \left\langle \nabla H \Phi - \mathring{A}\nabla\Phi, w \right\rangle \left[H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2 - 1}{2} \\ \frac{|\Phi|^2 + 1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi \rangle \\ \langle \vec{n}, \Phi \rangle \end{pmatrix} \right], \end{aligned}$$

while

$$\mu\epsilon \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Pw \\ -\langle a, w \rangle \\ -\langle b, w \rangle \end{pmatrix}.$$

Focusing on the first three coordinates yields

$$\begin{aligned} Pw &= \langle H\Phi + \vec{n}, w \rangle \left[\nabla H \Phi - \mathring{A}\nabla\Phi \right] - \left\langle \nabla H\Phi - \mathring{A}\nabla\Phi, w \right\rangle [H\Phi + \vec{n}] \\ &= w \times \left[\Phi \times \left(\nabla H \vec{n} + H\mathring{A}\nabla\Phi \right) + \vec{n} \times \mathring{A}\nabla\Phi \right] \\ &= -\frac{1}{2} w \times \left[\Phi \times V_{\text{tra}} + 2\mathring{A}\nabla\Phi \times \vec{n} \right] = -\frac{1}{2} w \times \tilde{V}_{\text{rot}}. \end{aligned}$$

With this valid for all $w \in \mathbb{R}^3$, we deduce

$$P = \frac{1}{2} \begin{pmatrix} 0 & -\tilde{V}_{\text{rot}\,3} & \tilde{V}_{\text{rot}\,2} \\ \tilde{V}_{\text{rot}\,3} & 0 & -\tilde{V}_{\text{rot}\,1} \\ -\tilde{V}_{\text{rot}\,2} & \tilde{V}_{\text{rot}\,1} & 0 \end{pmatrix}.$$

Similarly :

$$
\begin{aligned}
\mu\epsilon\begin{pmatrix}0\\1\\1\end{pmatrix} &= \left\langle Y, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\rangle \nabla Y - \left\langle \nabla Y, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\rangle Y \\
&= -H\left[\nabla H\begin{pmatrix}\Phi\\ \frac{|\Phi|^2-1}{2}\\ \frac{|\Phi|^2+1}{2}\end{pmatrix} - \mathring{A}\begin{pmatrix}\nabla\Phi\\ \langle\nabla\Phi,\Phi\rangle\\ \langle\nabla\Phi,\Phi\rangle\end{pmatrix}\right] + \nabla H\left[H\begin{pmatrix}\Phi\\ \frac{|\Phi|^2-1}{2}\\ \frac{|\Phi|^2+1}{2}\end{pmatrix} + \begin{pmatrix}\vec{n}\\ \langle\vec{n},\Phi\rangle\\ \langle\vec{n},\Phi\rangle\end{pmatrix}\right] \\
&= \nabla H\begin{pmatrix}\vec{n}\\ \langle\vec{n},\Phi\rangle\\ \langle\vec{n},\Phi\rangle\end{pmatrix} + H\mathring{A}\begin{pmatrix}\nabla\Phi\\ \langle\nabla\Phi,\Phi\rangle\\ \langle\nabla\Phi,\Phi\rangle\end{pmatrix},
\end{aligned}
$$

while

$$
\mu\epsilon\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}a-b\\-\omega\\-\omega\end{pmatrix}.
$$

Hence

$$
\begin{aligned}
a-b &= -\frac{V_{\text{tra}}}{2},\\
\omega &= \frac{V_{\text{dil}}}{2}.
\end{aligned}
$$

In a similar fashion, computing in two ways $\mu\epsilon\begin{pmatrix}0\\1\\-1\end{pmatrix}$ yields

$$
a+b = \frac{V_{\text{inv}}}{2}.
$$

Hence

$$
\begin{aligned}
a &= -\frac{V_{\text{tra}}-V_{\text{inv}}}{4}\\
b &= \frac{V_{\text{inv}}+V_{\text{tra}}}{4}.
\end{aligned}
$$

To conclude, we assemble all the previous results and reach

$$
2\mu = \begin{pmatrix} U & -\frac{V_{\text{tra}}-V_{\text{inv}}}{2} & \frac{V_{\text{tra}}+V_{\text{inv}}}{2}\\ \left(\frac{V_{\text{tra}}-V_{\text{inv}}}{2}\right)^T & 0 & V_{\text{dil}}\\ -\left(\frac{V_{\text{inv}}+V_{\text{tra}}}{2}\right)^T & -V_{\text{dil}} & 0\end{pmatrix}
$$

which is the desired result. $\square$

One of the advantages of this formulation is that it describes conveniently how these conserved quantities change under the action of diffeomorphisms.

Theorem 2.4.3. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Let μ be as in theorem 2.4.2. Let $\varphi \in \text{Conf}\left(\mathbb{R}^3\right)$ and $M \in SO(4,1)$ associated. Let Y_φ be its conformal Gauss map and μ_φ be as in theorem 2.4.2. Then

$$
\mu_\varphi = M\mu M^T.
$$

Proof. Using proposition 2.3.3 one has $Y_\varphi = MY$ and thus

$$\mu_\varphi = Y_\varphi \left(\nabla Y_\varphi\right)^T - \nabla Y_\varphi \left(Y_\varphi\right)^T = M\left(Y\nabla Y^T - \nabla Y Y^T\right)M^T = M\mu M^T.$$

□

As an example theorem 2.4.3 yields an alternative proof of a result by A. Michelat and T. Rivière (theorem 3.9 in [MR17]) that describes the exchange laws of conserved quantities under the action of the inversions.

Corollary 2.4.1. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Let $\iota : x \mapsto \frac{x}{|x|^2}$ be the inversion at the origin. Let $V_{*,\iota}$ be the conserved quantity corresponding to the transformation $*$ for $\iota \circ \Phi$. Then

$$\begin{aligned} V_{\mathrm{tra},\,\iota} &= V_{\mathrm{inv}} \\ V_{\mathrm{inv},\,\iota} &= V_{\mathrm{tra}} \\ V_{\mathrm{dil},\,\iota} &= -V_{\mathrm{dil}} \\ \tilde{V}_{\mathrm{rot},\,\iota} &= \tilde{V}_{\mathrm{rot}}. \end{aligned}$$

Proof. One need only apply theorem 2.4.3 with $\varphi = \iota$ and $M = M_\iota = \begin{pmatrix} -Id & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (see (2.2.7)), and interpret the result with theorem 2.4.2.

□

On non simply-connected domains, each conserved quantity yields a corresponding residue (as in (1.5.99)) which follows the exchange law presented in corollary 2.4.1. The exchange law of residues was in fact a result obtained by A. Michelat and T. Rivière in [MR17] through computations (theorem 3.9).

As was pointed out in conclusion of subsection 2.3.2, a sufficient condition for X^* to be conformal is X Willmore. In that case Y is the conformal Gauss map of X^*.

Theorem 2.4.4. Let $X : \mathbb{D} \to \mathbb{S}^3$ be a Willmore immersion, conformal, of conformal Gauss map Y. Then there exists a branched conformal Willmore immersion $X^* : \mathbb{D} \to \mathbb{S}^3$ such that Y is the conformal Gauss map of X^*. Then X^* is called the conformal dual immersion of X.

Proof. Taking X^* as in theorem 2.3.7, and recalling (2.3.33) with X Willmore, one finds X^* conformal and enveloped by Y. Theorem 2.3.1 concludes.

□

Another way to see this is to understand that Y minimal means there are two isotropic directions in which Y has zero mean curvature, meaning Y is the conformal Gauss map of two immersions, according to theorem 2.3.8. One is X, the other is its conformal dual.

2.4.2 Bryant's quartic

R. Bryant introduced in his seminal paper [Bry84] a holomorphic quartic with far-reaching properties.

Definition 2.4.1. Let Σ be a Riemann surface and $X : \Sigma \to \mathbb{S}^3$ be an immersion of representation Φ in $\mathbb{R}^3$ and Z in $\mathbb{H}^3$ and of conformal Gauss map Y. The Bryant quartic of X (respectively Φ, Z) is defined as

$$\mathcal{Q} := \left\langle \partial^2 Y, \partial^2 Y \right\rangle.$$

In a local complex chart one has $\mathcal{Q} = Qdz^4$, with $Q = \langle Y_{zz}, Y_{zz} \rangle$.

The results in this subsection are framed for conformal immersions of the unit disk for convenience. Indeed it means that we can work with the more familiar function, Q instead of the quartic $\mathcal{Q}$. They are generalizable to immersions of a Riemman surface by the same process that took us from definition 1.2.5 to 1.2.6 : systematically working in local conformal charts.

One can draw a parallel between constant mean curvature immersions and Willmore immersions. Indeed while for a CMC immersion, the Gauss map is harmonic, for a Willmore immersion the conformal Gauss map is. The Bryant's quartic allows us to further this comparison, as it is analogous to the Hopf differential. While the Hopf differential of a CMC immersion is holomorphic, the Bryant's quartic of a Willmore immersion is holomorphic.

Proposition 2.4.5. If X is Willmore then $\mathcal{Q}$ is holomorphic.

Proof. If X is Willmore then necessarily $Y_{z\bar{z}} = -\langle Y_z, Y_{\bar{z}} \rangle Y$, and then

$$Y_{zz\bar{z}} = (Y_{z\bar{z}})_z = -(\langle Y_z, Y_{\bar{z}} \rangle)_z Y - \langle Y_z, Y_{\bar{z}} \rangle Y_z$$

and since Y is conformal

$$\langle Y_{zz}, Y_z \rangle = \frac{1}{2} (\langle Y_z, Y_z \rangle)_z = 0,$$

and

$$\langle Y_{zz}, Y \rangle = (\langle Y_z, Y \rangle)_z - \langle Y_z, Y_z \rangle = 0.$$

Then

$$Q_{\bar{z}} = 2 \langle Y_{zz\bar{z}}, Y_{zz} \rangle = 0.$$

Hence $\overline{\partial} \mathcal{Q} = 0$. ☐

Using expression (A.2.60) in any orthonormal isotropic frame (ν, ν^*) (that is satisfying $\langle \nu, \nu^* \rangle = -1$) of the normal bundle of Y :

$$Y_{zz} = 2\mathcal{L}_z Y_z - \frac{\Omega_\nu}{2}\nu^* - \frac{\Omega_{\nu^*}}{2}\nu,$$

where $\mathcal{L}$ is the conformal factor of Y, one finds

$$Q = -\frac{\Omega_\nu \Omega_{\nu^*}}{2}. \tag{2.4.36}$$

Taking ν and ν^* as in susection 2.3.2 and using (2.3.31) and (2.3.32) further yields

$$\begin{aligned} Q &= \omega^2 e^{-2\Lambda} \left(\left(\frac{\omega_z}{\omega} \right)_{\bar{z}} + \frac{h^2+1}{4} e^{2\Lambda} \right) \\ &= \left(\omega_{z\bar{z}} \omega - \omega_z \omega_{\bar{z}} \right) e^{-2\Lambda} + \omega^2 \frac{h^2+1}{4}. \end{aligned} \tag{2.4.37}$$

Remark 2.4.2. The computations in $\mathbb{R}^3$ lead to a similar expression :

$$Q = \left(\Omega_{z\bar{z}} \Omega - \Omega_z \Omega_{\bar{z}} \right) e^{-2\lambda} + \Omega^2 \frac{H^2}{4}. \tag{2.4.38}$$

The converse of proposition 2.4.5 is not true.

Proposition 2.4.6. $\mathcal{Q}$ is holomorphic if and only if there exists a holomorphic function f on $\mathbb{D}$ such that

$$\mathcal{W}_{\mathbb{S}^3}\left(X\right) = \omega \overline{f} e^{-2\Lambda}. \tag{2.4.39}$$

Proof. We once again use the notations of subsection A.2.7 with ν and ν^* defined in (2.3.24) and (2.3.25). Then, as before,

$$Q_{\bar{z}} = 2\left\langle Y_{zz\bar{z}}, Y_{zz}\right\rangle,$$

and using (2.3.28) (2.3.30) and (A.2.61) :

$$\begin{aligned}\mathcal{Q}_{\bar{z}} &= 2\left\langle \left(\frac{\mathcal{W}_{\mathbb{S}^3}}{4}(X)\nu - \frac{|\omega|^2 e^{-2\Lambda}}{2}Y\right)_z, Y_{zz}\right\rangle \\ &= \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2}\left\langle \nu_z, Y_{zz}\right\rangle + \frac{(\mathcal{W}_{\mathbb{S}^3}(X))_z}{2}\left\langle \nu, Y_{zz}\right\rangle.\end{aligned}$$

Using (A.2.54) and $\nu = \begin{pmatrix} X \\ 1 \end{pmatrix}$ yields

$$\left\langle \nu_z, Y_{zz}\right\rangle = -\frac{1}{2}\left(\omega e^{-2\Lambda}\right)_z e^{2\Lambda}.$$

Further by (2.3.31) $\left\langle \nu, Y_{zz}\right\rangle = \frac{\omega}{2}$. Hence

$$\begin{aligned}Q_{\bar{z}} &= \frac{\left(\mathcal{W}_{\mathbb{S}^3}(X)\right)_z \omega}{4} - \frac{\mathcal{W}_{\mathbb{S}^3}(X)\left(\omega e^{-2\Lambda}\right)_z e^{2\Lambda}}{4} \\ &= e^{-2\Lambda}\omega^2\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4\omega e^{-2\Lambda}}\right)_z.\end{aligned}$$

To conclude $\mathcal{Q}$ holomorphic implies $\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{\omega e^{-2\Lambda}}\right)_z = 0$, which means there exists f holomorphic such that

$$\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{\omega e^{-2\Lambda}} = \overline{f}.$$

This concludes the proof. □

This result follows from the work of C. Bohle (see theorem 4.4 in [Boh12]). A. Michelat found an equivalent condition in [Miced].

Proposition 2.4.6 bears striking resemblance to the definition 1.2.5 of conformal Willmore immersions, with the added condition that $\overline{f}\omega \in \mathbb{R}$. This might be better understood with the notion of isothermic immersions, which we study in the fashion of T. Rivière ((I.4) in [Riv13]).

Definition 2.4.2. A conformal immersion Φ of the disk $\mathbb{D}$ into $\mathbb{R}^3$ (or equivalently X into $\mathbb{S}^3$) is said to be isothermic if around each point of $\mathbb{D}$ there exists a local conformal reparametrization such that $\Omega \in \mathbb{R}$ (equivalently $\omega \in \mathbb{R}$). Such a parametrization will be called isothermic, or in isothermic coordinates.

Isothermic immersions can be conveniently caracterized (as explained by proposition I.1 in [Riv13]).

Proposition 2.4.7. A conformal immersion Φ of the disk $\mathbb{D}$ into $\mathbb{R}^3$ (or equivalently X into $\mathbb{S}^3$) is isothermic if and only if there exists a non zero holomorphic function F on $\mathbb{D}$, such that

$$\Im\left(\overline{F}\Omega\right) = 0.$$

Equivalently X is isothermic if and only if there exists a non zero holomorphic function f on $\mathbb{D}$, such that

$$\Im\left(\overline{f}\omega\right) = 0.$$

In fact away from its zeros, $\sqrt{f}$ yields the conformal reparametrization into isothermic coordinates.

Of course when Φ is defined on any Riemann surface on Σ, it can be said to be isothermic if proposition 2.4.7 stands in any local conformal chart. There exists, however, a way to formulate it using tensors on the surface (following is definition I.1 from [Riv13]).

Definition 2.4.3. Let $\Phi : \Sigma \to \mathbb{R}^3$. One says that Φ is global isothermic if there exists an holomorphic quadratic form q such that :

$$\Im\left(\left\langle q, \vec{h}_0 \right\rangle_{WP}\right) = 0,$$

where $\langle u, v\rangle_{WP} = g^{-1} \otimes g^{-1} \otimes u \otimes \overline{v}$ is the Weil-Peterson product.

Then (2.4.39) not only yields that X is conformal Willmore, but either f is null and then X is Willmore, or there exists a non null holomorphic f such that $\overline{f}\omega \in \mathbb{R}$, that is $\Im\left(\overline{f}\omega\right) = 0$ i.e. X is isothermic.

Corollary 2.4.2. If $\mathcal{Q}$ is holomorphic then either X is Willmore, or X is conformal Willmore and isothermic.

2.4.3 The residues through the conformal Gauss map

This subsection aims at finding easy ways to compute the two relevant residues in theorem 1.5.1 when the surface is conformally minimal. Let us then consider Φ a branched Willmore immersion of Σ with a branch point at $b \in \Sigma$, and Ψ a minimal immersion with a branched end at b. We assume there exists an inversion ι such that $\Phi = \iota \circ \Psi$.

Applying corollary 2.4.1 one finds that $\gamma_{\vec{0}\Phi} = \gamma_{\vec{3},\Psi}$. Further since Ψ is assumed to be minimal, one can inject $H = 0$ into (1.2.42) and find

$$\gamma_{\vec{3},\Psi} = \frac{2}{\pi} \int \Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right).\nu.$$

Further, thanks to (1.2.42) and (A.2.5) we conclude :

$$\begin{aligned}\Phi \times \left(\vec{n} \times \mathring{A}\nabla\Phi\right) &= -\Phi \times (\vec{n} \times \nabla\vec{n}) = -\Phi \times \mathring{A}\nabla^{\perp}\Phi = -\Phi \times \nabla^{\perp}\vec{n} \\ &= -\nabla^{\perp}\left(\Phi \times \vec{n}\right) + \nabla^{\perp}\Phi \times \vec{n} = \nabla^{\perp}(\Phi \times \vec{n}) - \nabla\Phi.\end{aligned}$$

Thus $\gamma_{\vec{0},\Phi}$ is proportional to the flux of Ψ through its branched end at b. Further given that Ψ is minimal all the other residues are null. We then deduce the following result, already present in [MR17] :

Corollary 2.4.3. Inversions of vanishing flux minimal surfaces are *true* branched Willmore surfaces : all their residues vanish.

Example 2.4.1. The Enneper surface, the Chen-Gackstatter torus, the Bryant surface, the López surface are all of vanishing flux. Its inverses are true Willmore surfaces.
The Catenoïd is a minimal surface with flux, hence the inverted catenoïd offers an example of singular Willmore surface.

It is also interesting to notice that the second residue of Φ around b can be read on its conformal Gauss map Y. Indeed, since Y is defined as :

$$Y_\Phi = H_\Phi \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n}_\Phi \\ \langle \vec{n}_\Phi, \Phi \rangle \\ \langle \vec{n}_\Phi, \Phi \rangle \end{pmatrix},$$

and since Φ and $\vec{n}_\Phi$ are bounded around the branch point, necessarily

$$Y_\Phi \sim_b C_\Phi z^{-\alpha}.$$

This is true whether Φ is conformally minimal or not.

However, in the particular case where Φ is the conformal transform of the minimal branched immersion Ψ, it has been shown by proposition 2.3.4 that there exists a *fixed* matrix $M \in SO(4,1)$ such that $Y_\Psi = MY_\Phi$. This yields that necessarily $Y_\Psi \sim_b C_\Psi z^{-\alpha}$. Hence, considering that Ψ is minimal, we deduce that

$$Y_\Psi = \begin{pmatrix} \vec{n}_\Psi \\ \langle \vec{n}_\Psi, \Psi \rangle \\ \langle \vec{n}_\Psi, \Psi \rangle \end{pmatrix}.$$

Since $\vec{n}_\Psi$ is bounded,

$$\langle \vec{n}_\Psi, \Psi \rangle \sim_b C z^{-\alpha}. \tag{2.4.40}$$

We can apply this to compute rather easily the second residue, from the Enneper-Weierstrass representation. Following is a non trivial example : the Chen-Gackstatter torus.

Proposition 2.4.8. The inverted Chen-Gackstatter torus has a second residue $\alpha = 2$ at its branch point.

Proof. Let $\Psi : (\mathbb{C}\backslash\mathbb{Z}^2)/\mathbb{Z}^2 \to \mathbb{R}^3$ be a parametrization of the Chen-Gackstatter torus, $p \in \mathbb{R}^3$ such that $d(p,\Psi) > 1$, and $\Phi = \iota \circ (\Psi - p)$, the studied inverse.

We will now use the Enneper-Weierstrass parametrization of Ψ and (2.4.40) to compute the second residue of Ψ at its branch point. Chen-Gackstatter is a minimal surface of genus 1 and of Enneper-Weierstrass data centered on the branch point : $(f,g) = \left(2\mathfrak{p}(z), A\frac{\mathfrak{p}_z}{\mathfrak{p}}(z)\right)$ (see [CG82]) where $\mathfrak{p}$ is the Weierstrass elliptic function, of elliptic invariants

$$\begin{aligned} g_2 &= 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+ni)^4} > 0, \\ g_3 &= 0, \end{aligned}$$

and

$$A = \sqrt{\frac{3\pi}{2g_2}} \in \mathbb{R}_+.$$

Then, φ has the following expansion around 0 (see [Apo90]) :

$$\begin{aligned} \mathfrak{p}(z) &= \frac{1}{z^2} + O(z^2) \\ \mathfrak{p}_z(z) &= \frac{-2}{z^3} + O(z). \end{aligned}$$

Hence we can state that

$$\begin{aligned}
\Phi &= 2\Re\left(\int \frac{1}{z^2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{4A^2}{z^4}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{4A}{z^3}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(1)dz\right)\\
&= 2\Re\left(\frac{4A^2}{3z^3}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{1}{z}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{2A}{z^2}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(z)\right) \qquad (2.4.41)\\
&= \left(\frac{4A^2}{3\bar{z}^3} - \frac{1}{z}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \left(\frac{4A^2}{3z^3} - \frac{1}{\bar{z}}\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + 2A\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(r).
\end{aligned}$$

Similarly :

$$\begin{aligned}
\Phi_z \times \Phi_{\bar{z}} &= \left(\frac{1}{z^2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{4A^2}{z^4}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{4A}{z^3}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(1)\right) \times \left(\frac{1}{\bar{z}^2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{4A^2}{\bar{z}^4}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{4A}{\bar{z}^3}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(1)\right)\\
&= -\frac{4Ai}{z^2\bar{z}^3}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{32iA^4}{r^8}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} - \frac{16A^3i}{z^4\bar{z}^3}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{4Ai}{z^3\bar{z}^2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{16A^3i}{z^3\bar{z}^4}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + O\left(\frac{1}{r^4}\right)\\
&= \frac{32iA^4}{r^8}\left(-\left(\frac{z}{2A} + \frac{z^2\bar{z}}{8A^3}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \left(\frac{\bar{z}}{2A} + \frac{\bar{z}^2 z}{8A^3}\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O(r^4)\right),
\end{aligned}$$

and

$$\begin{aligned}
|\Phi_z|^2 &= \frac{32A^4}{r^8} + \frac{16A^2}{r^6} + O(r^{-4})\\
&= \frac{32A^4}{r^8}\left(1 + \frac{r^2}{2A^2} + O(r^4)\right),
\end{aligned}$$

which yields

$$\vec{n}_\Phi = \left(1 - \frac{r^2}{2A^2}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} - \left(\frac{z}{2A} - \frac{z^2\bar{z}}{8A^3}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \left(\frac{\bar{z}}{2A} - \frac{z\bar{z}^2}{8A^3}\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + O(r^4). \qquad (2.4.42)$$

Combining (2.4.41) and (2.4.42) ensures :

$$\langle \vec{n}_\Phi, \Phi\rangle = 2A\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right) - \frac{4A}{3z^2} - \frac{4A}{3\bar{z}^2} + O\left(\frac{1}{r}\right) = \frac{2A}{3z^2} + \frac{2A}{3\bar{z}^2} + O\left(\frac{1}{r}\right). \qquad (2.4.43)$$

Considering (2.4.43) in light of (2.4.40) yields $\alpha = 2$. □

2.5 Conformally CMC immersions : proof of theorems B and C

A quick study of proposition 2.3.2 and (2.3.15) reveals that the mean curvature in the three models can be written as a function of Y, with interesting geometric interpretations.

Corollary 2.5.1. Let Φ be a smooth conformal immersion on $\mathbb{D}$, and X (respectively Z) its representation in $\mathbb{S}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). Let Y be its conformal Gauss map. Then

$$\begin{aligned} H &= Y_5 - Y_4, \\ h &= Y_5, \\ H^Z &= -Y_4. \end{aligned} \tag{2.5.44}$$

We denote

$$v_s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; v_t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; v_l = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

One deduces immediately from this that Φ is minimal (respectively of constant mean curvature) if and only if $Y_4 = Y_5$ (respectively if there exists a constant $H_0 \in \mathbb{R}$ such that $Y_5 - Y_4 - H_0 = 0$), X is minimal (respectively of constant mean curvature) if and only $Y_5 = 0$ (respectively if there exists a constant $h_0 \in \mathbb{R}$ such that $Y_5 - h_0 = 0$), Z is minimal (respectively of constant mean curvature) if and only if $Y_4 = 0$ (respectively if there exists a constant $H_0^Z \in \mathbb{R}$ such that $Y_4 + H_0^Z = 0$). This can be reframed as : Φ is minimal (respectively CMC) if and only if Y is in a linear (respectively affine) hyperplane of lightlike normal v_l, X is minimal (respectively CMC) if and only if Y is in a linear (respectively affine) hyperplane of timelike normal v_t, Z is minimal (respectively CMC) if and only if Y is in a linear (respectively affine) hyperplane of spacelike normal v_s.

Then, given proposition 2.3.3 and its analogues, and since any $M \in SO(4,1)$ conserves hyperplanes in $\mathbb{R}^{4,1}$ and the type of vectors, we deduce the following theorem.

Theorem 2.5.1. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a smooth conformal immersion, and X (respectively Z) its representation in $\mathbb{S}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). Let Y be its conformal Gauss map. We assume the set of umbilic points of Φ (or equivalently, see (A.2.29) and (A.2.42), X or Z) to be nowhere dense.

We say that Φ (respectively X, Z) is conformally CMC (respectively minimal) if and only if there exists a conformal diffeomorphism φ of $\mathbb{R}^3 \cup \{\infty\}$ (respectively $\mathbb{S}^3$, $\mathbb{H}^3$) such that $\varphi \circ \Phi$ (respectively $\varphi \circ X$, $\varphi \circ Z$) has constant mean curvature (respectively is minimal) in $\mathbb{R}^3$ (respectively $\mathbb{S}^3$, $\mathbb{H}^3$).

Then

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with lightlike normal.

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with timelike normal.

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if Y lies in an affine (respectively linear) hyperplane of $\mathbb{R}^{4,1}$ with spacelike normal.

R. Bryant showed that its eponym quartic is highly relevant when considering this problem. We paraphrase below theorem C of [Bry84] below :

Theorem 2.5.2. Let $\Phi : \Sigma \to \mathbb{R}^3$ be a Willmore immersion of a compact connected surface Σ. Assume that Φ is not all umbilic but that $\mathcal{Q} = 0$. Then Φ is conformally minimal in $\mathbb{R}^3$.

This result is also presented in J-H. Eschenburg's and B. Palmer's surveys (respectively [Esc88] and[Pal91]). Many theorems describing conformally CMC immersions (even in higher codimensions) have been obtained using *DPW* (also called *loop groups*) methods by, among others N. Ejiri (see [Eji88]), S. Montiel (see [Mon00]), J. Richter ([Ric97]) or J. Dorfmeister and P. Wang (see [DW19]). We will however present a result reached through classical differential geometry.

We work with immersions of the disk for convenience of exposition. Let $X : \mathbb{D} \to \mathbb{S}^3$ of representation Φ in $\mathbb{R}^3$, Z in $\mathbb{H}^3$ without umbilic points and of conformal Gauss map Y. Our aim is to find a necessary and sufficient condition to have one of the three representations be conformally CMC in its immersion space.

Let us first focus on finding a set of necessary conditions. Thanks to theorem 2.5.1, we know it is equivalent to the fact that Y lies in a hyperplane of $\mathbb{R}^{4,1}$. That is, there exists constants $v \in \mathbb{R}^{4,1} \backslash \{0\}$ and $\eta \in \mathbb{R}$ such that

$$\langle Y, v \rangle = \eta. \tag{2.5.45}$$

Since v and η are constants, differentiating (2.5.45) yields

$$\langle Y_z, v \rangle = 0 \tag{2.5.46}$$

and

$$\langle Y_{\bar{z}}, v \rangle = 0. \tag{2.5.47}$$

One can write v in the moving frame $(Y, Y_z, Y_{\bar{z}}, \nu, \nu^*)$ with ν and ν^* defined in (2.3.24) and (2.3.25) :

$$v = lY + mY_z + nY_{\bar{z}} + a\nu + b\nu^*.$$

Applying (2.5.45), (2.5.46) and (2.5.47) yields

$$\begin{aligned} l &= \eta \\ m &= 0 \\ n &= 0 \end{aligned}$$

And thus

$$v = \eta Y + a\nu + b\nu^*. \tag{2.5.48}$$

v can be taken such that

$$\langle v, v \rangle = \kappa = \begin{cases} 1 \text{ if } v \text{ is spacelike} \\ 0 \text{ if } v \text{ is lightlike} \\ -1 \text{ if } v \text{ is timelike.} \end{cases}$$

From this decomposition we will deduce characterizations of a and b. Since v is constant one can differentiate (2.5.48) and put formulas (A.2.66) and (A.2.71) to effect :

$$\begin{aligned} 0 &= (\eta - aH_\nu - bH_{\nu^*})\, Y_z + (a_z - a\,\langle \nu_z, \nu^* \rangle)\,\nu + (b_z - b\,\langle \nu^*_z, \nu \rangle)\,\nu^* - \frac{(a\Omega_\nu + b\Omega_{\nu^*})}{|\omega|^2 e^{-2\Lambda}} Y_{\bar{z}} \\ &= \left(\eta + \frac{b\mathcal{W}_{\mathbb{S}^3}(X)}{2|\omega|^2 e^{-2\Lambda}}\right) Y_z + (a_z - a\,\langle \nu_z, \nu^* \rangle)\,\nu + (b_z - b\,\langle \nu^*_z, \nu \rangle)\,\nu^* - \frac{(a\Omega_\nu + b\Omega_{\nu^*})}{|\omega|^2 e^{-2\Lambda}} Y_{\bar{z}} \end{aligned}$$

with (2.3.28) and (2.3.30). Further since $\langle \nu^*_z, \nu \rangle = (\langle \nu, \nu^* \rangle)_z - \langle \nu_z, \nu^* \rangle = \frac{\overline{\omega}_z}{\overline{\omega}}$, using (2.3.26), we find

$$0 = \left(\eta + \frac{b\mathcal{W}_{\mathbb{S}^3}(X)}{2|\omega|^2 e^{-2\Lambda}}\right) Y_z + \left(a_z + a\frac{\overline{\omega}_z}{\overline{\omega}}\right)\nu + \left(b_z - b\frac{\overline{\omega}_z}{\overline{\omega}}\right)\nu^* - \frac{(a\Omega_\nu + b\Omega_{\nu^*})}{|\omega|^2 e^{-2\Lambda}} Y_{\bar{z}}.$$

Besides

$$\langle v, v\rangle = \eta^2 - 2ab,$$

and since Y, ν and ν^* are bounded in $\mathbb{R}^5$ away from umbilic points, $a, b < \infty$. Then a, b are real functions and η a real constant such that

$$a_z + a\frac{\overline{\omega}_z}{\overline{\omega}} = 0, \tag{2.5.49}$$

$$b_z - b\frac{\overline{\omega}_z}{\overline{\omega}} = 0, \tag{2.5.50}$$

$$b\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2} + \eta|\omega|^2 e^{-2\Lambda} = 0, \tag{2.5.51}$$

$$a\Omega_\nu + b\Omega_{\nu^*} = 0, \tag{2.5.52}$$

$$ab = -\frac{\langle v, v\rangle - \eta^2}{2} \text{ real constant.} \tag{2.5.53}$$

One can recast (2.5.49) as $a_z\overline{\omega} + a\overline{\omega}_z = 0$, or rather since $a \in \mathbb{R}$

$$a_{\bar{z}}\omega + a\omega_{\bar{z}} = 0.$$

This yields

$$(a\omega)_{\bar{z}} = 0,$$

i.e. there exists $f : \mathbb{D} \to \mathbb{C}$ holomorphic (since $a\omega < \infty$) such that

$$a\omega = f. \tag{2.5.54}$$

One then has $\overline{f}\omega = \overline{a\omega}\omega = a|\omega|^2 \in \mathbb{R}$ since $a \in \mathbb{R}$. Then according to proposition 2.4.7, unless $f = 0$ on $\mathbb{D}$, X is isothermic. Working similarly on (2.5.50) one finds there exists g holomorphic (since $b < \infty$ and $\omega \neq 0$ by hypothesis) on $\mathbb{D}$ such that

$$b = g\omega. \tag{2.5.55}$$

Then, if g is not null on $\mathbb{D}$, working away from its zeros yields

$$\overline{\frac{1}{g}}\omega = \frac{|\omega|^2}{\overline{b}} \in \mathbb{R}$$

since $b \in \mathbb{R}$. Then according to proposition 2.4.7 X is isothermic. So unless $f = g = 0$ on $\mathbb{D}$, X is isothermic. If $f = g = 0$, then (2.5.51) ensures $\eta = 0$ which in turn yields $v = 0$, a case excluded from the start of this reasoning. As a consequence we get our first necessary condition :

X is isothermic.

To go further one can reframe (2.5.51) in terms of f and g. Indeed

$$b\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2} + \eta|\omega|^2 e^{-2\Lambda} = \omega\left(g\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2} + \eta\overline{\omega}e^{-2\Lambda}\right)$$

with $\omega \neq 0$ ensuring that (2.5.51) is equivalent to

$$g\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2} + \eta\overline{\omega}e^{-2\Lambda} = 0.$$

This implies that if $g(z_0) = 0$ for any given z_0 in $\mathbb{D}$, then $\eta = 0$, and with (2.5.52) $f(z_0) = 0$. So $v(z_0) = 0$ and since v is a constant $v = 0$, which is a contradiction. Then g has no zero on $\mathbb{D}$. Letting $\varphi = \frac{1}{g}$ be a holomorphic function on $\mathbb{D}$, one finds (2.5.51) to be equivalent to

$$\mathcal{W}_{\mathbb{S}^3}(X) = -2\eta\varphi\overline{\omega}e^{-2\Lambda} = \overline{(-2\eta\varphi)}\omega e^{-2\Lambda}. \tag{2.5.56}$$

Consequently, proposition 2.4.6 implies our second necessary condition

$\mathcal{Q}$ is holomorphic.

Similarly

$$\begin{aligned} a\Omega_\nu + b\Omega_{\nu^*} &= a\omega\frac{\Omega_\nu}{\omega} + \frac{b}{\omega}\omega\Omega_{\nu^*} \\ &= a\omega + \frac{b}{\omega}\Omega_\nu\Omega_{\nu^*} \text{ using (2.3.31)} \\ &= a\omega - 2\frac{b}{\omega}Q \text{ using (2.4.36).} \end{aligned}$$

This yields that (2.5.52) is equivalent to

$$Q = \frac{a\omega^2}{2b} = \frac{f}{2g} = \frac{fg}{2g^2} = \frac{1}{2}ab\varphi^2 = \frac{\eta^2 - \kappa}{4}\varphi^2. \tag{2.5.57}$$

Summing up our analysis has given us two necessary conditions :

- **X is isothermic, with $\varphi\overline{\omega} \in \mathbb{R}$**
- **$\mathcal{Q}$ is holomorphic, with $Q = \frac{\eta^2-\kappa}{4}\varphi^2$.**

Let us show they are sufficient.

Let X be an isothermic immersion such that $\mathcal{Q}$ is holomorphic. Our aim is to write $\mathcal{Q}$ and $\mathcal{W}_{\mathbb{S}^3}(X)$ in the forms respectively of (2.5.57) and (2.5.56).

Since X is isothermic there exists a non null holomorphic function φ_0 such that

$$R := \overline{\varphi_0}\omega \in \mathbb{R}.$$

Claim 1 : there exists a constant $m \in \mathbb{R}$ such that $Q = m\varphi_0^2$.

Proof. We will write Q as a function of φ, using (2.4.37) :

$$Q = (\omega_{z\bar{z}}\omega - \omega_z\omega_{\bar{z}})\, e^{-2\Lambda} + \omega^2\frac{h^2+1}{4}$$

Since $\omega = \frac{R}{\overline{\varphi_0}}$,

$$\begin{aligned} \omega_z &= \frac{R_z}{\overline{\varphi_0}} \\ \omega_{\bar{z}} &= \frac{R_{\bar{z}}}{\overline{\varphi_0}} - \frac{\overline{\partial_z\varphi_0}R}{\overline{\varphi_0^2}} \\ \omega_{z\bar{z}} &= \frac{R_{z\bar{z}}}{\overline{\varphi_0}} - \frac{\overline{\partial_z\varphi_0}R_z}{\overline{\varphi_0^2}}. \end{aligned}$$

Thus

$$\begin{aligned}\omega_{z\bar{z}}\omega - \omega_z\omega_{\bar{z}} &= \frac{R}{\overline{\varphi_0}}\left(\frac{R_{z\bar{z}}}{\overline{\varphi_0}} - \frac{\overline{\partial_z\varphi_0}R_z}{\overline{\varphi_0^2}}\right) - \frac{R_z}{\overline{\varphi_0}}\left(\frac{R_{\bar{z}}}{\overline{\varphi_0}} - \frac{\overline{\partial_z\varphi_0}R}{\overline{\varphi_0^2}}\right)\\ &= \frac{R_{z\bar{z}}R - R_zR_{\bar{z}}}{\overline{\varphi_0^2}} \\ &= \left(\frac{R_{z\bar{z}}R - R_zR_{\bar{z}}}{|\varphi_0|^4}\right)\varphi_0^2.\end{aligned} \tag{2.5.58}$$

As announced Q can be expressed :

$$Q = \frac{(R_{z\bar{z}}R - R_zR_{\bar{z}})\,e^{-2\Lambda} + \frac{h^2+1}{4}R^2}{|\varphi_0|^4}\varphi_0^2. \tag{2.5.59}$$

Since $R \in \mathbb{R}$, $\frac{(R_{z\bar{z}}R - R_zR_{\bar{z}})e^{-2\Lambda} + \frac{h^2+1}{4}R^2}{|\varphi_0|^4}$ is real. Further

$$\left(\frac{(R_{z\bar{z}}R - R_zR_{\bar{z}})\,e^{-2\Lambda} + \frac{h^2+1}{4}R^2}{|\varphi_0|^4}\right)_{\bar{z}} = A_z = \left(\frac{Q}{\varphi_0^2}\right)_{\bar{z}} = 0$$

since Q (equivalently $\mathcal{Q}$) and φ_0 are holomorphic. As a real holomorphic function A is necessarily a constant that we will denote m. This proves claim 1. ☐

Claim 2 : There exists $n \in \mathbb{R}$ such that $\mathcal{W}_{\mathbb{S}^3}(X) = n\omega\overline{\varphi_0}e^{-2\Lambda}$.

Proof. Proposition 2.4.6 yields f holomorphic on $\mathbb{D}$ such that

$$\mathcal{W}_{\mathbb{S}^3}(X) = \omega\overline{f}e^{-2\Lambda}.$$

Using $\omega = \frac{R}{\overline{\varphi_0}}$ one deduces

$$\overline{\left(\frac{f}{\varphi_0}\right)} = \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{Re^{-2\Lambda}} \in \mathbb{R}.$$

Since $\frac{f}{\varphi_0}$ is holomorphic, there exists $n \in \mathbb{R}$ such that $f = n\varphi_0$, which proves claim 2. ☐

Claim 3 : There exists $\lambda \in \mathbb{R}$, $\kappa \in \{-1, 0, 1\}$ and $\eta \in \mathbb{R}$ such that $\mathcal{W}_{\mathbb{S}^3}(X) = -2\eta\omega\overline{\varphi_0}e^{-2\Lambda}$ and $Q = \frac{\eta^2-\kappa}{4}\lambda^2\varphi_0^2$.

Proof. If $n^2 - m \neq 0$, let $\lambda = 2\sqrt{\left|\left(\frac{n}{4}\right)^2 - m\right|}$, $\kappa = sg(\left(\frac{n}{4}\right)^2 - m)$ and $\eta = -\frac{n}{2\lambda}$. Then $n = -2\lambda\eta$ and

$$\begin{aligned} m &= -\left(\left(\frac{n}{4}\right)^2 - m\right) + \left(\frac{n}{4}\right)^2 \\ &= -\frac{\lambda^2\kappa}{4} + \lambda^2\frac{\eta^2}{4} \\ &= \lambda^2\frac{\eta^2 - \kappa}{4}. \end{aligned}$$

If $n^2 = 16m$, let $\kappa = 0$, $\lambda = 1$, $n = -2\eta$, which concludes the proof of claim 3. ☐

In the following we set $\varphi = \lambda \varphi_0$.

Claim 4 : $v = \eta Y + \frac{\eta^2-\kappa}{2}\frac{\varphi}{\omega}\nu + \frac{\omega}{\varphi}\nu^*$ **is a constant vector in** $\mathbb{R}^{4,1}$**.**

Proof. Since $\eta \in \mathbb{R}$,

$$a := \frac{\eta^2 - \kappa}{2}\frac{\varphi}{\omega} = \frac{\eta^2-\kappa}{2}\frac{|\varphi|^2}{\omega\overline{\varphi}} \in \mathbb{R}$$

and

$$b := \frac{\omega}{\varphi} = \frac{\omega\overline{\varphi}}{|\varphi|^2} \in \mathbb{R}.$$

Then v does belong in $\mathbb{R}^{4,1}$. Further

$$a_{\overline{z}} + a\frac{\omega_{\overline{z}}}{\omega} = \frac{\eta^2-\kappa}{2}\varphi\left(\frac{-\omega_{\overline{z}}}{\omega^2} + \frac{\omega_{\overline{z}}}{\omega^2}\right) = 0$$

and

$$b_{\overline{z}} - b\frac{\omega_{\overline{z}}}{\omega} = \frac{1}{\varphi}\left(\omega_{\overline{z}} - \omega_{\overline{z}}\right) = 0,$$

meaning that a and b satisfy (2.5.49) and (2.5.50). Besides

$$b\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{2} + \eta|\omega|^2 e^{-2\Lambda} = -2\frac{\eta}{2}\overline{\omega}\varphi e^{-2\Lambda}\frac{\omega}{\varphi} + \eta|\omega|^2 e^{-2\Lambda} = 0$$

since by design, see claim 3, $\mathcal{W}_{\mathbb{S}^3}(X) = -2\eta\omega\overline{\varphi}e^{-2\Lambda} = -2\eta\overline{\omega}\varphi e^{-2\Lambda}$. v must then satisfy (2.5.51). Once more, by construction, Q satisfies (2.5.57), which was shown to be equivalent to (2.5.52). v then satisfies : $v_z = 0$, and v is a constant in $\mathbb{R}^{4,1}$, which proves claim 4. □

Y is then hyperplanar and, according to theorem 2.5.1, X is conformally CMC in a space depending entirely on $\langle v, v\rangle = \kappa$. κ can be expressed explicitely from $\mathcal{Q}$ et $\mathcal{W}_{\mathbb{S}^3}(X)$. Indeed

$$\begin{aligned}\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} &= \frac{\eta^2}{4}\overline{\omega}^2\varphi^2 e^{-4\Lambda} - \frac{\eta^2-\kappa}{4}\varphi^2\overline{\omega}^2 e^{-4\Lambda} \text{ using Claim 3}\\ &= \kappa\left(\frac{\varphi\overline{\omega}e^{-2\Lambda}}{2}\right)^2.\end{aligned}$$

Since $\varphi\overline{\omega} \in \mathbb{R}^*$, $\left(\frac{\varphi\overline{\omega}e^{-2\Lambda}}{2}\right)^2 \in \mathbb{R}_+^*$ and necessarily :

$$\kappa = sg\left(\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}\right). \tag{2.5.60}$$

We deduce the following theorem.

Theorem B. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). We assume that X (or equivalently, see (A.2.29) and (A.2.42), Φ or Z) has no umbilic point. One of the representation of X is conformally CMC in its ambient space if and only if $\mathcal{Q}$ is holomorphic and X is isothermic. More precisely $\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}$ is then necessarily real and

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

Notice especially that according to our analysis X isothermic and $\mathcal{Q}$ holomorphic heavily determines $\mathcal{Q}$. As a matter of fact it ensures that $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$. Accordingly one can slightly change the hypotheses of theorem B.

Theorem C. Let X be a smooth conformal immersion on $\mathbb{D}$ in $\mathbb{S}^3$, and Φ (respectively Z) its representation in $\mathbb{R}^3$ (respectively $\mathbb{H}^3$) through π (respectively $\tilde{\pi}$). We assume X (or equivalently, see (A.2.29) and (A.2.42), Φ or Z) has no umbilic point. One of the representation of X is conformally *CMC* in its ambient space if and only if $\mathcal{Q}$ is holomorphic and $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$. More precisely

— Φ is conformally CMC (respectively minimal) in $\mathbb{R}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} = 0.$$

— X is conformally CMC (respectively minimal) in $\mathbb{S}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} < 0.$$

— Z is conformally CMC (respectively minimal) in $\mathbb{H}^3$ if and only if

$$\left(\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\right)^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q} > 0.$$

Conformally minimal immersions satisfy $\mathcal{W}_{\mathbb{S}^3}(X) = 0$.

Proof. If X is conformally CMC, then $\mathcal{Q}$ is holomorphic and $(\mathcal{W}_{\mathbb{S}^3}(X))^2 - \overline{\omega}^2 e^{-4\Lambda}\mathcal{Q}$ is real according to theorem B. Then since $\mathcal{W}_{\mathbb{S}^3}(X) \in \mathbb{R}$, $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$.

Conversely assume that $\mathcal{Q}$ is holomorphic and $\overline{\omega^2}\mathcal{Q} \in \mathbb{R}$. Then using corollary 2.4.2, X is isothermic and conformal Willmore or Willmore. If X is isothermic, the theorem is proved with theorem B. Let us then assume that X is Willmore. Let us first assume that $\mathcal{Q}$ is non null. Away from the zeros of $\mathcal{Q}$, $\overline{\omega^2}\mathcal{Q}$ does not cancel and is then of fixed sign, and $\sqrt{\mathcal{Q}}$ is holomorphic. Then

$$\left(\overline{\omega}\sqrt{\mathcal{Q}}\right)^2 \in \mathbb{R}^*,$$

and thus

$$\overline{\omega}\sqrt{\mathcal{Q}} \in \mathbb{R} \text{ or } i\mathbb{R}.$$

There exists then a non null holomorphic function ($\varphi = \sqrt{\mathcal{Q}}$ or $\varphi = i\sqrt{\mathcal{Q}}$) such that $\overline{\omega}\varphi \in \mathbb{R}$. The theorem is then proved with theorem B. The case X Willmore and $\mathcal{Q} = 0$ is now the only one left. Using theorem2.5.2 yields Φ conformally minimal in $\mathbb{R}^3$. This concludes the proof. $\square$

2.6 Classification of Willmore spheres

Let us consider Φ a branched Willmore immersion of a sphere in $\mathbb{R}^3$. Proposition 2.4.5 ensures that $\mathcal{Q}$ is a meromorphic quartic on the sphere. A glance at expression (2.4.38) ensure that poles of $\mathcal{Q}$ may only occur at branch points of the immersion. In the immersed case $\mathcal{Q}$ is then a holomorphic quartic on the sphere. Basic complex analysis then states that $\mathcal{Q}$ is null. From theorem 2.5.2 the following theorem by R. Bryant (theorem E in [Bry84]) follows.

Theorem 2.6.1. Let $\Phi : \mathbb{S}^2 \to \mathbb{R}^3$ be a Willmore immersion. Then $\Phi\left(\mathbb{S}^2\right)$ is conformally minimal in $\mathbb{R}^3$.

The branched case can be partially treated through the expansions of the quartic at the branch points offered by theorem 1.5.1. Following is a concatenation of a result by T. Lamm and H. Nguyen (theorem 3.1 from [LN15]) and A. Michelat and T. Rivière (theorem G from [MR17]).

Proposition 2.6.2. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a branched *true* Willmore immersion with a single branch point at 0 of multiplicity $\theta + 1$. Then $\mathcal{Q}$ is meromorphic on $\mathbb{D}$ with a single pole at the branch point of order *at most* 2.

Proof. Using (2.4.38), since $H^2\Omega^2$ is bounded across concentration points (this can be seen by combining (1.5.107) and (1.5.106)), the only possibly singular term is then

$$e^{-2\lambda}\left(\Omega_{z\bar{z}}\Omega - \Omega_z\Omega_{\bar{z}}\right).$$

Estimate (1.5.110) concludes the proof. □

From this we extend somewhat the classification of Willmore spheres to branched Willmore spheres. It has been partially found by T. Lamm and H. Nguyen (theorem I.2 in [LN15]) and extended to this form by A. Michelat and T. Rivière (theorem F in [MR17]). Where they used mostly Riemann-Roch type theorems, we will give a proof using only Liouville theorem.

Theorem 2.6.3. Let $\Phi : \mathbb{S}^2 \to \mathbb{R}^3$ be a branched true Willmore immersion, with at most 3 branch points. Then $\Phi\left(\mathbb{S}^2\right)$ is conformally minimal in $\mathbb{R}^3$.

Proof. We choose a two chart atlas on $\mathbb{S}^2$, $z = \frac{1}{h}$. In the z-chart, $\mathcal{Q} = Q(z)dz^4$ with, according to proposition 2.6.2 at most three poles of order at most 2, let us say at a, b and $c \in \mathbb{C}$. Then $(z-a)^2(z-b)^2(z-c)^2Q$ is a holomorphic function on $\mathbb{C}$. Further changing charts, one finds

$$\begin{aligned}\mathcal{Q} &= Q\left(\frac{1}{h}\right) d\left(\frac{1}{h}\right)^4 \\ &= \frac{Q\left(\frac{1}{h}\right)}{h^8} dh^4.\end{aligned}$$

Since $\mathcal{Q}$ has no pole at ∞, $\frac{Q\left(\frac{1}{h}\right)}{h^8}$ is *holomorphic* around $h = 0$. Then

$$\left(\frac{1}{h} - a\right)^2 \left(\frac{1}{h} - b\right)^2 \left(\frac{1}{h} - c\right)^2 Q\left(\frac{1}{h}\right) = h^2 \frac{Q\left(\frac{1}{h}\right)}{h^8} \left(1 - ah\right)^2 \left(1 - bh\right)^2 \left(1 - ch\right)^2.$$

Then $(z-a)^2(z-b)^2(z-c)^2Q$ is a holomorphic function on $\mathbb{C}$ which tends toward 0 at ∞. By Liouville theorem $(z-a)^2(z-b)^2(z-c)^2Q = 0$, which means $Q = 0$. This concludes the proof. □

2.7 Bryant's quartic at branch point of residue $\alpha \leq \theta - 1$

This subsection will study how the Bryant's quartic of a true Willmore immersion behaves around a branch point of multiplicity $\theta + 1$ when the immersion is better than expected, namely when the second residue satisfies $\alpha \leq \theta - 1$ (compared to the native control $\alpha \leq \theta$).Most of the computations, the theorems concluding them and the ideas are originally found in section 4 of [MR17]. We will only expose the broad strokes to give an idea of the phenomena, and sometimes rephrase and reframe their results. The main one is the following :

Theorem 2.7.1. Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a branched true Willmore conformal immersion with a single branch point of multiplicity $\theta + 1$ at the origin such that the second residue α at the branch point satisfies $\alpha \leq \theta - 1$.
Then the Bryant's quartic $\mathcal{Q}$ of Φ is holomorphic on $\mathbb{D}$. In addition either $\alpha \leq \theta - 2$ or 0 is an umbilic point.

Proof. We will use the formalism of section 1.5.2. Using estimate (1.5.110), the singular term of the Bryant's quartic at the origin has at most a pole of order 1. For $\alpha < \theta - 1$ the origin is a regular point for $\mathcal{Q}$ which is then holomorphic on the disk. We will thus consider only the $\alpha = \theta - 1$ case.

From the expressions (1.5.106) and (1.5.107) we deduce that this singular term has the overall $z^{-1}CV$ where C and V are the two complex constants in the expansion (1.5.102) (given below when $\alpha = \theta - 1$) :

$$\Phi_z = \frac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} z^{\theta} + \vec{A}_2 z^{\theta+2} + \left(V z^{\theta+1} + \frac{C}{\theta+1} z\bar{z}^{\theta+1} + \frac{\overline{C}}{2} z^{\theta}\bar{z}^2\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + \xi_z, \tag{2.7.61}$$

where ξ satisfies

$$\nabla^j \xi = O\left(|z|^{\theta+4-j-\nu}\right) \text{ for all } \upsilon > 0 \text{ and } j \leq 3,$$
$$|z|^{\theta}\nabla^4\xi \in L^p(\mathbb{D}) \quad \forall p \in \mathbb{N}.$$

Doing the same variable change as in section 1.5.2, we can assume that $\vec{A}_2$ has no component along $\begin{pmatrix}1\\-i\\0\end{pmatrix}$. Using the conformal equality, namely $\langle \Phi_z, \Phi_z \rangle = 0$, we can give a more accurate version of (2.7.61) :

$$\begin{aligned}\Phi_z = &\frac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} z^{\theta} - \frac{V^2}{2} z^{\theta+2}\begin{pmatrix}1\\i\\0\end{pmatrix}\\ &+ \left(V z^{\theta+1} + W z^{\theta+2} + \frac{C}{\theta+1} z\bar{z}^{\theta+1} + \frac{\overline{C}}{2} z^{\theta}\bar{z}^2\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + \xi_z.\end{aligned} \tag{2.7.62}$$

One can notice that terms of the shape CV appear as the first polynomial terms in both z and $\bar{z}$ when computing $\langle \Phi_z, \Phi_z \rangle$. Since Φ is conformal it would mean that this quantity cancels out. We however need to do an expansion up to $O(r^{\theta+4})$ to make sure no higher order term can compensate it.

With that goal in mind we use the first equation from (1.2.44) and write :

$$\vec{L}_z = -2i\left(H_z\vec{n} + H\Omega e^{-2\lambda}\Phi_{\bar{z}}\right) = -2i\vec{H}_z + 2iH\vec{n}_z - 2iH\Omega e^{-2\lambda}\Phi_{\bar{z}}.$$

Hence, using (1.5.106), (1.5.107) and (2.7.62) :

$$\begin{aligned}\left(\vec{H} + \frac{i}{2}\vec{L}\right)_z &= H^2\Phi_z + 2H\Omega e^{-2\lambda}\Phi_{\bar{z}} \\ &= 2\left(\overline{C}^2 z^{2-\theta} + C^2 z^\theta \bar{z}^{2-2\theta} + 2\,|C|^2\, z\bar{z}^{1-\theta}\right)\begin{pmatrix}1\\-i\\0\end{pmatrix} \\ &\quad + 2\left(\overline{C}z^{1-\theta} + C\bar{z}^{1-\theta}\right)V\begin{pmatrix}1\\i\\0\end{pmatrix} + O\left(r^{3-\theta-\upsilon}\right).\end{aligned} \tag{2.7.63}$$

To make the computations more palatable we have stopped giving the leftover terms names, and used the O formalism. While we cannot traditionnally differentiate these O, all the formalism developed by Y. Bernard and T. Rivière ensures that we can (see section 1.5.2 for examples, or [BR13] for the original). All our leftover estimated terms are then differentiable enough for our purposes.

When $\theta \geq 4$:

Multiplying (2.7.63) by $\bar{z}^\theta$ and applying $\partial_{\bar{z}}$ to the end result yields:

$$\begin{aligned}\Delta\Bigg(\bar{z}^\theta\left[\frac{i}{2}\vec{L} + \vec{H}\right] \\ - 2\bar{z}^\theta\left(\frac{\overline{C}^2}{3-\theta}z^{3-\theta} + \frac{C^2}{1+\theta}z^{\theta+1}\bar{z}^{2-2\theta} + |C|^2 z^2\bar{z}^{1-\theta}\right)e \\ -2\bar{z}^\theta\left(\frac{\overline{C}}{2-\theta}z^{2-\theta} + Cz\bar{z}^{1-\theta}\right)V\bar{e}\Bigg) = O(r^{2-\upsilon}),\end{aligned}$$

where $e = \begin{pmatrix}1\\-i\\0\end{pmatrix}$. Using theorem A.3.9 and integrating once, we deduce that there exists $P \in \mathbb{C}[X]$ such that

$$\begin{aligned}\frac{i}{2}\vec{L} + \vec{H} = P(\bar{z}) + 2\left(\frac{\overline{C}^2}{3-\theta}z^{3-\theta} + \frac{C^2}{1+\theta}z^{\theta+1}\bar{z}^{2-2\theta} + |C|^2 z^2\bar{z}^{1-\theta}\right)\begin{pmatrix}1\\-i\\0\end{pmatrix} \\ + 2\left(\frac{\overline{C}}{2-\theta}z^{2-\theta} + Cz\bar{z}^{1-\theta}\right)V\begin{pmatrix}1\\i\\0\end{pmatrix} + O\left(r^{4-\theta-\upsilon}\right).\end{aligned}$$

Since $\vec{L}$ is real we can take the real part of the previous equality and obtain an expansion of $\vec{H}$. Throwing away the superfluous terms it gives :

$$\begin{aligned}\vec{H} = 4\Re\left(\left(p_1 z^{2-\theta} + q_1\bar{z}^{2-\theta} + \frac{V\overline{C}}{2-\theta}z^{2-\theta} + VCz\bar{z}^{1-\theta}\right)\begin{pmatrix}1\\i\\0\end{pmatrix}\right) \\ + 2\left(\overline{C}z^{1-\theta} + C\bar{z}^{1-\theta} + h_1 z^{2-\theta} + \overline{h_1}\bar{z}^{2-\theta}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} + O\left(r^{3-\theta}\right).\end{aligned} \tag{2.7.64}$$

Here p_1, q_1, h_1 come from the decomposition of P in our working base of $\mathbb{C}^3$.

From (2.7.62) we get the following expansion for the conformal factor :

$$|\Phi_z|^2 = \frac{e^{2\lambda}}{2} = \frac{r^{2\theta}}{2} + |V|^2 r^{2\theta+2} + O\left(r^{2\theta+3-v}\right).$$

Injecting it into (2.7.64) yields :

$$\begin{aligned}\frac{\vec{H}e^{2\lambda}}{2} = 4\Re\Bigg(&\left(p_1 z^2\bar{z}^\theta + q_1 z^\theta \bar{z}^2 + \frac{V\overline{C}}{2-\theta} z^2\bar{z}^\theta + VCz^{\theta+1}\bar{z}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix}\Bigg)\\ &+\left(\overline{C}z\bar{z}^\theta + Cz^\theta\bar{z} + h_1 z^2\bar{z}^\theta + \overline{h_1}z^\theta\bar{z}^2\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O\left(r^{\theta+3}\right).\end{aligned} \tag{2.7.65}$$

When we inject (2.7.65) into the conformal equation

$$\Phi_{z\bar{z}} = \frac{\vec{H}e^{2\lambda}}{2},$$

and apply theorem A.3.9, we end up with an expansion of Φ_z. Namely

$$\Phi_z = A_1\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + A_2\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + A_3\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + O\left(r^{\theta+4}\right), \tag{2.7.66}$$

where

$$\begin{aligned}A_1 &= -\frac{|V|^2}{2}z^{\theta+2} + U_3 z^{\theta+3} + \frac{p_1}{1+\theta}z^2\bar{z}^{\theta+1} + \frac{q_1}{3}z^\theta\bar{z}^3 + \frac{V\overline{C}}{(1+\theta)(2-\theta)}z^2\bar{z}^{\theta+1} + \frac{VC}{2}z^{\theta+1}\bar{z}^2,\\ A_2 &= \frac{z^2}{2} + W_3 z^{\theta+3} + \frac{\overline{p_1}}{3}z^\theta\bar{z}^3 + \frac{\overline{q_1}}{\theta+1}z^2\bar{z}^{\theta+1} + \frac{\overline{V}C}{3(2-\theta)}z^\theta\bar{z}^3 + \frac{\overline{VC}}{\theta+2}z\bar{z}^{\theta+2},\\ A_3 &= Vz^{\theta+1} + Wz^{\theta+2} + \tilde{W}z^{\theta+3} + \frac{C}{2}z^\theta\bar{z}^2 + \frac{\overline{C}}{\theta+1}z\bar{z}^{\theta+1} + \frac{h_1}{\theta+1}z^2\bar{z}^{\theta+1} + \frac{\overline{h_1}}{3}z^\theta\bar{z}^3.\end{aligned}$$

We can then compute $\langle\Phi_z, \Phi_z\rangle$ and find :

$$\begin{aligned}\langle\Phi_z, \Phi_z\rangle &= 2z^{2\theta+3}\left(2U_3 + VW\right) + 2z^{2\theta+1}\bar{z}^2 CV + \frac{2}{3}z^{2\theta}\bar{z}^3\left(\overline{p_1} + \frac{C\overline{V}}{2-\theta}\right)\\ &+ \frac{2}{\theta+1}z^{2+\theta}\bar{z}^{\theta+1}\left(p_1 + \frac{\overline{C}V}{2-\theta}\right) + 2z^{\theta+2}\bar{z}^{\theta+1}\frac{V\overline{C}}{\theta+1} + \frac{2}{\theta+1}z^{\theta+1}\bar{z}^{\theta+2}\overline{C}V + O\left(r^{2\theta+4}\right).\end{aligned} \tag{2.7.67}$$

Since Φ is conformal, $\langle\Phi_z, \Phi_z\rangle = 0$ which implies that $CV = 0$.

When $\theta \leq 3$: The computations are actually very similar, with logarithmic terms added. Since those cannot be compensated by mere power functions, it only adds terms to (2.7.67), and thus the conclusion will remain the same.

The consequences are twofold. First is that $\mathcal{Q}$ is actually holomorphic across the branch point, since the singular term is CVz^{-1}. Second is that either $C = 0$ or $V = 0$. In the first case the second residue satisfies $\alpha \leq \theta - 2$. In the second case one has $\left(\Omega e^{-\lambda}\right)(0) = 0$, and then 0 is an umbilic point. This concludes the proof. ☐

Combining theorem 2.6.3 and theorem 2.7.1, we can state

Theorem 2.7.2. True branched Willmore spheres with at most three branch points of maximal second residues are conformally minimal.

A. Michelat's and T. Rivière's work then allows one to extend Bryant's classification result to some branched Willmore spheres.

Part II

Sequences of Willmore immersions

Table of contents

3 Compactness and bubbling

Abstract.

3.1 Introduction

A major part of this chapter is dedicated to recalling the state of the art concerning the compactness of weak immersions, and more specifically of weak Willmore immersions. Once more, since most of the theorems are already known, we will not detail the proofs but give the underlying ideas.

First subsection 3.2.1 will recall theorem 5.3 of T. Rivière's [Riv16], which offers a concentration-compactness result for weak immersions. In essence, when one manages to find charts of uniformly small energy, the local Harnack controls on the conformal factor obtained by corollary 1.3.2 ensure uniform bounds, and thus weak convergence. Whenever this cannot be done, it reveals a concentration point, around which the convergence is weaker. Whenever the immersions are assumed to be Willmore, the ε-regularity theorem (theorem 1.4.3) gives smooth controls and ensures the smooth convergence toward a smooth Willmore immersion, away from the concentration points. However at those points, the limit immersion may degenerate and become branched, which represents a loss of compactness for Willmore immersions. Studying the behavior of the immersions at the concentration points with a *blow-up* procedure sheds light on bubbling phenomena and the appearance of a tree of Willmore spherical bubbles. Due to the concentration, the Harnack inequality on the conformal factor is lost, and thus branch points might appear on the limit surface. Bubbles are glued to the surface on these concentration points thanks to *neck domains* which crucially will have no energy at the limit : this is the no-neck energy, which in turn yields an energy quantization. Subsection 3.2.2 will then give different versions of those results for weak Willmore immersions, either under hypotheses of compactness for the induced metric (from theorems I.2 and I.3 of [BR14]), or under a control of residues (theorem I.2 in [LR18a]).

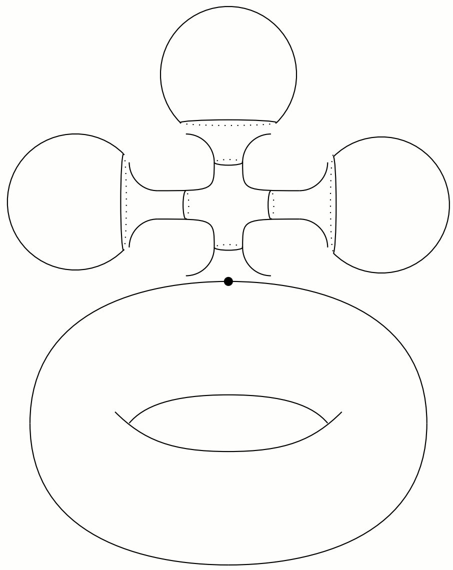

Figure 3.1 – A bubbling configuration : using a Bryant's minimal surface to glue a Clifford torus and 3 spheres.

The goal for what remains of the present work is to show a convergence result across the concentration points when the bubble is minimal, or in other words to eliminate as many bubbling configurations as possible. From the work of P. Laurain and T. Rivière in [LR18a] comes a first restriction on the surfaces involved in bubbling. Both the limit surfaces and the bubbles are *true* Willmore surfaces, which eliminates all possibility of catenoïd-like bubbles. From this they deduced a low energy compactness result. Similarly we will show a highly constraining result, linking the branched behavior of the bubbles and the surfaces around their linking points.

Theorem D. A Willmore bubble with a branched end of multiplicity $\theta+1$ at infinity can only appear on a branch point of multiplicity $\theta+1$.
A Willmore bubble with a branch point of multiplicity $\theta-1$ at infinity can only appear on a branched end of multiplicity $\theta-1$.

In essence this result is a consequence of the no-neck energy. The branched order of the bubble or the surface can be seen as a winding number of the immersion. Since the neck has no energy, it can neither wind up or unwind the immersion, and can only transmit the branched behavior from one to the other.

We will conclude this chapter by detailing, to our knowledge, the first explicit example of Willmore bubbling. It will consist in fusing three of the four ends of a Bryant's minimal surface. This can be done if we carefully respect a kind of equilibrium formula, and yields a López surface. Considering the process on the inversed surfaces, we fuse three sheets at their intersection to obtain, at the limit, a branch point of multiplicity 3. The appearance of such a branch point is symptomatic of concentration phenomena. Indeed, on this branch point, an Enneper bubble is glued on the concentration point in accordance with theorem D. With this example, we not only prove that high energy Willmore immersions may degenerate into branched Willmore immersions, and are thus not compact, we offer the first explicit instance of Willmore bubbling when its mere possibility seemed dubious at the beginning of my doctoral studies. Further section 4.4.4 will offer insight on why this example stems from the lack of compactness and commutativity of the conformal group.

Theorem E. There exists $\Phi_k : \mathbb{S}^2 \to \mathbb{R}^3$ a sequence of Willmore immersions such that

$$W(\Phi_k) = 16\pi,$$

and

$$\Phi_k \to \Phi_\infty,$$

smoothly on $\mathbb{S}^2 \backslash \{0\}$, where Φ_∞ is the inversion of a Lopez surface. Further

$$\lim_{k\to\infty} E(\Phi_k) = E(\Phi_\infty) + E(\Psi_\infty),$$

where $\Psi_\infty : \mathbb{C} \to \mathbb{R}^3$ is the immersion of an Enneper surface.

The goal is then to find where the compactness threshold lies. In the torus case, P. Laurain and T. Rivière introduced an a priori possible bubbling configuration, consisting in an Enneper bubble, glued on the branch point of an inverted Chen-Gackstatter torus. Eliminating this will be the goal of the final chapter of the present work.

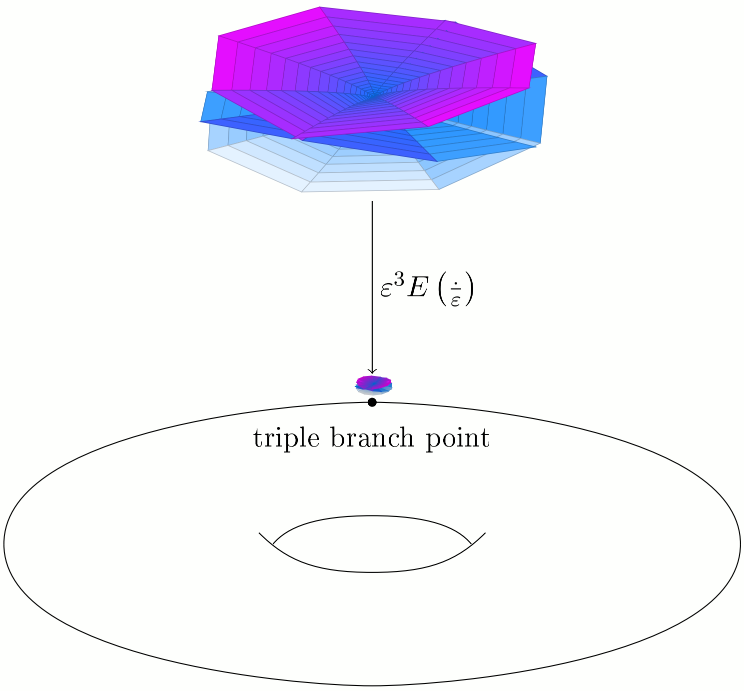

Figure 3.2 – Desingularizing the inversion of a Chen-Gackstatter surface with a piece of Enneper.

3.2 Compactness results

3.2.1 Compactness for weak immersions : the concentration-compactness dialectic

In this subsection we will briefly go over the main ideas of section 5 of [Riv16], dealing with the compactness of sequences of weak immersions of bounded energy E, and the introduction of concentration-compactness phenomena.

Since the energy E is a conformal invariant, any sequence of bounded E can only converge up to applying external conformal diffeomorphisms. Indeed, considering Φ an immersion of a Riemann surface remaining in a compact of $\mathbb{R}^3$, the sequences $\Phi_k = k\Phi$ $\tilde{\Phi}_k = \Phi + k\vec{a}$ respectively blow and drift to infinity, and thus cannot be expected to converge. In fact one needs to apply conformal diffeomorphisms to these examples (a dilation in the first case and a translation in the second) to ensure convergence. In that way any convergence result will have to take into consideration the lack of compactness of $\mathrm{Conf}(\mathbb{R}^3)$.

In a similar manner, taking Φ a likewise bounded immersion of a Riemann surface, and Ψ_k a non compact sequence of diffeomorphisms of Σ, $\Phi_k = \Phi \circ \Psi_k$ cannot be expected to converge as immersions. One must then compensate the possible loss of compactness of the parametrization, even though the image surface converges. To that aim, if the induced conformal class of the metrics is assumed to be in a compact subset of the moduli space $\mathcal{M}_\Sigma$, up to extraction the conformal classes converge. Then, up to applying Riemann's uniformization theorem the induced metric themselves converge which allows one to deal with this loss of compactness.

Taking these considerations into account we define a notion of *weak convergence*.

Definition 3.2.1. Let $\Phi_k \in \mathcal{E}_\Sigma$. Let g_k be the induced metric, and h_k the uniformized metric of constant scalar curvature. We assume that the induced conformal classes $[(\Sigma, h_k)]$ are contained in a compact subset of $\mathcal{M}_\Sigma$, the moduli space of Σ. Then up to extraction there exists a constant scalar curvature metric h_∞ such that $h_k \to h_\infty$. The sequence Φ_k is called *weakly convergent* if there exist Lipschitz diffeomorphisms Ψ_k of Σ, conformal transformations $\Theta_k \in \mathrm{Conf}(\mathbb{R}^3)$ with

$$\Phi_k(\Sigma) \cap \{\text{center of inversion of } \Theta_k\} = \emptyset$$

and finitely many points $a_1, \dots, a_N \in \Sigma$, called concentration points such that

$$\xi_k := \Theta_k \circ \Phi_k \circ \Psi_k \,:\, (\Sigma, h_k) \to \mathbb{R}^3$$

is conformal, and there exists a map $\xi_\infty \,:\, \Sigma \to \mathbb{R}^3$ such that

- $\xi_\infty \,:\, (\Sigma, h_\infty) \to \mathbb{R}^3$ is conformal,
- $\xi_k \rightharpoonup \xi_\infty$ weakly in $W^{2,2}_{\mathrm{loc}}\left(\Sigma \backslash \{a_1, \dots, a_N\}\right)$,
- $\ln |d\xi_k|^2 \rightharpoonup \ln |d\xi_\infty|^2$ weakly in $(L^\infty)^*_{\mathrm{loc}}\left(\Sigma \backslash \{a_1, \dots, a_N\}\right)$
- $\xi_k \rightharpoonup \xi_\infty$ weakly in $W^{1,2} \cap (L^\infty)^*(\Sigma)$.

We can then state the weak almost-closure theorem (theorem 5.3 of T. Rivière's [Riv16]).

Theorem 3.2.1. Let $\Phi_k \in \mathcal{E}_\Sigma$ such that

$$\sup_k \int_\Sigma |\nabla \vec{n}|^2 < \infty.$$

Let g_k be the induced metric, and h_k the uniformized metric of constant scalar curvature. We assume that the induced conformal classes $[(\Sigma, h_k)]$ are contained in a compact subset of $\mathcal{M}_\Sigma$, the moduli space of Σ. Then, up to extraction, Φ_k converges in the sense of definition 3.2.1.

Proof. Since the proof is extensively detailed in section 5 of [Riv16], we will only present its outline, which will be sufficient for understanding the involved ideas.

The diffeomorphisms Ψ_k are the conformal transformations given by the uniformization theorem. The convergence of the metrics then ensures the convergence of the harmonic atlas, and allows us to work in local conformal charts. Further considering any point $x \in \Sigma$, the question is whether one can find a disk centered on x such that the energy of $\nabla \vec{n}_k$ is *uniformly* low, in fact lower than the ε_0 in theorem 1.3.3. The points around which it is not possible are the concentration points. Since around each one a precise quantum of energy concentrates, and since, by hypothesis, $\int_\Sigma \left|\nabla \vec{n}^k\right|^2 < \infty$, they are in finite number. Away from these points one can apply theorem 1.3.3 and control the conformal factor up to a constant, which is in turn managed using conformal diffeomorphisms Θ_k of $\mathbb{R}^3 \cup \{\infty\}$. The sequence Φ_k is then uniformly bounded away from the concentration points, which yields the convergence in the sense of definition 3.2.1 thanks to classical Riesz compactness theorems. $\square$

In the specific case of sequences of weak Willmore immersions, the low energy condition of theorem 1.3.3 combines fairly well with the ε-regularity condition of theorem 1.4.3, as we will see in the following.

3.2.2 Energy quantization results for sequences of Willmore immersions

In the context of Willmore immersions, theorem 1.4.3 will ensure the smoothness of the convergence away from the concentration points, while bubbling extraction procedures (for two slightly different examples see [BR14] or [LR18a]) allows one to extract bubble trees of possibly branched, possibly non compact Willmore spheres. A key result is the *energy quantization*, which ensures that no energy is lost in the necks. Following is a combination of theorem I.2 and I.3 in [BR14]. Once more, we will not detail the proof, but give some of the overall ideas.

Theorem 3.2.2. Let Φ_k be a sequence of Willmore immersions of a closed surface Σ. Assume that

$$\limsup_{k\to\infty} W(\Phi_k) < \infty,$$

and that the conformal class of $\Phi_k^*\xi$ remains within a compact subdomain of the moduli space of Σ. Then modulo extraction of a subsequence, the following energy identity holds

$$\lim_{k\to\infty} W(\Phi_k) = W(\Phi_\infty) + \sum_{s=1}^{p} W(\eta_s) + \sum_{t=1}^{q} \left[W(\zeta_t) - 4\pi\theta_t\right],$$

where Φ_∞ (respectively η_s, ζ_t) is a possibly branched smooth immersion of Σ (respectively $\mathbb{S}^2$) and $\theta_t \in \mathbb{N}$. Further there exists $a^1 \dots a^n \in \Sigma$ such that

$$\Phi_k \to \Phi_\infty \text{ in } C^\infty_{\text{loc}}\left(\Sigma\backslash\{a^1, \dots, a^n\}\right)$$

up to conformal diffeomorphisms of $\mathbb{R}^3 \cup \{\infty\}$. Moreover there exists a sequence of radii ρ_k^s, points $x_k^s \in \mathbb{C}$ converging to one of the a^i such that up to conformal diffeomorphisms of $\mathbb{R}^3$

$$\Phi_k\left(\rho_k^s y + x_k^s\right) \to \eta_s \circ \pi^{-1}(y) \text{ in } C^\infty_{\text{loc}}\left(\mathbb{C}\backslash\{\text{finite set}\}\right).$$

Finally there exists a sequence of radii ρ_k^t, points $x_k^t \in \mathbb{C}$ converging to one of the a^i such that up to conformal diffeomorphisms of $\mathbb{R}^3$

$$\Phi_k\left(\rho_k^t y + x_k^t\right) \to \iota_{p_t} \circ \zeta_t \circ \pi^{-1}(y) \text{ in } C^\infty_{\text{loc}}\left(\mathbb{C}\backslash\{\text{finite set}\}\right).$$

Here ι_{p_t} is an inversion at $p \in \zeta_t(\mathbb{S}^2)$. The integer θ_t is the density of ζ_t at p_t.

While theorem 3.2.2 states an energy quantization for W, equality VIII.8 in [BR14] offers in fact a stronger energy quantization for E (and one for $\mathcal{E}$ follows). The a^i are the aforementioned concentration points and the η_s and $\iota_{p_t} \circ \zeta_t$ are the bubbles blown on those concentration points. More precisely, the η_s are the compact bubbles, while the $\iota_{p_t} \circ \zeta_t$ are the non compact ones. Non-compact bubbles stand out as a consequence of the conformal invariance of the problem (see [Lau12a] to compare with the bubble tree extraction in the constant mean curvature framework). One might notice that $W(\iota_{p_t} \circ \zeta_t) = W(\zeta_t) - 4\pi\theta_t$, and deduce that if $W(\zeta_t) = 4\pi\theta_t$, then the bubble $\iota_{p_t} \circ \zeta_t$ is minimal. This case, which we will refer to as *minimal bubbling* will be of special interest to us in this memoir. Further if there is only one bubble at a given concentration point we will call the bubbling *simple*. For simplicity's sake we will mostly consider simple bubbling.

The non-degeneracy hypothesis from theorem 3.2.2 makes sure that no compactness is lost on the source domain, as explained in the previous subsection. It can be left out, but at the price of another condition on the convergence of residues which ensure that the neck domain do not degenerate too violently, as explained in theorem 1.1 in [LR18a]. We quote it for completeness' sake.

Theorem 3.2.3. Let (Σ, h_k) be a sequence of closed surfaces with fixed genus, constant curvature and normalized volume if needed. We assume that this sequence converges to a nodal surface $(\tilde{\Sigma}, \tilde{h})$ and we denote $\left(c^i_j\right)$ the finite number of pinching geodesics of length l^i_k. Then let $\Phi_k : (\Sigma, h_k) \to \mathbb{R}^3$ be a sequence of conformal Willmore immersions with bounded energy :

$$\limsup_{k\to\infty} W(\Phi_k) < \infty$$

such that around every degenerating geodesic

$$\lim_{k\to\infty} \frac{\vec{\gamma_0}^k}{\sqrt{l_k}} = 0.$$

Then the conclusion of theorem 3.2.2 stands.

Since the proofs are both quite long and quite technical, we will not give them here and instead refer the reader to the original papers (theorems I.2 and I.3 in [BR14], and theorem 1.1 in [LR18a] respectively). We however wish to explain one key point : the control of the conformal factor.

Broadly speaking, the proof is in two parts. First is an extraction scheme designed to find all the bubbles and to decompose Σ into a limit surface, bubbles and neck domains (essentially annuli of degenerating conformal classes). Then comes the no-neck energy. The aim is to exploit the Willmore equations in order to have vanishing controls of the L^2 norm of $\nabla\vec{n}$ in the neck domains. The first step is controlling the conformal factor. It is clear that since the energy concentrates, theorem 1.3.3 and thus estimate (1.3.72) are no longer valid. The absence of (1.3.72) is in fact precisely what makes the developping of branch points possible. However, the extraction schemes presented in [BR14] and [LR18a] naturally yields $\left\|\nabla\vec{n}^k\right\|_{L^{2,\infty}} \to 0$. Modifying the proof of theorem 1.3.3 to work on annuli of degenerating conformal classes with an input small control of $L^{2,\infty}$ is one of the major advances in [BR14]. To that aim they introduced $L^{2,\infty}$ Coulomb frames in neck domains. Resulting is the following lemma.

Lemma 3.2.1. There exists a constant $\eta > 0$ with the following property. Let $0 < 4r < R < \infty$. If Φ is any (weak) conformal immersion of $\Omega := \mathbb{D}_R \backslash \mathbb{D}_r$ into $\mathbb{R}^3$ with L^2-bounded second fundamental form and satisfying

$$\|\nabla\vec{n}\|_{L^{2,\infty}(\Omega)} < \sqrt{\eta},$$

then there exist $\frac{1}{2} < \alpha < 1$ and $A \in \mathbb{R}$ depending on R, r, m and Φ such that

$$\|\lambda(x) - d\log|x| - A\|_{L^\infty\left(\mathbb{D}_{\alpha R}\setminus\mathbb{D}_{\frac{r}{\alpha}}\right)} \leq C\left(\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2\, dz^2\right), \tag{3.2.1}$$

where d satisfies

$$\begin{aligned}\left|2\pi d - \int_{\partial\mathbb{D}_r} \partial_r\lambda dl\right| \leq & C\left[\int_{\mathbb{D}_{2r}\setminus\mathbb{D}_r} |\nabla\vec{n}|^2\, dz^2\right.\\ & \left.+\frac{1}{\log\frac{R}{r}}\left(\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2\, dz^2\right)\right].\end{aligned} \tag{3.2.2}$$

Here the classical Harnack control around a constant given by (1.3.72) is replaced by a control around a power function. Indeed compare (1.3.72)

$$\frac{e^\Lambda}{C} \leq e^\lambda \leq Ce^\Lambda,$$

with this direct consequence of lemma 3.2.1 :

$$\frac{e^A r^d}{C} \leq e^\lambda \leq Ce^A r^d.$$

This lemma is, in the author's opinion, the centerpiece in the energy quantization. While the proof is far from over, one can now work with $\vec{L}$, S and $\vec{R}$ in a similar way as what has been done for the ε-regularity, but on annuli. One must also notice that to prove theorem 3.2.3 it is necessary to control the residues, which is an added difficulty to the proof. However, for brevity's sake, we will stop our look at the idea behind the proof there.

Remark 3.2.1. As has been noticed by P. Laurain and T. Rivière in [LR18a], when in the context of [BR14], while the immersions may degenerate around concentration points into branched immersions, the residues $\vec{\gamma}_0, \dots, \vec{\gamma}_3$ around the branch points are obtained as limit of residues defined on disks (and not punctured disks) $\vec{\gamma_0}^k, \dots, \vec{\gamma_3}^k$, and thus are necessarily null. Consequently all the surfaces involved in bubbling phenomena are *true* Willmore surfaces.

This excludes some bubbling configurations. For instance no catenoïd bubble can appear. In fact enough configurations have been eliminated for a compactness result for small energies (theorem I.2 in [LR18a]).

Theorem 3.2.4. Let Σ be a closed surface of genus $g \geq 1$, and let $\Phi_k : \Sigma \to \mathbb{R}^3$ be a sequence of conformal Willmore immersions such that the conformal class of the induced metric remains in a compact set of the moduli space and :

$$\limsup_{k\to\infty} W(\Phi_k) < 12\pi.$$

Then up to extraction and conformal diffeomorphisms of $\mathbb{R}^3 \cup \{\infty\}$ and Σ, Φ_k converges to a smooth Willmore immersion $\Phi_\infty : \Sigma \to \mathbb{R}^3$ in $C^\infty(\Sigma)$.

In their aforementioned paper, P. Laurain and T. Rivière also put forth a bubbling configuration of energy 12π where compactness might fail, namely an inverted Chen-Gackstatter torus whose branch point is desingularized by an Enneper surface. Finding additional constraints which one may exploit to eliminate this configuration is the prime motivator for what is to follow.

As we have throughout this thesis, we will favor working in local conformal charts (and in complex notations when convenient). We then present a standardized working version of simple bubbling on a disk.

Lemma 3.2.2. Let ξ_k be a sequence of Willmore immersions of a closed surface Σ satisfying the hypotheses of theorem 3.2.2. Then, in proper conformal charts around a concentration point on which a *simple* bubble is blown up ξ^k yields a sequence of Willmore conformal immersions $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$, of conformal factor $\lambda^\varepsilon = \frac{1}{2}\ln\left(\frac{|\nabla \Phi^\varepsilon|^2}{2}\right)$, Gauss map $\vec{n}^\varepsilon$, mean curvature H^ε and tracefree curvature $\Omega^\varepsilon := 2\langle \Phi^\varepsilon_{zz}, \vec{n}^\varepsilon\rangle$, satisfying the following set of hypotheses :

1. There exists $C_0 > 0$ such that
$$\|\Phi^\varepsilon\|_{L^\infty(\mathbb{D})} + \|\nabla \Phi^\varepsilon\|_{L^2(\mathbb{D})} + \|\nabla \lambda^\varepsilon\|_{L^{2,\infty}(\mathbb{D})} + \|\nabla \vec{n}^\varepsilon\|_{L^2(\mathbb{D})} \leq C_0.$$
2. $\Phi^\varepsilon \to \Phi^0$ $C^\infty_{\text{loc}}(\mathbb{D}\backslash\{0\})$, where Φ^0 is a true branched Willmore conformal immersion, with a unique branch point of multiplicity $\theta_0 + 1$ at 0, meaning that
$$\Phi^0_z \sim_0 \vec{A} z^{\theta_0}. \tag{3.2.3}$$
We denote λ^0 its conformal factor, $\vec{n}^0$ its Gauss map, H^0 its mean curvature and Ω^0 its tracefree curvature.
3. There exists a sequence of real numbers $C^\varepsilon > 0$ such that
$$\widetilde{\Phi}^\varepsilon := \frac{\Phi^\varepsilon(\varepsilon .) - \Phi^\varepsilon(0)}{C^\varepsilon} \to \Phi^1_z$$
$C^\infty_{\text{loc}}(\mathbb{C})$, where Φ^1 is assumed to be a conformal Willmore immersion of $\mathbb{C}$, possibly non compact, with a branched behavior at infinity : meaning that there exists $\theta_1 \in \mathbb{Z}\backslash\{-1\}$.
$$\Phi^1_z \sim_\infty \widetilde{A} z^{\theta_1}.$$
We denote λ^1 its conformal factor, $\vec{n}^1$ its Gauss map, H^1 its mean curvature and Ω^1 its tracefree curvature.
4. $$\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\frac{1}{R}}\backslash \mathbb{D}_{\varepsilon R}} |\nabla \vec{n}^\varepsilon|^2\, dz\right) = 0.$$

Proof. Such assumptions are natural if we consider ξ^k satisfying the hypotheses of theorem 3.2.2. Thanks to theorems I.2 and I.3 of [BR14] such a sequence ξ^k converges smoothly away from concentration points. In a conformal chart centered on a concentration point, ξ^k yields a sequence of conformal, weak Willmore immersions $\Phi^k : \mathbb{D} \to \mathbb{R}^3$ converging smoothly away from the origin toward a true Willmore surface (i.e. hypothesis 2). Hypothesis 1 stands if we choose proper conformal charts (see theorem 3.1 of P. Laurain and T. Rivière's [LR18b], recalled here in theorem 1.3.2, for a detailed explanation). Hypothesis 3 then specifies that we consider the case where there is only one simple bubble which concentrates on 0 in the aforementioned chart. We do not presume on the behavior of Φ^1 at infinity. It may have a branched end (if $\theta_1 > -1$) in which case it is of multiplicity $\theta + 1$, or a branch point (if $\theta < -1$) in which case it is of multiplicity $-\theta - 1$. Since $\theta = -1$ would induce some residues, remark 3.2.1 excludes this case. Hypothesis 4 is just the energy quantization once the whole bubble tree is extracted and corresponds to inequality VIII.8 in [BR14]. Further, by definition of a concentration point
$$\left\|\nabla \vec{n}^k\right\|_{L^\infty(\mathbb{D})} \to \infty.$$

We then define the concentration speed as

$$\varepsilon_k = \frac{1}{\|\nabla \vec{n}^k\|_{L^\infty(\mathbb{D})}} \to 0,$$

and we assume it is reached at the origin. For simplicity's sake we reparametrize this sequence by the concentration speed which we denote ε. □

3.2.3 Branch point-branched end correspondance, proof of theorem D

The goal of this subsection is to show that in lemma 3.2.2, Φ^1 has an end at infinity, and that is multiplicity equals the multiplicity of the branch point of the surface Φ^0.

Lemma 3.2.3. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ satisfying 1-4. Then $\theta_0 = \theta_1 = \theta$.

Proof. Since Φ^ε is conformal, the Liouville equation states

$$\Delta \lambda^\varepsilon = K^\varepsilon e^{2\lambda^\varepsilon}, \tag{3.2.4}$$

where K^ε is the Gauss curvature of Φ^ε. Then given $R \in \mathbb{R}_+$

$$\begin{aligned}
\int_{\mathbb{D}_{\frac{1}{R}} \setminus \mathbb{D}_{\varepsilon R}} K^\varepsilon e^{2\lambda^\varepsilon} dz &= \int_{\mathbb{D}_{\frac{1}{R}} \setminus \mathbb{D}_{\varepsilon R}} \Delta \lambda^\varepsilon dz \\
&= \int_{\partial \mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^\varepsilon d\sigma - \int_{\partial \mathbb{D}_{\varepsilon R}} \partial_r \lambda^\varepsilon d\sigma \\
&= \int_{\partial \mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^\varepsilon d\sigma - \int_{\partial \mathbb{D}_{R}} \varepsilon \partial_r \lambda^\varepsilon (\varepsilon .) d\sigma \\
&= \int_{\partial \mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^\varepsilon d\sigma - \int_{\partial \mathbb{D}_{R}} \partial_r \left[\lambda^\varepsilon (\varepsilon .) \right] d\sigma \\
&= \int_{\partial \mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^\varepsilon d\sigma - \int_{\partial \mathbb{D}_{R}} \partial_r \widetilde{\lambda}^\varepsilon d\sigma.
\end{aligned} \tag{3.2.5}$$

Besides, hypotheses 2 and 3 ensure that $\lambda^\varepsilon \to \lambda^0$ on $\partial \mathbb{D}_{\frac{1}{R}}$ and $\widetilde{\lambda}^\varepsilon \to \lambda^1$ on $\partial \mathbb{D}_R$. Further, since Φ^0 has a branch point of multiplicity $\theta_0 + 1$ at 0,

$$\lim_{R \to \infty} \int_{\mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^0 d\sigma \to 2\pi \theta_0. \tag{3.2.6}$$

Similarly, the behavior of Φ^1 at infinity (without assuming that it is an end or a branch point) implies :

$$\lim_{R \to \infty} \int_{\mathbb{D}_R} \partial \lambda^1 d\sigma \to 2\pi \theta_1. \tag{3.2.7}$$

Injecting (3.2.6) and (3.2.7) in (3.2.5) yields

$$\begin{aligned}
2\pi \left| \theta_0 - \theta_1 \right| &\leq \lim_{R \to \infty} \left(\lim_{\varepsilon \to 0} \left| \int_{\partial \mathbb{D}_{\frac{1}{R}}} \partial_r \lambda^\varepsilon d\sigma - \int_{\partial \mathbb{D}_R} \partial_r \widetilde{\lambda}^\varepsilon d\sigma \right| \right) \\
&\leq \lim_{R \to \infty} \left(\lim_{\varepsilon \to 0} \left| \int_{\mathbb{D}_{\frac{1}{R}} \setminus \mathbb{D}_{\varepsilon R}} K^\varepsilon e^{2\lambda^\varepsilon} dz \right| \right) \leq \lim_{R \to \infty} \left(\lim_{\varepsilon \to 0} \int_{\mathbb{D}_{\frac{1}{R}} \setminus \mathbb{D}_{\varepsilon R}} \left| K^\varepsilon e^{2\lambda^\varepsilon} \right| dz \right) \\
&\leq \lim_{R \to \infty} \left(\lim_{\varepsilon \to 0} \int_{\mathbb{D}_{\frac{1}{R}} \setminus \mathbb{D}_{\varepsilon R}} \left| \nabla \vec{n}^\varepsilon \right|^2 dz \right) = 0,
\end{aligned}$$

using hypothesis 1. As a conclusion $\theta_0 = \theta_1 = \theta$. $\square$

While we wrote the proof in the simple bubbling case, it remains valid for any behavior of Φ^0 and Φ^1 (branched points or ends) and relies solely on the energy quantization result. In a broader frame it yields a construction rule for Willmore bubble trees.

Theorem D. A Willmore bubble with a branched end of multiplicity $\theta+1$ at infinity can only appear on a branch point of multiplicity $\theta+1$.
A Willmore bubble with a branch point of multiplicity $\theta-1$ at infinity can only appear on a branched end of multiplicity $\theta-1$.

3.3 An explicit example of Willmore bubbling : non compactness above 16π

Before considering genus one sequences in order to remove the obstruction to compactness put forth by P. Laurain and T. Rivière, a study of the spherical case offers interesting perspectives. Indeed in his seminal work [Bry84], partly recalled in section 2.6, R. Bryant offered a classification of Willmore immersions of a sphere in $\mathbb{R}^3$, showed they were conformal transformations of minimal immersions (see theorem 2.6.1), and thus that their Willmore energy was 4π-quantized. Moreover while giving a complete description of the Willmore immersions of energy 16π (part 5), R. Bryant remarked :

"Surprisingly, this space [of Willmore immersions of energy 16π] is *not* compact."

It is then interesting to consider whether one can degenerate a sequence of 16π immersions into a bubble blown on a Willmore sphere. A quick study direct our search toward the most likely case : a sequence of four ended Bryant's surfaces (see example 1.2.2) degenerating into an Enneper immersion glued on the branch point of the inverse of a López minimal surface.

Theorem E. There exists $\Phi_k : \mathbb{S}^2 \to \mathbb{R}^3$ a sequence of Willmore immersions such that

$$W(\Phi_k) = 16\pi,$$

and

$$\Phi_k \to \Phi_\infty,$$

smoothly on $\mathbb{S}^2\backslash\{0\}$, where Φ_∞ is the inversion of a López surface. Further

$$\lim_{k\to\infty} E(\Phi_k) = E(\Phi_\infty) + E(\Psi_\infty),$$

where $\Psi_\infty : \mathbb{C} \to \mathbb{R}^3$ is the immersion of an Enneper surface.

Theorem E proves that minimal bubbles can appear and thus that Willmore immersions *are not compact*. It might also indicate the possibility of gluing an Enneper bubble on an inverted Chen-Gackstatter torus. However R. Bryant's classification result proves that one cannot glue an Enneper bubble on an inverted Enneper surface (the resulting surface would be of energy 12π, and thus limit of Willmore immersions of equal energy, which R. Bryant showed did not exist). The local behavior of the limit surface around its branch point needs then to be constrained in order to forbid this case. Since the Chen-Gackstatter torus and the Enneper surface are asymptotic near their branched end there is hope yet to eliminate this configuration.

Proof. We will build a sequence of Willmore immersions whose energy E concentrates on a point where an Enneper bubble blows up. Working from section 5 of R. Bryant's [Bry84], we study a family of four ended minimal immersions $\Psi_\mu : \mathbb{C}\backslash\{a_1, a_2, a_2\} \to \mathbb{R}^3$:

$$\begin{aligned} \Psi_\mu &= 2\Re\left(f_\mu\right) \\ f_\mu &= \frac{a_1}{z-\mu} + \frac{a_2}{z-\mu j} + \frac{a_3}{z-\mu j^2} + a_4 z, \end{aligned} \tag{3.3.8}$$

with a_1, a_2, a_3, $a_4 \in \mathbb{C}^3$, $j^3 = 1$, and μ a real parameter that will go toward 0. As explained in [Bry84] the (a_i) must be constrained for Ψ_μ to be a conformal immersion. Indeed :

$$\begin{aligned} \left\langle (\Psi_\mu)_z , (\Psi_\mu)_z \right\rangle &= \left\langle (f_\mu)_z , (f_\mu)_z \right\rangle \\ &= \frac{\langle a_1, a_1\rangle}{(z-\mu)^4} + \frac{\langle a_2, a_2\rangle}{(z-\mu j)^4} + \frac{\langle a_3, a_3\rangle}{(z-\mu j^2)^4} + \langle a_4, a_4\rangle \\ &+ \frac{2\langle a_1, a_2\rangle}{(z-\mu)^2 (z-\mu j)^2} + \frac{2\langle a_1, a_3\rangle}{(z-\mu)^2(z-\mu j^2)^2} + \frac{2\langle a_2, a_3\rangle}{(z-\mu j)^2(z-\mu j^2)^2} \\ &- \frac{2\langle a_1, a_4\rangle}{(z-\mu)^2} - \frac{2\langle a_2, a_4\rangle}{(z-\mu j)^2} - \frac{2\langle a_3, a_4\rangle}{(z-\mu j^2)^2}. \end{aligned}$$

Further since given $u, v \in \mathbb{C}$:

$$\frac{1}{(z-u)^2(z-v)^2} = \frac{1}{(u-v)^2}\frac{1}{(z-u)^2} + \frac{1}{(u-v)^2}\frac{1}{(z-v)^2} - \frac{2}{(u-v)^3}\frac{1}{z-u} + \frac{2}{(u-v)^3}\frac{1}{z-v},$$

we deduce that $\left\langle (\Psi_\mu)_z , (\Psi_\mu)_z \right\rangle = 0$ if and only if

$$\begin{aligned} &\langle a_1, a_1\rangle = \langle a_2, a_2\rangle = \langle a_3, a_3\rangle = \langle a_4, a_4\rangle = 0, \\ &\langle a_1, a_2\rangle = \langle a_1, a_3\rangle = \langle a_2, a_3\rangle, \\ &\text{and } a_4 = -\frac{1}{3\mu^2}\left(a_1 + j a_2 + j^2 a_3\right). \end{aligned} \tag{3.3.9}$$

One can check that under the conditions (3.3.9), (a_1, a_2, a_3) is a linearly independant family of $\mathbb{C}^3$ and thus that Ψ_μ is an immersion.

Here we take, with $b \in \mathbb{C}$ a parameter to be adjusted later,

$$\begin{aligned} a_1 &= \frac{1}{2\mu^2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}, \\ a_2 &= \frac{j}{2\mu^2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{\mu^2 b^2}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + b j^2\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}, \\ a_3 &= \frac{j^2}{2\mu^2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{\mu^2 b^2}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - b j\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \end{aligned}$$

One can check that these (a_i) satisfy (3.3.9). Computing, we find :

$$f_\mu = \frac{3}{z^3-\mu^3}\frac{1}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - b^2\left(\mu^2\frac{2z+\mu}{z^2+\mu z+\mu^2} + \frac{z}{3}\right)\frac{1}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + bj(j-1)\frac{z+\mu}{z^2+\mu z+\mu^2}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

To simplify this expression, we set $b = \frac{3a}{2j(j-1)}$ with $a \in \mathbb{C}$ to be fixed at the end of the reasoning, and reach :

$$f_\mu = \frac{3}{z^3-\mu^3}\frac{1}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{a^2}{4}\left(3\mu^2\frac{2z+\mu}{z^2+\mu z+\mu^2}+z\right)\frac{1}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + \frac{3a}{2}\frac{z+\mu}{z^2+\mu z+\mu^2}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{3.3.10}$$

Then $\Psi_\mu : \mathbb{S}^2 \to \mathbb{R}^3$ is a sequence of minimal immersions with four simple planar ends. Applying theorem 1.2.5 and proposition 1.2.4 we find :

$$\int_{\mathbb{S}^2} K_{\Psi_\mu} d\mathrm{vol}_{g_{\Psi_\mu}} = -12\pi, \tag{3.3.11}$$

$$\int_{\mathbb{S}^2} \left|\mathring{A}\right|^2_{\Psi_\mu} d\mathrm{vol}_{g_{\Psi_\mu}} = 24\pi. \tag{3.3.12}$$

Letting $\mu \to 0$ in (3.3.10) we find that, away from 0,

$$f_\mu \to f_0 = \frac{3}{2z^3}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{a^2 z}{8}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + \frac{3a}{2z}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

and deduce that $\Psi_\mu \to \Psi_0 := 2\Re(f_0)$ smoothly away from 0, where Ψ_0 is a branched minimal immersion of the sphere with one simple planar end and one planar end of multiplicity 3. This immersion is in fact the López minimal surface mentioned in theorem E, and described in example 1.2.3. Then

$$\int_{\mathbb{S}^2} K_{\Psi_0} d\mathrm{vol}_{g_{\Psi_0}} = -8\pi, \tag{3.3.13}$$

$$\int_{\mathbb{S}^2} \left|\mathring{A}\right|^2_{\Psi_0} d\mathrm{vol}_{g_{\Psi_0}} = 16\pi. \tag{3.3.14}$$

Let p be a point in $\mathbb{R}^3$ such that $d(p, \Psi_\mu) > 1$. We now introduced $\Phi_\mu := \iota_p \circ \Psi_\mu$ and $\Phi_0 := \iota_p \circ \Psi_0$, with $\iota(x) = \frac{x-p}{|x-p|^2}$ the inversion in $\mathbb{R}^3$ centered at p. Then Φ_μ is a sequence of closed Willmore conformal immersions of the sphere converging toward Φ_0 smoothly away from 0, and Φ_0 is a closed Willmore conformal branched immersion of the sphere with a single branch point of multiplicity 3 at 0. Thus

$$\int_{\mathbb{S}^2} K_{\Phi_\mu} d\mathrm{vol}_{g_{\Phi_\mu}} = 4\pi, \tag{3.3.15}$$

$$\int_{\mathbb{S}^2} K_{\Phi_0} d\mathrm{vol}_{g_{\Phi_0}} = 8\pi. \tag{3.3.16}$$

Since $\left|\mathring{A}\right|^2 d\mathrm{vol}_g$ is a conformal invariant, we deduce from (3.3.12) and (3.3.14) :

$$\int_{\mathbb{S}^2} \left|\mathring{A}\right|^2_{\Phi_\mu} d\mathrm{vol}_{g_{\Phi_\mu}} = 24\pi, \tag{3.3.17}$$

$$\int_{\mathbb{S}^2} \left|\mathring{A}\right|^2_{\Phi_0} d\mathrm{vol}_{g_{\Phi_0}} = 16\pi. \tag{3.3.18}$$

With proposition 1.2.4 we conclude with (3.3.15) and (3.3.17) :

$$\int_{\mathbb{S}^2} H^2_{\Phi_\mu} d\mathrm{vol}_{g_{\Phi_\mu}} = \frac{1}{2}\int_{\mathbb{S}^2} |\mathring{A}|^2_{\Phi_\mu} d\mathrm{vol}_{g_{\Phi_\mu}} + \int_{\mathbb{S}^2} K_{\Phi_\mu} d\mathrm{vol}_{g_{\Phi_\mu}} = 16\pi, \tag{3.3.19}$$

and with (3.3.16) and (3.3.18) :

$$\int_{\mathbb{S}^2} H^2_{\Phi_0} d\mathrm{vol}_{g_{\Phi_0}} = \frac{1}{2}\int_{\mathbb{S}^2} |\mathring{A}|^2_{\Phi_0} d\mathrm{vol}_{g_{\Phi_0}} + \int_{\mathbb{S}^2} K_{\Phi_0} d\mathrm{vol}_{g_{\Phi_0}} = 16\pi. \tag{3.3.20}$$

Comparing (3.3.17)-(3.3.20) reveals that while : $W(\Phi_\mu) \to W(\Phi_0)$, there is an energy gap of 8π in $\mathcal{E}$ (or equivalently in E). From this, and the energy quantization theorem (theorem 3.2.2, written above), we deduce that a simple minimal bubble of energy $E = 8\pi$ is blown. The only possible bubble is then an Enneper surface (see for instance [Oss86] or example 1.2.5), given by :

$$E(z) = 2\Re\left(\frac{z}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{z^2}{2}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} - \frac{z^3}{6}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\right). \tag{3.3.21}$$

This is enough to ensure that the immersions Φ_μ offer an exemple of an Enneper bubble appearing on a sequence of Willmore immersions, which proves theorem E. $\square$

We however wish to make the appearance of the Enneper bubble explicit in the computations. To do that we will perform a blow-up at the origin at scale μ^3. This concentration scale has been determined the classical way (see the bubble tree extraction procedure in [LR18a] or [BR14]) by computing $\left\|\nabla \vec{n}_{\Psi_\mu}\right\|_{L^\infty(\mathbb{S}^2)}$. Since these computations do not, by themselves, further the understanding of the bubbling phenomena, they are omitted. Considering (3.3.10) we find

$$\begin{aligned}
f_\mu\left(\mu^3 z\right) &= \frac{3}{2\mu^3\left(\mu^6 z^3 - 1\right)}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{a^2\mu}{8}\left(3\frac{2z\mu^2+1}{1+\mu^2 z+\mu^4 z^2} + \mu^2 z\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\\
&+ \frac{3a}{2\mu}\frac{1+\mu^2 z}{1+\mu^2 z+\mu^4 z^2}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\\
&= \left(-\frac{3}{2\mu^3} - \frac{3\mu^3 z^3}{2} + O\left(\mu^9\right)\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \frac{a^2}{2}\left(\frac{3\mu}{4} + z\mu^3 - +O(\mu^5)\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\\
&+ a\left(\frac{3}{2\mu} - \frac{3}{2}\mu^3 z^2 + O(\mu^5)\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.
\end{aligned}$$

Which means that

$$\begin{aligned}
\Psi_\mu(\mu^3 z) &= \frac{1}{2}\left(\frac{3\overline{a}^2\mu}{4} - \frac{3}{\mu^3} + \mu^3\left(\overline{a}^2\overline{z} - 3z^3\right) + O(\mu^5)\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix}\\
&+ \frac{1}{2}\left(\frac{3a^2\mu}{4} - \frac{3}{\mu^3} + \mu^3\left(a^2 z - 3\overline{z}^3\right) + O(\mu^5)\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\\
&+ \left(\frac{3\left(a+\overline{a}\right)}{2\mu} - \frac{3\mu^3}{2}\left(az^2 + \overline{a}\overline{z}^2\right) + O(\mu^5)\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.
\end{aligned}$$

With $p = \frac{p_1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix} + \frac{\overline{p_1}}{2}\begin{pmatrix}1\\i\\0\end{pmatrix} + p_3\begin{pmatrix}0\\0\\1\end{pmatrix}$, defined previously, we conclude :

$$\begin{aligned}\Psi_\mu(\mu^3 z) - p = \frac{1}{\mu^3}&\left(\frac{1}{2}\left(-3 - \mu^3 p_1 + \frac{3\overline{a}^2\mu^4}{4} + \mu^6\left(\overline{a}^2\overline{z} - 3z^3\right) + O(\mu^8)\right)\begin{pmatrix}1\\i\\0\end{pmatrix}\right.\\&+\frac{1}{2}\left(-3 - \mu^3\overline{p_1} + \frac{3a^2\mu^4}{4} + \mu^6\left(a^2 z - 3\overline{z}^3\right) + O(\mu^8)\right)\begin{pmatrix}1\\-i\\0\end{pmatrix}\\&\left.+\left(\frac{3(a+\overline{a})\mu^2}{2} - \mu^3 p_3 - \frac{3\mu^6}{2}(az^2 + \overline{a}\overline{z}^2) + O(\mu^8)\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\right).\end{aligned} \tag{3.3.22}$$

Here, the only relevant terms are the first non constant ones *i.e.* those in μ^6. This yields :

$$\begin{aligned}\left|\Psi_\mu(\mu^3 z) - p\right|^2 = \frac{1}{\mu^6}&\left(9 + 3\mu^3(p_1 + \overline{p_1}) + \mu^4\frac{9(a^2+\overline{a}^2)}{4} + \mu^4\frac{9(a+\overline{a})^2}{4}\right.\\&-\frac{3\mu^5(a+\overline{a})}{2}(p_3 + \overline{p_3}) - 3\mu^6(a^2 z + \overline{a}^2\overline{z}^- 3z^3 - 3\overline{z}^3 + |p_1|^2 + |p_3|^2)\\&\left.+O(\mu^7)\right).\end{aligned} \tag{3.3.23}$$

We can combine (3.3.22) and (3.3.23) :

$$\begin{aligned}\Phi_\mu\left(\mu^3 z\right) &= \frac{\Psi_\mu - p}{|\Psi_\mu - p|^2}\\&= \mu^3\left(\frac{1}{2}\left(-\frac{1}{3} - \frac{\mu^3}{9}p_1 + \frac{\overline{a}^2\mu^4}{12} + \frac{\mu^6}{9}\left(\overline{a}^2\overline{z} - 3z^3\right) + O(\mu^7)\right)\begin{pmatrix}1\\i\\0\end{pmatrix}\right.\\&+\frac{1}{2}\left(-\frac{1}{3} - \frac{\mu^3}{9}\overline{p_1} + \frac{a^2\mu^4}{12} + \frac{\mu^6}{9}\left(a^2 z - 3\overline{z}^3\right) + O(\mu^7)\right)\begin{pmatrix}1\\-i\\0\end{pmatrix}\\&\left.+\left(\frac{a+\overline{a}\mu^2}{6} - \frac{\mu^3}{9}p_3 - \frac{\mu^6}{6}(az^2 + \overline{a}\overline{z}^2) + O(\mu^7)\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\left(1 - \frac{1}{3}\mu^3(p_1 + \overline{p_1})\right.\\&-\mu^4\frac{(a^2+\overline{a}^2)}{4} - \mu^4\frac{(a+\overline{a})^2}{4} + \frac{\mu^5(a+\overline{a})}{6}(p_3 + \overline{p_3})\\&\left.+\frac{1}{3}\mu^6(a^2 z + \overline{a}^2\overline{z} - 3z^3 - 3\overline{z}^3 + |p_1|^2 + |p_3|^2 + \frac{1}{3}\left(p_1 + \overline{p_1}\right)^2) + O(\mu^7)\right).\end{aligned}$$

Since the constant terms are irrelevant, we can gather them into $\Phi_\mu(0)$ and simplify somewhat the expression into :

$$\begin{aligned}\Phi_\mu\left(\mu^3 z\right) &= \Phi_\mu(0) + \mu^9 \left(\frac{1}{9}\left(\overline{a}^2\overline{z} - 3z^3 - a^2 z - \overline{a}^2\overline{z} + 3z^3 + 3\overline{z}^3\right)\frac{1}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}\right.\\
&\left. + \frac{1}{9}\left(a^2 z - 3\overline{z}^3 - a^2 z - \overline{a}^2\overline{z} + 3z^3 + 3\overline{z}^3\right)\frac{1}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{az^2 + \overline{a}\overline{z}^2}{6}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right)\\
&+ O(\mu^{10})\\
&= \Phi_\mu(0) + \mu^9\left(\left(\frac{\overline{z}^3}{3} - \frac{a^2}{9}z\right)\frac{1}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \left(\frac{z^3}{3} - \frac{a^2}{9}\overline{z}\right)\frac{1}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\right.\\
&\left. - \left(\frac{a}{3}\frac{z^2}{2} + \frac{\overline{a}}{3}\frac{\overline{z}^2}{2}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right) + O(\mu^{10}).\end{aligned}$$

Taking $a = 3$ we find exactly

$$\Phi_\mu\left(\mu^3 z\right) = \Phi_\mu(0) - \mu^9 E(z) + O\left(\mu^{10}\right). \tag{3.3.24}$$

Hence we do have :

$$\frac{\Phi_\mu(\mu^3 z) - \Phi_\mu(0)}{-\mu^9} \to E(z)$$

smoothly on every compact of $\mathbb{C}$, which does illustrate theorem E.

Remark 3.3.1. Chosing another value for a would have led to another Enneper surface, with Enneper-Weierstrass data $(f, g) = (1, \frac{3}{a}z)$ instead of simply $(f, g) = (1, z)$.

Remark 3.3.2. One must notice the fundamentally asymetric role of Φ_0 (the surface) and E (the bubble). Indeed while we have compactly glued E on Φ_0 we cannot compactly glue $\Psi_0 = \iota \circ \Phi_0$ on an inverted Enneper using the same construction, since Ψ_0 has an end which is not on the concentration point. Doing so would require to glue a closed bubble tree on said planar end (and would necessarily add Willmore energy to the concentration point). Further theorem G ensures that no construction will ever enable us to do so, given that the second residue of the inverted Enneper surface is $\alpha = 2$ (see example 1.C of [BR13]).

Remark 3.3.3. It must also be pointed out that this counter-example explicitely illustrates the difference between the ε-regularity of E. Kuwert and R. Schatzle (theorem 2.10 of [KS01b]) and those obtained with the T. Rivière formalism (theorems I.5 in [Riv08] and I.1 in [BR14]) as was explained in [BWV18]. Indeed consider the end of the Bryant surfaces placed at infinity, sent to the origin after the inversion. This point is regular, that is, without concentration, and one can always find *intrinsic* neighborhoods of uniformly small energy. However when one takes an *extrinsic* neighborhood of the image point after inversion, one cannot help intersecting the image of the Enneper bubble, and thus containing a given quantum of energy. This point is thus singular for theorem 2.10 of [KS01b]. The two approaches are thus different not only in philosophy, but also in the results they yield.

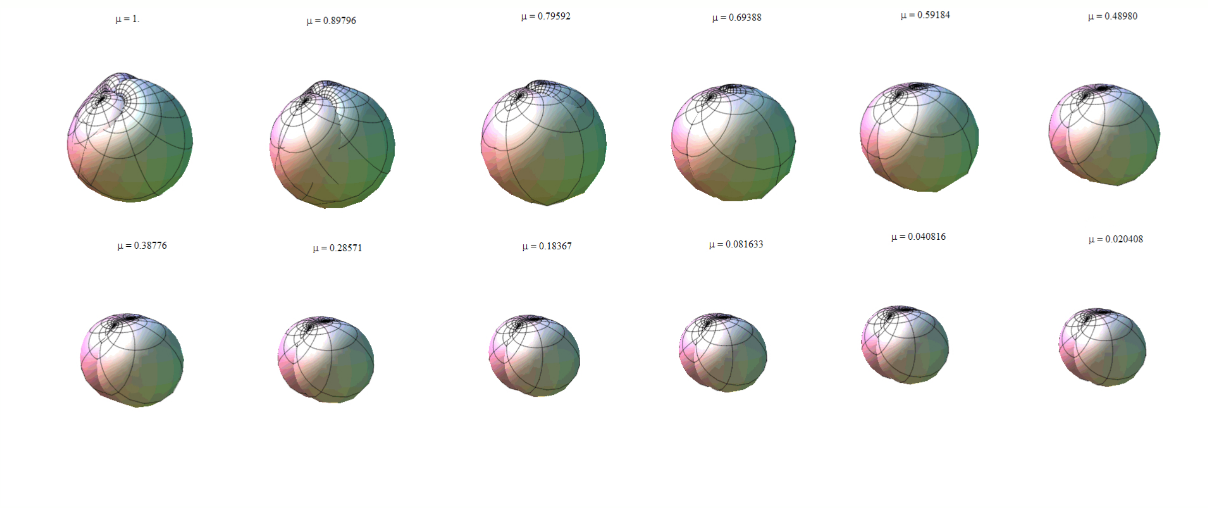

Figure 3.3 – The transformation of the Bryant's surfaces into a López surface

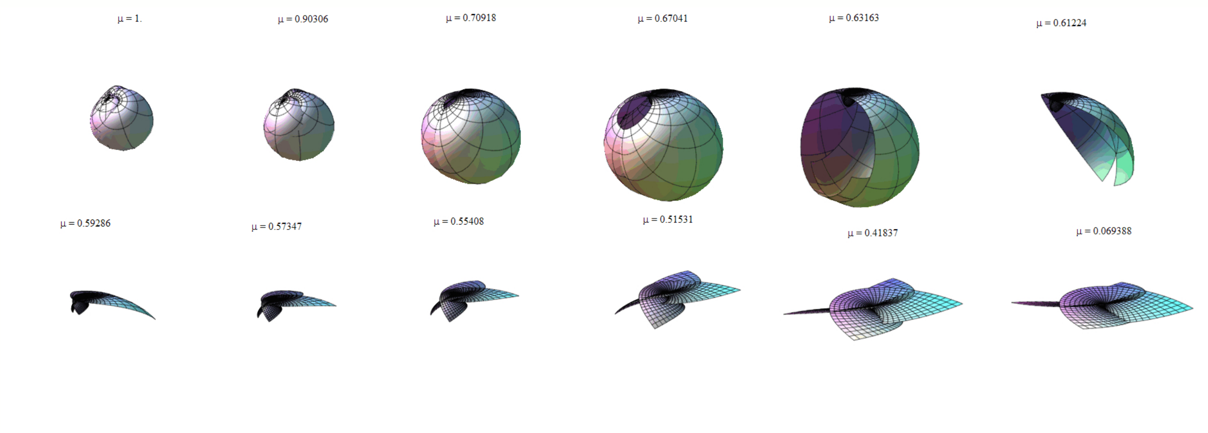

Figure 3.4 – The blow-up into an Enneper surface

4

A study of simple minimal bubbling

Abstract.

4.1 Introduction

This final chapter studies the case where only one bubble, which is minimal, concentrates on a given concentration point. We call it *simple minimal bubbling.* The aim is to show that it is then more regular than expected.

We start by modifying the ε-regularity theorem 1.4.3, making a relevant use of system (1.2.49), to replace the small $\|\nabla\vec{n}\|_{L^2}$ control by a smaller, less demanding (since (A.2.11) stands) control on $\|H\nabla\Phi\|_{L^2}$. The following theorem was one of the foci of [Mar19c].

Theorem F. Let $\Phi \in \mathcal{E}(\mathbb{D})$. We assume

$$\|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_\rho)} + \|\nabla\vec{n}\|_{L^2(\mathbb{D}_\rho)} \leq C_0.$$

Then there exists ε_0' depending only on C_0 such that if

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0'$$

then for any $r < 1$ there exists a constant $C \in \mathbb{R}$ depending on r, C_0, p and

$$r_0 = \frac{1}{\rho}\inf\left\{ s \,\middle|\, \int_{B_s(p)} |\nabla\vec{n}|^2 = \frac{8\pi}{6},\ \forall p \in \mathbb{D}_\rho \text{ s.t. } B_s(p) \subset \mathbb{D}_\rho \right\},$$

such that

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \le C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

and

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \le C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

Theorem F, thanks to the added regularity it provides, can then serve as a jumping point for successive expansions of increasing accuracy, whose end result is a control of the second residue of the limit surface at its branch points receiving simple minimal bubbling.

Theorem G. Let Φ_k be a sequence of Willmore immersions of a closed surface Σ. Assume that

$$\limsup_{k\to\infty} W(\Phi_k) < \infty,$$

and that the conformal class of $\Phi_k^*\xi$ remains within a compact subdomain of the moduli space of Σ. Then at each concentration point $p \in \Sigma$ of multiplicity $\theta_p + 1$ on which a simple minimal bubble is blown, the second residue α_p of the limit immersion Φ_∞ satisfies

$$\alpha_p \le \theta_p - 1.$$

Since the inverted Chen-Gackstatter torus has second residue 2, it cannot be the recipient of simple minimal bubbling, which will give compactness below 12π, as follows.

Theorem H. Let Σ be a closed surface of genus 1 and $\Phi_k : \Sigma \to \mathbb{R}^3$ a sequence of Willmore immersions such that the induced metric remains in a compact set of the moduli space and

$$\limsup_{k\to\infty} W(\Phi_k) \le 12\pi.$$

Then there exists a diffeomorphism ψ_k of Σ and a conformal transformation Θ_k of $\mathbb{R}^3\cup\{\infty\}$, such that $\Theta_k \circ \Phi_k \circ \psi_k$ converges up to a subsequence toward a smooth Willmore immersion $\Phi_\infty : \Sigma \to \mathbb{R}^3$ in $C^\infty(\Sigma)$.

In fact theorem G eliminates all Chen-Gackstatter surfaces as recipient of simple minimal bubbling, and even further, all inversions of minimal surfaces with ends asymptotic to an Enneper surface. Should someone prove the expected classification result ensuring that minimal immersions of critical total curvature are asymptotic to Enneper, theorem H could be extended to surfaces of higher genus.

We would now like to highlight the difficulties we will encounter in our path towards these three key theorems. Indeed theorem F by itself cannot yield control across a concentration point, precisely because those are the ones where $r_0 \to 0$, and thus where the resulting estimates degenerate. This is why, we will in fact first prove a more flexible result with a starting hypothesis on $\vec{L}$.

Theorem 4.1.1. Let $\Phi \in \mathcal{E}(\mathbb{D})$ satisfy the hypotheses of theorem 1.4.1. We assume there exists $r' < 1$ and $C_1 > 0$ such that

$$\left\|\vec{L}e^\lambda\right\|_{L^{2,\infty}(\mathbb{D}_{r'})} \le C_1 \|H\nabla\Phi\|_{L^2(\mathbb{D})}$$

where $\vec{L}$ is given by (1.2.43). Then there exists ε_0' depending only on C_0 such that if

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \le \varepsilon_0',$$

then, for any $r < r'$, there exists a constant $C \in \mathbb{R}$ depending on r, C_0, p and C_1 such that

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

and

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \leq C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

The proof will rely on a joint exploitation of systems (1.2.46) and (1.2.49), and an analytical ropewalking, where the regularity lost with (1.2.49) is exactly compensated by sharper estimates for (1.2.46). Further, the controls obtained on $\vec{L}$ are flexible enough to apply on the neck domains. Indeed, we will slightly modify inequality (VI.23) in [BR14] (in the fashion of theorem 1.4.1) to better suit our needs.

Theorem 4.1.2. Let $R > 0$ and $\Phi \in \mathcal{E}(\mathbb{D}_R)$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and λ its conformal factor. We assume

$$\|\nabla\vec{n}\|_{L^2(\mathbb{D}_R)} + \|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_R)} \leq C_0.$$

Then there exists $\varepsilon_0 > 0$ (independant of Φ) such that if $0 < 8r < R$ and

$$\sup_{r<s<\frac{R}{2}} \int_{\mathbb{D}_{2s}\backslash\mathbb{D}_s} |\nabla\vec{n}|^2 \leq \varepsilon_0,$$

then there exists $\vec{\mathcal{L}} \in \mathbb{R}^3$ and $C \in \mathbb{R}$ depending on C_0 but not on the conformal class of $\mathbb{D}_R\backslash\mathbb{D}_r$ such that

$$\left\| e^\lambda \left(\vec{L} - \vec{\mathcal{L}}\right)\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{R}{2}}\backslash\mathbb{D}_{2r}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D}_R)},$$

where $\vec{L}$ is given by (1.2.43).

One can then combine theorems 1.4.1 and 4.1.2 to enjoy estimates on $\vec{L}$ both on the neck-domain, and on the bubbles, which thanks to the flexibility of theorem 4.1.1 yields the true control on H across the concentration point.

To simplify the proceedings we will introduce a local formalism that we will show is equivalent to the simple minimal bubbling.

Theorem 4.1.3. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ a sequence of conformal, weak, Willmore immersions, of Gauss map $\vec{n}^\varepsilon$, mean curvature H^ε and conformal factor λ^ε, of parameter $\varepsilon > 0$. We assume

1. $\displaystyle\int_{\mathbb{D}} |\nabla\vec{n}^\varepsilon|^2\, dz \leq M < \infty$,
2. $\|\nabla\lambda^\varepsilon\|_{L^{2,\infty}(\mathbb{D})} \leq M$,
3. $\displaystyle\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\frac{1}{R}}\backslash\mathbb{D}_{\varepsilon R}} |\nabla\vec{n}^\varepsilon|^2\, dz\right) = 0$,
4. $\Phi^\varepsilon \to \Phi^0$ in $C^\infty_{loc}(\mathbb{D}\backslash\{0\})$, with Φ^0 a branched Willmore immersion on $\mathbb{D}$,
5. There exists $C^\varepsilon > 0$ such that $\frac{\Phi^\varepsilon(\varepsilon.)-\Phi^\varepsilon(0)}{C^\varepsilon} \to \Psi$ in $C^\infty_{loc}(\mathbb{C})$, with Ψ a minimal immersion (that is of mean curvature $H_\Psi = 0$).

Then $\Phi^\varepsilon \to \Phi^0$ $C^{2,\alpha}(\mathbb{D})$ for all $\alpha < 1$.

With these notations, the first notable expansion on our way to theorem G is a precise description of the conformal factor.

Theorem 4.1.4. Let Φ^ε be a sequence of Willmore conformal immersions. We assume there is only one concentration point on which a simple minimal bubble is blown, at scale ε. Then there exists $l^\varepsilon \in L^\infty(\mathbb{D})$ such that :

$$\begin{aligned}\lambda^\varepsilon &= \ln\left(\varepsilon^\theta + r^\theta\right) + l^\varepsilon,\\ \|l^\varepsilon\|_{L^\infty(\mathbb{D})} &\leq C(C_0).\end{aligned}$$

As a result if we denote $\chi = \sqrt{\varepsilon^2 + r^2}$, the immersion satisfies the following Harnack inequality :

$$\frac{\chi^\theta}{C(C_0)} \leq e^{\lambda^\varepsilon} \leq C(C_0)\chi^\theta.$$

Building on this expansion, we can find others on the immersion, the mean curvature, and even ∇S and $\nabla \vec{R}$. All these can be injected into the Willmore equations. Since those are non linear, each and every one will yield a linearized version and thus a constraint on the terms of the expansions. These constraints will add up, until enough regularity in the convergence is gained to prove theorem G.

While all the preceding results were prepublished in [Mar19c] and [Mar19b], the concluding section of the present chapter will present an unpublished analysis of the lack of compactness of Willmore immersions, explained by the properties of the conformal group. In a reasonable attempt to exploit the whole conformal group to control to control the mean curvature, we will prove that, because the conformal group is neither compact nor commutative, one can only tamper with inversions under a pointwise hypothesis at the concentration point. Under this assumption, the bubbling regularity jumps significantly. Namely we will show the following

Theorem 4.1.5. Let Σ be a compact Riemann surface of genus less than 1, and $\Phi^k : \Sigma \to \mathbb{R}^3$ a sequence of Willmore immersions of uniformly bounded total curvature and such that the conformal class of the induced metric is in a compact of the moduli space. We further assume that Φ^k has only a single concentration point p on which a simple Enneper bubble is blown, and that Φ^k converges smoothly away from p toward a branched immersion $\Phi^0 : \Sigma \to \mathbb{R}^3$. Then either

$$\frac{\nabla H^k(p)}{\|\nabla \vec{n}^k\|_{L^\infty(\Sigma)}} \to \infty, \tag{4.1.1}$$

or Φ^0 is the inversion of a branched minimal immersion, with second residue $\alpha \leq -2$.

The hypothesis (4.1.1) is the aforementioned pointwise control necessary to use inversions to control the mean curvature at the concentration point. It is significant that the example displayed in theorem E does not satisfy (4.1.1), and can then be seen as a consequence of the lack of commutativity and compactness of the conformal group.

4.2 H ε-regularity : proof of theorem F

4.2.1 Results starting with a control on $\vec{L}$

This subsection will focus on proving an ε-regularity result with control on H *starting with a control on* $\vec{L}$. Namely

Theorem 4.2.1. Let $\Phi \in \mathcal{E}(\mathbb{D})$ satisfy the hypotheses of theorem 1.4.1. We assume there exists $r' < 1$ and $C_1 > 0$ such that

$$\left\| \vec{L} e^{\lambda} \right\|_{L^{2,\infty}(\mathbb{D}_{r'})} \leq C_1 \left\| H \nabla \Phi \right\|_{L^2(\mathbb{D})}$$

where $\vec{L}$ is given by (1.2.43). Then there exists ε_0' depending only on C_0 such that if

$$\| H \nabla \Phi \| \leq \varepsilon_0'$$

then for any $r < r'$ there exists a constant $C \in \mathbb{R}$ depending on r, C_0, p and C_1 such that

$$\| H \nabla \Phi \|_{L^\infty(\mathbb{D}_r)} \leq C \| H \nabla \Phi \|_{L^2(\mathbb{D})},$$

and

$$\| \nabla \Phi \|_{W^{3,p}(\mathbb{D}_r)} \leq C \| \nabla \Phi \|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

Proof. Let $r < r' < 1$, we follow the outline given in the introduction.
Step 1 : $W^{1,(2,1)}$ control on the Willmore quantities
Let $\vec{L}$ satisfy our hypothesis. Theorem 1.4.2 gives :

$$\| \nabla S \|_{L^{2,1}\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \left\| \nabla \vec{R} \right\|_{L^{2,1}\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} \leq C_1 \| H \nabla \Phi \|_{L^2(\mathbb{D})}. \tag{4.2.2}$$

Step 2 : $W^{1,q}$ control on the Willmore quantities, for $q > 2$
Thanks to (1.2.46) and (1.2.49) we can decompose in any $B_t(p)$, with $p \in \mathbb{D}_{\frac{r+r'}{2}}$ and t sufficiently small, $S = \sigma + s$ and $\vec{R} = \vec{\rho} + \vec{r}$, with

$$\begin{cases} \Delta \sigma = \Delta S = \langle H \nabla \Phi, \nabla^\perp \vec{R} \rangle = -\langle \nabla \vec{n}, \nabla^\perp \vec{R} \rangle \text{ in } B_t(p) \\ \quad \sigma = 0 \text{ on } \partial B_t(p), \end{cases} \tag{4.2.3}$$

$$\begin{cases} \Delta \vec{\rho} = \Delta \vec{R} = -H \nabla \Phi \times \nabla^\perp \vec{R} - \nabla^\perp S H \nabla \Phi \\ \qquad = \nabla \vec{n} \times \nabla^\perp \vec{R} + \nabla^\perp S \nabla \vec{n} \text{ in } B_t(p) \\ \quad \vec{\rho} = 0 \text{ on } \partial B_t(p) \end{cases} \tag{4.2.4}$$

$$\begin{cases} \Delta s = 0 \text{ in } B_t(p) \\ \quad s = S \text{ on } \partial B_t(p), \end{cases} \tag{4.2.5}$$

$$\begin{cases} \Delta \vec{r} = 0 \text{ in } B_t(p) \\ \quad \vec{r} = \vec{R} \text{ on } \partial B_t(p). \end{cases} \tag{4.2.6}$$

Since s and $\vec{r}$ are harmonic functions, $l \to \frac{1}{l^2} \int_{B_l(p)} |\nabla s|^2$ and $l \to \frac{1}{l^2} \int_{B_l(p)} |\nabla \vec{r}|^2$ are classically non-decreasing (see lemma IV.1 in [Riv07]). It follows that

$$\begin{aligned} \| \nabla s \|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} &\leq \frac{1}{4} \| \nabla s \|^2_{L^2(B_t(p))}, \\ \| \nabla \vec{r} \|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} &\leq \frac{1}{4} \| \nabla \vec{r} \|^2_{L^2(B_t(p))}. \end{aligned} \tag{4.2.7}$$

Furthermore thanks to (4.2.3) and theorem A.3.6 we have

$$\|\nabla\sigma\|_{L^{2,1}(B_t(p))} \leq C\|\nabla\vec{R}\|_{L^2(B_t(p))}\|\nabla\vec{n}\|_{L^2(B_t(p))}. \tag{4.2.8}$$

Thanks to (4.2.3) and theorem A.3.3 we find

$$\begin{aligned}\|\nabla\sigma\|_{L^{2,\infty}(B_t(p))} &\leq C\left\|\langle H\nabla\Phi, \nabla^\perp\vec{R}\rangle\right\|_{L^1(B_t(p))}\\ &\leq C\left\|\nabla\vec{R}\right\|_{L^2(B_t(p))}\|H\nabla\Phi\|_{L^2(B_t(p))}.\end{aligned} \tag{4.2.9}$$

Exploiting the duality of $L^{2,1}$ and $L^{2,\infty}$, (4.2.8) and (4.2.9) yield

$$\begin{aligned}\|\nabla\sigma\|^2_{L^2(B_t(p))} &\leq \|\nabla\sigma\|_{L^{2,\infty}(B_t(p))}\|\nabla\sigma\|_{L^{2,1}(B_t(p))}\\ &\leq C\left(\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\right)\|\nabla\vec{R}\|^2_{L^2(B_t(p))}\|H\nabla\Phi\|_{L^2(\mathbb{D})}.\end{aligned} \tag{4.2.10}$$

Working similarly with $\vec{\rho}$ we find

$$\|\nabla\vec{\rho}\|^2_{L^2(B_t(p))} \leq C\left(\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\right)\left(\|\nabla\vec{R}\|^2_{L^2(B_t(p))} + \|\nabla S\|^2_{L^2(B_t(p))}\right)\|H\nabla\Phi\|_{L^2(\mathbb{D})}. \tag{4.2.11}$$

We remind the reader that the constant from theorems A.3.6 and A.3.3 are universal due to the scale invariance properties of the L^2, $L^{2,\infty}$ and $L^{2,1}$ norms. The constants in (4.2.10) and (4.2.11) then do depend solely on $\|\nabla\vec{n}\|_{L^2(\mathbb{D})}$.

We can combine (4.2.7), (4.2.10) and (4.2.11) to get

$$\begin{aligned}\|\nabla S\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} + \|\nabla\vec{R}\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} &\leq \frac{1}{2}\left(\|\nabla s\|^2_{L^2(B_t(p))} + \|\nabla\vec{r}\|^2_{L^2(B_t(p))}\right)\\ &\quad + 2C\left(\|\nabla\vec{n}\|_{L^2(\mathbb{D})}\right)\left(\|\nabla\vec{R}\|^2_{L^2(B_t(p))} + \|\nabla S\|^2_{L^2(B_t(p))}\right)\|H\nabla\Phi\|_{L^2(\mathbb{D})}\\ &\leq \left(\frac{1}{2} + \|H\nabla\Phi\|_{L^2(\mathbb{D})}C\right)\left(\|\nabla S\|^2_{L^2(B_t(p))} + \|\nabla\vec{R}\|^2_{L^2(B_t(p))}\right),\end{aligned} \tag{4.2.12}$$

where C depends solely on $\|\nabla\vec{n}\|_{L^2(\mathbb{D})}$. Should $\|H\nabla\Phi\|_{L^2(\mathbb{D})}$ be small enough then (4.2.12) would yield

$$\|\nabla S\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} + \|\nabla\vec{R}\|^2_{L^2\left(B_{\frac{t}{2}}(p)\right)} \leq \frac{3}{4}\left(\|\nabla S\|^2_{L^2(B_t(p))} + \|\nabla\vec{R}\|^2_{L^2(B_t(p))}\right). \tag{4.2.13}$$

Since the chosen ε_0' depends only of $\|\nabla\vec{n}\|_{L^2(\mathbb{D})}$, (4.2.13) is uniformly true for all $B_l(p) \subset \mathbb{D}_{\frac{2r+r'}{3}}$ and yields a Morrey-type estimate on $\mathbb{D}_{\frac{2r+r'}{3}}$. Through usual estimates on Riesz potentials, see for instance theorem 3.1 in [Ada75], it entails

$$\exists q > 2 \text{ s.t.} \quad \|\nabla S\|_{L^q\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} + \left\|\nabla\vec{R}\right\|_{L^q\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C_q\left(\|\nabla S\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \left\|\nabla\vec{R}\right\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)}\right). \tag{4.2.14}$$

Step 3 : L^∞ control on $H\nabla\Phi$

Thanks to Step 2 and (1.2.52) we deduce

$$\|H\nabla\Phi\|_{L^q\left(\mathbb{D}_{\frac{3r+r'}{4}}\right)} \leq C_q\left(\|\nabla S\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)} + \|\nabla\vec{R}\|_{L^2\left(\mathbb{D}_{\frac{r+r'}{2}}\right)}\right).$$

The criticality of system (1.2.49) is thus broken : $\Delta S, \Delta \vec{R}$ are in $L^{\frac{q}{2}}$ with $\frac{q}{2} > 1$. One can apply classic Calderón-Zygmund theory (see for instance theorem 9.9 and 9.11 of [GT01]) to start a bootstrap of limiting regularity L^∞ on $H\nabla\Phi$. *In fine* one has with estimate (4.2.2)

$$\|\nabla S\|_{W^{1,p}\left(\mathbb{D}_{\frac{4r+r'}{5}}\right)} + \left\|\nabla \vec{R}\right\|_{W^{1,p}\left(\mathbb{D}_{\frac{4r+r'}{5}}\right)} + \|H\nabla\Phi\|_{L^\infty\left(\mathbb{D}_{\frac{4r+r'}{5}}\right)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})} \quad (4.2.15)$$

for all $p < \infty$. Here C is a real constant which depends on r, r', C_0 and C_1.

Step 4 : $W^{3,p}$ control on Φ

The control on $\nabla\Phi$ is obtained by a similar Calderón-Zygmund bootstrap on equation

$$2\Delta\Phi = \nabla^\perp S \nabla\Phi + \nabla^\perp \vec{R} \times \nabla\Phi,$$

which achieves the proof. $\square$

One only needs to combine theorems 1.4.1 and 4.2.1 to prove :

Theorem F. Let $\Phi \in \mathcal{E}(\mathbb{D})$ satisfy the hypotheses of theorem 1.4.1. Then there exists ε_0' depending only on C_0 such that if

$$\|H\nabla\Phi\|_{L^2(\mathbb{D})} \leq \varepsilon_0',$$

then for any $r < 1$ there exists a constant $C \in \mathbb{R}$ depending on r, C_0, p and r_0 (defined in (1.3.67)) such that

$$\|H\nabla\Phi\|_{L^\infty(\mathbb{D}_r)} \leq C\|H\nabla\Phi\|_{L^2(\mathbb{D})},$$

and

$$\|\nabla\Phi\|_{W^{3,p}(\mathbb{D}_r)} \leq C\|\nabla\Phi\|_{L^2(\mathbb{D})}$$

for all $p < \infty$.

The dependance in r_0 is actually problematic for our blow-up analys purposes. Indeed as the energy concentrates, r_0 goes to 0, and the estimates in theorem F degenerate. However applied to a ball of radius ε (using the notations of lemma 3.2.2), theorem 1.4.1 will yield uniform estimates on $\vec{L}$. One then only has to control $\vec{L}$ on the so-called "neck area" : $\mathbb{D}\backslash\mathbb{D}_\varepsilon$.

4.2.2 Results on a "neck-type" domain

In this section we focus on a control of $\vec{L}$ on annuli of small energy, independantly of its conformal class. We modify a preexisting result ((VI.23) in [BR14]) into the following theorem.

Theorem 4.2.2. Let $R > 0$ and $\Phi \in \mathcal{E}(\mathbb{D}_R)$ be a conformal weak Willmore immersion. Let $\vec{n}$ denote its Gauss map, H its mean curvature and λ its conformal factor. We assume

$$\|\nabla\vec{n}\|_{L^2(\mathbb{D}_R)} + \|\nabla\lambda\|_{L^{2,\infty}(\mathbb{D}_R)} \leq C_0.$$

Then there exists $\varepsilon_0 > 0$ (independant of Φ) such that if $0 < 8r < R$ and

$$\sup_{r<s<\frac{R}{2}} \int_{\mathbb{D}_{2s}\backslash\mathbb{D}_s} |\nabla\vec{n}|^2 \leq \varepsilon_0,$$

then there exists $\vec{\mathcal{L}} \in \mathbb{R}^3$ and $C \in \mathbb{R}$ depending on C_0 but not on the conformal class of $\mathbb{D}_R \backslash \mathbb{D}_r$ such that

$$\left\| e^{\lambda} \left(\vec{L} - \vec{\mathcal{L}} \right) \right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{R}{2}} \backslash \mathbb{D}_{2r}\right)} \leq C \|H \nabla \Phi\|_{L^2(\mathbb{D}_R)},$$

where $\vec{L}$ is given by (1.2.43).

Once more, we will follow Y. Bernard and T. Rivière's proof, with a few tweaks in order to obtain a control of $\vec{L}e^{\lambda}$ by $H\nabla\Phi$ instead of $\nabla\vec{n}$. It is important for Φ to be well-defined, and the bound on its conformal factor and Gauss map to stand, on the whole disk and not merely on the annulus. We refer the reader to [LR18a] for a study of what can happen otherwise. In the context of theorem 3.2.2, theorem 4.2.2 gives controls on the neck regions around the concentration points.

Proof. **Step 1 : Pointwise estimates on $\vec{H}$ and $\nabla \vec{H}$**
We set ourselves in the setting of theorem 4.2.2 and consider $\Phi \in \mathcal{E}(\mathbb{D}_R)$ a conformal weak Willmore immersions of Gauss map $\vec{n}$, mean curvature H, conformal factor λ and tracefree second fundamental form $\mathring{A}$. We assume that

$$\|\nabla \vec{n}\|_{L^2(\mathbb{D}_R)} + \|\nabla \lambda\|_{L^{2,\infty}(\mathbb{D}_R)} \leq C_0 < \infty,$$

and that

$$\sup_{r<s<\frac{R}{2}} \int_{\mathbb{D}_{2s} \backslash \mathbb{D}_s} |\nabla \vec{n}|^2 \leq \varepsilon_0. \tag{4.2.16}$$

Consider $x \in \mathbb{D}_{\frac{R}{2}} \backslash \mathbb{D}_{2r}$, then $B_{\frac{|x|}{4}}(x) \subset \mathbb{D}_{2|x|} \backslash \mathbb{D}_{\frac{|x|}{2}}$ and thus (4.2.16) implies

$$\int_{B_{\frac{|x|}{4}}(x)} |\nabla \vec{n}|^2 \leq \varepsilon_0. \tag{4.2.17}$$

On $B_{\frac{|x|}{4}}(x)$ one can then apply either theorem 1.4.3, or theorem F (with $r_0 = 1$ since (4.2.17) stands) to deduce

$$\|\nabla \vec{n}\|_{L^\infty\left(B_{\frac{|x|}{8}}(x)\right)} \leq \frac{C}{|x|} \|\nabla \vec{n}\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}, \tag{4.2.18}$$

and

$$\|H \nabla \Phi\|_{L^\infty\left(B_{\frac{|x|}{8}}(x)\right)} \leq \frac{C}{|x|} \|H \nabla \Phi\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}. \tag{4.2.19}$$

Here C depends on C_0. Corollary 1.3.1 then ensures a Harnack inequality on $B_{\frac{|x|}{8}}(x)$, meaning there exists $\Lambda \in \mathbb{R}$ and C depending only on C_0 such that for all $p \in B_{\frac{|x|}{8}}(x)$ we have

$$\frac{e^{\Lambda}}{C} \leq e^{\lambda(p)} \leq C e^{\Lambda}. \tag{4.2.20}$$

This allows one to control H with (4.2.19) :

$$\|H\|_{L^\infty\left(B_{\frac{|x|}{8}}(x)\right)} \leq \frac{C e^{-\Lambda}}{|x|} \|H \nabla \Phi\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}. \tag{4.2.21}$$

Since Φ is Willmore, it satisfies (1.2.30) :

$$\Delta H + \left|\mathring{A}\right|^2 H = 0.$$

Combining (4.2.18), (4.2.21) and (A.2.10) yields

$$\left\| |\mathring{A}|^2 H \right\|_{L^\infty\left(B_{\frac{|x|}{8}}(x)\right)} \leq \frac{Ce^{-\Lambda}}{|x|^3} \left\| H\nabla\Phi \right\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}.$$

Then

$$\left\| \Delta H \right\|_{L^\infty\left(B_{\frac{|x|}{8}}(x)\right)} \leq \frac{Ce^{-\Lambda}}{|x|^3} \left\| H\nabla\Phi \right\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}.$$

Classic Calderón-Zygmund results (see for instance theorem 9.9 and 9.11 of [GT01]) ensure that

$$\left\| \nabla H \right\|_{L^\infty\left(B_{\frac{|x|}{16}}(x)\right)} \leq \frac{Ce^{-\Lambda}}{|x|^2} \left\| H\nabla\Phi \right\|_{L^2\left(B_{\frac{|x|}{4}}(x)\right)}. \tag{4.2.22}$$

Combining first (4.2.19) and (4.2.20), and then (4.2.22) and (4.2.20) yields when evaluated at x

$$e^{\lambda(x)} \left| H(x) \right| \leq C\delta(|x|), \tag{4.2.23}$$

$$e^{\lambda(x)} \left| \nabla H(x) \right| \leq \frac{C}{|x|}\delta(|x|), \tag{4.2.24}$$

where

$$\delta(s) = \frac{1}{s} \left\| H\nabla\Phi \right\|_{L^2\left(\mathbb{D}_{2s}\backslash \mathbb{D}_{\frac{s}{2}}\right)}.$$

Since $\nabla\vec{H} = \nabla H\vec{n} + H\nabla\vec{n}$, we can extend (4.2.23) and (4.2.24) to $\vec{H}$ and $\nabla\vec{H}$ thanks to (4.2.18), which yields the desired estimates.

Step 2 : Controls on δ

We have

$$s\delta(s) \leq \left\| H\nabla\Phi \right\|_{L^2\left(\mathbb{D}_R\backslash \mathbb{D}_{\frac{r}{2}}\right)}. \tag{4.2.25}$$

Further for any function positive function f :

$$\begin{aligned} \int_r^{\frac{R}{2}} \frac{1}{s} \int_{\frac{s}{2}}^{2s} f(t)dtds &\leq \int_{\frac{r}{2}}^{R} \int_{\frac{t}{2}}^{2t} \frac{1}{s} f(t)dsdt \\ &\leq \int_{\frac{r}{2}}^{R} f(t)\log\left(\frac{2t}{\frac{t}{2}}\right)dt \\ &\leq \log 4 \int_{\frac{r}{2}}^{R} f(t)dt. \end{aligned} \tag{4.2.26}$$

Applying (4.2.26) with $f(t) = \int_{\partial\mathbb{D}_t} |H\nabla\Phi|^2 \, d\sigma_{\partial\mathbb{D}_t}$ we find

$$\int_r^{\frac{R}{2}} s\delta^2(s)ds \leq \log 4 \left\| H\nabla\Phi \right\|^2_{L^2\left(\mathbb{D}_R\backslash \mathbb{D}_{\frac{r}{2}}\right)}, \tag{4.2.27}$$

while with $\tilde{f}(t) = \int_{\partial\mathbb{D}_t} |\nabla\vec{n}|^2 \, d\sigma_{\partial\mathbb{D}_t}$, it yields (VI.9) in [BR14] :

$$\int_r^{\frac{R}{2}} s\tilde{\delta}^2(s)ds \leq \log 4 \left\| \nabla\vec{n} \right\|^2_{L^2\left(\mathbb{D}_R\backslash \mathbb{D}_{\frac{r}{2}}\right)}, \tag{4.2.28}$$

where

$$\tilde{\delta}(s) = \frac{1}{s} \|\nabla \vec{n}\|_{L^2\left(\mathbb{D}_{2s}\setminus \mathbb{D}_{\frac{s}{2}}\right)}.$$

Step 3 : Exploitation and control of $\vec{L}$

Let $\vec{L}$ be a first Willmore quantity of Φ on $\mathbb{D}_R$, i.e. satisfying (1.3.74). From (1.3.74), (4.2.18), (4.2.23) and (4.2.24) we deduce for all $x \in \mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}$

$$\left|\nabla \vec{L}\right|(x) \leq \frac{Ce^{-\lambda(x)}}{|x|}\delta(|x|). \tag{4.2.29}$$

We consider for any $r \leq t \leq R$

$$\vec{L}_t := \frac{1}{|\partial \mathbb{D}_t|}\int_{\partial \mathbb{D}_t} \vec{L} d\sigma_{\partial \mathbb{D}_t}.$$

Then given $x \in \mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}$

$$\begin{aligned}\left|\vec{L}(x) - \vec{L}_{|x|}\right| &\leq \int_{\partial \mathbb{D}_{|x|}} \left|\nabla \vec{L}\right| d\sigma_{\partial\mathbb{D}_{|x|}} \\ &\leq \int_{\partial \mathbb{D}_{|x|}} \frac{Ce^{-\lambda(x)}}{|x|}\delta(|x|) d\sigma_{\partial\mathbb{D}_{|x|}} \\ &\leq C\delta(|x|)\int_0^{2\pi} e^{-\lambda(|x|e^{i\theta})}d\theta.\end{aligned} \tag{4.2.30}$$

Further, in our case lemma 3.2.1 implies the following Harnack inequality for all $x \in \mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}$

$$\frac{e^A|x|^d}{C} \leq e^{\lambda(x)} \leq Ce^A|x|^d, \tag{4.2.31}$$

with d, A in $\mathbb{R}$, and C a constant depending on C_0. Then (4.2.30) yields

$$\left|\vec{L}(x) - \vec{L}_{|x|}\right| \leq C\delta(|x|)e^{-\lambda(x)}, \tag{4.2.32}$$

with C depending on C_0. We can then estimate $\vec{L} - \vec{L}_{|x|}$ with (4.2.27) and (4.2.32) :

$$\int_{\mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}} e^{2\lambda}\left|\vec{L} - \vec{L}_{|x|}\right|^2 dx \leq C\int_{2r}^{\frac{R}{2}} r\delta^2(r)dr \leq C\|H\nabla\Phi\|^2_{L^2\left(\mathbb{D}_R\setminus\mathbb{D}_{\frac{r}{2}}\right)}. \tag{4.2.33}$$

We will control similarly $\frac{d\vec{L}_t}{dt} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial \vec{L}}{\partial t}(t,\theta)d\theta$. We use expression (1.4.76) of $\nabla\vec{L}$ and deduce

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial \vec{L}}{\partial t}(t,\theta)d\theta = \frac{3}{2\pi}\int_0^{2\pi} H\partial_\theta \vec{n} d\theta + \frac{1}{2\pi}\int_0^{2\pi}\partial_\nu \vec{n}\times\vec{H}d\theta.$$

Using (4.2.18), (4.2.23) and (4.2.31) we deduce from this

$$\left|\frac{d\vec{L}_t}{dt}\right| \leq Ce^{-A}\frac{\delta(t)\tilde{\delta}(t)}{t^d}. \tag{4.2.34}$$

Defining $a(t) = \left|\vec{L}_t\right|$ yields $\left|\frac{da}{dt}\right| \le \left|\frac{d\vec{L}_t}{dt}\right|$ which, combined with (4.2.31) and (4.2.34) ensures

$$\left|\frac{da}{dt}\right| \le Ce^{-A}\frac{\delta(t)\tilde{\delta}(t)}{t^d}. \tag{4.2.35}$$

Then

$$\begin{aligned}\int_{2r}^{\frac{R}{2}} s^{1+d}\left|\frac{da}{ds}\right|(s)ds &\le Ce^{-A}\int_{2r}^{\frac{R}{2}} s\delta(s)\tilde{\delta}(s)\\ &\le Ce^{-A}\left(\int_{2r}^{\frac{R}{2}} s\delta(s)^2 ds\right)^{\frac{1}{2}}\left(\int_{2r}^{\frac{R}{2}} s\tilde{\delta}(s)^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

We can then apply (4.2.27) and (4.2.28) and conclude

$$\begin{aligned}\int_{2r}^{\frac{R}{2}} s^{1+d}\left|\frac{da}{ds}\right|(s)ds &\le Ce^{-A}\left\|\nabla\vec{n}\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)}\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)}\\ &\le CC_0e^{-A}\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)}.\end{aligned} \tag{4.2.36}$$

An integration by parts gives for any $r<\tau<T<R$,

$$\int_\tau^T s^{1+d}\frac{da}{ds}(s)ds = T^{1+d}a(T) - \tau^{1+d}a(\tau) - (1+d)\int_\tau^T s^d a(s)ds.$$

Hence, since $a \ge 0$, we have

— if $d \le -1$, for all $2r < t < \frac{R}{2}$,

$$t^{1+d}a(t) \le (2r)^{1+d}a(2r) + \int_{2r}^{\frac{R}{2}} s^{1+d}\left|\frac{da}{ds}\right|(s)ds,$$

— if $d \ge -1$, for all $2r < t < \frac{R}{2}$,

$$t^{1+d}a(t) \le \left(\frac{R}{2}\right)^{1+d} a\left(\frac{R}{2}\right) + \int_{2r}^{\frac{R}{2}} s^{1+d}\left|\frac{da}{ds}\right|(s)ds.$$

Then, if $d \le -1$, we take $\int_{\partial\mathbb{D}_{2r}}\vec{L} = 0$, whereas if $d \ge -1$, we take $\int_{\partial\mathbb{D}_{\frac{R}{2}}}\vec{L} = 0$.

In both cases, for all $2r < |x| < \frac{R}{2}$, thanks to (4.2.36), we have

$$\begin{aligned}|x|e^{\lambda(x)}\left|\vec{L}_{|x|}\right| &\le |x|^{d+1}e^A a(|x|)\\ &\le e^A\int_{2r}^{\frac{R}{2}} s^{1+d}\left|\frac{da}{dt}\right|(s)ds\\ &\le C\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)},\end{aligned} \tag{4.2.37}$$

where C depends only on C_0. Since $\frac{1}{|x|}$ is in $L^{2,\infty}$, we conclude with

$$\left\|e^{\lambda(x)}\vec{L}_{|x|}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}\right)} \le C\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)}. \tag{4.2.38}$$

Combined with (4.2.33), this yields the desired result :

$$\left\|e^{\lambda}\vec{L}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{R}{2}}\setminus\mathbb{D}_{2r}\right)} \le C\left\|H\nabla\Phi\right\|_{L^2(\mathbb{D}_R\setminus\mathbb{D}_r)}.$$

The constant appearing in the theorem corresponds to the choice of $\int_{\partial\mathbb{D}_{2r}}\vec{L} = 0$ or $\int_{\partial\mathbb{D}_{\frac{R}{2}}}\vec{L} = 0$ depending on d. $\square$

This result combines fairly well with theorem 4.2.1. Indeed thinking in the context of simple minimal bubbling, one can slice the domain of study into a bubble domain, on which $\vec{L}$ is bounded with theorem 1.4.1, and a neck domain where one estimates $\vec{L}$ thanks to theorem 4.2.2. Then $\vec{L}$ will bounded on the whole set, which allows for better controls and increasingly regular convergences.

4.3 Constraints on minimal bubbling

4.3.1 Consequences of the ε-regularity

The following result, which was the core of the prepublication [Mar19c], stands.

Theorem 4.3.1. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ a sequence of conformal, weak, Willmore immersions, of Gauss map $\vec{n}^\varepsilon$, mean curvature H^ε and conformal factor λ^ε, of parameter $\varepsilon > 0$. We assume

1. $\displaystyle\int_{\mathbb{D}} |\nabla \vec{n}^\varepsilon|^2 \, dz \leq M < \infty,$
2. $\|\nabla \lambda^\varepsilon\|_{L^{2,\infty}(\mathbb{D})} \leq M,$
3. $\displaystyle\lim_{R\to\infty} \left(\lim_{\varepsilon \to 0} \int_{\mathbb{D}_{\frac{1}{R}} \backslash \mathbb{D}_{\varepsilon R}} |\nabla \vec{n}^\varepsilon|^2 \, dz \right) = 0,$
4. $\Phi^\varepsilon \to \Phi^0$ in $C^\infty_{loc}(\mathbb{D}\backslash\{0\})$, with Φ^0 a branched Willmore immersion on $\mathbb{D}$,
5. There exists $C^\varepsilon > 0$ such that $\frac{\Phi^\varepsilon(\varepsilon.) - \Phi^\varepsilon(0)}{C^\varepsilon} \to \Psi$ in $C^\infty_{loc}(\mathbb{C})$, with Ψ a minimal immersion (that is of mean curvature $H_\Psi = 0$).

Then $\Phi^\varepsilon \to \Phi^0$ $C^{2,\alpha}(\mathbb{D})$ for all $\alpha < 1$.

The assumptions of the theorem are natural. They correspond to the conclusions of lemma 3.2.2, and thus to the compactness theorem 3.2.2 in a good conformal chart. The only added assumption is that the bubble is *minimal*. We are considering *simple minimal* bubbling.

Proof. In the following $\tilde{\Phi}^\varepsilon := \frac{\Phi^\varepsilon(\varepsilon.) - \Phi^\varepsilon(0)}{C^\epsilon} : \mathbb{D}_{\frac{1}{\varepsilon}} \to \mathbb{R}^3$ and $\tilde{\vec{n}}^\varepsilon$, $\tilde{H}^\varepsilon$ $\tilde{\lambda}^\varepsilon$ will denote respectively its Gauss map, its mean curvature and its conformal factor. We can check :

$$\tilde{\vec{n}}^\varepsilon = \vec{n}^\varepsilon(\varepsilon.), \tag{4.3.39}$$

$$\tilde{H}^\varepsilon \nabla \tilde{\Phi}^\varepsilon = \varepsilon H^\varepsilon \nabla \Phi^\varepsilon(\varepsilon.). \tag{4.3.40}$$

Then for all $\frac{1}{\varepsilon} > R > 0$

$$\int_{\mathbb{D}_{\epsilon R}} |\nabla \vec{n}^\varepsilon|^2 \, dz = \int_{\mathbb{D}_R} \left| \nabla \tilde{\vec{n}}^\varepsilon \right|^2 dz, \tag{4.3.41}$$

and

$$\int_{\mathbb{D}_{\varepsilon R}} |H^\epsilon \nabla \Phi^\varepsilon|^2 \, dz = \int_{\mathbb{D}_R} \left| \tilde{H}^\varepsilon \nabla \tilde{\Phi}^\varepsilon \right|^2 dz. \tag{4.3.42}$$

Hypothesis 5 implies

$$\lim_{\epsilon\to 0} \int_{\mathbb{D}_{\varepsilon R}} |\nabla \vec{n}^\varepsilon|^2 \, dz = \int_{\mathbb{D}_R} |\nabla \vec{n}_\Psi|^2 \, dz, \tag{4.3.43}$$

$$\lim_{\epsilon\to 0} \int_{\mathbb{D}_{\varepsilon R}} |H^\varepsilon \nabla \Phi^\varepsilon|^2 \, dz = \int_{\mathbb{D}_R} |H_\Psi \nabla \Psi|^2 \, dz = 0. \tag{4.3.44}$$

Besides combining (A.2.11) and hypothesis 3 yields

$$\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\frac{1}{R}}\setminus\mathbb{D}_{\varepsilon R}}|H^\varepsilon\nabla\Phi^\varepsilon|^2\,dz\right)\leq\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\frac{1}{R}}\setminus\mathbb{D}_{\varepsilon R}}|\nabla\vec{n}^\varepsilon|^2\,dz\right)=0. \tag{4.3.45}$$

Together (4.3.44) and (4.3.45) ensure that for R sufficiently big and ε sufficiently small

$$\|H^\varepsilon\nabla\Phi^\varepsilon\|_{L^2\left(\mathbb{D}_{\frac{1}{R}}\right)}\leq\varepsilon_0'(M), \tag{4.3.46}$$

with $\varepsilon_0'(M)$ given by theorem 4.2.1. Up to a rescaling, and thus without loss of generality, we can assume that (4.3.46) stands on $\mathbb{D}$. We will find a uniform $L^{2,\infty}$ bound on a first Willmore quantity, theorem 4.2.1 then gives the uniform controls proving theorem 4.3.1.

Recalling (4.3.41) yields

$$\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\varepsilon R}}|\nabla\vec{n}^\varepsilon|^2\,dz=\int_{\mathbb{D}_R}|\nabla\vec{n}_\Psi|^2\,dz.$$

Then either Ψ parametrizes a plane, and classical ε-regularity results yield smooth convergence (and there is de facto no real bubbling) or for R big enough,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\varepsilon R}}|\nabla\vec{n}^\varepsilon|^2\,dz>\frac{8\pi}{3}.$$

Then

$$\inf\left\{s\left|\int_{B_s(p)}|\nabla\vec{n}^\varepsilon|^2=\frac{8\pi}{6},\ \forall p\in\mathbb{D}\text{ s.t. }B_s(p)\subset\mathbb{D}\right.\right\}\to 0.$$

This means that the estimates given by theorem 1.4.1 degenerates as ε goes to 0. Finding a uniform control on $\vec{L}e^\lambda$ will require a "bubble-neck" decomposition. The bubble region will be $\mathbb{D}_{4\varepsilon R}$ while the neck region will be $\mathbb{D}_{\frac{1}{R}}\setminus\mathbb{D}_{\varepsilon R}$, with a R that we determine in what follows. We consider $\vec{L}^\varepsilon$ a first Willmore quantity of Φ^ε on $\mathbb{D}$.

Step 1 : Neck estimates

By hypothesis 3, there exists $R_0>0$ such that for ε small enough,

$$\int_{\mathbb{D}_{\frac{1}{R_0}}\setminus\mathbb{D}_{\varepsilon R_0}}|\nabla\vec{n}^\varepsilon|\leq\varepsilon_0,$$

where ε_0 is given by theorem 4.2.2. In turn, this ensures that

$$\sup_{\varepsilon R_0<s<\frac{1}{2R_0}}\int_{\mathbb{D}_{2s}\setminus\mathbb{D}_s}|\nabla\vec{n}^\varepsilon|^2\leq\varepsilon_0.$$

We can then apply theorem 4.2.2 and find a sequence $\vec{\mathcal{L}}_1^\varepsilon\in\mathbb{R}^3$ such that

$$\left\|\left(\vec{L}^\varepsilon-\vec{\mathcal{L}}_1^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2R_0}}\setminus\mathbb{D}_{2\varepsilon R_0}\right)}\leq C\,\|H^\varepsilon\nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})}, \tag{4.3.47}$$

where C depends solely on M defined in 1 and 2.

Step 2 : Bubble estimates

Let $p^\varepsilon=\varepsilon x^\varepsilon\in\mathbb{D}_{4R_0\varepsilon}$ and $r^\varepsilon=\varepsilon s^\varepsilon$ such that $B_{r^\varepsilon}(p^\varepsilon)\subset\mathbb{D}_{4R_0\varepsilon}$ and

$$\int_{B_{r^\varepsilon}(p^\varepsilon)}|\nabla\vec{n}^\varepsilon|^2=\frac{8\pi}{6}.$$

Then $x^\varepsilon \in \mathbb{D}_{4R_0}$ and $s^\varepsilon \leq 4R_0$, meaning that there exists $x \in \overline{\mathbb{D}_{4R_0}}$ and $s \leq 4R_0$ such that (up to a subsequence)

$$\begin{aligned} x^\epsilon &\to x, \\ s^\epsilon &\to s, \\ B_s(x) &\subset \mathbb{D}_{4R_0}. \end{aligned}$$

Adapting slightly (4.3.41) we find

$$\lim_{\varepsilon\to 0}\int_{B_{r^\varepsilon}(p^\varepsilon)} |\nabla \vec{n}^\varepsilon|^2\,dz = \lim_{\varepsilon\to 0}\int_{B_{s^\varepsilon}(x^\varepsilon)} \left|\nabla \tilde{\vec{n}}^\varepsilon\right|^2 dz = \int_{B_s(x)} |\nabla \vec{n}_\Psi|^2\,dz = \frac{8\pi}{6}.$$

Necessarily

$$\frac{s}{4R_0} \geq r_0^\Psi := \frac{1}{4R_0}\inf\left\{t \,\middle|\, \int_{B_t(p)} |\nabla \vec{n}_\Psi|^2 = \frac{8\pi}{6},\ \forall p \in \mathbb{D}_{4R_0} \text{ s.t. } B_s(t) \subset \mathbb{D}_{4R_0}\right\} > 0.$$

Thus if we set

$$r_0^\varepsilon := \frac{1}{4\varepsilon R_0}\inf\left\{r \,\middle|\, \int_{B_r(p)} |\nabla \vec{n}^\varepsilon|^2 = \frac{8\pi}{6},\ \forall p \in \mathbb{D}_{4\varepsilon R_0} \text{ s.t. } B_r(t) \subset \mathbb{D}_{4\varepsilon R_0}\right\},$$

we deduce that for ε small enough r_0^ϵ is uniformly bounded from below :

$$r_0^\epsilon \geq \frac{1}{10} r_0^\Psi. \tag{4.3.48}$$

Inequality (4.3.48) translates the simple bubbling of Φ^ε. While Φ^ε concentrates at 0 at the scale ε, $\tilde{\Phi}^\varepsilon$ does not concentrate any further, everything happens at the same scale for $\tilde{\Phi}^\varepsilon$. For instance corollary 1.3.2 ensures that the conformal factor satisfies a Harnack inequality. Namely we find $\Lambda^\varepsilon \in \mathbb{R}$ such that

$$\forall x \in \mathbb{D}_{3\varepsilon R_0} \quad \frac{e^{\Lambda^\varepsilon}}{C} \leq e^{\lambda^\varepsilon(x)} \leq C e^{\Lambda^\varepsilon}. \tag{4.3.49}$$

Here C depends on M and r_0^Ψ. Theorem 1.4.1 then allows us to control the first Willmore quantity ; i.e. there exists $\vec{\mathcal{L}}_2^\varepsilon \in \mathbb{R}^3$ such that

$$\left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)} \leq C(M, r_0^\Psi)\left\|H^\varepsilon \nabla \Phi^\varepsilon\right\|_{L^2(\mathbb{D})}. \tag{4.3.50}$$

Step 3 : Estimates across the concentration point

We first wish to estimate $\left|\vec{\mathcal{L}}_1^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right|$. Using (4.3.47) and (4.3.50) we find

$$\begin{aligned} \left\|\left(\vec{\mathcal{L}}_1^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)} &\leq \left\|\left(\vec{\mathcal{L}}_1^\varepsilon - \vec{L}^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)} \\ &\quad + \left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)} \\ &\leq \left\|\left(\vec{\mathcal{L}}_1^\varepsilon - \vec{L}^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2R_0}}\setminus\mathbb{D}_{2R_0\varepsilon}\right)} \\ &\quad + \left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right)e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\right)} \\ &\leq C(M, r_0^\Psi)\left\|H^\epsilon \nabla \Phi^\epsilon\right\|_{L^2(\mathbb{D})}. \end{aligned}$$

Thus

$$\left|\vec{\mathcal{L}}_1^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right| \leq \frac{C(M, r_0^\Psi)}{\|e^{\lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)}} \|H^\epsilon \nabla \Phi^\varepsilon\|_{L^2(\mathbb{D})}. \tag{4.3.51}$$

We can now assemble our estimates on the neck and the bubble. Using successively (4.3.47), (4.3.50) and (4.3.51) we find

$$\begin{aligned}
\left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2R_0}}\right)} &\leq \left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2R_0}}\setminus\mathbb{D}_{2\varepsilon R_0}\right)} \\
&+ \left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)} \\
&\leq C(M)\,\|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})} \\
&+ \left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)} \\
&+ \left\|\left(\vec{\mathcal{L}}_2^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)} \\
&\leq C(M, r_0^\Psi)\,\|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})} + \left|\vec{\mathcal{L}}_1^\varepsilon - \vec{\mathcal{L}}_2^\varepsilon\right| \left\|e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)} \\
&\leq C(M, r_0^\Psi)\,\|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})} \left(1 + \frac{\|e^{\lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)}}{\|e^{\lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)}}\right).
\end{aligned}$$

With (4.3.49), we can simplify the last right-hand term in the inequality.

$$\begin{aligned}
\frac{\|e^{\lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)}}{\|e^{\lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)}} &\leq C(M, r_0^\Psi) \frac{\|e^{\Lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)}}{\|e^{\Lambda^\varepsilon}\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)}} \\
&\leq C(M, r_0^\Psi) \frac{\|1\|_{L^{2,\infty}\left(\mathbb{D}_{3\varepsilon R_0}\right)}}{\|1\|_{L^{2,\infty}\left(\mathbb{D}_{3R_0\varepsilon}\setminus\mathbb{D}_{2R_0\varepsilon}\right)}} \\
&\leq C(M, r_0^\Psi)
\end{aligned}$$

since Λ^ε is a constant. Accordingly there exists $C(M, r_0^\Psi) > 0$ such that the following estimate across the concentration point stands.

$$\left\|\left(\vec{L}^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon\right) e^{\lambda^\varepsilon}\right\|_{L^{2,\infty}\left(\mathbb{D}_{\frac{1}{2R_0}}\right)} \leq C(M, r_0^\Psi)\,\|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})}. \tag{4.3.52}$$

Step 4 : Conclusion

We have then found a first Willmore quantity, $\vec{L}^\varepsilon - \vec{\mathcal{L}}_1^\varepsilon$, with uniform $L^{2,\infty}$ control on a disk of fixed radius $\rho = \frac{1}{2R_0}$. Since (4.3.46) stands we can apply theorem 4.2.1 on $\mathbb{D}_\rho$ and find

$$\|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^\infty\left(\mathbb{D}_{\frac{\rho}{2}}\right)} \leq C \|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D}_\rho)}, \tag{4.3.53}$$

$$\|\nabla\Phi^\varepsilon\|_{W^{3,p}\left(\mathbb{D}_{\frac{\rho}{2}}\right)} \leq C \|\nabla\Phi^\varepsilon\|_{L^2(\mathbb{D}_\rho)}, \tag{4.3.54}$$

while the second and third Willmore quantities satisfy

$$\|\nabla S^\varepsilon\|_{W^{1,p}\left(\mathbb{D}_{\frac{\rho}{2}}\right)} + \|\nabla \vec{R}^\varepsilon\|_{W^{1,p}\left(\mathbb{D}_{\frac{\rho}{2}}\right)} \leq C \|H^\varepsilon \nabla\Phi^\varepsilon\|_{L^2(\mathbb{D}_\rho)} \tag{4.3.55}$$

for all $p < \infty$.

Theorem 4.3.1 then follows from classical compactness results. □

4.3.2 Drawing a local framework

We will slightly change the conclusions of lemma 3.2.2 to draw a framework that better fits simple minimal bubbling.

Lemma 4.3.1. Let ξ_k be a sequence of Willmore immersions of a closed surface Σ satisfying the hypotheses of theorem 3.2.2. Then, in proper conformal charts around a concentration point on which a simple minimal bubble is blown, ξ^k yields a sequence of Willmore conformal immersions $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$, of conformal factor $\lambda^\varepsilon = \frac{1}{2}\ln\left(\frac{|\nabla \Phi^\varepsilon|^2}{2}\right)$, Gauss map $\vec{n}^\varepsilon$, mean curvature H^ε and tracefree curvature $\Omega^\varepsilon := 2\left\langle \Phi^\varepsilon_{zz}, \vec{n}^\varepsilon\right\rangle$, satisfying the following set of hypotheses :

1. There exists $C_0 > 0$ such that
$$\|\Phi^\varepsilon\|_{L^\infty(\mathbb{D})} + \|\nabla\Phi^\varepsilon\|_{L^2(\mathbb{D})} + \|\nabla\lambda^\varepsilon\|_{L^{2,\infty}(\mathbb{D})} + \|\nabla\vec{n}^\varepsilon\|_{L^2(\mathbb{D})} \leq C_0.$$

2. $\Phi^\varepsilon \to \Phi^0$ $C^\infty_{\mathrm{loc}}(\mathbb{D}\backslash\{0\})$, where Φ^0 is a true branched Willmore conformal immersion, with a unique branch point of multiplicity $\theta+1$ at 0, meaning that
$$\Phi^0_z \sim_0 \vec{A} z^\theta. \tag{4.3.56}$$
We denote λ^0 its conformal factor, $\vec{n}^0$ its Gauss map, H^0 its mean curvature and Ω^0 its tracefree curvature.

3. There exists a sequence of real numbers $C^\varepsilon > 0$ such that
$$\widetilde{\Phi}^\varepsilon := \frac{\Phi^\varepsilon(\varepsilon .) - \Phi^\varepsilon(0)}{C^\varepsilon} \to \Phi^1_z$$
$C^\infty_{\mathrm{loc}}(\mathbb{C})$, where Φ^1 is assumed to be a minimal conformal immersion of $\mathbb{C}$ with a branched end of multiplicity $\theta+1$, meaning that :
$$\Phi^1_z \sim_\infty \widetilde{A} z^\theta.$$
We denote λ^1 its conformal factor, $\vec{n}^1$ its Gauss map, H^1 its mean curvature and Ω^1 its tracefree curvature.

4. $$\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{\frac{1}{R}}\backslash\mathbb{D}_{\varepsilon R}} |\nabla\vec{n}^\varepsilon|^2\, dz\right) = 0.$$

5. $\left|\Omega^\varepsilon e^{-\lambda^\varepsilon}\right|$ reaches its maximum at 0 and
$$\left|\Omega^\varepsilon e^{-\lambda^\varepsilon}\right|(0) = \frac{2}{\varepsilon}.$$

Proof. Hypotheses 1, 2 and 4 were already obtained in lemma 3.2.2. Hypothesis 3 then specifies that we consider the case where there is only one simple minimal bubble which concentrates on 0 in the aforementioned chart. The equality of the multiplicity of the end of the bubble and the branch point of the surface is merely a consequence of theorem 3.2.3. Further, by definition of a concentration point
$$\left\|\nabla\vec{n}^k\right\|_{L^\infty(\mathbb{D})} \to \infty.$$

On the other hand, theorem 4.3.1 states that

$$\left\| H^k \nabla \Phi^k \right\|_{L^\infty(\mathbb{D})} \leq C(C_0).$$

Since $\left|\nabla \vec{n}^k\right|^2 = \left|H^k \nabla \Phi^k\right|^2 + \left|\Omega^k e^{-\lambda^k}\right|^2$, necessarily

$$\left\| \Omega^k e^{-\lambda^k} \right\|_{L^\infty(\mathbb{D})} \to \infty.$$

We then redefine the concentration speed as

$$\varepsilon_k = \frac{2}{\left\| \Omega^k e^{-\lambda^k} \right\|_{L^\infty(\mathbb{D})}},$$

and we assume it is reached at the origin. For simplicity's sake we reparametrize this sequence by the concentration speed which we denote ε. Hypothesis 5 is then a consequence of this slight adjustment. □

This modification is done in order to more subtly detect the phenomena. Indeed for minimal simple bubbling, the mean curvature terms remain bounded, and thus concentration will happen entirely on the tracefree parts. Further, controls on Ω will be more easily reverberated onto Φ than controls on $\nabla \vec{n}$.

We can expand on hypotheses 1-5 up to, first macroscopic, then microscopic, adjustments.

Lemma 4.3.2. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ be a sequence of Willmore conformal immersions satisfying hypotheses 1-5. Then θ is even, and up to macroscopic adjustments we can assume that :

6.
$$\Phi^1_z = \frac{1}{2} \begin{pmatrix} Q^2 - P^2 \\ i\left(Q^2 + P^2\right) \\ 2PQ \end{pmatrix},$$

where $P, Q \in \mathbb{C}_{\frac{\theta}{2}}[X]$, $P \wedge Q = 1$, and

$$\begin{aligned} P(0) &= 0, \\ Q(0) = P'(0) &= 1, \\ P''(0) &= 2Q'(0). \end{aligned}$$

The end of multiplicity $\theta+1$ of Φ^1 can be highlighted as follows : there exists $\widetilde{A} \in \mathbb{C}^3 \backslash \{0\}$ and $V \in \mathbb{C}_{\theta-1}[X]$ such that

$$\Phi^1_z = \widetilde{A} z^\theta + V.$$

Proof. **Adjusting with homothetic transformations of $\mathbb{R}^3$:**
Since Φ^1 has no branch point on $\mathbb{C}$, up to a fixed rotation and a dilation in $\mathbb{R}^3$ one can assume :

$$\Phi^1_z(0) = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{4.3.57}$$

Adjusting the parametrization :

Taking $M_{-\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we set $\Psi := M_{-\theta} \Phi^1\left(e^{i\theta}.\right)$. We denote respectively

λ_Ψ, H_Ψ, Ω_Ψ and $\vec{n}_\Psi$ its conformal factor, its mean curvature, its tracefree curvature and its Gauss map. Then $\Psi_z = e^{i\theta} M_{-\theta} \Phi^1_z \left(e^{i\theta}.\right)$ which implies $\vec{n}_\Psi = M_{-\theta}\vec{n}$ and $e^{\lambda_\Psi} = e^{\lambda^1}$. Consequently using (4.3.57)

$$\Psi_z(0) = \frac{e^{i\theta}}{2} M_\theta \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \frac{e^{i\theta}}{2} \begin{pmatrix} e^{-i\theta} \\ ie^{-i\theta} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{4.3.58}$$

Further we can compute $\Psi_{zz} = e^{2i\theta} M_{-\theta} \Phi^1_{zz}\left(e^{i\theta}\right)$ and deduce

$$\Omega_\Psi = e^{2i\theta}\Omega^1,$$

which in turn implies

$$\left[\Omega_\Psi e^{-\lambda_\Psi}\right](0) = e^{2i\theta}\left[\Omega^1 e^{-\lambda^1}\right](0).$$

According to (4.3.78), one can choose θ such that $e^{2i\theta} = \frac{-2}{\left[\Omega^1 e^{-\lambda^1}\right](0)}$. In that case $\Omega_\Psi(0)e^{-\lambda_\Psi(0)} = -2$, which yields thanks to (4.3.58)

$$\Omega_\Psi(0) = -2. \tag{4.3.59}$$

The sequence $\Psi^\varepsilon = M_\theta \Phi^\varepsilon\left(e^{i\theta}.\right)$ satisfies hypotheses 1, 4 and 5, while

$$\begin{aligned} &\Psi^\varepsilon \to M_\theta \Phi^0\left(e^{i\theta}\right) \quad C^\infty_{\mathrm{loc}}\left(\mathbb{D}\backslash\{0\}\right), \\ &\widetilde{\Psi}^\varepsilon \to \Psi \quad C^\infty_{\mathrm{loc}}\left(\mathbb{C}\right). \end{aligned}$$

We will not change notations for simplicity's sake, and will merely assume, without loss of generality, that

$$\Omega^1(0) = -2. \tag{4.3.60}$$

Summing up :

$$\begin{aligned} &\Phi^1_z = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \\ &\Omega^1(0) = -2, \\ &\left(\left|\Omega^1\right|^2 e^{-2\lambda^1}\right)_z(0) = 0. \end{aligned} \tag{4.3.61}$$

Consequences on the Enneper-Weierstrass representation :

Since Φ^1 is a minimal immersion, we can use the Enneper-Weierstrass representation :

$$\Phi^1_z = \frac{f}{2}\begin{pmatrix} 1-g^2 \\ i\left(1+g^2\right) \\ 2g \end{pmatrix}$$

where f is a holomorphic function on $\mathbb{C}$ and g a meromorphic one. Since (according to (4.3.74)) Φ^1 has finite total curvature, g is a meromorphic function of finite degree on $\mathbb{C}$. Thus, there exists two polynomials P, $Q \in \mathbb{C}[X]$ such that $P \wedge Q = 1$ and $g = \frac{P}{Q}$. Since Φ^1 has no end on $\mathbb{C}$, f has a zero of order $2k$ at each pole of order k of g. Consequently there exists a holomorphic function $\widetilde{f}$ such that $f = Q^2\widetilde{f}$. Further, Φ^1 has no branch point on $\mathbb{C}$ and one finite end at ∞, thus $\widetilde{f}$ is a holomorphic function without zeros and of finite order at infinity, *i.e.* a constant. We can then write

$$\Phi^1_z = \frac{1}{2}\begin{pmatrix} Q^2 - P^2 \\ i\left(Q^2 + P^2\right) \\ 2PQ \end{pmatrix}. \tag{4.3.62}$$

Further since Φ^1 is assumed to have an end of multiplicity $\theta + 1$ one can expand (4.3.62) as

$$\Phi^1_z = \widetilde{A} z^\theta + O\left(z^{\theta-1}\right), \tag{4.3.63}$$

where $\widetilde{A} \in \mathbb{C}^3 \backslash \{0\}$. Comparing (4.3.62) and (4.3.63) notably implies that θ is even and $P, Q \in \mathbb{C}_{\frac{\theta}{2}}[X]$. From (4.3.62), we then successively deduce

$$\vec{n}^1 = \frac{1}{|P|^2 + |Q|^2}\begin{pmatrix} P\overline{Q} + \overline{P}Q \\ i\left(\overline{P}Q - P\overline{Q}\right) \\ |P|^2 - |Q|^2 \end{pmatrix}, \tag{4.3.64}$$

$$e^{2\lambda^1} = \left(|P|^2 + |Q|^2\right)^2, \tag{4.3.65}$$

$$\Phi^1_{zz} = Q'\begin{pmatrix} Q \\ Qi \\ P \end{pmatrix} - P'\begin{pmatrix} P \\ -iP \\ -Q \end{pmatrix}, \tag{4.3.66}$$

which implies

$$\Omega^1 = 2\left(PQ' - P'Q\right), \tag{4.3.67}$$

and in turn

$$\Omega^1 e^{-\lambda^1} = 2\frac{PQ' - P'Q}{|P|^2 + |Q|^2}, \tag{4.3.68}$$

$$\left|\Omega^1 e^{-\lambda^1}\right|^2 = 4\frac{|P|^2|Q'|^2 + |P'|^2|Q|^2 - P\overline{P'}Q'\overline{Q} - P'Q\overline{PQ'}}{(|P|^2 + |Q|^2)^2}. \tag{4.3.69}$$

Conditions (4.3.61) then translate on P and Q as

$$\begin{aligned} P(0) &= 0, \\ Q(0) &= P'(0) = 1, \\ P''(0) &= 2Q'(0). \end{aligned} \tag{4.3.70}$$

This concludes the proof. □

Below, we write and prove the infinitesimal counterpart of theorem 4.3.2.

Lemma 4.3.3. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ be a sequence of Willmore conformal immersions satisfying hypotheses 1-6.
Up to infinitesimal adjustments we can assume that :

7.

$$\Phi^\varepsilon_z(0) = \frac{C^\varepsilon}{\varepsilon}\Phi^1_z(0) = \frac{C^\varepsilon}{2\varepsilon}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix},$$

$$\left[\Omega^\varepsilon e^{-\lambda^\varepsilon}\right](0) = \frac{1}{\varepsilon}\left[\Omega^1 e^{-\lambda^1}\right](0) = -\frac{2}{\varepsilon}.$$

Proof. **Using homothetic transformations of $\mathbb{R}^3$:**
By hypothesis 3, $\widetilde{\Phi}^\varepsilon_z(0) \to \Phi^1_z(0)$, thus there exists a sequence of homothetic transformations $\sigma^\varepsilon \to Id$ such that $\sigma^\varepsilon \widetilde{\Phi}^\varepsilon_z(0) = \Phi^1_z(0)$. Since σ^ε tends toward the identity, hypotheses 1-3 are still satisfied, and hypothesis 5 still stands due to the conformal invariance properties of the tracefree curvature. We will then apply this sequence of transformations without changing the notations for simplicity's sake and assume $\widetilde{\Phi}^\varepsilon_z(0) = \Phi^1_z(0)$. Considering

$$\widetilde{\Phi}^\varepsilon_z = \frac{\varepsilon}{C^\varepsilon}\Phi^\varepsilon_z,$$

we deduce with (4.3.61),

$$\Phi^\varepsilon_z(0) = \frac{C^\varepsilon}{2\varepsilon}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}. \tag{4.3.71}$$

Adjusting the parametrization :
Using (4.3.78) and (4.3.76), we can set

$$e^{2i\theta^\varepsilon} = \frac{\left[\widetilde{\Omega}^\varepsilon e^{-\widetilde{\lambda}^\varepsilon}\right](0)}{\left[\Omega^1 e^{-\lambda^1}\right](0)} = \varepsilon\frac{\left[\Omega^\varepsilon e^{-\lambda^\varepsilon}\right](0)}{\left[\Omega^1 e^{-\lambda^1}\right](0)} \to 1.$$

We consider $\Psi^\varepsilon = M_{\theta^\varepsilon}\Phi^\varepsilon\left(e^{i\theta^\varepsilon}.\right)$. Since $e^{i\theta^\varepsilon} \to 1$, Ψ^ε satisfies 1-6 (5 is still satisfied due to the invariance properties of the tracefree curvature). As detailed in the previous section we have

$$\widetilde{\Omega}^\varepsilon_\Psi e^{-\widetilde{\lambda}^\varepsilon_\Psi} = -2,$$

which implies

$$\left[\Omega^\varepsilon_\Psi e^{-\lambda^\varepsilon_\Psi}\right](0) = -\frac{2}{\varepsilon}.$$

For simplicity's sake we will not change the notations and assume that Φ^ε satisfies

$$\left[\Omega^\varepsilon e^{-\lambda^\varepsilon}\right](0) = -\frac{2}{\varepsilon}. \tag{4.3.72}$$

This gives us the desired result. $\square$

An immediate consequence of hypotheses 2-4 is the following energy quantization result

$$\int_{\mathbb{D}} |\nabla \vec{n}^\varepsilon|^2\, dz \to \int_{\mathbb{D}} \left|\nabla \vec{n}^0\right|^2 dz + \int_{\mathbb{C}} \left|\nabla \vec{n}^1\right|^2 dz. \tag{4.3.73}$$

while hypothesis 1 ensures

$$\left\|\nabla \vec{n}^0\right\|_{L^2(\mathbb{D})} + \left\|\nabla \vec{n}^1\right\|_{L^2(\mathbb{C})} \leq C(C_0). \tag{4.3.74}$$

Further if we denote $\widetilde{\lambda}^\varepsilon$ the conformal factor of $\widetilde{\Phi}^\varepsilon$, $\widetilde{H}^\varepsilon$ its mean curvature, $\widetilde{\Omega}^\varepsilon$ its tracefree curvature and $\widetilde{\vec{n}}^\varepsilon$ its Gauss map, we have

$$\widetilde{\lambda}^\varepsilon = \lambda^\varepsilon(\varepsilon.) - \ln\left(\frac{C^\varepsilon}{\varepsilon}\right), \tag{4.3.75}$$

and

$$\widetilde{\Omega}^\varepsilon e^{-\widetilde{\lambda}^\varepsilon} = \varepsilon\left[\Omega^\varepsilon e^{-\lambda^\varepsilon}\right](\varepsilon.)\,. \tag{4.3.76}$$

Using hypothesis 5, one may conclude that

$$\left|\widetilde{\Omega}^{\varepsilon} e^{-\widetilde{\lambda}^{\varepsilon}}\right|(0) = 2. \tag{4.3.77}$$

Hypothesis 3 then yields

$$\left|\Omega^{1} e^{-\lambda^{1}}\right|(0) = 2. \tag{4.3.78}$$

Similarly we know, thanks to hypotheses 3 and 5,

$$\left(\left|\widetilde{\Omega}^{\varepsilon} e^{-\widetilde{\lambda}^{\varepsilon}}\right|\right)_z (0) = 0, \tag{4.3.79}$$

and :

$$\left(\left|\Omega^{1} e^{-\lambda^{1}}\right|\right)_z (0) = 0. \tag{4.3.80}$$

Finally, hypothesis 4 allows us to apply theorem 1.5.1 to Φ^0 : there exists $\alpha \leq \theta_0$, $\vec{A} \in \mathbb{C}^3 \backslash \{0\}$, $\left(\vec{B}_j\right)_{j=1..\theta_0+1-\alpha} \in \mathbb{C}^3$, $\vec{C}_\alpha \in \mathbb{C}^3\backslash\{0\}$ and $\xi : \mathbb{D} \to \mathbb{R}^3$ such that

$$\Phi^0_z = \vec{A} z^{\theta_0} + \sum_{j=1}^{\theta_0+1-\alpha} \vec{B}_j z^{\theta_0+j} + \frac{\vec{C}_\alpha}{\theta_0+1} z^{\theta_0-\alpha} \bar{z}^{\theta_0+1} + \frac{\overline{\vec{C}_\alpha}}{\theta_0+1-\alpha} z^{\theta_0} \bar{z}^{\theta_0+1-\alpha} + \xi_z, \tag{4.3.81}$$

where ξ satisfies :

$$\nabla^j \xi = O\left(r^{2\theta_0+3-\alpha-j-\upsilon}\right),$$

for all $\upsilon > 0$ and $j \leq \theta_0 + 2 - \alpha$. The second residue α is in fact defined as follows (see theorem I.8 in [BR13]) :

$$H^0 \sim C_\alpha |z|^{-\alpha}. \tag{4.3.82}$$

Our proofs will use the quantities $\vec{L}$, S and $\vec{R}$, stemming from the Willmore conservation laws (see for instance theorem I.4 in [Riv08]), which at the core, are a consequence of the conformal invariance of W (see [Ber16]). More precisely $\vec{L}$, S and $\vec{R}$ are defined as follows :

$$\begin{aligned} \nabla^\perp \vec{L} &= \nabla \vec{H} - 3\pi_{\vec{n}}\left(\nabla \vec{H}\right) + \nabla^\perp \vec{n} \times \vec{H}, \\ \nabla^\perp S &= \langle \vec{L}, \nabla^\perp \Phi \rangle, \\ \nabla^\perp \vec{R} &= \vec{L} \times \nabla^\perp \Phi + 2H \nabla^\perp \Phi. \end{aligned} \tag{4.3.83}$$

Exploiting these was key in T. Rivière's proof of the ε-regularity for Willmore surfaces.

Under hypotheses 1-5, the conclusion of [Mar19c] stands and yields (see (96)-(98) in the aforementioned paper) :

$$\|H^\varepsilon \nabla \Phi^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C(C_0), \tag{4.3.84}$$

$$\|\nabla \Phi^\varepsilon\|_{W^{3,p}(\mathbb{D})} \leq C(C_0), \tag{4.3.85}$$

while the second and third Willmore quantities satisfy

$$\|\nabla S^\varepsilon\|_{W^{1,p}(\mathbb{D})} + \|\nabla \vec{R}^\varepsilon\|_{W^{1,p}(\mathbb{D})} \leq C(C_0) \tag{4.3.86}$$

for all $p < \infty$. Up to an inconsequential translation one can further assume $\Phi^\varepsilon(0) = 0$.

4.3.3 Local expansion on the conformal factor

This section will prove the following expansion on the conformal factor, which will serve as a stepping point in the proof of theorem G.

Theorem 4.3.2. Let Φ^ε be a sequence of Willmore conformal immersions satisfying 1-7. Then there exists $l^\varepsilon \in L^\infty(\mathbb{D})$ such that :

$$\lambda^\varepsilon = \ln\left(\varepsilon^\theta + r^\theta\right) + l^\varepsilon,$$
$$\|l^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C(C_0).$$

As a result if we denote $\chi = \sqrt{\varepsilon^2 + r^2}$, the immersion satisfies the following Harnack inequality :

$$\frac{\chi^\theta}{C(C_0)} \leq e^{\lambda^\varepsilon} \leq C(C_0)\chi^\theta.$$

Proof. **Step 1 : Controls on the neck area**
Given hypothesis 4, for any $\varepsilon_0 > 0$ arbitrarily small there exists R big enough such that

$$\lim_{\varepsilon\to 0}\left(\int_{\mathbb{D}_{\frac{1}{R}}\backslash\mathbb{D}_{\varepsilon R}} |\nabla \vec{n}^\varepsilon|^2\, dz\right) \leq \varepsilon_0. \tag{4.3.87}$$

We first recall lemma V.3 of [BR14].

Lemma 4.3.4. There exists a constant $\eta > 0$ with the following property. Let $0 < 4r < R < \infty$. If Φ is any (weak) conformal immersion of $\Omega := \mathbb{D}_R\backslash\mathbb{D}_r$ into $\mathbb{R}^3$ with L^2-bounded second fundamental form and satisfying

$$\|\nabla \vec{n}\|_{L^{2,\infty}(\Omega)} < \sqrt{\eta},$$

then there exist $\frac{1}{2} < \alpha < 1$ and $A \in \mathbb{R}$ depending on R, r, m and Φ such that

$$\|\lambda(x) - d\ln|x| - A\|_{L^\infty\left(\mathbb{D}_{\alpha R}\backslash\mathbb{D}_{\frac{r}{\alpha}}\right)} \leq C\left(\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2\right), \tag{4.3.88}$$

where d satisfies

$$\begin{aligned}\left|2\pi d - \int_{\partial\mathbb{D}_r} \partial_r\lambda dl_{\partial\mathbb{D}_r}\right| \leq & C\left[\int_{\mathbb{D}_{2r}\backslash\mathbb{D}_r} |\nabla\vec{n}|^2\, dz\right.\\ &\left. + \frac{1}{\ln\frac{R}{r}}\left(\|\nabla\lambda\|_{L^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2\right)\right].\end{aligned} \tag{4.3.89}$$

Thus, according to (4.3.87), there exists R_0 such that for all $R \geq R_0$ and ε small enough, we can apply lemma 4.3.4 on $\mathbb{D}_{\frac{1}{R}}\backslash\mathbb{D}_{\varepsilon R}$ and conclude that there exists d^ε_R and $A^\varepsilon_R \in \mathbb{R}$ such that

$$\|\lambda^\varepsilon(x) - d^\varepsilon_R \ln r - A^\varepsilon_R\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R}}\backslash\mathbb{D}_{2\varepsilon R}\right)} \leq C_0, \tag{4.3.90}$$

$$\left|d^\varepsilon_R - \frac{1}{2\pi}\int_{\partial\mathbb{D}_{\varepsilon R}} \partial_r\lambda^\varepsilon dl_{\partial\mathbb{D}_{\varepsilon R}}\right| \leq C\left[\int_{\mathbb{D}_{2\varepsilon R}\backslash\mathbb{D}_{\varepsilon R}} |\nabla\vec{n}|^2\, dz + \frac{C_0}{-\ln(\varepsilon R^2)}\right]. \tag{4.3.91}$$

Here C_0 is the uniform bound given by hypothesis 1 (up to a multiplicative uniform constant). We saw in (3.2.7) that

$$\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\partial\mathbb{D}_{\varepsilon R}}\partial_r\lambda^\varepsilon dl_{\partial\mathbb{D}_{\varepsilon R}}\right)=\theta,$$

while hypothesis 4 ensures that

$$\lim_{R\to\infty}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{D}_{2\varepsilon R}\backslash\mathbb{D}_{\varepsilon R}}|\nabla\vec{n}|^2\,dz\right)=0.$$

Hence we can fix $R_1>0$ such that for ε small enough :

$$|d^\varepsilon_R-\theta|\leq\frac{1}{10^3}. \tag{4.3.92}$$

Since R_1 is fixed, we will get rid of the subscript on d^ε and A^ε. Then for any ε small enough :

$$\|\lambda^\varepsilon(x)-d^\varepsilon\ln r-A^\varepsilon\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}\right)}\leq C(C_0,R_1). \tag{4.3.93}$$

Step 2 : Estimates on the exterior boundary :

Hypothesis 2 ensures that on $\partial\mathbb{D}_{\frac{1}{2R_1}}$, $\lambda^\varepsilon\to\lambda^0$ smoothly, and that λ^0 is a bounded function away from 0, which implies

$$\|\lambda^\varepsilon\|_{L^\infty\left(\partial\mathbb{D}_{\frac{1}{2R_1}}\right)}\leq C(C_0,R_1). \tag{4.3.94}$$

On the other hand, (4.3.92) ensures

$$|d^\varepsilon\ln R_1|\leq C(C_0,R_1). \tag{4.3.95}$$

As a result, combining (4.3.93) on $\partial\mathbb{D}_{\frac{1}{2R_1}}$, (4.3.94), and (4.3.95) yields

$$|A^\varepsilon|\leq C(C_0,R_1),$$

which we can inject in (4.3.93) to obtain

$$\|\lambda^\varepsilon(x)-d^\varepsilon\ln r\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}\right)}\leq C(C_0,R_1). \tag{4.3.96}$$

Step 3 : Estimates on the interior boundary :

Estimate (4.3.96) implies

$$\|\lambda^\varepsilon(x)-d^\varepsilon\ln r\|_{L^\infty(\partial\mathbb{D}_{2\varepsilon R_1})}\leq C(C_0,R_1). \tag{4.3.97}$$

Further (4.3.75) yields

$$\|\lambda^\varepsilon(x)-d^\varepsilon\ln r\|_{L^\infty(\partial\mathbb{D}_{2\varepsilon R_1})}=\left\|\widetilde{\lambda}^\varepsilon(x)-d^\varepsilon\ln r+\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)-d^\varepsilon\ln\varepsilon\right\|_{L^\infty(\partial\mathbb{D}_{2R_1})}.$$

Hypothesis 3 then ensures that

$$\left\|\widetilde{\lambda}^\varepsilon\right\|_{L^\infty(\mathbb{D}_{2R_1})}\leq C(C_0,R_1), \tag{4.3.98}$$

and (4.3.92) that

$$\|d^\varepsilon \ln r\|_{L^\infty(\mathbb{D}_{2R_1})} \leq C(C_0, R_1). \tag{4.3.99}$$

Together (4.3.97), (4.3.98) and (4.3.99) yield

$$\left| \ln\left(\frac{C^\varepsilon}{\varepsilon^{d^\varepsilon+1}}\right)\right| \leq C(C_0, R_1). \tag{4.3.100}$$

A direct consequence of (4.3.92) and (4.3.100) is the following estimate :

$$\frac{\varepsilon^{\theta+1-10^{-3}}}{C(R_1,C_0)} \leq \frac{\varepsilon^{d^\varepsilon+1}}{C(C_0,R_1)} \leq C^\varepsilon \leq C(C_0,R_1)\varepsilon^{d^\varepsilon+1} \leq C(C_0,R_1)\varepsilon^{\theta+1+10^{-3}}. \tag{4.3.101}$$

Step 4 : Expanding the conformal factor on the whole disk :

We forcefully write $\lambda^\varepsilon = \ln\left(\varepsilon^{d^\varepsilon} + r^{d^\varepsilon}\right) + l^\varepsilon$. We aim to show that

$$\|l^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C(C_0, R_1).$$

On $\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}$:

Using hypothesis 2,

$$\|\lambda^\varepsilon\|_{L^\infty\left(\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}\right)} \leq C(C_0, R_1). \tag{4.3.102}$$

One might also notice that, thanks to (4.3.92), on $\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}$:

$$\left|\ln\left(\varepsilon^{d^\varepsilon} + r^{d^\varepsilon}\right)\right| \leq C(C_0,R_1)\left|\ln\left(\left(\frac{1}{2R_1}\right)^{m+\frac{1}{100}}\right)\right|. \tag{4.3.103}$$

Then using (4.3.102) and (4.3.103) :

$$\begin{aligned} \|l^\varepsilon\|_{L^\infty\left(\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}\right)} &\leq \|\lambda^\varepsilon\|_{L^\infty\left(\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}\right)} + \left\|\ln\left(\varepsilon^{d^\varepsilon} + r^{d^\varepsilon}\right)\right\|_{L^\infty\left(\mathbb{D}\backslash\mathbb{D}_{\frac{1}{4R_1}}\right)} \\ &\leq C(C_0, R_1). \end{aligned} \tag{4.3.104}$$

On $\mathbb{D}_{4\varepsilon R_1}$:

Using hypothesis 3

$$\left\|\widetilde{\lambda}^\varepsilon\right\|_{L^\infty(\mathbb{D}_{4R_1})} \leq C(C_0, R_1), \tag{4.3.105}$$

while thanks to (4.3.100)

$$\begin{aligned} \left\|\ln\left(\varepsilon^{d^\varepsilon} + r^{d^\varepsilon}\right) - \ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})} &\leq \left\|\ln\left(\varepsilon^{d^\varepsilon} + (\varepsilon r)^{d^\varepsilon}\right) - \ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4R_1})} \\ &\leq \left\|\ln\left(1 + r^{d^\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4R_1})} + \left\|\ln\left(\frac{C^\varepsilon}{\varepsilon^{d^\varepsilon+1}}\right)\right\|_{L^\infty(\mathbb{D}_{4R_1})} \\ &\leq C(C_0, R_1). \end{aligned} \tag{4.3.106}$$

To conclude, we deduce thanks to (4.3.105) and (4.3.106) :

$$\begin{aligned}\|l^\varepsilon\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})} &\leq \left\|\lambda^\varepsilon - \ln\left(\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}\right)+\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)-\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})}\\ &\leq \left\|\lambda^\varepsilon-\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})}+\left\|\ln\left(\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}\right)-\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})}\\ &\leq \left\|\widetilde{\lambda}^\varepsilon\right\|_{L^\infty(\mathbb{D}_{4R_1})}+\left\|\ln\left(\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}\right)-\ln\left(\frac{C^\varepsilon}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{D}_{4\varepsilon R_1})}\\ &\leq C(C_0,R_1).\end{aligned} \tag{4.3.107}$$

<u>On $\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}$:</u>
Thanks to (4.3.96) :

$$\begin{aligned}\|l^\varepsilon\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}\right)} &\leq \|\lambda^\varepsilon - d^\varepsilon\ln r\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}\right)}+\left\|\ln\left(\frac{r^{d^\varepsilon}}{\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}}\right)\right\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\backslash\mathbb{D}_{2\varepsilon R_1}\right)}\\ &\leq C(C_0,R_1).\end{aligned} \tag{4.3.108}$$

Combining (4.3.104), (4.3.107) and (4.3.108) yields

$$\|l^\varepsilon\|_{L^\infty(\mathbb{D})}\leq C(C_0,R_1), \tag{4.3.109}$$

which is as desired. We now wish to refine this first expansion by showing that d^ε converges toward θ fast enough to be replaced in (4.3.96).

<u>**Step 5 : Refinement :**</u>
A consequence of estimate (4.3.109) is the following Harnack inequality on the conformal factor :

$$\frac{\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}}{C(C_0,R_1)}\leq e^{\lambda^\varepsilon}\leq C(C_0,R_1)\left(\varepsilon^{d^\varepsilon}+r^{d^\varepsilon}\right)$$

which, using the notation $\chi=\sqrt{\varepsilon^2+r^2}$, we will rewrite in the more convenient form

$$\frac{\chi^{d^\varepsilon}}{C(C_0,R_1)}\leq e^{\lambda^\varepsilon}\leq C(C_0,R_1)\chi^{d^\varepsilon}. \tag{4.3.110}$$

Injecting (4.3.92) into (4.3.110) yields

$$e^{\lambda^\varepsilon}\leq C(C_0,R_1)\chi^{\theta-10^{-3}}. \tag{4.3.111}$$

Since Φ^ε is conformal,

$$\Delta\Phi^\varepsilon = 2H^\varepsilon e^{2\lambda^\varepsilon}\vec{n}^\varepsilon=\chi^{\theta-10^{-3}}2H^\varepsilon e^{\lambda^\varepsilon}\frac{e^{\lambda^\varepsilon}}{\chi^{\theta-10^{-3}}}\vec{n}^\varepsilon. \tag{4.3.112}$$

Noticing that (4.3.84) and(4.3.111) imply

$$\left\|2H^\varepsilon e^{\lambda^\varepsilon}\frac{e^{\lambda^\varepsilon}}{\chi^{\theta-10^{-3}}}\right\|_{L^\infty(\mathbb{D})}\leq C(C_0,R_1), \tag{4.3.113}$$

we can apply theorem A.3.9 to equation (4.3.112) and find :

$$\Phi^\varepsilon_z = P^\varepsilon(z)+\varphi^\varepsilon_0, \tag{4.3.114}$$

where $P^\varepsilon = \sum_{q=0}^{\theta} p_q^\varepsilon z^q \in \mathbb{C}_\theta[X]$ with $\left|p_q^\varepsilon\right| \leq C(C_0, R_1)$ for all $q \leq \theta$ and $\varphi_0^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ satisfies

$$\begin{aligned} &\forall \upsilon > 0 \quad \left\| \frac{\varphi_0^\varepsilon}{\chi^{\theta+1-10^{-3}-\upsilon}} \right\|_{L^\infty(\mathbb{D})} \leq C_\upsilon\left(C_0, R_1\right), \\ &\forall p < \infty \quad \left\| \frac{\nabla\varphi_0^\varepsilon}{\chi^{\theta-10^{-3}}} \right\|_{L^p(\mathbb{D})} \leq C_p\left(C_0, R_1\right). \end{aligned} \tag{4.3.115}$$

By convergence of Φ^ε away from zero (hypothesis 2), $p_q^\varepsilon \to p_q \in \mathbb{C}$ as ε goes to 0. Further (4.3.115) yields $\varphi_0^\varepsilon \to \varphi_0$ $W^{1,p}(\mathbb{D})$, with φ_0 satisfying

$$\begin{aligned} &\forall \upsilon > 0 \quad \left\| \frac{\varphi_0}{r^{\theta+1-10^{-3}-\upsilon}} \right\|_{L^\infty(\mathbb{D})} \leq C_\upsilon\left(C_0, R_1\right), \\ &\forall p < \infty \quad \left\| \frac{\nabla\varphi_0}{r^{\theta-10^{-3}}} \right\|_{L^p(\mathbb{D})} \leq C_p\left(C_0, R_1\right), \end{aligned} \tag{4.3.116}$$

since $\chi \to r$ as $\varepsilon \to 0$. Then (4.3.114) ensures that

$$\Phi_z^\varepsilon \to \sum_{q=0}^{\theta} p_q z^q + \varphi_0. \tag{4.3.117}$$

Since we assumed that $\Phi^\varepsilon \to \Phi^0$ away from 0, comparing (4.3.81) and (4.3.117) yields

$$\begin{aligned} &\forall q < \theta \quad p_q^\varepsilon \to 0 \\ &p_\theta^\varepsilon \to \vec{A} \neq 0. \end{aligned} \tag{4.3.118}$$

Further, (4.3.114) gives the following

$$\widetilde{\Phi}_z^\varepsilon = \sum_{q=0}^{\theta} p_q^\varepsilon \frac{\varepsilon^{q+1}}{C^\varepsilon} z^q + \frac{\varepsilon \varphi_0^\varepsilon(\varepsilon.)}{C^\varepsilon}. \tag{4.3.119}$$

One might also notice using (4.3.115)

$$\begin{aligned} \left|\varphi_0^\varepsilon\right|(\varepsilon z) &\leq C(C_0, R_1)\chi^{\theta+\frac{1}{2}}(\varepsilon z) \\ &\leq \varepsilon^{\theta+\frac{1}{2}} C(C_0, R_1)\sqrt{1+r^2}^{\theta+\frac{1}{2}}. \end{aligned} \tag{4.3.120}$$

This, along with (4.3.100), implies

$$\begin{aligned} \left|\frac{\varepsilon\varphi_0^\varepsilon(\varepsilon.)}{C^\varepsilon}\right| &\leq \left|\frac{\varepsilon^{\theta+1+\frac{1}{2}}}{C^\varepsilon}\right| C(C_0, R_1)\sqrt{1+r^2}^{\theta+\frac{1}{2}} \\ &\leq C(C_0, R_1)\sqrt{1+r^2}^{\theta+\frac{1}{2}}\varepsilon^{\frac{1}{2}}. \end{aligned}$$

Consequently

$$\frac{\varepsilon\varphi_0^\varepsilon(\varepsilon z)}{C^\varepsilon} \to 0 \quad L^\infty_{\text{loc}}(\mathbb{C}). \tag{4.3.121}$$

Since $\widetilde{\Phi}^\varepsilon$ is assumed to converge smoothly towards Φ^1 on compacts of $\mathbb{C}$, we deduce from (4.3.63), (4.3.119) and (4.3.121)

$$\sum_{q=0}^{\theta} p_q^\varepsilon \frac{\varepsilon^{q+1}}{C^\varepsilon} z^q \to \Phi_z^1 = \widetilde{A} z^\theta + O(z^{\theta-1}). \tag{4.3.122}$$

Hence

$$\frac{\varepsilon^{\theta+1}}{C^\varepsilon}p_\theta^\varepsilon \to \widetilde{A} \neq 0.$$

Further, given that $p_\theta^\varepsilon \to \vec{A} \neq 0$, there exists $C(C_0, R_1) > 0$ such that

$$\left|\ln\left(\frac{C^\varepsilon}{\varepsilon^{\theta+1}}\right)\right| \leq C(C_0, R_1). \tag{4.3.123}$$

Combining (4.3.100) and (4.3.123) yields

$$\left|\ln\left(\frac{\varepsilon^{d^\varepsilon}}{\varepsilon^\theta}\right)\right| \leq C(C_0, R_1),$$

which ensures

$$|(d^\varepsilon - \theta)\ln\varepsilon| \leq C(C_0, R_1). \tag{4.3.124}$$

Then, (4.3.96) and (4.3.124) combine and yield

$$\begin{aligned}\|\lambda^\varepsilon - \theta\ln r\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\setminus\mathbb{D}_{2\varepsilon R_1}\right)} &\leq \|\lambda^\varepsilon - \theta\ln r\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\setminus\mathbb{D}_{2\varepsilon R_1}\right)} + \|(\theta - d^\varepsilon)\ln r\|_{L^\infty\left(\mathbb{D}_{\frac{1}{2R_1}}\setminus\mathbb{D}_{2\varepsilon R_1}\right)}\\ &\leq C(C_0, R_1).\end{aligned} \tag{4.3.125}$$

Since inequality (4.3.125) is analogous to (4.3.96), we can do all the reasonings from (4.3.96) to (4.3.122) with $d^\varepsilon = \theta$. The conformal factor then satisfies :

$$\lambda^\varepsilon = \ln\left(\varepsilon^\theta + r^\theta\right) + l^\varepsilon, \tag{4.3.126}$$

with l^ε such that

$$\|l^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C(C_0, R_1).$$

$$\frac{\chi^\theta}{C(C_0, R_1)} \leq e^{\lambda^\varepsilon} \leq C(C_0, R_1)\chi^\theta. \tag{4.3.127}$$

This concludes the proof of the desired result since R_1 is fixed. □

Further for simplicity's sake we can, up to an inconsequential (thanks to (4.3.123)) adjustment, assume $C^\varepsilon = \varepsilon^{\theta+1}$. Then, exploiting (4.3.122) yields :

$$\widetilde{A} = \vec{A}. \tag{4.3.128}$$

$$\sum_{q=0}^{\theta} p_q^\varepsilon \varepsilon^{q-\theta} z^q \to \Phi_z^1.$$

We can then decompose

$$P^\varepsilon = \varepsilon^\theta \Phi_z^1\left(\frac{z}{\varepsilon}\right) + \varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right),$$

with $Q^\varepsilon \in \mathbb{C}_\theta[X]$ such that $Q^\varepsilon \to 0$.

Φ^ε then satisfies the following decomposition :

$$\Phi_z^\varepsilon = \varepsilon^\theta \Phi_z^1\left(\frac{z}{\varepsilon}\right) + \varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right) + \varphi_0^\varepsilon, \tag{4.3.129}$$

where

$$\begin{aligned}&\forall v > 0 \quad \left\|\frac{\varphi_0^\varepsilon}{\chi^{\theta+1-v}}\right\|_{L^\infty(\mathbb{D})} \leq C_v\left(C_0, R_1\right),\\ &\forall p < \infty \quad \left\|\frac{\nabla\varphi_0^\varepsilon}{\chi^\theta}\right\|_{L^p(\mathbb{D})} \leq C_p\left(C_0, R_1\right).\end{aligned} \tag{4.3.130}$$

□

Remark 4.3.1. One can compare the idea behind theorem 4.3.2 to the one in [MŠ95] : the Liouville equation ensures that the conformal factor behaves as some, z^d with $d \in \mathbb{R}$, and the fact that it comes form a conformal immersion, forces d to become an integer.

Remark 4.3.2. Equality (4.3.128) can be seen as prolonging theorem 3.2.3 : not only must the multiplicity of the end and the multiplicity of the branch point correspond, but so must the parametrization of the limit planes in both cases.

Remark 4.3.3. A. Michelat and T. Rivière have presented the author with another proof of the expansion which works in the more general framework of any simple bubble (in [MR19]).

4.4 Conditions on the limit surface :

4.4.1 First control of the second residue, proof of theorem G

The aim of this section is to prove :

Theorem G. Let Φ_k be a sequence of Willmore immersions of a closed surface Σ satisfying the hypotheses of theorem 3.2.2. Then at each concentration point $p \in \Sigma$ of multiplicity $\theta_p + 1$ on which a simple minimal bubble is blown, the second residue α_p of the limit immersion Φ_∞ satisfies

$$\alpha_p \leq \theta_p - 1.$$

As detailed in lemmas 3.2.2, 4.3.2 and 4.3.3 we can equivalently work in conformal parametrizations under hypotheses 1-7. We will then instead prove :

Theorem 4.4.1. Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ be a sequence of Willmore conformal immersions satisfying hypotheses 1-7. Then the second residue of Φ^0 at 0 satisfies

$$\alpha \leq \theta - 1.$$

Proof. <u>**Step 1 : Expansion of Φ^ε_{zz} :**</u>
We consider Φ^ε satisfying hypotheses 1-7, and thus (4.3.126)-(4.3.130). The system (7) of [Mar19c] states

$$\begin{cases} \Delta S^\varepsilon = \left\langle H^\varepsilon \nabla \Phi^\varepsilon, \nabla^\perp \vec{R}^\varepsilon \right\rangle \\ \Delta \vec{R}^\varepsilon = -H^\varepsilon \nabla \Phi^\varepsilon \times \nabla^\perp \vec{R}^\varepsilon - \nabla^\perp S^\varepsilon H^\varepsilon \nabla \Phi^\varepsilon \\ \Delta \Phi^\varepsilon = \frac{1}{2}\left(\nabla^\perp S^\varepsilon . \nabla \Phi^\varepsilon + \nabla^\perp \vec{R}^\varepsilon \times \nabla \Phi^\varepsilon \right). \end{cases} \tag{4.4.131}$$

Then (4.3.84), (4.3.86) and (4.4.131) yield :

$$\|\Delta S^\varepsilon\|_{L^\infty(\mathbb{D})} + \left\| \Delta \vec{R}^\varepsilon \right\|_{L^\infty(\mathbb{D})} \leq C(C_0). \tag{4.4.132}$$

Applying theorem A.3.9 gives the following decomposition on S^ε and $\vec{R}^\varepsilon$:

$$\begin{aligned} S^\varepsilon_z &= S^\varepsilon_z(0) + \sigma^\varepsilon_0 \\ \vec{R}^\varepsilon_z &= \vec{R}^\varepsilon_z(0) + \rho^\varepsilon_0, \end{aligned} \tag{4.4.133}$$

with σ^ε_0 and ρ^ε_0 satisfying

$$\begin{aligned} &\forall v > 0 \quad |\sigma^\varepsilon_0|(z) + |\rho^\varepsilon_0|(z) \leq C_v(C_0)\chi^{1-v}, \\ &\forall p < \infty \quad \|\nabla \sigma^\varepsilon_0\|_{L^p(\mathbb{D})} + \|\nabla \rho^\varepsilon_0\|_{L^p(\mathbb{D})} \leq C_p(C_0). \end{aligned} \tag{4.4.134}$$

Injecting (4.3.129) and (4.4.133) into the third equation of (4.4.131) yields

$$\Delta\Phi^\varepsilon = 2\Im\left(\vec{R}^\varepsilon_z(0)\times\overline{\left[\varepsilon^\theta\Phi^1_z\left(\frac{z}{\varepsilon}\right)+\varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right]}+S^\varepsilon_z(0)\overline{\left[\varepsilon^\theta\Phi^1_z\left(\frac{z}{\varepsilon}\right)+\varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right]}\right)+\Psi^\varepsilon_0, \tag{4.4.135}$$

where

$$\Psi^\varepsilon_0 := 2\Im\left(\rho^\varepsilon_0\times\overline{\Phi^\varepsilon_z}+\sigma^\varepsilon_0\overline{\Phi^\varepsilon_z}+\vec{R}^\varepsilon_z\times\overline{\varphi^\varepsilon_0}+S^\varepsilon_z\overline{\varphi^\varepsilon_0}\right)$$

satisfies

$$\begin{aligned}&\forall v>0 \quad \left|\Psi^\varepsilon_0\right|(z)\le C_v(C_0)\chi^{\theta+1-v},\\ &\forall p<\infty \quad \left\|\frac{\nabla\Psi^\varepsilon_0}{\chi^\theta}\right\|_{L^p(\mathbb{D})}\le C_p(C_0).\end{aligned} \tag{4.4.136}$$

One may notice that

$$\widetilde{h}^\varepsilon := 2\Im\left(\vec{R}^\varepsilon_z(0)\times\overline{\left[\varepsilon^\theta\Phi^1_z\left(\frac{z}{\varepsilon}\right)+\varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right]}+S^\varepsilon_z(0)\overline{\left[\varepsilon^\theta\Phi^1_z\left(\frac{z}{\varepsilon}\right)+\varepsilon^\theta Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right]}\right)$$

is the sum of a polynomial of degree θ in z and a polynomial of degree θ in $\overline{z}$, whose coefficients are uniformly bounded by a constant depending on C_0. Additionnally it is a $O(\chi^\theta)$ thanks to theorem 4.3.2. We can then find a polynomial h^ε in z and $\overline{z}$ of total degree $\theta+2$ such that

$$\begin{aligned}&h^\varepsilon(0)=h^\varepsilon_z(0)=h^\varepsilon_{\overline{z}}(0)=0,\\ &\Delta h^\varepsilon=\widetilde{h}^\varepsilon,\\ &h^\varepsilon=O\left(\chi^{\theta+2}\right).\end{aligned}$$

Then

$$\Delta\left(\Phi^\varepsilon-h^\varepsilon\right)=\Psi^\varepsilon_0. \tag{4.4.137}$$

Applying theorem A.3.3 to (4.4.137), with $a=\theta+1-v$ for v arbitrarily small yields

$$\Phi^\varepsilon_z = P^\varepsilon(z)+h^\varepsilon_z+\varphi^\varepsilon_1, \tag{4.4.138}$$

where P^ε is a polynomial of degree $\theta+1$ that we can split $P^\varepsilon = P^\varepsilon_\theta + p^\varepsilon z^{\theta+1}$ with $P^\varepsilon_\theta\in\mathbb{C}_\theta[X]$, and φ^ε_1 satisfies :

$$\begin{aligned}&\forall v>0 \quad \frac{|\varphi^\varepsilon_1|}{\chi^{\theta+2-v}}+\frac{|\nabla\varphi^\varepsilon_1|}{\chi^{\theta+1-v}}\le C_v(C_0),\\ &\forall p<\infty \quad \left\|\frac{\nabla^2\varphi^\varepsilon_1}{\chi^\theta}\right\|_{L^p(\mathbb{D})}\le C_p(C_0).\end{aligned} \tag{4.4.139}$$

Comparing (4.3.129) and (4.4.138) as in the proof of lemma A.3.2 yields :

$$\begin{aligned}&P^\varepsilon_\theta=\varepsilon^\theta\left[\Phi^1\left(\frac{z}{\varepsilon}\right)+Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right],\\ &\varphi^\varepsilon_0=p^\varepsilon z^{\theta+1}+h^\varepsilon_z+\varphi^\varepsilon_1.\end{aligned}$$

Consequently φ^ε_0 satisfies :

$$\begin{aligned}&\frac{|\varphi^\varepsilon_0|}{\chi^{\theta+1}}+\frac{|\nabla\varphi^\varepsilon_0|}{\chi^\theta}\le C(C_0)\\ &\forall p<\infty \quad \left\|\frac{\nabla^2\varphi^\varepsilon_0}{\chi^{\theta-1}}\right\|_{L^p(\mathbb{D})}\le C_p(C_0).\end{aligned} \tag{4.4.140}$$

Estimates (4.4.140) applied to (4.3.129) allow for a pointwise expansion of Φ^ε_{zz} :

$$\begin{aligned}
\Phi^\varepsilon_z &= \varepsilon^\theta \left[\Phi^1_z\left(\frac{z}{\varepsilon}\right) + Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right] + \varphi^\varepsilon_0,\\
\Phi^\varepsilon_{zz} &= \varepsilon^{\theta-1} \left[\Phi^1_{zz}\left(\frac{z}{\varepsilon}\right) + Q^\varepsilon_z\left(\frac{z}{\varepsilon}\right)\right] + \left(\varphi^\varepsilon_0\right)_z.
\end{aligned} \tag{4.4.141}$$

Step 2 : Initial conditions

The relations (4.4.141) yield when evaluated at 0

$$\begin{aligned}
\Phi^\varepsilon_z(0) &= \varepsilon^\theta\Phi^1_z(0) + \varepsilon^m Q^\varepsilon(0) + O\left(\varepsilon^{m+1}\right),\\
\Phi^\varepsilon_{zz}(0) &= \varepsilon^{\theta-1}\Phi^1_{zz}(0) + \varepsilon^{\theta-1}Q^\varepsilon_z(0) + O\left(\varepsilon^m\right).
\end{aligned} \tag{4.4.142}$$

There, hypothesis 7 stands as :

$$\begin{aligned}
\Phi^\varepsilon(0) &= 0,\\
\Phi^\varepsilon_z(0) &= \varepsilon^\theta\Phi^1_z(0) = \frac{\varepsilon^\theta}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\\
\left[\Omega^\varepsilon e^{-\lambda^\varepsilon}\right](0) &= \frac{1}{\varepsilon^\theta}\Omega^\varepsilon(0) = -\frac{2}{\varepsilon}.
\end{aligned}$$

This implies $\vec{n}^\varepsilon(0) = \begin{pmatrix}0\\0\\-1\end{pmatrix}$, and since $\frac{\Omega^\varepsilon}{2} = \langle \vec{n}^\varepsilon, \Phi^\varepsilon_{zz}\rangle$,

$$\begin{aligned}
\Phi^\varepsilon_z(0) &= \frac{\varepsilon^\theta}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\\
\left\langle \Phi^\varepsilon_{zz}(0), \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle &= \varepsilon^{\theta-1}.
\end{aligned} \tag{4.4.143}$$

Comparing (4.4.142) and (4.4.143) yields

$$\begin{aligned}
Q^\varepsilon(0) &= O(\varepsilon),\\
\left\langle Q^\varepsilon_z(0), \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle &= O(\varepsilon).
\end{aligned}$$

However as we pointed out in remark A.3.3, Q^ε is loosely defined. We can then evacuate the coefficients of order ε into φ^ε_0 (which we will do without changing the notations) to obtain :

$$\begin{aligned}
Q^\varepsilon(0) &= 0,\\
\left\langle Q^\varepsilon_z(0), \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle &= 0.
\end{aligned} \tag{4.4.144}$$

To conclude we write $Q^\varepsilon \in \mathbb{C}^3$ as

$$Q^\varepsilon := A^\varepsilon \begin{pmatrix}1\\ i\\ 0\end{pmatrix} + B^\varepsilon\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + C^\varepsilon\begin{pmatrix}0\\0\\1\end{pmatrix},$$

then (4.4.144) yields

$$\begin{aligned} &A^{\varepsilon}(0) = B^{\varepsilon}(0) = C^{\varepsilon}(0) = 0, \\ &C^{\varepsilon}_z(0) = 0. \end{aligned} \tag{4.4.145}$$

When taken at 0, (1.2.53) yields

$$\vec{R}^{\varepsilon}_z(0) = \varepsilon^{\theta}\left(H^{\varepsilon}(0) + iV^{\varepsilon}(0)\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + iS^{\varepsilon}_z(0)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{4.4.146}$$

Estimate (4.3.86) then ensures that $\upsilon^{\varepsilon} := \varepsilon^{\theta}\left(H^{\varepsilon}(0) + iV^{\varepsilon}(0)\right)$ is uniformly bounded :

$$|\upsilon^{\varepsilon}| \leq C(C_0).$$

Step 3 : Φ^{ε} is conformal

We will linearize the conformality condition :

$$\langle \Phi^{\varepsilon}_z, \Phi^{\varepsilon}_z \rangle = 0.$$

Injecting (4.3.129) in the former yields

$$Q^2 B^{\varepsilon} - P^2 A^{\varepsilon} + PQC^{\varepsilon} + A^{\varepsilon}B^{\varepsilon} + \frac{(C^{\varepsilon})^2}{2} = 0. \tag{4.4.147}$$

Applying hypothesis 6 and (4.4.145) then yields :

$$z^2 \text{ divides } B^{\varepsilon}. \tag{4.4.148}$$

Conformality also implies

$$\langle \Delta\Phi^{\varepsilon}, \Phi^{\varepsilon}_z \rangle = 0.$$

Injecting (4.4.135) and (4.4.141) into the former then yields

$$\left\langle \widetilde{h}^{\varepsilon}, \varepsilon^{\theta}\left[\Phi^1_z + Q^{\varepsilon}\right]\left(\frac{z}{\varepsilon}\right)\right\rangle = \left\langle \widetilde{h}^{\varepsilon}, \varphi^{\varepsilon}_0 \right\rangle + \langle \Delta\Phi^{\varepsilon}, \Psi^{\varepsilon}_0 \rangle =: \Psi^{\varepsilon}_1, \tag{4.4.149}$$

with Ψ^{ε}_1 satisfying, thanks to (4.4.136) and (4.4.140),

$$\forall \upsilon > 0 \quad |\Psi^{\varepsilon}_1| \leq C_{\upsilon}\chi^{2\theta+1-\upsilon}. \tag{4.4.150}$$

Considering that $\left\langle \widetilde{h}^{\varepsilon}, \varepsilon^{\theta}\left[\Phi^1_z + Q^{\varepsilon}\right]\left(\frac{z}{\varepsilon}\right)\right\rangle$ is a polynomial of degree at most 2θ in z and $\overline{z}$, we can state :

$$\left\langle \widetilde{h}^{\varepsilon}, \varepsilon^{\theta}\left[\Phi^1_z + Q^{\varepsilon}\right]\left(\frac{z}{\varepsilon}\right)\right\rangle = \sum_{p+q=0}^{2\theta} h^{\varepsilon}_{pq}\varepsilon^{2\theta-p-q}z^p\overline{z}^q.$$

Together (4.4.149) and (4.4.150) yield :

$$\forall \upsilon > 0 \quad \sum_{p+q=0}^{2\theta} h^{\varepsilon}_{pq}\varepsilon^{2\theta-p-q}z^p\overline{z}^q = O\left(\chi^{2\theta+1-\upsilon}\right).$$

Applying lemma A.3.4 then yields :

$$\forall p, q \quad \forall \upsilon > 0 \quad h^{\varepsilon}_{pq} = O\left(\varepsilon^{1-\upsilon}\right). \tag{4.4.151}$$

Step 4 : Computing $\widetilde{h}^\varepsilon$

We compute

$$\varepsilon^\theta \Phi^1_{\frac{z}{z}}\left(\frac{z}{\varepsilon}\right) + \varepsilon^\theta \overline{Q^\varepsilon\left(\frac{z}{\varepsilon}\right)} = \varepsilon^\theta \left(\overline{-\frac{P^2\left(\frac{z}{\varepsilon}\right)}{2} + B^\varepsilon\left(\frac{z}{\varepsilon}\right)}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + \varepsilon^\theta\left(\overline{\frac{Q^2\left(\frac{z}{\varepsilon}\right)}{2} + A^\varepsilon\left(\frac{z}{\varepsilon}\right)}\right)\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$
$$+\varepsilon^\theta\left(\overline{P\left(\frac{z}{\varepsilon}\right)Q\left(\frac{z}{\varepsilon}\right) + C^\varepsilon\left(\frac{z}{\varepsilon}\right)}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Hence

$$\vec{R}^\varepsilon_z(0) \times \overline{\varepsilon^\theta\left[\Phi^1_z + Q^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)} = -2iv^\varepsilon\varepsilon^\theta\overline{\left[\frac{Q^2}{2} + A^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + iv^\varepsilon\varepsilon^\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$$
$$+ S^\varepsilon_z(0)\varepsilon^\theta\overline{\left[-\frac{P^2}{2} + B^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - S^\varepsilon_z(0)\varepsilon^\theta\overline{\left[\frac{Q^2}{2} + A^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix},$$

and

$$S^\varepsilon_z(0)\overline{\varepsilon^\theta\left[\Phi^1_z + Q^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)} = S^\varepsilon_z(0)\varepsilon^\theta\overline{\left[-\frac{P^2}{2} + B^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + S^\varepsilon_z(0)\varepsilon^\theta\overline{\left[\frac{Q^2}{2} + A^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}$$
$$+ S^\varepsilon_z(0)\varepsilon^\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Then

$$\widetilde{h}^\varepsilon = 2\Im\left(\left(S^\varepsilon_z(0)\varepsilon^\theta\overline{[-P^2 + 2B^\varepsilon]\left(\frac{z}{\varepsilon}\right)} + iv^\varepsilon\varepsilon^\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix}\right.$$
$$\left.+\left(S^\varepsilon_z(0)\varepsilon^\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)} - iv^\varepsilon\varepsilon^\theta\overline{[Q^2 + 2A^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\right)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right).$$

From this we deduce

$$\begin{aligned}
\left\langle \widetilde{h}^\varepsilon, \varepsilon^\theta\left[\Phi^1_z + Q^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)\right\rangle &= \left(S^\varepsilon_z(0)\varepsilon^\theta\overline{[-P^2 + 2B^\varepsilon]\left(\frac{z}{\varepsilon}\right)} + iv^\varepsilon\varepsilon^\theta\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\right)\varepsilon^\theta\left[-P^2 + 2B^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)\\
&+\left(S^\varepsilon_z(0)\varepsilon^\theta\overline{[PQ + C^\varepsilon]\left(\frac{z}{\varepsilon}\right)} - iv^\varepsilon\varepsilon^\theta\overline{[Q^2 + 2A^\varepsilon]\left(\frac{z}{\varepsilon}\right)}\right)\varepsilon^\theta\left[PQ + C^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)\\
&= S^\varepsilon_z(0)\varepsilon^{2\theta}\left[|P|^2\left(|P|^2 + |Q|^2\right) + 2\Re\left(PQ\overline{C^\varepsilon} - 2P^2\overline{B^\varepsilon}\right) + 4\left|B^\varepsilon\right|^2 + \left|C^\varepsilon\right|^2\right]\left(\frac{z}{\varepsilon}\right)\\
&+ iv^\varepsilon\varepsilon^{2\theta}\left[-P\overline{Q}\left(|P|^2 + |Q|^2\right) - P^2\overline{C^\varepsilon} + 2B^\varepsilon\overline{PQ} - 2PQ\overline{A^\varepsilon}\right.\\
&\left.-C^\varepsilon\overline{Q^2} + 2B^\varepsilon\overline{C^\varepsilon} - 2C^\varepsilon\overline{A^\varepsilon}\right]\left(\frac{z}{\varepsilon}\right).
\end{aligned} \tag{4.4.152}$$

Studying (4.4.152) with (4.4.145), (4.4.148), (4.4.149), (4.4.150) and hypothesis 6 in mind, we can write

$$\left\langle \widetilde{h}^\varepsilon, \varepsilon^\theta\left[\Phi^1_z + Q^\varepsilon\right]\left(\frac{z}{\varepsilon}\right)\right\rangle = -iv^\varepsilon\varepsilon^{2\theta-1}z + O(r^2), \tag{4.4.153}$$

which implies $h^{\varepsilon}_{1,0} = -i\upsilon^{\varepsilon}$, and in turn thanks to (4.4.151) :

$$\forall s>0 \quad \upsilon^{\varepsilon} = O\left(\varepsilon^{1-s}\right). \tag{4.4.154}$$

Then, (4.4.149), (4.4.150) and (4.4.152) give us :

$$\begin{aligned}\left\langle \widetilde{h}^{\varepsilon}, \varepsilon^{\theta}\left[\Phi^1_z + Q^{\varepsilon}\right]\left(\frac{z}{\varepsilon}\right)\right\rangle = S^{\varepsilon}_z(0)\varepsilon^{2\theta}\Big[|P|^2\left(|P|^2+|Q|^2\right) + 2\Re\left(PQ\overline{C^{\varepsilon}} - 2P^2\overline{B^{\varepsilon}}\right) + 4\left|B^{\varepsilon}\right|^2 + \left|C^{\varepsilon}\right|^2\Big]\left(\frac{z}{\varepsilon}\right) \\ + O\left(\chi^{2\theta+1-\upsilon}\right).\end{aligned} \tag{4.4.155}$$

A similar process on the remaining polynomial allows us to state

$$\forall \upsilon>0 \quad S^{\varepsilon}_z(0) = O\left(\varepsilon^{1-\upsilon}\right). \tag{4.4.156}$$

Step 5 : Conclusion

From (4.4.133), (4.4.154) and (4.4.155) we deduce :

$$\begin{aligned}&\forall \upsilon>0 \quad \left|\vec{R}^{\varepsilon}_z\right| + \left|S^{\varepsilon}_z\right| \le C_{\upsilon}\chi^{1-\upsilon}, \\ &\forall p<\infty \quad \left\|\nabla \vec{R}^{\varepsilon}_z\right\|_{L^p(\mathbb{D})} + \left\|\nabla S^{\varepsilon}_z\right\|_{L^p(\mathbb{D})} \le C_p.\end{aligned} \tag{4.4.157}$$

Inequality (1.2.52) then yields :

$$\forall \upsilon>0 \quad \left|H^{\varepsilon}e^{\lambda^{\varepsilon}}\right| \le C_{\upsilon}\chi^{1-\upsilon}. \tag{4.4.158}$$

Letting (4.4.158) converge away from 0 gives, thanks to hypothesis 2, the following :

$$\forall \upsilon>0 \quad \left|H^0 e^{\lambda^0}\right| \le C_{\upsilon} r^{1-\upsilon}.$$

However since Φ^0 is assumed to have a branch point of order $\theta+1$ at 0, by definition, $e^{\lambda^0} \sim Cr^{\theta}$, which means

$$\forall \upsilon>0 \quad \left|H^0\right| \le C_{\upsilon} r^{1-\theta-\upsilon}. \tag{4.4.159}$$

By definition of α (see (4.3.82)), $H^0 \simeq r^{-\alpha}$. Since $\alpha \in \mathbb{Z}$, (4.4.159) ensures :

$$\alpha \le \theta - 1. \tag{4.4.160}$$

This concludes the proof of the desired result. □

In the continuity of the previous proof we can improve on the convergence obtained in theorem 4.3.1 :

Theorem 4.4.2. Let $\Phi^k : \Sigma \to \mathbb{R}^3$ be a sequence of Willmore immersions satisfying the hypotheses of theorem 3.2.2. Assume further that at each concentration point a simple minimal bubble is blown. Then $\Phi^k \to \Phi^0$ $C^{3,\eta}$ for all $\eta<1$.

Proof. As before we can reason locally, under hypotheses 1-7, and will continue from (4.4.160). Injecting (4.4.157) into (4.4.135) ensures :

$$\begin{aligned}&\forall \upsilon>0 \quad \left|\Delta\Phi^{\varepsilon}\right| \le C_{\upsilon}\chi^{\theta+1-\upsilon} \\ &\forall p<\infty \quad \left\|\frac{\nabla\left(\Delta\Phi^{\varepsilon}\right)}{\chi^{\theta}}\right\|_{L^p(\mathbb{D})} \le C_p.\end{aligned} \tag{4.4.161}$$

We can then compute

$$
\begin{aligned}
H^\varepsilon \Phi^\varepsilon_z &= \frac{\Phi^\varepsilon_{z\bar{z}} \times \Phi^\varepsilon_z}{i\,|\Phi^\varepsilon_z|^2},\\
\nabla\left(H^\varepsilon \Phi^\varepsilon_z\right) &= \frac{\nabla\left(\Phi^\varepsilon_{z\bar{z}}\right)\times \Phi^\varepsilon_z}{i\,|\Phi^\varepsilon_z|^2} + \frac{\Phi^\varepsilon_{z\bar{z}} \times \nabla\left(\Phi^\varepsilon_z\right)}{i\,|\Phi^\varepsilon_z|^2} - \left(\langle \nabla \Phi^\varepsilon_z, \Phi_{\bar{z}}\rangle + \langle \Phi_z, \nabla \Phi_{\bar{z}}\rangle\right) \frac{\Phi^\varepsilon_{z\bar{z}} \times \Phi^\varepsilon_z}{i\,|\Phi^\varepsilon_z|^4}.
\end{aligned}
\tag{4.4.162}
$$

Combining (4.3.127), (4.4.141) and (4.4.161) yields :

$$
\begin{aligned}
&\forall v > 0 \quad \left\| \frac{H^\varepsilon \Phi^\varepsilon_z}{\chi^{1-v}} \right\|_{L^\infty(\mathbb{D})} \leq C_v,\\
&\forall p < \infty \quad \left\| \nabla\left(H^\varepsilon \Phi^\varepsilon_z\right) \right\|_{L^p(\mathbb{D})} \leq C_p.
\end{aligned}
\tag{4.4.163}
$$

Consequently, injecting (4.4.163) into (4.4.131) and applying Calderon-Zygmund yields

$$
\forall p < \infty \quad \|\nabla S^\varepsilon\|_{W^{2,p}(\mathbb{D})} + \left\|\nabla \vec{R}^\varepsilon\right\|_{W^{2,p}(\mathbb{D})} + \|\nabla \Phi^\varepsilon\|_{W^{3,p}(\mathbb{D})} \leq C(C_0). \tag{4.4.164}
$$

Which proves theorem 4.4.2 thanks to classical embeddings. □

Remark 4.4.1. We can further our expansions to the next order. Indeed injecting (4.4.163) and (4.4.157) into (4.4.131) yields

$$
\begin{aligned}
&\forall v > 0 \quad \left\| \frac{\Delta S^\varepsilon}{\chi^{2-v}} \right\|_{L^\infty(\mathbb{D})} + \left\| \frac{\Delta \vec{R}^\varepsilon}{\chi^{2-v}} \right\|_{L^\infty(\mathbb{D})} \leq C_v,\\
&\forall p < \infty \quad \left\| \frac{\Delta \nabla S^\varepsilon}{\chi} \right\|_{L^p(\mathbb{D})} + \left\| \frac{\Delta \nabla \vec{R}^\varepsilon}{\chi} \right\|_{L^p(\mathbb{D})} \leq C_p.
\end{aligned}
$$

Applying corollary A.3.3 then yields

$$
\begin{aligned}
S^\varepsilon_z &= S^\varepsilon_z(0) + s^\varepsilon_1 z + s^\varepsilon_2 z^2 + \sigma^\varepsilon_1,\\
\vec{R}^\varepsilon_z &= \vec{R}^\varepsilon_z(0) + \vec{r}^{\,\varepsilon}_1 z + \vec{r}^{\,\varepsilon}_2 z^2 + \vec{\rho}^{\,\varepsilon}_1,
\end{aligned}
\tag{4.4.165}
$$

where the s^ε_j and the $\vec{r}^{\,\varepsilon}_j$ are uniformly bounded constants and σ^ε_1, ρ^ε_1 satisfy :

$$
\begin{aligned}
&\forall v > 0 \quad \left| \frac{\sigma^\varepsilon_1}{\chi^{3-v}} \right| + \left| \frac{\nabla \sigma^\varepsilon_1}{\chi^{2-v}} \right| + \left| \frac{\vec{\rho}^{\,\varepsilon}_1}{\chi^{3-v}} \right| + \left| \frac{\nabla \vec{\rho}^{\,\varepsilon}_1}{\chi^{2-v}} \right| \leq C_v,\\
&\forall p < \infty \quad \left\| \frac{\nabla^2 \sigma^\varepsilon_1}{\chi} \right\|_{L^p(\mathbb{D})} + \left\| \frac{\nabla^2 \vec{\rho}^{\,\varepsilon}_1}{\chi} \right\|_{L^p(\mathbb{D})} \leq C_p.
\end{aligned}
\tag{4.4.166}
$$

Setting $\sigma^\varepsilon_0 = s^\varepsilon_1 z + s^\varepsilon_2 z^2 + \sigma^\varepsilon_1$ and $\vec{\rho}^{\,\varepsilon}_0 = \vec{r}^{\,\varepsilon}_1 z + \vec{r}^{\,\varepsilon}_2 z^2 + \rho^\varepsilon_1$ yields

$$
\begin{aligned}
&\left| \frac{\sigma^\varepsilon_0}{\chi} \right| + |\nabla \sigma^\varepsilon_0| + \left| \frac{\vec{\rho}^{\,\varepsilon}_0}{\chi} \right| + |\nabla \vec{\rho}^{\,\varepsilon}_0| \leq C,\\
&\forall p < \infty \quad \left\| \nabla^2 \sigma^\varepsilon_0 \right\|_{L^p(\mathbb{D})} + \left\| \nabla^2 \vec{\rho}^{\,\varepsilon}_0 \right\|_{L^p(\mathbb{D})} \leq C_p.
\end{aligned}
\tag{4.4.167}
$$

We can then do all the reasonings from (4.4.135) to (4.4.163) for better controls :

$$
\left| \frac{H^\varepsilon \Phi^\varepsilon_z}{\chi} \right| + \left| \frac{S^\varepsilon_z}{\chi} \right| + \left| \frac{\vec{R}^\varepsilon_z}{\chi} \right| \leq C. \tag{4.4.168}
$$

Injecting this added regularity into the third equation of (4.4.131) ensures :

$$\begin{aligned} &\left|\frac{\Delta \Phi^{\varepsilon}}{\chi^3}\right| + \left|\frac{\Delta \nabla \Phi}{\chi^2}\right| \leq C \\ \forall p < \infty \quad &\left\|\frac{\Delta \nabla^2 \Phi}{\chi}\right\|_{L^p(\mathbb{D})} \leq C_p. \end{aligned} \tag{4.4.169}$$

With another application of corollary A.3.3 we can expand Φ^{ε}_z in the following manner :

$$\Phi^{\varepsilon}_z = \varepsilon^{\theta}\left[\Phi^1_z\left(\frac{z}{\varepsilon}\right) + Q^{\varepsilon}\left(\frac{z}{\varepsilon}\right)\right] + \varphi^{\varepsilon}_0, \tag{4.4.170}$$

with

$$\begin{aligned} &\frac{|\varphi^{\varepsilon}_0|}{\chi^{\theta+1}} + \frac{|\nabla \varphi^{\varepsilon}_0|}{\chi^{\theta}} + \frac{|\nabla^2 \varphi^{\varepsilon}_0|}{\chi^{\theta-1}} \leq C(C_0) \\ \forall p < \infty \quad &\left\|\frac{\nabla^3 \varphi^{\varepsilon}_0}{\chi^{\theta-2}}\right\|_{L^p(\mathbb{D})} \leq C_p(C_0). \end{aligned} \tag{4.4.171}$$

Remark 4.4.2. With subsection 2.7, theorem G implies that the Bryant's quartic of the limit immersion is holomorphic across concentration points where a simple minimal bubble is blown. Consequently it seems to suggest that surfaces involved in minimal bubbling, tend themselves to be inversions of minimal surfaces.

4.4.2 An exploration of consequences : proof of theorem H

First we will put aside the first bubbling case imagined by P. Laurain and T. Rivière in [LR18a] :

Corollary 4.4.1. The convergence of Willmore immersions cannot lead to a minimal bubble and an inverted Chen-Gackstatter torus.

Proof. Applying theorem G in light of proposition 2.4.8 concludes the proof. □

We conclude with a slight improvement of the threshold for compactness :

Theorem H. Let Σ be a closed surface of genus 1 and $\Phi_k : \Sigma \to \mathbb{R}^3$ a sequence of Willmore immersions such that the induced metric remains in a compact set of the moduli space and

$$\limsup_{k\to\infty} W(\Phi_k) \leq 12\pi.$$

Then there exists a diffeomorphism ψ_k of Σ and a conformal transformation Θ_k of $\mathbb{R}^3 \cup \{\infty\}$, such that $\Theta_k \circ \Phi_k \circ \psi_k$ converges up to a subsequence toward a smooth Willmore immersion $\Phi_\infty : \Sigma \to \mathbb{R}^3$ in $C^\infty(\Sigma)$.

Proof. We only have to exclude the case

$$\limsup_{k\to\infty} W(\Phi_k) = 12\pi. \tag{4.4.172}$$

Consider then Φ_k satisfying (4.4.172) and converging toward Φ_∞ away from a finite number of concentration points. We consider a concentration point and reason on its multiplicity θ_0+1. If $\theta_0 \geq 1$, using theorem D, the bubble glued on its concentration point is branched, with the same multiplicity. Using proposition C.1 in [LR18a] ensures that the multiplicity is odd, and then $\theta_0 \geq 2$. Given P. Li and S. Yau's inequality (see [LY82]) and (4.4.172),

Φ_∞ has a Willmore energy of exactly 12π meaning that the branch point is of multiplicity exactly 3, that the bubbles have no Willmore energy (i.e. they are minimal and more accurately Enneper). Using formulas from theorem 1.2.6, Φ_∞ is the inverse of a minimal torus of total curvature -8π. The main result of [L9́2] ensures that this minimal torus is a Chen-Gackstatter immersion. We are then in the case excluded by corollary 4.4.1.

If the concentration point is not branched, we refer the reader to the concluding remark of P. Laurain and T. Rivière's [LR18a] (found just before the appendix) which states that the energy is then at least $2\pi^2 + 12\pi$, which concludes the proof. □

There are different prospects to improve on this result. First is to extend it to surfaces of arbitrary geni. For any genus g, the 12π configuration is going to be a Willmore surface with a branch point of multiplicity 3 and an Enneper bubble. Applying theorem 1.2.6 we know the limit surface is the inversion of a minimal surface with a single end of multiplicity 3, and thus, thanks to theorem 1.2.3, of critical total curvature :

$$\int_\Sigma K d\mathrm{vol}_g = -4\pi\left(1+g\right).$$

Since [L9́2] states that the Chen-Gackstatter surface is the only torus of critical total curvature, we have the theorem. No such result is known for higher genus, even though it seems reasonable, and is conjectured, that Chen-Gackstatter surfaces of higher genus are the only examples, or that at least minimal surfaces of critical total curvature have an Enneper end (which is actually the first step in [L9́2]). Extension to higher genus is thus a classification of minimal surfaces issue.

The other way to deepen our result is to find the exact threshold for tori. Here we see two possibilities : either all the involved surfaces are conformally minimal, and in that case the threshold is immediately pushed back at 16π or higher, or they are not and in which case some degeneracies are possible. One could then imagine a torus of Willmore energy $12\pi + \delta$ and a branch point of multiplicity 3 with low second residue on which a simple Enneper bubble is blown. Given the current state of knowledge, it however seems somewhat unlikely. In both cases one should first try to classify Willmore tori in Bryant's fashion. The best available tool would then be F. Hélein's Weierstrass representation for Willmore surfaces presented in [Hél98]. However this representation was reached thanks to DPW methods, and is in not very explicit, limiting the interpretation possibilities. Understanding this representation, or finding an explicit equivalent, seems key to us, in finding exactly where the compactness threshold lies for sequences of immersed Willmore tori.

4.4.3 Adjustments on the conformal Gauss map

To obtain hypotheses 1-7, we have used translations, rotations and dilations to adjust the immersions. There remains one family of conformal transformations we can use for adjustments : the inversions. In fact, with inversions, we can act *on the mean curvature*, which may allow us to eliminate the problematic terms in the expansion (4.4.165). However as we we will see this is only possible if the mean curvature at the point does not degenerate too much. This section thus focuses on the proof of the following result.

Theorem 4.4.3. Let Φ^ε be a sequence of Willmore conformal immersions satisfying hypotheses 1-7. If

$$\begin{aligned} H^\varepsilon(0) &= O(1), \\ \varepsilon H^\varepsilon_z(0) &= O(1), \end{aligned} \tag{4.4.173}$$

then up to infinitesimal conformal adjustments we can assume that Φ^ε satisfies

8.
$$H^\varepsilon(0)=0,$$
$$H_z^\varepsilon(0)=0.$$

We will write the proof using the Conformal Gauss map formalism.

Proof. We focus on the conformal Gauss map $\widetilde{Y}^\varepsilon$ at 0 of $\widetilde{\Phi}^\varepsilon$. Hypotheses 3 and 6 ensure

$$\begin{aligned}\widetilde{Y}^\varepsilon(0)&=\widetilde{H}^\varepsilon(0)\begin{pmatrix}\widetilde{\Phi}^\varepsilon\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\end{pmatrix}(0)+\begin{pmatrix}\widetilde{\vec{n}}^\varepsilon\\ \left\langle\widetilde{\vec{n}}^\varepsilon,\widetilde{\Phi}^\varepsilon\right\rangle\\ \left\langle\widetilde{\vec{n}}^\varepsilon,\widetilde{\Phi}^\varepsilon\right\rangle\end{pmatrix}(0)\\ &=C^\varepsilon H^\varepsilon(0)\begin{pmatrix}0\\0\\0\\-\frac12\\ \frac12\end{pmatrix}+\begin{pmatrix}0\\0\\-1\\0\\0\end{pmatrix}\to Y^1(0)=\begin{pmatrix}0\\0\\-1\end{pmatrix},\end{aligned}\tag{4.4.174}$$

$$\begin{aligned}\widetilde{Y}_z^\varepsilon(0)&=\widetilde{H}_z^\varepsilon(0)\begin{pmatrix}\widetilde{\Phi}^\varepsilon\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\end{pmatrix}-\widetilde{\Omega}^\varepsilon(0)e^{-2\widetilde{\lambda}^\varepsilon(0)}\begin{pmatrix}\widetilde{\Phi}_z^\varepsilon(0)\\ \left\langle\widetilde{\Phi}_z^\varepsilon(0),\widetilde{\Phi}^\varepsilon(0)\right\rangle\\ \left\langle\widetilde{\Phi}_z^\varepsilon(0),\widetilde{\Phi}^\varepsilon(0)\right\rangle\end{pmatrix}\\ &=\varepsilon C^\varepsilon H_z^\varepsilon(0)\begin{pmatrix}0\\0\\0\\-\frac12\\ \frac12\end{pmatrix}+\begin{pmatrix}1\\-i\\0\\0\\0\end{pmatrix}\to Y_z^1(0)=\begin{pmatrix}1\\-i\\0\\0\\0\end{pmatrix},\end{aligned}\tag{4.4.175}$$

$$e_{\widetilde{\Phi}^\varepsilon}(0):=\begin{pmatrix}\widetilde{\Phi}^\varepsilon\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\\ \frac{|\widetilde{\Phi}^\varepsilon|^2-1}{2}\end{pmatrix}=\begin{pmatrix}1\\-i\\0\\0\\0\end{pmatrix}.\tag{4.4.176}$$

Then if we denote $\widetilde{f}^\varepsilon=\left(\widetilde{Y}^\varepsilon,\frac{\widetilde{Y}_x}{\sqrt{\langle\widetilde{Y}_x,\widetilde{Y}_x\rangle}},\frac{\widetilde{Y}_y}{\sqrt{\langle\widetilde{Y}_y,\widetilde{Y}_y\rangle}},e_{\widetilde{\Phi}^\varepsilon}\right)$ and $f^1=\left(Y^1,\frac{Y_x^1}{\sqrt{\langle Y_x^1,Y_x^1\rangle}},\frac{Y_y^1}{\sqrt{\langle Y_y^1,Y_y^1\rangle}},e_{\Phi^1}\right)$, we have

$$\widetilde{f}^\varepsilon(0)\to f^1(0).$$

Further since $\widetilde{f}^\varepsilon(0)$ and $f^1(0)$ are two orthonormal families for the Minkowski product, there exists $\widetilde{M}^\varepsilon\in SO(4,1)$ such that

$$\begin{aligned}\widetilde{M}^\varepsilon\widetilde{f}^\varepsilon&=f^1,\\ \widetilde{M}^\varepsilon&\to Id.\end{aligned}\tag{4.4.177}$$

Quick computations ensure that

$$\widetilde{M}^\varepsilon=\begin{pmatrix}Id&-\widetilde{\nu}^\varepsilon&-\widetilde{\nu}^\varepsilon\\ (\widetilde{\nu}^\varepsilon)^T&1-\frac{|\widetilde{\nu}^\varepsilon|^2}{2}&-\frac{|\widetilde{\nu}^\varepsilon|^2}{2}\\ -(\widetilde{\nu}^\varepsilon)^T&\frac{|\widetilde{\nu}^\varepsilon|^2}{2}&1+\frac{|\widetilde{\nu}^\varepsilon|^2}{2}\end{pmatrix},\tag{4.4.178}$$

with $\nu^\varepsilon = \begin{pmatrix} \frac{1}{4}\varepsilon C^\varepsilon \nabla H^\varepsilon(0) \\ -C^\varepsilon \frac{H^\varepsilon(0)}{2} \end{pmatrix}$. Thanks to proposition 2.3.4, $\widetilde{M^\varepsilon}\widetilde{Y}^\varepsilon$ is the conformal Gauss map of $\widetilde{\Psi}^\varepsilon := \varphi^\varepsilon \circ \widetilde{\Phi}^\varepsilon$, where φ^ε is a conformal diffeomorphism of $\mathbb{R}^3 \cup \infty$ corresponding to $\widetilde{M}^\varepsilon$ given by :

$$\varphi^\varepsilon(x) = \widetilde{M^\varepsilon}.x = \frac{y_\diamond}{y_5 - y_4}$$

where

$$y = \widetilde{M^\varepsilon} \begin{pmatrix} x \\ \frac{|x|^2-1}{2} \\ \frac{|x|^2+1}{2} \end{pmatrix} = \begin{pmatrix} y_\diamond \\ y_4 \\ y_5 \end{pmatrix}.$$

Hence we have an explicit formula for $\widetilde{\Psi}^\varepsilon$

$$\widetilde{\Psi}^\varepsilon(z) = \frac{\widetilde{\Phi}^\varepsilon - \left|\widetilde{\Phi}^\varepsilon\right|^2 \widetilde{\nu}^\varepsilon}{1 - 2\left\langle \widetilde{\nu}^\varepsilon, \widetilde{\Phi}^\varepsilon \right\rangle + |\widetilde{\Phi}^\varepsilon|^2 \, |\widetilde{\nu}^\varepsilon|^2}, \tag{4.4.179}$$

which implies that

$$\widetilde{\Psi}^\varepsilon_z(0) = \widetilde{\Phi}^\varepsilon_z(0) = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{4.4.180}$$

Further since $e_{\widetilde{\Psi}^\varepsilon}(0) = \widetilde{M^\varepsilon} e_{\widetilde{\Phi}^\varepsilon}(0) = e^1(0)$, $\widetilde{\Psi}^\varepsilon(0) = \widetilde{\Phi}^\varepsilon(0)$. Similarly $Y_{\widetilde{\Psi}^\varepsilon}(0) = \widetilde{M^\varepsilon}\widetilde{Y}^\varepsilon(0) = Y^1(0)$ and $\left(Y_{\widetilde{\Psi}^\varepsilon}\right)_z(0) = \widetilde{M^\varepsilon}\widetilde{Y}^\varepsilon_z(0) = Y^1_z(0)$ implies

$$\begin{aligned} H_{\widetilde{\Psi}^\varepsilon}(0) &= 0, \\ \left(H_{\widetilde{\Psi}^\varepsilon}\right)_z(0) &= 0, \\ \Omega_{\widetilde{\Psi}^\varepsilon}(0) &= -2. \end{aligned} \tag{4.4.181}$$

We introduce

$$\Psi^\varepsilon := C^\varepsilon \widetilde{\Psi}^\varepsilon \left(\frac{z}{\varepsilon}\right) = C^\varepsilon \widetilde{M^\varepsilon}.\left(\frac{1}{C^\varepsilon}\Phi^\varepsilon\right) = M^\varepsilon.\Phi^\varepsilon,$$

with $M^\varepsilon = D_{C^\varepsilon}\widetilde{M^\varepsilon} D_{\frac{1}{C^\varepsilon}}$, where

$$D_\lambda = \begin{pmatrix} Id & 0 & 0 \\ 0 & \frac{\lambda}{2} + \frac{1}{2\lambda} & \frac{\lambda}{2} - \frac{1}{2\lambda} \\ 0 & \frac{\lambda}{2} - \frac{1}{2\lambda} & \frac{\lambda}{2} + \frac{1}{2\lambda} \end{pmatrix}$$

is the matrix of the dilation of factor λ in $SO(4,1)$ (see (2.2.5)). Ψ^ε is a sequence of immersions satisfying hypotheses 4, 3, 5 , 6, 7 and

$$\begin{aligned} H_{\Psi^\varepsilon}(0) &= 0, \\ \left(H_{\Psi^\varepsilon}\right)_z(0) &= 0. \end{aligned} \tag{4.4.182}$$

If M^ε converges in $SO(4,1)$ toward a matrix which is not the representation of the inversion at the origin, Ψ^ε can be proven to satisfy hypothesis 1 and 2. To that end we can compute :

$$M^\varepsilon = \begin{pmatrix} Id & -\nu^\varepsilon & -\nu^\varepsilon \\ (\nu^\varepsilon)^T & 1 - \frac{|\nu^\varepsilon|^2}{2} & -\frac{|\nu^\varepsilon|^2}{2} \\ -(\nu^\varepsilon)^T & \frac{|\nu^\varepsilon|^2}{2} & 1 + \frac{|\nu^\varepsilon|^2}{2} \end{pmatrix}, \tag{4.4.183}$$

with
$$\nu^\varepsilon = \begin{pmatrix} \frac{1}{4}\varepsilon\nabla H^\varepsilon(0) \\ -\frac{1}{2}H^\varepsilon(0) \end{pmatrix}. \tag{4.4.184}$$
Since $\nu^\varepsilon = O(1)$, then up to extracting a subsequence $\nu^\varepsilon \to \nu \in \mathbb{R}^3$. Since
$$\Psi^\varepsilon = \frac{\Phi^\varepsilon - |\Phi^\varepsilon|^2 \nu^\varepsilon}{1 - 2\langle \nu^\varepsilon, \Phi^\varepsilon\rangle + |\Phi^\varepsilon|^2 |\nu^\varepsilon|^2},$$
then Ψ^ε converges toward
$$\Psi^0 = \frac{\Phi^0 - \left|\Phi^0\right|^2 \nu}{1 - 2\langle \nu, \Phi^0\rangle + |\Phi^0|^2 |\nu|^2}.$$
The application $x \to \frac{x - |x|^2\nu}{1 - 2\langle \nu, x\rangle + |x|^2|\nu|^2}$ is a conformal transform bounded away from $\frac{\nu}{|\nu|^2} \neq 0$ which is sent to ∞. Up to restricting the domain of study we can then assume that Ψ^ε satisfies 1 and 2.

On the other hand, if $\nu^\varepsilon \to \infty$, the conformal transform given by M^ε degenerates and
$$\Psi^\varepsilon \to 0.$$
These considerations conclude the proof of the theorem, given expression (4.4.184) of ν^ε. $\square$

Remark 4.4.3. Requiring (4.4.173) to properly adjust the mean curvature stems from the lack of compactness and commutativity of the invariance group in our problem. Indeed that may allow $D_{C^\varepsilon}\widetilde{M}^\varepsilon D_{\frac{1}{C^\varepsilon}}$ to degenerate even though $\widetilde{M}^\varepsilon$ converges toward the identity. In other words, while it is possible to adjust the mean curvature of the blown-up immersions, carrying this back to the immersion it is not automatic due to the non-compact rescaling between the two.

Following is an analysis of the gluing of an Enneper bubble when the mean curvature has been adjusted that way. It is meant as an illustration of how the lackluster properties of the conformal group (non-commutativity, non-compactness) allow for the bubbling example presented in subsection 3.3.

4.4.4 Specific case of an Enneper bubble

In this section we show how under a small control on the concentration point, one finds higher regularity for the surface receiving the minimal bubbling. While the computations necessary for the proof may be obscure and difficult to follow, drawing a parallel with the CMC case may clarify the phenomena. In [Lau12b] P. Laurain treated the case of CMC bubbling for spheres, by using the invariance group to erase all the solutions of the linearized equation. This is possible because of the properties of the CMC invariance group (the isometries). In the Willmore case, and more specifically in the Enneper simple bubbling configuration, in order to eliminate the solutions of the linearized equation, one must draw upon all of the conformal group, which is *non-compact* and *non-commutative*, meaning that we need additional pointwise controls to do these adjustments both at the microscopic (on the bubble) and macroscopic (on the surface) scale. If we enjoy these pointwise controls we experience a regularity jump corresponding to the main result of [Lau12b] (we do not, even in this case, eliminate Enneper bubbling due to the much greater flexibility of Willmore surfaces compared to CMC surfaces). The bubbling example given in section 3.3 lies just before this regularity jump, and makes clear that the lackluster topological properties of $\mathrm{Conf}(\mathbb{R}^3)$ do engender some non-compactness for Willmore immersions.

Theorem 4.4.4. Let Σ be a compact Riemann surface of genus less than 1, and Φ^k : $\Sigma \to \mathbb{R}^3$ a sequence of Willmore immersions of uniformly bounded total curvature and such that the conformal class of the induced metric is in a compact of the moduli space. We further assume that Φ^k has only a single concentration point p on which a simple Enneper bubble is blown, and that Φ^k converges smoothly away from p toward a branched immersion $\Phi^0 : \Sigma \to \mathbb{R}^3$. Then either

$$\frac{\nabla H^k(p)}{\|\nabla \vec{n}^k\|_{L^\infty(\Sigma)}} \to \infty, \tag{4.4.185}$$

or Φ^0 is the inversion of a branched minimal immersion, with second residue $\alpha \leq -2$.

Proof. As has been readily explained, looking at Φ^k in good conformal charts, and up to minor adjustments allows us to consider $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ satisfying hypotheses 1-7. Further, if (4.4.185) does not stand, we can apply theorem 4.4.3 and find a converging sequence of conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$, Θ^k, such that, while $\Psi_k := \Theta^k \circ \Phi^k$ may have ends, it is a uniformly bounded smooth immersion around the concentration point p, which satisfies hypothesis 8. Additionally, Ψ^k converges smoothly toward a branched Willmore immersion Ψ^0 with a single branch point at p and possibly a finite number of simple planar ends. When considering the immersion in local charts we will still denote it Φ^ε, to avoid multiplying notations. It then satisfies 1- 8. We follow from the proof of theorem 4.4.1, start with its conclusions and use the same formalism.

We are thus considering case where $\Phi^1 = E = \Re\left(\int \frac{1}{2}\begin{pmatrix} 1-z^2 \\ i\left(1+z^2\right) \\ 2z\end{pmatrix}\right)$, *i.e.* in the language of lemma 4.3.1 : $Q = 1$, $P = z$. Then $m = 2$, and (4.3.126), (4.3.127), (4.4.170), (4.4.171) (4.4.145) and (4.4.147) yield the following :

— The conformal factor satisfies

$$\lambda^\varepsilon = \ln\left(\varepsilon^2 + r^2\right) + l^\varepsilon, \tag{4.4.186}$$

with l^ε such that

$$\|l^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C(C_0, R_1).$$

Consequently

$$\frac{\chi^2}{C(C_0, R_1)} \leq e^{\lambda^\varepsilon} \leq C(C_0, R_1)\chi^2. \tag{4.4.187}$$

— The immersion is expanded as

$$\Phi^\varepsilon_z = \varepsilon^2\left[E_z\left(\frac{z}{\varepsilon}\right) + Q^\varepsilon\left(\frac{z}{\varepsilon}\right)\right] + \varphi^\varepsilon_0, \tag{4.4.188}$$

with

$$\begin{aligned} &\frac{|\varphi^\varepsilon_0|}{\chi^3} + \frac{|\nabla\varphi^\varepsilon_0|}{\chi^2} + \frac{\left|\nabla^2\varphi^\varepsilon_0\right|}{\chi} \leq C(C_0), \\ &\forall p < \infty \quad \left\|\nabla^3\varphi^\varepsilon_0\right\|_{L^p(\mathbb{D})} \leq C_p(C_0), \end{aligned} \tag{4.4.189}$$

and

$$Q^\varepsilon = A^\varepsilon\begin{pmatrix}1\\ i\\ 0\end{pmatrix} + B^\varepsilon\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + C^\varepsilon\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}, \tag{4.4.190}$$

where

$$\begin{aligned} A^\varepsilon &= a_1^\varepsilon z + a_2^\varepsilon z^2 \\ B^\varepsilon &= b_1^\varepsilon z + b_2^\varepsilon z^2 \\ C^\varepsilon &= c^\varepsilon z^2, \end{aligned} \tag{4.4.191}$$

which also satisfy

$$\varepsilon^4 B^\varepsilon\left(\frac{z}{\varepsilon}\right) - \varepsilon^2 z^2 A^\varepsilon\left(\frac{z}{\varepsilon}\right) + \varepsilon^3 z C^\varepsilon\left(\frac{z}{\varepsilon}\right) + \varepsilon^4 A^\varepsilon\left(\frac{z}{\varepsilon}\right) B^\varepsilon\left(\frac{z}{\varepsilon}\right) + \varepsilon^4 \frac{\left(C^\varepsilon\left(\frac{z}{\varepsilon}\right)\right)^2}{2} = 0. \tag{4.4.192}$$

<u>$c^\varepsilon = 0$:</u>

A direct consequence of (4.4.192) is

$$\begin{aligned} B^\varepsilon &= 0, \\ A^\varepsilon &= c^\varepsilon z + \frac{(c^\varepsilon)^2}{2} z^2. \end{aligned} \tag{4.4.193}$$

The immediate aim here is to use the only initial condition that we have yet to exploit (*i.e.* $\left(|\Omega^\varepsilon|\, e^{-\lambda^\varepsilon}\right)_z(0) = 0$) and find $c^\varepsilon = 0$. We compute

$$\left(|\Omega^\varepsilon|^2 e^{-2\lambda^\varepsilon}\right)_z = \Omega^\varepsilon_z \overline{\Omega^\varepsilon} e^{-2\lambda^\varepsilon} + \Omega^\varepsilon \overline{\Omega^\varepsilon_{\bar z}} e^{-2\lambda^\varepsilon} - 2\lambda^\varepsilon_z |\Omega^\varepsilon|^2 e^{-2\lambda^\varepsilon}.$$

Since $\Omega^\varepsilon = 2\langle \Phi^\varepsilon_{zz}, \vec n^\varepsilon\rangle$,

$$\begin{aligned} \Omega^\varepsilon_z &= 2\langle \Phi^\varepsilon_{zzz}, \vec n^\varepsilon\rangle - 2\left\langle \Phi^\varepsilon_{zz}, \Omega^\varepsilon e^{-2\lambda^\varepsilon} \Phi^\varepsilon_{\bar z}\right\rangle \\ &= 2\langle \Phi^\varepsilon_{zzz}, \vec n^\varepsilon\rangle - 2\lambda^\varepsilon_z \Omega^\varepsilon. \end{aligned}$$

Similarly

$$\begin{aligned} \Omega^\varepsilon_{\bar z} &= 2\langle \Phi^\varepsilon_{z\bar z z}, \vec n^\varepsilon\rangle - 2\langle \Phi^\varepsilon_{zz}, H^\varepsilon \Phi^\varepsilon_{\bar z}\rangle \\ &= 2\langle \Phi^\varepsilon_{z\bar z z}, \vec n^\varepsilon\rangle - 2\lambda^\varepsilon_z H^\varepsilon e^{2\lambda^\varepsilon}. \end{aligned}$$

Hence

$$\left(|\Omega^\varepsilon|^2 e^{-2\lambda^\varepsilon}\right)_z = 2\langle \Phi^\varepsilon_{zzz}, \vec n^\varepsilon\rangle \overline{\Omega^\varepsilon} e^{-2\lambda^\varepsilon} + 2\langle \Phi^\varepsilon_{z\bar z\bar z}, \vec n^\varepsilon\rangle \Omega^\varepsilon e^{-2\lambda^\varepsilon} - 4\lambda^\varepsilon_z |\Omega^\varepsilon|^2 e^{-2\lambda^\varepsilon} - 2\lambda^\varepsilon_{\bar z} H^\varepsilon \Omega^\varepsilon. \tag{4.4.194}$$

From (4.4.188) we find

$$\begin{aligned} |\Phi^\varepsilon_z|^2 &= \frac{\left(\varepsilon^2 + r^2\right)^2}{2} + 2\Re\left(\left\langle Q^\varepsilon, \overline{\varepsilon^2 E_z\left(\frac{z}{\varepsilon}\right)}\right\rangle\right) + \psi^\varepsilon \\ &= \frac{\left(\varepsilon^2 + r^2\right)^2}{2} + 2\Re\left(\varepsilon^4 A^\varepsilon\left(\frac{z}{\varepsilon}\right) + \varepsilon^3 \bar z C^\varepsilon\left(\frac{z}{\varepsilon}\right)\right) + \psi^\varepsilon \\ &= \frac{\left(\varepsilon^2 + r^2\right)^2}{2} + 2\Re\left(\varepsilon^3 z c^\varepsilon + \varepsilon z^2 \bar z c^\varepsilon + \varepsilon^2 z^2 \frac{(c^\varepsilon)^2}{2}\right) + \psi^\varepsilon, \end{aligned} \tag{4.4.195}$$

where ψ^ε satisfies

$$\begin{aligned} &\frac{|\psi^\varepsilon|}{\chi^5} + \frac{|\nabla\psi^\varepsilon|}{\chi^4} + \frac{\left|\nabla^2\psi^\varepsilon\right|}{\chi^3} \le C(C_0), \\ &\forall p < \infty \quad \left\|\frac{\nabla^3\psi^\varepsilon}{\chi^2}\right\|_{L^p(\mathbb{D})} \le C_p(C_0). \end{aligned} \tag{4.4.196}$$

Since $|\Phi^\varepsilon_z|^2 = \frac{e^{2\lambda^\varepsilon}}{2}$, we deduce :

$$\lambda^\varepsilon_z(0)e^{2\lambda^\varepsilon(0)} = \varepsilon^3 c^\varepsilon + O\left(\varepsilon^4\right).$$

By hypothesis 7, $e^{\lambda^\varepsilon}(0) = \varepsilon^2$, which yields

$$\lambda^\varepsilon_z(0) = \frac{c^\varepsilon}{\varepsilon} + O(1). \tag{4.4.197}$$

Then, thanks once more to our hypothesis 7

$$\left(4\lambda^\varepsilon_z \left|\Omega^\varepsilon\right|^2 e^{-2\lambda^\varepsilon}\right)(0) = \frac{16c^\varepsilon}{\varepsilon^3} + O\left(\frac{1}{\varepsilon^2}\right). \tag{4.4.198}$$

Thanks to (4.4.168),

$$H^\varepsilon(0) = O\left(\frac{1}{\varepsilon}\right).$$

This implies

$$\left(2\lambda^\varepsilon_{\bar{z}}\Omega^\varepsilon H^\varepsilon\right)(0) = O\left(\frac{1}{\varepsilon}\right). \tag{4.4.199}$$

Further, thanks to (4.4.169), $\Delta\nabla\Phi^\varepsilon(0) = O\left(\varepsilon^2\right)$, and hence

$$\left(2\left\langle \Phi^\varepsilon_{z\bar{z}\bar{z}}, \vec{n}^\varepsilon\right\rangle \Omega^\varepsilon e^{-2\lambda^\varepsilon}\right)(0) = O\left(\frac{1}{\varepsilon}\right). \tag{4.4.200}$$

To conclude

$$\begin{aligned}
\left(\left|\Omega^\varepsilon\right|^2 e^{-2\lambda^\varepsilon}\right)_z(0) &= \frac{4}{\varepsilon^3}\left\langle \Phi^\varepsilon_{zzz}(0), \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle - 16\frac{c^\varepsilon}{\varepsilon^3} + O\left(\frac{1}{\varepsilon^3}\right)\\
&= -8\frac{c^\varepsilon}{\varepsilon^3} + O\left(\frac{1}{\varepsilon^3}\right) = 0.
\end{aligned}$$

Hence

$$c^\varepsilon = O\left(\varepsilon\right), \tag{4.4.201}$$

which allows us to take $c^\varepsilon = 0$ (see remark A.3.3). Henceforth we will use the following expansion for Φ^ε :

$$\Phi^\varepsilon_z = \varepsilon^2 E_z\left(\frac{z}{\varepsilon}\right) + \varphi^\varepsilon_0, \tag{4.4.202}$$

where φ^ε_0 satisfies (4.4.189). The expansion (4.4.202) ensures

$$|\Phi^\varepsilon_z|^2 = \frac{\left(\varepsilon^2 + r^2\right)^2}{2} + \Psi^\varepsilon, \tag{4.4.203}$$

with

$$\begin{aligned}
&\frac{|\Psi^\varepsilon|}{\chi^5} + \frac{|\nabla\Psi^\varepsilon|}{\chi^4} + \frac{\left|\nabla^2\Psi^\varepsilon\right|}{\chi^3} \leq C,\\
\forall p < \infty \quad &\left\|\frac{\nabla^3\Psi^\varepsilon}{\chi^2}\right\|_{L^p(\mathbb{D})} \leq C_p.
\end{aligned} \tag{4.4.204}$$

Similarly

$$\vec{n}^\varepsilon = \omega\left(\frac{z}{\varepsilon}\right) + \nu^\varepsilon, \tag{4.4.205}$$

where $\omega(z) = \frac{1}{1+r^2}\begin{pmatrix} z+\bar{z} \\ i(\bar{z}-z) \\ r^2-1 \end{pmatrix}$ is the Gauss map of E, and ν^ε satisfies :

$$\begin{aligned} &\frac{|\nu^\varepsilon|}{\chi} + \chi\left|\nabla\nu^\varepsilon\right| + \chi^2\left|\nabla^2\nu^\varepsilon\right| \leq C, \\ &\forall p<\infty \quad \left\|\chi^3\nabla^3\nu^\varepsilon\right\|_{L^p(\mathbb{D})} \leq C_p. \end{aligned} \tag{4.4.206}$$

Expansion of H^ε :
Starting from (4.4.165) we expand

$$\begin{aligned} S_z^\varepsilon &= S_z^\varepsilon(0) + s_1^\varepsilon z + \sigma_2^\varepsilon, \\ \vec{R}_z^\varepsilon &= \vec{R}_z^\varepsilon(0) + \vec{r_1^\varepsilon} z + \vec{\rho_2^\varepsilon}, \end{aligned} \tag{4.4.207}$$

with

$$\begin{aligned} \sigma_2^\varepsilon &= s_2^\varepsilon z^2 + \sigma_1^\varepsilon, \\ \rho_2^\varepsilon &= \vec{r_2^\varepsilon} z^2 + \vec{\rho^\varepsilon}_1, \end{aligned}$$

which consequently satisfy

$$\begin{aligned} &\left|\frac{\sigma_2^\varepsilon}{\chi^2}\right| + \left|\frac{\nabla\sigma_2^\varepsilon}{\chi}\right| + \left|\frac{\vec{\rho_2^\varepsilon}}{\chi^2}\right| + \left|\frac{\nabla\vec{\rho_2^\varepsilon}}{\chi}\right| \leq C, \\ &\forall p<\infty \quad \left\|\nabla^2\sigma_2^\varepsilon\right\|_{L^p(\mathbb{D})} + \left\|\nabla^2\vec{\rho_2^\varepsilon}\right\|_{L^p(\mathbb{D})} \leq C_p. \end{aligned} \tag{4.4.208}$$

Given (4.4.207),

$$\begin{aligned} S_{zz}^\varepsilon &= s_1^\varepsilon + \sigma_{2,z}^\varepsilon, \\ \vec{R}_{zz}^\varepsilon &= \vec{r_1^\varepsilon} + \vec{\rho_{2,z}^\varepsilon}, \end{aligned}$$

and thus, thanks to (4.4.208),

$$\begin{aligned} S_{zz}^\varepsilon(0) &= s_1^\varepsilon + O(\varepsilon), \\ \vec{R}_{zz}^\varepsilon(0) &= \vec{r_1^\varepsilon} + O(\varepsilon). \end{aligned}$$

We can then, up to modifying σ_2^ε and $\vec{\rho_2^\varepsilon}$ without impacting (4.4.208) assume :

$$\begin{aligned} S_z^\varepsilon &= S_z^\varepsilon(0) + S_{zz}^\varepsilon(0)z + \sigma_2^\varepsilon, \\ \vec{R}_z^\varepsilon &= \vec{R}_z^\varepsilon(0) + \vec{R}_{zz}^\varepsilon(0)z + \vec{\rho_2^\varepsilon}. \end{aligned} \tag{4.4.209}$$

From (1.2.53), we find

$$\begin{aligned} \vec{R}_{zz}^\varepsilon = {} & 2\left(\left(H_z^\varepsilon + iV_z^\varepsilon\right) + 2\lambda_z^\varepsilon\left(H^\varepsilon + iV^\varepsilon\right)\right)\Phi_z^\varepsilon + \left(H^\varepsilon + iV^\varepsilon\right)\Omega^\varepsilon\vec{n}^\varepsilon \\ & - iS_{zz}^\varepsilon\vec{n}^\varepsilon + iS_z^\varepsilon H\Phi_z^\varepsilon + iS_z^\varepsilon\Omega^\varepsilon e^{-2\lambda^\varepsilon}\Phi_{\bar{z}}^\varepsilon. \end{aligned} \tag{4.4.210}$$

Further, since

$$i\vec{L}_z^\varepsilon = 2\left(H_z^\varepsilon\vec{n}^\varepsilon + H^\varepsilon\Omega^\varepsilon e^{-2\lambda^\varepsilon}\Phi_{\bar{z}}^\varepsilon\right), \tag{4.4.211}$$

we can compute

$$\begin{aligned} S_{zz}^\varepsilon &= \left\langle \vec{L}_z^\varepsilon, \Phi_z^\varepsilon \right\rangle + \left\langle \vec{L}^\varepsilon, 2\lambda_z^\varepsilon\Phi_z^\varepsilon + \frac{\Omega}{2}\vec{n}^\varepsilon \right\rangle \\ &= -iH^\varepsilon\Omega^\varepsilon + 2\lambda_z^\varepsilon S_z^\varepsilon + \Omega^\varepsilon V^\varepsilon. \end{aligned} \tag{4.4.212}$$

Similarly,

$$\begin{aligned} V_z^\varepsilon &= \frac{1}{2}\left\langle \vec{L}_z^\varepsilon, \vec{n}^\varepsilon \right\rangle - \frac{H^\varepsilon}{2} S_z^\varepsilon - \frac{\Omega^\varepsilon e^{-2\lambda^\varepsilon}}{2} S_{\bar{z}}^\varepsilon \\ &= -iH_z^\varepsilon - \frac{H^\varepsilon}{2} S_z^\varepsilon - \frac{\Omega^\varepsilon e^{-2\lambda^\varepsilon}}{2} S_{\bar{z}}^\varepsilon. \end{aligned} \tag{4.4.213}$$

Injecting (4.4.212) and (4.4.213) into (4.4.210) yields

$$\vec{R}_{zz}^\varepsilon = \left(4H_z^\varepsilon - i\Omega^\varepsilon e^{-2\lambda^\varepsilon} S_{\bar{z}}^\varepsilon + 4\lambda_z^\varepsilon \left[H^\varepsilon + iV^\varepsilon\right]\right) \Phi_z^\varepsilon + iS_z^\varepsilon \Omega^\varepsilon e^{-2\lambda^\varepsilon} \Phi_{\bar{z}}^\varepsilon + -2i\lambda_z^\varepsilon S_z^\varepsilon \vec{n}^\varepsilon. \tag{4.4.214}$$

The expansion (4.4.203) ensures

$$\lambda_z^\varepsilon e^{2\lambda^\varepsilon} = 2\bar{z}\left(\varepsilon^2 + r^2\right) + \Psi_z^\varepsilon,$$

which yields

$$\lambda_z^\varepsilon(0) = O(1). \tag{4.4.215}$$

With hypothesis 5, hypothesis 7, (4.4.168) and (4.4.215), (4.4.210) implies :

$$\vec{R}_{zz}^\varepsilon(0) = \left(2\varepsilon^2 H_z^\varepsilon(0) + \frac{iS_{\bar{z}}^\varepsilon(0)}{\varepsilon}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{iS_z^\varepsilon(0)}{\varepsilon}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} + O(\varepsilon). \tag{4.4.216}$$

Similarly,

$$S_{zz}^\varepsilon(0) = 2i\varepsilon\left(H^\varepsilon(0) + iV^\varepsilon(0)\right) + O(\varepsilon). \tag{4.4.217}$$

Denoting $\upsilon^\varepsilon = \varepsilon\left(H^\varepsilon(0) + iV^\varepsilon(0)\right)$, the following expansions stand :

$$\begin{aligned} S_z^\varepsilon &= S_z^\varepsilon(0) + 2i\upsilon^\varepsilon z + \widetilde{\sigma}_2^\varepsilon, \\ \vec{R}_z^\varepsilon &= \varepsilon\upsilon^\varepsilon \begin{pmatrix}1\\ i\\ 0\end{pmatrix} + iS_z^\varepsilon(0)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + \left[\left(2\varepsilon^2 H_z^\varepsilon(0) + \frac{iS_{\bar{z}}^\varepsilon(0)}{\varepsilon}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{iS_z^\varepsilon(0)}{\varepsilon}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\right] z + \widetilde{\rho}_2^\varepsilon. \end{aligned} \tag{4.4.218}$$

We can then exploit the relation $S_z^\varepsilon = i\left\langle \vec{n}^\varepsilon, \vec{R}_z^\varepsilon \right\rangle$, using (4.4.205) and (4.4.218) :

$$\begin{aligned} S_z^\varepsilon(0) + 2i\upsilon^\varepsilon z &= i\Bigg\langle \omega\left(\frac{z}{\varepsilon}\right), \varepsilon\upsilon^\varepsilon \begin{pmatrix}1\\ i\\ 0\end{pmatrix} + iS_z^\varepsilon(0)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \\ &\quad + \left[\left(2\varepsilon^2 H_z^\varepsilon(0) + \frac{iS_{\bar{z}}^\varepsilon(0)}{\varepsilon}\right)\begin{pmatrix}1\\ i\\ 0\end{pmatrix} - \frac{iS_z^\varepsilon(0)}{\varepsilon}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}\right] z \Bigg\rangle + O\left(\chi^2\right) \\ &= \frac{2i}{\varepsilon^2 + r^2}\Bigg(\varepsilon z\left(\varepsilon\upsilon^\varepsilon + \left(2\varepsilon^2 H_z^\varepsilon(0) + \frac{iS_{\bar{z}}^\varepsilon(0)}{\varepsilon}\right) z\right) \\ &\quad - z\bar{z} iS_z^\varepsilon(0) + \frac{r^2 - \varepsilon^2}{2} iS_z^\varepsilon(0)\Bigg) + O(\chi^2). \end{aligned}$$

This implies

$$\begin{aligned} \varepsilon^2 S_z^\varepsilon(0) + 2i\varepsilon^2 z\upsilon^\varepsilon + z\bar{z} S_z^\varepsilon(0) + 2iz^2\bar{z}\upsilon^\varepsilon = i\Big(&-\varepsilon^2 iS_z^\varepsilon(0) + 2\varepsilon^2\upsilon^\varepsilon z + \left(4\varepsilon^3 H_z^\varepsilon(0) + i2S_{\bar{z}}^\varepsilon(0)\right) z^2 \\ &- z\bar{z} iS^\varepsilon(0)\Big) + O\left(\chi^4\right), \end{aligned}$$

which, in turn, ensures :

$$\left(4\varepsilon^3 H_z^\varepsilon(0) + 2iS_{\bar{z}}^\varepsilon(0)\right) z^2 - 2z^2\bar{z}\upsilon^\varepsilon = O\left(\chi^4\right). \tag{4.4.219}$$

Applying lemma A.3.4 to (4.4.219) yields

$$\begin{aligned} v^\varepsilon &= O(\varepsilon), \\ 2\varepsilon^3 H^\varepsilon_z(0) + i S^\varepsilon_{\bar z}(0) &= O(\varepsilon). \end{aligned} \tag{4.4.220}$$

A consequence of (4.4.220) is that

$$H^\varepsilon(0) = O(1). \tag{4.4.221}$$

We can then modify (4.4.218) into

$$\begin{aligned} S^\varepsilon_z &= -2i\varepsilon^3 H^\varepsilon_{\bar z}(0) + \sigma^\varepsilon_3, \\ \vec{R}^\varepsilon_z &= 2\varepsilon^3 H^\varepsilon_{\bar z}(0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 2\varepsilon^2 H^\varepsilon_{\bar z}(0) z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} + \rho^\varepsilon_3, \end{aligned} \tag{4.4.222}$$

with σ^ε_3 and ρ^ε_3 satisfying (4.4.208). We can then compute with (4.4.202) and (4.4.222) :

$$\begin{aligned} \vec{R}^\varepsilon_z \times \Phi^\varepsilon_{\bar z} + S^\varepsilon_z \Phi^\varepsilon_{\bar z} &= i\varepsilon^2 H^\varepsilon_{\bar z}(0) \left(2\varepsilon \bar z^2 \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + 2\varepsilon z \bar z \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} - 2\left[\varepsilon^2 \bar z - z \bar z^2\right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \kappa^\varepsilon \\ &= 2i\varepsilon^2 H^\varepsilon_{\bar z}(0) \bar z \left(\varepsilon^2 + r^2\right) \omega \left(\frac{z}{\varepsilon}\right) + \kappa^\varepsilon, \end{aligned}$$

where κ^ε satisfies

$$\begin{aligned} &\left|\frac{\kappa^\varepsilon}{\chi^4}\right| + \left|\frac{\nabla \kappa^\varepsilon}{\chi^3}\right| \leq C, \\ \forall p < \infty \quad &\left\|\frac{\nabla^2 \kappa^\varepsilon}{\chi^2}\right\|_{L^p(\mathbb{D})} \leq C_p. \end{aligned} \tag{4.4.223}$$

The third equation of (4.4.131) then yields

$$\Delta \Phi^\varepsilon = 2\varepsilon^2 \left(H^\varepsilon_z(0) z + H^\varepsilon_{\bar z}(0) \bar z\right) \left(\varepsilon^2 + r^2\right) \omega^\varepsilon \left(\frac{z}{\varepsilon}\right) + \Im\left(\kappa^\varepsilon\right) \tag{4.4.224}$$

We can conclude using $\Delta\Phi^\varepsilon = 2e^{2\lambda^\varepsilon} H^\varepsilon \vec{n}^\varepsilon$ and expansions (4.4.203) and (4.4.205), finally reaching

$$H^\varepsilon = \frac{\varepsilon^2 H^\varepsilon_z(0) z + \varepsilon^2 H^\varepsilon_{\bar z}(0) \bar z}{\varepsilon^2 + r^2} + \zeta^\varepsilon, \tag{4.4.225}$$

where ζ^ε satisfies

$$\begin{aligned} &|\zeta^\varepsilon| + |\chi \nabla \zeta^\varepsilon| \leq C, \\ \forall p < \infty \quad &\left\|\chi^2 \nabla^2 \zeta^\varepsilon\right\|_{L^p(\mathbb{D})} \leq C_p. \end{aligned} \tag{4.4.226}$$

Let us notice that the expression obtained for H^ε corresponds to a solution of the linearized Willmore equation for an Enneper bubble. Indeed the Willmore equation $\Delta H + \left|\mathring{A}\right|^2 H = 0$ linearizes as

$$\Delta h + 2\left(\frac{2\varepsilon}{\varepsilon^2 + r^2}\right)^2 h = 0.$$

Rescaling at scale ε, this is exactly the Jacobi equation on the sphere, and the first term on the expansion of $H^\varepsilon(\varepsilon .)$ is a solution. In the following we write all the possible solutions of the equation.

Solutions of the linearized equation :

Proposition 4.4.5. Let h^ε such that

$$h^\varepsilon_{z\bar z} + \frac{2\varepsilon^2}{\left(\varepsilon^2+r^2\right)^2} h^\varepsilon = O\left(\chi^{\deg P^\varepsilon - 3}\right). \tag{4.4.227}$$

with $\left(\varepsilon^2+r^2\right)^2 h^\varepsilon \in \mathbb{C}_{\deg P^\varepsilon+2}[z,\bar z]$. This means

$$h^\varepsilon = \sum_{p+q=0}^{\deg P^\varepsilon+2} \frac{h^\varepsilon_{p,q} z^p \bar z^q}{\left(\varepsilon^2+r^2\right)^2}.$$

Then

$$\begin{aligned} h^\varepsilon = & \sum_{q=0}^{\deg P^\varepsilon-2} V^\varepsilon_q \bar z^q \frac{(q-1)r^2+(q+1)\varepsilon^2}{\varepsilon^2+r^2} \\ & + \sum_{q=0}^{\deg P^\varepsilon-2} W^\varepsilon_q z^q \frac{(q-1)r^2+(q+1)\varepsilon^2}{\varepsilon^2+r^2} + O\left(\chi^{\deg P^\varepsilon_2-1}\right). \end{aligned} \tag{4.4.228}$$

Proof. We can compute

$$h^\varepsilon_{z\bar z} = \sum_{p+q=0}^{\deg P^\varepsilon+2} h^\varepsilon_{p,q} \frac{(pq-2p-2q+4)\, z^{p+1}\bar z^{q+1} + 2\varepsilon^2(pq-p-q-1) z^p \bar z^q + \varepsilon^4 pq z^{p-1}\bar z^{q-1}}{\left(\varepsilon^2+r^2\right)^4}.$$

This yields :

$$\begin{aligned} LE &= h^\varepsilon_{z\bar z} + \frac{2\varepsilon^2}{\left(\varepsilon^2+r^2\right)^2} h^\varepsilon \\ &= \sum_{p+q=0}^{\deg P^\varepsilon+4} \frac{h^\varepsilon_{p-1,q-1}(p-3)(q-3) + 2\varepsilon^2 h^\varepsilon_{pq}(pq-p-q) + \varepsilon^4 h^\varepsilon_{p+1,q+1}(p+1)(q+1)}{\left(\varepsilon^2+r^2\right)^4} z^p \bar z^q. \end{aligned}$$

Consequently

$$\sum_{p+q=0}^{\deg P^\varepsilon_2+4} \left(h^\varepsilon_{p-1,q-1}(p-3)(q-3) + 2\varepsilon^2 h^\varepsilon_{pq}(pq-p-q) + \varepsilon^4 h^\varepsilon_{p+1,q+1}(p+1)(q+1)\right) z^p \bar z^q = O\left(\chi^{\deg P^\varepsilon+5}\right).$$

Applying lemma A.3.4 then yields for all $p+q \leq \deg P^\varepsilon + 4$:

$$h^\varepsilon_{p-1,q-1}(p-3)(q-3) + 2\varepsilon^2 h^\varepsilon_{pq}(pq-p-q) + \varepsilon^4 h^\varepsilon_{p+1,q+1}(p+1)(q+1) = O(\varepsilon^{\deg P^\varepsilon+5-p-q}). \tag{4.4.229}$$

A quick induction ensures that for all $p, q \geq 3$,

$$h^\varepsilon_{p,q} = O(\varepsilon^{\deg P^\varepsilon+3-p-q}).$$

Given $q \geq 2$, considering (4.4.229) with $p = 2$, q, yields :

$$2\varepsilon^2(q-2)h^\varepsilon_{2,q} = (q-3)h_{1,q-1} + O\left(\varepsilon^{\deg P^\varepsilon+3-q}\right), \tag{4.4.230}$$

while with $p = 1$, $q-1$, it ensures :

$$2\varepsilon^4 q h^\varepsilon_{2,q} = 2\varepsilon^2 h_{1,q-1} + 2(q-4)h^\varepsilon_{0,q-2} + O\left(\varepsilon^{\deg P^\varepsilon+5-q}\right). \tag{4.4.231}$$

Combining (4.4.230) and (4.4.231) yields :

$$\begin{aligned} h^\varepsilon_{1,q-1} &= \frac{2\varepsilon^2(q-2)}{q-3} h^\varepsilon_{2,q} + O\left(\varepsilon^{\deg P^\varepsilon + 3 - q}\right), \\ h^\varepsilon_{0,q-2} &= \frac{\varepsilon^4(q-1)}{q-3} h^\varepsilon_{2,q} + O\left(\varepsilon^{\deg P^\varepsilon + 5 - q}\right). \end{aligned} \tag{4.4.232}$$

This gives

$$\begin{aligned} h^\varepsilon_{2,q} z^2 \bar{z}^q + h^\varepsilon_{1,q-1} z \bar{z}^{q-1} + h^\varepsilon_{0,q-2} \bar{z}^{q-2} &= \frac{h^\varepsilon_{2,q} \bar{z}^{q-2}}{q-3} \left((q-3) z^2 \bar{z}^2 + 2\varepsilon^2(q-2) z\bar{z} + \varepsilon^4(q-1)\right) \\ &\quad + O\left(\chi^{\deg P^\varepsilon + 3}\right), \\ &= \frac{h^\varepsilon_{2,q} \bar{z}^{q-2}}{q-3} \left(\varepsilon^2 + r^2\right) \left((q-3) r^2 + (q-1)\varepsilon^2\right) \\ &\quad + O\left(\chi^{\deg P^\varepsilon + 3}\right). \end{aligned} \tag{4.4.233}$$

Working similarly with the $h^\varepsilon_{q,2}$ yields :

$$\begin{aligned} h^\varepsilon_{q,2} z^q \bar{z}^2 + h^\varepsilon_{q-1,1} z^{q-1} \bar{z} + h^\varepsilon_{q-2,0} z^{q-2} &= \frac{h^\varepsilon_{q,2} z^{q-2}}{q-3} \left((q-3) z^2 \bar{z}^2 + 2\varepsilon^2(q-2) z\bar{z} + \varepsilon^4(q-1)\right) \\ &\quad + O\left(\chi^{\deg P^\varepsilon + 3}\right), \\ &= \frac{h^\varepsilon_{q,2} z^{q-2}}{q-3} \left(\varepsilon^2 + r^2\right) \left((q-3) r^2 + (q-1)\varepsilon^2\right) \\ &\quad + O\left(\chi^{\deg P^\varepsilon + 3}\right). \end{aligned} \tag{4.4.234}$$

Hence we finally can write :

$$h^\varepsilon = \sum_{q=2}^{\deg P^\varepsilon} \frac{h^\varepsilon_{2,q} \bar{z}^{q-2}}{q-3} \frac{(q-3) r^2 + (q-1)\varepsilon^2}{\varepsilon^2 + r^2} + \sum_{q=2}^{\deg P^\varepsilon} \frac{h^\varepsilon_{q,2} z^{q-2}}{q-3} \frac{(q-3) r^2 + (q-1)\varepsilon^2}{\varepsilon^2 + r^2} + O\left(\chi^{\deg P^\varepsilon - 1}\right),$$

which we reframe as :

$$h^\varepsilon = \sum_{q=0}^{\deg P^\varepsilon - 2} V^\varepsilon_q \bar{z}^q \frac{(q-1) r^2 + (q+1)\varepsilon^2}{\varepsilon^2 + r^2} + \sum_{q=0}^{\deg P^\varepsilon - 2} W^\varepsilon_q z^q \frac{(q-1) r^2 + (q+1)\varepsilon^2}{\varepsilon^2 + r^2} + O\left(\chi^{\deg P^\varepsilon - 1}\right).$$

This is the desired result which concludes the proof. □

This is in accordance with the expansion (4.4.225) obtained for H^ε. Without adding any hypothesis on the mean curvature at 0, there is no hope for any better controls. However, with the assumption (4.4.173), there will be a jump in regularity.

Under good initial conditions :

Let $\Phi^\varepsilon : \mathbb{D} \to \mathbb{R}^3$ be a sequence of Willmore conformal immersions satisfying hypotheses 1 - 8. Then (4.4.223), (4.4.224), (4.4.225) and (4.4.226) imply :

$$\begin{aligned} S^\varepsilon_z &= \widetilde{\sigma}^\varepsilon_2, \\ \vec{R}^\varepsilon_z &= \widetilde{\rho}^\varepsilon_2, \\ H^\varepsilon \Phi^\varepsilon_z &= \theta^\varepsilon, \end{aligned} \tag{4.4.235}$$

where θ^ε satisfies the same estimates as $\widetilde{\sigma}_2^\varepsilon$ and $\widetilde{\rho}_2^\varepsilon$, detailed in (4.4.208). Proceeding as in (4.4.165) -(4.4.209) allows us to decompose :

$$\begin{aligned} S_z^\varepsilon &= P_1^\varepsilon + \sigma_3^\varepsilon \\ \vec{R}_z^\varepsilon &= P_2^\varepsilon + \vec{\rho}_3^\varepsilon, \end{aligned} \tag{4.4.236}$$

where $P_1^\varepsilon, P_2^\varepsilon \in \mathbb{C}_2\left[X\right]$, and

$$\begin{aligned} \frac{|\sigma_3^\varepsilon|}{\chi^{\deg P_1^\varepsilon+1}} + \frac{|\nabla\sigma_3^\varepsilon|}{\chi^{\deg P_1^\varepsilon}} + \frac{|\nabla^2\sigma_3^\varepsilon|}{\chi^{\deg P_1^\varepsilon-1}} &\leq C, \\ \frac{|\vec{\rho}_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon+1}} + \frac{|\nabla\vec{\rho}_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon}} + \frac{|\nabla^2\vec{\rho}_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon-1}} &\leq C, \\ \forall p<\infty \quad \left\| \frac{\nabla^3\sigma_3^\varepsilon}{\chi^{\deg P_1^\varepsilon-2}} \right\|_{L^p(\mathbb{D})} + \left\| \frac{\nabla^3\vec{\rho}_3^\varepsilon}{\chi^{\deg P_2^\varepsilon-2}} \right\|_{L^p(\mathbb{D})} &\leq C_p. \end{aligned} \tag{4.4.237}$$

Here we have written (4.4.237) depending on $\deg P_i^\varepsilon$ as it will be more convenient for later applications. For the first pass in this loop $\deg P_i^\varepsilon = 2$. The third equation of (4.4.131) yields

$$\Delta\Phi^\varepsilon = \Im\left(P_1^\varepsilon \overline{\varepsilon^2 E\left(\frac{z}{\varepsilon}\right)} + P_2^\varepsilon \times \overline{\varepsilon^2 E\left(\frac{z}{\varepsilon}\right)} \right) + \varphi_3^\varepsilon,$$

and

$$\begin{aligned} \frac{|\varphi_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon+3}} + \frac{|\nabla\varphi_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon+2}} + \frac{|\nabla^2\varphi_3^\varepsilon|}{\chi^{\deg P_2^\varepsilon+1}} &\leq C, \\ \forall p<\infty \quad \left\| \frac{\nabla^3\varphi_3^\varepsilon}{\chi^{\deg P_2^\varepsilon}} \right\|_{L^p(\mathbb{D})} &\leq C_p. \end{aligned} \tag{4.4.238}$$

Then since $H^\varepsilon = \frac{\langle \Delta\Phi^\varepsilon, \vec{n}^\varepsilon\rangle}{|\nabla\Phi^\varepsilon|^2}$, we find

$$H^\varepsilon = h^\varepsilon + O\left(\chi^{\deg P_2^\varepsilon - 1}\right), \tag{4.4.239}$$

where

$$h^\varepsilon = 2i\frac{\left\langle P_2^\varepsilon, \overline{\varepsilon^2 E_z\left(\frac{z}{\varepsilon}\right)} \right\rangle - \left\langle \overline{P_2^\varepsilon}, \varepsilon^2 E_z\left(\frac{z}{\varepsilon}\right) \right\rangle}{\left(\varepsilon^2+r^2\right)^2}, \tag{4.4.240}$$

and h^ε is a solution of (4.4.227), with $\deg P^\varepsilon = \deg P_2^\varepsilon$. The conclusion of proposition 4.4.5 then stands. According to 8, $h^\varepsilon(0) = h_z^\varepsilon(0) = 0$.

Consequently, as long as $\deg P_2^\varepsilon \leq 3$, $h^\varepsilon = O\left(\chi^{\deg P_2^\varepsilon-1}\right)$, which implies thanks to

$$H^\varepsilon = O\left(\chi^{\deg P_2^\varepsilon-1}\right). \tag{4.4.241}$$

Injecting (4.4.236) and (4.4.241) into (4.4.131) ensure

$$\begin{aligned} \Delta S^\varepsilon &= O\left(\chi^{\deg P_2^\varepsilon+3}\right), \\ \Delta\vec{R}^\varepsilon &= O\left(\chi^{\deg P_2^\varepsilon+3}\right). \end{aligned} \tag{4.4.242}$$

We can thus go through the process from (4.4.235) to (4.4.227) once more until we reach $\deg P_2^\varepsilon = 4$. The final estimate is then :

$$\begin{aligned} H^\varepsilon &= O\left(\chi^2\right), \\ \nabla H^\varepsilon &= O\left(\chi\right). \end{aligned} \tag{4.4.243}$$

Local consequences on the limit surface :

Letting $\varepsilon \to 0$ away from 0 in (4.4.243) yields the following estimates on the mean curvature H^0 of Φ^0 :

$$\begin{aligned} H^0 &= O(r^2) \\ \nabla H^0 &= O(r). \end{aligned} \tag{4.4.244}$$

Going back to (4.3.81) and (4.3.82), (4.4.244) is tantamount to

$$\alpha \leq -2. \tag{4.4.245}$$

Injecting (4.4.245) into (2.4.38) then ensures that

$$\mathcal{Q}^0 = O\left(r^2\right). \tag{4.4.246}$$

Then (4.4.246) implies that p is in fact a zero of order 2. The Bryant's functional of Φ^0 and Ψ^0 is then a holomorphic function on a compact Riemann surface, with at least a zero. Hence

$$\mathcal{Q}^0 = 0.$$

This concludes the proof. □

Remark 4.4.4. In the proof of theorem 4.4.4, nothing prevents Ψ^0 from being minimal, in which case Θ is simply the limit of the Θ^k, and is not centered on $\Phi^0(p)$.

Appendix

$\mathcal{A}$

Appendix

A.1 A brief reminder on Lorentz spaces

The following will recall basic notions concerning Lorentz spaces and is mostly extracted from chapter 3 of [Hél02]. It will contain no proof.

Definition A.1.1. Let O be an open subset of $\mathbb{R}^m$, $|O|$ be the lebesgue measure of O, and $f : O \to \mathbb{R}$ be a measurable function. The non-increasing rearrangement of $|f|$ on $[0, |O|)$ is the unique function, denoted by f^*, from $[0, |O|)$ to $\mathbb{R}$ which is non-increasing and such that

$$\text{measure}\{x \in O \,|\, |f(x)| \geq s\} = \text{measure}\{t \in (0, |O|) \,|\, f^*(t) \geq s\}.$$

Definition A.1.2. Let O be an open subset in $\mathbb{R}^m$, $1 < p < \infty$, $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}(O, \mathbb{R})$ is the set of measurable functions $f : O \to \mathbb{R}$ such that

$$|f|_{p,q} = \left[\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} < \infty, \text{ if } q < \infty,$$

or

$$|f|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty, \text{ if } q = \infty.$$

While the quantities $|f|_{p,q}$ are not norms (they do not satisfy the triangular inequality, if we define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds,$$

the quantities

$$\|f\|_{p,q} = \left[\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, \text{ if } q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t),$$

are space norms.

Theorem A.1.1. The Lorentz space $L^{p,q}(O, \mathbb{R})$ is a Banach space.

While the definition itself is complex and rather obscure, the Lorentz spaces can be understood as generalizations of the Lebesgue spaces. In fact one has for any $1 < q < p < q' < \infty$ and any $p > 1$

$$L^{p,1} \subset L^{p,q} \subset L^{p,p} = L^p \subset L^{p,q'} \subset L^{p,\infty}.$$

Actually one can interpolate the Lebesgue space in between Lorentz spaces thanks to Marcinkiewitz interpolation theorem (see theorem 3.3.3 from [Hél02]).

Theorem A.1.2. Let O be an open subset of $\mathbb{R}^m$ and U an open subset of $\mathbb{R}^n$. Let r_0, r_1, p_0, p_1 be real numbers such that

$$1 \leq r_0 < r_1 \leq \infty,$$

and

$$1 \leq p_0 \neq p_1 \leq \infty.$$

Let T be a linear operator whose domain D contains

$$\bigcup_{r_0 \leq r \leq r_1} L^r(O),$$

and which maps continuously $L^{r_0}(O)$ to $L^{p_0}(U)$, and $L^{r_1}(O)$ to $L^{p_1}(U)$ with the norms

$$\begin{aligned} \forall f \in L^{r_0}(O), \quad & \|Tf\|_{L^{p_0}(U)} \leq k_0 \|f\|_{L^{r_0}(O)}, \\ \forall f \in L^{r_1}(O), \quad & \|Tf\|_{L^{p_1}(U)} \leq k_0 \|f\|_{L^{r_1}(O)}. \end{aligned}$$

Then, for each $1 \leq q \leq \infty$, and for every pair (p, r) such that $\exists \theta \in (0,1)$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

f maps continuously $L^{r,q}(O)$ to $L^{p,q}(U)$, and moreover,

$$\forall f \in L^{r,q}(O), \quad \|Tf\|_{L^{p,q}(U)} \leq B_\theta \|f\|_{L^{r,q}(O)},$$

where

$$B_\theta = \left(\frac{r}{|\gamma| p} \right)^{\frac{1}{q}} 2^{\frac{1}{p}} \left(\frac{r k_0}{r - r_0} + \frac{r_1 k_1}{r_1 - r} \right),$$

and

$$\gamma = \left(\frac{1}{p_0} - \frac{1}{p} \right) \left(\frac{1}{r_0} - \frac{1}{r} \right)^{-1} = \left(\frac{1}{p_1} - \frac{1}{p} \right) \left(\frac{1}{r} - \frac{1}{r_1} \right)^{-1}.$$

In the present work we will only use $L^{2,\infty}$, $L^2 = L^{2,2}$ and $L^{2,\infty}$. In accordance with theorem A.1.2, $L^{2,1}$ is a slightly more restrictive space than L^2, whose elements are a bit smoother, while $L^{2,\infty}$ is a bigger space which allows for more singular behaviors. For instance :

Example A.1.1. In dimension 2, $\frac{1}{r} \in L^{2,\infty}(\mathbb{D})$, but $\frac{1}{r} \notin L^2(\mathbb{D})$.

Additionnally, theorems 3.3.4 and 3.3.5 of [Hél02] offer a perspective on $L^{2,\infty}$ and $L^{2,1}$ as some kind of closure of the L^2 space regarding Calderón-Zygmund theorems or Sobolev injections. Indeed while $W^{1,p} \hookrightarrow C^{0,\alpha}$ for all $p > 2$, $W^{1,2}$ only injects into L^q for $q < \infty$. Closing this gap is the Lorentz space $L^{2,1}$.

Theorem A.1.3. Let O be an open compact subset of $\mathbb{R}^3$ with smooth boundary. Let $f \in W^{1,2}(O)$ and suppose that $\nabla f \in L^{2,1}(O)$. Then f is continuous and uniformly bounded in O.

Besides while the Calderón-Zygmund theory works for L^p with $p > 1$, it classically fails for L^1. Closing this boundary is the $L^{2,\infty}$ space.

Theorem A.1.4. Let O be an open subset of $\mathbb{R}^2$, with smooth boundary. Let $f \in L^1(O)$, and ϕ be a solution of

$$\begin{cases} \Delta\phi = f \text{ in } O \\ \phi = 0 \text{ on } \partial O. \end{cases}$$

Then $\nabla\phi \in L^{2,\infty}(O)$ and

$$\|\nabla\phi\|_{L^{2,\infty}(O)} \leq C\,\|f\|_{L^1(O)}.$$

To sum up, the Lorentz spaces are a refinement of Lebesgue spaces, that can be seen as closure for the Sobolev embeddings, or the Calderón-Zygmund inequalities.

A.2 Formulas for a conformal immersion

In this section we show several formulas useful for the core of the thesis. Most are well known, but their proof is included for self-containedness, and to display the interplays that we will make use of throughout our reasonings.

A.2.1 Conformal immersions of a disk in $\mathbb{R}^3$

Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a conformal immersion, that is such that

$$|\Phi_x|^2 - |\Phi_y|^2 = \langle \Phi_x, \Phi_y \rangle = 0.$$

Its Gauss map is defined as $\vec{n} = \frac{\Phi_x \times \Phi_y}{|\Phi_x \times \Phi_y|}$ (with $\times$ the usual vectorial product in $\mathbb{R}^3$) and its conformal factor as $\lambda = \ln|\Phi_x| = \ln|\Phi_y|$. Its second fundamental form is then

$$A := \left\langle \nabla^2\Phi, \vec{n} \right\rangle =: \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

One can check

$$\nabla\vec{n} = -e^{-2\lambda} A \nabla\Phi = -e^{-2\lambda} \begin{pmatrix} e\Phi_x + f\Phi_y \\ f\Phi_x + g\Phi_y \end{pmatrix}, \tag{A.2.1}$$

and deduce immediately

$$\nabla^\perp\vec{n} = -e^{-2\lambda} \begin{pmatrix} -f\Phi_x - g\Phi_y \\ e\Phi_x + f\Phi_y \end{pmatrix}. \tag{A.2.2}$$

Defining the mean curvature as

$$H = \frac{e+g}{2e^{2\lambda}},$$

and the tracefree second fundamental form as

$$\mathring{A} = e^{-2\lambda} \begin{pmatrix} \frac{e-g}{2} & f \\ f & \frac{g-e}{2} \end{pmatrix},$$

one finds

$$\begin{aligned} \nabla\vec{n} &= -H\nabla\Phi - \mathring{A}\nabla\Phi, \\ \nabla^\perp\vec{n} &= -H\nabla^\perp\Phi + \mathring{A}\nabla^\perp\Phi. \end{aligned} \tag{A.2.3}$$

By definition of $\vec{n}$

$$\begin{aligned} \vec{n}\times\Phi_x &= \Phi_y, \\ \vec{n}\times\Phi_y &= -\Phi_x, \end{aligned}$$

which implies

$$\begin{aligned} \vec{n}\times\nabla\Phi &= -\nabla^\perp\Phi, \\ \vec{n}\times\nabla^\perp\Phi &= \nabla\Phi. \end{aligned} \tag{A.2.4}$$

Combining (A.2.3) and (A.2.4) yields

$$\begin{aligned} \vec{n}\times\nabla\vec{n} &= H\nabla^\perp\Phi + \mathring{A}\nabla^\perp\Phi, \\ \vec{n}\times\nabla^\perp\vec{n} &= -H\nabla\Phi + \mathring{A}\nabla\Phi. \end{aligned} \tag{A.2.5}$$

As a result, $H\nabla\Phi$ and $\mathring{A}\nabla\Phi$ can be deduced solely from $\nabla\vec{n}$:

$$\begin{aligned} H\nabla\Phi &= -\frac{\vec{n}\times\nabla^\perp\vec{n}+\nabla\vec{n}}{2}, \\ \mathring{A}\nabla\Phi &= \frac{\vec{n}\times\nabla^\perp\vec{n}-\nabla\vec{n}}{2}. \end{aligned} \tag{A.2.6}$$

It is well known that since Φ is conformal

$$\Delta\Phi = \vec{H}\left|\nabla\Phi\right|^2, \tag{A.2.7}$$

where $\vec{H} = H\vec{n}$, and

$$\Delta\lambda = Ke^{2\lambda} \tag{A.2.8}$$

where $K = e^{-4\lambda}\det A = e^{-4\lambda}\left(eg - f^2\right)$ is the Gauss curvature. Equation (A.2.8) is known as the Liouville equation.

With (A.2.3), one can compute

$$\begin{aligned} |\nabla\vec{n}|^2 &= \left|H\nabla\Phi + \mathring{A}\nabla\Phi\right|^2 \\ &= |H\nabla\Phi|^2 + \left|\mathring{A}\nabla\Phi\right|^2, \end{aligned} \tag{A.2.9}$$

since $\mathring{A}$ is tracefree. From (A.2.9) we deduce

$$\left|\mathring{A}\right| \leq |\nabla\vec{n}|, \tag{A.2.10}$$

and

$$|H\nabla\Phi| \leq |\nabla\vec{n}|. \tag{A.2.11}$$

A.2.2 Conformal immersions of a disk in $\mathbb{R}^3$: complex notations

In this context, it is most convenient to use complex notations. Let

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) = \frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}.\nabla = \frac{i}{2}\begin{pmatrix}1\\-i\end{pmatrix}.\nabla^\perp. \tag{A.2.12}$$

Then, Φ conformal translates as

$$\begin{aligned}\langle \Phi_z, \Phi_z \rangle &= 0,\\ |\Phi_z|^2 &= \frac{e^{2\lambda}}{2}.\end{aligned} \tag{A.2.13}$$

The Gauss map can be written in coplex notations in the following way

$$\vec{n} = \frac{\Phi_z \times \Phi_{\bar{z}}}{i\,|\Phi_z|^2}, \tag{A.2.14}$$

which gives the complex counterpart to (A.2.4) :

$$\begin{aligned}\vec{n} \times \Phi_z &= i\Phi_z\\ \vec{n} \times \Phi_{\bar{z}} &= -i\Phi_z.\end{aligned} \tag{A.2.15}$$

If we define the tracefree curvature as $\Omega = \frac{e-g}{2} - if = 2\,\langle \Phi_{zz}, \vec{n} \rangle$, (A.2.3) becomes

$$\vec{n}_z = -H\Phi_z - \Omega e^{-2\lambda}\Phi_{\bar{z}}, \tag{A.2.16}$$

while (A.2.9) turns into

$$|\vec{n}_z|^2 = \frac{H^2 e^{2\lambda} + |\Omega|^2 e^{-2\lambda}}{2}. \tag{A.2.17}$$

Similarly, (A.2.7) translates to

$$\Phi_{z\bar{z}} = \frac{He^{2\lambda}}{2}\vec{n}. \tag{A.2.18}$$

Exploiting (A.2.13), one finds

$$\begin{aligned}\langle \Phi_{zz}, \Phi_z \rangle &= 0\\ \langle \Phi_{zz}, \Phi_{\bar{z}} \rangle &= \left(\langle \Phi_z, \Phi_{\bar{z}} \rangle\right)_z - \langle \Phi_z, \Phi_{z\bar{z}} \rangle\\ &= \lambda_z e^{2\lambda}.\end{aligned}$$

Subsequently,

$$\Phi_{zz} = 2\lambda_z\Phi_z + \frac{\Omega}{2}\vec{n}. \tag{A.2.19}$$

We can then compute

$$\begin{aligned}\vec{n}_{z\bar{z}} &= -H_{\bar{z}}\Phi_z - \frac{H^2e^{2\lambda}}{2}\vec{n} - \left(\Omega_{\bar{z}}e^{-2\lambda} - 2\lambda_{\bar{z}}\Omega e^{-2\lambda}\right)\Phi_{\bar{z}} - 2\lambda_{\bar{z}}\left(\Omega e^{-2\lambda}\right)\Phi_{\bar{z}} - \frac{|\Omega|^2 e^{-2\lambda}}{2}\vec{n}\\ &= -H_{\bar{z}}\Phi_z - \Omega_{\bar{z}}e^{-2\lambda}\Phi_{\bar{z}} - \frac{H^2e^{2\lambda} + |\Omega|^2 e^{-2\lambda}}{2}\vec{n}.\end{aligned}$$

However, $\vec{n}_{z\bar{z}} \in \mathbb{R}^3$ since $\vec{n} \in \mathbb{R}^3$. Then necessarily $\Omega_{\bar{z}}e^{-2\lambda} = \overline{H_{\bar{z}}}$ i.e.

$$H_z = \Omega_{\bar{z}}e^{-2\lambda}. \tag{A.2.20}$$

Equation (A.2.20) is the Gauss-Codazzi equation in complex notations.

Using (A.2.20) and (A.2.17), we find

$$\vec{n}_{z\bar{z}} + |\vec{n}_z|^2\,\vec{n} + 2\Re\left(H_z\Phi_{\bar{z}}\right) = 0. \tag{A.2.21}$$

While the complex notations are most convenient for computations, the resulting equations are not always telling. We will then translate (A.2.21) back to its classic real form :

$$\vec{n}_{z\bar{z}} + |\vec{n}_z|^2\,\vec{n} + 2\Re\left(H_z\Phi_{\bar{z}}\right) = \frac{1}{4}\left(\Delta\vec{n} + |\nabla\vec{n}|^2\,\vec{n} + 2\left(H_x\Phi_x + H_y\Phi_y\right)\right).$$

The Gauss map $\vec{n}$ then satisfies

$$\Delta\vec{n} + |\nabla\vec{n}|^2\,\vec{n} + 2\nabla H\nabla\Phi = 0. \tag{A.2.22}$$

This can be slightly changed to better suit our needs

$$\begin{aligned}
\Delta\vec{n} + |\nabla\vec{n}|^2\,\vec{n} + 2\nabla H\nabla\Phi &= \Delta\vec{n} + |\nabla\vec{n}|^2\,\vec{n} + 2\mathrm{div}\,(H\nabla\Phi) - 2H\Delta\Phi \\
&= \Delta\vec{n} + \left(|\nabla\vec{n}|^2 - 2\,|H\nabla\Phi|^2\right)\vec{n} + 2\mathrm{div}\,(H\nabla\Phi) \\
&= \Delta\vec{n} + \left(\left|\mathring{A}\nabla\Phi\right|^2 - |H\nabla\Phi|^2\right)\vec{n} + 2\mathrm{div}\,(H\nabla\Phi)\,.
\end{aligned}$$

The second equality is obtained with (A.2.7), and the third with (A.2.9). Now we compute

$$\begin{aligned}
\nabla\vec{n}\times\nabla^{\perp}\vec{n} &= -\vec{n}_x\times\vec{n}_y + \vec{n}_y\times\vec{n}_x = -2\vec{n}_x\times\vec{n}_y \\
&= -2e^{-4\lambda}\left(e\Phi_x + f\Phi_y\right)\times\left(f\Phi_x + g\Phi_y\right) \\
&= -2e^{-2\lambda}\left(eg\vec{n} - f^2\vec{n}\right) \\
&= -2e^{-2\lambda}\left(\left(\frac{e+g}{2}\right)^2 - \left(\frac{e-g}{2}\right)^2 - f^2\right)\vec{n} \\
&= -2H^2e^{2\lambda}\vec{n} + 2\left(\left(\frac{e-g}{2}\right)^2 + f^2\right)e^{-2\lambda}\vec{n} \\
&= -\,|H\nabla\Phi|^2\,\vec{n} + \left|\mathring{A}\nabla\Phi\right|^2\vec{n}.
\end{aligned}$$

We then find

$$\Delta\vec{n} + \nabla^{\perp}\vec{n}\times\nabla\vec{n} + 2\mathrm{div}\,(H\nabla\Phi) = 0. \tag{A.2.23}$$

A.2.3 Formulas in $\mathbb{S}^3$

Let $\Phi : \mathbb{D}\to\mathbb{R}^3$ be a smooth conformal immersion and $X = \pi^{-1}\circ\Phi : \mathbb{D}\to\mathbb{S}^3$. Let $\Lambda := \frac{1}{2}\log\left(2\,|X_z|^2\right)$ be its conformal factor, $\vec{N}$ such that $\left(X, e^{-\Lambda}X_x, e^{-\Lambda}X_y, \vec{N}\right)$ is a direct orthonormal basis of $\mathbb{R}^4$ its Gauss map, $h = \left\langle \frac{X_{z\bar{z}}}{|X_z|^2}, \vec{N}\right\rangle$ its mean curvature and $\omega := 2\,\langle X_{zz}, \vec{n}\rangle$ its tracefree curvature. Then

$$X := \frac{1}{1+|\Phi|^2}\begin{pmatrix} 2\Phi \\ |\Phi|^2 - 1\end{pmatrix}, \tag{A.2.24}$$

which yields

$$X_z = d\pi^{-1}\left(\Phi_z\right) = \frac{2}{1+|\Phi|^2}\begin{pmatrix}\Phi_z \\ 0\end{pmatrix} - \frac{4\langle\Phi_z,\Phi\rangle_3}{\left(1+|\Phi|^2\right)^2}\begin{pmatrix}\Phi \\ -1\end{pmatrix}. \tag{A.2.25}$$

Since π is conformal, $\left\langle d\pi^{-1}\left(\Phi_z\right), d\pi^{-1}\left(\vec{n}\right)\right\rangle = \langle\Phi_z,\vec{n}\rangle = 0$. Then $\vec{N} = \frac{d\pi^{-1}(\vec{n})}{|d\pi^{-1}(\vec{n})|}$ and thus

$$\vec{N} = \begin{pmatrix}\vec{n} \\ 0\end{pmatrix} - \frac{2\langle\vec{n},\Phi\rangle}{1+|\Phi|^2}\begin{pmatrix}\Phi \\ -1\end{pmatrix}. \tag{A.2.26}$$

Using the corresponding definitions we successively deduce

$$e^{2\Lambda} = 2\,\langle X_z, X_{\bar{z}}\rangle = \frac{4}{\left(1+|\Phi|^2\right)^2}e^{2\lambda}, \tag{A.2.27}$$

$$h = \left\langle \frac{X_{z\bar{z}}}{|X_z|^2}, \vec{N} \right\rangle = \frac{|\Phi|^2+1}{2} H + \langle \vec{n}, \Phi \rangle \tag{A.2.28}$$

$$\omega = 2\left\langle X_{zz}, \vec{N} \right\rangle = \frac{2\Omega}{1+|\Phi|^2}. \tag{A.2.29}$$

Then one can compute

$$\begin{aligned}
h\begin{pmatrix} X \\ 1 \end{pmatrix} + \begin{pmatrix} \vec{N} \\ 0 \end{pmatrix} &= \left(\frac{|\Phi|^2+1}{2} H + \langle \vec{n}, \Phi \rangle\right) \begin{pmatrix} \frac{2\Phi}{1+|\Phi|^2} \\ \frac{|\Phi|^2-1}{1+|\Phi|^2} \\ 1 \end{pmatrix} + \begin{pmatrix} \vec{n} - \frac{2\langle \vec{n}, \Phi \rangle}{1+|\Phi|^2}\Phi \\ \frac{2\langle \vec{n}, \Phi \rangle}{1+|\Phi|^2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} H\Phi + \frac{2\langle \vec{n}, \Phi \rangle}{1+|\Phi|^2}\Phi \\ H\frac{|\Phi|^2-1}{2} + \langle \vec{n}, \Phi \rangle \frac{|\Phi|^2-1}{1+|\Phi|^2} \\ H\frac{|\Phi|^2+1}{2} + \langle \vec{n}, \Phi \rangle \end{pmatrix} + \begin{pmatrix} \vec{n} - \frac{2\langle \vec{n}, \Phi \rangle}{1+|\Phi|^2}\Phi \\ \frac{2\langle \vec{n}, \Phi \rangle}{1+|\Phi|^2} \\ 0 \end{pmatrix} \\
&= H\begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi \rangle \\ \langle \vec{n}, \Phi \rangle \end{pmatrix}.
\end{aligned} \tag{A.2.30}$$

Which shows that

$$Y = h\begin{pmatrix} X \\ 1 \end{pmatrix} + \begin{pmatrix} N \\ 0 \end{pmatrix}. \tag{A.2.31}$$

One may wish to compute in $\mathbb{S}^3$ without going through Φ. The relevant formulas then are

$$\vec{N}_z = -hX_z - \omega e^{-2\Lambda} X_{\bar{z}}, \tag{A.2.32}$$

$$X_{z\bar{z}} = h\frac{e^{2\Lambda}}{2}\vec{N} - \frac{e^{2\Lambda}}{2}X, \tag{A.2.33}$$

$$X_{zz} = 2\Lambda_z X_z + \frac{\omega}{2}\vec{N}, \tag{A.2.34}$$

and Gauss-Codazzi can be written

$$\omega_{\bar{z}} e^{-2\Lambda} = h_z. \tag{A.2.35}$$

A.2.4 Application : mean curvature of a sphere in $\mathbb{S}^3$

Let σ be a sphere in $\mathbb{S}^3$. Up to an isometry of $\mathbb{S}^3$ σ can be assumed to be a sphere centered on the south pole S of radius $r \leq \frac{\pi}{2}$. Then $\pi \circ \sigma$ is a sphere of $\mathbb{R}^3$ centered on the origin of radius $R \leq 1$. It can be conformally parametrized over $\mathbb{R}^2 \cup \infty$ by $\Phi(x,y) = \frac{R}{1+x^2+y^2}\begin{pmatrix} 2x \\ 2y \\ x^2+y^2-1 \end{pmatrix}$, of constant mean curvature $H = \frac{1}{R}$. Then σ is conformally parametrized by

$$X = \frac{1}{1+R^2}\begin{pmatrix} \frac{2R}{1+x^2+y^2}\begin{pmatrix} 2x \\ 2y \\ x^2+y^2-1 \end{pmatrix} \\ R^2-1 \end{pmatrix}.$$

One can easily compute using basic trigonometry the tangent of r and find

$$\tan(r) = \frac{2R}{1-R^2}.$$

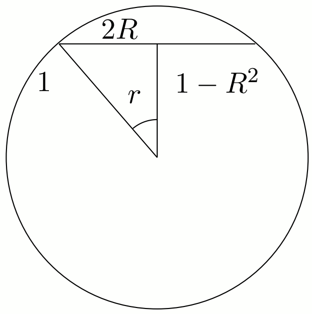

Figure A.1 – 2D illustration

Computing h at any point (x, y) using (A.2.28) yields with $H = \frac{1}{R}$, $\vec{n} = -\frac{\Phi}{R}$

$$h = \frac{R^2+1}{2R} - R = \frac{1}{\tan(r)}$$

for any (x, y).

Since neither h nor r change under the action of isometries, any sphere σ of $\mathbb{S}^3$ of radius r has constant mean curvature

$$h = \operatorname{cotan}(r). \tag{A.2.36}$$

A.2.5 Formulas in $\mathbb{H}^3$

Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a smooth conformal immersion and $Z = \tilde{\pi}^{-1} \circ \Phi : \mathbb{D} \to \mathbb{H}^3$ (see section 2.2.1 for the definition of the projection $\tilde{\pi}$). Then

$$Z := \frac{1}{1-|\Phi|^2}\begin{pmatrix} 2\Phi \\ |\Phi|^2+1 \end{pmatrix} \tag{A.2.37}$$

which yields

$$Z_z = \frac{2}{1-|\Phi|^2}\begin{pmatrix} \Phi_z \\ 0 \end{pmatrix} + \frac{4\langle \Phi_z, \Phi\rangle}{(1-|\Phi|^2)^2}\begin{pmatrix} \Phi \\ 1 \end{pmatrix}. \tag{A.2.38}$$

Since $\tilde{\pi}$ is conformal, $\left\langle d\tilde{\pi}^{-1}(\vec{n}), Z_z\right\rangle = \langle \Phi_z, \vec{n}\rangle = 0$. Then $\vec{n}^Z = \frac{d\tilde{\pi}^{-1}(\vec{n})}{|d\tilde{\pi}^{-1}(\vec{n})|}$ and thus

$$\vec{n}^Z = \begin{pmatrix} \vec{n} \\ 0 \end{pmatrix} + \frac{2\langle \vec{n}, \Phi\rangle}{1-|\Phi|^2}\begin{pmatrix} \Phi \\ 1 \end{pmatrix}. \tag{A.2.39}$$

Using the corresponding definition we successively deduce

$$e^{2\lambda^Z} = \frac{4}{(1-|\Phi|^2)^2} e^{2\lambda}, \tag{A.2.40}$$

$$H^Z = \frac{1-|\Phi|^2}{2} H - \langle \vec{n}, \Phi\rangle, \tag{A.2.41}$$

$$\Omega^Z = \frac{2\Omega}{1-|\Phi|^2}. \tag{A.2.42}$$

Then, one can compute

$$\begin{aligned}
H^Z \begin{pmatrix} Z_h \\ -1 \\ Z_4 \end{pmatrix} + \begin{pmatrix} \vec{n}_h^Z \\ 0 \\ \vec{n}_4^Z \end{pmatrix} &= \left(\frac{1-|\Phi|^2}{2} H - \langle \vec{n}, \Phi\rangle \right) \begin{pmatrix} \frac{2\Phi}{1-|\Phi|^2} \\ -1 \\ \frac{|\Phi|^2+1}{1-|\Phi|^2} \end{pmatrix} + \begin{pmatrix} \vec{n} + \frac{2\langle \vec{n},\Phi\rangle}{1-|\Phi|^2}\Phi \\ 0 \\ \frac{2\langle \vec{n},\Phi\rangle}{1-|\Phi|^2} \end{pmatrix} \\
&= \begin{pmatrix} H\Phi - \frac{2\langle \vec{n},\Phi\rangle}{1-|\Phi|^2}\Phi \\ -\frac{1-|\Phi|^2}{2} H + \langle \vec{n}, \Phi\rangle \\ H\frac{|\Phi|^2+1}{2} - \langle \vec{n}, \Phi\rangle \frac{|\Phi|^2+1}{1-|\Phi|^2} \end{pmatrix} + \begin{pmatrix} \vec{n} + \frac{2\langle \vec{n},\Phi\rangle}{1-|\Phi|^2}\Phi \\ 0 \\ \frac{2\langle \vec{n},\Phi\rangle}{1-|\Phi|^2} \end{pmatrix} \\
&= H \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi\rangle \\ \langle \vec{n}, \Phi\rangle \end{pmatrix}.
\end{aligned} \tag{A.2.43}$$

Which shows that

$$Y = H^Z \begin{pmatrix} Z_h \\ -1 \\ Z_4 \end{pmatrix} + \begin{pmatrix} \vec{n}_h^Z \\ 0 \\ \vec{n}_4^Z \end{pmatrix}. \tag{A.2.44}$$

A.2.6 Computations for the conformal Gauss map

Let $\Phi : \mathbb{D} \to \mathbb{R}^3$ be a smooth conformal immersion of representation X in $\mathbb{S}^3$ and of conformal Gauss map Y.

Let us first use the expression (2.3.15). Then

$$Y_z = H_z \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} + H \begin{pmatrix} \Phi_z \\ \langle \Phi_z, \Phi\rangle \\ \langle \Phi_z, \Phi_z\rangle \end{pmatrix} + \begin{pmatrix} \vec{n}_z \\ \langle \vec{n}_z, \Phi\rangle \\ \langle \vec{n}_z, \Phi\rangle \end{pmatrix}$$

and using (A.2.16)

$$Y_z = H_z \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} - \Omega e^{-2\lambda} \begin{pmatrix} \Phi_{\bar{z}} \\ \langle \Phi_{\bar{z}}, \Phi\rangle \\ \langle \Phi_{\bar{z}}, \Phi\rangle \end{pmatrix}. \tag{A.2.45}$$

Using (A.2.20) and (A.2.19) we compute

$$\begin{aligned}
Y_{z\bar{z}} &= H_{z\bar{z}} \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} - \frac{|\Omega|^2}{2} e^{-2\lambda} \begin{pmatrix} \vec{n} \\ \langle \vec{n}, \Phi\rangle \\ \langle \vec{n}, \Phi\rangle \end{pmatrix} \\
&= \frac{\mathcal{W}(\Phi)}{4} \begin{pmatrix} \Phi \\ \frac{|\Phi|^2-1}{2} \\ \frac{|\Phi|^2+1}{2} \end{pmatrix} - \frac{|\Omega|^2 e^{-2\lambda}}{2} Y
\end{aligned} \tag{A.2.46}$$

where

$$\frac{\mathcal{W}(\Phi)}{4} = H_{z\bar{z}} + \frac{|\Omega|^2 e^{-2\lambda}}{2} H \in \mathbb{R}. \tag{A.2.47}$$

On the other hand,

$$\begin{aligned}
Y_{zz} = {} & H_{zz}\begin{pmatrix}\Phi\\ \frac{|\Phi|^2-1}{2}\\ \frac{|\Phi|^2+1}{2}\end{pmatrix} + H_z\begin{pmatrix}\Phi_z\\ \langle\Phi_z,\Phi\rangle\\ \langle\Phi_z,\Phi\rangle\end{pmatrix} - \left(\Omega e^{-2\lambda}\right)_z\begin{pmatrix}\Phi_{\bar z}\\ \langle\Phi_{\bar z},\Phi\rangle\\ \langle\Phi_{\bar z},\Phi\rangle\end{pmatrix}\\
& -\Omega\left(\frac{H}{2}\begin{pmatrix}\vec n\\ \langle\vec n,\Phi\rangle\\ \langle\vec n,\Phi\rangle\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}\right)
\end{aligned} \tag{A.2.48}$$

using (A.2.18). Then, if we define Bryant's functional as $\mathcal{Q} = \langle Y_{zz}, Y_{zz}\rangle$ we find

$$\begin{aligned}
\mathcal{Q} &= H_{zz}\Omega - H_z\left(\Omega e^{-2\lambda}\right)_z e^{2\lambda} + \Omega\frac{H^2}{4}\\
&= \left(\Omega_{\bar z}e^{-2\lambda}\right)_z\Omega - \Omega_{\bar z}\left(\Omega e^{-2\lambda}\right)_z + \Omega\frac{H^2}{4}\text{using (A.2.20)}\\
&= \left(\Omega_{z\bar z}\Omega - \Omega_z\Omega_{\bar z}\right)e^{-2\lambda} + \Omega\frac{H^2}{4}\\
&= \Omega^2e^{-2\lambda}\left(\frac{\Omega_z}{\Omega}\right)_{\bar z} + \Omega\frac{H^2}{4} = \Omega^2e^{-2\lambda}\left(\frac{\Omega_{\bar z}}{\Omega}\right)_z + \Omega\frac{H^2}{4}.
\end{aligned} \tag{A.2.49}$$

We will now compute using expression (A.2.44). Then

$$Y_z = h_z\begin{pmatrix}X\\1\end{pmatrix} + h\begin{pmatrix}X_z\\1\end{pmatrix} + \begin{pmatrix}\vec N_z\\0\end{pmatrix}$$

and using (A.2.32)

$$Y_z = h_z\begin{pmatrix}X\\1\end{pmatrix} - \omega e^{-2\Lambda}\begin{pmatrix}X_{\bar z}\\0\end{pmatrix}. \tag{A.2.50}$$

Using (A.2.35) and (A.2.34), we compute

$$\begin{aligned}
Y_{z\bar z} &= h_{z\bar z}\begin{pmatrix}X\\1\end{pmatrix} - \frac{|\omega|^2}{2}e^{-2\Lambda}\begin{pmatrix}\vec N\\0\end{pmatrix}\\
&= \frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4}\begin{pmatrix}X\\1\end{pmatrix} - \frac{|\omega|^2e^{-2\Lambda}}{2}Y
\end{aligned} \tag{A.2.51}$$

where

$$\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4} = h_{z\bar z} + \frac{|\omega|^2e^{-2\Lambda}}{2}h \in \mathbb{R}. \tag{A.2.52}$$

Notice that using (A.2.27), (A.2.28) and (A.2.29)

$$
\begin{aligned}
\frac{\mathcal{W}_{\mathbb{S}^3}(X)}{4} &= \left(\frac{|\Phi|^2+1}{2}H + \langle \vec{n},\Phi\rangle\right)_{z\bar{z}} + \left(\frac{|\Phi|^2+1}{2}H + \langle \vec{n},\Phi\rangle\right)\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&= \left(\frac{|\Phi|^2+1}{2}H_z + \langle \Phi_z,\Phi\rangle H + \langle \vec{n}_z,\Phi\rangle\right)_{\bar{z}} + \frac{|\Phi|^2+1}{2}H\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&+ \langle \vec{n},\Phi\rangle\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&= \left(\frac{|\Phi|^2+1}{2}H_z - \Omega e^{-2\lambda}\langle \Phi_{\bar{z}},\Phi\rangle\right)_{\bar{z}} + \frac{|\Phi|^2+1}{2}H\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&+ \langle \vec{n},\Phi\rangle\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&= \frac{|\Phi|^2+1}{2}\frac{\mathcal{W}(\Phi)}{4} + \langle \Phi_{\bar{z}},\Phi\rangle H_z - \Omega_{\bar{z}}e^{-2\lambda}\langle \Phi_{\bar{z}},\Phi\rangle - \frac{|\Omega|^2e^{-2\lambda}}{2}\langle \vec{n},\Phi\rangle\\
&+ \langle \vec{n},\Phi\rangle\frac{|\Omega|^2e^{-2\lambda}}{2}\\
&= \frac{|\Phi|^2+1}{2}\frac{\mathcal{W}(\Phi)}{4},
\end{aligned}
\tag{A.2.53}
$$

using (A.2.16) to obtain the third equality and (A.2.35) to conclude. On the other hand

$$
Y_{zz} = h_{zz}\begin{pmatrix} X\\ 1\end{pmatrix} + h_z\begin{pmatrix} X_z\\ 0\end{pmatrix} - \left(\omega e^{-2\Lambda}\right)_z\begin{pmatrix} X_{\bar{z}}\\ 0\end{pmatrix} - \omega\left(\frac{h}{2}\begin{pmatrix} \vec{N}\\ 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix} X\\ 0\end{pmatrix}\right) \tag{A.2.54}
$$

using (A.2.18). Then if we define $\mathcal{Q} = \langle Y_{zz}, Y_{zz}\rangle$ we find, once more by applying (A.2.35),

$$
\begin{aligned}
\mathcal{Q} &= h_{zz}\omega - h_z\left(\omega e^{-2\Lambda}\right)_z e^{2\Lambda} + \omega^2\frac{h^2+1}{4}\\
&= \left(\omega_{\bar{z}}e^{-2\Lambda}\right)_z\omega - \omega_{\bar{z}}\left(\omega e^{-2\Lambda}\right)_z + \omega^2\frac{h^2+1}{4}\\
&= \left(\omega_{z\bar{z}}\omega - \omega_z\omega_{\bar{z}}\right)e^{-2\Lambda} + \omega^2\frac{h^2+1}{4}\\
&= \omega^2e^{-2\Lambda}\left(\frac{\omega_z}{\omega}\right)_{\bar{z}} + \omega^2\frac{h^2+1}{4} = \omega^2e^{-2\Lambda}\left(\frac{\omega_{\bar{z}}}{\omega}\right)_z + \omega^2\frac{h^2+1}{4}.
\end{aligned}
\tag{A.2.55}
$$

A.2.7 Formulas in $\mathbb{S}^{4,1}$

This section is devoted to computations for spacelike immersions in $\mathbb{S}^{4,1}$ without relying on their being the conformal Gauss map of a given immersion.

Let $Y : D \to \mathbb{S}^{4,1}$ be a smooth-spacelike conformal immersion, that is Y satisfies

$$\langle Y_z, Y_z\rangle = 0$$

and

$$\langle Y_z, Y_{\bar{z}}\rangle =: \frac{e^{2\mathcal{L}}}{2} > 0.$$

Let ν, $\nu^* \in \mathcal{C}^{4,1}$ such that $e = (Y, Y_z, Y_{\bar{z}}, \nu, \nu^*)$ is an orthogonal frame of $\mathbb{R}^{4,1}$, that is

$$\langle Y,\nu\rangle = \langle Y_z,\nu\rangle = \langle Y_{\bar{z}},\nu\rangle = \langle \nu,\nu\rangle = 0$$

and

$$\langle Y,\nu^*\rangle = \langle Y_z,\nu^*\rangle = \langle Y_{\bar{z}},\nu^*\rangle = \langle \nu^*,\nu^*\rangle = 0.$$

We define successively the tracefree curvature in the direction ν

$$\Omega_\nu = 2\left\langle Y_{zz}, \nu\right\rangle, \tag{A.2.56}$$

the tracefree curvature in the direction ν^*

$$\Omega_{\nu^*} = 2\left\langle Y_{zz}, \nu^*\right\rangle, \tag{A.2.57}$$

the mean curvature in the direction ν

$$H_\nu = 2e^{-2\mathcal{L}}\left\langle Y_{z\bar{z}}, \nu\right\rangle, \tag{A.2.58}$$

and the mean curvature in the direction ν^*

$$H_{\nu^*} = 2e^{-2\mathcal{L}}\left\langle Y_{zz}, \nu^*\right\rangle. \tag{A.2.59}$$

Then

$$Y_{zz} = 2\mathcal{L}_z Y_z + \frac{\Omega_\nu}{2\langle\nu,\nu^*\rangle}\nu^* + \frac{\Omega_{\nu^*}}{2\langle\nu,\nu^*\rangle}\nu, \tag{A.2.60}$$

and

$$Y_{z\bar{z}} = \frac{H_\nu e^{2\mathcal{L}}}{2\langle\nu,\nu^*\rangle}\nu^* + \frac{H_{\nu^*} e^{2\mathcal{L}}}{2\langle\nu,\nu^*\rangle}\nu - \frac{e^{2\mathcal{L}}}{2}Y. \tag{A.2.61}$$

Further

$$\langle\nu_z, Y\rangle = \left(\langle\nu, Y\rangle\right)_z - \langle\nu, Y_z\rangle = 0, \tag{A.2.62}$$

and with (A.2.60),

$$\begin{aligned}\langle\nu_z, Y_z\rangle &= \left(\langle\nu, Y_z\rangle\right)_z - \langle\nu, Y_{zz}\rangle \\ &= -2\mathcal{L}_z\left\langle\nu, Y_z\right\rangle - \frac{\Omega_\nu}{2\langle\nu,\nu^*\rangle}\langle\nu,\nu^*\rangle - \frac{\Omega_\nu^*}{2\langle\nu,\nu^*\rangle}\langle\nu,\nu\rangle \\ &= -\frac{\Omega_\nu}{2},\end{aligned} \tag{A.2.63}$$

while with (A.2.61),

$$\begin{aligned}\langle\nu_z, Y_{\bar{z}}\rangle &= \left(\langle\nu, Y_{\bar{z}}\rangle\right)_z - \langle\nu, Y_{z\bar{z}}\rangle \\ &= -\frac{H_\nu e^{2\mathcal{L}}}{2\langle\nu,\nu^*\rangle}\langle\nu,\nu^*\rangle \\ &= -\frac{H_\nu e^{2\mathcal{L}}}{2},\end{aligned} \tag{A.2.64}$$

and

$$\langle\nu_z, \nu\rangle = \left(\langle\nu, \nu\rangle\right)_z - \langle\nu, \nu_z\rangle,$$

meaning

$$\langle\nu_z, \nu\rangle = 0. \tag{A.2.65}$$

Combining (A.2.62), (A.2.63), (A.2.64) and (A.2.65) yields

$$\nu_z = -\left\langle\nu_z, \nu^*\right\rangle\nu - H_\nu Y_z - \Omega_\nu e^{-2\mathcal{L}} Y_{\bar{z}}. \tag{A.2.66}$$

Similarly

$$\langle\nu_z^*, Y\rangle = \left(\langle\nu^*, Y\rangle\right)_z - \langle\nu^*, Y_z\rangle = 0, \tag{A.2.67}$$

and with (A.2.60),

$$\begin{aligned}\langle \nu_z^*, Y_z\rangle &= \left(\langle \nu^*, Y_z\rangle\right)_z - \langle \nu^*, Y_{zz}\rangle \\ &= -2\mathcal{L}_z \langle \nu^*, Y_z\rangle - \frac{\Omega_{\nu^*}}{2\langle \nu, \nu^*\rangle}\langle \nu, \nu^*\rangle - \frac{\Omega_{\nu}}{2\langle \nu, \nu^*\rangle}\langle \nu^*, \nu^*\rangle \\ &= -\frac{\Omega_{\nu^*}}{2},\end{aligned} \tag{A.2.68}$$

while with (A.2.61)

$$\begin{aligned}\langle \nu_z^*, Y_{\bar z}\rangle &= \left(\langle \nu^*, Y_{\bar z}\rangle\right)_z - \langle \nu^*, Y_{z\bar z}\rangle \\ &= -\frac{H_{\nu^*} e^{2\mathcal{L}}}{2\langle \nu, \nu^*\rangle}\langle \nu, \nu^*\rangle \\ &= -\frac{H_{\nu^*} e^{2\mathcal{L}}}{2},\end{aligned} \tag{A.2.69}$$

$$\langle \nu_z^* \nu^*\rangle = 0, \tag{A.2.70}$$

$$\nu_z^* = -\langle \nu_z^*, \nu\rangle \nu^* - H_{\nu^*} Y_z - \Omega_{\nu^*} e^{-2\mathcal{L}} Y_{\bar z}. \tag{A.2.71}$$

Then

$$\begin{aligned}\langle \nu_z, \nu_z\rangle &= H_\nu \Omega_\nu \\ \langle \nu_z^*, \nu_z^*\rangle &= H_{\nu^*}\Omega_{\nu^*}.\end{aligned} \tag{A.2.72}$$

A.3 Analytic lemmas

A.3.1 Low-regularity estimates

Following is a sequence of low regularity auxiliary theorems needed in our proofs.

Theorem A.3.1 (Theorem 3.5 in [Riv16])**.** Let $X \in L^1\left(\mathbb{D}, \mathbb{R}^2\right)$, if f is the $W_0^{1,1}$ solution in a distributional sense of

$$\begin{cases} \Delta f = \operatorname{div} X \text{ in } \mathbb{D}, \\ \quad f = 0 \text{ on } \partial\mathbb{D}, \end{cases}$$

then $f \in L^{2,\infty}(\mathbb{D})$ with

$$\|f\|_{L^{2,\infty}(\mathbb{D})} \leq C\|X\|_{L^1(\mathbb{D})}.$$

Theorem A.3.2. Let $V \in \mathcal{D}'\left(\mathbb{R}^3\right)$ such that $\nabla V = \nabla^\perp a + B$ with $\nabla^\perp a \in H^{-1}\left(\mathbb{D}, \mathbb{R}^2\right)$ and $B \in L^1\left(\mathbb{D}, \mathbb{R}^2\right)$. Then for any $r < 1$ there exists $c \in \mathbb{R}$ a constant and $C(r) > 0$ such that

$$\|V - c\|_{L^{2,\infty}(\mathbb{D}_r)} \leq C(r)\left(\|A\|_{H^{-1}(\mathbb{D})} + \|B\|_{L^1(\mathbb{D})}\right).$$

Proof. We write $\nabla V = \nabla b + H$ with

$$\begin{cases} \Delta b = \operatorname{div} B \text{ in } \mathbb{D} \\ \quad b = 0 \text{ on } \partial\mathbb{D}. \end{cases}$$

Since

$$\operatorname{div}(H) = \Delta V - \Delta b = \operatorname{div}\left(\nabla^\perp a + B\right) - \operatorname{div}(B) = 0$$

and

$$\operatorname{curl}(H) = \operatorname{curl}\left(\nabla V - \nabla b\right) = 0$$

in $\mathcal{D}'(\mathbb{D})$, one finds $\operatorname{div}(H)=\operatorname{curl}(H)=0$, that is H is harmonic. We will write it $H=\nabla h$ with h harmonic.Using theorem A.3.1, we find

$$\|b\|_{L^{2,\infty}(\mathbb{D})}\le C\|B\|_{L^1(\mathbb{D})}. \tag{A.3.73}$$

Besides, given $\phi\in C_c^\infty(\mathbb{D})$:

$$\begin{aligned}|\langle H,\phi\rangle| &= \left|\langle\nabla^\perp a+B-\nabla b,\phi\rangle\right|\\ &\le \left|\langle\nabla^\perp a,\phi\rangle\right|+|\langle B,\phi\rangle|+|\langle b,\nabla\phi\rangle|\\ &\le \|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}\|\phi\|_{H^1(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\|\phi\|_{L^\infty(\mathbb{D})}\\ &+\|b\|_{L^{2,\infty}(\mathbb{D})}\|\nabla\phi\|_{L^{2,1}(\mathbb{D})}\end{aligned}$$

since $L^{2,1}=\left(L^{2,\infty}\right)^*$. Now using (A.3.73) and the continuous injection $L^{2,1}\hookrightarrow L^2$ we find

$$|\langle H,\phi\rangle|\le C\left(\|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)\left(\|\phi\|_{L^\infty(\mathbb{D})}+\|\nabla\phi\|_{L^{2,1}(\mathbb{D})}\right).$$

This yields

$$|\langle\nabla h,\phi\rangle|\le C\left(\|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)\left(\|\phi\|_{L^\infty(\mathbb{D})}+\|\nabla\phi\|_{L^{2,1}(\mathbb{D})}\right). \tag{A.3.74}$$

Since h is harmonic, we write $h_z=\displaystyle\sum_{p\in\mathbb{Z}}h_pz^p$, and we apply (A.3.74) with $\phi_p=r\eta(r)e^{-ip\theta}$, where η is a smooth positive cut-off function on $\mathbb{D}$ with support in $\left[0,\frac{3}{4}\right]$, $\eta=1$ on $\left[0,\frac{1}{2}\right]$.

$$\begin{aligned}\left|\int_{\mathbb{D}}h_p\eta(r)r^{p+2}drd\theta\right| &\le C\left(\|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)\left(\|\phi_p\|_{L^\infty(\mathbb{D})}+\|\nabla\phi_p\|_{L^{2,1}(\mathbb{D})}\right)\\ &\le C\|\eta\|_{C^1(\mathbb{D})}\left(\|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)p.\end{aligned}$$

However, since $\eta\ge 0$ and $\eta=1$ on $\mathbb{D}_{\frac{1}{2}}$:

$$\begin{aligned}\left|\int_{\mathbb{D}}\eta(r)r^{p+2}drd\theta\right| &\ge \int_{\mathbb{D}_{\frac{1}{2}}}\eta(r)r^{p+2}drd\theta,\\ &\ge \int_{\mathbb{D}_{\frac{1}{2}}}r^{p+2}dr.\end{aligned}$$

This means that for $p\ge -2$

$$|h_p|\le C\|\eta\|_{C^1(\mathbb{D})}\left(\|\nabla^\perp a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)p(p+3) \tag{A.3.75}$$

and for $p<-2$

$$|h_p|=0.$$

Thus h_p grows at most quadratically and as a consequence $\displaystyle\sum_{p\in\mathbb{Z}}h_pz^p$ converges on $\mathbb{D}_r$ with $r<1$. Then there exists $c_1\in\mathbb{R}$ such that h can be written

$$h(z)-c_1=2\Re\left(\frac{h_{-2}}{z}+\frac{h_{-1}}{2}\log(r)+\sum_{p\ge 0}\frac{h_p}{p+1}z^{p+1}\right)$$

which converges smoothly on $\mathbb{D}_r$ for any $r<1$. Then

$$\begin{aligned}\|h-c_1\|_{L^{2,\infty}(\mathbb{D}_r)} &\leq C\left(|h_{-2}|\left\|\frac{1}{r}\right\|_{L^{2,\infty}(\mathbb{D}_r)}+|h_{-1}|\,\|\log r\|_{L^{2,\infty}(\mathbb{D}_r)}+\sum_{p\geq 0}\left|\frac{h_p}{p+1}\right|\left\|r^{p+1}\right\|_{L^{2,\infty}(\mathbb{D}_r)}\right)\\ &\leq C\left(|h_{-2}|+|h_{-1}|+\sum_{p\geq 0}\left|\frac{h_p}{(p+1)\,(2p+4)}\right|r^{2p+4}\right)\\ &\leq C\left(\|\nabla^{\perp}a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)\left(1+\sum_{p\geq 0}\frac{p(p+3)}{(p+1)\,(2p+4)}r^{2p+4}\right),\end{aligned}$$

using (A.3.75). This yields

$$\|h-c_1\|_{L^{2,\infty}(\mathbb{D}_r)}\leq C(r)\left(\|\nabla^{\perp}a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right). \tag{A.3.76}$$

Here $C(r)<\infty$ as soon as $r<1$. Since by definition $\nabla V=\nabla b+\nabla h$, there exists a constant $c\in\mathbb{R}$ such that

$$V-c=b+h-c_1.$$

Using (A.3.73) and (A.3.76) we then deduce

$$\|V-c\|_{L^{2,\infty}(\mathbb{D}_r)}\leq C\left(\|\nabla^{\perp}a\|_{H^{-1}(\mathbb{D})}+\|B\|_{L^1(\mathbb{D})}\right)$$

with C depending only on r, which concludes the proof. □

We conclude this subsection by recalling an extension of Calderon-Zygmund with Lorentz spaces (theorem 3.3.6 in [Hél02]).

Theorem A.3.3. Let Ω be an open subset of $\mathbb{R}^2$ with C^1 boundary. Let $f\in L^1(\Omega)$ and φ solution of

$$\begin{cases}\Delta\varphi=f \text{ in } \Omega\\ \quad\varphi=0 \text{ on } \partial\Omega.\end{cases}$$

then there exists a constant $C(\Omega)$ such that

$$\|\varphi\|_{L^{2,\infty}(\Omega)}\leq C(\Omega)\,\|f\|_{L^1(\Omega)}.$$

A.3.2 Integrability by compensation

Following are a few variations on Wente's theorem, which will prove useful in the core of the article. First is Wente's inequality, originally presented in [Wen71], we here follow see also 3.1.2 in [Hél02]

Theorem A.3.4. Let $a,b\in W^{1,2}(\mathbb{D},\mathbb{R})$ and u a solution of

$$\begin{cases}\Delta u=\nabla a.\nabla^{\perp}b \text{ in } \mathbb{D}\\ u=0 \text{ on } \partial\mathbb{D}.\end{cases}$$

Then $u\in C^0(\mathbb{D},\mathbb{R})\cap W^{1,2}(\mathbb{D},\mathbb{R})$, and there exists $C>0$

$$\|u\|_{L^{\infty}(\mathbb{D})}+\|\nabla u\|_{L^2(\mathbb{D})}\leq C\,\|\nabla a\|_{L^2(\mathbb{D})}\,\|\nabla b\|_{L^2(\mathbb{D})}.$$

Following are two refinements for Lorentz spaces, first theorem 3.4.5 then 3.4.1 of [Hél02]

Theorem A.3.5. Let Ω be a bounded domain of $\mathbb{R}^2$, with C^2 boundary. Suppose a and b such that $\nabla a \in L^{2,\infty}(\Omega)$ and $\nabla b \in L^2(\Omega)$. Let φ be the solution of

$$\begin{cases} \Delta\varphi = \nabla a.\nabla^{\perp} b \text{ in } \Omega \\ \quad \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

Then $\varphi \in W^{1,2}(\Omega)$, and there exists $C(\Omega) > 0$ such that

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla a\|_{L^{2,\infty}(\Omega)}\|\nabla b\|_{L^2(\Omega)}.$$

Theorem A.3.6. Let Ω be a bounded domain of $\mathbb{R}^2$, with C^2 boundary. Suppose a and b such that $a \in W^{1,2}(\Omega)$ and $b \in W^{1,2}(\Omega)$. Let φ be the solution of

$$\begin{cases} \Delta\varphi = \nabla a.\nabla^{\perp} b \text{ in } \Omega \\ \quad \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

Then $\varphi \in W^{1,(2,1)}(\Omega)$, and there exists $C(\Omega) > 0$ such that

$$\|\nabla\varphi\|_{L^{2,1}(\Omega)} \leq C(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}.$$

Remark A.3.1. One must notice that, since $L^{2,\infty}$ and $L^{2,1}$ are scale-invariant, but not conformal invariant, the constant $C(\Omega)$ in theorems A.3.5 and A.3.6 depends on the shape of Ω, but not its size. The same constant C then works for all disks $\mathbb{D}_r$. Since L^2 is a conformal invariant the constant in theorem A.3.4 does not depend on Ω. We refer the reader to [BG93] for more details.

A.3.3 Hodge decomposition

In this subsection we briefly recall results on the Hodge decomposition and recast them in our framework.

Theorem A.3.7 (L^p decomposition, theorem 10.5.1 in [IM01])**.** Let Ω be a smoothly bounded domain in $\mathbb{R}^n$ and $1 < p < \infty$. Then for any l-differential form $\omega \in L^p$ there exists a $l-1$ differential form α, a $l+1$-differential form β and a l-differential form h such that :

$$\omega = d\alpha + d^*\beta + h$$

with $dh = d^*h = 0$ and

$$\|\alpha\|_{W^{1,p}(\Omega)} + \|\beta\|_{W^{1,p}(\Omega)} \leq C_p(\Omega)\, \|\omega\|_{L^p(\Omega)}\,.$$

Theorem 10.5.1 in [IM01] is in fact more accurate and actually goes into much more details about the boundary conditions. However quoting it in a comprehensive manner would require to introduce new notations. We thus restrict ourselves to this partial result, which will satisfy our current needs. Taking $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in L^p(\mathbb{D}_r, \mathbb{R}\times\mathbb{R})$, and $\omega = X_1 dx + X_2 dy$, one can apply theorem A.3.7 and find a function α, a volume form β and a harmonic 1-form h on $\mathbb{D}_r$ such that :

$$\omega = d\alpha + d^*\beta + h,$$

$$\|\alpha\|_{W^{1,p}(\mathbb{D}_r)} + \|\beta\|_{W^{1,p}(\mathbb{D}_r)} \leq C_p(\mathbb{D}_r)\, \|\omega\|_{L^p(\mathbb{D}_r)} \leq C_p(r)\, \|X\|_{L^p(\mathbb{D}_r)}\,.$$

Since $div(X) = d^*\omega = \Delta\alpha$ we deduce

Corollary A.3.1. Let $r > 0$ and $1 < p < \infty$. For any $X \in L^p(\mathbb{D}_r, \mathbb{R} \times \mathbb{R})$ there exists $\alpha \in W^{1,p}(\mathbb{D}_r)$ such that

$$\operatorname{div}(X) = \Delta\alpha$$

and

$$\|\alpha\|_{W^{1,p}(\mathbb{D}_r)} \leq C_p(r)\|X\|_{L^p(\mathbb{D}_r)}.$$

Using Marcinkiewitz interpolation theorem (see for example theorem 3.3.3 of [Hél02]) enables us to write

Corollary A.3.2. Let $r > 0$, for any $X \in L^{2,1}(\mathbb{D}_r, \mathbb{R}^2)$ there exists $\alpha \in W^{1,(2,1)}(\mathbb{D}_r)$ such that

$$\Delta\alpha = \operatorname{div}(X)$$

and

$$\|\alpha\|_{W^{1,(2,1)}(\mathbb{D}_r)} \leq C(r)\|X\|_{L^{2,1}(\mathbb{D}_r)}.$$

A.3.4 Weighted Calderon-Zygmund

Theorems A.3.8 and A.3.9 are taken from Y. Bernard and T. Rivière's [BR13] (Proposition C.2 and C.3).

Theorem A.3.8. Let $u \in C^2\left(\overline{\mathbb{D}}\backslash\{0\}\right)$ solve

$$\Delta u(z) = \mu(z)f(z) \text{ in } \mathbb{D},$$

with $f \in L^p(\mathbb{D})$ for $2 < p \leq \infty$ and the weight μ satisfying for some $a \in \mathbb{N}$

$$|\mu(z)| = O(|z|^a).$$

Then

$$u_z(z) = P(z) + |z|^a T(z)$$

with $P \in \mathbb{C}_a[X]$ and $T = O(|z|^{1-\frac{2}{p}-\upsilon})$ for all $\upsilon > 0$. More precisely one has

$$\left\| \frac{T}{|z|^{1-\frac{2}{p}-\upsilon}} \right\|_{L^\infty(\mathbb{D})} \leq C_\upsilon \left(\left\| \frac{\mu}{|z|^a} \right\|_{L^\infty(\mathbb{D})} \|f\|_{L^p(\mathbb{D})} + \|u\|_{C^1(\partial\mathbb{D})} \right).$$

Additionally if $\mu \in C^1(\mathbb{D}\backslash\{0\})$, $a \neq 0$ and

$$\nabla\mu(z) = O\left(|z|^{a-1}\right)$$

Then :

$$u_{zz}(z) = P_z + |z|^a Q$$

with $Q \in L^{p'}(\mathbb{D})$ for all $p' < p$ and

$$\|Q\|_{L^{p'}(\mathbb{D})} \leq C_{p'} \left(\left(\left\| \frac{\mu}{|z|^a} \right\|_{L^\infty(\mathbb{D})} + \left\| \frac{\nabla\mu}{|z|^{a-1}} \right\|_{L^\infty(\mathbb{D})} \right) \|f\|_{L^p(\mathbb{D})} + \|u\|_{C^1(\partial\mathbb{D})} \right).$$

In fact $Q = \frac{(|z|^a T(z))_z}{|z|^a}$.

Remark A.3.2. Theorem A.3.8 works with $a = 0$, it is the classic Calderon-Zygmund theorem.

Proof. We will write the proof for $p = \infty$ to paint a picture of the involved reasonings and refer the reader to the original results for the general case ($p < \infty$). Such an estimate can be written freely away from 0. Then one can assume $|z| \leq \frac{1}{2}$. Using Green's formula for the Laplacian and denoting ν the outer normal unit vector to $\partial\mathbb{D}$, one writes explicitely u :

$$\begin{aligned} u_z(z) &= \frac{1}{2\pi}\left(\int_{\partial\mathbb{D}}\left(\frac{\bar{z}-\bar{x}}{|z-x|^2}\partial_\nu u(x) - u(x)\partial\frac{\bar{z}-\bar{x}}{|z-x|^2}\right)d\sigma(x) - \int_{\mathbb{D}}\frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx\right) \\ &=: J_0(z) + J_1(z). \end{aligned} \tag{A.3.77}$$

We first point out that for $|x| > |z|$ one term can be expanded :

$$\frac{\bar{x}-\bar{z}}{|x-z|^2} = \sum_{m\geq 0} z^m x^{-(m+1)}.$$

Then we find :

$$\begin{aligned} J_0(z) &= \frac{1}{2\pi}\sum_{m\geq 0}\int_{\partial\mathbb{D}}\left(z^m x^{-(m+1)}\partial_\nu u(x) - u(x)\partial_\nu\left(z^m x^{-(m+1)}\right)\right)d\sigma(x) \\ &= \frac{1}{2\pi}\sum_{m\geq 0} z^m\int_0^{2\pi}\left((m+1)u(e^{i\theta}) - (\partial_\nu u)\left(e^{i\theta}\right)\right)e^{i(m+1)\theta}d\theta \\ &= \sum_{m\geq 0} C_m z^m \end{aligned}$$

where the C_m are complex valued constants depending only on the C^1 norm of u along $\partial\mathbb{D}$. Since u is by hypothesis bounded C^1 on the boundary of the unit disk by hypothesis, $\int_0^{2\pi} u(e^{i\theta})e^{i(m+1)\theta}d\theta$ and $\int_0^{2\pi}\partial_\nu u\left(e^{i\theta}\right)e^{i(m+1)\theta}d\theta$ are bounded by the C^1 norm of u and thus the C_m are growing at most linearly. Thus there exists a $\delta > 0$ such that for $|z| \leq \delta$, and a $C > 0$

$$\begin{aligned} J_0(z) &= \sum_{m=0}^{a} C_m z^m + \sum_{m=a+1}^{\infty} C_m z^m \\ \left|\sum_{m=a+1}^{\infty} C_m z^m\right| &\leq C|z|^{a+1}. \end{aligned}$$

Then, one writes

$$\begin{aligned} J_0(z) &= \sum_{m=0}^{a} C_m z^m + |z|^{a+1}T_0(z) \text{ in } \mathbb{D}_\delta, \\ \text{with } |z|^{a+1}T_0 &= \sum_{m=a+1}^{\infty} C_m z^m, \\ |T_0| &\leq C\|u\|_{C^1(\partial\mathbb{D})} < \infty \text{ in } \mathbb{D}_\delta. \end{aligned} \tag{A.3.78}$$

Now we notice J_0 is uniformly bounded, with bounds depending only on $\|u\|_{C^1(\partial\mathbb{D}))}$, on $\mathbb{D}_{\frac{1}{2}}$. We can then extend (A.3.78) to the whole of $\mathbb{D}_{\frac{1}{2}}$ up to a constant adjustment.

One must now control J_1. We start by writing :

$$\begin{aligned} J_1(z) &= \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}}\frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx + \frac{1}{2\pi}\int_{\mathbb{D}\backslash\mathbb{D}_{2|z|}}\frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx \\ &= \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}}\frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx + \frac{1}{2\pi}\int_{\mathbb{D}\backslash\mathbb{D}_{2|z|}}\sum_{m=0}^{\infty} z^m x^{-(m+1)}\mu(x)f(x)dx. \end{aligned}$$

Now, since on $\mathbb{D}\backslash\mathbb{D}_{2|z|}$,

$$\sum_{m=0}^{\infty}\left(\frac{|z|}{|x|}\right)^m \leq \sum_{m=0}^{\infty}\frac{1}{2^m} < \infty,$$

we deduce,

$$J_1(z) = \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}} \frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx + \sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{\mathbb{D}\backslash\mathbb{D}_{2|z|}} z^m x^{-(m+1)}\mu(x)f(x)dx.$$

We then introduce the following decomposition :

$$J_1(z) = I_1(z) + \sum_{m=0}^{a} I_1^m(z) + I_2^m(z) - \sum_{m=0}^{a} I_1^m(z) + \sum_{m=a+1}^{\infty} I_2^m(z), \tag{A.3.79}$$

where :

$$\begin{aligned}
I_1(z) &:= \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}} \frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx,\\
I_1^m(z) &:= \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}} z^m x^{-(m+1)}\mu(x)f(x)dx,\\
I_2^m(z) &:= \frac{1}{2\pi}\int_{\mathbb{D}\backslash\mathbb{D}_{2|z|}} z^m x^{-(m+1)}\mu(x)f(x)dx.
\end{aligned}$$

We notice

$$\sum_{m=0}^{a} I_1^m(z) + I_2^m(z) = \sum_{m=0}^{a}\frac{z^m}{2\pi}\int_{\mathbb{D}} x^{-(m+1)}\mu(x)f(x)dx,$$

and for $m \leq a$

$$\begin{aligned}
\left|\int_{\mathbb{D}} x^{-(m+1)}\mu(x)f(x)dx\right| &\leq \left\|\frac{\mu}{|z|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}}|x|^{a-m-1}dx\\
&\leq C\left\|\frac{\mu}{|z|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}.
\end{aligned}$$

which yields

$$\sum_{m=0}^{a} I_1^m(z) + I_2^m(z) = \sum_{m=0}^{a} A_m z^m \tag{A.3.80}$$

with $A_m = \frac{1}{2\pi}\int_{\mathbb{D}} x^{-(m+1)}\mu(x)f(x)dx$ a sequence of finite coefficients. Besides :

$$\begin{aligned}
|I_1(z)| &\leq \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}}\frac{1}{|z-x|}|\mu(x)||f(x)|dx \leq \frac{1}{2\pi}\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}_{2|z|}}\frac{|x|^a}{|z-x|}dx\\
&\leq C_a|z|^a\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}_{2|z|}}\frac{1}{|z-x|}dx\\
&\leq C|z|^{a+1}\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})},
\end{aligned} \tag{A.3.81}$$

and for $m \leq a$,

$$\begin{aligned}
|I_1^m(z)| &\leq \frac{1}{2\pi}\int_{\mathbb{D}_{2|z|}}|z|^m|x|^{-(m+1)}|\mu(x)||f(x)|dx\\
&\leq C\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}|z|^m\int_{\mathbb{D}_{2|z|}}|x|^{a-(m+1)}dx \leq C_a|z|^{a+1}\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}.
\end{aligned} \tag{A.3.82}$$

Finally, for $a+2 \leq m$ we write

$$\begin{aligned}
|I_2^m(z)| &\leq C|z|^m \|f\|_{L^\infty(\mathbb{D})} \int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}} |x|^{-(m+1)}|\mu(x)|dx \\
&\leq C|z|^m \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}} |x|^{a-(m+1)}dx \\
&\leq C|z|^m \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \int_{2|z|}^1 r^{a-m}dr \\
&\leq C|z|^m \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \frac{|2z|^{a+1-m}-1}{m-a-1} \\
&\leq C\frac{1}{2^{m-a-1}(m-a-1)}|z|^{a+1} \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})},
\end{aligned} \tag{A.3.83}$$

while I_2^{a+1} is controlled in the following way

$$\begin{aligned}
|I_2^{a+1}(z)| &\leq C|z|^{a+1} \|f\|_{L^\infty(\mathbb{D})} \int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}} |x|^{-(a+2)}|\mu(x)|dx \\
&\leq C|z|^{a+1} \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}} |x|^{-2}dx \\
&\leq C|z|^{a+1}\ln|z| \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \\
&\leq C_v|z|^{a+1-v} \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \quad \forall v>0.
\end{aligned} \tag{A.3.84}$$

Consequently

$$\begin{aligned}
\left|\sum_{m=a+1}^{\infty} I_2^m(z)\right| &\leq C_v|z|^{a+1-v} \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})} \sum \frac{1}{2^{m-a-1}} \\
&\leq C_v|z|^{a+1-v} \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})}.
\end{aligned} \tag{A.3.85}$$

Injecting (A.3.80)-(A.3.85) into (A.3.79) shows J_1 satisfies

$$\begin{aligned}
J_1(z) &= \sum_{m=0}^{a} A_m z^m + |z|^{a+1-v}T_1(z) \text{ in } \mathbb{D} \\
|T_1| &\leq C_v \left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})} \|f\|_{L^\infty(\mathbb{D})}.
\end{aligned} \tag{A.3.86}$$

To conclude, (A.3.78) and (A.3.86) yield the desired result on u_z when applied to (A.3.77). To prove the next part of the theorem one need only notice that necessarily

$$\begin{aligned}
|z|^aQ(z) &= (|z|^aT(z))_z \\
&= \left(\sum_{m\geq a+1} C_m z^m\right)_z + (I_1)_z(z) + \sum_{m\geq a+1} (I_2^m)_z(z) - \sum_{0\leq m\leq a} (I_1^m)_z(z)
\end{aligned} \tag{A.3.87}$$

Now, since we have shown that the C_m have a mere linear growth, $\left(\sum_{m\geq a+1} C_m z^m\right)_z = \sum_{m\geq a+1} mC_m z^{m-1}$ has the same strictly positive convergence radius. The same argument as before applies and yields the wanted control on the first term of (A.3.87). The other terms are estimated as before. Indeed :

$$\begin{aligned}|\left(I_1^m\right)_z(z)| &\leq \left|\frac{1}{2\pi}\int_{\mathbb{D}\cap\mathbb{D}_{2|x|}} m z^{m-1}x^{-m-1}\mu(x)f(x)dx + \frac{1}{2\pi}\frac{\bar{z}}{|z|}\int_{\partial\mathbb{D}_{2|z|}} z^m x^{-m-1}\mu(x)f(x)dx\right| \\ &\leq C_m|z|^{m-1}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}_{2|x|}}|x|^{a-m-1}dx + C|z|^a\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})} \\ &\leq C_a|z|^a\|f\|_{L^\infty(\mathbb{D})},\end{aligned} \tag{A.3.88}$$

as long as $m \leq a$. Similarly :

$$\begin{aligned}|\left(I_2^m\right)(z)| &\leq \left|\frac{1}{2\pi}\int_{\mathbb{D}\setminus\mathbb{D}_{2|x|}} m z^{m-1}x^{-m-1}\mu(x)f(x)dx + \frac{1}{2\pi}\frac{\bar{z}}{|z|}\int_{\partial\left(\mathbb{D}\setminus\mathbb{D}_{2|z|}\right)} z^m x^{-m-1}\mu(x)f(x)dx\right| \\ &\leq Cm|z|^{m-1}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}}|x|^{a-m-1}dx + \frac{C}{2^m}|z|^a\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})} \\ &\leq \frac{C}{2^m}|z|^a\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\end{aligned} \tag{A.3.89}$$

for $m \geq a+2$; while

$$\begin{aligned}|\left(I_2^{a+1}\right)_z(z)| &\leq C_a|z|^a\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\left(\int_{\mathbb{D}\setminus\mathbb{D}_{2|z|}}|x|^{-2}dx + 1\right) \\ &\leq C_a|z|^a\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}\ln|z|.\end{aligned} \tag{A.3.90}$$

The I_1 estimate is slightly more difficult to obtain. Differentiating we find $I_{1\,z} = \frac{1}{2\pi}\left(L(z)+K(z)\right)$ with

$$K(z) = \frac{\bar{z}}{|z|}\int_{\partial\mathbb{D}_{2|z|}(0)}\frac{\bar{z}-\bar{x}}{|z-x|^2}\mu(x)f(x)dx$$

and

$$L(z) = \left(\Omega * f\mu\chi_{\mathbb{D}\cap\mathbb{D}_{2|z|}}\right)(z)$$

where $\Omega(y) = -2\frac{\bar{y}^2}{|y|^4}$. One clearly finds :

$$|K(z)| \leq C\|f\|_{L^\infty(\mathbb{D})}\int_{\partial\mathbb{D}_{2|z|}}|\mu(x)| \leq C|z|^a\|f\|_{L^\infty(\mathbb{D})}\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}, \tag{A.3.91}$$

and

$$L(z) - \mu(z)\left(\Omega * f\chi_{\mathbb{D}_{2|z|}}\right)(z) = \int_{\mathbb{D}_{2|z|}}\Omega(z-x)f(x)\left(\mu(x)-\mu(z)\right)dx.$$

Given z in $\mathbb{D}$, let S_z be the cone with apex $\frac{z}{2}$ such that it contains $D_{\frac{|z|}{2}}$. For $x \in S_z$, we have $2|z-x| > |z|$. Hence :

$$\begin{aligned}\int_{S_x\cap\mathbb{D}_{2|z|}}\Omega(z-x)f(x)\left(\mu(x)-\mu(z)\right)dx &\leq C\left(\frac{|\mu(z)|}{|z|^2}\int_{D_{2|z|}}|f(x)|dx + \frac{1}{|z|^2}\int_{D_{2|z|}}|f(x)||\mu(x)|dx\right) \\ &\leq C\left\|\frac{\mu}{|x|^a}\right\|_{L^\infty(\mathbb{D})}\|f\|_{L^\infty(\mathbb{D})}|z|^a.\end{aligned}$$

Since $\mu \in C^1(\mathbb{D}\backslash\{0\})$, $\mu \in C^1(S_z^c)$. Thus for all $x \in S_z^c$ one can write :

$$|\mu(z) - \mu(x)| \leq C \left\| \frac{\nabla\mu}{|x|^{a-1}} \right\|_{L^\infty(\mathbb{D})} |z|^{a-1}|x-z|.$$

Accordingly :

$$\begin{aligned}\left| \int_{S_x^c \cap D_{2|z|}} \Omega(z-x) f(x) \left(\mu(z) - \mu(x)\right) dx \right| &\leq C \left\| \frac{\nabla\mu}{|x|^{a-1}} \right\|_{L^\infty(\mathbb{D})} |z|^{a-1} \int_{D_{2|z|}} \frac{|f(z)|}{|z-x|} dx \\ &\leq C \left\| \frac{\nabla\mu}{|x|^{a-1}} \right\|_{L^\infty(\mathbb{D})} |z|^{a-1} \|f\|_{L^\infty(\mathbb{D})} \int_{D_{2|z|}} \frac{1}{|z-x|} dx \\ &\leq C \left\| \frac{\nabla\mu}{|x|^{a-1}} \right\|_{L^\infty(\mathbb{D})} |z|^{a-1} \|f\|_{L^\infty(\mathbb{D})} \int_{B_{3|z|}(z)} \frac{1}{|z-x|} dx \\ &\leq C \left\| \frac{\nabla\mu}{|x|^{a-1}} \right\|_{L^\infty(\mathbb{D})} |z|^{a} \|f\|_{L^\infty(\mathbb{D})}.\end{aligned} \tag{A.3.92}$$

Combining (A.3.87), (A.3.88), (A.3.89), (A.3.91) and (A.3.92) yields the desired result and concludes the proof. □

In the core we will use weights with non integer exponents. The same proof allows for this slight adaptation, already presented in [BR13].

Theorem A.3.9. Let $u \in C^2(\mathbb{D}\backslash\{0\})$ solve

$$\Delta u(z) = \mu(z) f(z) \text{ in } \mathbb{D},$$

with $f \in L^p(\mathbb{D})$ for $2 < p \leq \infty$ and the weight μ satisfying for some $a \in \mathbb{R}_+$

$$|\mu(z)| = O\left(|z|^a\right).$$

Then

$$u_z(z) = P(z) + |z|^a T(z)$$

with $P \in \mathbb{C}_{\lceil a \rceil}[X]$ and $T = O(|z|^{1-\frac{2}{p}-v})$ for all $v > 0$. Here $\lceil a \rceil$ is the upper integral part of a. More precisely one has

$$\left\| \frac{T}{|z|^{1-\frac{2}{p}-v}} \right\|_{L^\infty(\mathbb{D})} \leq C_v \left(\left\| \frac{\mu}{|z|^a} \right\|_{L^\infty(\mathbb{D})} \|f\|_{L^p(\mathbb{D})} + \|u\|_{C^1(\partial\mathbb{D})} \right).$$

Additionally if $\mu \in C^1(\mathbb{D}\backslash\{0\})$, $a \neq 0$ and

$$\nabla\mu(z) = O\left(|z|^{a-1}\right)$$

Then :

$$u_{zz}(z) = P_z + |z|^a Q$$

with $Q \in L^{p'}(\mathbb{D})$ for all $p' < p$ and

$$\|Q\|_{L^{p'}(\mathbb{D})} \leq C_{p'} \left(\left(\left\| \frac{\mu}{|z|^a} \right\|_{L^\infty(\mathbb{D})} + \left\| \frac{\nabla\mu}{|z|^{a-1}} \right\|_{L^\infty(\mathbb{D})} \right) \|f\|_{L^p(\mathbb{D})} + \|u\|_{C^1(\partial\mathbb{D})} \right).$$

In fact $Q = \frac{(|z|^a T(z))_z}{|z|^a}$.

Proof. The proof is the same as in theorem A.3.8, if a is not an integer we simply split the terms in the sums at $\lceil a \rceil$, and we do not have to treat the $a+1$ term separately in that case (as we did in (A.3.84)). $\square$

Given the nature of the bubbling phenomena, we will need a version of such theorems with sliding weights : $\chi = \sqrt{\varepsilon^2 + r^2}$. The parameter ε will represent the concentration speed.

Theorem A.3.10. Let $(u^\varepsilon)_{\varepsilon>0} \in C^2(\mathbb{D}\backslash\{0\})$ solve

$$\Delta u^\varepsilon(z) = \chi^a f^\varepsilon(z) \text{ in } \mathbb{D},$$

with $f^\varepsilon \in L^p(\mathbb{D})$ for $2 \leq p \leq \infty$, $a \in \mathbb{R}$ and $\chi := \sqrt{\varepsilon^2 + r^2}$. Then

$$u_z^\varepsilon(z) = P^\varepsilon(z) + \chi^a T^\varepsilon(z)$$

with $P^\varepsilon \in \mathbb{C}_{\lceil a \rceil}[X]$ and $T^\varepsilon = O(\chi^{1-\frac{2}{p}-\upsilon})$ for all $\upsilon > 0$. Here $\lceil a \rceil$ is the upper integral part of a. More precisely one has

$$\left\| \frac{T^\varepsilon}{\chi^{1-\frac{2}{p}-\upsilon}} \right\|_{L^\infty(\mathbb{D})} \leq C_\upsilon \left(\|f^\varepsilon\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\partial\mathbb{D})} \right).$$

Additionally :

$$u_{zz}^\varepsilon(z) = P_z^\varepsilon + \chi^a Q^\varepsilon$$

with $Q^\varepsilon \in L^{p'}(\mathbb{D})$ for all $p' < p$ and

$$\|Q^\varepsilon\|_{L^{p'}(\mathbb{D})} \leq C_{p'} \left(\|f^\varepsilon\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\partial\mathbb{D})} \right).$$

In fact $Q^\varepsilon = \frac{(\chi^a T^\varepsilon(z))_z}{\chi^a}$.

Proof. We first state that for all $a \in \mathbb{R}_+$, there exists $C_a \in \mathbb{R}_+^*$ such that

$$\frac{1}{C_a} \leq \frac{\varepsilon^a + r^a}{\chi^a} \leq C_a. \tag{A.3.93}$$

Here C_a depends solely on a, and not on ε or r.

We then write

$$\Delta u^\varepsilon = (\varepsilon^a + r^a) \frac{\chi^a}{\varepsilon^a + r^a} f^\varepsilon = (\varepsilon^a + r^a) \widetilde{f^\varepsilon},$$

where $\widetilde{f^\varepsilon} = \frac{\chi^a}{\varepsilon^a + r^a} f^\varepsilon$ satisfies, thanks to (A.3.93),

$$\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} \leq C_a \|f^\varepsilon\|_{L^p(\mathbb{D})}. \tag{A.3.94}$$

We can then use Green's formula to write

$$\begin{aligned} u_z^\varepsilon(z) &= \frac{1}{2\pi} \int_{\partial\mathbb{D}} \left(\frac{\bar{z}-\bar{x}}{|z-x|^2} \partial_\nu u^\varepsilon(x) - u^\varepsilon(x) \partial \frac{\bar{z}-\bar{x}}{|z-x|^2} \right) d\sigma(x) \\ &\quad -\frac{1}{2\pi} \int_{\mathbb{D}} \frac{\bar{z}-\bar{x}}{|z-x|^2} (\varepsilon^a + r^a) \widetilde{f^\varepsilon}(x) dx \\ &= \frac{1}{2\pi} \int_{\partial\mathbb{D}} \left(\frac{\bar{z}-\bar{x}}{|z-x|^2} \partial_\nu u^\varepsilon(x) - u^\varepsilon(x) \partial \frac{\bar{z}-\bar{x}}{|z-x|^2} \right) d\sigma(x) \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{\bar{z}-\bar{x}}{|z-x|^2} \varepsilon^a \widetilde{f^\varepsilon}(x) dx - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{\bar{z}-\bar{x}}{|z-x|^2} r^a \widetilde{f^\varepsilon}(x) dx \\ &= I_0^\varepsilon(z) + I_1^\varepsilon(z) + I_2^\varepsilon(z). \end{aligned} \tag{A.3.95}$$

We can then successively estimate the three terms as in the proof of theorem A.3.9 and write

$$I_0^\varepsilon(z) = P_0^\varepsilon(z) + z^{\lceil a \rceil + 1} T_0^\varepsilon, \tag{A.3.96}$$

where P_0^ε is a polynomial of degree at most $\lceil a \rceil$ and whose coefficients are bounded by $\|u^\varepsilon\|_{C^1(\partial\mathbb{D})}$, and

$$\|T_0^\varepsilon\|_{L^\infty(\mathbb{D})} \leq C \, \|u^\varepsilon\|_{C^1(\mathbb{D})} \, .$$

Working as for (A.3.86) we write

$$\begin{aligned} I_1^\varepsilon(z) &= C^\varepsilon + \varepsilon^a T_1^\varepsilon(z) \\ I_2^\varepsilon(z) &= P_2^\varepsilon + r^a T_2^\varepsilon(z) \end{aligned} \tag{A.3.97}$$

where C^ε is a constant and P_2^ε a polynomial of degree at most $\lceil a \rceil$, both bounded by $\|u^\varepsilon\|_{C^1(\partial\mathbb{D})}$, while

$$\begin{aligned} \left\| \frac{T_1^\varepsilon}{r^{1-\frac{2}{p}-v}} \right\|_{L^\infty(\mathbb{D})} &\leq C_v \left(\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\mathbb{D})} \right) \\ \left\| \frac{T_2^\varepsilon}{r^{1-\frac{2}{p}-v}} \right\|_{L^\infty(\mathbb{D})} &\leq C_v \left(\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\mathbb{D})} \right) . \end{aligned}$$

In the end, combining (A.3.96) and (A.3.97) yields

$$u_z^\varepsilon = P^\varepsilon + \varepsilon^a T_1^\varepsilon + r^a T_3^\varepsilon, \tag{A.3.98}$$

where P^ε is a polynomial of degree at most $\lceil a \rceil$, T_1^ε is as previously stated and T_3^ε still satisfies

$$\left\| \frac{T_3^\varepsilon}{r^{1-\frac{2}{p}-v}} \right\|_{L^\infty(\mathbb{D})} \leq C_v \left(\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\mathbb{D})} \right) .$$

Proceeding similarly then ensures that

$$u_{zz}^\varepsilon = P_z^\varepsilon + \varepsilon^a Q_1^\varepsilon + r^a Q_2^\varepsilon, \tag{A.3.99}$$

where $Q_1^\varepsilon = T_{1,z}^\varepsilon$ and $Q_2^\varepsilon = \frac{\left(r^a T_3^\varepsilon\right)_z}{r^a}$ satisfy for all $p' < p$

$$\begin{aligned} \|Q_1^\varepsilon\|_{L^{p'}(\mathbb{D})} &\leq C_{p'} \left(\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\partial\mathbb{D})} \right), \\ \|Q_3^\varepsilon\|_{L^{p'}(\mathbb{D})} &\leq C_{p'} \left(\left\| \widetilde{f^\varepsilon} \right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\partial\mathbb{D})} \right). \end{aligned}$$

Let us notice that the estimate on Q_1^ε is not stricto sensu derived from the proof of theorem A.3.9, but from similar classical Calderon-Zygmund estimates.

From (A.3.98), we write $u_z^\varepsilon = P^\varepsilon + \chi^a T^\varepsilon$ with

$$T^\varepsilon = \frac{\varepsilon^a}{\chi^a} T_1^\varepsilon + \frac{r^a}{\chi^a} T_3^\varepsilon$$

which then satisfies

$$\begin{aligned}\left\|\frac{T^\varepsilon}{\chi^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})} &\leq \left\|\frac{\varepsilon^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{T_1^\varepsilon}{\chi^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})} + \left\|\frac{r^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{T_3^\varepsilon}{\chi^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})}\\ &\leq \left\|\frac{\varepsilon^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{T_1^\varepsilon}{r^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{r^{1-\frac{2}{p}-v}}{\chi^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})}\\ &+ \left\|\frac{r^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{T_3^\varepsilon}{r^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})}\left\|\frac{r^{1-\frac{2}{p}-v}}{\chi^{1-\frac{2}{p}-v}}\right\|_{L^\infty(\mathbb{D})}\\ &\leq C_v\left(\left\|\widetilde{f}^\varepsilon\right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\mathbb{D})}\right),\end{aligned} \tag{A.3.100}$$

using lemma A.3.1.

From (A.3.99) we write $u^\varepsilon_{zz} = P^\varepsilon_z + \chi^a Q^\varepsilon$ with

$$Q^\varepsilon = \frac{\varepsilon^a}{\chi^a}Q_1^\varepsilon + \frac{r^a}{\chi^a}Q_3^\varepsilon = \frac{(\chi^a T^\varepsilon)_z}{\chi^a}$$

which then satisfies

$$\begin{aligned}\|Q^\varepsilon\|_{L^{p'}(\mathbb{D})} &\leq \left\|\frac{\varepsilon^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\|Q_1^\varepsilon\|_{L^{p'}(\mathbb{D})} + \left\|\frac{r^a}{\chi^a}\right\|_{L^\infty(\mathbb{D})}\|Q_3^\varepsilon\|_{L^{p'}(\mathbb{D})}\\ &\leq C_{p'}\left(\left\|\widetilde{f}^\varepsilon\right\|_{L^p(\mathbb{D})} + \|u^\varepsilon\|_{C^1(\mathbb{D})}\right),\end{aligned} \tag{A.3.101}$$

using lemma A.3.1. □

Remark A.3.3. We must point out that the expansion offered by theorem A.3.10 is by no means unique. Indeed if for instance $u_z = P^\varepsilon + \chi^m T^\varepsilon$, then one could readily write

$$u_z = P^\varepsilon + \varepsilon^{m+1} + \chi^m\left(T^\varepsilon + \frac{\varepsilon^{m+1}}{\chi^m}\right)$$

with $T^\varepsilon + \frac{\varepsilon^{m+1}}{\chi^m}$ still satisfying (A.3.100).

We give here a small lemma which can help one to understand the essence of χ :

Lemma A.3.1. For all $a,b \in \mathbb{R}_+$, there exists a constant $C_{a,b}$ such that

$$\frac{\varepsilon^a r^b}{\chi^{a+b}} \leq C_{a,b}.$$

Theorem A.3.10 can be applied several times to prove an increased regularity on the higher order terms :

Lemma A.3.2. Let $u^\varepsilon \in C^2(\mathbb{D}\backslash\{0\})$ such that

$$\Delta u^\varepsilon = \chi^a f^\varepsilon,$$

with $f^\varepsilon \in L^\infty$ and

$$\Delta(\nabla u^\varepsilon) = \chi^{a-1}g^\varepsilon,$$

with $g^\varepsilon \in L^p$. Then

$$u^\varepsilon_z = P^\varepsilon + \mu^\varepsilon,$$

where P^ε is a complex polynomial of degree at most $\lceil a\rceil$, and μ^ε such that

$$\frac{|\mu^\varepsilon|}{\chi^{a+1-\upsilon}}+\frac{|\nabla\mu^\varepsilon|}{\chi^{a-\frac{2}{p}-\upsilon}}\leq C_\upsilon\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right),$$

and

$$\left\|\frac{\nabla^2\mu^\varepsilon}{\chi^{a-1}}\right\|_{L^{p'}(\mathbb{D})}\leq C_{p'}\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right).$$

Proof. We apply theorem A.3.10 twice and decompose u_z and $(u_z)_z$:

$$\begin{aligned}u_z^\varepsilon&=P_1^\varepsilon+\mu_1^\varepsilon\\(u_z^\varepsilon)_z&=P_2^\varepsilon+\mu_2^\varepsilon,\end{aligned}\tag{A.3.102}$$

where

$$\begin{aligned}\frac{|\mu_1^\varepsilon|}{\chi^{a+1-\upsilon}}+\frac{|\mu_2^\varepsilon|}{\chi^{a-\frac{2}{p}-\upsilon}}&\leq C_\upsilon\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right)\\\left\|\frac{\nabla\mu_1^\varepsilon}{\chi^{a}}\right\|_{L^{p'_1}(\mathbb{D})}+\left\|\frac{\nabla\mu_2^\varepsilon}{\chi^{a-1}}\right\|_{L^{p'_2}(\mathbb{D})}&\leq C_{p'_1,p'_2}\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right),\end{aligned}\tag{A.3.103}$$

for all $p'_1<\infty$ and $p'_2<p$. We then enjoy two expressions for u_{zz} :

$$u_{zz}=P^\varepsilon_{1,z}+\mu^\varepsilon_{1,z}=P_2^\varepsilon+\mu_2^\varepsilon.$$

Consequently,

$$P^\varepsilon_{1,z}-P^\varepsilon_2=\mu^\varepsilon_2-\mu^\varepsilon_{1,z},$$

which in turn, combined with (A.3.103), implies that

$$\int_{\mathbb{D}}\left|\frac{P^\varepsilon_{1,z}-P^\varepsilon_2}{\chi^a}\right|^s dz\leq C_s\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right),$$

for all $s<\infty$. We decompose

$$P^\varepsilon_{1,z}-P^\varepsilon_2=\sum_{q=0}^{\lfloor a\rfloor}p_q^\varepsilon z^q,$$

and can state for a given $R_0>0$

$$\int_{\mathbb{D}_{\epsilon R_0}}\left|\frac{\sum_{q=0}^{\lfloor a\rfloor}p_q^\varepsilon z^q}{\chi^a}\right|^s dz\leq\int_{\mathbb{D}}\left|\frac{\sum_{q=0}^{\lfloor a\rfloor}p_q^\varepsilon z^q}{\chi^a}\right|^p\leq C_s\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right).$$

Changing variables yields

$$\int_{\mathbb{D}_{R_0}}\left|\frac{\sum_{q=0}^{\lfloor a\rfloor}\frac{p_q^\varepsilon}{\varepsilon^{a-q-\frac{2}{p}}}z^q}{\sqrt{1+r^2}^a}\right|^s dz\leq C_s\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right).$$

And since on $\mathbb{D}_{\mathbb{R}_0}$, $\frac{1}{1+r^2}\geq\frac{1}{1+R_0^2}$, we deduce

$$\int_{\mathbb{D}_{R_0}}\left|\sum_{q=0}^{\lfloor a\rfloor}\frac{p_q^\varepsilon}{\varepsilon^{a-q-\frac{2}{p}}}z^q\right|^s dz\leq C_{s,R_0}\left(\|f^\varepsilon\|_{L^\infty(\mathbb{D})}+\|g^\varepsilon\|_{L^p(\mathbb{D})}+\|u^\varepsilon\|_{C^2(\partial\mathbb{D})}\right).\tag{A.3.104}$$

It is now important to notice that the left-hand term in (A.3.104) is in fact a polynomial in R_0, which is uniformly bounded in ε on compacts of $\mathbb{C}$. All its coefficients are thus uniformly bounded in ε, and straightforward computations then yield :

$$\forall s<\infty \quad \forall j \leq \lfloor a \rfloor \quad \forall \varepsilon>0 \quad \left|\frac{p_q^{\varepsilon}}{\varepsilon^{a-q-\frac{2}{s}}}\right| \leq C_s\left(\|f^{\varepsilon}\|_{L^{\infty}(\mathbb{D})}+\|g^{\varepsilon}\|_{L^p(\mathbb{D})}+\|u^{\varepsilon}\|_{C^2(\partial\mathbb{D})}\right)$$

which thanks to lemma A.3.1 translates on $P^{\varepsilon}_{1,z}-P^{\varepsilon}_2$ as

$$\forall s<\infty \quad \left|\frac{P^{\varepsilon}_{1,z}-P^{\varepsilon}_2}{\chi^{a-\frac{2}{s}}}\right| \leq C_s\left(\|f^{\varepsilon}\|_{L^{\infty}(\mathbb{D})}+\|g^{\varepsilon}\|_{L^p(\mathbb{D})}+\|u^{\varepsilon}\|_{C^2(\partial\mathbb{D})}\right), \tag{A.3.105}$$

and

$$\forall s<\infty \quad \left|\frac{\left(P^{\varepsilon}_{1,z}-P^{\varepsilon}_2\right)_z}{\chi^{a-\frac{2}{s}-1}}\right| \leq C_s\left(\|f^{\varepsilon}\|_{L^{\infty}(\mathbb{D})}+\|g^{\varepsilon}\|_{L^p(\mathbb{D})}+\|u^{\varepsilon}\|_{C^2(\partial\mathbb{D})}\right). \tag{A.3.106}$$

Now since $\mu^{\varepsilon}_{1,z}=\mu^{\varepsilon}_2-\left(P^{\varepsilon}_{1,z}-P^{\varepsilon}_2\right)$ we can combine (A.3.103) and (A.3.105) to find for all $v>0$

$$\left|\frac{\mu^{\varepsilon}_{1\,z}}{\chi^{a-\frac{2}{p}-v}}\right| \leq C_v\left(\|f^{\varepsilon}\|_{L^{\infty}(\mathbb{D})}+\|g^{\varepsilon}\|_{L^p(\mathbb{D})}+\|u^{\varepsilon}\|_{C^2(\partial\mathbb{D})}\right). \tag{A.3.107}$$

Further since $\mu^{\varepsilon}_{1\,zz}=\mu^{\varepsilon}_{2\,z}-\left(P^{\varepsilon}_{1,z}-P^{\varepsilon}_2\right)_z$, (A.3.103) and (A.3.106) yield for all $p'<p$:

$$\left\|\frac{\mu^{\varepsilon}_{1\,zz}}{\chi^{a-1}}\right\|_{L^{p'}(\mathbb{D})} \leq C_{p'}\left(\|f^{\varepsilon}\|_{L^{\infty}(\mathbb{D})}+\|g^{\varepsilon}\|_{L^p(\mathbb{D})}+\|u^{\varepsilon}\|_{C^2(\partial\mathbb{D})}\right). \tag{A.3.108}$$

Applying similarly theorem A.3.10 to $u^{\varepsilon}_{\bar{z}}$ yields controls akin to (A.3.107) and (A.3.108) on the missing terms in the gradient and the Hessian, which concludes the proof. □

A cautious reader might have noticed that we have additionnaly shown the following lemma :

Lemma A.3.3. Let $u\in\mathbb{N}$, $v\geq u$ and $P^{\varepsilon}=\sum_{j=0}^{u}p_j^{\varepsilon}z^j\in\mathbb{C}_u[X]$ such that

$$\forall p<\infty \quad \frac{P^{\varepsilon}}{\chi^v}\in L^p.$$

Then

$$\forall \nu>0 \quad \forall j\leq u \quad \left|\frac{p_j^{\varepsilon}}{\varepsilon^{v-j-\nu}}\right|\leq C_{\nu}.$$

We will also use a corresponding result for polynomials in z and $\bar{z}$:

Lemma A.3.4. Let $u\in\mathbb{N}$, $v\geq u$ and $P^{\varepsilon}=\sum_{i+j=0}^{u}p_{i,j}^{\varepsilon}z^i\bar{z}^j$ such that

$$\forall p<\infty \quad \frac{P^{\varepsilon}}{\chi^v}\leq C.$$

Then

$$\forall \nu>0 \quad \forall i+j\leq u \quad \left|\frac{p_{i,j}^{\varepsilon}}{\varepsilon^{v-i-j}}\right|\leq C_{\nu}.$$

Applying lemma A.3.2 several times yields :

Corollary A.3.3. Let $u^\varepsilon \in C^2(\mathbb{D}\backslash\{0\})$ such that, for $a \geq t$

$$\begin{aligned}
\Delta u^\varepsilon &= \chi^a f_0^\varepsilon,\\
\Delta \nabla u^\varepsilon &= \chi^{a-1} f_1^\varepsilon\\
&\cdots\\
\Delta \nabla^t u^\varepsilon &= \chi^{a-t} f_t^\varepsilon
\end{aligned}$$

with $f_j^\varepsilon \in L^\infty(\mathbb{D})$ for $j \leq t-1$ and $f_t^\varepsilon \in L^p(\mathbb{D})$. Then

$$u_z^\varepsilon = P^\varepsilon + \mu^\varepsilon,$$

where P^ε is a complex polynomial of degree at most $\lceil a \rceil$, and μ^ε such that

$$\frac{|\mu^\varepsilon|}{\chi^{a+1-v}} + \frac{|\nabla \mu^\varepsilon|}{\chi^{a-v}} + \cdots + \frac{|\nabla^t \mu^\varepsilon|}{\chi^{a+1-t-\frac{2}{p}-v}} \leq C_v \left(\sum_{q=0}^{t} \left\| f_q^\varepsilon \right\|_{L^\infty(\mathbb{D})} + \|u^\varepsilon\|_{C^{t+1}(\partial\mathbb{D})} \right),$$

and

$$\left\| \frac{\nabla^{t+1}\mu^\varepsilon}{\chi^{a-t}} \right\|_{L^{p'}(\mathbb{D})} \leq C_{p'} \left(\sum_{q=0}^{t} \left\| f_q^\varepsilon \right\|_{L^\infty(\mathbb{D})} + \|u^\varepsilon\|_{C^{t+1}(\partial\mathbb{D})} \right).$$

Proof. The proof is a recurrence whose initialization is theorem A.3.10 and whose heredity is obtained by applying lemma is A.3.2 to the $\nabla^s u^\varepsilon$. $\square$

Bibliography

[Ada75] D. Adams. A note on Riesz potentials. *Duke Math. J.*, 42(4):765–778, 1975. ↑ 63, 134

[AG96] M. A. Akivis and V. Goldberg. *Conformal differential geometry and its generalizations*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1996. A Wiley-Interscience Publication. ↑ 29, 75

[Apo90] T. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. ↑ 36, 97

[BC85] H. Brezis and J.-M. Coron. Convergence of solutions of H-systems or how to blow bubbles. *Arch. Rational Mech. Anal.*, 89(1):21–56, 1985. ↑ 9, 16

[Ber16] Y. Bernard. Noether's theorem and the Willmore functional. *Adv. Calc. Var.*, 9(3):217–234, 2016. ↑ 9, 15, 26, 37, 44, 47, 149

[BG93] F. Bethuel and J.-M. Ghidaglia. Improved regularity of solutions to elliptic equations involving Jacobians and applications. *J. Math. Pures Appl. (9)*, 72(5):441–474, 1993. ↑ 196

[Bla55] W. Blaschke. *Vorlesungen über Integralgeometrie*. Deutscher Verlag der Wissenschaften, Berlin, 1955. 3te Aufl. ↑ 8, 14, 30, 34

[Boh12] C. Bohle. Constant mean curvature tori as stationary solutions to the Davey-Stewartson equation. *Math. Z.*, 271(1-2):489–498, 2012. ↑ 10, 17, 71, 95

[BR13] Y. Bernard and T. Rivière. Singularity removability at branch points for Willmore surfaces. *Pacific J. Math.*, 265(2):257–311, 2013. ↑ 9, 15, 28, 64, 65, 67, 108, 127, 149, 197, 202

[BR14] Y. Bernard and T. Rivière. Energy quantization for Willmore surfaces and applications. *Ann. of Math. (2)*, 180(1):87–136, 2014. ↑ 9, 16, 28, 63, 90, 113, 117, 118, 119, 120, 125, 127, 131, 135, 137, 150

[Bry84] R. Bryant. A duality theorem for Willmore surfaces. *J. Differential Geom.*, 20(1):23–53, 1984. ↑ 8, 9, 10, 15, 16, 17, 34, 36, 72, 93, 99, 106, 122, 123

[BWV18] Y. Bernard, G. Wheeler, and Wheeler V. Analysis of the inhomogeneous willmore equation. *arXiv:1811.08546*, 2018. ↑ 64, 127

[CG82] C. Chen and F. Gackstatter. Elliptische und hyperelliptische Funktionen und vollständige Minimalflächen vom Enneperschen Typ. *Math. Ann.*, 259(3):359–369, 1982. ↑ 36, 97

[DPW98] J. Dorfmeister, F. Pedit, and H. Wu. Weierstrass type representation of harmonic maps into symmetric spaces. *Comm. Anal. Geom.*, 6(4):633–668, 1998. ↑ 10, 17, 69

[DW19] J. Dorfmeister and P. Wang. Willmore surfaces in spheres: the DPW approach via the conformal gauss map. *arXiv:1903.00883*, 2019. ↑ 10, 17, 69, 100

[EG15] L. Evans and R. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015. ↑ 55, 58

[Eji88] N. Ejiri. Willmore surfaces with a duality in $S^N(1)$. *Proc. London Math. Soc. (3)*, 57(2):383–416, 1988. ↑ 10, 17, 69, 100

[Esc88] J.-H. Eschenburg. *Willmore surfaces and Moebius geometry*. 1988. ↑ 10, 16, 99

[Eva10] L. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. ↑ 60

[Ger31] S. Germain. Mémoire sur la coubure des surfaces. *J. Reine Angew. Math.*, 7:1–29, 1831. ↑ 7, 14, 29

[GT01] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. ↑ 135, 137

[Hél98] F. Hélein. Willmore immersions and loop groups. *J. Differential Geom.*, 50(2):331–385, 1998. ↑ 10, 16, 164

[Hél02] F. Hélein. *Harmonic maps, conservation laws and moving frames*, volume 150 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 2002. Translated from the 1996 French original, With a foreword by James Eells. ↑ 37, 53, 54, 60, 90, 181, 182, 195, 196, 197

[IM01] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001. ↑ 196

[KS01a] E. Kuwert and R. Schätzle. The Willmore flow with small initial energy. *J. Differential Geom.*, 57(3):409–441, 2001. ↑ 8, 15

[KS01b] E. Kuwert and R. Schätzle. The Willmore flow with small initial energy. *J. Differential Geom.*, 57(3):409–441, 2001. ↑ 64, 65, 127

[KS07] Ernst Kuwert and Reiner Schätzle. Branch points of Willmore surfaces. *Duke Math. J.*, 138(2):179–201, 2007. ↑ 65

[Kus89] R. Kusner. Comparison surfaces for the Willmore problem. *Pacific J. Math.*, 138(2):317–345, 1989. ↑ 33

[L9́2] F. López. The classification of complete minimal surfaces with total curvature greater than -12π. *Trans. Amer. Math. Soc.*, 334(1):49–74, 1992. ↑ 164

[Lau12a] P. Laurain. Asymptotic analysis for surfaces with large constant mean curvature and free boundaries. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(1):109–129, 2012. ↑ 118

[Lau12b] P Laurain. Concentration of CMC surfaces in a 3-manifold. *Int. Math. Res. Not. IMRN*, (24):5585–5649, 2012. ↑ 9, 16, 167

[Law70] H. Blaine Lawson, Jr. Complete minimal surfaces in S^3. *Ann. of Math. (2)*, 92:335–374, 1970. ↑ 33

[LN15] T. Lamm and H. Nguyen. Branched Willmore spheres. *J. Reine Angew. Math.*, 701:169–194, 2015. ↑ 31, 36, 67, 106

[LR18a] P. Laurain and T. Rivière. Energy quantization of Willmore surfaces at the boundary of the moduli space. *Duke Math. J.*, 167(11):2073–2124, 2018. ↑ 9, 16, 52, 113, 114, 117, 118, 119, 125, 136, 163, 164

[LR18b] P. Laurain and T. Rivière. Optimal estimate for the gradient of Green's function on degenerating surfaces and applications. *Comm. Anal. Geom.*, 26(4):887–913, 2018. ↑ 53, 120

[LY82] P. Li and S. Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.*, 69(2):269–291, 1982. ↑ 33, 163

[Mar19a] N. Marque. Conformal gauss map geometry and application to Willmore surfaces in model spaces. *arXiv:1903.07475*, 2019. ↑ 13, 20, 69

[Mar19b] N. Marque. Minimal bubbling for willmore surfaces. *arXiv:1906.00248*, 2019. ↑ 13, 20, 132

[Mar19c] N. Marque. An ε-regularity result with mean curvature control for Willmore immersions and application to minimal bubbling. *arXiv:1904.05215*, 2019. ↑ 13, 20, 28, 47, 48, 55, 57, 129, 132, 140, 149, 156

[Miced] A. Michelat. *PhD thesis.* to be published. ↑ 95

[MN16] F. Marques and A. Neves. Min-max theory and a proof of the Willmore conjecture. In *Geometric analysis*, volume 22 of *IAS/Park City Math. Ser.*, pages 277–300. Amer. Math. Soc., Providence, RI, 2016. ↑ 8, 15, 26, 32

[MN18] A. Mondino and H. Nguyen. Global conformal invariants of submanifolds. *Ann. Inst. Fourier (Grenoble)*, 68(6):2663–2695, 2018. ↑ 31

[Mon00] S. Montiel. Willmore two-spheres in the four-sphere. *Trans. Amer. Math. Soc.*, 352(10):4469–4486, 2000. ↑ 100

[MR17] A. Michelat and T. Rivière. The classification of branched Willmore spheres in the 3-sphere and the 4-sphere. *arXiv:1706.01405*, 2017. ↑ 10, 16, 26, 36, 44, 65, 70, 72, 93, 96, 106, 107

[MR19] A. Michelat and T. Rivière. Private communications. 2019. ↑ 156

[MŠ95] S. Müller and V. Šverák. On surfaces of finite total curvature. *J. Differential Geom.*, 42(2):229–258, 1995. ↑ 156

[Oss86] R. Osserman. *A survey of minimal surfaces.* Dover Publications, Inc., New York, second edition, 1986. ↑ 125

[Pal91] B. Palmer. The conformal Gauss map and the stability of Willmore surfaces. *Ann. Global Anal. Geom.*, 9(3):305–317, 1991. ↑ 10, 16, 99

[Pit81] J. Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds*, volume 27 of *Mathematical Notes.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981. ↑ 8, 15

[Poi14] S. D. Poisson. Mémoire sur les surfaces élastiques. *Mémoire de l'Institut*, 1814. ↑ 29

[Ric97] J. Richter. *Conformal maps of a riemannian surface into the space of quaternion.* 1997. ↑ 10, 17, 69, 100

[Riv07] T. Rivière. Conservation laws for conformally invariant variational problems. *Invent. Math.*, 168(1):1–22, 2007. ↑ 133

[Riv08] T. Rivière. Analysis aspects of Willmore surfaces. *Invent. Math.*, 174(1):1–45, 2008. ↑ 9, 15, 26, 28, 42, 44, 45, 46, 47, 52, 53, 54, 63, 127, 149

[Riv12] T. Rivière. *Conformally Invariant Variational Problems.* 2012. ↑ 51

[Riv13] T. Rivière. Sequences of smooth global isothermic immersions. *Comm. Partial Differential Equations*, 38(2):276–303, 2013. ↑ 95, 96

[Riv16] T. Rivière. Weak immersions of surfaces with L^2-bounded second fundamental form. In *Geometric analysis*, volume 22 of *IAS/Park City Math. Ser.*, pages 303–384. Amer. Math. Soc., Providence, RI, 2016. ↑ 9, 16, 28, 54, 57, 113, 115, 116, 193

[SU81] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Ann. of Math. (2)*, 113(1):1–24, 1981. ↑ 9, 16

[Tho23] G. Thomsen. *Über Konforme Geometrie I: Grundlagen der konformen Flachentheorie,.* Abh. Math. Sem. Hamburg, 1923. 3te Aufl. ↑ 8, 14

[Wen71] H. Wente. An existence theorem for surfaces of constant mean curvature. *Bull. Amer. Math. Soc.*, 77:200–202, 1971. ↑ 195

[Wil65] T. Willmore. Note on embedded surfaces. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)*, 11B:493–496, 1965. ↑ 8, 15, 32, 38

[Wil93] T. J. Willmore. *Riemannian geometry.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. ↑ 8, 15, 30, 32

[Zhu13] M. Zhu. Regularity for harmonic maps into certain pseudo-Riemannian manifolds. *J. Math. Pures Appl. (9)*, 99(1):106–123, 2013. ↑ 90