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On coarse embeddings among groups and metric spaces

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Résumé

Cette thèse est consacrée à l'étude des plongements grossiers entre groupes et espaces métriques. La notion de plongement grossier a été introduite par Gromov dans les années 80. C'est une généralisation des plongements quasi-isométriques quand les fonctions de contrôle ne sont pas nécessairement affines. On s'intéresse particulièrement aux obstructions à l'existence de tels plongements entre espaces symétriques de type non-compact, immeubles Euclidiens, espaces CAT(0) cocompacts et groupes modulaires de surfaces.
Il est bien connu, par des résultats d'Anderson–Schroeder et de Kleiner, que le rang (la dimension maximale d'un plat/quasi-plat) des espaces CAT(0) propres et cocompacts est monotone par plongements quasi-isométriques. Ce n'est plus le cas pour les plongements grossiers, comme le montrent les plongements horosphériques dans les espaces hyperboliques.
Nous montrons que si l'espace de départ est un produit d'espaces métriques géodésiques à croissance exponentielle, ou un produit d'espaces symétriques de type non-compact et d'immeubles Euclidiens sans facteur Euclidien, le rang est toujours monotone par plongements grossiers. L'espace d'arrivée peut être un espace CAT(0) propre cocompact ou un groupe modulaire de surface. Ceci répond à une question de David Fisher et Kevin Whyte pour les espaces symétriques. Nous pouvons aussi affaiblir la condition sur l'espace de départ en lui permettant de contenir un facteur Euclidien de dimension 1, répondant ainsi à une question de Gromov. La preuve fait intervenir les fonctions de remplissage homologiques de dimensions supérieures.

Mots-clés

Plongements grossiers, espaces symétriques, immeubles Euclidiens, espaces CAT(0), groupes modulaires de surfaces, fonctions de remplissage.

Abstract

This thesis is devoted to the study of coarse embeddings between groups and metric spaces. The notion of coarse embedding was introduced by Gromov in the 80s. It is a generalization of quasi-isometric embeddings when the control functions are not necessarily affine. We are particularly interested in the obstructions to the existence of such embeddings between symmetric spaces of noncompact type, Euclidean buildings, CAT(0) spaces and mapping class groups.

It is well known, by results of Anderson–Schroeder and Kleiner, that the rank (the maximal dimension of a flat/quasi-flat) of proper cocompact CAT(0) spaces is monotonous under quasi-isometric embeddings. This is no longer the case for coarse embeddings as shown by horospherical embeddings in hyperbolic spaces.

We show that if the domain is either a product of geodesic metric spaces of exponential growth, or a product of symmetric spaces of noncompact type and Euclidean buildings with no Euclidean factor, the rank is still monotonous under coarse embeddings. The target space can be a proper cocompact CAT(0) space or a mapping class group. This answers a question by David Fisher and Kevin Whyte for symmetric spaces. We also show that we can relax the condition on the domain by allowing it to contain a Euclidean factor of dimension 1, thus answering a question by Gromov. The proof involves higher dimensional homological filling functions.

Keywords

Coarse embeddings, symmetric spaces, Euclidean buildings, CAT(0) spaces, mapping class groups, filling functions.

Contents

Introduction Française

La théorie géométrique des groupes et la géométrie grossière sont des domaines très étroitement liés et relativement récents qui ont été popularisés dans les années 80, principalement grâce à Mikhail Gromov dans [Gro87] et [Gro93]. Leur but est l'étude des groupes infinis de type fini et des espaces sur lesquels ils agissent, en exploitant les connexions entre les propriétés algébriques des groupes et les propriétés topologiques et géométriques de ces espaces. L'un des principaux objectifs de la théorie géométrique des groupes est de classer les groupes de type fini par leur "géométrie à grande échelle". Cela revient à les classer par leur métrique des mots à quasi-isométrie près, et c'est là que la géométrie grossière entre en jeu. En effet, l'étude de la géométrie à grande échelle d'un espace revient souvent à étudier les propriétés qui sont soit invariantes, soit monotones sous plongements quasi-isométriques ou plongements grossiers. Ces propriétés sont les outils qui nous permettent de comparer les géométries à grande échelle, comme le taux de croissance, la dimension asymptotique ou les fonctions isopérimétriques.
Dans cette thèse, plutôt que les plongements quasi-isométriques, nous considérons les plongements grossiers.

Une application $f : (X, d_X) \to (Y, d_Y)$ est un plongement grossier s'il existe des fonctions $\rho_\pm : [0, \infty) \to [0, \infty)$ telles que $\rho_-(r) \to \infty$ quand $r \to \infty$ et pour tout $x, y \in X$

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).$$

Les plongement grossier, ou coarse embeddings, ont d'abord été introduits sous le nom de "placements" ou "placings" dans [Gro88]. Ils apparaissent également dans la littérature sous le nom de plongements uniformes dans [Gro93],[Sha04],[FW18], "effectively proper Lipschitz maps" dans [BW92] ou "uniformly proper embeddings" [MSW03].

Les plongements grossiers de groupes ou de graphes infinis dans des espaces de Hilbert (ou des espaces de Banach plus généraux) ont fait l'objet d'une grande attention en raison de leur lien avec les conjectures de Baum-Connes et de Novikov (voir [Yu00], [STY02], [KY06]). Cependant, ils ont été à l'origine beaucoup moins étudiés que les plongements quasi-isométriques parmi les groupes finiment engendrés, probablement en raison de leur plus grande flexibilité. Par exemple, une inclusion de sous-groupe entre des groupes finiment engendrés est toujours un plongement grossier, alors qu'il s'agit d'un plongement quasi-isométrique uniquement lorsque le sous-groupe en question n'est pas distordu. Très peu d'invariants géométriques sont connus pour être monotones sous des plongements grossiers; les seuls exemples (qui prennent une infinité de valeurs différentes) étant la croissance du volume des boules, la dimension asymptotique [Gro93], le profil de séparation [BST12] et les profils de Poincaré [HMT20].

La croissance du volume des boules permet de dire par exemple qu'un espace métrique à croissance exponentielle ne peut pas être plongé grossièrement dans un espace à croissance polynomiale, mais elle ne permet pas de distinguer les espaces à croissance exponentielle. Plus récemment, il a été démontré que la fonction de Følner est monotone sous plongements grossiers entre groupes moyennables [DKLMT20]. Ceci fournit un invariant beaucoup plus fin pour distinguer les groupes résolubles.

La dimension asymptotique a été introduite par Gromov dans [Gro88] et [Gro93] et est un analogue à grande échelle de la dimension topologique, ou la dimension de recouvrement de Lebesgue. Pour les espaces symétriques de courbure non positive et les immeubles Euclidiens, elle coïncide simplement avec la dimension. Par exemple, elle permet d'exclure les plongements grossiers de $\mathbb{H}^n$ dans $\mathbb{H}^p$ pour $p < n$, mais elle n'empêche pas par exemple l'existence d'un plongement grossier du plan hyperbolique complexe dans $\mathbb{H}^4$.

Le profil de séparation est un puissant invariant monotone introduit par Benjamini, Schramm et Timár [BST12]. Le profil de séparation d'un graphe infini de degré borné en $n \in \mathbb{N}$ est le supremum sur tous les sous-graphes de taille $\leq n$, du nombre de sommets qu'il faut retirer du sous-graphe afin que chaque composante connexe ait une taille au plus $n/2$. Le profil de séparation d'un espace métrique à géométrie bornée peut être défini comme le profil de séparation de tout graphe qui lui est quasi-isométrique. Pour les espaces symétriques de rang 1, il a été démontré qu'il permet de détecter la dimension conforme du bord à l'infini [HMT20], fournissant par exemple une obstruction à l'existence d'un plongement grossier du plan hyperbolique complexe dans $\mathbb{H}^4$. Plus tard, Hume–Mackay–Tessera [HMT20] ont introduit les profils L^p-Poincaré, généralisant le profil de séparation. Ces invariants sont très efficaces pour distinguer les espaces de la forme $X \times \mathbb{R}^n$ où X est un espace hyperbolique. Cependant, pour les espaces symétriques de rang supérieur, tous les profils L^p-Poincaré sont $\simeq n/\log n$: ceci fournit une obstruction aux plongements grossiers de rang supérieur dans $X \times \mathbb{R}^n$, où X est de rang un, mais ne fournit aucune obstruction parmi les espaces symétriques de rang supérieur.

Hume–Sisto ont donné dans [HS17] une obstruction pour qu'un groupe admette un plongement grossier dans un espace hyperbolique réel : admettre un nombre exponentiel de gros "bigons". Ils montrent par exemple que les produits directs de deux groupes infinis dont l'un a une croissance exponentielle, les groupes résolubles qui ne sont pas virtuellement nilpotents et les réseaux uniformes en rang supérieur ne se plongent pas grossièrement dans un espace hyperbolique.

En utilisant une combinaison de certains des résultats mentionnés ci-dessus et d'un résultat de Le Coz et Gournay [CG19], Tessera a récemment prouvé l'affirmation suivante [Tes20] : un groupe moyennable admet un plongement grossier dans un groupe hyperbolique si et seulement s'il est virtuellement nilpotent.
Le résultat précédent a une application intéressante en géométrie pseudo-riemmanienne. L'histoire commence avec une observation de Gromov dans [Gro88, Section 4.1] selon laquelle le groupe d'isométrie d'une variété lorentzienne compacte de dimension $n+1$ admet un plongement grossier dans l'espace hyperbolique réel $\mathbb{H}^n$. Frances [Fra21] a exploité ce point de vue et le résultat ci-dessus pour prouver une alternative de Tits pour les sous-groupes discrets du groupe d'isométrie des variétés lorentziennes compactes. A savoir, un sous-groupe discret finiment engendré du groupe d'isométrie d'une variété lorentzienne compacte de dimension $n \geq 2$ contient soit un sous-groupe libre à deux générateurs, auquel cas il est virtuellement isomorphe à un sous-groupe discret de $\mathrm{PO}(1, d)$, soit est virtuellement nilpotent de degré de croissance $\leq n-1$.

Nous concluons de ce qui précède que la seule obstruction connue pour les plongements grossiers parmi les espaces symétriques de rang supérieur est donnée par leur dimension. Le but de cette thèse est de traiter ce problème en montrant que, sous certaines conditions, le rang est aussi un invariant monotone.

Chapter 1

Introduction

Geometric group theory and coarse geometry are very closely related and relatively new fields which were popularized in the 80's, mainly thanks to the two monographs [Gro87] and [Gro93] of Mikhail Gromov. These fields are devoted to the study of infinite and finitely generated groups and of the spaces on which such groups act, by exploring the connections between the algebraic properties of the groups and the topological and geometric properties of these spaces. One of the main goals of geometric group theory is to classify infinite and finitely generated groups by their "large-scale geometry". This amounts to classifying them by their word metric up to quasi-isometry, and this is where coarse geometry comes into play. Indeed, the study of the large-scale geometry of a space often comes down to studying the properties that are either invariant or monotonous under quasi-isometric embeddings or coarse embeddings. These properties are the tools that allow us to compare the large-scale geometries, such as the growth rate, the asymptotic dimension or the isoperimetric functions.

In this thesis, rather than quasi-isometric embeddings, we consider coarse embeddings.

A map $f : (X, d_X) \to (Y, d_Y)$ is a coarse embedding if there exist functions $\rho_\pm : [0, \infty) \to [0, \infty)$ such that $\rho_-(r) \to \infty$ as $r \to \infty$ and for all $x, y \in X$

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).$$

Coarse embeddings were first introduced under the name of "placements" or "placings" in [Gro88]. They also appear in the litterature as uniform embeddings in [Gro93],[Sha04],[FW18], effectively proper Lipschitz maps in [BW92] or uniformly proper embeddings [MSW03].

Coarse embeddings from groups or infinite graphs into Hilbert spaces (or more general Banach spaces) have received a lot of attention due to their connection with the Baum–Connes and the Novikov conjecture (see [Yu00], [STY02], [KY06]). However, they have originally been much less studied than quasi-isometric embeddings among finitely generated groups, probably due to their much greater flexibility. As an example, a subgroup inclusion between finitely generated groups is always a coarse embedding, while it is a quasi-isometric embedding only when the subgroup in question is undistorted. Very few geometric invariants are known to be monotonous under coarse embeddings; the only examples (which takes infinitely many different values) being the volume growth, the asymptotic dimension [Gro93], the separation profile [BST12] and the Poincaré profiles [HMT20].

The volume growth allows one to say for example that a metric space with exponential growth cannot be coarsely embedded into a space with polynomial growth, but it does not distinguish between spaces of exponential growth. More recently the Følner function has been shown to be monotonous under coarse embeddings between amenable groups [DKLMT20]. This provides a much more refined invariant to distinguish among solvable groups.

The asymptotic dimension was introduced by Gromov in [Gro88] and [Gro93] and is a large-scale analogue of Lebesgue covering dimension. For symmetric spaces of nonpositive curvature and Euclidean buildings, it simply coincides with the dimension. For instance it allows one to rule out coarse embeddings from $\mathbb{H}^n$ to $\mathbb{H}^p$ for $p < n$, but it does not prevent for instance the existence of a coarse embedding of the complex hyperbolic plane to $\mathbb{H}^4$.

The separation profile is a powerful monotone coarse invariant introduced by Benjamini, Schramm and Timár [BST12]. The separation profile of an infinite, bounded degree graph at $n \in \mathbb{N}$ is the supremum over all subgraphs of size $\leq n$, of the number of vertices needed to be removed from the subgraph, in order to cut it into connected pieces of size at most $n/2$. The separation profile of a metric space with bounded geometry can be defined as the separation profile of any graph that is quasi-isometric to it. For rank 1 symmetric spaces, it has been shown to detect the conformal dimension of the boundary [HMT20], providing for instance an obstruction to the existence of a coarse embedding from the complex hyperbolic plane to $\mathbb{H}^4$. Later Hume–Mackay–Tessera [HMT20] introduced the L^p-Poincaré profiles, generalizing the separation profile. These invariants are very efficient in distinguishing between spaces of the form $X \times \mathbb{R}^n$ where X is a hyperbolic space. However, for higher rank symmetric spaces all L^p-Poincaré profiles are $\simeq n/\log n$: this provides an obstruction to coarse embeddings from higher rank to $X \times \mathbb{R}^n$, where X is rank one, but does not provide any obstruction among higher rank symmetric spaces.

Hume–Sisto gave in [HS17] an obstruction for a group to admit a coarse embedding into a real hyperbolic space: admitting exponentially many fat bigons. It is shown for instance that direct products of two infinite groups one of which has exponential growth, solvable groups that are not virtually nilpotent, and uniform higher-rank lattices do not coarsely embed into a hyperbolic space.

Using a combination of some of the above-mentioned results and a result by Le Coz and Gournay [CG19], Tessera recently proved the following statement [Tes20]: an amenable group admits a coarse embedding into a hyperbolic group if and only if it is virtually nilpotent.
The previous result has an interesting application in pseudo-riemmanian geometry. The story starts with an early observation of Gromov [Gro88, Section 4.1] according to which the isometry group of a compact $(n+1)$-dimensional Lorentzian manifold admits a coarse embedding into the real hyperbolic space $\mathbb{H}^n$. Frances [Fra21] exploited this point of view and the above result to prove a Tits alternative for discrete subgroups of the isometry group of compact Lorentz manifolds. Namely, a discrete, finitely generated subgroup of the isometry group of a compact Lorentz manifold of dimension $n \geq 2$ either contains a free subgroup in two generators, in which case it is virtually isomorphic to a discrete subgroup of $\mathrm{PO}(1,d)$, or is virtually nilpotent of growth degree $\leq n-1$.

We conclude from the above that the only known obstruction for coarse embeddings among higher rank symmetric spaces is given by their dimension. The aim of this thesis is to address this problem by showing that, under certain conditions, the rank is also a coarse monotonous invariant.

Let us now turn to the spaces that we will consider in this thesis. Let S be a product of symmetric spaces of non-compact type, and B a product of thick Euclidean buildings with cocompact affine Weyl group, with bounded geometry and no Euclidean factor. Let us call the spaces of the form $X = \mathbb{R}^n \times S \times B$ *model spaces.*
Below are a few known examples of embeddings between such model spaces:

- Every regular tree T_d quasi-isometrically embeds into the hyperbolic plane $\mathbb{H}^2$.
- There are quasi-isometric embeddings $\mathbb{H}^3 \to \mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{H}^5 \to \mathbb{H}^3 \times \mathbb{H}^3$ (see [BF98] for a more general statement).
- There exist coarse embeddings $\mathbb{H}^2 \to \mathbb{H}^3$ with arbitrarily small lower control [BH21].
- $\mathbb{H}^n$ quasi-isometrically embeds into a product of n binary trees $T_3 \times \cdots \times T_3$ [BDS07].

We will also consider general CAT(0) spaces that are proper (i.e. where closed balls are compact) and cocompact, and mapping class groups $\mathcal{MCG}(S_{g,p})$ of orientable compact connected surfaces of genus g and p boundary components. Recall that the mapping class group of a surface S, $\mathcal{MCG}(S)$, is defined to be the group of orientation-preserving homeomorphisms up to isotopy. It is finitely-generated [Deh12],[FM11], and for any finite generating set one considers the word metric, whence yielding a metric space which is unique up to quasi-isometry.

We define the *rank* of a metric space as the maximal dimension of an isometrically embedded copy of a Euclidean space. Similarly, we define the *geometric rank* (or the *quasi-flat rank* or the *quasi-rank*) of a metric space X as the maximal dimension of a quasi-isometrically embedded copy of a Euclidean space. We will denote it by $\operatorname{grank}(X)$. We introduce the geometric rank especially for mapping class groups since they are only defined up to quasi-isometry. For all the other spaces, the geometric rank is actually equal to the rank as we will see below. Therefore, by abuse of notation, when we consider the rank of a metric space, it should be understood that it is the geometric rank when the metric space is a mapping class group.

We are interested in the following natural question.

Question. Is the rank monotonous under coarse embeddings? In other words, if X and Y are such spaces and there is a coarse embedding $f : X \to Y$, does it imply that $\operatorname{rank}(X) \leq \operatorname{rank}(Y)$?

The answer is positive and well-known if one replaces the word coarse by quasi-isometric. It was shown by Anderson–Schroeder [AS86] that if X is a symmetric space of non-compact type and $f : \mathbb{R}^n \to X$ is a quasi-isometric embedding, then X contains an n-flat, i.e. there exists an isometric embedding of $\mathbb{R}^n$ into X. Later, Kleiner [Kle99] generalized this to all locally compact cocompact Hadamard spaces. In particular, for proper cocompact CAT(0) spaces, the geometric rank is equal to the rank. This answers the question for quasi-isometric embeddings. Indeed, if $f : X \to Y$ is a quasi-isometric embedding between two proper cocompact CAT(0) and $\operatorname{rank}(X) = p$ then $\mathbb{R}^p$ quasi-isometrically embeds into Y, hence $\operatorname{rank}(Y) \geq p$. Moreover, it was shown by Kleiner–Leeb [KL97] and by Eskin–Farb [EF97] that maximal quasi-flats in higher rank symmetric spaces of non-compact type satisfy a generalization of the Morse Lemma: if Y is a symmetric space of non-compact type of rank $n \geq 2$, and $f : \mathbb{R}^n \to Y$ is a quasi-isometric embedding then there exist $\delta > 0$ and $k \in \mathbb{N}$ (that depends on the quasi-isometry constants) such that $f(\mathbb{R}^n)$ lies in the δ-neighborhood of a union of k flats in Y. Note that this is a general phenomenon that occurs in all asymphoric hierarchically hyperbolic spaces [BHS21].

The monotonicity of the rank is no longer satisfied in the setting of coarse embeddings in general. Counter-examples are given by horospherical embeddings. Let $\mathbb{H}^2$ denote the upper half-plane model for the hyperbolic plane. We have

$$d_{\mathbb{H}^2}((x,1),(x+r,1)) \;=\; \operatorname{arg}\cosh(1+r^2/2) \underset{\ell\to\infty}{\sim} 2\ln r.$$

So the map $f : \mathbb{R} \longrightarrow \mathbb{H}^2$, $x \longmapsto (x, 1)$, which is a parametrization of a horocycle, is a coarse embedding that is exponentially distorted. This carries over to the hyperbolic space of any dimension $n \geq 1$, and provides coarse embeddings $f : \mathbb{R}^n \longrightarrow \mathbb{H}^{n+1}$ whose images are horospheres. These embeddings do not respect the monotonicity of the rank since $\text{rank}(\mathbb{R}^n) = n$ and $\text{rank}(\mathbb{H}^{n+1}) = 1$. Therefore, as long as the domain X has a Euclidean factor $\mathbb{R}^p$ with $p \geq 2$, it can be embedded into a space with lower rank. Indeed, if $X = \mathbb{R}^p \times Z$ and $\text{rank}(X) = p + r$, then it coarsely embeds into $\mathbb{H}^{p+1} \times Z$, whose rank is $1 + r$. This naturally leads us to ask the following questions.

Question. When X has no Euclidean factor, can we still have a coarse embedding from X to Y such that $\text{rank}(X) > \text{rank}(Y)$? What about when X has a Euclidean factor of dimension 1? i.e. if $X = \mathbb{R} \times Z$, can we have a coarse embedding from X to Y such that $\text{rank}(X) > \text{rank}(Y)$?

Main results

We can show that, in the absence of a Euclidean factor in the domain X, the rank is monotonous under coarse embeddings in many cases, especially between symmetric spaces, thus answering a question of Fisher and Whyte [FW18], see section 5.

In all the following results, the Euclidean buildings in the model space in the target are not necessarily thick, nor have cocompact affine Weyl group. It is only required for the Euclidean buildings in the domain.

Theorem 1.1 (See Theorem 1.7)**.** *Let $X = X_1 \times \cdots \times X_k$ be a product of geodesic metric spaces of exponential growth. If Y is a proper cocompact CAT(0) space of rank $< k$, or a mapping class group of geometric rank $< k$, then there is no coarse embedding from X to Y.*

This result has been originally announced by Gromov in [Gro93], who outlined a strategy of proof. Our proof relies mainly on his sketch. Moreover, we can extend this strategy to prove a similar result when X is a general model space. Namely, we show

Theorem 1.2 (see Theorem 1.8)**.** *Let $X = S \times B$ a model space of rank k. If Y is a proper cocompact CAT(0) space of rank $< k$, or a mapping class group of geometric rank $< k$, then there is no coarse embedding from X to Y.*

This implies for example that there is no coarse embedding from $T_3 \times T_3 \times T_3$ into $\mathbb{H}^p \times \mathbb{H}^q$ for any integers $p, q \geq 1$, nor into the symmetric space $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$.

When X has a Euclidean factor of dimension 1, we can show that the monotonicity of the rank still holds when the target is a model space, answering a question by Gromov [Gro93], see section 7.E_2.

Theorem 1.3 (see Theorem 1.9)**.** *Let X be either $X_1 \times \cdots \times X_k$ as in Theorem 1.1, or a model space $S \times B$ of rank k as in Theorem 1.2, and let $Y = \mathbb{R}^n \times S' \times B'$ be of rank $\leq k$. Then there is no coarse embedding from $X \times \mathbb{R}$ to Y.*

We also give a result about coarse embeddings of Euclidean spaces into symmetric spaces of lower rank, which uses the generalization of the Morse lemma for higher-rank symmetric spaces:

Theorem 1.4. *Let $p > k \geq 1$ be integers. Let Y be a symmetric space of non-compact type such that* $\operatorname{rank}(Y) = k$*, and let $f : \mathbb{R}^p \to Y$ be a coarse embedding. Then no Euclidean subspace $E \simeq \mathbb{R}^k \subset \mathbb{R}^p$ is sent quasi-isometrically by f.*

In the rank one case, we can extend the result to Gromov-hyperbolic geodesic metric spaces with bounded geometry thanks to a result by Bonk–Schramm [BS11]. This result was also suggested by Gromov in [Gro93].

Corollary 1.5. *Let Y be a Gromov-hyperbolic geodesic metric space with bounded geometry, and let $p \geq 2$ be an integer. A coarse embedding $f : \mathbb{R}^p \to Y$ is uniformly compressing, i.e. it admits a sublinear upper control function ρ^+.*

Now, one can naturally ask: If X does not coarsely embed into Y, does adding a Euclidean factor in the target makes the embedding possible?
We can answer this question when $\operatorname{rank}(X) > \operatorname{rank}(Y) + 1$. In fact, when $\operatorname{rank}(X) > \operatorname{rank}(Y) + 1$, even adding a nilpotent factor in the target does not make this embedding possible:

Corollary 1.6. *Let X be either $X_1 \times \cdots \times X_k$ as in Theorem 1.1, or a model space $S \times B$ of rank k as in Theorem 1.2, and let $Y = \mathbb{R}^n \times S' \times B'$ be a model space of rank $< k - 1$. Then for any nilpotent connected Lie group P, there is no coarse embedding from X to $Y \times P$.*

Proof. By Assouad's embedding theorem [Ass82], P coarsely embeds into some $\mathbb{Z}^d$, which in turn coarsely embeds into $\mathbb{H}^{d+1}$. Therefore, we get a coarse embedding from X to $Y \times \mathbb{H}^{d+1}$, but $\operatorname{rank}(Y \times \mathbb{H}^{d+1}) = \operatorname{rank}(Y) + 1 < k$. Such coarse embedding is not possible by Theorems 1.1 and 1.2. □

The main tools that we are going to use are the homological filling functions, using Lipschitz chains. They measure the difficulty of filling cycles of a given size. The key point is that filling functions detect the rank of proper cocompact CAT(0) spaces, and the geometric rank of mapping class groups. In other words, they behave differently whether the dimension of the cycle is smaller or greater than the rank of the space. In fact, if X is either a proper cocompact CAT(0) space or a mapping class group, a cycle Σ in X with dimension $k < \operatorname{rank}(X)$ and $\operatorname{Vol}_k^X(\Sigma) = l$ can be filled similarly as in a Euclidean space, i.e. it has a filling with $(k+1)$-volume $\precsim l^{\frac{k+1}{k}}$. On the other hand, if $k \geq \operatorname{rank}(X)$, then it can be filled in a sub-Euclidean way, i.e. it admits a filling with $(k+1)$-volume $= o\left(l^{\frac{k+1}{k}}\right)$. This result is due to Wenger [Wen11] for proper cocompact CAT(0) spaces, and to Behrstock–Drutu [BD19b] for mapping class groups, and is the main tool for the proof of Theorem 1.1 and Theorem 1.2. For Theorem 1.3 and Theorem 1.4, we need a stronger condition on the filling of the target space, that is a linear filling above the rank. This is a result of Leuzinger [Leu14] who proved that if X is a model space, and Σ is a k-cycle such that $k \geq \operatorname{rank}(X)$ and $\operatorname{Vol}_k^X(\Sigma) = l$, then it can be filled similarly as in a hyperbolic space, i.e. it has a filling with $(k+1)$-volume $\precsim l$.

Our proof will also rely on the co-area formula which requires some slicing techniques. However, Lipschitz chains do not behave well under slicing, that is why we will consider metric currents in complete metric spaces that were introduced by Ambrosio–Kirchheim [AK+00]. They can be seen as a generalization of Lipschitz chains since any Lipschitz chain induces naturally a current, which behaves well under slicing.

We denote by FV_k^X the k-filling function of a metric space X. We can restate our results in terms of filling functions, and in a more general framework, allowing the target space Y to be any complete uniformly contractible metric space (see Section 2.5 for a definition) with at most exponential growth:

Theorem 1.7. *Let $X = X_1 \times \cdots \times X_k$ be a product of geodesic metric spaces with exponential growth, $k \geq 2$ an integer, and let Y be a uniformly contractible complete metric space with at most exponential growth. If $\operatorname{FV}_k^Y(l) = o\left(l^{\frac{k}{k-1}}\right)$, then there is no coarse embedding from X to Y.*

Theorem 1.8. *Let $X = S \times B$ be a model space of rank $k \geq 2$, and let Y be a uniformly contractible complete metric space with at most exponential growth.*
If $\operatorname{FV}_k^Y(l) = o\left(l^{\frac{k}{k-1}}\right)$, then there is no coarse embedding from X to Y.

Theorem 1.9. *Let X be either $X_1 \times \cdots \times X_{k-1}$ as in Theorem 1.1, or a model space $S \times B$ of rank k, and let Y be a uniformly contractible complete metric space with at most exponential growth. If $\operatorname{FV}_k^Y(l) \sim l$, then there is no coarse embedding from $X \times \mathbb{R}$ to Y.*

Theorem 1.1 and Theorem 1.2 immediately follow from Theorem 1.7 and Theorem 1.8 by results of Wenger [Wen11] and Behrstock-Drutu [BD19b]. Theorem 1.9 follows from Theorem 1.3 by Leuzinger's result [Leu14].

Idea of the proof

Let us first look at Theorem 1.7. The case $k = 1$ is a simple volume obstruction observation: a space with exponential growth cannot be coarsely embedded with sublinear upper control ρ_+ in a space with at most exponential growth. Let us give a sketch of the proof when $k = 2$, which already contains most of the conceptual difficulties of the general case. When $X = X_1 \times X_2$ is a product of two geodesic metric spaces of exponential growth and the 2-dimensional filling function of Y is sub-Euclidean, suppose that a coarse embedding $f : X \to Y$ exists. By our observation, no copy of X_1 inside of X is sent sublinearly, which means that there exist "undistorted directions", more precisely there exist pairs of points $(a_n)_n$, $(b_n)_n$ in X_1 such that $d_n := d_X(a_n, b_n) \to \infty$ and which are mapped in Y quasi-isometrically. Now consider the rectangles R_n with one geodesic segment from a_n to b_n in X_1 as one side, and a geodesic segment in a copy of X_2 of length at most $\varphi(d_n)$ as the orthogonal side, where φ is a sublinear function that depends on FV_2^Y and such that $f(\partial R_n)$ can be filled in Y by a chain of volume $= o(d_n\, \varphi(d_n))$. Since (a_n) and (b_n) are sent quasi-isometrically, the height of $f(\partial R_n)$ is comparable to that of ∂R_n, however the filling of $f(\partial R_n)$ in Y is much smaller than that of ∂R_n in X. This implies, by the co-area formula for currents, that the width of $f(\partial R_n)$ is highly compressed. From this observation, we construct a sequence of sets in X whose "coarse"-volume, that will be defined in the background material, is not coarsely preserved by f, which is not possible for coarse embeddings. For $k \geq 3$, we proceed by induction on k. Suppose that we have proved the result for some $k \geq 2$, and that there exists a coarse embedding f from $X = X_1 \times \cdots \times X_{k+1}$ to Y that satisfies a sub-Euclidean $(k+1)$-filling. By the induction hypothesis, there exists a sequence of k-dimensional rectangles in a copy of $X_1 \times \cdots \times X_k$, such that the image of their boundaries have fillings comparable to that in the domain. The sub-Euclidean $(k + 1)$-filling implies that there exist, orthogonally to these k-rectangles, sets that are highly compressed by f, which contradicts the coarse volume preservation.

To prove Theorem 1.8, which implies Theorem 1.2, we will adapt the previous proof. Since the domain is no longer a product space, we will consider parallelograms instead of rectangles, and adapt the size of its sides to make the proof work. More importantly, instead of looking for "undistorted directions", we need to find "maximally singular undistorted directions". Maximally singular directions are directions following which a space of rank k contains a subset that factorizes as a product $\mathbb{R} \times X'$, where X' has rank $k - 1$. To find these directions, we will decompose the cycle along the walls of a suitable Weyl sector, and use a pigeonhole-like reasoning.

Theorem 1.9 is a consequence of the first two theorems, using the linear filling above the rank and Theorem 5.1, which is a result about coarse embeddings of Euclidean spaces into symmetric spaces of lower rank. This result allows us to derive an upper bound for fillings of images of $(k-1)$-parallelograms, starting from the linear filling of dimension $k+1$ in Y.

The proof of Theorem 1.4 is done by contradiction. If there exists such quasi-isometric embedding, then by the quasi-flats Theorem of Kleiner–Leeb [KL97] and Eskin–Farb [EF97], such a quasi-flat is a bounded distance from a finite union of maximal flats. This implies that there exist arbitrarily big balls that are sent uniformly close to a maximal flat. By considering parallelograms inside these balls and the filling of the projection of their images in this flat, we get a contradiction. Namely, we get a sequence of images of parallelograms that are sent quasi-isometrically in a Euclidean space, and that are filled linearly, which gives a contradiction by applying the co-area formula. Finally, a result of Bonk–Schramm [BS11] says that any Gromov hyperbolic geodesic metric space with bounded geometry quasi-isometrically embeds into a hyperbolic space $\mathbb{H}^n$. By composing by this quasi-isometry and applying the previous theorem we get the Corollary 1.5.

The manuscript is organized as follows. We will start in Chapter 2 by giving some background material. More precisely, section 2.1 is dedicated to coarse embeddings. In section 2.2 we collect definitions of the model spaces and mapping class groups, and give some large-scale properties, like parallel sets and cross sections in CAT(0) spaces. In 2.3, we give a brief overview of the theory of metric currents introduced by Ambrosio–Kirchheim, and state the Slicing Theorem that will play a crucial role in the proofs. We define the filling functions in subsection 2.4 and state results of Wenger [Wen11], Behrstock-Drutu [BD19b] and Leuzinger [Leu14] that shows that the filling functions can detect the rank of the spaces we are considering. Finally, connect the dots argument in 2.5 will allow us to always assume that our coarse embeddings are Lipschitz, which is very convenient when working with Lipschitz chains or currents. We start chapter 3 by giving some useful lemmas about coarse embeddings and volume preservation in section 3.1. Then we will consider Lipschitz maps with sub-Euclidean filling of the image of the boundary of some rectangles in section 3.2, leading to the proof of Theorem 1.7, which implies Theorem 1.1, in section 3.3. In chapter 4, we start in 4.1 by a preliminary result about Weyl sectors in either symmetric spaces of non-compact type or Euclidean buildings. In 4.2 we define parallelepipeds and parallelograms, and we prove a decomposition result for parallelograms that will be used in the section 4.3 where we consider Lipschitz maps with sub-Euclidean filling of the image of the boundary of some parallelograms. This leads to the proof of Theorem 1.8 in section 4.4. The goal of chapter 5 is to give a proof of Theorem 1.4 and Theorem 1.9, thus implying Theorem 1.3. It starts first by giving a result about coarse embeddings of Euclidean spaces into symmetric spaces of lower rank in 5.1, which implies Theorem 1.4 and Corollary 1.5 in 5.2, and implies Theorem 1.9 in 5.3. Finally, we collect some further questions in chapter 6.

Chapter 2

Coarse geometry and filling functions

2.1 Coarse embeddings

A map $f : (X, d_X) \to (Y, d_Y)$ is a coarse embedding if there exist functions $\rho_\pm : [0, \infty) \to [0, \infty)$ such that $\rho_-(r) \to \infty$ as $r \to \infty$ and for all $x, y \in X$

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).$$

Equivalently, for every pair $(x_n)_{n\geq 0}$, $(x'_n)_{n\geq 0}$ of sequences of points in X,

$$\lim_{n\to\infty} d_X(x_n, x'_n) = \infty \iff \lim_{n\to\infty} d_Y(f(x_n), f(x'_n)) = \infty.$$

When the control functions ρ_-, ρ_+ are affine, f is said to be a *quasi-isometric embedding*. The map f is said to be *large-scale Lipschitz* if it admits an affine upper control, without necessarily having a lower control.

Definition 2.1. A metric space (X, d) is *large-scale geodesic* if there exist constants $\lambda, c > 0$ and $b \geq 0$ such that, for every pair (x, x') of points of X, there exists a sequence $x_0 = x, x_1, \ldots, x_n = x'$ of points in X such that $d(x_{i-1}, x_i) \leq c$ for $i = 1, \ldots, n$ and $n \leq \lambda d(x, x') + b$.

If X is large-scale geodesic, then f admits a control function ρ_+ which is affine.

Definition 2.2. A metric space X has *bounded geometry* if there exists $R_0 \geq 0$ such that, for every $R \geq 0$, there exists an integer N such that every ball of radius R in X can be covered by N balls of radius R_0.

Remark 2.3. An example of a metric space that is not of bounded geometry is an infinite-dimensional Hilbert space, with the metric given by the norm.

Definition 2.4. Let X be a metric space with bounded geometry and let R_0 as in definition

2.2. Let $\varepsilon \geq R_0$. We define the *ε-volume* of a subset $A \subset X$, that we will denote $\mathrm{Vol}_X^{\varepsilon}(A)$, as the minimal number of balls of radius ε needed to cover A.

We will denote by $B(x,r)$ the closed ball $B(x,r) := \{y \in X : d(y,x) \leq r\}$.

Definition 2.5. Let $\varepsilon \geq R_0$ as in the previous definitions. We define the *ε-growth function* of a metric space (X,d) with bounded geometry to be

$$\beta_X^{\varepsilon}(r) = \sup\{\mathrm{Vol}_X^{\varepsilon}(B(x,r)) \mid x \in X\}.$$

We say that X has *polynomial growth* if there exist $\varepsilon > 0$ and $D \geq 0$ such that $\beta_X^{\varepsilon}(r) \preccurlyeq r^D$. We say that X has *exponential growth* if there exists $\varepsilon > 0$ such that $\beta_X^{\varepsilon}(r) \approx e^r$.
The comparison on functions $\mathbb{N} \to \mathbb{R}$ is defined as follows: $f \preccurlyeq g$ if there exists a constant $c > 0$ such that $f(n) \leq cg(cn+c)+c$ for all $n \in \mathbb{N}$, and $f \approx g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Remark 2.6. If X is large-scale geodesic and has bounded geometry, then it has at most exponential growth $\beta_X^{\varepsilon}(r) \preccurlyeq e^r$.

2.2 Coarse geometry of model spaces and mapping class groups

This section is devoted to recall the definition of the spaces we will be considering. We will define model spaces as products of symmetric spaces and Euclidean buildings with a Euclidean factor, and recall the definition of mapping class groups. We will give some properties of these spaces that will be useful.
Symmetric spaces and Euclidean buildings can be seen as leading examples of CAT(0) spaces. In fact, they seem to be the most rigid among all proper CAT(0) spaces [CM09][Lee].

2.2.1 Symmetric spaces

A *symmetric space* is a connected riemannian manifold M such that for all $x \in M$, there exists an isometry $\sigma_x \in \mathrm{Isom}(M)$ such that $\sigma_x(x) = x$ and $T_x\sigma_x = -\mathrm{Id}_{T_xM}$. It is said to be *of nonpositive curvature* if it has non-positive sectional curvature. If moreover it has no non-trivial Euclidean factor, M is called a *symmetric space of non-compact type*. All symmetric spaces of non-compact type can be obtained as coset spaces G/K, where G is a connected semi-simple Lie group with trivial center and no compact factors, and K is a maximal compact subgroup of G. The metric on G/K comes from the Killing form of $\mathrm{Lie}(G)$. When M is a symmetric space of non-compact type, G is the identity component of the isometry group $\mathrm{Isom}_0(M)$.
An important example of symmetric spaces of non-compact type is given by $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. When $n = 2$, it corresponds to the hyperbolic plane $\mathbb{H}^2$. More generally, if M is a symmetric space of non-compact type such that $\dim(\mathrm{Isom}_0(M)) = n \geq 2$, then after rescaling

the metrics of the de Rham factors of M by positive constants, M can be isometrically embedded in $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ [Ebe96]. The symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ can also be seen as the collection of scalar products on $\mathbb{R}^n$, for which the unit ball has volume 1.

2.2.2 Euclidean buildings

Euclidean buildings are non-Archimedean analogues of symmetric spaces of non-compact type. In fact, to any semi-simple algebraic group over a local field, $\mathrm{SL}_n(\mathbb{Q}_p)$ for example, we can associate a Euclidean building, called its Bruhat-Tits Building, on which the group acts isometrically in a very transitive way. Let us define Euclidean buildings in general. A general reference for buildings is [AB08], and for an introductory course see [Cap14].

Let $W < \mathrm{Isom}(\mathbb{R}^n)$ be a discrete reflection group, i.e. a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ generated by orthogonal reflections through a collection $\mathcal{H}$ of hyperplanes, called *walls*, and such that this collection is locally finite. The pattern determined by the set of walls defines a cellular decomposition Σ of $\mathbb{R}^n$, called a *Euclidean Coxeter complex*. The group W is called the *affine Weyl group* (or the *affine Coxeter group*) of the Euclidean Coxeter complex Σ. A *chamber* (or *alcove*) in that complex is defined as a connected component of $\mathbb{R}^n - \cup_{h\in\mathcal{H}} h$. The affine Weyl group W acts transitively on the set of chambers. The top-dimensional cells of Σ are the closures of chambers, also called *closed chambers*, which are compact if and only if the group W acts cocompactly. A lower dimensional cell is the intersection of a closed chamber with a set of walls.

Definition 2.7. Let W be an affine Weyl group of $\mathbb{R}^n$, and Σ its associated Euclidean Coxeter complex. A *(discrete) Euclidean building* modeled on (Σ, W) is a cell complex B, which is covered by subcomplexes all isomorphic to Σ, called the *apartments* of B, and such that the following incidences properties hold:

1. Any two cells of B lie in some apartment.
2. For any two apartments, there is an isomorphism between them fixing their intersection pointwise.

Euclidean buildings are frequently called *Affine buildings* in the literature.

A Euclidean building is *thick* if each wall belongs to at least 3 half-apartments with disjoint interiors. If the affine Weyl group acts cocompactly on the apartments, then the chambers of the Euclidean Coxeter complex are polysimplices. This induces on the Euclidean building a structure of a polysimplicial complex. If the building is moreover irreducible, then this complex is a simplicial complex.

A Euclidean building always possesses a geometric realization, which is a complete CAT(0) metric space, and such that the restriction of its distance to each apartment is the Euclidean metric.

2.2.3 Model spaces

We call *model spaces* the spaces of the form $X = \mathbb{R}^n \times S \times B$, where S is a product of symmetric spaces of non-compact type and B a product of thick Euclidean buildings with cocompact affine Weyl group, with bounded geometry and no Euclidean factor. Note that such spaces are CAT(0), thus uniformly contractible.

2.2.4 Mapping class groups

References for what follows are [FM11],[Ham05], [BM08].
We consider orientable compact connected surfaces $S = S_{g,p}$ of genus g and p boundary components. The mapping class group, $\mathcal{MCG}(S)$, is defined to be $Homeo^+(S)/Homeo_0(S)$, the group of isotopy classes of homeomorphisms of S. Boundary components are not required to be fixed by the mapping class group, so each boundary should be considered as a puncture. This group is finitely-generated and for any finite generating set one considers the word metric in the usual way, yielding a metric space which is unique up to quasi-isometry. It is known that mapping class groups are not CAT(0) because they cannot act geometrically on a CAT(0) space when the surface has genus ≥ 3, or genus 2 and at least one puncture [KL96]. Nonetheless, their filling functions behave like those of CAT(0) spaces, as we will see in section 2.4.
Throughout the remainder, we assume that $3g-3+p \geq 2$, i.e. we exclude the sphere with at most 4 punctures and the torus with at most 1 puncture. The mapping class group of these 7 surfaces is either finite or virtually free, so quasi-isometric to a point or a locally finite regular tree, which is already covered by the Euclidean buildings that we consider.
By a result of [Mos95], mapping class groups are automatic, so they are combable in the sens that their Cayley graphs admit a bounded quasi-geodesic combing, see [BH13]. Therefore, they are of type $\mathcal{F}_\infty$ [ECH+91]. In particular, for every $n \in \mathbb{N}$, there exists an n-connected CW-complex X_S on which $\mathcal{MCG}(S)$ acts freely, properly discontinuously and cocompactly.

2.2.5 Rank and geometric rank

Definition 2.8. Let X be a metric space. We define its *rank* as
$\mathrm{rank}(X) = \max\{k \in \mathbb{N} \mid \exists\, g : \mathbb{R}^k \to X \text{ an isometric embedding}\}$.

Similarly, we define the geometric rank, that we also call quasi-flat rank or quasi-rank.

Definition 2.9. Let X be a metric space. We define its *geometric rank* as
$\mathrm{grank}(X) = \max\{k \in \mathbb{N} \mid \exists\, g : \mathbb{R}^k \to X \text{ a quasi-isometric embedding}\}$.

We recall that, by a result of [AS86] and [Kle99], the geometric rank is equal to the rank for proper cocompact CAT(0) spaces. For mapping class groups, it was shown by [BM08] and by [Ham05] that the geometric rank of a mapping class group $\mathcal{MCG}(S_{g,p})$ is equal to the maximal rank of its free abelian subgroups, which is equal to $3g-3+p$ by [BLM83]. In particular, this rank is realized by any subgroup generated by Dehn twists on a maximal set of disjoint essential simple closed curves.

Example 2.10. *For all $p \geq 1$, $\operatorname{rank}(\mathbb{R}^p) = p$.*
For all $n \geq 2$, $\operatorname{rank}(\mathbb{H}^n) = 1$ and $\operatorname{rank}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})) = \operatorname{rank}(\mathrm{BT}(\mathrm{SL}_n(\mathbb{Q}_p))) = n-1$, where $\mathrm{BT}(\mathrm{SL}_n(\mathbb{Q}_p))$ is the Bruhat-Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$.

Remark 2.11. Note that the rank of a Euclidean building is equal to its dimension, and $\operatorname{rank}(\mathbb{R}^n \times S \times B) = n + \operatorname{rank}(S) + \dim(B)$.

2.2.6 Parallel sets in model spaces

References for this subsection are [Ebe96],[Hel01] for symmetric spaces, and [KL97], [Lee] for Euclidean buildings.

Let X be a model space of rank k. A subspace of X is a *flat* if it is isometric to some Euclidean space. A *maximal flat* is a flat of dimension k, and flats of dimension 1 are the geodesics. A geodesic γ is said to be *regular* if it is contained in a unique maximal flat. Otherwise it is called *singular*. It is said to be *maximally singular* if it belongs to k half-flats of maximal dimension, with disjoint interiors. A *singular flat* is a flat which is the intersection of maximal flats.

Let X be a model space with no Euclidean factor, and let $F \subset X$ be a flat. If another flat F' is at finite Hausdorff distance from F then by the Flat Strip Theorem [BH13], since X is CAT(0), there exists a segment I such that the convex hull of the union $F \cup F'$ is isometric to $F \times I$. In that case, the flats F and F' are called *parallel*. The flat F is a closed convex subset with extendible geodesics (i.e. each geodesic segment is contained in a bi-infinite geodesic in F), therefore we can consider its *parallel set* $P_X(F)$, which is the union of flats that are parallel to F. It is a closed convex subset of X isometric to a product

$$P_X(F) = F \times CS_X(F),$$

where $CS_X(F)$ is a model space. It is called the *cross section* of F. Moreover, if F is a singular flat then its cross section has no Euclidean factor and $\operatorname{rank}(CS_X(F)) = \operatorname{rank}(X) - \dim(F)$. The flats in X being just products of flats in its factors, and singular flats being products of singular flats in the factors, the cross section of a product is the product of cross sections (see [Lee] and [KL97]).

Example 2.12. *Let X be the Bruhat-Tits building associated to $\mathrm{SL}_3(\mathbb{Q}_p)$ and let γ be a maximally singular geodesic. We then have that $CS_X(\gamma)$ is isometric to the $(p+1)$-regular simplicial tree T_{p+1}, and*

$$P_X(\gamma) = \mathbb{R} \times T_{p+1}.$$

Example 2.13. *Let $Y = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, and let γ be a maximally singular geodesic. We then have that $CS_Y(\gamma)$ is isometric to the hyperbolic plane $\mathbb{H}^2$, and*

$$P_Y(\gamma) = \mathbb{R} \times \mathbb{H}^2.$$

In the proof of Theorem 1.8, we will need a uniform lower bound on the volume growth of cross sections of singular flats. It is obvious for symmetric spaces since cross sections of singular flats are symmetric spaces of lower rank, and there are only finitely many isometry classes. For Euclidean buildings we need the following result.

Proposition 2.14. *Let X be a thick Euclidean building with cocompact affine Weyl group. Then for every singular flat F, there exists $D = D(F) > 0$ such that the 3-regular tree of edge length D isometrically embeds into the cross section of F. This implies that there exists a uniform lower bound on the volume growth of cross sections of singular flats in X.*

Proof. Let X be such Euclidean building, and let $n = \dim X$. If $n = 1$, X is a tree an the singular flats are the vertices. Thus the cross sections are the whole building. Let $D > 0$ be the distance between vertices in an apartment. Since X is thick, it contains a 3-regular tree of edge length D.

Suppose that $n \geq 2$, and let F be a singular flat of dimension $k \leq n-1$. Let F' be a singular $(n-1)$-flat (i.e. a wall) that contains F. Any $(n-1)$-flat parallel to F' contains a k-flat parallel to F, therefore $P_X(F') \subset P_X(F)$, which implies that $\mathbb{R}^{n-1-k} \times CS_X(F') \subset CS_X(F)$. So it is enough to show it for $k = n-1$.

The affine Weyl group W is a semi-direct product $W = W_r \ltimes T$, where W_r is the associated finite Coxeter group and T is a group of translations generated by vectors orthogonal to some walls. Since it is cocompact, T is isomorphic to $\mathbb{Z}^n$. Let $F \subset X$ be a wall, and fix a generating set of T. Since the affine Weyl group is cocompact, there exists $D = D(F) > 0$ such that the walls parallel to F in an apartment are at distance an integer multiple of D, see also [KL97] Corollary 5.1.3. In particular, every wall parallel to F admits at least three parallel walls in X at distance D. Consider the graph G whose vertex set is the set of walls parallel to F, and whose edges are pairs of walls that differ by a translation by a generator of T in an apartment. If moreover the length of the edges is the distance in X between the corresponding walls, then the graph G is isometric the the cross section of F, which is called the *wall tree* of F. It is a thick tree, see [Wei08] Chapter 10 and [KW14], and contains a subtree which is 3-regular of edge length D, obtained by only considering

for each vertex two parallel walls at distance D.
Moreover, since there is only a finite number of singular flats up to translation, there is a uniform lower bound on the volume growth of such wall trees. □

2.3 Metric currents

References for this section are [AK+00],[HKS22] and [Wen05].
Ambrosio and Kirchheim extended the classical theory of normal and integral currents developped by Federer and Fleming [FF60] to arbitrarily complete metric spaces.

2.3.1 An overview of the theory

Let (X, d) be a complete metric space and $k \geq 0$ and let $\mathcal{D}^k(X)$ denote the set of $(k+1)$-tuples $(f, \pi_1, \ldots, \pi_k)$ of Lipschitz functions on X with f bounded.

Definition 2.15. A *k-dimensional metric current* T on X is a multi-linear functional on $\mathcal{D}^k(X)$ satisfying the following properties:

1. If π_i^j converges point-wise to π_i as $j \to \infty$ and if $\sup_{i,j} \mathrm{Lip}(\pi_i^j) < \infty$ then
$$T(f, \pi_1^j, \ldots, \pi_k^j) \to T(f, \pi_1, \ldots, \pi_k).$$
2. If $\{x \in X : f(x) \neq 0\}$ is contained in the union $\bigcup_{i=1}^k B_i$ of Borel sets B_i and if π_i is constant on B_i then
$$T(f, \pi_1, \ldots, \pi_k) = 0.$$
3. There exists a finite Borel measure μ on X such that
$$|T(f, \pi_1, \ldots, \pi_k)| \leq \prod_{i=1}^k \mathrm{Lip}(\pi_i) \int_X |f| d\mu \tag{2.3.1}$$
for all $(f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(X)$.

The space of k-dimensional metric currents on X is denoted by $\mathbf{M}_k(X)$ and the minimal Borel measure μ satisfying (2.3.1) is called *mass* of T and written as $\|T\|$. Let us denote by $\mathbf{M}(T) = \|T\|(X)$ the *total mass* of T. *The support of* T is, by definition, the closed set spt$T$ of points $x \in X$ such that $\|T\|(B(x,r)) > 0$ for all $r > 0$.
Every function $g \in L^1_{loc}(\mathbb{R}^k)$ induces a current $[\![g]\!] \in \mathbf{M}_k(\mathbb{R}^k)$. Indeed, by Rademacher's theorem, Lipschitz function are differentiable almost everywhere, so if $(f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(\mathbb{R}^k)$, $[\![g]\!]$ is defined by: :

$$[\![g]\!](f, \pi_1, \ldots, \pi_k) := \int_{\mathbb{R}^k} gf \det\left(\frac{\partial \pi_i}{\partial x_j}\right) d\mathcal{L}^k$$

which corresponds to the integration of the differential form $f d\pi_1 \wedge \cdots \wedge d\pi_k$, weighted by f. Therefore, every borel set $A \subset \mathbb{R}^k$ induces a current $[\![A]\!] := [\![\chi_A]\!]$.

The restriction of $T \in \mathbf{M}_k(X)$ to a Borel set $A \subset X$ is given by

$$(T \llcorner A)(f, \pi_1, \ldots, \pi_k) := T(f\chi_A, \pi_1, \ldots, \pi_k).$$

This expression is well-defined since T can be extended to a functional on tuples for which the first argument lies in $L^\infty(X, \|T\|)$.

The boundary of $T \in \mathcal{D}_k(X)$ is defined by analogy with Stokes formula:

$$\partial T(f, \pi_1, \ldots, \pi_{k-1}) := T(1, f, \pi_1, \ldots, \pi_{k-1}).$$

It is clear that ∂T satisfies conditions (1) and (2) in the above definition. If ∂T also has finite mass (condition (3)) then T is called a *normal current.* The space of normal currents is denoted by $\mathbf{N}_k(X)$.
The push-forward of $T \in \mathbf{M}_k(X)$ under a Lipschitz map φ from X to another complete metric space Y is given by

$$\varphi_\# T(g, \tau_1, \ldots, \tau_k) := T(g \circ \varphi, \tau_1 \circ \varphi, \ldots, \tau_k \circ \varphi),$$

for $(g, \tau_1, \ldots, \tau_k) \in \mathcal{D}^k(Y)$. This defines a k-dimensional current on Y.

Let $\mathcal{H}^k$ denote the Hausdorff k-dimensional measure. An $\mathcal{H}^k$-measurable set $A \subset X$ is said to be *countably $\mathcal{H}^k$-rectifiable* if there exist Lipschitz maps $f_i : B_i \longrightarrow X$ from subsets $B_i \subset \mathbb{R}^k$ such that

$$\mathcal{H}^k(A - \bigcup f_i(B_i)) = 0.$$

Definition 2.16. A current $T \in \mathbf{M}_k(X)$ with $k \geq 1$ is said to be *rectifiable* if

1. $\|T\|$ is concentrated on a countably $\mathcal{H}^k$-rectifiable set and
2. $\|T\|$ vanishes on $\mathcal{H}^k$-negligible sets.

T is called *integer rectifiable* if, in addition, the following property holds:

(3) For any Lipschitz map $\varphi \colon X \longrightarrow \mathbb{R}^k$ and any open set $U \subset X$ there exists $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$ such that

$$\varphi_\#(T \llcorner U) = [\![\theta]\!].$$

We denote the space of rectifiable currents by $\mathcal{R}_k(X)$, and $\mathcal{I}_k(X)$ for integer rectifiable currents. Integer rectifiable normal currents are called *integral currents*, denoted by $\mathbf{I}_k(X)$.

An element $T \in \mathbf{I}_k(X)$ is called a *cycle* $\partial T = 0$.

Before moving to the slicing theorem, let us show the following lemma.

Lemma 2.17. *Let X be a complete metric space, and $x, y \in X$. Let $T \in \mathbf{I}_1(X)$ such that $\partial T = [\![x]\!] - [\![y]\!]$, then*

$$\mathbf{M}(T) \geq d_X(x, y),$$

and

$$\mathrm{FillVol}_1^{X,\mathrm{cr}}([\![x]\!] - [\![y]\!]) \geq d_X(x, y).$$

If moreover there is a geodesic segment from x to y, then

$$\mathrm{FillVol}_1^{X,\mathrm{cr}}([\![x]\!] - [\![y]\!]) = d_X(x, y).$$

Proof. $\partial T = [\![x]\!] - [\![y]\!]$ means that for all $f : X \to \mathbb{R}$ Lipschitz and bounded, we have

$$\partial T(f) = f(x) - f(y).$$

By definition, we have $\partial T(f) = T(1, f)$. So $T(1, f) = f(x) - f(y)$.
T being a 1-current, there exists a finite Borel measure μ on X such that for all $g : X \to \mathbb{R}$ Lipschitz and bounded and for all $f : X \to \mathbb{R}$ Lipschitz

$$|T(g, f)| \leq \mathrm{Lip}(f) \int_X |g(x)|\, d\mu(x).$$

Let $\mu = \|T\|$ be the minimal Borel measure. In particular, we have that $\mu(X) = \|T\|(X) = \mathbf{M}(T)$.
For $g = 1$ and for all $f : X \to \mathbb{R}$ Lipschitz, we have

$$|f(x) - f(y)| = |T(1, f)| \leq \mathrm{Lip}(f) \int_X |1(x)|\, d\mu(x) = \mathrm{Lip}(f)\mathbf{M}(T).$$

So $\mathbf{M}(T)$ satisfies : $\forall f : X \to \mathbb{R}$ Lipschitz, $|f(x) - f(y)| \leq \mathrm{Lip}(f)\mathbf{M}(T)$. This implies that

$$\sup_{f\, 1-\mathrm{Lip}} \frac{|f(x) - f(y)|}{\mathrm{Lip}(f)} \leq \mathbf{M}(T).$$

Consider the map $d_y : X \to \mathbb{R}$, $\forall z \in X$, $d_y(z) := d_X(y, z)$. It is a 1-Lipschitz map that satisfies:

$$\frac{|d_y(x) - d_y(y)|}{\mathrm{Lip}(f)} = d_X(x, y).$$

Therefore

$$\sup_{f\, 1-\mathrm{Lip}} \frac{|f(x) - f(y)|}{\mathrm{Lip}(f)} = d_X(x, y),$$

and

$$\mathbf{M}(T) \geq d_X(x, y).$$

By taking the infimum of the mass of all such 1-currents,

$$\mathrm{FillVol}_1^{X,\mathrm{cr}}([\![x]\!] - [\![y]\!]) \geq d_X(x, y).$$

If there is a geodesic segment γ from y to x, consider the Lipschitz 1-chain realized by γ. Then $\partial\gamma = [\![x]\!] - [\![y]\!]$ and $\mathbf{M}(\gamma) = d_X(x, y)$, and the infimum is attained. □

2.3.2 The slicing theorem

The main motivation for using currents instead of Lipschitz chains is the following Slicing Theorem due to Ambrosio–Kirchheim [AK+00] that will play a crucial role in this paper.

Theorem 2.18. *Let be $T \in \mathbf{I}_k(X)$ and π a Lipschitz function on X. Then there exists for almost every $r \in \mathbb{R}$ an integral current $\langle T, \pi, r\rangle \in \mathbf{I}_{k-1}(X)$ with the following properties:*

1. $\langle T, \pi, r\rangle = \partial(T \llcorner \{\pi \leq r\}) - (\partial T) \llcorner \{\pi \leq r\}$,
2. $\|\langle T, \pi, r\rangle\|$ *and* $\|\partial\langle T, \pi, r\rangle\|$ *are concentrated on* $\pi^{-1}(\{r\})$,
3. $\mathbf{M}(\langle T, \pi, r\rangle) \leq \mathrm{Lip}(\pi)\frac{d}{dr}\mathbf{M}(T \llcorner \{\pi \leq r\})$, *which is just a reformulation of the co-area formula.*

2.4 Homological filling functions

The basic idea of a filling function is to measure the difficulty of filling a boundary of a given size. There are several ways to make this rigorous, depending on the type of boundary and the type of filling. We will use the Homological filling, as in [Gro93][Leu14], which consists on filling Lipschitz cycles by Lipschitz chains, instead of the homotopical filling where we fill spheres by balls.

2.4.1 Definition and examples

An integral Lipschitz k-chain in a complete metric space X is a finite linear combination $\Sigma = \sum_i a_i\sigma_i$, with $a_i \in \mathbb{Z}$, of Lipschitz maps $\sigma_i : \Delta^k \to X$ from the Euclidean k-dimensional simplex Δ^k to X. We will often call this simply a *Lipschitz k-chain*. The boundary operator is defined as in the case of singular chains.
Note that by the push-forward of metric currents, every Lipschitz chain in X induces an integral current. Indeed, if $\sigma_i : \Delta^k \to X$ is Lipschitz, then $f_\#([\![\Delta^k]\!])$ is an integral k-current

in X. We define the *k-volume of* σ_i as the total mass of $f_\#([\![\Delta^k]\!])$:

$$\mathrm{Vol}_k\sigma_i = \mathbf{M}(f_\#([\![\Delta^k]\!])).$$

Note that if X is a riemannian manifold then, by Rademacher's theorem, σ_i is differentiable almost everywhere and $\mathrm{Vol}_k\sigma_i$ is also equal to the integral of the magnitude of its Jacobian. Note also that it is not necessarily equal to the volume of the image (its hausdorff measure), unless the map is injective, because the volume is counted with multiplicity.
We then define the *k-volume* of a Lipschitz k-chain as

$$\mathrm{Vol}_k^X \Sigma := \sum_i |a_i|\mathrm{vol}_k\sigma_i.$$

We wish to measure the difficulty to fill Lipschitz k-cycles by Lipschitz $(k+1)$-chains.
To ensure that such a filling exists, the space considered must be k-connected, i.e. all homotopy groups $\pi_k(X)$ are trivial (which is the case for CAT(0) spaces and for the corresponding CW-complexes X_S associated to $\mathcal{MCG}(S)$), hence the corresponding homology groups are also trivial.
More precisely, for an integral Lipschitz k-cycle Σ in k-connected space we define its *filling volume*

$$\mathrm{FillVol}_{k+1}^X(\Sigma) := \inf\{\mathrm{Vol}_{k+1}(\Omega) \mid \Omega \text{ is a Lipschitz } (k+1)\text{-chain with } \partial\Omega = \Sigma\}.$$

The $(k+1)$*-dimensional filling function* of X is then given by

$$\mathrm{FV}_{k+1}^X(l) := \sup\{\mathrm{FillVol}_{k+1}^X(\Sigma) \mid \Sigma \text{ is a Lipschitz } k\text{-cycle in X with } \mathrm{Vol}_k^X(\Sigma) \leq l\}.$$

We are only interested in the asymptotic behaviour of these filling functions. We have that if X and Y are two k-connected manifolds or simplicial complexes which are quasi-isometric, then by [APW99]

$$\mathrm{FV}_k^X(l) \sim \mathrm{FV}_k^Y(l).$$

The equivalence relation on functions $\mathbb{R} \to \mathbb{R}$ is define as follows: we write $f \precsim g$ if there is a constant $C > 0$ such that $f(x) \leq Cg(Cx + C) + Cx + C$. We write $f \sim g$ and say that they are *asymptotically equivalent* if $f \precsim g$ and $g \precsim f$.

Example 2.19.
• *A hyperbolic space $X = \mathbb{H}^n$ satisfies for all $2 \leq k \leq n$, $\mathrm{FV}_X^k(l) \sim l$ [Gro87] [Lan00].*
• *A Euclidean space $X = \mathbb{R}^n$ satisfies for all $2 \leq k \leq n$, $\mathrm{FV}_X^k(l) \sim l^{\frac{k}{k-1}}$ [FF60].*
• *More generally, [Wen05] showed that a complete CAT(0) space satisfies $\mathrm{FV}_k^X(l) \precsim l^{\frac{k}{k-1}}$ for all dimensions k.*
• *Mapping class groups are combable, so by [BD19a] they satisfy for every k, $\mathrm{FV}^k(l) \precsim$*

$l^{\frac{k}{k-1}}$. *The filling functions are defined in the corresponding CW-complex associated to the mapping class group.*

Note that by the push-forward of metric currents, every Lipschitz chain in X induces an integral current. Moreover, if σ is a Lipschitz k-chain $\mathrm{Vol}^k_X(\sigma) = \mathbf{M}(\sigma)$. Therefore, the the chains that we will consider will be seen as currents, so it is convenient to define filling functions in this setting.

Definition 2.20. Let (X, d) be a complete metric space and $k \geq 0$ an integer. For a k-cycle $\Sigma \in \mathbf{I}_k(X)$, we define its $(k+1)$-*dimensional current-filling volume* to be

$$\mathrm{FillVol}^{X,\mathrm{cr}}_{k+1}(\Sigma) := \inf\{\mathbf{M}(\Omega) \mid \Omega \in \mathbf{I}_{k+1} \text{ with } \partial\Omega = \Sigma\}.$$

Since every Lipschitz chain induces a metric current, it is clear that if Σ is a Lipschitz k-cycle seen in $\mathbf{I}_k(X)$, then $\mathrm{FillVol}^{X,\mathrm{cr}}_{k+1}(\Sigma) \leq \mathrm{FillVol}^X_{k+1}(\Sigma)$.

2.4.2 Filling functions detect the rank

The following key theorems say that the filling functions can detect the rank of these spaces.

Theorem 2.21 ([Wen11]).
Let X be a proper cocompact CAT(0) space. If $k > \mathrm{rank}(X)$, then $\mathrm{FV}^X_k(l) = o\left(l^{\frac{k}{k-1}}\right)$.

Wenger Actually proved it for all complete quasi-geodesic metric spaces admitting cone-type inequalities up to dimension k. He also showed that the filling functions are asymptotically equal to the Euclidean filling functions below the rank.

Theorem 2.22 ([BD19b]).
Let $X = \mathcal{MCG}(S_{g,p})$. If $k > \mathrm{grank}(X) = 3g - 3 + p$, then $\mathrm{FV}^X_k(l) = o\left(l^{\frac{k}{k-1}}\right)$.

It is known that below the geometric rank, the filling functions are asymptotically equal to the Euclidean filling functions.

Theorem 2.23 ([Leu14]).
Let $X = \mathbb{R}^d \times S \times B$ be a model space. Then

(i) *X has Euclidean filling functions below the rank:*

$$\mathrm{FV}^X_k(l) \sim l^{\frac{k}{k-1}} \quad \text{if } \ 2 \leq k \leq \mathrm{rank}\ (X);$$

(ii) *X has linear filling functions above the rank:*

$$\mathrm{FV}^X_k(l) \sim l \quad \text{if } \ \mathrm{rank}\ (X) < k \leq \dim(X).$$

When X is a symmetric space of non-compact type, this result has been correctly asserted by Gromov in [Gro93], section 5.D, and proposed a possible different proof for the upper bound by projecting the cycle to a maximal flat, see [Leu14] for more details. The proof of Leuzinger uses a different approach by projecting to a suitable horosphere.

2.5 Connect the dots argument

Definition 2.24. A metric space X is *uniformly contractible* if there is a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x \in X$ and $r > 0$, the ball $B_X(x, r)$ is contractible in the ball $B_X(x, \phi(r))$.

Lemma 2.25. *Let $f : X \to Y$ be a large-scale Lipschitz map, X a CW complex where the size of the cells is globally bounded, and Y is uniformly contractible. Then f is a bounded distance from a Lipshitz map $g : X \to Y$. That is, $\sup_{x\in X} d_Y(f(x), g(x)) \leq C$ for some constant $C > 0$.*

Proof. Let us prove it by induction on the dimension of the cells of X, as suggested in [BW97]. Let $D = \sup_\sigma \operatorname{diam}(\sigma)$ over all the cells of X, and let (λ, c) be the large-scale Lipschitz constants of f. For any vertex $v \in X^{(0)}$, define $g(v) := f(v)$. Given an edge $e = (u, v) \in X^{(1)}$, $f(\partial e) \subset B_Y(f(u), \lambda D + c)$, so $f(\partial e)$ can be filled inside of $B_Y(f(u), \phi(\lambda D + c))$. Let γ be such filling, and define $g(e) := \gamma$ in a continuous way that respects the boundary. Then g ,defined at this stage on $X^{(1)}$, is uniformly close to f on $X^{(1)}$. Indeed, if $x \in X^{(1)}$ is in an edge $e = (u, v)$ so

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), f(u)) + d_Y(g(u), g(x)) \\ &\leq \lambda D + c + \phi(\lambda D + c) \end{aligned}$$

Suppose that g is defined on $X^{(k)}$. Let σ be a $(k+1)$-cell in X, and take $x \in \sigma$. $f(\partial\sigma) \subset B_Y(f(x), \lambda D + c)$, so it can be filled inside of $B_Y(f(x), \phi(\lambda D + c))$. Let Ω be such filling, and define $g(\sigma) := \Omega$ in a continuous way that respects the boundary. Same as before, we can easily check that g is a bounded distance from f. $\square$

This lemma applies to all the coarse embeddings that we will be considering, so we can always assume that they are actually Lipschitz.

Chapter 3

The domain is a product of spaces of exponential growth

The goal of this chapter is to prove Theorem 1.7, which implies Theorem 1.1. We start by giving some useful lemmas about coarse embeddings and coarse preservation of the ε-volume in section 3.1. Then, we will consider Lipschitz maps with sub-Euclidean filling of the image of the boundary of some rectangles in section 3.2, leading to the proof of Theorem 1.7 in section 3.3.

3.1 Coarse preservation of the ε-volume

Let us give some useful lemmas first. One crucial property that we are going to use is that the ε-volume is coarsely preserved by coarse embeddings:

Lemma 3.1. *Let X and Y be metric spaces with bounded geometry, $f : X \to Y$ a coarse embedding, and let $\varepsilon > 0$ big enough. There exist $\alpha, \beta > 0$ such that for all $A \subset X$ with $\mathrm{Vol}_X^\varepsilon(A) < \infty$,*

$$\alpha \, \mathrm{Vol}_X^\varepsilon(A) \leq \mathrm{Vol}_Y^\varepsilon(f(A)) \leq \beta \, \mathrm{Vol}_X^\varepsilon(A).$$

Proof. Let f be such coarse embedding with control functions ρ_- and ρ_+. X and Y have bounded geometry, so there exist R_0 and R_0' as in definition 2.2. Take $\varepsilon \geq \max(R_0, R_0')$.
• Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that

$$A \subset B_X(x_1, \varepsilon) \cup \ldots \cup B_X(x_n, \varepsilon).$$

Then

$$f(A) \subset f(B_X(x_1, \varepsilon)) \cup \ldots \cup f(B_X(x_n, \varepsilon)) \subset B_Y(f(x_1), \rho^+(\varepsilon)) \cup \ldots \cup B_Y(f(x_n), \rho^+(\varepsilon)).$$

Since Y has bounded geometry, there exists $\beta = \mathrm{Vol}_X^\varepsilon(B(\rho^+(\varepsilon))) \in \mathbb{N}$ such that every ball of radius $\rho^+(\varepsilon)$ is in the union of p balls of radius ε. Therefore $f(A)$ is in the union of $\beta \times n$ balls of radius ε. Since its volume is the minimum among such number of balls, we have

$$\mathrm{Vol}_Y^\varepsilon(f(A)) \leq \beta n.$$

By taking n to be minimal, we have

$$\mathrm{Vol}_Y^\varepsilon(f(B_X(x,R))) \leq \beta \, \mathrm{Vol}_X^\varepsilon(B_X(x,R)).$$

• Now let $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in Y$ such that

$$f(A) \subset B_Y(y_1, \varepsilon) \cup \cdots \cup B_Y(y_n, \varepsilon).$$

Then

$$A \subset f^{-1}(f(A)) \subset f^{-1}\left(B_Y(y_1,\varepsilon)\right) \cup \cdots \cup f^{-1}(B_Y(y_n,\varepsilon)\big).$$

Note that for any $i = 1, \ldots, n$, $f^{-1}(B_Y(y_i,\varepsilon))$ has diameter less than $\rho_-^{-1}(2\varepsilon)$: $\forall a, b \in f^{-1}(B_Y(y_i,\varepsilon))$, $d_Y(f(a), f(b)) \leq 2\varepsilon$, so $d_X(a,b) \leq \rho_-^{-1}(2\varepsilon)$. For any $i = 1, \ldots, n$ such that $f^{-1}(B_Y(y_i,\varepsilon))$ is non-empty, take some x_i in it. So $f^{-1}(B_Y(y_i,\varepsilon)) \subset B_X(x_i, \rho_-^{-1}(2\varepsilon))$. Let us denote $D = \rho_-^{-1}(2\varepsilon)$. Therefore

$$A \subset B_X(x_1, D) \cup \ldots \cup B_X(x_n, D).$$

X has bounded geometry, so there exists $\gamma = \mathrm{Vol}_X^\varepsilon(B(D)) \in \mathbb{N}$ such that every ball of radius D is in the union of γ balls of radius D. So A is in the union of $\gamma \times n$ balls of radius ε. Since its volume is the minimum among such number of balls, we have

$$\mathrm{Vol}_X^\varepsilon(A) \leq \gamma n.$$

By taking n to be minimal and by denoting $\alpha = \gamma^{-1}$, we have

$$\mathrm{Vol}_Y^\varepsilon(f(A)) \geq \alpha \, \mathrm{Vol}_X^\varepsilon(A).$$

$\square$

Let us also prove the following lemma.

Lemma 3.2. *Let X be a metric space with bounded geometry, $\varepsilon > 0$ big enough, and let $\delta > 0$. If $A \subset X$ with $\mathrm{Vol}_X^\varepsilon(A) < \infty$, then*

$$\mathrm{Vol}_X^\varepsilon(\mathrm{N}_\delta(A)) \leq \beta_X^\varepsilon(\delta + \varepsilon) \times \mathrm{Vol}_X^\varepsilon(A).$$

Proof. Let $\varepsilon \geq R_0$ as in the previous lemma. X has bounded geometry, so there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that $A \subset B_X(x_1, \varepsilon) \cup \cdots \cup B_X(x_n, \varepsilon)$.
Let $z \in N_\delta(A)$, i.e. there exists $a \in A$ such that $d_X(z, a) \leq \delta$. So there exists $i \in \{1, \ldots, n\}$ such that $d_X(z, x_i) \leq \delta + \varepsilon$, therefore $N_\delta(A) \subset B_X(x_1, \varepsilon + \delta) \cup .. \cup B_X(x_n, \varepsilon + \delta)$.
There exists $p = \beta_X^\varepsilon(\delta + \varepsilon)$ such that every ball of radius $\varepsilon + \delta$ can be covered by p balls of radius ε. Therefore $N_\delta(A)$ can be covered by $p \times n$ balls of radius ε, so

$$\mathrm{Vol}_X^\varepsilon(\mathrm{N}_\delta(A)) \leq \beta_X^\varepsilon(\delta + \varepsilon) \times n.$$

By taking n to be minimal, we have the result. □

3.2 Maps with sub-Euclidean fillings of hyperrectangles

Let us prove a more general result where we do not actually require the k-filling function of the target space to be sub-Euclidean, but we only need a weaker condition on the filling of the boundaries of some k-rectangles.
For all that follows, if f and g are functions from $\mathbb{R}$ to $\mathbb{R}$, we will denote $f \ll g$ if $f = o(g)$ at $+\infty$, and $f \gg g$ if $g = o(f)$ at $+\infty$.

Theorem 3.3. *Let $(\varphi_i)_{i \in \mathbb{N}^*}$ be a sequence of functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\varphi_1(d) = d$ for all d, and for all $i \in \mathbb{N}^*$ $\varphi_i \gg \varphi_{i+1}$, and $\varphi_i(d)$ tends to $+\infty$ at $+\infty$.*
Let $k \geq 1$ be an integer, and let $X = X_1 \times ... \times X_k$ be a product of geodesic metric spaces with bounded geometry, and let Y be a uniformly contractible complete metric space with at most exponential growth. Let $f : X \to Y$ be a large-scale Lipschitz map. If
• for all $i = 1, \ldots, k$, X_i has exponential growth,
• there exists a sublinear function ϕ such that for every $d > 0$ big enough and every k-dimensional rectangle R_d, whose sides are geodesic segments in the X_i's, with side lengths $l_1, \ldots, l_k$ that satisfy $l_1 = \varphi_1(d), l_2 \leq \varphi_2(d), \ldots, l_k \leq \varphi_k(d)$, we have

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}\left(f(\partial R_d)\right) \leq \phi(\varphi_1(d) \times \cdots \times \varphi_k(d)),$$

then f is not a coarse embedding.

Remark 3.4. This condition means that the filling of the image of the boundary of such rectangles in Y is sub-Euclidean, since their filling in the domain is the product of their side lengths.

Remark 3.5. This result can be seen as a generalization of the fact, for $k = 1$, that a space with exponential growth cannot be coarsely embedded with a sublinear ρ^+ in a space with at most exponential growth, because the second condition means that f admits a sublinear ρ^+. Indeed, R_d is just a geodesic segment in X. Let x and y be its extremities. So the second condition can be rewritten as

$$\text{FillVol}_1^{Y,\text{cr}}([\![f(x)]\!] - [\![f(y)]\!]) \leq \phi(d_X(x,y)).$$

By lemma 2.17, we have

$$d_Y(f(x), f(y)) \leq \text{FillVol}_1^{Y,\text{cr}}([\![f(x)]\!] - [\![f(y)]\!]),$$

therefore

$$d_Y(f(x), f(y)) \leq \phi(d_X(x,y)).$$

Proof. Let us prove the theorem by induction on k.

• For $k = 1$, let X be a metric space with exponential growth, and suppose that such a coarse embedding $f : X \to Y$ exists for a sublinear ϕ.

Let $x \in X$ and $R > 0$. Then

$$f(B_X(x,R)) \subset B_Y(f(x), \phi(R)).$$

Let $\varepsilon > 0$. By taking the ε-volumes :

$$\text{Vol}_Y^\varepsilon(f(B_X(x,R))) \leq \text{Vol}_Y^\varepsilon(B_Y(f(x), \phi(R))).$$

Y has at most exponential growth so there exists $\lambda \geq 0$ such that for all $y \in Y$ and for all $R > 0$,

$$\text{Vol}_Y^\varepsilon(B_Y(y,R)) \leq e^{\lambda R}.$$

But, on one hand f is a coarse embedding, so by lemma 3.1 it preserves the volume coarsely. In particular there exists $\alpha > 0$ such that $\forall x \in X$

$$\alpha \, \text{Vol}_X^\varepsilon(B_X(x,R)) \leq \text{Vol}_Y^\varepsilon(f(B_X(x,R))).$$

On the other hand, X has exponential growth, so there exists $\gamma > 0$ such that for all $x \in X$ and for all $R > 0$

$$\text{Vol}_X^\varepsilon(B_X(x,R)) \geq e^{\gamma R}.$$

So we finally get

$$\alpha e^{\gamma R} \leq \alpha \, Vol_X^\varepsilon(B_X(x,R)) \leq \text{Vol}_Y^\varepsilon(f(B_X(x,R))) \leq \text{Vol}_Y^\varepsilon(B_Y(f(x), \phi(R))) \leq e^{\lambda \phi(R)}.$$

Which is not possible for a sublinear function ϕ.

Note that for $k = 1$ we did not require X to be geodesic.

Before proving it for all $k \geq 2$, and just for the sake of clarity, let us first prove it for

$k = 2$. Note that it already contains most of the conceptual difficulties of the general case.

• Let $X = X_1 \times X_2$ be a product of two geodesic metric spaces of exponential growth and Y be a metric space with at most exponential growth. Suppose that $f : X \to Y$ is a coarse embedding that satisfies the second condition, i.e. there exists a sublinear function ϕ such that for every $d > 0$ and every 2-dimensional rectangle R_d, whose sides are geodesic segments in the $X_i\,'s$, of side lengths l_1, l_2 with $l_1 = d$ and $l_2 \leq \varphi_2(d)$, we have

$$\mathrm{FillVol}_2^{Y,\mathrm{cr}}(f(\partial R_d)) \leq \phi\left(d \times \varphi_2(d)\right).$$

Following the case $k = 1$, for all $x_2 \in X_2$, the copy $X_1 \times \{x_2\}$ cannot be sent sublinearly. Therefore, if we fix some $x_2 \in X_2$, there exist $\lambda > 0$ and two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $X_1 \times \{x_2\}$ such that

$$d_n := d_X(a_n, b_n) \underset{n\to+\infty}{\longrightarrow} +\infty \qquad \text{and} \qquad d_Y(f(a_n), f(b_n)) \geq \lambda\, d_X(a_n, b_n).$$

Let $n \in \mathbb{N}$ and take $x_2' \in X_2$ such that $d_{X_2}(x_2, x_2') \leq \varphi_2(d_n)$. Consider $a_n' = (\mathrm{proj}_{X_1}(a_n), x_2')$, and $b_n' = (\mathrm{proj}_{X_1}(b_n), x_2')$.
Let c_1 be a geodesic segment in X_1 going from $\mathrm{proj}_{X_1}(a_n)$ to $\mathrm{proj}_{X_1}(b_n)$, and c_2 be a geodesic segment in X_2 from x_2 to x_2'.
Now consider the four following geodesic segments in X :

$$\gamma_1 = (c_1, x_2), \qquad \gamma_2 = (\mathrm{proj}_{X_1}(b_n), c_2), \qquad \gamma_3 = (-c_1, x_2), \qquad \gamma_4 = (\mathrm{proj}_{X_1}(a_n), -c_2).$$

By concatenating them, i.e. taking their formal sum as Lipschitz chains, we get a Lipschitz 1-cycle that we will denote R_n.
The second condition on the filling volume implies that

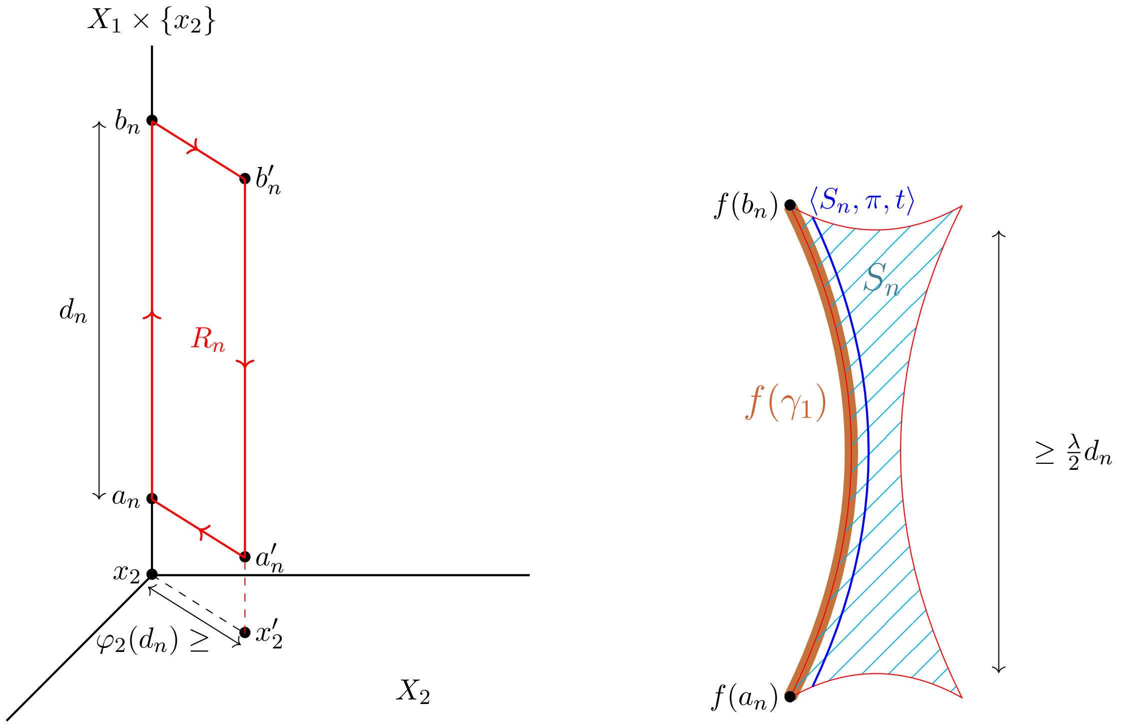
$$\mathrm{FillVol}_2^{Y,\mathrm{cr}}(f(R_n)) \leq \phi\left(d_n\, \varphi_2(d_n)\right).$$

This implies that there exists $S_n \in \mathbf{I}_2(Y)$ in Y such that $\partial S_n = f(R_n)$ and

$$\mathbf{M}(S_n) \leq \phi\left(d_n\, \varphi_2(d_n)\right).$$

Now consider the 1-Lipschitz map $\pi : Y \to \mathbb{R}$, $\pi(y) = d_Y(y, f(\gamma_1))$.
By the Slicing Theorem, we have that for a.e. $t \in \mathbb{R}$, there exists $< S_n, \pi, t > \in \mathbf{I}_1(Y)$ such that $\langle S_n, \pi, t\rangle = \partial(S_n \llcorner \{\pi \leq t\}) - (\partial S_n) \llcorner \{\pi \leq t\}$, and by integrating the co-area formula over the distance t, we have

$$\mathbf{M}(\langle S_n, \pi, t\rangle) \leq \frac{d}{dt}\mathbf{M}(S_n \llcorner \{\pi \leq t\}),$$



(a) R_n in $X_1 \times X_2$ constructed from γ_1 and x'_2

(b) A slice of S_n at distance t

Figure 3.1: The cycle R_n in X, and a slice of the filling of its image in Y

$$\int_0^{+\infty} \mathbf{M}(\langle S_n, \pi, t\rangle) dt \leq \int_0^{+\infty} \frac{d}{dt} \mathbf{M}(S_n \llcorner \{\pi \leq t\}),$$

$$\int_0^{+\infty} \mathbf{M}(\langle S_n, \pi, t\rangle) dt \leq \mathbf{M}(S_n).$$

Since $\mathbf{M}(S_n) \leq \phi\left(d_n\, \varphi_2(d_n)\right)$, we get

$$\int_0^{D} \mathbf{M}(\langle S_n, \pi, t\rangle) dt \leq \int_0^{+\infty} \mathbf{M}(\langle S_n, \pi, t\rangle) dt \leq \phi\left(d_n\, \varphi_2(d_n)\right), \tag{3.2.1}$$

where $D = d_Y(f(\gamma_1), f(\gamma_3))$. However, for a.e. $t \in]0, D[$, $\mathbf{M}(\langle S_n, \pi, t\rangle)$ cannot be too small since the current $\langle S_n, \pi, t\rangle$ almost gives a filling for the 0-cycle $[\![f(b_n)]\!] - [\![f(a_n)]\!]$.

Claim 3.6. For n big enough and for a.e. $t \in]0, D[$, $\mathbf{M}(\langle S_n, \pi, t\rangle) \geq \frac{\lambda}{2} d_n$.

Proof. For a.e. $t \in]0, D[$,

$$\partial(S_n) \llcorner \{\pi \leq t\} = \partial(S_n) \llcorner \{\pi = 0\} + \partial(S_n) \llcorner \{0 < \pi \leq t\}$$

$$= f(\gamma_1) + H_t.$$

Where H_t is the 1-current $\partial(S_n) \llcorner \{0 < \pi \leq t\}$.

Since $\|\langle S_n, \pi, t\rangle\|$ is concentrated on $\pi^{-1}(\{t\})$,

$$\partial(S_n \llcorner \{\pi \leq t\}) = \langle S_n, \pi, t\rangle + f(\gamma_1) + H_t.$$

Which means that $\langle S_n, \pi, t\rangle + f(\gamma_1) + H_t$ is a 1-current that is actually a cycle.
So $-(\langle S_n, \pi, t\rangle + H_t)$ is a 1-chain that fills the 0-cycle $\partial f(\gamma_1) = f(\partial \gamma_1) = [\![f(b_n)]\!] - [\![f(a_n)]\!]$.
Therefore

$$\mathbf{M}(-(\langle S_n, \pi, t\rangle + H_t)) \geq \mathrm{FillVol}_1^{Y,\mathrm{cr}}([\![f(b_n)]\!] - [\![f(a_n)]\!]).$$

Since, by the lemma 2.17 , $\mathrm{FillVol}_1^{Y,\mathrm{cr}}([\![f(b_n)]\!] - [\![f(a_n)]\!]) \geq d_Y(f(a_n), f(b_n))$, we have

$$\mathbf{M}(-(\langle S_n, \pi, t\rangle + H_t)) \geq \lambda\, d_n.$$

So

$$\mathbf{M}(\langle S_n, \pi, t\rangle) + \mathbf{M}(H_t) \geq \lambda\, d_n.$$

Note that

$$\begin{aligned} H_t &= \partial S_n \llcorner \{0 < \pi \leq t\} \\ &= \big(f(\gamma_2) + f(\gamma_4)\big) \llcorner \{0 < \pi \leq t\}, \end{aligned}$$

because $0 < \pi < D$. So

$$H_t = f(\gamma_2) \llcorner \{0 < \pi \leq t\} + f(\gamma_4) \llcorner \{0 < \pi \leq t\}.$$

By taking the mass

$$\begin{aligned} \mathbf{M}(H_t) &\leq \mathbf{M}(f(\gamma_2) \llcorner \{0 < \pi \leq t\}) + \mathbf{M}(f(\gamma_4) \llcorner \{0 < \pi \leq t\}) \\ &\leq \mathbf{M}(f(\gamma_2)) + \mathbf{M}(f(\gamma_4)) \\ &\leq \mathrm{Lip}(f)\big(\mathbf{M}(\gamma_2) + \mathbf{M}(\gamma_4)\big). \end{aligned}$$

Since the $\gamma_i' s$ are Lipschitz chains, their mass is equal to their volume.
So $\mathbf{M}(\gamma_2) = \mathbf{M}(\gamma_4) \leq \varphi_2(d_n)$, and $\mathbf{M}(H_t) \leq 2\mathrm{Lip}(f)\varphi_2(d_n)$.

Therefore, since $\varphi_2(d) \ll d$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for a.e. $t \in]0, D[$,

$$\mathbf{M}(\langle S_n, \pi, t\rangle) \geq \lambda d_n - 2\mathrm{Lip}(f)\varphi_2(d_n) \geq \frac{\lambda}{2} d_n. \qquad \square$$

Thus, by (3.2.1),

$$\begin{aligned}\phi\left(d_n\,\varphi_2(d_n)\right) &\geq \int_0^D \mathbf{M}(\langle S_n,\pi,t\rangle)\,dt\\ &\geq D\,\frac{\lambda}{2}d_n.\end{aligned}$$

Which implies that

$$D \leq \frac{2}{\lambda}\frac{\phi\left(d_n\,\varphi_2(d_n)\right)}{d_n}.$$

Let us denote $\psi(\varphi_2(d_n)) = \frac{2}{\lambda}\frac{\phi(d_n\,\varphi_2(d_n))}{d_n}$. Note that ψ is sublinear.

Since $D = d_Y(f(\gamma_1), f(\gamma_3))$, the last inequality implies that there exists $z \in X$ such that $z \in \gamma_3$ and $d_Y(f(\gamma_1), f(z)) \leq \psi(\varphi_2(d_n))$.
But $z \in \gamma_3$ implies that $d_X(z, \gamma_1) = d_X(x_2, x_2')$.

By doing this process for all $n \geq N$ and all $x_2' \in B_{X_2}(x_2, \varphi_2(d_n))$, we get subsets $C_n \subset X$ that projects onto $B_{X_2}(x_2, \varphi_2(d_n))$, i.e.

$$B_{X_2}(x_2, \varphi_2(d_n)) \subset \mathrm{proj}_{X_2}(C_n),$$

and such that

$$f(C_n) \subset N_{\psi(\varphi_2(d_n))}(f(\gamma_1)), \tag{3.2.2}$$

since $\forall z \in C_n$, $d_Y(f(z), f(\gamma_1)) \leq \psi(\varphi_2(d_n))$.

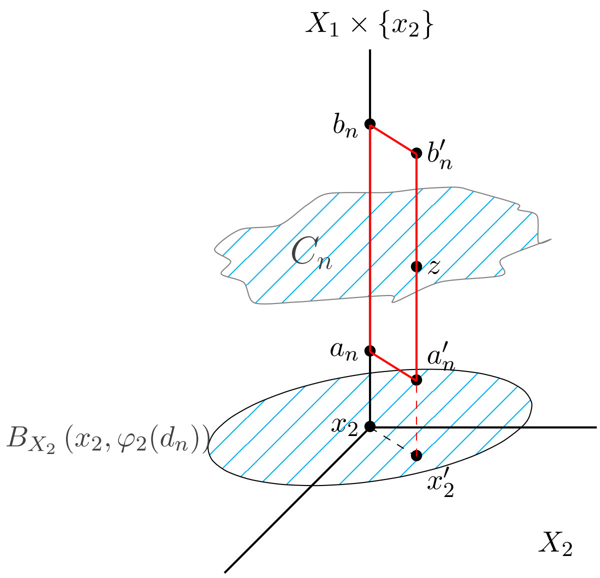


Figure 3.2: The subsets C_n in X that are highly compressed by f

The projection onto X_2 is 1-Lipschitz, so if we fix $\varepsilon > 0$,

$$\mathrm{Vol}_X^\varepsilon(C_n) \geq \mathrm{Vol}_X^\varepsilon(\mathrm{proj}_{X_2}(C_n)) \geq \mathrm{Vol}_X^\varepsilon\left(B_{X_2}\left(x_2, \varphi_2(d_n)\right)\right).$$

X_2 has exponential growth, so there exists $\alpha > 0$ such that $\forall R > 0$, $\mathrm{Vol}_{X_2}^\varepsilon(B_{X_2}(R)) \geq e^{\alpha R}$. So

$$\mathrm{Vol}_X^\varepsilon(C_n) \geq e^{\alpha\, \varphi_2(d_n)}.$$

While the equation (3.2.2) implies that

$$\mathrm{Vol}_Y^\varepsilon(f(C_n)) \leq \mathrm{Vol}_Y^\varepsilon(N_{\psi(\varphi_2(d_n))}(f(\gamma_1))).$$

But, like we saw for the case $k = 1$, f coarsely preserves volumes, i.e. there exist $\delta, \delta' > 0$ such that

$$\delta\, \mathrm{Vol}_X^\varepsilon(C_n) \leq \mathrm{Vol}_Y^\varepsilon(f(C_n)) \leq \delta'\, \mathrm{Vol}_X^\varepsilon(C_n).$$

Also, by lemma 3.2

$$\mathrm{Vol}_Y^\varepsilon\left(N_{\psi(\varphi_2(d_n))}(f(\gamma_1))\right) \leq \beta_Y^\varepsilon\left(\varepsilon + \psi(\varphi_2(d_n)\right) \times \mathrm{Vol}_Y^\varepsilon(f(\gamma_1)).$$

Y has at most exponential growth, so there exists $\beta > 0$ such that for all $R > 0$, $\beta_Y^\varepsilon\left(R\right) \leq e^{\beta R}$. Therefore, we have on one hand

$$\beta_Y^\varepsilon\left(\varepsilon + \psi(\varphi_2(d_n)\right) \leq \exp\left(\varepsilon + \psi(\varphi_2(d_n))\right).$$

On the other hand, by taking a partition of the geodesic segment γ_1 into sub-intervals of length ε, we get :

$$\mathrm{Vol}_X^\varepsilon(\gamma_1) \leq \frac{d_n}{\varepsilon} + 1 \leq \frac{2d_n}{\varepsilon}.$$

Hence

$$\mathrm{Vol}_Y^\varepsilon(f(\gamma_1)) \leq \delta'\, \mathrm{Vol}_X^\varepsilon(\gamma_1) \leq \frac{2\delta'}{\varepsilon} d_n.$$

So we have

$$\mathrm{Vol}_Y^\varepsilon\left(N_{\psi(\varphi_2(d_n))}(f(\gamma_1))\right) \leq \exp\left(\varepsilon + \psi(\varphi_2(d_n))\right) \times \frac{2\delta'}{\varepsilon} d_n.$$

We conclude from the previous inequalities that

$$\delta e^{\alpha\, \varphi_2(d_n)} \leq \delta\, \mathrm{Vol}_X^\varepsilon(C_n) \leq \mathrm{Vol}_Y^\varepsilon(f(C_n)) \leq \exp\left(\varepsilon + \psi(\varphi_2(d_n))\right) \times \frac{2\delta'}{\varepsilon} d_n.$$

Which implies finally that for all $n \geq N$

$$\delta \exp\left(\alpha\, \varphi_2(d_n)\right) \leq \exp\left(\varepsilon + \psi(\varphi_2(d_n))\right) \times \frac{2\delta'}{\varepsilon} d_n.$$

Which is not possible when $d_n \to \infty$ because ψ is sublinear. This completes the proof for $k = 2$.

• Suppose now that the result holds for some $k \geq 2$, and let us prove it for $k+1$.
Suppose that there is a coarse embedding $f : X = X_1 \times \cdots \times X_{k+1} \to Y$ that satisfies the second condition of the theorem, and let us fix some $x_{k+1} \in X_{k+1}$.
The induction hypothesis implies that the restriction of f to the copy $X_1 \times \cdots \times X_k \times \{x_{k+1}\}$ of $X_1 \times \cdots \times X_k$ does not satisfy the second condition of the theorem. Which means that there exists a sequence $(d_n)_n$ such that $d_n \to \infty$ and a sequence of k-dimensional rectangles R_{d_n},whose sides are geodesic segments in the X_i's, of lengths $l_1, \ldots, l_k$, such that $l_1 = d_n, l_2 \leq \varphi_2(d_n), \ldots, l_k \leq \varphi_k(d_n)$ and there exists $\lambda > 0$ such that for all n

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(\partial R_{d_n})) \geq \lambda\, d_n \times \varphi_2(d_n) \times \cdots \times \varphi_k(d_n).$$

Let $n \in \mathbb{N}$ and let us take $x'_{k+1} \in X_{k+1}$ such that $d_{X_{k+1}}(x_{k+1}, x'_{k+1}) \leq \varphi_{k+1}(d_n)$, and consider the $(k+1)$-dimensional rectangle $R'_{d_n} = R_{d_n} \times [x_{k+1}, x'_{k+1}]$.
Since f satisfies the second condition of the theorem, there exists a sublinear function ϕ such that

$$\mathrm{FillVol}_{k+1}^{Y,\mathrm{cr}}(f(\partial R'_{d_n})) \leq \phi(d_n \times \varphi_2(d_n) \times \cdots \times \varphi_{k+1}(d_n)).$$

This implies that there exists $V_n \in \mathbf{I}_{k+1}(Y)$ in Y such that $\partial V_n = f(\partial R'_{d_n})$ and

$$\mathbf{M}(V_n) \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right).$$

Now consider the 1-Lipschitz map $\pi : Y \to \mathbb{R}$, $\pi(y) = d_Y\big(y, f(R_{d_n} \times \{x_{k+1}\})\big)$, i.e. the map that gives the distance from the image of the basis of the $(k+1)$-dimensional rectangle R'_{d_n}.
Again, by the Slicing Theorem, we have that for a.e. $t \in \mathbb{R}$, there exists $\langle V_n, \pi, t\rangle \in \mathbf{I}_k(Y)$ such that $\langle V_n, \pi, t\rangle = \partial(V_n \llcorner \{\pi \leq t\}) - (\partial V_n) \llcorner \{\pi \leq t\}$, and by integrating the co-area formula over the distance t, we have

$$\int_0^{+\infty} \mathbf{M}(\langle V_n, \pi, t\rangle) dt \leq \mathbf{M}(V_n).$$

Since $\mathbf{M}(V_n) \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right)$, we get

$$\int_0^{D} \mathbf{M}(\langle V_n, \pi, t\rangle) dt \leq \int_0^{+\infty} \mathbf{M}(\langle V_n, \pi, t\rangle) dt \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right), \tag{3.2.3}$$

where $D = d_Y\big(f(R_{d_n} \times \{x_{k+1}\}), f(R_{d_n} \times \{x'_{k+1}\})\big)$.

However, for a.e. $t \in]0, D[$, $\mathbf{M}(\langle V_n, \pi, t\rangle)$ cannot be too small since $\langle V_n, \pi, t\rangle$ almost gives a filling of the $(k-1)$-cycle $\partial f(R_{d_n} \times \{x_{k+1}\}) = f(\partial(R_{d_n} \times \{x_{k+1}\}))$, that we will just denote by $f(\partial R_{d_n})$. We will also denote $R_{d_n} \times \{x_{k+1}\}$ by R_{d_n}.

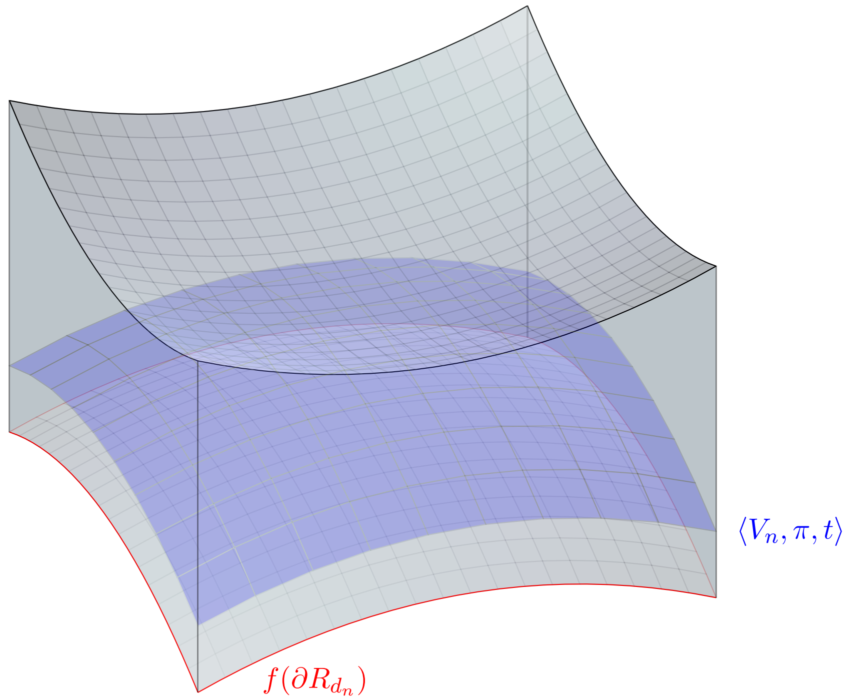


Figure 3.3: A slice of V_n at distance t from $f(R_{d_n})$

Claim 3.7. For n big enough and for a.e. $t \in]0, D[$,

$$\mathbf{M}(\langle V_n, \pi, t\rangle) \geq \frac{\lambda}{2} \prod_{i=1}^{k} \varphi_i(d_n).$$

Proof. For a.e. $t \in]0, D[$,

$$\partial(V_n) \llcorner \{\pi \leq t\} = \partial(V_n) \llcorner \{\pi = 0\} + \partial(V_n) \llcorner \{0 < \pi \leq t\}$$

$$= f(R_{d_n}) + H_t.$$

Where H_t is the k-current $\partial(V_n) \llcorner \{0 < \pi \leq t\}$.

Since $\|\langle V_n, \pi, t\rangle\|$ is concentrated on $\pi^{-1}(\{t\})$,

$$\partial(V_n \llcorner \{\pi \leq t\}) = \langle V_n, \pi, t\rangle + f(R_{d_n}) + H_t.$$

Which means that $\langle V_n, \pi, t\rangle + f(R_{d_n}) + H_t$ is actually a cycle.
So $-(\langle V_n, \pi, t\rangle + H_t)$ is a k-chain that fills the $(k-1)$-cycle $\partial f(R_{d_n}) = f(\partial R_{d_n})$.
Therefore, by the induction hypothesis

$$\mathbf{M}(-(\langle V_n, \pi, t\rangle + H_t)) \geq \mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(\partial R_{d_n})) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n).$$

So

$$\mathbf{M}(\langle V_n, \pi, t\rangle) + \mathbf{M}(H_t) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n).$$

Note that

$$\begin{aligned} H_t &= \partial(V_n) \llcorner \{0 < \pi \leq t\} \\ &= \big(\sum_{F \in \Delta} f(F)\big) \llcorner \{0 < \pi \leq t\}, \end{aligned}$$

where Δ is the set of side faces of R'_{d_n}, i.e. faces of R'_{d_n} except $R_{d_n} \times \{x_{k+1}\}$ and $R_{d_n} \times \{x'_{k+1}\}$, because $0 < \pi < D$. So by taking the mass

$$\begin{aligned} \mathbf{M}(H_t) &= \mathbf{M}(\big(\sum_{F \in \Delta} f(F)\big) \llcorner \{0 < \pi \leq t\}) \\ &\leq \sum_{F \in \Delta} \mathbf{M}((f(F)) \llcorner \{0 < \pi \leq t\}) \\ &\leq \sum_{F \in \Delta} \mathbf{M}(f(F)) \\ &\leq \mathrm{Lip}(f) \sum_{F \in \Delta} \mathbf{M}(F). \end{aligned}$$

Since every side face F satisfies $\mathbf{M}(F) \leq \frac{\prod_{i=1}^{k+1} \varphi_i(d_n)}{\varphi_s(d_n)}$, where $s \in \{1, \ldots, k\}$, so every $F \in \Delta$ satisfies $\mathbf{M}(F) \leq \frac{\prod_{i=1}^{k+1} \varphi_i(d_n)}{\varphi_k(d_n)}$. There are $2k$ side faces, So

$$\mathbf{M}(H_t) \leq 2k \, \mathrm{Lip}(f) \prod_{\substack{i=1 \\ i \neq k}}^{k+1} \varphi_i(d_n).$$

Hence

$$\begin{aligned}\mathbf{M}(\langle V_n,\pi,t\rangle) &\geq \lambda \prod_{i=1}^{k}\varphi_i(d_n) - \mathbf{M}(H_t)\\ &\geq \lambda \prod_{i=1}^{k}\varphi_i(d_n) - 2k\,\mathrm{Lip}(f)\prod_{\substack{i=1\\ i\neq k}}^{k+1}\varphi_i(d_n).\end{aligned}$$

Since $\varphi_{k+1} \ll \varphi_k$, it implies that

$$\prod_{\substack{i=1\\ i\neq k}}^{k+1}\varphi_i(d_n) \ll \prod_{i=1}^{k}\varphi_i(d).$$

So by taking n big enough we have

$$\lambda \prod_{i=1}^{k}\varphi_i(d_n) - 2k\,\mathrm{Lip}(f)\prod_{\substack{i=1\\ i\neq k}}^{k+1}\varphi_i(d_n) \geq \frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n).$$

We conclude that for sufficiently big n and for a.e. $t \in]0, D[$

$$\mathbf{M}(\langle V_n,\pi,t\rangle) \geq \frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n). \qquad \square$$

Therefore, by (3.2.3)

$$\begin{aligned}\phi\left(\prod_{i=1}^{k+1}\varphi_i(d_n)\right) &\geq \int_0^D \mathbf{M}(\langle V_n,\pi,t\rangle)dt\\ &\geq D\,\frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n).\end{aligned}$$

This implies that

$$\frac{2}{\lambda}\frac{\phi\left(\prod_{i=1}^{k+1}\varphi_i(d_n)\right)}{\prod_{i=1}^{k}\varphi_i(d_n)} \geq D.$$

Denote $\psi(\varphi_{k+1}(d_n)) = \frac{2}{\lambda}\frac{\phi(\prod_{i=1}^{k+1}\varphi_i(d_n))}{\prod_{i=1}^{k}\varphi_i(d_n)}$. Note that ψ is sublinear : $\psi(\varphi_{k+1}(d_n)) = o(\varphi_{k+1}(d_n))$ because ϕ is sublinear.

Since $D = d_Y(f(R_{d_n}\times\{x_{k+1}\}), f(R_{d_n}\times\{x'_{k+1}\}))$, the last inequality implies that $\exists z \in X$ such that $z \in R_{d_n}\times\{x'_{k+1}\}$ and $d_Y(f(R_{d_n}\times\{x_{k+1}\}), f(z)) \leq \psi(\varphi_{k+1}(d_n))$.
But $z \in R_{d_n}\times\{x'_{k+1}\}$ implies that $d_X(z, R_{d_n}\times\{x_{k+1}\}) = d_{X_{k+1}}(x_{k+1}, x'_{k+1})$.

By doing this process for all $x'_{k+1} \in B_{X_{k+1}}(x_{k+1}, \varphi_{k+1}(d_n))$, we get a subset $C_n \subset X$ that projects onto $B_{X_{k+1}}(x_{k+1}, \varphi_{k+1}(d_n))$, i.e.

$$B_{X_{k+1}}(x_{k+1}, \varphi_{k+1}(d_n)) \subset \mathrm{proj}_{X_{k+1}}(C_n), \tag{3.2.4}$$

and such that

$$f(C_n) \subset N_{\psi(\varphi_{k+1}(d_n))}(f(R_{d_n} \times \{x_{k+1}\})), \tag{3.2.5}$$

since $\forall z \in C_n$, $d_Y(f(z), f(R_{d_n} \times \{x_{k+1}\})) \leq \psi(\varphi_{k+1}(d_n))$.
The projection onto X_{k+1} is 1-Lipschitz, so (3.2.4) implies that, if we fix $\varepsilon > 0$,

$$\mathrm{Vol}^{\varepsilon}_X(C_n) \geq \mathrm{Vol}^{\varepsilon}_X(\mathrm{proj}_{X_{k+1}}(C_n)) \geq \mathrm{Vol}^{\varepsilon}_X(B_{X_{k+1}}(x_{k+1}, \varphi_{k+1}(d_n))).$$

X_{k+1} has exponential growth, so $\exists \alpha > 0$ such that $\forall R > 0$, $\mathrm{Vol}^{\varepsilon}_{X_{k+1}}(B_{X_{k+1}}(R)) \geq e^{\alpha R}$. So

$$\mathrm{Vol}^{\varepsilon}_X(C_n) \geq \exp(\alpha\, \varphi_{k+1}(d_n)).$$

While (3.2.5) implies that

$$\mathrm{Vol}^{\varepsilon}_Y(f(C_n)) \leq \mathrm{Vol}^{\varepsilon}_Y\left(N_{\psi(\varphi_{k+1}(d_n))}(f(R_{d_n} \times \{x_{k+1}\}))\right).$$

Since f coarsely preserves volumes, there exist $\delta, \delta' > 0$ such that

$$\delta\, \mathrm{Vol}^{\varepsilon}_X(C_n) \leq \mathrm{Vol}^{\varepsilon}_Y(f(C_n)) \leq \delta'\, \mathrm{Vol}^{\varepsilon}_X(C_n).$$

By lemma 3.2, we have

$$\mathrm{Vol}^{\varepsilon}_Y\left(N_{\psi(\varphi_{k+1}(d_n))}(f(R_{d_n} \times \{x_{k+1}\}))\right) \leq \beta^{\varepsilon}_Y\,(\varepsilon + \psi(\varphi_{k+1}(d_n)) \times \{x_{k+1}\})).$$

Y has at most exponential growth, so there exists $\beta > 0$ such that for all $R > 0$,

$$\beta^{\varepsilon}_Y\,(R) \leq e^{\beta R}.$$

In particular, we have on hand that

$$\beta^{\varepsilon}_Y\,(\varepsilon + \psi(\varphi_{k+1}(d_n)) \leq \exp\left(\beta(\varepsilon + \psi(\varphi_{k+1}(d_n)))\right) \leq \exp\left(2\beta\psi(\varphi_{k+1}(d_n))\right).$$

On the other hand, by taking a partition of each side vector of R_{d_n} into sub-intervals of length ε, we get :

$$\mathrm{Vol}^{\varepsilon}_X(R_{d_n} \times \{x_{k+1}\}) \leq \prod_{i=1}^{k}\left(\frac{\varphi_i(d_n)}{\varepsilon} + 1\right) \leq \prod_{i=1}^{k}\left(\frac{2\varphi_i(d_n)}{\varepsilon}\right) \leq (2/\varepsilon)^k \prod_{i=1}^{k} \varphi_i(d_n).$$

So, by denoting $A = \left(2/\varepsilon\right)^k$, we have

$$\begin{aligned} \mathrm{Vol}_Y^\varepsilon(f(R_{d_n} \times \{x_{k+1}\})) &\le \delta' \,\mathrm{Vol}_X^\varepsilon\left(R_{d_n} \times \{x_{k+1}\}\right) \\ &\le \delta' A \prod_{i=1}^{k} \varphi_i(d_n). \end{aligned}$$

Therefore

$$\mathrm{Vol}_Y^\varepsilon\left(N_{\psi(\varphi_{k+1}(d_n))}(f(R_{d_n} \times \{x_{k+1}\}))\right) \le \exp(2\beta\psi(\varphi_{k+1}(d_n)) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

We conclude from all the previous inequalities that

$$\delta \exp(\alpha\, \varphi_{k+1}(d_n)) \le \delta\, \mathrm{Vol}_X^\varepsilon(C_n) \le \mathrm{Vol}_Y^\varepsilon(f(C_n)) \le \exp(2\beta\psi(\varphi_{k+1}(d_n)) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

Which implies finally that

$$\delta \exp(\alpha\, \varphi_{k+1}(d_n)) \le \exp(2\beta\psi(\varphi_{k+1}(d_n))) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

Which is not possible when $d_n \to \infty$ because ψ is sublinear. This completes the proof. $\square$

3.3 Proof of Theorem 1.7

Let us show that Theorem 1.7 follows from Theorem 3.3.
Let $X = X_1 \times \cdots \times X_k$ be a product of geodesic metric spaces with exponential growth, and let Y be a uniformly contractible complete metric space with at most exponential growth such that $\mathrm{FV}_k^Y(l) = o\left(l^{\frac{k}{k-1}}\right)$. Let us show that there exists a sequence $(\varphi_i)_i$ as in Theorem 3.3 such that for every k-dimensional rectangle R_d, whose sides are geodesic segments in the X_i's, with side lengths $l_1, \ldots, l_k$ that satisfy $l_1 = \varphi_1(d), l_2 \le \varphi_2(d), \ldots, l_k \le \varphi_k(d)$, we have

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(\partial R_d)) \ll \varphi_1(d) \times \cdots \times \varphi_k(d).$$

Actually, we do not need an infinite sequence $(\varphi_i)_{i\in\mathbb{N}^*}$ but only $\varphi_1, \ldots, \varphi_k$ that satisfy the same conditions, because we proceed by induction from $p = 1$ to $p = k$.

$\mathrm{FV}_k^Y(l) = o\left(l^{\frac{k}{k-1}}\right)$ implies that there exists a function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that $a = o(1)$, and $\mathrm{FV}_k^Y(l) \le l^{\frac{k}{k-1}} a(l)$ for all $l > 0$. Without loss of generality, we can assume that the function a is slowly decreasing: it is deceasing and that $a(d) \ge \frac{1}{d}$ for all $d > 0$.
Let us consider the following sequence: for all $i \in \mathbb{N}^*$, $\alpha_i = 1 - \frac{1}{i}$. Let us denote its partial

sums by $S_n = \sum_{i=1}^{n} \alpha_i = n - H_n$, where H_n is the partial sum of the harmonic series.

We define the sequence $(\varphi_i)_{i\in\mathbb{N}^*}$ as follows: for all $i \in \mathbb{N}^*$, and for all $d > 0$, $\varphi_i(d) = a(d)^{\alpha_i} d$. Therefore $\varphi_1(d) = d$ for all $d > 0$. It is clear that $\varphi_i \gg \varphi_{i+1}$ for all $i \in \mathbb{N}^*$. We also have that $\varphi_i(d)$ tends to $+\infty$ for all $i \in \mathbb{N}^*$. Indeed, for all $d > 0$, $a(d) \geq \frac{1}{d}$, so $a(d)^{\alpha_i} \geq \frac{1}{d^{\alpha_i}}$. Therefore $\varphi_i(d) \geq d^{\frac{1}{i}}$ that tends to $+\infty$.

Let us now show the inequality.

$$\begin{aligned}\mathrm{FillVol}_k^{Y,\mathrm{cr}}\left(f(\partial R_d)\right) &\leq \mathrm{FV}_k^Y\left(\mathrm{Vol}_{k-1}^Y\left(f(\partial R_d)\right)\right)\\ &\leq \mathrm{FV}_k^Y\left(2k\mathrm{Lip}(f) l_1 \times \cdots \times l_{k-1}\right).\end{aligned}$$

Without loss of generality, we can assume that $\mathrm{Lip}(f) = \frac{1}{2k}$. So

$$\begin{aligned}\mathrm{FillVol}_k^{Y,\mathrm{cr}}\left(f(\partial R_d)\right) &\leq \mathrm{FV}_k^Y\left(\varphi_1(d) \times \cdots \times \varphi_{k-1}(d)\right)\\ &\leq \left(\varphi_1(d) \times \cdots \times \varphi_{k-1}(d)\right)^{\frac{k}{k-1}} a\left(\varphi_1(d) \times \cdots \times \varphi_{k-1}(d)\right).\end{aligned}$$

Note that $\varphi_1(d) \times \cdots \times \varphi_{k-1}(d) = d^{k-1} a(d)^{S_{k-1}}$. So, we have on one hand

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}\left(f(\partial R_d)\right) \leq d^k a(d)^{\frac{k}{k-1} S_{k-1}} a\left(d^{k-1} a(d)^{S_{k-1}}\right).$$

On the other hand

$$\varphi_1(d) \times \cdots \times \varphi_k(d) = d^k a(d)^{S_k}.$$

Therefore, to get the desired inequality, let us show that

$$a\left(d^{k-1} a(d)^{S_{k-1}}\right) \ll a(d)^{S_k - \frac{k}{k-1} S_{k-1}}.$$

• Let us first consider the left hand side. For all $d > 0$, $a(d) \geq \frac{1}{d}$, so $d^{k-1} a(d)^{S_{k-1}} \geq d^{k-1-S_{k-1}} = d^{H_{k-1}}$. For $k \geq 2$, $H_{k-1} \geq 1$. So $d^{k-1} a(d)^{S_{k-1}} \geq d$. Since a is decreasing, we get for all $k \geq 2$ and for all $d > 0$

$$a\left(d^{k-1} a(d)^{S_{k-1}}\right) \leq a(d).$$

• Let us now consider the right hand side. We have

$$\begin{aligned} S_k - \frac{k}{k-1}S_{k-1} &= k - H_k - \frac{k}{k-1}((k-1) - H_{k-1}) \\ &= -H_k + \frac{k}{k-1}(H_k - \frac{1}{k}) \\ &= -H_k + \frac{k}{k-1}H_k - \frac{1}{k-1} \\ &= \frac{1}{k-1}(H_k - 1). \end{aligned}$$

For all $k \geq 2$, $H_k < k$. So $\frac{1}{k-1}(H_k - 1) < 1$. Therefore, for all $k \geq 2$, $S_k - \frac{k}{k-1}S_{k-1} < 1$. This implies that

$$a(d) \ll a(d)^{S_k - \frac{k}{k-1}S_{k-1}}.$$

We conclude that

$$a\left(d^{k-1}a(d)^{S_{k-1}}\right) \ll a(d)^{S_k - \frac{k}{k-1}S_{k-1}}.$$

This concludes the proof of the theorem.

Chapter 4

The domain is a product of symmetric spaces and Euclidean buildings

4.1 Preliminary results on Weyl sectors

The goal of this section is to prove a structure result for Weyl sectors in model spaces. In both symmetric spaces and Euclidean buildings, singular hyperplanes in a maximal flat give rise to a root system. Let us first define the Weyl sectors associated to a root system.

Definition 4.1. Let Φ be a root system in a Euclidean space $(E, \langle,\rangle)$ (see [Bou81] Chap.6), and let Δ be a basis of Φ. Consider the hyperplane perpendicular to each root in Δ. The complement of this finite set of hyperplanes is disconnected, and each connected component is called a *Weyl sector*. Equivalently, we may define Weyl sectors as the equivalence classes of the following relation:

$$u \sim v \quad \text{if} \quad \langle \alpha, u\rangle \cdot \langle \alpha, v\rangle > 0 \qquad \text{for every } \alpha \in \Delta.$$

Remark 4.2. They are also called Weyl chambers in the literature. We prefer Weyl sectors to avoid any confusion with chambers in Euclidean buildings.

Let us now define the root system associated to a maximal flat in a model space.

• Let X be a Euclidean building of rank p. Let $x_0 \in X$ be a vertex, and let A be an apartment containing it. Let $\mathcal{H}$ denote the set of walls in A that contains x_0. Recall that $\mathcal{H}$ is stable under reflections by walls in $\mathcal{H}$. So if we take for each wall in $\mathcal{H}$ an orthogonal vector with a suitable size, we get a root system. The *Weyl sectors with tip at x_0 of the apartment A* are defined as the Weyl sectors of this root system. They are also the connected components of $A - \cup_{h\in\mathcal{H}} h$.

• Let M is a symmetric space of non-compact type of rank p. A reference for what follows is [Ebe96], Chap.2.
Let $x_0 \in M$ and let F be a a maximal flat containing x_0. Let $G = \mathrm{Isom}_0(M)$ be the identity component of the isometry group of M, $\mathfrak{g}$ its Lie algebra and $K = \mathrm{Stab}_G(x_0)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition determined by x_0, and $\theta = \theta_{x_0}$ be the Cartan involution of $\mathfrak{g}$ induced by x_0.
Consider $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace such that $F = \exp(\mathfrak{a}).x_0$, and the root space decomposition of $\mathfrak{g}$ determined by $\mathfrak{a}$

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{o}} \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

where $\Lambda \subset \mathfrak{a}^*$ is a finite subset called the restricted root system determined by $\mathfrak{a}$, and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [A, X] = \alpha(A)X,\ \forall A \in \mathfrak{a}\}.$$

The bilinear form $\langle X, Y\rangle := -B(\theta(X), Y)$, where B is the Killing form of $\mathfrak{g}$, defines a positive definite inner product on $\mathfrak{g}$. This inner product induces an isomorphism between $\mathfrak{a}$ and $\mathfrak{a}^*$, so for all $\alpha \in \Lambda$, let us denote h_α the unique vector in $\mathfrak{a}$ such that we have

$$\alpha(A) = \langle h_\alpha, A\rangle, \qquad \forall A \in \mathfrak{a}.$$

A *wall* in F, or equivalently in $\mathfrak{a}$, is a hyperplane of the form $\ker(\alpha)$. Note that we can identify $\mathfrak{a}$ with F by the exponential map composed with the orbital map at x_0, which is an isometry when restricted to $\mathfrak{a}$. For every $\alpha \in \Lambda$, consider the reflection S_α in the hyperplane $\mathfrak{a}_\alpha = \ker(\alpha)$, that is orthogonal to h_α. In particular, we have $S_\alpha(A) = A - 2\frac{\langle h_\alpha, A\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha$, $\forall A \in \mathfrak{a}$.
The collection $\{h_\alpha, \alpha \in \Lambda\}$ form a root system. We define the *Weyl sectors with tip at x_0 of the maximal flat F* as the Weyl sectors of this root system. They are also the connected components of $\mathfrak{a} \setminus \cup_{\alpha\in\Lambda} \ker(\alpha)$.

Proposition 4.3. *Let X be a model space. Then its Weyl sectors are acute open simplicial cones generated by maximally singular vectors. In other words, if $x_0 \in X$ and F is a maximal flat containing x_0, and if C is a Weyl sector of F with tip at x_0, then after identifying x_0 with the origin of F:*
• *There exist $u_1, \dots, u_p$ maximally singular vectors in $\overline{C}$ such that $\forall x \in C$, $\exists\, x_1, \dots, x_p \geq 0$ such that $x = \sum_{i=1}^p x_i u_i$, where $p = \mathrm{rank}\ X$.*
• *For all $x, y \in C$, $\quad \langle x, y\rangle > 0$.*

Proof. Since maximal flats in a product space are products of maximal flats in the factors, Weyl sectors of a product space are products of Weyl sectors of the factors. Therefore,

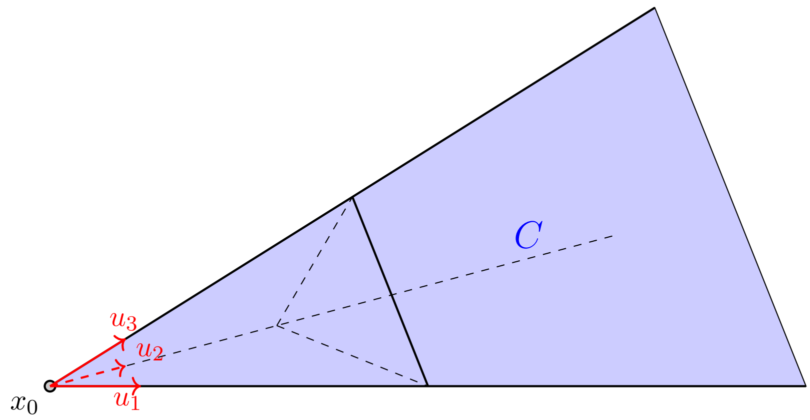


Figure 4.1: A Weyl sector C generated by the maximal singular vectors u_1, u_2, u_3.

it is enough to show the result for a irreducible space, i.e. for a general root system in a Euclidean space.

• Let Φ be a root system in a Euclidean space $E = \mathbb{R}^p$. Let W_r be its Weyl group, i.e. the group generated by reflections through the walls associated to the roots of Φ.

Let C be a Weyl sector with tip at the origin. By [Bou81] Theorem 2 p.153 and section 6 p.64, there exist $\alpha_1, ..., \alpha_p$ a basis of Φ^* (equivalently there exist $h_1, \ldots, h_p$ a basis of Φ) such that C is an open simplicial cone, whose walls are exactly the hyperplanes $\{\alpha_i = 0\}$ for $i = 1 \ldots p$ (i.e. the subsets $\{x \in \mathbb{R}^p, \langle x, h_i \rangle = 0\}$ for $i = 1 \ldots p$). Which means that $C = \{x = \sum_{i=1}^p x_i e_i \mid x_i \geq 0\}$, where

$$e_i \in \bigcap_{\substack{k=1 \\ k \neq i}}^{p} \ker(\alpha_k).$$

Since $\{\alpha_1, \ldots, \alpha_p\}$ form a basis of Φ^*, they form a basis of $(\mathbb{R}^p)^*$, and the intersection is one dimensional. So the e_i's are maximally singular vectors.

• Now let us show that C is acute. Again by [Bou81] Theorem 2 p.153, we can choose the basis $\{h_1, \ldots, h_p\}$ in a unique way, up to permutation, such that the walls of C are again $\mathrm{Ker}(\alpha_i)$ and such that h_i and C are both in the same side of the wall $\mathrm{Ker}(\alpha_i)$. In that case, we have

$$C = \cap_{i=1}^p (\alpha_i > 0) = \{x \in E \mid \forall i \in \{1, \ldots, p\}, \quad \langle h_i, x \rangle > 0\}.$$

Since $\{h_1, \ldots, h_p\}$ is a basis of the root system Λ, then $\langle h_i, h_j \rangle \leq 0$ for all $i \neq j$.
Therefore if $x, y \in C$, i.e. for every $i \in \{1, \ldots, p\}$, $\langle x, h_i \rangle > 0$ and $\langle y, h_i \rangle > 0$, then by [Bou81] p.79 lemma 6, we have $\langle x, y \rangle > 0$. So C is acute. □

Corollary 4.4. *Let C be a Weyl sector in a model space of rank p, and $u_1, ..., u_p \in \overline{C}$ maximally singular vectors generating it. Then if $x = \sum_{i=1}^{p} x_i u_i \in C$, we have $\forall J \subset \{1, \ldots, p\}$*

$$\|x\|^2 \geq \sum_{j \in J} \|x_j u_j\|^2.$$

Proof. Since $u_1, ..., u_p \in \overline{C}$, by the proposition and the continuity of the inner product, we have $\forall i, j \in \{1, ..., p\}$ $\langle u_i, u_j \rangle \geq 0$.
If $x = \sum_{i=1}^{p} x_i u_i \in C$, then

$$\|x\|^2 = \sum_{i=1}^{p} \|x_i u_i\|^2 + 2 \sum_{i,j=1}^{p} x_i x_j \langle u_i, u_j \rangle \geq \sum_{i=1}^{p} \|x_i u_i\|^2,$$

since for any $1 \leq i \leq p$, $x_i \geq 0$. □

4.2 Parallelepipeds and Parallelograms

In the proof of Theorem 1.1, we considered rectangles in the product space. However, in the case of irreducible symmetric space of non-compact type or Euclidean buildings, the space does not factorize and we can no longer consider rectangles. More precisely, maximally singular geodesics are not necessarily orthogonal in a maximal flat. Therefore, we will have to work with cycles that are higher dimensional parallelograms in a maximal flat. We will define them as boundaries of higher dimensional parallelepipeds by induction, as sum of simplices.

4.2.1 Definitions and lemmas

Let E be a Euclidean space and $x_1, \ldots, x_n \in E$. We denote by $[x_1 \ldots x_n]$ the $(n-1)$-simplex which is the convex hull of these points and the orientation is given by the order in which the vertices are listed. Its boundary $\partial[x_1 \ldots x_n]$ is

$$\partial[x_1 \ldots x_n] = \sum_{k=1}^{n} (-1)^{k+1} [x_1, \ldots, \hat{x_k}, \ldots, x_n],$$

where $[x_1, \ldots, \hat{x_k}, \ldots, x_n]$ is the $(n-2)$-simplex given by omitting x_k.
Let us define the 0-*dimensional parallelepiped* in x, $C^0(x)$, as the 0-simplex $[x]$.
A *1-dimensional parallelepiped* is defined from a point x and a vector $u \in E$ by

$$C^1(x; u) := [x, x+u].$$

Suppose that we defined $C^{n-1}(x; u_1, \ldots, u_{n-1})$ as a sum of $(n-1)$-simplices. To define the n-dimensional parallelepiped $C^n(x; u_1, \ldots, u_n)$, we use its standard triangulation: x has n hyper-faces opposite to it. Each one of them is an $(n-1)$-dimensional parallelepiped written as $C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots u_n)$, for $k \in \{1, \ldots, n\}$, which is a sum of $(n-1)$-simplices. We take its convex hull with x, that we can write as $[x; C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots u_n)]$, which is just the sum of the n-simplices that we get by taking the convex hull of x with the $(n-1)$-simplices in $C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots u_n)$.

We define the *n-dimensional parallelepiped* $C^n(x; u_1, \ldots, u_n)$ as the alternating sum of these terms:

$$C^n(x; u_1, \ldots, u_n) := \sum_{k=1}^{n} (-1)^{k+1} [x, C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots u_n)].$$

We define the *$(n-1)$-dimensional parallelogram* as its boundary

$$P^{n-1}(x; u_1, \ldots, u_n) := \partial C^n(x; u_1, \ldots, u_n).$$

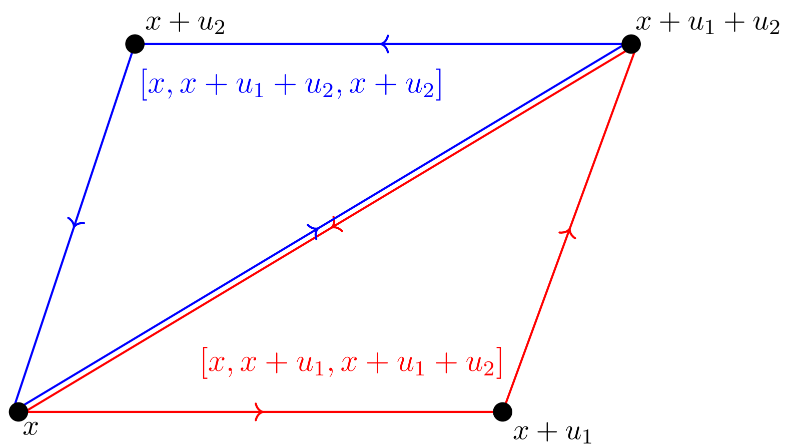


Figure 4.2: $P^1(x; u_1, u_2)$ as sum of two simplices

Lemma 4.5. *Let $y, u_1, \ldots, u_n \in E$.*
- *We have*

$$C^n(y; u_1, \ldots, u_k, \ldots, u_n) = -C^n(y + u_k; u_1, \ldots, -u_k, \ldots, u_n),$$

therefore

$$C^n(y; u_1, \ldots, u_n) = (-1)^n C^n(y + \sum_{k=1}^{n} u_k; -u_1, \ldots, -u_n).$$

- *For all $x \in E$,*

$$\partial[x, C^n(y; u_1, \ldots, u_{n-1})] = C^n(y; u_1, \ldots, u_{n-1}) - [x, \partial C^n(y; u_1, \ldots, u_{n-1})]$$

Proof. By the boundary formula and by linearity of the boundary operator. □

Lemma 4.6. *Let $x, u_1, \ldots, u_n \in E$. We can write the $(n-1)$-dimensional parallelogram as a sum of $(n-1)$-dimensional parallelepipeds :*

$$\begin{aligned} P^{n-1}(x; u_1, \ldots, u_n) &= \sum_{k=1}^{n} (-1)^k C^{n-1}(x; u_1, \ldots, \hat{u_k}, \ldots, u_n) \\ &\quad + (-1)^n \sum_{k=1}^{n} (-1)^k C^{n-1}(x + \sum_{i=1}^{n} u_i; -u_1 \ldots, -\hat{u_k}, \ldots, -u_n). \end{aligned}$$

Proof. By the preceding lemma, we have

$$\begin{aligned} P^{n-1}(x; u_1, \ldots, u_n) &= \partial C^n(x; u_1, \ldots, u_n) \\ &= \partial\Big(\sum_{k=1}^{n} (-1)^{k+1} [x, C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n)]\Big) \\ &= \sum_{k=1}^{n} (-1)^{k+1} C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n) \\ &- \sum_{k=1}^{n} (-1)^{k+1} [x, \partial C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n)]. \end{aligned}$$

Also by lemma 4.5, we have

$$C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n) = (-1)^{n-1} C^{n-1}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_k}, \ldots, , -u_n).$$

So

$$\begin{aligned} P^{n-1}(x; u_1, \ldots, u_n) &= \sum_{k=1}^{n} (-1)^k (-1)^n C^{n-1}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_k}, \ldots, , -u_n) \\ &- \sum_{k=1}^{n} (-1)^{k+1} [x, \partial C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n)]. \end{aligned}$$

Let us denote

$$\begin{aligned} S_1 &= (-1)^n \sum_{k=1}^{n} (-1)^k C^{n-1}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_k}, \ldots, , -u_n), \\ S_2 &= \sum_{k=1}^{n} (-1)^{k+1} [x, \partial C^{n-1}(x + u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n)], \end{aligned}$$

so that $P^{n-1}(x; u_1, \ldots, u_n) = S_1 - S_2$. Let us first develop S_2. We have

$$\partial C^{n-1}(x+u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n) = \sum_{\substack{p=1\\p\neq k}}^{n} (-1)^p \varepsilon(p,k) C^{n-2}(x+u_k; u_1, \ldots, \hat{u_p}, \ldots, \hat{u_k}, \ldots, u_n)$$
$$+ (-1)^{n-1} \sum_{\substack{p=1\\p\neq k}}^{n} (-1)^p \varepsilon(p,k) C^{n-2}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_p}, \ldots, -\hat{u_k}, \ldots, -u_n),$$

where $\varepsilon(p,k) = 1$ if $p < k$ and $\varepsilon(p,k) = -1$ if $p > k$. Therefore

$$\begin{aligned}
S_2 &= \sum_{k=1}^{n} (-1)^{k+1} [x, \partial C^{n-1}(x+u_k; u_1, \ldots, \hat{u_k}, \ldots, u_n)] \\
&= \sum_{k=1}^{n} (-1)^{k+1} \Big[x, \sum_{\substack{p=1\\p\neq k}}^{n} (-1)^p \varepsilon(p,k) C^{n-2}(x+u_k; u_1, \ldots, \hat{u_p}, \ldots, \hat{u_k}, \ldots, u_n)\Big] \\
&+ \sum_{k=1}^{n} (-1)^{k+1} \Big[x, (-1)^{n-1} \sum_{\substack{p=1\\p\neq k}}^{n} (-1)^p \varepsilon(p,k) C^{n-2}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_p}, \ldots, -\hat{u_k}, \ldots, -u_n)\Big] \\
&= \sum_{\substack{p=1,k=1\\p\neq k}}^{n} (-1)^{k+1}(-1)^p \varepsilon(p,k) \Big[x, C^{n-2}(x+u_k; u_1, \ldots, \hat{u_p}, \ldots, \hat{u_k}, \ldots, u_n)\Big] \\
&+ (-1)^{n-1} \sum_{\substack{p=1,k=1\\p\neq k}}^{n} (-1)^{k+1}(-1)^p \varepsilon(p,k) \Big[x, C^{n-2}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_p}, \ldots, -\hat{u_k}, \ldots, -u_n)\Big] \\
&= \sum_{p=1}^{n} (-1)^p \sum_{\substack{k=1\\k\neq p}}^{n} (-1)^{k+1} \varepsilon(p,k) \Big[x, C^{n-2}(x+u_k; u_1, \ldots, \hat{u_p}, \ldots, \hat{u_k}, \ldots, u_n)\Big] \\
&+ (-1)^{n-1} \sum_{\substack{p=1,k=1\\p\neq k}}^{n} (-1)^{p+k+1} \varepsilon(p,k) \Big[x, C^{n-2}(x + \sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_p}, \ldots, -\hat{u_k}, \ldots, -u_n)\Big].
\end{aligned}$$

Remark that

$$\sum_{\substack{k=1\\k\neq p}}^{n} (-1)^{k+1} \varepsilon(p,k) \Big[x, C^{n-2}(x+u_k; u_1, \ldots, \hat{u_p}, \ldots, \hat{u_k}, \ldots, u_n)\Big] = -C^{n-1}(x; u_1, \ldots, \hat{u_p}, \ldots, u_n).$$

So the first sum is just equal to $-\sum_{p=1}^{n} (-1)^p C^{n-1}(x; u_1, \ldots, \hat{u_p}, \ldots, u_n)$.
By denoting $E(p,k) := (-1)^{p+k+1}\Big[x, C^{n-2}(x+\sum_{i=1}^{n} u_i; -u_1, \ldots, -\hat{u_p}, \ldots, -\hat{u_k}, \ldots, -u_n)\Big]$,

the second sum can be written as

$$(-1)^{n-1} \sum_{\substack{p=1,k=1 \\ p\neq k}}^{n} \varepsilon(p,k)E(p,k).$$

Since $E(p,k) = E(k,p)$, and $\varepsilon(p,k) = -\varepsilon(k,p)$, this second sum is equal to zero. So

$$\begin{aligned} P^{n-1}(x;u_1,\ldots,u_n) &= S_1 - S_2 \\ &= (-1)^n \sum_{k=1}^{n} (-1)^k C^{n-1}(x + \sum_{i=1}^{n} u_i; -u_1,\ldots,-\hat{u_k},\ldots,,-u_n) \\ &+ \sum_{k=1}^{n} (-1)^k C^{n-1}(x;u_1,\ldots,\hat{u_k},\ldots,u_n). \end{aligned}$$ □

4.2.2 A decomposition result for parallelograms

Proposition 4.7. *Let $x, u_1, \ldots, u_{n+1} \in E$. If $u_{n+1} = \sum_{i=1}^{p} a_i$, with $a_1, \ldots, a_p \in E$, Then $P^n(x;u_1,\ldots,u_{n+1}) - \sum_{i=1}^{p} P^n(x + \sum_{k=1}^{i-1} a_k; u_1,\ldots,u_n,a_i)$ is an n-cycle, sum of $2n(p+1)$ n-parallelepipeds which are of the form $C^n(y;w_1,\ldots,w_n)$, where $w_n \in \{u_{n+1}, a_1,\ldots,a_p\}$ and $(w_1,\ldots,w_{n-1}) \in \{(u_1,\ldots,\hat{u_k},\ldots,u_n) \mid k \in \{1,\ldots,n\}\}$.*

In particular, if $\|u_1\| > \|u_2\| > \cdots > \|u_{n+1}\|$, which is the case that we will be interested in, then the cycle obtained has small volume since all its parallelepipeds have vectors $\neq (u_1,\ldots,u_n)$. In other words, we decompose the cycle $P^n(x;u_1,\ldots,u_{n+1})$ along the vectors $a_1,\ldots,a_p$, plus a residual cycle with a very small volume compared to the other cycles.

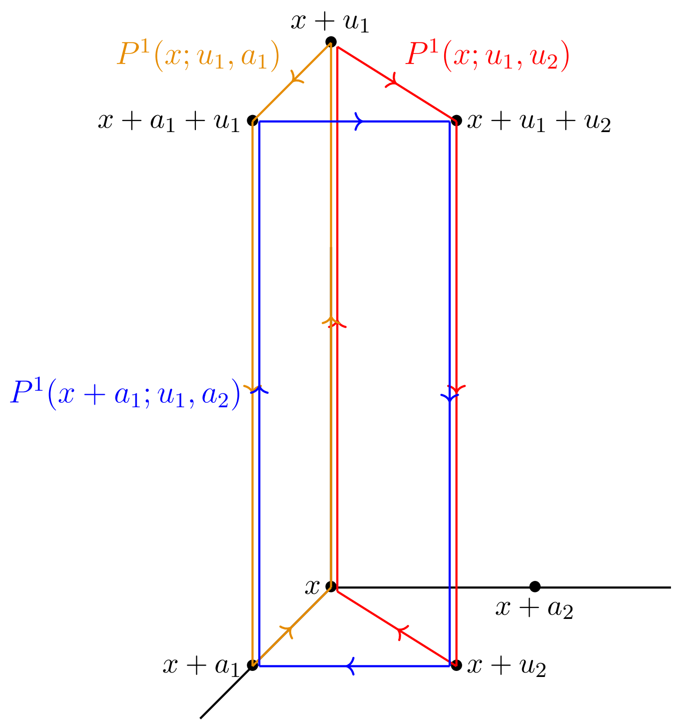


Figure 4.3: A decomposition of $P^1(x;u_1,u_2)$ along the vectors a_1 and a_2.
The residual cycle is the sum of the boundaries of the upper and lower triangles.

Proof. Let $i \in \{1,\dots,p\}$,

$$\begin{aligned}
P^n(x+\sum_{k=1}^{i-1} a_k; u_1,\dots,u_n,a_i) &= \sum_{s=1}^{n+1}(-1)^s C^n(x+\sum_{k=1}^{i-1} a_k; u_1,\dots,\hat{u_s},\dots,u_n,a_i)\\
&+(-1)^{n+1}\sum_{s=1}^{n+1}(-1)^s C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i; -u_1\dots,-\hat{u_s},\dots,-u_n,-a_i)\\
&=\sum_{s=1}^{n}(-1)^s C^n(x+\sum_{k=1}^{i-1} a_k; u_1,\dots,\hat{u_s},\dots,u_n,a_i)\\
&+(-1)^{n+1}\sum_{s=1}^{n}(-1)^s C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i; -u_1\dots,-\hat{u_s},\dots,-u_n,-a_i)\\
&+(-1)^{n+1}C^n(x+\sum_{k=1}^{i-1} a_k; u_1,\dots,u_n)\\
&+(-1)^{n+1}(-1)^{n+1}C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i; -u_1,\dots,-u_n).
\end{aligned}$$

By lemma 4.5, we have

$$C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i;-u_1,\dots,-u_n)=(-1)^n C^n(x+\sum_{k=1}^{i} a_k;u_1,\dots,u_n).$$

So

$$\begin{aligned}
&(-1)^{n+1}C^n(x+\sum_{k=1}^{i-1} a_k;u_1,\dots,u_n)\\
&+(-1)^{n+1}(-1)^{n+1}C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i;-u_1,\dots,-u_n)\\
=&(-1)^{n+1}C^n(x+\sum_{k=1}^{i-1} a_k;u_1,\dots,u_n)\\
&+(-1)^n C^n(x+\sum_{k=1}^{i} a_k;u_1,\dots,u_n).
\end{aligned}$$

By adding them, for $i\in\{1,\dots,p\}$, we get

$$\begin{aligned}
&\sum_{i=1}^{p}\Big((-1)^{n+1}C^n(x+\sum_{k=1}^{i-1} a_k;u_1,\dots,u_n)+(-1)^n C^n(x+\sum_{k=1}^{i} a_k;u_1,\dots,u_n)\Big)\\
&=(-1)^n\Big(C^n(x+u_{n+1};u_1,\dots,u_n)-C^n(x;u_1,\dots,u_n)\Big).
\end{aligned}$$

So the sum of the p parallelograms $\sum_{i=1}^{p} P^n(x+\sum_{k=1}^{i-1} a_k;u_1,\dots,u_n,a_i)$ gives

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{s=1}^{n}(-1)^s C^n(x+\sum_{k=1}^{i-1} a_k;u_1,\dots,\hat{u_s},\dots,u_n,a_i)\\
&+(-1)^{n+1}\sum_{i=1}^{p}\sum_{s=1}^{n}(-1)^s C^n(x+\sum_{k=1}^{i} a_k+\sum_{i=1}^{n} u_i;-u_1\dots,-\hat{u_s},\dots,-u_n,-a_i)\\
&+(-1)^n\Big(C^n(x+u_{n+1};u_1,\dots,u_n)-C^n(x;u_1,\dots,u_n)\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
P^n(x;u_1,\dots,u_{n+1}) &= \sum_{k=1}^{n+1}(-1)^k C^n(x;u_1,\dots,\hat{u}_k,\dots,u_{n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n+1}(-1)^k C^n(x+\sum_{i=1}^{n+1}u_i;-u_1\dots,-\hat{u}_k,\dots,-u_{n+1})\\
&=\sum_{k=1}^{n}(-1)^k C^n(x;u_1,\dots,\hat{u}_k,\dots,u_{n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n}(-1)^k C^n(x+\sum_{i=1}^{n+1}u_i;-u_1\dots,-\hat{u}_k,\dots,-u_{n+1})\\
&+(-1)^{n+1}C^n(x;u_1,\dots,u_n)\\
&+(-1)^n C^n(x+u_{n+1};u_1\dots,u_n).
\end{aligned}$$

Since $(-1)^{n+1}(-1)^{n+1}C^n(x+\sum_{i=1}^{n+1}u_i;-u_1\dots,-u_n)=(-1)^nC^n(x+u_{n+1};u_1\dots,u_n)$. Finally, by taking the difference, all the chains whose vectors are $(u_1,\dots,u_n)$ simplifies and we get:

$$\begin{aligned}
&P^n(x;u_1,\dots,u_{n+1})-\sum_{i=1}^{p}P^n(x+\sum_{k=1}^{i-1}a_k;u_1,\dots,u_n,a_i)\\
&=\sum_{k=1}^{n}(-1)^kC^n(x;u_1,\dots,\hat{u}_k,\dots,u_{n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n}(-1)^kC^n(x+\sum_{i=1}^{n+1}u_i;-u_1\dots,-\hat{u}_k,\dots,-u_{n+1})\\
&-\sum_{i=1}^{p}\sum_{s=1}^{n}(-1)^sC^n(x+\sum_{k=1}^{i-1}a_k;u_1,\dots,\hat{u}_s,\dots,u_n,a_i)\\
&-(-1)^{n+1}\sum_{i=1}^{p}\sum_{s=1}^{n}(-1)^sC^n(x+\sum_{k=1}^{i}a_k+\sum_{i=1}^{n}u_i;-u_1\dots,-\hat{u}_s,\dots,-u_n,-a_i).
\end{aligned}$$

This is a sum of $2n(p+1)$ n-parallelepipeds, whose last vector is either u_{n+1} or an a_i, and whose first $(n-1)$-vectors are in $\{(u_1,\dots,\hat{u}_k,\dots,u_n)\mid k\in\{1,\dots,n\}\}$. □

4.3 Maps with sub-Euclidean fillings of parallelograms

We saw in subsection 2.2.6 that the cross section of a product is the product of cross sections. Therefore, to prove Theorem 1.8, it is enough to prove it when X is an irreducible symmetric space of non-compact type or a Euclidean building with no Euclidean factor. Let us prove the following more general result.

Theorem 4.8. *Let $r \geq 2$ be an integer. Let $\varphi_1, \varphi_2, \ldots, \varphi_r$ be functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\varphi_1(d) = d$ for all d, and for all i $\varphi_i \gg \varphi_{i+1}$, and $\varphi_i(d)$ tends to $+\infty$ at $+\infty$. Suppose moreover that for all $p = 2, \ldots, r$, $\left(\prod_{\substack{i=1\\ i\neq p-1}}^{p} \varphi_i(d)\right)^{\frac{p}{p-1}} \ll \prod_{i=1}^{p} \varphi_i(d)$.*
Let X be a symmetric space of non-compact type of rank $\geq r$, or a thick Euclidean building with cocompact affine Weyl group, with bounded geometry and no Euclidean factor, of dimension $\geq r$. Let Y be a uniformly contractible complete metric space with at most exponential growth, and let $f : X \to Y$ a large-scale Lipschitz map. If there exists a sublinear map ϕ such that
$\forall x \in X$, for every F maximal flat that contains x, $\forall d > 0$, $\forall u_1, \ldots, u_r \in T_x F \simeq F$ that satisfy
- *$u_1, \ldots, u_{r-1}$ are maximally singular,*
- *$\|u_1\| = d$ and $\forall i = 2, \ldots, r$, $\|u_i\| \leq \varphi_i(d)$,*

we have

$$\mathrm{FillVol}_r^{Y,\mathrm{cr}}(f(P^{r-1}(x; u_1, \ldots, u_r))) \leq \phi\left(\prod_{i=1}^{r} \varphi_i(d)\right),$$

then f is not a coarse embedding.

Proof. Let $r \geq 2$ be fixed, and let $\varphi_1, \varphi_2, \ldots, \varphi_r$ be functions satisfying the four conditions. We will prove the theorem by induction on k, from $k = 2$ to $k = r$.
- Let us start by the case $k = 2$.

Let X be a symmetric space of non-compact type of rank $p \geq 2$, or a Euclidean building of rank $p \geq 2$. Suppose there exists a sublinear map ϕ such that $\forall x \in X$, for every F maximal flat that contains x, $\forall d > 0$, $\forall u_1, u_2 \in T_x F \simeq F$ that satisfy: u_1 is maximally singular, $\|u_1\| = \varphi_1(d) = d$ and $\|u_2\| \leq \varphi_2(d)$, we have

$$\mathrm{FillVol}_2^{Y,\mathrm{cr}}(f(P^1(x; u_1, u_2))) \leq \phi(d\,\varphi_2(d)).$$

Since X has exponential growth and Y has at most exponential growth, the case $k = 1$ in Theorem 3.3 implies that X is not sent sublinearly, i.e. there exist $\lambda > 0$ and two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X such that $d_n := d_X(a_n, b_n) \to \infty$, and $\forall n \in \mathbb{N}$

$$d_Y(f(a_n), f(b_n)) \geq \lambda d_X(a_n, b_n).$$

For all $n \in \mathbb{N}$, let γ_n denote the geodesic segment in X going from a_n to b_n, which is unique since X is CAT(0). If γ_n is maximally singular, then there is nothing to do at this step. If not, consider a maximal flat F_n containing γ_n and we will denote $w_n := \frac{d_n}{\|\gamma_n'(0)\|}\gamma_n'(0)$ the directing vector of γ_n with magnitude d_n.
Let C be a Weyl sector containing γ_n, i.e. a Weyl sector of F_n with tip at a_n, such that $b_n \in \overline{C}$. So there exist $u_1, \ldots, u_p$ maximally singular vectors at a_n generating C, in par-

ticular there exist $\delta_1, \dots, \delta_p \geq 0$ such that $w_n = \sum_{i=1}^{p} \delta_i u_i$.

Now let us consider the geodesic segment γ_n as the 1-chain $C^1(a_n; w_n)$ in F_n, and its boundary $\partial C^1(a_n; w_n) = P^0(a_n; w_n)$. We have that

$$P^0(a_n; w_n) = P^0(a_n; \delta_1 u_1) + P^0(a_n + \delta_1 u_1; \delta_2 u_2) + \cdots + P^0(a_n + \sum_{i=1}^{p-1} \delta_i u_i; \delta_p u_p).$$

So, by applying f which is functorial, we have

$$f(P^0(a_n; w_n)) = \sum_{i=1}^{p} f(P^0(a_n + \sum_{s=1}^{i-1} \delta_s u_s; \delta_i u_i)).$$

So

$$\begin{aligned}\mathrm{FillVol}_1^{Y,\mathrm{cr}}(f(P^0(a_n; w_n))) &= \mathrm{FillVol}_1^{Y,\mathrm{cr}}\big(\sum_{i=1}^{p} f(P^0(a_n + \sum_{s=1}^{i-1} \delta_s u_s; \delta_i u_i))\big) \\ &\leq \sum_{i=1}^{p} \mathrm{FillVol}_1^{Y,\mathrm{cr}}\big(f(P^0(a_n + \sum_{s=1}^{i-1} \delta_s u_s; \delta_i u_i))\big).\end{aligned}$$

Since $f(P^0(a_n; w_n)) = [f(b_n)] - [f(a_n)]$, by lemma 2.17

$$\mathrm{FillVol}_1^{Y,\mathrm{cr}}(f(P^0(a_n; w_n))) = d_Y(f(a_n), f(b_n)).$$

So there exists $i_0 \in \{1, \dots, p\}$ such that

$$\begin{aligned}\frac{1}{p} d_Y(f(a_n), f(b_n)) &\leq \mathrm{FillVol}_1^{Y,\mathrm{cr}}\big(f(P^0(a_n + \sum_{s=1}^{i_0-1} \delta_s u_s; \delta_{i_0} u_{i_0}))\big) \\ &= d_Y(f(a_n + \sum_{s=1}^{i_0-1} \delta_s u_s), f(a_n + \sum_{s=1}^{i_0} \delta_s u_s)).\end{aligned}$$

Let us denote $a'_n = a_n + \sum_{s=1}^{i_0-1} \delta_s u_s$ and $b'_n = a_n + \sum_{s=1}^{i_0} \delta_s u_s$. So we have

$$\frac{\lambda}{p} d_X(a_n, b_n) \leq \frac{1}{p} d_Y(f(a_n), f(b_n)) \leq d_Y(f(a'_n), f(b'_n)).$$

This implies in particular that $d_Y(f(a'_n), f(b'_n)) \to \infty$, which implies that $d_X(a'_n, b'_n) \to \infty$ because f is a coarse embedding.
Since C is an acute simplicial cone, we use corollary 4.4. We have

$$d_X(a_n, b_n) = \|w_n\| \geq \|\delta_{i_0} u_{i_0}\| = d_X(a'_n, b'_n).$$

We finally get

$$\frac{\lambda}{p} d_X(a'_n, b'_n) \leq d_Y(f(a'_n), f(b'_n)).$$

Now we have two sequences $(a'_n)_n$ and $(b'_n)_n$ in X that satisfy $d_X(a'_n, b'_n) \to \infty$, and $\forall n \in \mathbb{N}$

$$d_Y(f(a'_n), f(b'_n)) \geq \lambda' d_X(a'_n, b'_n).$$

Where $\lambda' = \frac{\lambda}{p}$, and more importantly, the geodesic between a'_n and b'_n is maximally singular.

To simplify the notations for the rest of the proof, we will just denote a'_n by a_n, b'_n by b_n and λ' by λ.

For every $n \in \mathbb{N}$, let us denote γ_n the bi-infinite geodesic in X that contains a_n and b_n, and consider $P_X(\gamma_n)$ the parallel set of γ_n. $P_X(\gamma_n)$ is a convex subset that splits metrically as

$$P_X(\gamma_n) = \mathbb{R} \times C_X(\gamma_n).$$

The cross section $C_X(\gamma_n)$ is either :

• A symmetric space of non-compact type of rank ≥ 1, when X is a symmetric space,

• Or a Euclidean building with bounded geometry of dimension ≥ 1, with no Euclidean factor, when X is a Euclidean building.

Let us denote $C_X(\gamma_n)$ by X'_n. Note that all such cross sections have exponential growth, and by Proposition 2.14, there is a uniform lower bound on their growth. Therefore, if we fix $\varepsilon > 0$, there exists $\mu > 0$ such that for all $n \in \mathbb{N}$ and for all $R > 0$

$$\mathrm{Vol}^{\varepsilon}\left(B_{X'_n}(R)\right) \geq \exp(\mu R).$$

Now we will work in this product space and apply the same strategy as when X is a product space.

$$P_X(\gamma_n) = \mathbb{R} \times X'_n.$$

Note that the slices $\mathbb{R} \times \{x'\}$, with $x' \in X'$, are the geodesics that are parallel to γ. Let $x'_n \in X'$ such that $\gamma_n = \mathbb{R} \times \{x'_n\}$.

For all $y \in B_{X'_n}\left(x'_n, \varphi_2(d_n)\right)$, consider the geodesic $[x'_n, y]$ in X'_n. Denote γ'_n the (unique since X'_n is CAT(0)) bi-infinite geodesic extension of $[x'_n, y]$, and $F''_n = F'_n \times \gamma'_n$ the 2-flat. Consider the 1-parallelogram $P^1(a_n; u_1^n, u_2^n)$ in F''_n, where u_1 is the (maximally singular) directing vector of γ_n that satisfies $\|u_1^n\| = d_X(a_n, b_n)$, and u_2^n is the unique vector in $T_{x_n}F''_n$ that is parallel to γ'_n and satisfies $\|u_2^n\| = d_{X'_n}(x'_n, y)$. This 1-parallelogram satisfies the induction hypothesis:

$$\mathrm{FillVol}_2^{Y,\mathrm{cr}}(f(P^1(a_n;u_1^n,u_2^n))) \leq \phi(d_n\,\varphi_2(d_n)).$$

Which implies that there exists $V_n \in \mathbf{I}_2(Y)$ in Y such that $\partial V_n = f(P^1(a_n;u_1^n,u_2^n))$ and

$$\mathbf{M}(V_n) \leq \phi(d_n\,\varphi_2(d_n)).$$

Now consider the 1-Lipschitz map $\pi : Y \to \mathbb{R}$, $\pi(z) = d_Y(z, f(C^1(a_n;u_1^n)))$.
By the Slicing Theorem, we have that for a.e. $t \in \mathbb{R}$, there exists $< V_n, \pi, t > \in \mathbf{I}_1(Y)$ such that $\langle V_n, \pi, t\rangle = \partial(V_n \llcorner \{\pi \leq t\}) - (\partial V_n) \llcorner \{\pi \leq t\}$, and by integrating the co-area formula over the distance t, we have

$$\mathbf{M}(\langle V_n,\pi,t\rangle) \leq \frac{d}{dt}\mathbf{M}(V_n \llcorner \{\pi \leq t\}),$$

$$\int_0^{+\infty} \mathbf{M}(\langle V_n,\pi,t\rangle)dt \leq \int_0^{+\infty} \frac{d}{dt}\mathbf{M}(V_n \llcorner \{\pi \leq t\}),$$

$$\int_0^{+\infty} \mathbf{M}(\langle V_n,\pi,t\rangle)dt \leq \mathbf{M}(V_n).$$

Since $\mathbf{M}(V_n) \leq \phi(d_n\,\varphi_2(d_n))$, we get

$$\int_0^{D} \mathbf{M}(\langle V_n,\pi,t\rangle)dt \leq \int_0^{+\infty} \mathbf{M}(\langle V_n,\pi,t\rangle)dt \leq \phi(d_n\,\varphi_2(d_n)), \tag{4.3.1}$$

where $D = d_Y\big(f(C^1(a_n;u_1^n)), f(C^1(a_n+u_2^n;u_1^n))\big)$.
However, for a.e. $t \in]0,D[$, $\mathbf{M}(\langle V_n,\pi,t\rangle)$ cannot be too small since $\langle V_n,\pi,t\rangle$ almost gives a filling of the 0-cycle $\partial\left(f(C^1(a_n;u_1^n))\right) = [\![f(b_n)]\!] - [\![f(a_n)]\!]$.

Claim 4.9. For n big enough: for a.e. $t \in]0,D[$, $\mathbf{M}(\langle V_n,\pi,t\rangle) \geq \frac{\lambda}{2}d_n$.

Proof. For a.e. $t \in]0,D[$,

$$\begin{aligned}\partial(V_n) \llcorner \{\pi \leq t\} &= \partial(V_n) \llcorner \{\pi = 0\} + \partial(V_n) \llcorner \{0 < \pi \leq t\}\\ &= \big(f(b_n) - f(a_n)\big) + H_t.\end{aligned}$$

Where H_t is the 1-current $\partial(V_n) \llcorner \{0 < \pi \leq t\}$.
Since $\|\langle V_n,\pi,t\rangle\|$ is concentrated on $\pi^{-1}(\{t\})$,

$$\partial(V_n \llcorner \{\pi \leq t\}) = \langle V_n,\pi,t\rangle + \big(f(b_n) - f(a_n)\big) + H_t.$$

Which means that $\langle V_n,\pi,t\rangle + \big(f(b_n) - f(a_n)\big) + H_t$ is a 1-current that is actually a cycle.

So $-(\langle V_n, \pi, t\rangle + H_t)$ is a 1-chain that fills the 0-cycle $[f(b_n)] - [f(a_n)]$. Therefore,

$$\mathbf{M}(-(\langle V_n, \pi, t\rangle + H_t)) \geq \mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(b_n) - f(a_n)) \geq \lambda d_n.$$

So

$$\mathbf{M}(\langle V_n, \pi, t\rangle) + \mathbf{M}(H_t) \geq \lambda\, d_n.$$

Note that

$$\begin{aligned} H_t &= \partial(V_n) \llcorner \{0 < \pi \leq t\} \\ &= \big(f(C^1(a_n; u_2^n)) + f(C^1(b_n; u_2^n))\big) \llcorner \{0 < \pi \leq t\}. \end{aligned}$$

By taking the mass

$$\begin{aligned} \mathbf{M}(H_t) &= \mathbf{M}((\ f(C^1(a_n; u_2^n)) + f(C^1(b_n; u_2^n))) \llcorner \{0 < \pi \leq t\}) \\ &\leq \mathbf{M}(f(C^1(a_n; u_2^n)) \llcorner \{0 < \pi \leq t\}) + \mathbf{M}(f(C^1(b_n; u_2^n)) \llcorner \{0 < \pi \leq t\}) \\ &\leq \mathbf{M}(f(C^1(a_n; u_2^n))) + \mathbf{M}(f(C^1(b_n; u_2^n))) \\ &\leq \mathrm{Lip}(f)\big(\mathbf{M}(C^1(a_n; u_2^n)) + \mathbf{M}(C^1(b_n; u_2^n))\big). \end{aligned}$$

Since $\mathbf{M}(C^1(a_n; u_2^n)) = \mathbf{M}(C^1(b_n; u_2^n)) \leq \varphi_2(d_n)$, so

$$\mathbf{M}(H_t) \leq 2\mathrm{Lip}(f)\, \varphi_2(d_n).$$

So

$$\mathbf{M}(\langle V_n, \pi, t\rangle) \geq \lambda\, d_n - \mathbf{M}(H_t) \geq \lambda\, d_n - 2\mathrm{Lip}(f)\, \varphi_2(d_n).$$

Since $\varphi_2(d_n) = o(d_n)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\lambda\, d_n - 2\mathrm{Lip}(f)\, \varphi_2(d_n) \geq \frac{\lambda}{2}\, d_n.$$

So we conclude that for all $n \geq N$ and for a.e. $t \in]0, D[$,

$$\mathbf{M}(\langle V_n, \pi, t\rangle) \geq \frac{\lambda}{2} d_n. \qquad \square$$

So, by (4.3.1), we have

$$\begin{aligned} \phi(d_n\, \varphi_2(d_n)) &\geq \int_0^D \mathbf{M}(\langle V_n, \pi, t\rangle) dt \\ &\geq D\, \frac{\lambda}{2} d_n. \end{aligned}$$

Which implies that

$$D \leq \frac{2}{\lambda} \frac{\phi(d_n\, \varphi_2(d_n))}{d_n}.$$

Let us denote $\psi(\varphi_2(d_n)) = \frac{2}{\lambda}\frac{\phi(d_n\,\varphi_2(d_n))}{d_n}$. Note that ψ is sublinear: $\frac{\psi(\varphi_2(d_n))}{\varphi_2(d_n)}$ tends to 0. Since $D = d_Y\big(f(C^1(a_n;u_1^n)), f(C^1(a_n+u_2^n;u_1^n))\big)$, the last inequality implies that there exists $z \in C^1(a_n+u_2^n;u_1^n)$ such that

$$d_Y(f(C^1(a_n;u_1^n)), f(z)) \leq \psi(\varphi_2(d_n)).$$

But $z \in C^1(a_n+u_2^n;u_1^n)$ implies that $\mathrm{proj}_{X'_n}(z) = y$.

If we choose another $y \in B_{X'_n}\big(x'_n, \varphi_2(d_n)\big)$, we get another z such that

$$\mathrm{proj}_{X'_n}(z) = y,$$

$$d_Y(f(C^1(a_n;u_1^n)), f(z)) \leq \psi(\varphi_2(d_n)).$$

By doing this process $\forall y \in B_{X'_n}\big(x'_n, \varphi_2(d_n)\big)$, we get a subset $C_n \subset X$ that projects onto $B_{X'_n}\big(x'_n, \varphi_2(d_n)\big)$, i.e.

$$B_{X'_n}\big(x'_n, \varphi_2(d_n)\big) \subset \mathrm{proj}_{X'_n}(C_n),$$

and such that

$$f(C_n) \subset N_{\psi(\varphi_2(d_n))}(f(C^1(a_n;u_1^n))).$$

Since the projection onto X'_n is 1-Lipschitz, it implies that

$$\mathrm{Vol}_Y^{\varepsilon}(f(C_n)) \leq \mathrm{Vol}_Y^{\varepsilon}\left(N_{\psi(\varphi_2(d_n))}(f(C^1(a_n;u_1^n)))\right).$$

f coarsely preserves volumes, so there exist $\delta, \delta' > 0$ such that

$$\delta\,\mathrm{Vol}_X^{\varepsilon}(C_n) \leq \mathrm{Vol}_Y^{\varepsilon}(f(C_n)) \leq \delta'\,\mathrm{Vol}_X^{\varepsilon}(C_n).$$

And

$$\mathrm{Vol}_Y^{\varepsilon}\left(N_{\psi(\varphi_2(d_n))}(f(C^1(a_n;u_1^n)))\right) \leq \beta_Y^{\varepsilon}\left(\varepsilon + \psi(\varphi_2(d_n)\right) \times \mathrm{Vol}_Y^{\varepsilon}(f(C^1(a_n;u_1^n))).$$

Y has at most exponential growth, so $\exists \beta > 0$ such that $\forall R > 0$, $\beta_Y^{\varepsilon}(R) \leq e^{\beta R}$. In particular, we have on hand that

$$\beta_Y^{\varepsilon}\left(\varepsilon + \psi(\varphi_2(d_n)\right) \leq \exp(2\beta\psi(\varphi_2(d_n))).$$

On the other hand, by taking a partition of the geodesic segment $[a_n, b_n]$ into sub-intervals of length ε, we get:

$$\mathrm{Vol}_X^\varepsilon\big(C^1(a_n;u_1^n)\big) \leq \frac{\|u_1^n\|}{\varepsilon}+1 \leq \frac{2\|u_1^n\|}{\varepsilon} \leq \big(2/\varepsilon\big)d_n.$$

So, by denoting $A=2/\varepsilon$, we have

$$\begin{aligned}\mathrm{Vol}_Y^\varepsilon(f(C^1(a_n;u_1^n))) &\leq \delta' \,\mathrm{Vol}_X^\varepsilon\big(C^1(a_n;u_1^n)\big)\\ &\leq \delta' A\, d_n.\end{aligned}$$

Therefore

$$\mathrm{Vol}_Y^\varepsilon\left(N_{\psi(\varphi_2(d_n))}(f(C^1(a_n;u_1^n)))\right) \leq \exp(2\beta\psi(\varphi_2(d_n)))\times \delta' A\, d_n.$$

We conclude from all the previous inequalities that

$$\delta\exp(\mu\varphi_2(d_n)) \leq \delta\,\mathrm{Vol}_X^\varepsilon(C_n) \leq \mathrm{Vol}_Y^\varepsilon(f(C_n)) \leq \exp(2\beta\psi(\varphi_2(d_n)))\times\delta' A\, d_n.$$

Which implies finally that for all $n\geq N$

$$\delta\exp(\mu\varphi_2(d_n)) \leq \exp(2\beta\psi(\varphi_2(d_n)))\times\delta' A\, d_n.$$

Which is not possible when $d_n\to\infty$ because ψ is sublinear.
This completes the proof of the case "rank $X\geq 2$" .

Now suppose that it is true for rank $X\geq k$ for some $k\in\{2,\ldots,r-1\}$, and let us prove it for $k+1$.
Let X be of rank $p\geq k+1$ and suppose that such a coarse embedding $f:X\to Y$ exists, i.e. there exists a sublinear function ϕ such that $\forall x\in X$, for every maximal flat F that contains x, $\forall d>0$, $\forall u_1,\ldots,u_{k+1}\in T_xF\simeq F$ that satisfy
- $u_1,\ldots,u_k$ are maximally singular,
- $\|u_1\|=\varphi_1(d)=d$ and $\forall i=2,\ldots,k+1$, $\|u_i\|\leq\varphi_i(d)$, we have

$$\mathrm{FillVol}_{k+1}^{Y,\mathrm{cr}}(f(P^k(x;u_1,\ldots,u_{k+1}))) \leq \phi(\varphi_1(d)\times\cdots\times\varphi_{k+1}(d)).$$

X is of rank $\geq k+1$, so it is of rank $\geq k$ and coarsely embeds into Y, therefore it does not satisfy the "rank $\geq k$" case. This means that there exist a constant $\lambda>0$, a sequence $(x_n)_n\in X$, maximal flats F_n containing x_n, a sequence d_n that goes to $+\infty$ and $u_1^n,\ldots,u_k^n\in T_{x_n}F_n$ such that
- $u_1^n,\ldots,u_{k-1}^n$ are maximally singular,

- $\|u_1^n\| = \varphi_1(d_n) = d_n$ and $\forall i = 2, \ldots, k$, $\|u_i^n\| \leq \varphi_i(d_n)$, and

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n))) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n).$$

The goal now is to find $\lambda' > 0$ and for every $n \in \mathbb{N}$: $x_n' \in F_n$ and maximally singular vectors $u_1^{n\prime}, \ldots, u_k^{n\prime} \in F_n$ such that $\|u_1^{n\prime}\| = d_n$ and $\forall i = 2, \ldots, k$, $\|u_i^{n\prime}\| \leq \varphi_i(d_n)$, and they all tends to $+\infty$, that satisfy

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n'; u_1^{n\prime}, \ldots, u_k^{n\prime}))) \geq \lambda' \prod_{i=1}^{k} \varphi_i(d_n).$$

Let $n \in \mathbb{N}$. If u_k^n is maximally singular, there is nothing to do at this step. If not, take F_n a maximal flat at x_n that contains $u_1^n, \ldots, u_k^n$, which is possible since rank $X = p \geq k+1$. Let C be a Weyl sector of F_n with tip at x_n that contains u_k^n, i.e. such that $x_n + u_k^n \in \overline{C}$. Since C is a simplicial open cone generated by maximally singular vectors, there exist $e_1, \ldots, e_p \in F_n$ maximally singular vectors generating C and $\delta_1, \ldots, \delta_p \geq 0$ such that $u_k^n = \sum_{i=1}^p \delta_i e_i$.
Now let us consider the $(k-1)$-parallelogram $P^{k-1}(x_n; u_1^n, \ldots, u_k^n)$ seen as a $(k-1)$-cycle in $F_n \simeq \mathbb{R}^p$. By the decomposition lemma 4.7, we decompose our parallelogram along the walls of the Weyl sector:

$$P^{k-1}(x_n; u_1^n, \ldots, u_k^n) = \sum_{i=1}^{p} P^{k-1}(x_n + \sum_{s=1}^{i-1} \delta_s e_s; u_1^n, \ldots, u_{k-1}^n, \delta_i e_i) + R_n^{k-1},$$

where R_n^{k-1} is a $(k-1)$-cycle in F_n, sum of $2(k-1)(p+1)$ $(k-1)$-chains, all of the form $C^{k-1}(y; w_1, \ldots, w_{k-1})$, where $w_{k-1} \in \{u_k^n, \delta_1 e_1, \ldots, \delta_p e_p\}$ and $(w_1, \ldots, w_{k-2}) \in \{(u_1^n, \ldots, \hat{u_j^n}, \ldots, u_{k-1}^n) \mid j \in \{1, \ldots, k-1\}\}$.
So every $(k-1)$-chain $\Sigma = C^{k-1}(y, w_1, \ldots, w_{k-1})$ in R_n^{k-1} satisfies

$$\begin{aligned}
\mathrm{Vol}_{F_n}^{k-1}(\Sigma) &\leq \|w_1\| \cdots \|w_{k-2}\| \cdot \|w_{k-1}\| \\
&\leq \|u_1^n\| \cdots \|u_{k-2}^n\| \cdot \|u_k^n\| \\
&\leq \varphi_1(d_n) \times \cdots \times \varphi_{k-2}(d_n) \times \varphi_k(d_n) \\
&= \frac{\prod_{i=1}^k \varphi_i(d_n)}{\varphi_{k-1}(d_n)}
\end{aligned}$$

Therefore

$$\mathrm{Vol}_{F_n}^{k-1}(R_n^{k-1}) \leq 2(k-1)(p+1) \prod_{\substack{i=1 \\ i \neq k-1}}^{k} \varphi_i(d_n).$$

By applying f to the parallelogram decomposition:

$$f(P^{k-1}(x_n;u_1^n,\dots,u_k^n)) = \sum_{i=1}^{p} f(P^{k-1}(x_n+\sum_{s=1}^{i-1}\delta_s e_s;u_1^n,\dots,u_{k-1}^n,\delta_i e_i)) + f(R_n^{k-1}).$$

Since a filling of every cycle of the right-hand side gives a filling of the left-hand side, we have

$$\begin{aligned}\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n;u_1^n,\dots,u_k^n))) \le \sum_{i=1}^{p} &\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(P^{k-1}(x_n+\sum_{s=1}^{i-1}\delta_s e_s;u_1^n,\dots,u_{k-1}^n,\delta_i e_i))\big)\\ &+ \mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(R_n^{k-1})\big).\end{aligned}$$

However, $\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(R_n^{k-1})\big)$ is very small compared to $\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n;u_1^n,\dots,u_k^n)))$. Indeed, by considering the images by f of fillings of R_n^{k-1} in $F_n \subset X$, we have

$$\begin{aligned}\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(R_n^{k-1})\big) &\le \mathrm{Lip}(f)\,\mathrm{FillVol}_k^{X,\mathrm{cr}}\big(R_n^{k-1}\big)\\ &\le \mathrm{Lip}(f)\,\mathrm{FillVol}_k^{F_n,\mathrm{cr}}\big(R_n^{k-1}\big).\end{aligned}$$

By the Euclidean filling of R_n^{k-1} in the flat $F_n \simeq \mathbb{R}^p$, we have

$$\begin{aligned}\mathrm{FillVol}_k^{F_n,\mathrm{cr}}\big(R_n^{k-1}\big) &\le \big(\mathrm{Vol}_{F_n}^{k-1}(R_n^{k-1})\big)^{\frac{k}{k-1}}\\ &\le \left(2(k-1)(p+1)\prod_{i\neq k-1}^{k}\varphi_i(d_n)\right)^{\frac{k}{k-1}}.\end{aligned}$$

So

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(R_n^{k-1})\big) \le \mathrm{Lip}(f)\,(2(k-1)(p+1))^{\frac{k}{k-1}}\left(\prod_{i\neq k-1}^{k}\varphi_i(d_n)\right)^{\frac{k}{k-1}}.$$

This is where the additional assumption on the functions is needed.
Indeed, since $\left(\prod_{i\neq k-1}^{k}\varphi_i(d_n)\right)^{\frac{k}{k-1}} \ll \prod_{i=1}^{k}\varphi_i(d_n)$, there exists $N\in\mathbb{N}$ such that for all $n\ge N$:

$$\lambda\prod_{i=1}^{k}\varphi_i(d_n) - \mathrm{Lip}(f)\,[2(k-1)(p+1)]^{\frac{k}{k-1}}\left(\prod_{i\neq k-1}^{k}\varphi_i(d_n)\right)^{\frac{k}{k-1}} \ge \frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n).$$

In particular, for $n\ge N$:

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n;u_1^n,\dots,u_k^n))) - \mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(R_n^{k-1})\big) \ge \frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n).$$

i.e. for $n \geq N$:

$$\sum_{i=1}^{p} \mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(P^{k-1}(x_n + \sum_{s=1}^{i-1} \delta_s e_s; u_1^n, \dots, u_{k-1}^n, \delta_i e_i))\big) \geq \frac{\lambda}{2} \prod_{i=1}^{k} \varphi_i(d_n).$$

So there exists $i_0 \in \{1, \dots, p\}$ such that

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(P^{k-1}(x_n + \sum_{s=1}^{i_0-1} \delta_s e_s; u_1^n, \dots, u_{k-1}^n, \delta_{i_0} e_{i_0}))\big) \geq \frac{\lambda}{2p} \prod_{i=1}^{k} \varphi_i(d_n).$$

Let us denote $x_n' = x_n + \sum_{s=1}^{i_0-1} \delta_s e_s$, $u_1^{n\prime} = u_1^n, \dots, {u_{k-1}^n}' = u_{k-1}^n, u_k^{n\prime} = \delta_{i_0} e_{i_0}$, which are all maximally singular.
Remark that $\forall i \in \{2, \dots, k-1\}$, $\|u_i^{n\prime}\| \to \infty$ with a speed comparable to that of $\varphi_i(d_n)$. Indeed, on one hand we have

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(P^{k-1}(x_n'; u_1^{n\prime}, \dots, u_k^{n\prime}))\big) \geq \frac{\lambda}{2p} \prod_{i=1}^{k} \varphi_i(d_n).$$

Note also that

$$\begin{aligned}
\mathrm{FillVol}_k^{Y,\mathrm{cr}}\big(f(P^{k-1}(x_n'; u_1^{n\prime}, \dots u_k^{n\prime}))\big) &\leq \mathrm{Lip}(f)\, \mathrm{FillVol}_k^{X,\mathrm{cr}}\big(P^{k-1}(x_n'; u_1^{n\prime}, \dots, u_k^{n\prime})\big) \\
&\leq \mathrm{Lip}(f)\, \mathrm{FillVol}_k^{F_n,\mathrm{cr}}\big(P^{k-1}(x_n'; u_1^{n\prime}, \dots, u_k^{n\prime})\big) \\
&\leq \mathrm{Lip}(f)\, \|u_1^{n\prime}\| \times \cdots \times \|u_k^{n\prime}\|.
\end{aligned}$$

Therefore

$$\frac{\lambda}{2p\,\mathrm{Lip}(f)} \prod_{i=1}^{k} \varphi_i(d_n) \leq \|u_1^{n\prime}\| \times \cdots \times \|u_k^{n\prime}\|.$$

Since $\|u_1^{n\prime}\| = \varphi_1(d_n)$ and $\forall i \in \{2, \dots, k\}$, $\|u_i^{n\prime}\| \leq \varphi_i(d_n)$, we get $\forall i \in \{2, \dots, k\}$:

$$\frac{\lambda}{2p\,\mathrm{Lip}(f)}\, \varphi_i(d_n) \leq \|u_i^{n\prime}\| \leq \varphi_i(d_n).$$

In particular, $\|u_k^{n\prime}\| \gg \varphi_{k+1}(d_n)$.

Now that we found the point $x_n' \in F_n$, the maximally singular vectors $u_1^{n\prime}, \dots, u_k^{n\prime} \in F_n$, and the positive constant $\lambda' = \frac{\lambda}{2p}$ that satisfy the desired inequality, and to simplify the notations for the rest of the proof, we will just denote x_n' by x_n, λ' by λ and $u_i^{n\prime}$ by u_i^n. So that we have

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n; u_1^n, \dots, u_k^n))) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n),$$

with $\|u_1^n\| = \varphi_1(d_n)$ and $\forall i = 2, \dots, k$, $\frac{\lambda}{\mathrm{Lip(f)}} \varphi_i(d_n) \leq \|u_i^n\| \leq \varphi_i(d_n)$.

Let $n \geq N$. Consider the singular k-flat $F'_n \subset F_n$ at x_n generated by the maximally singular vectors $u_1^n, \ldots, u_k^n$, and let $P_X(F'_n)$ be its parallel set. It splits metrically as

$$P_X(F'_n) = \mathbb{R}^k \times C_X(F'_n).$$

Where the cross section $C_X(F'_n)$, that we will denote X'_n, is either :
• a symmetric space of non-compact type of rank ≥ 1, when X is a symmetric space,
• or a Euclidean building with bounded geometry of dimension ≥ 1, with no Euclidean factor, when X is a Euclidean building.
By the same argument as before, if we fix $\varepsilon > 0$, there exists $\mu > 0$ such that for all $n \in \mathbb{N}$ and for all $R > 0$

$$\mathrm{Vol}^{\varepsilon}\left(B_{X'_n}(R)\right) \geq \exp(\mu R). \tag{4.3.2}$$

Now we will work in this product space and apply the same strategy as when X is a product space. We have

$$P_X(F'_n) = \mathbb{R}^k \times X'_n.$$

Note that the slices $\mathbb{R}^k \times \{x'\}$, with $x' \in X'$, are the flats that are parallel to F'_n. So the k-flat F'_n is such a slice. Let $x'_n \in X'$ such that $F'_n = \mathbb{R}^k \times \{x'_n\}$.
For all $y \in B_{X'_n}\left(x'_n, d_n^{\alpha_{k+1}}\right)$, consider the geodesic $[x'_n, y]$ in X'_n. Denote γ'_n the (unique) bi-infinite geodesic extension of $[x'_n, y]$, and $F''_n = F'_n \times \gamma'_n$ the $(k+1)$-flat. Consider the k-parallelogram $P^k(x_n; u_1^n, \ldots, u_k^n, u_{k+1}^n)$ in F''_n, where u_{k+1}^n is the unique vector in $T_{x_n}F''_n$ that is parallel to γ'_n and satisfies $\|u_{k+1}^n\| = d_{X'_n}(x'_n, y)$. This k-parallelogram satisfies the induction hypothesis:

$$\mathrm{FillVol}_{k+1}^{Y,\mathrm{cr}}(f(P^k(x_n; u_1^n, \ldots, u_{k+1}^n))) \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right).$$

Which implies that there exists $V_n \in \mathbf{I}_{k+1}(Y)$ in Y such that $\partial V_n = f(P^k(x_n; u_1^n, \ldots, u_{k+1}^n))$ and

$$\mathbf{M}(V_n) \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right).$$

Now consider the 1-Lipschitz map $\pi : Y \to \mathbb{R}$, $\pi(z) = d_Y(z, f(C^k(x_n; u_1^n, \ldots, u_k^n)))$.
Again, by the Slicing Theorem, we have that for a.e. $t \in \mathbb{R}$, there exists $\langle V_n, \pi, t\rangle \in \mathbf{I}_k(Y)$ such that $\langle V_n, \pi, t\rangle = \partial(V_n \llcorner \{\pi \leq t\}) - (\partial V_n) \llcorner \{\pi \leq t\}$, and by integrating the co-area formula over the distance t, we have that

$$\int_0^{+\infty} \mathbf{M}(\langle V_n, \pi, t\rangle) dt \leq \mathbf{M}(V_n).$$

Since $\mathbf{M}(V_n) \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right)$, we get

$$\int_0^D \mathbf{M}(\langle V_n, \pi, t\rangle)dt \leq \int_0^{+\infty} \mathbf{M}(\langle V_n, \pi, t\rangle)dt \leq \phi\left(\prod_{i=1}^{k+1} \varphi_i(d_n)\right), \tag{4.3.3}$$

where $D = d_Y\big(f(C^k(x_n; u_1^n, \ldots, u_k^n)), f(C^k(x_n + u_{k+1}^n; u_1^n, \ldots, u_k^n))\big)$.
However, for a.e. $t \in]0, D[$, $\mathbf{M}(\langle V_n, \pi, t\rangle)$ cannot be too small since it gives a filling of the $(k-1)$-cycle $\partial f(C^k(x_n; u_1^n, \ldots, u_k^n)) = f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n))$.

Claim 4.10. For n big enough: for a.e. $t \in]0, D[$, $\mathbf{M}(\langle V_n, \pi, t\rangle) \geq \frac{\lambda}{2} \prod_{i=1}^{k} \varphi_i(d_n)$.

Proof. For every $t \in]0, D[$,

$$\begin{aligned}\partial(V_n) \llcorner \{\pi \leq t\} &= \partial(V_n) \llcorner \{\pi = 0\} + \partial(V_n) \llcorner \{0 < \pi \leq t\} \\ &= f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n)) + H_t\end{aligned}$$

Where H_t is the k-current $\partial(V_n) \llcorner \{0 < \pi \leq t\}$.
Since $\|\langle V_n, \pi, t\rangle\|$ is concentrated on $\pi^{-1}(\{t\})$,

$$\partial(V_n \llcorner \{\pi \leq t\}) = \langle V_n, \pi, t\rangle + f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n)) + H_t.$$

Which means that $\langle V_n, \pi, t\rangle + f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n)) + H_t$ is a k-current that is actually a cycle.
So $-(\langle V_n, \pi, t\rangle + H_t)$ is a k-chain that fills the $(k-1)$-cycle $f(P^{k-1}(x_n; u_1^n, \ldots, u_k^n))$. Therefore,

$$\mathbf{M}(-(\langle V_n, \pi, t\rangle + H_t)) \geq \mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x; u_1, \ldots, u_k))) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n).$$

So

$$\mathbf{M}(\langle V_n, \pi, t\rangle) + \mathbf{M}(H_t) \geq \lambda \prod_{i=1}^{k} \varphi_i(d_n).$$

Note that

$$\begin{aligned}H_t &= \partial(V_n) \llcorner \{0 < \pi \leq t\} \\ &= \Big(\sum_{F \in \Delta} f(F)\Big) \llcorner \{0 < \pi \leq t\},\end{aligned}$$

where Δ is the set of side faces of $P^k(x_n; u_1^n, \ldots, u_{k+1}^n)$, i.e. faces whose vectors are not

$(u_1^n, \ldots, u_k^n)$, because $0 < \pi < D$. Indeed, by lemma 4.6

$$P^k(x_n; u_1^n, \ldots, u_{k+1}^n) = \sum_{s=1}^{k+1} (-1)^s C^k(x; u_1^n, \ldots, \hat{u_s^n}, \ldots, u_{k+1}^n)$$
$$+ (-1)^{k+1} \sum_{k=1}^{k+1} (-1)^s C^k(x + \sum_{i=1}^{k+1} u_i^n; -u_1^n \ldots, -\hat{u_s^n}, \ldots, -u_{k+1}^n).$$

Therefore

$$\Delta = \{C^k(x; u_1^n, \ldots, \hat{u_s^n}, \ldots, u_{k+1}^n), C^k(x + \sum_{i=1}^{k+1} u_i^n; -u_1^n \ldots, -\hat{u_s^n}, \ldots, -u_{k+1}^n) \mid s \in \{1, \ldots, k\}\}.$$

So by taking the mass

$$\begin{aligned}
\mathbf{M}(H_t) &= \mathbf{M}\big(\big(\sum_{F \in \Delta} f(F)\big) \llcorner \{0 < \pi \le t\}\big) \\
&\le \sum_{F \in \Delta} \mathbf{M}((f(F)) \llcorner \{0 < \pi \le t\}) \\
&\le \sum_{F \in \Delta} \mathbf{M}(f(F)) \\
&\le \mathrm{Lip}(f) \sum_{F \in \Delta} \mathbf{M}(F).
\end{aligned}$$

Since every side face F satisfies $\mathbf{M}(F) \le \frac{\prod_{i=1}^{k+1} \varphi_i(d_n)}{\varphi_s(d_n)}$, where $s \in \{1, \ldots, k\}$, so every $F \in \Delta$ satisfies $\mathbf{M}(F) \le \frac{\prod_{i=1}^{k+1} \varphi_i(d_n)}{\varphi_k(d_n)}$. And there are $2k$ side faces, so

$$\mathbf{M}(H_t) \le 2k\mathrm{Lip}(f) \prod_{i \ne k}^{k+1} \varphi_i(d_n).$$

So

$$\mathbf{M}(\langle V_n, \pi, t\rangle) \ge \lambda \prod_{i=1}^{k} \varphi_i(d_n) - \mathbf{M}(H_t) \ge \lambda \prod_{i=1}^{k} \varphi_i(d_n) - 2k\mathrm{Lip}(f) \prod_{i \ne k}^{k+1} \varphi_i(d_n).$$

Since for all i, $\varphi_i \gg \varphi_{i+1}$, so $\prod_{i=1}^{k} \varphi_i(d_n) \gg \prod_{i \ne k}^{k+1} \varphi_i(d_n)$. There exists $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\lambda \prod_{i=1}^{k} \varphi_i(d_n) - 2k\mathrm{Lip}(f) \prod_{i \ne k}^{k+1} \varphi_i(d_n) \ge \frac{\lambda}{2} \prod_{i=1}^{k} \varphi_i(d_n).$$

We conclude that for all $n \ge N$ and for a.e. $t \in]0, D[$,

$$\mathbf{M}(\langle V_n, \pi, t\rangle) \ge \frac{\lambda}{2} \prod_{i=1}^{k} \varphi_i(d_n). \qquad \square$$

Therefore, by (4.3.3), we have

$$\begin{aligned}\phi\left(\prod_{i=1}^{k+1}\varphi_i(d_n)\right) &\geq \int_0^D \mathbf{M}(\langle V_n,\pi,t\rangle)dt\\ &\geq D\,\frac{\lambda}{2}\prod_{i=1}^{k}\varphi_i(d_n).\end{aligned}$$

This implies that

$$D \leq \frac{2}{\lambda}\frac{\phi\left(\prod_{i=1}^{k+1}\varphi_i(d_n)\right)}{\prod_{i=1}^{k}\varphi_i(d_n)}$$

Let us denote $\psi(\varphi_{k+1}(d_n)) = \frac{2}{\lambda}\frac{\phi(\prod_{i=1}^{k+1}\varphi_i(d_n))}{\prod_{i=1}^{k}\varphi_i(d_n)}$. Note that ψ is sublinear: $\frac{\psi(\varphi_{k+1}(d_n))}{\varphi_{k+1}(d_n)}$ tends to 0.

Since $D = d_Y\big(f(C^k(x_n;u_1^n,\dots,u_k^n)), f(C^k(x_n+u_{k+1}^n;u_1^n,\dots,u_k^n))\big)$, the last inequality implies that there exists $z \in C^k(x_n+u_{k+1}^n;u_1^n,\dots,u_k^n)$ such that

$$d_Y(f(C^k(x_n;u_1^n,\dots,u_k^n)), f(z)) \leq \psi(\varphi_{k+1}(d_n)).$$

But $z \in C^k(x_n+u_{k+1}^n;u_1^n,\dots,u_k^n)$ implies that $\mathrm{proj}_{X'_n}(z) = y$.

If we choose another $y \in B_{X'_n}\big(x'_n,\varphi_{k+1}(d_n)\big)$, we get another $z \in C^k(x_n+u_{k+1}^n;u_1^n,\dots,u_k^n)$ such that

$$\begin{gathered}\mathrm{proj}_{X'_n}(z) = y,\\ d_Y(f(C^k(x_n;u_1^n,\dots,u_k^n)), f(z)) \leq \psi(\varphi_{k+1}(d_n)).\end{gathered}$$

By doing this process for all $y \in B_{X'_n}\big(x'_n,\varphi_{k+1}(d_n)\big)$, we get subsets $C_n \subset X$ that projects onto $B_{X'_n}\big(x'_n,\varphi_{k+1}(d_n)\big)$, i.e.

$$B_{X'_n}\big(x'_n,\varphi_{k+1}(d_n)\big) \subset \mathrm{proj}_{X'_n}(C_n),$$

and such that

$$f(C_n) \subset N_{\psi(\varphi_{k+1}(d_n))}(f(C^k(x_n;u_1^n,\dots,u_k^n))).$$

Since the projection onto X'_n is 1-Lipschitz, if we fix $\varepsilon > 0$

$$\mathrm{Vol}_X^\varepsilon(B_{X'_n}\big(x'_n,\varphi_{k+1}(d_n)\big)) \leq \mathrm{Vol}_X^\varepsilon(\mathrm{proj}_{X'_n}(C_n)) \leq \mathrm{Vol}_X^\varepsilon(C_n).$$

By the uniform lower bound on the volume growth of all such cross sections (4.3.2), we get

$$\exp(\mu\varphi_{k+1}(d_n)) \leq \mathrm{Vol}_X^{\varepsilon}(B_{X_n'}(x_n', \varphi_{k+1}(d_n))) \leq \mathrm{Vol}_X^{\varepsilon}(C_n).$$

On the other hand, we get

$$\mathrm{Vol}_Y^{\varepsilon}(f(C_n)) \leq \mathrm{Vol}_Y^{\varepsilon}(N_{\psi(\varphi_{k+1}(d_n))}(f(C^k(x_n; u_1^n, \ldots, u_k^n)))).$$

f coarsely preserves volumes, i.e. there exist $\delta, \delta' > 0$ such that

$$\delta \,\mathrm{Vol}_X^{\varepsilon}(C_n) \leq \mathrm{Vol}_Y^{\varepsilon}(f(C_n)) \leq \delta' \,\mathrm{Vol}_X^{\varepsilon}(C_n)$$

By lemma 3.2, we have

$$\mathrm{Vol}_Y^{\varepsilon}(N_{\psi(\varphi_{k+1}(d_n))}(f(C^k(x_n; u_1^n, \ldots, u_k^n)))) \leq \beta_Y^{\varepsilon}\,(\varepsilon + \psi(\varphi_{k+1}(d_n)) \times \mathrm{Vol}_Y^{\varepsilon}(f(C^k(x_n; u_1^n, \ldots, u_k^n))).$$

Y has at most exponential growth, so there exists $\beta > 0$ such that $\forall R > 0$

$$\beta_Y^{\varepsilon}\,(R) \leq e^{\beta R}.$$

In particular, we have on hand that

$$\beta_Y^{\varepsilon}\,(\varepsilon + \psi(\varphi_{k+1}(d_n)) \leq \exp(2\beta\psi(\varphi_{k+1}(d_n))).$$

On the other hand, by taking a partition of each side vector into sub-intervals of length ε, we get a partition of the k-parallelepiped $C^k(x_n; u_1^n, \ldots, u_k^n)$:

$$\mathrm{Vol}_X^{\varepsilon}\big(C^k(x_n; u_1^n, \ldots, u_k^n)\big) \leq \prod_{i=1}^{k} \Big(\frac{\|u_i^n\|}{\varepsilon} + 1\Big) \leq \prod_{i=1}^{k} \Big(\frac{2\|u_i^n\|}{\varepsilon}\Big) \leq \big(2/\varepsilon\big)^k \prod_{i=1}^{k} \varphi_i(d_n).$$

So, by denoting $A = \big(2/\varepsilon\big)^k$, we have

$$\begin{aligned}\mathrm{Vol}_Y^{\varepsilon}(f(C^k(x_n; u_1^n, \ldots, u_k^n))) &\leq \delta' \,\mathrm{Vol}_X^{\varepsilon}\big(C^k(x_n; u_1^n, \ldots, u_k^n)\big) \\ &\leq \delta' A \prod_{i=1}^{k} \varphi_i(d_n).\end{aligned}$$

Therefore

$$\mathrm{Vol}_Y^{\varepsilon}(N_{\psi(\varphi_{k+1}(d_n))}(f(C^k(x_n; u_1^n, \ldots, u_k^n)))) \leq \exp(2\beta\psi(\varphi_{k+1}(d_n))) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

We conclude from all the previous inequalities that

$$\delta \exp(\mu d_n^{\alpha_{k+1}}) \leq \delta \operatorname{Vol}_X^{\varepsilon}(C_n) \leq \operatorname{Vol}_Y^{\varepsilon}(f(C_n)) \leq \exp(2\beta\psi(\varphi_{k+1}(d_n))) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

Which implies finally that for all $n \geq N$

$$\delta \exp(\mu\varphi_{k+1}(d_n)) \leq \exp(2\beta\psi(\varphi_{k+1}(d_n))) \times \delta' A \prod_{i=1}^{k} \varphi_i(d_n).$$

Which is not possible when $d_n \to \infty$ because ψ is sublinear. This completes the induction. □

4.4 Proof of Theorem 1.8

Similarly, let us show that Theorem 1.8 follows from Theorem 4.8.

Proof. Let $X = S \times B$ be a model space of rank $k \geq 2$, and let Y be a uniformly contractible complete metric space with at most exponential growth such that $\mathrm{FV}_k^Y(l) = o\left(l^{\frac{k}{k-1}}\right)$. Let us show that there exist $\varphi_1, \varphi_2, \ldots, \varphi_k$ functions as in Theorem 4.8 that satisfies: $\forall x \in X$, for every F maximal flat that contains x, $\forall d > 0$, $\forall u_1, \ldots, u_r \in T_x F \simeq F$ that satisfy
- $u_1, \ldots, u_{r-1}$ are maximally singular,
- $\|u_1\| = d$ and $\forall i = 2, \ldots, r$, $\|u_i\| \leq \varphi_i(d)$,

we have

$$\operatorname{FillVol}_r^{Y,\mathrm{cr}}(f(P^{r-1}(x; u_1, \ldots, u_r))) \ll \prod_{i=1}^{r} \varphi_i(d).$$

To do so, we can apply the same strategy to get $\varphi_1, \varphi_2, \ldots, \varphi_k$ that satisfy the first three conditions, i.e. by taking for all $i = 1 \ldots k$, $\varphi_i(d) = a(d)^{1-\frac{1}{i}} d$, where $a(d)$ is defined as in the proof of Theorem 1.7. However, this sequence do not satisfy the fourth condition. Let us replace the sequence $((\frac{1}{i}))_i$ by a sequence $(\beta_i)_i$, i.e. $\varphi_i(d) = a(d)^{1-\beta_i} d$. To get the first three conditions, this sequence should satisfy $\beta_1 = 1$, it should be decreasing, and $\beta_i > 0$ for all i. Let us do the computations to see what condition on this sequence does the fourth one imply. Let us denote $S_n = \sum_{i=1}^{n} \beta_i$. We have for all $p = 2, \ldots, k$,

$$\left(\prod_{i \neq p-1}^{p} \varphi_i(d) \right)^{\frac{p}{p-1}} \ll \prod_{i=1}^{p} \varphi_i(d).$$

It implies that for all $p = 2, \ldots, k$,

$$\left(\frac{d^{p-1} a(d)^{p-S_p}}{a(d)^{1-\beta_{p-1}}} \right)^{\frac{p}{p-1}} \ll d^p a(d)^{p-S_p}.$$

Hence
$$d^p \left(a(d)^{(p-1)-(S_p-\beta_{p-1})}\right)^{\frac{p}{p-1}} \ll d^p a(d)^{p-S_p}.$$
Therefore
$$a(d)^{p-\frac{p}{p-1}((S_p-\beta_{p-1}))} \ll a(d)^{p-S_p}.$$
Since $a(d)$ tends to zero, this implies that for all $p = 2, \ldots, k$,
$$\frac{p}{p-1}(S_p - \beta_{p-1}) < S_p.$$
Equivalently, for all $p = 2, \ldots, k$,
$$\frac{S_p}{p} < \beta_{p-1}.$$
We can construct arbitrarily long finite sequences that satisfy the required properties. We will use the condition $\frac{p}{p-1}(S_p - \beta_{p-1}) < S_p$ to define our sequence by induction.
Let us consider the sequence $(\beta_n)_{n\in\mathbb{N}^*}$ defined by induction :
$$\beta_1 \in \mathbb{R}, \quad \text{and} \quad \forall n \in \mathbb{N}^*, \quad \beta_{n+1} = (n+1)\beta_n - S_n - 1,$$
where $S_n = \sum_{k=1}^n \beta_k$. For all $n \in \mathbb{N}^*$
$$\beta_{n+1} - \beta_n = (n+1)\beta_n - S_n - 1 - n\beta_{n-1} + S_{n-1} + 1 = n(\beta_n - \beta_{n-1}).$$
So
$$\beta_{n+1} - \beta_n = (\beta_2 - \beta_1)\, n!$$
Therefore
$$\beta_n = \beta_1 - (\beta_1 - \beta_2)\sum_{k=1}^{n-1} k!$$
Since $\beta_2 = \beta_1 - 1$, we get
$$\beta_n = \beta_1 - \sum_{k=1}^{n-1} k!$$
So $(\beta_n)_{n\in\mathbb{N}^*}$ is strictly decreasing and tends to $-\infty$. For any $k \in \mathbb{N}^*$, there exists $\beta_1 \in \mathbb{N}$ such that the first k-terms of the sequence are positive. Up to re-normalizing, we can suppose that $\beta_1 = 1$. We conclude that $\beta_1, \ldots, \beta_k$ satisfy the desired conditions and that the functions $\varphi_i(d) = a(d)^{1-\beta_i} d$, for $i = 1, \ldots, k$, satisfy the conditions of Theorem 4.8. □

Remark 4.11. It turns out that such an infinite sequence does not exist. We thank Mingkun Liu for having pointed it out to us. That is why, unlike in Theorem 3.3, we did not ask for an infinite sequence of functions $(\varphi_i)_i$, but only a finite one. Indeed, such infinite sequence

$(\beta_n)_n$ must satisfy for all $n \in \mathbb{N}^*$

$$(\beta_1 - \beta_n) + \cdots + (\beta_{n-1} - \beta_n) + 0 + (\beta_{n+1} - \beta_n) < 0.$$

Since the sequence is decreasing, for all $n \geq 2$

$$(\beta_{n-1} - \beta_n) + (\beta_{n+1} - \beta_n) < 0.$$

This implies that the sequence $(\beta_n - \beta_{n+1})_n$ is increasing, thus for all $n \geq 2$

$$\beta_1 - \beta_2 < \beta_n - \beta_{n+1}.$$

This is not possible since $\beta_1 - \beta_2 > 0$, and $(\beta_n - \beta_{n+1})_n$ converges to 0.

Chapter 5

Coarse embeddings of Euclidean spaces into lower rank

The goal of this chapter is to prove Theorem 1.9, which implies Theorem 1.3, and Theorem 1.4.

5.1 From Euclidean spaces to spaces with linear filling

Both theorems follow from the following one. This result allows us to derive an upper bound for fillings of images of $(k-2)$-parallelograms, starting from the linear filling of dimension k in Y.

Theorem 5.1. *Let $p \geq k \geq 2$ be integers. Let Y be a complete uniformly contractible metric space with linear k-dimensional filling function $\mathrm{FV}_Y^k(l) \sim l$, and let $f : \mathbb{R}^p \to Y$ be a coarse embedding. Then for any $x \in \mathbb{R}^p$, and every linearly independent vectors $u_1, \ldots, u_{k-1} \in \mathbb{R}^p$,*

$$\frac{\mathrm{FillVol}_{k-1}^{Y,\mathrm{cr}}(f(P^{k-2}(x; u_1, \ldots, u_{k-1})))}{\|u_1\| \times \cdots \times \|u_{k-1}\|} \to 0, \text{when } \min\{\|u_i\|\} \text{ tends to } +\infty\ .$$

Proof. Let $C > 0$ such that for any $(k-1)$-cycle Σ we have $\mathrm{FillVol}_k^{Y,\mathrm{cr}}(\Sigma) \leq C \,\mathrm{Vol}_Y^{\varepsilon}(\Sigma)$. Let $x \in \mathbb{R}$, and $u_1, \ldots, u_{k-1} \in \mathbb{R}^p$ linearly independent vectors. Without loss of generality, we can assume that $\|u_1\| \geq \cdots \geq \|u_{k-1}\|$. Let $u_k \in \mathbb{R}^p$ be a vector orthogonal to $\mathrm{span}\{u_1, \ldots, u_{k-1}\}$ such that $\|u_k\| = \|u_{k-1}\|^{1/2}$. Then the $(k-1)$-paralellogram $P^{k-1}(x; u_1, \ldots, u_k)$ is a $(k-1)$-cycle. Therefore

$$\begin{aligned}
\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x; u_1, \ldots, u_k))) &\leq C \,\mathrm{Vol}_{k-1}^{Y}(f(P^{k-1}(x; u_1, \ldots, u_k))) \\
&\leq C \,\mathrm{Lip}(f)\, \mathrm{Vol}_{k-1}^{\mathbb{R}^p}(P^{k-1}(x; u_1, \ldots, u_k)) \\
&\leq C \,\mathrm{Lip}(f)\, 2(k+1)\, \|u_1\| \times \cdots \times \|u_{k-1}\|,
\end{aligned}$$

because $P^{k-1}(x; u_1, \ldots, u_k)$ contains $2(k+1)$ faces, and their volume is at most $\|u_1\| \times \cdots \times \|u_{k-1}\|$ since $\|u_1\| \geq \cdots \geq \|u_{k-1}\|$.
Let $\Omega \in \mathbf{I}_k(Y)$ be a k-current in Y such that $\partial\Omega = f(P^{k-1}(x_n; u_1, \ldots, u_k))$ and

$$\mathbf{M}(\Omega) \leq C \operatorname{Lip}(f)\, 2(k+1)\, \|u_1\| \times \cdots \times \|u_{k-1}\|.$$

Consider the 1-Lipschitz map $\pi : Y \to \mathbb{R}$, $\pi(z) = d_Y(z, f(C^{k-1}(x; u_1, \ldots, u_{k-1})))$.
By the Slicing Theorem, we have that for a.e. $t \in \mathbb{R}$, there exists $\langle \Omega, \pi, t\rangle \in \mathbf{I}_{k-1}(Y)$ such that $\langle \Omega, \pi, t\rangle = \partial(\Omega \llcorner \{\pi \leq t\}) - (\partial\Omega) \llcorner \{\pi \leq t\}$.
By integrating the co-area formula over the distance t, we have that

$$\int_0^{+\infty} \mathbf{M}(\langle \Omega, \pi, t\rangle) dt \leq \mathbf{M}(\Omega).$$

So

$$\int_0^{D} \mathbf{M}(\langle \Omega, \pi, t\rangle) dt \leq \int_0^{+\infty} \mathbf{M}(\langle \Omega, \pi, t\rangle) dt \leq C \operatorname{Lip}(f)\, 2(k+1)\, \|u_1\| \times \cdots \times \|u_{k-1}\|.$$

Where $D = d_Y\left(f(C^{k-1}(x; u_1, \ldots, u_{k-1})), f(C^{k-1}(x + u_k; u_1, \ldots, u_{k-1}))\right)$. However:

$$D \geq \rho^- \left(d_{\mathbb{R}^p}(C^{k-1}(x; u_1, \ldots, u_{k-1}), C^{k-1}(x + u_k; u_1, \ldots, u_{k-1}))\right).$$

u_k is orthogonal to $\operatorname{span}\{u_1, \ldots, u_{k-1}\}$, so

$$d_{\mathbb{R}^p}(C^{k-1}(x; u_1, \ldots, u_{k-1}), C^{k-1}(x + u_k; u_1, \ldots, u_{k-1})) = \|u_k\| = \|u_{k-1}\|^{1/2}.$$

Therefore $D \geq \rho^- \left(\|u_{k-1}\|^{1/2}\right)$. Let us denote $D' = \rho^- \left(\|u_{k-1}\|^{1/2}\right)$. So

$$\int_0^{D'} \mathbf{M}(\langle \Omega, \pi, t\rangle) dt \leq C \operatorname{Lip}(f)\, 2(k+1)\, \|u_1\| \times \cdots \times \|u_{k-1}\|.$$

So there exists $t_0 \in]0, D'[$ that satisfies the Slicing Theorem and such that

$$\frac{D'}{2} \mathbf{M}(\langle \Omega, \pi, t_0\rangle) \leq C \operatorname{Lip}(f)\, 2(k+1)\, \|u_1\| \times \cdots \times \|u_{k-1}\|.$$

However $\mathbf{M}(\langle \Omega, \pi, t_0\rangle)$ almost gives a filling of the basis $f(P^{k-2}(x; u_1, \ldots, u_{k-1}))$. Indeed, by (1) of the Slicing Theorem 2.18, we have

$$\partial(\Omega \llcorner \{\pi \leq t_0\}) = \langle \Omega, \pi, t_0\rangle + f(P^{k-2}(x; u_1, \ldots, u_{k-1})) + H_{t_0},$$

where H_{t_0} is the $(k-1)$-current $\partial(\Omega) \llcorner \{0 < \pi < t_0\}$. This means that $-(\langle \Omega, \pi, t_0\rangle + H_{t_0})$

is a $(k-1)$-chain that fills $f(P^{k-2}(x;u_1,\dots,u_{k-1}))$. Therefore,

$$\begin{aligned}\mathrm{FillVol}^{Y,\mathrm{cr}}_{k-1}(f(P^{k-2}(x;u_1,\dots,u_{k-1}))) &\leq \mathbf{M}(-(\langle\Omega,\pi,t_0\rangle+H_t)) \\ &\leq \mathbf{M}((\langle\Omega,\pi,t_0\rangle)+\mathbf{M}(H_{t_0}).\end{aligned}$$

Note that

$$\begin{aligned}H_{t_0} &= \partial(\Omega)\llcorner\{0<\pi<t_0\} \\ &= \Big(\sum_{F\in\Delta} f(F)\Big)\llcorner\{0<\pi<t_0\},\end{aligned}$$

where Δ is the set of side faces of $P^{k-1}(x_n;u_1^n,\dots,u_k^n)$, i.e. faces whose vectors are not $(u_1^n,\dots,u_{k-1}^n)$, because $0<\pi<D$. Indeed, by lemma 4.6

$$\begin{aligned}P^{k-1}(x_n;u_1^n,\dots,u_k^n) &= \sum_{s=1}^{k}(-1)^s C^{k-1}(x;u_1^n,\dots,\hat{u_s^n},\dots,u_k^n) \\ &\quad+(-1)^k\sum_{s=1}^{k}(-1)^s C^{k-1}(x+\sum_{i=1}^{k}u_i^n;-u_1^n\dots,-\hat{u_s^n},\dots,-u_k^n).\end{aligned}$$

So $\Delta=\{C^{k-1}(x;u_1^n,\dots,\hat{u_s^n},\dots,u_k^n),C^{k-1}(x+\sum_{i=1}^k u_i^n;-u_1^n\dots,-\hat{u_s^n},\dots,-u_{k+1}^n)\mid s\in\{1,\dots,k-1\}\}$.
By taking the mass

$$\begin{aligned}\mathbf{M}(H_{t_0}) &= \mathbf{M}\Big(\Big(\sum_{F\in\Delta} f(F)\Big)\llcorner\{0<\pi\leq t_0\}\Big) \\ &\leq \sum_{F\in\Delta}\mathbf{M}((f(F))\llcorner\{0<\pi\leq t_0\}) \\ &\leq \sum_{F\in\Delta}\mathbf{M}(f(F)) \\ &\leq \mathrm{Lip}(f)\sum_{F\in\Delta}\mathbf{M}(F).\end{aligned}$$

Since every side face F satisfies $\mathbf{M}(F)\leq\frac{\|u_1\|\times\cdots\times\|u_k\|}{\|u_s\|}$, where $s\in\{1,\dots,k-1\}$, every $F\in\Delta$ satisfies $\mathbf{M}(F)\leq\|u_1\|\times\cdots\times\|u_{k-2}\|\|u_k\|$. There are $2k$ side faces, So

$$\mathbf{M}(H_{t_0})\leq 2k\mathrm{Lip}(f)\|u_1\|\times\cdots\times\|u_{k-2}\|\|u_k\|.$$

So

$$\mathbf{M}((\langle\Omega,\pi,t_0\rangle)+\mathbf{M}(H_{t_0})\leq\|u_1\|\times\cdots\times\|u_{k-1}\|\Big(\frac{2C\,\mathrm{Lip}(f)\,2(k+1)}{D'}+\frac{2k\,\mathrm{Lip}(f)}{\|u_{k-1}\|^{1/2}}\Big).$$

Therefore

$$\frac{\mathrm{FillVol}_{k-1}^{Y,\mathrm{cr}}(f(P^{k-2}(x;u_1,\ldots,u_{k-1})))}{\|u_1\|\times\cdots\times\|u_{k-1}\|}\leq\Big(\frac{2C\,\mathrm{Lip}(f)\,2(k+1)}{\rho^-\left(\|u_{k-1}\|^{1/2}\right)}+\frac{2k\,\mathrm{Lip}(f)}{\|u_{k-1}\|^{1/2}}\Big).$$

and the right hand side clearly tends to $+\infty$ when $\|u_{k-1}\|$ tends to $+\infty$. □

5.2 Proof of Theorem 1.4

Let us now prove Theorem 1.4 using the quasi-flats Theorem of [KL97] and [EF97].

Proof. Let us suppose that there exists a subspace $E\simeq\mathbb{R}^k\subset\mathbb{R}^p$ that is sent quasi-isometrically with constants (λ,c). By Lemma 2.25, we may assume that f is λ-Lipschitz. Let us also denote by f its restriction to E. So $f(E)$ is a maximal quasi-flat in Y. By the quasi-flats Theorem in [KL97] and [EF97], there exist $\delta>0$ and maximal flats $F_1,\ldots,F_r$ in Y such that $f(E)\subset N_\delta\left(F_1\cup\cdots\cup F_r\right)$. We can assume that the union $F_1\cup\cdots\cup F_r$ is minimal in the sens that for all $i\in\{1,\ldots,r\}$, $f(E)$ is not in a bounded neighborhood of $F_1\cup\cdots\cup F_r\backslash F_i$ (elsewhere we can just remove F_i and modify δ). Therefore by minimality of this union, there exists a sequence $(x_n)_n\in E$ such that $d_Y\left(f(x_n),F_1\cup\cdots\cup F_{r-1}\right)\geq n^2$, i.e. $(f(x_n))_n$ only stays in the δ-neighborhood of F_r.
Let $z\in B_E(x_n,n)$, so $d_Y(f(z),f(x_n))\leq\lambda n$. On one hand we have

$$d_Y\big(f(x_n),F_1\cup\cdots\cup F_{r-1}\big)\leq d_Y(f(z),f(x_n))+d_Y\big(f(z),F_1\cup\cdots\cup F_{r-1}\big).$$

So, for n big enough we have

$$\delta<n^2-\lambda n\leq d_Y\big(f(z),F_1\cup\cdots\cup F_{r-1}\big).$$

On the other hand $f(z)\in N_\delta\left(F_1\cup\cdots\cup F_r\right)$. So for n big enough, we have $f(z)\in N_\delta(F_r)$. Thus there exists $N\in\mathbb{N}$ such that for all $n\geq N$, $f\left(B_E(x_n,n)\right)\subset N_\delta(F_r)$.
Let us consider the projection map $\pi:Y\to F_r$, which is 1-Lipschitz since Y is CAT(0) and F_r is a closed convex subset. Let us denote $A:=\cup_{n\geq N}B_E(x_n,n)$ and consider the map $g:A\to F_r$, $g(a):=\pi\circ f(a)$. It is a quasi-isometric embedding. Indeed, it is clearly λ-Lipschitz, and for $x,y\in A$ we have

$$\begin{aligned}d_Y(f(x),f(y))&\leq d_Y(f(x),\pi(f(x)))+d_Y(\pi(f(x)),\pi(f(y)))+d_Y(\pi(f(y)),f(y))\\&\leq d_{F_r}(\pi(f(x)),\pi(f(y)))+2\delta.\end{aligned}$$

Therefore, for all $x,y\in A$

$$\lambda\,d_A(x,y)-c-2\delta\leq d_{F_r}(\pi(f(x)),\pi(f(y)))\leq\lambda\,d_A(f(x),f(y)).$$

Let us denote $c' = c + 2\delta$.

For all $n \geq N$, let $u_1^n, \dots, u_k^n \in E$ be orthogonal vectors such that $\|u_i\| = n^{1/i}$, and consider the $(k-1)$-parallelograms $P^{k-1}(x_n; u_1^n, \dots, u_k^n)$ in E, which are actually boundaries of k-dimensional rectangles in this case. Note that for all $n \geq N$, $P^{k-1}(x_n; u_1^n, \dots, u_k^n)$ is in $B_E(x_n, n)$, so all the parallelograms are in A. Moreover, by the previous theorem, there exists a sublinear function ϕ such that for all $n \geq N$

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x_n; u_1^n, \dots, u_k^n))) \leq \phi(n^{H_k}),$$

where $H_k = \sum_{i=1}^k \frac{1}{i}$. π is 1-Lipschitz so

$$\mathrm{FillVol}_k^{F_r,\mathrm{cr}}(g(P^{k-1}(x_n; u_1^n, \dots, u_k^n))) \leq \phi(n^{H_k}).$$

Let $V_n \in \mathbf{I}_k(F_r)$ such that $\partial V_n = g(P^{k-1}(x_n; u_1^n, \dots, u_k^n))$ and $\mathbf{M}(V) \leq \phi(n^{H_k})$, and consider the 1-Lipschitz map $d_1 : F_r \to \mathbb{R}$, $d_1(z) = d_{F_r}(z, g(C^{k-1}(x_n; u_1^n, \dots, u_{k-1}^n)))$. By the Slicing Theorem,

$$\int_0^D \mathbf{M}(\langle V_n, \pi, t\rangle)dt \leq \int_0^{+\infty} \mathbf{M}(\langle V_n, \pi, t\rangle)dt \leq \mathbf{M}(V_n) \leq \phi(n^{H_k}).$$

Where $D = d_{F_r}\big(g(C^{k-1}(x_n; u_1^n, \dots, u_{k-1}^n)), g(C^{k-1}(x_n + u_k^n; u_1^n, \dots, u_{k-1}^n))\big)$. g is a (λ, c') quasi-isometric embedding and u_k^n is orthogonal to $\mathrm{span}\{u_1^n, \dots, u_{k-1}^n\}$, so

$$D \geq \lambda d_X\big(C^{k-1}(x_n; u_1^n, \dots, u_{k-1}^n), C^{k-1}(x_n + u_k^n; u_1^n, \dots, u_{k-1}^n)\big) - c' \geq \lambda n^{1/k} - c'.$$

Up to taking bigger N, we can assume that for all $n \geq N$ and for all $i \in \{1 \dots k\}$, $\lambda n^{1/i} - c' \geq \frac{\lambda}{2} n^{1/i}$.
Therefore there exists $t_1 \in]0, \frac{\lambda}{2} n^{1/k}[$ such that

$$\frac{\lambda}{4} n^{1/k}\, \mathbf{M}(\langle V_n, \pi, t_1\rangle) \leq \mathbf{M}(V_n).$$

Again, by adding a small current to $\langle V_n, \pi, t_1\rangle$ of mass less than $2\lambda(k-1)n^{H_k - \frac{1}{k-1}}$, we get a filling of the cycle $g(P^{k-2}(x_n; u_1^n, \dots, u_{k-1}^n))$. So

$$\mathrm{FillVol}_{k-1}^{F_r,\mathrm{cr}}(g(P^{k-2}(x_n; u_1^n, \dots, u_{k-1}^n))) \leq \frac{4}{\lambda} \frac{\phi(n^{1/k})}{n^{1/k}} + 2\lambda(k-1)n^{H_k - \frac{1}{k-1}}.$$

Since $H_k - \frac{1}{k-1} < H_{k-1}$, the right hand side is $\ll n^{H_{k-1}}$. Let us denote by ψ the sublinear function such that

$$\mathrm{FillVol}_{k-1}^{F_r,\mathrm{cr}}(g(P^{k-2}(x_n; u_1^n, \dots, u_{k-1}^n))) \leq \psi(n^{H_{k-1}}).$$

We can apply the same strategy again by considering the 1-Lipschitz map $d_2 : F_r \to \mathbb{R}$, $d_2(z) = d_{F_r}(z, g(C^{k-2}(x_n; u_1^n, \dots, u_{k-2}^n)))$, and using the Slicing Theorem. By repeating this process $(k-1)$ times, we get a sublinear function φ such that for all $n \geq N$

$$\mathrm{FillVol}_1^{F_r,\mathrm{cr}}(g(P^0(x_n; u_1^n))) \leq \varphi(n).$$

If we denote $a_n = x_n$ and $b_n = x_n + u_1^n$, this implies that

$$\frac{\lambda}{2} n \leq d_{F_r}(g(a_n), g(b_n)) \leq \varphi(n).$$

Which contradicts the fact that g is a quasi-isometric embedding. $\square$

Remark 5.2. In higher rank, Theorem 1.4 does not necessarily imply that the coarse embedding is uniformly compressing, as shown by the following example:

$$\begin{aligned} \psi : \quad \mathbb{R}^3 &\longrightarrow \mathbb{H}^3 \times \mathbb{R} \\ (x, y, z) &\longmapsto \big((x, y, 1), z\big) \end{aligned}$$

Where $\mathbb{H}^3$ is the upper half-space model. ψ is the product of a horospherical embedding with the identity map on $\mathbb{R}$. It is clearly a coarse embedding, $\mathrm{rank}(\mathbb{H}^3 \times \mathbb{R}) = 2$, but it is not uniformly compressing since the z-axis is sent isometrically.

Let us give a proof of Corollary 1.5.

Proof. By a result of Bonk–Schramm [BS11], there exists $n \in \mathbb{N}$ such that Y quasi-isometrically embeds into $\mathbb{H}^n$. Let $g : Y \to \mathbb{H}^n$ be such an embedding with constants (λ, c), and consider the coarse embedding $h = g \circ f : \mathbb{R}^p \to \mathbb{H}^n$. By Theorem 5.1, for all $x \in \mathbb{R}^p$, and every $u \in \mathbb{R}^p$

$$\frac{\mathrm{FillVol}_1^{\mathbb{H}^n,\mathrm{cr}}(h(P^0(x; u)))}{\|u\|} \to 0, \text{when } \|u\| \text{ tends to } +\infty\ .$$

In other words, and by lemma 2.17, there exists a sublinear function ϕ such that for all $x, y \in \mathbb{R}^p$, $d_{\mathbb{H}^n}(h(x), h(y)) \leq \phi(d_{\mathbb{R}^p}(x, y))$. However, g is a (λ, c)-quasi-isometric embedding so

$$\frac{1}{\lambda} d_Y(f(x), f(y)) - c \leq d_{\mathbb{H}^n}(h(x), h(y)).$$

Therefore, for all $x, y \in \mathbb{R}^p$

$$d_Y(f(x), f(y)) \leq \lambda\, \phi(d_{\mathbb{R}^p}(x, y)) + \lambda c,$$

and $\psi(t) := \lambda\phi(t) + \lambda c$ is sublinear. $\square$

5.3 When the domain has a one-dimensional Euclidean factor

Theorem 1.9 is an immediate consequence of Theorem 5.1.

Proof. Let $k \in \mathbb{N}$, X, Y be as in Theorem 1.9, and let $f : X \times \mathbb{R} \to Y$ be a coarse embedding.
Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be functions as in Theorem 4.8 (therefore as in Theorem 3.3). Let $x \in X$ and let F be a maximal flat in X containing x. Consider the maximal flat $E = F \times \mathbb{R}$ in $X \times \mathbb{R}$. By Theorem 5.1, there exists a sublinear function ϕ such that for all $d > 0$, for all $u_1, \ldots, u_k$ linearly independent vectors in $T_xE \simeq E$ such that $\|u_i\| \leq \varphi_i(d)$,

$$\mathrm{FillVol}_k^{Y,\mathrm{cr}}(f(P^{k-1}(x; u_1, \ldots, u_k))) \leq \phi\left(\prod_{i=1}^{k} \varphi_i(d)\right).$$

In particular, $u_1, \ldots, u_k$ can be chosen in $T_xF \subset T_xE$. Therefore, when restricted to a copy $X \times \{z\}$ for some $z \in \mathbb{R}$, the coarse embedding $\tilde{f} := f \restriction_X : X \to Y$ satisfies the condition of Theorem 4.8, or Theorem 3.3, which is not possible. Therefore, a coarse embedding $f : X \times \mathbb{R} \to Y$ cannot exist. $\square$

Chapter 6

Conclusion and further questions

We conclude this dissertation by some questions that follow from, or are related to our results

Question 6.1. Does every cocompact geodesic metric space with exponential growth admit a coarse embedding of a binary tree?

This question was already asked by Shalom in [Sha04]. Note that if the answer is positive, then Theorem 1.1 is just a consequence of Theorem 1.2. Indeed, Theorem 1.2 already treats the case when the domain X is a product of regular trees, since the $(p+1)$-regular tree can be seen as the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)$. Moreover, all regular trees de degree ≥ 3 are quasi-isometric. So if $X = X_1 \times \cdots \times X_k$ is a product of geodesic metric spaces of exponential growth, then X contains an isometric copy of a model space of rank k with no Euclidean factor. Therefore Theorem 1.1 follows from Theorem 1.2. This question in full generality is still open even for groups. In [DCT08], Cornulier and Tessera showed that if a group G is either a connected Lie group, or a finitely generated solvable group with exponential growth, then it contains a quasi-isometrically embedded free sub-semigroup on 2 generators. Thus it contains quasi-isometrically embedded binary tree. However, it is not known whether every locally compact compactly generated (or finitely generated) group with exponential growth contains a quasi-isometrically (or even coarsely) embedded copy of a binary tree.

Question 6.2. Let X be either $X_1 \times \cdots \times X_k$ as in Theorem 1.1, or a model space $S \times B$ of rank k as in Theorem 1.2, and let $Y = \mathbb{R}^n \times S' \times B'$ be a model space of rank $= k-1$. We saw that there is no coarse embedding $X \to Y$. Can adding a Euclidean factor in the target make it possible? i.e. can we embed X into $Y \times \mathbb{R}^p$ for some $p \in \mathbb{N}$?

We answered this question negatively in Corollary 1.6 when $\mathrm{rank}(Y) < k-1$. But we do not know if the case $\mathrm{rank}(Y) = k-1$ is different. For example, there is no coarse embedding from the symmetric space $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ of rank 2 into a hyperbolic space $\mathbb{H}^n$. Can we have a coarse embedding $SL_3(\mathbb{R})/SO_3(\mathbb{R}) \to \mathbb{H}^n \times \mathbb{R}^p$ for some $p \in \mathbb{N}$?

Question 6.3. What can we say about the limit set $\overline{f(X)} \cap \partial Y$ of a coarse embedding $f : X \to Y$ between model spaces, and in particular about coarse embeddings of $\mathbb{R}^n$ into $\mathbb{H}^{n+1}$? For instance, the limit sets of horospherical embeddings are reduced to a point. Is it the case for all coarse embeddings from $\mathbb{R}^n$ into $\mathbb{H}^{n+1}$? Is the image of such embedding always contained in a horoball?

Note that Bowditch [Bow17] showed that when Y is Gromov-hyperbolic with bounded geometry and X is a geodesic metric space that has "fast growth", in particular if it has exponential growth, then the limit set does not contain isolated points.

Our results extends to uniform lattices in symmetric spaces, since they are quasi-isometric to them. A natural question to ask is the following.

Question 6.4. What can be said about nonuniform lattices? For example, can we embed $SL_n(\mathbb{Z})$ into a symmetric space of rank $< n-1$?
Fisher–Whyte [FW18] proved that $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ quasi-isometrically embeds into $SL_3(\mathbb{R})$. Can we have a coarse embedding from $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ into $SL_3(\mathbb{Z})$? Note that Leuzinger and Young managed recently to give higher filling functions of some nonuniform lattices [LY21].

Question 6.5. Can we extend Theorem 1.3 and Theorem 1.9 to all proper cocompact CAT(0) spaces in the target?

To do so, we need an analogue of Leuzinger's result, i.e. is the filling of a proper cocompact CAT(0) space linear above the rank? This is still an open question. Note that Goldhirsh-Lang [GL21] recently proved that it holds for cycles with controlled density.

Bibliography

[AB08] Peter Abramenko and Kenneth S Brown. *Buildings: theory and applications*, volume 248. Springer Science & Business Media, 2008.

[AK+00] Luigi Ambrosio, Bernd Kirchheim, et al. Currents in metric spaces. *Acta Mathematica*, 185(1):1–80, 2000.

[APW99] JM Alonso, SJ Pride, and X Wang. Higher dimensional isoperimetric (or dehn) functions of groups. *Journal of Group Theory*, 1999.

[AS86] Michael T Anderson and Viktor Schroeder. Existence of flats in manifolds of nonpositive curvature. *Inventiones mathematicae*, 85(2):303–315, 1986.

[Ass82] Patrice Assouad. Sur la distance de nagata. *CR Acad. Paris*, 294:31–34, 1982.

[BD19a] Jason Behrstock and Cornelia Druţu. Combinatorial higher dimensional isoperimetry and divergence. *Journal of Topology and Analysis*, 11(03):499–534, 2019.

[BD19b] Jason Behrstock and Cornelia Druţu. Higher dimensional divergence for mapping class groups. *Groups, Geometry, and Dynamics*, 13(3):1035–1056, 2019.

[BDS07] Sergei Buyalo, Alexander Dranishnikov, and Viktor Schroeder. Embedding of hyperbolic groups into products of binary trees. *Inventiones mathematicae*, 169(1):153–192, 2007.

[BF98] Noel Brady and Benson Farb. Filling-invariants at infinity for manifolds of nonpositive curvature. *Transactions of the American Mathematical Society*, 350(8):3393–3405, 1998.

[BH13] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[BH21] Yves Benoist and Dominique Hulin. Harmonic quasi-isometric maps ii: negatively curved manifolds. *Journal of the European Mathematical Society*, 2021.

[BHS21] Jason Behrstock, Mark F Hagen, and Alessandro Sisto. Quasiflats in hierarchically hyperbolic spaces. *Duke Mathematical Journal*, 170(5):909–996, 2021.

[BLM83] Joan S Birman, Alex Lubotzky, and John McCarthy. Abelian and solvable subgroups of the mapping class groups. *Duke Mathematical Journal*, 50(4):1107–1120, 1983.

[BM08] Jason A Behrstock and Yair N Minsky. Dimension and rank for mapping class groups. *Annals of Mathematics*, pages 1055–1077, 2008.

[Bou81] Nicolas Bourbaki. *Groupes et algèbres de Lie: Chapitres 4, 5 et 6*. Masson, 1981.

[Bow17] Brian H Bowditch. Limit sets of coarse embeddings. 2017.

[BS11] Mario Bonk and Oded Schramm. Embeddings of gromov hyperbolic spaces. In *Selected Works of Oded Schramm*, pages 243–284. Springer, 2011.

[BST12] Itai Benjamini, Oded Schramm, and Ádám Timár. On the separation profile of infinite graphs. *Groups, Geometry, and Dynamics*, 6(4):639–658, 2012.

[BW92] Jonathan Block and Shmuel Weinberger. Aperiodic tilings, positive scalar curvature, and amenability of spaces. *Journal of the American Mathematical Society*, 5(4):907–918, 1992.

[BW97] Jonathan Block and Shmuel Weinberger. Large scale homology theories and geometry. *AMS/IP Stud. in Adv. Math*, 2:522–569, 1997.

[Cap14] Pierre-Emmanuel Caprace. Lectures on CAT(0) spaces and their isometry groups. 2014.

[CG19] Corentin Le Coz and Antoine Gournay. Separation profiles, isoperimetry, growth and compression. *arXiv preprint:1910.11733*, 2019.

[CM09] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of nonpositively curved spaces: structure theory. *Journal of topology*, 2(4):661–700, 2009.

[DCT08] Yves De Cornulier and Romain Tessera. Quasi-isometrically embedded free sub-semigroups. *Geometry & Topology*, 12(1):461–473, 2008.

[Deh12] Max Dehn. *Papers on group theory and topology*. Springer Science & Business Media, 2012.

[DKLMT20] Thiebout Delabie, Juhani Koivisto, François Le Maître, and Romain Tessera. Quantitative measure equivalence between finitely generated groups. 2020.

[Ebe96] Patrick Eberlein. *Geometry of nonpositively curved manifolds.* University of Chicago Press, 1996.

[ECH+91] DBA Epstein, JW Cannon, DF Holt, MS Paterson, and WP Thurston. *Word processing and group theory, no. 30 in Inst.* 1991.

[EF97] Alex Eskin and Benson Farb. Quasi-flats and rigidity in higher rank symmetric spaces. *Journal of the American Mathematical Society*, pages 653–692, 1997.

[FF60] Herbert Federer and Wendell H Fleming. Normal and integral currents. *Annals of Mathematics*, pages 458–520, 1960.

[FM11] Benson Farb and Dan Margalit. *A primer on mapping class groups (pms-49).* Princeton university press, 2011.

[Fra21] Charles Frances. Isometry group of lorentz manifolds: A coarse perspective. *arXiv preprint:2102.09213*, 2021.

[FW18] David Fisher and Kevin Whyte. Quasi-isometric embeddings of symmetric spaces. *Geometry & Topology*, 22(5):3049–3082, 2018.

[GL21] Tommaso Goldhirsch and Urs Lang. Characterizations of higher rank hyperbolicity. *arXiv preprint:2112.15565*, 2021.

[Gro87] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, 1987.

[Gro88] Mikhail Gromov. Rigid transformation groups, in "géometrie différentielle"(d. bernard, choquet-bruhat, ed.). *Travaux encours*, 33, 1988.

[Gro93] Mikhael Gromov. Asymptotic invariants of infinite groups. *Geometric group theory*, 2:1–295, 1993.

[Ham05] Ursula Hamenstaedt. Geometry of the mapping class groups iii: Quasi-isometric rigidity. *arXiv preprint math/0512429*, 2005.

[Hel01] Sigurdur Helgason. *Differential geometry and symmetric spaces*, volume 341. American Mathematical Soc., 2001.

[HKS22] Jingyin Huang, Bruce Kleiner, and Stephan Stadler. Morse quasiflats i. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2022.

[HMT20] David Hume, John M Mackay, and Romain Tessera. Poincaré profiles of groups and spaces. *Revista Matemática Iberoamericana*, 36(6):1835–1886, 2020.

[HS17] David Hume and Alessandro Sisto. Groups with no coarse embeddings into hyperbolic groups. *New York Journal of Mathematics*, 23:1657–1670, 2017.

[KL96] Michael Kapovich and Bernhard Leeb. Actions of discrete groups on nonpositively curved spaces. *Mathematische Annalen*, 306(1):341–352, 1996.

[KL97] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and euclidean buildings. *Publications mathematiques de l'IHES*, 86:115–197, 1997.

[Kle99] Bruce Kleiner. The local structure of length spaces with curvature bounded above. *Mathematische Zeitschrift*, 231(3):409–456, 1999.

[KW14] Linus Kramer and Richard M Weiss. Coarse equivalences of euclidean buildings. *Advances in Mathematics*, 253:1–49, 2014.

[KY06] Gennadi Kasparov and Guoliang Yu. The coarse geometric novikov conjecture and uniform convexity. *Advances in Mathematics*, 206(1):1–56, 2006.

[Lan00] Urs Lang. Higher-dimensional linear isoperimetric inequalities in hyperbolic groups. *International Mathematics Research Notices*, 2000(13):709–717, 2000.

[Lee] Bernhard Leeb. A characterization of irreducible symmetric spaces and euclidean buildings of higher rank by their asymptotic geometry. bonner mathematische schriften. *Bonn Mathematical Publications*, 326:105–113.

[Leu14] Enrico Leuzinger. Optimal higher-dimensional dehn functions for some cat (0) lattices. *Groups, Geometry, and Dynamics*, 8(2):441–466, 2014.

[LY21] Enrico Leuzinger and Robert Young. Filling functions of arithmetic groups. *Annals of Mathematics*, 193(3):733–792, 2021.

[Mos95] Lee Mosher. Mapping class groups are automatic. *Annals of Mathematics*, 142(2):303–384, 1995.

[MSW03] Lee Mosher, Michah Sageev, and Kevin Whyte. Quasi-actions on trees i. bounded valence. *Annals of mathematics*, pages 115–164, 2003.

[Sha04] Yehuda Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Mathematica*, 192(2):119, 2004.

[STY02] Georges Skandalis, Jean-Louis Tu, and Guoliang Yu. The coarse Baum–Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.

[Tes20] Romain Tessera. On coarse embeddings of amenable groups into hyperbolic graphs. *arXiv preprint:2010.07205*, 2020.

[Wei08] Richard M Weiss. *The Structure of Affine Buildings.(AM-168).* Princeton University Press, 2008.

[Wen05] Stefan Wenger. Isoperimetric inequalities of euclidean type in metric spaces. *GAFA, Geom. funct. anal.*, 5:534–554, 2005.

[Wen11] Stefan Wenger. The asymptotic rank of metric spaces. *Commentarii Mathematici Helvetici*, 86(2):247–275, 2011.

[Yu00] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into hilbert space. *, Invent. Math*, 139, 2000.