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Bases canoniques et graduations associées aux algèbres de Hecke doublement affines rationnelles

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献给我最亲爱的爸爸妈妈

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Introduction

Cette thèse concerne l'étude de la catégorie $\mathcal{O}$ des algèbres de Hecke doublement affines rationnelles cyclotomiques ainsi que celle des représentations d'algèbres de Hecke affines de type D. Elle se compose de trois chapitres. Bien qu'ils soient indépendants, ils ont tous en commun d'étudier ces catégories de représentations via des catégorifications des algèbres de Kac-Moody.

1 Contexte

1.1 Algèbres de Hecke et un théorème d' Ariki

Fixons une fois pour toute un entier positif l. Soit $\mathbf{q} = (q, q_1, \ldots, q_l)$ un $l+1$-uplet de nombres complexes. L'algèbre de Hecke cyclotomique $\mathscr{H}_{\mathbf{q},n}$ de paramètre $\mathbf{q}$ est la $\mathbb{C}$-algèbre engendrée par T_0, T_1, $\ldots$, T_{n-1} avec les relations suivantes :

$$\begin{aligned}
&T_0T_1T_0T_1 = T_1T_0T_1T_0, && \\
&T_iT_j = T_jT_i, && j \geqslant i+2, \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && 1 \leqslant i \leqslant n-2, \\
&(T_0 - q_1)\cdots(T_0 - q_l) = (T_i + 1)(T_i - q) = 0, && 1 \leqslant i \leqslant n-1.
\end{aligned}$$

C'est un cas particulier des algèbres de Hecke associées aux groupes de réflexions complexes introduites par Broué-Malle-Rouquier [BMR98]. Le groupe en question est le produit en couronne $B_n(l)$ d'un groupe symétrique $\mathfrak{S}_n$ et d'un groupe cyclique d'ordre l. En particulier, si $l = 1$, on retrouve les algèbres de Hecke associées aux groupes symétriques. L'algèbre $\mathscr{H}_{\mathbf{q},n}$ peut être aussi se réalisée comme un quotient d'une algèbre de Hecke affine de type $\mathrm{A_{n-1}}$.

Les représentations des algèbres de Hecke cyclotomiques ont été beaucoup étudiées. Si $\mathscr{H}_{\mathbf{q},n}$ est semi-simple, ses modules simples (de dimension finie) sont paramétrés par les représentations irréductibles du groupe $B_n(l)$, qui en outre sont paramétrées par les l-partitions. Dans le cas non semi-simple, une question importante et difficile est de comprendre la paramétrisation des $\mathscr{H}_{\mathbf{q},n}$-modules simples de dimension finie. Ariki a répondu cette question quand le paramètre $\mathbf{q}$ est de la forme

$$q = \exp(2\pi\sqrt{-1}/e), \quad q_p = q^{s_p}, \quad s_p \in \mathbb{Z}/e\mathbb{Z},\ 1 \leqslant p \leqslant l. \tag{1.1.1}$$

Plus précisément, soit $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod la catégorie de tous les $\mathscr{H}_{\mathbf{q},n}$-modules de dimension finie avec n varié dans $\mathbb{N}$. On considère les endo-fonteurs exacts $E^{\mathscr{H}}$, $F^{\mathscr{H}}$ sur $\mathscr{H}_{\mathbf{q},\mathbb{N}}$-mod donnés respectivement par les sommes des foncteurs de restriction et d'induction de $\mathscr{H}_{\mathbf{q},n}$ à $\mathscr{H}_{\mathbf{q},n+1}$. Ariki a introduit les foncteurs de i-restriction $E_i^{\mathscr{H}}$ et de i-induction $F_i^{\mathscr{H}}$ en décomposant les foncteurs $E^{\mathscr{H}}$, $F^{\mathscr{H}}$ en espaces propres généralisés pour l'action de certain éléments centraux. Avec ces foncteurs, il a démontré le résultat suivant dans [Ari96].

Theorem 1.1.1. (Ariki) Soit $\mathbf{q}$ comme dans (1.1.1). Soit $\widetilde{\mathfrak{sl}}_e$ l'algèbre de Kac-Moody de type $\mathrm{A}^{(1)}_{\mathrm{e}-1}$. Alors le groupe de Grothendieck (complexifié) $[\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod}]$ admet un structure de module sur $\widetilde{\mathfrak{sl}}_e$ dont les actions des générateurs de Chevalley sont données par les opérateurs $E_i^{\mathscr{H}}$, $F_i^{\mathscr{H}}$. En plus, le $\widetilde{\mathfrak{sl}}_e$-module $[\mathscr{H}_{\mathbf{q},\mathbb{N}}\text{-mod}]$ est simple de plus haut poids $\Lambda_{\mathbf{q}}$, où $\Lambda_{\mathbf{q}}$ dépend de $\mathbf{q}$. Sa base canonique (duale) est donnée par les classes des modules simples.

1.2 Espace de Fock et sa base canonique

Le point de départ du théorème d'Ariki était une conjecture de Lascoux-Leclerc-Thibon qui relie les représentations des algèbres de Hecke associées aux groupes symétriques avec un modèle combinatoire, à savoir l'espace de Fock de niveau 1. Étant une démonstration et généralisation de cette conjecture au niveau l arbitraire, le théorème d'Ariki fait intervenir l'espace de Fock $\mathcal{F}_{\mathbf{s}}$ de niveau l avec un multi-charge $\mathbf{s} = (s_1, \ldots, s_l)$, qui est liée à $\mathbf{q}$ via (1.1.1). En tant qu'espace vectoriel $\mathcal{F}_{\mathbf{s}}$ a une base $\{|\lambda\rangle\}$ indexée par les l-partitions λ. Plus intéressant, pour chaque $e > 0$, l'espace de Fock $\mathcal{F}_{\mathbf{s}}$ admet une structure de $\widetilde{\mathfrak{sl}}_e$-module intégrable dont les actions des générateurs de Chevalley sont données par des règles combinatoires. Le sous module de $\mathcal{F}_{\mathbf{s}}$ engendré par le vecteur vide $|\emptyset\rangle$ est le $\widetilde{\mathfrak{sl}}_e$-module simple qui intervient dans le théorème d'Ariki. De plus, l'action de $\widetilde{\mathfrak{sl}}_e$ peut être déformée en une action de l'algèbre enveloppante quantique $\mathbf{U}_v(\widetilde{\mathfrak{sl}}_e)$ sur $\mathcal{F}_{\mathbf{s}}(v) = \mathcal{F}_{\mathbf{s}} \otimes_{\mathbb{C}} \mathbb{C}(v)$. En généralisant les travaux de [LT96], Uglov [Ugl00] a défini et étudié une base canonique $\mathbf{B}_{\mathbf{s}}$ pour $\mathcal{F}_{\mathbf{s}}(v)$. Il a donné un algorithme explicite pour calculer la matrice de décomposition de $\mathcal{F}_{\mathbf{s}}(v)$. La base $\mathbf{B}_{\mathbf{s}}$ spécialisée à $v = 1$ est la base canonique de $\mathcal{F}_{\mathbf{s}}$. Un cristal est la donnée d'un ensemble avec des flèches $\tilde{e}_i$, $\tilde{f}_i$ satisfaisant certains axiomes. La base $\mathbf{B}_{\mathbf{s}}$ spécialisée à $v = 0$ donne un cristal sur l'ensemble des l-partitions. On l'appelle le cristal de l'espace de Fock.

1.3 DAHA rationnelles cyclotomiques

À chaque groupe de réflexions complexes W et chaque paramètre c, on peut associer une algèbre de Hecke doublement affine(=DAHA) rationnelle. Ces algèbres sont des cas particuliers des algèbres de réflexions symplectiques introduites par Etingof-Ginzburg [EG02]. Leurs théorie des représentations a une certaine similarité avec celle des algèbres de Lie semi-simples. En particulier, elle possède une catégorie $\mathcal{O}$ introduite par Ginzburg-Guay-Opdam-Rouquier [GGOR03]. Cette catégorie est intéressante. D'une part elle est une catégorie de plus haut poids, dont les modules standards sont paramétrés par les représentations irréductibles de W. D'autre part, elle est étroitement liée aux algèbres de Hecke associées au même groupe W via un foncteur KZ. L'un des problèmes les plus importants pour les représentations des DAHA rationnelles est de comprendre les caractères des modules simples. Pour l'instant, la réponse à cette question n'est connue que dans très peu de cas, y compris celui du groupe symétrique. Dans le cas cyclotomique, une répose conjecturale à cette question est liée à un analogue du théorème de Ariki dont nous allons discuter dans la section suivante.

2 Présentation des résultats

2.1 Cristaux et DAHA rationnelles cyclotomiques

Dans le chapitre I, nous donnons un analogue de la construction d'Ariki pour les catégories $\mathcal{O}$ des DAHA rationnelles cyclotomiques avec paramètres rationnels. Notre construction est basée sur les foncteurs d'induction/restriction paraboliques sur la catégorie $\mathcal{O}$ de DAHA rationnelles, introduits très récemment par Bezrukavnikov-Etingof [BE09]. Nous démontrons, dans le cas général des groupes de réflexions complexes quelconques, que ces foncteurs correspondent via le foncteur KZ aux foncteurs d'induction/restriction des algèbres de Hecke correspondantes. Cela permet, dans le cas cyclotomique, de les décomposer en foncteurs de i-induction/i-restriction, qui sont analogues à ceux d'Ariki pour les algèbres de Hecke. Plus précisément, reprenons les notations de la section 1.1. Fixons un l-uplet d'entiers $\mathbf{s} = (s_1, \ldots s_l)$. Soit $\mathbf{h} = (h, h_1, \ldots, h_{l-1})$ avec

$$h = \frac{-1}{e}, \quad h_p = \frac{s_{p+1} - s_p}{e} - \frac{1}{l}, \quad 1 \leqslant p \leqslant l-1\,.$$

Soit $\mathcal{O}_{\mathbf{h},n}$ la catégorie $\mathcal{O}$ de DAHA rationnelle associée au groupe $B_n(l)$ et le paramètre $\mathbf{h}$. Le foncteur KZ envoie $\mathcal{O}_{\mathbf{h},n}$ à la catégorie des $\mathscr{H}_{\mathbf{q},n}$-modules. Soit $\mathcal{O}_{\mathbf{h},\mathbb{N}} = \bigoplus_{n\in\mathbb{N}} \mathcal{O}_{\mathbf{h},n}$. Les modules standards dans $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ sont paramétrés par les l-partitions. Donc on a un isomorphisme θ d'espaces vectoriels entre le groupe de Grothendieck $[\mathcal{O}_{\mathbf{h},\mathbb{N}}]$ et l'espace de Fock $\mathcal{F}_{\mathbf{s}}$ de niveau l. Il envoie la classe du module standard associé à λ sur le vecteur $|\lambda\rangle$. Nous nous intéressons aux endo-foncteurs E, F de $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ donnés par les sommes des foncteurs de restriction/induction du $\mathcal{O}_{\mathbf{h},n}$ à $\mathcal{O}_{\mathbf{h},n+1}$. Ces sont des analogues des foncteurs $E^{\mathscr{H}}$, $F^{\mathscr{H}}$ pour les algèbres de Hecke. Le foncteur KZ induit un isomorphisme entre le centre de $\mathcal{O}$ et celui de l'algèbre de Hecke. Cela nous permet de décomposer les foncteurs E, F en des foncteurs de i-restriction/i-induction E_i, F_i de la même manière que fait Ariki pour décomposer les foncteurs $E^{\mathscr{H}}$, $F^{\mathscr{H}}$. En calculant les actions des E_i, F_i sur les classes des modules standards, nous démontrons qu'ils définissent une action de $\widetilde{\mathfrak{sl}}_e$ sur $[\mathcal{O}_{\mathbf{h},\mathbb{N}}]$ telle que θ devienne un isomorphisme de $\widetilde{\mathfrak{sl}}_e$-modules. Ensuite, nous définissons via ces foncteurs une $\widetilde{\mathfrak{sl}}_e$-catégorification au sens de Chuang-Rouquier [CR08]. Les propriétés générales de ces catégorifications nous permettent de donner une structure de cristal sur l'ensemble des classes des modules simples dans $\mathcal{O}_{\mathbf{h},\mathbb{N}}$ et de l'identifier au cristal de l'espace de Fock. C'est le résultat principal du chapitre.

Il est conjecturé par Rouquier que les classes des modules simples correspondent via θ à la base canonique de $\mathcal{F}_{\mathbf{s}}$. La confirmation de cette conjecture donnerait les caractères des modules simples. Cette conjecture ne peut pas être démontrée uniquement par les techniques de notre construction.

2.2 Algèbres de Hecke affines de type D et bases canoniques

Le chapitre II est un travail en collaboration avec Michela Varagnolo et Eric Vasserot. Nous démontrons une conjecture de Kashiwara-Miemietz qui donne un analogue du théorème d'Ariki pour les algèbres de Hecke affines de type D.

Soit $\mathbf{f}$ la partie négative de l'algèbre enveloppante quantique de type $A^{(1)}$. La construction géométrique des modules simples des algèbres de Hecke affines utilisée par Kazhdan-Lusztig dans la démonstration de la conjecture de Deligne-Lusztig donne une identification naturelle entre ces modules simples et la base canonique de $\mathbf{f}$. Une partie de la démonstration du Théorème 1.1.1 est basé sur cette construction. Plus précisément, il y a un isomorphisme linéaire entre $\mathbf{f}$ et le groupe de Grothendieck de la catégorie des modules de dimension finie sur algèbres de Hecke affines de type A. Les foncteurs d'induction/restriction donnent l'action des générateurs de Chevalley et leurs transposés par rapport à une certain forme bilinéaire symétrique sur $\mathbf{f}$.

Récemment, une nouvelle famille d'algèbres, que l'on appelle maintenant les algèbres KLR, ont été introduites par Khovanov-Lauda [KL09], et indépendemment par Rouquier [Rou08a]. Ces algèbres sont Morita équivalentes à des algèbres de Hecke affines de type A. Elles donnent lieu à une nouvelle interprétation de la catégorification de $\mathbf{f}$. Ces algèbres sont graduées. Donc elles sont mieux adaptées que les algèbres de Hecke affines pour décrire la base canonique. Plus précisément, le lien entre algèbres KLR et bases canoniques est donné par un isomorphisme explicite d'algèbres, due à Varagnolo-Vasserot [VV09b], entre les algèbres KLR et les algèbres d'extension de certains complexes de faisceaux constructibles qui interviennent dans la construction de Kazhdan-Lusztig.

Les règles de branchement pour les algèbres de Hecke affines de type B ont été étudiées très récemment par [Eno09], [EK06, EK08a, EK08b], [Mie08] et [VV09a]. En particulier, dans [Eno09], [EK06, EK08a, EK08b] un analogue de la construction d'Ariki ont été conjecturé et étudié. Ici $\mathbf{f}$ est remplacé par un module ${}^{\theta}\mathbf{V}(\lambda)$ d'une algèbre ${}^{\theta}\boldsymbol{\mathcal{B}}$. Plus précisément, il était conjecturé que ${}^{\theta}\mathbf{V}(\lambda)$ admet une base canonique qui s'identifie de façon canonique avec l'ensemble des classes d'isomorphisme des objets simples de la catégorie des modules d'algèbres de Hecke affines de type B. Sous cette identification les règles de branchement pour les algèbres de Hecke affines de type B doivent être donné par la ${}^{\theta}\boldsymbol{\mathcal{B}}$-action sur ${}^{\theta}\mathbf{V}(\lambda)$. Cette conjecture a été démontré par Varagnolo-Vasserot [VV09a]. La preuve utilise à la fois les approches géométriques qui étaient introduites dans [Eno09] et un nouveau type d'algèbres graduées qui sont des analogues des algèbres KLR.

Une description similaire pour les algèbres de Hecke affines de type D a été conjecturé dans [KM07]. Dans ce cas $\mathbf{f}$ est remplacé par un autre module ${}^{\circ}\mathbf{V}$ sur l'algèbre ${}^{\theta}\boldsymbol{\mathcal{B}}$ (qui est la même algèbre que dans le cas de type B). Dans le chapitre II, nous donnons une preuve de cette conjecture. La méthode est la même que celle de [VV09a]. D'abord, nous introduisons une famille d'algèbres graduées ${}^{\circ}\mathbf{R}_m$ pour m un entier non négatif. Elles peuvent être vues comme des algèbres d'extension de certains complexes de faisceaux constructibles qui sont naturellement attachés à l'algèbre de Lie du groupe $SO(2m)$, et elles sont Morita équivalent aux algèbres de Hecke affines de type D. Nous identifions ensuite ${}^{\circ}\mathbf{V}$ avec groupes de Grothendieck des algèbres graduées ${}^{\circ}\mathbf{R}_m$ en tant que ${}^{\theta}\boldsymbol{\mathcal{B}}$-modules. Le résultat principal du chapitre est le théorème II.3.9.1, où nous démontrons que le ${}^{\theta}\boldsymbol{\mathcal{B}}$-module ${}^{\circ}\mathbf{V}$ admet une base canonique, elle est donnée par les classes des ${}^{\circ}\mathbf{R}_m$-modules projectives gradués auto-duaux.

2.3 Graduation sur la catégorie $\mathcal{O}$ et filtration de Jantzen

L'action de l'algèbre enveloppante de type $\mathrm{A}^{(1)}$ sur l'espace de Fock $\mathcal{F}_{\mathbf{s}}$ se quantifie en une action de l'algèbre enveloppante quantique $\mathbf{U}_v(\widetilde{\mathfrak{sl}}_e)$. Du point de vue de la catégorification, le paramètre v ici correspond à une graduation sur la catégorie $\mathcal{O}$. Cette graduation est connue pour la catégorie $\mathcal{O}$ (parabolique) des algèbres de Lie semisimples depuis les travaux de Beilinson-Ginzburg-Soergel [BGS96]. Elle a été beaucoup étudiée. À ce jour, la graduation sur la catégorie $\mathcal{O}$ des DAHA rationnelles est inconnue. Mais dans le cas cyclotomique, il est conjecturé par Varagnolo-Vasserot [VV08] que la catégorie $\mathcal{O}_{\mathbf{h},n}$ est équivalente à une catégorie $\mathcal{O}$ parabolique d'une algèbre de Lie affine de $\mathfrak{gl}$ au niveau négatif, dont le type parabolique dépend de la multi-charge $\mathbf{s}$. Au niveau $l=1$ (le cas du groupe symétrique), cette équivalence est connue. Elle est donnée par un foncteur explicite défini par Suzuki. L'avantage de la catégorie $\mathcal{O}$ affine parabolique est qu'elle admet une théorie de localisation à la Beilinson-Bernstein. Ceci donne lieu à une graduation d'origine géométrique sur cette catégorie, qui est l'analogue de celle apparue dans [BGS96]. Donc l'analogie avec le cas du type fini nous indique que la graduation sur les modules standards est donnée par la filtration de Jantzen.

Dans le chapitre III nous nous plaçons dans le cas $l = 1$ et nous étudions cette filtration. Nous introduisons d'abord la filtration de Jantzen pour un objet standard dans une catégorie de plus haut poids associée à une déformation donnée de la catégorie. Les catégories déformées équivalentes donnent la même filtration. À l'aide d'une version déformée du foncteur de Suzuki, nous construisons ensuite une équivalence de catégories de plus haut poids entre la catégorie $\mathcal{O}$ affine parabolique déformée et la catégorie $\mathcal{O}_{\mathbf{h},n}$ déformée. D'après une équivalence de Rouquier [Rou08b], ces deux catégories sont aussi équivalentes en tant que catégories de plus haut poids à la catégorie des modules de dimension finie sur une algèbre de q-Schur. Toutes ces équivalences permettent donc d'identifier les filtrations de Jantzen sur les modules standards dans les trois cas. Enfin, nous calculons la filtration des modules de Verma paraboliques en utilisant les $\mathscr{D}$-modules sur la variété de drapeaux affines. Cet argument est une généralisation des techniques utilisées par Beilinson-Bernstein dans la preuve des conjectures de Jantzen dans le cas du type fini (nonparabolique) [BB93]. Le résutat final montre que les multiplicités (graduées) des modules simples dans cette filtration sont données par certains polynômes de Kazhdan-Lusztig paraboliques. Ce sont aussi les polynômes qui apparaissent dans la matrice de décomposition de l'espace de Fock (quantifié) de niveau 1. En particulier, ceci confirme une conjecture de Leclerc-Thibon [LT96] sur la filtration de Jantzen de modules de Weyl, dont la version nongraduée était auparavant démontré dans [VV99].

Dans la suite, il serait intéressant de comprendre l'équivalence de catégories entre $\mathcal{O}_{\mathbf{h},n}$ et la catégorie $\mathcal{O}$ affine parabolique au niveaux supérieurs comme conjecturé dans [VV08]. En particulier, une telle équivalence confirmerait la conjecture de Rouquier mentionée à la fin de la section 2.1 de cette introduction. De plus, une version déformée appropriée d'une telle équivalence nous permettrait de généraliser les résultats du chapitre III et d'étudier les graduations sur les catégories $\mathcal{O}_{\mathbf{h},n}$ pour l quelconque. Il serait aussi très intéressant de comparer les différents types de catégorifications sur les deux catégories $\mathcal{O}$.

Chapter I

Crystals and rational DAHA's

In [Ari96], Ariki defined the i-restriction and i-induction functors for cyclotomic Hecke algebras. He showed that the Grothendieck group of the category of finitely generated modules of these algebras admits a module structure over the affine Lie algebra of type $A^{(1)}$, with the action of Chevalley generators given by the i-restriction and i-induction functors.

In this chapter, we give an analogue of Ariki's construction for the category $\mathcal{O}$ of cyclotomic rational DAHA's. More precisely, we define the i-restriction and i-induction functors in this setting by refining the parabolic restriction and induction functors of rational DAHA's introduced by Bezrukavnikov and Etingof [BE09]. We show that the action of these functors make the Grothendieck group of this category $\mathcal{O}$ a representation of the type $A^{(1)}$ affine Lie algebra and it is isomorphic to a Fock space representation. We also construct a crystal on the set of isomorphism classes of simple modules in the category $\mathcal{O}$. It is isomorphic to the crystal of the Fock space.

The result of this chapter has been prepublished in [Sha08].

Notation

For an algebra A, we will write A-mod for the category of finitely generated A-modules. For $f : A \to B$ an algebra homomorphism from A to another algebra B such that B is finitely generated over A, we will write

$$f_* : B\text{-mod} \to A\text{-mod}$$

for the restriction functor and we write

$$f^* : A\text{-mod} \to B\text{-mod}, \quad M \mapsto B \otimes_A M.$$

A $\mathbb{C}$-linear category $\mathcal{A}$ is called *artinian* if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and each object has a finite length. Given an object M in $\mathcal{A}$, we denote by $\mathrm{soc}(M)$ (resp. $\mathrm{head}(M)$) the *socle* (resp. the *head*) of M, which is the largest semi-simple subobject (quotient) of M.

Let $\mathcal{C}$ be an abelian category. The *Grothendieck group* of $\mathcal{C}$ is the quotient of the free abelian group generated by objects in $\mathcal{C}$ modulo the relations $M = M' + M''$ for all objects M, M', M'' in $\mathcal{C}$ such that there is an exact sequence $0 \to M' \to M \to M'' \to 0$. Let $[\mathcal{C}]$ denote the complexified Grothendieck group, a $\mathbb{C}$-vector space. For each object M in $\mathcal{C}$, let $[M]$ be its class in $[\mathcal{C}]$. Any exact functor $F : \mathcal{C} \to \mathcal{C}'$ between two abelian categories induces a vector space homomorphism $[\mathcal{C}] \to [\mathcal{C}']$, which we will denote by F again. Given an algebra A we will abbreviate $[A] = [A\text{-mod}]$.

Denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$. For $F \in \mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ write $\mathrm{End}(F)$ for the ring of endomorphisms of the functor F. We denote by $1_F : F \to F$ the identity element in $\mathrm{End}(F)$. Let $G \in \mathrm{Fct}(\mathcal{C}', \mathcal{C}'')$ be a functor from $\mathcal{C}'$ to another category $\mathcal{C}''$. For any $X \in \mathrm{End}(F)$ and any $X' \in \mathrm{End}(G)$ we write $X'X : G \circ F \to G \circ F$ for the morphism of functors given by $X'X(M) = X'(F(M)) \circ G(X(M))$ for any $M \in \mathcal{C}$.

1 Hecke algebras

In this section, we give some reminders on Hecke algebras and their parabolic restriction and induction functors.

1.1 Definition

Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$. A *pseudo-reflection* on $\mathfrak{h}$ is a non trivial element s of $\mathrm{GL}(\mathfrak{h})$ which acts trivially on a hyperplane, called the *reflecting hyperplane* of s. A *complex refection group* W is a finite subgroup of $\mathrm{GL}(\mathfrak{h})$ generated by pseudo-reflections. Let $\mathcal{S}$ be the set of pseudo-reflections in W and let $\mathcal{A}$ be the set of reflecting hyperplanes. Let

$$\mathfrak{h}_{reg} = \mathfrak{h} - \bigcup_{H \in \mathcal{A}} H.$$

It is stable under the W-action on $\mathfrak{h}$. Fix a point $x_0 \in \mathfrak{h}_{reg}$. We will denote its image in $\mathfrak{h}_{reg}/W$ again by x_0. The *braid group* $B(W, \mathfrak{h})$ is the fundamental group $\pi_1(\mathfrak{h}_{reg}/W, x_0)$.

For $H \in \mathcal{A}$, let W_H be the pointwise stabilizer of H in W. It is a cyclic group. Let e_H be the order of W_H and $\zeta_H = \exp(2\pi\sqrt{-1}/e_H)$. We denote by s_H the unique element in W_H with determinant ζ_H. For $x \in \mathfrak{h}$ we set $x = \mathrm{pr}_H(x) + \mathrm{pr}_H^{\perp}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_H^{\perp}(x) \in \mathrm{Im}(s_H - \mathrm{Id}_{\mathfrak{h}})$. For $t \in \mathbb{R}$ we denote by s_H^t the element in $\mathrm{GL}(\mathfrak{h})$ defined by

$$s_H^t(x) = \zeta_H^t \,\mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x),$$

where $\zeta_H^t = \exp(2\pi t\sqrt{-1}/e_H)$. For $x \in \mathfrak{h}$ we denote by $\sigma_{H,x}$ the path in $\mathfrak{h}$ from x to $s_H(x)$ defined by

$$\sigma_{H,x} : [0,1] \to V, \quad t \mapsto s_H^t(x).$$

Whenever γ is a path in $\mathfrak{h}_{reg}$ with initial point x_0 and terminal point x_H, we define the path $\sigma_{H,\gamma}$ from x_0 to $s_H(x_0)$ by

$$\sigma_{H,\gamma} = s_H(\gamma^{-1}) \cdot \sigma_{H,x_H} \cdot \gamma.$$

The homotopy class of $\sigma_{H,\gamma}$ does not depend on the choice of x_H provided x_H is close enough to H. The element in $B(W, \mathfrak{h})$ induced by this homotopy class is called a *generator of the monodromy* around H, see [BMR98, Section 2B]. The following definition is due to [BMR98, Definition 4.21].

Definition 1.1.1. For any map $q : \mathcal{S} \to \mathbb{C}^*$ that is constant on the W-conjugacy classes, the *Hecke algebra* $\mathscr{H}_q(W, \mathfrak{h})$ the quotient of the group algebra $\mathbb{C}B(W, \mathfrak{h})$ by the relations:

$$(T_{s_H} - 1) \prod_{s \in W_H \cap \mathcal{S}} (T_{s_H} - q(s)) = 0, \tag{1.1.1}$$

where $H \in \mathcal{A}$ and $T_{s_H} \in B(W, \mathfrak{h})$ is a generator of the monodromy around H.

Let $\mathfrak{h}^W$ denote the subspace of fixed points of W in $\mathfrak{h}$. If $\mathfrak{h}^W = 1$ we will abbreviate

$$B_W = B(W, \mathfrak{h}), \quad \mathscr{H}_q(W) = \mathscr{H}_q(W, \mathfrak{h}).$$

1.2 Parabolic restriction and induction functors for Hecke algebras

In this section we will assume that $\mathfrak{h}^W = 1$. A *parabolic subgroup* W' of W is by definition the stabilizer of a point $b \in \mathfrak{h}$. By a theorem of Steinberg, the group W' is also generated by pseudo-reflections. Let q' be the restriction of q to $\mathcal{S}' = W' \cap \mathcal{S}$. There is an explicit inclusion

$$\imath_q : \mathscr{H}_{q'}(W') \hookrightarrow \mathscr{H}_q(W)$$

given by [BMR98, Section 2D]. The *restriction* functor

$${}^{\mathscr{H}}\mathrm{Res}^W_{W'} : \mathscr{H}_q(W)\text{-mod} \to \mathscr{H}_{q'}(W')\text{-mod}$$

is the functor $(\imath_q)_*$. The *induction* functor

$${}^{\mathscr{H}}\mathrm{Ind}^W_{W'} = \mathscr{H}_q(W) \otimes_{\mathscr{H}_{q'}(W')} -$$

is left adjoint to ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$. The *coinduction* functor

$${}^{\mathscr{H}}\mathrm{coInd}^W_{W'} = \mathrm{Hom}_{\mathscr{H}_{q'}(W')}(\mathscr{H}_q(W), -),$$

is right adjoint to ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$. All of the three functors above are exact functors.

Now, let us recall the definition of $\imath_q$. It is induced from an inclusion

$$\imath : B_{W'} \hookrightarrow B_W,$$

which is in turn the composition of three morphisms ℓ, κ, $\jmath$ defined as follows. First, let $\mathcal{A}' \subset \mathcal{A}$ be the set of reflecting hyperplanes of W'. Write

$$\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}^{W'}, \quad \overline{\mathcal{A}} = \{\overline{H} = H/\mathfrak{h}^{W'} \mid H \in \mathcal{A}'\}, \quad \overline{\mathfrak{h}}_{reg} = \overline{\mathfrak{h}} - \bigcup_{\overline{H} \in \overline{\mathcal{A}}} \overline{H}, \quad \mathfrak{h}'_{reg} = \mathfrak{h} - \bigcup_{H \in \mathcal{A}'} H.$$

The canonical epimorphism $p : \mathfrak{h} \to \overline{\mathfrak{h}}$ induces a trivial W'-equivariant fibration $p : \mathfrak{h}'_{reg} \to \overline{\mathfrak{h}}_{reg}$, which yields an isomorphism

$$\ell : B_{W'} = \pi_1(\overline{\mathfrak{h}}_{reg}/W', p(x_0)) \xrightarrow{\sim} \pi_1(\mathfrak{h}'_{reg}/W', x_0). \tag{1.2.1}$$

Next, we equip $\mathfrak{h}$ with a W-invariant hermitian scalar product. Let $|| \cdot ||$ be the associated norm. Set

$$\Omega = \{x \in \mathfrak{h} \mid ||x - b|| < \varepsilon\}, \tag{1.2.2}$$

where ε is a positive real number such that the closure of Ω does not intersect any hyperplane that is in the complement of $\mathcal{A}'$ in $\mathcal{A}$. Let $\gamma : [0,1] \to \mathfrak{h}$ be a path such that $\gamma(0) = x_0$, $\gamma(1) = b$ and $\gamma(t) \in \mathfrak{h}_{reg}$ for $0 < t < 1$. Let $u \in [0, 1[$ such that $x_1 = \gamma(u)$ belongs to Ω, write γ_u for the restriction of γ to $[0, u]$. Consider the homomorphism

$$\sigma : \pi_1(\Omega \cap \mathfrak{h}_{reg}, x_1) \to \pi_1(\mathfrak{h}_{reg}, x_0), \quad \lambda \mapsto \gamma_u^{-1} \cdot \lambda \cdot \gamma_u.$$

The canonical inclusion $\mathfrak{h}_{reg} \hookrightarrow \mathfrak{h}'_{reg}$ induces a homomorphism $\pi_1(\mathfrak{h}_{reg}, x_0) \to \pi_1(\mathfrak{h}'_{reg}, x_0)$. Composing it with σ gives an invertible homomorphism

$$\pi_1(\Omega \cap \mathfrak{h}_{reg}, x_1) \to \pi_1(\mathfrak{h}'_{reg}, x_0).$$

Since Ω is W'-invariant, its inverse gives an isomorphism

$$\kappa : \pi_1(\mathfrak{h}'_{reg}/W', x_0) \xrightarrow{\sim} \pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1). \tag{1.2.3}$$

Finally, we see from above that σ is injective. So it induces an inclusion

$$\pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W', x_0).$$

Composing it with the canonical inclusion $\pi_1(\mathfrak{h}_{reg}/W', x_0) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W, x_0)$ gives an injective homomorphism

$$\jmath : \pi_1((\Omega \cap \mathfrak{h}_{reg})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{reg}/W, x_0) = B_W. \tag{1.2.4}$$

By composing ℓ, κ, $\jmath$ we get the inclusion

$$\imath = \jmath \circ \kappa \circ \ell : B_{W'} \hookrightarrow B_W. \tag{1.2.5}$$

It is proved in [BMR98, Section 4C] that $\imath$ preserves the relations in (1.1.1). So it induces an inclusion of Hecke algebras which is the desired inclusion

$$\imath_q : \mathscr{H}_{q'}(W') \hookrightarrow \mathscr{H}_q(W).$$

Note that if $\imath$, $\imath' : B_{W'} \hookrightarrow B_W$ are two inclusions defined as above via different choices of the path γ, then there exists an element $\rho \in \pi_1(\mathfrak{h}_{reg}, x_0)$ such that for any $a \in B_{W'}$ we have $\imath(a) = \rho\imath'(a)\rho^{-1}$. In particular, the functors $\imath_*$ and $(\imath')_*$ from B_W-mod to $B_{W'}$-mod are isomorphic. Also, we have $(\imath_q)_* \simeq (\imath'_q)_*$. So there is a unique restriction functor ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$ up to isomorphisms.

1.3 Biadjointness of ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$ and ${}^{\mathscr{H}}\mathrm{Ind}^W_{W'}$

We say that a finite dimensional $\mathbb{C}$-algebra A is *symmetric* if A is isomorphic to $A^* = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ as (A, A)-bimodules.

Proposition 1.3.1. *Assume that $\mathscr{H}_q(W)$ and $\mathscr{H}_{q'}(W')$ are symmetric algebras. Then the functors ${}^{\mathscr{H}}\mathrm{Ind}^W_{W'}$ and ${}^{\mathscr{H}}\mathrm{coInd}^W_{W'}$ are isomorphic, i.e., the functor ${}^{\mathscr{H}}\mathrm{Ind}^W_{W'}$ is biadjoint to ${}^{\mathscr{H}}\mathrm{Res}^W_{W'}$.*

Proof. We abbreviate $\mathscr{H} = \mathscr{H}_q(W)$ and $\mathscr{H}' = \mathscr{H}_{q'}(W')$. Since $\mathscr{H}$ is free as a left $\mathscr{H}'$-module, for any $\mathscr{H}'$-module M the map

$$\mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathscr{H}') \otimes_{\mathscr{H}'} M \to \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, M) \tag{1.3.1}$$

given by multiplication is an isomorphism of $\mathscr{H}$-modules. By assumption $\mathscr{H}'$ is isomorphic to $(\mathscr{H}')^*$ as $(\mathscr{H}', \mathscr{H}')$-bimodules. Thus we have the following $(\mathscr{H}, \mathscr{H}')$-bimodule isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathscr{H}') &= \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, (\mathscr{H}')^*) \\ &= \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}' \otimes_{\mathscr{H}'} \mathscr{H}, \mathbb{C}) \\ &= \mathscr{H}^* \\ &= \mathscr{H}. \end{aligned}$$

The last isomorphism follows from the fact the $\mathscr{H}$ is symmetric. Thus, by (1.3.1) the functors ${}^{\mathscr{H}}\mathrm{Ind}^W_{W'}$ and ${}^{\mathscr{H}}\mathrm{coInd}^W_{W'}$ are isomorphic. □

Remark 1.3.2. The Hecke algebra $\mathscr{H}_q(W)$ is known to be symmetric for all irreducible complex reflection group W except for some of the 34 exceptional groups in the Shephard-Todd classification. See [BMM99, Section 2A] for details.

2 Category $\mathcal{O}$ of rational DAHA's

2.1 Rational double affine Hecke algerbas.

The rational double affine Hecke algebras (=rational DAHA's), also called rational Cherednik algebras, have been introduced by Etingof and Ginzburg [EG02]. Let us recall their definition.

Definition 2.1.1. Let c be a map from $\mathcal{S}$ to $\mathbb{C}$ that is constant on the W-conjugacy classes. The *rational double affine Hecke algebra* $H_c(W,\mathfrak{h})$ is the quotient of the smash product of $\mathbb{C}W$ and the tensor algebra of $\mathfrak{h}\oplus\mathfrak{h}^*$ by the relations

$$[x,x']=0,\quad [y,y']=0,\quad [y,x]=\langle x,y\rangle-\sum_{s\in\mathcal{S}}c_s\langle\alpha_s,y\rangle\langle x,\alpha_s^\vee\rangle s,$$

for all $x,x'\in\mathfrak{h}^*$, $y,y'\in\mathfrak{h}$. Here $\langle\cdot,\cdot\rangle$ is the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$, the element α_s is a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*}-1)$ and $\alpha_s^\vee$ is the generator of $\mathrm{Im}(s|_{\mathfrak{h}}-1)$ such that $\langle\alpha_s,\alpha_s^\vee\rangle=2$.

By definition the algebra $H_c(W,\mathfrak{h})$ contains three subalgebras $\mathbb{C}W$, $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. We have the following Poincaré-Birkhoff-Witt type theorem [EG02, Theorem 1.3].

Proposition 2.1.2. *The multiplication map yields an isomorphism of* $\mathbb{C}$*-vector spaces*

$$\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}}\mathbb{C}W\otimes_{\mathbb{C}}\mathbb{C}[\mathfrak{h}^*]\xrightarrow{\sim}H_c(W,\mathfrak{h}).$$

For $s\in\mathcal{S}$ write λ_s for the non trivial eigenvalue of s in $\mathfrak{h}^*$. The element

$$\mathbf{eu}_0=\sum_{s\in\mathcal{S}}\frac{2c_s}{1-\lambda_s}s-\frac{\dim(\mathfrak{h})}{2}. \tag{2.1.1}$$

is in the center of $\mathbb{C}W$. Let $\{x_i\}$ be a basis of $\mathfrak{h}^*$ and let $\{y_i\}$ be the dual basis. The *Euler element* in $H_c(W,\mathfrak{h})$ is given by

$$\mathbf{eu}=\sum_i x_iy_i-\mathbf{eu}_0 \tag{2.1.2}$$

Its definition is independent of the choice of the basis $\{x_i\}$. We have

$$[\mathbf{eu},x_i]=x_i,\quad [\mathbf{eu},y_i]=-y_i,\quad [\mathbf{eu},s]=0. \tag{2.1.3}$$

2.2 The category $\mathcal{O}$

A category $\mathcal{O}$ for $H_c(W,\mathfrak{h})$ has been introduced in [GGOR03]. Let us recall its definition and some basic properties.

Definition 2.2.1. The *category* $\mathcal{O}$ of $H_c(W,\mathfrak{h})$ is the full subcategory $\mathcal{O}_c(W,\mathfrak{h})$ of the category of $H_c(W,\mathfrak{h})$-modules consisting of objects that are finitely generated as $\mathbb{C}[\mathfrak{h}]$-modules and $\mathfrak{h}$-locally nilpotent.

Note that by definition for any object M in $\mathcal{O}_c(W,\mathfrak{h})$ the action of $\mathbf{eu}$ on M is locally finite. Let $\mathrm{Irr}(W)$ be the set of irreducible representations of W over $\mathbb{C}$. We will view each $\xi\in\mathrm{Irr}(W)$ as a $\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}]$-module via the pull back of the projection

$$\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}]\to\mathbb{C}W,\quad (w,x)\mapsto w,\quad \forall\ x\in\mathfrak{h}^*,\ w\in W.$$

The *standard module* associated with ξ is the induced $H_c(W,\mathfrak{h})$-module

$$\Delta(\xi) = H_c(W,\mathfrak{h}) \otimes_{\mathbb{C}W\ltimes\mathbb{C}[\mathfrak{h}^*]} \xi. \tag{2.2.1}$$

It is an indecomposable object of $\mathcal{O}_c(W,\mathfrak{h})$ with a simple head $L(\xi)$. As ξ varies in $\mathrm{Irr}(W)$ the $L(\xi)$'s give a complete set of nonisomorphic simple objects in $\mathcal{O}_c(W,\mathfrak{h})$.

We say that an object M in $\mathcal{O}_c(W,\mathfrak{h})$ *has a standard filtration* if there is a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_d = M$$

by objects in $\mathcal{O}_c(W,\mathfrak{h})$ such that each quotient M_i/M_{i-1} is isomorphic to a standard module. Let $\mathcal{O}_c^\Delta(W,\mathfrak{h})$ be the full subcategory of $\mathcal{O}_c(W,\mathfrak{h})$ consisting of such objects. The following is due to [GGOR03, Proposition 2.21].

Lemma 2.2.2. *An object in $\mathcal{O}_c(W,\mathfrak{h})$ has a standard filtration if and only if it is free as a $\mathbb{C}[\mathfrak{h}]$-module.*

The category $\mathcal{O}_c(W,\mathfrak{h})$ is an artinian category with enough projective objects. For each $L(\xi)$ let $P(\xi)$ be a projective cover of $L(\xi)$ in $\mathcal{O}_c(W,\mathfrak{h})$. For $\xi \in \mathrm{Irr}(W)$ let $c_\xi \in \mathbb{C}$ be the scalar by which the central element $\mathbf{eu}_0 \in \mathbb{C}W$ acts on ξ. Following [GGOR03, Theorem 2.19], we equip $\mathrm{Irr}(W)$ with a partial order $\prec_c$ given by

$$\xi \prec_c \eta \iff c_\eta - c_\xi \text{ is a positive integer.}$$

Proposition 2.2.3. *The category $\mathcal{O}_c(W,\mathfrak{h})$ is a highest weight category with respect to the standard objects $\Delta(\xi)$ and the partial order $\prec_c$, i.e., for $\xi \in \mathrm{Irr}(W)$ we have*

(a) The Jordan-Hölder factors in the kernel of the quotient map $\Delta(\xi) \to L(\xi)$ are of the form $L(\eta)$ with $\eta \prec_c \xi$.

(b) The object $P(\xi)$ has a standard filtration

$$P(\xi)_0 = 0 \subset P(\xi)_1 \subset \ldots \subset P(\xi)_d = P(\xi)$$

such that $P(\xi)_d/P(\xi)_{d-1} \cong \Delta(\xi)$ and $P(\xi)_i/P(\xi)_{i-1} \cong \Delta(\eta)$ with $\eta \succ \xi$ for $1 \leqslant i \leqslant d-1$. In particular, any projective object in $\mathcal{O}_c(W,\mathfrak{h})$ has a standard filtration.

See [GGOR03, Corollaries 2.10, 2.14, Theorem 2.19].

Corollary 2.2.4. *(a) The sets*

$$\{\,[\Delta(\xi)]\,|\,\xi \in \mathrm{Irr}(W)\,\} \quad \textit{and} \quad \{\,[L(\xi)]\,|\,\xi \in \mathrm{Irr}(W)\,\}$$

give two bases of the $\mathbb{C}$-vector space $[\mathcal{O}_c(W,\mathfrak{h})]$.

(b) The category $\mathcal{O}_c(W,\mathfrak{h})$ has finite homological dimension.

Proof. Since $\mathcal{O}_c(W,\mathfrak{h})$ is artinian, the classes of simple objects form a basis of $[\mathcal{O}_c(W,\mathfrak{h})]$. By Proposition 2.2.3(a), this implies that the classes of standard objects also form a basis. This proves part (a). Part (b) is a general fact about highest weight categories, see e.g., [Don98, Appendix, Proposition A2.3]. □

2.3 Hom-space for functors on $\mathcal{O}_c(W, \mathfrak{h})$.

Let $\mathrm{Proj}_c(W, \mathfrak{h})$ denote the full subcategory of $\mathcal{O}_c(W, \mathfrak{h})$ consisting of projective objects. Let

$$I : \mathrm{Proj}_c(W, \mathfrak{h}) \to \mathcal{O}_c(W, \mathfrak{h})$$

denote the canonical embedding functor. The following lemma will be useful to us.

Lemma 2.3.1. *For any abelian category $\mathcal{A}$ and any right exact functors F_1, F_2 from $\mathcal{O}_c(W, \mathfrak{h})$ to $\mathcal{A}$, the homomorphism of vector spaces*

$$r_I : \mathrm{Hom}(F_1, F_2) \to \mathrm{Hom}(F_1 \circ I, F_2 \circ I), \quad \gamma \mapsto \gamma 1_I$$

is an isomorphism.

Proof. We need to show that for any morphism of functors $\nu : F_1 \circ I \to F_2 \circ I$ there is a unique morphism $\tilde{\nu} : F_1 \to F_2$ such that $\tilde{\nu} 1_I = \nu$. Since $\mathcal{O}_c(W, \mathfrak{h})$ has enough projectives, for any $M \in \mathcal{O}_c(W, \mathfrak{h})$ there exists P_0, $P_1 \in \mathrm{Proj}_c(W, \mathfrak{h})$ and an exact sequence in $\mathcal{O}_c(W, \mathfrak{h})$

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0. \tag{2.3.1}$$

Applying the right exact functors F_1, F_2 to this sequence we get the two exact sequences in the diagram below. The morphism of functors $\nu : F_1 \circ I \to F_2 \circ I$ yields well defined morphisms $\nu(P_1)$, $\nu(P_0)$ such that the square commutes

$$\begin{tikzcd}
F_1(P_1) \arrow[r, "F_1(d_1)"] \arrow[d, "\nu(P_1)"] & F_1(P_0) \arrow[r, "F_1(d_0)"] \arrow[d, "\nu(P_0)"] & F_1(M) \arrow[r] & 0 \\
F_2(P_1) \arrow[r, "F_2(d_1)"] & F_2(P_0) \arrow[r, "F_2(d_0)"] & F_2(M) \arrow[r] & 0.
\end{tikzcd}$$

Define $\tilde{\nu}(M)$ to be the unique morphism $F_1(M) \to F_2(M)$ that makes the diagram commute. Its definition is independent of the choice of P_0, P_1, and it is independent of the choice of the exact sequence (2.3.1). The assignment $M \mapsto \tilde{\nu}(M)$ gives a morphism of functor $\tilde{\nu} : F_1 \to F_2$ such that $\tilde{\nu} 1_I = \nu$. It is unique by the uniqueness of the morphism $\tilde{\nu}(M)$. $\square$

Corollary 2.3.2. *Let F_1, F_2 be two functors as in the lemma above. Then we have*

$$F_1 \cong F_2 \quad \textit{if and only if} \quad F_1 \circ I \cong F_2 \circ I.$$

2.4 The Knizhnik-Zamolodchikov functor

The category $\mathcal{O}_c(W, \mathfrak{h})$ is closely related to a Hecke algebra $\mathscr{H}_q(W, \mathfrak{h})$ via the so called Knizhnik-Zamolodchikov functor. Let us recall its definition from [GGOR03, Section 5.3].

Let $\mathcal{D}(\mathfrak{h}_{reg})$ be the algebra of differential operators on $\mathfrak{h}_{reg}$. We abbreviate

$$H_c(W, \mathfrak{h}_{reg}) = H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}].$$

There is an isomorphism of algebras

$$H_c(W, \mathfrak{h}_{reg}) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{reg}) \rtimes \mathbb{C}W,$$

called the *Dunkl isomorphism*. It is given by the assignment

$$\begin{array}{rcl} x & \mapsto & x, \\ w & \mapsto & w, \\ y & \mapsto & \partial_y + \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(s-1), \end{array}$$

for $x \in \mathfrak{h}^*$, $w \in W$, and $y \in \mathfrak{h}$. For $M \in \mathcal{O}_c(W, \mathfrak{h})$ let

$$M_{\mathfrak{h}_{reg}} = M \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}].$$

It identifies via the Dunkl isomorphism as a $\mathcal{D}(\mathfrak{h}_{reg}) \rtimes W$-module which is finitely generated over $\mathbb{C}[\mathfrak{h}_{reg}]$. So $M_{\mathfrak{h}_{reg}}$ is a W-equivariant vector bundle on $\mathfrak{h}_{reg}$ with an integrable connection ∇ given by $\nabla_y(m) = \partial_y m$ for $m \in M$, $y \in \mathfrak{h}$. Further, the connection ∇ has regular singularities, see [GGOR03, Proposition 5.7]. Now, regard $\mathfrak{h}_{reg}$ as a complex manifold endowed with the transcendental topology. Let $\mathcal{O}^{an}_{\mathfrak{h}_{reg}}$ be the sheaf of holomorphic functions on $\mathfrak{h}_{reg}$. For any free $\mathbb{C}[\mathfrak{h}_{reg}]$-module N of finite rank, we consider

$$N^{an} = N \otimes_{\mathbb{C}[\mathfrak{h}_{reg}]} \mathcal{O}^{an}_{\mathfrak{h}_{reg}}.$$

It is an analytic locally free sheaf on $\mathfrak{h}_{reg}$. For ∇ an integrable connection on N, the sheaf of holomorphic horizontal sections

$$N^{\nabla} = \{n \in N^{an} \mid \nabla_y(n) = 0 \text{ for all } y \in \mathfrak{h}\}$$

is a W-equivariant local system on $\mathfrak{h}_{reg}$. Hence it identifies with a local system on $\mathfrak{h}_{reg}/W$. So it yields a finite dimensional representation of $\mathbb{C}B(W, \mathfrak{h})$. For $M \in \mathcal{O}_c(W, \mathfrak{h})$ it is proved in [GGOR03, Theorem 5.13] that the action of $\mathbb{C}B(W, \mathfrak{h})$ on $(M_{\mathfrak{h}_{reg}})^{\nabla}$ factors through a Hecke algebra $\mathscr{H}_q(W, \mathfrak{h})$ with the parameter q given in [GGOR03, Section 5.2].

Definition 2.4.1. The *Knizhnik-Zamolodchikov functor* is the exact functor given by

$$\mathrm{KZ}(W, \mathfrak{h}) : \mathcal{O}_c(W, \mathfrak{h}) \to \mathscr{H}_q(W, \mathfrak{h})\text{-mod}, \quad M \mapsto (M_{\mathfrak{h}_{reg}})^{\nabla}.$$

We may abbreviate $\mathrm{KZ} = \mathrm{KZ}(W, \mathfrak{h})$ if this does not create any confusion.

2.5 Properties of the functor KZ

In this section, we assume that

the algebras $\mathscr{H}_q(W, \mathfrak{h})$ and $\mathbb{C}W$ have the same dimension over $\mathbb{C}$.

We recall some properties of KZ from [GGOR03]. The functor KZ is represented by a projective object P_{KZ} in $\mathcal{O}_c(W, \mathfrak{h})$. More precisely, there is an algebra homomorphism

$$\rho : \mathscr{H}_q(W, \mathfrak{h}) \to \mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}})^{\mathrm{op}}$$

such that KZ is isomorphic to the functor $\mathrm{Hom}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}}, -)$. By [GGOR03, Theorem 5.15] the homomorphism ρ is an isomorphism. In particular $\mathrm{KZ}(P_{\mathrm{KZ}})$ is isomorphic to $\mathscr{H}_q(W, \mathfrak{h})$ as $\mathscr{H}_q(W, \mathfrak{h})$-modules.

Next, let $Z(\mathcal{O}_c(W, \mathfrak{h}))$ be the center of the category $\mathcal{O}_c(W, \mathfrak{h})$, that is the algebra of endomorphisms of the identity functor $Id_{\mathcal{O}_c(W,\mathfrak{h})}$. Then we have a canonical map

$$Z(\mathcal{O}_c(W, \mathfrak{h})) \to \mathrm{End}_{\mathcal{O}_c(W,\mathfrak{h})}(P_{\mathrm{KZ}}).$$

The composition of this map with ρ^{-1} yields an algebra homomorphism

$$\gamma : Z(\mathcal{O}_c(W,\mathfrak{h})) \to Z(\mathscr{H}_q(W,\mathfrak{h})),$$

where $Z(\mathscr{H}_q(W,\mathfrak{h}))$ denotes the center of $\mathscr{H}_q(W,\mathfrak{h})$.

Lemma 2.5.1. *(a) The homomorphism γ is an isomorphism.*
(b) For $M \in \mathcal{O}_c(W,\mathfrak{h})$ and $f \in Z(\mathcal{O}_c(W,\mathfrak{h}))$ the morphism

$$\mathrm{KZ}(f(M)) : \mathrm{KZ}(M) \to \mathrm{KZ}(M)$$

is the multiplication by $\gamma(f)$.

See [GGOR03, Corollary 5.18] for (a). Part (b) follows from the definition of γ.

The functor KZ is a quotient functor, see [GGOR03, Theorem 5.14]. Therefore it has a right adjoint $S : \mathscr{H}_q(W,\mathfrak{h})\text{-mod} \to \mathcal{O}_c(W,\mathfrak{h})$ such that the canonical adjunction map

$$\mathrm{KZ} \circ S \to \mathrm{Id}_{\mathscr{H}_q(W,\mathfrak{h})}$$

is an isomorphism of functors. We have the following proposition.

Proposition 2.5.2. *Let Q be a projective object in $\mathcal{O}_c(W,\mathfrak{h})$.*
(a) For any object $M \in \mathcal{O}_c(W,\mathfrak{h})$, the following morphism of $\mathbb{C}$-vector spaces is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_c(W,\mathfrak{h})}(M,Q) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{H}_q(W)}(\mathrm{KZ}(M),\mathrm{KZ}(Q)), \quad f \mapsto \mathrm{KZ}(f).$$

In particular, the functor KZ *is fully faithful over* $\mathrm{Proj}_c(W,\mathfrak{h})$*.*
(b)The canonical adjunction map gives an isomorphism $Q \xrightarrow{\sim} S \circ \mathrm{KZ}(Q)$.

See [GGOR03, Theorems 5.3, 5.16].

3 Parabolic restriction and induction for rational DAHA's.

From now on we assume that $\mathfrak{h}^W = 1$. Fix a point $b \in \mathfrak{h}$. Let W' be the parabolic subgroup of W given by the stabilizer of b. We will use the same notation as in Section 1.2. Let $c' : \mathcal{S}' \to \mathbb{C}$ be the map given by the restriction of $c : \mathcal{S} \to \mathbb{C}$, and consider the rational DAHA $H_{c'}(W',\overline{\mathfrak{h}})$ and the category $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$. In [BE09], Bezrukavnikov and Etingof defined the parabolic restriction and induction functors

$$\mathrm{Res}_b : \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_{c'}(W',\overline{\mathfrak{h}})\,, \quad \mathrm{Ind}_b : \mathcal{O}_{c'}(W',\overline{\mathfrak{h}}) \to \mathcal{O}_c(W,\mathfrak{h}).$$

In this section, we will first review their constructions (Sections 3.1–3.3), then we give some further properties of these functors (Sections 3.4–3.8).

3.1 The equivalence ζ.

We first explain the relation between the category $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$ and the category $\mathcal{O}_{c'}(W',\mathfrak{h})$. Let $\mathfrak{h}^{*W'}$ be the subspace of $\mathfrak{h}^*$ consisting of fixed points of W'. Set

$$(\mathfrak{h}^{*W'})^{\perp} = \{v \in \mathfrak{h} \mid f(v) = 0 \text{ for all } f \in \mathfrak{h}^{*W'}\}.$$

We have a W'-invariant decomposition

$$\mathfrak{h} = (\mathfrak{h}^{*W'})^{\perp} \oplus \mathfrak{h}^{W'}.$$

The W'-space $(\mathfrak{h}^{*W'})^\perp$ is canonically identified with $\bar{\mathfrak{h}}$. Since the action of W' on $\mathfrak{h}^{W'}$ is trivial, we have an obvious algebra isomorphism

$$H_{c'}(W',\mathfrak{h}) = H_{c'}(W',\bar{\mathfrak{h}}) \otimes \mathcal{D}(\mathfrak{h}^{W'}). \tag{3.1.1}$$

It maps an element y in the subset $\mathfrak{h}^{W'}$ of $H_{c'}(W',\mathfrak{h})$ to the operator ∂_y in $\mathcal{D}(\mathfrak{h}^{W'})$. Write $\mathcal{O}(1,\mathfrak{h}^{W'})$ for the category of finitely generated $\mathcal{D}(\mathfrak{h}^{W'})$-modules that are ∂_y-locally nilpotent for all $y \in \mathfrak{h}^{W'}$. Then the algebra isomorphism above yields an equivalence of categories

$$\mathcal{O}_{c'}(W',\mathfrak{h}) = \mathcal{O}_{c'}(W',\bar{\mathfrak{h}}) \otimes \mathcal{O}(1,\mathfrak{h}^{W'}). \tag{3.1.2}$$

Upon taking Fourier transform, Kashiwara's lemma (see e.g., [HTT08, Theorem 1.6.1]) implies that the functor

$$\mathcal{O}(1,\mathfrak{h}^{W'}) \xrightarrow{\sim} \mathbb{C}\text{-mod}, \quad M \to \{m \in M \mid \partial_y(m) = 0,\ \forall\, y \in \mathfrak{h}^{W'}\},$$

is an equivalence of categories. Composing it with the equivalence (3.1.2) we get an equivalence of categories

$$\zeta : \mathcal{O}_{c'}(W',\mathfrak{h}) \to \mathcal{O}_{c'}(W',\bar{\mathfrak{h}}), \quad M \mapsto \{v \in M \mid yv = 0,\ \forall\, y \in \mathfrak{h}^{W'}\}. \tag{3.1.3}$$

It has a quasi-inverse given by

$$\zeta^{-1} :\ \mathcal{O}_{c'}(W',\bar{\mathfrak{h}}) \to \mathcal{O}_{c'}(W',\mathfrak{h}), \quad N \mapsto N \otimes \mathbb{C}[\mathfrak{h}^{W'}], \tag{3.1.4}$$

where $\mathbb{C}[\mathfrak{h}^{W'}] \in \mathcal{O}(1,\mathfrak{h}^{W'})$ is the polynomial representation of $\mathcal{D}(\mathfrak{h}^{W'})$. Moreover, the functor ζ maps a standard module in $\mathcal{O}_{c'}(W',\mathfrak{h})$ to a standard module in $\mathcal{O}_{c'}(W',\bar{\mathfrak{h}})$. Indeed, for any $\xi \in \mathrm{Irr}(W')$, we have an isomorphism of $H_{c'}(W',\mathfrak{h})$-modules

$$H_{c'}(W',\mathfrak{h}) \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes W'} \xi = \big(H_{c'}(W',\bar{\mathfrak{h}}) \otimes_{\mathbb{C}[(\bar{\mathfrak{h}})^*]\rtimes W'} \xi\big) \otimes \big(\mathcal{D}(\mathfrak{h}^{W'}) \otimes_{\mathbb{C}[(\mathfrak{h}^{W'})^*]} \mathbb{C}\big).$$

On the right hand side $\mathbb{C}$ denotes the trivial module of $\mathbb{C}[(\mathfrak{h}^{W'})^*]$, and the latter is identified with the subalgebra of $\mathcal{D}(\mathfrak{h}^{W'})$ generated by ∂_y for all $y \in \mathfrak{h}^{W'}$. We have

$$\mathcal{D}(\mathfrak{h}^{W'}) \otimes_{\mathbb{C}[(\mathfrak{h}^{W'})^*]} \mathbb{C} = \mathbb{C}[\mathfrak{h}^{W'}]$$

as $\mathcal{D}(\mathfrak{h}^{W'})$-modules. So ζ maps the standard module $\Delta(\xi)$ for $H_{c'}(W',\mathfrak{h})$ to the standard module $\Delta(\xi)$ for $H_{c'}(W',\bar{\mathfrak{h}})$.

3.2 The isomorphism Θ and the equivalence R.

For a point $p \in \mathfrak{h}$ we write $\mathbb{C}[[\mathfrak{h}]]_p$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at p, and we write $\widehat{\mathbb{C}[\mathfrak{h}]}_p$ for the completion of $\mathbb{C}[\mathfrak{h}]$ at the W-orbit of p in $\mathfrak{h}$. Note that we have $\mathbb{C}[[\mathfrak{h}]]_0 = \widehat{\mathbb{C}[\mathfrak{h}]}_0$. For any $\mathbb{C}[\mathfrak{h}]$-module M we write

$$\widehat{M}_p = \widehat{\mathbb{C}[\mathfrak{h}]}_p \otimes_{\mathbb{C}[\mathfrak{h}]} M.$$

The completions $\widehat{H}_c(W,\mathfrak{h})_b$, $\widehat{H}_{c'}(W',\mathfrak{h})_0$ are well defined algebras. We denote by $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$ the category of $\widehat{H}_c(W,\mathfrak{h})_b$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_b$, and we denote by $\widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0$ the category of $\widehat{H}_{c'}(W',\mathfrak{h})_0$-modules that are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_0$. Let

$$P = \mathrm{Fun}_{W'}(W, \widehat{H}_c(W',\mathfrak{h})_0)$$

be the set of W'-invariant maps from W to $\widehat{H}_c(W',\mathfrak{h})_0$. Let $Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)$ be the ring of endomorphisms of the right $\widehat{H}_c(W',\mathfrak{h})_0$-module P. The following is due to Bezrukavnikov and Etingof [BE09, Theorem 3.2].

Proposition 3.2.1. *There is an isomorphism of algebras*

$$\Theta : \widehat{H}_c(W,\mathfrak{h})_b \longrightarrow Z(W, W', \widehat{H}_{c'}(W',\mathfrak{h})_0)$$

defined as follows: for $f \in P$, $\alpha \in \mathfrak{h}^*$, $a \in \mathfrak{h}$, $u \in W$,

$$\begin{array}{rcl} (\Theta(u)f)(w) & = & f(wu), \\ (\Theta(x_\alpha)f)(w) & = & (x^{(b)}_{w\alpha} + \alpha(w^{-1}b))f(w), \\ (\Theta(y_a)f)(w) & = & y^{(b)}_{wa}f(w) + \displaystyle\sum_{s\in\mathcal{S}, s\notin W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s}+\alpha_s(b)}(f(sw)-f(w)), \end{array}$$

where $x_\alpha \in \mathfrak{h}^* \subset H_c(W,\mathfrak{h})$, $x^{(b)}_\alpha \in \mathfrak{h}^* \subset H_{c'}(W',\mathfrak{h})$, $y_a \in \mathfrak{h} \subset H_c(W,\mathfrak{h})$, $y^{(b)}_a \in \mathfrak{h} \subset H_{c'}(W',\mathfrak{h})$.

Using Θ we will identify $\widehat{H}_c(W,\mathfrak{h})_b$-modules with $Z(W,W',\widehat{H}_{c'}(W',\mathfrak{h})_0)$-modules. So $P = \mathrm{Fun}_{W'}(W, \widehat{H}_c(W',\mathfrak{h})_0)$ becomes an $(\widehat{H}_c(W,\mathfrak{h})_b, \widehat{H}_{c'}(W',\mathfrak{h})_0)$-bimodule. Hence for any $N \in \widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0$ the module $P \otimes_{\widehat{H}_{c'}(W',\mathfrak{h})_0} N$ lives in $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$. It is naturally identified with $\mathrm{Fun}_{W'}(W,N)$, the set of W'-invariant maps from W to N. By Morita theory, the functor

$$J : \widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0 \to \widehat{\mathcal{O}}_c(W,\mathfrak{h})_b, \quad N \mapsto \mathrm{Fun}_{W'}(W,N),$$

is an equivalence of categories. Let us give a quasi-inverse of J. To this end, fix elements $1 = u_1, u_2, \ldots, u_r$ in W such that $W = \bigsqcup_{i=1}^r W'u_i$. Let $\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$ be the algebra of $r \times r$ matrices with coefficients in $\widehat{H}_{c'}(W',\mathfrak{h})_0$. We have an algebra isomorphism

$$\begin{array}{rcl} \Phi : Z(W,W',\widehat{H}_{c'}(W',\mathfrak{h})_0) & \to & \mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0), \\ A & \mapsto & (\Phi(A)_{ij})_{1\leqslant i,j\leqslant r} \end{array} \tag{3.2.1}$$

such that

$$(Af)(u_i) = \sum_{j=1}^r \Phi(A)_{ij} f(u_j), \quad \forall\, f \in P,\ 1 \leqslant i \leqslant r.$$

Denote by E_{ij}, $1 \leqslant i,j \leqslant r$, the elementary matrix in $\mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$ with coefficient 1 in the position (i,j) and zero elsewhere. Then the algebra isomorphism

$$\Phi\circ\Theta : \widehat{H}_c(W,\mathfrak{h})_b \xrightarrow{\sim} \mathrm{Mat}_r(\widehat{H}_{c'}(W',\mathfrak{h})_0)$$

restricts to an isomorphism of subalgebras

$$\widehat{\mathbb{C}[\mathfrak{h}]}_b \cong \bigoplus_{i=1}^r \mathbb{C}[[\mathfrak{h}]]_0 E_{ii}. \tag{3.2.2}$$

Indeed, there is a unique isomorphism of algebras

$$\varpi : \widehat{\mathbb{C}[\mathfrak{h}]}_b \cong \bigoplus_{i=1}^r \mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b}. \tag{3.2.3}$$

extending the algebra homomorphism

$$\mathbb{C}[\mathfrak{h}] \to \bigoplus_{i=1}^r \mathbb{C}[\mathfrak{h}], \quad x \mapsto (x,x,\ldots,x), \quad \forall\, x \in \mathfrak{h}^*.$$

For each i consider the isomorphism of algebras

$$\phi_i : \mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b} \to \mathbb{C}[[\mathfrak{h}]]_0, \quad x \mapsto u_i x + x(u_i^{-1}b), \quad \forall\, x \in \mathfrak{h}^*.$$

The isomorphism (3.2.2) is exactly the composition of ϖ with the direct sum $\oplus_{i=1}^r \phi_i$, and E_{ii} is given by the image of the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to the component $\mathbb{C}[[\mathfrak{h}]]_{u_i^{-1}b}$. We will denote by x_{pr} the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to $\mathbb{C}[[\mathfrak{h}]]_b$, i.e., $\Phi \circ \Theta(x_{\mathrm{pr}}) = E_{11}$. Then a quasi-inverse of J is given by the functor

$$R : \widehat{\mathcal{O}}_c(W, \mathfrak{h})_b \to \widehat{\mathcal{O}}_{c'}(W', \mathfrak{h})_0, \quad M \mapsto x_{\mathrm{pr}} M,$$

where the action of $\widehat{H}_{c'}(W', \mathfrak{h})_0$ on $R(M) = x_{\mathrm{pr}} M$ is given by the following formulas. For any $\alpha \in \mathfrak{h}^*$, $w \in W'$, $a \in \mathfrak{h}^*$, $m \in M$, we have

$$x_\alpha^{(b)} x_{\mathrm{pr}}(m) = x_{\mathrm{pr}}((x_\alpha - \alpha(b))m), \tag{3.2.4}$$

$$w x_{\mathrm{pr}}(m) = x_{\mathrm{pr}}(wm), \tag{3.2.5}$$

$$y_a^{(b)} x_{\mathrm{pr}}(m) = x_{\mathrm{pr}}((y_a + \sum_{s \in \mathcal{S},\, s \notin W'} \frac{2c_s}{1-\lambda_s} \frac{\alpha_s(a)}{x_{\alpha_s}})m). \tag{3.2.6}$$

In particular, we have

$$R(M) = \phi_1^*(x_{\mathrm{pr}}(M)) \tag{3.2.7}$$

as $\mathbb{C}[[\mathfrak{h}]]_0 \rtimes W'$-modules. Finally, note that the following equality holds in $\widehat{H}_c(W, \mathfrak{h})_b$

$$x_{\mathrm{pr}} u x_{\mathrm{pr}} = 0, \quad \forall\, u \in W - W'. \tag{3.2.8}$$

3.3 Definition and basic properties of Res_b and Ind_b.

For any $\mathbb{C}[\mathfrak{h}^*]$-module M write $E(M) \subset M$ for the locally nilpotent part of M under the action of $\mathfrak{h}$. It is proved in [BE09, Theorem 2.3] that the functor

$$\widehat{}_0 : \mathcal{O}_{c'}(W', \mathfrak{h}) \to \widehat{\mathcal{O}}_{c'}(W', \mathfrak{h})_0, \quad N \mapsto \widehat{N}_0,$$

is an equivalence of categories. A quasi-inverse is given by

$$E : \widehat{\mathcal{O}}_{c'}(W', \mathfrak{h})_0 \to \mathcal{O}_{c'}(W', \mathfrak{h}), \quad M \mapsto E(M).$$

Moreover, the canonical inclusion $N \subset E(\widehat{N}_0)$ is an equality for any $N \in \mathcal{O}_{c'}(W', \mathfrak{h})$. We will also consider the exact functor

$$\widehat{}_b : \mathcal{O}_c(W, \mathfrak{h}) \to \widehat{\mathcal{O}}_c(W, \mathfrak{h})_b, \quad M \mapsto \widehat{M}_b.$$

It has an exact right adjoint

$$E^b : \widehat{\mathcal{O}}_c(W, \mathfrak{h})_b \to \mathcal{O}_c(W, \mathfrak{h}), \quad N \to E(N).$$

Now, we can give the definition of Res_b and Ind_b [BE09, Section 3.5].

Definition 3.3.1. The *restriction* functor Res_b and the *induction* functor Ind_b are defined by

$$\begin{aligned} \mathrm{Res}_b(M) &= \zeta \circ E \circ R(\widehat{M}_b), && M \in \mathcal{O}_c(W, \mathfrak{h}), \\ \mathrm{Ind}_b(N) &= E^b \circ J(\widehat{\zeta^{-1}(N)}_0), && N \in \mathcal{O}_{c'}(W', \overline{\mathfrak{h}}). \end{aligned} \tag{3.3.1}$$

For any $\mathcal{O}_c(W,\mathfrak{h})$ we consider the following map between Grothendieck groups

$$\omega : [\mathcal{O}_c(W,\mathfrak{h})] \to [\mathbb{C}W], \quad [\Delta(\xi)] \mapsto [\xi], \quad \forall\, \xi \in \mathrm{Irr}(W).$$

Then the functors Res_b and Ind_b have the following properties.

Proposition 3.3.2. *(a) Both functors* Res_b *and* Ind_b *are exact. The functor* Res_b *is left adjoint to* Ind_b*. In particular* Res_b *preserves projective objects and* Ind_b *preserves injective objects.*

(b) Let $\mathrm{Res}^W_{W'}$ *and* $\mathrm{Ind}^W_{W'}$ *be respectively the restriction and induction functors of groups. Then the following diagram commute*

$$\begin{tikzcd}
{[\mathcal{O}_c(W,\mathfrak{h})]} \arrow[r, "\omega", "\sim"'] \arrow[d, shift left, "\mathrm{Res}_b"] & {[\mathbb{C}W]} \arrow[d, shift left, "\mathrm{Res}^W_{W'}"] \\
{[\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})]} \arrow[r, "\omega", "\sim"'] \arrow[u, shift left, "\mathrm{Ind}_b"] & {[\mathbb{C}W'].} \arrow[u, shift left, "\mathrm{Ind}^W_{W'}"]
\end{tikzcd}$$

See [BE09, Proposition 3.9, Theorem 3.10] for part (a) and [BE09, Proposition 3.14] for (b).

3.4 Restriction of modules having a standard filtration

In this section, we study the action of the restriction functors on modules with a standard filtration in $\mathcal{O}_c(W,\mathfrak{h})$. We will need the following lemmas.

Lemma 3.4.1. *Let M be an object of* $\mathcal{O}^{\Delta}_c(W,\mathfrak{h})$.

(a) There is a finite dimensional subspace V of M such that V is stable under the action of $\mathbb{C}W$ *and the map*

$$\mathbb{C}[\mathfrak{h}] \otimes V \to M, \quad p \otimes v \mapsto pv$$

is an isomorphism of $\mathbb{C}[\mathfrak{h}] \rtimes W$*-modules.*

(b) We have the following equality in $[\mathbb{C}W]$

$$\omega([M]) = [V].$$

Proof. Let

$$0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$$

be a filtration of M such that for any $1 \leqslant i \leqslant l$ we have $M_i/M_{i-1} \cong \Delta(\xi_i)$ for some $\xi_i \in \mathrm{Irr}(W)$. We prove (a) and (b) by recurrence on l. If $l = 1$, then M is a standard module. Both (a) and (b) hold by definition. For $l > 1$, by induction we may assume that there is a subspace V' of M_{l-1} such that the lemma holds for M_{l-1} and V'. Now, consider the exact sequence

$$0 \longrightarrow M_{l-1} \longrightarrow M \xrightarrow{j} \Delta(\xi_l) \longrightarrow 0$$

From the isomorphism of $\mathbb{C}[\mathfrak{h}] \rtimes W$-modules $\Delta(\xi_l) \cong \mathbb{C}[\mathfrak{h}] \otimes \xi$ we see that $\Delta(\xi_l)$ is a projective $\mathbb{C}[\mathfrak{h}] \rtimes W$-module. Hence there exists a morphism of $\mathbb{C}[\mathfrak{h}] \rtimes W$-modules $s : \Delta(\xi_l) \to M$ providing a section of j. Let $V = V' \oplus s(\xi_l) \subset M$. It is stable under the action of $\mathbb{C}W$. The map $\mathbb{C}[\mathfrak{h}] \otimes V \to M$ in (a) is an injective morphism of $\mathbb{C}[\mathfrak{h}] \rtimes W$-modules. Its image is $M_{l-1} \oplus s(\Delta(\xi)) = M$. So it is an isomorphism. Since

$$\omega([M]) = \omega([M_{l-1}]) + \omega([\Delta(\xi_l)]),$$

and $\omega([M_{l-1}]) = [V']$ by assumption, we deduce that $\omega([M]) = [V'] + [\xi_l] = [V]$. $\square$

Lemma 3.4.2. *(a) Let M be a $\widehat{H}_c(W,\mathfrak{h})_0$-module free over $\mathbb{C}[[\mathfrak{h}]]_0$. If there exist generalized eigenvectors $v_1,\ldots v_n$ of* **eu** *which form a basis of M over $\mathbb{C}[[\mathfrak{h}]]_0$, then for $f_1,\ldots,f_n\in\mathbb{C}[[\mathfrak{h}]]_0$ the element $m=\sum_{i=1}^n f_iv_i$ is* **eu***-finite if and only if $f_1,\ldots,f_n$ all belong to $\mathbb{C}[\mathfrak{h}]$.*

(b) Let $N\in\mathcal{O}_c(W,\mathfrak{h})$. If $\widehat{N}_0$ is a free $\mathbb{C}[[\mathfrak{h}]]_0$-module, then N is a free $\mathbb{C}[\mathfrak{h}]$-module. Further, it admits a basis consisting of generalized eigenvectors $v_1,\ldots,v_n$ of **eu***.*

Proof. Part (a) follows from the proof of [BE09, Theorem 2.3]. Let us concentrate on part (b). Since N belongs to $\mathcal{O}_c(W,\mathfrak{h})$, it is finitely generated over $\mathbb{C}[\mathfrak{h}]$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{C}[[\mathfrak{h}]]_0$. The canonical map $N\to\widehat{N}_0/\mathfrak{m}\widehat{N}_0$ is surjective. So there exist $v_1,\ldots,v_n$ in N such that their images form a basis of $\widehat{N}_0/\mathfrak{m}\widehat{N}_0$ over $\mathbb{C}$. Moreover, we may choose $v_1,\ldots,v_n$ to be generalized eigenvectors of **eu**, because the **eu**-action on N is locally finite. Since $\widehat{N}_0$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$, Nakayama's lemma yields that $v_1,\ldots,v_n$ form a basis of $\widehat{N}_0$ over $\mathbb{C}[[\mathfrak{h}]]_0$. By part (a) the set N' of **eu**-finite elements in $\widehat{N}_0$ is the free $\mathbb{C}[\mathfrak{h}]$-submodule generated by $v_1,\ldots,v_n$. Since $\widehat{N}_0$ belongs to $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_0$, by [BE09, Proposition 2.4] an element in $\widehat{N}_0$ is $\mathfrak{h}$-nilpotent if and only if it is **eu**-finite. So $N'=E(\widehat{N}_0)$. On the other hand, the canonical inclusion $N\subset E(\widehat{N}_0)$ is an equality by [BE09, Theorem 3.2]. Hence $N=N'$. This implies that N is free over $\mathbb{C}[\mathfrak{h}]$, with a basis given by $v_1,\ldots,v_n$, which are generalized eigenvectors of **eu**. The lemma is proved. □

Proposition 3.4.3. *Let M be an object of $\mathcal{O}_c^{\Delta}(W,\mathfrak{h})$.*

(a) The object $\mathrm{Res}_b(M)$ has a standard filtration.

(b) Let V be a subspace of M as in Lemma 3.4.1(a). Then there is an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes\mathrm{Res}^W_{W'}(V).$$

Proof. We see from the end of Section 3.1 the equivalence ζ maps a standard module in $\mathcal{O}_{c'}(W',\mathfrak{h})$ to a standard one in $\mathcal{O}_{c'}(W',\overline{\mathfrak{h}})$. Hence to prove that $\mathrm{Res}_b(M)=\zeta\circ E\circ R(\widehat{M}_b)$ has a standard filtration, it is enough to show that $N=E\circ R(\widehat{M}_b)$ has one. We claim that the module N is free over $\mathbb{C}[\mathfrak{h}]$. To prove this, recall from (3.2.7) that $R(\widehat{M}_b)=\phi_1^*(x_{\mathrm{pr}}\widehat{M}_b)$ as $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules. Using the isomorphism of $\mathbb{C}[\mathfrak{h}]\rtimes W$-modules $M\cong\mathbb{C}[\mathfrak{h}]\otimes V$ given in Lemma 3.4.1(a), we deduce an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0\rtimes W'$-modules

$$\begin{aligned} R(\widehat{M}_b) &\cong \phi_1^*(x_{\mathrm{pr}}(\widehat{\mathbb{C}[\mathfrak{h}]}_b\otimes V)) \\ &\cong \mathbb{C}[[\mathfrak{h}]]_0\otimes V. \end{aligned}$$

So the module $R(\widehat{M}_b)$ is free over $\mathbb{C}[[\mathfrak{h}]]_0$. The completion of the module N at 0 is isomorphic to $R(\widehat{M}_b)$. So Lemma 3.4.2(b) implies that N is free over $\mathbb{C}[\mathfrak{h}]$. The claim is proved. Now, part (a) follows from Lemma 2.2.2. For part (b), note that since $\mathrm{Res}_b(M)$ has a standard filtration, by Lemma 3.4.1 there exists a finite dimensional vector space $V'\subset\mathrm{Res}_b(M)$ such that V' is stable under the action of $\mathbb{C}W'$ and we have an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes V'.$$

Moreover, we have $\omega([\mathrm{Res}_b(M)])=[V']$ in $[\mathbb{C}W']$. So Proposition 3.3.2(b) yields that $\mathrm{Res}^W_{W'}(\omega[M])=\omega([\mathrm{Res}_b(M)])$. Since $\omega([M])=[V]$ by Lemma 3.4.1(b), the $\mathbb{C}W'$-module V' is isomorphic to $\mathrm{Res}^W_{W'}(V)$. So we have an isomorphism of $\mathbb{C}[\overline{\mathfrak{h}}]\rtimes W'$-modules

$$\mathrm{Res}_b(M)\cong\mathbb{C}[\overline{\mathfrak{h}}]\otimes\mathrm{Res}^W_{W'}(V).$$

□

3.5 KZ commutes with restriction functors

Now, we relate the restriction and induction functors for rational DAHA's to the corresponding functors for Hecke algebras via the functor KZ. We will work under the same assumption as in Section 2.5. Let W be a complex reflection group acting on $\mathfrak{h}$. Let b be a point in $\mathfrak{h}$ and let W' be its stabilizer in W. We abbreviate $\mathrm{KZ} = \mathrm{KZ}(W, \mathfrak{h})$, $\mathrm{KZ}' = \mathrm{KZ}(W', \overline{\mathfrak{h}})$.

Theorem 3.5.1. *There is an isomorphism of functors*

$$\mathrm{KZ}' \circ \mathrm{Res}_b \cong {}^{\mathscr{H}}\mathrm{Res}_{W'}^{W} \circ \mathrm{KZ}\,.$$

Proof. We will regard $\mathrm{KZ} : \mathcal{O}_c(W, \mathfrak{h}) \to \mathscr{H}_q(W)$-mod as a functor from $\mathcal{O}_c(W, \mathfrak{h})$ to B_W-mod in the obvious way. Similarly we will regard KZ' as a functor to $B_{W'}$-mod. Recall the inclusion $\imath : B_{W'} \hookrightarrow B_W$ from (1.2.5). The theorem amounts to prove that for any $M \in \mathcal{O}_c(W, \mathfrak{h})$ there is a natural isomorphism of $B_{W'}$-modules

$$\mathrm{KZ}' \circ \mathrm{Res}_b(M) \cong \imath_* \circ \mathrm{KZ}(M). \tag{3.5.1}$$

Step 1. Recall the functor $\zeta : \mathcal{O}_{c'}(W', \mathfrak{h}) \to \mathcal{O}_{c'}(W', \overline{\mathfrak{h}})$ from (3.1.3) and its quasi-inverse ζ^{-1} in (3.1.4). Let

$$N = \zeta^{-1}(\mathrm{Res}_b(M)).$$

We have $N \cong \mathrm{Res}_b(M) \otimes \mathbb{C}[\mathfrak{h}^{W'}]$. Since the canonical epimorphism $\mathfrak{h} \to \overline{\mathfrak{h}}$ induces a fibration $\mathfrak{h}'_{reg} \to \overline{\mathfrak{h}}_{reg}$, see Section 1.2, we have

$$N_{\mathfrak{h}'_{reg}} \cong \mathrm{Res}_b(M)_{\overline{\mathfrak{h}}_{reg}} \otimes \mathbb{C}[\mathfrak{h}^{W'}]. \tag{3.5.2}$$

By Dunkl isomorphisms, the left hand side is a $\mathcal{D}(\mathfrak{h}'_{reg}) \rtimes W'$-module while the right hand side is a $(\mathcal{D}(\overline{\mathfrak{h}}_{reg}) \rtimes W') \otimes \mathcal{D}(\mathfrak{h}^{W'})$-module. Identify these two algebras in the obvious way. The isomorphism (3.5.2) is compatible with the W'-equivariant $\mathcal{D}$-module structures. Hence we have

$$(N_{\mathfrak{h}'_{reg}})^{\nabla} \cong (\mathrm{Res}_b(M)_{\overline{\mathfrak{h}}_{reg}})^{\nabla} \otimes \mathbb{C}[\mathfrak{h}^{W'}]^{\nabla}.$$

Since $\mathbb{C}[\mathfrak{h}^{W'}]^{\nabla} = \mathbb{C}$, this yields a natural isomorphism

$$\ell_* \circ \mathrm{KZ}(W', \mathfrak{h})(N) \cong \mathrm{KZ}' \circ \mathrm{Res}_b(M),$$

where ℓ is the homomorphism defined in (1.2.1).

Step 2. Consider the W'-equivariant algebra isomorphism

$$\phi : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}], \quad x \mapsto x + x(b) \text{ for } x \in \mathfrak{h}^*.$$

It induces an isomorphism $\hat{\phi} : \mathbb{C}[[\mathfrak{h}]]_b \xrightarrow{\sim} \mathbb{C}[[\mathfrak{h}]]_0$. The latter yields an algebra isomorphism

$$\mathbb{C}[[\mathfrak{h}]]_b \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}] \simeq \mathbb{C}[[\mathfrak{h}]]_0 \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}'_{reg}].$$

To see this, note first that by definition the left hand side is $\mathbb{C}[[\mathfrak{h}]]_b[\alpha_s^{-1}, s \in \mathcal{S}]$. For $s \in \mathcal{S}$, $s \notin W'$ the element α_s is invertible in $\mathbb{C}[[\mathfrak{h}]]_b$, so we have

$$\mathbb{C}[[\mathfrak{h}]]_b \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}] = \mathbb{C}[[\mathfrak{h}]]_b[\alpha_s^{-1}, s \in \mathcal{S} \cap W'].$$

For $s \in \mathcal{S} \cap W'$ we have $\alpha_s(b) = 0$, so $\hat{\phi}(\alpha_s) = \alpha_s$. Hence

$$\begin{aligned}\hat{\phi}(\mathbb{C}[[\mathfrak{h}]]_b)[\hat{\phi}(\alpha_s)^{-1}, s \in \mathcal{S} \cap W'] &= \mathbb{C}[[\mathfrak{h}]]_0[\alpha_s^{-1}, s \in \mathcal{S} \cap W'] \\ &= \mathbb{C}[[\mathfrak{h}]]_0 \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}'_{reg}].\end{aligned}$$

Step 3. We will assume in Steps 3, 4, 5 that M is an object of $\mathcal{O}_c^{\Delta}(W, \mathfrak{h})$. In this step we prove that N is isomorphic to $\phi^*(M)$ as $\mathbb{C}[\mathfrak{h}] \rtimes W'$-modules. Let V be a subspace of M as in Lemma 3.4.1(a). Then we have an isomorphism of $\mathbb{C}[\mathfrak{h}] \rtimes W$-modules

$$M \cong \mathbb{C}[\mathfrak{h}] \otimes V. \tag{3.5.3}$$

Also, by Proposition 3.4.3(b) there is an isomorphism of $\mathbb{C}[\mathfrak{h}] \rtimes W'$-modules

$$N \cong \mathbb{C}[\mathfrak{h}] \otimes \mathrm{Res}_{W'}^{W}(V).$$

So N is isomorphic to $\phi^*(M)$ as $\mathbb{C}[\mathfrak{h}] \rtimes W'$-modules.

Step 4. In this step we compare $((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}}$ and $(\widehat{N}_0)_{\mathfrak{h}'_{reg}}$ as $\widehat{\mathcal{D}(\mathfrak{h}'_{reg})}_0$-modules. By (3.3.1) we have $N = E \circ R(\widehat{M}_b)$, so we have $\widehat{N}_0 \cong R(\widehat{M}_b)$. Next, by (3.2.7) we have an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0 \rtimes W'$-modules

$$\begin{aligned}R(\widehat{M}_b) &= \hat{\phi}^*(x_{\mathrm{pr}}(\widehat{M}_b)) \\ &= (\widehat{\phi^*(M)})_0.\end{aligned}$$

So we get an isomorphism of $\mathbb{C}[[\mathfrak{h}]]_0 \rtimes W'$-modules

$$\hat{\Psi} : (\widehat{\phi^*(M)})_0 \to \widehat{N}_0.$$

Now, let us consider connections on these modules. Note that by Step 2 we have

$$((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}} = \hat{\phi}^*(x_{\mathrm{pr}}(\widehat{M}_b)_{\mathfrak{h}_{reg}}).$$

Write ∇ for the connection on $M_{\mathfrak{h}_{reg}}$ given by the Dunkl isomorphism for $H_c(W, \mathfrak{h}_{reg})$. We equip $((\widehat{\phi^*(M)})_0)_{\mathfrak{h}'_{reg}}$ with the connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_a(x_{\mathrm{pr}} m) = x_{\mathrm{pr}}(\nabla_a(m)), \quad \forall\, m \in (\widehat{M}_b)_{\mathfrak{h}_{reg}},\ a \in \mathfrak{h}.$$

Let $\nabla^{(b)}$ be the connection on $N_{\mathfrak{h}'_{reg}}$ given by the Dunkl isomorphism for $H_{c'}(W', \mathfrak{h}'_{reg})$. This restricts to a connection on $(\widehat{N}_0)_{\mathfrak{h}'_{reg}}$. We claim that Ψ is compatible with these connections, i.e., we have

$$\nabla_a^{(b)}(x_{\mathrm{pr}} m) = x_{\mathrm{pr}} \nabla_a(m), \quad \forall\, m \in (\widehat{M}_b)_{\mathfrak{h}_{reg}}. \tag{3.5.4}$$

Recall the subspace V of M from Step 3. By Lemma 3.4.1(a) the map

$$(\widehat{\mathbb{C}[\mathfrak{h}]}_b \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}]) \otimes V \to (\widehat{M}_b)_{\mathfrak{h}_{reg}}, \quad p \otimes v \mapsto pv$$

is a bijection. So it is enough to prove (3.5.4) for $m = pv$ with $p \in \widehat{\mathbb{C}[\mathfrak{h}]}_b \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}]$, $v \in V$. We have

$$\begin{aligned}
\nabla_a^{(b)}(x_{\mathrm{pr}}pv) &= (y_a^{(b)} - \sum_{s\in\mathcal{S}\cap W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}^{(b)}}(s-1))(x_{\mathrm{pr}}pv) \\
&= x_{\mathrm{pr}}(y_a + \sum_{s\in\mathcal{S}, s\notin W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}} - \\
&\quad - \sum_{s\in\mathcal{S}\cap W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}}(s-1))(x_{\mathrm{pr}}pv) \\
&= x_{\mathrm{pr}}(\nabla_a + \sum_{s\in\mathcal{S}, s\notin W'} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{x_{\alpha_s}}s)(x_{\mathrm{pr}}pv) \\
&= x_{\mathrm{pr}}\nabla_a(x_{\mathrm{pr}}pv). \qquad (3.5.5)
\end{aligned}$$

Here the first equality is by the Dunkl isomorphism for $H_{c'}(W', \mathfrak{h}'_{reg})$. The second is by (3.2.4), (3.2.5), (3.2.6) and the fact that $x_{\mathrm{pr}}^2 = x_{\mathrm{pr}}$. The third is by the Dunkl isomorphism for $H_c(W, \mathfrak{h}_{reg})$. The last is by (3.2.8). Next, since x_{pr} is the idempotent in $\widehat{\mathbb{C}[\mathfrak{h}]}_b$ corresponding to the component $\mathbb{C}[[\mathfrak{h}]]_b$ in the decomposition (3.2.3), we have

$$\begin{aligned}
\nabla_a(x_{\mathrm{pr}}pv) &= (\partial_a(x_{\mathrm{pr}}p))v + x_{\mathrm{pr}}p\,(\nabla_a v) \\
&= x_{\mathrm{pr}}(\partial_a(p))v + x_{\mathrm{pr}}p\,(\nabla_a v) \\
&= x_{\mathrm{pr}}\nabla_a(pv).
\end{aligned}$$

Together with (3.5.5) this implies that

$$\nabla_a^{(b)}(x_{\mathrm{pr}}pv) = x_{\mathrm{pr}}\nabla_a(pv).$$

So the claim is proved.

Step 5. Now, we prove the isomorphism (3.5.1) for $M \in \mathcal{O}_c^{\Delta}(W, \mathfrak{h})$. Here we need some more notation. For $X = \mathfrak{h}$ or $\mathfrak{h}'_{reg}$, let U be an open analytic subvariety of X, write $i : U \hookrightarrow X$ for the canonical embedding. For F an analytic coherent sheaf on X we write $i^*(F)$ for the restriction of F to U. If U contains 0, for an analytic locally free sheaf E over U, we write $\widehat{E}$ for the restriction of E to the formal disc at 0.

Let $\Omega \subset \mathfrak{h}$ be the open ball defined in (1.2.2). Let $f : \mathfrak{h} \to \mathfrak{h}$ be the morphism defined by ϕ. The preimage of Ω via f is an open ball Ω_0 in $\mathfrak{h}$ centered at 0. We have

$$f(\Omega_0 \cap \mathfrak{h}'_{reg}) = \Omega \cap \mathfrak{h}_{reg}.$$

Let $u : \Omega_0 \cap \mathfrak{h}'_{reg} \hookrightarrow \mathfrak{h}$ and $v : \Omega \cap \mathfrak{h}_{reg} \hookrightarrow \mathfrak{h}$ be the canonical embeddings. By Step 3 there is an isomorphism of W'-equivariant analytic locally free sheaves over $\Omega_0 \cap \mathfrak{h}'_{reg}$

$$u^*(N^{an}) \cong \phi^*(v^*(M^{an})).$$

By Step 4 the isomorphism $\hat{\Psi}$ yields an isomorphism

$$\widehat{u^*(N^{an})} \xrightarrow{\sim} \phi^*(\widehat{v^*(M^{an})})$$

which is compatible with their connections. It follows from Lemma 3.5.2 below that there is an isomorphism

$$\left(u^*(N^{an})\right)^{\nabla^{(b)}} \cong \phi^*\left((v^*(M^{an}))^{\nabla}\right).$$

Since $\Omega_0 \cap \mathfrak{h}'_{reg}$ is homotopy equivalent to $\mathfrak{h}'_{reg}$ via u, the left hand side is isomorphic to $(N_{\mathfrak{h}'_{reg}})^{\nabla^{(b)}}$. So we have

$$\mathrm{KZ}(W', \mathfrak{h})(N) \cong \kappa_* \circ \jmath_* \circ \mathrm{KZ}(M),$$

where κ, $\jmath$ are as in (1.2.3), (1.2.4). Combined with Step 1 we have the following isomorphisms

$$\begin{aligned} \mathrm{KZ}' \circ \mathrm{Res}_b(M) &\cong \ell_* \circ \mathrm{KZ}(W', \mathfrak{h})(N) \\ &\cong \ell_* \circ \kappa_* \circ \jmath_* \circ \mathrm{KZ}(M) \\ &= \imath_* \circ \mathrm{KZ}(M). \end{aligned} \tag{3.5.6}$$

They are functorial on M.

Lemma 3.5.2. *Let E be an analytic locally free sheaf over the complex manifold $\mathfrak{h}'_{reg}$. Let ∇_1, ∇_2 be two integrable connections on E with regular singularities. If there exists an isomorphism $\hat{\psi} : (\widehat{E}, \nabla_1) \to (\widehat{E}, \nabla_2)$, then the local systems E^{∇_1} and E^{∇_2} are isomorphic.*

Proof. Write $\mathrm{End}(E)$ for the sheaf of endomorphisms of E. Then $\mathrm{End}(E)$ is a locally free sheaf over $\mathfrak{h}'_{reg}$. The connections ∇_1, ∇_2 define a connection ∇ on $\mathrm{End}(E)$ as follows,

$$\nabla : \mathrm{End}(E) \to \mathrm{End}(E), \quad f \mapsto \nabla_2 \circ f - f \circ \nabla_1.$$

So the isomorphism $\hat{\psi}$ is a horizontal section of $(\widehat{\mathrm{End}(E)}, \nabla)$. Let $(\mathrm{End}(E)^{\nabla})_0$ be the set of germs of horizontal sections of $(\mathrm{End}(E), \nabla)$ on zero. By the Comparison theorem [KK81, Theorem 6.3.1] the canonical map $(\mathrm{End}(E)^{\nabla})_0 \to (\widehat{\mathrm{End}(E)})^{\nabla}$ is bijective. Hence there exists a holomorphic isomorphism $\psi : (E, \nabla_1) \to (E, \nabla_2)$ which maps to $\hat{\psi}$. Now, let U be an open ball in $\mathfrak{h}'_{reg}$ centered at 0 with radius ε small enough such that the holomorphic isomorphism ψ converges in U. Write E_U for the restriction of E to U. Then ψ induces an isomorphism of local systems $(E_U)^{\nabla_1} \cong (E_U)^{\nabla_2}$. Since $\mathfrak{h}'_{reg}$ is homotopy equivalent to U, we have

$$E^{\nabla_1} \cong E^{\nabla_2}.$$

□

Step 6. Finally, recall that I is the inclusion of $\mathrm{Proj}_c(W, \mathfrak{h})$ into $\mathcal{O}_c(W, \mathfrak{h})$. By Proposition 2.2.3(b), any projective object in $\mathcal{O}_c(W, \mathfrak{h})$ has a standard filtration. So (3.5.6) yields an isomorphism of functors

$$\mathrm{KZ}' \circ \mathrm{Res}_b \circ I \to \imath_* \circ \mathrm{KZ} \circ I.$$

Applying Corollary 2.3.2 to the exact functors $\mathrm{KZ}' \circ \mathrm{Res}_b$ and $\imath_* \circ \mathrm{KZ}$ yields that there is an isomorphism of functors

$$\mathrm{KZ}' \circ \mathrm{Res}_b \cong \imath_* \circ \mathrm{KZ}.$$

□

An immediate corollary of Theorem 3.5.1 is the following.

Corollary 3.5.3. *There is an isomorphism of functors*

$$\mathrm{KZ} \circ \mathrm{Ind}_b \cong {}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'.$$

Proof. To simplify notation let us write

$$\mathcal{O} = \mathcal{O}_c(W, \mathfrak{h}), \quad \mathcal{O}' = \mathcal{O}_{c'}(W', \overline{\mathfrak{h}}), \quad \mathscr{H} = \mathscr{H}_q(W), \quad \mathscr{H}' = \mathscr{H}_{q'}(W').$$

Recall that the functor KZ is represented by a projective object P_{KZ} in $\mathcal{O}$. So for any $N \in \mathcal{O}'$ we have a morphism of $\mathscr{H}$-modules

$$\begin{aligned} \mathrm{KZ} \circ \mathrm{Ind}_b(N) &\cong \mathrm{Hom}_{\mathcal{O}}(P_{\mathrm{KZ}}, \mathrm{Ind}_b(N)) \\ &\cong \mathrm{Hom}_{\mathcal{O}'}(\mathrm{Res}_b(P_{\mathrm{KZ}}), N) \\ &\to \mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}'(\mathrm{Res}_b(P_{\mathrm{KZ}})), \mathrm{KZ}'(N)). \end{aligned} \tag{3.5.7}$$

By Theorem 3.5.1 we have

$$\mathrm{KZ}' \circ \mathrm{Res}_b(P_{\mathrm{KZ}}) \cong {}^{\mathscr{H}}\mathrm{Res}_{W'}^{W} \circ \mathrm{KZ}(P_{\mathrm{KZ}}).$$

Recall from Section 2.4 that the $\mathscr{H}$-module $\mathrm{KZ}(P_{\mathrm{KZ}})$ is isomorphic to $\mathscr{H}$. So as $\mathscr{H}'$-modules $\mathrm{KZ}'(\mathrm{Res}_b(P_{\mathrm{KZ}}))$ is also isomorphic to $\mathscr{H}$. Therefore the morphism (3.5.7) can be rewritten as

$$\chi(N) : \mathrm{KZ} \circ \mathrm{Ind}_b(N) \to \mathrm{Hom}_{\mathscr{H}'}(\mathscr{H}, \mathrm{KZ}'(N)). \tag{3.5.8}$$

It yields a morphism of functors

$$\chi : \mathrm{KZ} \circ \mathrm{Ind}_b \to {}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'\,.$$

Note that if N is a projective object in $\mathcal{O}'$, then $\chi(N)$ is an isomorphism by Proposition 2.5.2(a). So Corollary 2.3.2 implies that χ is an isomorphism of functors, because both functors $\mathrm{KZ} \circ \mathrm{Ind}_b$ and ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} \circ \mathrm{KZ}'$ are exact. □

3.6 A useful lemma.

Lemma 3.6.1. *Let K, L be two right exact functors from $\mathcal{O}_1$ to $\mathcal{O}_2$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ can be either $\mathcal{O}_c(W, \mathfrak{h})$ or $\mathcal{O}_{c'}(W', \overline{\mathfrak{h}})$. Let KZ_2 denote the KZ-functor on $\mathcal{O}_2$. Suppose that K, L map projective objects to projective ones. Then the vector space homomorphism*

$$\mathrm{Hom}(K, L) \to \mathrm{Hom}(\mathrm{KZ}_2 \circ K, \mathrm{KZ}_2 \circ L), \quad f \mapsto 1_{\mathrm{KZ}_2} f, \tag{3.6.1}$$

is an isomorphism.

Note that if $K = L$, this is even an isomorphism of rings.

Proof. Let Proj_1, Proj_2 be respectively the subcategory of projective objects in $\mathcal{O}_1$, $\mathcal{O}_2$. Let $\tilde{K}$, $\tilde{L}$ be the functors from Proj_1 to Proj_2 given by the restrictions of K, L, respectively. Let $\mathscr{H}_2$ be the Hecke algebra corresponding to $\mathcal{O}_2$. Since the functor KZ_2 is fully faithful over Proj_2 by Proposition 2.5.2(a), the following functor

$$\mathrm{Fct}(\mathrm{Proj}_1, \mathrm{Proj}_2) \to \mathrm{Fct}(\mathrm{Proj}_1, \mathscr{H}_2\text{-mod})\,, \quad G \mapsto \mathrm{KZ}_2 \circ G$$

is also fully faithful. Therefore we have an isomorphism

$$\mathrm{Hom}(\tilde{K}, \tilde{L}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{KZ}_2 \circ \tilde{K}, \mathrm{KZ}_2 \circ \tilde{L}), \quad f \mapsto 1_{\mathrm{KZ}_2} f.$$

Next, by Lemma 2.3.1 the canonical morphisms

$$\mathrm{Hom}(K, L) \to \mathrm{Hom}(\tilde{K}, \tilde{L}), \quad \mathrm{Hom}(\mathrm{KZ}_2 \circ K, \mathrm{KZ}_2 \circ L) \to \mathrm{Hom}(\mathrm{KZ}_2 \circ \tilde{K}, \mathrm{KZ}_2 \circ \tilde{L})$$

are isomorphisms. So the map (3.6.1) is also an isomorphism. □

3.7 Transitivity of restriction and induction functors

Let $b(W, W'')$ be a point in $\mathfrak{h}$ whose stabilizer is W''. Let $b(W', W'')$ be its image in $\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}^{W'}$ via the canonical projection. Write $b(W, W') = b$.

Proposition 3.7.1. *There are isomorphisms of functors*

$$\begin{aligned}\operatorname{Res}_{b(W',W'')} \circ \operatorname{Res}_{b(W,W')} &\cong \operatorname{Res}_{b(W,W'')},\\ \operatorname{Ind}_{b(W,W')} \circ \operatorname{Ind}_{b(W',W'')} &\cong \operatorname{Ind}_{b(W,W'')}.\end{aligned}$$

Proof. Since the restriction functors map projective objects to projective ones by Proposition 3.3.2(a), Lemma 3.6.1 applied to the categories $\mathcal{O}_1 = \mathcal{O}_c(W, \mathfrak{h})$, $\mathcal{O}_2 = \mathcal{O}_{c''}(W'', \mathfrak{h}/\mathfrak{h}^{W''})$ yields an isomorphism

$$\begin{aligned}&\operatorname{Hom}(\operatorname{Res}_{b(W',W'')} \circ \operatorname{Res}_{b(W,W')}, \operatorname{Res}_{b(W,W'')})\\ &= \operatorname{Hom}(\mathrm{KZ}'' \circ \operatorname{Res}_{b(W',W'')} \circ \operatorname{Res}_{b(W,W')}, \mathrm{KZ}'' \circ \operatorname{Res}_{b(W,W'')}).\end{aligned}$$

By Theorem 3.5.1 the set on the second row is

$$\operatorname{Hom}({}^{\mathscr{H}}\operatorname{Res}^{W'}_{W''} \circ {}^{\mathscr{H}}\operatorname{Res}^{W}_{W'} \circ \mathrm{KZ}, {}^{\mathscr{H}}\operatorname{Res}^{W}_{W''} \circ \mathrm{KZ}). \tag{3.7.1}$$

By the presentations of Hecke algebras in [BMR98, Proposition 4.22], there is an isomorphism

$$\sigma : {}^{\mathscr{H}}\operatorname{Res}^{W'}_{W''} \circ {}^{\mathscr{H}}\operatorname{Res}^{W}_{W'} \xrightarrow{\sim} {}^{\mathscr{H}}\operatorname{Res}^{W}_{W''}.$$

Hence the element $\sigma 1_{\mathrm{KZ}}$ in the set (3.7.1) maps to an isomorphism

$$\operatorname{Res}_{b(W',W'')} \circ \operatorname{Res}_{b(W,W')} \cong \operatorname{Res}_{b(W,W'')}.$$

This proves the first isomorphism in the corollary. The second one follows from the uniqueness of right adjoint functor. □

3.8 Biadjointness of Res_b and Ind_b.

In this section, we prove the biadjointness of Res_b and Ind_b (Proposition 3.8.2). This result was conjectured in [BE09, Remark 3.18]. Let us first consider the following lemma.

Lemma 3.8.1. *(a) Let A, B be noetherian algebras and T be a functor*

$$T : A\text{-mod} \to B\text{-mod}.$$

If T is right exact and commutes with direct sums, then it has a right adjoint.

(b) The functor

$$\operatorname{Res}_b : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{c'}(W', \overline{\mathfrak{h}})$$

has a left adjoint.

Proof. Consider the (B, A)-bimodule $M = T(A)$. We claim that the functor T is isomorphic to the functor $M \otimes_A -$. Indeed, by definition we have $T(A) \cong M \otimes_A A$ as B modules. Now, for any $N \in A$-mod, since N is finitely generated and A is noetherian there exists m, $n \in \mathbb{N}$ and an exact sequence

$$A^{\oplus n} \longrightarrow A^{\oplus m} \longrightarrow N \longrightarrow 0.$$

Since both T and $M \otimes_A -$ are right exact and they commute with direct sums, the fact that $T(A) \cong M \otimes_A A$ implies that $T(N) \cong M \otimes_A N$ as B-modules. This proved the claim. Now, the functor $M \otimes_A -$ has a right adjoint $\mathrm{Hom}_B(M, -)$, so T also has a right adjoint. This proves part (a). Let us concentrate on part (b). Recall that for any complex reflection group W, a contravariant duality functor

$$(-)^\vee : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*)$$

was defined in [GGOR03, Section 4.2], here $c^\dagger : \mathcal{S} \to \mathbb{C}$ is another parameter explicitly determined by c. Consider the functor

$$\mathrm{Res}_b^\vee = (-)^\vee \circ \mathrm{Res}_b \circ (-)^\vee : \mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*) \to \mathcal{O}_{c'^\dagger}(W', (\bar{\mathfrak{h}})^*).$$

The category $\mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*)$ has a projective generator P. The algebra $\mathrm{End}_{\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)}(P)^{\mathrm{op}}$ is finite dimensional over $\mathbb{C}$ and by Morita theory we have an equivalence of categories

$$\mathcal{O}_{c^\dagger}(W, \mathfrak{h}^*) \cong \mathrm{End}_{\mathcal{O}_{c^\dagger}(W,\mathfrak{h}^*)}(P)^{\mathrm{op}}\text{-mod}\,.$$

Since the functor $\mathrm{Res}_b^\vee$ is exact and obviously commutes with direct sums, by part (a) it has a right adjoint Ψ. Then it follows that $(-)^\vee \circ \Psi \circ (-)^\vee$ is left adjoint to Res_b. The lemma is proved. $\square$

Proposition 3.8.2. [1] *Assume that the algebras $\mathscr{H}_q(W)$ and $\mathscr{H}_{q'}(W')$ are symmetric. Then the functor Ind_b is left adjoint to Res_b.*

Proof. We will prove the proposition in two steps.

Step 1. We abbreviate $\mathcal{O} = \mathcal{O}_c(W, \mathfrak{h})$, $\mathcal{O}' = \mathcal{O}_{c'}(W', \bar{\mathfrak{h}})$, $\mathscr{H} = \mathscr{H}_q(W)$, $\mathscr{H}' = \mathscr{H}_{q'}(W')$, and write $\mathrm{Id}_{\mathcal{O}}$, $\mathrm{Id}_{\mathcal{O}'}$, $\mathrm{Id}_{\mathscr{H}}$, $\mathrm{Id}_{\mathscr{H}'}$ for the identity functor on the corresponding categories. We also abbreviate $E^{\mathscr{H}} = {}^{\mathscr{H}}\mathrm{Res}_{W'}^W$, $F^{\mathscr{H}} = {}^{\mathscr{H}}\mathrm{Ind}_{W'}^W$ and $E = \mathrm{Res}_b$. By Lemma 3.8.1 the functor E has a left adjoint. We denote it by $F : \mathcal{O}' \to \mathcal{O}$. Recall the functors

$$\mathrm{KZ} : \mathcal{O} \to \mathscr{H}\text{-mod}, \quad \mathrm{KZ}' : \mathcal{O}' \to \mathscr{H}'\text{-mod}\,.$$

The goal of this step is to show that there exists an isomorphism of functors

$$\mathrm{KZ} \circ F \cong F^{\mathscr{H}} \circ \mathrm{KZ}'\,.$$

To this end, let S, S' be respectively the right adjoints of KZ, KZ$'$, see Section 2.5. We will first give an isomorphism of functors

$$F^{\mathscr{H}} \cong \mathrm{KZ} \circ F \circ S'.$$

Let $M \in \mathscr{H}'$-mod and $N \in \mathscr{H}$-mod. Consider the following equalities given by adjunctions

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ} \circ F \circ S'(M), N) &= \mathrm{Hom}_{\mathcal{O}}(F \circ S'(M), S(N)) \\ &= \mathrm{Hom}_{\mathcal{O}'}(S'(M), E \circ S(N)).\end{aligned}$$

The functor KZ$'$ yields a map

$$a(M,N) : \mathrm{Hom}_{\mathcal{O}'}(S'(M), E \circ S(N)) \to \mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}' \circ S'(M), \mathrm{KZ}' \circ E \circ S(N)). \tag{3.8.1}$$

1. This result has been independently obtained by Iain Gordon and Maurizio Martino.

Since the canonical adjunction maps $\mathrm{KZ}' \circ S' \to \mathrm{Id}_{\mathscr{H}'}$, $\mathrm{KZ} \circ S \to \mathrm{Id}_{\mathscr{H}}$ are isomorphisms (see Section 2.5) and since we have an isomorphism of functors $\mathrm{KZ}' \circ E \cong E^{\mathscr{H}} \circ \mathrm{KZ}$ by Theorem 3.5.1, we get the following equalities

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}'}(\mathrm{KZ}' \circ S'(M), \mathrm{KZ}' \circ E \circ S(N)) &= \mathrm{Hom}_{\mathscr{H}'}(M, E^{\mathscr{H}} \circ \mathrm{KZ} \circ S(N)) \\ &= \mathrm{Hom}_{\mathscr{H}'}(M, E^{\mathscr{H}}(N)) \\ &= \mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M), N).\end{aligned}$$

In the last equality we used that $F^{\mathscr{H}}$ is left adjoint to $E^{\mathscr{H}}$. So the map (3.8.1) can be rewritten into the following form

$$a(M,N) : \mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ} \circ F \circ S'(M), N) \to \mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M), N).$$

Now, take $N = \mathscr{H}$. Recall that $\mathscr{H}$ is isomorphic to $\mathrm{KZ}(P_{\mathrm{KZ}})$ as $\mathscr{H}$-modules. Since P_{KZ} is projective, by Proposition 2.5.2(b) we have a canonical isomorphism in $\mathcal{O}$

$$P_{\mathrm{KZ}} \cong S(\mathrm{KZ}(P_{\mathrm{KZ}})) = S(\mathscr{H}).$$

Further E maps projectives to projectives by Proposition 3.3.2(a), so $E \circ S(\mathscr{H})$ is also projective. Hence Proposition 2.5.2(a) implies that in this case (3.8.1) is an isomorphism for any M, i.e., we get an isomorphism

$$a(M,\mathscr{H}) : \mathrm{Hom}_{\mathscr{H}}(\mathrm{KZ} \circ F \circ S'(M), \mathscr{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{H}}(F^{\mathscr{H}}(M), \mathscr{H}).$$

Further this is an isomorphism of right $\mathscr{H}$-modules with respect to the right action of $\mathscr{H}$ on itself. Now, the fact that $\mathscr{H}$ is a symmetric algebra yields that for any finite dimensional $\mathscr{H}$-module N we have isomorphisms of right $\mathscr{H}$-modules

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}}(N, \mathscr{H}) &\cong \mathrm{Hom}_{\mathscr{H}}(N, \mathrm{Hom}_{\mathbb{C}}(\mathscr{H}, \mathbb{C})) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(N, \mathbb{C}).\end{aligned}$$

Therefore $a(M, \mathscr{H})$ yields an isomorphism of right $\mathscr{H}$-modules

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{KZ} \circ F \circ S'(M), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(F^{\mathscr{H}}(M), \mathbb{C}).$$

We deduce a natural isomorphism of left $\mathscr{H}$-modules

$$\mathrm{KZ} \circ F \circ S'(M) \cong F^{\mathscr{H}}(M)$$

for any $\mathscr{H}'$-module M. This gives an isomorphism of functors

$$\psi : \mathrm{KZ} \circ F \circ S' \xrightarrow{\sim} F^{\mathscr{H}}.$$

Finally, consider the canonical adjunction map $\eta : \mathrm{Id}_{\mathcal{O}'} \to S' \circ \mathrm{KZ}'$. We have a morphism of functors

$$\phi = (1_{\mathrm{KZ} \circ F}\eta) \circ (\psi 1_{\mathrm{KZ}'}) : \mathrm{KZ} \circ F \to F^{\mathscr{H}} \circ \mathrm{KZ}'.$$

Note that $\psi 1_{\mathrm{KZ}'}$ is an isomorphism of functors. If Q is a projective object in $\mathcal{O}'$, then by Proposition 2.5.2(b) the morphism $\eta(Q) : Q \to S' \circ \mathrm{KZ}'(Q)$ is also an isomorphism, so $\phi(Q)$ is an isomorphism. This implies that ϕ is an isomorphism of functors by Corollary 2.3.2, because both $\mathrm{KZ} \circ F$ and $F^{\mathscr{H}} \circ \mathrm{KZ}'$ are right exact functors. Here the right exactness of F follows from that it is left adjoint to E. So we get the desired isomorphism of functors

$$\mathrm{KZ} \circ F \cong F^{\mathscr{H}} \circ \mathrm{KZ}'.$$

Step 2. Let us now prove that F is right adjoint to E. By uniqueness of adjoint functors, this will imply that F is isomorphic to Ind_b. First, by Proposition 1.3.1 the functor $F^{\mathscr{H}}$ is isomorphic to ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W}$. So $F^{\mathscr{H}}$ is right adjoint to $E^{\mathscr{H}}$. In other words, we have morphisms of functors

$$\varepsilon^{\mathscr{H}}: E^{\mathscr{H}} \circ F^{\mathscr{H}} \to \mathrm{Id}_{\mathscr{H}'}, \quad \eta^{\mathscr{H}}: \mathrm{Id}_{\mathscr{H}} \to F^{\mathscr{H}} \circ E^{\mathscr{H}}$$

such that

$$(\varepsilon^{\mathscr{H}} 1_{E^{\mathscr{H}}}) \circ (1_{E^{\mathscr{H}}} \eta^{\mathscr{H}}) = 1_{E^{\mathscr{H}}}, \quad (1_{F^{\mathscr{H}}} \varepsilon^{\mathscr{H}}) \circ (\eta^{\mathscr{H}} 1_{F^{\mathscr{H}}}) = 1_{F^{\mathscr{H}}}.$$

Next, both F and E have exact right adjoints, given respectively by E and Ind_b. Therefore F and E map projective objects to projective ones. Applying Lemma 3.6.1 to $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}'$, $K = E \circ F$, $L = \mathrm{Id}_{\mathcal{O}'}$ yields that the following map is bijective

$$\mathrm{Hom}(E \circ F,\ \mathrm{Id}_{\mathcal{O}'}) \to \mathrm{Hom}(\mathrm{KZ}' \circ E \circ F,\ \mathrm{KZ}' \circ \mathrm{Id}_{\mathcal{O}}), \quad f \mapsto 1_{\mathrm{KZ}'} f. \tag{3.8.2}$$

By Theorem 3.5.1 and Step 1 there exist isomorphisms of functors

$$\phi_E : E^{\mathscr{H}} \circ \mathrm{KZ} \xrightarrow{\sim} \mathrm{KZ}' \circ E, \quad \phi_F : F^{\mathscr{H}} \circ \mathrm{KZ}' \xrightarrow{\sim} \mathrm{KZ} \circ F.$$

Let

$$\begin{aligned} \phi_{EF} &= (\phi_E 1_F) \circ (1_{E^{\mathscr{H}}} \phi_F) : E^{\mathscr{H}} \circ F^{\mathscr{H}} \circ \mathrm{KZ}' \xrightarrow{\sim} \mathrm{KZ}' \circ E \circ F, \\ \phi_{FE} &= (\phi_F 1_E) \circ (1_{F^{\mathscr{H}}} \phi_E) : F^{\mathscr{H}} \circ E^{\mathscr{H}} \circ \mathrm{KZ} \xrightarrow{\sim} \mathrm{KZ} \circ F \circ E. \end{aligned}$$

Identify

$$\mathrm{KZ} \circ \mathrm{Id}_{\mathcal{O}} = \mathrm{Id}_{\mathscr{H}} \circ \mathrm{KZ}, \quad \mathrm{KZ}' \circ \mathrm{Id}_{\mathcal{O}'} = \mathrm{Id}_{\mathscr{H}'} \circ \mathrm{KZ}'.$$

We have a bijective map

$$\mathrm{Hom}(\mathrm{KZ}' \circ E \circ F,\ \mathrm{KZ}' \circ \mathrm{Id}_{\mathcal{O}'}) \xrightarrow{\sim} \mathrm{Hom}(E^{\mathscr{H}} \circ F^{\mathscr{H}} \circ \mathrm{KZ}',\ \mathrm{Id}_{\mathscr{H}'} \circ \mathrm{KZ}'), \quad g \mapsto g \circ \phi_{EF}.$$

Together with (3.8.2), it implies that there exists a unique morphism $\varepsilon : E \circ F \to \mathrm{Id}_{\mathcal{O}'}$ such that

$$(1_{\mathrm{KZ}'}\varepsilon) \circ \phi_{EF} = \varepsilon^{\mathscr{H}} 1_{\mathrm{KZ}'}.$$

Similarly, there exists a unique morphism $\eta : \mathrm{Id}_{\mathcal{O}} \to F \circ E$ such that

$$(\phi_{FE})^{-1} \circ (1_{\mathrm{KZ}}\eta) = \eta^{\mathscr{H}} 1_{\mathrm{KZ}}.$$

Now, we have the following commutative diagram

$$\begin{tikzcd}
E^{\mathscr{H}} \circ \mathrm{KZ} \arrow[r, equal] \arrow[d, "1_{E^{\mathscr{H}}}\eta^{\mathscr{H}}1_{\mathrm{KZ}}"'] & E^{\mathscr{H}} \circ \mathrm{KZ} \arrow[r, "\phi_E"] \arrow[d, "1_{E^{\mathscr{H}}}1_{\mathrm{KZ}}\eta"] & \mathrm{KZ}' \circ E \arrow[d, "1_{\mathrm{KZ}'}1_E\eta"] \\
E^{\mathscr{H}} \circ F^{\mathscr{H}} \circ E^{\mathscr{H}} \circ \mathrm{KZ} \arrow[r, "1_{E^{\mathscr{H}}}\phi_{FE}"] \arrow[d, equal] & E^{\mathscr{H}} \circ \mathrm{KZ} \circ F \circ E \arrow[r, "\phi_E 1_F 1_E"] & \mathrm{KZ}' \circ E \circ F \circ E \arrow[d, equal] \\
E^{\mathscr{H}} \circ F^{\mathscr{H}} \circ E^{\mathscr{H}} \circ \mathrm{KZ} \arrow[r, "1_{E^{\mathscr{H}}}1_{F^{\mathscr{H}}}\phi_E"] \arrow[d, "\varepsilon^{\mathscr{H}}1_{E^{\mathscr{H}}}1_{\mathrm{KZ}}"'] & E^{\mathscr{H}} \circ F^{\mathscr{H}} \circ \mathrm{KZ}' \circ E \arrow[r, "\phi_{EF}1_E"] \arrow[u, "1_{E^{\mathscr{H}}}\phi_F 1_E"'] \arrow[d, "\varepsilon^{\mathscr{H}}1_{\mathrm{KZ}'}1_E"] & \mathrm{KZ}' \circ E \circ F \circ E \arrow[d, "1_{\mathrm{KZ}'}\varepsilon 1_E"] \\
E^{\mathscr{H}} \circ \mathrm{KZ} \arrow[r, "\phi_E"] & \mathrm{KZ}' \circ E \arrow[r, equal] & \mathrm{KZ}' \circ E.
\end{tikzcd}$$

It yields that

$$(1_{\mathrm{KZ}'}\varepsilon 1_E)\circ(1_{\mathrm{KZ}'}1_E\eta)=\phi_E\circ(\varepsilon^{\mathscr{H}}1_{E^{\mathscr{H}}}1_{\mathrm{KZ}})\circ(1_{E^{\mathscr{H}}}\eta^{\mathscr{H}}1_{\mathrm{KZ}})\circ(\phi_E)^{-1}.$$

We deduce that

$$\begin{aligned}1_{\mathrm{KZ}'}((\varepsilon 1_E)\circ(1_E\eta)) &= \phi_E\circ(1_{E^{\mathscr{H}}}1_{\mathrm{KZ}})\circ(\phi_E)^{-1}\\ &= 1_{\mathrm{KZ}'}1_E.\end{aligned}\tag{3.8.3}$$

By applying Lemma 3.6.1 to $\mathcal{O}_1=\mathcal{O}$, $\mathcal{O}_2=\mathcal{O}'$, $K=L=E$, we deduce that the following map

$$\mathrm{End}(E)\to\mathrm{End}(\mathrm{KZ}'\circ E),\quad f\mapsto 1_{\mathrm{KZ}'}f,$$

is bijective. Hence (3.8.3) implies that

$$(\varepsilon 1_E)\circ(1_E\eta)=1_E.$$

Similarly, we have $(1_F\varepsilon)\circ(\eta 1_F)=1_F$. So E is left adjoint to F. By uniqueness of adjoint functors this implies that F is isomorphic to Ind_b. Therefore Ind_b is biadjoint to Res_b. □

4 Fock spaces and cyclotomic rational DAHA's

In this section, we construct the i-restriction and i-induction functors on the category $\mathcal{O}$ of cyclotomic rational DAHA's. We show that these functors yield a type $A^{(1)}$ affine Lie algebra action on the Grothendieck group of the category $\mathcal{O}$, which is isomorphic to a Fock space.

4.1 The affine Lie algebra $\widetilde{\mathfrak{sl}}_e$

Let $e\geqslant 2$ be an integer and let t be a formal parameter. Let $\mathfrak{sl}_e$ be the Lie algebra of traceless $e\times e$ complex matrices. Consider the affine Lie algebra of type $A^{(1)}$

$$\widetilde{\mathfrak{sl}}_e=\mathfrak{sl}_e\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}\partial$$

with the Lie bracket given by

$$[\xi\otimes t^m+xc+y\partial,\xi'\otimes t^n+x'c+y'\partial]=[\xi,\xi']\otimes t^{m+n}+m\delta_{m,-n}\,\mathrm{tr}(\xi\xi')c+ny\xi'\otimes t^n-my'\xi\otimes t^m,$$

where $\mathrm{tr}:\mathfrak{sl}_e\to\mathbb{C}$ is the trace map. Let

$$\widehat{\mathfrak{sl}}_e=\mathfrak{sl}_e\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c.$$

It is the Lie subalgebra of $\widetilde{\mathfrak{sl}}_e$ generated by the Chevalley generators

$$\begin{aligned}&e_i=E_{i,i+1}\otimes 1, && f_i=F_{i+1,i}\otimes 1, && h_i=(E_{ii}-E_{i+1,i+1})\otimes 1, && 1\leqslant i\leqslant e-1,\\ &e_0=E_{e1}\otimes t, && f_0=E_{1e}\otimes t^{-1}, && h_0=(E_{ee}-E_{11})\otimes 1+c.\end{aligned}$$

Here E_{ij} is the elementary matrix with 1 in the position (i,j) and 0 elsewhere. We consider the Cartan subalgebra

$$\mathfrak{t}=\bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}}\mathbb{C}h_i\oplus\mathbb{C}\partial,$$

and its dual $\mathfrak{t}^*$. For $i\in\mathbb{Z}/e\mathbb{Z}$ let $\alpha_i\in\mathfrak{t}^*$ (resp. $\alpha_i^\vee\in\mathfrak{t}$) be the simple root (resp. coroot) corresponding to e_i. The fundamental weights are $\Lambda_i\in\mathfrak{t}^*$, $i\in\mathbb{Z}/e\mathbb{Z}$, with $\Lambda_j(\alpha_i^\vee)=\delta_{ij}$ for any $i, j \in \mathbb{Z}/e\mathbb{Z}$. Let $\delta \in \mathfrak{t}^*$ be the element given by $\delta(h_i) = 0$ for all i and $\delta(\partial) = 1$. Let P be the weight lattice of $\widetilde{\mathfrak{sl}}_e$. It is the free abelian group generated by the fundamental weights and δ. For any $\widetilde{\mathfrak{sl}}_e$-module V and $\mu \in \mathfrak{t}^*$ let

$$V_\mu = \{v \in V \,|\, hv = \mu(h)v, \ \forall\ h \in \mathfrak{t}\}.$$

An element $v \in V$ is called *a weight vector* if it belongs to V_μ for some μ.

4.2 Fock spaces.

In the rest of the chapter, we will fix once for all a positive integer l. Let us introduce the following notation. For any positive integer n, a *partition* μ of n is a sequence of integers $\mu_1 \geqslant \cdots \geqslant \mu_k > 0$ such that the sum $|\mu| = \sum_{i=1}^k \mu_i = n$. An *l-partition* of n is an l-tuple of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ such that $|\lambda| = \sum_{i=1}^l |\lambda^{(i)}| = n$. We denote by $\mathcal{P}_{n,l}$ the set of l-partitions of n. To an l-partition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ we attach the set

$$\Upsilon_\lambda = \{(a, b, j) \in \mathbb{N} \times \mathbb{N} \times (\mathbb{Z}/l\mathbb{Z}) \,|\, 0 < b \leqslant (\lambda^{(j)})_a\}.$$

If μ is an l-partition such that Υ_μ contains Υ_λ, then we write μ/λ for the complement of Υ_λ in Υ_μ. Let $|\mu/\lambda|$ be the number of elements in this set.

Given an l tuple of integers $\mathbf{s} = (s_1, \ldots, s_l)$, the *Fock space* with multi-charge $\mathbf{s}$ is given as follows. As a $\mathbb{C}$-vector space, it is spanned by the l-partitions, i.e., we have

$$\mathcal{F}_\mathbf{s} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathcal{P}_{n,l}} \mathbb{C}|\lambda\rangle.$$

Moreover, following [JMMO91], the space $\mathcal{F}_\mathbf{s}$ carries an integrable $\widetilde{\mathfrak{sl}}_e$-action given by

$$e_i(|\lambda\rangle) = \sum_{|\lambda/\mu|=1, \mathrm{res}(\lambda/\mu)=i} |\mu\rangle, \qquad f_i(|\lambda\rangle) = \sum_{|\mu/\lambda|=1, \mathrm{res}(\mu/\lambda)=i} |\mu\rangle, \quad \forall\ i \in \mathbb{Z}/e\mathbb{Z}. \tag{4.2.1}$$

$$\partial(|\lambda\rangle) = -n_0\ |\lambda\rangle.$$

Here we have used the notation that

$$\mathrm{res}((a, b, j)) = b - a + s_j \in \mathbb{Z}/e\mathbb{Z}, \quad \forall\ (a, b, j) \in \Upsilon_\lambda,$$

and n_i denotes the number of elements in the set $\{(a, b, j) \in \Upsilon_\lambda \,|\, \mathrm{res}((a, b, j)) = i\}$ for any $i \in \mathbb{Z}/e\mathbb{Z}$. For $k \in \mathbb{Z}$ we set $\Lambda_k = \Lambda_{k(mod\ e)}$. Consider the weight

$$\Lambda_\mathbf{s} = \Lambda_{s_1} + \ldots + \Lambda_{s_l}.$$

Then each vector $|\lambda\rangle$ in $\mathcal{F}_\mathbf{s}$ is a weight vector of weight

$$\mathrm{wt}(|\lambda\rangle) = \Lambda_\mathbf{s} - \sum_{i \in \mathbb{Z}/e\mathbb{Z}} n_i \alpha_i. \tag{4.2.2}$$

We call $\mathrm{wt}(|\lambda\rangle)$ the *weight* of λ.

4.3 The wreath product $B_n(l)$

Let $\varepsilon = \exp(2\pi\sqrt{-1}/l)$. Let n be a positive integer. We consider the complex reflection group $B_n(l)$ given by the wreath product of the symmetric group $\mathfrak{S}_n$ and the cyclic group $\mathbb{Z}/l\mathbb{Z}$. In other words, let $\{y_1, \ldots, y_n\}$ be the standard basis of $\mathfrak{h}_n = \mathbb{C}^n$. For $1 \leqslant i, j, k \leqslant n$ with i, j, k distinct, let ε_k, s_{ij} be the following elements of $GL(\mathfrak{h}_n)$:

$$\varepsilon_k(y_k) = \varepsilon y_k, \quad \varepsilon_k(y_j) = y_j, \quad s_{ij}(y_i) = y_j, \quad s_{ij}(y_k) = y_k.$$

Then $B_n(l)$ is the subgroup of $GL(\mathfrak{h}_n)$ generated by the elements ε_k, $1 \leqslant k \leqslant n$, and s_{ij}, $1 \leqslant i < j \leqslant n$. The set of reflections in $B_n(l)$ is

$$\mathcal{S}_n = R \sqcup (\bigsqcup_{p=1}^{l-1} Z^p),$$

where R and Z^p are conjugacy classes given by

$$R = \{s_{ij}^{(p)} = s_{ij}\varepsilon_i^p\varepsilon_j^{-p} \,|\, 1 \leqslant i < j \leqslant n, 1 \leqslant p \leqslant l\}, \quad Z^p = \{\varepsilon_i^p \,|\, 1 \leqslant i \leqslant n\}, \quad 1 \leqslant p \leqslant l-1.$$

Note that there is an obvious inclusion $\mathcal{S}_{n-1} \hookrightarrow \mathcal{S}_n$. It yields an embedding

$$B_{n-1}(l) \hookrightarrow B_n(l). \tag{4.3.1}$$

This embedding identifies $B_{n-1}(l)$ with the parabolic subgroup of $B_n(l)$ given by the stabilizer of the point $b_n = (0, \ldots, 0, 1) \in \mathbb{C}^n$.

Given a partition λ of n, we denote the corresponding irreducible representation of $\mathfrak{S}_n$ again by λ. The irreducible representations of $B_n(l)$ are labeled by the l-partitions of n. Indeed, for $\lambda \in \mathcal{P}_{n,l}$, let

$$I_\lambda(p) = \{\sum_{i=1}^{p-1}|\lambda^{(i)}| + 1, \ \sum_{i=1}^{p-1}|\lambda^{(i)}| + 2, \ \ldots, \sum_{i=1}^{p}|\lambda^{(i)}|\}, \quad 1 \leqslant p \leqslant l.$$

We put $\mathfrak{S}_\lambda = \mathfrak{S}_{I_\lambda(1)} \times \cdots \times \mathfrak{S}_{I_\lambda(l)}$ and $B_\lambda(l) = (\mathbb{Z}/l\mathbb{Z})^n \rtimes \mathfrak{S}_\lambda$. Consider the character $\psi : \mathbb{Z}/l\mathbb{Z} \to \mathbb{C}^*$, $a \mapsto \varepsilon^a$. We denote by $\psi^{(p)}$ the one dimensional character of $(\mathbb{Z}/l\mathbb{Z})^{I_\lambda(p)} \rtimes \mathfrak{S}_{I_\lambda(p)}$ whose restriction to $(\mathbb{Z}/l\mathbb{Z})^{I_\lambda(p)}$ is $(\psi^{p-1})^{\otimes|\lambda^{(p)}|}$ and whose restriction to $\mathfrak{S}_{I_\lambda(p)}$ is trivial. Then we have a bijection

$$\mathcal{P}_{n,l} \xrightarrow{\sim} \mathrm{Irr}(B_n(l)), \quad \lambda \mapsto \mathrm{Ind}_{B_\lambda(l)}^{B_n(l)}(\psi^{(1)}\lambda^{(1)} \otimes \cdots \otimes \psi^{(l)}\lambda^{(l)}),$$

see e.g. [Rou08b, Section 6.1.1]. Below, we will always identify $\mathcal{P}_{n,l}$ and $\mathrm{Irr}(B_n(l))$ in this way.

4.4 Cyclotomic Hecke algebras

The Hecke algebras attached to the group $B_n(l)$ are called *cyclotomic Hecke algebras*, or *Ariki-Koike algebras*. Given a parameter $\mathbf{q} = (q, q_1, \ldots, q_l)$, the corresponding cyclotomic Hecke algebra $\mathscr{H}_{\mathbf{q},n}$ has the following presentation:

- Generators: $T_0, T_1, \ldots, T_{n-1}$,
- Relations:

$$\begin{aligned}
&(T_0 - q_1)\cdots(T_0 - q_l) = (T_i + 1)(T_i - q) = 0, && 1 \leqslant i \leqslant n-1,\\
&T_0T_1T_0T_1 = T_1T_0T_1T_0, &&\\
&T_iT_j = T_jT_i, && j \geqslant i+2,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, && 1 \leqslant i \leqslant n-2.
\end{aligned} \tag{4.4.1}$$

The algebra $\mathscr{H}_{\mathbf{q},n}$ satisfies the assumption of Section 2.5, i.e., it has the same dimension as the group algebra $\mathbb{C}B_n(l)$. We will abbreviate

$$\mathcal{C}_{\mathbf{q},n} = \mathcal{C}_{\mathbf{q},n}, \quad \mathcal{C}_{\mathbf{q}} = \bigoplus_{n\in\mathbb{N}} \mathcal{C}_{\mathbf{q},n}.$$

For each l-partition λ of n, let S_λ be the corresponding Specht module in $\mathcal{C}_{\mathbf{q},n}$, see [Ari02, Definition 13.22] for its definition. The classes $[S_\lambda]$ span the vector space $[\mathcal{C}_{\mathbf{q},n}]$.

Following Section 1.2, the embedding (4.3.1) of $B_{n-1}(l)$ into $B_n(l)$ yields an embedding of Hecke algebras

$$\imath_{\mathbf{q}} : \mathscr{H}_{\mathbf{q},n-1} \hookrightarrow \mathscr{H}_{\mathbf{q},n}.$$

By [BMR98, Proposition 2.29] this embedding is given by

$$\imath_{\mathbf{q}}(T_i) = T_i, \quad \forall\ 0 \leqslant i \leqslant n-2.$$

We will consider the following restriction and induction functors:

$$E(n)^{\mathscr{H}} = {}^{\mathscr{H}}\mathrm{Res}_{B_{n-1}(l)}^{B_n(l)}, \quad F(n)^{\mathscr{H}} = {}^{\mathscr{H}}\mathrm{Ind}_{B_{n-1}(l)}^{B_n(l)}.$$

The algebra $\mathscr{H}_{\mathbf{q},n}$ is symmetric, see Remark 1.3.2. Hence by Proposition 1.3.1 we have

$$F(n)^{\mathscr{H}} \cong {}^{\mathscr{H}}\mathrm{coInd}_{B_{n-1}(l)}^{B_n(l)}.$$

Let

$$E^{\mathscr{H}} = \bigoplus_{n\geqslant 1} E(n)^{\mathscr{H}}, \quad F^{\mathscr{H}} = \bigoplus_{n\geqslant 1} F(n)^{\mathscr{H}}.$$

Then $(E^{\mathscr{H}}, F^{\mathscr{H}})$ is a pair of biadjoint endo-functors of $\mathcal{C}_{\mathbf{q}}$.

4.5 The i-Restriction and i-induction functors of cyclotomic Hecke algebras.

Fix an l-tuple $\mathbf{s} = (s_1, \ldots, s_l)$ and fix a positive integer $e \geqslant 2$. Let the parameter $\mathbf{q} = (q, q_1, \ldots, q_l)$ be given by

$$q = \exp(\frac{2\pi\sqrt{-1}}{e}), \quad q_p = q^{s_p}, \quad 1 \leqslant p \leqslant l. \tag{4.5.1}$$

In this case, Ariki defined the i-restriction and i-induction functors on $\mathcal{C}_{\mathbf{q}}$ as follows.

First, consider the following elements in $\mathscr{H}_{\mathbf{q},n}$,

$$J_0 = T_0, \quad J_i = q^{-1}T_iJ_{i-1}T_i \quad \text{for } 1 \leqslant i \leqslant n-1.$$

They are called the *Jucy-Murphy* elements. Recall that $Z(\mathscr{H}_{\mathbf{q},n})$ is the center of $\mathscr{H}_{\mathbf{q},n}$. For any symmetric polynomial σ of n variables, the element $\sigma(J_0, \ldots, J_{n-1})$ belongs to $Z(\mathscr{H}_{\mathbf{q},n})$, see [Ari02, Section 13.1]. In particular, let z be a formal variable. Then the polynomial

$$C_n(z) = \prod_{i=0}^{n-1}(z - J_i) \in \mathscr{H}_{\mathbf{q},n}[z]$$

has coefficients in $Z(\mathscr{H}_{\mathbf{q},n})$. Next, let $\mathbb{C}(z)$ be the fraction field of $\mathbb{C}[z]$. To any $a(z) \in \mathbb{C}(z)$ we associate an exact functor

$$P_{n,a(z)} : \mathcal{C}_{\mathbf{q},n} \to \mathcal{C}_{\mathbf{q},n}, \quad M \mapsto P_{n,a(z)}(M),$$

such that $P_{n,a(z)}(M)$ is the generalized eigenspace of $C_n(z)$ acting on M with the eigenvalue $a(z)$.

Definition 4.5.1. Let $i \in \mathbb{Z}/e\mathbb{Z}$. The *i-restriction* functor and *i-induction* functor

$$E_i(n)^{\mathscr{H}} : \mathcal{C}_{\mathbf{q},n} \to \mathcal{C}_{\mathbf{q},n-1}, \quad F_i(n)^{\mathscr{H}} : \mathcal{C}_{\mathbf{q},n-1} \to \mathcal{C}_{\mathbf{q},n}$$

are given by

$$E_i(n)^{\mathscr{H}} = \bigoplus_{a(z)\in\mathbb{C}(z)} P_{n-1,a(z)/(z-q^i)} \circ E(n)^{\mathscr{H}} \circ P_{n,a(z)},$$
$$F_i(n)^{\mathscr{H}} = \bigoplus_{a(z)\in\mathbb{C}(z)} P_{n,a(z)(z-q^i)} \circ F(n)^{\mathscr{H}} \circ P_{n-1,a(z)}.$$

See [Ari02, Definition 13.33]. We abbreviate

$$E_i^{\mathscr{H}} = \bigoplus_{n\geqslant 1} E_i(n)^{\mathscr{H}}, \quad F_i^{\mathscr{H}} = \bigoplus_{n\geqslant 1} F_i(n)^{\mathscr{H}}.$$

They are endo-functors of $\mathcal{C}_{\mathbf{q}}$. Further, let

$$a_\lambda(z) = \prod_{v\in\Upsilon_\lambda} (z - q^{\mathrm{res}(v)}), \quad \forall\ \lambda \in \mathcal{P}_{n,l}.$$

Then the functors $E_i^{\mathscr{H}}$, $F_i^{\mathscr{H}}$ have the following properties.

Proposition 4.5.2. *(a) The functors $E_i(n)^{\mathscr{H}}$, $F_i(n)^{\mathscr{H}}$ are exact biadjoint functors.*

(b) For $\lambda \in \mathcal{P}_{n,l}$ the element $C_n(z)$ has a unique eigenvalue on the Specht module S_λ. It is equal to $a_\lambda(z)$.

(c) We have

$$E_i(n)^{\mathscr{H}}([S_\lambda]) = \sum_{\mathrm{res}(\lambda/\mu)=i} [S_\mu], \qquad F_i(n)^{\mathscr{H}}([S_\lambda]) = \sum_{\mathrm{res}(\mu/\lambda)=i} [S_\mu].$$

(d) We have

$$E(n)^{\mathscr{H}} = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i(n)^{\mathscr{H}}, \quad F(n)^{\mathscr{H}} = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} F_i(n)^{\mathscr{H}}.$$

Proof. Part (a) follows from the fact that $E(n)^{\mathscr{H}}$, $F(n)^{\mathscr{H}}$ are exact and biadjoint. See [Ari02, Theorem 13.21(2)] for part (b) and [Ari02, Lemma 13.37] for part (c). Part (d) follows from part (c) and [Ari02, Lemma 13.32]. □

4.6 Cyclotomic rational DAHA's

Now, let us consider the cyclotomic rational DAHA's, i.e., the rational DAHA's associated with the complex reflection group $B_n(l)$'s. Given an l-tuple complex numbers $\mathbf{h} = (h, h_1, \ldots, h_{l-1})$, we define a W-invariant function $c : \mathcal{S}_n \to \mathbb{C}$ by setting

$$c(R) = -h, \quad c(Z^p) = \frac{-1}{2}\sum_{p'=1}^{l-1} (\varepsilon^{-pp'} - 1)h_{p'}, \quad \forall\ 1 \leqslant p \leqslant l-1.$$

We write $H_{\mathbf{h},n} = H_c(B_n(l), \mathfrak{h}_n)$, and write $\mathcal{O}_{\mathbf{h},n}$ for the corresponding category $\mathcal{O}$. Note that the standard objects in $\mathcal{O}_{\mathbf{h},n}$ are indexed by l-partitions of n.

In the rest of the chapter, we will fix once for all an l-tuple of integers $\mathbf{s} = (s_1, \ldots, s_l)$ and a positive integer $e \geqslant 2$. We will always let the parameter $\mathbf{h}$ be given by

$$h = \frac{-1}{e}, \quad h_p = \frac{s_{p+1} - s_p}{e} - \frac{1}{l}, \quad 1 \leqslant p \leqslant l-1\,. \tag{4.6.1}$$

In this case, the Knizhnik-Zamolodchikov functor has the following form

$$\mathrm{KZ}_{\mathbf{h},n} = \mathrm{KZ}(B_n(l), \mathfrak{h}_n) : \mathcal{O}_{\mathbf{h},n} \to \mathcal{C}_{\mathbf{q},n},$$

where $\mathbf{q}$ is as in (4.5.1). The operator $\mathrm{KZ}_{\mathbf{h},n} : [\mathcal{O}_{\mathbf{h},n}] \to [\mathscr{H}_{\mathbf{q},n}]$ has the following property.

Lemma 4.6.1. *For $\lambda \in \mathcal{P}_{n,l}$ we have*

$$\mathrm{KZ}_{\mathbf{h},n}([\Delta(\lambda)]) = [S_\lambda].$$

Proof. We abbreviate $\mathrm{KZ} = \mathrm{KZ}_{\mathbf{h},n}$. Let R be any commutative ring over $\mathbb{C}$. For any l-tuple $\mathbf{z} = (z, z_1, \ldots, z_{l-1})$ of elements in R one defines the rational DAHA over R attached to $B_n(l)$ with parameter $\mathbf{z}$ in the same way as before. Denote it by $H_{R,\mathbf{z},n}$. The standard modules $\Delta_R(\lambda)$ are also defined as before. For any $(l+1)$-tuple $\mathbf{u} = (u, u_1, \ldots, u_l)$ of invertible elements in R the Hecke algebra $\mathscr{H}_{R,\mathbf{u},n}$ over R attached to $B_n(l)$ with parameter $\mathbf{u}$ is defined by the same presentation as in Section 4.4. The Specht modules $S_{R,\lambda}$ are also well-defined, see [Ari02]. If R is a field, we will write $\mathrm{Irr}(\mathscr{H}_{R,\mathbf{u},n})$ for the set of isomorphism classes of simple $\mathscr{H}_{R,\mathbf{u},n}$-modules.

Now, fix R to be the ring of holomorphic functions of one variable ϖ. We choose $\mathbf{z} = (z, z_1, \ldots, z_{l-1})$ to be given by

$$z = l\varpi, \quad z_p = (s_{p+1} - s_p)l\varpi + e\varpi, \quad 1 \leqslant p \leqslant l-1.$$

Write $x = \exp(-2\pi\sqrt{-1}\varpi)$. Let $\mathbf{u} = (u, u_1, \ldots, u_l)$ be given by

$$u = x^l, \quad u_p = \varepsilon^{p-1} x^{s_p l - (p-1)e}, \quad 1 \leqslant p \leqslant l.$$

By [BMR98, Theorem 4.12] the same definition as in Section 2.4 yields a well defined $\mathscr{H}_{R,\mathbf{u},n}$-module

$$T_R(\lambda) = \mathrm{KZ}_R(\Delta_R(\lambda)).$$

It is a free R-module of finite rank and it commutes with the base change functor by the existence and unicity theorem for linear differential equations, i.e., for any ring homomorphism $R \to R'$ over $\mathbb{C}$, we have a canonical isomorphism of $\mathscr{H}_{R',\mathbf{u},n}$-modules

$$T_{R'}(\lambda) = \mathrm{KZ}_{R'}(\Delta_{R'}(\lambda)) = T_R(\lambda) \otimes_R R'. \tag{4.6.2}$$

In particular, for any ring homomorphism $a : R \to \mathbb{C}$. Write $\mathbb{C}_a$ for the vector space $\mathbb{C}$ equipped with the R-module structure given by a. Let $a(\mathbf{z})$, $a(\mathbf{u})$ denote the images of $\mathbf{z}$, $\mathbf{u}$ by a. Note that we have $H_{a(\mathbf{z}),n} = H_{R,\mathbf{z},n} \otimes_R \mathbb{C}_a$ and $\mathscr{H}_{a(\mathbf{u}),n} = \mathscr{H}_{R,\mathbf{u},n} \otimes_R \mathbb{C}_a$. Denote the Knizhnik-Zamolodchikov functor of $H_{a(\mathbf{z}),n}$ by $\mathrm{KZ}_{a(\mathbf{z})}$ and the standard module corresponding to λ by $\Delta_{a(\mathbf{z})}(\lambda)$. Then we have an isomorphism of $\mathscr{H}_{a(\mathbf{u}),n}$-modules

$$T_R(\lambda) \otimes_R \mathbb{C}_a = \mathrm{KZ}_{a(\mathbf{z})}(\Delta_{a(\mathbf{z})}(\lambda)).$$

Let K be the fraction field of R. By [GGOR03, Theorem 2.19] the category $\mathcal{O}_{K,\mathbf{z},n}$ is split semisimple. In particular, the standard modules are simple. We have

$$\{T_K(\lambda) \,|\, \lambda \in \mathcal{P}_{n,l}\} = \mathrm{Irr}(\mathscr{H}_{K,\mathbf{u},n}).$$

The Hecke algebra $\mathscr{H}_{K,\mathbf{u},n}$ is also split semisimple and we have

$$\{S_{K,\lambda} \,|\, \lambda \in \mathcal{P}_{n,l}\} = \mathrm{Irr}(\mathscr{H}_{K,\mathbf{u},n}),$$

see e.g., [Ari02, Corollary 13.9]. Thus there is a bijection $\varphi : \mathcal{P}_{n,l} \to \mathcal{P}_{n,l}$ such that $T_K(\lambda)$ is isomorphic to $S_{K,\varphi(\lambda)}$ for all λ. We claim that φ is identity. To see this, consider the algebra homomorphism $a_0 : R \to \mathbb{C}$ given by $\varpi \mapsto 0$. Then $\mathscr{H}_{a_0(\mathbf{u}),n}$ is canonically isomorphic to the group algebra $\mathbb{C}B_n(l)$. In particular, it is semi-simple. Let $\overline{K}$ be the algebraic closure of K. Let $\overline{R}$ be the integral closure of R in $\overline{K}$ and fix an extension $\overline{a}_0$ of a_0 to $\overline{R}$. By Tit's deformation theorem, see e.g., [CR87, Section 68A], there is a bijection

$$\psi : \mathrm{Irr}(\mathscr{H}_{\overline{K},\mathbf{u},n}) \xrightarrow{\sim} \mathrm{Irr}(\mathscr{H}_{a_0(\mathbf{u}),n})$$

such that

$$\psi(T_{\overline{K}}(\lambda)) = T_{\overline{R}}(\lambda) \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}, \quad \psi(S_{\overline{K},\lambda}) = S_{\overline{R},\lambda} \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0}.$$

By the definition of Specht modules we have $S_{\overline{R},\lambda} \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0} \cong \lambda$ as $\mathbb{C}B_n(l)$-modules. On the other hand, since $a_0(\mathbf{z}) = 0$, by (4.6.2) we have the following isomorphisms

$$\begin{aligned} T_{\overline{R}}(\lambda) \otimes_{\overline{R}} \mathbb{C}_{\overline{a}_0} &= T_R(\lambda) \otimes_R \mathbb{C}_{a_0} \\ &= \mathrm{KZ}_0(\Delta_0(\lambda)) \\ &= \lambda. \end{aligned}$$

So $\psi(T_{\overline{K}}(\lambda)) = \psi(S_{\overline{K},\lambda})$. Hence we have $T_{\overline{K}}(\lambda) = S_{\overline{K},\lambda}$. Since $T_{\overline{K}}(\lambda) = T_K(\lambda) \otimes_K \overline{K}$ is isomorphic to $S_{\overline{K},\varphi(\lambda)} = S_{K,\varphi(\lambda)} \otimes_K \overline{K}$, we deduce that $\varphi(\lambda) = \lambda$. The claim is proved.

Finally, let $\mathfrak{m}$ be the maximal ideal of R consisting of the functions vanishing at $\varpi = -1/el$. Let $\widehat{R}$ be the completion of R at $\mathfrak{m}$. It is a discrete valuation ring with residue field $\mathbb{C}$. Let $a_1 : \widehat{R} \to \widehat{R}/\mathfrak{m}\widehat{R} = \mathbb{C}$ be the quotient map. We have $a_1(\mathbf{z}) = \mathbf{h}$ and $a_1(\mathbf{u}) = \mathbf{q}$. Let $\widehat{K}$ be the fraction field of $\widehat{R}$. The *decomposition map* is given by

$$d : [\mathscr{H}_{\widehat{K},\mathbf{u},n}] \to [\mathscr{H}_{\mathbf{q},n}], \quad [M] \mapsto [L \otimes_{\widehat{R}} \mathbb{C}_{a_1}].$$

Here L is any $\mathscr{H}_{\widehat{R},\mathbf{u},n}$-submodule of M, free over $\widehat{R}$, such that $L \otimes_{\widehat{R}} \widehat{K} = M$. The choice of L does not affect the class $[L \otimes_{\widehat{R}} \mathbb{C}_{a_1}]$ in $[\mathscr{H}_{\mathbf{q},n}]$. See [Ari02, Section 13.3] for details on this map. So we have

$$d([S_{\widehat{K},\lambda}]) = [S_{\widehat{R},\lambda} \otimes_{\widehat{R}} \mathbb{C}_{a_1}] = [S_\lambda],$$
$$d([T_{\widehat{K}}(\lambda)]) = [T_{\widehat{R}}(\lambda) \otimes_{\widehat{R}} \mathbb{C}_{a_1}] = [\mathrm{KZ}(\Delta(\lambda))].$$

Since $\widehat{K}$ is an extension of K, by the last paragraph we have $[S_{\widehat{K},\lambda}] = [T_{\widehat{K}}(\lambda)]$. We deduce that $[\mathrm{KZ}(\Delta(\lambda))] = [S_\lambda]$. □

4.7 The i-Restricion and i induction of cyclotomic rational DAHA's

We abbreviate

$$\mathscr{O}_{\mathbf{h}} = \bigoplus_{n\in\mathbb{N}} \mathscr{O}_{\mathbf{h},n}, \quad \mathrm{KZ} = \bigoplus_{n\in\mathbb{N}} \mathrm{KZ}_{\mathbf{h},n}\,.$$

We have the following exact functors on $\mathscr{O}_{\mathbf{h}}$

$$E = \bigoplus_{n\geqslant 1} E(n), \quad F = \bigoplus_{n\geqslant 1} F(n).$$

By Proposition 3.8.2, they are biadjoint. We have isomorphisms of functors

$$E^{\mathscr{H}} \circ \mathrm{KZ} \cong \mathrm{KZ} \circ E, \quad F^{\mathscr{H}} \circ \mathrm{KZ} \cong \mathrm{KZ} \circ F \tag{4.7.1}$$

by Theorem 3.5.1 and Corollary 3.5.3.

Recall from Lemma 2.5.1(a) that we have an algebra isomorphism

$$\gamma : Z(\mathcal{O}_{\mathbf{h},n}) \xrightarrow{\sim} Z(\mathscr{H}_{\mathbf{q},n}).$$

So there are unique elements $K_1, \ldots, K_n \in Z(\mathcal{O}_{\mathbf{h},n})$ such that the polynomial

$$D_n(z) = z^n + K_1 z^{n-1} + \cdots + K_n$$

maps to $C_n(z)$ by γ. Since the elements $K_1, \ldots, K_n$ act on simple modules by scalars and the category $\mathcal{O}_{\mathbf{h},n}$ is artinian, every module M in $\mathcal{O}_{\mathbf{h},n}$ is a direct sum of generalized eigenspaces of $D_n(z)$. For $a(z) \in \mathbb{C}(z)$ let $Q_{n,a(z)}$ be the exact functor

$$Q_{n,a(z)} : \mathcal{O}_{\mathbf{h},n} \to \mathcal{O}_{\mathbf{h},n}, \quad M \mapsto Q_{n,a(z)}(M),$$

such that $Q_{n,a(z)}(M)$ is the generalized eigenspace of $D_n(z)$ acting on M with the eigenvalue $a(z)$.

Definition 4.7.1. The *i-restriction* functor and the *i-induction* functor

$$E_i(n) : \mathcal{O}_{\mathbf{h},n} \to \mathcal{O}_{\mathbf{h},n-1}, \quad F_i(n) : \mathcal{O}_{\mathbf{h},n-1} \to \mathcal{O}_{\mathbf{h},n}$$

are given by

$$E_i(n) = \bigoplus_{a(z)\in\mathbb{C}(z)} Q_{n-1,a(z)/(z-q^i)} \circ E(n) \circ Q_{n,a(z)},$$
$$F_i(n) = \bigoplus_{a(z)\in\mathbb{C}(z)} Q_{n,a(z)(z-q^i)} \circ F(n) \circ Q_{n-1,a(z)}.$$

We abbreviate

$$E_i = \bigoplus_{n\geqslant 1} E_i(n), \quad F_i = \bigoplus_{n\geqslant 1} F_i(n). \tag{4.7.2}$$

Lemma 4.7.2. *We have isomorphisms of functors*

$$\mathrm{KZ} \circ E_i(n) \cong E_i(n)^{\mathscr{H}} \circ \mathrm{KZ}, \quad \mathrm{KZ} \circ F_i(n) \cong F_i(n)^{\mathscr{H}} \circ \mathrm{KZ}, \quad \forall\, i \in \mathbb{Z}/e\mathbb{Z}.$$

Proof. Since $\gamma(D_n(z)) = C_n(z)$, by Lemma 2.5.1(b) we have

$$\mathrm{KZ} \circ Q_{n,a(z)} \cong P_{n,a(z)} \circ \mathrm{KZ}, \quad \forall\, a(z) \in \mathbb{C}(z).$$

So the proposition follows from (4.7.1). □

Now, let us give some properties of the functors E_i, F_i in parallel to Proposition 4.5.2.

Proposition 4.7.3. *(a) The functors $E_i(n)$, $F_i(n)$ are exact biadjoint functors.*

(b) For any $\lambda \in \mathcal{P}_{n,l}$ the unique eigenvalue of $D_n(z)$ on the standard module $\Delta(\lambda)$ is $a_\lambda(z)$.

(c) We have

$$E_i(n)([\Delta(\lambda)]) = \sum_{\mathrm{res}(\lambda/\mu)=i} [\Delta(\mu)], \qquad F_i(n)([\Delta(\lambda)]) = \sum_{\mathrm{res}(\mu/\lambda)=i} [\Delta(\mu)]. \tag{4.7.3}$$

(d) We have

$$E(n) = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i(n), \quad F(n) = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} F_i(n).$$

Proof. By definition, the functors $E_i(n)$, $F_i(n)$ are exact. The biadjointness in part (a) follows from Proposition 3.8.2. For part (b), note that a standard module is indecomposable, so the element $D_n(z)$ has a unique eigenvalue on $\Delta(\lambda)$. By Lemma 4.6.1 this eigenvalue is the same as the eigenvalue of $C_n(z)$ on S_λ. So part (b) follows from Proposition 4.5.2(b). Next, let us prove the equality for $E_i(n)$ in part (c). The Pieri rule for the group $B_n(l)$ together with Proposition 3.3.2(b) yields

$$E(n)([\Delta(\lambda)]) = \sum_{|\lambda/\mu|=1} [\Delta(\mu)], \quad F(n)([\Delta(\lambda)]) = \sum_{|\mu/\lambda|=1} [\Delta(\mu)]. \tag{4.7.4}$$

So we have

$$\begin{aligned} E_i(n)([\Delta(\lambda)]) &= \bigoplus_{a(z)\in\mathbb{C}[z]} Q_{n-1,a(z)/(z-q^i)}(E(n)(Q_{n,a(z)}([\Delta(\lambda)]))) \\ &= Q_{n-1,a_\lambda(z)/(z-q^i)}(E(n)(Q_{n,a_\lambda(z)}([\Delta(\lambda)]))) \\ &= Q_{n-1,a_\lambda(z)/(z-q^i)}(E(n)([\Delta(\lambda)])) \\ &= Q_{n-1,a_\lambda(z)/(z-q^i)}(\sum_{|\lambda/\mu|=1} [\Delta(\mu)]) \\ &= \sum_{\mathrm{res}(\lambda/\mu)=i} [\Delta(\mu)]. \end{aligned}$$

The last equality follows from the fact that for any l-partition μ such that $|\lambda/\mu| = 1$ we have $a_\lambda(z) = a_\mu(z)(z - q^{\mathrm{res}(\lambda/\mu)})$. The equality for $F_i(n)$ is proved in the same way. So part (c) is proved. Finally, part (d) follows from part (c) and (4.7.4). $\square$

Now, let us consider the following map

$$\theta : [\mathcal{O}_{\mathbf{h}}] \xrightarrow{\sim} \mathcal{F}_{\mathbf{s}}, \quad [\Delta(\lambda)] \mapsto |\lambda\rangle. \tag{4.7.5}$$

It is an isomorphism of vector spaces.

Proposition 4.7.4. *Under the isomorphism θ, the operators E_i, F_i on $[\mathcal{O}_{\mathbf{h}}]$ go respectively to the operators e_i, f_i on $\mathcal{F}_{\mathbf{s}}$. When i runs over $\mathbb{Z}/e\mathbb{Z}$ they yield an action of $\widehat{\mathfrak{sl}}_e$ on $[\mathcal{O}_{\mathbf{h}}]$ such that θ is an isomorphism of $\widehat{\mathfrak{sl}}_e$-modules.*

Proof. This is clear from Proposition 4.7.3(c) and (4.2.1). $\square$

5 Categorifications and crystals

In this section, we present the main result of the chapter. We first construct an $\widetilde{\mathfrak{sl}}_e$-categorification on $\mathcal{O}_{\mathbf{h}}$. Then we use it to construct a crystal on the classes of simple objects of $\mathcal{O}_{\mathbf{h}}$, which is isomorphic to the crystal of the Fock space $\mathcal{F}_{\mathbf{s}}$.

5.1 Categorifications

Let $q = \exp(\frac{2\pi\sqrt{-1}}{e})$. Recall that P is the weight lattice for the Lie algebra $\widetilde{\mathfrak{sl}}_e$. Let $\mathcal{C}$ be a $\mathbb{C}$-linear artinian abelian category. For any functor $F : \mathcal{C} \to \mathcal{C}$ and any morphism $X \in \mathrm{End}(F)$, the generalized eigenspace of X acting on F with eigenvalue $a \in \mathbb{C}$ is called *the a-eigenspace of X in F.*

Definition 5.1.1. An $\widetilde{\mathfrak{sl}}_e$*-categorification* on $\mathcal{C}$ is the data of

(i) an adjoint pair (U, V) of exact functors $\mathcal{C} \to \mathcal{C}$,
(ii) $X \in \mathrm{End}(U)$ and $T \in \mathrm{End}(U^2)$,
(iii) a decomposition $\mathcal{C} = \bigoplus_{\tau \in P} \mathcal{C}_\tau$.

Let U_i (resp. V_i) be the q^i-eigenspace of X in U (resp. in V) [2] for $i \in \mathbb{Z}/e\mathbb{Z}$. We require that

(a) $U = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} U_i$,
(b) the endomorphisms X and T satisfy

$$\begin{gathered}(1_U T) \circ (T 1_U) \circ (1_U T) = (T 1_U) \circ (1_U T) \circ (T 1_U), \\ (T + 1_{U^2}) \circ (T - q 1_{U^2}) = 0, \\ T \circ (1_U X) \circ T = q X 1_U,\end{gathered} \tag{5.1.1}$$

(c) the action of $e_i = U_i$, $f_i = V_i$ on $[\mathcal{C}]$ with i running over $\mathbb{Z}/e\mathbb{Z}$ gives an integrable representation of $\widehat{\mathfrak{sl}}_e$.
(d) $U_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau+\alpha_i}$ and $V_i(\mathcal{C}_\tau) \subset \mathcal{C}_{\tau-\alpha_i}$,
(e) V is isomorphic to a left adjoint of U.

See [Rou08a, Definition 5.29], and also [CR08, Section 5.2.1].

5.2 Crystals arising from categorifications

Now, we explain how to use categorifications to construct crystals.

Definition 5.2.1. A *crystal* (or more precisely, an $\widetilde{\mathfrak{sl}}_e$-*crystal*) is a set B together with maps

$$\mathrm{wt} : B \to P, \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \quad \epsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\},$$

such that

- $\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$,
- if $\tilde{e}_i b \in B$, then $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$, $\quad \epsilon_i(\tilde{e}_i(b)) = \epsilon_i(b) - 1$, $\quad \varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$,
- if $\tilde{f}_i(b) \in B$, then $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$, $\quad \epsilon_i(\tilde{f}_i(b)) = \epsilon_i(b) + 1$, $\quad \varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$,
- let $b, b' \in B$, then $\tilde{f}_i(b) = b'$ if and only if $\tilde{e}_i(b') = b$,
- if $\varphi_i(b) = -\infty$, then $\tilde{e}_i(b) = 0$ and $\tilde{f}_i(b) = 0$.

Let $\mathfrak{b}$ be the Lie subalgebra of $\widetilde{\mathfrak{sl}}_e$ generated by $\mathfrak{t}$ and the elements e_i, $i \in \mathbb{Z}/e\mathbb{Z}$. We say that an $\widetilde{\mathfrak{sl}}_e$-module V is $\mathfrak{b}$-*locally finite* if

- $V = \bigoplus_{\mu \in P} V_\mu$,
- for any $v \in V$, the $\mathfrak{b}$-submodule of V generated by v is finite dimensional.

Let V be a $\mathfrak{b}$-locally finite $\widetilde{\mathfrak{sl}}_e$-module. For any nonzero vector $v \in V$ and any $i \in \mathbb{Z}/e\mathbb{Z}$ we set

$$l_i(v) = \max\{l \in \mathbb{N} \mid e_i^l(v) \neq 0\}.$$

Write $l_i(0) = -\infty$. For $l \geqslant 0$ let

$$V_i^{<l} = \{v \in V \mid l_i(v) < l\}.$$

A *weight basis* of V is a basis B of V such that each element of B is a weight vector. The following definition is due to A. Berenstein and D. Kazhdan [BK07, Definition 5.30].

2. Here X acts on V via the isomorphism $\mathrm{End}(U) \cong \mathrm{End}(V)^{op}$ given by adjunction, see [CR08, Section 4.1.2] for the precise definition.

Definition 5.2.2. A *perfect basis* of V is a weight basis B together with maps $\tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}$ for $i \in \mathbb{Z}/e\mathbb{Z}$ such that

- for $b, b' \in B$ we have $\tilde{f}_i(b) = b'$ if and only if $\tilde{e}_i(b') = b$,
- we have $\tilde{e}_i(b) \neq 0$ if and only if $e_i(b) \neq 0$,
- if $e_i(b) \neq 0$ then we have

$$e_i(b) \in \mathbb{C}^* \tilde{e}_i(b) + V_i^{<l_i(b)-1}. \tag{5.2.1}$$

Given a perfect basis $(B, \tilde{e}_i, \tilde{f}_i)$ and $b \in B$ let $\mathrm{wt}(b)$ be the weight of b. Set also $\epsilon_i(b) = l_i(b)$ and

$$\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle.$$

Then the data

$$(B, \mathrm{wt}, \tilde{e}_i, \tilde{f}_i, \epsilon_i, \varphi_i) \tag{5.2.2}$$

is a crystal. We will always attach this crystal structure to $(B, \tilde{e}_i, \tilde{f}_i)$.

An element $b \in B$ is called *primitive* if $e_i(b) = 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. Let B^+ be the set of primitive elements in B. Let V^+ be the vector space spanned by all the primitive vectors in V. Then we have the following lemma.

Lemma 5.2.3. *For any perfect basis $(B, \tilde{e}_i, \tilde{f}_i)$ the set B^+ is a basis of V^+.*

Proof. This is [BK07, Claim 5.32]. We give a proof for completeness. By definition we have $B^+ \subset V^+$. Given a vector $v \in V^+$, there exist $\zeta_1, \ldots, \zeta_r \in \mathbb{C}^*$ and distinct elements $b_1, \ldots, b_r \in B$ such that $v = \sum_{j=1}^r \zeta_j b_j$. For any $i \in \mathbb{Z}/e\mathbb{Z}$ let $l_i = \max\{l_i(b_j) \,|\, 1 \leqslant j \leqslant r\}$ and $J = \{j \,|\, l_i(b_j) = l_i,\ 1 \leqslant j \leqslant r\}$. Then by the third property of perfect basis there exist $\eta_j \in \mathbb{C}^*$ for $j \in J$ and a vector $w \in V^{<l_i-1}$ such that $0 = e_i(v) = \sum_{j \in J} \zeta_j \eta_j \tilde{e}_i(b_j) + w$. For distinct $j, j' \in J$, we have $b_j \neq b_{j'}$, so $\tilde{e}_i(b_j)$ and $\tilde{e}_i(b_{j'})$ are different unless they are zero. Moreover, since $l_i(\tilde{e}_i(b_j)) = l_i - 1$, the equality yields that $\tilde{e}_i(b_j) = 0$ for all $j \in J$. So $l_i = 0$. Hence $b_j \in B^+$ for $j = 1, \ldots, r$. $\square$

Consider an $\widetilde{\mathfrak{sl}}_e$-categorification on a $\mathbb{C}$-linear artinian abelian category $\mathcal{C}$ given by an adjoint pair of endo-functors (U, V), $X \in \mathrm{End}(U)$ and $T \in \mathrm{End}(U^2)$. Assume that the $\widetilde{\mathfrak{sl}}_e$-module $[\mathcal{C}]$ is $\mathfrak{b}$-locally finite, then one can construct a perfect basis of $[\mathcal{C}]$ as follows. For $i \in \mathbb{Z}/e\mathbb{Z}$ let U_i, V_i be the q^i-eigenspaces of X in U and V. By definition, the action of X restricts to each U_i. The endomorphism T of U^2 also restricts to endomorphism of $(U_i)^2$, see e.g., the beginning of Section 7 in [CR08]. It follows that the data (U_i, V_i, X, T) gives an $\mathfrak{sl}_2$-categorification on $\mathcal{C}$ in the sense of [CR08, Section 5.21]. By [CR08, Proposition 5.20] this implies that for any simple object L in $\mathcal{C}$, the object $\mathrm{head}(U_i(L))$ (resp. $\mathrm{soc}(V_i L)$) is simple unless it is zero.

Let $B_{\mathcal{C}}$ be the set of isomorphism classes of simple objects in $\mathcal{C}$. As part of the data of the $\widetilde{\mathfrak{sl}}_e$-categorification, we have a decomposition $\mathcal{C} = \oplus_{\tau \in P} \mathcal{C}_\tau$. For a simple module $L \in \mathcal{C}_\tau$, the weight of $[L]$ in $[\mathcal{C}]$ is τ. Hence $B_{\mathcal{C}}$ is a weight basis of $[\mathcal{C}]$. Now for $i \in \mathbb{Z}/e\mathbb{Z}$ define the maps

$$\begin{aligned} \tilde{e}_i : \quad & B_{\mathcal{C}} \to B_{\mathcal{C}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{head}(U_i L)], \\ \tilde{f}_i : \quad & B_{\mathcal{C}} \to B_{\mathcal{C}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{soc}(V_i L)]. \end{aligned}$$

Proposition 5.2.4. *The data $(B_{\mathcal{C}}, \tilde{e}_i, \tilde{f}_i)$ is a perfect basis of $[\mathcal{C}]$.*

Proof. Fix $i \in \mathbb{Z}/e\mathbb{Z}$. Let us check the conditions in the definition. First, for two simple modules L, $L' \in \mathcal{C}$, we have $\tilde{e}_i([L]) = [L']$ if and only if $0 \neq \mathrm{Hom}(U_i L, L') = \mathrm{Hom}(L, V_i L')$, if and only if $\tilde{f}_i([L']) = [L]$. The second condition follows from the fact that any non trivial module has a non trivial head. Finally, the third condition follows from [CR08, Proposition 5.20(d)]. □

Example 5.2.5. We have the following well known $\widetilde{\mathfrak{sl}}_e$-categorification on $\mathcal{C}_{\mathbf{q}}$, see e.g., [CR08, Section 7.2.2]. Consider the biadjoint endofunctors $E^{\mathscr{H}}$, $F^{\mathscr{H}}$ on $\mathcal{C}_{\mathbf{q}}$. Let $X^{\mathscr{H}}$ be the endomorphism of $E^{\mathscr{H}}$ given on $E(n)^{\mathscr{H}}$ as the multiplication by the Jucy-Murphy element J_{n-1}. Let $T^{\mathscr{H}}$ be the endomorphism of $(E^{\mathscr{H}})^2$ given on $E(n)^{\mathscr{H}} \circ E(n-1)^{\mathscr{H}}$ as the multiplication by the element T_{n-1} in $\mathscr{H}_{\mathbf{q},n}$. Note that the endomorphisms $X^{\mathscr{H}}$ and $T^{\mathscr{H}}$ satisfy the relations (5.1.1). Moreover, the q^i-eigenspace of $X^{\mathscr{H}}$ in $E^{\mathscr{H}}$ and $F^{\mathscr{H}}$ gives respectively the i-restriction functor $E_i^{\mathscr{H}}$ and the i-induction functor $F_i^{\mathscr{H}}$ for $i \in \mathbb{Z}/e\mathbb{Z}$. Finally, by [LM07, Theorem 2.11] the block decomposition of the category $\mathcal{C}_{\mathbf{q}}$ has the form

$$\mathcal{C}_{\mathbf{q}} = \bigoplus_{\tau \in P} (\mathcal{C}_{\mathbf{q}})_\tau, \tag{5.2.3}$$

where $(\mathcal{C}_{\mathbf{q}})_\tau$ is the subcategory generated by the composition factors of the Specht modules S_λ with λ running over l-partitions of weight τ. By convention $(\mathcal{C}_{\mathbf{q}})_\tau$ is zero if such λ does not exist. Then the data $(E^{\mathscr{H}}, F^{\mathscr{H}}, X^{\mathscr{H}}, T^{\mathscr{H}})$ and the decomposition (5.2.3) is an $\widetilde{\mathfrak{sl}}_e$-categorification on $\mathcal{C}_{\mathbf{q}}$.

5.3 An $\widetilde{\mathfrak{sl}}_e$-categorification on $\mathcal{O}_{\mathbf{h}}$.

We construct an $\widetilde{\mathfrak{sl}}_e$-categorification on $\mathcal{O}_{\mathbf{h}}$ as follows. The adjoint pair will be given by (E, F). To construct the endomorphisms X, T, consider the isomorphism of functors

$$\mathrm{KZ} \circ E \cong E^{\mathscr{H}} \circ \mathrm{KZ}$$

given by Theorem 3.5.1. It yields an isomorphism of rings

$$\mathrm{End}(\mathrm{KZ} \circ E) \cong \mathrm{End}(E^{\mathscr{H}} \circ \mathrm{KZ}).$$

By Proposition 3.3.2(a), the functor E maps projective objects to projective ones, so Lemma 3.6.1 applied to $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_{\mathbf{h}}$ and $K = L = E$ yields an isomorphism

$$\mathrm{End}(E) \cong \mathrm{End}(\mathrm{KZ} \circ E).$$

Composing it with the isomorphism above gives a ring isomorphism

$$\sigma_E : \mathrm{End}(E) \xrightarrow{\sim} \mathrm{End}(E^{\mathscr{H}} \circ \mathrm{KZ}). \tag{5.3.1}$$

Replacing E by E^2 we get another isomorphism

$$\sigma_{E^2} : \mathrm{End}(E^2) \xrightarrow{\sim} \mathrm{End}((E^{\mathscr{H}})^2 \circ \mathrm{KZ}).$$

Consider the $X^{\mathscr{H}}$, $T^{\mathscr{H}}$ defined in Example 5.2.5. We define the endomorphisms $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$ by

$$X = \sigma_E^{-1}(X^{\mathscr{H}} 1_{\mathrm{KZ}}), \quad T = \sigma_{E^2}^{-1}(T^{\mathscr{H}} 1_{\mathrm{KZ}}).$$

Finally, recall from Lemma 2.5.1 that the functor KZ induces a bijection between the blocks of the category $\mathcal{O}_{\mathbf{h}}$ and the blocks of $\mathcal{C}_{\mathbf{q}}$. So by (5.2.3), the block decomposition of $\mathcal{O}_{\mathbf{h}}$ is

$$\mathcal{O}_{\mathbf{h}} = \bigoplus_{\tau \in P} (\mathcal{O}_{\mathbf{h}})_\tau,$$

where $(\mathcal{O}_{\mathbf{h}})_\tau$ is the block corresponding to $(\mathcal{C}_{\mathbf{q}})_\tau$ via KZ.

Theorem 5.3.1. *The data of*

(i) *the adjoint pair* (E, F)*,*

(ii) *the endomorphisms* $X \in \mathrm{End}(E)$*,* $T \in \mathrm{End}(E^2)$*,*

(iii) *the decomposition* $\mathcal{O}_{\mathbf{h}} = \bigoplus_{\tau \in P} (\mathcal{O}_{\mathbf{h}})_\tau$

is an $\widetilde{\mathfrak{sl}}_e$*-categorification on* $\mathcal{O}_{\mathbf{h}}$*.*

Proof. First, let us show that for $i \in \mathbb{Z}/e\mathbb{Z}$ the q^i-generalized eigenspaces of X in E, F are respectively the functors E_i, F_i defined in (4.7.2). Recall from Proposition 4.5.2(d) and Proposition 4.7.3(d) that we have

$$E = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i \quad \text{and} \quad E^{\mathscr{H}} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i^{\mathscr{H}}.$$

By the proof of Lemma 4.7.2 we see that any isomorphism

$$\mathrm{KZ} \circ E \cong E^{\mathscr{H}} \circ \mathrm{KZ}$$

restricts to an isomorphism $\mathrm{KZ} \circ E_i \cong E_i^{\mathscr{H}} \circ \mathrm{KZ}$ for all $i \in \mathbb{Z}/e\mathbb{Z}$. So the isomorphism σ_E in (5.3.1) maps $\mathrm{Hom}(E_i, E_j)$ to $\mathrm{Hom}(E_i^{\mathscr{H}} \circ \mathrm{KZ}, E_j^{\mathscr{H}} \circ \mathrm{KZ})$. Write

$$X = \sum_{i,j \in \mathbb{Z}/e\mathbb{Z}} X_{ij}, \quad X^{\mathscr{H}} 1_{\mathrm{KZ}} = \sum_{i,j \in \mathbb{Z}/e\mathbb{Z}} (X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij}$$

with $X_{ij} \in \mathrm{Hom}(E_i, E_j)$ and $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij} \in \mathrm{Hom}(E_i^{\mathscr{H}} \circ \mathrm{KZ}, E_j^{\mathscr{H}} \circ \mathrm{KZ})$. We have

$$\sigma_E(X_{ij}) = (X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij}.$$

Since $E_i^{\mathscr{H}}$ is the q^i-eigenspace of $X^{\mathscr{H}}$ in $E^{\mathscr{H}}$, we have $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ij} = 0$ for $i \neq j$ and $(X^{\mathscr{H}} 1_{\mathrm{KZ}})_{ii} - q^i$ is nilpotent for $i \in \mathbb{Z}/e\mathbb{Z}$. Since σ_E is an isomorphism of rings, this implies that $X_{ij} = 0$ and $X_{ii} - q^i$ is nilpotent in $\mathrm{End}(E)$. So E_i is the q^i-eigenspace of X in E. The fact that F_i is the q^i-eigenspace of X in F follows from adjunction.

Now, let us check the conditions (a)–(e):

(a) It is given by Proposition 4.7.3(d).

(b) Since $X^{\mathscr{H}}$ and $T^{\mathscr{H}}$ satisfy relations in (5.1.1), the endomorphisms X and T also satisfy them. Because these relations are preserved by ring homomorphisms.

(c) It follows from Proposition 4.7.4.

(d) By the definition of $(\mathcal{O}_{\mathbf{h}})_\tau$ and Lemma 4.6.1, the standard modules in $(\mathcal{O}_{\mathbf{h}})_\tau$ are all the $\Delta(\lambda)$ such that $\mathrm{wt}(\lambda) = \tau$. By (4.2.2) if μ is an l-partition such that $\mathrm{res}(\lambda/\mu) = i$ then $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + \alpha_i$. Now, the result follows from (4.7.3).

(e) This is Proposition 3.8.2. $\square$

5.4 Crystals of Fock spaces.

Let $B_{\mathcal{F}_{\mathbf{s}}}$ be the set of l-partitions. In [JMMO91] this set is given a crystal structure. We will call it *the crystal of the Fock space* $\mathcal{F}_{\mathbf{s}}$.

Theorem 5.4.1. [3] *(a) The set*

$$B_{\mathcal{O}_{\mathbf{h}}} = \{[L(\lambda)] \in [\mathcal{O}_{\mathbf{h}}] : \lambda \in \mathcal{P}_{n,l}, n \in \mathbb{N}\}$$

and the maps

$$\begin{aligned} \tilde{e}_i: \quad & B_{\mathcal{O}_{\mathbf{h}}} \to B_{\mathcal{O}_{\mathbf{h}}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{head}(E_i L)], \\ \tilde{f}_i: \quad & B_{\mathcal{O}_{\mathbf{h}}} \to B_{\mathcal{O}_{\mathbf{h}}} \sqcup \{0\}, \qquad [L] \mapsto [\mathrm{soc}(F_i L)]. \end{aligned}$$

define a crystal structure on $B_{\mathcal{O}_{\mathbf{h}}}$.

(b) The crystal $B_{\mathcal{O}_{\mathbf{h}}}$ *given by (a) is isomorphic to the crystal* $B_{\mathcal{F}_{\mathbf{s}}}$.

Proof. The Fock space $\mathcal{F}_{\mathbf{s}}$ is a $\mathfrak{b}$-locally finite $\widetilde{\mathfrak{sl}}_e$-module. So applying Proposition 5.2.4 to the $\widetilde{\mathfrak{sl}}_e$-categorification in Theorem 5.3.1 yields that $(B_{\mathcal{O}_{\mathbf{h}}}, \tilde{e}_i, \tilde{f}_i)$ is a perfect basis. Therefore it defines a crystal structure on $B_{\mathcal{O}_{\mathbf{h}}}$ by (5.2.2). This proves part (a). Now, let us concentrate on part (b). It is known that $B_{\mathcal{F}_{\mathbf{s}}}$ is a perfect basis of $\mathcal{F}_{\mathbf{s}}$. We identify the $\widetilde{\mathfrak{sl}}_e$-modules $\mathcal{F}_{\mathbf{s}}$ and $[\mathcal{O}_{\mathbf{h}}]$ via θ. By Lemma 5.2.3 the set $B^+_{\mathcal{F}_{\mathbf{s}}}$ and $B^+_{\mathcal{O}_{\mathbf{h}}}$ are two weight bases of $\mathcal{F}^+_{\mathbf{s}}$. So there is a bijection $\psi : B^+_{\mathcal{F}_{\mathbf{s}}} \to B^+_{\mathcal{O}_{\mathbf{h}}}$ such that $\mathrm{wt}(b) = \mathrm{wt}(\psi(b))$. Since $\mathcal{F}_{\mathbf{s}}$ is a direct sum of highest weight simple $\widetilde{\mathfrak{sl}}_e$-modules, this bijection extends to an automorphism ψ of the $\widetilde{\mathfrak{sl}}_e$-module $\mathcal{F}_{\mathbf{s}}$. By [BK07, Main Theorem 5.37] any automorphism of $\mathcal{F}_{\mathbf{s}}$ which maps $B^+_{\mathcal{F}_{\mathbf{s}}}$ to $B^+_{\mathcal{O}_{\mathbf{h}}}$ induces an isomorphism of crystals $B_{\mathcal{F}_{\mathbf{s}}} \cong B_{\mathcal{O}_{\mathbf{h}}}$. □

Remark 5.4.2. One can prove that if $n < e$ then a simple module $L \in \mathcal{O}_{\mathbf{h},n}$ is finite dimensional over $\mathbb{C}$ if and only if the class $[L]$ is a primitive element in $B_{\mathcal{O}_{\mathbf{h}}}$. In the case $n = 1$, we have $B_n(l) = \mu_l$, the cyclic group, and the primitive elements in the crystal $B_{\mathcal{F}_{\mathbf{s}}}$ have explicit combinatorial descriptions. This yields another proof of the classification of finite dimensional simple modules of $H_{\mathbf{h}}(\mu_l)$, which was first given by W. Crawley-Boevey and M. P. Holland. See type A case of [CBH98, Theorem 7.4].

3. This result has been independently obtained by Iain Gordon and Maurizio Martino.

Chapter II

Canonical bases and affine Hecke algebras of type D.

In this chapter, we prove a conjecture of Kashiwara and Miemietz on canonical bases and branching rules of affine Hecke algebras of type D.

This chapter is a joint work with Michela Varagnolo and Eric Vasserot. It has been prepublished in [SVV09].

Introduction

Let $\mathbf{f}$ be the negative part of the quantized enveloping algebra of type $\mathrm{A}^{(1)}$. Lusztig's description of the canonical basis of $\mathbf{f}$ implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [Gro94], and was used in [Ari96]. More precisely, there is a linear isomorphism between $\mathbf{f}$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [Ari96] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $\mathbf{f}$.

The branching rules for affine Hecke algebras of type B have been investigated quite recently, see [Eno09], [EK06, EK08a, EK08b], [Mie08] and [VV09a]. In particular, in [Eno09], [EK06, EK08a, EK08b] an analogue of Ariki's construction is conjectured and studied for affine Hecke algebras of type B. Here $\mathbf{f}$ is replaced by a module ${}^{\theta}\mathbf{V}(\lambda)$ over an algebra ${}^{\theta}\boldsymbol{\mathcal{B}}$. More precisely it is conjectured there that ${}^{\theta}\mathbf{V}(\lambda)$ admits a canonical basis which is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B should be given by the ${}^{\theta}\boldsymbol{\mathcal{B}}$-action on ${}^{\theta}\mathbf{V}(\lambda)$. This conjecture has been proved [VV09a]. It uses both the geometric picture introduced in [Eno09] (to prove part of the conjecture) and a new kind of graded algebras similar to the KLR algebras from [KL09], [Rou08a].

A similar description of the branching rules for affine Hecke algebras of type D has also been conjectured in [KM07]. In this case $\mathbf{f}$ is replaced by another module ${}^{\circ}\mathbf{V}$ over the algebra ${}^{\theta}\boldsymbol{\mathcal{B}}$ (the same algebra as in the type B case). The purpose of this chapter is to prove the type D case. The method of the proof is the same as in [VV09a]. First we introduce a family of graded algebras ${}^{\circ}\mathbf{R}_m$ for m a non negative integer. They can be viewed as the Ext-algebras of some complex of constructible sheaves naturally attached to the Lie algebra of the group $SO(2m)$. This complex enters in the Kazhdan-Lusztig classification of the simple modules of the affine Hecke algebra of the group $Spin(2m)$. Then we identify $^\circ\mathbf{V}$ with the sum of the Grothendieck groups of the graded algebras $^\circ\mathbf{R}_m$.

The plan of this chapter is the following. In Section 1 we introduce a graded algebra $^\circ\mathbf{R}(\Gamma)_\nu$. It is associated with a quiver Γ with an involution θ and with a dimension vector ν. In Section 2 we consider a particular choice of pair (Γ, θ). The graded algebras $^\circ\mathbf{R}(\Gamma)_\nu$ associated with this pair (Γ, θ) are denoted by the symbol $^\circ\mathbf{R}_m$. Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group $SO(2m)$, and we prove that it is Morita equivalent to $^\circ\mathbf{R}_m$. In Section 3 we categorify the module $^\circ\mathbf{V}$ from [KM07] using the graded algebras $^\circ\mathbf{R}_m$, see Theorem 3.8.2. The main result of the chapter is Theorem 3.9.1.

Notation

0.1 Graded modules over graded algebras

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. By a graded $\mathbf{k}$-algebra $\mathbf{R} = \bigoplus_d \mathbf{R}_d$ we will always mean a $\mathbb{Z}$-graded associative $\mathbf{k}$-algebra. Let $\mathbf{R}$**-mod** be the category of finitely generated graded $\mathbf{R}$-modules, $\mathbf{R}$**-fmod** be the full subcategory of finite-dimensional graded modules and $\mathbf{R}$**-proj** be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We will abbreviate

$$K(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{proj}], \quad G(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{fmod}].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the $\mathbf{k}$-vector spaces $\mathbf{R}_d$ are finite dimensional for each d. Then $K(\mathbf{R})$ is a free abelian group with a basis formed by the isomorphism classes of the indecomposable objects in $\mathbf{R}$**-proj**, and $G(\mathbf{R})$ is a free abelian group with a basis formed by the isomorphism classes of the simple objects in $\mathbf{R}$**-fmod**. Given an object M of $\mathbf{R}$**-proj** or $\mathbf{R}$**-fmod** let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M = [M]$. We will write $[M : N]$ for the composition multiplicity of the $\mathbf{R}$-module N in the $\mathbf{R}$-module M. Consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. If the grading of $\mathbf{R}$ is bounded below then the $\mathcal{A}$-modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here $\mathcal{A}$ acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM = M[1], \quad v^{-1}M = M[-1].$$

For any M, N in $\mathbf{R}$**-mod** let

$$\mathrm{hom}_{\mathbf{R}}(M, N) = \bigoplus_d \mathrm{Hom}_{\mathbf{R}}(M, N[d])$$

be the $\mathbb{Z}$-graded $\mathbf{k}$-vector space of all $\mathbf{R}$-module homomorphisms. If $\mathbf{R} = \mathbf{k}$ we will omit the subscript $\mathbf{R}$ in hom's and in tensor products. For any graded $\mathbf{k}$-vector space $M = \bigoplus_d M_d$ we will write

$$\mathrm{gdim}(M) = \sum_d v^d \mathrm{dim}(M_d),$$

where dim is the dimension over $\mathbf{k}$.

0.2 Quivers with involutions

Recall that a *quiver* Γ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where I is the set of vertices, H is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of h respectively. Note that the set I may be infinite. We will assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write

$$H_{i,j} = \{h \in H \mid h' = i, h'' = j\}.$$

We will abbreviate $i \to j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

An *involution* θ on Γ is a pair of involutions on I and H, both denoted by θ, such that the following properties hold for each h in H

- $\theta(h)' = \theta(h'')$ and $\theta(h)'' = \theta(h')$,
- $\theta(h') = h''$ if and only if $\theta(h) = h$.

We will always assume that θ has no fixed points in I, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we will say that θ has no fixed point. Let

$${}^\theta\mathbb{N}I = \{\nu = \sum_i \nu_i i \in \mathbb{N}I \mid \nu_{\theta(i)} = \nu_i, \ \forall i\}.$$

For any $\nu \in {}^\theta\mathbb{N}I$ set $|\nu| = \sum_i \nu_i$. It is an even integer. Write $|\nu| = 2m$ with $m \in \mathbb{N}$. We will denote by ${}^\theta I^\nu$ the set of sequences

$$\mathbf{i} = (i_{1-m}, \ldots, i_{m-1}, i_m)$$

of elements in I such that $\theta(i_l) = i_{1-l}$ and $\sum_k i_k = \nu$. For any such sequence $\mathbf{i}$ we will abbreviate $\theta(\mathbf{i}) = (\theta(i_{1-m}), \ldots, \theta(i_{m-1}), \theta(i_m))$. Finally, we set

$${}^\theta I^m = \bigcup_\nu {}^\theta I^\nu, \quad \nu \in {}^\theta\mathbb{N}I, \quad |\nu| = 2m.$$

0.3 The wreath product

Given a positive integer m, let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. Write $s_1, \ldots, s_{m-1}$ for the simple reflections in $\mathfrak{S}_m$. For each $l = 1, 2, \ldots m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l-th position. There is a unique action of W_m on the set $\{1-m, \ldots, m-1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \ldots m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^\theta I^\nu$. Indeed, view a sequence $\mathbf{i}$ as the map

$$\{1-m, \ldots, m-1, m\} \to I, \quad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$. For each ν we fix once for all a sequence

$$\mathbf{i}_e = (i_{1-m}, \ldots, i_m) \in {}^\theta I^\nu.$$

Let W_e be the centralizer of $\mathbf{i}_e$ in W_m. Then there is a bijection

$$W_e \backslash W_m \to {}^\theta I^\nu, \quad W_e w \mapsto w^{-1}(\mathbf{i}_e).$$

Now, assume that $m > 1$. We set $s_0 = \varepsilon_1 s_1 \varepsilon_1$. Let ${}^\circ W_m$ be the subgroup of W_m generated by $s_0, \ldots, s_{m-1}$. We will regard it as a Weyl group of type D_m such that $s_0, \ldots, s_{m-1}$ are the simple reflections. Note that W_e is a subgroup of ${}^\circ W_m$. Indeed, if $W_e \not\subset {}^\circ W_m$ there should exist l such that ε_l belongs to W_e. This would imply that $i_l = \theta(i_l)$, contradicting the fact that θ has no fixed point. Therefore ${}^\theta I^\nu$ decomposes into two ${}^\circ W_m$-orbits. We will denote them by ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$. For $m = 1$ we set ${}^\circ W_1 = \{e\}$ and we choose again ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$ in a obvious way.

1 The graded k-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$. Set $|\nu| = 2m$.

1.1 Definition of the graded k-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$

Assume that $m > 1$. We define a graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 0, 1, \dots, m-1$ modulo the following defining relations

(a) $1_\mathbf{i}\, 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}, \quad \sigma_k 1_\mathbf{i} = 1_{s_k \mathbf{i}} \sigma_k, \quad \varkappa_l 1_\mathbf{i} = 1_\mathbf{i} \varkappa_l,$

(b) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l,$

(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) 1_\mathbf{i},$

(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $1 \leqslant k < k' - 1 < m - 1$ or $0 = k < k' \neq 2,$

(e) $(\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_\mathbf{i} =$

$$= \begin{cases} \frac{Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_k) - Q_{i_k, i_{s_k(k)}}(\varkappa_{s_k(k)}, \varkappa_{s_k(k)+1})}{\varkappa_k - \varkappa_{s_k(k)+1}} 1_\mathbf{i} & \text{if } i_k = i_{s_k(k)+1}, \\ 0 & \text{else.} \end{cases}$$

$$(f) \quad (\sigma_k \varkappa_l - \varkappa_{s_k(l)} \sigma_k) 1_\mathbf{i} = \begin{cases} -1_\mathbf{i} & \text{if } l = k,\ i_k = i_{s_k(k)}, \\ 1_\mathbf{i} & \text{if } l = s_k(k),\ i_k = i_{s_k(k)}, \\ 0 & \text{else.} \end{cases}$$

Here we have set $\varkappa_{1-l} = -\varkappa_l$ and

$$Q_{i,j}(u,v) = \begin{cases} (-1)^{h_{i,j}} (u-v)^{-i \cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases} \tag{1.1.1}$$

If $m = 0$ we set ${}^\circ\mathbf{R}(\Gamma)_0 = \mathbf{k} \oplus \mathbf{k}$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $\mathbf{i} = i\theta(i)$, and

$${}^\circ\mathbf{R}(\Gamma)_\nu = \mathbf{k}[\varkappa_1] 1_\mathbf{i} \oplus \mathbf{k}[\varkappa_1] 1_{\theta(\mathbf{i})}.$$

We will abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_\mathbf{i}$ and $\varkappa_{\mathbf{i},l} = \varkappa_l 1_\mathbf{i}$. The grading on ${}^\circ\mathbf{R}(\Gamma)_0$ is the trivial one. For $m \geqslant 1$ the grading on ${}^\circ\mathbf{R}(\Gamma)_\nu$ is given by the following rules :

$$\deg(1_\mathbf{i}) = 0,$$
$$\deg(\varkappa_{\mathbf{i},l}) = 2,$$
$$\deg(\sigma_{\mathbf{i},k}) = -i_k \cdot i_{s_k(k)}.$$

We define ω to be the unique anti-involution of the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}(\Gamma)_\nu$ which fixes $1_\mathbf{i}$, $\varkappa_l$, σ_k. We set ω to be identity on ${}^\circ\mathbf{R}(\Gamma)_0$.

1.2 Relation with the graded k-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$

A family of graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ has been introduced in [VV09a, Section 5.1], for λ an arbitrary dimension vector in $\mathbb{N}I$. Here we will only consider the special case $\lambda = 0$, and we abbreviate ${}^\theta\mathbf{R}(\Gamma)_\nu = {}^\theta\mathbf{R}(\Gamma)_{0,\nu}$. Recall that if $\nu \neq 0$ then ${}^\theta\mathbf{R}(\Gamma)_\nu$ is the graded $\mathbf{k}$-algebra with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, π_1, with $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, 2, \dots, m$, $k = 1, \dots, m-1$ such that $1_\mathbf{i}$, $\varkappa_l$ and σ_k satisfy the same relations as before and

$$\pi_1^2 = 1, \quad \pi_1 1_\mathbf{i} \pi_1 = 1_{\varepsilon_1 \mathbf{i}}, \quad \pi_1 \varkappa_l \pi_1 = \varkappa_{\varepsilon_1(l)},$$
$$(\pi_1 \sigma_1)^2 = (\sigma_1 \pi_1)^2, \quad \pi_1 \sigma_k \pi_1 = \sigma_k \text{ if } k \neq 1.$$

If $\nu = 0$ then ${}^\theta\mathbf{R}(\Gamma)_0 = \mathbf{k}$. The grading is given by setting $\deg(1_\mathbf{i})$, $\deg(\varkappa_{\mathbf{i},l})$, $\deg(\sigma_{\mathbf{i},k})$ to be as before and $\deg(\pi_1 1_\mathbf{i}) = 0$. *In the rest of Section 1 we will assume $m > 0$.* Then there is a canonical inclusion of graded $\mathbf{k}$-algebras

$$^\circ\mathbf{R}(\Gamma)_\nu \subset {}^\theta\mathbf{R}(\Gamma)_\nu \tag{1.2.1}$$

such that $1_\mathbf{i}, \varkappa_l, \sigma_k \mapsto 1_\mathbf{i}, \varkappa_l, \sigma_k$ for $\mathbf{i} \in {}^\theta I^\nu$, $l = 1, \dots, m$, $k = 1, \dots, m-1$ and such that $\sigma_0 \mapsto \pi_1\sigma_1\pi_1$. From now on we will write $\sigma_0 = \pi_1\sigma_1\pi_1$ whenever $m > 1$. The assignment $x \mapsto \pi_1 x \pi_1$ defines an involution of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_\nu$ which normalizes $^\circ\mathbf{R}(\Gamma)_\nu$. Thus it yields an involution

$$\gamma : {}^\circ\mathbf{R}(\Gamma)_\nu \to {}^\circ\mathbf{R}(\Gamma)_\nu. \tag{1.2.2}$$

Let $\langle\gamma\rangle$ be the group of two elements generated by γ. The smash product $^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle$ is a graded $\mathbf{k}$-algebra such that $\deg(\gamma) = 0$. There is an unique isomorphism of graded $\mathbf{k}$-algebras

$$^\circ\mathbf{R}(\Gamma)_\nu \rtimes \langle\gamma\rangle \to {}^\theta\mathbf{R}(\Gamma)_\nu \tag{1.2.3}$$

which is identity on $^\circ\mathbf{R}(\Gamma)_\nu$ and which takes γ to π_1.

1.3 The polynomial representation and the PBW theorem

For any $\mathbf{i}$ in ${}^\theta I^\nu$ let ${}^\theta\mathbf{F}_\mathbf{i}$ be the subalgebra of $^\circ\mathbf{R}(\Gamma)_\nu$ generated by $1_\mathbf{i}$ and $\varkappa_{\mathbf{i},l}$ with $l = 1, 2, \dots, m$. It is a polynomial algebra. Let

$${}^\theta\mathbf{F}_\nu = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathbf{F}_\mathbf{i}.$$

The group W_m acts on ${}^\theta\mathbf{F}_\nu$ via $w(\varkappa_{\mathbf{i},l}) = \varkappa_{w(\mathbf{i}),w(l)}$ for any $w \in W_m$. Consider the fixed points set

$$^\circ\mathbf{S}_\nu = ({}^\theta\mathbf{F}_\nu)^{^\circ W_m}.$$

Regard ${}^\theta\mathbf{R}(\Gamma)_\nu$ and $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ as ${}^\theta\mathbf{F}_\nu$-algebras via the left multiplication. In [VV09a, Proposition 5.4] is given an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism ${}^\theta\mathbf{R}(\Gamma)_\nu \to \mathrm{End}({}^\theta\mathbf{F}_\nu)$. It restricts via (1.2.1) to an injective graded ${}^\theta\mathbf{F}_\nu$-algebra morphism

$$^\circ\mathbf{R}(\Gamma)_\nu \to \mathrm{End}({}^\theta\mathbf{F}_\nu). \tag{1.3.1}$$

Next, recall that $^\circ W_m$ is the Weyl group of type D_m with simple reflections $s_0, \dots, s_{m-1}$. For each w in $^\circ W_m$ we choose a reduced decomposition $\dot{w}$ of w. It has the following form

$$w = s_{k_1}s_{k_2}\cdots s_{k_r}, \quad 0 \leqslant k_1, k_2, \dots, k_r \leqslant m-1.$$

We define an element $\sigma_{\dot{w}}$ in $^\circ\mathbf{R}(\Gamma)_\nu$ by

$$\sigma_{\dot{w}} = \sum_\mathbf{i} 1_\mathbf{i}\sigma_{\dot{w}}, \quad 1_\mathbf{i}\sigma_{\dot{w}} = \begin{cases} 1_\mathbf{i} & \text{if } r = 0 \\ 1_\mathbf{i}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else,}\end{cases} \tag{1.3.2}$$

Observe that the element $\sigma_{\dot{w}}$ may depend on the choice of the reduced decomposition $\dot{w}$. Let ${}^\theta\mathbf{F}'_\nu = \bigoplus_\mathbf{i} {}^\theta\mathbf{F}'_\mathbf{i}$, where ${}^\theta\mathbf{F}'_\mathbf{i}$ is the localization of the ring ${}^\theta\mathbf{F}_\mathbf{i}$ with respect to the multiplicative system generated by

$$\{\varkappa_{\mathbf{i},l} \pm \varkappa_{\mathbf{i},l'} \,|\, 1 \leqslant l \neq l' \leqslant m\} \cup \{\varkappa_{\mathbf{i},l} \,|\, l = 1, 2, \dots, m\}.$$

Proposition 1.3.1. *The* $\mathbf{k}$*-algebra* $^\circ\mathbf{R}(\Gamma)_\nu$ *is a free (left or right)* $^\theta\mathbf{F}_\nu$*-module with basis* $\{\sigma_{\dot w}\,|\,w\in {}^\circ W_m\}$*. Its rank is* $2^{m-1}m!$*. The operator* $1_\mathbf{i}\sigma_{\dot w}$ *is homogeneous and its degree is independent of the choice of the reduced decomposition* $\dot w$*. The inclusion* $^\circ\mathbf{R}(\Gamma)_\nu\subset \mathrm{End}(^\theta\mathbf{F}_\nu)$ *yields an isomorphism of* $^\theta\mathbf{F}'_\nu$*-algebras* $^\theta\mathbf{F}'_\nu\otimes_{^\theta\mathbf{F}_\nu}{}^\circ\mathbf{R}(\Gamma)_\nu\to {}^\theta\mathbf{F}'_\nu\rtimes{}^\circ W_m$*, such that for each* $\mathbf{i}$ *and each* $l=1,2,\dots,m$*,* $k=0,1,2,\dots,m-1$ *we have*

$$\begin{aligned}
&1_\mathbf{i}\mapsto 1_\mathbf{i},\\
&\varkappa_{\mathbf{i},l}\mapsto \varkappa_l 1_\mathbf{i},\\
&\sigma_{\mathbf{i},k}\mapsto\begin{cases}(\varkappa_k-\varkappa_{s_k(k)})^{-1}(s_k-1)1_\mathbf{i} & \text{if } i_k=i_{s_k(k)},\\ (\varkappa_k-\varkappa_{s_k(k)})^{h_{i_{s_k(k)},i_k}}s_k1_\mathbf{i} & \text{if } i_k\neq i_{s_k(k)}.\end{cases}
\end{aligned}\tag{1.3.3}$$

Proof. Following the proof of [VV09a, Proposition 5.5], we filter the algebra $^\circ\mathbf{R}(\Gamma)_\nu$ with $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$ in degree 1. The *Nil Hecke algebra* of type D_m is the $\mathbf{k}$-algebra $^\circ\mathbf{NH}_m$ generated by $\bar\sigma_0,\bar\sigma_1,\dots,\bar\sigma_{m-1}$ with relations

$$\begin{gathered}
\bar\sigma_k\bar\sigma_{k'}=\bar\sigma_{k'}\bar\sigma_k\quad\text{if } 1\leqslant k<k'-1<m-1 \text{ or } 0=k<k'\neq 2,\\
\bar\sigma_{s_k(k)}\bar\sigma_k\bar\sigma_{s_k(k)}=\bar\sigma_k\bar\sigma_{s_k(k)}\bar\sigma_k,\quad \bar\sigma_k^2=0.
\end{gathered}$$

We can form the semidirect product $^\theta\mathbf{F}_\nu\rtimes{}^\circ\mathbf{NH}_m$, which is generated by $1_\mathbf{i}$, $\bar\varkappa_l$, $\bar\sigma_k$ with the relations above and

$$\bar\sigma_k\bar\varkappa_l=\bar\varkappa_{s_k(l)}\bar\sigma_k,\quad \bar\varkappa_l\bar\varkappa_{l'}=\bar\varkappa_{l'}\bar\varkappa_{l'}.$$

We have a surjective $\mathbf{k}$-algebra morphism

$$^\theta\mathbf{F}_\nu\rtimes{}^\circ\mathbf{NH}_m\to \mathrm{gr}\,{}^\circ\mathbf{R}(\Gamma)_\nu,\quad 1_\mathbf{i}\mapsto 1_\mathbf{i},\ \bar\varkappa_l\mapsto\varkappa_l,\ \bar\sigma_k\mapsto\sigma_k.$$

Thus the elements $\sigma_{\dot w}$ with $w\in{}^\circ W_m$ generate $^\circ\mathbf{R}(\Gamma)_\nu$ as a $^\theta\mathbf{F}_\nu$-module. We claim that they form a $^\theta\mathbf{F}_\nu$-basis of $^\circ\mathbf{R}(\Gamma)_\nu$. Indeed, by [VV09a, Corollary 5.6], the inclusion (1.3.1) yields a surjective algebra homomorphism

$$\phi: {}^\theta\mathbf{F}'_\nu\otimes_{^\theta\mathbf{F}_\nu}{}^\circ\mathbf{R}(\Gamma)_\nu\to{}^\theta\mathbf{F}'_\nu\rtimes{}^\circ W_m,$$

such that for each $\mathbf{i}$ and each $l=1,2,\dots,m$, $k=0,1,2,\dots,m-1$ we have

$$\begin{aligned}
&\phi(1_\mathbf{i})=1_\mathbf{i},\\
&\phi(\varkappa_{\mathbf{i},l})=\varkappa_l1_\mathbf{i},\\
&\phi(\sigma_{\mathbf{i},k})=\begin{cases}(\varkappa_k-\varkappa_{s_k(k)})^{-1}(s_k-1)1_\mathbf{i} & \text{if } i_k=i_{s_k(k)},\\ (\varkappa_k-\varkappa_{s_k(k)})^{h_{i_{s_k(k)},i_k}}s_k1_\mathbf{i} & \text{if } i_k\neq i_{s_k(k)}.\end{cases}
\end{aligned}$$

Therefore we have

$$\phi(\sigma_{\dot w}1_\mathbf{i})=\sum_{v\in{}^\circ W_m,\ v\leqslant w} vp_{v,w}1_\mathbf{i},$$

for some element $p_{v,w}$ in the ring $^\theta\mathbf{F}'_\mathbf{i}$, with $p_{w,w}$ invertible. This proves the claim. The rest of the proposition follows. □

Restricting the $^\theta\mathbf{F}_\nu$-action on $^\circ\mathbf{R}(\Gamma)_\nu$ to the $\mathbf{k}$-subalgebra $^\circ\mathbf{S}_\nu$ we get a structure of graded $^\circ\mathbf{S}_\nu$-algebra on $^\circ\mathbf{R}(\Gamma)_\nu$.

Proposition 1.3.2. *(a) The algebra* $^\circ\mathbf{S}_\nu$ *is isomorphic to the center of* $^\circ\mathbf{R}(\Gamma)_\nu$*.*

(b) The algebra $^\circ\mathbf{R}(\Gamma)_\nu$ *is a free graded module over* $^\circ\mathbf{S}_\nu$ *of rank* $(2^{m-1}m!)^2$*.*

Proof. Part (a) is clear by the $^\theta\mathbf{F}'_\nu$-algebra isomorphism $^\theta\mathbf{F}'_\nu\otimes_{^\theta\mathbf{F}_\nu}{}^\circ\mathbf{R}(\Gamma)_\nu\xrightarrow{\sim}{}^\theta\mathbf{F}'_\nu\rtimes{}^\circ W_m$ in Proposition 1.3.1. Part (b) follows from (a) and Proposition 1.3.1. □

2 Affine Hecke algebras of type D

2.1 Affine Hecke algebras of type D

Fix p in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the extended affine Hecke algebra $\mathbf{H}_m$ of type D_m as follows. If $m > 1$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra with 1 generated by

$$T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \ldots, m-1, \quad l = 1, 2, \ldots, m$$

satisfying the following defining relations :

(a) $X_l X_{l'} = X_{l'} X_l$,

(b) $T_k T_{s_k(k)} T_k = T_{s_k(k)} T_k T_{s_k(k)}$, $T_k T_{k'} = T_{k'} T_k$ if $1 \leqslant k < k'-1$ or $k = 0$, $k' \neq 2$,

(c) $(T_k - p)(T_k + p^{-1}) = 0$,

(d) $T_0 X_1^{-1} T_0 = X_2$, $T_k X_k T_k = X_{s_k(k)}$ if $k \neq 0$, $T_k X_l = X_l T_k$ if $k \neq 0, l, l-1$ or $k = 0$, $l \neq 1, 2$.

Finally, we set $\mathbf{H}_0 = \mathbf{k} \oplus \mathbf{k}$ and $\mathbf{H}_1 = \mathbf{k}[X_1^{\pm 1}]$.

Remark 2.1.1. (a) The extended affine Hecke algebra $\mathbf{H}_m^{\mathrm{B}}$ of type B_m with parameters $p, q \in \mathbf{k}^\times$ such that $q = 1$ is generated by P, T_k, $X_l^{\pm 1}$, $k = 1, \ldots, m-1$, $l = 1, \ldots, m$ such that T_k, $X_l^{\pm 1}$ satisfy the relations as above and

$$P^2 = 1, \quad (PT_1)^2 = (T_1 P)^2, \quad PT_k = T_k P \text{ if } k \neq 1,$$
$$PX_1^{-1} P = X_1, \quad PX_l = X_l P \text{ if } l \neq 1.$$

See e.g., [VV09a, Section 6.1]. There is an obvious $\mathbf{k}$-algebra embedding $\mathbf{H}_m \subset \mathbf{H}_m^{\mathrm{B}}$. Let γ denote also the involution $\mathbf{H}_m \to \mathbf{H}_m$, $a \mapsto PaP$. We have a canonical isomorphism of $\mathbf{k}$-algebras

$$\mathbf{H}_m \rtimes \langle \gamma \rangle \simeq \mathbf{H}_m^{\mathrm{B}}.$$

Compare Section 1.2.

(b) Given a connected reductive group G we call *affine Hecke algebra of G* the Hecke algebra of the extended affine Weyl group $W \ltimes P$, where W is the Weyl group of (G, T), P is the group of characters of T, and T is a maximal torus of G. Then $\mathbf{H}_m$ is the affine Hecke algebra of the group $SO(2m)$. Let $\mathbf{H}_m^e$ be the affine Hecke algebra of the group $Spin(2m)$. It is generated by $\mathbf{H}_m$ and an element X_0 such that

$$X_0^2 = X_1 X_2 \ldots X_m, \quad T_k X_0 = X_0 T_k \text{ if } k \neq 0, \quad T_0 X_0 X_1^{-1} X_2^{-1} T_0 = X_0.$$

Thus $\mathbf{H}_m$ is the fixed point subset of the $\mathbf{k}$-algebra automorphism of $\mathbf{H}_m^e$ taking T_k, X_l to $T_k, (-1)^{\delta_{l,0}} X_l$ for all k, l. Therefore, if p is not a root of 1 the simple $\mathbf{H}_m$-modules can be recovered from the Kazhdan-Lusztig classification of the simple $\mathbf{H}_m^e$-modules via Clifford theory, see e.g., [Ree02].

2.2 Intertwiners and blocks of $\mathbf{H}_m$

We define

$$\mathbf{A} = \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \quad \mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_l X_{l'}^{\pm 1}, \quad 1 - p^2 X_l^{\pm 1} X_{l'}^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k is the element of $\mathbf{H}'_m$ given by the following formulas

$$\varphi_k - 1 = \frac{X_k - X_{s_k(k)}}{pX_k - p^{-1}X_{s_k(k)}}\,(T_k - p). \tag{2.2.1}$$

The group ${}^\circ W_m$ acts on $\mathbf{A}'$ as follows

$$(s_k a)(X_1, \dots, X_m) = a(X_1, \dots, X_{k+1}, X_k, \dots, X_m) \text{ if } k \neq 0,$$
$$(s_0 a)(X_1, \dots, X_m) = a(X_2^{-1}, X_1^{-1}, \dots, X_m).$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes {}^\circ W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z} \rtimes \{-1, 1\}$ acts on $\mathbf{k}^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \dots, X_m$ is locally finite with eigenvalues in I. We associate to the set I and to the element $p \in \mathbf{k}^\times$ a quiver Γ as follows. The set of vertices is I, and there is one arrow $p^2 i \to i$ whenever i lies in I. We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. *We will assume that the set I does not contain* 1 *nor* -1 *and that* $p \neq 1, -1$. Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

Remark 2.2.1. We may assume that $I = \pm\{p^n;\, n \in \mathbb{Z}_{\text{odd}}\}$. See the discussion in [KM07]. Then Γ is of type A_∞ if p has infinite order and Γ is of type $\mathrm{A}_r^{(1)}$ if p^2 is a primitive r-th root of unity.

2.3 The $\mathbf{H}_m$-modules versus the ${}^\circ\mathbf{R}_m$-modules

Assume that $m \geqslant 1$. We define the graded $\mathbf{k}$-algebra

$${}^\theta\mathbf{R}_{I,m} = \bigoplus_\nu {}^\theta\mathbf{R}_{I,\nu}, \quad {}^\theta\mathbf{R}_{I,\nu} = {}^\theta\mathbf{R}(\Gamma)_\nu, \quad {}^\circ\mathbf{R}_{I,m} = \bigoplus_\nu {}^\circ\mathbf{R}_{I,\nu}, \quad {}^\circ\mathbf{R}_{I,\nu} = {}^\circ\mathbf{R}(\Gamma)_\nu, \quad {}^\theta I^m = \bigsqcup_\nu {}^\theta I^\nu,$$

where ν runs over the set of all dimension vectors in ${}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate

$${}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}_{I,\nu}, \quad {}^\theta\mathbf{R}_m = {}^\theta\mathbf{R}_{I,m}, \quad {}^\circ\mathbf{R}_\nu = {}^\circ\mathbf{R}_{I,\nu}, \quad {}^\circ\mathbf{R}_m = {}^\circ\mathbf{R}_{I,m}.$$

Note that ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_m$ are the same as in [VV09a, Section 6.4], with $\lambda = 0$. Note also that the $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_m$ may not have 1, because the set I may be infinite. We define ${}^\circ\mathbf{R}_m$-$\mathbf{Mod}_0$ as the category of all (non-graded) ${}^\circ\mathbf{R}_m$-modules such that the elements $\varkappa_1, \varkappa_2, \dots, \varkappa_m$ act locally nilpotently. Let ${}^\circ\mathbf{R}_m$-$\mathbf{fMod}_0$ and $\mathbf{H}_m$-$\mathbf{fMod}_I$ be the full subcategories of finite dimensional modules in ${}^\circ\mathbf{R}_m$-$\mathbf{Mod}_0$ and $\mathbf{H}_m$-$\mathbf{Mod}_I$ respectively. Fix a formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo $(\varkappa^2)$.

Theorem 2.3.1. *We have an equivalence of categories*

$${}^\circ\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

(a) *X_l acts on $1_\mathbf{i}M$ by $i_l^{-1} f(\varkappa_l)$ for each $l = 1, 2, \dots, m$,* (*b*) *if* $m > 1$ *then* T_k *acts on* $1_{\mathbf{i}}M$ *as follows for each* $k = 0, 1, \dots, m-1$,

$$\begin{cases} \frac{(pf(\varkappa_k)-p^{-1}f(\varkappa_{s_k(k)}))(\varkappa_k-\varkappa_{s_k(k)})}{f(\varkappa_k)-f(\varkappa_{s_k(k)})}\sigma_k + p, & \text{if } i_{s_k(k)} = i_k, \\ \frac{f(\varkappa_k)-f(\varkappa_{s_k(k)})}{(p^{-1}f(\varkappa_k)-pf(\varkappa_{s_k(k)}))(\varkappa_k-\varkappa_{s_k(k)})}\sigma_k + \frac{(p^{-2}-1)f(\varkappa_{s_k(k)})}{pf(\varkappa_k)-p^{-1}f(\varkappa_{s_k(k)})}, & \text{if } i_{s_k(k)} = p^2 i_k, \\ \frac{pi_kf(\varkappa_k)-p^{-1}i_{s_k(k)}f(\varkappa_{s_k(k)})}{i_kf(\varkappa_k)-i_{s_k(k)}f(\varkappa_{s_k(k)})}\sigma_k + \frac{(p^{-1}-p)i_kf(\varkappa_{s_k(k)})}{i_{s_k(k)}f(\varkappa_k)-i_kf(\varkappa_{s_k(k)})}, & \text{if } i_{s_k(k)} \neq i_k, p^2 i_k. \end{cases}$$

Proof. This follows from [VV09a, Theorem 6.5] by Section 1.2 and Remark 2.1.1(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.3.3) and (2.2.1). □

Corollary 2.3.2. *There is an equivalence of categories*

$$\Psi : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \quad M \mapsto M.$$

Remark 2.3.3. The results in Section 2.3 are still true if $\mathbf{k}$ is any field. Set $f(\varkappa) = 1 + \varkappa$ for instance.

2.4 Induction and restriction of $\mathbf{H}_m$-modules

For $i \in I$ we define functors

$$\begin{aligned} E_i &: \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \\ F_i &: \mathbf{H}_m\text{-}\mathbf{fMod}_I \to \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I, \end{aligned} \tag{2.4.1}$$

where $E_iM \subset M$ is the generalized i^{-1}-eigenspace of the X_{m+1}-action, and where

$$F_iM = \mathrm{Ind}_{\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]}^{\mathbf{H}_{m+1}}(M \otimes \mathbf{k}_i).$$

Here $\mathbf{k}_i$ is the 1-dimensional representation of $\mathbf{k}[X_{m+1}^{\pm 1}]$ defined by $X_{m+1} \mapsto i^{-1}$.

3 Global bases of °V and projective graded °R-modules

3.1 The Grothendieck groups of ${}^\circ\mathbf{R}_m$

The graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_m$ is free of finite rank over its center, a commutative graded $\mathbf{k}$-subalgebra. See Proposition 1.3.2. Therefore any simple object of ${}^\circ\mathbf{R}_m$-**mod** is finite-dimensional and there is a finite number of isomorphism classes of simple modules in ${}^\circ\mathbf{R}_m$-**mod**. The abelian group $G({}^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of ${}^\circ\mathbf{R}_m$-**mod**. The abelian group $K({}^\circ\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G({}^\circ\mathbf{R}_m)$ and $K({}^\circ\mathbf{R}_m)$ are free $\mathcal{A}$-modules, where v shifts the grading by 1. We consider the following $\mathcal{A}$-modules

$$\begin{aligned} {}^\circ\mathbf{K}_I &= \bigoplus_{m \geqslant 0} {}^\circ\mathbf{K}_{I,m}, \quad {}^\circ\mathbf{K}_{I,m} = K({}^\circ\mathbf{R}_m), \\ {}^\circ\mathbf{G}_I &= \bigoplus_{m \geqslant 0} {}^\circ\mathbf{G}_{I,m}, \quad {}^\circ\mathbf{G}_{I,m} = G({}^\circ\mathbf{R}_m). \end{aligned}$$

We will also abbreviate

$${}^\circ\mathbf{K}_{I,*} = \bigoplus_{m > 0} {}^\circ\mathbf{K}_{I,m}, \quad {}^\circ\mathbf{G}_{I,*} = \bigoplus_{m > 0} {}^\circ\mathbf{G}_{I,m}.$$

From now on, to unburden the notation we may abbreviate ${}^\circ\mathbf{R} = {}^\circ\mathbf{R}_m$, hoping it will not create any confusion. For any M, N in ${}^\circ\mathbf{R}$**-mod** we set

$$(M : N) = \mathrm{gdim}(M^\omega \otimes_{{}^\circ\mathbf{R}} N), \quad \langle M : N\rangle = \mathrm{gdim}\,\mathrm{hom}_{{}^\circ\mathbf{R}}(M, N),$$

where ω is the anti-involution defined in Section 1.1. The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$${}^\circ\mathbf{K}_I \times {}^\circ\mathbf{G}_I \to \mathcal{A}, \quad (P, M) \mapsto \langle P : M\rangle.$$

First, we concentrate on the $\mathcal{A}$-module ${}^\circ\mathbf{G}_I$. Consider the duality

$${}^\circ\mathbf{R}\text{-}\mathbf{fmod} \to {}^\circ\mathbf{R}\text{-}\mathbf{fmod}, \quad M \mapsto M^\flat = \mathrm{hom}(M, \mathbf{k}),$$

with the action and the grading given by

$$(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \mathrm{Hom}(M_{-d}, \mathbf{k}).$$

This duality functor yields an $\mathcal{A}$-antilinear map

$${}^\circ\mathbf{G}_I \to {}^\circ\mathbf{G}_I, \quad M \mapsto M^\flat.$$

Let ${}^\circ B$ denote the set of isomorphism classes of simple objects of ${}^\circ\mathbf{R}\text{-}\mathbf{fMod}_0$. We can now define the upper global basis of ${}^\circ\mathbf{G}_I$ as follows. The proof is given in Section 3.6.

Proposition/Definition 3.1.1. *For each b in ${}^\circ B$ there is a unique selfdual irreducible graded ${}^\circ\mathbf{R}$-module ${}^\circ G^{up}(b)$ which is isomorphic to b as a (non graded) ${}^\circ\mathbf{R}$-module. We set ${}^\circ G^{up}(0) = 0$ and ${}^\circ\mathbf{G}^{up} = \{{}^\circ G^{up}(b) \,|\, b \in {}^\circ B\}$. Hence ${}^\circ\mathbf{G}^{up}$ is an $\mathcal{A}$-basis of ${}^\circ\mathbf{G}_I$.*

Now, we concentrate on the $\mathcal{A}$-module ${}^\circ\mathbf{K}_I$. We equip ${}^\circ\mathbf{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$${}^\circ\mathbf{K}_I \times {}^\circ\mathbf{K}_I \to \mathcal{A}, \quad (M, N) \mapsto (M : N). \tag{3.1.1}$$

Consider the duality

$${}^\circ\mathbf{R}\text{-}\mathbf{proj} \to {}^\circ\mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \mathrm{hom}_{{}^\circ\mathbf{R}}(P, {}^\circ\mathbf{R}),$$

with the action and the grading given by

$$(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \mathrm{Hom}_{{}^\circ\mathbf{R}}(P[-d], {}^\circ\mathbf{R}).$$

This duality functor yields an $\mathcal{A}$-antilinear map

$${}^\circ\mathbf{K}_I \to {}^\circ\mathbf{K}_I, \quad P \mapsto P^\sharp.$$

Set $\mathcal{K} = \mathbb{Q}(v)$. Let $\mathcal{K} \to \mathcal{K}$, $f \mapsto \bar{f}$ be the unique involution such that $\bar{v} = v^{-1}$.

Definition 3.1.2. For each b in ${}^\circ B$ let ${}^\circ G^{\mathrm{low}}(b)$ be the unique indecomposable graded module in ${}^\circ\mathbf{R}$**-proj** whose top is isomorphic to ${}^\circ G^{\mathrm{up}}(b)$. We set ${}^\circ G^{\mathrm{low}}(0) = 0$ and ${}^\circ\mathbf{G}^{\mathrm{low}} = \{{}^\circ G^{\mathrm{low}}(b \,|\, b \in {}^\circ B\}$. The latter is an $\mathcal{A}$-basis of ${}^\circ\mathbf{K}_I$.

Proposition 3.1.3. *(a) We have $\langle {}^\circ G^{low}(b) : {}^\circ G^{up}(b')\rangle = \delta_{b,b'}$ for each b, b' in ${}^\circ B$.*

(b) We have $\langle P^\sharp : M\rangle = \overline{\langle P : M^\flat\rangle}$ for each P, M.

(c) We have ${}^\circ G^{low}(b)^\sharp = {}^\circ G^{low}(b)$ for each b in ${}^\circ B$.

The proof is the same as in [VV09a, Proposiiton 8.4].

Example 3.1.4. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Consider the graded $^\circ\mathbf{R}_\nu$-modules

$$^\circ\mathbf{R}_{\mathbf{i}} = {}^\circ\mathbf{R}1_{\mathbf{i}} = 1_{\mathbf{i}}{}^\circ\mathbf{R}, \quad {}^\circ\mathbf{L}_{\mathbf{i}} = \mathrm{top}({}^\circ\mathbf{R}_{\mathbf{i}}).$$

The global bases are given by

$$^\circ\mathbf{G}^{\mathrm{low}}_\nu = \{{}^\circ\mathbf{R}_{\mathbf{i}},\, {}^\circ\mathbf{R}_{\theta(\mathbf{i})}\}, \quad {}^\circ\mathbf{G}^{\mathrm{up}}_\nu = \{{}^\circ\mathbf{L}_{\mathbf{i}},\, {}^\circ\mathbf{L}_{\theta(\mathbf{i})}\}.$$

For $m = 0$ we have $^\circ\mathbf{R}_0 = \mathbf{k} \oplus \mathbf{k}$. Set $\phi_+ = \mathbf{k} \oplus 0$ and $\phi_- = 0 \oplus \mathbf{k}$. We have

$$^\circ\mathbf{G}^{\mathrm{low}}_0 = {}^\circ\mathbf{G}^{\mathrm{up}}_0 = \{\phi_+,\, \phi_-\}.$$

3.2 Definition of the operators e_i, f_i, e'_i, f'_i

In this section we will always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in $^\circ W_{m+1}$ of the cosets in $^\circ W_m \backslash {}^\circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$

For each element w of $D_{m,1;m,1}$ we set

$$W(w) = {}^\circ W_m \cap w({}^\circ W_m)w^{-1}.$$

Let $\mathbf{R}_1$ be the $\mathbf{k}$-algebra generated by elements 1_i, $\varkappa_i$, $i \in I$, satisfying the defining relations $1_i\, 1_{i'} = \delta_{i,i'} 1_i$ and $\varkappa_i = 1_i \varkappa_i 1_i$. We equip $\mathbf{R}_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\varkappa_i) = 2$. Let

$$\mathbf{R}_i = 1_i\mathbf{R}_1 = \mathbf{R}_1 1_i, \quad \mathbf{L}_i = \mathrm{top}(\mathbf{R}_i) = \mathbf{R}_i/(\varkappa_i).$$

Then $\mathbf{R}_i$ is a graded projective $\mathbf{R}_1$-module and $\mathbf{L}_i$ is simple. We abbreviate

$$^\theta\mathbf{R}_{m,1} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_1, \quad {}^\circ\mathbf{R}_{m,1} = {}^\circ\mathbf{R}_m \otimes \mathbf{R}_1.$$

There is an unique inclusion of graded $\mathbf{k}$-algebras

$$\begin{aligned} {}^\theta\mathbf{R}_{m,1} &\to {}^\theta\mathbf{R}_{m+1}, \\ 1_{\mathbf{i}} \otimes 1_i &\mapsto 1_{\mathbf{i}'}, \\ 1_{\mathbf{i}} \otimes \varkappa_{i,l} &\mapsto \varkappa_{\mathbf{i}',m+l}, \\ \varkappa_{\mathbf{i},l} \otimes 1_i &\mapsto \varkappa_{\mathbf{i}',l}, \\ \pi_{\mathbf{i},1} \otimes 1_i &\mapsto \pi_{\mathbf{i}',1}, \\ \sigma_{\mathbf{i},k} \otimes 1_i &\mapsto \sigma_{\mathbf{i}',k}, \end{aligned}$$

where, given $\mathbf{i} \in {}^\theta I^m$ and $i \in I$, we have set $\mathbf{i}' = \theta(i)\mathbf{i}i$, a sequence in $^\theta I^{m+1}$. This inclusion restricts to an inclusion $^\circ\mathbf{R}_{m,1} \subset {}^\circ\mathbf{R}_{m+1}$.

Lemma 3.2.1. *The graded $^\circ\mathbf{R}_{m,1}$-module $^\circ\mathbf{R}_{m+1}$ is free of rank $2(m+1)$.*

Proof. For each w in $D_{m,1}$ we have the element $\sigma_{\dot w}$ in $^\circ\mathbf{R}_{m+1}$ defined in (1.3.2). Using filtered/graded arguments it is easy to see that

$$^\circ\mathbf{R}_{m+1} = \bigoplus_{w \in D_{m,1}} {}^\circ\mathbf{R}_{m,1}\sigma_{\dot w}.$$

□

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$\psi^* : {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod}$$

is the restriction and $\psi_!$, ψ_* are given by

$$\psi_! : \begin{cases} {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod}, \\ (M, M') \mapsto {}^\circ\mathbf{R}_{m+1} \otimes_{{}^\circ\mathbf{R}_{m,1}} (M \otimes M'), \end{cases}$$

$$\psi_* : \begin{cases} {}^\circ\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_1\text{-}\mathbf{mod} \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{mod}, \\ (M, M') \mapsto \hom_{{}^\circ\mathbf{R}_{m,1}}({}^\circ\mathbf{R}_{m+1}, M \otimes M'). \end{cases}$$

First, note that the functors $\psi_!$, ψ^*, ψ_* commute with the shift of the grading. Next, the functor ψ^* is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is free of finite rank. Thus $\psi_!$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded ${}^\circ\mathbf{R}_{m,1}$-module ${}^\circ\mathbf{R}_{m+1}$ is also free of finite rank. Thus the functor ψ_* is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further $\psi_!$ and ψ^* take projective graded modules to projective ones, because they are left adjoint to the exact functors ψ^*, ψ_* respectively. To summarize, the functors $\psi_!$, ψ^*, ψ_* are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\psi_!$, ψ^* take projective graded modules to projective ones.

For any graded ${}^\circ\mathbf{R}_m$-module M we write

$$\begin{aligned} f_i(M) &= {}^\circ\mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \\ e_i(M) &= {}^\circ\mathbf{R}_{m-1} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} 1_{m-1,i} M. \end{aligned} \tag{3.2.1}$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$1_{m-1,i} M = \bigoplus_{\mathbf{i}} 1_{\theta(i)\mathbf{i}i} M, \quad \mathbf{i} \in {}^\theta I^{m-1}.$$

Note that $f_i(M)$ is a graded ${}^\circ\mathbf{R}_{m+1}$-module, while $e_i(M)$ is a graded ${}^\circ\mathbf{R}_{m-1}$-module. The tensor product in the definition of $e_i(M)$ is relative to the graded $\mathbf{k}$-algebra homomorphism

$${}^\circ\mathbf{R}_{m-1,1} \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_1 \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{R}_i \to {}^\circ\mathbf{R}_{m-1} \otimes \mathbf{L}_i = {}^\circ\mathbf{R}_{m-1}.$$

In other words, let $e'_i(M)$ be the graded ${}^\circ\mathbf{R}_{m-1}$-module obtained by taking the direct summand $1_{m-1,i}M$ and restricting it to ${}^\circ\mathbf{R}_{m-1}$. Observe that if M is finitely generated then $e'_i(M)$ may not lie in ${}^\circ\mathbf{R}_{m-1}$-$\mathbf{mod}$. To remedy this, since $e'_i(M)$ affords a ${}^\circ\mathbf{R}_{m-1}\otimes\mathbf{R}_i$-action we consider the graded ${}^\circ\mathbf{R}_{m-1}$-module

$$e_i(M) = e'_i(M)/\varkappa_i e'_i(M).$$

Definition 3.2.2. The functors e_i, f_i preserve the category ${}^\circ\mathbf{R}$-$\mathbf{proj}$, yielding $\mathcal{A}$-linear operators on ${}^\circ\mathbf{K}_I$ which act on ${}^\circ\mathbf{K}_{I,*}$ by the formula (3.2.1) and satisfy also

$$f_i(\phi_+) = {}^\circ\mathbf{R}_{\theta(i)i}, \quad f_i(\phi_-) = {}^\circ\mathbf{R}_{i\theta(i)}, \quad e_i(\mathbf{R}_{\theta(j)j}) = \delta_{i,j}\phi_+ + \delta_{i,\theta(j)}\phi_-.$$

Let e_i, f_i denote also the $\mathcal{A}$-linear operators on ${}^\circ\mathbf{G}_I$ which are the transpose of f_i, e_i with respect to the Cartan pairing.

Note that the symbols $e_i(M)$, $f_i(M)$ have a different meaning if M is viewed as an element of $^\circ\mathbf{K}_I$ or if M is viewed as an element of $^\circ\mathbf{G}_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [VV09a, Lemma 8.9].

Lemma 3.2.3. *(a) The operators e_i, f_i on $^\circ\mathbf{G}_I$ are given by*

$$e_i(M) = 1_{m-1,i}M \quad f_i(M) = \hom_{^\circ\mathbf{R}_{m,1}}(^\circ\mathbf{R}_{m+1}, M \otimes \mathbf{L}_i), \quad M \in {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod}.$$

*(b) For each $M \in {}^\circ\mathbf{R}_m$-**mod**, $M' \in {}^\circ\mathbf{R}_{m+1}$-**mod** we have*

$$(e'_i(M') : M) = (M' : f_i(M)).$$

*(c) We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\circ\mathbf{R}$-**proj**.*

*(d) We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\circ\mathbf{R}$-**fmod**.*

3.3 Induction of $\mathbf{H}_m$-modules versus induction of $^\circ\mathbf{R}_m$-modules

Recall the functors E_i, F_i on $\mathbf{H}\text{-}\mathbf{fMod}_I$ defined in (2.4.1). We have also the functors

$$\mathbf{for} : {}^\circ\mathbf{R}_m\text{-}\mathbf{fmod} \to {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0, \quad \Psi : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I,$$

where **for** is the forgetting of the grading. Finally we define functors

$$\begin{aligned} &E_i : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m-1}\text{-}\mathbf{fMod}_0, \quad E_iM = 1_{m-1,i}M, \\ &F_i : {}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\circ\mathbf{R}_{m+1}\text{-}\mathbf{fMod}_0, \quad F_iM = \psi_!(M, \mathbf{L}_i). \end{aligned} \tag{3.3.1}$$

Proposition 3.3.1. *There are canonical isomorphisms of functors*

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \mathbf{for} = \mathbf{for} \circ e_i, \quad F_i \circ \mathbf{for} = \mathbf{for} \circ f_{\theta(i)}.$$

Proof. The proof is the same as in [VV09a, Proposition 8.17]. □

Proposition 3.3.2. *(a) The functor Ψ yields an isomorphism of abelian groups*

$$\bigoplus_{m \geqslant 0} [{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0] = \bigoplus_{m \geqslant 0} [\mathbf{H}_m\text{-}\mathbf{fMod}_I].$$

The functors E_i, F_i yield endomorphisms of both sides which are intertwined by Ψ.

(b) The functor **for** *factors to a group isomorphism*

$$^\circ\mathbf{G}_I/(v-1) = \bigoplus_{m \geqslant 0} [{}^\circ\mathbf{R}_m\text{-}\mathbf{fMod}_0].$$

Proof. Claim (a) follows from Corollary 2.3.2 and Proposition 3.3.1. Claim (b) follows from Proposition 3.1.1. □

3.4 Type D versus type B

We can compare the previous constructions with their analogues in type B. Let ${}^\theta\mathbf{K}$, ${}^\theta B$, ${}^\theta G^{\mathrm{low}}$, etc., denote the type B analogues of ${}^\circ\mathbf{K}$, ${}^\circ B$, ${}^\circ G^{\mathrm{low}}$, etc., see [VV09a] for details. We will use the same notation for the functors ψ^*, $\psi_!$, ψ_*, e_i, f_i, etc., on the type B side and on the type D side. Fix $m > 0$ and $\nu \in {}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. The inclusion of graded $\mathbf{k}$-algebras ${}^\circ\mathbf{R}_\nu \subset {}^\theta\mathbf{R}_\nu$ in (1.2.1) yields a restriction functor

$$\mathrm{res} : {}^\theta\mathbf{R}_\nu\textbf{-mod} \to {}^\circ\mathbf{R}_\nu\textbf{-mod}$$

and an induction functor

$$\mathrm{ind} : {}^\circ\mathbf{R}_\nu\textbf{-mod} \to {}^\theta\mathbf{R}_\nu\textbf{-mod}, \quad M \mapsto {}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M.$$

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $\mathcal{A}$-modules

$$\begin{aligned} \mathrm{res} : {}^\theta\mathbf{K}_{I,m} \to {}^\circ\mathbf{K}_{I,m}, \quad \mathrm{res} : {}^\theta\mathbf{G}_{I,m} \to {}^\circ\mathbf{G}_{I,m}, \\ \mathrm{ind} : {}^\circ\mathbf{K}_{I,m} \to {}^\theta\mathbf{K}_{I,m}, \quad \mathrm{ind} : {}^\circ\mathbf{G}_{I,m} \to {}^\theta\mathbf{G}_{I,m}. \end{aligned}$$

Moreover, for any $P \in {}^\theta\mathbf{K}_{I,m}$ and any $L \in {}^\theta\mathbf{G}_{I,m}$ we have

$$\begin{aligned} \mathrm{res}(P^\sharp) = (\mathrm{res}\, P)^\sharp, \quad \mathrm{ind}(P^\sharp) = (\mathrm{ind}\, P)^\sharp \\ \mathrm{res}(L^\flat) = (\mathrm{res}\, L)^\flat, \quad \mathrm{ind}(L^\flat) = (\mathrm{ind}\, L)^\flat. \end{aligned} \tag{3.4.1}$$

Note also that ind and res are left and right adjoint functors, because

$${}^\theta\mathbf{R}_\nu \otimes_{{}^\circ\mathbf{R}_\nu} M = \mathrm{hom}_{{}^\circ\mathbf{R}_\nu}({}^\theta\mathbf{R}_\nu, M)$$

as graded ${}^\theta\mathbf{R}_\nu$-modules. Recall the involution γ of ${}^\circ\mathbf{R}_\nu$ from (5.9.4).

Definition 3.4.1. For any graded ${}^\circ\mathbf{R}_\nu$-module M we define M^γ to be the graded ${}^\circ\mathbf{R}_\nu$-module with the same underlying graded $\mathbf{k}$-vector space as M such that the action of ${}^\circ\mathbf{R}_\nu$ is twisted by γ, i.e., the graded $\mathbf{k}$-algebra ${}^\circ\mathbf{R}_\nu$ acts on M^γ by $a\, m = \gamma(a)m$ for $a \in {}^\circ\mathbf{R}_\nu$ and $m \in M$.

Note that $(M^\gamma)^\gamma = M$, and that M^γ is simple (resp. projective, indecomposable) if M has the same property. For any graded ${}^\circ\mathbf{R}_m$-module M we have canonical isomorphisms of ${}^\circ\mathbf{R}$-modules

$$(f_i(M))^\gamma = f_i(M^\gamma), \quad (e_i(M))^\gamma = e_i(M^\gamma).$$

The first isomorphism is given by

$${}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M \to {}^\circ\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\circ\mathbf{R}_m} M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$

The second one is the identity map on the vector space $1_{m,i}M$.

Recall that ${}^\theta I^\nu$ is the disjoint union of ${}^\theta I^\nu_+$ and ${}^\theta I^\nu_-$. We set

$$1_{\nu,+} = \sum_{\mathbf{i} \in {}^\theta I^\nu_+} 1_{\mathbf{i}}, \quad 1_{\nu,-} = \sum_{\mathbf{i} \in {}^\theta I^\nu_-} 1_{\mathbf{i}}.$$

Lemma 3.4.2. *Let M be a graded ${}^\circ\mathbf{R}_\nu$-module.*

(*a*) *Both* $1_{\nu,+}$ *and* $1_{\nu,-}$ *are central idempotents in* $^\circ\mathbf{R}_\nu$. *We have* $1_{\nu,+} = \gamma(1_{\nu,-})$.
(*b*) *There is a decomposition of graded* $^\circ\mathbf{R}_\nu$*-modules* $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.
(*c*) *We have a canonical isomorphism of* $^\circ\mathbf{R}_\nu$*-modules* $\mathrm{res} \circ \mathrm{ind}(M) = M \oplus M^\gamma$.
(*d*) *If there exists* $a \in \{+,-\}$ *such that* $1_{\nu,-a}M = 0$, *then there are canonical isomorphisms of graded* $^\circ\mathbf{R}_\nu$*-modules*

$$M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma.$$

Proof. Part (*a*) follows from Proposition 1.3.2 and the equality $\varepsilon_1(^\theta I_+^\nu) = {}^\theta I_-^\nu$. Part (*b*) follows from (*a*), (*c*) is given by definition, and (*d*) follows from (*a*), (*b*). □

Now, let m and ν be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group W_m acts on $^\theta I^{\nu'}$, and the map

$$^\theta I^\nu \to {}^\theta I^{\nu'}, \quad \mathbf{i} \mapsto \theta(i)\mathbf{i}i$$

is W_m-equivariant. Thus there is $a_i \in \{+,-\}$ such that the image of $^\theta I_+^\nu$ is contained in $^\theta I_{a_i}^{\nu'}$, and the image of $^\theta I_-^\nu$ is contained in $^\theta I_{-a_i}^{\nu'}$.

Lemma 3.4.3. *Let* M *be a graded* $^\circ\mathbf{R}_\nu$*-module such that* $1_{\nu,-a}M = 0$, *with* $a = +,-$. *Then we have*

$$1_{\nu',-a_ia}f_i(M) = 0, \quad 1_{\nu',a_ia}f_{\theta(i)}(M) = 0.$$

Proof. We have

$$\begin{aligned} 1_{\nu',-a_ia}f_i(M) &= 1_{\nu',-a_ia}{}^\circ\mathbf{R}_{\nu'}1_{\nu,i} \otimes_{^\circ\mathbf{R}_\nu} M && (3.4.2)\\ &= {}^\circ\mathbf{R}_{\nu'}1_{\nu',-a_ia}1_{\nu,i}1_{\nu,a} \otimes_{^\circ\mathbf{R}_\nu} M. && (3.4.3)\end{aligned}$$

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a}, 1_i)$ via the inclusion (3.2.1). The definition of this inclusion and that of a_i yield that

$$1_{\nu',a_ia}1_{\nu,i}1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a_ia}1_{\nu,i}1_{\nu,a} = 0.$$

The first equality follows. Next, note that for any $\mathbf{i} \in {}^\theta I^\nu$, the sequences $\theta(i)\mathbf{i}i$ and $i\mathbf{i}\theta(i) = \varepsilon_{m+1}(\theta(i)\mathbf{i}i)$ always belong to different $^\circ W_{m+1}$-orbits. Thus, we have $a_{\theta(i)} = -a_i$. So the second equality follows from the first. □

Consider the following diagram

$$\begin{tikzcd} {}^\circ\mathbf{R}_\nu\textbf{-mod} \times \mathbf{R}_i\textbf{-mod} \arrow[r, shift left, "\psi_!"] \arrow[d, shift left, "\mathrm{ind}\times\mathrm{Id}"] & {}^\circ\mathbf{R}_{\nu'}\textbf{-mod} \arrow[l, shift left, "\psi^*"] \arrow[d, shift left, "\mathrm{ind}"] \\ {}^\theta\mathbf{R}_\nu\textbf{-mod} \times \mathbf{R}_i\textbf{-mod} \arrow[r, shift left, "\psi_!"] \arrow[u, shift left, "\mathrm{res}\times\mathrm{Id}"] & {}^\theta\mathbf{R}_{\nu'}\textbf{-mod}. \arrow[l, shift left, "\psi^*"] \arrow[u, shift left, "\mathrm{res}"] \end{tikzcd}$$

Lemma 3.4.4. *There are canonical isomorphisms of functors*

$$\begin{aligned} &\mathrm{ind} \circ \psi_! = \psi_! \circ (\mathrm{ind} \times \mathrm{Id}), \quad \psi^* \circ \mathrm{ind} = (\mathrm{ind} \times \mathrm{Id}) \circ \psi^*, \quad \mathrm{ind} \circ \psi_* = \psi_* \circ (\mathrm{ind} \times \mathrm{Id}),\\ &\mathrm{res} \circ \psi_! = \psi_! \circ (\mathrm{res} \times \mathrm{Id}), \quad \psi^* \circ \mathrm{res} = (\mathrm{res} \times \mathrm{Id}) \circ \psi^*, \quad \mathrm{res} \circ \psi_* = \psi_* \circ (\mathrm{res} \times \mathrm{Id}).\end{aligned}$$

Proof. The functor ind is left and right adjoint to res. Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

$$\mathrm{ind}\circ\psi_! = \psi_!\circ(\mathrm{ind}\times\mathrm{Id})$$

comes from the associativity of the induction. Let us prove that

$$\psi^*\circ\mathrm{ind} = (\mathrm{ind}\times\mathrm{Id})\circ\psi^*.$$

For any M in ${}^\circ\mathbf{R}_{\nu'}$-**mod**, the obvious inclusion ${}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i\subset{}^\theta\mathbf{R}_{\nu'}$ yields a map

$$(\mathrm{ind}\times\mathrm{Id})\,\psi^*(M) = ({}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i)\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}\psi^*(M)\to\psi^*({}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}M).$$

Combining it with the obvious map

$${}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_\nu\otimes\mathbf{R}_i}M\to{}^\theta\mathbf{R}_{\nu'}\otimes_{{}^\circ\mathbf{R}_{\nu'}}M$$

we get a morphism of ${}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i$-modules

$$(\mathrm{ind}\times\mathrm{Id})\,\psi^*(M)\to\psi^*\,\mathrm{ind}(M).$$

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

$$M\oplus(\pi_{1,\nu}\otimes M)\to M\oplus(\pi_{1,\nu'}\otimes M),\quad m+\pi_{1,\nu}\otimes n\mapsto m+\pi_{1,\nu'}\otimes n.$$

Here $\pi_{1,\nu}$ and $\pi_{1,\nu'}$ denote the element π_1 in ${}^\theta\mathbf{R}_\nu$ and ${}^\theta\mathbf{R}_{\nu'}$ respectively. □

Corollary 3.4.5. *(a) The operators e_i, f_i on ${}^\circ\mathbf{K}_{I,*}$ and on ${}^\theta\mathbf{K}_{I,*}$ are intertwined by the maps* ind, res, *i.e., we have*

$$e_i\circ\mathrm{ind}=\mathrm{ind}\circ e_i,\quad f_i\circ\mathrm{ind}=\mathrm{ind}\circ f_i,\quad e_i\circ\mathrm{res}=\mathrm{res}\circ e_i,\quad f_i\circ\mathrm{res}=\mathrm{res}\circ f_i.$$

(b) The same result holds for the operators e_i, f_i on ${}^\circ\mathbf{G}_{I,*}$ and on ${}^\theta\mathbf{G}_{I,*}$.

3.5 Non graded case

Now, we concentrate on non graded irreducible modules. First, let

$$\mathrm{Res}:{}^\theta\mathbf{R}_\nu\text{-}\mathbf{Mod}\to{}^\circ\mathbf{R}_\nu\text{-}\mathbf{Mod},\quad\mathrm{Ind}:{}^\circ\mathbf{R}_\nu\text{-}\mathbf{Mod}\to{}^\theta\mathbf{R}_\nu\text{-}\mathbf{Mod}$$

be the (non graded) restriction and induction functors. We have

$$\mathbf{for}\circ\mathrm{res}=\mathrm{Res}\circ\mathbf{for},\quad\mathbf{for}\circ\mathrm{ind}=\mathrm{Ind}\circ\mathbf{for}.$$

Lemma 3.5.1. *Let L, L' be irreducible ${}^\circ\mathbf{R}_\nu$-modules.*

(a) The ${}^\circ\mathbf{R}_\nu$-modules L and L^γ are not isomorphic.

(b) $\mathrm{Ind}(L)$ *is an irreducible ${}^\theta\mathbf{R}_\nu$-module, and every irreducible ${}^\theta\mathbf{R}_\nu$-module is obtained in this way.*

(c) $\mathrm{Ind}(L)\simeq\mathrm{Ind}(L')$ *if and only if $L'\simeq L$ or L^γ.*

Proof. For any ${}^\theta\mathbf{R}_\nu$-module $M\neq 0$, both $1_{\nu,+}M$ and $1_{\nu,-}M$ are nonzero. Indeed, we have $M=1_{\nu,+}M\oplus 1_{\nu,-}M$, and we may suppose that $1_{\nu,+}M\neq 0$. The automorphism $M\to M$, $m\mapsto\pi_1 m$ takes $1_{\nu,+}M$ to $1_{\nu,-}M$ by Lemma 3.15(a). Hence $1_{\nu,-}M\neq 0$.

Now, to prove part (a), suppose that $\phi:L\to L^\gamma$ is an isomorphism of ${}^\circ\mathbf{R}_\nu$-modules. We can regard ϕ as a γ-antilinear map $L\to L$. Since L is irreducible, by Schur's lemma we may assume that $\phi^2=\mathrm{Id}_L$. Then L admits a ${}^\theta\mathbf{R}_\nu$-module structure such that the ${}^\circ\mathbf{R}_\nu$-action is as before and π_1 acts as ϕ. Thus, by the discussion above, neither $1_{\nu,+}L$ nor $1_{\nu,-}L$ is zero. This contradicts the fact that L is an irreducible ${}^\circ\mathbf{R}_\nu$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR03, appendix]. □

3.6 Proof of Proposition 3.1.1

Now, let us prove Proposition 3.1.1. Let $b \in {}^\circ B$. We may suppose that $b = 1_{\nu,+}b$. By Lemma 3.5.1(b) the module ${}^\theta b = \mathrm{Ind}(b)$ lies in ${}^\theta B$. By [VV09a, Proposition 8.2] there exists a unique selfdual irreducible graded ${}^\theta\mathbf{R}$-module ${}^\theta G^{\mathrm{up}}({}^\theta b)$ which is isomorphic to ${}^\theta b$ as a non graded module. Set

$$ {}^\circ G^{\mathrm{up}}(b) = 1_{\nu,+}\,\mathrm{res}({}^\theta G^{\mathrm{up}}({}^\theta b)). $$

By Lemma 3.4.2(d) we have ${}^\circ G^{\mathrm{up}}(b) = b$ as a non graded $^\circ\mathbf{R}$-module, and by (3.4.1) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that M is another module with the same properties. Then $\mathrm{ind}(M)$ is a selfdual graded ${}^\theta\mathbf{R}$-module which is isomorphic to ${}^\theta b$ as a non graded ${}^\theta\mathbf{R}$-module. Thus we have $\mathrm{ind}(M) = {}^\theta G^{\mathrm{up}}({}^\theta b)$ by loc. cit. By Lemma 3.4.2(d) we have also

$$ M = 1_{\nu,+}\,\mathrm{res}({}^\theta G^{\mathrm{up}}({}^\theta b)). $$

So M is isomorphic to ${}^\circ G^{\mathrm{up}}(b)$. □

3.7 The crystal operators on $^\circ\mathbf{G}_I$ and $^\circ B$

Fix a vertex i in I. For each irreducible graded $^\circ\mathbf{R}_m$-module M we define

$$ \tilde{e}_i(M) = \mathrm{soc}\,(e_i(M)), \quad \tilde{f}_i(M) = \mathrm{top}\,\psi_!(M, \mathbf{L}_i), \quad \varepsilon_i(M) = \max\{n \geqslant 0 \,|\, e_i^n(M) \neq 0\}. $$

Lemma 3.7.1. *Let M be an irreducible graded $^\circ\mathbf{R}$-module such that $e_i(M)$, $f_i(M)$ belong to $^\circ\mathbf{G}_{I,*}$. We have*

$$ \mathrm{ind}(\tilde{e}_i(M)) = \tilde{e}_i(\mathrm{ind}(M)), \quad \mathrm{ind}(\tilde{f}_i(M)) = \tilde{f}_i(\mathrm{ind}(M)), \quad \varepsilon_i(M) = \varepsilon_i(\mathrm{ind}(M)). $$

In particular, $\tilde{e}_i(M)$ is irreducible or zero and $\tilde{f}_i(M)$ is irreducible.

Proof. By Corollary 3.4.5 we have $\mathrm{ind}(e_i(M)) = e_i(\mathrm{ind}(M))$. Thus, since ind is an exact functor we have $\mathrm{ind}(\tilde{e}_i(M)) \subset e_i(\mathrm{ind}(M))$. Since ind is an additive functor, by Lemma 3.5.1(b) we have indeed

$$ \mathrm{ind}(\tilde{e}_i(M)) \subset \tilde{e}_i(\mathrm{ind}(M)). $$

Note that $\mathrm{ind}(M)$ is irreducible by Lemma 3.5.1(b), thus $\tilde{e}_i(\mathrm{ind}(M))$ is irreducible by [VV09a, Proposition 8.21]. Since $\mathrm{ind}(\tilde{e}_i(M))$ is nonzero, the inclusion is an isomorphism. The fact that $\mathrm{ind}(\tilde{e}_i(M))$ is irreducible implies in particular that $\tilde{e}_i(M)$ is simple. The proof of the second isomorphism is similar. The third equality is obvious. □

Similarly, for each irreducible $^\circ\mathbf{R}$-module b in $^\circ B$ we define

$$ \tilde{E}_i(b) = \mathrm{soc}(E_i(b)), \quad \tilde{F}_i(b) = \mathrm{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{n \geqslant 0 \,|\, E_i^n(b) \neq 0\}. $$

Hence we have

$$ \mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}, \quad \mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}, \quad \varepsilon_i = \varepsilon_i \circ \mathbf{for}. $$

Proposition 3.7.2. *For each b, b' in $^\circ B$ we have*

(a) $\tilde{F}_i(b) \in {}^\circ B$,

(b) $\tilde{E}_i(b) \in {}^\circ B \cup \{0\}$,

(c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,

(d) $\varepsilon_i(b) = \max\{n \geqslant 0 \,|\, \tilde{E}_i^n(b) \neq 0\}$,

(e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,

(f) if $\tilde{E}_i(b) = 0$ *for all* i *then* $b = \phi_\pm$.

Proof. Part (c) follows from adjunction. The other parts follow from [VV09a, Proposition 3.14] and Lemma 3.7.1. □

Remark 3.7.3. For any $b \in {}^\circ B$ and any $i \neq j$ we have $\tilde{F}_i(b) \neq \tilde{F}_j(b)$. This is obvious if $j \neq \theta(i)$. Because in this case $\tilde{F}_i(b)$ and $\tilde{F}_j(b)$ are $^\circ\mathbf{R}_\nu$-modules for different ν. Now, consider the case $j = \theta(i)$. We may suppose that $\tilde{F}_i(b) = 1_{\nu,+}\tilde{F}_i(b)$ for certain ν. Then by Lemma 3.4.3 we have $1_{\nu,+}\tilde{F}_{\theta(i)}(b) = 0$. In particular $\tilde{F}_i(b)$ is not isomorphic to $\tilde{F}_{\theta(i)}(b)$.

3.8 The algebra ${}^\theta\boldsymbol{\mathcal{B}}$ and the ${}^\theta\boldsymbol{\mathcal{B}}$-module $^\circ\mathbf{V}$

Following [EK06, EK08a, EK08b] we define a $\mathcal{K}$-algebra ${}^\theta\boldsymbol{\mathcal{B}}$ as follows.

Definition 3.8.1. Let ${}^\theta\boldsymbol{\mathcal{B}}$ be the $\mathcal{K}$-algebra generated by e_i, f_i and invertible elements t_i, $i \in I$, satisfying the following defining relations

(a) $t_i t_j = t_j t_i$ and $t_{\theta(i)} = t_i$ for all i, j,

(b) $t_i e_j t_i^{-1} = v^{i\cdot j + \theta(i)\cdot j} e_j$ and $t_i f_j t_i^{-1} = v^{-i\cdot j - \theta(i)\cdot j} f_j$ for all i, j,

(c) $e_i f_j = v^{-i\cdot j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ for all i, j,

(d) $\displaystyle\sum_{a+b=1-i\cdot j} (-1)^a e_i^{(a)} e_j e_i^{(b)} = \sum_{a+b=1-i\cdot j} (-1)^a f_i^{(a)} f_j f_i^{(b)} = 0$ if $i \neq j$.

Here and below we use the following notation

$$\theta^{(a)} = \theta^a / \langle a \rangle !, \quad \langle a \rangle = \sum_{l=1}^{a} v^{a+1-2l}, \quad \langle a \rangle ! = \prod_{l=1}^{m} \langle l \rangle.$$

We can now construct a representation of ${}^\theta\boldsymbol{\mathcal{B}}$ as follows. By base change, the operators e_i, f_i in Definition 3.2.2 yield $\mathcal{K}$-linear operators on the $\mathcal{K}$-vector space

$$^\circ\mathbf{V} = \mathcal{K} \otimes_{\mathcal{A}} {}^\circ\mathbf{K}_I.$$

We equip $^\circ\mathbf{V}$ with the $\mathcal{K}$-bilinear form given by

$$(M : N)_{KE} = (1 - v^2)^m \,(M : N), \quad \forall\, M, N \in {}^\circ\mathbf{R}_m\textbf{-proj}.$$

Theorem 3.8.2. *(a) The operators e_i, f_i define a representation of ${}^\theta\boldsymbol{\mathcal{B}}$ on $^\circ\mathbf{V}$. The ${}^\theta\boldsymbol{\mathcal{B}}$-module $^\circ\mathbf{V}$ is generated by linearly independent vectors ϕ_+ and ϕ_- such that for each $i \in I$ we have*

$$e_i\phi_\pm = 0, \quad t_i\phi_\pm = \phi_\mp, \quad \{x \in {}^\circ\mathbf{V} \,|\, e_j x = 0, \forall j\} = \mathbf{k}\,\phi_+ \oplus \mathbf{k}\,\phi_-.$$

(b) The symmetric bilinear form on $^\circ\mathbf{V}$ is non-degenerate. We have $(\phi_a : \phi_{a'})_{KE} = \delta_{a,a'}$ for $a, a' = +, -$, and $(e_i x : y) = (x : f_i y)_{KE}$ for $i \in I$ and $x, y \in {}^\circ\mathbf{V}$.

Proof. For each i in I we define the $\mathcal{A}$-linear operator t_i on ${}^\circ\mathbf{K}_I$ by setting

$$t_i\phi_\pm = \phi_\mp \quad \text{and} \quad t_iP = v^{-\nu\cdot(i+\theta(i))}P^\gamma, \quad \forall\ P \in {}^\circ\mathbf{R}_\nu\text{-}\mathbf{proj}.$$

We must prove that the operators e_i, f_i, and t_i satisfy the relations of ${}^\theta\boldsymbol{\mathcal{B}}$. The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey's induction-restriction theorem. Note that for $m > 1$ we have

$$D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1}\varepsilon_1\},$$
$$W(e) = {}^\circ W_m, \quad W(s_m) = {}^\circ W_{m-1}, \quad W(\varepsilon_{m+1}\varepsilon_1) = {}^\circ W_m.$$

Recall also that for $m = 1$ we have set ${}^\circ W_1 = \{e\}$.

Lemma 3.8.3. *Fix i, j in I. Let μ, ν in ${}^\theta\mathbb{N}I$ be such that $\nu + i + \theta(i) = \mu + j + \theta(j)$. Put $|\nu| = |\mu| = 2m$. The graded $({}^\circ\mathbf{R}_{m,1}, {}^\circ\mathbf{R}_{m,1})$-bimodule $1_{\nu,i}{}^\circ\mathbf{R}_{m+1}1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to*

$$\delta_{i,j}({}^\circ\mathbf{R}_\nu \otimes \mathbf{R}_i) \oplus \delta_{\theta(i),j}\big(({}^\circ\mathbf{R}_\nu)^\gamma \otimes \mathbf{R}_{\theta(i)}\big)[d'] \oplus A[d],$$

where A is equal to

$$\begin{array}{lr} ({}^\circ\mathbf{R}_m 1_{\nu',i} \otimes \mathbf{R}_i) \otimes_{\mathbf{R}'} (1_{\nu',i}{}^\circ\mathbf{R}_m \otimes \mathbf{R}_i) & \text{if } m > 1, \\ ({}^\circ\mathbf{R}_{\theta(\mathbf{j})} \otimes \mathbf{R}_i \otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1} {}^\circ\mathbf{R}_{\theta(\mathbf{i})} \otimes \mathbf{R}_j) \oplus ({}^\circ\mathbf{R}_{\mathbf{j}} \otimes \mathbf{R}_i \otimes_{{}^\circ\mathbf{R}_1\otimes\mathbf{R}_1} {}^\circ\mathbf{R}_{\mathbf{i}} \otimes \mathbf{R}_j) & \text{if } m = 1. \end{array}$$

Here we have set $\nu' = \nu - j - \theta(j)$, $\mathbf{R}' = {}^\circ\mathbf{R}_{m-1,1} \otimes \mathbf{R}_1$, $\mathbf{i} = i\theta(i)$, $\mathbf{j} = j\theta(j)$, $d = -i\cdot j$, and $d' = -\nu\cdot(i+\theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have

$$f_j(P) = {}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1), \quad e'_i(P) = 1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\circ\mathbf{R}_{m-1}$-module. Therefore we have

$$e'_if_j(P) = 1_{m,i}{}^\circ\mathbf{R}_{m+1}1_{m,j} \otimes_{{}^\circ\mathbf{R}_{m,1}} (P \otimes \mathbf{R}_1),$$
$$f_je'_i(P) = {}^\circ\mathbf{R}_m 1_{m-1,j} \otimes_{{}^\circ\mathbf{R}_{m-1,1}} (1_{m-1,i}P \otimes \mathbf{R}_1).$$

Therefore, up to some filtration we have the following identities

- $e'_if_i(P) = P \otimes \mathbf{R}_i + f_ie'_i(P)[-2]$,
- $e'_if_{\theta(i)}(P) = P^\gamma \otimes \mathbf{R}_{\theta(i)}[-\nu\cdot(i+\theta(i))/2] + f_{\theta(i)}e'_i(P)[-i\cdot\theta(i)]$,
- $e'_if_j(P) = f_je'_i(P)[-i\cdot j]$ if $i \neq j, \theta(j)$.

These identities also hold for $m = 1$ and $P = {}^\circ\mathbf{R}_{\theta(i)i}$ for any $i \in I$. The first claim of part (a) follows because $\mathbf{R}_i = \mathbf{k} \oplus \mathbf{R}_i[2]$. The fact that ${}^\circ\mathbf{V}$ is generated by $\phi_\pm$ is a corollary of Proposition 3.8.5 below. Part (b) of the theorem follows from [KM07, Proposition 2.2(ii)] and Lemma 3.2.3(b). □

Remark 3.8.4. (a) The ${}^\theta\boldsymbol{\mathcal{B}}$-module ${}^\circ\mathbf{V}$ is the same as the ${}^\theta\boldsymbol{\mathcal{B}}$-module V_θ from [KM07, Proposition 2.2]. The involution $\sigma : {}^\circ\mathbf{V} \to {}^\circ\mathbf{V}$ in [KM07, Remark 2.5(ii)] is given by $\sigma(P) = P^\gamma$. It yields an involution of ${}^\circ B$ in the obvious way. Note that Lemma 3.5.1(a) yields $\sigma(b) \neq b$ for any $b \in {}^\circ B$.

(b) Let ${}^\theta\mathbf{V}$ be the ${}^\theta\boldsymbol{\mathcal{B}}$-module $\mathcal{K}\otimes_{\mathcal{A}}{}^\theta\mathbf{K}_I$ and let ϕ be the class of the trivial ${}^\theta\mathbf{R}_0$-module $\mathbf{k}$, see [VV09a, Theorem 8.30]. We have an inclusion of ${}^\theta\boldsymbol{\mathcal{B}}$-modules

$${}^\theta\mathbf{V} \to {}^\circ\mathbf{V}, \quad \phi \mapsto \phi_+ \oplus \phi_-, \quad P \mapsto \mathrm{res}(P).$$

Proposition 3.8.5. *For any $b \in {}^\circ B$ the following holds*

$$(a) \qquad \begin{cases} f_i({}^\circ G^{low}(b)) = \langle \varepsilon_i(b)+1\rangle\, {}^\circ G^{low}(\tilde{F}_i b) + \sum_{b'} f_{b,b'}\, {}^\circ G^{low}(b'), \\ b' \in {}^\circ B, \quad \varepsilon_i(b') > \varepsilon_i(b)+1, \quad f_{b,b'} \in v^{2-\varepsilon_i(b')}\mathbb{Z}[v], \end{cases}.$$

$$(b) \qquad \begin{cases} e_i({}^\circ G^{low}(b)) = v^{1-\varepsilon_i(b)}\, {}^\circ G^{low}(\tilde{E}_i b) + \sum_{b'} e_{b,b'}\, {}^\circ G^{low}(b'), \\ b' \in {}^\circ B, \quad \varepsilon_i(b') \geqslant \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v]. \end{cases}.$$

Proof. We prove part (a), the proof for (b) is similar. If ${}^\circ G^{\mathrm{low}}(b) = \phi_\pm$ this is obvious. So we assume that ${}^\circ G^{\mathrm{low}}(b)$ is a ${}^\circ\mathbf{R}_m$-module for $m \geqslant 1$. Fix $\nu \in {}^\theta\mathbb{N}I$ such that $f_i({}^\circ G^{\mathrm{low}}(b))$ is a ${}^\circ\mathbf{R}_\nu$-module. We will abbreviate $1_{\nu,a} = 1_a$ for $a \in \{+,-\}$. Since ${}^\circ G^{\mathrm{low}}(b)$ is indecomposable, it fulfills the condition of Lemma 3.4.3. So there exists $a \in \{+,-\}$ such that $1_{-a} f_i({}^\circ G^{\mathrm{low}}(b)) = 0$. Thus, by Lemma 3.4.2$(c),(d)$ and Corollary 3.4.5 we have

$$f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \operatorname{res}\operatorname{ind} f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \operatorname{res}\, f_i\, \operatorname{ind}({}^\circ G^{\mathrm{low}}(b)).$$

Note that ${}^\theta b = \mathrm{Ind}(b)$ belongs to ${}^\theta B$ by Lemma 3.5.1(b). Hence (3.4.1) yields

$$\operatorname{ind}({}^\circ G^{\mathrm{low}}(b)) = {}^\theta G^{\mathrm{low}}({}^\theta b).$$

We deduce that

$$f_i({}^\circ G^{\mathrm{low}}(b)) = 1_a \operatorname{res} f_i({}^\theta G^{\mathrm{low}}({}^\theta b)).$$

Now, write

$$f_i({}^\theta G^{\mathrm{low}}({}^\theta b)) = \sum f_{{}^\theta b, {}^\theta b'}\, {}^\theta G^{\mathrm{low}}({}^\theta b'), \quad {}^\theta b' \in {}^\theta B.$$

Then we have

$$f_i({}^\circ G^{\mathrm{low}}(b)) = \sum f_{{}^\theta b, {}^\theta b'} 1_a \operatorname{res}({}^\theta G^{\mathrm{low}}({}^\theta b')).$$

For any ${}^\theta b' \in {}^\theta B$ the ${}^\circ\mathbf{R}$-module $1_a\,\mathrm{Res}({}^\theta b')$ belongs to ${}^\circ B$. Thus we have

$$1_a \operatorname{res}({}^\theta G^{\mathrm{low}}({}^\theta b')) = {}^\circ G^{\mathrm{low}}(1_a\,\mathrm{Res}({}^\theta b')).$$

If ${}^\theta b' \neq {}^\theta b''$ then $1_a\,\mathrm{Res}({}^\theta b') \neq 1_a\,\mathrm{Res}({}^\theta b'')$, because ${}^\theta b' = \mathrm{Ind}(1_a\,\mathrm{Res}({}^\theta b'))$. Thus

$$f_i({}^\circ G^{\mathrm{low}}(b)) = \sum f_{{}^\theta b, {}^\theta b'}\, {}^\circ G^{\mathrm{low}}(1_a\,\mathrm{Res}({}^\theta b')),$$

and this is the expansion of the left hand side in the lower global basis. Finally, we have

$$\varepsilon_i(1_a\,\mathrm{Res}({}^\theta b')) = \varepsilon_i({}^\theta b')$$

by Lemma 3.7.1. So part (a) follows from [VV09a, Propositions 10.11(b), 10.16]. $\square$

3.9 The global bases of $^\circ\mathbf{V}$

Since the operators e_i, f_i on $^\circ\mathbf{V}$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ on the ${}^\theta\boldsymbol{\mathcal{B}}$-module $^\circ\mathbf{V}$ as follows. For each u in $^\circ\mathbf{V}$ we write

$$u = \sum_{n\geqslant 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,$$

$$\tilde{\mathbf{e}}_i(u) = \sum_{n\geqslant 1} f_i^{(n-1)} u_n, \quad \tilde{\mathbf{f}}_i(u) = \sum_{n\geqslant 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let $^\circ\mathbf{L}$ be the $\mathcal{R}$-submodule of $^\circ\mathbf{V}$ spanned by the elements $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$ with $l \geqslant 0$, $i_1, \dots, i_l \in I$. The following is the main result of this chapter.

Theorem 3.9.1. *(a) We have*

$$^\circ\mathbf{L} = \bigoplus_{b\in {}^\circ B} \mathcal{R}\, {}^\circ G^{low}(b), \quad \tilde{\mathbf{e}}_i(^\circ\mathbf{L}) \subset {}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i(^\circ\mathbf{L}) \subset {}^\circ\mathbf{L},$$

$$\tilde{\mathbf{e}}_i({}^\circ G^{low}(b)) = {}^\circ G^{low}(\tilde{E}_i(b)) \mod v\,{}^\circ\mathbf{L}, \quad \tilde{\mathbf{f}}_i({}^\circ G^{low}(b)) = {}^\circ G^{low}(\tilde{F}_i(b)) \mod v\,{}^\circ\mathbf{L}.$$

(b) The assignment $b \mapsto {}^\circ G^{low}(b) \mod v\,{}^\circ\mathbf{L}$ yields a bijection from ${}^\circ B$ to the subset of ${}^\circ\mathbf{L}/v{}^\circ\mathbf{L}$ consisting of the $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\pm)$'s. Further ${}^\circ G^{low}(b)$ is the unique element $x \in {}^\circ\mathbf{V}$ such that $x^\sharp = x$ and $x = {}^\circ G^{low}(b) \mod v\,{}^\circ\mathbf{L}$.

(c) For each b, b' in ${}^\circ B$ let $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ be the coefficients of ${}^\circ G^{low}(b')$ in $e_{\theta(i)}({}^\circ G^{low}(b))$, $f_i({}^\circ G^{low}(b))$ respectively. Then we have

$$E_{i,b,b'}|_{v=1} = [F_i \Psi\mathbf{for}({}^\circ G^{up}(b')) : \Psi\mathbf{for}({}^\circ G^{up}(b))],$$

$$F_{i,b,b'}|_{v=1} = [E_i \Psi\mathbf{for}({}^\circ G^{up}(b')) : \Psi\mathbf{for}({}^\circ G^{up}(b))].$$

Proof. Part (a) follows from [EK08b, Theorem 4.1, Corollary 4.4], [Eno09, Section 2.3], and Proposition 3.8.5. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.3.1. More precisely, by duality we can regard $E_{i,b,b'}$, $F_{i,b,b'}$ as the coefficients of ${}^\circ G^{\mathrm{up}}(b)$ in $f_{\theta(i)}({}^\circ G^{\mathrm{up}}(b'))$ and $e_i({}^\circ G^{\mathrm{up}}(b'))$ respectively. Therefore, by Proposition 3.3.1 we can regard $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b))$ in $F_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ and $E_i\Psi\mathbf{for}({}^\circ G^{\mathrm{up}}(b'))$ respectively. □

Chapter III

The v-Schur algebras and Jantzen filtration

In this chapter, we prove that certain parabolic Kazhdan-Lusztig polynomials calculate the graded decomposition matrices of v-Schur algebras given by the Jantzen filtration of Weyl modules. This confirms a conjecture of Leclerc and Thibon [LT96].

The result of this chapter has been prepublished in [Sha10].

1 Statement of the main result

Let v be a r-th root of unity in $\mathbb{C}$. The v-Schur algebra $\mathbf{S}_v(n)$ over $\mathbb{C}$ is a finite dimensional quasi-hereditary algebra. Its standard modules are the Weyl modules $W_v(\lambda)$ indexed by partitions λ of n. The module $W_v(\lambda)$ has a simple quotient $L_v(\lambda)$. See Section 3.9 for more details.

The decomposition matrix of $\mathbf{S}_v(n)$ is given by the following algorithm. Recall from Section I.4.2 that the Fock space $\mathcal{F}$ of level one is the $\mathbb{C}$-vector space with a basis $\{|\lambda\rangle\}$ indexed by the set of partitions. Let $\mathcal{F}_q = \mathcal{F} \otimes \mathbb{C}(q)$ be its q-version. It carries an action of the quantum enveloping algebra $U_q(\widehat{\mathfrak{sl}}_r)$. Let L^+ (resp. L^-) be the $\mathbb{Z}[q]$-submodule (resp. $\mathbb{Z}[q^{-1}]$-submodule) in $\mathcal{F}_q$ spanned by $\{|\lambda\rangle\}$. Following Leclerc and Thibon [LT96, Theorem 4.1], the Fock space $\mathcal{F}_q$ admits two particular bases $\{G^+_\lambda\}$, $\{G^-_\lambda\}$ with the properties that

$$G^+_\lambda \equiv |\lambda\rangle \bmod qL^+, \qquad G^-_\lambda \equiv |\lambda\rangle \bmod q^{-1}L^-.$$

Let $d_{\lambda\mu}(q), e_{\lambda\mu}(q)$ be elements in $\mathbb{Z}[q]$ such that

$$G^+_\mu = \sum_\lambda d_{\lambda\mu}(q)|\lambda\rangle, \qquad G^-_\lambda = \sum_\mu e_{\lambda\mu}(-q^{-1})|\mu\rangle.$$

For any partition λ write λ' for the transposed partition. Then the Jordan-Hölder multiplicity of $L_v(\mu)$ in $W_v(\lambda)$ is equal to the value of $d_{\lambda'\mu'}(q)$ at $q = 1$. This result was conjectured by Leclerc and Thibon [LT96, Conjecture 5.2] and has been proved by Varagnolo and Vasserot [VV99].

We are interested in the Jantzen filtration of $W_v(\lambda)$ [JM97]

$$W_v(\lambda) = J^0W_v(\lambda) \supset J^1W_v(\lambda) \supset \dots.$$

It is a filtration by $\mathbf{S}_v(n)$-submodules. The graded decomposition matrix of $\mathbf{S}_v(n)$ counts the multiplicities of $L_v(\mu)$ in the associated graded module of $W_v(\lambda)$. The graded version of the above algorithm was also conjectured by Leclerc and Thibon [LT96, Conjecture 5.3]. The main result of this chapter is a proof of this conjecture under a mild restriction on v.

Theorem 1.0.1. *Suppose that* $v = \exp(2\pi i/\kappa)$ *with* $\kappa \in \mathbb{Z}$ *and* $\kappa \leqslant -3$. *Let* λ, μ *be partitions of* n. *Then*

$$d_{\lambda'\mu'}(q) = \sum_{i\geqslant 0}[J^i W_v(\lambda)/J^{i+1}W_v(\lambda) : L_v(\mu)]q^i. \tag{1.0.1}$$

Let us outline the idea of the proof. We first show that certain equivalence of highest weight categories preserves the Jantzen filtrations of standard modules (Proposition 2.5.1). By constructing such an equivalence between the module category of the v-Schur algebra and a subcategory of the affine parabolic category $\mathcal{O}$ of negative level, we then transfer the problem of computing the Jantzen filtration of Weyl modules into the same problem for parabolic Verma modules (Corollary 3.11.3). The latter is solved using Beilinson-Bernstein's technics (Sections 5, 6, 7).

2 Jantzen filtration of standard modules

2.1 Notation

We will denote by A-mod the category of finitely generated modules over an algebra A, and by A-proj its subcategory consisting of projective objects. Let R be a commutative noetherian $\mathbb{C}$-algebra. By a *finite projective R-algebra* we mean a R-algebra that belongs to R-proj.

A R-category $\mathcal{C}$ is a category whose Hom sets are R-modules. All the functors between R-categories will be assumed to be R-linear, i.e., they induce morphisms of R-modules on the Hom sets. Unless otherwise specified, all the functors will be assumed to be covariant. If $\mathcal{C}$ is abelian, we will write $\mathcal{C}$-proj for the full subcategory consisting of projective objects. If there exists a finite projective algebra A together with an equivalence of R-categories $F : \mathcal{C} \cong A$-mod, then we define $\mathcal{C} \cap R$-proj to be the full subcategory of $\mathcal{C}$ consisting of objects M such that $F(M)$ belongs to R-proj. By Morita theory, the definition of $\mathcal{C}\cap R$-proj is independent of A or F. Further, for any $\mathbb{C}$-algebra homomorphism $R \to R'$ we will abbreviate $R'\mathcal{C}$ for the category $(R' \otimes_R A)$-mod. The definition of $R'\mathcal{C}$ is independent of the choice of A up to equivalence of categories.

For any abelian category $\mathcal{C}$ we will write $[\mathcal{C}]$ for the Grothendieck group of $\mathcal{C}$. Any exact functor F from $\mathcal{C}$ to another abelian category $\mathcal{C}'$ yields a group homomorphism $[\mathcal{C}] \to [\mathcal{C}']$, which we will again denote by F.

A $\mathbb{C}$-category $\mathcal{C}$ is called *artinian* if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and every object has a finite length. The Jordan-Hölder multiplicity of a simple object L in an object M of $\mathcal{C}$ will be denoted by $[M : L]$.

We abbreviate $\otimes = \otimes_{\mathbb{C}}$ and $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{C}}$.

2.2 Highest weight categories

Let $\mathcal{C}$ be a R-category that is equivalent to the category A-mod for some finite projective R-algebra A. Let Δ be a finite set of objects of $\mathcal{C}$ together with a partial order $<$. Let $\mathcal{C}^\Delta$ be the full subcategory of $\mathcal{C}$ consisting of objects which admit a finite filtration such that the successive quotients are isomorphic to objects in

$$\{D \otimes U \,|\, D \in \Delta,\ U \in R\text{-proj}\}.$$

We have the following definition, see [Rou08b, Definition 4.11].

Definition 2.2.1. The pair $(\mathcal{C}, \Delta)$ is called a *highest weight R-category* if the following conditions hold:

- the objects of Δ are projective over R,
- we have $\mathrm{End}_{\mathcal{C}}(D) = R$ for all $D \in \Delta$,
- given D_1, $D_2 \in \Delta$, if $\mathrm{Hom}_{\mathcal{C}}(D_1, D_2) \neq 0$, then $D_1 \leqslant D_2$,
- if $M \in \mathcal{C}$ satisfies $\mathrm{Hom}_{\mathcal{C}}(D, M) = 0$ for all $D \in \Delta$, then $M = 0$,
- given $D \in \Delta$, there exists $P \in \mathcal{C}$-proj and a surjective morphism $f : P \to D$ such that $\ker f$ belongs to $\mathcal{C}^{\Delta}$. Moreover, in the filtration of $\ker f$ only $D' \otimes U$ with $D' > D$ appears.

The objects in Δ are called *standard*. We say that an object has *a standard filtration* if it belongs to $\mathcal{C}^{\Delta}$. There is another set ∇ of objets in $\mathcal{C}$, called *costandard* objects, given by the following proposition.

Proposition 2.2.2. *Let $(\mathcal{C}, \Delta)$ be a highest weight R-category. Then there is a set $\nabla = \{D^{\vee} \mid D \in \Delta\}$ of objects of $\mathcal{C}$, unique up to isomorphism, with the following properties:*

(a) the pair $(\mathcal{C}^{\mathrm{op}}, \nabla)$ is a highest weight R-category, where ∇ is equipped with the same partial order as Δ,

(b) for D_1, $D_2 \in \Delta$ we have $\mathrm{Ext}^i_{\mathcal{C}}(D_1, D_2^{\vee}) \cong \begin{cases} R & \text{if } i = 0 \text{ and } D_1 = D_2 \\ 0 & \text{else.} \end{cases}$

See [Rou08b, Proposition 4.19].

2.3 Base change for highest weight categories.

From now on, unless otherwise specified we will fix $R = \mathbb{C}[[s]]$, the ring of formal power series in the variable s. Let $\wp$ be its maximal ideal and let K be its fraction field. For any R-module M, any morphism f of R-modules and any $i \in \mathbb{N}$ we will write

$$M(\wp^i) = M \otimes_R (R/\wp^i R), \qquad M_K = M \otimes_R K,$$
$$f(\wp^i) = f \otimes_R (R/\wp^i R), \qquad f_K = f \otimes_R K.$$

We will abbreviate

$$\mathcal{C}(\wp) = R(\wp)\mathcal{C}, \quad \mathcal{C}_K = K\mathcal{C}.$$

Let us first recall the following basic facts.

Lemma 2.3.1. *Let A be a finite projective R-algebra. Let $P \in A$-mod.*

(a) The A-module P is projective if and only if P is a projective R-module and $P(\wp)$ belongs to $A(\wp)$-proj.

(b) If P belongs to A-proj, then we have a canonical isomorphism

$$\mathrm{Hom}_A(P, M)(\wp) \xrightarrow{\sim} \mathrm{Hom}_{A(\wp)}(P(\wp), M(\wp)), \quad \forall\, M \in A\text{-mod}.$$

Further, if M belongs to R-proj then $\mathrm{Hom}_A(P, M)$ also belongs to R-proj.

We will also need the following theorem of Rouquier [Rou08b, Theorem 4.15].

Proposition 2.3.2. *Let $\mathcal{C}$ be a R-category that is equivalent to A-mod for some finite projective R-algebra A. Let Δ be a finite poset of objects of $\mathcal{C} \cap R$-proj. Then the category $(\mathcal{C}, \Delta)$ is a highest weight R-category if and only if $(\mathcal{C}(\wp), \Delta(\wp))$ is a highest weight $\mathbb{C}$-category.*

Finally, the costandard objects can also be characterized in the following way.

Lemma 2.3.3. *Let $(\mathcal{C}, \Delta)$ be a highest weight R-category. Assume that $\nabla' = \{{}^\vee D \mid D \in \Delta\}$ is a set of objects of $\mathcal{C} \cap R$-proj such that for any $D \in \Delta$ we have*

$$({}^\vee D)(\wp) \cong D(\wp)^\vee, \quad ({}^\vee D)_K \cong D_K.$$

Then we have ${}^\vee D \cong D^\vee \in \nabla$.

Proof. We prove the lemma by showing that ∇' has the properties (a), (b) in Proposition 2.2.2 with ${}^\vee D$ playing the role of $D^\vee$. This will imply that ${}^\vee D \cong D^\vee \in \nabla$. To check (a) note that $\nabla'(\wp)$ is the set of costandard modules of $\mathcal{C}(\wp)$ by assumption. So $(\mathcal{C}(\wp)^{\mathrm{op}}, \nabla'(\wp))$ is a highest weight $\mathbb{C}$-category. Therefore $(\mathcal{C}^{\mathrm{op}}, \nabla')$ is a highest weight R-category by Proposition 2.3.2. Now, let us concentrate on (b). Given D_1, $D_2 \in \Delta$, let $P_\bullet = 0 \to P_n \to \cdots \to P_0$ be a projective resolution of D_1 in $\mathcal{C}$. Then $\mathrm{Ext}^i_{\mathcal{C}}(D_1, {}^\vee D_2)$ is the cohomology of the complex

$$C_\bullet = \mathrm{Hom}_{\mathcal{C}}(P_\bullet, {}^\vee D_2).$$

Since D_1 and all the P_i belong to R-proj and R is a discrete valuation ring, by the Universal Coefficient Theorem the complex

$$P_\bullet(\wp) = 0 \to P_n(\wp) \to \cdots \to P_0(\wp)$$

is a resolution of $D_1(\wp)$ in $\mathcal{C}(\wp)$. Further, each $P_i(\wp)$ is a projective object in $\mathcal{C}(\wp)$ by Lemma 2.3.1(a). So $\mathrm{Ext}^i_{\mathcal{C}(\wp)}(D_1(\wp), {}^\vee D_2(\wp))$ is given by the cohomology of the complex

$$C_\bullet(\wp) = \mathrm{Hom}_{\mathcal{C}(\wp)}(P_\bullet(\wp), {}^\vee D_2(\wp)).$$

Again, by the Universal Coefficient Theorem, the canonical map

$$H_i(C_\bullet)(\wp) \longrightarrow H_i(C(\wp)_\bullet)$$

is injective. In other words we have a canonical injective map

$$\mathrm{Ext}^i_{\mathcal{C}}(D_1, {}^\vee D_2)(\wp) \longrightarrow \mathrm{Ext}^i_{\mathcal{C}(\wp)}(D_1(\wp), {}^\vee D_2(\wp)). \tag{2.3.1}$$

Note that each R-module C_i is finitely generated. Therefore $\mathrm{Ext}^i_{\mathcal{C}}(D_1, {}^\vee D_2)$ is also finitely generated over R. Note that if $i > 0$, or $i = 0$ and $D_1 \neq D_2$, then the right hand side of (2.3.1) is zero by assumption. So $\mathrm{Ext}^i_{\mathcal{C}}(D_1, {}^\vee D_2)(\wp) = 0$, and hence $\mathrm{Ext}^i_{\mathcal{C}}(D_1, {}^\vee D_2) = 0$ by Nakayama's lemma. Now, let us concentrate on the R-module $\mathrm{Hom}_{\mathcal{C}}(D, {}^\vee D)$ for $D \in \Delta$. First, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(D, {}^\vee D) \otimes_R K &= \mathrm{Hom}_{\mathcal{C}_K}(D_K, ({}^\vee D)_K) \\
&= \mathrm{End}_{\mathcal{C}_K}(D_K) \\
&= \mathrm{End}_{\mathcal{C}}(D) \otimes_R K \\
&= K.
\end{aligned} \tag{2.3.2}$$

Here the second equality is given by the isomorphism $D_K \cong ({}^\vee D)_K$ and the last equality follows from $\mathrm{End}_{\mathcal{C}}(D) = R$. Next, note that $\mathrm{Hom}_{\mathcal{C}}(D, {}^\vee D)(\wp)$ is included into the vector space $\mathrm{Hom}_{\mathcal{C}(\wp)}(D(\wp), {}^\vee D(\wp)) = \mathbb{C}$ by (2.3.1). So its dimension over $\mathbb{C}$ is less than one. Together with (2.3.2) this yields an isomorphism of R-modules $\mathrm{Hom}_{\mathcal{C}}(D, {}^\vee D) \cong R$, because R is a discrete valuation ring. So we have verified that ∇' satisfies both property (a) and (b) in Proposition 2.2.2. Therefore it coincides with ∇ with ${}^\vee D$ isomorphic to $D^\vee$. $\square$

2.4 The Jantzen filtration of standard modules

Let $(\mathcal{C}_{\mathbb{C}}, \Delta_{\mathbb{C}})$ be a highest weight $\mathbb{C}$-category and $(\mathcal{C}, \Delta)$ be a highest weight R-category such that $(\mathcal{C}_{\mathbb{C}}, \Delta_{\mathbb{C}}) \cong (\mathcal{C}(\wp), \Delta(\wp))$. Then any standard module in $\Delta_{\mathbb{C}}$ admits a Jantzen type filtration associated with $(\mathcal{C}, \Delta)$. It is given as follows.

Definition 2.4.1. For any $D \in \Delta$ let $\phi : D \to D^\vee$ be a morphism in $\mathcal{C}$ such that $\phi(\wp) \neq 0$. For any positive integer i let

$$\pi_i : D^\vee \longrightarrow D^\vee/\wp^i D^\vee \tag{2.4.1}$$

be the canonical quotient map. Set

$$D^i = \ker(\pi_i \circ \phi) \subset D, \quad J^i(D(\wp)) = (D^i + \wp D)/\wp D.$$

Below, we will abbreviate $J^i(D(\wp)) = J^i D(\wp)$. The *Jantzen filtration* of $D(\wp)$ is the filtration

$$D(\wp) = J^0 D(\wp) \supset J^1 D(\wp) \supset \cdots.$$

To see that the Jantzen filtration is well defined, one notices first that the morphism ϕ always exists because $\mathrm{Hom}_{\mathcal{C}}(D, D^\vee)(\wp) \cong R(\wp)$. Further, the filtration is independent of the choice of ϕ. Because if $\phi' : D \to D^\vee$ is another morphism such that $\phi'(\wp) \neq 0$, the fact that $\mathrm{Hom}_{\mathcal{C}}(D, D^\vee) \cong R$ and $\phi(\wp) \neq 0$ implies that there exists an element a in R such that $\phi' = a\phi$. Moreover $\phi'(\wp) \neq 0$ implies that a is invertible in R. So ϕ and ϕ' define the same filtration.

Remark 2.4.2. If the category $\mathcal{C}_K$ is semi-simple, then the Jantzen filtration of any standard module $D(\wp)$ is finite. In fact, since $\mathrm{End}_{\mathcal{C}}(D) = R$ we have $\mathrm{End}_{\mathcal{C}_K}(D_K) = K$. Therefore D_K is an indecomposable object in $\mathcal{C}_K$. The fact that $\mathcal{C}_K$ is semi-simple implies that the object D_K is simple. Similarly $D_K^\vee$ is also simple. So the morphism $\phi_K : D_K \to D_K^\vee$ is an isomorphism. In particular ϕ is injective. Now, consider the intersection

$$\bigcap_i J^i D(\wp) = \bigcap_i (D^i + \wp D)/\wp D.$$

Since we have $D^i \supset D^{i+1}$, the intersection on the right hand side is equal to $((\bigcap_i D^i) + \wp D)/\wp D$. The injectivity of ϕ implies that $\bigcap_i D^i = \ker \phi$ is zero. Hence $\bigcap_i J^i D(\wp) = 0$. Since $D(\wp) \in \mathcal{C}(\wp)$ has a finite length, we have $J^i D(\wp) = 0$ for i large enough.

2.5 Equivalences of highest weight categories and Jantzen filtrations.

Let $(\mathcal{C}_1, \Delta_1)$, $(\mathcal{C}_2, \Delta_2)$ be highest weight R-categories (resp. $\mathbb{C}$-categories or K-categories). A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an *equivalence of highest weight categories* if it is an equivalence of categories and if for any $D_1 \in \Delta_1$ there exists $D_2 \in \Delta_2$ such that $F(D_1) \cong D_2$. Note that for such an equivalence F we also have

$$F(D_1^\vee) \cong D_2^\vee, \tag{2.5.1}$$

because the two properties in Proposition 2.2.2 which characterize the costandard objects are preserved by F.

Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor. Since $\mathcal{C}_1$ is equivalent to A-mod for some finite projective R-algebra A, the functor F is represented by a projective object P in $\mathcal{C}_1$, i.e., we have $F \cong \mathrm{Hom}_{\mathcal{C}_1}(P, -)$. Set

$$F(\wp) = \mathrm{Hom}_{\mathcal{C}_1(\wp)}(P(\wp), -) : \mathcal{C}_1(\wp) \to \mathcal{C}_2(\wp).$$

Note that the functor $F(\wp)$ is unique up to equivalence of categories. It is an exact functor, and it is isomorphic to the functor $\mathrm{Hom}_{\mathcal{C}_1}(P,-)(\wp)$, see Lemma 2.3.1. In particular, for $D \in \Delta_1$ there are canonical isomorphisms

$$F(D)(\wp) \cong F(\wp)(D(\wp)), \quad F(D^\vee)(\wp) \cong F(\wp)(D^\vee(\wp)). \tag{2.5.2}$$

Proposition 2.5.1. *Let $(\mathcal{C}_1, \Delta_1)$, $(\mathcal{C}_2, \Delta_2)$ be two equivalent highest weight R-categories. Fix an equivalence $F : \mathcal{C}_1 \to \mathcal{C}_2$. Then the following holds.*

(a) The functor $F(\wp)$ is an equivalence of highest weight categories.

(b) The functor $F(\wp)$ preserves the Jantzen filtration of standard modules, i.e., for any $D_1 \in \Delta_1$ let $D_2 = F(D_1) \in \Delta_2$, then

$$F(\wp)(J^i D_1(\wp)) = J^i D_2(\wp), \quad \forall\ i \in \mathbb{N}.$$

Proof. (a) If $G : \mathcal{C}_2 \to \mathcal{C}_1$ is a quasi-inverse of F then $G(\wp)$ is a quasi-inverse of $F(\wp)$. So $F(\wp)$ is an equivalence of categories. It maps a standard object to a standard one because of the first isomorphism in (2.5.2).

(b) The functor F yields an isomorphism of R-modules

$$\mathrm{Hom}_{\mathcal{C}_1}(D_1, D_1^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_2}(F(D_1), F(D_1^\vee)),$$

where the right hand side identifies with $\mathrm{Hom}_{\mathcal{C}_2}(D_2, D_2^\vee)$ via the isomorphism (2.5.1). Let ϕ_1 be an element in $\mathrm{Hom}_{\mathcal{C}_1}(D_1, D_1^\vee)$ such that $\phi_1(\wp) \neq 0$. Let

$$\phi_2 = F(\phi_1) : D_2 \to D_2^\vee.$$

Then we also have $\phi_2(\wp) \neq 0$.

For $a = 1, 2$ and $i \in \mathbb{N}$ let $\pi_{a,i} : D_a^\vee \to D_a^\vee(\wp^i)$ be the canonical quotient map. Since F is R-linear and exact, the isomorphism $F(D_1^\vee) \cong D_2^\vee$ maps $F(\wp^i D_1^\vee)$ to $\wp^i D_2^\vee$ and induces an isomorphism

$$F(D_1^\vee(\wp^i)) \cong D_2^\vee(\wp^i).$$

Under these isomorphisms the morphism $F(\pi_{1,i})$ is identified with $\pi_{2,i}$. So we have

$$\begin{aligned} F(D_1^i) &= F(\ker(\pi_{1,i} \circ \phi_1)) \\ &= \ker(F(\pi_{1,i}) \circ F(\phi_1)) \\ &\cong \ker(\pi_{2,i} \circ \phi_2) \\ &= D_2^i. \end{aligned}$$

Now, apply F to the short exact sequence

$$0 \to \wp D_1 \to D_1^i + \wp D_1 \to J^i D_1(\wp) \to 0, \tag{2.5.3}$$

we get

$$\begin{aligned} F(J^i D_1(\wp)) &\cong (F(D_1^i) + \wp F(D_1))/\wp F(D_1) \\ &\cong J^i D_2(\wp). \end{aligned}$$

Since $F(J^i D_1(\wp)) = F(\wp)(J^i D_1(\wp))$, the proposition is proved. $\square$

3 Affine parabolic category $\mathcal{O}$ and v-Schur algebras

3.1 The affine Lie algebra

Fix an integer $m > 1$. Let $G_0 \supset B_0 \supset T_0$ be respectively the linear algebraic group $GL_m(\mathbb{C})$, the Borel subgroup of upper triangular matrices and the maximal torus of diagonal matrices. Let $\mathfrak{g}_0 \supset \mathfrak{b}_0 \supset \mathfrak{t}_0$ be their Lie algebras. Let

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$$

be the affine Lie algebra of $\mathfrak{g}_0$. Its Lie bracket is given by

$$[\xi \otimes t^a + x\mathbf{1} + y\partial, \xi' \otimes t^b + x'\mathbf{1} + y'\partial] = [\xi, \xi'] \otimes t^{a+b} + a\delta_{a,-b}\,\mathrm{tr}(\xi\xi')\mathbf{1} + by\xi' \otimes t^b - ay'\xi \otimes t^a,$$

where $\mathrm{tr} : \mathfrak{g}_0 \to \mathbb{C}$ is the trace map. Set $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$.

For any Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, let $\mathcal{U}(\mathfrak{a})$ be its enveloping algebra. For any $\mathbb{C}$-algebra R, we will abbreviate $\mathfrak{a}_R = \mathfrak{a} \otimes R$ and $\mathcal{U}(\mathfrak{a}_R) = \mathcal{U}(\mathfrak{a}) \otimes R$.

In the rest of the chapter, we will fix once for all an integer c such that

$$\kappa = c + m \in \mathbb{Z}_{<0}. \tag{3.1.1}$$

Let $\mathcal{U}_\kappa$ be the quotient of $\mathcal{U}(\mathfrak{g})$ by the two-sided ideal generated by $\mathbf{1} - c$. The $\mathcal{U}_\kappa$-modules are precisely the $\mathfrak{g}$-modules of level c.

Given a $\mathbb{C}$-linear map $\lambda : \mathfrak{t} \to R$ and a $\mathfrak{g}_R$-module M we set

$$M_\lambda = \{v \in M \mid hv = \lambda(h)v, \ \forall\ h \in \mathfrak{t}\}. \tag{3.1.2}$$

Whenever M_λ is non zero, we call λ a *weight* of M.

We equip $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ with the basis $\epsilon_1, \ldots, \epsilon_m$, ω_0, δ such that $\epsilon_1, \ldots, \epsilon_m \in \mathfrak{t}_0^*$ is dual to the canonical basis of $\mathfrak{t}_0$,

$$\delta(\partial) = \omega_0(\mathbf{1}) = 1, \quad \omega_0(\mathfrak{t}_0 \oplus \mathbb{C}\partial) = \delta(\mathfrak{t}_0 \oplus \mathbb{C}\mathbf{1}) = 0.$$

Let $\langle - : - \rangle$ be the symmetric bilinear form on $\mathfrak{t}^*$ such that

$$\langle \epsilon_i : \epsilon_j \rangle = \delta_{ij}, \quad \langle \omega_0 : \delta \rangle = 1, \quad \langle \mathfrak{t}_0^* \oplus \mathbb{C}\delta : \delta \rangle = \langle \mathfrak{t}_0^* \oplus \mathbb{C}\omega_0 : \omega_0 \rangle = 0.$$

For $h \in \mathfrak{t}^*$ we will write $||h||^2 = \langle h : h \rangle$. The weights of a $\mathcal{U}_\kappa$-module belong to

$${}_\kappa\mathfrak{t}^* = \{\lambda \in \mathfrak{t}^* \mid \langle \lambda : \delta \rangle = c\}.$$

Let a denote the projection from $\mathfrak{t}^*$ to $\mathfrak{t}_0^*$. Consider the map

$$z : \mathfrak{t}^* \to \mathbb{C} \tag{3.1.3}$$

such that $\lambda \mapsto z(\lambda)\delta$ is the projection $\mathfrak{t}^* \to \mathbb{C}\delta$.

Let Π be the root system of $\mathfrak{g}$ with simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i \leqslant m-1$ and $\alpha_0 = \delta - \sum_{i=1}^{m-1} \alpha_i$. The root system Π_0 of $\mathfrak{g}_0$ is the root subsystem of Π generated by $\alpha_1, \ldots, \alpha_{m-1}$. We will write Π^+, Π_0^+ for the sets of positive roots in Π, Π_0 respectively.

The affine Weyl group $\mathfrak{S}$ is a Coxeter group with simple reflections s_i for $0 \leqslant i \leqslant m-1$. It is isomorphic to the semi-direct product of the symmetric group $\mathfrak{S}_0$ with the lattice $\mathbb{Z}\Pi_0$. There is a linear action of $\mathfrak{S}$ on $\mathfrak{t}^*$ such that $\mathfrak{S}_0$ fixes ω_0, δ, and acts on $\mathfrak{t}_0^*$ by permuting ϵ_i's, and an element $\tau \in \mathbb{Z}\Pi_0$ acts by

$$\tau(\delta) = \delta, \quad \tau(\omega_0) = \tau + \omega_0 - \langle \tau : \tau \rangle \delta/2, \quad \tau(\lambda) = \lambda - \langle \tau : \lambda \rangle \delta, \quad \forall\ \lambda \in \mathfrak{t}_0^*. \tag{3.1.4}$$

Let ρ_0 be the half sum of positive roots in Π_0 and $\rho = \rho_0 + m\omega_0$. The dot action of $\mathfrak{S}$ on $\mathfrak{t}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$. For $\lambda \in \mathfrak{t}^*$ we will denote by $\mathfrak{S}(\lambda)$ the stabilizer of λ in $\mathfrak{S}$ under the dot action. Let $l : \mathfrak{S} \to \mathbb{N}$ be the length function.

3.2 The parabolic Verma modules and their deformations

The subset Π_0 of Π defines a standard parabolic Lie subalgebra of $\mathfrak{g}$, which is given by

$$\mathfrak{q} = \mathfrak{g}_0 \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial.$$

It has a Levi subalgebra

$$\mathfrak{l} = \mathfrak{g}_0 \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial.$$

The parabolic Verma modules of $\mathcal{U}_\kappa$ associated with $\mathfrak{q}$ are given as follows.

Let λ be an element in

$$\Lambda^+ = \{\lambda \in {}_\kappa\mathfrak{t}^* \,|\, \langle \lambda : \alpha \rangle \in \mathbb{N},\ \forall\ \alpha \in \Pi_0^+\}.$$

Then there is a unique finite dimensional simple $\mathfrak{g}_0$-module $V(\lambda)$ of highest weight $a(\lambda)$. It can be regard as a $\mathfrak{l}$-module by letting $h \in \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$ act by the scalar $\lambda(h)$. It is further a $\mathfrak{q}$-module if we let the nilpotent radical of $\mathfrak{q}$ act trivially. The *parabolic Verma module* of highest weight λ is given by

$$M_\kappa(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V(\lambda).$$

It has a unique simple quotient, which we denote by $L_\kappa(\lambda)$.

Recall that $R = \mathbb{C}[[s]]$ and $\wp$ is its maximal ideal. Set

$$\mathbf{c} = c + s \qquad \text{and} \qquad \mathbf{k} = \kappa + s.$$

They are elements in R. Write $\mathcal{U}_\mathbf{k}$ for the quotient of $\mathcal{U}(\mathfrak{g}_R)$ by the two-sided ideal generated by $\mathbf{1} - \mathbf{c}$. So if M is a $\mathcal{U}_\mathbf{k}$-module, then $M(\wp)$ is a $\mathcal{U}_\kappa$-module. Now, note that R admits a $\mathfrak{q}_R$-action such that $\mathfrak{g}_0 \otimes \mathbb{C}[t]$ acts trivially and $\mathfrak{t}$ acts by the weight $s\omega_0$. Denote this $\mathfrak{q}_R$-module by $R_{s\omega_0}$. For $\lambda \in \Lambda^+$ the *deformed parabolic Verma module* $M_\mathbf{k}(\lambda)$ is the $\mathfrak{g}_R$-module induced from the $\mathfrak{q}_R$-module $V(\lambda) \otimes R_{s\omega_0}$. It is a $\mathcal{U}_\mathbf{k}$-module of highest weight $\lambda + s\omega_0$, and we have a canonical isomorphism

$$M_\mathbf{k}(\lambda)(\wp) \cong M_\kappa(\lambda).$$

We will abbreviate $\lambda_s = \lambda + s\omega_0$ and will write

$${}_\mathbf{k}\mathfrak{t}^* = \{\lambda_s \,|\, \lambda \in {}_\kappa\mathfrak{t}^*\}.$$

Lemma 3.2.1. *The $\mathfrak{g}_K$-module $M_\mathbf{k}(\lambda)_K = M_\mathbf{k}(\lambda) \otimes_R K$ is simple.*

Proof. Assume that $M_\mathbf{k}(\lambda)_K$ is not simple. Then it contains a nontrivial submodule. This submodule must have a highest weight vector of weight μ_s for some $\mu \in \Lambda^+$, $\mu \neq \lambda$. By the linkage principle, there exists $w \in \mathfrak{S}$ such that $\mu_s = w \cdot \lambda_s$. Therefore w fixes ω_0, so it belongs to $\mathfrak{S}_0$. But then we must have $w = 1$, because λ, $\mu \in \Lambda^+$. So $\lambda = \mu$. This is a contradiction. $\square$

3.3 The Jantzen filtration of parabolic Verma modules

For $\lambda \in \Lambda^+$ the Jantzen filtration of $M_\kappa(\lambda)$ is given as follows. Let σ be the R-linear anti-involution on $\mathfrak{g}_R$ such that

$$\sigma(\xi \otimes t^n) = {}^t\xi \otimes t^{-n}, \quad \sigma(\mathbf{1}) = \mathbf{1}, \quad \sigma(\partial) = \partial.$$

Here $\xi \in \mathfrak{g}_0$ and ${}^t\xi$ is the transposed matrix. Let $\mathfrak{g}_R$ act on $\mathrm{Hom}_R(M_{\mathbf{k}}(\lambda), R)$ via $(xf)(v) = f(\sigma(x)v)$ for $x \in \mathfrak{g}_R$, $v \in M_{\mathbf{k}}(\lambda)$. Then

$$\mathbf{D}M_{\mathbf{k}}(\lambda) = \bigoplus_{\mu \in {}_{\mathbf{k}}\mathfrak{t}^*} \mathrm{Hom}_R(M_{\mathbf{k}}(\lambda)_\mu, R) \tag{3.3.1}$$

is a $\mathfrak{g}_R$-submodule of $\mathrm{Hom}_R(M_{\mathbf{k}}(\lambda), R)$. It is the *deformed dual parabolic Verma module* with highest weight λ_s. The λ_s-weight spaces of $M_{\mathbf{k}}(\lambda)$ and $\mathbf{D}M_{\mathbf{k}}(\lambda)$ are both free R-modules of rank one. Any isomorphism between them yields, by the universal property of Verma modules, a $\mathfrak{g}_R$-module morphism

$$\phi : M_{\mathbf{k}}(\lambda) \to \mathbf{D}M_{\mathbf{k}}(\lambda)$$

such that $\phi(\wp) \neq 0$. The Jantzen filtration $\big(J^i M_\kappa(\lambda)\big)$ of $M_\kappa(\lambda)$ defined by [Jan79] is the filtration given by Definition 2.4.1 using the morphism ϕ above.

3.4 The deformed parabolic category $\mathcal{O}$

The *deformed parabolic category* $\mathcal{O}$, denoted by $\mathcal{O}_{\mathbf{k}}$, is the category of $\mathcal{U}_{\mathbf{k}}$-modules M such that

- $M = \bigoplus_{\lambda \in {}_{\mathbf{k}}\mathfrak{t}^*} M_\lambda$ with $M_\lambda \in R$-mod,
- for any $m \in M$ the R-module $\mathcal{U}(\mathfrak{q}_R)m$ is finitely generated.

It is an abelian category and contains deformed parabolic Verma modules. Replacing $\mathbf{k}$ by κ and R by $\mathbb{C}$ we get the usual parabolic category $\mathcal{O}$, denoted $\mathcal{O}_\kappa$.

Recall the map z in (3.1.3). For any integer r set

$${}^r_\kappa\mathfrak{t}^* = \{\mu \in {}_\kappa\mathfrak{t}^* \,|\, r - z(\mu) \in \mathbb{Z}_{\geqslant 0}\}.$$

Define ${}^r_{\mathbf{k}}\mathfrak{t}^*$ in the same manner. Let ${}^r\mathcal{O}_\kappa$ (resp. ${}^r\mathcal{O}_{\mathbf{k}}$) be the Serre subcategory of $\mathcal{O}_\kappa$ (resp. $\mathcal{O}_{\mathbf{k}}$) consisting of objects M such that $M_\mu \neq 0$ implies that μ belongs to ${}^r_\kappa\mathfrak{t}^*$ (resp. ${}^r_{\mathbf{k}}\mathfrak{t}^*$). Write ${}^r\Lambda^+ = \Lambda^+ \cap {}^r_\kappa\mathfrak{t}^*$. We have the following lemma.

Lemma 3.4.1. *(a) For any finitely generated projective object P in ${}^r\mathcal{O}_{\mathbf{k}}$ and any $M \in {}^r\mathcal{O}_{\mathbf{k}}$ the R-module* $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}(P, M)$ *is finitely generated and the canonical map*

$$\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}(P, M)(\wp) \to \mathrm{Hom}_{{}^r\mathcal{O}_\kappa}(P(\wp), M(\wp))$$

is an isomorphism. Moreover, if M is free over R, then $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}(P, M)$ *is also free over R.*

(b) The assignment $M \mapsto M(\wp)$ yields a functor

$${}^r\mathcal{O}_{\mathbf{k}} \to {}^r\mathcal{O}_\kappa.$$

This functor gives a bijection between the isomorphism classes of simple objects and a bijection between the isomorphism classes of indecomposable projective objects.

For any $\lambda \in {}^r\Lambda^+$ there is a unique finitely generated projective cover ${}^rP_\kappa(\lambda)$ of $L_\kappa(\lambda)$ in ${}^r\mathcal{O}_\kappa$, see [RCW82, Lemma 4.12]. Let $L_{\mathbf{k}}(\lambda)$, ${}^rP_{\mathbf{k}}(\lambda)$ be respectively the simple object and the indecomposable projective object in ${}^r\mathcal{O}_{\mathbf{k}}$ that map respectively to $L_\kappa(\lambda)$, $P_\kappa(\lambda)$ by the bijections in Lemma 3.4.1(b). Then we have the following lemma.

Lemma 3.4.2. *The object ${}^rP_{\mathbf{k}}(\lambda)$ is, up to isomorphism, the unique finitely generated projective cover of $L_{\mathbf{k}}(\lambda)$ in ${}^r\mathcal{O}_{\mathbf{k}}$. It has a filtration by deformed parabolic Verma modules. In particular, it is a free R-module.*

The proof of Lemmas 3.4.1, 3.4.2 can be given by imitate [Fie03, Section 2]. There, Fiebig proved the analogue of these results for the (nonparabolic) deformed category $\mathcal{O}$ by adapting arguments of [RCW82]. The proof here goes in the same way, because the parabolic case is also treated in [RCW82]. We left the details to the reader. Note that the deformed parabolic category $\mathcal{O}$ for reductive Lie algebras has also been studied in [Str09].

3.5 The highest weight category $\mathcal{E}_{\mathbf{k}}$

Fix a positive integer $n \leqslant m$. Let $\mathcal{P}_n$ denote the set of partitions of n. Recall that a partition λ of n is a sequence of integers $\lambda_1 \geqslant \ldots \geqslant \lambda_m \geqslant 0$ such that $\sum_{i=1}^m \lambda_i = n$. For such a λ denote the element $\sum_{i=1}^m \lambda_i \epsilon_i$ in $\mathfrak{t}_0^*$ again by λ. We will identify $\mathcal{P}_n$ with a subset of Λ^+ by the following inclusion

$$\mathcal{P}_n \to \Lambda^+, \quad \lambda \mapsto \lambda + c\omega_0 - \frac{\langle \lambda : \lambda + 2\rho_0 \rangle}{2\kappa}\delta. \tag{3.5.1}$$

We will also fix an integer r large enough such that $\mathcal{P}_n$ is contained in ${}^r\Lambda^+$. Equip Λ^+ with the partial order $\preceq$ given by $\lambda \preceq \mu$ if and only if there exists $w \in \mathfrak{S}$ such that $\mu = w \cdot \lambda$ and $a(\mu) - a(\lambda) \in \mathbb{N}\Pi_0^+$. Let $\trianglelefteq$ denote the dominance order on $\mathcal{P}_n$ given by

$$\lambda \trianglelefteq \mu \iff \sum_{j=1}^{i} \lambda_j \leqslant \sum_{j=1}^{i} \mu_j, \quad \forall\, 1 \leqslant i \leqslant m.$$

Note that for λ, $\mu \in \mathcal{P}_n$ we have

$$\lambda \preceq \mu \Longrightarrow \lambda \trianglelefteq \mu, \tag{3.5.2}$$

because $\lambda \preceq \mu$ implies that $\mu - \lambda \in \mathbb{N}\Pi_0^+$, which implies that

$$\sum_{j=1}^{i} \mu_j - \sum_{j=1}^{i} \lambda_j = \langle \mu - \lambda, \epsilon_1 + \cdots + \epsilon_i \rangle \geqslant 0, \quad \forall\, 1 \leqslant i \leqslant m.$$

Now consider the following subset of ${}^r\Lambda^+$

$$E = \{\mu \in {}^r\Lambda^+ \mid \mu = w \cdot \lambda \text{ for some } w \in \mathfrak{S}, \lambda \in \mathcal{P}_n\}.$$

Lemma 3.5.1. *The set E is finite.*

Proof. Since $\mathcal{P}_n$ is finite, it is enough to show that for each $\lambda \in \mathcal{P}_n$ the set $\mathfrak{S} \cdot \lambda \cap {}^r\Lambda^+$ is finite. Note that for $w \in \mathfrak{S}_0$ and $\tau \in \mathbb{Z}\Pi_0$ we have $z(w\tau \cdot \lambda) = z(\tau \cdot \lambda)$. By (3.1.4) we have

$$z(\tau \cdot \lambda) = z(\lambda) - \frac{\kappa}{2}(||\tau + \frac{\lambda + \rho}{\kappa}||^2 - ||\frac{\lambda + \rho}{\kappa}||^2).$$

If $z(\tau \cdot \lambda) \leqslant r$, then

$$||\tau + \frac{\lambda + \rho}{\kappa}||^2 \leqslant \frac{2}{-\kappa}(r - z(\lambda)) + ||\frac{\lambda + \rho}{\kappa}||^2.$$

There exists only finitely many $\tau \in \mathbb{Z}\Pi_0$ which satisfies this condition, hence the set E is finite. $\square$

Let $\mathcal{E}_\kappa$ be the full subcategory of ${}^r\mathcal{O}_\kappa$ consisting of objects M such that

$$\mu \in {}^r\Lambda^+, \ \mu \notin E \quad \Longrightarrow \quad \mathrm{Hom}_{{}^r\mathcal{O}_\kappa}({}^rP_\kappa(\mu), M) = 0.$$

Note that since ${}^rP_\kappa(\mu)$ is projective in ${}^r\mathcal{O}_\kappa$, an object $M \in {}^r\mathcal{O}_\kappa$ is in $\mathcal{E}_\kappa$ if and only if each simple subquotient of M is isomorphic to $L_\kappa(\mu)$ for $\mu \in E$. In particular $\mathcal{E}_\kappa$ is abelian and it is a Serre subcategory of ${}^r\mathcal{O}_\kappa$. Further $\mathcal{E}_\kappa$ is also an artinian category. In fact, each object $M \in \mathcal{E}_\kappa$ has a finite length because E is finite and for each $\mu \in E$ the multiplicity of $L_\kappa(\mu)$ in M is finite because $\dim_{\mathbb{C}} M_\mu < \infty$. Let $\mathfrak{g}'$ denote the Lie subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{g}' = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}.$$

Forgetting the ∂-action yields an equivalence of categories from $\mathcal{E}_\kappa$ to a category of $\mathfrak{g}'$-modules, see [Soe98, Proposition 8.1] for details. Since κ is negative, this category of $\mathfrak{g}'$-modules is equal to the category studied in [KL93, KL94a, KL94b].

Lemma 3.5.2. *(a) For $\lambda \in E$, $\mu \in {}^r\Lambda^+$ such that $[M_\kappa(\lambda) : L_\kappa(\mu)] \neq 0$ we have $\mu \in E$ and $\mu \preceq \lambda$.*

(b) The module ${}^rP_\kappa(\lambda)$ admits a filtration by $\mathcal{U}_\kappa$-modules

$${}^rP_\kappa(\lambda) = P_0 \supset P_1 \supset \cdots \supset P_l = 0$$

such that P_0/P_1 is isomorphic to $M_\kappa(\lambda)$ and $P_i/P_{i+1} \cong M_\kappa(\mu_i)$ for some $\mu_i \succ \lambda$.

(c) The category $\mathcal{E}_\kappa$ is a highest weight $\mathbb{C}$-category with standard objects $M_\kappa(\lambda)$, $\lambda \in E$. The indecomposable projective objects in $\mathcal{E}_\kappa$ are the modules ${}^rP_\kappa(\lambda)$ with $\lambda \in E$.

Proof. Let $\mathbf{U}_v$ be the quantized enveloping algebra of $\mathfrak{g}_0$ with the parameter $v = \exp(2\pi i/\kappa)$. Then the Kazhdan-Lusztig's tensor equivalence [KL93, Theorem IV.38.1] identifies $\mathcal{E}_\kappa$ with a full subcategory of the category of finite dimensional $\mathbf{U}_v$-modules. It maps the module $M_\kappa(\lambda)$ to the Weyl module of $\mathbf{U}_v$ with highest weight $a(\lambda)$. Since v is a root of unity, part (a) follows from the strong linkage principle for $\mathbf{U}_v$, see [And03, Theorem 3.1]. Part (b) follows from (a) and [KL93, Proposition I.3.9]. Finally, part (c) follows directly from parts (a), (b). □

Now, let us consider the deformed version. Let $\mathcal{E}_{\mathbf{k}}$ be the full subcategory of ${}^r\mathcal{O}_{\mathbf{k}}$ consisting of objects M such that

$$\mu \in {}^r\Lambda^+, \ \mu \notin E \quad \Longrightarrow \quad \mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), M) = 0.$$

Lemma 3.5.3. *An object $M \in {}^r\mathcal{O}_{\mathbf{k}}$ belongs to $\mathcal{E}_{\mathbf{k}}$ if and only if $M(\wp)$ belongs to $\mathcal{E}_\kappa$. In particular, we have $M_{\mathbf{k}}(\lambda)$ and ${}^rP_{\mathbf{k}}(\lambda)$ belong to $\mathcal{E}_{\mathbf{k}}$ for $\lambda \in E$.*

Proof. By Lemma 3.4.1(a) for any $\mu \in {}^r\Lambda^+$ the R-module $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), M)$ is finitely generated and we have

$$\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), M)(\wp) = \mathrm{Hom}_{{}^r\mathcal{O}_\kappa}({}^rP_\kappa(\mu), M(\wp)).$$

Therefore $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), M)$ is nonzero if and only if $\mathrm{Hom}_{{}^r\mathcal{O}_\kappa}({}^rP_\kappa(\mu), M(\wp))$ is nonzero by Nakayama's lemma. So the first statement follows from the definition of $\mathcal{E}_{\mathbf{k}}$ and $\mathcal{E}_\kappa$. The rest follows from Lemma 3.5.2(c). □

Let

$$P_{\mathbf{k}}(E) = \bigoplus_{\lambda \in E} {}^rP_{\mathbf{k}}(\lambda), \quad P_\kappa(E) = \bigoplus_{\lambda \in E} {}^rP_\kappa(\lambda).$$

We have the following corollary.

Corollary 3.5.4. *(a) The category $\mathcal{E}_{\mathbf{k}}$ is abelian.*

(b) For $M \in \mathcal{E}_{\mathbf{k}}$ there exists a positive integer d and a surjective map

$$P_{\mathbf{k}}(E)^{\oplus d} \longrightarrow M.$$

(c) The functor $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}(P_{\mathbf{k}}(E), -)$ *yields an equivalence of R-categories*

$$\mathcal{E}_{\mathbf{k}} \cong \mathrm{End}_{{}^r\mathcal{O}_{\mathbf{k}}}(P_{\mathbf{k}}(E))^{\mathrm{op}}\text{-mod}\,.$$

Proof. Let $M \in \mathcal{E}_{\mathbf{k}}$, $N \in {}^r\mathcal{O}_{\mathbf{k}}$. First assume that $N \subset M$. For $\mu \in {}^r\Lambda^+$ if $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), N) \neq 0$, then $\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}({}^rP_{\mathbf{k}}(\mu), M) \neq 0$, so μ belongs to E. Hence N belongs to $\mathcal{E}_{\mathbf{k}}$. Now, if N is a quotient of M, then $N(\wp)$ is a quotient of $M(\wp)$. Since $M(\wp)$ belongs to $\mathcal{E}_\kappa$, we also have $N(\wp) \in \mathcal{E}_\kappa$. Hence N belongs to $\mathcal{E}_{\mathbf{k}}$ by Lemma 3.5.3. This proves part (a). Let us concentrate on (b). Since $M \in \mathcal{E}_{\mathbf{k}}$ we have $M(\wp) \in \mathcal{E}_\kappa$. The category $\mathcal{E}_\kappa$ is artinian with $P_\kappa(E)$ a projective generator. Hence there exists a positive integer d and a surjective map

$$f : P_\kappa(E)^{\oplus d} \longrightarrow M(\wp).$$

Since $P_{\mathbf{k}}(E)^{\oplus d}$ is projective in ${}^r\mathcal{O}_{\mathbf{k}}$, this map lifts to a map of $\mathcal{U}_{\mathbf{k}}$-modules $\tilde{f} : P_{\mathbf{k}}(E)^{\oplus d} \to M$ such that the following diagram commute

$$\begin{tikzcd} P_{\mathbf{k}}(E)^{\oplus d} \arrow[r, "\tilde{f}"] \arrow[d] & M \arrow[d] \\ P_\kappa(E)^{\oplus d} \arrow[r, two heads, "f"] & M(\wp). \end{tikzcd}$$

Now, since the map $\tilde{f}$ preserves weight spaces and all the weight spaces of $P_{\mathbf{k}}(E)^{\oplus r}$ and M are finitely generated R-modules, by Nakayama's lemma, the surjectivity of f implies that $\tilde{f}$ is surjective. This proves (b). Finally part (c) is a direct consequence of parts (a), (b) by Morita theory. $\square$

Proposition 3.5.5. *The category $\mathcal{E}_{\mathbf{k}}$ is a highest weight R-category with standard modules $M_{\mathbf{k}}(\mu)$, $\mu \in E$.*

Proof. Note that $\mathrm{End}_{{}^r\mathcal{O}_{\mathbf{k}}}(P_{\mathbf{k}}(E))^{\mathrm{op}}$ is a finite projective R-algebra by Lemmas 3.4.1, 3.4.2. Since $\mathcal{E}_\kappa$ is a highest weight $\mathbb{C}$-category by Lemma 3.5.2(c), the result follows from Proposition 2.3.2. $\square$

3.6 The highest weight category $\mathcal{A}_{\mathbf{k}}$

By definition $\mathcal{P}_n$ is a subset of E. Let $\mathcal{A}_{\mathbf{k}}$ be the full subcategory of $\mathcal{E}_{\mathbf{k}}$ consisting of the objects M such that

$$\mathrm{Hom}_{{}^r\mathcal{O}_{\mathbf{k}}}(M_{\mathbf{k}}(\lambda), M) = 0, \quad \forall\ \lambda \in E,\ \lambda \notin \mathcal{P}_n.$$

We define the subcategory $\mathcal{A}_\kappa$ of $\mathcal{E}_\kappa$ in the same way. Let

$$\Delta_{\mathbf{k}} = \{M_{\mathbf{k}}(\lambda) \,|\, \lambda \in \mathcal{P}_n\}, \quad \Delta_\kappa = \{M_\kappa(\lambda) \,|\, \lambda \in \mathcal{P}_n\}.$$

Recall that $E \subset {}^r\Lambda^+$ is equipped with the partial order $\preceq$, and that $\mathcal{P}_n \subset E$. We have the following lemma

Lemma 3.6.1. *The set $\mathcal{P}_n$ is an ideal in E, i.e., for $\lambda \in E$, $\mu \in \mathcal{P}_n$, if $\lambda \preceq \mu$ then we have $\lambda \in \mathcal{P}_n$.*

Proof. Let $\lambda \in E$ and $\mu \in \mathcal{P}_n$ and assume that $\lambda \preceq \mu$. Recall that $E \subset {}_\kappa\mathfrak{t}^*$, so we can write $a(\lambda) = \sum_{i=1}^m \lambda_i \epsilon_i$. Since $E \subset {}^r\Lambda^+$ we have $\lambda_i \in \mathbb{Z}$ and $\lambda_i \geqslant \lambda_{i+1}$. We need to show that $\lambda_m \in \mathbb{N}$. Since $\lambda \preceq \mu$ there exist $r_i \in \mathbb{N}$ such that $a(\mu) - a(\lambda) = \sum_{i=1}^{m-1} r_i \alpha_i$. Therefore we have $\lambda_m = \mu_m + r_{m-1} \geqslant 0$. □

Now, we can prove the following proposition.

Proposition 3.6.2. *The category $(\mathcal{A}_\mathbf{k}, \Delta_\mathbf{k})$ is a highest weight R-category with respect to the partial order $\trianglelefteq$ on $\mathcal{P}_n$. The highest weight category $(\mathcal{A}_\mathbf{k}(\wp), \Delta_\mathbf{k}(\wp))$ given by base change is equivalent to $(\mathcal{A}_\kappa, \Delta_\kappa)$.*

Proof. Since $\mathcal{E}_\mathbf{k}$ is a highest weight R-category and $\mathcal{P}_n$ is an ideal of E, [Rou08b, Proposition 4.14] implies that $(\mathcal{A}_\mathbf{k}, \Delta_\mathbf{k})$ is a highest weight R-category with respect to the partial order $\preceq$ on $\mathcal{P}_n$. By (3.5.2) this implies that $(\mathcal{A}_\mathbf{k}, \Delta_\mathbf{k})$ is also a highest weight R-category with respect to $\trianglelefteq$. Finally, the equivalence $\mathcal{A}_\mathbf{k}(\wp) \cong \mathcal{A}_\kappa$ follows from the equivalence $\mathcal{E}_\mathbf{k}(\wp) \cong \mathcal{E}_\kappa$ and loc. cit. □

3.7 Costandard objects of $\mathcal{A}_\mathbf{k}$

Consider the (contravariant) *duality* functor $\mathbf{D}$ on $\mathcal{O}_\mathbf{k}$ given by

$$\mathbf{D}M = \bigoplus_{\mu \in {}_\mathbf{k}\mathfrak{t}^*} \mathrm{Hom}_R(M_\mu, R), \tag{3.7.1}$$

where the action of $\mathcal{U}_\mathbf{k}$ on $\mathbf{D}M$ is given as in Section 3.3, with the module $M_\mathbf{k}(\lambda)$ there replaced by M. Similarly, we define the (contravariant) *duality* functor $\mathbf{D}$ on $\mathcal{O}_\kappa$ by

$$\mathbf{D}M = \bigoplus_{\mu \in {}_\kappa\mathfrak{t}^*} \mathrm{Hom}(M_\mu, \mathbb{C}), \tag{3.7.2}$$

with the $\mathcal{U}_\kappa$-action given in the same way. This functor fixes the simple modules in $\mathcal{O}_\kappa$. Hence it restricts to a duality functor on $\mathcal{A}_\kappa$, because $\mathcal{A}_\kappa$ is a Serre subcategory of $\mathcal{O}_\kappa$. Therefore $(\mathcal{A}_\kappa, \Delta_\kappa)$ is a highest weight category with duality in the sense of [CPS89]. It follows from [CPS89, Proposition 1.2] that the costandard module $M_\kappa(\lambda)^\vee$ in $\mathcal{A}_\kappa$ is isomorphic to $\mathbf{D}M_\kappa(\lambda)$.

Lemma 3.7.1. *The costandard module $M_\mathbf{k}(\lambda)^\vee$ in $\mathcal{A}_\mathbf{k}$ is isomorphic to $\mathbf{D}M_\mathbf{k}(\lambda)$ for any $\lambda \in \mathcal{P}_n$.*

Proof. By definition we have a canonical isomorphism

$$(\mathbf{D}M_\mathbf{k}(\lambda))(\wp) \cong \mathbf{D}(M_\kappa(\lambda)) \cong M_\kappa(\lambda)^\vee.$$

Recall from Lemma 3.2.1 that $M_\mathbf{k}(\lambda)_K$ is a simple $\mathcal{U}_{\mathbf{k},K}$-module. Therefore we have $(\mathbf{D}M_\mathbf{k}(\lambda))_K \cong M_\mathbf{k}(\lambda)_K$. So the lemma follows from Lemma 2.3.3 applied to the highest weight category $(\mathcal{A}_\mathbf{k}, \Delta_\mathbf{k})$ and the set $\{\mathbf{D}M_\mathbf{k}(\lambda) \mid \lambda \in \mathcal{P}_n\}$. □

3.8 Comparison of the Jantzen filtrations

By Definition 2.4.1 for any $\lambda \in \mathcal{P}_n$ there is a Jantzen filtration of $M_\kappa(\lambda)$ associated with the highest weight category $(\mathcal{A}_\mathbf{k}, \Delta_\mathbf{k})$. Lemma 3.7.1 implies that this Jantzen filtration coincides with the one given in Section 3.3.

3.9 The v-Schur algebra

In this section let R denote an arbitrary integral domain. Let $\mathbf{v}$ be an invertible element in R. The Hecke algebra $\mathscr{H}_{\mathbf{v}}$ over R is a R-algebra, which is free as a R-module with basis $\{T_w \,|\, w \in \mathfrak{S}_0\}$, the multiplication is given by

$$\begin{aligned} T_{w_1}T_{w_2} &= T_{w_1w_2}, &\quad \text{if } l(w_1w_2) = l(w_1) + l(w_2),\\ (T_{s_i}+1)(T_{s_i} - \mathbf{v}) &= 0, &\quad 1 \leqslant i \leqslant m-1. \end{aligned}$$

Next, recall that a composition of n is a sequence $\mu = (\mu_1, \ldots, \mu_d)$ of positive integers such that $\sum_{i=1}^{d} \mu_i = n$. Let $\mathcal{X}_n$ be the set of compositions of n. For $\mu \in \mathcal{X}_n$ let $\mathfrak{S}_\mu$ be the subgroup of $\mathfrak{S}_0$ generated by s_i for all $1 \leqslant i \leqslant d-1$ such that $i \neq \mu_1 + \cdots + \mu_j$ for any j. Write

$$x_\mu = \sum_{w \in \mathfrak{S}_\mu} T_w \quad \text{and} \quad y_\mu = \sum_{w\in\mathfrak{S}_\mu} (-\mathbf{v})^{-l(w)} T_w.$$

The *v-Schur algebra* $\mathbf{S}_{\mathbf{v}}$ of parameter $\mathbf{v}$ is the endomorphism algebra of the right $\mathscr{H}_{\mathbf{v}}$-module $\bigoplus_{\mu \in \mathcal{X}_n} x_\mu \mathscr{H}_{\mathbf{v}}$. We will abbreviate

$$\mathcal{A}_{\mathbf{v}} = \mathbf{S}_{\mathbf{v}}\text{-mod}\,.$$

Consider the composition ϖ of n such that $\varpi_i = 1$ for $1 \leqslant i \leqslant n$. Then $x_\varpi \mathscr{H}_{\mathbf{v}} = \mathscr{H}_{\mathbf{v}}$. So the Hecke algebra $\mathscr{H}_{\mathbf{v}}$ identifies with a subalgebra of $\mathbf{S}_{\mathbf{v}}$ via the canonical isomorphism $\mathscr{H}_{\mathbf{v}} \cong \mathrm{End}_{\mathscr{H}_{\mathbf{v}}}(\mathscr{H}_{\mathbf{v}})$.

For $\lambda \in \mathcal{P}_n$ let λ' be the transposed partition of λ. Let φ_λ be the element in $\mathbf{S}_{\mathbf{v}}$ given by $\varphi_\lambda(h) = x_\lambda h$ for $h \in x_\varpi \mathscr{H}_{\mathbf{v}}$ and $\varphi_\lambda(x_\mu \mathscr{H}_{\mathbf{v}}) = 0$ for any composition $\mu \neq \varpi$. Then there is a particular element $w_\lambda \in \mathfrak{S}_0$ associated with λ such that the *Weyl module* $W_{\mathbf{v}}(\lambda)$ is the left ideal in $\mathbf{S}_{\mathbf{v}}$ generated by the element

$$z_\lambda = \varphi_\lambda T_{w_\lambda} y_{\lambda'} \in \mathbf{S}_{\mathbf{v}}.$$

See [JM97] for details. We will write

$$\Delta_{\mathbf{v}} = \{W_{\mathbf{v}}(\lambda) \,|\, \lambda \in \mathcal{P}_n\}.$$

3.10 The Jantzen filtration of Weyl modules

Now, set again $R = \mathbb{C}[[s]]$. Fix

$$v = \exp(2\pi i/\kappa) \in \mathbb{C} \qquad \text{and} \qquad \mathbf{v} = \exp(2\pi i/\mathbf{k}) \in R.$$

Below we will consider the v-Schur algebra over $\mathbb{C}$ with the parameter v, and the v-Schur algebra over R with the parameter $\mathbf{v}$. The category $(\mathcal{A}_v, \Delta_v)$ is a highest weight $\mathbb{C}$-category. Write $L_v(\lambda)$ for the simple quotient of $W_v(\lambda)$. The canonical algebra isomorphism $\mathbf{S}_{\mathbf{v}}(\wp) \cong \mathbf{S}_v$ implies that $(\mathcal{A}_{\mathbf{v}}, \Delta_{\mathbf{v}})$ is a highest weight R-category and there is a canonical equivalence

$$(\mathcal{A}_{\mathbf{v}}(\wp), \Delta_{\mathbf{v}}(\wp)) \cong (\mathcal{A}_v, \Delta_v).$$

We define the Jantzen filtration $(J^i W_v(\lambda))$ of $W_v(\lambda)$ by applying Definition 2.4.1 to $(\mathcal{A}_{\mathbf{v}}, \Delta_{\mathbf{v}})$. This filtration coincides with the one defined in [JM97], because the contravariant form on $W_{\mathbf{v}}(\lambda)$ used in [JM97]'s definition is equivalent to a morphism from $W_{\mathbf{v}}(\lambda)$ to the dual standard module $W_{\mathbf{v}}(\lambda)^\vee = \mathrm{Hom}_R(W_{\mathbf{v}}(\lambda), R)$.

3.11 Equivalence of $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{A}_{\mathbf{v}}$

In this section we will show that the highest weight R-categories $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{A}_{\mathbf{v}}$ are equivalent. The proof uses rational double affine Hecke algebras and Suzuki's functor. Let us first give some reminders. Let $\mathfrak{h} = \mathbb{C}^n$, let $y_1, \ldots, y_n$ be its standard basis and $x_1, \ldots, x_n \in \mathfrak{h}^*$ be the dual basis. Let $\mathbf{H}_{1/\kappa}$ be the rational double affine Hecke algebra associated with $\mathfrak{S}_n$ with parameter $1/\kappa$. Recall from Definition I.2.1.1 that $\mathbf{H}_{1/\kappa}$ is the quotient of the smash product of the tensor algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ with $\mathbb{C}\mathfrak{S}_n$ by the relations

$$[y_i, x_i] = 1 + \frac{1}{\kappa}\sum_{j \neq i} s_{ij}, \quad [y_i, x_j] = \frac{-1}{\kappa} s_{ij}, \quad 1 \leqslant i, j \leqslant n,\ i \neq j.$$

Here s_{ij} denotes the element of $\mathfrak{S}_n$ that permutes i and j. Denote by $\mathcal{B}_\kappa$ the category $\mathcal{O}$ of $\mathbf{H}_{1/\kappa}$, see Section I.2.2. It is a highest weight $\mathbb{C}$-category. Let $\{B_\kappa(\lambda) \,|\, \lambda \in \mathcal{P}_n\}$ be the set of standard modules.

Now, let $V = \mathbb{C}^m$ be the dual of the vectorial representation of $\mathfrak{g}_0$. For any object M in $\mathcal{A}_\kappa$ consider the action of the Lie algebra $\mathfrak{g}_0 \otimes \mathbb{C}[z]$ on the vector space

$$T(M) = V^{\otimes n} \otimes M \otimes \mathbb{C}[\mathfrak{h}]$$

given by

$$(\xi \otimes z^a)(v \otimes m \otimes f) = \sum_{i=1}^{n} \xi_{(i)}(v) \otimes m \otimes x_i^a f + v \otimes (-1)^a (\xi \otimes t^{-a}) m \otimes f$$

for $\xi \in \mathfrak{g}_0$, $a \in \mathbb{N}$, $v \in V^{\otimes n}$, $m \in M$, $f \in \mathbb{C}[\mathfrak{h}]$. Here $\xi_{(i)}$ is the operator on $V^{\otimes n}$ that acts on the i-th copy of V by ξ and acts on the other copies of V by identity. Suzuki defined a natural action of $\mathbf{H}_{1/\kappa}$ on the space of coinvariants

$$\mathfrak{E}_\kappa(M) = H_0(\mathfrak{g}_0 \otimes \mathbb{C}[z], T(M)).$$

The assignment $M \mapsto \mathfrak{E}_\kappa(M)$ gives a right exact functor

$$\mathfrak{E}_\kappa : \mathcal{A}_\kappa \to \mathcal{B}_\kappa.$$

See [Suz06] or [VV08, Section 2] for details. We have

$$\mathfrak{E}_\kappa(M_\kappa(\lambda)) = B_\kappa(\lambda),$$

and $\mathfrak{E}_\kappa$ is an equivalence of highest weight categories [VV08, Theorem A.6.1].

Next, we consider the rational double affine Hecke algebra $\mathbf{H}_{1/\mathbf{k}}$ over R with parameter $1/\mathbf{k}$. The category $\mathcal{O}$ of $\mathbf{H}_{1/\mathbf{k}}$ is defined in the obvious way. It is a highest weight R-category. We will denote it by $\mathcal{B}_{\mathbf{k}}$. The standard modules will be denoted by $B_{\mathbf{k}}(\lambda)$. The Suzuki functor over R

$$\mathfrak{E}_{\mathbf{k}} : \mathcal{A}_{\mathbf{k}} \to \mathcal{B}_{\mathbf{k}}, \quad M \mapsto H_0(\mathfrak{g}_0 \otimes \mathbb{C}[z], T(M))$$

is defined in the same way. It has the following properties.

Lemma 3.11.1. *(a) We have $\mathfrak{E}_{\mathbf{k}}(M_{\mathbf{k}}(\lambda)) = B_{\mathbf{k}}(\lambda)$ for $\lambda \in \mathcal{P}_n$.*

(b) The functor $\mathfrak{E}_{\mathbf{k}}$ restricts to an exact functor $\mathcal{A}_{\mathbf{k}}^{\Delta} \to \mathcal{B}_{\mathbf{k}}^{\Delta}$.

(c) The functor $\mathfrak{E}_{\mathbf{k}}$ maps a projective generator of $\mathcal{A}_{\mathbf{k}}$ to a projective generator of $\mathcal{B}_{\mathbf{k}}$.

Proof. The proof of part (a) is the same as in the nondeformed case. For part (b), since $\mathfrak{E}_{\mathbf{k}}$ is right exact over $\mathcal{A}_{\mathbf{k}}$, it is enough to prove that for any injective morphism $f : M \to N$ with M, $N \in \mathcal{A}_{\mathbf{k}}^{\Delta}$ the map

$$\mathfrak{E}_{\mathbf{k}}(f) : \mathfrak{E}_{\mathbf{k}}(M) \to \mathfrak{E}_{\mathbf{k}}(N)$$

is injective. Recall from Lemma 3.2.1 that the $\mathcal{U}_{\mathbf{k},K}$-module $M_{\mathbf{k}}(\lambda)_K$ is simple for any λ. So the functor

$$\mathfrak{E}_{\mathbf{k},K} : \mathcal{A}_{\mathbf{k},K} \to \mathcal{B}_{\mathbf{k},K}$$

is an equivalence. Hence the map

$$\mathfrak{E}_{\mathbf{k}}(f) \otimes_R K : \mathfrak{E}_{\mathbf{k},K}(M_K) \to \mathfrak{E}_{\mathbf{k},K}(N_K)$$

is injective. Since both $\mathfrak{E}_{\mathbf{k}}(M)$ and $\mathfrak{E}_{\mathbf{k}}(N)$ are free R-modules, this implies that $\mathfrak{E}_{\mathbf{k}}(f)$ is also injective. Now, let us concentrate on (c). Let P be a projective generator of $\mathcal{A}_{\mathbf{k}}$. Then $P(\wp)$ is a projective generator of $\mathcal{A}_{\kappa}$. Since $\mathfrak{E}_{\kappa}$ is an equivalence of categories, we have $\mathfrak{E}_{\kappa}(P(\wp))$ is a projective generator of $\mathcal{B}_{\kappa}$. By (b) the object $\mathfrak{E}_{\mathbf{k}}(P)$ belongs to $\mathcal{B}_{\mathbf{k}}^{\Delta}$, so it is free over R. Therefore by the Universal Coefficient Theorem we have

$$(\mathfrak{E}_{\mathbf{k}}(P))(\wp) \cong \mathfrak{E}_{\kappa}(P(\wp)).$$

Hence $\mathfrak{E}_{\mathbf{k}}(P)$ is a projective object of $\mathcal{B}_{\mathbf{k}}$. Note that for any $\lambda \in \mathcal{P}_n$ there is a surjective map $P \to M_{\mathbf{k}}(\lambda)$. The right exact functor $\mathfrak{E}_{\mathbf{k}}$ sends it to a surjective map $\mathfrak{E}_{\mathbf{k}}(P) \to B_{\mathbf{k}}(\lambda)$. This proves that $\mathfrak{E}_{\mathbf{k}}(P)$ is a projective generator of $\mathcal{B}_{\mathbf{k}}$. □

Proposition 3.11.2. *Assume that $\kappa \leqslant -3$. Then there exists an equivalence of highest weight R-categories*

$$\mathcal{A}_{\mathbf{k}} \xrightarrow{\sim} \mathcal{A}_{\mathbf{v}},$$

which maps $M_{\mathbf{k}}(\lambda)$ to $W_{\mathbf{v}}(\lambda)$ for any $\lambda \in \mathcal{P}_n$.

Proof. We first give an equivalence of highest weight categories

$$\Phi : \mathcal{A}_{\mathbf{k}} \to \mathcal{B}_{\mathbf{k}}$$

as follows. Let P be a projective generator of $\mathcal{A}_{\mathbf{k}}$. Then $Q = \mathfrak{E}_{\mathbf{k}}(P)$ is a projective generator of $\mathcal{B}_{\mathbf{k}}$ by Lemma 3.11.1(c). By Morita theory we have equivalences of categories

$$\mathrm{Hom}_{\mathcal{A}_{\mathbf{k}}}(P,-) : \mathcal{A}_{\mathbf{k}} \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P)^{\mathrm{op}}\text{-mod},$$
$$\mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q,-) : \mathcal{B}_{\mathbf{k}} \xrightarrow{\sim} \mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q)^{\mathrm{op}}\text{-mod}.$$

We claim that the algebra homomorphism

$$\mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P) \to \mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q), \quad f \mapsto \mathfrak{E}_{\mathbf{k}}(f), \tag{3.11.1}$$

is an isomorphism. To see this, note that we have

$$Q(\wp) = \mathfrak{E}_{\kappa}(P(\wp)), \quad (\mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P))(\wp) = \mathrm{End}_{\mathcal{A}_{\kappa}}(P(\wp)), \quad (\mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q))(\wp) = \mathrm{End}_{\mathcal{B}_{\kappa}}(Q(\wp)).$$

Since $\mathfrak{E}_{\kappa}$ is an equivalence, it yields an isomorphism

$$\mathrm{End}_{\mathcal{A}_{\kappa}}(P(\wp)) \xrightarrow{\sim} \mathrm{End}_{\mathcal{B}_{\kappa}}(Q(\wp)), \quad f \mapsto \mathfrak{E}_{\kappa}(f).$$

Since both $\mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P)$ and $\mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q)$ are finitely generated free R-modules, by Nakayama's lemma the morphism (3.11.1) is an isomorphism. In particular, it yields an equivalence of categories

$$\mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P)^{\mathrm{op}}\text{-mod} \cong \mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q)^{\mathrm{op}}\text{-mod}\,.$$

Combined with the other two equivalences above, we get an equivalence of categories

$$\Phi : \mathcal{A}_{\mathbf{k}} \to \mathcal{B}_{\mathbf{k}}.$$

It remains to show that

$$\Phi(M_{\mathbf{k}}(\lambda)) \cong B_{\mathbf{k}}(\lambda), \quad \lambda \in \mathcal{P}_n.$$

Note that the functor $\mathfrak{E}_{\mathbf{k}}$ yields a morphism of finitely generated R-modules

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, \Phi(M_{\mathbf{k}}(\lambda))) &= \mathrm{End}_{\mathcal{B}_{\mathbf{k}}}(Q)^{\mathrm{op}} \otimes_{\mathrm{End}_{\mathcal{A}_{\mathbf{k}}}(P)^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}_{\mathbf{k}}}(P, M_{\mathbf{k}}(\lambda)) \\
&\to \mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, \mathfrak{E}_{\mathbf{k}}(M_{\mathbf{k}}(\lambda))) \\
&= \mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, B_{\mathbf{k}}(\lambda)).
\end{aligned}$$

Let us denote it by φ. Note also that we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}_{\mathbf{k}}}(P, M_{\mathbf{k}}(\lambda))(\wp) &= \mathrm{Hom}_{\mathcal{A}_{\kappa}}(P(\wp), M_{\kappa}(\lambda)), \\
\mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, B_{\mathbf{k}}(\lambda))(\wp) &= \mathrm{Hom}_{\mathcal{B}_{\kappa}}(Q(\wp), B_{\kappa}(\lambda)),
\end{aligned}$$

and note that $\mathfrak{E}_{\kappa}$ is an equivalence of categories. So the map $\varphi(\wp)$ is an isomorphism. Further $\mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, B_{\mathbf{k}}(\lambda))$ is free over R, so Nakayama's lemma implies that φ is also an isomorphism. The preimage of φ under the equivalence $\mathrm{Hom}_{\mathcal{B}_{\mathbf{k}}}(Q, -)$ yields an isomorphism

$$\Phi(M_{\mathbf{k}}(\lambda)) \simeq B_{\mathbf{k}}(\lambda).$$

Finally, if $v \neq -1$, i.e., $\kappa \leqslant -3$, then by [Rou08b, Theorem 6.8] the categories $\mathcal{B}_{\mathbf{k}}$ and $\mathcal{A}_{\mathbf{v}}$ are equivalent highest weight R-categories with $B_{\mathbf{k}}(\lambda)$ corresponding to $W_{\mathbf{v}}(\lambda)$. This equivalence composed with Φ gives the desired equivalence in the proposition. □

Corollary 3.11.3. *Assume that $\kappa \leqslant -3$. Then for any λ, $\mu \in \mathcal{P}_n$ and $i \in \mathbb{N}$ we have*

$$[J^i M_{\kappa}(\lambda)/J^{i+1} M_{\kappa}(\lambda) : L_{\kappa}(\mu)] = [J^i W_v(\lambda)/J^{i+1} W_v(\lambda) : L_v(\mu)]. \tag{3.11.2}$$

Proof. This follows from the proposition above and Proposition 2.5.1. □

To prove the main theorem, it remains to compute the left hand side of (3.11.2). This will be done by generalizing the approach of [BB93] to the affine parabolic case. To this end, we first give some reminders on $\mathscr{D}$-modules on affine flag varieties.

4 Generalities on $\mathscr{D}$-modules on ind-schemes

In this section, we first recall basic notion for $\mathscr{D}$-modules on (possibly singular) schemes. We will also discuss twisted $\mathscr{D}$-modules and holonomic $\mathscr{D}$-modules. Then we introduce the notion of $\mathscr{D}$-modules on ind-schemes following [BD00] and [KV04].

4.1 Reminders on $\mathscr{D}$-modules

Unless specified otherwise, all the schemes will be assumed to be of finite type over $\mathbb{C}$, quasi-separated and quasi-projective. Although a large number of statements are true in a larger generality, we will only use them for quasi-projective schemes. For any scheme Z, let $\mathscr{O}_Z$ be the structure sheaf over Z. We write $\mathbf{O}(Z)$ for the category of quasi-coherent $\mathscr{O}_Z$-modules on Z. Note that we abbreviate $\mathscr{O}_Z$-module for sheaf of $\mathscr{O}_Z$-modules over Z. For $f : Z \to Y$ a morphism of schemes, we write f_*, f^* for the functors of direct and inverse images on $\mathbf{O}(Z)$, $\mathbf{O}(Y)$. If f is a closed embedding and $\mathscr{M} \in \mathbf{O}(Y)$, we consider the quasi-coherent $\mathscr{O}_Z$-module

$$f^! \mathscr{M} = f^{-1}\, \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_Z, \mathscr{M}).$$

It is the restriction to Z of the subsheaf of $\mathscr{M}$ consisting of sections supported scheme-theoretically on $f(Z) \subset Y$.

Let Z be a smooth scheme. Let $\mathscr{D}_Z$ be the ring of differential operators on Z. We denote by $\mathbf{M}(Z)$ the category of right $\mathscr{D}_Z$-modules that are quasi-coherent as $\mathscr{O}_Z$-modules. It is an abelian category. Let Ω_Z denote the sheaf of differential forms of highest degree on Z. The category of right $\mathscr{D}_Z$-modules is equivalent to the category of left $\mathscr{D}_Z$-modules via $\mathscr{M} \mapsto \Omega_Z \otimes_{\mathscr{O}_Z} \mathscr{M}$. Let $i : Y \to Z$ be a morphism of smooth schemes. We consider the $(\mathscr{D}_Y, i^{-1}\mathscr{D}_Z)$-bimodule

$$\mathscr{D}_{Y\to Z} = i^*\mathscr{D}_Z = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_Z} i^{-1}\mathscr{D}_Z.$$

We define the following functors

$$\begin{aligned} i^* : \mathbf{M}(Z) \to \mathbf{M}(Y), &\qquad \mathscr{M} \mapsto \Omega_Y \otimes_{\mathscr{O}_Y} \big(\mathscr{D}_{Y\to Z} \otimes_{\mathscr{D}_Z} (\Omega_Z \otimes_{\mathscr{O}_Z} \mathscr{M})\big), \\ i_\bullet : \mathbf{M}(Y) \to \mathbf{M}(Z), &\qquad \mathscr{M} \mapsto i_*(\mathscr{M} \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to Z}). \end{aligned}$$

For any $\mathscr{M} \in \mathbf{M}(Y)$ let $\mathscr{M}^{\mathscr{O}}$ denote the underlying $\mathscr{O}_Y$-module of $\mathscr{M}$. Then we have

$$i^*(\mathscr{M}^{\mathscr{O}}) = i^*(\mathscr{M})^{\mathscr{O}}.$$

If the morphism i is a locally closed affine embedding, then the functor $i_\bullet$ is exact. For any closed subscheme Z' of Z, we denote by $\mathbf{M}(Z, Z')$ the full subcategory of $\mathbf{M}(Z)$ consisting of $\mathscr{D}_Z$-modules supported set-theoretically on Z'. If $i : Y \to Z$ is a closed embedding of smooth varieties, then by a theorem of Kashiwara, the functor $i_\bullet$ yields an equivalence of categories

$$\mathbf{M}(Y) \cong \mathbf{M}(Z, Y). \tag{4.1.1}$$

For more details on $\mathscr{D}$-modules on smooth schemes, see [HTT08] for instance.

Now, let Z be a possibly singular scheme. We consider the abelian category $\mathbf{M}(Z)$ of right $\mathscr{D}$-modules on Z with a faithful *forgetful* functor

$$\mathbf{M}(Z) \to \mathbf{O}(Z), \quad \mathscr{M} \mapsto \mathscr{M}^{\mathscr{O}}$$

as in [BD00, 7.10.3]. If Z is smooth, it is equivalent to the category $\mathbf{M}(Z)$ above, see [BD00, 7.10.12]. For any closed embedding $i : Z \to X$ there is a left exact functor

$$i^! : \mathbf{M}(X) \to \mathbf{M}(Z)$$

such that $(i^!(\mathscr{M}))^{\mathscr{O}} = i^!(\mathscr{M}^{\mathscr{O}})$ for all $\mathscr{M}$. It admits an exact left adjoint functor

$$i_\bullet : \mathbf{M}(Z) \to \mathbf{M}(X).$$

In the smooth case these functors coincide with the one before. If X is smooth, then $i_\bullet$ and $i^!$ yield mutually inverse equivalences of categories

$$\mathbf{M}(Z) \cong \mathbf{M}(X, Z). \tag{4.1.2}$$

such that $i^! \circ i_\bullet = \mathrm{Id}$, see [BD00, 7.10.11]. Note that when Z is smooth, this is Kashiwara's equivalence (4.1.1). In this chapter, we will always consider $\mathscr{D}$-modules on a (possibly singular) scheme Z embedded into a smooth scheme. Finally, if $j : Y \to Z$ is a locally closed affine embedding and Y is smooth, then we have the following exact functor

$$j_\bullet = i^! \circ (i \circ j)_\bullet : \mathbf{M}(Y) \to \mathbf{M}(Z). \tag{4.1.3}$$

Its definition is independent of the choice of i.

4.2 Holonomic $\mathscr{D}$-modules

Let Z be a scheme. If Z is smooth, we denote by $\mathbf{M}_h(Z)$ the category of holonomic $\mathscr{D}_Z$-modules, see e.g., [HTT08, Definition 2.3.6]. Otherwise, let $i : Z \to X$ be a closed embedding into a smooth scheme X. We define $\mathbf{M}_h(Z)$ to be the full subcategory of $\mathbf{M}(Z)$ consisting of objects $\mathscr{M}$ such that $i_\bullet \mathscr{M}$ is holonomic. The category $\mathbf{M}_h(Z)$ is abelian. There is a (contravariant) *duality* functor on $\mathbf{M}_h(Z)$ given by

$$\mathbb{D} : \mathbf{M}_h(Z) \to \mathbf{M}_h(Z), \quad \mathscr{M} \mapsto i^!(\Omega_X \otimes_{\mathscr{O}_X} \mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(i_\bullet \mathscr{M}, \mathscr{D}_X)).$$

For a locally closed affine embedding $i : Y \to Z$ with Y a smooth scheme, the functor $i_\bullet$ given by (4.1.3) maps $\mathbf{M}_h(Y)$ to $\mathbf{M}_h(Z)$. We put

$$i_! = \mathbb{D} \circ i_\bullet \circ \mathbb{D} : \ \mathbf{M}_h(Y) \to \mathbf{M}_h(Z).$$

There is a canonical morphism of functors

$$\psi : i_! \to i_\bullet.$$

The *intermediate extension* functor is given by

$$i_{!\bullet} : \mathbf{M}_h(Y) \to \mathbf{M}_h(Z), \quad \mathscr{M} \mapsto \mathrm{Im}(\psi(\mathscr{M}) : i_! \mathscr{M} \to i_\bullet \mathscr{M}).$$

Let us give some properties of these functors, see e.g. [HTT08] for details.

Lemma 4.2.1. *Let Y be a smooth scheme and let $i : Y \to Z$ be a locally closed affine embedding.*

(a) The functors $i_\bullet$, $i_!$ are exact.

(b) If i is a closed embedding, then ψ is an isomorphism of functors $i_! \cong i_\bullet$.

(c) If i is an open embedding and the scheme Z is smooth, then the functor $i^!$ is exact. Further we have $i^! = i^*$ and

$$(i_!, i^! = i^*, i_\bullet)$$

form a triple of adjoint functors between the categories $\mathbf{M}_h(Y)$ and $\mathbf{M}_h(Z)$. Finally, for $\mathscr{M} \in \mathbf{M}_h(Z)$ we have

$$(i_\bullet(\mathscr{M}))^{\mathscr{O}} = i_*(\mathscr{M}^{\mathscr{O}}).$$

4.3 Weakly equivariant $\mathscr{D}$-modules

Let T be a linear group. For any T-scheme Z there is an abelian category $\mathbf{M}^T(Z)$ of weakly T-equivariant right $\mathscr{D}$-modules on Z with a faithful forgetful functor

$$\mathbf{M}^T(Z) \to \mathbf{M}(Z). \tag{4.3.1}$$

If Z is smooth, an object $\mathscr{M}$ of $\mathbf{M}^T(Z)$ is an object $\mathscr{M}$ of $\mathbf{M}(Z)$ equipped with a structure of T-equivariant $\mathscr{O}_Z$-module such that the action map $\mathscr{M} \otimes_{\mathscr{O}_Z} \mathscr{D}_Z \to \mathscr{M}$ is T-equivariant. For any T-scheme Z with a T-equivariant closed embedding $i : Z \to X$ into a smooth T-scheme X, the functor $i_\bullet$ yields an equivalence $\mathbf{M}^T(Z) \cong \mathbf{M}^T(X, Z)$, where $\mathbf{M}^T(X, Z)$ is the subcategory of $\mathbf{M}^T(X)$ consisting of objects supported set-theoretically on Z.

4.4 Twisted $\mathscr{D}$-modules

Let T be a torus, and let $\mathfrak{t}$ be its Lie algebra. Let $\pi : Z^\dagger \to Z$ be a right T-torsor over the scheme Z. For any object $\mathscr{M} \in \mathbf{M}^T(Z^\dagger)$ the $\mathscr{O}_Z$-module $\pi_*(\mathscr{M}^{\mathscr{O}})$ carries a T-action. Let

$$\mathscr{M}^\dagger = \pi_*(\mathscr{M}^{\mathscr{O}})^T$$

be the $\mathscr{O}_Z$-submodule of $\pi_*(\mathscr{M}^{\mathscr{O}})$ consisting of the T-invariant local sections. We have

$$\Gamma(Z, \mathscr{M}^\dagger) = \Gamma(Z^\dagger, \mathscr{M})^T.$$

For any weight $\lambda \in \mathfrak{t}^*$ we define the categories $\mathbf{M}^{\tilde{\lambda}}(Z)$, $\mathbf{M}^\lambda(Z)$ as follows.

First, assume that Z is a smooth scheme. Then $Z^\dagger$ is also smooth. So we have a sheaf of algebras on Z given by

$$\mathscr{D}_Z^\dagger = (\mathscr{D}_{Z^\dagger})^\dagger,$$

and $\mathscr{M}^\dagger$ is a right $\mathscr{D}_Z^\dagger$-module for any $\mathscr{M} \in \mathbf{M}^T(Z^\dagger)$. For any open subscheme $U \subset Z$ the T-action on $\pi^{-1}(U)$ yields an algebra homomorphism

$$\delta_r : \mathcal{U}(\mathfrak{t}) \to \Gamma(U, \mathscr{D}_Z^\dagger), \tag{4.4.1}$$

whose image lies in the center of the right hand side. Thus there is also an action of $\mathcal{U}(\mathfrak{t})$ on $\mathscr{M}^\dagger$ commuting with the $\mathscr{D}_Z^\dagger$-action. For $\lambda \in \mathfrak{t}^*$ let $\mathfrak{m}_\lambda \subset \mathcal{U}(\mathfrak{t})$ be the ideal generated by

$$\{h + \lambda(h) \mid h \in \mathfrak{t}\}.$$

We define $\mathbf{M}^\lambda(Z)$ (resp. $\mathbf{M}^{\tilde{\lambda}}(Z)$) to be the full subcategory of $\mathbf{M}^T(Z^\dagger)$ consisting of the objects $\mathscr{M}$ such that the action of $\mathfrak{m}_\lambda$ on $\mathscr{M}^\dagger$ is zero (resp. nilpotent). In particular $\mathbf{M}^\lambda(Z)$ is a full subcategory of $\mathbf{M}^{\tilde{\lambda}}(Z)$ and both categories are abelian. We will write

$$\Gamma(Z, \mathscr{M}) \;=\; \Gamma(Z, \mathscr{M}^\dagger), \quad \forall\, \mathscr{M} \in \mathbf{M}^{\tilde{\lambda}}(Z). \tag{4.4.2}$$

Now, let Z be any scheme. We say that a T-torsor $\pi : Z^\dagger \to Z$ is *admissible* if there exists a T-torsor $X^\dagger \to X$ with X smooth and a closed embedding $i : Z \to X$ such that $Z^\dagger \cong X^\dagger \times_X Z$ as a T-scheme over Z. We will only use admissible T-torsors. Let $\mathbf{M}^\lambda(X, Z)$, $\mathbf{M}^{\tilde{\lambda}}(X, Z)$ be respectively the subcategories of $\mathbf{M}^\lambda(X)$, $\mathbf{M}^{\tilde{\lambda}}(X)$ consisting of objects supported on $Z^\dagger$. We define $\mathbf{M}^\lambda(Z)$, $\mathbf{M}^{\tilde{\lambda}}(Z)$ to be the full subcategories of $\mathbf{M}^T(Z^\dagger)$ consisting of objects $\mathscr{M}$ such that $i_\bullet(\mathscr{M})$ belongs to $\mathbf{M}^\lambda(X, Z)$, $\mathbf{M}^{\tilde{\lambda}}(X, Z)$ respectively. Their definition only depends on the T-torsor π.

Remark 4.4.1. Let Z be a smooth scheme. Let $\mathbf{M}(\mathscr{D}^\dagger_Z)$ be the category of right $\mathscr{D}^\dagger_Z$-modules on Z that are quasi-coherent as $\mathscr{O}_Z$-modules. The functor

$$\mathbf{M}^T(Z^\dagger) \xrightarrow{\sim} \mathbf{M}(\mathscr{D}^\dagger_Z), \quad \mathscr{M} \mapsto \mathscr{M}^\dagger \tag{4.4.3}$$

is an equivalence of categories. A quasi-inverse is given by π^*, see e.g., [BB93, Lemma 1.8.10].

Remark 4.4.2. We record the following fact for a further use. For any smooth T-torsor $\pi : Z^\dagger \to Z$, the exact sequence of relative differential 1-forms

$$\pi^*(\Omega^1_Z) \longrightarrow \Omega^1_{Z^\dagger} \longrightarrow \Omega^1_{Z^\dagger/Z} \longrightarrow 0$$

yields an isomorphism

$$\Omega_{Z^\dagger} = \pi^*(\Omega_Z) \otimes_{\mathscr{O}_{Z^\dagger}} \Omega_{Z^\dagger/Z}.$$

Since π is a T-torsor we have indeed

$$\Omega_{Z^\dagger/Z} = \mathscr{O}_{Z^\dagger}$$

as a line bundle. Therefore we have an isomorphism of $\mathscr{O}_{Z^\dagger}$-modules

$$\Omega_{Z^\dagger} = \pi^*(\Omega_Z).$$

Below, we will identify them whenever needed.

4.5 Twisted holonomic $\mathscr{D}$-modules and duality functors

Let $\pi : Z^\dagger \to Z$ be an admissible T-torsor. We define $\mathbf{M}^T_h(Z^\dagger)$ to be the full subcategory of $\mathbf{M}^T(Z^\dagger)$ consisting of objects $\mathscr{M}$ whose image via the functor (4.3.1) belongs to $\mathbf{M}_h(Z^\dagger)$. We define the categories $\mathbf{M}^\lambda_h(Z)$, $\mathbf{M}^{\tilde\lambda}_h(Z)$ in the same manner.

Assume that Z is smooth. Then the category $\mathbf{M}^T(Z^\dagger)$ has enough injective objects, see e.g., [Kas08, Proposition 3.3.5] and the references there. We define a (contravariant) duality functor on $\mathbf{M}^T_h(Z^\dagger)$ by

$$\mathbb{D}' : \mathbf{M}^T_h(Z^\dagger) \to \mathbf{M}^T_h(Z^\dagger), \quad \mathscr{M} \mapsto \Omega_{Z^\dagger} \otimes_{\mathscr{O}_{Z^\dagger}} \mathscr{E}xt^{\dim Z^\dagger}_{\mathbf{M}^T(Z^\dagger)}(\mathscr{M}, \mathscr{D}_{Z^\dagger}).$$

We may write $\mathbb{D}' = \mathbb{D}'_Z$. Note that by Remark 4.4.2 and the equivalence (4.4.3) we have

$$(\mathbb{D}'_Z\mathscr{M})^\dagger = \Omega_Z \otimes_{\mathscr{O}_Z} \mathscr{E}xt^{\dim Z^\dagger}_{\mathscr{D}^\dagger_Z}(\mathscr{M}^\dagger, \mathscr{D}^\dagger_Z), \quad \forall\, \mathscr{M} \in \mathbf{M}^T(Z^\dagger). \tag{4.5.1}$$

For any $\lambda \in \mathfrak{t}^*$ the functor $\mathbb{D}'$ restricts to (contravariant) equivalences of categories

$$\mathbb{D}' : \mathbf{M}^{\tilde\lambda}_h(Z) \to \mathbf{M}^{\widetilde{-\lambda}}_h(Z), \qquad \mathbb{D}' : \mathbf{M}^\lambda_h(Z) \to \mathbf{M}^{-\lambda}_h(Z), \tag{4.5.2}$$

see e.g., [BB93, Remark 2.5.5(iv)]. In particular, if $\lambda = 0$ then $\mathbb{D}'$ yields a duality on $\mathbf{M}^0_h(Z^\dagger)$. Further (4.4.3) yields an equivalence of categories

$$\Phi : \mathbf{M}^0_h(Z) \to \mathbf{M}_h(Z), \quad \mathscr{M} \mapsto \mathscr{M}^\dagger.$$

Recall the duality functor $\mathbb{D}$ on $\mathbf{M}_h(Z)$ defined in Section 4.2. The following result is standard.

Lemma 4.5.1. *We have $\Phi \circ \mathbb{D}' = \mathbb{D} \circ \Phi$.*

For an admissible T-torsor $\pi : Z^\dagger \to Z$ with an embedding i into a smooth T-torsor $X^\dagger \to X$, we define

$$\mathbb{D}' : \mathbf{M}_h^T(Z^\dagger) \to \mathbf{M}_h^T(Z^\dagger), \quad \mathscr{M} \mapsto i^! \mathbb{D}'_X(i_\bullet(\mathscr{M})).$$

The definition of $\mathbb{D}'$ only depends on π. The equivalence (4.5.2) and Lemma 4.5.1 hold again.

A weight $\lambda \in \mathfrak{t}^*$ is *integral* if it is given by the differential of a character $e^\lambda : T \to \mathbb{C}^*$. For an integral weight λ we consider the invertible sheaf $\mathscr{L}_Z^\lambda$ in $\mathbf{O}(Z)$ defined by

$$\Gamma(U, \mathscr{L}_Z^\lambda) = \{\gamma \in \Gamma(\pi^{-1}(U), \mathscr{O}_{Z^\dagger}) \,|\, \gamma(xh^{-1}) = e^\lambda(h)\gamma(x), \quad (x,h) \in \pi^{-1}(U) \times T\}$$

for any open set $U \subset Z$. Then we define the following *translation functor*

$$\Theta^\lambda : \mathbf{M}_h^T(Z^\dagger) \to \mathbf{M}_h^T(Z^\dagger), \quad \mathscr{M} \mapsto \mathscr{M} \otimes_{\mathscr{O}_{Z^\dagger}} \pi^*(\mathscr{L}_Z^\lambda).$$

It is an equivalence of categories. A quasi-inverse is given by $\Theta^{-\lambda}$. For any $\mu \in \mathfrak{t}^*$ the restriction of Θ^λ yields equivalences of categories

$$\Theta^\lambda : \mathbf{M}_h^{\tilde{\mu}}(Z) \to \mathbf{M}_h^{\widetilde{\mu+\lambda}}(Z), \quad \Theta^\lambda : \mathbf{M}_h^{\mu}(Z) \to \mathbf{M}_h^{\mu+\lambda}(Z). \tag{4.5.3}$$

We define the *duality* functor on $\mathbf{M}_h^{\tilde{\lambda}}(Z)$ to be

$$\mathbb{D} : \mathbf{M}_h^{\tilde{\lambda}}(Z) \to \mathbf{M}_h^{\tilde{\lambda}}(Z), \quad \mathscr{M} \mapsto \Theta^{2\lambda} \circ \mathbb{D}'(\mathscr{M}).$$

It restricts to a duality functor on $\mathbf{M}_h^{\lambda}(Z)$, which we denote again by $\mathbb{D}$. To avoid any confusion, we may write $\mathbb{D} = \mathbb{D}^\lambda$. The equivalence Θ^λ intertwines the duality functors, i.e., we have

$$\mathbb{D}^{\lambda+\mu} \circ \Theta^\lambda = \Theta^\lambda \circ \mathbb{D}^\mu. \tag{4.5.4}$$

For any locally closed affine embedding of T-torsors $i : Z \to Y$ with Z smooth, we define the functor

$$i_! = \mathbb{D} \circ i_\bullet \circ \mathbb{D} : \mathbf{M}_h^{\tilde{\lambda}}(Z) \to \mathbf{M}_h^{\tilde{\lambda}}(Y).$$

As in Section 4.2, we have a morphism of functors $\psi : i_! \to i_\bullet$ which is an isomorphism if i is a closed embedding. The intermediate extension functor $i_{!\bullet}$ is defined in the same way. Lemma 4.2.1 holds again.

Remark 4.5.2. Assume that Z is smooth. Let $\mathscr{M} \in \mathbf{M}_h^\lambda(Z)$. Put $\mu = 0$ in (4.5.3). Using the equivalence Φ we see that $\mathscr{M}^\dagger$ is a right module over the sheaf of algebras

$$\mathscr{D}_Z^\lambda = \mathscr{L}_Z^{-\lambda} \otimes_{\mathscr{O}_Z} \mathscr{D}_Z \otimes_{\mathscr{O}_Z} \mathscr{L}_Z^\lambda.$$

Further, we have

$$\mathbb{D}(\mathscr{M})^\dagger = \Omega_Z \otimes_{\mathscr{O}_Z} \otimes_{\mathscr{O}_Z} \mathscr{L}_Z^{2\lambda} \otimes_{\mathscr{O}_Z} \mathscr{E}xt^{\dim Z}_{\mathscr{D}_Z^\lambda}(\mathscr{M}^\dagger, \mathscr{D}_Z^\lambda)$$

by Lemma 4.5.1 and (4.5.4), compare [KT95, (2.1.2)].

4.6 Injective and projective limit of categories

Let us introduce the following notation. Let A be a filtering poset. For any inductive system of categories $(\mathcal{C}_\alpha)_{\alpha\in A}$ with functors $i_{\alpha\beta}: \mathcal{C}_\alpha \to \mathcal{C}_\beta$, $\alpha \leqslant \beta$, we denote by $2\varinjlim \mathcal{C}_\alpha$ its *inductive limit*, i.e., the category whose objects are pairs (α, M_α) with $\alpha \in A$, $M_\alpha \in \mathcal{C}_\alpha$ and

$$\mathrm{Hom}_{2\varinjlim \mathcal{C}_\alpha}((\alpha, M_\alpha), (\beta, N_\beta)) = \varinjlim_{\gamma \geqslant \alpha,\beta} \mathrm{Hom}_{\mathcal{C}_\gamma}(i_{\alpha\gamma}(M_\alpha), i_{\beta\gamma}(N_\beta)).$$

For any projective system of categories $(\mathcal{C}_\alpha)_{\alpha\in A}$ with functors $j_{\alpha\beta}: \mathcal{C}_\beta \to \mathcal{C}_\alpha$, $\alpha \leqslant \beta$, we denote by $2\varprojlim \mathcal{C}_\alpha$ its *projective limit*, i.e., the category whose objects are systems consisting of objects $M_\alpha \in \mathcal{C}_\alpha$ given for all $\alpha \in A$ and isomorphisms $j_{\alpha\beta}(M_\beta) \to M_\alpha$ for each $\alpha \leqslant \beta$ and satisfying the compatibility condition for each $\alpha \leqslant \beta \leqslant \gamma$. Morphisms are defined in the obvious way. See e.g., [KV04, 3.2, 3.3].

4.7 The $\mathscr{O}$-modules on ind-schemes

An *ind-scheme* X is a filtering inductive system of schemes $(X_\alpha)_{\alpha\in A}$ with closed embeddings $i_{\alpha\beta}: X_\alpha \to X_\beta$ for $\alpha \leqslant \beta$ such that X represents the ind-object "$\varinjlim$"X_α. See [KS94, 1.11] for details on ind-objects. Below we will simply write $\varinjlim$ for "$\varinjlim$", hoping this does not create any confusion. The categories $\mathbf{O}(X_\alpha)$ form a projective system via the functors $i^!_{\alpha\beta}: \mathbf{O}(X_\beta) \to \mathbf{O}(X_\alpha)$. Following [BD00, 7.11.4] and [KV04, 3.3] we define the category of $\mathscr{O}$-modules on X as

$$\mathbf{O}(X) = 2\varprojlim \mathbf{O}(X_\alpha).$$

It is an abelian category. An object $\mathscr{M}$ of $\mathbf{O}(X)$ is represented by

$$\mathscr{M} = (\mathscr{M}_\alpha,\ \varphi_{\alpha\beta}: i^!_{\alpha\beta}\mathscr{M}_\beta \to \mathscr{M}_\alpha)$$

where $\mathscr{M}_\alpha$ is an object of $\mathbf{O}(X_\alpha)$ and $\varphi_{\alpha\beta}$, $\alpha \leqslant \beta$, is an isomorphism in $\mathbf{O}(X_\alpha)$.

Note that any object $\mathscr{M}$ of $\mathbf{O}(X)$ is an inductive limit of objects from $\mathbf{O}(X_\alpha)$. More precisely, we first identify $\mathbf{O}(X_\alpha)$ as a full subcategory of $\mathbf{O}(X)$ in the following way: since the poset A is filtering, to any $\mathscr{M}_\alpha \in \mathbf{O}(X_\alpha)$ we may associate a canonical object $(\mathscr{N}_\beta)$ in $\mathbf{O}(X)$ such that $\mathscr{N}_\beta = i_{\alpha\beta *}(\mathscr{M}_\alpha)$ for $\alpha \leqslant \beta$ and the structure isomorphisms $\varphi_{\beta\gamma}$, $\beta \leqslant \gamma$, are the obvious ones. Let us denote this object in $\mathbf{O}(X)$ again by $\mathscr{M}_\alpha$. Given any object $\mathscr{M} \in \mathbf{O}(X)$ represented by $\mathscr{M} = (\mathscr{M}_\alpha, \varphi_{\alpha\beta})$, these $\mathscr{M}_\alpha \in \mathbf{O}(X)$, $\alpha \in A$, form an inductive system via the canonical morphisms $\mathscr{M}_\alpha \to \mathscr{M}_\beta$. Then, the ind-object $\varinjlim \mathscr{M}_\alpha$ of $\mathbf{O}(X)$ is represented by $\mathscr{M}$. So, we define the space of global sections of $\mathscr{M}$ to be the inductive limit of vector spaces

$$\Gamma(X, \mathscr{M}) = \varinjlim \Gamma(X_\alpha, \mathscr{M}_\alpha). \tag{4.7.1}$$

We will also use the category $\hat{\mathbf{O}}(X)$ defined as the limit of the projective system of categories $(\mathbf{O}(X_\alpha), i^*_{\alpha\beta})$, see [BD00, 7.11.3] or [KV04, 3.3]. Note that the canonical isomorphisms $i^*_{\alpha\beta}\mathscr{O}_{X_\beta} = \mathscr{O}_{X_\alpha}$ yield an object $(\mathscr{O}_{X_\alpha})_{\alpha\in A}$ in $\hat{\mathbf{O}}(X)$. We denote this object by $\mathscr{O}_X$. An object $\mathscr{F} \in \hat{\mathbf{O}}(X)$ is said to be *flat* if each $\mathscr{F}_\alpha$ is a flat $\mathscr{O}_{X_\alpha}$-module. Such a $\mathscr{F}$ yields an exact functor

$$\mathbf{O}(X) \to \mathbf{O}(X), \quad \mathscr{M} \mapsto \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{F} = (\mathscr{M}_\alpha \otimes_{\mathscr{O}_{X_\alpha}} \mathscr{F}_\alpha). \tag{4.7.2}$$

For $\mathscr{F} \in \hat{\mathbf{O}}(X)$ the vector spaces $\Gamma(X_\alpha, \mathscr{F}_\alpha)$ form a projective system with the structure maps induced by the functors $i^*_{\alpha\beta}$. We set

$$\Gamma(X, \mathscr{F}) = \varprojlim \Gamma(X_\alpha, \mathscr{F}_\alpha). \tag{4.7.3}$$

4.8 The $\mathscr{D}$-modules on ind-schemes

The category of $\mathscr{D}$-modules on the ind-scheme X is defined as the limit of the inductive system of categories $(\mathbf{M}(X_\alpha), i_{\alpha\beta\bullet})$, see e.g., [KV04, 3.3]. We will denote it by $\mathbf{M}(X)$. Since $\mathbf{M}(X_\alpha)$ are abelian categories and $i_{\alpha\beta\bullet}$ is an exact functor, the category $\mathbf{M}(X)$ is abelian. Recall that an object of $\mathbf{M}(X)$ is represented by a pair $(\alpha, \mathscr{M}_\alpha)$ with $\alpha \in A$, $\mathscr{M}_\alpha \in \mathbf{M}(X_\alpha)$. There is an exact and faithful forgetful functor

$$\mathbf{M}(X) \to \mathbf{O}(X), \quad \mathscr{M} = (\alpha, \mathscr{M}_\alpha) \to \mathscr{M}^{\mathscr{O}} = (i_{\alpha\beta\bullet}(\mathscr{M}_\alpha)^{\mathscr{O}})_{\beta\geqslant\alpha}.$$

The *global sections* functor on $\mathbf{M}(X)$ is defined by

$$\Gamma(X, \mathscr{M}) = \Gamma(X, \mathscr{M}^{\mathscr{O}}).$$

Next, we say that X is a *T-ind-scheme* if $X = \varinjlim X_\alpha$ with each X_α being a T-scheme and $i_{\alpha\beta} : X_\alpha \to X_\beta$ being T-equivariant. We define $\mathbf{M}^T(X)$ to be the abelian category given by the limit of the inductive system of categories $(\mathbf{M}^T(X_\alpha), i_{\alpha\beta\bullet})$. The functors (4.3.1) for each X_α yield an exact and faithful functor

$$\mathbf{M}^T(X) \to \mathbf{M}(X). \tag{4.8.1}$$

The functor Γ on $\mathbf{M}^T(X)$ is given by the functor Γ on $\mathbf{M}(X)$.

Finally, given a T-ind-scheme $X = \varinjlim X_\alpha$ let $\pi : X^\dagger \to X$ be a T-torsor over X, i.e., π is the limit of an inductive system of T-torsors $\pi_\alpha : X^\dagger_\alpha \to X_\alpha$. We say that π is *admissible* if each of the π_α is admissible. Assume this is the case. Then the categories $\mathbf{M}^\lambda(X_\alpha)$, $\mathbf{M}^{\tilde\lambda}(X_\alpha)$ form, respectively, two inductive systems of categories via $i_{\alpha\beta\bullet}$. Let

$$\mathbf{M}^\lambda(X) = 2\varinjlim \mathbf{M}^\lambda(X_\alpha), \qquad \mathbf{M}^{\tilde\lambda}(X) = 2\varinjlim \mathbf{M}^{\tilde\lambda}(X_\alpha).$$

They are abelian subcategories of $\mathbf{M}^T(X^\dagger)$. For any object $\mathscr{M} = (\alpha, \mathscr{M}_\alpha)$ of $\mathbf{M}^T(X^\dagger)$, the $\mathscr{O}_{X_\beta}$-modules $(i_{\alpha\beta\bullet}\mathscr{M}_\alpha)^\dagger$ with $\beta \geqslant \alpha$ give an object of $\mathbf{O}(X)$. We will denote it by $\mathscr{M}^\dagger$. The functor

$$\mathbf{M}^T(X^\dagger) \to \mathbf{O}(X), \quad \mathscr{M} \mapsto \mathscr{M}^\dagger$$

is exact and faithful. For $\mathscr{M} \in \mathbf{M}^{\tilde\lambda}(X)$ we will write

$$\Gamma(\mathscr{M}) = \Gamma(X, \mathscr{M}^\dagger). \tag{4.8.2}$$

Note that it is also equal to $\Gamma(X^\dagger, \mathscr{M})^T$. We will also consider the following categories

$$\mathbf{M}_h^{\tilde\lambda}(X) = 2\varinjlim \mathbf{M}_h^{\tilde\lambda}(X_\alpha), \quad \mathbf{M}_h^{\lambda}(X) = 2\varinjlim \mathbf{M}_h^{\lambda}(X_\alpha).$$

Let Y be a smooth scheme. A locally closed affine embedding $i : Y \to X$ is the composition of an affine open embedding $i_1 : Y \to \overline{Y}$ with a closed embedding $i_2 : \overline{Y} \to X$. For such a morphism the functor $i_\bullet : \mathbf{M}_h^{\tilde\lambda}(Y) \to \mathbf{M}_h^{\tilde\lambda}(X)$ is defined by $i_\bullet = i_{2\bullet} \circ i_{1\bullet}$, and the functor $i_! : \mathbf{M}_h^{\tilde\lambda}(Y) \to \mathbf{M}_h^{\tilde\lambda}(X)$ is defined by $i_! = i_{2\bullet} \circ i_{1!}$.

4.9 The sheaf of differential operators on a formally smooth ind-scheme.

Let X be a *formally smooth* ind-scheme, i.e., for any commutative $\mathbb{C}$-algebra A and any nilpotent ideal $I \subset A$, any morphism of schemes $f : \mathrm{Spec}(A/I) \to X$ is given by the composition of $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ and a morphism $f' : \mathrm{Spec}(A) \to X$, see e.g., [BD00, 7.11.1] and the references there. Fix $\beta \geqslant \alpha$ in A let $\mathscr{D}\!\mathit{iff}_{\beta,\alpha}$ be the $\mathscr{O}_{X_\beta \times X_\alpha}$-submodule of $\mathscr{H}\!om_{\mathbb{C}}(\mathscr{O}_{X_\beta}, i_{\alpha\beta *}\mathscr{O}_{X_\alpha})$ consisting of local sections supported set-theoretically on the diagonal $X_\alpha \subset X_\beta \times X_\alpha$. Here $\mathscr{H}\!om_{\mathbb{C}}(\mathscr{O}_{X_\beta}, i_{\alpha\beta *}\mathscr{O}_{X_\alpha})$ denotes the sheaf of morphisms between the sheaves of $\mathbb{C}$-vector spaces associated with $\mathscr{O}_{X_\beta}$ and $i_{\alpha\beta *}\mathscr{O}_{X_\alpha}$. As a left $\mathscr{O}_{X_\alpha}$-module $\mathscr{D}\!\mathit{iff}_{\beta,\alpha}$ is quasi-coherent, see e.g., [BB93, Section 1.1]. So it is an object in $\mathbf{O}(X_\alpha)$. For $\beta \leqslant \gamma$ the functor $i_{\beta\gamma *}$ and the canonical map $\mathscr{O}_{X_\gamma} \to i_{\beta\gamma *}\mathscr{O}_{X_\beta}$ yield a morphism of $\mathscr{O}_{X_\alpha}$-modules

$$\mathscr{H}\!om_{\mathbb{C}}(\mathscr{O}_{X_\beta}, i_{\alpha\beta *}\mathscr{O}_{X_\alpha}) \to \mathscr{H}\!om_{\mathbb{C}}(\mathscr{O}_{X_\gamma}, i_{\alpha\gamma *}\mathscr{O}_{X_\alpha}).$$

It induces a morphism $\mathscr{D}\!\mathit{iff}_{\beta,\alpha} \to \mathscr{D}\!\mathit{iff}_{\gamma,\alpha}$ in $\mathbf{O}(X_\alpha)$. The $\mathscr{O}_{X_\alpha}$-modules $\mathscr{D}\!\mathit{iff}_{\beta,\alpha}$, $\beta \geqslant \alpha$, together with these maps form an inductive system. Let

$$\mathscr{D}\!\mathit{iff}_{\alpha} = \varinjlim_{\beta \geqslant \alpha} \mathscr{D}\!\mathit{iff}_{\beta,\alpha} \in \mathbf{O}(X_\alpha).$$

The system consisting of the $\mathscr{D}\!\mathit{iff}_{\alpha}$'s and the canonical isomorphisms $i^*_{\alpha\beta}\mathscr{D}\!\mathit{iff}_{\beta} \to \mathscr{D}\!\mathit{iff}_{\alpha}$ is a flat object in $\hat{\mathbf{O}}(X)$, see [BD00, 7.11.11]. We will call it the *sheaf of differential operators* on X and denote it by $\mathscr{D}_X$. It carries canonically a structure of $\mathscr{O}_X$-bimodules, and a structure of algebra given by

$$\mathscr{D}\!\mathit{iff}_{\gamma,\beta} \otimes_{\mathscr{O}_{X_\beta}} \mathscr{D}\!\mathit{iff}_{\beta,\alpha} \to \mathscr{D}\!\mathit{iff}_{\gamma,\alpha}, \quad (g,f) \mapsto g \circ f, \quad \alpha \leqslant \beta \leqslant \gamma.$$

Any object $\mathscr{M} \in \mathbf{M}(X)$ admits a canonical right $\mathscr{D}_X$-action given by a morphism

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M} \tag{4.9.1}$$

in $\mathbf{O}(X)$ which is compatible with the multiplication in $\mathscr{D}_X$.

5 Localization theorem for affine Lie algebras of negative level

In this section we first consider the affine localization theorem which relates right $\mathscr{D}$-modules on the affine flag variety (an ind-scheme) to a category of modules over the affine Lie algebra with integral weights and a negative level. When the weight is regular, we compute the image of standard $\mathscr{D}$-modules using Kashiwara-Tanisaki's construction (via the Kashiwara affine flag scheme). Next, we give a geometric construction of the translation functor for the affine category $\mathcal{O}$ inspired from [BG99], and we apply this to singular blocks. Finally, we consider the $\mathscr{D}$-modules corresponding to the parabolic Verma modules. All these constructions hold for a general simple linear group. We will only use the case of SL_m, since the multiplicities on the left hand side of (3.11.2) that we want to compute are the same for $\mathfrak{sl}_m$ and $\mathfrak{gl}_m$. We will use, for $\mathfrak{sl}_m$, the same notation as in Section 3 for $\mathfrak{gl}_m$. In particular $\mathfrak{g}_0 = \mathfrak{sl}_m$ and $\mathfrak{t}_0^*$ is now given the basis consisting of the weights $\epsilon_i - \epsilon_{i+1}$ with $1 \leqslant i \leqslant m-1$. We identify $\mathcal{P}_n$ as a subset of $\mathfrak{t}_0^*$ via the map

$$\mathcal{P}_n \to \mathfrak{t}_0^*, \quad \lambda = (\lambda_1, \ldots, \lambda_m) \mapsto \sum_{i=1}^m (\lambda_i - n/m)\epsilon_i.$$

Finally, we will modify slightly the definition of $\mathfrak{g}$ by extending $\mathbb{C}[t, t^{-1}]$ to $\mathbb{C}((t))$, i.e., from now on we set

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}((t)) \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial.$$

The bracket is given in the same way as before. We will again denote by $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{q}$, etc., the corresponding Lie subalgebras of $\mathfrak{g}$.

5.1 The affine Kac-Moody group

Consider the group ind-scheme $LG_0 = G_0(\mathbb{C}((t)))$ and the group scheme $L^+G_0 = G_0(\mathbb{C}[[t]])$. Let $I \subset L^+G_0$ be the Iwahori subgroup. It is the preimage of B_0 via the canonical map $L^+G_0 \to G_0$. For $z \in \mathbb{C}^*$ the loop rotation $t \mapsto zt$ yields a $\mathbb{C}^*$-action on LG_0. Write

$$\widehat{LG}_0 = \mathbb{C}^* \ltimes LG_0.$$

Let G be the Kac-Moody group associated with $\mathfrak{g}$. It is a group ind-scheme which is a central extension

$$1 \to \mathbb{C}^* \to G \to \widehat{LG}_0 \to 1,$$

see e.g., [Kum02, Section 13.2]. There is an obvious projection $\mathrm{pr} : G \to LG_0$. We set

$$B = \mathrm{pr}^{-1}(I), \quad Q = \mathrm{pr}^{-1}(L^+G_0), \quad T = \mathrm{pr}^{-1}(T_0).$$

Finally, let N be the prounipotent radical of B. We have

$$\mathfrak{g} = \mathrm{Lie}(G), \quad \mathfrak{b} = \mathrm{Lie}(B), \quad \mathfrak{q} = \mathrm{Lie}(Q), \quad \mathfrak{t} = \mathrm{Lie}(T), \quad \mathfrak{n} = \mathrm{Lie}(N).$$

5.2 The affine flag variety

Let $X = G/B$ be the affine flag variety. It is a formally smooth ind-scheme. The enhanced affine flag variety $X^\dagger = G/N$ is a T-torsor over X via the canonical projection

$$\pi : X^\dagger \to X. \tag{5.2.1}$$

The T-action on $X^\dagger$ is given by $gN \mapsto gh^{-1}N$ for $h \in T$, $g \in G$. The T-torsor π is admissible, see the end of Section 5.5. The ind-scheme $X^\dagger$ is also formally smooth. For any subscheme Z of X we will write $Z^\dagger = \pi^{-1}(Z)$. The B-orbit decomposition of X is

$$X = \bigsqcup_{w \in \mathfrak{S}} X_w, \qquad X_w = B\dot{w}B/B,$$

where $\dot{w}$ is a representative of w in the normalizer of T in G. Each X_w is an affine space of dimension $l(w)$. Its closure $\overline{X}_w$ is an irreducible projective variety. We have

$$\overline{X}_w = \bigsqcup_{w' \leqslant w} X_{w'}, \qquad X = \varinjlim_{w} \overline{X}_w.$$

5.3 Localization theorem

Recall the sheaf of differential operators $\mathscr{D}_{X^\dagger} \in \hat{\mathbf{O}}(X^\dagger)$. The space of sections of $\mathscr{D}_{X^\dagger}$ is defined as in (4.7.3). The left action of G on $X^\dagger$ yields an algebra homomorphism

$$\delta_l : \mathcal{U}(\mathfrak{g}) \to \Gamma(X^\dagger, \mathscr{D}_{X^\dagger}). \tag{5.3.1}$$

Since the G-action on $X^\dagger$ commutes with the right T-action, the image of the map above lies in the T-invariant part of $\Gamma(X^\dagger, \mathscr{D}_{X^\dagger})$. So for $\mathscr{M} \in \mathbf{M}^T(X^\dagger)$ the $\mathscr{D}_{X^\dagger}$-action on $\mathscr{M}$ given by (4.9.1) induces a $\mathfrak{g}$-action[1] on $\mathscr{M}^\dagger$ via δ_l. In particular the vector space $\Gamma(\mathscr{M})$ as defined in (4.8.2) is a $\mathfrak{g}$-module. Let $\mathbf{M}(\mathfrak{g})$ be the category of $\mathfrak{g}$-modules. We say that a weight $\lambda \in \mathfrak{t}^*$ is *antidominant* (resp. *dominant*, *regular*) if for any $\alpha \in \Pi^+$ we have $\langle \lambda : \alpha \rangle \leqslant 0$ (resp. $\langle \lambda : \alpha \rangle \geqslant 0$, $\langle \lambda : \alpha \rangle \neq 0$).

1. More precisely, here by $\mathfrak{g}$-action we mean the $\mathfrak{g}$-action on the associated sheaf of vector spaces $(\mathscr{M}^\dagger)^{\mathbb{C}}$, see Step 1 of the proof of Proposition 5.9.1 for details.

Proposition 5.3.1. *(a) The functor*

$$\Gamma : \mathbf{M}^{\lambda}(X) \to \mathbf{M}(\mathfrak{g}), \quad \mathscr{M} \mapsto \Gamma(\mathscr{M})$$

is exact if $\lambda + \rho$ is antidominant.

(b) The functor

$$\Gamma : \mathbf{M}^{\tilde{\lambda}}(X) \to \mathbf{M}(\mathfrak{g}), \quad \mathscr{M} \mapsto \Gamma(\mathscr{M})$$

is exact if $\lambda + \rho$ is antidominant.

Proof. A proof of part (a) is sketched in [BD00, Theorem 7.15.6]. A detailed proof can be given using similar technics as in the proof of the Proposition 5.9.1 below. This is left to the reader. See also [FG04, Theorem 2.2] for another proof of this result. Now, let us concentrate on part (b). Let $\mathscr{M} = (\alpha, \mathscr{M}_\alpha)$ be an object in $\mathbf{M}^{\tilde{\lambda}}(X)$. By definition the action of $\mathfrak{m}_\lambda$ on $\mathscr{M}^\dagger$ is nilpotent. Let $\mathscr{M}_n$ be the maximal subobject of $\mathscr{M}$ such that the ideal $(\mathfrak{m}_\lambda)^n$ acts on $\mathscr{M}_n^\dagger$ by zero. We have $\mathscr{M}_{n-1} \subset \mathscr{M}_n$ and $\mathscr{M} = \varinjlim \mathscr{M}_n$. Write $R^k\Gamma(X, -)$ for the k-th derived functor of the global sections functor $\Gamma(X, -)$. Given $n \geqslant 1$, suppose that

$$R^k\Gamma(X, \mathscr{M}_n^\dagger) = 0, \qquad \forall\, k > 0.$$

Since $\mathscr{M}_{n+1}/\mathscr{M}_n$ is an object of $\mathbf{M}^{\lambda}(X)$, by part (a) we have

$$R^k\Gamma(X, (\mathscr{M}_{n+1}/\mathscr{M}_n)^\dagger) = 0, \quad \forall\, k > 0.$$

The long exact sequence for $R\Gamma(X, -)$ applied to the short exact sequence

$$0 \longrightarrow \mathscr{M}_n^\dagger \longrightarrow \mathscr{M}_{n+1}^\dagger \longrightarrow (\mathscr{M}_{n+1}/\mathscr{M}_n)^\dagger \longrightarrow 0$$

implies that $R^k\Gamma(X, \mathscr{M}_{n+1}^\dagger) = 0$ for any $k > 0$. Therefore by induction the vector space $R^k\Gamma(X, \mathscr{M}_n^\dagger)$ vanishes for any $n \geqslant 1$ and $k > 0$. Finally, since the functor $R^k\Gamma(X, -)$ commutes with direct limits, see e.g., [TT90, Lemma B.6], we have

$$R^k\Gamma(X, \mathscr{M}^\dagger) = \varinjlim R^k\Gamma(X, \mathscr{M}_n^\dagger) = 0, \qquad \forall\ k > 0.$$

□

5.4 The category $\tilde{\mathscr{O}}_\kappa$ and Verma modules

For a $\mathfrak{t}$-module M and $\lambda \in \mathfrak{t}^*$ we set

$$M_{\tilde{\lambda}} = \{m \in M \,|\, (h - \lambda(h))^N m = 0,\ \forall\ h \in \mathfrak{t},\ N \gg 0\}. \tag{5.4.1}$$

We call a $\mathfrak{t}$-module M a *generalized weight module* if it satisfies the conditions

$$M = \bigoplus_{\lambda \in_\kappa \mathfrak{t}^*} M_{\tilde{\lambda}},$$

$$\dim_{\mathbb{C}} M_{\tilde{\lambda}} < \infty, \qquad \forall\ \lambda \in \mathfrak{t}^*.$$

Its character $\mathrm{ch}(M)$ is defined as the formal sum

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathfrak{t}^*} \dim_{\mathbb{C}}(M_{\tilde{\lambda}}) e^{\lambda}. \tag{5.4.2}$$

Let $\tilde{\mathscr{O}}$ be the category consisting of the $\mathcal{U}(\mathfrak{g})$-modules M such that

- as a $\mathfrak{t}$-module M is a generalized weight module,
- there exists a finite subset $\Xi \subset \mathfrak{t}^*$ such that $M_{\tilde{\lambda}} \neq 0$ implies that $\lambda \in \Xi + \sum_{i=0}^{m-1} \mathbb{Z}_{\leqslant 0}\alpha_i$.

The category $\tilde{\mathcal{O}}$ is an abelian category. We define the duality functor $\mathbf{D}$ on $\tilde{\mathcal{O}}$ by

$$\mathbf{D}M = \bigoplus_{\lambda \in \mathfrak{t}^*} \mathrm{Hom}(M_{\tilde{\lambda}}, \mathbb{C}), \tag{5.4.3}$$

with the action of $\mathfrak{g}$ given by the involution σ, see Section 3.3. Let $\tilde{\mathcal{O}}_\kappa$ be the full subcategory of $\tilde{\mathcal{O}}$ consisting of the $\mathfrak{g}$-modules M such that $\mathbf{1} - c$ acts on M locally nilpotently, where $c = \kappa - m$. The category $\tilde{\mathcal{O}}_\kappa$ is also abelian. It is stable under the duality functor, because $\sigma(\mathbf{1}) = \mathbf{1}$. The category $\mathcal{O}_\kappa$ is a full Serre subcategory of $\tilde{\mathcal{O}}_\kappa$.

For $\lambda \in {}_\kappa\mathfrak{t}^*$ we consider the *Verma module*

$$N_\kappa(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

Here $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module such that $\mathfrak{n}$ acts trivially and $\mathfrak{t}$ acts by λ. It is an object of $\tilde{\mathcal{O}}_\kappa$. Let $L_\kappa(\lambda)$ be the unique simple quotient of $N_\kappa(\lambda)$. We have $\mathbf{D}L_\kappa(\lambda) = L_\kappa(\lambda)$ for any λ. A simple subquotient of a module $M \in \tilde{\mathcal{O}}_\kappa$ is isomorphic to $L_\kappa(\lambda)$ for some $\lambda \in {}_\kappa\mathfrak{t}^*$. The classes $[L_\kappa(\lambda)]$ form a basis of the vector space $[\tilde{\mathcal{O}}_\kappa]$, because the characters of the $L_\kappa(\lambda)$'s are linearly independent.

Denote by Λ the set of integral weights in ${}_\kappa\mathfrak{t}^*$. Let $\lambda \in \Lambda$ and $w \in \mathfrak{S}$. Recall the line bundle $\mathscr{L}^\lambda_{X_w}$ from Section 4.5. Let

$$\mathscr{A}^\lambda_w = \Omega_{X^\dagger_w} \otimes_{\mathscr{O}_{X^\dagger_w}} \pi^*(\mathscr{L}^\lambda_{X_w}). \tag{5.4.4}$$

It is an object of $\mathbf{M}^\lambda_h(X_w)$ such that

$$\mathbb{D}(\mathscr{A}^\lambda_w) = \mathscr{A}^\lambda_w.$$

Let $i_w : X^\dagger_w \to X^\dagger$ be the canonical embedding. It is locally closed and affine. We have the following objects in $\mathbf{M}^\lambda_h(X)$,

$$\mathscr{A}^\lambda_{w!} = i_{w!}(\mathscr{A}^\lambda_w), \quad \mathscr{A}^\lambda_{w!\bullet} = i_{w!\bullet}(\mathscr{A}^\lambda_w), \quad \mathscr{A}^\lambda_{w\bullet} = i_{w\bullet}(\mathscr{A}^\lambda_w)$$

We will consider the Serre subcategory $\mathbf{M}^\lambda_0(X)$ of $\mathbf{M}^\lambda_h(X)$ generated by the simple objects $\mathscr{A}^\lambda_{w!\bullet}$ for $w \in \mathfrak{S}$. It is an artinian category. Since $\mathbb{D}(\mathscr{A}^\lambda_{w!\bullet}) = \mathscr{A}^\lambda_{w!\bullet}$, the category $\mathbf{M}^\lambda_0(X)$ is stable under the duality. We have the following proposition.

Proposition 5.4.1. *Let $\lambda \in \Lambda$ be such that $\lambda + \rho$ is antidominant and regular. Then we have isomorphisms of $\mathfrak{g}$-modules*

$$\Gamma(\mathscr{A}^\lambda_{v!}) = N_\kappa(v \cdot \lambda), \quad \Gamma(\mathscr{A}^\lambda_{v\bullet}) = \mathbf{D}N_\kappa(v \cdot \lambda), \quad \Gamma(\mathscr{A}^\lambda_{v!\bullet}) = L_\kappa(v \cdot \lambda), \quad \forall\, v \in \mathfrak{S}. \tag{5.4.5}$$

This is essentially due to [KT95]. However, the setting of loc. cit. is slightly different from the one used here. Let us recall their construction and adapt it to our setting.

5.5 The Kashiwara affine flag variety

We first introduce some more notation. Recall that Π is the root system of $\mathfrak{g}$ and Π^+ is the set of positive root. Write $\Pi^- = -\Pi^+$. For $\alpha \in \Pi$ we write

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \quad \forall\, h \in \mathfrak{t}\}.$$

For any subset Υ of Π^+, Π^- we set respectively

$$\mathfrak{n}(\Upsilon) = \bigoplus_{\alpha\in\Upsilon} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^-(\Upsilon) = \bigoplus_{\alpha\in\Upsilon} \mathfrak{g}_\alpha.$$

For $\alpha = \sum_{i=0}^{m-1} h_i\alpha_i \in \Pi$ we write $\mathrm{ht}(\alpha) = \sum_{i=0}^{m-1} h_i$ and for $l \in \mathbb{N}$ we set

$$\Pi_l^- = \{\alpha \in \Pi^- \mid \mathrm{ht}(\alpha) \leqslant -l\}, \qquad \mathfrak{n}_l^- = \mathfrak{n}^-(\Pi_l^-).$$

Consider the group scheme $L^-G_0 = G_0(\mathbb{C}[[t^{-1}]])$. Let B^- be the preimage of B_0^- by the map

$$L^-G_0 \to G_0, \qquad t^{-1} \mapsto 0,$$

where B_0^- is the Borel subgroup of G_0 opposite to B_0. Let $N_l^- \subset B^-$ be the group subscheme given by

$$N_l^- = \varprojlim_k \exp(\mathfrak{n}_l^-/\mathfrak{n}_k^-).$$

Let $\mathfrak{X}$ be the Kashiwara affine flag variety, see [Kas89]. It is a quotient scheme $\mathfrak{X} = G_\infty/B$, where G_∞ is a coherent scheme with a locally free left action of B^- and a locally free right action of B. The scheme $\mathfrak{X}$ is coherent, prosmooth, non quasi-compact, locally of countable type, with a left action of B^-. There is a right T-torsor

$$\pi : \mathfrak{X}^\dagger = G_\infty/N \to \mathfrak{X}.$$

For any subscheme Z of $\mathfrak{X}$ let $Z^\dagger$ be its preimage by π. Let

$$\mathfrak{X} = \bigsqcup_{w\in\mathfrak{S}} \mathring{\mathfrak{X}}^w.$$

be the B^--orbit decomposition. The scheme $\mathfrak{X}$ is covered by the following open sets

$$\mathfrak{X}^w = \bigsqcup_{v\leqslant w} \mathring{\mathfrak{X}}^v.$$

For each w there is a canonical closed embedding $\overline{X}_w \to \mathfrak{X}^w$. Moreover, for any integer l that is large enough, the group N_l^- acts locally freely on $\mathfrak{X}^w$, $\mathfrak{X}^{w\dagger}$, the quotients

$$\mathfrak{X}_l^w = N_l^-\backslash\mathfrak{X}^w, \qquad \mathfrak{X}_l^{w\dagger} = N_l^-\backslash\mathfrak{X}^{w\dagger}$$

are smooth schemes [2], and the induced morphism

$$\overline{X}_w \to \mathfrak{X}_l^w \tag{5.5.1}$$

is a closed immersion. See [KT95, Lemma 2.2.1]. Further we have

$$\overline{X}_w^\dagger = \overline{X}_w \times_{\mathfrak{X}_l^w} \mathfrak{X}_l^{w\dagger}.$$

In particular, we get a closed embedding of the T-torsor $\overline{X}_w^\dagger \to \overline{X}_w$ into the T-torsor $\mathfrak{X}_l^{w\dagger} \to \mathfrak{X}_l^w$. This implies that the T-torsor $\pi : X^\dagger \to X$ is admissible. Finally, let

$$p_{l_1l_2} : \mathfrak{X}_{l_1}^{w\dagger} \to \mathfrak{X}_{l_2}^{w\dagger}, \quad p_l : \mathfrak{X}^{w\dagger} \to \mathfrak{X}_l^{w\dagger}, \quad l_1 \geqslant l_2$$

be the canonical projections. They are affine morphisms.

2. For l large enough the scheme $\mathfrak{X}_l^w$ is separated (hence quasi-separated). To see this, one uses the fact that $\mathfrak{X}^w$ is separated and applies [TT90, Proposition C.7].

5.6 The category $\mathbf{H}^{\tilde{\lambda}}(X)$.

Fix $w, y \in \mathfrak{S}$ with $y \geqslant w$. For $l_1 \geqslant l_2$ large enough, the functor

$$(p_{l_1 l_2})_\bullet : \mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_{l_1}^y, \overline{X}_w) \to \mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_{l_2}^y, \overline{X}_w)$$

yields a filtering projective system of categories, and we set

$$\mathbf{H}^{\tilde{\lambda}}(\mathfrak{X}^y, \overline{X}_w) = 2\varprojlim_{l} \mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^y, \overline{X}_w).$$

For $z \geqslant y$ let $j_{yz} : \mathfrak{X}^{y\dagger} \to \mathfrak{X}^{z\dagger}$ be the canonical open embedding. It yields a map $j_{yz} : \mathfrak{X}_l^{y\dagger} \to \mathfrak{X}_l^{z\dagger}$ for each l. The pull-back functors by these maps yield, by base change, a morphism of projective systems of categories

$$(\mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^z, \overline{X}_w))_l \to (\mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^y, \overline{X}_w))_l.$$

Hence we get a map

$$\mathbf{H}^{\tilde{\lambda}}(\mathfrak{X}^z, \overline{X}_w) \to \mathbf{H}^{\tilde{\lambda}}(\mathfrak{X}^y, \overline{X}_w).$$

As y, z varies these maps yield again a projective system of categories and we set

$$\mathbf{H}^{\tilde{\lambda}}(\overline{X}_w) = 2\varprojlim_{y \geqslant w} \mathbf{H}^{\tilde{\lambda}}(\mathfrak{X}^y, \overline{X}_w).$$

Finally, for $w \leqslant v$ the category $\mathbf{H}^{\tilde{\lambda}}(\overline{X}_w)$ is canonically a full subcategory of $\mathbf{H}^{\tilde{\lambda}}(\overline{X}_v)$. We define

$$\mathbf{H}^{\tilde{\lambda}}(X) = 2\varinjlim_{w} \mathbf{H}^{\tilde{\lambda}}(\overline{X}_w).$$

This definition is inspired from [KT95], where the authors considered the categories $\mathbf{M}_h^{\lambda}(\mathfrak{X}_l^y, \overline{X}_w)$ instead of the categories $\mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^y, \overline{X}_w)$. Finally, note that since the category $\mathbf{H}^{\tilde{\lambda}}(\overline{X}_w)$ is equivalent to $\mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^y, \overline{X}_w)$ for y, l large enough, and since the latter is equivalent to $\mathbf{M}_h^{\tilde{\lambda}}(\overline{X}_w)$, see Section 4.1, we have an equivalence of categories

$$\mathbf{H}^{\tilde{\lambda}}(X) \cong \mathbf{M}_h^{\tilde{\lambda}}(X).$$

5.7 The functors $\hat{\Gamma}$ and $\overline{\Gamma}$

For an object $\mathscr{M}$ of $\mathbf{H}^{\tilde{\lambda}}(X)$, there exists $w \in \mathfrak{S}$ such that $\mathscr{M}$ is an object of the subcategory $\mathbf{H}^{\tilde{\lambda}}(\overline{X}_w)$. Thus $\mathscr{M}$ is represented by a system $(\mathscr{M}_l^y)_{y \geqslant w, l}$, with $\mathscr{M}_l^y \in \mathbf{M}_h^{\tilde{\lambda}}(\mathfrak{X}_l^y, \overline{X}_w)$ and l large enough. For $l_1 \geqslant l_2$ there is a canonical map

$$(p_{l_1 l_2})_*(\mathscr{M}_{l_1}^y) \to (p_{l_1 l_2})_\bullet(\mathscr{M}_{l_1}^y) = \mathscr{M}_{l_2}^y.$$

It yields a map (see (4.4.2) for the notation)

$$\Gamma(\mathfrak{X}_{l_1}^y, \mathscr{M}_{l_1}^y) \to \Gamma(\mathfrak{X}_{l_2}^y, \mathscr{M}_{l_2}^y).$$

Next, for y, $z \geqslant w$ and l large enough, we have a canonical isomorphism

$$\Gamma(\mathfrak{X}_l^y, \mathscr{M}_l^y) = \Gamma(\mathfrak{X}_l^z, \mathscr{M}_l^z).$$

Following [KT95], we choose a $y \geqslant w$ and we set

$$\hat{\Gamma}(\mathscr{M}) = \varprojlim_l \Gamma(\mathfrak{X}_l^y, \mathscr{M}_l^y).$$

This definition does not depend on the choice of w, y. Now, regard $\mathscr{M}$ as an object of $\mathbf{M}_h^{\tilde{\lambda}}(X)$. Recall the object $\mathscr{M}^\dagger \in \mathbf{O}(X)$ from Section 4.4. Suppose that $\mathscr{M}^\dagger$ is represented by a system $(\mathscr{M}_y^{\mathscr{O}})_{y \geqslant w}$ with $\mathscr{M}_y^{\mathscr{O}} \in \mathbf{O}(\overline{X}_y)$. By definition we have $\mathscr{M}_y^{\mathscr{O}} = (i^! \mathscr{M}_l^y)^\dagger$, where i denotes the closed embedding $\overline{X}_y^\dagger \to \mathfrak{X}_l^{y\dagger}$, see (5.5.1). Therefore we have

$$\begin{aligned} \Gamma(\overline{X}_y, \mathscr{M}_y^{\mathscr{O}}) &= \Gamma(\overline{X}_y, i^!(\mathscr{M}_l^y)^\dagger) \\ &\subset \Gamma(\mathfrak{X}_l^y, \mathscr{M}_l^y). \end{aligned} \tag{5.7.1}$$

Next, recall that we have

$$\Gamma(\mathscr{M}) = \Gamma(X, \mathscr{M}^\dagger) = \varinjlim_y \Gamma(\overline{X}_y, \mathscr{M}_y^{\mathscr{O}}).$$

So by first taking the projective limit on the right hand side of (5.7.1) with respect to l and then taking the inductive limit on the left hand side with respect to y we get an inclusion

$$\Gamma(\mathscr{M}) \subset \hat{\Gamma}(\mathscr{M}).$$

It identifies $\Gamma(\mathscr{M})$ with the subset of $\hat{\Gamma}(\mathscr{M})$ consisting of the sections supported on subschemes (of finite type) of X.

The vector space $\hat{\Gamma}(\mathscr{M})$ has a $\mathfrak{g}$-action, see [KT95, Section 2.3]. The vector space $\Gamma(\mathscr{M})$ has also a $\mathfrak{g}$-action by Section 5.3. The inclusion is compatible with these $\mathfrak{g}$-actions. Following loc. cit., let

$$\overline{\Gamma}(\mathscr{M}) \subset \hat{\Gamma}(\mathscr{M})$$

be the set of $\mathfrak{t}$-finite elements. It is a $\mathfrak{g}$-submodule of $\hat{\Gamma}(\mathscr{M})$.

5.8 Proof of Proposition 5.4.1

By [KT95, Theorem 3.4.1] under the assumption of the proposition we have isomorphisms of $\mathfrak{g}$-modules.

$$\overline{\Gamma}(\mathscr{A}_{v!}^\lambda) = N_\kappa(v \cdot \lambda), \qquad \overline{\Gamma}(\mathscr{A}_{v\bullet}^\lambda) = \mathbf{D}N_\kappa(v \cdot \lambda), \qquad \overline{\Gamma}(\mathscr{A}_{v!\bullet}^\lambda) = L_\kappa(v \cdot \lambda), \quad \forall v \in \mathfrak{S}.$$

We must check that for $\sharp =!$, $\bullet$, or $!\bullet$, the $\mathfrak{g}$-submodules $\Gamma(\mathscr{A}_{v\sharp}^\lambda)$ and $\overline{\Gamma}(\mathscr{A}_{v\sharp}^\lambda)$ of $\hat{\Gamma}(\mathscr{A}_{v\sharp}^\lambda)$ are equal. Let us prove this for $\sharp = \bullet$. We will do this in several steps.

Step 1. Following [KT95] we first define a particular section ϑ in $\hat{\Gamma}(\mathscr{A}_{v\bullet}^\lambda)$. Let ω be a nowhere vanishing section of Ω_{X_v}. It is unique up to a nonzero scalar. Let t^λ be the nowhere vanishing section of $\mathscr{L}_{X_v}^\lambda$ such that $t^\lambda(u\dot{v}b) = e^{-\lambda}(b)$ for $u \in N$, $b \in B$. Then $\omega \otimes t^\lambda$ is a nowhere vanishing section of $\mathscr{A}_v^{\lambda,\dagger}$ over X_v. Now, for $y \geqslant v$ and l large enough, let $i_l^v : X_v \to \mathfrak{X}_l^y$ be the composition of the locally closed embedding $X_v \to \overline{X}_y$ and the closed embedding $\overline{X}_y \to \mathfrak{X}_l^y$ in (5.5.1). We will denote the corresponding embedding $X_v^\dagger \to \mathfrak{X}_l^{y\dagger}$ again by i_l^v. Note that $(i_{l\bullet}^v(\mathscr{A}_v^\lambda))_l$ represents the object $\mathscr{A}_{v\bullet}^\lambda$ in $\mathbf{H}^{\tilde{\lambda}}(X)$. Therefore we have

$$\hat{\Gamma}(\mathscr{A}_{v\bullet}^\lambda) = \varprojlim_l \Gamma(\mathfrak{X}_l^y, i_{l\bullet}^v(\mathscr{A}_v^\lambda)).$$

Consider the canonical inclusion of $\mathscr{O}_{\mathfrak{X}_l^y}$-modules

$$i_{l*}^v(\mathscr{A}_v^\lambda)^\dagger \longrightarrow i_{l\bullet}^v(\mathscr{A}_v^\lambda)^\dagger.$$

Let $\vartheta_l \in \Gamma(\mathfrak{X}^y_l, i^v_{l\bullet}(\mathscr{A}^\lambda_v)^\dagger)$ be the image of $\omega \otimes t^\lambda$ under this map. The family (ϑ_l) defines an element

$$\vartheta \in \hat{\Gamma}(\mathscr{A}^\lambda_{v\bullet}).$$

Step 2. Let $V^y = yB^- \cdot B/B$. It is an affine open set in $\mathfrak{X}^y$. For l large enough, let V^y_l be the image of V^y in $\mathfrak{X}^y_l$ via the canonical projection $\mathfrak{X}^y \to \mathfrak{X}^y_l$. Write $j^y_l : V^y_l \to \mathfrak{X}^y_l$ for the inclusion. Note that $V^y_l \cong N^-/N^-_l$ as affine spaces. Therefore, if l is large enough such that $\Pi^-_l \subset \Pi^- \cap v\Pi^-$, then the right $\mathscr{D}^\dagger_{X_v}$-module structure on $\mathscr{A}^{\lambda,\dagger}_v$ yields an isomorphism of sheaves of $\mathbb{C}$-vector spaces over V^y_l

$$\begin{array}{rcl} j^{y*}_l(i^v_{l*}\mathscr{O}_{X_v}) \otimes \mathcal{U}(\mathfrak{n}^-(\Pi^- \cap v(\Pi^-))/\mathfrak{n}^-_l) & \stackrel{\sim}{\to} & j^{y*}_l(i^v_{l\bullet}(\mathscr{A}^\lambda_v)^\dagger), \\ f \otimes p & \mapsto & (\vartheta \cdot f) \cdot \delta_l(p). \end{array}$$

This yields an isomorphism of $\mathfrak{t}$-modules

$$\Gamma(\mathfrak{X}^y_l, i^v_{l\bullet}(\mathscr{A}^\lambda_v)) = \mathcal{U}(\mathfrak{n}^-/\mathfrak{n}^-_l) \otimes \mathbb{C}_{v\cdot\lambda}, \tag{5.8.1}$$

see [KT95, Lemma 3.2.1]. By consequence we have an isomorphism of $\mathfrak{t}$-modules

$$\hat{\Gamma}(\mathscr{A}^\lambda_{v\bullet}) = (\varprojlim_l \mathcal{U}(\mathfrak{n}^-/\mathfrak{n}^-_l)) \otimes \mathbb{C}_{v\cdot\lambda}. \tag{5.8.2}$$

Step 3. Now, let us prove $\Gamma(\mathscr{A}^\lambda_{v\bullet}) = \overline{\Gamma}(\mathscr{A}^\lambda_{v\bullet})$. First, by (5.8.1) the space $\Gamma(\mathfrak{X}^y_l, i^v_{l\bullet}(\mathscr{A}^\lambda_v))$ is $\mathfrak{t}$-locally finite. So (5.7.1) implies that $\Gamma(\mathscr{A}^\lambda_{v\bullet})$ is the inductive limit of a system of $\mathfrak{t}$-locally finite submodules. Therefore it is itself $\mathfrak{t}$-locally finite. Hence we have

$$\Gamma(\mathscr{A}^\lambda_{v\bullet}) \subset \overline{\Gamma}(\mathscr{A}^\lambda_{v\bullet}).$$

To see that this is indeed an equality, note that if $m \in \hat{\Gamma}(\mathscr{A}^\lambda_{v\bullet})$ is not $\mathfrak{t}$-locally finite, then by (5.8.2) the section m is represented by an element in

$$\varprojlim_l \mathcal{U}(\mathfrak{n}^-/\mathfrak{n}^-_l) \otimes \mathbb{C}_{v\cdot\lambda}$$

which does not come from $\mathcal{U}(\mathfrak{n}^-) \otimes \mathbb{C}_{v\cdot\lambda}$ via the obvious map. Then one sees that m can not be supported on a finite dimensional scheme, i.e., it can not belong to $\Gamma(\mathscr{A}^\lambda_{v\bullet})$. This proves that

$$\Gamma(\mathscr{A}^\lambda_{v\bullet}) = \overline{\Gamma}(\mathscr{A}^\lambda_{v\bullet}).$$

Now, we can prove the other two equalities. Since $\lambda + \rho$ is antidominant, by Proposition 5.3.1(a) the functor Γ is exact on $\mathbf{M}^\lambda(X)$. So $\Gamma(\mathscr{A}^\lambda_{v!\bullet})$ is a $\mathfrak{g}$-submodule of $\Gamma(\mathscr{A}^\lambda_{v\bullet})$. Therefore all the elements in $\Gamma(\mathscr{A}^\lambda_{v!\bullet})$ are $\mathfrak{t}$-finite, i.e., we have

$$\Gamma(\mathscr{A}^\lambda_{v!\bullet}) \subset \overline{\Gamma}(\mathscr{A}^\lambda_{v!\bullet}).$$

On the other hand, by [KT95, Theorem 3.4.1] we have

$$\overline{\Gamma}(\mathscr{A}^\lambda_{v!\bullet}) \subset \overline{\Gamma}(\mathscr{A}^\lambda_{v\bullet}).$$

Therefore, Step 3 yields that each section in $\overline{\Gamma}(\mathscr{A}^\lambda_{v!\bullet})$ is supported on a finite dimensional scheme, and hence belongs to $\Gamma(\mathscr{A}^\lambda_{v!\bullet})$. We deduce that

$$\Gamma(\mathscr{A}^\lambda_{v!\bullet}) = \overline{\Gamma}(\mathscr{A}^\lambda_{v!\bullet}). \tag{5.8.3}$$

Finally, since $\mathscr{A}^\lambda_{v!}$ has a finite composition series whose constituents are given by $\mathscr{A}^\lambda_{w!\bullet}$ for $w \leqslant v$. Since both Γ and $\overline{\Gamma}$ are exact functors on $\mathbf{M}^\lambda_0(X)$, see Proposition 5.3.1 and [KT95, Corollary 3.3.3, Theorem 3.4.1]. We deduce from (5.8.3) that $\Gamma(\mathscr{A}^\lambda_{v!})$ is $\mathfrak{t}$-locally finite, and the sections of $\overline{\Gamma}(\mathscr{A}^\lambda_{v!})$ are supported on finite dimensional subschemes. Therefore we have

$$\Gamma(\mathscr{A}^\lambda_{v!}) = \overline{\Gamma}(\mathscr{A}^\lambda_{v!}).$$

The proposition is proved. □

5.9 Translation functors

In order to compute the images of $\mathscr{A}^\lambda_{v!}$ and $\mathscr{A}^\lambda_{v\bullet}$ in the case when $\lambda+\rho$ is not regular, we need the translation functors. For $\lambda \in {}_\kappa\mathfrak{t}^*$ such that $\lambda+\rho$ is anti-dominant, we define $\tilde{\mathcal{O}}_{\kappa,\lambda}$ to be the Serre subcategory of $\tilde{\mathcal{O}}_\kappa$ generated by $L_\kappa(w\cdot\lambda)$ for all $w \in \mathfrak{S}$. The same argument as in the proof of [DGK82, Theorem 4.2] yields that each $M \in \tilde{\mathcal{O}}_\kappa$ admits a decomposition

$$M = \bigoplus M^\lambda, \qquad M^\lambda \in \tilde{\mathcal{O}}_{\kappa,\lambda}, \tag{5.9.1}$$

where λ runs over all the weights in ${}_\kappa\mathfrak{t}^*$ such that $\lambda+\rho$ is antidominant. The projection

$$\mathrm{pr}_\lambda : \tilde{\mathcal{O}}_\kappa \to \tilde{\mathcal{O}}_{\kappa,\lambda}, \quad M \mapsto M^\lambda,$$

is an exact functor. Fix two integral weights λ, μ in $\mathfrak{t}^*$ such that $\lambda+\rho$, $\mu+\rho$ are antidominant and the integral weight $\nu = \lambda-\mu$ is dominant. Assume that $\lambda \in {}_\kappa\mathfrak{t}^*$, then μ belongs to ${}_{\kappa'}\mathfrak{t}^*$ for an integer $\kappa' < \kappa$. Let $V(\nu)$ be the simple $\mathfrak{g}$-module of highest weight ν. Then for any $M \in \tilde{\mathcal{O}}_{\kappa'}$ the module $M \otimes V(\nu)$ belongs to $\tilde{\mathcal{O}}_\kappa$. Therefore we can define the following translation functor

$$\theta^\nu : \tilde{\mathcal{O}}_{\kappa',\mu} \to \tilde{\mathcal{O}}_{\kappa,\lambda}, \quad M \mapsto \mathrm{pr}_\lambda(M \otimes V(\nu)),$$

see [Kum94]. Note that the subcategory $\tilde{\mathcal{O}}_{\kappa,\lambda}$ of $\tilde{\mathcal{O}}$ is stable under the duality $\mathbf{D}$, because $\mathbf{D}$ fixes simples modules. We have a canonical isomorphism of functors

$$\theta^\nu \circ \mathbf{D} = \mathbf{D} \circ \theta^\nu. \tag{5.9.2}$$

Indeed, it follows from (5.4.3) that $\mathbf{D}(M \otimes V(\nu)) = \mathbf{D}(M) \otimes \mathbf{D}(V(\nu))$ as $\mathfrak{g}$-modules. Since $V(\nu)$ is simple, we have $\mathbf{D}V(\nu) = V(\nu)$. The equality (5.9.2) follows.

On the geometric side, recall the T-torsor $\pi : X^\dagger \to X$. For any integral weight $\lambda \in \mathfrak{t}^*$ the family of line bundles $\mathscr{L}^\lambda_{X_w}$ (see Section 4.5) with $w \in \mathfrak{S}$ form a projective system of $\mathscr{O}$-modules under restriction, yielding a flat object $\mathscr{L}^\lambda$ of $\hat{\mathbf{O}}(X)$. Note that $\pi^*(\mathscr{L}^\lambda)$ is a line bundle on $X^\dagger$. For integral weights λ, μ in $\mathfrak{t}^*$ the translation functor

$$\Theta^{\lambda-\mu} : \mathbf{M}^\mu_0(X) \to \mathbf{M}^\lambda_0(X), \quad \mathscr{M} \mapsto \mathscr{M} \otimes_{\mathscr{O}_{X^\dagger}} \pi^*(\mathscr{L}^{\lambda-\mu}),$$

is an equivalence of categories. A quasi-inverse is given by $\Theta^{\mu-\lambda}$. By the projection formula we have

$$\Theta^{\lambda-\mu}(\mathcal{A}^\mu_{w\sharp}) = \mathcal{A}^\lambda_{w\sharp}, \quad \text{for} \quad \sharp =!,\ !\bullet,\ \bullet. \tag{5.9.3}$$

Now, assume that $\mu+\rho$ is antidominant. Consider the exact functor

$$\Gamma : \mathbf{M}^\mu(X) \to \mathbf{M}(\mathfrak{g}), \quad \mathscr{M} \mapsto \Gamma(\mathscr{M})$$

as in Proposition 5.3.1. Note that if $\mu+\rho$ is regular, then Γ maps $\mathscr{A}^{\mu}_{v!\bullet}$ to $L_\kappa(v\cdot\mu)$ by Proposition 5.4.1. Since the subcategory $\tilde{\mathcal{O}}_\kappa$ of $\mathbf{M}(\mathfrak{g})$ is stable under extension, the exact functor Γ restricts to a functor

$$\Gamma : \mathbf{M}^{\mu}_0(X) \to \tilde{\mathcal{O}}_{\kappa,\mu}.$$

The next proposition is an affine analogue of [BG99, Proposition 2.8].

Proposition 5.9.1. *Let λ, μ be integral weights in $\mathfrak{t}^*$ such that $\lambda+\rho$, $\mu+\rho$ are antidominant and $\nu=\lambda-\mu$ is dominant. Assume further that $\mu+\rho$ is regular. Then the functors*

$$\theta^{\nu}\circ\Gamma :\ \mathbf{M}^{\mu}_0(X)\to\tilde{\mathcal{O}}_{\kappa,\lambda}\subset\mathbf{M}(\mathfrak{g})\quad \textit{and}\quad \Gamma\circ\Theta^{\nu} :\ \mathbf{M}^{\mu}_0(X)\to\mathbf{M}(\mathfrak{g})$$

are isomorphic.

Proof. We will prove the proposition in several steps.

Step 1. First, we define a category $\mathbf{Sh}(X)$ of sheaves of $\mathbb{C}$-vector spaces on X and we consider $\mathfrak{g}$-modules in this category. To do this, for $w\in\mathfrak{S}$ let $\mathbf{Sh}(\overline{X}_w)$ be the category of sheaves of $\mathbb{C}$-vector spaces on $\overline{X}_w$. For $w\leqslant x$ we have a closed embedding $i_{w,x}:\overline{X}_w\to\overline{X}_x$, and an exact functor

$$i^!_{w,x}:\mathbf{Sh}(\overline{X}_x)\to\mathbf{Sh}(\overline{X}_w),\quad \mathscr{F}\mapsto i^!_{w,x}(\mathscr{F}),$$

where $i^!_{w,x}(\mathscr{F})$ is the subsheaf of $\mathscr{F}$ consisting of the local sections supported set-theoretically on $\overline{X}_w$. We get a projective system of categories

$$(\mathbf{Sh}(\overline{X}_w),\ i^!_{w,x}).$$

Following [BD00, 7.15.10] we define the category of sheaves of $\mathbb{C}$-vector spaces on X to be the projective limit

$$\mathbf{Sh}(X)=2\varprojlim\mathbf{Sh}(\overline{X}_w).$$

This is an abelian category. By the same arguments as in the second paragraph of Section 4.7, the category $\mathbf{Sh}(\overline{X}_w)$ is canonically identified with a full subcategory of $\mathbf{Sh}(X)$, and each object $\mathscr{F}\in\mathbf{Sh}(X)$ is a direct limit

$$\mathscr{F}=\varinjlim\mathscr{F}_w,\quad \mathscr{F}_w\in\mathbf{Sh}(\overline{X}_w).$$

The space of global sections of an object of $\mathbf{Sh}(X)$ is given by

$$\Gamma(X,\mathscr{F})=\varinjlim\Gamma(\overline{X}_w,\mathscr{F}_w).$$

Next, consider the *forgetful* functor

$$\mathbf{O}(\overline{X}_w)\to\mathbf{Sh}(\overline{X}_w),\quad \mathscr{N}\mapsto\mathscr{N}^{\mathbb{C}}.$$

Recall that for $\mathscr{M}\in\mathbf{O}(X)$ we have $\mathscr{M}=\varinjlim\mathscr{M}_w$ with $\mathscr{M}_w\in\mathbf{O}(\overline{X}_w)$. The tuple of sheaves of $\mathbb{C}$-vector spaces

$$\varinjlim_{x\geqslant w} i^!_{w,x}(\mathscr{M}^{\mathbb{C}}_x),\ w\in\mathfrak{S},$$

gives an object in $\mathbf{Sh}(X)$. Let us denote it by $\mathscr{M}^{\mathbb{C}}$. The assignment $\mathscr{M}\mapsto\mathscr{M}^{\mathbb{C}}$ yields a faithful exact functor

$$\mathbf{O}(X)\to\mathbf{Sh}(X)$$

such that

$$\Gamma(X, \mathscr{M}) = \Gamma(X, \mathscr{M}^{\mathbb{C}}), \tag{5.9.4}$$

see (4.7.1) for the definition of the left hand side. Now, let $\mathscr{F} = (\mathscr{F}_w)$ be an object in $\mathbf{Sh}(X)$. The vector spaces $\mathrm{End}(\mathscr{F}_w)$ form a projective system via the maps

$$\mathrm{End}(\mathscr{F}_x) \to \mathrm{End}(\mathscr{F}_w), \quad f \mapsto i^!_{w,x}(f).$$

We set

$$\mathrm{End}(\mathscr{F}) = \varprojlim_w \mathrm{End}(\mathscr{F}_w). \tag{5.9.5}$$

We say that an object $\mathscr{F}$ of $\mathbf{Sh}(X)$ is a $\mathfrak{g}$-*module* if it is equipped with an algebra homomorphism

$$\mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathscr{F}).$$

For instance, for $\mathscr{M} \in \mathbf{M}^T(X^\dagger)$ the object $(\mathscr{M}^\dagger)^{\mathbb{C}}$ of $\mathbf{Sh}(X)$ is a $\mathfrak{g}$-module via the algebra homomorphism

$$\delta_l : \mathcal{U}(\mathfrak{g}) \to \Gamma(X^\dagger, \mathscr{D}_{X^\dagger}) \tag{5.9.6}$$

See also the beginning of Section 5.3.

Step 2. Next, we define G-modules in $\hat{\mathbf{O}}(X)$. A *standard parabolic subgroup* of G is a group scheme of the form $P = Q \times_{G_0} P_0$ with P_0 a parabolic subgroup of G_0. Here the morphism $Q \to G_0$ is the canonical one. We fix a subposet ${}^P\mathfrak{S} \subset \mathfrak{S}$ such that for $w \in {}^P\mathfrak{S}$ the subscheme $\overline{X}_w \subset X$ is stable under the P-action and

$$X = \varinjlim_{w \in {}^P\mathfrak{S}} \overline{X}_w.$$

We say that an object $\mathscr{F} = (\mathscr{F}_w)$ of $\hat{\mathbf{O}}(X)$ has an *algebraic P-action* if $\mathscr{F}_w$ has the structure of a P-equivariant quasicoherent $\mathscr{O}_{\overline{X}_w}$-module for $w \in {}^P\mathfrak{S}$ and if the isomorphism $i^*_{w,x}\mathscr{F}_x \cong \mathscr{F}_w$ is P-equivariant for $w \leqslant x$. Finally, we say that $\mathscr{F}$ is a *G-module* if it is equipped with an action of the (abstract) group G such that for any standard parabolic subgroup P, the P-action on $\mathscr{F}$ is algebraic.

We are interested in a family of G-modules $\mathscr{V}^i$ in $\hat{\mathbf{O}}(X)$ defined as follows. Fix a basis $(m_i)_{i\in\mathbb{N}}$ of $V(\nu)$ such that each m_i is a weight vector of weight ν_i and $\nu_j > \nu_i$ implies $j < i$. By assumption we have $\nu_0 = \nu$. For each i let V^i be the subspace of $V(\nu)$ spanned by the vectors m_j for $j \leqslant i$. Then

$$V^0 \subset V^1 \subset V^2 \subset \dots$$

is a sequence of B-submodule of $V(\nu)$. We write $V^\infty = V(\nu)$. For $0 \leqslant i \leqslant \infty$ we define a $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{V}^i_{\mathfrak{X}}$ on $\mathfrak{X}$ such that for any open set $U \subset \mathfrak{X}$ we have

$$\Gamma(U, \mathscr{V}^i_{\mathfrak{X}}) = \{f : p^{-1}(U) \to V^i \mid f(gb^{-1}) = bf(g)\},$$

where $p : G_\infty \to \mathfrak{X}$ is the quotient map. Let $\mathscr{V}^i_w$ be the restriction of $\mathscr{V}^i_{\mathfrak{X}}$ to $\overline{X}_w$. Then $(\mathscr{V}^i_w)_{w\in\mathfrak{S}}$ is a flat G-module in $\hat{\mathbf{O}}(X)$. We will denote it by $\mathscr{V}^i$. Note that since $V(\nu)$ admits a G-action, the G-module $\mathscr{V}^\infty \in \hat{\mathbf{O}}(X)$ is isomorphic to the G-module $\mathscr{O}_X \otimes V(\nu)$ with G acting diagonally. Therefore, for $\mathscr{M} \in \mathbf{M}^\mu_0(X)$ the projection formula yields a canonical isomorphism of vector spaces

$$\begin{aligned} \Gamma(\mathscr{M}) \otimes V(\nu) &= \Gamma(X, \mathscr{M}^\dagger) \otimes V(\nu) \\ &= \Gamma(X, \mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^\infty). \end{aligned} \tag{5.9.7}$$

On the other hand, we have

$$\begin{aligned}\Gamma(\Theta^\nu(\mathscr{M})) &= \Gamma(X, (\mathscr{M} \otimes_{\mathscr{O}_{X^\dagger}} \pi^*(\mathscr{L}^\nu))^\dagger) \\ &= \Gamma(X, \mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^\nu).\end{aligned} \tag{5.9.8}$$

Our goal is to compare the $\mathfrak{g}$-modules $\Gamma(\Theta^\nu(\mathscr{M}))$ and the direct factor $\theta^\nu(\Gamma(\mathscr{M}))$ of $\Gamma(\mathscr{M}) \otimes V(\nu)$. To this end, we first define in Step 3 a $\mathfrak{g}$-action on $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i)^\mathbb{C}$ for each i, then we prove in Steps 4-6 that the inclusion

$$(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^\nu)^\mathbb{C} \to (\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^\infty)^\mathbb{C} \tag{5.9.9}$$

induced by the inclusion $\mathscr{L}^\nu = \mathscr{V}^0 \subset \mathscr{V}^\infty$ splits as a $\mathfrak{g}$-module homomorphism in $\mathbf{Sh}(X)$.

Step 3. Let P be a standard parabolic subgroup of G, and let $\mathfrak{p}$ be its Lie algebra. Let ${}^P\mathfrak{S} \subset \mathfrak{S}$ be as in Step 2. The P-action on $\mathscr{V}^i$ yields a Lie algebra homomorphism

$$\mathfrak{p} \to \mathrm{End}(\mathscr{V}^i_w), \quad \forall\, w \in {}^P\mathfrak{S}.$$

Consider the $\mathfrak{g}$-action on $(\mathscr{M}^\dagger)^\mathbb{C}$ given by the map δ_l in (5.9.6). Note that for $w \leqslant x$ in ${}^P\mathfrak{S}$, any element $\xi \in \mathfrak{p}$ maps a local section of $\mathscr{M}^\dagger_x$ supported on $\overline{X}_w$ to a local section of $\mathscr{M}^\dagger_x$ with the same property. In particular, for $w \in {}^P\mathfrak{S}$ we have a Lie algebra homomorphism

$$\mathfrak{p} \to \mathrm{End}\big((\mathscr{M}^\dagger_w \otimes_{\mathscr{O}_{\overline{X}_w}} \mathscr{V}^i_w)^\mathbb{C}\big), \quad \xi \mapsto (m \otimes v \mapsto \xi m \otimes v + m \otimes \xi v), \tag{5.9.10}$$

where m denotes a local section of $\mathscr{M}^\dagger_w$, v denotes a local section of $\mathscr{V}^i_w$. These maps are compatible with the restriction

$$\mathrm{End}\big((\mathscr{M}^\dagger_x \otimes_{\mathscr{O}_{\overline{X}_x}} \mathscr{V}^i_x)^\mathbb{C}\big) \to \mathrm{End}\big((\mathscr{M}^\dagger_w \otimes_{\mathscr{O}_{\overline{X}_w}} \mathscr{V}^i_w)^\mathbb{C}\big), \quad f \mapsto i^!_{w,x}(f).$$

They yield a Lie algebra homomorphism

$$\mathfrak{p} \to \mathrm{End}\big((\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i)^\mathbb{C}\big).$$

As P varies, these maps glue together yielding a Lie algebra homomorphism

$$\mathfrak{g} \to \mathrm{End}\big((\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i)^\mathbb{C}\big). \tag{5.9.11}$$

This defines a $\mathfrak{g}$-action on $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i)^\mathbb{C}$ such that the obvious inclusions

$$(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^0)^\mathbb{C} \subset (\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^1)^\mathbb{C} \subset \cdots$$

are $\mathfrak{g}$-equivariant. So (5.9.9) is a $\mathfrak{g}$-module homomorphism. Note that the flatness of $\mathscr{V}^i$ yields an isomorphism in $\mathbf{O}(X)$

$$\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i / \mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^{i-1} \cong \mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^{\nu_i}. \tag{5.9.12}$$

Step 4. In order to show that the $\mathfrak{g}$-module homomorphism (5.9.9) splits, we consider the generalized Casimir operator of $\mathfrak{g}$. Identify $\mathfrak{t}$ and $\mathfrak{t}^*$ via the pairing $\langle - : - \rangle$. Let $\rho^\vee \in \mathfrak{t}$ be the image of ρ. Let h_i be a basis of $\mathfrak{t}_0$, and let h^i be its dual basis in $\mathfrak{t}_0$ with respect to the pairing $\langle - : - \rangle$. For $\xi \in \mathfrak{g}_0$ and $n \in \mathbb{Z}$ we will abbreviate $\xi^{(n)} = \xi \otimes t^n$ and $\xi = \xi^{(0)}$. The generalized Casimir operator is given by the formal sum

$$\mathfrak{C} = 2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1} + \sum_{i<j} e_{ji}e_{ij} + \sum_{n\geqslant 1}\sum_{i\neq j} e_{ij}^{(-n)} e_{ji}^{(n)} + \sum_{n\geqslant 1}\sum_i h^{i,(-n)} h_i^{(n)}, \tag{5.9.13}$$

see e.g., [Kac90, Section 2.5]. Let $\delta_l(\mathfrak{C})$ be the formal sum given by applying δ_l term by term to the right hand side of (5.9.13). We claim that $\delta_l(\mathfrak{C})$ is a well defined element in $\Gamma(X^\dagger, \mathscr{D}_{X^\dagger})$, i.e., the sum is finite at each point of $X^\dagger$. More precisely, let

$$\Sigma = \{e_{ij} \,|\, i < j\} \cup \{e_{ji}^{(n)}, h_i^{(n)} \,|\, i \neq j, n \geqslant 1\},$$

and let e be the base point of $X^\dagger$. We need to prove that the sets

$$\Sigma_g = \{\xi \in \Sigma \,|\, \delta_l(\xi)(ge) \neq 0\}, \quad g \in G,$$

are finite. To show this, consider the adjoint action of G on $\mathfrak{g}$

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g,$$

and the G-action on the object $\mathscr{D}_{X^\dagger}$ of $\hat{\mathbf{O}}(X^\dagger)$ coming from the G-action on $X^\dagger$. The map δ_l is G-equivariant with respect to these actions. So for $\xi \in \mathfrak{g}$ and $g \in G$ we have

$$\delta_l(\xi)(ge) \neq 0 \quad \Longleftrightarrow \quad \delta_l(\mathrm{Ad}_{g^{-1}}(\xi))(e) \neq 0.$$

Further the right hand side holds if and only if $\mathrm{Ad}_{g^{-1}}(\xi) \notin \mathfrak{n}$. Therefore

$$\Sigma_g = \{\xi \in \Sigma \mid \ \mathrm{Ad}_{g^{-1}}(\xi) \notin \mathfrak{n}\}$$

is a finite set, the claim is proved. By consequence $\mathfrak{C}$ acts on the $\mathfrak{g}$-module $(\mathscr{M}^\dagger)^{\mathbb{C}}$ for any $\mathscr{M} \in \mathbf{M}^T(X^\dagger)$. Next, we claim that the action of $\mathfrak{C}$ on the $\mathfrak{g}$-module $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i)^{\mathbb{C}}$ is also well defined. It is enough to prove this for $(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{V}^\infty)^{\mathbb{C}}$. By (5.9.10) the action of $\mathfrak{C}$ on $(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{V}^\infty)^{\mathbb{C}}$ is given by the operator

$$\mathfrak{C} \otimes 1 + 1 \otimes \mathfrak{C} - \sum_{n \in \mathbb{Z}, i \neq j} e_{ij}^{(-n)} \otimes e_{ji}^{(n)} - \sum_{n \in \mathbb{Z}, i} h^{i,(-n)} \otimes h_i^{(n)}.$$

Since for both $\mathscr{M}^\dagger$ and $\mathscr{V}^\infty$, at each point, there are only finitely many elements from Σ which act nontrivially on it, the action of $\mathfrak{C}$ on the tensor product is well defined.

Step 5. Now, let us calculate the action of $\mathfrak{C}$ on $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^{\nu_i})^{\mathbb{C}}$. We have

$$\mathrm{Ad}_{g^{-1}}(\mathfrak{C}) = \mathfrak{C}, \quad \forall\, g \in G.$$

Therefore the global section $\delta_l(\mathfrak{C})$ is G-invariant and its value at e is

$$\delta_l(\mathfrak{C})(e) = \delta_l(2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1})(e).$$

On the other hand, the right T-action on $X^\dagger$ yields a map

$$\delta_r : \mathfrak{t} \to \Gamma(X^\dagger, \mathscr{D}_{X^\dagger}).$$

Since the right T-action commutes with the left G-action, for any $h \in \mathfrak{t}$ the global section $\delta_r(h)$ is G-invariant. We have $\delta_r(h)(e) = -\delta_l(h)(e)$ because the left and right T-actions on the point e are inverse to each other. Therefore the global sections $\delta_l(\mathfrak{C})$ and $\delta_r(-2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1})$ takes the same value at the point e. Since both of them are G-invariant, we deduce that

$$\delta_l(\mathfrak{C}) = \delta_r(-2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1}).$$

Recall from Section 4.4 that for $\lambda \in \mathfrak{t}^*$ and $\mathscr{M} \in \mathbf{M}^\lambda(X)$ the operator $\delta_r(-2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1})$ acts on $\mathscr{M}^\dagger$ by the scalar

$$-\lambda(-2\rho^\vee + \sum_i h^i h_i + 2\partial\mathbf{1}) = ||\lambda+\rho||^2 - ||\rho||^2.$$

Therefore $\mathfrak{C}$ acts on $\mathscr{M}^\dagger$ by the same scalar. In particular, for $\mathscr{M} \in \mathbf{M}^\mu(X)$ and $i \in \mathbb{N}$, the element $\mathfrak{C}$ acts on $\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^{\nu_i}$ by $||\mu+\nu_i+\rho||^2 - ||\rho||^2$. Note that the isomorphism (5.9.12) is compatible with the $\mathfrak{g}$-actions. So $\mathfrak{C}$ also acts by $||\mu+\nu_i+\rho||^2 - ||\rho||^2$ on $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^i / \mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^{i-1})^{\mathbb{C}}$.

Step 6. Now, we can complete the proof of the proposition. First, we claim that

$$||\lambda+\rho||^2 - ||\rho||^2 = ||\mu+\nu_i+\rho||^2 - ||\rho||^2 \iff \nu_i = \nu. \tag{5.9.14}$$

The "if" part is trivial. For the "only if" part, we have by assumption

$$\begin{aligned} ||\mu+\nu+\rho||^2 &= ||\mu+\nu_i+\rho||^2 \\ &= ||\mu+\nu+\rho||^2 + ||\nu-\nu_i||^2 - 2\langle\mu+\nu+\rho : \nu-\nu_i\rangle. \end{aligned}$$

Since $\nu - \nu_i \in \mathbb{N}\Pi^+$ and $\mu+\nu+\rho = \lambda+\rho$ is antidominant, the term $-2\langle\lambda+\rho : \nu-\nu_i\rangle$ is positive. Hence the equality implies that $||\nu-\nu_i||^2 = 0$. So $\nu-\nu_i$ belongs to $\mathbb{N}\delta$. But $\langle\lambda+\rho : \delta\rangle = \kappa < 0$. So we have $\nu = \nu_i$. This proves the claim in (5.9.14). A direct consequence of this claim and of Step 5 is that the $\mathfrak{g}$-module monomorphism (5.9.9) splits. It induces an isomorphism of $\mathfrak{g}$-modules

$$\Gamma(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^\nu) = \mathrm{pr}_\lambda \Gamma(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^\infty), \quad \mathscr{M} \in \mathbf{M}_0^\mu(X). \tag{5.9.15}$$

Finally, note that the vector spaces isomorphisms (5.9.7) and (5.9.8) are indeed isomorphisms of $\mathfrak{g}$-modules by the definition of the $\mathfrak{g}$-actions on $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{V}^\infty)^{\mathbb{C}}$ and $(\mathscr{M}^\dagger \otimes_{\mathscr{O}_X} \mathscr{L}^\nu)^{\mathbb{C}}$. Therefore (5.9.15) yields an isomorphism of $\mathfrak{g}$-modules

$$\Gamma(\Theta^\nu(\mathscr{M})) = \theta^\nu(\Gamma(\mathscr{M})).$$

$\square$

Remark 5.9.2. We have assumed $\mu+\rho$ regular in Proposition 5.9.1 in order to have $\Gamma(\mathbf{M}_0^\mu(X)) \subset \tilde{\mathcal{O}}_{\kappa,\mu}$. It follows from Proposition 5.9.1 that this inclusion still holds if $\mu+\rho$ is not regular. So Proposition 5.9.1 makes sense without this regularity assumption, and the proof is the same in this case.

Corollary 5.9.3. *Let $\lambda \in \Lambda$ such that $\lambda+\rho$ is antidominant. Then*

(a) $\Gamma(\mathscr{A}^\lambda_{w!}) = N_\kappa(w\cdot\lambda)$,

(b) $\Gamma(\mathscr{A}^\lambda_{w\bullet}) = \mathbf{D}N_\kappa(w\cdot\lambda)$,

(c) $\Gamma(\mathscr{A}^\lambda_{w!\bullet}) = \begin{cases} L_\kappa(w\cdot\lambda) & \text{if } w \text{ is the shortest element in } w\mathfrak{S}(\lambda), \\ 0 & \text{else.} \end{cases}$

Proof. By Proposition 5.4.1 it is enough to prove the corollary in the case when $\lambda+\rho$ is not regular. Let ω_i, $0 \leqslant i \leqslant m-1$, be the fundamental weights in $\mathfrak{t}^*$. Let

$$\nu = \sum \omega_i,$$

where the sum runs over all $i = 0, \dots, m-1$ such that $\langle \lambda + \rho : \alpha_i \rangle = 0$. The weight ν is dominant. Let $\mu = \lambda - \nu$. Then $\mu + \rho$ is an antidominant weight. It is moreover regular, because we have

$$\langle \mu + \rho : \alpha_i \rangle = \langle \lambda + \rho : \alpha_i \rangle - \langle \nu : \alpha_i \rangle < 0, \quad 0 \leqslant i \leqslant m-1.$$

Let $\kappa' = \langle \mu + \rho : \delta \rangle$. So Propositions 5.4.1, 5.9.1 and the equation (5.9.3) implies that

$$\Gamma(\mathscr{A}^\lambda_{w!}) = \theta^\nu(N_{\kappa'}(w \cdot \mu)), \quad \Gamma(\mathscr{A}^\lambda_{w!\bullet}) = \theta^\nu(L_{\kappa'}(w \cdot \mu)), \quad \Gamma(\mathscr{A}^\lambda_{w\bullet}) = \theta^\nu(\mathbf{D}N_{\kappa'}(w \cdot \mu)).$$

So parts (a), (c) follow from the properties of the translation functor θ^ν given in [Kum94, Proposition 1.7]. Part (b) follows from (a) and the equality (5.9.2). $\square$

5.10 The parabolic Verma modules.

Let ${}^Q\mathfrak{S}$ be the set of the longest representatives of the cosets $\mathfrak{S}_0 \backslash \mathfrak{S}$. Let w_0 be the longest element in $\mathfrak{S}_0$. Recall the following basic facts.

Lemma 5.10.1. *For $w \in \mathfrak{S}$ if $w \cdot \lambda \in \Lambda^+$ for some $\lambda \in \Lambda$ with $\lambda + \rho$ antidominant, then $w \in {}^Q\mathfrak{S}$. Further, if $w \in {}^Q\mathfrak{S}$ then we have*

(a) the element w is the unique element v in $\mathfrak{S}_0 w$ such that $\Pi_0^+ \subset -v(\Pi^+)$,

(b) for any $v \in \mathfrak{S}_0$ we have $l(vw) = l(w) - l(v)$,

(c) the element $w_0 w$ is the shortest element in $\mathfrak{S}_0 w$.

The Q-orbit decomposition of X is given by

$$X = \bigsqcup_{w \in {}^Q\mathfrak{S}} Y_w, \qquad Y_w = Q\dot{w}B/B.$$

Each Y_w is a smooth subscheme of X, and X_w is open and dense in Y_w. The closure of Y_w in X is a projective irreducible variety of dimension $l(w)$ given by

$$\overline{Y}_w = \bigsqcup_{w' \in {}^Q\mathfrak{S},\, w' \leqslant w} Y_{w'}.$$

Recall that $Y_w^\dagger = \pi^{-1}(Y_w)$. The canonical embedding $j_w : Y_w^\dagger \to X^\dagger$ is locally closed and affine, see Remark 6.2.2(b). For $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$ let

$$\mathscr{B}^\lambda_w = \Omega_{Y_w^\dagger} \otimes_{\mathscr{O}_{Y_w^\dagger}} \pi^*(\mathscr{L}^\lambda_{Y_w}). \tag{5.10.1}$$

We have the following objects in $\mathbf{M}^\lambda_h(X)$

$$\mathscr{B}^\lambda_{w!} = j_{w!}(\mathscr{B}^\lambda_w), \quad \mathscr{B}^\lambda_{w!\bullet} = j_{w!\bullet}(\mathscr{B}^\lambda_w), \quad \mathscr{B}^\lambda_{w\bullet} = j_{w\bullet}(\mathscr{B}^\lambda_w).$$

Now, consider the canonical embedding $r : X_w^\dagger \to Y_w^\dagger$. Consider the triple of adjoint functors $(r_!, r^*, r_\bullet)$ between the categories $\mathbf{M}^{\tilde{\lambda}}_h(Y_w)$ and $\mathbf{M}^{\tilde{\lambda}}_h(X_w)$, see Lemma 4.2.1(c). Note that $r^*(\mathscr{B}^\lambda_w) \cong \mathscr{A}^\lambda_w$. We have the following lemma.

Lemma 5.10.2. *For $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$ the following holds.*

(a) The unit map $r_! r^* \to \mathrm{Id}$ yields a surjective morphism in $\mathbf{M}^\lambda_h(Y_w)$

$$r_!(\mathscr{A}^\lambda_w) \to \mathscr{B}^\lambda_w. \tag{5.10.2}$$

(b) The counit map $\mathrm{Id} \to r_\bullet r^*$ yields an injective morphism in $\mathbf{M}^\lambda_h(Y_w)$

$$\mathscr{B}^\lambda_w \to r_\bullet(\mathscr{A}^\lambda_w). \tag{5.10.3}$$

Proof. We begin by considering part (b). It is enough to prove that the $\mathscr{O}_{Y_w^\dagger}$-module morphism

$$(\mathscr{B}_w^\lambda)^{\mathscr{O}} \to \big(r_\bullet r^*(\mathscr{B}_w^\lambda)\big)^{\mathscr{O}} \tag{5.10.4}$$

is injective. By Lemma 4.2.1 the right hand side is equal to $r_*r^*\big((\mathscr{B}_w^\lambda)^{\mathscr{O}}\big)$. Now, consider the closed embedding

$$i : Y_w^\dagger - X_w^\dagger \longrightarrow Y_w^\dagger.$$

The morphism (5.10.4) can be completed into the following exact sequence in $\mathbf{O}(Y_w^\dagger)$,

$$0 \to i_*i^!\big((\mathscr{B}_w^\lambda)^{\mathscr{O}}\big) \to (\mathscr{B}_w^\lambda)^{\mathscr{O}} \to r_*r^*\big((\mathscr{B}_w^\lambda)^{\mathscr{O}}\big),$$

see e.g., [HTT08, Proposition 1.7.1]. Note that the $\mathscr{O}_{Y_w^\dagger}$-module $(\mathscr{B}_w^\lambda)^{\mathscr{O}}$ is locally free. So it has no subsheaf supported on the closed subscheme $Y_w^\dagger - X_w^\dagger$. We deduce that $i_*i^!\big((\mathscr{B}_w^\lambda)^{\mathscr{O}}\big) = 0$. So the morphism (5.10.4) is injective. This proves part (b). Now, consider the (contravariant) duality functor $\mathbb{D}$ on $\mathbf{M}_h^\lambda(Y_w)$, see Section 4.5. We have

$$\mathbb{D}(\mathscr{B}_w^\lambda) = \mathscr{B}_w^\lambda, \quad \mathbb{D}(r_!(\mathscr{A}_w^\lambda)) = r_\bullet(\mathbb{D}(\mathscr{A}_w^\lambda)) = r_\bullet(\mathscr{A}_w^\lambda).$$

So applying $\mathbb{D}$ to the morphism (5.10.3) we get the morphism (5.10.2). So part (b) implies part (a). ☐

Lemma 5.10.3. *For $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$ we have*

$$\mathscr{A}_{w!\bullet}^\lambda \cong \mathscr{B}_{w!\bullet}^\lambda.$$

Proof. By applying the exact functor $j_{w\bullet}$ to the map (5.10.3) we see that $\mathscr{B}_{w\bullet}^\lambda$ is a subobject of $\mathscr{A}_{w\bullet}^\lambda$ in $\mathbf{M}_h^\lambda(X)$. In particular $\mathscr{B}_{w!\bullet}^\lambda$ is a simple subobject of $\mathscr{A}_{w\bullet}^\lambda$. So it is isomorphic to $\mathscr{A}_{w!\bullet}^\lambda$. ☐

Proposition 5.10.4. *Let $\lambda \in \Lambda$ such that $\lambda + \rho$ is antidominant, and let $w \in {}^Q\mathfrak{S}$.*

(a) If there exists $\alpha \in \Pi_0^+$ such that $\langle w(\lambda+\rho) : \alpha\rangle = 0$, then

$$\Gamma(\mathscr{B}_{w!}^\lambda) = 0.$$

(b) We have

$$\langle w(\lambda+\rho) : \alpha\rangle \neq 0, \quad \forall\ \alpha \in \Pi_0^+ \quad \iff \quad w \cdot \lambda \in \Lambda^+.$$

In this case, we have

$$\Gamma(\mathscr{B}_{w!}^\lambda) = M_\kappa(w\cdot\lambda), \qquad \Gamma(\mathscr{B}_{w\bullet}^\lambda) = \mathbf{D}M_\kappa(w\cdot\lambda).$$

(c) We have

$$\Gamma(\mathscr{B}_{w!\bullet}^\lambda) = \begin{cases} L_\kappa(w\cdot\lambda) & \textit{if } w \textit{ is the shortest element in } w\mathfrak{S}(\lambda), \\ 0 & \textit{else.}\end{cases}$$

Proof. The proof is inspired by the proof in the finite type case, see e.g., [Mil93, Theorem G.2.10]. First, by Kazhdan-Lusztig's algorithm, see Remark 7.2.3, in the Grothendieck group $[\mathbf{M}_0^\lambda(X)]$ we have

$$[\mathscr{B}_{w!}^\lambda] = \sum_{y\in\mathfrak{S}_0} (-1)^{l(y)}[\mathscr{A}_{yw!}^\lambda]. \tag{5.10.5}$$

Since $\lambda+\rho$ is antidominant, the functor Γ is exact on $\mathbf{M}_0^\lambda(X)$ by Proposition 5.3.1(a). Therefore we have the following equalities in $[\tilde{\mathcal{O}}_\kappa]$

$$\begin{aligned}[\Gamma(\mathscr{B}^\lambda_{w!})] &= \sum_{y\in\mathfrak{S}_0}(-1)^{l(y)}[\Gamma(\mathscr{A}^\lambda_{yw!})] \\ &= \sum_{y\in\mathfrak{S}_0}(-1)^{l(y)}[N_\kappa(yw\cdot\lambda)]. \end{aligned} \tag{5.10.6}$$

Here the second equality is given by Corollary 5.9.3. Now, suppose that there exists $\alpha\in\Pi_0^+$ such that $\langle w(\lambda+\rho):\alpha\rangle=0$. Let s_α be the corresponding reflection in $\mathfrak{S}_0$. Then we have

$$s_\alpha w\cdot\lambda = w\cdot\lambda\,.$$

By Lemma 5.10.1(b) we have $l(w)=l(s_\alpha w)+1$. So the right hand side of (5.10.6) vanishes. Therefore we have $\Gamma(\mathscr{B}^\lambda_{w!})=0$. This proves part (a). Now, let us concentrate on part (b). Note that $\lambda+\rho$ is antidominant. Thus by Lemma 5.10.1(a) we have $\langle w(\lambda+\rho):\alpha\rangle\in\mathbb{N}$ for any $\alpha\in\Pi_0^+$. Hence

$$\langle w(\lambda+\rho):\alpha\rangle\neq 0 \quad\Longleftrightarrow\quad \langle w(\lambda+\rho):\alpha\rangle\geqslant 1 \quad\Longleftrightarrow\quad \langle w\cdot\lambda:\alpha\rangle\geqslant 0.$$

By consequence $\langle w(\lambda+\rho):\alpha\rangle\neq 0$ for all $\alpha\in\Pi_0^+$ if and only if $w\cdot\lambda$ belongs to Λ^+. In this case, the right hand side of (5.10.6) is equal to $[M_\kappa(w\cdot\lambda)]$ by the BGG-resolution. We deduce that

$$[\Gamma(\mathscr{B}^\lambda_{w!})] = [M_\kappa(w\cdot\lambda)]. \tag{5.10.7}$$

Now, applying the exact functor $j_!$ to the surjective morphism in (5.10.2) yields a quotient map $\mathscr{B}^\lambda_{w!}\to\mathscr{A}^\lambda_{w!}$ in $\mathbf{M}^\lambda(X)$. The exactness of Γ implies that $\Gamma(\mathscr{B}^\lambda_{w!})$ is a quotient of $N_\kappa(w\cdot\lambda)=\Gamma(\mathscr{A}^\lambda_{w!})$. Since $M_\kappa(w\cdot\lambda)$ is the maximal $\mathfrak{q}$-locally-finite quotient of $N_\kappa(w\cdot\lambda)$ and $\Gamma(\mathscr{B}^\lambda_{w!})$ is $\mathfrak{q}$-locally finite, we deduce that $\Gamma(\mathscr{B}^\lambda_{w!})$ is a quotient of $M_\kappa(w\cdot\lambda)$. So the first equality in part (b) follows from (5.10.7). The proof of the second one is similar. Finally, part (c) follows from Lemma 5.10.3 and Corollary 5.9.3. □

Remark 5.10.5. Note that if $w\in{}^{\mathfrak{Q}}\mathfrak{S}$ is a shortest element in $w\mathfrak{S}(\lambda)$, then we have $\langle w(\lambda+\rho):\alpha\rangle\neq 0$ for all $\alpha\in\Pi_0^+$. Indeed, if there exists $\alpha\in\Pi_0^+$ such that $\langle w(\lambda+\rho):\alpha\rangle=0$. Let $s'=w^{-1}s_\alpha w$. Then s' belongs to $\mathfrak{S}(\lambda)$. Therefore we have $l(ws')>l(w)$. But $ws'=s_\alpha w$ and $s_\alpha\in\mathfrak{S}_0$, by Lemma 5.10.1 we have $l(ws')=l(s_\alpha w)<l(w)$. This is a contradiction.

6 The geometric construction of the Jantzen filtration

In this part, we give the geometric construction of the Jantzen filtration in the affine parabolic case by generalizing the result of [BB93].

6.1 Notation

Let R be any noetherian $\mathbb{C}$-algebra. To any abelian category $\mathcal{C}$ we associate a category $\mathcal{C}_R$ whose objects are the pairs (M,μ_M) with M an object of $\mathcal{C}$ and $\mu_M:R\to\mathrm{End}_{\mathcal{C}}(M)$ a ring homomorphism. A morphism $(M,\mu_M)\to(N,\mu_N)$ is a morphism $f:M\to N$ in $\mathcal{C}$ such that $\mu_N(r)\circ f=f\circ\mu_M(r)$ for $r\in R$. The category $\mathcal{C}_R$ is also abelian. We have a faithful forgetful functor

$$for:\mathcal{C}_R\to\mathcal{C},\qquad (M,\mu_M)\to M. \tag{6.1.1}$$

Any functor $F : \mathcal{C} \to \mathcal{C}'$ gives rise to a functor

$$F_R : \mathcal{C}_R \to \mathcal{C}'_R, \quad (M, \mu_M) \mapsto (F(M),\, \mu_{F(M)})$$

such that $\mu_{F(M)}(r) = F(\mu_M(r))$ for $r \in R$. The functor F_R is R-linear. If F is exact, then F_R is also exact. We have $for \circ F_R = F \circ for$. Given an inductive system of categories $(\mathcal{C}_\alpha, i_{\alpha\beta})$, it yields an inductive system $((\mathcal{C}_\alpha)_R, (i_{\alpha\beta})_R)$, and we have a canonical equivalence

$$(2\varinjlim \mathcal{C}_\alpha)_R = 2\varinjlim((\mathcal{C}_\alpha)_R).$$

6.2 The function f_w

Let $Q' = (Q, Q)$ be the commutator subgroup of Q. It acts transitively on $Y^\dagger_w$ for $w \in {}^Q\mathfrak{S}$. We have the following lemma.

Lemma 6.2.1. *For any $w \in {}^Q\mathfrak{S}$ there exists a regular function $f_w : \overline{Y}^\dagger_w \to \mathbb{C}$ such that $f_w^{-1}(0) = \overline{Y}^\dagger_w - Y^\dagger_w$ and*

$$f_w(qxh^{-1}) = e^{w^{-1}\omega_0}(h) f_w(x), \qquad q \in Q',\ x \in Y^\dagger_w,\ h \in T.$$

Proof. Let V denote the simple $\mathfrak{g}$-module of highest weight ω_0. It is integrable, hence it admits an action of G. Let $v_0 \in V$ be a nonzero vector in the weight space V_{ω_0}. It is fixed under the action of Q'. So the map

$$\varphi : G \to V, \quad g \mapsto g^{-1} v_0$$

maps $Q\dot{w}B$ to $B\dot{w}^{-1}v_0$ for any $w \in {}^Q\mathfrak{S}$. Let $V(w^{-1})$ be the $\mathcal{U}(\mathfrak{b})$-submodule of V generated by the weight space $V_{w^{-1}\omega_0}$. We have $B\dot{w}^{-1}v_0 \subset V(w^{-1})$. Recall that for $w' \in {}^Q\mathfrak{S}$ we have

$$\begin{aligned} w' < w &\iff (w')^{-1} < w^{-1} \\ &\iff (\dot{w}')^{-1} v_0 \in \mathfrak{n}\dot{w}^{-1} v_0, \end{aligned} \tag{6.2.1}$$

see e.g., [Kum02, Proposition 7.1.20]. Thus, if $w' \leqslant w$ then $\varphi(Q\dot{w}'B) \subset V(w^{-1})$. The $\mathbb{C}$-vector space $V(w^{-1})$ is finite dimensional. We choose a linear form $l_w : V(w^{-1}) \to \mathbb{C}$ such that

$$l_w(\dot{w}^{-1}v_0) \neq 0 \quad \text{and} \quad l_w(\mathfrak{n}\dot{w}^{-1}v_0) = 0.$$

Set $\tilde{f}_w = l_w \circ \varphi$. Then for $q \in Q'$, $h \in T$, $u \in N$ we have

$$\begin{aligned} \tilde{f}_w(q\dot{w}h^{-1}u) &= l_w(u^{-1}h\dot{w}^{-1}v_0) \\ &= e^{w^{-1}\omega_0}(h) l_w(\dot{w}^{-1}v_0) \\ &= e^{w^{-1}\omega_0}(h) \tilde{f}_w(\dot{w}^{-1}). \end{aligned}$$

A similar calculation together with (6.2.1) yields that $\tilde{f}_w(Q\dot{w}'B) = 0$ for $w' < w$. Hence $\tilde{f}_w$ defines a regular function on $\bigsqcup_{w' \leqslant w} Q\dot{w}'B$ which is invariant under the right action of N. By consequence it induces a regular function f_w on $\overline{Y}^\dagger_w$ which has the required properties. $\square$

Remark 6.2.2. (a) The function f_w above is completely determined by its value on the point $\dot{w}N/N$, hence is unique up to scalar.

(b) The lemma implies that the embedding $j_w : Y_w^\dagger \to X^\dagger$ is affine.

(c) The function f_w is an analogue of the function defined in [BB93, Lemma 3.5.1] in the finite type case. Below we will use it to define the Jantzen filtration on $\mathscr{B}_{w!}^\lambda$. Note that [BB93]'s function is defined on the whole enhanced flag variety (which is a smooth scheme). Although our f_w is only defined on the singular scheme $\overline{Y}_w^\dagger$, this does not create any problem, because the definition of the Jantzen filtration is local (see Section 6.6), and each point of $\overline{Y}_w^\dagger$ admits a neighborhood V which can be embedded into a smooth scheme U such that f_w extends to U. The choice of such an extension will not affect the filtration, see [BB93, Remark 4.2.2(iii)].

6.3 The $\mathscr{D}$-module $\mathscr{B}^{(n)}$

Fix $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$. In the rest of Section 6, we will abbreviate

$$j = j_w, \quad f = f_w, \quad \mathscr{B} = \mathscr{B}_w^\lambda, \quad \mathscr{B}_! = \mathscr{B}_{w!}^\lambda, \quad \text{etc.}$$

Following [BB93] we introduce the deformed version of $\mathscr{B}$. Recall that $R = \mathbb{C}[[s]]$ and $\wp$ is the maximal ideal. Let x denote a coordinate on $\mathbb{C}$. For each integer $n > 0$ set $R^{(n)} = R(\wp^n)$. Consider the left $\mathscr{D}_{\mathbb{C}^*}$-module

$$\mathscr{I}^{(n)} = (\mathscr{O}_{\mathbb{C}^*} \otimes R^{(n)})x^s.$$

It is a rank one $\mathscr{O}_{\mathbb{C}^*} \otimes R^{(n)}$-module generated by a global section x^s such that the action of $\mathscr{D}_{\mathbb{C}^*}$ is given by $x\partial_x(x^s) = s(x^s)$. The restriction of f yields a map $Y_w^\dagger \to \mathbb{C}^*$. Thus $f^*\mathscr{I}^{(n)}$ is a left $\mathscr{D}_{Y_w^\dagger} \otimes R^{(n)}$-module. So we get a right $\mathscr{D}_{Y_w^\dagger} \otimes R^{(n)}$-module

$$\mathscr{B}^{(n)} = \mathscr{B} \otimes_{\mathscr{O}_{Y_w^\dagger}} f^*\mathscr{I}^{(n)}.$$

Lemma 6.3.1. *The right $\mathscr{D}_{Y_w^\dagger} \otimes R^{(n)}$-module $\mathscr{B}^{(n)}$ is an object of $\mathbf{M}_h^{\tilde{\lambda}}(Y_w)$.*

Proof. Since $R^{(n)}$ is a $\mathbb{C}$-algebra of dimension n and $\mathscr{B}$ is locally free of rank one over $\mathscr{O}_{Y_w^\dagger}$, the $\mathscr{O}_{Y_w^\dagger}$-module $\mathscr{B}^{(n)}$ is locally free of rank n. Hence it is a holonomic $\mathscr{D}_{Y_w^\dagger}$-module. Note that the $\mathscr{D}_{\mathbb{C}^*}$-module $\mathscr{I}^{(n)}$ is weakly T-equivariant such that x^s is a T-invariant global section. Since the map f is T-equivariant, we deduce that the $\mathscr{D}_{Y_w^\dagger}$-module $f^*\mathscr{I}^{(n)}$ is weakly T-equivariant. Let f^s be the global section of $f^*\mathscr{I}^{(n)}$ given by the image of x^s under the inclusion

$$\Gamma(\mathbb{C}^*, \mathscr{I}^{(n)}) \subset \Gamma(Y_w^\dagger, f^*\mathscr{I}^{(n)}).$$

Then f^s is T-invariant. It is nowhere vanishing on $Y_w^\dagger$, and thus yields an isomorphism of $\mathscr{O}_{Y_w^\dagger} \otimes R^{(n)}$-modules

$$f^*\mathscr{I}^{(n)} \cong \mathscr{O}_{Y_w^\dagger} \otimes R^{(n)}.$$

By consequence we have the following isomorphism

$$\begin{aligned}
(\mathscr{B}^{(n)})^\dagger &= \pi_*(\pi^*(\Omega_{Y_w} \otimes_{\mathscr{O}_{Y_w}} \mathscr{L}_{Y_w}^\lambda) \otimes_{\mathscr{O}_{Y_w^\dagger}} f^*\mathscr{I}^{(n)})^T \\
&= \Omega_{Y_w} \otimes_{\mathscr{O}_{Y_w}} \mathscr{L}_{Y_w}^\lambda \otimes_{\mathscr{O}_{Y_w}} \pi_*(f^*\mathscr{I}^{(n)})^T \\
&\cong \Omega_{Y_w} \otimes_{\mathscr{O}_{Y_w}} \mathscr{L}_{Y_w}^\lambda \otimes_{\mathscr{O}_{Y_w}} (\mathscr{O}_{Y_w} \otimes R^{(n)}).
\end{aligned}$$

See Remark 4.4.2 for the first equality. Next, recall from (4.4.1) that the right T-action on $Y_w^\dagger$ yields a morphism of Lie algebras

$$\delta_r : \mathfrak{t} \to \Gamma(Y_w^\dagger, \mathscr{D}_{Y_w^\dagger}).$$

The right $\mathscr{D}_{Y_w}^\dagger$-module structure of $f^*(\mathscr{I}^{(n)})$ is such that

$$(f^s \cdot \delta_r(h))(m) = sw^{-1}\omega_0(-h)f^s(m), \quad \forall\ m \in Y_w^\dagger.$$

So the action of the element

$$h + \lambda(h) + sw^{-1}\omega_0(h) \in \mathcal{U}(\mathfrak{t}) \otimes R^{(n)}$$

on $(\mathscr{B}^{(n)})^\dagger$ via the map δ_r vanishes. Since the multiplication by s on $(\mathscr{B}^{(n)})^\dagger$ is nilpotent, the action of the ideal $\mathfrak{m}_\lambda$ is also nilpotent. Therefore $\mathscr{B}^{(n)}$ belongs to the category $\mathbf{M}_h^{\tilde{\lambda}}(Y_w)$. □

It follows from the lemma that we have the following objects in $\mathbf{M}_h^{\tilde{\lambda}}(X)$

$$\mathscr{B}_!^{(n)} = j_!(\mathscr{B}^{(n)}), \quad \mathscr{B}_{!\bullet}^{(n)} = j_{!\bullet}(\mathscr{B}^{(n)}), \quad \mathscr{B}_\bullet^{(n)} = j_\bullet(\mathscr{B}^{(n)}).$$

Further, for $v = xw \in \mathfrak{S}$ with $x \in \mathfrak{S}_0$, let $r_x : X_v^\dagger \to Y_w^\dagger$ be the canonical inclusion. We have $i_v = j \circ r_x$. Then the tensor product

$$\mathscr{A}_v^{(n)} = \mathscr{A}_v^\lambda \otimes_{\mathscr{O}_{X_v^\dagger}} r_x^* f^* \mathscr{I}^{(n)} \tag{6.3.1}$$

is an object of $\mathbf{M}_h^{\tilde{\lambda}}(X_v)$, and we have the following objects in $\mathbf{M}_h^{\tilde{\lambda}}(X)$

$$\mathscr{A}_{v!}^{(n)} = i_{v!}(\mathscr{A}_v^{(n)}), \quad \mathscr{A}_{v!\bullet}^{(n)} = i_{v!\bullet}(\mathscr{A}_v^{(n)}), \quad \mathscr{A}_{v\bullet}^{(n)} = i_{v\bullet}(\mathscr{A}_v^{(n)}).$$

6.4 Deformed Verma modules

Fix $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$. For $\mu \in {}_\kappa\mathfrak{t}^*$ we have defined the Verma module $N_\kappa(\mu)$ in Section 5.4. The deformed Verma module is the $\mathcal{U}_{\mathbf{k}}$-module given by

$$N_{\mathbf{k}}(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} R_{\mu+s\omega_0}.$$

Here the $\mathfrak{b}$-module $R_{\mu+s\omega_0}$ is a rank one R-module over which $\mathfrak{t}$ acts by $\mu + s\omega_0$, and $\mathfrak{n}$ acts trivially. The deformed dual Verma module is

$$\mathbf{D}N_{\mathbf{k}}(\mu) = \bigoplus_{\lambda \in {}_{\mathbf{k}}\mathfrak{t}^*} \mathrm{Hom}_R(N_{\mathbf{k}}(\mu)_\lambda, R),$$

see (3.3.1). Let $n > 0$. We will abbreviate

$$N_{\mathbf{k}}^{(n)}(\mu) = N_{\mathbf{k}}(\mu)(\wp^n), \quad \mathbf{D}N_{\mathbf{k}}^{(n)}(\mu) = \mathbf{D}N_{\mathbf{k}}(\mu)(\wp^n).$$

For any $R^{(n)}$-module (resp. R-module) M let $\mu(s^i) : M \to M$ be the multiplication by s^i and write s^iM for the image of $\mu(s^i)$. We define a filtration

$$F^\bullet M = (F^0M \supset F^1M \supset F^2M \supset \dots)$$

on M by putting $F^iM = s^iM$. We say that it is of length n if $F^nM = 0$ and $F^{n-1}M \neq 0$. We set

$$\mathrm{gr}\, M = \bigoplus_{i \geqslant 0} \mathrm{gr}^i M, \quad \mathrm{gr}^i M = F^iM/F^{i+1}M.$$

For any $\mathfrak{g}_{R^{(n)}}$-module M let $\mathbf{ch}(M)$ be the $\mathfrak{t}_{R^{(n)}}$-module image of M by the forgetful functor.

Lemma 6.4.1. *If $\lambda+\rho$ is antidominant then we have an isomorphism of $\mathfrak{t}_{R^{(n)}}$-modules*

$$\mathbf{ch}(\Gamma(\mathscr{A}_{v\bullet}^{(n)})) = \mathbf{ch}(\mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda)).$$

Proof. The proof is very similar to the proof of Proposition 5.4.1. We will use the notation introduced there.

Step 1. Consider the nowhere vanishing section f^s of $(f^*\mathscr{I}^{(n)})^\dagger$ over Y_w. Its restriction to X_v yields an isomorphism

$$(\mathscr{A}_v^{(n)})^\dagger \cong \Omega_{X_v}\otimes_{\mathscr{O}_{X_v}}\mathscr{L}_{X_v}^\lambda\otimes R^{(n)}.$$

Let ω be a nowhere vanishing section of Ω_{X_v}, and let t^λ be the nowhere vanishing section of $\mathscr{L}_{X_v}^\lambda$ over X_v such that $t^\lambda(u\dot{v}b)=e^{-\lambda}(b)$ for $u\in N$, $b\in B$. Then the global section $\omega\otimes t^\lambda\otimes f^s$ of $\mathscr{A}_v^{(n)}$ defines an element

$$\vartheta^s\in\hat{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})$$

in the same way as ϑ is defined in the first step of the proof of Proposition 5.4.1.

Step 2. In this step, we show that

$$\mathbf{ch}(\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})) = \mathbf{ch}(\mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda)).$$

The proof is the same as in the second step of the proof of Proposition 5.4.1. The right $\mathscr{D}_{X_v}^\dagger$-module structure on $(\mathscr{A}_v^{(n)})^\dagger$ yields an isomorphism of sheaves of $\mathbb{C}$-vector spaces over V_l^y

$$\begin{array}{rcl} j_l^{y*}(i_{l*}^v\mathscr{O}_{X_v})\otimes\mathcal{U}(\mathfrak{n}^-(\Pi^-\cap v(\Pi^-))/\mathfrak{n}_l^-)\otimes R^{(n)} & \xrightarrow{\sim} & j_l^{y*}(i_{l\bullet}^v(\mathscr{A}_v^{(n)})^\dagger),\\ f\otimes p\otimes r & \mapsto & ((\vartheta^s\cdot f)\cdot\delta_l(p))r.\end{array}$$

This yields an isomorphism of $\mathfrak{t}_{R^{(n)}}$-modules

$$\mathbf{ch}\big(\Gamma(\mathfrak{X}_l^y, i_{l\bullet}^v(\mathscr{A}^{(n)}))\big) = \mathbf{ch}\big(\mathcal{U}(\mathfrak{n}^-/\mathfrak{n}_l^-)\otimes R^{(n)}_{v\cdot\lambda+s\omega_0}\big).$$

Therefore we have

$$\mathbf{ch}(\hat{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})) = \mathbf{ch}\big((\varprojlim_l\mathcal{U}(\mathfrak{n}^-/\mathfrak{n}_l^-))\otimes R^{(n)}_{v\cdot\lambda+s\omega_0}\big), \tag{6.4.1}$$

and

$$\begin{aligned}\mathbf{ch}(\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})) &= \mathbf{ch}\big(\mathcal{U}(\mathfrak{n}^-)\otimes R^{(n)}_{v\cdot\lambda+s\omega_0}\big)\\ &= \mathbf{ch}(\mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda)).\end{aligned}\tag{6.4.2}$$

Step 3. In this step, we prove that $\Gamma(\mathscr{A}_{v\bullet}^{(n)})=\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})$ as $\mathfrak{g}_{R^{(n)}}$-modules. Since both of them are $\mathfrak{g}_{R^{(n)}}$-submodules of $\hat{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})$. It is enough to prove that they are equal as vector spaces. Consider the filtration $F^\bullet(\mathscr{A}_v^{(n)})$ on $\mathscr{A}_v^{(n)}$. It is a filtration in $\mathbf{M}^{\tilde{\lambda}}(X_v)$ of length n and

$$\mathrm{gr}^i(\mathscr{A}_v^{(n)})=\mathscr{A}_v^\lambda,\quad 0\leqslant i\leqslant n-1.$$

Since $i_{v\bullet}$ is exact and

$$R^i\Gamma(\mathscr{A}_{v\bullet}^\lambda)=R^i\overline{\Gamma}(\mathscr{A}_{v\bullet}^\lambda)=0,\quad\forall\ i>0,$$

the functor $\Gamma \circ i_{v\bullet}$ commute with the filtration. Therefore both the filtrations $F^\bullet\Gamma(\mathscr{A}_{v\bullet}^{(n)})$ and $F^\bullet\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})$ have length n and

$$\mathrm{gr}^i\,\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = \Gamma(\mathscr{A}_{v\bullet}^{\lambda}), \quad \mathrm{gr}^i\,\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)}) = \overline{\Gamma}(\mathscr{A}_{v\bullet}^{\lambda}), \quad 0 \leqslant i \leqslant n-1.$$

By Step 3 of the proof of Proposition 5.4.1, we have $\Gamma(\mathscr{A}_v^\lambda) = \overline{\Gamma}(\mathscr{A}_v^\lambda)$. We deduce that all the sections in $\Gamma(\mathscr{A}_{v\bullet}^{(n)})$ are $\mathfrak{t}$-finite and all the sections in $\overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)})$ are supported on finite dimensional subschemes. This proves that

$$\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = \overline{\Gamma}(\mathscr{A}_{v\bullet}^{(n)}).$$

We are done by Step 2. □

Lemma 6.4.2. *If $\lambda+\rho$ is antidominant there is an isomorphism of $\mathfrak{g}_{R^{(n)}}$-modules*

$$\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = \mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda).$$

Proof. Note that

$$\begin{aligned} \mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda) &= \bigoplus_{\lambda\in{}_{\mathbf{k}}\mathfrak{t}^*} \mathrm{Hom}_R(N_{\mathbf{k}}(v\cdot\lambda)_\lambda, R)(\wp^n) \\ &= \bigoplus_{\lambda\in{}_{\mathbf{k}}\mathfrak{t}^*} \mathrm{Hom}_{R^{(n)}}(N_{\mathbf{k}}^{(n)}(v\cdot\lambda)_\lambda, R^{(n)}). \end{aligned}$$

For $\lambda \in {}_{\mathbf{k}}\mathfrak{t}^*$ let $\Gamma(\mathscr{A}_{v\bullet}^{(n)})_\lambda$ be the weight space as defined in (3.1.2). By Lemma 6.4.1 we have

$$\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = \bigoplus_{\lambda\in{}_{\mathbf{k}}\mathfrak{t}^*} \Gamma(\mathscr{A}_{v\bullet}^{(n)})_\lambda,$$

because the same equality holds for $N_{\mathbf{k}}^{(n)}(v\cdot\lambda)$. So we can consider the following $\mathfrak{g}_{R^{(n)}}$-module

$$\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = \bigoplus_{\lambda\in{}_{\mathbf{k}}\mathfrak{t}^*} \mathrm{Hom}_{R^{(n)}}(\Gamma(\mathscr{A}_{v\bullet}^{(n)})_\lambda, R^{(n)}). \tag{6.4.3}$$

It is enough to prove that we have an isomorphism of $\mathfrak{g}_{R^{(n)}}$-modules

$$\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)}) = N_{\mathbf{k}}^{(n)}(v\cdot\lambda).$$

By (6.4.3) we have

$$\mathbf{ch}\big(\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)})\big) = \mathbf{ch}\big(\Gamma(\mathscr{A}_{v\bullet}^{(n)})\big). \tag{6.4.4}$$

Together with Lemma 6.4.1, this yields an isomorphism of $R^{(n)}$-modules

$$N_{\mathbf{k}}^{(n)}(v\cdot\lambda)_{v\cdot\lambda+s\omega_0} = \big(\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)})\big)_{v\cdot\lambda+s\omega_0}.$$

By the universal property of Verma modules, such an isomorphism induces a morphism of $\mathfrak{g}_{R^{(n)}}$-module

$$\varphi : N_{\mathbf{k}}^{(n)}(v\cdot\lambda) \to \mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)}).$$

We claim that for each $\mu \in {}_{\mathbf{k}}\mathfrak{t}^*$ the $R^{(n)}$-module morphism

$$\varphi_\mu : N_{\mathbf{k}}^{(n)}(v\cdot\lambda)_\mu \to \big(\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)})\big)_\mu$$

given by the restriction of φ is invertible. Indeed, by Lemma 6.4.1 and (6.4.4), we have

$$\mathrm{ch}(\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)})) = \mathrm{ch}(\mathbf{D}N_{\mathbf{k}}^{(n)}(v\cdot\lambda) = \mathrm{ch}(N_{\mathbf{k}}^{(n)}(v\cdot\lambda).$$

So

$$N_{\mathbf{k}}^{(n)}(v\cdot\lambda)_\mu = \big(\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{(n)})\big)_\mu$$

as $R^{(n)}$-modules. On the other hand, Corollary 5.9.3 yields that the map

$$\varphi(\wp) = \varphi\otimes_{R^{(n)}}(R^{(n)}/\wp R^{(n)}):\ N_\kappa(v\cdot\lambda)\to\mathbf{D}\Gamma(\mathscr{A}_{v\bullet}^{\lambda})$$

is an isomorphism of $\mathfrak{g}$-modules. So $\varphi_\mu(\wp)$ is also an isomorphism. By Nakayama's lemma this implies that φ_μ is an isomorphism. So φ is an isomorphism. The lemma is proved. □

Lemma 6.4.3. *If $\lambda+\rho$ is antidominant and v is a shortest element in $v\mathfrak{S}(\lambda)$, then there is an isomorphism of $\mathfrak{g}_{R^{(n)}}$-modules*

$$\Gamma(\mathscr{A}_{v!}^{(n)}) = N_{\mathbf{k}}^{(n)}(v\cdot\lambda).$$

Proof. We abbreviate $\nu = v\cdot\lambda$. The lemma will be proved in three steps.

Step 1. Recall the character map from (5.4.2). Note that since Γ and $i_!$ are exact, and

$$\mathrm{gr}\,\mathscr{A}_v^{(n)} = (\mathscr{A}_v^{\lambda})^{\oplus n},$$

we have an isomorphism of $\mathfrak{t}$-modules

$$\mathrm{gr}\,\Gamma(\mathscr{A}_{v!}^{(n)}) = \Gamma(\mathscr{A}_{v!}^{\lambda})^{\oplus n}. \tag{6.4.5}$$

Next, since the action of s on $\Gamma(\mathscr{A}_{v!}^{(n)})$ is nilpotent, for any $\mu\in\mathfrak{t}^*$ we have

$$\dim_{\mathbb{C}}(\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde\mu}) = \dim_{\mathbb{C}}((\mathrm{gr}\,\Gamma(\mathscr{A}_{v!}^{(n)}))_{\tilde\mu}).$$

We deduce that as a $\mathfrak{t}$-module $\Gamma(\mathscr{A}_{v!}^{(n)})$ is a generalized weight module and

$$\mathrm{ch}(\Gamma(\mathscr{A}_{v!}^{(n)})) = \mathrm{ch}(\mathrm{gr}\,\Gamma(\mathscr{A}_{v!}^{(n)})) = n\,\mathrm{ch}\,\Gamma(\mathscr{A}_{v!}^{\lambda}). \tag{6.4.6}$$

On the other hand, we have the following isomorphism of $\mathfrak{t}$-modules

$$\mathrm{gr}\,N_{\mathbf{k}}^{(n)}(\nu) = N_\kappa(\nu)^{\oplus n}. \tag{6.4.7}$$

Therefore we have

$$\mathrm{ch}(N_{\mathbf{k}}^{(n)}(\nu)) = n\,\mathrm{ch}(N_\kappa(\nu)).$$

Since $\Gamma(\mathscr{A}_{v!}^{\lambda}) = N_\kappa(\nu)$ as $\mathfrak{g}$-modules by Corollary 5.9.3, this yields

$$\mathrm{ch}(\Gamma(\mathscr{A}_{v!}^{(n)})) = \mathrm{ch}(N_{\mathbf{k}}^{(n)}(\nu)). \tag{6.4.8}$$

Further, we claim that there is an isomorphism of $R^{(n)}$-module

$$\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde\mu} = N_{\mathbf{k}}^{(n)}(\nu)_{\tilde\mu},\quad \forall\ \mu\in\mathfrak{t}^*. \tag{6.4.9}$$

Note that $\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde\mu}$ is indeed an $R^{(n)}$-module because the action of s on $\Gamma(\mathscr{A}_{v!}^{(n)})$ is nilpotent. To prove the claim, it suffices to notice that for any finitely generated $R^{(n)}$-modules M, M' we have that M is isomorphic to M' as $R^{(n)}$-modules if and only if $\mathrm{gr}^i M = \mathrm{gr}^i M'$

for all i. So the claim follows from the isomorphisms of $\mathfrak{t}$-modules (6.4.5), (6.4.7) and Corollary 5.9.3.

Step 2. In this step, we prove that as a $\mathfrak{t}_{R^{(n)}}$-module

$$\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde{\nu}} = R^{(n)}_{\nu+s\omega_0}$$

where $R^{(n)}_{\nu+s\omega_0}$ is the rank one $R^{(n)}$-module over which $\mathfrak{t}$ acts by the weight $\nu + s\omega_0$. Let us consider the canonical morphisms in $\mathbf{M}_h^{\tilde{\lambda}}(X)$

$$\mathscr{A}_{v!}^{(n)} \twoheadrightarrow \mathscr{A}_{v!\bullet}^{(n)} \hookrightarrow \mathscr{A}_{v\bullet}^{(n)}.$$

Since Γ is exact on $\mathbf{M}^{\tilde{\lambda}}(X)$, we deduce the following chain of $\mathfrak{g}_{R^{(n)}}$-module morphisms

$$\Gamma(\mathscr{A}_{v!}^{(n)}) \xrightarrow{\alpha} \Gamma(\mathscr{A}_{v!\bullet}^{(n)}) \xhookrightarrow{\beta} \Gamma(\mathscr{A}_{v\bullet}^{(n)}).$$

Consider the following $\mathfrak{t}_{R^{(n)}}$-morphisms given by the restrictions of α, β

$$\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde{\nu}} \xrightarrow{\alpha_\nu} \Gamma(\mathscr{A}_{v!\bullet}^{(n)})_{\tilde{\nu}} \xhookrightarrow{\beta_\nu} \Gamma(\mathscr{A}_{v\bullet}^{(n)})_{\tilde{\nu}}.$$

We claim that α_ν and β_ν are isomorphisms. Note that by (6.4.6) we have

$$\dim_{\mathbb{C}}\Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde{\nu}} = n\dim_{\mathbb{C}}\Gamma(\mathscr{A}_{v!}^{\lambda})_{\tilde{\nu}} = n.$$

By Lemma 6.4.2 we also have $\dim_{\mathbb{C}}\Gamma(\mathscr{A}_{v\bullet}^{(n)})_{\tilde{\nu}} = n$. Next, consider the exact sequence in $\mathbf{M}_h^{\tilde{\lambda}}(X_v)$,

$$0 \to F^{i+1}\mathscr{A}_v^{(n)} \to F^i\mathscr{A}_v^{(n)} \to \operatorname{gr}^i \mathscr{A}_v^{(n)} \to 0.$$

Applying the functor $i_{v!\bullet}$ to it yields a surjective morphism

$$i_{v!\bullet}(F^i\mathscr{A}_v^{(n)})/i_{v!\bullet}(F^{i+1}\mathscr{A}_v^{(n)}) \to i_{v!\bullet}(\operatorname{gr}^i \mathscr{A}_v^{(n)}).$$

Since $i_{v!\bullet}(F^i\mathscr{A}_v^{(n)}) = F^i(i_{v!\bullet}(\mathscr{A}_v^{(n)}))$ and $\operatorname{gr}^i \mathscr{A}_v^{(n)} = \mathscr{A}_v^{\lambda}$, we deduce a surjective morphism

$$\operatorname{gr}^i \mathscr{A}_{v!\bullet}^{(n)} \to \mathscr{A}_{v!\bullet}^{\lambda}, \quad 0 \leqslant i \leqslant n-1.$$

Applying the exact functor Γ to this morphism and summing over i gives a surjective morphism of $\mathfrak{g}$-modules

$$\gamma : \operatorname{gr}\Gamma(\mathscr{A}_{v!\bullet}^{(n)}) \to \Gamma(\mathscr{A}_{v!\bullet}^{\lambda})^{\oplus n}.$$

Since v is minimal in $v\mathfrak{S}(\lambda)$, by Corollary 5.9.3(c) the right hand side is equal to $L_\kappa(\nu)$. We deduce from the surjectivity of γ that

$$\begin{aligned} \dim_{\mathbb{C}}\Gamma(\mathscr{A}_{v!\bullet}^{(n)})_{\tilde{\nu}} &= \dim_{\mathbb{C}} \operatorname{gr}\Gamma(\mathscr{A}_{v!\bullet}^{(n)})_{\tilde{\nu}} \\ &\geqslant \dim_{\mathbb{C}}(L_\kappa(\nu)_{\tilde{\nu}})^{\oplus n} \\ &= n \end{aligned}$$

It follows that the monomorphism α_ν and the epimorphism β_ν are isomorphisms. The claim is proved. So we have an isomorphisms of $\mathfrak{t}_{R^{(n)}}$-modules

$$\beta_\nu \circ \alpha_\nu : \Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde{\nu}} \to \Gamma(\mathscr{A}_{v\bullet}^{(n)})_{\tilde{\nu}}.$$

In particular, we deduce an isomorphisms of $\mathfrak{t}_{R^{(n)}}$-modules

$$\Gamma(\mathscr{A}_{v!}^{(n)})_{\nu+s\omega_0} \to \Gamma(\mathscr{A}_{v\bullet}^{(n)})_{\nu+s\omega_0},$$

because

$$\Gamma(\mathscr{A}_{v\sharp}^{(n)})_{\nu+s\omega_0} \subset \Gamma(\mathscr{A}_{v\sharp}^{(n)})_{\tilde{\nu}}, \quad \text{for } \sharp =!, \bullet.$$

By Lemma 6.4.2, we have

$$\Gamma(\mathscr{A}_{v\bullet}^{(n)})_{\nu+s\omega_0} = R_{\nu+s\omega_0}^{(n)}.$$

We deduce an isomorphism of $\mathfrak{t}_{R^{(n)}}$-modules

$$\Gamma(\mathscr{A}_{v!}^{(n)})_{\nu+s\omega_0} = R_{\nu+s\omega_0}^{(n)}.$$

Step 3. By the universal property of Verma modules and Step 2, there exists a morphism of $\mathfrak{g}_{R^{(n)}}$-modules

$$\varphi : N_{\mathbf{k}}^{(n)}(\nu) \to \Gamma(\mathscr{A}_{v!}^{(n)}).$$

For any $\mu \in \mathfrak{t}^*$ this map restricts to a morphism of $R^{(n)}$-modules

$$\varphi_\mu : N_{\mathbf{k}}^{(n)}(\nu)_{\tilde{\mu}} \to \Gamma(\mathscr{A}_{v!}^{(n)})_{\tilde{\mu}}.$$

By Step 1, the $R^{(n)}$-modules on the two sides are finitely generated and they are isomorphic. Further, the induced morphism

$$\varphi(\wp) : N_\kappa(\nu) \to \Gamma(\mathscr{A}_{v!}^{\lambda})$$

is an isomorphism by Corollary 5.9.3. So by Nakayama's lemma, the morphism φ_μ is an isomorphism for any μ. Therefore φ is an isomorphism. The lemma is proved. □

Remark 6.4.4. The hypothesis that v is a shortest element in $v\mathfrak{S}(\lambda)$ is probably not necessary but this is enough for our purpose.

6.5 Deformed parabolic Verma modules

Fix $\lambda \in \Lambda$ and $w \in {}^Q\mathfrak{S}$ as before. Let $n > 0$. We will abbreviate

$$M_\kappa = M_\kappa(w \cdot \lambda), \quad M_{\mathbf{k}} = M_{\mathbf{k}}(w \cdot \lambda), \quad M_{\mathbf{k}}^{(n)} = M_{\mathbf{k}}(\wp^n), \quad \mathbf{D}M_{\mathbf{k}}^{(n)} = (\mathbf{D}M_{\mathbf{k}})(\wp^n).$$

Lemma 6.5.1. *Assume that $\lambda + \rho$ is antidominant and that $w \cdot \lambda$ belongs to Λ^+. Then there are isomorphisms of $\mathfrak{g}_{R^{(n)}}$-modules*

$$\Gamma(\mathscr{B}_!^{(n)}) = M_{\mathbf{k}}^{(n)}, \quad \Gamma(\mathscr{B}_\bullet^{(n)}) = \mathbf{D}M_{\mathbf{k}}^{(n)}.$$

Proof. Consider the canonical embedding $r : X_w^\dagger \to Y_w^\dagger$. We claim that the adjunction map yields a surjective morphism

$$r_! r^*(\mathscr{B}^{(n)}) \to \mathscr{B}^{(n)}. \tag{6.5.1}$$

Indeed, an easy induction shows that it is enough to prove that $\mathrm{gr}^i(r_! r^*(\mathscr{B}^{(n)})) \to \mathrm{gr}^i \mathscr{B}^{(n)}$ is surjective for each i. Since the functors $r_!$, r_* are exact and $\mathrm{gr}^i \mathscr{B}^{(n)} = \mathscr{B}$, this follows from Lemma 5.10.2(a). Note that $r^*(\mathscr{B}^{(n)}) = \mathscr{A}_w^{(n)}$. So image of (6.5.1) by the exact functor $\Gamma \circ j_!$ is a surjective morphism

$$\Gamma(\mathscr{A}_{w!}^{(n)}) \to \Gamma(\mathscr{B}_!^{(n)}). \tag{6.5.2}$$

By Lemma 6.4.3 we have $\Gamma(\mathscr{A}_{w!}^{(n)}) = N_{\mathbf{k}}^{(n)}$. Since the $\mathfrak{g}_{R^{(n)}}$-module $\Gamma(\mathscr{B}_!^{(n)})$ is $\mathfrak{q}$-locally finite and $M_{\mathbf{k}}^{(n)}$ is the largest quotient of $N_{\mathbf{k}}^{(n)}$ in $\mathcal{O}_{\mathbf{k}}$, the morphism (6.5.2) induces a surjective morphism

$$\varphi : M_{\mathbf{k}}^{(n)} \to \Gamma(\mathscr{B}_!^{(n)}).$$

Further, by Proposition 5.10.4(b) the map

$$\varphi(\wp) : M_\kappa \to \Gamma(\mathscr{B}_!)$$

is an isomorphism. The same argument as in Step 1 of the proof of Lemma 6.4.3 shows that for each $\mu \in \mathbf{t}^*$ the generalized weight spaces $(M_{\mathbf{k}}^{(n)})_{\tilde{\mu}}$ and $\Gamma(\mathscr{B}_!^{(n)})_{\tilde{\mu}}$ are isomorphic as $R^{(n)}$-modules. We deduce that φ is an isomorphism by Nakayama's lemma. This proves the first equality. The proof for the second equality is similar. We consider the adjunction map

$$\mathscr{B}^{(n)} \to r_\bullet r^*(\mathscr{B}^{(n)}). \tag{6.5.3}$$

It is injective by Lemma 5.10.2(b) and the same arguments as above. So by applying the exact functor $\Gamma \circ j_\bullet$, we get an injective morphism

$$\varphi' : \ \Gamma(\mathscr{B}_\bullet^{(n)}) \to \mathbf{D}M_{\mathbf{k}}^{(n)}.$$

Again, by using Proposition 5.10.4(b) and Nakayama's lemma, we prove that φ' is an isomorphism. $\square$

6.6 The geometric Jantzen filtration

Now, we define the Jantzen filtration on $\mathscr{B}_!$ following [BB93, Sections 4.1,4.2]. Recall that $\mathscr{B}^{(n)}$ is an object of $\mathbf{M}_h^{\tilde{\lambda}}(Y_w)$. Consider the map

$$\mu : R^{(n)} \to \mathrm{End}_{\mathbf{M}_h^{\tilde{\lambda}}(Y_w)}(\mathscr{B}^{(n)}), \qquad \mu(r)(m) = rm,$$

where m denotes a local section of $\mathscr{B}^{(n)}$. Then the pair $(\mathscr{B}^{(n)}, \mu)$ is an object of the category $\mathbf{M}_h^{\tilde{\lambda}}(Y_w)_{R^{(n)}}$, see Section 6.1. We will abbreviate $\mathscr{B}^{(n)} = (\mathscr{B}^{(n)}, \mu)$. Fix some integer $a \geqslant 0$. Recall the morphism of functors $\psi : j_! \to j_\bullet$. We consider the morphism

$$\psi(a,n) : \mathscr{B}_!^{(n)} \to \mathscr{B}_\bullet^{(n)}$$

in the category $\mathbf{M}_h^{\tilde{\lambda}}(X)_{R^{(n)}}$ given by the composition of the chain of maps

$$\mathscr{B}_!^{(n)} \xrightarrow{j_!(\mu(s^a))} \mathscr{B}_!^{(n)} \xrightarrow{\psi(\mathscr{B}^{(n)})} \mathscr{B}_\bullet^{(n)} . \tag{6.6.1}$$

The category $\mathbf{M}_h^{\tilde{\lambda}}(X)_{R^{(n)}}$ is abelian. The obvious projection $R^{(n)} \to R^{(n-1)}$ yields a canonical map

$$\mathrm{Coker}(\psi(a,n)) \to \mathrm{Coker}(\psi(a,n-1)).$$

By [Beĭ87, Lemma 2.1] this map is an isomorphism when n is sufficiently large. We define

$$\pi^a(\mathscr{B}) = \mathrm{Coker}(\psi(a,n)), \quad n \gg 0. \tag{6.6.2}$$

This is an object of $\mathbf{M}_h^{\tilde{\lambda}}(X)_{R^{(n)}}$. We view it as an object of $\mathbf{M}_h^{\tilde{\lambda}}(X)$ via the forgetful functor (6.1.1). Now, let us consider the maps

$$\alpha : \mathscr{B}_! \to \pi^1(\mathscr{B}), \quad \beta : \pi^1(\mathscr{B}) \to \pi^0(\mathscr{B})$$

in $\mathbf{M}_h^{\tilde{\lambda}}(X)$ given as follows. First, since

$$\pi^0(\mathscr{B}) = \mathrm{Coker}(\psi(\mathscr{B}^{(n)})) \quad \text{and} \quad \pi^1(\mathscr{B}) = \mathrm{Coker}(\psi(\mathscr{B}^{(n)}) \circ j_!(\mu(s)))$$

by (6.6.1), there is a canonical projection $\pi^1(\mathscr{B}) \to \pi^0(\mathscr{B})$. We define β to be this map. Next, the morphism $\psi(\mathscr{B}^{(n)})$ maps $j_!(s(\mathscr{B}^{(n)}))$ to $\mathrm{Im}(\psi(1,n))$. Hence it induces a map

$$j_!(\mathscr{B}^{(n)}/s(\mathscr{B}^{(n)})) \to \pi^1(\mathscr{B}), \qquad n \gg 0.$$

Composing it with the isomorphism $\mathscr{B} \cong \mathscr{B}^{(n)}/s(\mathscr{B}^{(n)})$ we get the map α. Let μ^1 denote the $R^{(n)}$-action on $\pi^1(\mathscr{B})$. Then by [Beĭ87] the sequence

$$0 \longrightarrow \mathscr{B}_! \xrightarrow{\alpha} \pi^1(\mathscr{B}) \xrightarrow{\beta} \pi^0(\mathscr{B}) \longrightarrow 0, \tag{6.6.3}$$

is exact and α induces an isomorphism

$$\mathscr{B}_! \to \mathrm{Ker}(\mu^1(s) : \pi^1(\mathscr{B}) \to \pi^1(\mathscr{B})).$$

The *Jantzen filtration* of $\mathscr{B}_!$ is defined by

$$J^i(\mathscr{B}_!) = \mathrm{Ker}(\mu^1(s)) \cap \mathrm{Im}(\mu^1(s)^i), \qquad \forall\, i \geqslant 0. \tag{6.6.4}$$

6.7 Comparison of the Jantzen filtrations

Fix $\lambda \in \Lambda$ and $w \in {}^{Q}\mathfrak{S}$ as before. Consider the Jantzen filtration (J^iM_κ) on M_κ as defined in Section 3.3. The following proposition compares it with the geometric Jantzen filtration on $\mathscr{B}_!$.

Proposition 6.7.1. *Assume that $\lambda + \rho$ is antidominant and that w is a shortest element in $w\mathfrak{S}(\lambda)$. Then we have*

$$J^iM_\kappa = \Gamma(J^i\mathscr{B}_!), \quad \forall\, i \geqslant 0.$$

Proof. Recall from Remark 5.10.5 that the assumption of the proposition implies that $w \cdot \lambda \in \Lambda^+$. Therefore, by Proposition 5.10.4(b) and Lemma 6.5.1 we have

$$\Gamma(\mathscr{B}_!) = M_\kappa, \quad \Gamma(\mathscr{B}_!^{(n)}) = M_{\mathbf{k}}^{(n)}, \quad \Gamma(\mathscr{B}_\bullet^{(n)}) = \mathbf{D}M_{\mathbf{k}}^{(n)}.$$

So the map

$$\phi^{(n)} = \Gamma(\psi(\mathscr{B}^{(n)})) : \Gamma(\mathscr{B}_!^{(n)}) \to \Gamma(\mathscr{B}_\bullet^{(n)}).$$

identifies with a $\mathfrak{g}_{R^{(n)}}$-module homomorphism

$$\phi^{(n)} : M_{\mathbf{k}}^{(n)} \to \mathbf{D}M_{\mathbf{k}}^{(n)}.$$

Consider the projective systems $(M_{\mathbf{k}}^{(n)})$, $(\mathbf{D}M_{\mathbf{k}}^{(n)})$, $n > 0$, induced by the quotient map $R^{(n)} \to R^{(n-1)}$. Their limits are respectively $M_{\mathbf{k}}$ and $\mathbf{D}M_{\mathbf{k}}$. The morphisms $\phi^{(n)}$, $n > 0$, yield a morphism of $\mathfrak{g}_R$-modules

$$\phi = \varprojlim \phi^{(n)} : M_{\mathbf{k}} \to \mathbf{D}M_{\mathbf{k}}$$

such that

$$\phi(\wp) = \phi^{(1)} = \Gamma(\psi(\mathscr{B})).$$

The functor Γ is exact by Proposition 5.3.1. So the image of $\phi(\wp)$ is $\Gamma(\mathscr{B}_{!\bullet})$. It is non zero by Proposition 5.10.4(c). Hence ϕ satisfies the condition of Definition 2.4.1 and we have

$$J^i M_\kappa = (\{x \in M_{\mathbf{k}} \mid \phi(x) \in s^i \mathbf{D} M_{\mathbf{k}}\} + sM_{\mathbf{k}})/sM_{\mathbf{k}}.$$

By Lemma 3.2.1 and Remark 2.4.2 the map ϕ is injective. So the equality above can be rewritten as

$$J^i M_\kappa = (\phi(M_{\mathbf{k}}) \cap s^i \mathbf{D} M_{\mathbf{k}} + s\phi(M_{\mathbf{k}}))/s\phi(M_{\mathbf{k}}).$$

Now, for $a \geqslant 0$ let

$$\phi(a,n) : M_{\mathbf{k}}^{(n)} \to \mathbf{D} M_{\mathbf{k}}^{(n)}$$

be the $\mathfrak{g}_{R^{(n)}}$-module homomorphism given by the composition

$$M_{\mathbf{k}}^{(n)} \xrightarrow{\mu(s^a)} M_{\mathbf{k}}^{(n)} \xrightarrow{\phi^{(n)}} \mathbf{D} M_{\mathbf{k}}^{(n)}. \tag{6.7.1}$$

Then we have $\Gamma(\psi(a,n)) = \phi(a,n)$. Since Γ is exact, we have

$$\mathrm{Coker}(\phi(a,n)) = \Gamma\big(\mathrm{Coker}(\psi(a,n))\big).$$

So the discussion in the last section and the exactness of Γ yields that the canonical map

$$\mathrm{Coker}(\phi(a,n)) \to \mathrm{Coker}(\phi(a,n-1))$$

is an isomorphism if n is large enough. We deduce that

$$\mathbf{D} M_{\mathbf{k}}/s^a M_{\mathbf{k}} = \mathrm{Coker}(\phi(a,n)) = \Gamma(\pi^a(\mathscr{B})), \quad n \gg 0,$$

see (6.6.2). The action of $\mu(s)$ on $\mathbf{D} M_{\mathbf{k}}/s^a \phi(M_{\mathbf{k}})$ is nilpotent, because $\mu(s)$ is nilpotent on $\mathbf{D} M_{\mathbf{k}}^{(n)}$. Further Γ maps the exact sequence (6.6.3) to an exact sequence

$$0 \longrightarrow M_\kappa \longrightarrow \mathbf{D} M_{\mathbf{k}}/s\phi(M_{\mathbf{k}}) \longrightarrow \mathbf{D} M_{\mathbf{k}}/\phi(M_{\mathbf{k}}) \longrightarrow 0, \tag{6.7.2}$$

and the first map yields an isomorphism

$$M_\kappa = \mathrm{Ker}\,\big(\mu(s) : \mathbf{D} M_{\mathbf{k}}/s\phi(M_{\mathbf{k}}) \to \mathbf{D} M_{\mathbf{k}}/s\phi(M_{\mathbf{k}})\big).$$

Note that since $\mathbf{D} M_{\mathbf{k}}$ is a free R-module, for $x \in \mathbf{D} M_{\mathbf{k}}$ if $sx \in s\phi(M_{\mathbf{k}})$ then $x \in \phi(M_{\mathbf{k}})$. So by (6.6.4) and the exactness of Γ, we have for $i \geqslant 0$,

$$\begin{aligned} \Gamma(J^i \mathscr{B}_!) &= \mathrm{Ker}(\mu(s)) \cap \mathrm{Im}(\mu(s)^i) \\ &= (\phi(M_{\mathbf{k}}) \cap s^i \mathbf{D} M_{\mathbf{k}} + s\phi(M_{\mathbf{k}}))/s\phi(M_{\mathbf{k}}) \\ &= J^i M_\kappa. \end{aligned}$$

The proposition is proved. □

7 Proof of the main theorem

7.1 Mixed Hodge modules

Let Z be a smooth scheme. Let $\mathbf{MHM}(Z)$ be the category of mixed Hodge modules on Z [Sai90]. It is an abelian category. Each object $\mathscr{M}$ of $\mathbf{MHM}(Z)$ carries a canonical filtration

$$W^\bullet \mathscr{M} = \cdots W^k \mathscr{M} \supset W^{k-1} \mathscr{M} \cdots,$$

called the *weight filtration*. For each $k \in \mathbb{Z}$ the *Tate twist* is an auto-equivalence

$$(k) : \mathbf{MHM}(Z) \to \mathbf{MHM}(Z), \quad \mathscr{M} \mapsto \mathscr{M}(k)$$

such that $W^\bullet(\mathscr{M}(k)) = W^{\bullet+2k}(\mathscr{M})$. Let $\mathbf{Perv}(Z)$ be the category of perverse sheaves on Z with coefficient in $\mathbb{C}$. There is an exact forgetful functor

$$\varrho : \mathbf{MHM}(Z) \to \mathbf{Perv}(Z).$$

For any locally closed affine embedding $i : Z \to Y$ we have exact functors

$$i_!,\ i_\bullet : \mathbf{MHM}(Z) \to \mathbf{MHM}(Y)$$

which correspond via ϱ to the same named functors on the categories of perverse sheaves.

If Z is not smooth we embed it into a smooth variety Y and we define $\mathbf{MHM}(Z)$ as the full subcategory of $\mathbf{MHM}(Y)$ consisting of the objects supported on Z. It is independent of the choice of the embedding for the same reason as for $\mathscr{D}$-modules.

7.2 The graded multiplicities of $\mathscr{B}^\lambda_{x!\bullet}$ in $\mathscr{B}^\lambda_{w!}$

Now, let us calculate the multiplicities of a simple object $\mathscr{B}^\lambda_{x!\bullet}$ in the successive quotients of the Jantzen filtration of $\mathscr{B}^\lambda_{w!}$ for $x,\ w \in {}^Q\mathfrak{S}$ with $x \leqslant w$.

We fix once for all an element $v \in \mathfrak{S}$, and we consider the Serre subcategory $\mathbf{M}^\lambda_0(\overline{X}_v)$ of $\mathbf{M}^\lambda_h(\overline{X}_v)$ generated by the objects $\mathscr{A}^\lambda_{w!\bullet}$ with $w \leqslant v,\ w \in \mathfrak{S}$. The De Rham functor yields an exact fully faithful functor

$$\mathrm{DR} : \mathbf{M}^\lambda_0(\overline{X}_v) \longrightarrow \mathbf{Perv}(\overline{X}_v).$$

See e.g., [KT95, Section 4]. Let $\mathbf{MHM}_0(\overline{X}_v)$ be the full subcategory of $\mathbf{MHM}(\overline{X}_v)$ consisting of objects whose image by ϱ belong to the image of the functor DR. There exists a unique exact functor

$$\eta : \mathbf{MHM}_0(\overline{X}_v) \to \mathbf{M}^\lambda_0(\overline{X}_v)$$

such that $\mathrm{DR} \circ \eta = \varrho$. An object $\mathscr{M}$ in $\mathbf{MHM}_0(\overline{X}_v)$ is *pure of weight* i if we have $W^k\mathscr{M}/W^{k-1}\mathscr{M} = 0$ for any $k \neq i$. For any $w \in \mathfrak{S},\ w \leqslant v$, there is a unique simple object $\tilde{\mathscr{A}}^\lambda_w$ in $\mathbf{MHM}(X_w)$ pure of weight $l(w)$ such that $\eta(\tilde{\mathscr{A}}^\lambda_w) = \mathscr{A}^\lambda_w$, see e.g. [KT02]. Let

$$\tilde{\mathscr{A}}^\lambda_{w!} = (i_w)_!(\tilde{\mathscr{A}}^\lambda_w), \qquad \tilde{\mathscr{A}}^\lambda_{w!\bullet} = (i_w)_{!\bullet}(\tilde{\mathscr{A}}^\lambda_w).$$

They are objects of $\mathbf{MHM}_0(\overline{X}_v)$ such that

$$\eta(\tilde{\mathscr{A}}^\lambda_{w!}) = \mathscr{A}^\lambda_{w!}, \qquad \eta(\tilde{\mathscr{A}}^\lambda_{w!\bullet}) = \mathscr{A}^\lambda_{w!\bullet}.$$

Now, assume that $w \in {}^Q\mathfrak{S}$ and $w \leqslant v$. Recall that $\mathscr{B}^\lambda_w \in \mathbf{M}^\lambda(Y_w)$, and that $\mathscr{B}^\lambda_{w!} \in \mathbf{M}^\lambda(X)$ can be viewed as an object of $\mathbf{M}^\lambda(\overline{X}_v)$. We define similarly the objects $\tilde{\mathscr{B}}^\lambda_w \in \mathbf{MHM}(Y_w)$ and $\tilde{\mathscr{B}}^\lambda_{w!},\ \tilde{\mathscr{B}}^\lambda_{w!\bullet} \in \mathbf{MHM}_0(\overline{X}_v)$ such that

$$\eta(\tilde{\mathscr{B}}^\lambda_{w!}) = \mathscr{B}^\lambda_{w!}, \qquad \eta(\tilde{\mathscr{B}}^\lambda_{w!\bullet}) = \mathscr{B}^\lambda_{w!\bullet}.$$

The object $\tilde{\mathscr{B}}^\lambda_{w!}$ has a canonical weight filtration $W^\bullet$. We set $J^k\mathscr{B}^\lambda_{w!} = \mathscr{B}^\lambda_{w!}$ for $k < 0$. The following proposition is due to Gabber and Beilinson-Bernstein [BB93, Theorem 5.1.2, Corollary 5.1.3].

Proposition 7.2.1. *We have* $\eta(W^{l(w)-k}\tilde{\mathscr{B}}^\lambda_{w!}) = J^k\mathscr{B}^\lambda_{w!}$ *in* $\mathbf{M}^\lambda_0(\overline{X}_v)$ *for all* $k\in\mathbb{Z}$.

So the problem that we posed at the beginning of the section reduces to calculate the multiplicities of $\tilde{\mathscr{B}}^\lambda_{x!\bullet}$ in $\tilde{\mathscr{B}}^\lambda_{w!}$ in the category $\mathbf{MHM}_0(\overline{X}_v)$. Let q be a formal parameter. The Hecke algebra $\mathscr{H}_q(\mathfrak{S})$ of $\mathfrak{S}$ is a $\mathbb{Z}[q,q^{-1}]$-algebra with a $\mathbb{Z}[q,q^{-1}]$-basis $\{T_w\}_{w\in\mathfrak{S}}$ whose multiplication is given by

$$T_{w_1}T_{w_2} = T_{w_1w_2}, \qquad \text{if } l(w_1w_2) = l(w_1)+l(w_2),$$

$$(T_{s_i}+1)(T_{s_i}-q) = 0, \quad 0\leqslant i\leqslant m-1.$$

On the other hand, the Grothendieck group $[\mathbf{MHM}_0(\overline{X}_v)]$ is a $\mathbb{Z}[q,q^{-1}]$-module such that

$$q^k[\mathscr{M}] = [\mathscr{M}(-k)], \qquad k\in\mathbb{Z},\ \mathscr{M}\in\mathbf{MHM}_0(\overline{X}_v).$$

For $x\in\mathfrak{S}$ with $x\leqslant v$ consider the closed embedding

$$c_x : \mathrm{pt}\to\overline{X}_v, \quad \mathrm{pt}\mapsto \dot{x}B/B.$$

There is an injective $\mathbb{Z}[q,q^{-1}]$-module homomorphism, see e.g., [KT02, (5.4)],

$$\begin{aligned}\Psi : [\mathbf{MHM}_0(\overline{X}_v)] &\longrightarrow \mathscr{H}_q(\mathfrak{S}),\\ [\mathscr{M}] &\longmapsto \sum_{x\leqslant v}\sum_{k\in\mathbb{Z}}(-1)^k[H^kc_x^*(\mathscr{M})]T_x.\end{aligned}$$

The desired multiplicities are given by the following lemma.

Lemma 7.2.2. *For* $w\in{}^Q\mathfrak{S}$ *we have*

$$\Psi([\tilde{\mathscr{B}}^\lambda_{w!\bullet}]) = \sum_{x\in{}^Q\mathfrak{S},x\leqslant w}(-1)^{l(w)-l(x)}P_{x,w}\Psi([\tilde{\mathscr{B}}^\lambda_{x!}]),$$

where $P_{x,w}\in\mathbb{Z}[q,q^{-1}]$ *is the Kazhdan-Lusztig polynomial.*

Proof. Since the choice for the element v above is arbitrary, we may assume that $w\leqslant v$. By the definition of Ψ we have

$$\Psi([\tilde{\mathscr{A}}^\lambda_{w!}]) = (-1)^{l(w)}T_w. \tag{7.2.1}$$

By [KL80], [KT95], we have

$$\Psi([\tilde{\mathscr{A}}^\lambda_{w!\bullet}]) = (-1)^{l(w)}\sum_{x\in\mathfrak{S}}P_{x,w}T_x. \tag{7.2.2}$$

Next, for $x\in{}^Q\mathfrak{S}$ with $x\leqslant v$ we have

$$\begin{aligned}\Psi([\tilde{\mathscr{B}}^\lambda_{x!}]) &= \sum_{y\in\mathfrak{S}_0}\sum_{k\in\mathbb{Z}}(-1)^k[H^kc_{yx}^*(\tilde{\mathscr{B}}^\lambda_{x!})]T_{yx}\\ &= (-1)^{l(x)}\sum_{y\in\mathfrak{S}_0}T_{yx}.\end{aligned} \tag{7.2.3}$$

Since by Lemma 5.10.3 we have

$$\tilde{\mathscr{A}}^\lambda_{w!\bullet} = \tilde{\mathscr{B}}^\lambda_{w!\bullet},$$

the following equalities hold

$$\begin{aligned}
\Psi([\tilde{\mathscr{B}}^{\lambda}_{w!\bullet}]) &= \Psi([\tilde{\mathscr{A}}^{\lambda}_{w!\bullet}]) \\
&= (-1)^{l(w)} \sum_{x\in{}^{Q}\mathfrak{S}, x\leqslant w} \sum_{y\in\mathfrak{S}_0} P_{yx,w} T_{yx} \\
&= (-1)^{l(w)} \sum_{x\in{}^{Q}\mathfrak{S}, x\leqslant w} P_{x,w} \sum_{y\in\mathfrak{S}_0} T_{yx} \\
&= \sum_{x\in{}^{Q}\mathfrak{S}, x\leqslant w} (-1)^{l(w)-l(x)} P_{x,w} \Psi([\tilde{\mathscr{B}}^{\lambda}_{x!}]).
\end{aligned}$$

Here the third equality is given by the well known identity:

$$P_{yx,w} = P_{x,w}, \quad y \in \mathfrak{S}_0,\ x \in {}^{Q}\mathfrak{S},\ x \leqslant w.$$

□

Remark 7.2.3. Let $x \in {}^{Q}\mathfrak{S}$. Since Ψ is injective, the equation (7.2.3) yields that

$$[\tilde{\mathscr{B}}^{\lambda}_{x!}] = \sum_{y\in\mathfrak{S}_0} (-1)^{l(y)} [\tilde{\mathscr{A}}^{\lambda}_{yx!}].$$

By applying the functor η we get the following equality in $[\mathbf{M}^{\lambda}_0(X)]$

$$[\mathscr{B}^{\lambda}_{x!}] = \sum_{y\in\mathfrak{S}_0} (-1)^{l(y)} [\mathscr{A}^{\lambda}_{yx!}]. \tag{7.2.4}$$

7.3 Proof of Theorem 1.0.1.

Recall from (3.5.1) that we view $\mathcal{P}_n$ as a subset of Λ^+. By Corollary 3.11.3, Theorem 1.0.1 is a consequence of the following theorem.

Theorem 7.3.1. *Let λ, μ be partitions of n. Then for any negative integer κ we have*

$$d_{\lambda'\mu'}(q) = \sum_{i\geqslant 0} [J^i M_\kappa(\lambda)/J^{i+1}M_\kappa(\lambda) : L_\kappa(\mu)] q^i. \tag{7.3.1}$$

Here $d_{\lambda'\mu'}(q)$ is the polynomial defined in Section 1 with $v = \exp(2\pi i/\kappa)$.

Proof. By (5.9.1) we may assume that μ, λ belong to the same orbit of a weight ν under the dot action of $\mathfrak{S}$ such that $\nu + \rho$ is antidominant. For any $\mu \in \Lambda^+ \cap (\mathfrak{S}\cdot\nu)$ let $w(\mu)_\nu$ be the shortest element in the set

$$w(\mu)_\nu \mathfrak{S}(\nu) = \{w \in \mathfrak{S} \mid \mu = w\cdot\nu\}.$$

Note that $w(\mu)_\nu\mathfrak{S}(\nu)$ is contained in ${}^{Q}\mathfrak{S}$ by Lemma 5.10.1. We fix $v \in \mathfrak{S}$ such that $v \geqslant w(\gamma)_\nu$ for any $\gamma \in \mathcal{P}_n$. Let $q^{1/2}$ be a formal variable. We identify $q = (q^{1/2})^2$. Let $\tilde{P}_{x,w}$ be the Kazhdan-Lusztig polynomial normalized as follows

$$P_{x,w}(q) = q^{(l(w)-l(x))/2} \tilde{P}_{x,w}(q^{-1/2}).$$

Let $\tilde{Q}_{x,w}$ be the inverse Kazhdan-Lusztig polynomial given by

$$\sum_{x\in\mathfrak{S}} \tilde{Q}_{x,z}(-q)\tilde{P}_{x,w}(q) = \delta_{z,w}, \quad z, w \in \mathfrak{S}.$$

Then by (7.2.1), (7.2.2) we have

$$[\tilde{\mathscr{A}}^\nu_{x!}] = \sum_{w\in\mathfrak{S}} q^{(l(x)-l(w))/2}\tilde{Q}_{x,w}(q^{-1/2})[\tilde{\mathscr{A}}^\nu_{w!\bullet}], \quad \forall\, x\in\mathfrak{S}.$$

By Remark 7.2.3 we see that

$$[\tilde{\mathscr{B}}^\nu_{x!}] = \sum_{w\in{}^Q\mathfrak{S}} \Big(\sum_{s\in\mathfrak{S}_0}(-1)^{l(s)}q^{(l(sx)-l(w))/2}\tilde{Q}_{sx,w}(q^{-1/2})\Big)[\tilde{\mathscr{B}}^\nu_{w!\bullet}], \quad \forall\, x\in{}^Q\mathfrak{S}. \tag{7.3.2}$$

Now, let

$$[\mathbf{M}^\nu_0(\overline{X}_v)]_q = [\mathbf{M}^\nu_0(\overline{X}_v)]\otimes_{\mathbb{Z}}\mathbb{Z}[q^{1/2},q^{-1/2}], \qquad [\tilde{\mathcal{O}}_\kappa]_q = [\tilde{\mathcal{O}}_\kappa]\otimes_{\mathbb{Z}}\mathbb{Z}[q^{1/2},q^{-1/2}].$$

We have a $\mathbb{Z}[q,q^{-1}]$-module homomorphism

$$\begin{aligned}\varepsilon : [\mathbf{MHM}_0(\overline{X}_v)] &\longrightarrow [\mathbf{M}^\nu_0(\overline{X}_v)]_q,\\ [\mathscr{M}] &\longmapsto \sum_{i\in\mathbb{Z}}[\eta(W^i\mathscr{M}/W^{i-1}\mathscr{M})]q^{i/2}.\end{aligned}$$

Note that $\varepsilon([\tilde{\mathscr{B}}^\nu_{x!\bullet}]) = q^{l(x)/2}[\mathscr{B}^\nu_{x!\bullet}]$ and by Proposition 7.2.1 we have

$$\varepsilon([\tilde{\mathscr{B}}^\nu_{x!}]) = \sum_{i\in\mathbb{N}}[J^i\mathscr{B}^\nu_{x!}/J^{i+1}\mathscr{B}^\nu_{x!}]q^{(l(x)-i)/2}, \quad x\in{}^Q\mathfrak{S},\ x\leqslant v.$$

Next, let

$$[M_\kappa(\lambda)]_q = \sum_{i\in\mathbb{N}}[J^iM_\kappa(\lambda)/J^{i+1}M_\kappa(\lambda)]q^{-i/2}.$$

Then by Proposition 6.7.1, we have

$$\Gamma\varepsilon([\tilde{\mathscr{B}}^\nu_{w(\lambda)_\nu!}]) = q^{l(w(\lambda)_\nu)/2}[M_\kappa(\lambda)]_q.$$

On the other hand, by (7.3.2) we have

$$\begin{aligned}\Gamma\varepsilon([\tilde{\mathscr{B}}^\nu_{w(\lambda)_\nu!}]) &= \sum_{w\in{}^Q\mathfrak{S}}\Big(\sum_{s\in\mathfrak{S}_0}(-1)^{l(s)}q^{(l(sw(\lambda)_\nu)-l(w))/2}\tilde{Q}_{sw(\lambda)_\nu,w}(-q^{1/2})\Big)\Gamma\varepsilon[\tilde{\mathscr{B}}^\nu_{w!\bullet}]\\ &= \sum_{\mu\in\mathcal{P}_n}\Big(\sum_{s\in\mathfrak{S}_0}(-1)^{l(s)}q^{(l(sw(\lambda)_\nu)-l(w(\mu)_\nu))/2}\tilde{Q}_{sw(\lambda)_\nu,w(\mu)_\nu}(q^{-1/2})\Big)q^{l(w(\mu)_\nu)/2}[L_\kappa(\mu)].\end{aligned}$$

Here in the second equality we have used Proposition 5.10.4(c) and the fact that $\mathcal{P}_n$ is an ideal in Λ^+. Note that $l(sw(\lambda)_\nu) = l(w(\lambda)_\nu) - l(s)$ for $s\in\mathfrak{S}_0$ by Lemma 5.10.1(b). We deduce that

$$[M_\kappa(\lambda)]_q = \sum_{\mu\in\mathcal{P}_n}\Big(\sum_{s\in\mathfrak{S}_0}(-q^{-1/2})^{l(s)}\tilde{Q}_{sw(\lambda)_\nu,w(\mu)_\nu}(q^{-1/2})\Big)[L_\kappa(\mu)].$$

By [Lec00, Proposition 5], we have

$$d_{\lambda',\mu'}(q^{-1/2}) = \sum_{s\in\mathfrak{S}_0}(-q^{-1/2})^{l(s)}\tilde{Q}_{sw(\lambda)_\nu,w(\mu)_\nu}(q^{-1/2}),$$

see also the beginning of the proof of Proposition 6 in loc. cit., and [LT00, Lemma 2.2] for instance. We deduce that

$$\sum_{i\in\mathbb{N}}[J^iM_\kappa(\lambda)/J^{i+1}M_\kappa(\lambda)]q^i = \sum_{\mu\in\mathcal{P}_n}d_{\lambda',\mu'}(q)[L_\kappa(\mu)].$$

The theorem is proved. $\square$

Remark 7.3.2. The q-multiplicities of the Weyl modules $W_v(\lambda)$ have also been considered in [Ari09] and [RT10]. Both papers are of combinatorial nature, and are very different from the approach used here. In [Ari09] Ariki defined a grading on the q-Schur algebra and he proved that the q-multiplicities of the Weyl module with respect to this grading is also given by the same polynomials $d_{\lambda',\mu'}$. However, it not clear to us how to relate this grading to the Jantzen filtration.

Remark 7.3.3. The *radical filtration* $C^\bullet(M)$ of an object M in an abelian category $\mathcal{C}$ is given by putting $C^0(M) = M$ and $C^{i+1}(M)$ to be the radical of $C^i(M)$ for $i \leqslant 0$. It follows from [BB93, Lemma 5.2.2] and Proposition 6.7.1 that the Jantzen filtration of $\mathscr{B}_!$ coincides with the radical filtration. If $\lambda \in \Lambda$ such that $\lambda + \rho$ is antidominant and regular, then the exact functor Γ is faithful, see [BD00, Theorem 7.15.6]. In this case, we have

$$\Gamma(C^\bullet(\mathscr{B}_!)) = C^\bullet(\Gamma(\mathscr{B}_!)) = C^\bullet M_\kappa(\lambda).$$

So the Jantzen filtration on $M_\kappa(\lambda)$ coincides with the radical filtration. If we have further $\lambda \in \mathcal{P}_n$ and $\kappa \leqslant -3$, then by the equivalence in Proposition 3.11.2 we deduce that the Jantzen filtration of $W_v(\lambda)$ also coincides with the radical filtration. This is compatible with recent result of Parshall-Scott [PS09], where they computed the radical filtration of $W_v(\lambda)$ under the same assumption of regularity here but without assuming $\kappa \leqslant -3$. We conjecture that for any λ the Jantzen filtration on $M_\kappa(\lambda)$ coincides with the radical filtration.

Remark 7.3.4. The results of Sections 5, 6, 7 hold for any standard parabolic subgroup Q of G with the same proof. In particular, it allows us to calculate the graded decomposition matrices associated with the Jantzen filtration of the parabolic Verma modules in more general cases.

Appendix A

List of Notation

Note that the notation of the three chapters are independent.

Chapter I

1.1: $\mathfrak{h}$, $\mathcal{S}$, $\mathcal{A}$, $\mathfrak{h}_{reg}$, $B(W,\mathfrak{h})$, $\mathscr{H}_q(W,\mathfrak{h})$, B_W, $\mathscr{H}_q(W)$.

1.2: ${}^{\mathscr{H}}\mathrm{Res}_{W'}^{W}$, ${}^{\mathscr{H}}\mathrm{Ind}_{W'}^{W}$, ${}^{\mathscr{H}}\mathrm{coInd}_{W'}^{W}$, $\mathcal{A}'$, $\overline{\mathcal{A}}$, $\overline{\mathfrak{h}}$, $\overline{\mathfrak{h}}_{reg}$, $\mathfrak{h}'_{reg}$, Ω.

2.1: $H_c(W,\mathfrak{h})$, α_s, $\alpha_s^\vee$, $\mathbf{eu}_0$, $\mathbf{eu}$.

2.2: $\mathcal{O}_c(W,\mathfrak{h})$, $\mathrm{Irr}(W)$, $\Delta(\xi)$, $L(\xi)$, $\mathcal{O}_c^\Delta(W,\mathfrak{h})$, $P(\xi)$, $\prec_c$.

2.3: $\mathrm{Proj}_c(W,\mathfrak{h})$, $I:\mathrm{Proj}_c(W,\mathfrak{h})\to\mathcal{O}_c(W,\mathfrak{h})$.

2.4: $\mathcal{D}(\mathfrak{h}_{reg})$, $H_c(W,\mathfrak{h}_{reg})$, M_{reg}, $\mathcal{O}^{an}_{\mathfrak{h}_{reg}}$, N^{an}, N^∇, $\mathrm{KZ}=\mathrm{KZ}(W,\mathfrak{h})$.

2.5: P_{KZ}, $Z(\mathcal{O}_c(W,\mathfrak{h}))$, $Z(\mathscr{H}_q(W,\mathfrak{h}))$, γ, S.

3.1: b, W', c', ζ, ζ^{-1}.

3.2: $\mathbb{C}[[\mathfrak{h}]]_p$, $\widehat{\mathbb{C}[\mathfrak{h}]}_p$, $\widehat{M}_p$, $\widehat{H}_c(W,\mathfrak{h})_b$, $\widehat{H}_{c'}(W',\mathfrak{h})_0$, $\widehat{\mathcal{O}}_c(W,\mathfrak{h})_b$, $\widehat{\mathcal{O}}_{c'}(W',\mathfrak{h})_0$, $P=\mathrm{Fun}_{W'}(W,\widehat{H}_c(W',\mathfrak{h})_0)$, $Z(W,W',\widehat{H}_c(W',\mathfrak{h})_0)$, Θ, $x_\alpha^{(b)}$, $y_a^{(b)}$, J, x_{pr}, R.

3.3: $\widehat{\ }_0$, E, $\widehat{\ }_b$, E^b, Res_b, Ind_b.

3.5: $\mathrm{KZ}'=\mathrm{KZ}(W',\overline{\mathfrak{h}})$.

4.1: $\widetilde{\mathfrak{sl}}_e$, $\widehat{\mathfrak{sl}}_e$, e_i, f_i, h_i, $\mathfrak{t}$, α_i, $\alpha_i^\vee$, Λ_i, P, V_μ.

4.2: $|\lambda|$, $\mathcal{P}_{n,l}$, Υ_λ, $\mathbf{s}$, $\mathcal{F}_{\mathbf{s}}$, $|\lambda\rangle$, $\mathrm{res}(\mu/\lambda)$, $\Lambda_{\mathbf{s}}$, $\mathrm{wt}(|\lambda\rangle)$.

4.3: $B_n(l)$, $\mathcal{S}_n$, $b_n\in\mathbb{C}^n$, $\mathcal{P}_{n,l}\xrightarrow{\sim}\mathrm{Irr}(B_n(l))$.

4.4: $\mathscr{H}_{\mathbf{q},n}$, $\mathcal{C}_{\mathbf{q},n}$, $\mathcal{C}_{\mathbf{q}}$, S_λ, $E(n)^{\mathscr{H}}$, $F(n)^{\mathscr{H}}$, $E^{\mathscr{H}}$, $F^{\mathscr{H}}$.

4.5: $\mathbf{q}$, J_i, $C_n(z)$, $P_{n,a(z)}$, $E_i(n)^{\mathscr{H}}$, $F_i(n)^{\mathscr{H}}$, $E_i^{\mathscr{H}}$, $F_i^{\mathscr{H}}$, $a_\lambda(z)$.

4.6: $\mathbf{h}$, $H_{\mathbf{h},n}$, $\mathcal{O}_{\mathbf{h},n}$, $\mathrm{KZ}_{\mathbf{h},n}$.

4.7: $\mathcal{O}_{\mathbf{h}}$, KZ, E, F, $D_n(z)$, $Q_{n,a(z)}$, $E_i(n)$, $F_i(n)$, E_i, F_i,

5.1: $q=\exp(2\pi\sqrt{-1}/e)$.

5.2: B, wt, $\tilde{e}_i$, $\tilde{f}_i$, ϵ_i, φ_i, $\mathfrak{b}$, $l_i(v)$, $V_i^{<l}$, B^+, V^+.

5.4: $B_{\mathcal{F}_{\mathbf{s}}}$, $B_{\mathcal{O}_{\mathbf{h}}}$.

Chapter II

0.1: $\mathbf{k}$, $K(\mathbf{R})$, $G(\mathbf{R})$, $\mathcal{A}$, $\hom_{\mathbf{R}}$, gdim.

0.2: Γ, $H_{i,j}$, $i \to j$, $h_{i,j}$, $i \cdot j$, θ, ${}^\theta\mathbb{N}I$, $|\nu|$, ${}^\theta I^\nu$, $\theta(\mathbf{i})$, ${}^\theta I^m$.

0.3: $\mathfrak{S}_m$, $\mathbb{Z}_2$, W_m, $w(\mathbf{i})$, $\mathbf{i}_e$, W_e, s_0, ${}^\circ W_m$, ${}^\theta I^\nu_+$, ${}^\theta I^\nu_-$.

1.1: ${}^\circ\mathbf{R}(\Gamma)_\nu$, $1_{\mathbf{i}}$, $\varkappa_l$, σ_k, $Q_{i,j}(u,v)$, $\sigma_{\mathbf{i},k}$, $\varkappa_{\mathbf{i},l}$, ω.

1.2: ${}^\theta\mathbf{R}(\Gamma)_\nu$, π_1, σ_0, γ, $\langle\gamma\rangle$.

1.3: ${}^\theta\mathbf{F}_{\mathbf{i}}$, ${}^\theta\mathbf{F}_\nu$, ${}^\circ\mathbf{S}_\nu$, $\dot{w}$, $\sigma_{\dot{w}}$, ${}^\theta\mathbf{F}'_{\mathbf{i}}$, ${}^\theta\mathbf{F}'_\nu$.

2.1: $\mathbf{H}_m$, T_k, X_l.

2.2: $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{H}'_m$, φ_k, $\mathbf{H}_m$-$\mathbf{Mod}_I$.

2.3: ${}^\theta\mathbf{R}_\nu$, ${}^\theta\mathbf{R}_m$, ${}^\circ\mathbf{R}_\nu$, ${}^\circ\mathbf{R}_m$, ${}^\circ\mathbf{R}_m$-$\mathbf{Mod}_0$, ${}^\circ\mathbf{R}_m$-$\mathbf{fMod}_0$, $\mathbf{H}_m$-$\mathbf{fMod}_I$, Ψ.

2.4: E_i, F_i, $\mathbf{k}_i$.

3.1: ${}^\circ\mathbf{K}_I$, ${}^\circ\mathbf{G}_I$, ${}^\circ\mathbf{K}_{I,*}$, ${}^\circ\mathbf{G}_{I,*}$, $(\bullet:\bullet)$, $\langle\bullet:\bullet\rangle$, $M^\flat$, ${}^\circ B$, ${}^\circ\mathbf{G}^{\mathrm{up}}$, $P^\sharp$, $\mathcal{K}$, $\bar{f}$,${}^\circ\mathbf{G}^{\mathrm{low}}$, ϕ_+, ϕ_-.

3.2: $D_{m,1}$, $D_{m,1;m,1}$, $W(w)$, $\mathbf{R}_1$, $\mathbf{R}_i$, $\mathbf{L}_i$, ${}^\theta\mathbf{R}_{m,1}$, ${}^\circ\mathbf{R}_{m,1}$, $\psi_!$, ψ^*, ψ_*, f_i, e_i, e'_i.

3.3: **for**.

3.4: res, ind, M^γ, $1_{\nu,+}$, $1_{\nu,-}$.

3.5: Res, Ind.

3.7: $\tilde{e}_i$, $\tilde{f}_i$, ε_i.

3.8: ${}^\theta\boldsymbol{\mathcal{B}}$, $\theta^{(a)}$, $\langle a\rangle$, $\langle a\rangle!$, ${}^\circ\mathbf{V}$.

3.9: $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$, $\mathcal{R}$, ${}^\circ\mathbf{L}$.

Chapter III

1: $d_{\lambda\mu}(q)$, λ'

2.1: $\mathcal{C} \cap R$-proj, $R'\mathcal{C}$.

2.2: $\mathcal{C}^\Delta$, Δ, ∇, $D^\vee$.

2.3: $R = \mathbb{C}[[s]]$, $\wp$, K, $M(\wp^i)$, M_K, $f(\wp^i)$, f_K, $\mathcal{C}(\wp)$, $\mathcal{C}_K$.

2.4: $J^iD(\wp)$.

2.5: $F(\wp)$.

3.1: G_0, B_0, T_0, $\mathfrak{g}_0$, $\mathfrak{b}_0$, $\mathfrak{t}_0$, $\mathfrak{g}$, $\mathfrak{t}$, $\mathbf{1}$, ∂, c, $\kappa = c+m$, $\mathfrak{a}_R$, $\mathcal{U}_\kappa$, M_λ, $\mathfrak{t}_0^*$, $\mathfrak{t}^*$, δ, ω_0, ϵ_i, $\langle - : - \rangle$, $||h||^2$, ${}_\kappa\mathfrak{t}^*$, a, z, Π, Π_0, Π^+, Π_0^+, α_i, $\mathfrak{S}$, $\mathfrak{S}_0$, $w \cdot \lambda$, ρ_0, ρ, $\mathfrak{S}(\lambda)$, $l : \mathfrak{S} \to \mathbb{N}$.

3.2: $\mathfrak{q}$, $\mathfrak{l}$, Λ^+, $M_\kappa(\lambda)$, $L_\kappa(\lambda)$, $\mathbf{c}$, $\mathbf{k}$, $\mathcal{U}_\mathbf{k}$, $M_\mathbf{k}(\lambda)$, λ_s, ${}_\mathbf{k}\mathfrak{t}^*$.

3.3: σ, $\mathbf{D}M_\mathbf{k}(\lambda)$, $J^iM_\kappa(\lambda)$.

3.4: $\mathcal{O}_\mathbf{k}$, $\mathcal{O}_\kappa$, ${}^r_\kappa\mathfrak{t}^*$, ${}^r_\mathbf{k}\mathfrak{t}^*$, ${}^r\mathcal{O}_\kappa$, ${}^r\mathcal{O}_\mathbf{k}$, ${}^r\Lambda^+$, ${}^rP_\kappa(\lambda)$, $L_\mathbf{k}(\lambda)$, ${}^rP_\mathbf{k}(\lambda)$.

3.5: $\mathcal{P}_n$, $\preceq$, $\trianglelefteq$, E, $\mathcal{E}_\kappa$, $\mathfrak{g}'$, $\mathcal{E}_\mathbf{k}$, $P_\mathbf{k}(E)$, $P_\kappa(E)$.

3.6: $\mathcal{A}_\mathbf{k}$, $\mathcal{A}_\kappa$, $\Delta_\mathbf{k}$, Δ_κ.

3.7: $\mathbf{D}$.

3.9: $\mathscr{H}_\mathbf{v}$, $\mathbf{S}_\mathbf{v}$, $\mathcal{A}_\mathbf{v}$, $W_\mathbf{v}(\lambda)$, $\Delta_\mathbf{v}$.

3.10: $v = \exp(2\pi i/\kappa)$, $\mathbf{v} = \exp(2\pi i/\mathbf{k})$, $J^iW_v(\lambda)$.

3.11: $\mathbf{H}_{1/\kappa}$, $\mathcal{B}_\kappa$, $B_\kappa(\lambda)$, $\mathfrak{E}_\kappa$, $\mathbf{H}_{1/\mathbf{k}}$, $\mathcal{B}_\mathbf{k}$, $\mathfrak{E}_\mathbf{k}$, $B_\mathbf{k}(\lambda)$.

4.1: $\mathscr{O}_Z$, $\mathbf{O}(Z)$, f_*, f^*, $f^!$, $\mathscr{D}_Z$, $\mathbf{M}(Z)$, Ω_Z, $\mathscr{D}_{Y\to Z}$, $\mathscr{M}^\mathscr{O}$, $\mathbf{M}(Z, Z')$, i^*, $i_\bullet$, $i^!$.

4.2: $\mathbf{M}_h(Z)$, $\mathbb{D}$, $i_!$, $i_{!\bullet}$.

4.3: $\mathbf{M}^T(Z)$, $\mathbf{M}^T(X, Z)$.

4.4: $\mathscr{M}^\dagger$, $\mathscr{D}_Z^\dagger$, δ_r, $\mathfrak{m}_\lambda$, $\mathbf{M}^\lambda(Z)$, $\mathbf{M}^{\tilde\lambda}(Z)$, $\mathbf{M}^\lambda(X, Z)$, $\mathbf{M}^{\tilde\lambda}(X, Z)$, $\mathbf{M}(\mathscr{D}_Z^\dagger)$.

4.5: $\mathbf{M}_h^T(Z^\dagger)$, $\mathbf{M}_h^\lambda(Z)$, $\mathbf{M}_h^{\tilde\lambda}(Z)$, $\mathbb{D}'$, $\mathscr{L}_Z^\lambda$, Θ^λ, $\mathbb{D} = \mathbb{D}^\lambda$, $\mathscr{D}_Z^\lambda$.

4.6: $2\varinjlim \mathcal{C}_\alpha$, $2\varprojlim \mathcal{C}_\alpha$.

4.7: $X = \varinjlim_\alpha X_\alpha$, $\mathbf{O}(X)$, $\Gamma(X, \mathscr{M})$ (for $\mathscr{M} \in \mathbf{O}(X)$), $\hat{\mathbf{O}}(X)$, $\mathscr{O}_X$, $- \otimes_{\mathscr{O}_X} \mathscr{F}$, $\Gamma(X, \mathscr{F})$ (for $\mathscr{F} \in \hat{\mathbf{O}}(X)$).

4.8: $\mathbf{M}(X)$ (with X an ind-scheme), $\mathscr{M}^\mathscr{O}$, $\Gamma(X, \mathscr{M})$ (for $\mathscr{M} \in \mathbf{M}(X)$), $\mathbf{M}^T(X)$, $\mathbf{M}^\lambda(X)$, $\mathbf{M}^{\tilde\lambda}(X)$, $\Gamma(\mathscr{M})$ (for $\mathscr{M} \in \mathbf{M}^{\tilde\lambda}(X)$), $\mathbf{M}_h^\lambda(X)$, $\mathbf{M}_h^{\tilde\lambda}(X)$, $i_!$, $i_\bullet$.

4.9: $\mathscr{D}_X$ (with X a formally smooth ind-scheme).

5.1: G, B, Q, T, N, $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$.

5.2: $X = G/B$, $X^\dagger = G/N$, $\pi : X^\dagger \to X$, X_w, $\dot{w}$, $\overline{X}_w$.

5.3: $\delta_l : \mathcal{U}(\mathfrak{g}) \to \Gamma(X^\dagger, \mathscr{D}_{X^\dagger})$, $\mathbf{M}(\mathfrak{g})$, Γ

5.4: $M_{\tilde\lambda}$, $\mathrm{ch}(M)$, $\tilde{\mathcal{O}}$, $\tilde{\mathcal{O}}_\kappa$, $N_\kappa(\lambda)$, $L_\kappa(\lambda)$, Λ, $\mathscr{A}_w^\lambda$, i_w, $\mathscr{A}_{w!}^\lambda$, $\mathscr{A}_{w!\bullet}^\lambda$, $\mathscr{A}_{w\bullet}^\lambda$.

5.5: Π^-, $\mathfrak{n}(\Upsilon)$, $\mathfrak{n}^-(\Upsilon)$, Π_l^-, $\mathfrak{n}_l^-$, N_l^-, $\mathfrak{X}$, $\mathfrak{X}^\dagger$, $\mathfrak{X}^w$, $\mathfrak{X}_l^w$, $\mathfrak{X}_l^{w\dagger}$, $p_{l_1l_2}$, p_l.

5.6: $\mathbf{H}^{\tilde{\lambda}}(\mathfrak{X}^y, \overline{X}_w)$, $\mathbf{H}^{\tilde{\lambda}}(\overline{X}_w)$, $\mathbf{H}^{\tilde{\lambda}}(X)$.

5.7: $\hat{\Gamma}(\mathscr{M})$, $\overline{\Gamma}(\mathscr{M})$.

5.9: $\tilde{\mathcal{O}}_{\kappa,\lambda}$, pr_λ, $V(\nu)$, θ^ν, $\mathscr{L}^\lambda$, $\Theta^{\lambda-\mu}$.

5.10: ${}^Q\mathfrak{S}$, w_0, Y_w, $\overline{Y}_w$, j_w, $\mathscr{B}_w^\lambda$, $\mathscr{B}_{w!}^\lambda$, $\mathscr{B}_{w!\bullet}^\lambda$, $\mathscr{B}_{w\bullet}^\lambda$, $r : X_w^\dagger \to Y_w^\dagger$.

6.1: $\mathcal{C}_R$, μ_M, for, F_R.

6.2: Q', f_w.

6.3: j, f, $\mathscr{B}$, $\mathscr{B}_!$, $\mathscr{I}^{(n)}$, x^s, f^s, $\mathscr{B}^{(n)}$, $\mathscr{B}_!^{(n)}$, $\mathscr{B}_{!\bullet}^{(n)}$, $\mathscr{B}_\bullet^{(n)}$, $\mathscr{A}_v^{(n)}$, $\mathscr{A}_{v!}^{(n)}$, $\mathscr{A}_{v!\bullet}^{(n)}$, $\mathscr{A}_{v\bullet}^{(n)}$.

6.4: $N_{\mathbf{k}}(\mu)$, $R_{\mu+s\omega_0}$, $\mathbf{D}N_{\mathbf{k}}(\mu)$, $N_{\mathbf{k}}^{(n)}(\mu)$, $\mathbf{D}N_{\mathbf{k}}^{(n)}(\mu)$, s^iM, $F^\bullet M$, $\mathrm{gr}\, M$, $\mathbf{ch}(M)$.

6.5: M_κ, $M_{\mathbf{k}}$, $M_{\mathbf{k}}^{(n)}$, $\mathbf{D}M_{\mathbf{k}}^{(n)}$.

6.6: $\psi(a,n)$, $\pi^a(\mathscr{B})$, $J^i(\mathscr{B}_!)$.

7.1: $\mathbf{MHM}(Z)$, $W^\bullet\mathscr{M}$, (k), $\mathbf{Perv}(Z)$, ϱ.

7.2: $\mathbf{M}_0^\lambda(\overline{X}_v)$, DR, $\mathbf{MHM}_0(\overline{X}_v)$, η, $\tilde{\mathscr{A}}_w^\lambda$, $\tilde{\mathscr{A}}_{w!}^\lambda$, $\tilde{\mathscr{A}}_{w!\bullet}^\lambda$, $\tilde{\mathscr{B}}_w^\lambda$, $\tilde{\mathscr{B}}_{w!}^\lambda$, $\tilde{\mathscr{B}}_{w!\bullet}^\lambda$, q, $\mathscr{H}_q(\mathfrak{S})$, T_w, Ψ, $P_{x,w}$,

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Canonical bases and gradings associated with rational double affine Hecke algebras

Abstract

This thesis consists of three chapters. In Chapter I, we define the i-restriction and i-induction functors on the category $\mathcal{O}$ of the cyclotomic rational double affine Hecke algebras. Using these functors, we construct a crystal on the set of isomorphism classes of simple modules, which is isomorphic to the crystal of a Fock space. Chapter II is a joint work with Michela Varagnolo and Eric Vasserot. We prove a conjecture of Miemietz and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type D. In Chapter III, we prove a conjecture of Leclerc and Thibon on the graded multiplicities associated with the Jantzen filtration of Weyl modules over v-Schur algebras.

Résumé

Cette thèse se compose de trois chapitres. Dans le chapitre I, nous définissons les foncteurs de i-restriction et i-induction sur la catégorie $\mathcal{O}$ des algèbres de Hecke doublement affine rationnelles cyclotomiques. En utilisant ces foncteurs, nous construisons un cristal sur l'ensemble des classes d'isomorphisme des modules simples, qui est isomorphe au cristal de l'espace de Fock. Le chapitre II est un travail en collaboration avec Michela Varagnolo et Eric Vasserot. Nous démontrons une conjecture de Kashiwara et Miemietz sur bases canoniques et règles de branchement pour les algèbres de Hecke affines de type D. Dans le chapitre III, nous démontrons une conjecture de Leclerc et Thibon sur les multiplicités graduées associées à la filtration de Jantzen de modules de Weyl sur algèbres de v-Schur.