Université Paris Diderot - Paris 7

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Pierre-Guy Plamondon

Catégories amassées aux espaces de morphismes de dimension infinie, applications

---

dirigée par Bernhard Keller

Soutenue le 5 juillet 2011 devant le jury composé de :

M. Ibrahim Assem	(Université de Sherbrooke)	
M. Christof Geiss	(UNAM)	
M. David Hernandez	(Université Paris 7)	
M. Bernhard Keller	(Université Paris 7)	directeur
M. Bernard Leclerc	(Université de Caen)	rapporteur
M. Georges Maltsiniotis	(Université Paris 7)	

Rapporteur absent lors de la soutenance : **M. Thomas Brüstle** (Université de Sherbrooke et Bishop's University)

Institut de Mathématiques de Jussieu
175, rue du chevaleret
75 013 Paris

École doctorale Paris centre Case 188
4 place Jussieu
75 252 Paris cedex 05

Remerciements

Je remercie Bernhard Keller d'avoir accepté de diriger ma thèse. D'autres avant moi auront souligné sa patience, sa disponibilité, sa générosité et son enthousiasme à partager son grand savoir mathématique. Je lui suis extrêmement reconnaissant pour m'avoir guidé pendant ces trois années.

Je remercie Thomas Brüstle d'avoir accepté le rôle de rapporteur pour cette thèse, et pour ses nombreux conseils pendant mes études. Merci également à Bernard Leclerc pour avoir accepté d'écrire un rapport, et de m'accueillir à l'université de Caen l'année prochaine. Ibrahim Assem a guidé mes débuts en algèbre ; après qu'il ait dirigé ma maîtrise au Québec, je suis honoré qu'il fasse partie du jury de ma thèse. Merci à Christof Geiss pour les discussions enrichissantes en Allemagne. Merci enfin à David Hernandez et Georges Maltsiniotis d'avoir accepté de faire partie du jury.

Je dois beaucoup à Yann Palu, mon « grand frère de thèse », qui a passé tant de temps à discuter avec moi de mathématiques, à Paris et à Leeds, et qui m'a, dès le début, intégré parmi les doctorants. Merci à Nabil Kahouadji pour son amitié et pour m'avoir introduit au BDD. Merci à Dong Yang pour m'avoir patiemment expliqué tant de mathématiques. Merci à Fan Qin et à Lingyan Guo pour les nombreux moments passés devant un tableau à faire des mathématiques ; merci aussi à Lingyan pour les cours de chinois. Merci à Laurent Demonet pour de nombreuses et intéressantes discussions.

Si ma vie de doctorant a été aussi agréable, c'est en grande partie grâce aux gens que j'ai côtoyés tous les jours au sein (et hors) de l'institut. Au risque d'en oublier plusieurs (qui, je l'espère, me le pardonneront), j'aimerais citer ici, en ordre vaguement chronologique : Selene, Juliette, Johan, Farid, Laura, Nicolas, Nicolas, Delphine, Giovanni, Giovanni, Peng, Vincent, Fabien, Olivier, Thomas, François, Hendrik, Élodie, Paloma, Louis-Hadrien, Martin, Lukas, Daniel, Victoria, Xin, David, Ismaël, Anne-Sandrine, Clément, Pedro, Charles, Robert, Christophe, Sarah, Dragos, Yohan, Alexandre, Mathieu et Hoel.

Yann Palu et Dong Yang m'ont donné de précieux commentaires sur le contenu de ma thèse pendant ces trois années ; je les en remercie. Merci aussi à Nicolas Chevalier pour avoir repéré des coquilles sur une version précédente.

Partir pour un autre continent est une grande aventure. Je ne sais si j'aurais osé l'entreprendre sans les encouragements et le support de mes parents, Patrice et Diane, et de mes frère et soeurs, Philippe, Christine et Jacynthe. Je ne puis exprimer ici, en si peu de mots, toute ma gratitude envers eux.

Finalement, merci à Julie, pour sa présence et son amour, et grâce à qui, même au bout du monde, je ne suis pas seul.

Résumé

Résumé

Cette thèse est consacrée au développement et à l'utilisation d'outils catégoriques pour l'étude des algèbres amassées de S. Fomin et A. Zelevinsky. La catégorie amassée généralisée de C. Amiot est une catégorie triangulée ayant été utilisée, dans le cas où elle est Hom-finie, pour catégorifier certaines algèbres amassées au moyen de caractères amassés au sens de Y. Palu. Dans cette thèse, nous généralisons les méthodes connues au cas où la catégorie amassée n'est pas Hom-finie, obtenant ainsi une catégorification de toute algèbre amassée antisymétrique. Pour ce faire, nous nous restreignons à une sous-catégorie de la catégorie amassée qui est stable par mutation et possède une propriété analogue à la condition 2-Calabi–Yau. Nous prouvons l'existence d'un caractère amassé sur cette sous-catégorie. Nous utilisons ensuite ces outils pour interpréter la combinatoire des algèbres amassées au moyen de la catégorie amassée. Notamment, nous démontrons une correspondance entre les **g**-vecteurs et les indices, donnons une interprétation des F-polynômes, et prouvons que les définitions de mutation dans l'algèbre et dans la catégorie sont cohérentes entre elles. Ces propriétés nous permettent de donner une nouvelle démonstration à de nombreuses conjectures pour les algèbres amassées antisymétriques. Finalement, en nous inspirant d'un travail récent de C. Geiss, B. Leclerc et J. Schröer, nous montrons comment l'ensemble des indices, en bijection avec l'ensemble des **g**-vecteurs, permet la construction d'une base de certaines algèbres amassées. Nous expliquons pourquoi cette construction fournit un bon candidat pour l'obtention d'une base de l'algèbre amassée supérieure en général.

Mots-clefs

Catégories triangulées, Catégories amassées, Algèbres amassées.

Cluster categories with infinite-dimensional morphism spaces, applications

Abstract

This thesis is concerned with the development and application of categorical tools in the study of the cluster algebras of S. Fomin and A. Zelevinsky. C. Amiot's generalized cluster category is a triangulated category which has been used, in the case where it is Hom-finite, to categorify a certain class of cluster algebras, using cluster characters in the sense of Y. Palu. In this thesis, we generalize these results to the case where the cluster category is not Hom-finite, thus obtaining a categorification of any skew-symmetric cluster algebra. In order to do so, we restrict ourselves to a subcategory of the cluster category which is stable under mutation and satisfies an analogue of the 2-Calabi–Yau condition. We prove the existence of a cluster character on this subcategory. We then use these tools to interpret the combinatorics of cluster algebras inside the cluster category. In particular, we prove a correspondence between $\mathbf{g}$-vectors and indices, provide an interpretation of F-polynomials, and show that the definition of mutation in the algebra and in the category are consistent with each other. These properties allow us to give new proofs of numerous conjectures for skew-symmetric cluster algebras. Finally, starting from recent work by C. Geiss, B. Leclerc and J. Schröer, we show how the set of indices parametrizes a basis for a class of cluster algebras. We then show that this construction provides us with a good candidate for a basis of the upper cluster algebra in general.

Keywords

Triangulated categories, Cluster categories, Cluster algebras.

Table des matières

1 Préliminaires **1**

1.1 Algèbres amassées 1

1.1.1 Mutation 1

1.1.2 Graines 2

1.1.3 Algèbres amassées 3

1.2 Catégories triangulées 3

1.2.1 Définition des catégories triangulées 4

1.2.2 Quotients triangulés 5

1.3 Catégorie dérivée d'une algèbre différentielle graduée 7

1.4 Foncteurs dérivés 8

2 Summary of results **11**

2.1 Cluster categories with infinite-dimensional morphism spaces 11

2.2 Cluster characters 13

2.3 Application to cluster algebras 13

2.4 Link with decorated representations 14

2.5 Indices and generic bases 15

3 Cluster characters **17**

3.1 Introduction 17

3.2 Cluster category 19

3.2.1 Skew-symmetric cluster algebras 19

3.2.2 Quivers with potentials and their mutation 19

3.2.3 Complete Ginzburg dg algebras 21

3.2.4 Cluster category 21

3.2.5 Mutation in $\mathcal{C}$ 22

3.2.6 The subcategory $\mathrm{pr}_{\mathcal{C}}\Gamma$ 23

3.2.7 Properties of $\mathrm{pr}_{\mathcal{C}}\Gamma$ 25

3.2.8 Mutation of Γ in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ 32

3.3 Cluster character 35

3.3.1 Modules 35

3.3.2 Presentations and index 38

3.3.3 Cluster character : definition 42

3.3.4 Dichotomy 43

3.3.5 Multiplication formula 46

3.4 Application to cluster algebras 48

4 Applications to cluster algebras **51**
4.1 Introduction 51
4.2 Recollections 52
4.2.1 Background on cluster algebras with coefficients 52
4.2.2 Decorated representations of quivers with potentials 55
4.2.3 Some invariants of decorated representations 56
4.2.4 More on mutations as derived equivalences 56
4.3 Application to skew-symmetric cluster algebras 58
4.3.1 Rigid objects are determined by their index 58
4.3.2 Index and **g**-vectors 61
4.3.3 Cluster characters and F-polynomials 67
4.4 Link with decorated representations 69
4.4.1 Mutations 69
4.4.2 Interpretation of F-polynomials, **g**-vectors and **h**-vectors 73
4.4.3 Extensions and the E-invariant 78

5 Indices and generic bases for cluster algebras **81**
5.1 Introduction 81
5.2 Generic value of cluster characters 82
5.3 Varieties and projective presentations 83
5.3.1 Varieties of representations 84
5.3.2 Decomposition of projective presentations 85
5.4 Indices and strongly reduced components 87
5.4.1 Morphic cokernels 87
5.4.2 Codimensions of orbits 88
5.4.3 Orbits and the E-invariant 91
5.4.4 Proof of Theorem 5.1.2 92
5.5 Invariance under mutation 93
5.6 Proof of Theorem 5.1.1 97
5.6.1 Link with a conjecture of V. Fock and A. Goncharov 99

Bibliography **103**

Chapitre 1

Préliminaires

1.1 Algèbres amassées

Dans leur article [29] publié en 2002, S. Fomin et A. Zelevinsky introduisent la notion d'algèbre amassée. Ils espèrent ainsi fournir un cadre combinatoire pour l'étude de la positivité totale dans les variétés algébriques, d'après G. Lusztig [64], et la construction de bases canoniques pour les groupes quantiques, d'après M. Kashiwara [49] et G. Lusztig [63]. Malgré l'apparition relativement récente des algèbres amassées dans la littérature, une quantité surprenante de domaines des mathématiques leur sont aujourd'hui liés. Le lecteur intéressé se voit offrir un vaste choix d'articles proposant un tour d'horizon ce cette théorie et de ses applications ; citons ici [75] [40] [50] [28] [62] [71] et [1].

Dans cette section, nous définirons la notion d'algèbre amassée (ou, selon la terminologie de [31], d'algèbre amassée avec coefficients de type géométrique), énoncerons des propriétés fondamentales, puis mentionnerons quelques exemples.

1.1.1 Mutation

Un *carquois* Q est un graphe orienté ; on écrit $Q = (Q_0, Q_1, s, t)$, où Q_0 est l'ensemble des sommets du carquois, Q_1 est l'ensemble de ses flèches, et s (ou t) est l'application associant à chaque flèche sa source (ou son but). Un carquois est *fini* s'il ne possède qu'un nombre fini de sommets et de flèches.

Définition 1.1.1. Soit Q un carquois fini sans cycles orientés de longueur ≤ 2, et soit i un de ses sommets. La *mutation de Q en i* est le carquois $\mu_i(Q)$ construit à partir de Q comme suit :

1. pour chaque sous-carquois $j \xrightarrow{a} i \xrightarrow{b} \ell$, ajouter une flèche $j \xrightarrow{[ba]} \ell$;
2. remplacer chaque flèche a ayant i comme source ou but par une flèche a^* allant dans le sens opposé ;
3. retirer toutes les flèches d'un ensemble maximal de cycles de longueur 2 disjoints deux à deux.

Exemple 1.1.2. Le diagramme ci-dessous donne le résultat d'une mutation au sommet 2 d'un carquois.

$$Q = \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd} \qquad \mu_2(Q) = \quad \begin{tikzcd} 1 \arrow[dr] & 2 \arrow[l] \\ 4 \arrow[u] & 3 \arrow[l] \arrow[u] \end{tikzcd}$$

Exemple 1.1.3. Soit Q le carquois ci-dessous.

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll, Rightarrow] \end{tikzcd}$$

Alors tout carquois obtenu par mutations successives de Q est une rotation de l'un des quatre carquois

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ll, Rightarrow] \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dl] & \\ \bullet \arrow[rr, Rightarrow] & & \bullet \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dl] & \\ \bullet & & \bullet \arrow[ul] \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd} & \bullet & \\ \bullet \arrow[ur] & & \bullet \arrow[ul] \arrow[ll] \end{tikzcd}.$$

Le lecteur souhaitant expérimenter davantage avec l'opération de mutation trouvera sur la page personnelle de B. Keller un programme [53] offrant cette possibilité.

Remarque 1.1.4. La mutation en un sommet est une involution : $\mu_i^2(Q) = Q$.

Nous avons défini la mutation, qui est l'opération combinatoire à la base de la définition des algèbres amassées. Nous aurons à considérer des carquois et leurs mutations successives. Cependant, il nous faudra interdire la mutation en certains sommets ; ceci nous mène à la définition de carquois glacé, d'après [32].

Définition 1.1.5. Un *carquois glacé* est un couple (Q, F), où Q est un carquois fini sans cycles orientés de longueur ≤ 2 et F est un ensemble de sommets de Q, appelés *sommets gelés* de Q.

La mutation des carquois glacés est définie comme celle des carquois non glacés, à l'exception du fait que la mutation en un sommet gelé sera toujours une opération interdite. La mutation d'un carquois glacé préserve l'ensemble F des sommets gelés. Lorsque F est vide, on écrit Q au lieu de (Q, F).

1.1.2 Graines

Définition 1.1.6. Une *graine* est un couple $\big((Q, F), \underline{u}\big)$, où (Q, F) est un carquois glacé, et $\underline{u} = \{u_1, \ldots, u_n\}$ est une famille libre et génératrice du corps $\mathbb{Q}(x_1, \ldots, x_n)$.

Soit $\big((Q, F), \underline{u}\big)$ une graine, et soit i un sommet non gelé de Q. La *mutation de la graine* $\big((Q, F), \underline{u}\big)$ *en* i est la graine $\mu_i\big((Q, F), \underline{u}\big) = \big((Q', F'), \underline{u}'\big)$, où
- l'ensemble des sommets gelés F' est égal à F,
- le carquois Q' est la mutation de Q au sommet i, et
- l'ensemble $\underline{u}'$ est obtenu à partir de $\underline{u}$ en remplaçant l'élément u_i par un élément u_i' vérifiant la *relation d'échange* ci-dessous :

$$u_i u_i' = \prod_{s(a)=i} u_{t(a)} + \prod_{t(a)=i} u_{s(a)}.$$

La *graine initiale* associée à un carquois glacé (Q, F) est la graine $\big((Q, F), \{x_1, \ldots, x_n\}\big)$.

Remarque 1.1.7. La mutation des graines en un sommet est une involution.

Exemple 1.1.8. Le diagramme ci-dessous donne le résultat d'une mutation au sommet 2 d'une graine.

$$Q = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd} \qquad \mu_2(Q) = \begin{tikzcd} 1 \arrow[dr] & 2 \arrow[l] \\ 4 \arrow[u] & 3 \arrow[l] \arrow[u] \end{tikzcd}$$

$$\underline{u} = \{x_1, x_2, x_3, x_4\} \qquad \underline{u}' = \{x_1, \tfrac{x_1+x_3}{x_2}, x_3, x_4\}$$

1.1.3 Algèbres amassées

Définition 1.1.9. Soit (Q, F) un carquois glacé. On numérote de 1 à n les sommets de Q, et de $r+1$ à n les sommets gelés.

- Les éléments $x_{r+1}, \ldots, x_n$ de $\mathbb{Q}(x_1, \ldots, x_n)$ sont les *coefficients*.
- Les ensembles $\underline{u}$ dans les graines $((R, F), \underline{u})$ obtenues par mutations successives de la graine initiale associée à (Q, F) sont les *amas*.
- Les éléments des amas qui ne sont pas des coefficients sont les *variables d'amas*.
- La $\mathbb{Q}$-sous-algèbre $\mathcal{A}_{Q,F}$ de $\mathbb{Q}(x_1, \ldots, x_n)$ engendrée par les variables d'amas et les coefficients est *l'algèbre amassée* (avec coefficients, de type géométrique) associée à (Q, F).

De nombreux exemples d'algèbres amassées sont connus et peuvent être trouvés dans l'une des références mentionnées au premier paragraphe de ce chapitre. Une importante classe d'exemples est donnée par les anneaux de coordonnées de sous-groupes unipotents maximaux d'un groupe algébrique semi-simple complexe, voir [5].

Une propriété fondamentale des variables d'amas de toute algèbre amassée est le *phénomène de Laurent*.

Théorème 1.1.10 ([29], Théorème 3.1)**.** *Soit $\mathcal{A}_{Q,F}$ l'algèbre amassée associée à un carquois glacé (Q, F), et soit $\underline{u} = \{u_1, \ldots, u_n\}$ un amas quelconque de $\mathcal{A}_{Q,F}$. Alors toute variable d'amas de $\mathcal{A}_{Q,F}$ est un polynôme de Laurent à coefficients entiers en les $u_1, \ldots, u_n$.*

Conjecture 1.1.11 ([29])**.** *Avec les notations du Théorème 1.1.10, toute variable d'amas est un polynôme de Laurent à coefficients entiers* positifs *en les $u_1, \ldots, u_n$.*

On ignore à ce jour si cette conjecture est vraie.

Une des motivations principales pour la création des algèbres amassées est l'étude des bases canoniques de M. Kashiwara et G. Lusztig. Dès leur premier article [29] sur le sujet, S. Fomin et A. Zelevinsky conjecturent que les *monômes d'amas*, c'est-à-dire les monômes ayant pour facteurs des variables d'amas issues d'un même amas, font partie d'une base de l'algèbre amassée. Cette conjecture est confirmée pour une grande classe d'exemples par des résultats de C. Geiss, B. Leclerc et J. Schröer [35]. Nous montrons dans cette thèse (voir le Théorème 4.3.7 (2)) que les monômes d'amas d'une algèbre amassée avec assez de coefficients sont toujours linéairement indépendants, même s'ils ne suffisent pas à en former une base.

1.2 Catégories triangulées

Dans cette section, nous rappelons les définitions de catégories triangulées, catégories dérivées et foncteurs dérivés, que nous utiliserons dans cette thèse.

1.2.1 Définition des catégories triangulées

Définition 1.2.1 ([74])**.** Soit k un anneau commutatif. Une *catégorie triangulée* sur k est une k-catégorie additive $\mathcal{T}$ munie

- d'un automorphisme de k-catégories $\Sigma : \mathcal{T} \to \mathcal{T}$, appelé *foncteur de suspension*, et
- d'une collection de triplets de morphismes $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, appelés *triangles*,

de telle sorte que les axiomes (TR1) à (TR4) soient vérifiés.

Un peu de terminologie supplémentaire est nécessaire pour énoncer les axiomes. Soient $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ et $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ deux triangles de $\mathcal{T}$. Un *morphisme de triangles* du premier vers le second est la donnée de trois morphismes $X \xrightarrow{a} X'$, $Y \xrightarrow{b} Y'$ et $Z \xrightarrow{c} Z'$ de $\mathcal{T}$ tels que le diagramme

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

soit commutatif. Un *isomorphisme de triangles* est un morphisme de triangles admettant un inverse.

Soit n un entier, et soient X et Y deux objets de $\mathcal{T}$. Un *morphisme de degré* n de X vers Y est un morphisme de X vers $\Sigma^n Y$.

Nous pouvons maintenant énoncer les axiomes (TR1) à (TR4).

(TR1) – Tout triplet de morphismes isomorphe à un triangle est un triangle.
- Pour tout morphisme $X \xrightarrow{f} Y$ de $\mathcal{T}$, il existe un triangle contenant f, c'est-à-dire un triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.
- Pour tout objet X de $\mathcal{T}$, $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow \Sigma X$ est un triangle.

(TR2) Le triplet $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ est un triangle si, et seulement si, le triplet $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ en est un.

(TR3) Soient $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ et $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ deux triangles. Tout couple (a, b) de morphismes tels que $bf = f'a$ se complète en un morphisme

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

de triangles.

(TR4) (Axiome de l'octaèdre) Soient

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{j} \Sigma X$$

$$Y \xrightarrow{v} Z \xrightarrow{i'} X' \xrightarrow{j'} \Sigma Y$$

$$X \xrightarrow{vu} Z \xrightarrow{i''} Y' \xrightarrow{j''} \Sigma X$$

trois triangles. Il existe deux morphismes $Z' \xrightarrow{f} Y'$ et $Y' \xrightarrow{g} X'$ tels que

- $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(\Sigma i)j'} \Sigma Z'$ soit un triangle,
- $j'' \circ f = j$ et $g \circ i'' = i'$, et
- $f \circ i = i'' \circ v$ et $(\Sigma u) \circ j'' = j' \circ g$.

L'axiome (TR4) est avantageusement représenté par un octaèdre

$$\begin{tikzcd}
 & & Y' \arrow[ddl, "j''"] \arrow[dr, dotted, "g"] & & \\
Z' \arrow[urr, "f"] \arrow[dr, "j"] & & & X' \arrow[lll, dotted, "(\Sigma i)j'"'] \arrow[ddl, dotted, "j'"] & \\
 & X \arrow[rrr, "vu"] \arrow[dr, "u"] & & & Z \arrow[uull, "i''"'] \arrow[ul, dotted, "i'"'] \\
 & & Y \arrow[uull, "i"] \arrow[urr, "v"'] & &
\end{tikzcd}$$

dont quatre des faces sont des diagrammes commutatifs, quatre des faces sont des triangles, et dont les deux grands carrés reliant la base et le sommet sont commutatifs. La notation $A \longrightarrow\!\!\!\!\!+\!\!> B$ désigne un morphisme de degré 1 de A vers B.

Définition 1.2.2. Soient $\mathcal{T}$ et $\mathcal{T}'$ deux catégories triangulées. Un *foncteur triangulé* de $\mathcal{T}$ vers $\mathcal{T}'$ est la donnée d'un foncteur k-additif $F : \mathcal{T} \to \mathcal{T}'$ et d'un isomorphisme de foncteurs $\Phi : F\Sigma \to \Sigma F$ tels que si $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ est un triangle de $\mathcal{T}$, alors

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\Phi_X \circ Fh} \Sigma FX$$

est un triangle de $\mathcal{T}'$.

1.2.2 Quotients triangulés

Soit $\mathcal{T}$ une catégorie triangulée, et soit $\mathcal{N}$ une sous-catégorie triangulée pleine, *stricte* (c'est-à-dire stable par isomorphismes) et *épaisse* (c'est-à-dire stable sous l'action de prendre des facteurs directs) de $\mathcal{T}$.

Proposition 1.2.3 ([74]). *Il existe une catégorie triangulée $\mathcal{T}/\mathcal{N}$ et un foncteur triangulé $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ tels que tous les objets de $\mathcal{N}$ soient envoyés vers l'objet nul par Q, et pour tout foncteur triangulé $F : \mathcal{T} \to \mathcal{T}'$ envoyant les objets de $\mathcal{N}$ vers l'objet nul, il existe un unique foncteur triangulé $F' : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$ tel que $F' \circ Q = F$, comme sur le diagramme :*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{T}/\mathcal{N} \arrow[d, dotted, "F'"] \\
 & \mathcal{T}'.
\end{tikzcd}$$

On peut construire la catégorie $\mathcal{T}/\mathcal{N}$ comme suit. Les objets de $\mathcal{T}/\mathcal{N}$ sont ceux de $\mathcal{T}$. Soient X et Y deux objets de $\mathcal{T}/\mathcal{N}$. Les morphismes de X vers Y sont les classes d'équivalences de diagrammes de la forme

$$\begin{tikzcd}
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y,
\end{tikzcd}$$

où s et f sont des morphismes de $\mathcal{T}$, et s est contenu dans un triangle

$$X' \xrightarrow{s} X \longrightarrow N \longrightarrow \Sigma X' ,$$

où N est un objet de $\mathcal{N}$. Ces diagrammes sont soumis à la relation d'équivalence suivante :

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} & X'' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

sont équivalents s'il existe un troisième tel diagramme

$$\begin{tikzcd} & X''' \arrow[dl, "s''"'] \arrow[dr, "f''"] & \\ X & & Y, \end{tikzcd}$$

et un diagramme commutatif

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X''' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u] \arrow[d] & Y \\ & X''. \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

La composition de deux morphismes

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

se calcule comme suit. Il existe un diagramme commutatif comme ci-dessous, où les lignes sont des triangles et N est un objet de $\mathcal{N}$:

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "h"] & X' \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, equal] & \Sigma W \arrow[d, "\Sigma h"] \\ Y' \arrow[r, "t"] & Y \arrow[r] & N \arrow[r] & \Sigma Y'. \end{tikzcd}$$

On obtient alors un diagramme commutatif

$$\begin{tikzcd} & & W \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z. \end{tikzcd}$$

On vérifie que le diagramme

$$\begin{tikzcd} & W \arrow[dl, "su"'] \arrow[dr, "gh"] & \\ X & & Z \end{tikzcd}$$

est un morphisme dans $\mathcal{T}/\mathcal{N}$; c'est la composition des deux morphismes de départ. Ceci confère à $\mathcal{T}/\mathcal{N}$ une structure de catégorie triangulée, dont les triangles sont les triplets de morphismes $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ isomorphes à l'image d'un triangle de $\mathcal{T}$ par le foncteur canonique $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ envoyant chaque objet sur lui-même et chaque morphisme $f : X \to Y$ sur

$$\begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[dr, "f"] & \\ X & & Y. \end{tikzcd}$$

De plus le foncteur Q satisfait aux hypothèse de la proposition 1.2.3.

Une grande classe d'exemples de quotients triangulés est donnée par les catégories dérivées, comme nous le verrons dans la prochaine section.

1.3 Catégorie dérivée d'une algèbre différentielle graduée

Dans cette thèse, un exemple essentiel de catégorie triangulée est donné par la catégorie dérivée d'une algèbre différentielle graduée. Notre référence principale pour cette section est l'article [54] de B. Keller. Nous fixons un anneau commutatif k pour toute la section.

Définition 1.3.1. Une *algèbre différentielle graduée* (ou algèbre dg pour simplifier) est une k-algèbre graduée $A = \bigoplus_{i\in\mathbb{Z}} A^i$ munie d'une application k-linéaire homogène $d : A \to A$ de degré 1, appelée *différentielle*, telle que

$$\forall a \in A^i, \forall b \in A, \quad d(ab) = d(a)b + (-1)^i ad(b),$$

et telle que $d \circ d = 0$.

Toute k-algèbre est une algèbre dg concentrée en degré 0. Toute k-algèbre graduée est une algèbre dg de différentielle nulle. La *cohomologie* d'une algèbre dg A est définie en chaque degré par

$$H^i(A) = \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}}$$

Définition 1.3.2. Soit A une k-algèbre dg. Un *A-module différentiel gradué* (ou A-module dg pour simplifier) est un A-module gradué à droite $M = \oplus_{i\in\mathbb{Z}} M^i$ muni d'une application k-linéaire homogène $d : M \to M$ de degré 1, appelée *différentielle*, telle que

$$\forall a \in A, \forall m \in M^i, \quad d(ma) = d(m)a + (-1)^i md(a),$$

et telle que $d \circ d = 0$.

Les modules dg sur une algèbre A concentrée en degré 0 sont les complexes de A-modules dont la différentielle est A-linéaire. La cohomologie des modules dg est définie comme pour celle des algèbres dg.

Nous allons définir plusieurs catégories ayant pour objets les modules dg sur une algèbre dg A donnée. La définition finale sera celle de la catégorie dérivée de A.

Définition 1.3.3. Soient M et N deux A-modules dg. Le k-module dg $\mathcal{H}om_A(M, N)$ est défini ainsi :

– pour chaque entier i, $\mathcal{H}om_A(M, N)^i$ est le sous-ensemble de $\prod_{n\in\mathbb{Z}} \operatorname{Hom}_k(M^n, N^{n+i})$ dont les éléments sont les $(f_n)_{n\in\mathbb{Z}}$ tels que

$$\forall a \in A^j, \forall m \in M^i, \quad f_j(m)a = f_{j+i}(ma);$$

– la différentielle de $\mathcal{H}om_A(M, N)$ est définie par

$$\forall f \in \mathcal{H}om_A(M, N)^i, \quad d(f) = d_N \circ f - (-1)^i f \circ d_M.$$

Remarquons que le noyau de d^0, que l'on désigne par $Z^0(\mathcal{H}om_A(M, N))$, est exactement l'ensemble des applications A-linéaires homogènes de degré 0 de M vers N qui commutent avec les différentielles. Notons également que l'homologie en degré 0, désignée par $H^0(\mathcal{H}om_A(M, N))$, est exactement le quotient de $Z^0(\mathcal{H}om_A(M, N))$ par la relation d'homotopie (deux morphismes A-linéaires f et g de degré 0 de M vers N sont *homotopes* s'il existe un morphisme A-linéaire s de degré -1 tel que $d_N \circ s + s \circ d_M = f - g$).

Définition 1.3.4. Soit A une k-algèbre dg. La *catégorie des A-modules dg* est la k-catégorie $\mathcal{C}A$ dont les objets sont les A-modules dg et dont l'ensemble de morphismes de M vers N est donné par $Z^0(\mathcal{H}om_A(M, N))$, pour tous A-modules dg M et N. La *catégorie d'homotopie de A* est la catégorie $\mathcal{H}A$ définie de façon similaire, où les ensembles de morphismes sont donnés par $H^0(\mathcal{H}om_A(M, N))$.

Tout morphisme de $Z^0(\mathcal{H}om_A(M, N))$ induit naturellement un morphisme en cohomologie $H^i(M) \to H^i(N)$ pour tout entier i. Deux morphismes homotopes induisent les mêmes morphismes en cohomologie ; on peut donc dire que les morphismes pris dans $H^0(\mathcal{H}om_A(M, N))$ induisent les morphismes en cohomologie. Un *quasi-isomorphisme* est un morphisme de $\mathcal{H}A$ qui induit des isomorphismes en cohomologie en chaque degré.

La définition de la catégorie dérivée de A repose sur le fait que la catégorie $\mathcal{H}A$ est triangulée (son foncteur de suspension étant le décalage habituel des complexes), et que les objets N tels qu'il existe un triangle

$$X \xrightarrow{s} Y \longrightarrow N \longrightarrow \Sigma X$$

où s est un quasi-isomorphisme forment une sous-catégorie triangulée pleine, stricte et épaisse $\mathcal{N}$ de $\mathcal{H}A$. Ces objets N sont caractérisés par le fait que leur cohomologie est nulle en chaque degré.

Définition 1.3.5. Soit A une k-algèbre dg. Sa *catégorie dérivée* est le quotient triangulé

$$\mathcal{D}A = \mathcal{H}A/\mathcal{N}.$$

Dans cette thèse, nous utiliserons des quotients triangulés de sous-catégories de $\mathcal{D}A$.

1.4 Foncteurs dérivés

Il s'avère souvent nécessaire de savoir construire des foncteurs entre catégories dérivées. Nous présentons ici une méthode pour le faire. Nos principales références pour cette section sont [54] et [55].

Soient A et B deux k-algèbres dg. Soit M un B-A-bimodule dg, c'est-à-dire un B-A-bimodule gradué $M = \oplus_{i \in \mathbb{Z}} M^i$ muni d'une différentielle d faisant de M un A-module dg à droite et un B-module dg à gauche tel que

$$\forall a \in A, \forall b \in B, \forall m \in M, \quad (bm)a = b(ma).$$

Ce bimodule donne naissance à deux foncteurs

$$\begin{tikzcd} \mathcal{C}A \arrow[r, shift left, "\mathcal{H}om_A(M,?)"] & \mathcal{C}B. \arrow[l, shift left, "?\otimes_B M"] \end{tikzcd}$$

Pour tout A-module dg X, le B-module dg $\mathcal{H}om_A(M,?)$ est défini comme dans la définition 1.3.3 ; sa structure de B-module est déduite de celle de M. Pour tout B-module dg Y, le A-module dg $Y \otimes_B M$ est le quotient du k-module dg $Y \otimes_k M$ défini comme suit :

– pour tout entier i, $(Y \otimes_k M)^i = \bigoplus_{j+\ell=i} Y^j \otimes M^\ell$;
– la différentielle est donnée par $d = d_Y \otimes id_M + id_Y \otimes d_M$,

par le sous-module dg engendré par les éléments de la forme $y \otimes bx - yb \otimes x$, avec $y \in Y$, $b \in B$ et $x \in X$.

Ces deux foncteurs forment une adjonction $(? \otimes_B M, \mathcal{H}om_A(M,?))$. On vérifie qu'ils induisent des foncteurs

$$\mathcal{H}A \underset{? \otimes_B M}{\overset{\mathcal{H}om_A(M,?)}{\rightleftarrows}} \mathcal{H}B.$$

entre les catégories d'homotopie. Cependant, de manière générale, ils n'induisent pas de foncteurs entre les catégories dérivées. Un outil pour résoudre ce problème est la notion de remplacements fibrant et cofibrant.

Définition 1.4.1. 1. Un A-module dg P est *cofibrant* si, pour tout quasi-isomorphisme $s : X \to Y$ qui est surjectif en chaque degré, on a que

$$\mathrm{Hom}_{\mathcal{C}A}(P, X) \xrightarrow{s_*} \mathrm{Hom}_{\mathcal{C}A}(P, Y)$$

est surjectif.

2. Un A-module dg I est *fibrant* si, pour tout quasi-isomorphisme $i : X \to Y$ qui est injectif en chaque degré, on a que

$$\mathrm{Hom}_{\mathcal{C}A}(Y, I) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{C}A}(X, I)$$

est surjectif.

Proposition 1.4.2 ([54])**.** *Le foncteur naturel $\mathcal{H}A \to \mathcal{D}A$ admet un adjoint à gauche* $\mathbf{p}$ *et un adjoint à droite* $\mathbf{i}$ *tels que, pour tout objet M de $\mathcal{D}A$,*

– *$\mathbf{p}M$ est cofibrant et $\mathbf{i}M$ est fibrant, et*
– *il existe des quasi-isomorphismes $\mathbf{p}M \to M$ et $M \to \mathbf{i}M$.*

L'objet $\mathbf{p}M$ de $\mathcal{H}A$ est un *remplacement cofibrant* de M, et l'objet $\mathbf{i}M$ est un *remplacement fibrant* de M.

Définition 1.4.3. Soit M un B-A-bimodule dg.

– Le *produit tensoriel dérivé à gauche* est le foncteur $? \otimes^L_B M$ donné par la composition

$$\mathcal{D}B \xrightarrow{\mathbf{p}} \mathcal{H}B \xrightarrow{? \otimes_B M} \mathcal{H}A \longrightarrow \mathcal{D}A.$$

– Le *foncteur Hom dérivé à droite* est le foncteur $\mathrm{RHom}_A(M,?)$ donné par la composition

$$\mathcal{D}A \xrightarrow{\mathbf{i}} \mathcal{H}A \xrightarrow{\mathrm{RHom}_A(M,?)} \mathcal{H}B \longrightarrow \mathcal{D}B.$$

Notons que ces deux définitions donnent une adjonction $(? \otimes^L_B M, \mathrm{RHom}_A(M,?))$ de foncteurs triangulés.

Chapter 2

Summary of results

This thesis takes part in the categorification of S. Fomin's and A. Zelevinsky's cluster algebras by means of triangulated categories. The main contribution that it contains is the study of the generalized cluster category of C. Amiot in the case when its morphism spaces are of infinite dimension. In this chapter, we give a short summary of the main results of this work.

The thesis is organized as follows. In Chapter 3, we study the cluster category in the case where its morphism spaces are infinite-dimensional. We prove the existence of a cluster character in the sense of Y. Palu. In Chapter 4, we apply the results thus obtained to prove several conjectures of S. Fomin and A. Zelevinsky for any skew-symmetric cluster algebra. In Chapter 5, we turn to the problem of the construction of generic bases for cluster algebras by using our setup.

The results of Chapters 3 and 4 were published in [70] and [69], respectively. Those of Chapter 5 were the subject of a short talk given at the Oberwolfach workshop *Representation Theory of Quivers and Finite Dimensional Algebras.*

2.1 Cluster categories with infinite-dimensional morphism spaces

Let (Q, W) be a quiver with potential in the sense of [22]. Then one can define a differential graded algebra Γ from (Q, W), called the *complete Ginzburg dg algebra*, following [42]. Let $\mathcal{D}\Gamma$ be the derived category of Γ, $\operatorname{per}\Gamma$ be its perfect derived category, and $\mathcal{D}_{fd}\Gamma$ be the full subcategory of $\mathcal{D}\Gamma$ consisting of objects with finite-dimensional total homology. Then, following [2], we define the *(generalized) cluster category* of (Q, W) as the triangulated quotient

$$\mathcal{C}_{Q,W} = \operatorname{per}\Gamma/\mathcal{D}_{fd}\Gamma.$$

If (Q, W) is Jacobi-finite, then it was proved in [2] that $\mathcal{C}_{Q,W}$ has finite-dimensional morphism spaces, that it is 2-Calabi–Yau (that is, we have bifunctorial isomorphisms $\operatorname{Ext}^1_{\mathcal{C}}(X, Y) \cong D\operatorname{Ext}^1_{\mathcal{C}}(Y, X)$) and that the object Γ is cluster-tilting (that is, it has no self-extensions, and if $\operatorname{Ext}^1_{\mathcal{C}}(\Gamma, X) = 0$, then X belongs to $\operatorname{add}\Gamma$). However, none of these properties hold if (Q, W) is not Jacobi-finite.

The approach that we use to study the cluster categories with infinite-dimensional morphism spaces is to restrict ourselves to subcategories of $\mathcal{C}_{Q,W}$.

Definition (Section 3.2.6)**.** *Let $\mathcal{T}$ be any triangulated category, and let T be an object of $\mathcal{T}$. Define $\operatorname{pr}_{\mathcal{T}}T$ to be the full subcategory of $\mathcal{T}$ whose objects are those X such that there* *exists a triangle*

$$T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_1,$$

with T_0 *and* T_1 *in* $\operatorname{add} T$.

Definition (3.3.9). *The subcategory* $\mathcal{D}$ *of* $\mathcal{C}_{Q,W}$ *is the full subcategory of* $\operatorname{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma \cap \operatorname{pr}_{\mathcal{C}}\Gamma$ *whose objects are those* X *such that* $\operatorname{Ext}^1_{\mathcal{C}}(\Gamma, X)$ *is finite-dimensional.*

These subcategories allow us to recover the good properties that we had in the case where (Q, W) was Jacobi-finite.

Proposition (3.2.7). *The subcategory* $\operatorname{pr}_{\mathcal{C}}\Gamma$ *depends only on the mutation class of* Γ.

Proposition (3.2.16). *Let* X *be an object of* $\operatorname{pr}_{\mathcal{C}}\Gamma \cup \operatorname{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ *and* Y *be an object of* $\operatorname{pr}_{\mathcal{C}}\Gamma$. *Then there exists a bifunctorial bilinear form*

$$\overline{\beta}_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X, Y) \times \operatorname{Hom}_{\mathcal{C}}(Y, \Sigma^2 X) \longrightarrow k.$$

which is non-degenerate. In particular, if one of the two spaces is finite-dimensional, then so is the other.

We also get a theorem which allows us to gain some control over the mutation in the derived category $\mathcal{D}\Gamma$. As proved in [58], mutations can be viewed as derived equivalences. Let Γ' be the complete Ginzburg dg algebra of the mutated quiver with potential $\tilde{\mu}_i(Q, W)$. For any vertex j of Q, let $\Gamma_j = e_j\Gamma$ and $\Gamma'_j = e_j\Gamma'$.

Theorem ([58], Theorem 3.2).

1. *There is a triangle equivalence* $\widetilde{\mu}_i^+$ *from* $\mathcal{D}(\Gamma')$ *to* $\mathcal{D}(\Gamma)$ *sending* Γ'_j *to* Γ_j *if* $i \neq j$ *and to the cone* Γ_i^* *of the morphism*

$$\Gamma_i \longrightarrow \bigoplus_{\alpha} \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α *if* $i = j$. *The functor* $\widetilde{\mu}_i^+$ *restricts to triangle equivalences from* $\operatorname{per}\Gamma'$ *to* $\operatorname{per}\Gamma$ *and from* $\mathcal{D}_{fd}\Gamma'$ *to* $\mathcal{D}_{fd}\Gamma$.

2. *Let* Γ_{red} *and* Γ'_{red} *be the complete Ginzburg dg algebra of the reduced part of* (Q, W) *and* $\tilde{\mu}_i(Q, W)$, *respectively. The functor* $\widetilde{\mu}_i^+$ *induces a triangle equivalence* $\mu_i^+ : \mathcal{D}(\Gamma'_{red}) \longrightarrow \mathcal{D}(\Gamma_{red})$ *which restricts to triangle equivalences from* $\operatorname{per}\Gamma'_{red}$ *to* $\operatorname{per}\Gamma_{red}$ *and from* $\mathcal{D}_{fd}\Gamma'_{red}$ *to* $\mathcal{D}_{fd}\Gamma_{red}$.

Denote the quasi-inverse of $\widetilde{\mu}_i^+$ by $\widetilde{\mu}_i^-$. Then we get the following theorem.

Theorem (3.2.18). *Let* Γ *be the complete Ginzburg dg algebra of a non-degenerate quiver with potential* (Q, W). *Let* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r-1})$ *be a sequence of signs. Let* $(i_1, \ldots, i_r)$ *be a sequence of vertices, and let* $T = \bigoplus_{j \in Q_0} T_j$ *be the image of* $\Gamma^{(r)}$ *by the sequence of equivalences*

$$\mathcal{D}\Gamma^{(r)} \xrightarrow{\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}} \ldots \xrightarrow{\widetilde{\mu}_{i_1}^{\varepsilon_1}} \mathcal{D}\Gamma^{(1)} = \mathcal{D}\Gamma.$$

Suppose that T_j *lies in* $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$ *for all vertices* j *of* Q. *Then there exists a unique sign* ε_r *such that all summands of the image of* $\Gamma^{(r+1)}$ *by* $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_r}^{\varepsilon_r}$ *lie in* $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$.

2.2 Cluster characters

Let (Q, W) be a quiver with potential. On the subcategory $\mathcal{D}$ of $\mathcal{C}_{Q,W}$, we can define a *cluster character* in the sense of Y. Palu [68].

Definition (3.3.10)**.** *A* cluster character *on $\mathcal{C}_{Q,W}$ with values in a commutative ring A is a map*

$$\chi : obj(\mathcal{D}) \longrightarrow A$$

satisfying the following conditions :

- *if X and Y are two isomorphic objects in $\mathcal{D}$, then we have $\chi(X) = \chi(Y)$;*
- *for all objects X and Y of $\mathcal{D}$, $\chi(X \oplus Y) = \chi(X)\chi(Y)$;*
- *(multiplication formula) for all objects X and Y of $\mathcal{D}$ such that $\dim \operatorname{Ext}^1_{\mathcal{C}}(X, Y) = 1$, the equality*

  $$\chi(X)\chi(Y) = \chi(E) + \chi(E')$$

  holds, where $X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X$ and $Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$ are non split triangles.

For any object X of $\mathcal{D}$, define the *index of X* (with respect to Γ) as the element of $K_0(\operatorname{add}\Gamma)$ given by

$$\operatorname{ind}_\Gamma X = [T_0] - [T_1],$$

where T_0 and T_1 are objects of $\operatorname{add}\Gamma$ such that there exists a triangle

$$T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_1.$$

Lemma (3.3.6)**.** *Let X be an object in $\mathcal{D}$. Then the sum $\operatorname{ind}_T X + \operatorname{ind}_T \Sigma^{-1} X$ only depends on the dimension vector $\mathbf{d}$ of $\operatorname{Ext}^1_{\mathcal{C}}(\Gamma, X)$ viewed as an $\operatorname{End}_{\mathcal{C}}(\Gamma)$-module. Denote this sum by $\iota(\mathbf{d})$.*

Define the map

$$X'_? : obj(\mathcal{D}) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

as follows: for any object M of $\mathcal{D}$, put

$$X'_M = x^{\operatorname{ind}_\Gamma X} \sum_{\mathbf{e}} \chi\Big(\operatorname{Gr}_{\mathbf{e}}\big(\operatorname{Ext}^1_{\mathcal{C}}(\Gamma, M)\big)\Big) x^{-\iota(\mathbf{e})},$$

where χ is the Euler–Poincaré characteristic.

Theorem (3.3.12)**.** *The map $X'_?$ defined above is a cluster character on $\mathcal{C}$.*

2.3 Application to cluster algebras

We can use the cluster character defined above to interpret the combinatorics of cluster algebras inside the cluster category. This can be done for arbitrary skew-symmetric cluster algebras.

Theorem (3.4.1)**.** *Let (Q, W) be a non-degenerate quiver with potential. Then the cluster character $X'_?$ induces a surjection from the set of isomorphism classes of indecomposable reachable objects of $\mathcal{D}$ to the set of cluster variables of the cluster algebra associated with Q.*

In [31] were defined combinatorial objects related to cluster algebras, namely $\mathbf{g}$-vectors and F-polynomials. The authors then formulated several conjectures:

(5.4) every F-polynomial has constant term 1;

(6.13) the $\mathbf{g}$-vectors of the cluster variables of any given seed are *sign-coherent* in a sense to be defined;

(7.2) cluster monomials are linearly independent;

(7.10) different cluster monomials have different $\mathbf{g}$-vectors, and the $\mathbf{g}$-vectors of the cluster variables of any cluster form a basis of $\mathbb{Z}^r$;

(7.12) the mutation rule for $\mathbf{g}$-vectors can be expressed using a certain piecewise-linear transformation.

We now work with cluster algebras *with coefficients* (of geometric type). That is, we consider an *ice quiver* (in the sense of [32]) with mutable vertices 1 to r and frozen vertices $r+1$ to n. We let F be the set of frozen vertices. The cluster variables are then obtained by iterated mutations at mutable vertices.

Definition (4.3.2)**.** *Let $\mathcal{U}$ be the full subcategory of $\mathcal{D}$ whose objects are those X such that $\operatorname{Ext}^1_{\mathcal{C}}(\Gamma_j, X)$ vanishes, for $j = r+1, \ldots, n$.*

Using this definition, we get that $\mathbf{g}$-vectors in the cluster algebra correspond to indices in the cluster category.

Proposition (4.3.6)**.** *Let M be an object of $\mathcal{U}$. Then X'_M admits a $\mathbf{g}$-vector. This $\mathbf{g}$-vector is $(g_1, \ldots, g_r)$, where $g_j = [\operatorname{ind}_\Gamma M : \Gamma_j]$.*

Moreover, we have the following property.

Proposition (4.3.1)**.** *Objects of $\operatorname{pr}_{\mathcal{C}}\Gamma$ which have no self-extensions are determined by their index.*

This allows us to prove several conjectures of [31].

Theorem (4.3.7)**.** *Let (Q, F) be an ice quiver whose matrix if of full rank r. Then conjectures (6.13), (7.2), (7.10) and (7.12) hold for the associated cluster algebra.*

Theorem (4.3.13)**.** *Conjecture (5.4) holds.*

2.4 Link with decorated representations

Decorated representations of quivers with potential and their mutations were introduced in [22], along with their F-polynomials, $\mathbf{g}$-vectors, $\mathbf{h}$-vectors and E-invariants. Let (Q, W) be a non-degenerate quiver with potential. We define two maps Φ and Ψ between the set of isomorphism classes of objects of $\mathcal{D}$ and the set of isomorphism classes of decorated representations of (Q, W) as follows:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{isoclasses of} \\ \text{objects of } \mathcal{D} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{isoclasses of decorated} \\ \text{representations of } (Q, W) \end{array} \right\} \\ X = X' \oplus \displaystyle\bigoplus_{i=1}^{n} (e_i\Gamma)^{m_i} & \longmapsto & \Phi(X) = (FX', \displaystyle\bigoplus_{i=1}^{n} (S_i)^{m_i}) \\ \Psi(\mathcal{M}) = \overline{M} \oplus \displaystyle\bigoplus_{i=1}^{n} (e_i\Gamma)^{m_i} & \longleftarrow & \mathcal{M} = (M, \displaystyle\bigoplus_{i=1}^{n} S_i^{m_i}), \end{array}$$

where $F = \operatorname{Ext}^1_{\mathcal{C}}(\Gamma, ?)$, and $\overline{M}$ is a lift of M through F having no direct summands in $\operatorname{add}\Gamma$.

Then we get several properties showing that the theory of decorated representations and the theory of cluster categories are consistent with one another.

Proposition (4.4.1)**.** *With the above notations, Φ and Ψ are mutual inverse maps. Moreover, if $i \in Q_0$ is not on any cycle of length ≤ 2, and if $(Q', W') = \widetilde{\mu}_i(Q, W)$, then for any object X of $\mathcal{D}$, we have that*

$$\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X)),$$

where the functor $\widetilde{\mu}_i^-$ is as defined after Theorem 3.2.6.

Proposition (4.4.6)**.** *Let X be an object of $\mathcal{D}$. Then we have the equality*

$$F_X(x_{r+1}, \ldots, x_n) = F_{\Phi(X)}(x_{r+1}, \ldots, x_n).$$

Proposition (4.4.8)**.** *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X be an object of $\mathcal{D}$. Let $\mathbf{g}_{\Phi(X)} = (g_1, \ldots, g_n)$ be the $\mathbf{g}$-vector of the decorated representation $\Phi(X)$. Then we have the equality*

$$g_i = [\operatorname{ind}_\Gamma X : \Gamma_i]$$

for any vertex i of Q.

Corollary (4.4.9)**.** *For any decorated representation $\mathcal{M} = (M, V)$ of a quiver with potential (Q, W), we have the equality*

$$h_i = -\dim \operatorname{Hom}_{J(Q,W)}(S_i, M)$$

for any vertex i of Q.

Proposition (4.4.15)**.** *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X and Y be objects of $\mathcal{D}$. Then we have the following equalities:*

1. $E^{inj}(\Phi(X), \Phi(Y)) = \dim(\Sigma\Gamma)(X, \Sigma Y)$;
2. $E^{sym}(\Phi(X), \Phi(Y)) = \dim(\Sigma\Gamma)(X, \Sigma Y) + \dim(\Sigma\Gamma)(Y, \Sigma X)$;
3. $E(\Phi(X)) = (1/2) \dim \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X)$,

where $(\Sigma\Gamma)(X, Y)$ is the subspace of $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ containing all morphisms factoring through an object of $\operatorname{add} \Sigma\Gamma$.

2.5 Indices and generic bases

In the last chapter of this thesis, we show how, in some cases, we can construct a basis of the cluster algebra from the set $K_0(\operatorname{add} \Gamma)$ of indices in the cluster category. For this section, we assume that (Q, W) is non-degenerate and Jacobi-finite.

Definition (5.2.2)**.** *Define the map*

$$I : K_0(\operatorname{add} \Gamma) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

as follows: for any $[T_0] - [T_1] \in \operatorname{add} \Gamma$, let $I([T_0] - [T_1])$ be the generic value taken by the constructible function $X'_{cone(f)}$, for f taken in $\operatorname{Hom}_{\mathcal{C}}(T_1, T_0)$.

Theorem (5.1.1)**.** *The image of I lies in the upper cluster algebra associated with Q. If the matrix of Q is of full rank, then the elements in the image of I are linearly independent over $\mathbb{Z}$. If (Q, W) arises from the setting of [35], then the image of I is the basis of the cluster algebra $\mathcal{A}_Q$ found in that paper.*

In their construction of a basis for cluster algebras, the authors of [35] consider *strongly reduced components* of the variety $\operatorname{rep}(A)$ of finite-dimensional representations of the Jacobian algebra A of (Q, W). These components can be obtained from the set of indices.

Theorem (5.1.2)**.** *There exists a canonical surjection*

$$\Psi : K_0(\operatorname{add} A) \longrightarrow \{\textit{strongly reduced components of } \operatorname{rep}(A)\}.$$

Two elements δ and δ' have the same image by Ψ if, and only if, their canonical decompositions (in the sense of H. Derksen and J. Fei [20]) can be written as

$$\delta = \delta_1 \oplus \overline{\delta} \quad \textit{and} \quad \delta_1' \oplus \overline{\delta},$$

with δ_1 and δ_1' non-negative.

One can define the operation of mutation on the set of indices, and with this definition, the following result holds.

Theorem (5.1.3)**.** *The map I commutes with mutation.*

Finally, these results allow us to prove part of Conjecture 4.1 of [27] in the case where (Q, W) is Jacobi-finite. This conjecture states that there exists a bijection

$$\mathbb{Z}^n \longrightarrow E(\mathcal{A}),$$

where $E(\mathcal{A})$ is the subset of the cluster algebra consisting of elements which are Laurent polynomials with positive coefficients in the cluster variables of every cluster, and which cannot be written as a sum of two or more such elements, such that:

1. the bijection commutes with mutation (where the mutation in $\mathbb{Z}^n$ is defined in [27]);
2. an element $(a_1, \ldots, a_n)$ of $\mathbb{Z}^n$ with non-negative coefficients is sent to the element $\prod_{j=1}^n x_j^{a_j}$;
3. the set $E(\mathcal{A})$ is a $\mathbb{Z}$-basis of the upper cluster algebra $\mathcal{A}_Q^+$.

Theorem (5.6.2)**.** *Let (Q, W) be a non-degenerate, Jacobi-finite quiver with potential. Then the map*

$$I : \mathbb{Z}^n \cong \operatorname{add}\Gamma \longrightarrow \mathcal{A}_Q^+$$

satisfies conditions 1 and 2 above. If, moreover, (Q, W) arises from the setting of [35], then the image of I satisfies condition 3.

Chapter 3

Cluster characters

3.1 Introduction

In their series of papers [29], [30], [5] and [31] published between 2002 and 2007, S. Fomin and A. Zelevinsky, together with A. Berenstein for the third paper, introduced and developped the theory of cluster algebras. They were motivated by the search for a combinatorial setting for total positivity and canonical bases. Cluster algebras are a class of commutative algebras endowed with a distinguished set of generators, the cluster variables. The cluster variables are grouped into finite subsets, called clusters, and are defined recursively from initial variables by repeatedly applying an operation called mutation on the clusters. Recent surveys of the subject include [75], [40] and [50].

Cluster categories were introduced by A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov in [10], and by P. Caldero, F. Chapoton and R. Schiffler in [11] for the A_n case, in order to give a categorical interpretation of mutation of cluster variables. In [12], P. Caldero and F. Chapoton used the geometry of quiver Grassmannians to define a map which, as they showed, yields a bijection from the set of isomorphism classes of indecomposable objects of the cluster category of a Dynkin quiver to the set of cluster variables in the associated cluster algebra. It was proved by P. Caldero and B. Keller in [13] that, for cluster algebras associated with acyclic quivers, the Caldero-Chapoton map induces a bijection between the set of isomorphism classes of indecomposable rigid objects and the set of cluster variables.

Using the notion of quiver with potential as defined in [22], C. Amiot generalized the definition of cluster category in [2]. In the case where the quiver with potential is Jacobi-finite, the cluster character of Y. Palu introduced in [68] sends reachable indecomposable rigid objects of the (generalized) cluster category to cluster variables.

Another approach for the categorification of cluster algebras is studied by C. Geiss, B. Leclerc and J. Schröer in [37], [38], [39] and [34] where the authors use the category of modules over preprojective algebras of acyclic type.

In both cases, the categories encountered enjoy the following properties: (1) they are Hom-finite, meaning that the spaces of morphisms between any two objects is finite-dimensional; and (2) they are 2-Calabi–Yau in the sense that for any two objects X and Y, there is a bifunctorial isomorphism

$$\operatorname{Ext}^1(X,Y) \cong D\operatorname{Ext}^1(Y,X).$$

In this chapter, we study a version of Y. Palu's cluster characters for Hom-infinite cluster categories, that is, cluster categories with possibly infinite-dimensional morphism spaces. This cluster character $L \mapsto X'_L$ is not defined for all objects L but only for those in a suitable subcategory $\mathcal{D}$, which we introduce. We show that $\mathcal{D}$ is mutation-invariant (in a sense to be defined) and that, for any objects X and Y of $\mathcal{D}$, there is a bifunctorial non-degenarate bilinear form

$$\operatorname{Hom}(X, \Sigma Y) \times \operatorname{Hom}(Y, \Sigma X) \longrightarrow k$$

(this can be thought of as an adapted version of the 2-Calabi–Yau property).

The category $\mathcal{D}$ is equivalent to a k-linear subcategory of a certain derived category (the analogue of C. Amiot's fundamental domain $\mathcal{F}$ in [2]). We show that this subcategory also enjoys a certain property of invariance under mutation, as was first formulated as a "hope" by K. Nagao in [66].

The main feature of the definition of the subcategory $\mathcal{D}$ is the requirement that for any object X of $\mathcal{D}$, there exists a triangle

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$$

where T_0^X and T_1^X are direct sums of direct summands of a certain fixed rigid object T. This allows a definition of the *index* of X, as in [18] and [68].

The main result of this chapter, besides the definition and study of the subcategory $\mathcal{D}$, is the proof of a multiplication formula analogous to that of [68]: if X and Y are two objetcts of $\mathcal{D}$ such that the spaces $\operatorname{Hom}(X, \Sigma Y)$ and $\operatorname{Hom}(Y, \Sigma X)$ are one-dimensional, and if

$$X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X \quad \text{and} \quad Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$$

are two non-split triangles, then we have the equality

$$X'_X X'_Y = X'_E + X'_{E'}.$$

This cluster character is in particular defined for the cluster category of any non-degenerate quiver with potential in the sense of [22] (be it Jacobi-finite or not), and thus gives a categorification of any skew-symmetric cluster algebra. Applications to cluster algebras will be the subject of a subsequent paper by the author.

In a different setting, using decorated representations of quivers with potentials, a categorification of any skew-symmetric cluster algebra was obtained by H. Derksen, J. Weyman and A. Zelevinsky in the papers [22] and [21], and these results were used by the authors to prove almost all of the conjectures formulated in [31].

The chapter is organized as follows.

In Section 3.2, the main results concerning cluster categories of a quiver with potential and mutation are recalled. In particular, we include the interpretation of mutation as derived equivalence, after [58]. The subcategory $\operatorname{pr}_{\mathcal{C}}\Gamma$, needed to define the subcategory $\mathcal{D}$, is introduced and studied from Subsection 3.2.6 up to the next section. In Subsection 3.2.8, we prove a result on the mutation of objects in the derived category, confirming K. Nagao's hope in [66]. With hindsight, a precursor of this result is [46, Corollary 5.7].

Section 3.3 is devoted to the definition of the cluster character $X'_?$. After some preliminary results, it is introduced in Subsection 3.3.3 together with the subcategory $\mathcal{D}$. The multiplication formula is then proved in Subsection 3.3.5.

Finally, a link with skew-symmetric cluster algebras is given in Section 3.4.

Throughout the chapter, the symbol k will denote an algebraically closed field. When working with any triangulated category, we will use the symbol Σ to denote its suspension functor. An object X of any such category is *rigid* if the space $\operatorname{Hom}_{\mathcal{C}}(X, \Sigma X)$ vanishes.

3.2 Cluster category

In this section, after a brief reminder on quivers with potentials, the cluster category of a quiver with potential is defined after [2]. Mutation in the cluster category is then recalled. Finally, we construct a subcategory on which a version of the cluster character of [68] will be defined in Section 3.3.

3.2.1 Skew-symmetric cluster algebras

We briefly review the definition of (skew-symmetric) cluster algebras (the original definition appeared in [29] using mutation of matrices; the use of quivers was described, for example, in [30, Definition 7.3] in a slightly different way than the one used here, and in [41, Section 1.1]). This material will be used in Section 3.4.

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a set Q_0 of vertices, a set Q_1 of arrows, and two maps $s, t : Q_1 \longrightarrow Q_0$ which send each arrow to its source or target. A quiver is *finite* if it has finitely many vertices and arrows.

Let Q be a finite quiver without oriented cycles of length at most 2. We will denote the vertices of Q by the numbers $1, 2, \ldots, n$. Let i be a vertex of Q. One defines the *mutation of Q at i* to be the quiver $\mu_i(Q)$ obtained from Q in three steps :

1. for each subquiver of the form $j \xrightarrow{a} i \xrightarrow{b} \ell$, add an arrow $[ba]$ from j to ℓ;
2. for each arrow a such that $s(a) = i$ or $t(a) = i$, delete a and add an arrow a^* from $t(a)$ to $s(a)$ (that is, in the opposite direction);
3. delete the arrows of a maximal set of pairwise disjoint oriented cycles of length 2 (which may have appeared in the first step).

A *seed* is a pair $(Q, \mathbf{u})$, where Q is a finite quiver without oriented cycles of length at most 2, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is an (ordered) free generating set of $\mathbb{Q}(x_1, x_2, \ldots, x_n)$. Recall that n is the number of vertices of Q.

If i is a vertex of Q, the *mutation* of the seed $(Q, \mathbf{u})$ is a new seed, say $(Q', \mathbf{u}') = (u'_1, u'_2, \ldots, u'_n))$, where

– Q' is the mutated quiver $\mu_i(Q)$;
– $u'_j = u_j$ whenever $j \neq i$;
– u'_i is given by the equality

$$u'_i u_i = \prod_{a \in Q_1, t(a)=i} x_{s(a)} + \prod_{b \in Q_1, s(b)=i} x_{t(b)}.$$

Definition 3.2.1. Let Q be a finite quiver without oriented cycles of length at most 2. Define the *initial seed* as the seed $(Q, \mathbf{x} = (x_1, x_2, \ldots, x_n))$.

– A *cluster* is any set $\mathbf{u}$ appearing in a seed $(R, \mathbf{u})$ obtained from the initial seed by a finite sequence of mutation.
– A *cluster variable* is any element of a cluster.
– The *cluster algebra* associated with Q is the $\mathbb{Q}$-subalgebra of the field of rational functions $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ generated by the set of all cluster variables.

3.2.2 Quivers with potentials and their mutation

We recall the notion of quiver with potential from [22]. Let Q be a finite quiver. Denote by $\widehat{kQ}$ its *completed path algebra*, that is, the k-algebra whose underlying k-vector space is

$$\prod_{w \text{ path}} kw$$

and whose multiplication is deduced from the composition of paths by distributivity (by convention, we compose paths from right to left). It is a topological algebra for the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the ideal of $\widehat{kQ}$ generated by the arrows of Q. A *potential* on Q is an element W of the space

$$Pot(Q) = \widehat{kQ}/C,$$

where C is the closure of the commutator subspace $[\widehat{kQ}, \widehat{kQ}]$ in $\widehat{kQ}$. In other words, it is a (possibly infinite) linear combination of cyclically inequivalent oriented cycles of Q. The pair (Q, W) is a *quiver with potential.*

Given any arrow a of Q, the *cyclic derivative with respect to a* is the continuous linear map ∂_a from $Pot(Q)$ to $\widehat{kQ}$ whose action on (equivalence classes of) oriented cycles is given by

$$\partial_a(b_r \cdots b_2 b_1) = \sum_{b_i = a} b_{i-1} b_{i-2} \cdots b_1 b_r b_{r-1} \cdots b_{i+1}.$$

The *Jacobian algebra* $J(Q, W)$ of a quiver with potential (Q, W) is the quotient of the algebra $\widehat{kQ}$ by the closure of the ideal generated by the cyclic derivatives $\partial_a W$, as a ranges over all arrows of Q. In case $J(Q, W)$ is finite-dimensional, (Q, W) is *Jacobi-finite.*

The above map is generalized as follows. For any path p of Q, define ∂_p as the linear map from $Pot(Q)$ to $\widehat{kQ}$ whose action on any (equivalence class of) oriented cycle c is given by

$$\partial_p(c) = \sum_{c = upv} vu + \sum_{\substack{c = p_1 w p_2 \\ p = p_2 p_1}} w,$$

where the sums are taken over all decompositions of c into paths of smaller length, with u, v and w possibly trivial paths, and p_1 and p_2 non-trivial paths.

Let (Q, W) be a quiver with potential. In order to define the *mutation of* (Q, W) *at a vertex* ℓ, we must recall the process of reduction of a quiver with potential. Let Λ be the k-algebra given by $\bigoplus_{i \in Q_0} k e_i$, where e_i is the idempotent associated with the vertex i. Two quivers with potentials (Q, W) and (Q', W') are *right-equivalent* if $Q_0 = Q'_0$ and there exists an Λ-algebra isomorphism $\varphi : \widehat{kQ} \longrightarrow \widehat{kQ'}$ sending the class of W to the class of W' in $Pot(Q')$. In that case, it is shown in [22] that the Jacobian algebras of the two quivers with potentials are isomorphic.

A quiver with potential (Q, W) is *trivial* if W is a (possibly infinite) linear combination of paths of length at least 2, and $J(Q, W)$ is isomorphic to Λ. It is *reduced* if W has no terms which are cycles of length at most 2.

The direct sum of two quivers with potentials (Q, W) and (Q', W') such that $Q_0 = Q'_0$ is defined as being $(Q'', W + W')$, where Q'' is the quiver with the same set of vertices as Q and Q' and whose set of arrows is the union of those of Q and Q'.

Theorem 3.2.2 (([22], Theorem 4.6 and Proposition 4.5))**.** *Any quiver with potential* (Q, W) *is right equivalent to a direct sum of a reduced one* (Q_{red}, W_{red}) *and a trivial one* (Q_{triv}, W_{triv})*, both unique up to right-equivalence. Moreover,* $J(Q, W)$ *and* $J(Q_{red}, W_{red})$ *are isomorphic.*

We can now define the mutation of quivers with potentials. Let (Q, W) be a quiver with potential, and let ℓ be a vertex of Q not involved in any cycle of length ≤ 2. Assume that W is written as a series of oriented cycles which do not begin or end in ℓ (W is always cyclically equivalent to such a potential). The *mutation of* (Q, W) *at vertex* ℓ is the new quiver with potential $\mu_\ell(Q, W)$ obtained from (Q, W) as follows.

1. For any subquiver $i \xrightarrow{a} \ell \xrightarrow{b} j$ of Q, add an arrow $i \xrightarrow{[ba]} j$.

2. Delete any arrow a incident with ℓ and replace it by an arrow $a^\star$ going in the opposite direction; the first two steps yield a new quiver $\widetilde{Q}$.
3. Let $\widetilde{W}$ be the potential on $\widetilde{Q}$ defined by $\widetilde{W} = [W] + \sum a^\star b^\star [ba]$, where the sum is taken over all subquivers of Q of the form $i \xrightarrow{a} \ell \xrightarrow{b} j$, and where $[W]$ is obtained from W by replacing each occurence of ba in its terms by $[ba]$. These three steps yield a new quiver with potential $\widetilde{\mu}_\ell(Q, W) = (\widetilde{Q}, \widetilde{W})$.

The mutation $\mu_\ell(Q, W)$ is then defined as the reduced part of $\widetilde{\mu}_\ell(Q, W)$. Note that $\mu_\ell(Q, W)$ might contain oriented cycles of length 2, even if (Q, W) did not. This prevents us from performing iterated mutations following an arbitrary sequence of vertices.

A vertex i of (Q, W) which is not involved in any oriented cycle of length ≤ 2 (and thus at which mutation can be performed) is an *admissible vertex*. An *admissible sequence of vertices* is a sequence $\underline{i} = (i_1, \ldots, i_s)$ of vertices of Q such that i_1 is an admissible vertex of (Q, W), and i_m is an admissible vertex of $\mu_{m-1}\mu_{m-2} \cdots \mu_1(Q, W)$, for $1 < m \leq s$. In that case, we denote by $\mu_{\underline{i}}(Q, W)$ the mutated quiver with potential $\mu_s\mu_{s-1} \cdots \mu_1(Q, W)$.

A quiver with potential is *non-degenerate* if any sequence of vertices is admissible. Since we work over an algebraically closed field, the following existence result applies.

Proposition 3.2.3 ([22], Corollary 7.4)**.** *Suppose that Q is a finite quiver without oriented cycles of length at most 2. If the field k is uncountable, then there exists a potential W on Q such that (Q, W) is non-degenerate.*

3.2.3 Complete Ginzburg dg algebras

Let (Q, W) be a quiver with potential. Following Ginzburg in [42], we construct a differential graded (dg) algebra $\Gamma = \Gamma_{Q,W}$ as follows.

First construct a new graded quiver $\overline{Q}$ from Q. The vertices of $\overline{Q}$ are those of Q; its arrows are those of Q (these have degree 0), to which we add

– for any arrow $a : i \longrightarrow j$ of Q, an arrow $a^* : j \longrightarrow i$ of degree -1;

– for any vertex i of Q, a loop $t_i : i \longrightarrow i$ of degree -2.

Then, for any integer i, let

$$\Gamma^i = \prod_{\omega \text{ path of degree } i} k\omega.$$

This defines the graded k-algebra structure of Γ. Its differential d is defined from its action on the arrows of $\overline{Q}$. We put

– $d(a) = 0$, for each arrow a of Q;

– $d(a^*) = \partial_a W$, for each arrow a of Q;

– $d(t_i) = e_i\Big(\sum_{a \in Q_1}(aa^* - a^*a)\Big)e_i$, for each vertex i of Q.

The differential graded algebra Γ thus defined is the *complete Ginzburg dg algebra* of (Q, W). It is linked to the Jacobian algebra of (Q, W) as follows.

Lemma 3.2.4 ([58], Lemma 2.8)**.** *With the above notations, $J(Q, W)$ is isomorphic to* $\mathrm{H}^0\Gamma$.

3.2.4 Cluster category

Keep the notations of Section 3.2.3.

Denote by $\mathcal{D}\Gamma$ the derived category of Γ (see [54] or [58] for background material on the derived category of a dg algebra). Consider Γ as an object of $\mathcal{D}\Gamma$. The *perfect derived* *category* of Γ is the smallest full triangulated subcategory of $\mathcal{D}\Gamma$ containing Γ and closed under taking direct summands. It is denoted by $\operatorname{per}\Gamma$.

Denote by $\mathcal{D}_{fd}\Gamma$ the full subcategory of $\mathcal{D}\Gamma$ whose objects are those of $\mathcal{D}\Gamma$ with finite-dimensional total homology. This means that homology is zero except in finitely many degrees, where it is of finite dimension. As shown in [58, Theorem 2.17], the category $\mathcal{D}_{fd}\Gamma$ is a triangulated subcategory of $\operatorname{per}\Gamma$.

Moreover, we have the following *relative 3-Calabi–Yau property* of $\mathcal{D}_{fd}\Gamma$ in $\mathcal{D}\Gamma$.

Theorem 3.2.5 ([56], Lemma 4.1 and [51], Theorem 6.3)**.** *For any objects L of $\mathcal{D}\Gamma$ and M of $\mathcal{D}_{fd}\Gamma$, there is a canonical isomorphism*

$$D\operatorname{Hom}_{\mathcal{D}\Gamma}(M,L) \longrightarrow \operatorname{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-3}L,M)$$

functorial in both M and L.

Following [2, Definition 3.5] (and [58, Section 4] in the non Jacobi-finite case), we define the *cluster category* of (Q,W) as the idempotent completion of the triangulated quotient $(\operatorname{per}\Gamma)/\mathcal{D}_{fd}\Gamma$, and denote it by $\mathcal{C}=\mathcal{C}_{Q,W}$.

In case (Q,W) is Jacobi-finite, $\mathcal{C}_{Q,W}$ enjoys the following properties ([2, Theorem 3.6] and [57, Proposition 2.1]) :

- it is Hom-finite;
- it is 2-Calabi–Yau;
- the object Γ is *cluster-tilting* in the sense that it is rigid and any object X of $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(\Gamma,\Sigma X)=0$ is in $\operatorname{add}\Gamma$;
- any object X of $\mathcal{C}$ admits an $(\operatorname{add}\Gamma)$-*presentation*, that is, there exists a triangle $T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$, with T_1^X and T_0^X in $\operatorname{add}\Gamma$.

As we shall see later, most of these properties do not hold when (Q,W) is not Jacobi-finite.

3.2.5 Mutation in $\mathcal{C}$

Keep the notations of the previous section. Let i be a vertex of Q not involved in any oriented cycle of length 2. As seen in Section 3.2.2, one can mutate (Q,W) at the vertex i.

In the cluster category, this corresponds to changing a direct factor of Γ. Let Γ' be the complete Ginzburg dg algebra of $\tilde{\mu}_i(Q,W)$. For any vertex j of Q, let $\Gamma_j=e_j\Gamma$ and $\Gamma'_j=e_j\Gamma'$.

Theorem 3.2.6 ([58], Theorem 3.2)**.** *1. There is a triangle equivalence $\widetilde{\mu}_i^+$ from $\mathcal{D}(\Gamma')$ to $\mathcal{D}(\Gamma)$ sending Γ'_j to Γ_j if $i\neq j$ and to the cone Γ_i^* of the morphism*

$$\Gamma_i \longrightarrow \bigoplus_{\alpha}\Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α if $i=j$. The functor $\widetilde{\mu}_i^+$ restricts to triangle equivalences from $\operatorname{per}\Gamma'$ to $\operatorname{per}\Gamma$ and from $\mathcal{D}_{fd}\Gamma'$ to $\mathcal{D}_{fd}\Gamma$.

2. Let Γ_{red} and Γ'_{red} be the complete Ginzburg dg algebra of the reduced part of (Q,W) and $\tilde{\mu}_i(Q,W)$, respectively. The functor $\widetilde{\mu}_i^+$ induces a triangle equivalence μ_i^+ : $\mathcal{D}(\Gamma'_{red})\longrightarrow\mathcal{D}(\Gamma_{red})$ which restricts to triangle equivalences from $\operatorname{per}\Gamma'_{red}$ to $\operatorname{per}\Gamma_{red}$ and from $\mathcal{D}_{fd}\Gamma'_{red}$ to $\mathcal{D}_{fd}\Gamma_{red}$.

The object $\Gamma_i^* \oplus \bigoplus_{j\neq i} \Gamma_j$ in $\mathcal{D}\Gamma$ is the *mutation of* Γ *at the vertex* i, and we denote it by $\mu_i(\Gamma)$.

Note that in part (2) of the above theorem, $e_j\Gamma'_{red}$ is still sent to $e_j\Gamma_{red}$ if $i \neq j$ and to the cone $\Gamma^*_{red,i}$ of the morphism

$$e_i\Gamma_{red} \longrightarrow \bigoplus_{\alpha} e_{t(\alpha)}\Gamma_{red}$$

whose components are given by left multiplication by α if $i = j$.

For instance, if $(i_1, i_2, \dots, i_r)$ is an admissible sequence of vertices, then we get a sequence of triangle equivalences

$$\mathcal{D}\Gamma^{(r)} \longrightarrow \dots \longrightarrow \mathcal{D}\Gamma^{(1)} \longrightarrow \mathcal{D}\Gamma,$$

where $\Gamma^{(j)}$ is the complete Ginzburg dg-algebra of $\mu_{i_j}\mu_{i_{j-1}}\dots\mu_{i_1}(Q,W)$, for any $j \in \{1,2,\dots,r\}$. We denote the image of $\Gamma^{(r)}$ in $\mathcal{D}\Gamma$ by $\mu_{i_r}\mu_{i_{r-1}}\dots\mu_{i_1}(\Gamma)$.

We now remark some consequences of 3.2.6 on the level of cluster categories. First, there are induced triangle equivalence $\mathcal{C}_{\tilde{\mu}_i(Q,W)} \longrightarrow \mathcal{C}_{Q,W}$ and $\mathcal{C}_{\mu_i(Q,W)} \longrightarrow \mathcal{C}_{Q,W}$.

Moreover, as shown in Section 4 of [58], the cone of the morphism

$$\bigoplus_{\beta: j\to i} \Gamma_j \longrightarrow \Gamma_i$$

whose components are given by left multiplication by β is isomorphic to $\Sigma\Gamma_i^*$ in $\mathcal{C}$. Hence we have triangles in $\mathcal{C}$

$$\Gamma_i \longrightarrow \bigoplus_{\alpha: i\to j} \Gamma_j \longrightarrow \Gamma_i^* \longrightarrow \Sigma\Gamma_i \quad \text{and} \quad \Gamma_i^* \longrightarrow \bigoplus_{\alpha: j\to i} \Gamma_j \longrightarrow \Gamma_i \longrightarrow \Sigma\Gamma_i^*,$$

and $\dim \operatorname{Hom}_{\mathcal{C}}(\Gamma_j, \Sigma\Gamma_i^*) = \delta_{i,j}$ (see [58, Section 4]).

If (Q,W) is non-degenerate and reduced, then any sequence of vertices $i_1,\dots,i_r$ yields a sequence of triangle equivalences

$$\mathcal{C}_{\mu_{i_r}\cdots\mu_{i_1}(Q,W)} \longrightarrow \dots \longrightarrow \mathcal{C}_{\mu_{i_1}(Q,W)} \longrightarrow \mathcal{C}_{Q,W}$$

sending $\Gamma_{\mu_{i_r}\cdots\mu_{i_1}(Q,W)}$ to $\mu_{i_r}\dots\mu_{i_1}(\Gamma_{Q,W})$.

3.2.6 The subcategory $\operatorname{pr}_{\mathcal{C}}\Gamma$

Since in general the cluster category does not enjoy the properties listed in Section 3.2.4, we will need to restrict ourselves to a subcategory of it.

Let $\mathcal{T}$ be any triangulated category. For any subcategory $\mathcal{T}'$ of $\mathcal{T}$, define $\operatorname{ind}\mathcal{T}'$ as the set of isomorphism classes of indecomposable objects of $\mathcal{T}$ contained in $\mathcal{T}'$. Denote by $\operatorname{add}\mathcal{T}'$ the full subcategory of $\mathcal{T}$ whose objects are all finite direct sums of direct summands of objects in $\mathcal{T}'$. The subcategory $\mathcal{T}'$ is *rigid* if, for any two objects X and Y of $\mathcal{T}'$, $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma Y) = 0$.

Finally, define $\operatorname{pr}_{\mathcal{T}}\mathcal{T}'$ as the full subcategory of $\mathcal{T}$ whose objects are cones of morphisms in $\operatorname{add}\mathcal{T}'$ (the letters “pr” stand for *presentation*, as all objects of $\operatorname{pr}_{\mathcal{T}}\mathcal{T}'$ admit an $(\operatorname{add}\mathcal{T}')$-presentation). In the notations of [4, Section 1.3.9], this subcategory is written as $(\operatorname{add}\mathcal{T}')*(\operatorname{add}\Sigma\mathcal{T}')$.

As we shall now prove, the category $\operatorname{pr}_{\mathcal{T}}\mathcal{T}'$ is invariant under “mutation” of $\mathcal{T}'$.

Recall that a category $\mathcal{T}'$ is *Krull–Schmidt* if any object can be written as a finite direct sum of objects whose endomorphism rings are local. Note that in that case, we have $\mathcal{T}' = \operatorname{add}\mathcal{T}'$.

Proposition 3.2.7. *Let $\mathcal{R}$ and $\mathcal{R}'$ be rigid Krull–Schmidt subcategories of a triangulated category $\mathcal{T}$. Suppose that there exist indecomposable objects R of $\mathcal{R}$ and R^* of $\mathcal{R}'$ such that $\operatorname{ind}\mathcal{R} \setminus \{R\} = \operatorname{ind}\mathcal{R}' \setminus \{R^*\}$. Suppose, furthermore, that $\dim \operatorname{Hom}_{\mathcal{T}}(R, \Sigma R^*) = \dim \operatorname{Hom}_{\mathcal{T}}(R^*, \Sigma R) = 1$. Let*

$$R \longrightarrow E \longrightarrow R^* \longrightarrow \Sigma R \quad \text{and} \quad R^* \longrightarrow E' \longrightarrow R \longrightarrow \Sigma R^*$$

be non-split triangles, and suppose that E and E' lie in $\mathcal{R} \cap \mathcal{R}'$.

Then $\operatorname{pr}_{\mathcal{T}}\mathcal{R} = \operatorname{pr}_{\mathcal{T}}\mathcal{R}'$.

PROOF In view of the symmetry of the hypotheses, we only have to prove that any object of $\operatorname{pr}_{\mathcal{T}}\mathcal{R}$ is an object of $\operatorname{pr}_{\mathcal{T}}\mathcal{R}'$.

Let X be an object of $\operatorname{pr}_{\mathcal{T}}\mathcal{R}$. Let $T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_1$ be a triangle, with T_1 and T_0 in $\mathcal{R}$.

The category $\mathcal{R}$ being Krull–Schmidt, one can write (in a unique way up to isomorphism) $T_0 = \overline{T}_0 \oplus R^m$, where R is not a direct summand of $\overline{T}_0$.

The composition $\overline{T}_0 \oplus (E')^m \longrightarrow \overline{T}_0 \oplus R^m \longrightarrow X$ yields an octahedron

$$\begin{tikzcd}
 & & \Sigma W \arrow[ddl, "+"] \arrow[dr, dotted] & & \\
(\Sigma R^*)^m \arrow[urr] \arrow[dr, "+"] & & & \Sigma T_1 \arrow[lll, dotted, "+"] \arrow[ddl, dotted, "+"] & \\
 & \overline{T}_0 \oplus (E')^m \arrow[rrr] \arrow[dr] & & & X. \arrow[uull] \arrow[ul, dotted] \\
 & & \overline{T}_0 \oplus R^m \arrow[uull] \arrow[urr] & &
\end{tikzcd}$$

Now write $T_1 = \overline{T}_1 \oplus R^n$. Then we have a triangle

$$(R^*)^m \longrightarrow W \longrightarrow \overline{T}_1 \oplus R^n \xrightarrow{\ \varepsilon\ } (\Sigma R^*)^m .$$

Since $\operatorname{Hom}_{\mathcal{T}}(\overline{T}_1, \Sigma R^*) = 0$ and $\dim \operatorname{Hom}_{\mathcal{T}}(R, \Sigma R^*) = 1$, by a change of basis, we can write ε in matrix form as

$$\left(\begin{array}{c|c} I_r x & 0 \\ \hline 0 & 0 \end{array}\right)$$

where x is a non-zero element of $\operatorname{Hom}_{\mathcal{T}}(R, \Sigma R^*)$. Therefore W is isomorphic to $(E')^r \oplus R^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1$.

Now, we have a triangle $W \longrightarrow \overline{T}_0 \oplus (E')^m \longrightarrow X \longrightarrow \Sigma W$. Compose $\Sigma^{-1}X \longrightarrow W$ with $W = (E')^r \oplus R^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1 \longrightarrow (E')^r \oplus E^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1$ (the second term is changed) to get an octahedron

$$\begin{tikzcd}
 & & V \arrow[ddl, "+"] \arrow[d, dotted] & & \\
\overline{T}_0 \oplus (E')^m \arrow[urr] \arrow[dr, "+"] & & (R^*)^{n-r} \arrow[ll, dotted, "+"] \arrow[dd, dotted, "+"] & & \\
 & \Sigma^{-1}X \arrow[rr] \arrow[dr] & & (E')^r \oplus E^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1. \arrow[uul] \arrow[ul, dotted] & \\
 & & (E')^r \oplus R^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1 \arrow[uull] \arrow[ur] & &
\end{tikzcd}$$

The morphism $(R^*)^{n-r} \longrightarrow \overline{T}_0 \oplus (E')^m$ is zero, so the triangle $\overline{T}_0 \oplus (E')^m \longrightarrow V \longrightarrow (R^*)^{n-r} \longrightarrow \Sigma(\overline{T}_0 \oplus (E')^m)$ splits, and V is isomorphic to $(R^*)^{n-r} \oplus \overline{T}_0 \oplus (E')^m$.

Hence we have a triangle

$$(E')^r \oplus E^{n-r} \oplus (R^*)^{m-r} \oplus \overline{T}_1 \longrightarrow (R^*)^{n-r} \oplus \overline{T}_0 \oplus (E')^m \longrightarrow X \longrightarrow \ldots,$$

proving that X belongs to $\mathrm{pr}_{\mathcal{T}}\mathcal{R}'$. This finishes the proof. □

Corollary 3.2.8. *Let $\mathcal{C}$ be the cluster category of a quiver with potential (Q, W). For any admissible sequence $(i_1, \ldots, i_r)$ of vertices of Q, the following equality holds :*

$$\mathrm{pr}_{\mathcal{C}}\Gamma = \mathrm{pr}_{\mathcal{C}}\Big(\mu_{i_r} \ldots \mu_{i_1}(\Gamma)\Big).$$

PROOF We apply Proposition 3.2.7 and use induction on r. That $\mathrm{add}\,\Gamma$ is a Krull–Schmidt category is shown in Corollary 3.2.12 below. We also need that Γ is a rigid object of $\mathcal{C}$; this follows from Proposition 3.2.10 below. □

3.2.7 Properties of $\mathrm{pr}_{\mathcal{C}}\Gamma$

Let $\mathcal{C}$ be the cluster category of a quiver with potential (Q, W). We will prove in this section that the subcategory $\mathrm{pr}_{\mathcal{C}}\Gamma$ enjoys versions of some of the properties listed in Section 3.2.4.

We denote by $\mathcal{D}_{\leq 0}$ (and $\mathcal{D}_{\geq 0}$ respectively) the full subcategory of $\mathcal{D}\Gamma$ whose objects are those X whose homology is concentrated in non-positive (and non-negative, respectively) degrees. Recall that $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ form a *t-structure*; in particular, $\mathrm{Hom}_{\mathcal{D}\Gamma}(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 1})$ vanishes, and for each object X of $\mathcal{D}\Gamma$, there exists a unique (up to a unique triangle isomorphism) triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \longrightarrow \Sigma\tau_{\leq 0}X$$

with $\tau_{\leq 0}X$ in $\mathcal{D}_{\leq 0}$ and $\tau_{\geq 1}X$ in $\mathcal{D}_{\geq 1}$.

Lemma 3.2.9. *If X and Y lie in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$, then the quotient functor* $\mathrm{per}\,\Gamma \longrightarrow \mathcal{C}$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}\Gamma}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

PROOF Let X and Y be as in the statement. In particular, X and Y lie in $\mathcal{D}_{\leq 0}\Gamma$.

First suppose that a morphism $f : X \longrightarrow Y$ is sent to zero in $\mathcal{C}$. This means that f factors as

$$X \xrightarrow{g} M \xrightarrow{h} Y \ ,$$

with M in $\mathcal{D}_{fd}\Gamma$. Now, $X = \tau_{\leq 1}X$, so g factors through $\tau_{\leq 1}M$, which is still in $\mathcal{D}_{fd}$. Using Theorem 3.2.5, we have an isomorphism

$$D\operatorname{Hom}_{\mathcal{D}\Gamma}(\tau_{\leq 1}M, Y) \longrightarrow \operatorname{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma^3\tau_{\leq 1}M).$$

The right hand side of this equation is zero, since $\operatorname{Hom}_{\mathcal{D}\Gamma}(Y, \mathcal{D}_{\leq -2}\Gamma) = 0$. Hence $f = 0$. This shows injectivity.

To prove surjectivity, consider a fraction

$$X \xrightarrow{f} Y' \xleftarrow{s} Y,$$

where the cone of s is an object N of $\mathcal{D}_{fd}\Gamma$.

The following diagram will be helpful.

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d, "s"'] & Y \arrow[d, "t"] \\
X \arrow[r, "f"] & Y' \arrow[r, "g"] \arrow[d] & Y'' \arrow[d] \\
\tau_{\leq 0}N \arrow[r] & N \arrow[r] \arrow[d] & \tau_{\geq 1}N \arrow[d, "h"] \\
 & \Sigma Y \arrow[r, equal] & \Sigma Y
\end{tikzcd}$$

We have that $\operatorname{Hom}_{\mathcal{D}\Gamma}(\tau_{\leq 0}N, \Sigma Y)$ is isomorphic to $D\operatorname{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma^2\tau_{\leq 0}N)$ because of Theorem 3.2.5, and this space is zero since $\operatorname{Hom}_{\mathcal{D}\Gamma}(Y, \mathcal{D}_{\leq -2})$ vanishes. Thus there exists a morphism $h : \tau_{\geq 1}N \to \Sigma Y$ such that the lower right square of the above diagram commute. We embed h in a triangle; this triangle is the rightmost column of the diagram.

We get a new fraction

$$X \xrightarrow{gf} Y'' \xleftarrow{t} Y$$

which is equal to the one we started with. But since X is in $\mathcal{D}_{\leq 0}$ and $\tau_{\geq 1}N$ is in $\mathcal{D}_{\geq 1}$, the space $\operatorname{Hom}_{\mathcal{D}\Gamma}(X, \tau_{\geq 1}N)$ vanishes. Thus there exists a morphism $\ell : X \to Y$ such that $gf = t\ell$. It is easily seen that the fraction is then the image of ℓ under the quotient functor.

Thus the map is surjective. $\square$

Proposition 3.2.10. *The quotient functor* $\operatorname{per}\Gamma \longrightarrow \mathcal{C}$ *restricts to an equivalence of (k-linear) categories* $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma \longrightarrow \operatorname{pr}_{\mathcal{C}}\Gamma$*.*

PROOF It is a consequence of Lemma 3.2.9 that the functor is fully faithful.

It remains to be shown that it is dense. Let Z be an object of $\operatorname{pr}_{\mathcal{C}}\Gamma$, and let $T_1 \longrightarrow T_0 \longrightarrow Z \longrightarrow \Sigma T_1$ be an $\operatorname{add}\Gamma$-presentation. The functor being fully faithful, the morphism $T_1 \longrightarrow T_0$ lifts in $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$ to a morphism $P_1 \longrightarrow P_0$, with P_0 and P_1 in $\operatorname{add}\Gamma$. Its cone is clearly sent to Z in $\mathcal{C}$. This finishes the proof of the equivalence. $\square$

As in [2], we have the following characterization of $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$, which we shall prove after Corollary 3.2.12.

Lemma 3.2.11. *We have that that* $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma = \mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \operatorname{per}\Gamma$*.*

Corollary 3.2.12. *The category* $\mathrm{pr}_{\mathcal{C}}\Gamma$ *is a Krull–Schmidt category.*

Proof In view of Proposition 3.2.10, it suffices to prove that $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ is a Krull–Schmidt category. It is shown in [58, Lemma 2.17] that the category $\mathrm{per}\,\Gamma$ is a Krull–Schmidt category. Since $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ is a full subcategory of $\mathrm{per}\,\Gamma$, it is sufficient to prove that any direct summand of an object in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ is also in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$. The equality $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma = \mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \mathrm{per}\,\Gamma$ of Lemma 3.2.11 implies this property. Note that it also follows from [47, Proposition 2.1], whose proof does not depend on the Hom-finiteness assumption. $\square$

In order to prove Lemma 3.2.11, we will need the following definition.

Definition 3.2.13. A dg Γ-module M is *minimal perfect* if its underlying graded module is of the form

$$\bigoplus_{j=1}^{N} R_j,$$

where each R_j is a finite direct sum of shifted copies of direct summands of Γ, and if its differential is of the form $d_{int} + \delta$, where d_{int} is the direct sum of the differential of the R_j, and δ, as a degree 1 map from $\bigoplus_{j=1}^{N} R_j$ to itself, is a strictly upper triangular matrix whose entries are in the ideal of Γ generated by the arrows.

Lemma 3.2.14. *Let M be a dg Γ-module such that M is perfect in $\mathcal{D}\Gamma$. Then M is quasi-isomorphic to a minimal perfect dg module.*

Proof We will apply results of [6]. Using the notation of [6, Section 6.2], $\mathrm{per}\,\Gamma$ is equivalent to the category $\mathrm{Tr}(C)$, where C is the dg category whose objects are vertices of the quiver Q and morphisms dg vector spaces are given by the paths of Q. Thus any object of $\mathrm{per}\,\Gamma$ is quasi-isomorphic to a dg module as in Definition 3.2.13, where the entries of δ do not necessarily lie in the ideal generated by the arrows.

As a graded Γ-module, any such object can be written in the form $\Sigma^{i_1}\Gamma_{j_1} \oplus \ldots \oplus \Sigma^{i_r}\Gamma_{j_r}$, where each j_ℓ is a vertex of Q and each i_ℓ is an integer. Assume that $i_1 \leq \ldots \leq i_r$. The subcategory of objects wich can be written in this form, with $a \leq i_1 \leq i_r \leq b$, is denoted by $\mathcal{C}^{[a,b]}$. According to [6, Lemma 5.2.1], $\mathcal{C}^{[a,b]}$ is closed under taking direct summands.

Let X be an object of $\mathrm{per}\,\Gamma$. Then there are integers a and b such that X lies in $\mathcal{C}^{[a,b]}$. We prove the Lemma by induction on $b - a$.

If $a = b$, then δ has to be zero, and X is minimal perfect.

Suppose that all objects of $\mathcal{C}^{[a,b]}$ are isomorphic to a minimal perfect dg module whenever $b - a$ is less or equal to some integer $n \geq 0$.

Let X be an object of $\mathcal{C}^{[a,b]}$, with $b - a = n + 1$. We can assume that X is of the form $\Sigma^{i_1}\Gamma_{j_1} \oplus \ldots \oplus \Sigma^{i_r}\Gamma_{j_r}$ and that its differential is written in matrix form as

$$\begin{pmatrix} d_{\Sigma^{i_1}\Gamma_{j_1}} & f_{12} & \cdots & f_{1r} \\ 0 & d_{\Sigma^{i_2}\Gamma_{j_2}} & \cdots & f_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{\Sigma^{i_r}\Gamma_{j_r}} \end{pmatrix},$$

where all the f_{uv} are in the ideal generated by the arrows.

Suppose that $i_q = i_{q+1} = \ldots = i_r$, but $i_{q-1} < i_q$. Then X is the cone of the morphism from $\Sigma^{i_q - 1}\Gamma_{j_q} \oplus \ldots \oplus \Sigma^{i_r - 1}\Gamma_{j_r}$ to the submodule X' of X whose underlying graded module is $\Sigma^{i_1}\Gamma_{j_1} \oplus \ldots \oplus \Sigma^{i_{q-1}}\Gamma_{j_{q-1}}$ given by the matrix

$$\begin{pmatrix} \Sigma^{-1}f_{1,q} & \Sigma^{-1}f_{1,q+1} & \dots & \Sigma^{-1}f_{1,r} \\ \Sigma^{-1}f_{2,q} & \Sigma^{-1}f_{2,q+1} & \dots & \Sigma^{-1}f_{2,r} \\ \vdots & \vdots & & \vdots \\ \Sigma^{-1}f_{q-1,q} & \Sigma^{-1}f_{q-1,q+1} & \dots & \Sigma^{-1}f_{q-1,r} \end{pmatrix},$$

whose entries are still elements of Γ. Note that X' lies in $\mathcal{C}^{[a,b-1]}$. By the induction hypothesis, X' is quasi-isomorphic to a minimal perfect dg module. Thus we can assume that f_{ij} is in the ideal generated by the arrows, for $i = 1, 2, \dots q-1$ and $j = 1, 2, \dots, q-1$.

The rest of the proof is another induction, this time on the number of summands of X of the form $\Sigma^m\Gamma_\ell$ (this number is $r - q + 1$).

If this number is 1, then X is the cone of a morphism given in matrix form by a column. If this column contains no isomorphisms, then X is minimal perfect. Otherwise, we can suppose that the lowest term of the column is an isomorphism ϕ (by reordering the terms; note that if X' contained any term of the form $\Sigma^m\Gamma_\ell$, we could not suppose this, because by reordering the terms, the differential of X' could then not be triangular anymore). In this case, the morphism is a section, whose retraction is given by the matrix $(0, 0, \dots, \phi^{-1})$. Thus X is quasi-isomorphic to a summand of X', and is thus in $\mathcal{C}^{[a,b-1]}$. By induction hypothesis, it is quasi-isomorphic to a minimal perfect.

If $r - q + 1$ is greater than one, then X is obtained from X' in the following recursive fashion. Put $X_0 = X'$, and for an integer $k > 0$, let X_k be the cone of the morphism

$$\begin{pmatrix} \Sigma^{-1}f_{1,q+k-1} \\ \Sigma^{-1}f_{2,q+k-1} \\ \vdots \\ \Sigma^{-1}f_{q+k-2,q+k-1} \end{pmatrix}$$

into X_{k-1}. Then X is equal to X_{r-q+1}.

If one of these columns contains an isomorphism, we can reorder the terms so that the isomorphism is contained in the first of these columns. Then, by the above reasoning, this first column is a section, X_1 is quasi-isomorphic to a dg module which has no summands of the form $\Sigma^m\Gamma_\ell$, and X has only $r - q$ summands of this form. By induction, X is quasi-isomorphic to a minimal perfect dg module. This finishes the proof.

□

PROOF (of Lemma 3.2.11.) It is easily seen that $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ in contained in $\mathcal{D}_{\leq 0} \cap {}^\perp\mathcal{D}_{\leq -2} \cap \operatorname{per}\Gamma$. Let X be in $\mathcal{D}_{\leq 0} \cap {}^\perp\mathcal{D}_{\leq -2} \cap \operatorname{per}\Gamma$. Then X is quasi-isomorphic to a minimal perfect dg module. Thus suppose that X is minimal perfect.

Let S_i be the simple dg module at the vertex i. Since X is minimal perfect, the dimension of $\mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma^p S_i)$ is equal to the number of summands of X isomorphic to $\Sigma^p\Gamma_i$, as a graded Γ-module. Since X is in $\mathcal{D}_{\leq 0} \cap {}^\perp\mathcal{D}_{\leq -2}$, this number is zero unless i is 0 or 1. This proves that X is the cone of a morphism between objects of $\operatorname{add}\Gamma$, and thus X is in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$. □

We will need a particular result on the calculus of fractions in $\mathcal{C}$ for certain objects. Recall that, for any two objects X and Y of $\operatorname{per}\Gamma$, the space $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ is the colimit of the direct system $(\mathrm{Hom}_{\mathcal{D}\Gamma}(X', \Sigma Y))$ taken over all morphisms $f : X' \to X$ whose cone is in $\mathcal{D}_{fd}\Gamma$.

Lemma 3.2.15. *Let X and Y be objects of $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$. Then the space $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ is the colimit of the direct system $(\mathrm{Hom}_{\mathcal{D}\Gamma}(X', \Sigma Y))$ taken over all morphisms $f : X' \to X$ whose cone is in $\mathcal{D}_{fd}\Gamma \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$ and such that X' lies in $\mathcal{D}_{\leq 0}$.*

PROOF There is a natural map

$$\operatorname{colim} \operatorname{Hom}_{\mathcal{D}\Gamma}(X', \Sigma Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, \Sigma Y),$$

where the colimit is taken over all morphisms $f : X' \to X$ whose cone is in $\mathcal{D}_{fd}\Gamma \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$ and such that X' lies in $\mathcal{D}_{\leq 0}$.

We first prove that it is surjective. Let $X \xleftarrow{s} X' \xrightarrow{f} \Sigma Y$ be a morphism in $\mathcal{C}$, with $N = cone(s)$ in $\mathcal{D}_{fd}\Gamma$.

Using the canonical morphism $N \to \tau_{\geq 0}N$, we get a commuting diagram whose two lower rows and two leftmost columns are triangles:

$$\begin{tikzcd}
\Sigma^{-1}\tau_{<0}N \arrow[r, equal] \arrow[d] & \Sigma^{-1}\tau_{<0}N \arrow[d] & & \\
\Sigma^{-1}N \arrow[r] \arrow[d] & X' \arrow[r, "s"] \arrow[d, "a"] & X \arrow[r] \arrow[d, equal] & N \arrow[d] \\
\Sigma^{-1}\tau_{\geq 0}N \arrow[r] \arrow[d] & X'' \arrow[r, "t"] \arrow[d] & X \arrow[r] & \tau_{\geq 0}N. \\
\tau_{<0}N \arrow[r, equal] & \tau_{<0}N & &
\end{tikzcd}$$

Thanks to the 3-Calabi–Yau property, the space $\operatorname{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}\tau_{<0}N, \Sigma Y)$ is isomorphic to $D\operatorname{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma\tau_{<0}N)$, and this is zero since $\tau_{<0}N$ is in $\mathcal{D}_{\leq -2}$. Therefore f factors through a, and there exists a morphism $g : X'' \longrightarrow \Sigma Y$ such that $ga = f$. The fraction $X \xleftarrow{t} X'' \xrightarrow{g} \Sigma Y$ is equal to $X \xleftarrow{s} X' \xrightarrow{f} \Sigma Y$, and the cone of t is in $\mathcal{D}_{fd} \cap \mathcal{D}_{\geq 0}$.

Using the canonical morphism $\tau_{\leq 0}\tau_{\geq 0}N \to \tau_{\geq 0}N$, we get a commuting diagram whose rows are triangles:

$$\begin{tikzcd}
\Sigma^{-1}\tau_{\leq 0}\tau_{\geq 0}N \arrow[r] \arrow[d] & X''' \arrow[r, "u"] \arrow[d, "b"] & X \arrow[r] \arrow[d, equal] & \tau_{\leq 0}\tau_{\geq 0}N \arrow[d] \\
\Sigma^{-1}\tau_{\geq 0}N \arrow[r] & X'' \arrow[r, "t"] & X \arrow[r] & \tau_{\geq 0}N.
\end{tikzcd}$$

Taking $h = bg$, we get a fraction $X \xleftarrow{u} X''' \xrightarrow{h} \Sigma Y$ which is equal to the fraction $X \xleftarrow{t} X'' \xrightarrow{g} \Sigma Y$ and is such that the cone of u lies in $\mathcal{D}_{fd} \cap \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$.

However, X''' has no reason to lie in $\mathcal{D}_{\leq 0}$. Using the canonical morphism $\tau_{\leq 0}X''' \to X'''$, we get another commuting diagram whose middle rows and leftmost columns are triangles:

$$\begin{tikzcd}
\Sigma^{-1}\tau_{>0}X''' \arrow[r, equal] \arrow[d] & \Sigma^{-1}\tau_{>0}X''' \arrow[d] & & \\
\Sigma^{-1}M \arrow[r] \arrow[d] & \tau_{\leq 0}X''' \arrow[r, "v"] \arrow[d, "c"] & X \arrow[r] \arrow[d, equal] & M \arrow[d] \\
\Sigma^{-1}\tau_{\leq 0}\tau_{\geq 0}N \arrow[r] \arrow[d] & X''' \arrow[r, "u"] \arrow[d] & X \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}N. \\
\tau_{>0}X''' \arrow[r, equal] & \tau_{>0}X''' & &
\end{tikzcd}$$

Since X and $\tau_{\leq 0}X'''$ are in $\mathcal{D}_{\leq 0}$, then so is M. Moreover, $\tau_{>0}X''' = \mathrm{H}^1X'''$ is in $\mathcal{D}_{fd}$; indeed, the lower triangle gives an exact sequence $\mathrm{H}^0\tau_{\leq 0}\tau_{\geq 0}N \to \mathrm{H}^1X''' \to \mathrm{H}^1X$ whose leftmost term is finite-dimensional and whose rightmost term is zero. Therefore, since $\tau_{>0}X'''$ and $\tau_{\leq 0}\tau_{\geq 0}N$ are in $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{fd}$, then so is M, thanks to the leftmost triangle.

Hence, if we put $j = hc$, we have a new fraction $X \xleftarrow{v} \tau_{\leq 0}X''' \xrightarrow{j} \Sigma Y$ which is equal to $X \xleftarrow{u} X''' \xrightarrow{h} \Sigma Y$, and which is such that $\tau_{\leq 0}X'''$ is in $\mathcal{D}_{\leq 0}$ and $cone(v)$ is in $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. This proves surjectivity of the map.

We now prove that the map is injective. Let $X \xleftarrow{s} X' \xrightarrow{f} \Sigma Y$ be a fraction with X' in $\mathcal{D}_{\leq 0}$ and $cone(s)$ in $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. Suppose it is zero in $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$, that is, f factors through an object of $\mathcal{D}_{fd}$. We must prove that it factors through an object of $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$.

Put $f = hg$, with $g : X' \to M$ and $h : M \to \Sigma Y$, and M an object of $\mathcal{D}_{fd}$. Consider the following diagram:

$$\begin{tikzcd}
X' \arrow[rr, dotted, "\varphi"] \arrow[dr, "g"] & & \tau_{\leq 0}\tau_{\geq 0}M \arrow[d, "c"] \\
\tau_{<0}M \arrow[r, "a"] & M \arrow[r, "b"] \arrow[dl, "h"'] & \tau_{\geq 0}M \arrow[d, "d"] \arrow[dll, dotted] \\
\Sigma Y & & \tau_{>0}M.
\end{tikzcd}$$

By the 3-Calabi–Yau property, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}\Gamma}(\tau_{<0}M, \Sigma Y) \cong D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma^2\tau_{<0}M),$$

and this is zero since $\Sigma^2\tau_{<0}M$ is in $\mathcal{D}_{\leq -3}$. Hence h factors through b.

Moreover, $\mathrm{Hom}_{\mathcal{D}\Gamma}(X', \tau_{>0}M)$ is zero, since X' is in $\mathcal{D}_{\leq 0}$ and $\tau_{>0}M$ is in $\mathcal{D}_{>0}$. Hence bg factors through c.

This shows that $f = hg$ factors through $\tau_{\leq 0}\tau_{\geq 0}M$, which is an object of $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. Embed φ in a triangle

$$X'' \xrightarrow{\varepsilon} X' \xrightarrow{\varphi} \tau_{\leq 0}\tau_{\geq 0}M \longrightarrow \Sigma X''.$$

Then the fraction $(s\varepsilon)^{-1}(f\varepsilon)$ is equal to $s^{-1}f$. Since f factors through $\tau_{\leq 0}\tau_{\geq 0}M$, $f\varepsilon$ is zero.

Consider finally the natural morphism $\sigma : \tau_{\leq 0}X'' \to X''$. Its cone $\tau_{>0}X''$ is isomorphic to $\Sigma^{-1}\tau_{\leq 0}\tau_{\geq 0}M$, and is thus in $\mathcal{D}_{fd}$. Therefore the cone of $s\varepsilon\sigma$ is also in $\mathcal{D}_{fd}$ by composition, and we have a fraction $(s\varepsilon\sigma)^{-1}(f\varepsilon\sigma)$ which is equal to $s^{-1}f$, and is such that $f\varepsilon\sigma = 0$, $\tau_{\leq 0}X'' \in \mathcal{D}_{\leq 0}$ and $cone(s\varepsilon\sigma) \in \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0} \cap \mathcal{D}_{fd}$. This proves injectivity of the map. $\square$

Using the isomorphism of Theorem 3.2.5, we get a bifunctorial non-degenerate bilinear form

$$\beta_{M,L} : \mathrm{Hom}_{\mathcal{D}\Gamma}(M, L) \times \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-3}L, M) \longrightarrow k$$

for M in $\mathcal{D}_{fd}$ and L in $\mathrm{per}\,\Gamma$. Using this, C. Amiot constructs in [2, Section 1.1] a bifunctorial bilinear form

$$\overline{\beta}_{X,Y} : \mathrm{Hom}_{\mathcal{D}\Gamma}(X, Y) \times \mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma^2 X) \longrightarrow k$$

for X and Y in $\mathcal{C}$ in the following way.

Using the calculus of left fractions, let $s^{-1}f : X \to Y$ and $t^{-1}g : Y \to \Sigma^2 X$ be morphisms in $\mathcal{C}$. Composing them, we get a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"] & & Y \arrow[dl, "s"'] \arrow[dr, "g"] & & \Sigma^2 X \arrow[dl, "t"'] \\
& Y' \arrow[dr, "h"] & & \Sigma^2 X' \arrow[dl, "s'"'] & \\
& & \Sigma^2 X''. & &
\end{tikzcd}$$

Put $\Sigma^2 u = s't$. Then one gets a commuting diagram, where rows are triangles:

$$\begin{tikzcd}
& N \arrow[r, "a"] & X \arrow[r, "u"] \arrow[d, "f"] & X'' \arrow[r] & \Sigma N \\
& & Y \arrow[d, "h"] & & \\
\Sigma^2 X' \arrow[rr] & & \Sigma^2 X'' \arrow[r, "b"] & \Sigma^2 N \arrow[r] & \Sigma^3 X'.
\end{tikzcd}$$

Note that N is in $\mathcal{D}_{fd}$. We put $\overline{\beta}_{X,Y}(s^{-1}f, t^{-1}g) = \beta_{N,Y'}(fa, bh)$.

Proposition 3.2.16. *Let X be an object of $\mathrm{pr}_{\mathcal{C}}\Gamma \cup \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ and Y be an object of $\mathrm{pr}_{\mathcal{C}}\Gamma$. Then the bifunctorial bilinear form*

$$\overline{\beta}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 X) \longrightarrow k.$$

is non-degenerate. In particular, if one of the two spaces is finite-dimensional, then so is the other.

Proof Let X and Y be objects in $\mathrm{pr}_{\mathcal{C}}\Gamma \cup \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ and in $\mathrm{pr}_{\mathcal{C}}\Gamma$, respectively. In view of Proposition 3.2.10, there exist lifts $\overline{X}$ and $\overline{Y}$ of X and Y in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma \cup \mathrm{pr}_{\mathcal{D}\Gamma}\Sigma^{-1}\Gamma$ and $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$, respectively. In particular, $\overline{X}$ and $\overline{Y}$ lie in $\mathcal{D}_{\leq 1}\Gamma$.

Using the calculus of right-fractions, let $f \circ s^{-1}$ be non-zero a morphism from X to Y in $\mathcal{C} = \mathrm{per}\,\Gamma/\mathcal{D}_{fd}\Gamma$, with $f : \overline{X}' \longrightarrow \overline{Y}$ and $s : \overline{X}' \longrightarrow \overline{X}$ morphisms in $\mathcal{D}\Gamma$ such that the cone of s is in $\mathcal{D}_{fd}\Gamma$.

If $\overline{X}$ lies in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$, then Lemma 3.2.9 allows us to suppose that $\overline{X}' = \overline{X}$ and $s = id_{\overline{X}}$. If $\overline{X}$ lies in $\mathrm{pr}_{\mathcal{D}\Gamma}\Sigma^{-1}\Gamma$, then Lemma 3.2.15 allows us to suppose that $\overline{X}'$ lies in $\mathcal{D}_{\leq 1}$. In both case, $\overline{X}'$ lies in $\mathcal{D}_{\leq 1}$.

We now use [58, Proposition 2.19] : the (contravariant) functor

$$\begin{array}{rccc} \Phi : & \mathrm{per}\,\Gamma & \longrightarrow & \mathrm{Mod}\,(\mathcal{D}_{fd}(\Gamma)^{op}) \\ & P & \longmapsto & \mathrm{Hom}_{\mathcal{D}\Gamma}(P, ?)|_{\mathcal{D}_{fd}\Gamma} \end{array}$$

is fully faithful. Thus $\Phi(f) \neq 0$, meaning there exist N in $\mathcal{D}_{fd}\Gamma$ and a morphism $h : \overline{Y} \longrightarrow N$ such that its composition with f is non-zero.

Recall from Theorem 3.2.5 that we have a non-degenerate bilinear form

$$\beta : \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-3}N, X) \times \mathrm{Hom}_{\mathcal{D}\Gamma}(X, N) \longrightarrow k,$$

so there exists a morphism $j : \Sigma^{-3}N \longrightarrow X$ such that $\beta(j, h \circ f) \neq 0$.

All the morphisms can be arranged in the following commuting diagram, where the upper and lower row are triangles in $\mathcal{D}\Gamma$.

$$\begin{tikzcd}
\Sigma^{-3}N \arrow[r, "j"] & \overline{X}' \arrow[r, "g"] \arrow[d, "f"'] & X'' \arrow[r] & \Sigma^{-2}N \\
& \overline{Y} \arrow[r, "h"] \arrow[d, dotted, "\ell"'] & N \arrow[d, equal] & \\
\Sigma^2\overline{X}' \arrow[r, "\Sigma^2 g"] & \Sigma^2 X'' \arrow[r] & N \arrow[r, "\Sigma^3 j"] & \Sigma^3\overline{X}'.
\end{tikzcd}$$

We will show the existence of a morphism $\ell : \overline{Y} \longrightarrow \Sigma^2 X''$ making the above diagram commute. Once this is shown, the construction of [2] gives that

$$\overline{\beta}_{X,Y}(fs^{-1}, (\Sigma^2 s)(\Sigma^2 g)^{-1} \circ \ell) \;=\; \overline{\beta}_{X,Y}(f, (\Sigma^2 g)^{-1} \circ \ell) \;=\; \beta(j, h \circ f) \;\neq\; 0,$$

and shows that $\overline{\beta}_{X,Y}$ is non-degenerate (here the first equality follows from the bifunctoriality of the bilinear form, and the second follows from its definition).

The existence of ℓ follows from the fact that $\operatorname{Hom}_{\mathcal{D}\Gamma}(\overline{Y}, \mathcal{D}_{\leq -2}\Gamma) = 0$, so that $(\Sigma^3 j)\circ h = 0$. □

To end this section, we will prove that, in general, $\operatorname{pr}_{\mathcal{C}}\Gamma$ is not equal to the whole cluster category.

Lemma 3.2.17. *Let (Q, W) be a quiver with potential which is not Jacobi-finite. Then $\Sigma^2\Gamma$ is not in $\operatorname{pr}_{\mathcal{C}}\Gamma$.*

PROOF Suppose that $\Sigma^2\Gamma$ lies in $\operatorname{pr}_{\mathcal{C}}\Gamma$. Then, by Proposition 3.2.10, it lifts to an object X in $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$. We have that

$$\operatorname{Hom}_{\mathcal{C}}(\Gamma, \Sigma^2\Gamma) = \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma, X) = \mathrm{H}^0 X.$$

Now, since X and $\Sigma^2\Gamma$ have the same image in $\mathcal{C}$, and since $\mathrm{H}^0\Sigma^2\Gamma$ is zero (and thus finite-dimensional), $\mathrm{H}^0 X$ must be finite-dimensional.

By Proposition 3.2.16, this implies that $\operatorname{Hom}_{\mathcal{C}}(\Gamma, \Gamma)$ is also finite-dimensional, contradicting the hypothesis that $\mathrm{H}^0\Gamma = J(Q, W)$ is of infinite dimension.

Thus $\Sigma^2\Gamma$ cannot be in $\operatorname{pr}_{\mathcal{C}}\Gamma$. □

3.2.8 Mutation of Γ in $\operatorname{pr}_{\mathcal{D}\Gamma}\Gamma$

Recall the equivalence $\widetilde{\mu}_i^+$ of Theorem 3.2.6. Denote by $\widetilde{\mu}_i^-$ the quasi-inverse of the functor $\mathcal{D}\Gamma \longrightarrow \mathcal{D}\Gamma'$ obtained by applying Theorem 3.2.6 to the mutation of $\mu_i(Q, W)$ at the vertex i. Then $\widetilde{\mu}_i^-(\Gamma'_i)$ is isomorphic to the cone of the morphism

$$\Sigma^{-1} \bigoplus_{\alpha} \Gamma_{s(\alpha)} \longrightarrow \Sigma^{-1}\Gamma_i$$

whose components are given by right multiplication by α.

In this subsection, we prove the following theorem, which was first formulated as a "hope" by K. Nagao in his message [66] and which is used in [65]. In a more restrictive setup, an analogous result was obtained in [46, Corollary 5.7].

Theorem 3.2.18. *Let Γ be the complete Ginzburg dg algebra of a quiver with potential (Q, W). Let $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r-1})$ be a sequence of signs. Let $(i_1, \ldots, i_r)$ be an admissible sequence of vertices, and let $T = \bigoplus_{j \in Q_0} T_j$ be the image of $\Gamma^{(r)}$ by the sequence of equivalences*

$$\mathcal{D}\Gamma^{(r)} \xrightarrow{\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}} \ldots \xrightarrow{\widetilde{\mu}_{i_1}^{\varepsilon_1}} \mathcal{D}\Gamma^{(1)} = \mathcal{D}\Gamma.$$

Suppose that T_j lies in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ for all vertices j of Q. Then there exists a sign ε_r such that all summands of the image of $\Gamma^{(r+1)}$ by $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_r}^{\varepsilon_r}$ lie in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$.

We start by proving a result relating morphisms in the cluster category and in the derived category, first proved in [2, Proposition 2.12] in the Hom-finite case.

Proposition 3.2.19. *Let X and Y be objects of $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ such that $\mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma Y)$ is finite-dimensional. Then there is an exact sequence of vector spaces*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y) \longrightarrow D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma X) \longrightarrow 0.$$

The proof of the proposition requires some preparation. First a lemma on limits.

Lemma 3.2.20. *Let (V_i) be an inverse system of finite-dimensional vector spaces with finite-dimensional limit. Then the canonical arrow*

$$\mathrm{colim}\,(DV_i) \longrightarrow D(\lim V_i)$$

is an isomorphism.

Proof This follows by duality from the isomorphisms

$$D\mathrm{colim}\,(DV_i) \cong \lim\,(DDV_i) \cong \lim V_i.$$

□

We can now prove Proposition 3.2.19.

Proof (of Proposition 3.2.19.) Let $X' \longrightarrow X \longrightarrow N \longrightarrow \Sigma X'$ be a triangle in $\mathcal{D}\Gamma$, with X' in $\mathcal{D}_{\leq 0}$ and N in $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$.

By the 3-Calabi–Yau property, $\mathrm{Hom}_{\mathcal{D}\Gamma}(N, \Sigma Y) \cong D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma^2 N)$, and this is zero since $\Sigma^2 N$ is in $\mathcal{D}_{\leq -2}$. Moreover, $\mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}X, \Sigma Y) \cong \mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma^2 Y)$, and this is also zero since $\Sigma^2 Y$ is in $\mathcal{D}_{\leq -2}$.

The above triangle thus gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}\Gamma}(X', \Sigma Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}N, \Sigma Y) \longrightarrow 0.$$

We want to take the colimit of this exact sequence with respect to all morphisms $f'' : X'' \to X$ whose cone is in $\mathcal{D}_{fd} \cap \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$ and with X'' in $\mathcal{D}_{\leq 0}$. The colimit will still be a short exact sequence, since, as we shall prove, all the spaces involved and their colimits are finite-dimensional.

The leftmost term is constant; its colimit is itself.

Consider the rightmost term. Since Y is in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$, there is a triangle

$$P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow \Sigma P_1$$

with P_0 and P_1 in $\mathrm{add}\,\Gamma$. Noticing that $\mathrm{Hom}_{\mathcal{D}\Gamma}(P_i, N) = \mathrm{Hom}_{\mathcal{D}\Gamma}(\mathrm{H}^0 P_i, \mathrm{H}^0 N)$ for $i \in \{1, 2\}$, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(Y, N) \to \mathrm{Hom}_{\mathcal{D}}(\mathrm{H}^0 P_0, \mathrm{H}^0 N) \to \mathrm{Hom}_{\mathcal{D}}(\mathrm{H}^0 P_1, \mathrm{H}^0 N) \to \mathrm{Hom}_{\mathcal{D}}(Y, \Sigma N) \to 0.$$

Since the two middle spaces are finite-dimensional, so are the other two, and the limit of this sequence is still exact.

Now H^0X is an $(\mathrm{End}_{\mathcal{D}\Gamma}\Gamma)$-module. Since $\mathrm{End}_{\mathcal{D}\Gamma}\Gamma$ is the jacobian algebra of a quiver with potential, H^0X is the limit of all its finite-dimensional quotients. The system given by the H^0N is a system of all the finite-dimensional quotients of H^0X; its limit is thus H^0X.

Hence the limit of $\mathrm{Hom}_{\mathcal{D}\Gamma}(\mathrm{H}^0P_i, \mathrm{H}^0N)$ is $\mathrm{Hom}_{\mathcal{D}\Gamma}(\mathrm{H}^0P_i, \mathrm{H}^0X)$, which is isomorphic to $\mathrm{Hom}_{\mathcal{D}\Gamma}(P_i, \mathrm{H}^0X)$. We thus have an exact sequence

$$\mathrm{Hom}_{\mathcal{D}\Gamma}(P_0, \mathrm{H}^0X) \to \mathrm{Hom}_{\mathcal{D}\Gamma}(P_1, \mathrm{H}^0X) \to \lim \mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma N) \to 0.$$

This implies the isomorphisms

$$\lim \mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma N) \cong \mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma \mathrm{H}^0X) \cong \mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma X).$$

Using Lemma 3.2.20, we thus get that $\operatorname{colim} D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma N) = D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma X)$, and the 3-Calabi–Yau property of Theorem 3.2.5 implies that $D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma N)$ is isomorphic to $\mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}N, \Sigma Y)$. Therefore the colimit of the $\mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}N, \Sigma Y)$ is the space $D\,\mathrm{Hom}_{\mathcal{D}\Gamma}(Y, \Sigma X)$ as desired.

It remains to be shown that the colimit of the terms of the form $\mathrm{Hom}_{\mathcal{D}\Gamma}(X', \Sigma Y)$ is $\mathrm{Hom}_{\mathcal{D}\Gamma}(X, \Sigma Y)$. This is exactly Lemma 3.2.15. This finishes the proof of the Proposition. □

This enables us to formulate a result on the lifting of triangles from the cluster category to the derived category.

Proposition 3.2.21. *Let X and Y be objects of $\mathrm{pr}_{\mathcal{C}}\Gamma$, with $\dim \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y) = 1$ (and so $\dim \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma X) = 1$ by Proposition 3.2.16). Let*

$$X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X \text{ and } Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$$

be non-split triangles (they are unique up to isomorphism). Then one of the two triangles lifts to a triangle $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ in $\mathrm{per}\,\Gamma$, with A, B and C in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$.

PROOF According to Proposition 3.2.10, we can lift X and Y to objects $\overline{X}$ and $\overline{Y}$ of $\mathrm{pr}_{\mathcal{D}}\Gamma$. Using the short exact sequence of Proposition 3.2.19, we have that one of $\mathrm{Hom}_{\mathcal{D}\Gamma}(\overline{X}, \Sigma\overline{Y})$ and $\mathrm{Hom}_{\mathcal{D}\Gamma}(\overline{Y}, \Sigma\overline{X})$ is one-dimensional.

Suppose that $\mathrm{Hom}_{\mathcal{D}\Gamma}(\overline{Y}, \Sigma\overline{X})$ is one-dimensional. Let

$$\overline{X} \longrightarrow \overline{E} \longrightarrow \overline{Y} \longrightarrow \Sigma\overline{X}$$

be a non-split triangle. Since $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma = \mathcal{D}_{\leq 0} \cap {}^{\perp}\mathcal{D}_{\leq -2} \cap \mathrm{per}\,\Gamma$ is closed under extensions, $\overline{E}$ lies in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$. Thus the equivalence of Proposition 3.2.10 implies that the triangle descends to a non-split triangle in $\mathcal{C}$. Up to isomorphism, this non-split triangle is $X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X$.

The proof is similar if $\mathrm{Hom}_{\mathcal{D}\Gamma}(\overline{X}, \Sigma\overline{Y})$ is one-dimensional; in this case, the triangle $Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$ is the one which can be lifted. □

We can now prove the main theorem of this subsection.

PROOF (of Theorem 3.2.18.)

Put $i = i_r$. For any vertex $j \neq i$, the image of $\Gamma^{(r+1)}$ by $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}\widetilde{\mu}_{i_r}^{\varepsilon}$ is isomorphic to T_j for any sign ε, and is in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ by hypothesis. Now, the images of

$\Gamma_i^{(r+1)}$ by $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}\widetilde{\mu}_{i_r}^{+}$ and by $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}\widetilde{\mu}_{i_r}^{-}$ become isomorphic in the cluster category $\mathcal{C}_{Q,W}$, and they lie in $\mathrm{pr}_{\mathcal{C}}\Gamma$. Denote these images by T_i^*. We have that $\dim \mathrm{Hom}_{\mathcal{C}}(T_i, \Sigma T_i^*) = 1$. Thus we can apply Proposition 3.2.21 and get that T_i^* is lifted in $\mathrm{pr}_{\mathcal{D}\Gamma}\Gamma$ either to $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}\widetilde{\mu}_{i_r}^{+}(\Gamma_i^{r+1})$ or to $\widetilde{\mu}_{i_1}^{\varepsilon_1}\widetilde{\mu}_{i_2}^{\varepsilon_2}\cdots\widetilde{\mu}_{i_{r-1}}^{\varepsilon_{r-1}}\widetilde{\mu}_{i_r}^{-}(\Gamma_i^{r+1})$. □

3.3 Cluster character

Let $\mathcal{C}$ be a (not necessarily Hom–finite) triangulated category with suspension functor Σ. Let $T = \bigoplus_{i=1}^n T_i$ be a basic rigid object in $\mathcal{C}$ (with each T_i indecomposable), that is, an object T such that $\mathrm{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$ and $i \neq j$ implies that T_i and T_j are not isomorphic. We will assume the following :

1. $\mathrm{pr}_{\mathcal{C}}T$ is a Krull–Schmidt category;
2. $B = \mathrm{End}_{\mathcal{C}}(T)$ is the (completed) Jacobian algebra of a quiver with potential (Q, W);
3. the simple B–module at each vertex can be lifted to an object in $\mathrm{pr}_{\mathcal{C}}(T) \cap \mathrm{pr}_{\mathcal{C}}(\Sigma T)$ through the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$;
4. for all objects X of $\mathrm{pr}_{\mathcal{C}}(\Sigma T) \cup \mathrm{pr}_{\mathcal{C}}(T)$ and Y of $\mathrm{pr}_{\mathcal{C}}(\Sigma T)$, there exists a non-degenerate bilinear form
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 X) \longrightarrow k$$
which is functorial in both variables.

Lemma 3.3.1. *The above hypotheses hold for the cluster category $\mathcal{C}_{Q,W}$ of a quiver with potential (Q, W), where T is taken to be $\Sigma^{-1}\Gamma$.*

Proof Condition (1) is proved in Corollary 3.2.12, since $\mathrm{pr}_{\mathcal{C}}\Gamma$ is equivalent to $\mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$. Condition (2) follows from Proposition 3.2.10, since $\mathrm{End}_{\mathcal{C}}(\Sigma^{-1}\Gamma)$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(\Gamma)$, which is in turn isomorphic to $\mathrm{End}_{\mathcal{D}\Gamma}(\Gamma) = \mathrm{H}^0\Gamma$, and this is the completed Jacobian algebra of (Q, W). Condition (3) follows from the fact that $\mathrm{Hom}_{\mathcal{C}}(\Gamma_i, \Sigma\Gamma_i^*) = \mathrm{Hom}_{\mathcal{D}\Gamma}(\Gamma_i, \Sigma\Gamma_i^*)$ is one-dimensional (see [58, Section 4]). Finally, condition (4) is exactly the contents of Proposition 3.2.16. □

As in [18] and [68], define the *index with respect to T* of an object X of $\mathrm{pr}_{\mathcal{C}}T$ as the element of $\mathrm{K}_0(\mathrm{add}\, T)$ given by
$$\mathrm{ind}_T\, X = [T_0^X] - [T_1^X],$$
where $T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$ is an $(\mathrm{add}\, T)$-presentation of X. One can show as in [68] that the index is well-defined, that is, does not depend on the choice of a presentation. We write $\mathrm{ind}_T\, X = \sum_{i \in Q_0} [\mathrm{ind}_T\, X : T_i][T_i]$, where $[\mathrm{ind}_T\, X : T_i]$ is an integer for all $i \in Q_0$.

3.3.1 Modules

Consider the functors $F = \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \longrightarrow \mathrm{Mod}\, B$ and $G = \mathrm{Hom}_{\mathcal{C}}(-, \Sigma^2 T) : \mathcal{C} \longrightarrow \mathrm{Mod}\, B^{op}$, where $\mathrm{Mod}\, B$ is the category of right B–modules.

For an object U of $\mathcal{C}$, let (U) be the ideal of morphisms in $\mathcal{C}$ factoring through an object of $\mathrm{add}\, U$.

This subsection is devoted to proving some useful properties of the functors F and G.

Lemma 3.3.2. *Let X and Y be objects in $\mathcal{C}$.*

1. *If X lies in* $\mathrm{pr}_{\mathcal{C}}T$*, then F induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)/(\Sigma T) \longrightarrow \mathrm{Hom}_B(FX,FY).$$

If Y lies in $\mathrm{pr}_{\mathcal{C}}\Sigma T$*, then G induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)/(\Sigma T) \longrightarrow \mathrm{Hom}_{B^{op}}(GY,GX).$$

2. *F induces an equivalence of categories*

$$\mathrm{pr}_{\mathcal{C}}T/(\Sigma T) \longrightarrow \mathrm{mod}\, B,$$

where $\mathrm{mod}\, B$ *denotes the category of finitely presented B–modules.*

3. *Any finite-dimensional B–module can be lifted through F to an object in* $\mathrm{pr}_{\mathcal{C}}T \cap \mathrm{pr}_{\mathcal{C}}\Sigma T$*. Any short exact sequence of finite-dimensional B-modules can be lifted through F to a triangle of $\mathcal{C}$, whose three terms are in* $\mathrm{pr}_{\mathcal{C}}T \cap \mathrm{pr}_{\mathcal{C}}\Sigma T$*.*

PROOF (1) We only prove the first isomorphism; the proof of the second one is dual. First, suppose that $X = T_i$ is an indecomposable summand of T. Let $f : FT_i \longrightarrow FY$ be a morphism of B–modules. Note that any element g of $FT_i = \mathrm{Hom}_{\mathcal{C}}(T, T_i)$ is of the form pg', where $p : T \longrightarrow T_i$ is the canonical projection and g' is an endomorphism of T. Hence $f(g) = f(p)g'$. Moreover, consider the idempotent e_i in $\mathrm{End}_{\mathcal{C}}\, T$ associated with T_i. We have that $f(p) = f(pe_i) = f(p)e_i$. Hence $f(p)$ can be viewed as a morphism from T_i to Y, and $f = F(f(p))$. This shows that there is a bijection $\mathrm{Hom}_{\mathcal{C}}(T_i, Y) \longrightarrow \mathrm{Hom}_B(FT_i, FY)$.

One easily sees that this bijection will also hold if X is a direct sum of direct summands of T.

Now, let X be in $\mathrm{pr}_{\mathcal{C}}T$, and let $T_1^X \xrightarrow{\alpha} T_0^X \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma T_1^X$ be a triangle in $\mathcal{C}$, with $T_0^X, T_1^X \in \mathrm{add}\, T$. Let $f : FX \longrightarrow FY$ be a morphism of B–modules. We have that $fF\beta$ belongs to $\mathrm{Hom}_B(FT_0^X, FY)$, and by the above lifts to a morphism $\omega : T_0^X \longrightarrow Y$.

Moreover, $F(\omega\alpha) = F\omega F\alpha = fF\beta F\alpha = 0$, and by injectivity, $\omega\alpha = 0$. Hence there exists $\phi : X \longrightarrow Y$ such that $\phi\beta = \omega$, so $F\phi F\beta = fF\beta$. Since $F\beta$ is surjective, this gives $F\phi = f$. Therefore the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_B(FX, FY)$ is surjective.

Suppose now that $u : X \longrightarrow Y$ is such that $Fu = 0$. Then $F(u\beta) = FuF\beta = 0$, and by the injectivity proved above, $u\beta = 0$, and u factors through ΣT_1^X. This finishes the proof.

(2) It follows from part (1) that the functor is fully faithful. Let now $M \in \mathrm{mod}\, B$, and let $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a projective presentation. By part (1), $P_1 \longrightarrow P_0$ lifts to a morphism $T_1 \longrightarrow T_0$ in $\mathcal{C}$, with $T_0, T_1 \in \mathrm{add}\, T$. We can embed this morphism in a triangle $T_1 \longrightarrow T_0 \longrightarrow X \longrightarrow \Sigma T_1$, and we see that FX is isomorphic to M. This proves the equivalence.

(3) By our hypothesis, the statement is true for the simple modules at each vertex. Let M be a finite-dimensional B–module. According to a remark following Definition 10.1 of [22], M is nilpotent. Therefore it can be obtained from the simple modules by repeated extensions. All we have to do is show that the property is preserved by extensions in $\mathrm{Mod}\, B$.

Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence, with L and N in $\mathrm{mod}\, B$ admitting lifts $\overline{L}$ and $\overline{N}$ in $\mathrm{pr}_{\mathcal{C}}T \cap \mathrm{pr}_{\mathcal{C}}\Sigma T$, respectively. Using projective presentations of L and N, we consctruct one for M and obtain a diagram as below, where the upper two rows are split.

$$\begin{tikzcd}
0 \arrow[r] & P_1^L \arrow[r] \arrow[d] & P_1^L \oplus P_1^N \arrow[r] \arrow[d] & P_1^N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0^L \arrow[r] \arrow[d] & P_0^L \oplus P_0^N \arrow[r] \arrow[d] & P_0^N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Thanks to part (2), the upper left square can be lifted into a commutative diagram

$$\begin{tikzcd}
T_1^L \arrow[r] \arrow[d] & T_1^L \oplus T_1^N \arrow[d] \\
T_0^L \arrow[r] & T_0^L \oplus T_0^N
\end{tikzcd}$$

which in turn embeds in a nine-diagram as follows.

$$\begin{tikzcd}
T_1^L \arrow[r] \arrow[d] & T_1^L \oplus T_1^N \arrow[r] \arrow[d] & T_1^N \arrow[r] \arrow[d] & \Sigma T_1^L \\
T_0^L \arrow[r] \arrow[d] & T_0^L \oplus T_0^N \arrow[r] \arrow[d] & T_0^N \arrow[r] \arrow[d] & \Sigma T_0^L \\
\overline{L} \arrow[r] \arrow[d] & \overline{M} \arrow[r] \arrow[d] & \overline{N} \arrow[r] \arrow[d] & \Sigma \overline{L} \\
\Sigma T_1^L \arrow[r] & \Sigma T_1^L \oplus \Sigma T_1^N \arrow[r] & \Sigma T_1^N &
\end{tikzcd}$$

Hence $\overline{M}$ is a lift of M in $\mathrm{pr}_{\mathcal{C}} T$. Now, since $\overline{N}$ lies in $\mathrm{pr}_{\mathcal{C}}\Sigma T$, it follows from part (1) that the morphism $\Sigma^{-1}\overline{N} \longrightarrow \overline{L}$ is in (ΣT), and thus from Lemma 3.3.4 below that $\overline{M}$ is also in $\mathrm{pr}_{\mathcal{C}}\Sigma T$. This finishes the proof. □

Lemma 3.3.3. *Let X, Y and Z be objects in $\mathcal{C}$. Suppose that Y and Z lie in $\mathrm{pr}_{\mathcal{C}}\Sigma T$ and that FY is finite-dimensional. Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X$$

be a triangle. If $Ff = 0$, then $f \in (\Sigma T)$.

PROOF The equality $Ff = 0$ means that Fg is injective. Using the non-degenerate bilinear form, we get a commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(T, Y) \arrow[r, hook, "Fg"] \arrow[d, hook] & \mathrm{Hom}_{\mathcal{C}}(T, Z) \arrow[d, hook] \\
D\,\mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 T) \arrow[r, "DGg"] & D\,\mathrm{Hom}_{\mathcal{C}}(Z, \Sigma^2 T),
\end{tikzcd}$$

where the top horizontal morphism and the two vertical ones are injective. Since FY is finite-dimensional, the left morphism is an isomorphism. Thus DGg is injective, and Gg is surjective. But this means that $Gf = 0$, and by part (1) of Lemma 3.3.2, $f \in (\Sigma T)$. $\square$

3.3.2 Presentations and index

Let us now study some closure properties of $\mathrm{pr}_{\mathcal{C}}T$, and deduce some relations between triangles and indices.

Lemma 3.3.4. *Let* $X \longrightarrow Y \longrightarrow Z \xrightarrow{\varepsilon} \Sigma X$ *be a triangle in* $\mathcal{C}$ *such that* ε *is in* (ΣT). *Then*

1. *If two of* X, Y *and* Z *lie in* $\mathrm{pr}_{\mathcal{C}}T$, *then so does the third one.*
2. *If* $X, Y, Z \in \mathrm{pr}_{\mathcal{C}}T$, *then we have an equality* $\mathrm{ind}_T X + \mathrm{ind}_T Z = \mathrm{ind}_T Y$.

PROOF Let us first suppose that X and Z lie in $\mathrm{pr}_{\mathcal{C}}T$. Let $T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$ and $T_1^Z \longrightarrow T_0^Z \longrightarrow Z \longrightarrow \Sigma T_1^Z$ be two triangles, with T_0^X, T_1^X, T_0^Z and T_1^Z in $\mathrm{add}\, T$.

Since $\mathrm{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$, the composition $T_0^Z \longrightarrow Z \longrightarrow \Sigma X$ vanishes, so $T_0^Z \longrightarrow Z$ factors through Y. This gives a commutative square

$$\begin{tikzcd}
T_0^X \oplus T_0^Z \arrow[r] \arrow[d] & T_0^Z \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

which can be completed into a nine-diagram

$$\begin{tikzcd}
T_1^X \arrow[r] \arrow[d] & T_1^X \oplus T_1^Z \arrow[r] \arrow[d] & T_1^Z \arrow[r] \arrow[d] & \Sigma T_1^X \\
T_0^X \arrow[r] \arrow[d] & T_0^X \oplus T_0^Z \arrow[r] \arrow[d] & T_0^Z \arrow[r] \arrow[d] & \Sigma T_0^X \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \\
\Sigma T_1^X & \Sigma T_1^X \oplus \Sigma T_1^Z & \Sigma T_1^Z &
\end{tikzcd}$$

showing that Y is in $\mathrm{pr}_{\mathcal{C}}T$ and that assertion 2 is true.

Now suppose that X and Y lie in $\mathrm{pr}_{\mathcal{C}}T$. Since the composition $Z \longrightarrow \Sigma X \longrightarrow \Sigma^2 T_1^X$ is zero, the morphism $Z \longrightarrow \Sigma X$ factors through ΣT_0^X. This yields an octahedron

$$\begin{tikzcd}
 & & \Sigma Y \arrow[ddl, "+"] \arrow[dr, dotted] & & \\
\Sigma W \arrow[urr] \arrow[dr, "+"] & & & \Sigma^2 T_1^X \arrow[lll, dotted, "+"] \arrow[ddl, dotted, "+"] & \\
 & Z \arrow[rrr] \arrow[dr] & & & \Sigma X \arrow[uull] \arrow[ul, dotted] \\
 & & \Sigma T_0^X \arrow[uull] \arrow[urr] & &
\end{tikzcd}$$

which produces a triangle $T_1^X \longrightarrow W \longrightarrow Y \longrightarrow \Sigma T_1^X$. Composing with $Y \longrightarrow \Sigma T_1^Y$, we get a second octahedron

$$\begin{tikzcd}
 & & \Sigma U \arrow[ddl, "+"] \arrow[dr, dotted] & & \\
\Sigma T_1^X \arrow[urr] \arrow[dr, "+"] & & & \Sigma T_0^Y \arrow[lll, dotted, "+"] \arrow[ddl, dotted, "+"] & \\
 & W \arrow[rrr] \arrow[dr] & & & \Sigma T_1^Y \arrow[uull] \arrow[ul, dotted] \\
 & & Y \arrow[uull] \arrow[urr] & &
\end{tikzcd}$$

which gives triangles $T_1^X \longrightarrow U \longrightarrow T_0^Y \longrightarrow \Sigma T_1^X$ and $T_1^Y \longrightarrow U \longrightarrow W \longrightarrow \Sigma T_1^Y$. Note that, since $\mathrm{Hom}_{\mathcal{C}}(T, \Sigma T) = 0$, the first triangle is split, so U is isomorphic to $T_1^X \oplus T_0^Y$.

From the first octahedron, one gets a triangle $T_0^X \longrightarrow W \longrightarrow Z \longrightarrow \Sigma T_0^X$. Construct one last octahedron with the composition $U \longrightarrow W \longrightarrow Z$.

$$\begin{tikzcd}
 & & \Sigma V \arrow[ddl, "+"] \arrow[dr, dotted] & & \\
\Sigma T_1^Y \arrow[urr] \arrow[dr, "+"] & & & \Sigma T_0^X \arrow[lll, dotted, "+"] \arrow[ddl, dotted, "+"] & \\
 & U \arrow[rrr] \arrow[dr] & & & Z \arrow[uull] \arrow[ul, dotted] \\
 & & W \arrow[uull] \arrow[urr] & &
\end{tikzcd}$$

As was the case for U, V is in a split triangle, and is thus isomorphic to $T_1^Y \oplus T_0^X$. Hence there is a triangle $V \longrightarrow U \longrightarrow Z \longrightarrow \Sigma V$, with U and V in $\mathrm{add}\, T$. This proves that Z lies in $\mathrm{pr}_{\mathcal{C}} T$.

Finally, suppose that Y and Z are in $\mathrm{pr}_{\mathcal{C}} T$. Notice that since $\Sigma^{-1}\varepsilon$ factors through $\mathrm{add}\, T$, the composition $\Sigma^{-1} T_0^Z \longrightarrow \Sigma^{-1} Z \longrightarrow X$ vanishes. Applying a reasonning dual to that of the preceding case, one proves that X lies in $\mathrm{pr}_{\mathcal{C}} T$. $\square$

The next lemma is an adapted version of Proposition 6 of [68].

Lemma 3.3.5. *Let* $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ *be a triangle in* $\mathcal{C}$*, with* $X, Z \in \mathrm{pr}_{\mathcal{C}} T$ *such that* $\operatorname{Coker} F\beta$ *is finite-dimensional. Let* $C \in \mathrm{pr}_{\mathcal{C}} T \cap \mathrm{pr}_{\mathcal{C}} \Sigma T$ *be such that* $FC = \operatorname{Coker} F\beta$*. Then* $Y \in \mathrm{pr}_{\mathcal{C}} T$*, and* $\mathrm{ind}_T X + \mathrm{ind}_T Z = \mathrm{ind}_T Y + \mathrm{ind}_T C + \mathrm{ind}_T \Sigma^{-1} C$*.*

PROOF Note that since $\operatorname{Coker} F\beta$ is finite-dimensional, it can be lifted to $C \in \mathrm{pr}_{\mathcal{C}} T \cap \mathrm{pr}_{\mathcal{C}} \Sigma T$ thanks to Lemma 3.3.2.

The case where γ factors through $\mathrm{add}\, \Sigma T$ was treated in Lemma 3.3.4. In that case, $\operatorname{Coker} F\beta = 0$, and $C \in \mathrm{add}\, \Sigma T$, so that $\mathrm{ind}_T C = -\,\mathrm{ind}_T \Sigma^{-1} C$.

Suppose now that γ is not in (ΣT). In $\mathrm{mod}\, B$, there is a commutative triangle

$$\begin{tikzcd} FZ \arrow[rr] \arrow[dr] & & F\Sigma X \\ & \operatorname{Coker} F\beta \arrow[ur] & \end{tikzcd}$$

which, thanks to Lemma 3.3.2, we can lift to a commutative triangle

$$\begin{tikzcd} Z \arrow[rr, "a"] \arrow[dr, "b"'] & & \Sigma X \\ & C \oplus \Sigma\overline{T} \arrow[ur, "c"'] & \end{tikzcd}$$

in $\mathcal{C}$, where $\overline{T}$ lies in $\mathrm{add}\, T$. Form an octahedron

$$\begin{tikzcd} & & \Sigma Y \arrow[ddl, "+"] \arrow[dr, dotted] & & \\ U \arrow[urr] \arrow[dr, "+"] & & & U' \arrow[llll, dotted, "+"'] \arrow[ddl, dotted, "+"] & \\ & Z \arrow[rrr] \arrow[dr] & & & \Sigma X. \arrow[uull] \arrow[ul, dotted] \\ & & C \oplus \Sigma\overline{T} \arrow[uull] \arrow[urr] & & \end{tikzcd}$$

Since Fb is an epimorphism, the morphism $C \oplus \Sigma\overline{T} \longrightarrow U$ must lie in (ΣT), by Lemma 3.3.2, part (1). Since Fc is a monomorphism, the same must hold for $\Sigma^{-1}U' \longrightarrow C \oplus \Sigma\overline{T}$, by Lemma 3.3.3. By composition, the morphism $\Sigma^{-1}U' \longrightarrow U$ is also in (ΣT).

We thus have three triangles

$$\Sigma^{-1}U \longrightarrow Z \longrightarrow C \oplus \Sigma\overline{T} \longrightarrow U$$

$$\Sigma^{-1}C \oplus \overline{T} \longrightarrow X \longrightarrow \Sigma^{-1}U' \longrightarrow C \oplus \Sigma\overline{T}$$

$$\Sigma^{-1}U \longrightarrow Y \longrightarrow \Sigma^{-1}U' \longrightarrow U$$

whose third morphism factors through ΣT. Applying Lemma 3.3.4, we get that $\Sigma^{-1}U$, $\Sigma^{-1}U'$ and Y are in $\mathrm{pr}_{\mathcal{C}} T$, and that

$$\mathrm{ind}_T \Sigma^{-1}U + \mathrm{ind}_T C + \mathrm{ind}_T \Sigma\overline{T} = \mathrm{ind}_T Z,$$

$$\mathrm{ind}_T \Sigma^{-1}C + \mathrm{ind}_T \overline{T} + \mathrm{ind}_T \Sigma^{-1}U' = \mathrm{ind}_T X, \text{ and}$$

$$\operatorname{ind}_T Y = \operatorname{ind}_T \Sigma^{-1}U + \operatorname{ind}_T \Sigma^{-1}U'.$$

Summing up, and noticing that $\operatorname{ind}_T \overline{T} = -\operatorname{ind}_T \Sigma\overline{T}$, we get the desired equality. □

Lemma 3.3.6. *Let X be an object in $\operatorname{pr}_{\mathcal{C}}T \cap \operatorname{pr}_{\mathcal{C}}\Sigma T$ such that FX is finite-dimensional. Then the sum $\operatorname{ind}_T X + \operatorname{ind}_T \Sigma^{-1}X$ only depends on the dimension vector of FX.*

PROOF First, notice that $FX = 0$ if, and only if, X is in $\operatorname{add}\Sigma T$.

Second, suppose X is indecomposable. If Y is another such object such that FX and FY are isomorphic and non-zero, then X and Y are isomorphic in $\mathcal{C}$. Indeed, in view of Lemma 3.3.2, part (2), there exist morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_X + t$ and $g \circ f = id_Y + t'$, with $t, t' \in (\Sigma T)$. But since the endomorphism rings of X and Y are local and not contained in (ΣT), this implies that $f \circ g$ and $g \circ f$ are isomorphisms.

Third, let us show that the sum depends only on the isomorphism class of FX. Let Y be another such object such that FX and FY are isomorphic. Write $X = \overline{X} \oplus \Sigma T^X$ and $Y = \overline{Y} \oplus \Sigma T^Y$, where $T^X, T^Y \in \operatorname{add} T$ and $\overline{X}, \overline{Y}$ have no direct summand in $\operatorname{add}\Sigma T$. Then $F\overline{X} = F\overline{Y}$, and by the above $\overline{X}$ and $\overline{Y}$ are isomorphic. We have

$$\begin{aligned}
\operatorname{ind}_T X + \operatorname{ind}_T \Sigma^{-1}X &= \operatorname{ind}_T \overline{X} + \operatorname{ind}_T \Sigma^{-1}\overline{X} + \operatorname{ind}_T \Sigma T^X + \operatorname{ind}_T T^X \\
&= \operatorname{ind}_T \overline{X} + \operatorname{ind}_T \Sigma^{-1}\overline{X} \\
&= \operatorname{ind}_T \overline{Y} + \operatorname{ind}_T \Sigma^{-1}\overline{Y} + \operatorname{ind}_T \Sigma T^Y + \operatorname{ind}_T T^Y \\
&= \operatorname{ind}_T Y + \operatorname{ind}_T \Sigma^{-1}Y.
\end{aligned}$$

Finally, we prove that the sum only depends on the dimension vector of FX. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $\operatorname{mod} B$. As in the proof of part (3) of Lemma 3.3.2, lift it to a triangle $\overline{L} \longrightarrow \overline{M} \longrightarrow \overline{N} \longrightarrow \Sigma\overline{L}$, where the last morphism in in $(\Sigma T) \cap (\Sigma^2 T)$. Using Lemma 3.3.4, we get the equality

$$\operatorname{ind}_T \overline{M} + \operatorname{ind}_T \Sigma^{-1}\overline{M} = \operatorname{ind}_T \overline{L} + \operatorname{ind}_T \Sigma^{-1}\overline{L} + \operatorname{ind}_T \overline{N} + \operatorname{ind}_T \Sigma^{-1}\overline{N}.$$

This gives the independance on the dimension vector. □

Notation 3.3.7. For a dimension vector e, denote by $\iota(e)$ the sum $\operatorname{ind}_T X + \operatorname{ind}_T \Sigma^{-1}X$, where $\underline{\dim}\, FX = e$ (by the above Lemma, this does not depend on the choice of such an X).

Lemma 3.3.8. *If $X \in \operatorname{pr}_C(T)$ and $Y \in \operatorname{pr}_C(\Sigma T)$, then the bilinear form induces a non-degenerate bilinear form*

$$(\Sigma T)(X, Y) \times \operatorname{Hom}_{\mathcal{C}}(Y, \Sigma^2 X)/(\Sigma^2 T) \longrightarrow k.$$

PROOF Let $T_1^X \longrightarrow T_0^X \longrightarrow X \xrightarrow{\eta} \Sigma T_1^X$ be a triangle, with T_0^X, T_1^X in $\operatorname{add} T$. Consider the following diagram.

$$\begin{array}{ccccc}
\operatorname{Hom}_C(X, Y) & \times & \operatorname{Hom}_C(Y, \Sigma^2 X) & \longrightarrow & k \\
\uparrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \Sigma^2\eta_*} & & \\
\operatorname{Hom}_C(\Sigma T_1^X, Y) & \times & \operatorname{Hom}_C(Y, \Sigma^3 T_1^X) & \longrightarrow & k
\end{array}$$

The bifunctoriality of the bilinear form (call it β) implies that for each morphism f in $\mathrm{Hom}_{\mathcal{C}}(\Sigma T_1^X, Y)$ and each g in $\mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 X)$, $\beta(\eta^* f, g) = \beta(f, \Sigma^2 \eta_* g)$.

As a consequence, there is an induced non-degenerate bilinear form

$$\mathrm{Im}\ \eta^* \times \mathrm{Im}\ \Sigma^2\eta_* \longrightarrow k.$$

Since $\mathrm{Im}\ \eta^*$ is isomorphic to $(\Sigma T)(X, Y)$, and since $\mathrm{Im}\ \Sigma^2\eta_*$ is in turn isomorphic to $\mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^2 X)/(\Sigma^2 T)$, we get the desired result.

□

3.3.3 Cluster character : definition

In [68], Y. Palu defined the notion of *cluster character* for a Hom–finite 2-Calabi–Yau triangulated category with a cluster-tilting object. In our context, the category $\mathcal{C}$ is not Hom–finite nor 2-Calabi–Yau, and the object T is only assumed to be rigid. However, the definition can be adapted to this situation as follows.

Definition 3.3.9. Let $\mathcal{C}$ be a triangulated category and T be a rigid object as above. The category $\mathcal{D}$ is the full subcategory of $\mathrm{pr}_{\mathcal{C}}T \cap \mathrm{pr}_{\mathcal{C}}\Sigma T$ whose objects are those X such that FX is a finite-dimensional B–module.

Under the hypotheses of this Section, the subcategory $\mathcal{D}$ is Krull–Schmidt and stable under extensions. Moreover, in the special case where $\mathcal{C} = \mathcal{C}_{Q,W}$ and $T = \Sigma^{-1}\Gamma$, the subcategory $\mathcal{D}$ does not depend on the mutation class of T; that is, replacing Γ by any $\mu_r \ldots \mu_1\Gamma$ in the definition of T yields the same subcategory $\mathcal{D}$ (this is a consequence of the nearly Morita equivalence of [58, Corollary 4.6] and of Corollary 3.2.8).

The subcategory $\mathcal{D}$ allows us to extend the notion of cluster characters.

Definition 3.3.10. Let $\mathcal{C}$ be a triangulated category and T be a rigid object as above. Let $\mathcal{D}$ be as in Definition 3.3.9.

A *cluster character* on $\mathcal{C}$ (with respect to T) with values in a commutative ring A is a map

$$\chi : obj(\mathcal{D}) \longrightarrow A$$

satisfying the following conditions :

– if X and Y are two isomorphic objects in $\mathcal{D}$, then we have $\chi(X) = \chi(Y)$;

– for all objects X and Y of $\mathcal{D}$, $\chi(X \oplus Y) = \chi(X)\chi(Y)$;

– (multiplication formula) for all objects X and Y of $\mathcal{D}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 1$, the equality

$$\chi(X)\chi(Y) = \chi(E) + \chi(E')$$

holds, where $X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X$ and $Y \longrightarrow E' \longrightarrow X \longrightarrow \Sigma Y$ are non split triangles.

Note that $\mathrm{Ext}^1_{\mathcal{C}}(Y, X)$ is one-dimensional, thanks to the non-degenerate bilinear form. Also note that E and E' are in $\mathcal{D}$, thanks to Lemma 3.3.5, so $\chi(E)$ and $\chi(E')$ are defined.

Remark 3.3.11. If the category $\mathcal{C}$ happens to be Hom–finite and 2-Calabi–Yau, and if T is a cluster-tilting object, then this definition is equivalent to the one given in [68]. Indeed, in that case, it was shown in [57], Proposition 2.1, that $\mathrm{pr}_{\mathcal{C}}T = \mathrm{pr}_{\mathcal{C}}\Sigma T = \mathcal{C}$, so $\mathcal{D} = \mathcal{C}$.

Let $\mathcal{D}$ be as in Definition 3.3.10 and ι be as in Notation 3.3.7. Define the map

$$X'_? : obj(\mathcal{D}) \longrightarrow \mathbb{Q}(x_1, x_2, \ldots, x_n)$$

as follows : for any object X of $\mathcal{D}$, put

$$X'_X = x^{\mathrm{ind}_T \Sigma^{-1} X} \sum_e \chi\Big(\mathrm{Gr}_e(FX)\Big) x^{-\iota(e)}.$$

Here, χ is the Euler–Poincaré characteristic.

This definition is exactly the one in [68] and is a "shift" of the one in [32], both of which were given in a Hom-finite setup.

Theorem 3.3.12. *The map $X'_?$ defined above is a cluster character on $\mathcal{C}$ with respect to T.*

It is readily seen that the first two conditions of Definition 3.3.10 are satisfied by $X'_?$. We thus need to show that the multiplication formula holds in order to prove Theorem 3.3.12.

3.3.4 Dichotomy

This subsection mimics Section 4 of [68]. Our aim here is to prove the following *dichotomy* phenomenon.

Let X and Y be objects of $\mathcal{D}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(X,Y) = 1$. This implies that $\dim \mathrm{Ext}^1_{\mathcal{C}}(Y,X) = 1$. Let

$$X \xrightarrow{i} E \xrightarrow{p} Y \xrightarrow{\varepsilon} \Sigma X$$

$$Y \xrightarrow{i'} E' \xrightarrow{p'} X \xrightarrow{\varepsilon'} \Sigma Y$$

be non-split triangles. Recall that Lemma 3.3.5 implies that E and E' are in $\mathcal{D}$.

Let U and V be submodules of FX and FY, respectively. Define

$$G_{U,V} = \Big\{ W \in \bigcup_e \mathrm{Gr}_e(FE) \;\Big|\; (Fi)^{-1}(W) = U,\ Fp(W) = V \Big\} \text{ and}$$

$$G'_{U,V} = \Big\{ W \in \bigcup_e \mathrm{Gr}_e(FE') \;\Big|\; (Fi')^{-1}(W) = V,\ Fp'(W) = U \Big\}.$$

Proposition 3.3.13 (Dichotomy)**.** *Let U and V be as above. Then exactly one of $G_{U,V}$ and $G'_{U,V}$ is non-empty.*

In order to prove this Proposition, a few lemmata are needed.

Using Lemma 3.3.2, lift the inclusions $U \subseteq FX$ and $V \subseteq FY$ to morphisms $i_U : \overline{U} \longrightarrow X$ and $i_V : \overline{V} \longrightarrow Y$, where $\overline{U}$ and $\overline{V}$ lie in $\mathrm{pr}_{\mathcal{C}} T \cap \mathrm{pr}_{\mathcal{C}} \Sigma T$. Keep these notations for the rest of this Section and for the next.

The first lemma is about finiteness.

Lemma 3.3.14. *Let X and $\overline{U}$ be as above. Let M be an object of $\mathcal{C}$ such that FM and $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma M)$ are finite-dimensional. Then $\mathrm{Hom}_{\mathcal{C}}(\overline{U}, \Sigma M)$ is also finite-dimensional.*

PROOF Embed i_U in a triangle $\Sigma^{-1}X \xrightarrow{\alpha} Z \xrightarrow{\beta} \overline{U} \xrightarrow{i_U} X$. From this triangle, one gets the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, \Sigma M) \xrightarrow{i_U^*} \mathrm{Hom}_{\mathcal{C}}(\overline{U}, \Sigma M) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathcal{C}}(Z, \Sigma M).$$

The image of β^* is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(\overline{U}, \Sigma M)/\operatorname{Im} i_U^*$. Since $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma M)$ is finite-dimensional by hypothesis, it suffices to show that $\operatorname{Im} \beta^*$ is finite-dimensional to prove the Lemma.

Since $\Sigma^{-1}X$ and $\overline{U}$ are in $\mathrm{pr}_{\mathcal{C}}T$ and $\operatorname{Coker} F\beta$ is finite-dimensional, Lemma 3.3.5 can be applied to get that Z is in $\mathrm{pr}_{\mathcal{C}}T$. Let $T_1^Z \longrightarrow T_0^Z \longrightarrow Z \longrightarrow \Sigma T_1^Z$ be a triangle, with T_0^Z and T_1^Z in $\mathrm{pr}_{\mathcal{C}}T$.

Now, Fi_U is a monomorphism, so $F\beta = 0$, and Lemma 3.3.2 implies that β lies in (ΣT). Therefore $\operatorname{Im} \beta^*$ is contained in $(\Sigma T)(Z, \Sigma M)$. It is thus sufficient to show that the latter is finite-dimensional.

We have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(\Sigma T_1^Z, \Sigma M) \xrightarrow{\gamma} \mathrm{Hom}_{\mathcal{C}}(Z, \Sigma M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T_0^Z, \Sigma M).$$

Since T is rigid, we have that $(\Sigma T)(Z, \Sigma M) = \operatorname{Im} \gamma$. It is thus finite-dimensional. Indeed, $FM = \mathrm{Hom}_{\mathcal{C}}(T, M)$ is finite-dimensional, and this implies that the same property holds for $\mathrm{Hom}_{\mathcal{C}}(\Sigma T_1^Z, \Sigma M)$. This finishes the proof. □

The second lemma is a characterization in $\mathcal{C}$ of the non-emptiness of $G_{U,V}$. It is essentially [68, Lemma 14], where the proof differs in that not every object lies in $\mathrm{pr}_{\mathcal{C}}T$.

Lemma 3.3.15. *With the above notations, the following are equivalent:*

1. *$G_{U,V}$ is non-empty;*
2. *there exist morphisms $e : \Sigma^{-1}\overline{V} \longrightarrow \overline{U}$ and $f : \Sigma^{-1}Y \longrightarrow \overline{U}$ such that*
   - *(a) $(\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) = i_U e$*
   - *(b) $e \in (T)$*
   - *(c) $i_U f - \Sigma^{-1}\varepsilon \in (\Sigma T)$;*
3. *condition (2) where, moreover, $e = f\Sigma^{-1}i_V$.*

PROOF Let us first prove that (2) implies (1). The commutative square given by (a) gives a morphism of triangles

$$\begin{tikzcd} \overline{U} \arrow[r] \arrow[d, "i_U"] & \overline{W} \arrow[r] \arrow[d, "\phi"] & \overline{V} \arrow[r, "\Sigma e"] \arrow[d, "i_V"] & \Sigma\overline{U} \arrow[d, "\Sigma i_U"] \\ X \arrow[r, "i"] & E \arrow[r, "p"] & Y \arrow[r, "\varepsilon"] & \Sigma X. \end{tikzcd}$$

Applying the functor F, we get a commutative diagram in $\operatorname{mod} B$:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "Fi_U"] & F\overline{W} \arrow[r] \arrow[d, "F\phi"] & V \arrow[r, "\Sigma e"] \arrow[d, "Fi_V"] & 0 \\ FX \arrow[r, "Fi"] & FE \arrow[r, "Fp"] & FY. \end{tikzcd}$$

An easy diagram chasing shows that the image of $F\phi$ is in $G_{U,V}$, using the morphism f.

Let us now prove that (1) implies (2). Let W be in $G_{U,V}$. Notice that U contains $\operatorname{Ker} Fi = \operatorname{Im} F\Sigma^{-1}\varepsilon$, so $F\Sigma^{-1}\varepsilon$ factors through U. Since $\Sigma^{-1}Y \in \operatorname{pr}_{\mathcal{C}}T$, Lemma 3.3.2 allows us to find a lift $f : \Sigma^{-1}Y \longrightarrow \overline{U}$ of this factorization such that $i_U f - \Sigma^{-1}\varepsilon \in (\Sigma T)$.

Let us define e. Since $\overline{V} \in \operatorname{pr}_{\mathcal{C}}T$, there exists a triangle

$$T_1^V \longrightarrow T_0^V \longrightarrow \overline{V} \longrightarrow \Sigma T_1^V \ .$$

Since FT_0^V is projective, and since $W \xrightarrow{Fp} V$ is surjective, $FT_0^V \longrightarrow V$ factors through Fp. Composing with the inclusion of W in FE, we get a commutative square

$$\begin{tikzcd}
FT_0^V \arrow[r] \arrow[d] & V \arrow[d] \\
FE \arrow[r] & FY
\end{tikzcd}$$

which lifts to a morphism of triangles (thanks to Lemma 3.3.2)

$$\begin{tikzcd}
\Sigma^{-1}\overline{V} \arrow[r] \arrow[d] & T_1^V \arrow[r] \arrow[d] & T_0^V \arrow[r] \arrow[d] & \overline{V} \arrow[d] \\
\Sigma^{-1}Y \arrow[r] & X \arrow[r] & E \arrow[r] & Y.
\end{tikzcd}$$

Now $FT_0^V \longrightarrow FE$ factors through W, and since $U = (Fi)^{-1}(W)$, then the image of $FT_1^V \longrightarrow FX$ is contained in U. Thus $T_1^V \longrightarrow X$ factors through $\overline{U}$, and we take e to be the composition $\Sigma^{-1}V \longrightarrow T_1^V \longrightarrow U$. By construction, conditions (a) and (b) are satisfied.

Obviously, (3) implies (2). Let us show that (2) implies (3).

First, since $(\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) = i_U e$ and $i_U f\Sigma^{-1}i_V - (\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) \in (\Sigma T)$, we get that $i_U(f\Sigma^{-1}i_V - e) \in (\Sigma T)$. Since Fi_U is a monomorphism, and since $\Sigma^{-1}V \in \operatorname{pr}_{\mathcal{C}}T$, we get that $h := f\Sigma^{-1}i_V - e \in (\Sigma T)$.

Embed the morphism $\Sigma^{-1}i_V$ into a triangle

$$\Sigma^{-1}\overline{V} \xrightarrow{\Sigma^{-1}i_V} \Sigma^{-1}Y \longrightarrow C \longrightarrow \overline{V} \ .$$

Using Lemma 3.3.5, we see that C lies in $\operatorname{pr}_{\mathcal{C}}T$, and since Fi_V is a monomorphism, this implies that $C \longrightarrow \overline{V}$ lies in (ΣT), by Lemma 3.3.2.

Now, $h \in (\Sigma T)$ and $\Sigma^{-1}C \longrightarrow \Sigma^{-1}\overline{V} \in (T)$, so their composition vanishes. Therefore there exists a morphism $\Sigma^{-1}Y \xrightarrow{\ell} \overline{U}$ such that $\ell\Sigma^{-1}i_V = h$.

Since $\overline{V}$ is in $\operatorname{pr}_{\mathcal{C}}\Sigma T$, there is a triangle $T_V^1 \longrightarrow T_V^0 \longrightarrow \Sigma^{-1}\overline{V} \longrightarrow \Sigma T_V^1$. Now, since $\Sigma^{-1}C \longrightarrow \Sigma^{-1}\overline{V} \in (T)$ and $\ell\Sigma^{-1}i_V \in (\Sigma T)$, we have morphisms of triangles

$$\begin{tikzcd}
\Sigma^{-1}C \arrow[r] \arrow[d] & \Sigma^{-1}\overline{V} \arrow[r, "\Sigma^{-1}i_V"] \arrow[d, equal] & \Sigma^{-1}Y \arrow[r, "c"] \arrow[d, "v"] & C \arrow[d] \\
T_V^0 \arrow[r, "u"] \arrow[d] & \Sigma^{-1}\overline{V} \arrow[r] \arrow[d, "\Sigma^{-1}i_V"] & \Sigma T_V^1 \arrow[r] \arrow[d, "w"] & \Sigma T_V^0 \arrow[d] \\
\Sigma^{-1}C' \arrow[r] & \Sigma^{-1}Y \arrow[r, "\ell"] & \overline{U} \arrow[r] & C'.
\end{tikzcd}$$

Since the composition $(\ell - wv)\Sigma^{-1}i_V$ vanishes, there exists a morphism ℓ' from C to $\overline{U}$ such that $\ell' c = \ell - wv$.

Put $f_0 = f - wv$. Then

$$f_0\Sigma^{-1}i_V = f\Sigma^{-1}i_V - \ell\Sigma^{-1}i_V + \ell' c\Sigma^{-1}i_V = f\Sigma^{-1}i_V - h + 0 = e.$$

Moreover, $i_U f_0 - \Sigma^{-1}\varepsilon = (i_U f - \Sigma^{-1}\varepsilon) - wv \in (\Sigma T)$. This finishes the proof. □

PROOF (of Proposition 3.3.13) The proof is similar to that of Proposition 15 of [68]. Consider the linear map

$$\begin{array}{lcl} \alpha : (\Sigma^{-1}Y, \overline{U}) \oplus (\Sigma^{-1}Y, X) & \to & (\Sigma^{-1}\overline{V}, X) \oplus (\Sigma^{-1}\overline{V}, \overline{U})/(T) \oplus (\Sigma^{-1}Y, X)/(\Sigma T) \\ \qquad (x, y) & \mapsto & (y(\Sigma^{-1}i_V) - i_U x(\Sigma^{-1}i_V),\ x\Sigma^{-1}i_V, i_U x - y), \end{array}$$

where we write (X, Y) instead of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then $f \in \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}Y, \overline{U})$ satisfies condition (3) of Lemma 3.3.15 if, and only if, $(f, \Sigma^{-1}\varepsilon)$ is in $\mathrm{Ker}\,\alpha$. Since $\mathrm{Hom}_{\mathcal{C}}(Y, \Sigma^{-1}X)$ is one-dimensional, the existence of such an f is equivalent to the statement that the map

$$\beta : \mathrm{Ker}\,\alpha \hookrightarrow (\Sigma^{-1}Y, \overline{U}) \oplus (Y, \Sigma X) \twoheadrightarrow (Y, \Sigma X)$$

does not vanish, where the second map is the canonical projection.

Now, the emptiness of $G_{U,V}$ is equivalent to the vanishing of β, which is equivalent to the vanishing of its dual

$$D\beta : D(Y, \Sigma X) \hookrightarrow D(\Sigma^{-1}Y, \overline{U}) \oplus D(Y, \Sigma X) \twoheadrightarrow \mathrm{Coker}\, D\alpha,$$

which is in turn equivalent to the fact that any element of the space $D(\Sigma^{-1}Y, \overline{U}) \oplus D(Y, \Sigma X)$ of the form $(0, z)$ lies in Im $D\alpha$.

Using Lemma 3.3.14 and the non-degenerate bilinear form, we see that all five spaces involved in the definition of α are finite-dimensional. Therefore, Lemma 3.3.8 yields the following commutative diagram, where the vertical morphisms are componentwise isomorphisms :

$$\begin{tikzcd} D(\Sigma^{-1}\overline{V}, X) \oplus D(\Sigma^{-1}\overline{V}, \overline{U})/(T) \oplus D(\Sigma^{-1}Y, X)/(\Sigma T) \arrow[r, "D\alpha"] & D(\Sigma^{-1}Y, \overline{U}) \oplus D(\Sigma^{-1}Y, X) \\ (X, \Sigma\overline{V}) \oplus (\Sigma T)(\overline{U}, \Sigma\overline{V}) \oplus (\Sigma^2 T)(X, \Sigma Y) \arrow[u] \arrow[r, "\alpha'"] & (\overline{U}, \Sigma Y) \oplus (X, \Sigma Y). \arrow[u] \end{tikzcd}$$

Here, the action of α' is given by $\alpha'(x, y, z) = ((\Sigma i_V)x i_U + (\Sigma i_V)y + z i_U, -(\Sigma i_V)x - z)$.

Now, any element of the form $(0, w)$ is in Im α' if, and only if, $(0, \varepsilon')$ is in Im α', which in turn is equivalent to the fact that condition (2) of Lemma 3.3.15 is satisfied, and thus to the fact that $G'_{U,V}$ is non-empty. This finishes the proof. □

3.3.5 Multiplication formula

The main goal of this section is to prove the following Proposition, using the results of the previous sections.

Proposition 3.3.16 (Multiplication formula)**.** *The map $X'_?$ satisfies the multiplication formula given in Definition 3.3.10.*

In order to prove this result, some notation is in order. Let X and Y be objects of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ is one-dimensional. Let

$$X \xrightarrow{i} E \xrightarrow{p} Y \xrightarrow{\varepsilon} \Sigma X \text{ and}$$

$$Y \xrightarrow{i'} E' \xrightarrow{p'} X \xrightarrow{\varepsilon'} \Sigma X$$

be non-split triangles in $\mathcal{C}$. For any submodules U of FX and V of FY, define $G_{U,V}$ and $G'_{U,V}$ as in Section 3.3.4.

For any dimension vectors e, f and g, define the following varieties :

$$G_{e,f} = \bigcup_{\substack{\underline{\dim}\, U = e \\ \underline{\dim}\, V = f}} G_{U,V}$$

$$G'_{e,f} = \bigcup_{\substack{\underline{\dim}\, U = e \\ \underline{\dim}\, V = f}} G'_{U,V}$$

$$G^g_{e,f} = G_{e,f} \cap \mathrm{Gr}_g(FE)$$

$$G'^g_{e,f} = G'_{e,f} \cap \mathrm{Gr}_g(FE').$$

We first need an equality on Euler characteristics.

Lemma 3.3.17. *With the above notation, we have that*

$$\chi(\mathrm{Gr}_e(FX))\chi(\mathrm{Gr}_f(FY)) = \sum_g \Big(\chi(G^g_{e,f}) + \chi(G'^g_{e,f})\Big).$$

Proof The Lemma is a consequence of the following successive equalities:

$$\begin{aligned}
\chi(\mathrm{Gr}_e(FX))\chi(\mathrm{Gr}_f(FY)) &= \chi(\mathrm{Gr}_e(FX) \times \mathrm{Gr}_f(FY)) \\
&= \chi(G_{e,f} \sqcup G'_{e,f}) \\
&= \chi(G_{e,f}) + \chi(G'_{e,f}) \\
&= \sum_g \Big(\chi(G^g_{e,f}) + \chi(G'^g_{e,f})\Big).
\end{aligned}$$

The only equality requiring explanation is the second one. Consider the map

$$\begin{array}{rcl}
G_{e,f} \sqcup G'_{e,f} & \longrightarrow & \mathrm{Gr}_e(FX) \times \mathrm{Gr}_f(FY) \\
W & \longmapsto & \begin{cases} \Big((Fi)^{-1}(W), (Fp)(W)\Big) & \text{if } W \in G_{e,f} \\ \Big((Fi')^{-1}(W), (Fp')(W)\Big) & \text{if } W \in G'_{e,f}. \end{cases}
\end{array}$$

As a consequence of Proposition 3.3.13, the map is surjective, and as shown in [12], its fibers are affine spaces. The Euler characteristic of all its fibers is thus 1, and we have the desired equality. □

Secondly, we need an interpretation of the dimension vectors e, f and g.

Lemma 3.3.18. *If $G^g_{e,f}$ is not empty, then*

$$\underline{\dim}\,(\mathrm{Coker}\, F\Sigma^{-1}p) = e + f - g.$$

PROOF We have the following commuting diagram in mod B, where the rows are exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & U \arrow[r] \arrow[d, "Fi_U"] & W \arrow[r] \arrow[d, "Fi_W"] & V \arrow[r] \arrow[d, "Fi_V"] & 0 \\ & & FX \arrow[r, "Fi"] & FE \arrow[r, "Fp"] & FY \arrow[r, "F\varepsilon"] & F\Sigma X. \end{tikzcd}$$

In this diagram, $\underline{\dim}\, U = e$, $\underline{\dim}\, W = g$ and $\underline{\dim}\, V = f$. The existence of such a diagram is guaranteed by the non-emptiness of $G^g_{e,f}$.

Now $\mathrm{Coker}\, F\Sigma^{-1}p$ is isomorphic to $\mathrm{Ker}\, Fi$, which is in turn isomorphic to $\mathrm{Ker}\,(Fi \circ Fi_U)$, since $U = (Fi)^{-1}(W)$. This last kernel is isomorphic to K. Hence the equality $\underline{\dim}\,(\mathrm{Coker}\, F\Sigma^{-1}p) = \underline{\dim}\, K$ holds.

Finally, the upper exact sequence gives the equality $\underline{\dim}\, K + \underline{\dim}\, W = \underline{\dim}\, U + \underline{\dim}\, V$. By rearranging and substituting terms, we get the desired equality. □

Everything is now in place to prove the multiplication formula.

PROOF (of Proposition 3.3.16) The result is a consequence of the following successive equalities (explanations follow).

$$\begin{aligned} X'_X X'_Y &= x^{\mathrm{ind}_T \Sigma^{-1}X + \mathrm{ind}_T \Sigma^{-1}Y} \sum_{e,f} \chi\Big(\mathrm{Gr}_e(FX)\Big)\chi\Big(\mathrm{Gr}_f(FY)\Big) x^{-\iota(e+f)} \\ &= x^{\mathrm{ind}_T \Sigma^{-1}X + \mathrm{ind}_T \Sigma^{-1}Y} \sum_{e,f,g} \Big(\chi(G^g_{e,f}) + \chi(G'^g_{e,f})\Big) x^{-\iota(e+f)} \\ &= x^{\mathrm{ind}_T \Sigma^{-1}X + \mathrm{ind}_T \Sigma^{-1}Y - \iota(\mathrm{Coker}\, F\Sigma^{-1}p) - \iota(g)} \sum_{e,f,g} \chi(G^g_{e,f}) + \\ &\quad + \; x^{\mathrm{ind}_T \Sigma^{-1}X + \mathrm{ind}_T \Sigma^{-1}Y - \iota(\mathrm{Coker}\, F\Sigma^{-1}p') - \iota(g)} \sum_{e,f,g} \chi(G'^g_{e,f}) \\ &= x^{\mathrm{ind}_T \Sigma^{-1}E} \sum_g \chi\Big(\mathrm{Gr}_g(FE)\Big) x^{-\iota(g)} + \\ &\quad + \; x^{\mathrm{ind}_T \Sigma^{-1}E'} \sum_g \chi\Big(\mathrm{Gr}_g(FE')\Big) x^{-\iota(g)} \\ &= X'_E + X'_{E'}. \end{aligned}$$

The first equality is just the definition of $X'_?$. The second one is a consequence of Lemma 3.3.17, and the third one of Lemma 3.3.18. The fourth follows from Lemma 3.3.5. The fifth equality is obtained by definition of $G^g_{e,f}$ and $G'^g_{e,f}$. The final equality is, again, just the definition of $X'_?$. □

3.4 Application to cluster algebras

In this section, we apply the cluster character developped in Section 3.3 to any skew-symmetric cluster algebra, taking $T = \Sigma^{-1}\Gamma$.

An object X of a triangulated category is *rigid* if $\mathrm{Hom}(X, \Sigma X)$ vanishes. Let $\mathcal{C}_{Q,W}$ be the cluster category of a quiver with potential (Q, W). A *reachable object* of $\mathcal{C}_{Q,W}$ is a direct factor of a direct sum of copies $\mu_{i_r} \dots \mu_{i_1}(\Gamma)$ for some admissible sequence of vertices $(i_1, \dots, i_r)$ of Q. Notice that each reachable object is rigid.

Theorem 3.4.1. *Let (Q, W) be a quiver with potential. Then the cluster character $X'_?$ defined in 3.3.10 gives a surjection from the set of isomorphism classes of indecomposable reachable objects of $\mathcal{C}_{Q,W}$ to the set of clusters variables of the cluster algebra associated with Q obtained by mutating the initial seed at admissible sequences of vertices.*

More precisely, $X'_?$ sends the indecomposable summands of $\mu_{i_r} \dots \mu_{i_1}(\Gamma)$ to the elements of the mutated cluster $\mu_{i_r} \dots \mu_{i_1}(x_1, \dots, x_r)$, where $(x_1, \dots, x_r)$ is the initial cluster.

PROOF Let $(i_1, \dots, i_r)$ be an admissible sequence of vertices. It is easily seen that $X'_{\Gamma_i} = x_i$ for all vertices i. It is a consequence of [58, Corollary 4.6] that the space $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, \mu_{i_r} \dots \mu_{i_1}(\Gamma))$ is finite dimensional. Moreover, $\mu_{i_r} \dots \mu_{i_1}(\Gamma)$ is obviously in $\mathrm{pr}_{\mathcal{C}}(\mu_{i_r} \dots \mu_{i_1}(\Gamma)) \cap \mathrm{pr}_{\mathcal{C}}(\Sigma^{-1}\mu_{i_r} \dots \mu_{i_1}(\Gamma))$, which is equal to $\mathrm{pr}_{\mathcal{C}}\Gamma \cap \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ by Corollary 3.2.8. Therefore $\mu_{i_r} \dots \mu_{i_1}(\Gamma)$ is in the subcategory $\mathcal{D}$ of Definition 3.3.9, and we can apply $X'_?$ to $\mu_{i_r} \dots \mu_{i_1}(\Gamma)$.

We prove the result by induction on r.

First notice that $\mu_{i_r} \dots \mu_{i_1}(\Gamma)_i = \mu_{i_{r-1}} \dots \mu_{i_1}(\Gamma)_i$ if $i \neq i_r$. Now, for $i = i_r$, using the triangle equivalence $\mathcal{C}_{\mu_{i_r} \cdots \mu_{i_1}(Q,W)} \longrightarrow \mathcal{C}_{Q,W}$, we get triangles

$$\mu_{i_{r-1}} \dots \mu_{i_1}(\Gamma)_{i_r} \longrightarrow \bigoplus_{i_r \to j} \mu_{i_{r-1}} \dots \mu_{i_1}(\Gamma)_j \longrightarrow \mu_{i_r} \dots \mu_{i_1}(\Gamma)_{i_r} \longrightarrow \dots$$

$$\mu_{i_r} \dots \mu_{i_1}(\Gamma)_{i_r} \longrightarrow \bigoplus_{j \to i_r} \mu_{i_{r-1}} \dots \mu_{i_1}(\Gamma)_j \longrightarrow \mu_{i_{r-1}} \dots \mu_{i_1}(\Gamma)_{i_r} \longrightarrow \dots$$

to which we can apply the multiplication formula of Proposition 3.3.16. In this way, we obtain the mutation of variables in the cluster algebra. This proves the result. □

Remark 3.4.2. In some cases, the surjection of Theorem 3.4.1 is known to be a bijection, namely if the quiver Q is mutation-equivalent to an acyclic quiver ([13, Theorem 4]) or if the skew-symmetric matrix associated with Q is of full rank (Corollary 4.3.9). We do not know whether it is a bijection in general.

Chapter 4

Applications to cluster algebras

4.1 Introduction

Since their introduction by S. Fomin and A. Zelevinsky in [29], cluster algebras have been found to enjoy connections with several branches of mathematics, see for instance the survey papers [75], [40] and [50], or the talks of the ICM 2010 [28], [62] and [71]. Cluster algebras are commutative algebras generated by *cluster variables* grouped into sets of fixed finite cardinality called *clusters*. Of particular importance are cluster algebras *with coefficients*, as most known examples of cluster algebras are of this kind. In this chapter, we will work with cluster algebras of geometric type with coefficients.

In [31], the authors developped a combinatorial framework allowing the study of coefficients in cluster algebras. Important tools that the authors introduced are the F-polynomials and $\mathbf{g}$-vectors. In particular, they proved that the behaviour of the coefficients in any cluster algebra is governed by that of the coefficients in a cluster algebra *with principal coefficients*, using the F-polynomials (see [31, Theorem 3.7]).

The authors phrased a number of conjectures, mostly regarding F-polynomials and $\mathbf{g}$-vectors. We list some of them here:

(5.4) every F-polynomial has constant term 1;

(6.13) the $\mathbf{g}$-vectors of the cluster variables of any given seed are *sign-coherent* in a sense to be defined;

(7.2) cluster monomials are linearly independent;

(7.10) different cluster monomials have different $\mathbf{g}$-vectors, and the $\mathbf{g}$-vectors of the cluster variables of any cluster form a basis of $\mathbb{Z}^r$;

(7.12) the mutation rule for $\mathbf{g}$-vectors can be expressed using a certain piecewise-linear transformation.

Work on these conjectures includes

- a proof of (7.2) by P. Sherman and A. Zelevinsky [73] for Dynkin and affine type of rank 2;
- a proof of (7.2) by P. Caldero and B. Keller [14] for Dynkin type;
- a proof of (7.2) by G. Dupont [25] for affine type A;
- a proof of (7.2) by M. Ding, J. Xiao and F. Xu [23] for affine types;
- a proof of (7.2) by G. Cerulli Irelli [15] in type $A_2^{(1)}$ by explicit computations;
- a proof of (5.4) by R. Schiffler [72] for cluster algebras arising from unpunctured surfaces;
- a proof of (7.2) by L. Demonet [19] for certain skew-symmetrizable cluster algebras;

- a proof of all five conjectures by C. Fu and B. Keller [32] for cluster algebras categorified by Hom-finite 2-Calabi–Yau Frobenius or triangulated categories, using work of R. Dehy and B. Keller [18];
- a proof of (7.2) by C. Geiss, B. Leclerc and J. Schröer [34] for acyclic cluster algebras;
- a proof of (5.4), (6.13), (7.10) and (7.12) in full generality by H. Derksen, J. Weyman and A. Zelevinsky [21] using decorated representations of quivers with potentials;
- a proof of (5.4), (6.13), (7.10) and (7.12) in full generality by K. Nagao [65] using Donaldson–Thomas theory (see for instance [48], [59] and [9]).

In this chapter, we use (generalized) cluster categories to give a new proof of (5.4), (6.13), (7.10) and (7.12) in full generality, and to prove (7.2) for any skew-symmetric cluster algebra of geometric type whose associated matrix is of full rank.

More precisely, we use the cluster category introduced by the group of authors A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov in [10] (and independently by P. Caldero, F. Chapoton and R. Schiffler in [11] in the A_n case) and generalized to any quiver with potential by C. Amiot in [2]. Note that this category can be Hom-infinite. We obtain applications to cluster algebras via the cluster character of Y. Palu [68], which generalized the map introduced by P. Caldero and F. Chapoton in [12]. It was extended in Chapter 3 to possibly Hom-infinite cluster categories. In particular, we have to restrict the cluster character to a suitable subcategory $\mathcal{D}$ of the cluster category.

Using this cluster character, we give categorical interpretations of F-polynomials and $\mathbf{g}$-vectors which allow us to prove the conjectures mentioned above. We prove (7.2), (6.13), (7.10) and (7.12) in section 4.3.2 and (5.4) in section 4.3.3. Some of our results concerning rigid objects in section 4.3.1 and indices in section 4.3.2 are used in [44] and [45]. The methods we use are mainly generalizations of those used for the Hom-finite case in [18] and [32].

The key tool that we use in our proofs is the multiplication formula proved in Proposition 3.3.16, while the proofs of H. Derksen, J. Weyman and A. Zelevinsky rely on a substitution formula [21, Lemma 5.2].

We also show in section 4.4 that the setup used in [21] is closely related to the cluster-categorical approach. We prove in section 4.4.1 that (isomorphism classes of) decorated representations over a quiver with potential are in bijection with (isomorphism classes of) objects in the subcategory $\mathcal{D}$ of the cluster category. In sections 4.4.2 and 4.4.3, we give an interpretation of the F-polynomial, $\mathbf{g}$-vector, $\mathbf{h}$-vector and E-invariant of a decorated representation in cluster-categorical terms. In particular, we prove a stronger version of [21, Lemma 9.2] in Corollary 4.4.16. Using the substitution formula for F-polynomials [21, Lemma 5.2], we also obtain a substitution formula for cluster characters of not necessarily rigid object (Corollary 4.4.14).

4.2 Recollections

4.2.1 Background on cluster algebras with coefficients

We give a brief summary of the definitions and results we will need concerning cluster algebras with coefficients. Our main source for the material in this section is [31].

Cluster algebras with coefficients

We will first recall the definition of (skew-symmetric) cluster algebras (of geometric type).

The *tropical semifield* $\mathrm{Trop}(u_1, u_2, \dots, u_n)$ is the abelian group (written multiplicatively) freely generated by the u_i's, with an *addition* $\oplus$ defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

An *ice quiver* (see [32]) is a pair (Q, F), where Q is a quiver and F is a subset of Q_0. The elements of F are the *frozen vertices* of Q. The ice quiver is *finite* if Q is finite.

Let r and n be integers such that $1 \leq r \leq n$. Let us denote by $\mathbb{P}$ the tropical semifield $\mathrm{Trop}(x_{r+1}, \dots, x_n)$. Let $\mathcal{F}$ be the field of fractions of the ring of polynomials in r indeterminates with coefficients in $\mathbb{QP}$.

Let (Q, F) be a finite ice quiver, where Q has no oriented cycles of length ≤ 2, and F and Q_0 have r and n elements respectively. We will denote the elements of $Q_0 \setminus F$ by the numbers $1, 2, \dots, r$ and those of F by $(r+1), (r+2), \dots, n$. Let i be in $Q_0 \setminus F$. One defines the *mutation of* (Q, F) *at* i as the ice quiver $\mu_i(Q, F) = (Q', F')$ constructed from (Q, F) as follows:

– the sets Q'_0 and F' are equal to Q_0 and F, respectively;

– the quiver Q' is the mutated quiver $\mu_i(Q)$ defined in section 3.2.1.

A *seed* is a pair $\big((Q, F), \mathbf{x}\big)$, where (Q, F) is an ice quiver as above, and $\mathbf{x} = \{x_1, \dots, x_r\}$ is a free generating set of the field $\mathcal{F}$. Given a vertex i of $Q_0 \setminus F$, the *mutation* of the seed $\big((Q, F), \mathbf{x}\big)$ at the vertex i is the pair $\mu_i\big((Q, F), \mathbf{x}\big) = \big((Q', F'), \mathbf{x}'\big)$, where

– (Q', F') is the mutated ice quiver $\mu_i(Q, F)$;

– $\mathbf{x}' = \mathbf{x} \setminus \{x_i\} \cup \{x'_i\}$, where x'_i is obtained from the *exchange relation*

$$x_i x'_i = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = i}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} x_{s(\alpha)}.$$

The mutation of a seed is still a seed, and the mutation at a fixed vertex is an involution.

Fix an *initial seed* $\big((Q, F), \mathbf{x}\big)$.

– The sets $\mathbf{x}'$ obtained by repeated mutation of the initial seed are the *clusters*.

– The elements of the clusters are the *cluster variables*.

– The $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables is the *cluster algebra* $\mathcal{A} = \mathcal{A}\big((Q, F), \mathbf{x}\big)$.

Suppose that $n = 2r$. A cluster algebra has *principal coefficients at a seed* $\big((Q', F'), \mathbf{x}'\big)$ if there is exactly one arrow from $(r+\ell)$ to ℓ (for $\ell = 1, 2, \dots, r$), and if these are the only arrows whose source or target lies in F'.

Cluster monomials and g-vectors

Given an ice quiver (Q, F), we associate to it an $(n \times r)$-matrix $\tilde{B} = (b_{ij})$, where each entry b_{ij} is the number of arrows from i to j minus the number or arrows from j to i.

Let $\big((Q, F), \mathbf{x}\big)$ be a seed of a cluster algebra $\mathcal{A}$. A *cluster monomial* in $\mathcal{A}$ is a product of cluster variables lying in the same cluster.

To define $\mathbf{g}$-*vectors*, we shall need a bit of notation. For any integer j between 1 and r, let $\hat{y}_j$ be defined as

$$\hat{y}_j = \prod_{\ell \in Q_0} x_\ell^{b_{\ell j}}.$$

Let $\mathcal{M}$ be the set of non-zero elements of $\mathcal{A}$ which can be written in the form

$$z = R(\hat{y}_1, \dots, \hat{y}_r) \prod_{j=1}^{n} x_j^{g_j},$$

where $R(u_1, \ldots, u_r)$ is an element of $\mathbb{Q}(u_1, \ldots, u_r)$. Note that all cluster monomials belong to $\mathcal{M}$. By [31, Proposition 7.8], if the matrix $\tilde{B}$ is of full rank r, then any element of $\mathcal{M}$ can be written in a unique way in the form above, with R *primitive* (that is, R can be written as a ratio of two polynomials, none of which is divisible by any of the u_j's). In that case, if z is an element of $\mathcal{M}$ written as above with R primitive, the vector

$$\mathbf{g}(z) = (g_1, \ldots, g_r)$$

is the $\mathbf{g}$-*vector* of z.

Let us now state Conjectures 7.2, 7.10 and 7.12 of [31].

7.2 Cluster monomials are linearly independent over $\mathbb{ZP}$.

7.10 Different cluster monomials have different $\mathbf{g}$-vectors; the $\mathbf{g}$-vectors of the cluster variables of any cluster form a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.

7.12 Let $\mathbf{g} = (g_1, \ldots, g_r)$ and $\mathbf{g}' = (g'_1, \ldots, g'_r)$ be the $\mathbf{g}$-vectors of one cluster monomial with respect to two clusters t and t' related by one mutation at the vertex i. Then we have

$$g'_j = \begin{cases} -g_i & \text{if } j = i \\ g_j + [b_{ji}]_+ g_i - b_{ji} \min(g_i, 0) & \text{if } j \neq i \end{cases}$$

where $B = (b_{j\ell})$ is the matrix associated with the seed t, and we set $[x]_+ = \max(x, 0)$ for any real number x.

F-polynomials

Let $\mathcal{A}$ be a cluster algebra with principal coefficients at a given seed $\big((Q, F), \mathbf{x}\big)$. Let t be a seed of $\mathcal{A}$ and ℓ be a vertex of Q that is not in F. Then the ℓ-th cluster variable of t can be written as a subtraction-free rational function in variables $x_1, \ldots, x_{2r}$. Following [31, Definition 3.3], we define the F-*polynomial* $F_{\ell,t}$ as the specialization of this rational function at $x_1 = \ldots = x_r = 1$. It was proved in [31, Proposition 3.6] that $F_{\ell,t}$ is indeed a polynomial.

We now state Conjecture 5.4 of [31] : *Every F-polynomial has constant term* 1.

Y-seeds and their mutations

We now recall the notion of Y-seeds from [31]. As above, let $1 \leq r \leq n$ be integers, and let $\mathbb{P}$ be the tropical semifield in the variables $x_{r+1}, \ldots, x_n$.

A Y-*seed* is a pair $(Q, \mathbf{y})$, where

– Q is a finite quiver without oriented cycles of length ≤ 2; and

– $\mathbf{y} = (y_1, \ldots, y_r)$ is an element of $\mathbb{P}^r$.

Let $(Q, \mathbf{y})$ be a Y-seed, and let i be a vertex of Q. The *mutation of* $(Q, \mathbf{y})$ *at the vertex* i is the Y-seed $\mu_i(Q, \mathbf{y}) = (Q', \mathbf{y}')$, where

– Q is the mutated quiver $\mu_i(Q)$; and

– $\mathbf{y}' = (y'_1, \ldots, y'_r)$ is obtained from $\mathbf{y}$ using the mutation rule

$$y'_j = \begin{cases} y_i^{-1} & \text{if } i = j \\ y_j y_i^m (y_i \oplus 1)^{-m} & \text{if there are } m \text{ arrows from } i \text{ to } j \\ y_j (y_i \oplus 1)^m & \text{if there are } m \text{ arrows from } j \text{ to } i. \end{cases}$$

If, to any seed $\big((Q, F), \mathbf{x}\big)$ of a cluster algebra, we associate a Y-seed $(Q, \mathbf{y})$ defined by

$$y_j = \prod_{i=r+1}^{n} x_i^{b_{ij}},$$

then for any such seed and its associated Y-seed, and for any vertex i of Q, we have that the Y-seed associated to $\mu_i\big((Q,F),\mathbf{x}\big)$ is $\mu_i(Q,\mathbf{y})$. This was proved in [31] after Definition 2.12.

4.2.2 Decorated representations of quivers with potentials

We defined quivers with potentials in section 3.2.2, after [22]. We now recall from [22, Section 10] the notion of decorated representation of a quiver with potential.

Let (Q,W) be a quiver with potential, and let $J(Q,W)$ be its Jacobian algebra. A *decorated representation* of (Q,W) is a pair $\mathcal{M}=(M,V)$, where M is a finite-dimensional nilpotent $J(Q,W)$-right module and V is a finite-dimensional Λ-module (recall that Λ is given by $\bigoplus_{i\in Q_0} ke_i$).

We now turn to the *mutation* of decorated representations. Given a decorated representation $\mathcal{M}=(M,V)$ of (Q,W), and given any admissible vertex ℓ of (Q,W), we construct a decorated representation $\widetilde{\mu}_\ell(\mathcal{M})=(\overline{M},\overline{V})$ of $\widetilde{\mu}_\ell(Q,W)$ as follows.

We view M as a representation of the opposite quiver Q^{op} (we must work over the opposite quiver, since we use *right* modules). In particular, to each vertex i is associated a vector space M_i (which is equal to Me_i), and to each arrow $a:i\to j$ is associated a linear map $M_a:M_j\to M_i$. For any path $p=a_r\cdots a_2a_1$, we denote by M_p the linear map $M_{a_1}M_{a_2}\cdots M_{a_r}$, and for any (possibly infinite) linear combination $\sigma=\sum_{i\in I}\lambda_i p_i$ of paths, we denote by M_σ the linear map $\sum_{i\in I}\lambda_i M_{p_i}$ (this sum is finite since M is nilpotent). If σ is zero in $J(Q,W)$, then M_σ is the zero map. Define the vector spaces M_{in} and M_{out} by

$$M_{in}=\bigoplus_{\substack{a\in Q_1\\ s(a)=\ell}} M_{t(a)} \quad\text{and}\quad M_{out}=\bigoplus_{\substack{b\in Q_1\\ t(b)=\ell}} M_{s(b)}.$$

Define the linear map $\alpha:M_{in}\longrightarrow M_\ell$ as the map given in matrix form by the line vector $\big(M_a:M_{t(a)}\to M_\ell \;\big|\; a\in Q_1,\ s(a)=\ell\big)$. Similarly, define $\beta:M_\ell\longrightarrow M_{out}$ as the map given in matrix form by the column vector $\big(M_b:M_\ell\to M_{s(b)} \;\big|\; b\in Q_1,\ t(b)=\ell\big)$. Define a third map $\gamma:M_{out}\longrightarrow M_{in}$ as the map given in matrix form by

$$\big(M_{\partial_{ab}W}:M_{s(b)}\to M_{t(a)} \;\big|\; a,b\in Q_1,\ s(a)=t(b)=\ell\big).$$

Now construct $\widetilde{\mu}_\ell(\mathcal{M})=(\overline{M},\overline{V})$ as follows.

- For any vertex $i\neq\ell$, set $\overline{M}_i=M_i$ and $\overline{V}_i=V_i$.
- Define $\overline{M}_\ell$ and $\overline{V}_\ell$ by

$$\overline{M}_\ell=\frac{\operatorname{Ker}\gamma}{\operatorname{Im}\beta}\oplus\operatorname{Im}\gamma\oplus\frac{\operatorname{Ker}\alpha}{\operatorname{Im}\gamma}\oplus V_\ell \quad\text{and}\quad \overline{V}_\ell=\frac{\operatorname{Ker}\beta}{\operatorname{Ker}\beta\cap\operatorname{Im}\alpha}.$$

- For any arrow a of Q not incident with ℓ, set $\overline{M}_a=M_a$.
- For any subquiver of the form $i\xrightarrow{a}\ell\xrightarrow{b}j$, set $\overline{M}_{[ba]}=M_{ba}$.
- the actions of the remaining arrows are encoded in the maps

$$\overline{\alpha}=\begin{pmatrix}-\pi\rho\\ -\gamma\\ 0\\ 0\end{pmatrix} \quad\text{and}\quad \overline{\beta}=\begin{pmatrix}0 & \iota & \iota\sigma & 0\end{pmatrix},$$

where
- the map $\rho:M_{out}\to\operatorname{Ker}\gamma$ is such that its composition with the inclusion map of $\operatorname{Ker}\gamma$ gives the identity map of $\operatorname{Ker}\gamma$;

– the map $\pi : \operatorname{Ker}\gamma \to \operatorname{Ker}\gamma/\operatorname{Im}\beta$ is the natural projection map;
– the map $\sigma : \operatorname{Ker}\alpha/\operatorname{Im}\gamma \to \operatorname{Ker}\alpha$ is such that its composition with the projection map $\operatorname{Ker}\alpha \to \operatorname{Ker}\alpha/\operatorname{Im}\gamma$ gives the identity map of $\operatorname{Ker}\alpha/\operatorname{Im}\gamma$;
– the map $\iota : \operatorname{Im}\gamma \to M_{in}$ is the natural inclusion map.

It is shown in [22, Proposition 10.7] that $\widetilde{\mu}_\ell(\mathcal{M})$ is indeed a decorated representation of $\widetilde{\mu}_\ell(Q,W)$.

4.2.3 Some invariants of decorated representations

In this section, we recall from [22] and [21] the definitions of F-polynomial, $\mathbf{g}$-vector, $\mathbf{h}$-vector and E-invariant of a decorated representation.

We fix a quiver with potential (Q,W) and a decorated representation $\mathcal{M} = (M,V)$ of (Q,W). We number the vertices of Q from 1 to n.

The *F-polynomial* of $\mathcal{M}$ is the polynomial of $\mathbb{Z}[u_1,\dots,u_n]$ defined by

$$F_{\mathcal{M}}(u_1,\dots,u_n) = \sum_e \chi(\mathrm{Gr}_e(M)) \prod_{i=1}^n u_i^{e_i}.$$

The *$\mathbf{g}$-vector* of $\mathcal{M}$ is the vector $\mathbf{g}_{\mathcal{M}} = (g_1,\dots,g_n)$ of $\mathbb{Z}^n$, where

$$g_i = \dim \operatorname{Ker}\gamma_i - \dim M_i + \dim V_i,$$

where γ_i is the map $\gamma : M_{out} \longrightarrow M_{in}$ defined in section 4.2.2.

The *$\mathbf{h}$-vector* of $\mathcal{M}$ is the vector $\mathbf{h}_{\mathcal{M}} = (h_1,\dots,h_n)$ of $\mathbb{Z}^n$, where

$$h_i = -\dim \operatorname{Ker}\beta_i$$

where β_i is the map $\beta : M_i \longrightarrow M_{out}$ defined in section 4.2.2.

The *E-invariant* of $\mathcal{M}$ is the integer

$$E(\mathcal{M}) = \dim \operatorname{Hom}_{J(Q,W)}(M,M) + \sum_{i=1}^n g_i \dim M_i,$$

where $(g_1,\dots,g_n)$ is the $\mathbf{g}$-vector of $\mathcal{M}$.

Let $\mathcal{N} = (N,U)$ be another decorated representation of (Q,W). The E-invariant can also be defined using the two integer-valued invariants

$$E^{inj}(\mathcal{M},\mathcal{N}) = \dim \operatorname{Hom}_{J(Q,W)}(M,N) + \sum_{i=1}^n (\dim M_i) g_i(\mathcal{N}) \quad \text{and}$$

$$E^{sym}(\mathcal{M},\mathcal{N}) = E^{inj}(\mathcal{M},\mathcal{N}) + E^{inj}(\mathcal{N},\mathcal{M}).$$

We have that $E(\mathcal{M}) = E^{inj}(\mathcal{M},\mathcal{M}) = (1/2)E^{sym}(\mathcal{M},\mathcal{M})$.

4.2.4 More on mutations as derived equivalences

Let (Q,W) be a quiver with potential. Assume that Q has no loops, and that i is a vertex of Q not contained in a cycle of length 2. Let (Q',W') be the mutated quiver with potential $\widetilde{\mu}_i(Q,W)$. Let Γ and Γ' be the complete Ginzburg dg algebras associated with (Q,W) and (Q',W'), respectively.

We recall here some results of [58] on the mutation of Γ in $\mathcal{D}\Gamma$. Let Γ_i^* be the cone in $\mathcal{D}\Gamma$ of the morphism

$$\Gamma_i \longrightarrow \bigoplus_\alpha \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α. Similarly, let $\Sigma\overline{\Gamma}_i^*$ be the cone of the morphism

$$\bigoplus_{\beta} \Gamma_{s(\beta)} \longrightarrow \Gamma_i$$

whose components are given by left multiplication by β.

Then it is proved in the discussion after [58, Lemma 4.4] that the morphism $\varphi_i : \Sigma\Gamma_i^* \longrightarrow \Sigma\overline{\Gamma}_i^*$ given in matrix form by

$$\begin{pmatrix} -\beta^* & -\partial_{\alpha\beta}W \\ t_i & a^* \end{pmatrix}$$

becomes an isomorphism in $\mathcal{C}$.

Remark 4.2.1. In particular, the composition of the morphisms

$$\bigoplus_{\alpha} \Gamma_{t(\alpha)} \longrightarrow \Gamma_i^* \xrightarrow{\Sigma^{-1}\varphi_i} \overline{\Gamma}_i^* \longrightarrow \bigoplus_{\beta} \Gamma_{s(\beta)}$$

is given in matrix form by $\big(-\partial_{ab}W \big)$.

Recall the quasi-inverse equivalences $\widetilde{\mu}_i^+$ and $\widetilde{\mu}_i^-$ of theorem 3.2.6 and section 3.2.8. Note that these equivalences induce equivalences on the level of cluster categories, which we will also denote by $\widetilde{\mu}_i^+$ and $\widetilde{\mu}_i^-$.

In Section 4.4.1, we will need a concrete description of $\widetilde{\mu}_i^+$ and $\widetilde{\mu}_i^-$. The functor $\widetilde{\mu}_i^+$ is the derived functor $? \otimes^L_{\Gamma'} T$, where T is the Γ'-Γ-bimodule described below. The functor $\widetilde{\mu}_i^-$ is then $\mathcal{H}om_{\Gamma}(T, ?)$.

As a right Γ-module, T is a direct sum $\bigoplus_{j=1}^n T_j$, where T_j is isomorphic to $e_j\Gamma$ if $i \neq j$ and T_i is the cone of the morphism

$$e_i\Gamma \longrightarrow \bigoplus_{\substack{\alpha\in Q_1 \\ s(\alpha)=i}} e_{t(\alpha)}\Gamma,$$

whose components are given by left multiplication by α. Thus, as a graded module, T_i is isomorphic to

$$P_{\Sigma i} \oplus \bigoplus_{\substack{\alpha\in Q_1 \\ s(\alpha)=i}} P_\alpha,$$

where $P_{\Sigma i}$ is a copy of $\Sigma(e_i\Gamma)$, and each P_α is a copy of $e_t(\alpha)\Gamma$. We will denote by $e_{\Sigma i}$ the idempotent of $P_{\Sigma i}$ and by e_α the idempotent of P_α.

The left Γ'-module structure of T is described in terms of a homomorphism of dg algebras $\Gamma' \longrightarrow \mathcal{H}om_{\Gamma}(T,T)$, using the left $\mathcal{H}om_{\Gamma}(T,T)$-module structure of T. We will need the description of the image of some elements of Γ' under this homomorphism. This description is given below.

For any vertex j of Q, the element e_j is sent to the identity of T_j.

Any arrow δ not incident with i is sent to the map which is left multiplication by δ.

For any arrow α of Q such that $s(\alpha) = i$, the element $\alpha^\star$ is sent to the map $f_{\alpha^\star} : T_{t(\alpha)} \longrightarrow T_i$ defined by $f_{\alpha^\star}(a) = e_\alpha a$.

For any arrow β of Q such that $t(\beta) = i$, the element $\beta^\star$ is sent to the map $f_{\beta^\star} : T_i \longrightarrow T_{s(\beta)}$ defined by $f_{\beta^\star}(e_{\Sigma i}a_i + \sum_{s(\rho)=i} e_\rho a_\rho) = -\beta^* a_i - \sum_{s(\rho)=i}(\partial_{\rho\beta}W)a_\rho$.

4.3 Application to skew-symmetric cluster algebras

4.3.1 Rigid objects are determined by their index

This section is the Hom-infinite equivalent of [18, Section 2].

Let $\mathcal{C}$ be a triangulated category, and let $T = \bigoplus_{i=1}^{n} T_i$ be a rigid object of $\mathcal{C}$, where the T_i's are indecomposable and pairwise non-isomorphic. Assume that $\mathrm{pr}_{\mathcal{C}}T$ is a Krull–Schmidt category, and that $B = \mathrm{End}_{\mathcal{C}}\, T$ is the completed Jacobian algebra $J(Q,W)$ of a quiver with potential (Q,W). An example of such a situation is the cluster category $\mathcal{C}_{Q,W}$, with $T = \Sigma^{-1}\Gamma$.

The main result of this section is the following.

Proposition 4.3.1. *With the above assumptions, if X and Y are rigid objects in $\mathrm{pr}_{\mathcal{C}}T$ such that $\mathrm{ind}_T\, X = \mathrm{ind}_T\, Y$, then X and Y are isomorphic.*

The rest of the Section is devoted to the proof of the Proposition.

Let X be an object of $\mathrm{pr}_{\mathcal{C}}T$, and let the triangle $T_1^X \xrightarrow{f^X} T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$ be an $(\mathrm{add}\, T)$-presentation of X. The group $\mathrm{Aut}_{\mathcal{C}}(T_1^X) \times \mathrm{Aut}_{\mathcal{C}}(T_0^X)$ acts on the space $\mathrm{Hom}_{\mathcal{C}}(T_1^X, T_0^X)$, with action defined by $(g_1, g_0)f' = g_0 f'(g_1)^{-1}$. The orbit of f^X under this action is the image of the map

$$\begin{array}{rcl} \Phi : \mathrm{Aut}_{\mathcal{C}}(T_1^X) \times \mathrm{Aut}_{\mathcal{C}}(T_0^X) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(T_1^X, T_0^X) \\ (g_1, g_0) & \longmapsto & g_0 f^X (g_1)^{-1}. \end{array}$$

Our strategy is to show that if Y is another rigid object of $\mathrm{pr}_{\mathcal{C}}T$, then the orbits of f^X and f^Y must intersect (and thus coincide), proving that X and Y are isomorphic.

It was proved in Lemma 3.3.2 that the functor $F = \mathrm{Hom}_{\mathcal{C}}(T, ?)$ induces an equivalence of categories

$$\mathrm{pr}_{\mathcal{C}}T/(\Sigma T) \longrightarrow \mathrm{mod}\, B,$$

where $\mathrm{mod}\, B$ is the category of finitely presented right B-modules. Since T is rigid, this implies that F induces a fully faithful functor

$$\mathrm{add}\, T \longrightarrow \mathrm{mod}\, B.$$

Thus we can often consider automorphisms and morphisms in the category $\mathrm{mod}\, B$ instead of in $\mathcal{C}$.

Now, let $\mathfrak{m}$ be the ideal of $J(Q,W)$ generated by the arrows of Q.

The group $A = \mathrm{Aut}_B(FT_1^X) \times \mathrm{Aut}_B(FT_0^X)$ is the limit of the finite-dimensional affine algebraic groups

$$A_n = \mathrm{Aut}_B(FT_1^X/(FT_1^X\mathfrak{m}^n)) \times \mathrm{Aut}_B(FT_0^X/(FT_0^X\mathfrak{m}^n))$$

with respect to the natural projection maps from A_{n+1} to A_n, for $n \in \mathbb{N}$.

Similarly, the vector space $H = \mathrm{Hom}_B(FT_1^X, FT_0^X)$ is the limit of the spaces

$$H_n = \mathrm{Hom}_B\left(FT_1^X/(FT_1^X\mathfrak{m}^n),\ FT_0^X/(FT_0^X\mathfrak{m}^n)\right)$$

with respect to the natural projections. All the H_n are finite-dimensional spaces, and they are endowed with the Zariski topology. The projection maps are then continuous, and H is endowed with the limit topology.

Finally, for any integer n, we define a morphism $\Phi_n : A_n \to H_n$ which sends any element (g_1, g_0) of A_n to $g_0 f_n^X (g_1)^{-1}$, where f_n^X is the image of f^X in H_n under the canonical projection. Then the morphism Φ is the limit of the Φ_n's.

The situation is summarized in the following commuting diagram.

$$\begin{tikzcd}
A = \lim A_n \arrow[d, "\Phi"] & \dots \arrow[r] & \dots \arrow[r] & A_3 \arrow[r] \arrow[d, "\Phi_3"] & A_2 \arrow[r] \arrow[d, "\Phi_2"] & A_1 \arrow[d, "\Phi_1"] \\
H = \lim H_n & \dots \arrow[r] & \dots \arrow[r] & H_3 \arrow[r] & H_2 \arrow[r] & H_1.
\end{tikzcd}$$

The next step is the following : we will prove that the image of Φ is the limit of the images of the Φ_n's. This will follow from the Lemma below.

Lemma 4.3.2. *Let $(X_i)_{i\in\mathbb{N}}$ be a family of topological spaces. Let $(f_i : X_i \to X_{i-1})_{i\geq 1}$ be a family of continuous maps, and let $X = \lim X_i$. Let $(X'_i)_{i\in\mathbb{N}}$ be another family of topological spaces, with continuous maps $(f'_i : X'_i \to X'_{i-1})_{i\geq 1}$, and let $X' = \lim X'_i$. Let $(u_i : X_i \to X'_i)$ be a familiy of continuous maps such that $f'_i u_i = u_{i-1} f_i$ for all $i \geq 1$, and let $u = \lim u_i$. Denote by $p_i : X \to X_i$ and $p'_i : X' \to X'_i$ the canonical projections. For integers $i < j$, denote by f_{ij} (respectively f'_{ij}) the composition $f_j f_{j-1} \dots f_{i+1}$ (respectively $f'_j f'_{j-1} \dots f'_{i+1}$). Let x' be an element of X' with the property that for all $i \in \mathbb{N}$, there exists $j \geq i$ such that for all $\ell \geq j$, $f_{i\ell}(u_\ell^{-1}(p'_\ell(x'))) = f_{ij}(u_j^{-1}(p'_j(x')))$.*

Then x' admits a preimage in X, that is, there exists $x \in X$ such that $u(x) = x'$.

Proof This is a consequence of the Mittag-Leffler theorem, see for instance [8, Corollary II.5.2]. □

The above Lemma implies that the image of Φ is the limit of the images of the Φ_n. Indeed, the universal property of the limit gives an inclusion from the image of Φ to the limit of the images of the Φ_n. Let now x' be in the image of Φ, and let x'_n be its projection in the image of Φ_n. The set $\Phi_n^{-1}(x_n)$ is a closed subset of A_n, and for any $m \geq n$, the image of $\Phi_m^{-1}(x_m)$ in $\Phi_n^{-1}(x_n)$ is closed. Since A_n has finite dimension as a variety, the sequences of images of the $\Phi_m^{-1}(x_m)$ in $\Phi_n^{-1}(x_n)$ eventually becomes constant. Applying the above Lemma, we get that x' has a preimage in A by Φ. This proves that the image of Φ is the limit of the images of the Φ_n.

We will now prove that the image of each Φ_n is open (and thus dense, since H_n is irreducible). To prove this, we pass to the level of Lie algebras. To lighten notations, we let $E_n = \mathrm{End}_B(FT_1^X / FT_1^X \mathfrak{m}) \times \mathrm{End}_B(FT_0^X / FT_0^X \mathfrak{m})$ be the Lie algebra of A_n for all positive integers n. To prove that the image of Φ_n is open, it is sufficient to show that the map

$$\begin{array}{rcl} \Psi_n : E_n & \longrightarrow & H_n \\ (g_1, g_0) & \longmapsto & g_0 f_n^X - f_n^X g_1 \end{array}$$

is surjective.

The limit of the E_n's is $E = \mathrm{End}_B(FT_1^X) \times \mathrm{End}_B(FT_0^X)$, and the limit of the Ψ_n's is the map

$$\begin{array}{rcl} \Psi : E & \longrightarrow & H \\ (g_1, g_0) & \longmapsto & g_0 f^X - f^X g_1. \end{array}$$

The diagram below summarizes the situation.

$$\begin{tikzcd} E = \lim E_n \arrow[d, "\Psi"] & \dots \arrow[r] & \dots \arrow[r] & E_3 \arrow[r] \arrow[d, "\Psi_3"] & E_2 \arrow[r] \arrow[d, "\Psi_2"] & E_1 \arrow[d, "\Psi_1"] \\ H = \lim H_n & \dots \arrow[r] & \dots \arrow[r] & H_3 \arrow[r] & H_2 \arrow[r] & H_1. \end{tikzcd}$$

All the canonical projections are surjective.

Lemma 4.3.3. *The map Ψ defined above is surjective.*

PROOF This proof is contained in the proof of [18, Lemma 2.1] □ As a consequence,

all the Ψ_n's are surjective. Hence the images of the Φ_n's are open.

From this, we deduce that if Y is another rigid object of $\mathrm{pr}_{\mathcal{C}}T$ with $(\mathrm{add}\, T)$-presentation $T_0^X \xrightarrow{f^Y} T_1^X \longrightarrow Y \longrightarrow \Sigma T_1^X$, then X and Y are isomorphic. Indeed, by the above reasoning, the orbit of f^Y is the limit of the orbits of its projections in the H_n's. But these orbits are open, and so they intersect (and coincide) with the images of the Φ_n defined above. Hence the orbit of f^Y in H is the limit of the images of the Φ_n's, and this is exactly the orbit of f^X. Therefore X and Y are isomorphic.

The last step in proving Proposition 4.3.1 is to show that given $\mathrm{ind}_T X$, we can “deduce" T_1^X and T_0^X.

An $(\mathrm{add}\, T)$-approximation $T_1^X \to T_0^X \to X \to \Sigma T_1^X$ is *minimal* if one of the following conditions hold.

– The above triangle does not admit a direct summand of the form

$$R \xrightarrow{id_R} R \longrightarrow 0 \longrightarrow \Sigma R.$$

– The morphism $f : T_0^X \to X$ in the presentation has the property that for any $g : T_0^X \to T_0^X$, the equality $fg = f$ implies that g is an isomorphism.

In fact, any of these two conditions implies the other.

Lemma 4.3.4. *The above two conditions are equivalent if* $\mathrm{pr}_{\mathcal{C}}T$ *is Krull–Schmidt.*

PROOF First suppose that the presentation has the form

$$T_1' \oplus R \xrightarrow{u \oplus 1_R} T_0' \oplus R \xrightarrow{(f',0)} X \longrightarrow \Sigma T_1^X,$$

where $f = (f', 0)$ in matrix form. Then the endomorphism g of $T_0' \oplus R$ given by $g = 1_{T_0'} \oplus 0$ is not an isomorphism, and $fg = f$.

Now suppose that the presentation admits no direct summand of the form

$$R \xrightarrow{id_R} R \longrightarrow 0 \longrightarrow \Sigma R.$$

Using the Krull–Schmidt property of $\mathrm{pr}_{\mathcal{C}}T$, we can decompose bot T_0^X and T_1^X as a finite direct sum of objects with local endomorphism rings. In that case, the morphism f written in matrix form (in any basis) has no non-zero entries.

Let g be an endomorphism of T_0^X such that $fg = f$. Then $f(1_{T_0^X} - g) = 0$. Consider the morphism $(1_{T_0^X} - g)$ written in matrix form. If one of its entries is an isomorphism, then by a change of basis we can write $(1_{T_0^X} - g)$ as the matrix

$$\left(\begin{array}{c|c} * & 0 \\ \hline 0 & \phi \end{array} \right),$$

where ϕ is an isomorphism. In that case, it is impossible that $f(1_{T_0^X} - g) = 0$, since f has no non-zero entries. This implies that none of the entries of the matrix of $(1_{T_0^X} - g)$ is invertible. Therefore the diagonal entries of g are invertible (since for any element x of a local ring, if $(1-x)$ is not invertible, then x is), while the other entries are not, and g is an isomorphism. $\square$

Lemma 4.3.5. *If X is rigid and $T_1^X \xrightarrow{\alpha} T_0^X \to X \xrightarrow{\gamma} \Sigma T_1^X$ is a minimal* $(\operatorname{add} T)$*-presentation, then T_1^X and T_0^X have no direct summand in common.*

Proof The first proof of [18, Proposition 2.2] works in this setting. We include here a similar argument for the convenience of the reader.

Suppose that T_i is a direct factor of T_0^X. Let us prove that it is not a direct factor of T_1^X.

Applying $F = \operatorname{Hom}_{\mathcal{C}}(T, ?)$ to the triangle above, we get a minimal projective presentation of FX. This yields an exact sequence

$$(FX, S_i) \longrightarrow (FT_0^X, S_i) \xrightarrow{F\alpha^*} (FT_1^X, S_i),$$

where S_i is the simple at the vertex i. Since the presentation is minimal, $F\alpha^*$ vanishes, and there exists a non-zero morphism $f : FX \to S_i$. In particular, f is surjective.

Let $g : FT_1^X \longrightarrow S_i$ be a morphism. Since FT_1^X is projective, there exists a morphism $h : FT_1^X \longrightarrow FX$ such that $fh = g$.

Lift S_i to an object ΣT_i^* of $\mathcal{C}$, and lift f, g, and h to morphisms $\overline{f} : X \to \Sigma T_i^*$, $\overline{g} : T_1^X \to \Sigma T_i^*$ and $\overline{h} : T_1^X \to X$ of $\mathcal{C}$. Using Lemma 3.3.2, we get that $\overline{fh} = \overline{g}$.

$$\begin{tikzcd}
\Sigma^{-1}X \arrow[r, "\Sigma^{-1}\gamma"] & T_1^X \arrow[r, "\alpha"] \arrow[dl, "\overline{h}"'] \arrow[d, "\overline{g}"] & T_0^X \arrow[dl, dotted, "\sigma"] \\
X \arrow[r, "\overline{f}"] & \Sigma T_i^* &
\end{tikzcd}$$

Since X is rigid, $\overline{h}\Sigma^{-1}\gamma$ vanishes, and thus so does $\overline{g}\Sigma^{-1}\gamma$. Then there exists a morphism $\sigma : T_0^X \to \Sigma T_i^*$ such that $\sigma\alpha = \overline{g}$. But since $F\alpha^* = 0$, we get that $g = (F\sigma)(F\alpha)$ vanishes.

We have thus shown that there are no non-zero morphisms from FT_1^X to S_i. Therefore T_i is not a direct factor of T_1^X. $\square$

By the above Lemma, the knowledge of $\operatorname{ind}_T X$ is sufficient to deduce the isomorphism classes of T_1^X and T_0^X in any minimal (add)-presentation of X. Therefore, if Y is another rigid object of $\operatorname{pr}_{\mathcal{C}}T$ with $\operatorname{ind}_T X = \operatorname{ind}_T Y$, all of the above reasoning implies that X and Y are isomorphic. This finishes the proof of Proposition 4.3.1.

4.3.2 Index and g-vectors

It was proved in [32, Proposition 6.2] that, inside a certain Hom-finite cluster category $\mathcal{C}$, the index of an object M with respect to a cluster-tilting object T gives the **g**-vector of X'_M with respect to the associated cluster. The authors then used this result to prove conjectures of [31] in this case. In this section, we will prove a similar result, dropping the assumption of Hom-finiteness.

Let (Q, F) be a finite ice quiver, where Q has no oriented cycles of length ≤ 2. Suppose that the associated matrix B has full rank r. Denote by $\mathcal{A}$ the associated cluster algebra. Let W be a potential on Q, and let $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category.

Denote by $\mathcal{D}$ the full subcategory of $\mathrm{pr}_{\mathcal{C}}\Gamma \cap \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ whose objects are those X such that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, X)$ is finite-dimensional.

Following [32], let $\mathcal{U}$ be the full subcategory of $\mathcal{D}$ defined by

$$\mathcal{U} = \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_j, X) = 0 \text{ for } r+1 \leq j \leq n\}.$$

Note that $\mathcal{U}$ is invariant under iterated mutation of Γ at vertices $1, 2, \ldots, r$.

Let $T = \bigoplus_{j=1}^n T_j = \bigoplus_{j=1}^r T_j \oplus \bigoplus_{j=r+1}^n \Gamma_j$ be a rigid object of $\mathcal{D}$ reachable from Γ by mutation at an admissible sequence of vertices of Q not in F, and let G be the functor $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, ?)$ from $\mathcal{C}$ to the category of $\mathrm{End}_{\mathcal{C}}(T)$-modules. Let $X'_?$ be the associated cluster character, defined by

$$X'_M = x^{\mathrm{ind}_T M} \sum_e \Big(\chi(\mathrm{Gr}_e(GM))\Big) x^{-\iota(e)},$$

where $\iota(e)$ is the vector $\mathrm{ind}_T Y + \mathrm{ind}_T \Sigma Y$ for any Y such that the dimension vector of GY is e (it was proved in Lemma 3.3.6 that this vector is independent of the choice of such a Y, see also [68]).

Since we only allow mutations at vertices not in F, the Gabriel quiver of T can be thought of as an ice quiver (Q^T, F) with same set of frozen vertices as (Q, F). Let $B^T = (b^T_{j\ell})$ be the matrix associated to (Q^T, F). According to [41, Lemma 1.2] and [5, Lemma 3.2] , B^T is of full rank r if B is.

Suppose now that M is an object of $\mathcal{U}$. Let us prove that X'_M then admits a $\mathbf{g}$-vector, that is, X'_M is in the set $\mathcal{M}$ defined in Section 4.2.1. In order to do this, let us compute $-\iota(\delta_j)$, where δ_j is the vector whose j-th coordinate is 1 and all others are 0, for $j = 1, 2, \ldots, r$.

Let T_j^* be an indecomposable object of $\mathcal{D}$ such that GT_j^* is the simple $\mathrm{End}_{\mathcal{C}}(T)$-module at the vertex j. It follows from the derived equivalence in [58, Theorem 3.2] that we have triangles

$$T_j \to \bigoplus_{\substack{\alpha \in Q_1^T \\ s(\alpha)=j}} T_{t(\alpha)} \to T_j^* \to \Sigma T_j \quad \text{and} \quad T_j^* \to \bigoplus_{\substack{\alpha \in Q_1^T \\ t(\alpha)=j}} T_{s(\alpha)} \to T_j \to \Sigma T_j^*.$$

We deduce from those triangles that for any $0 \leq \ell \leq n$, the ℓ-th entry of $-\iota(\delta_j)$ is the number of arrows in Q^T from ℓ to j minus the number of arrows from j to ℓ. This number is $b^T_{\ell j}$. Thus, with the notations of Section 4.2.1, we have that $x^{-\iota(\delta_j)} = \prod_{\ell=1}^n x_\ell^{b_{\ell j}} = \hat{y}_j$.

Therefore, since ι is additive, for M in $\mathcal{U}$, we have the equality

$$X'_M = x^{\mathrm{ind}_T M} \sum_e \Big(\chi(\mathrm{Gr}_e(GM))\Big) \prod_{j=1}^r \hat{y}_j^{e_j}$$

(notice that if M is in $\mathcal{U}$, then $\mathrm{Gr}_e(GM)$ is empty for all vectors e such that one of $e_{r+1}, \ldots, e_n$ is non-zero). Moreover, the rational function

$$R(u_1, \ldots, u_r) = \sum_e \Big(\chi(\mathrm{Gr}_e(GM))\Big) \prod_{j=1}^r u_j^{e_j}$$

is in fact a polynomial with constant coefficient 1, and is thus primitive.

We have proved the following result.

Proposition 4.3.6. *Any object M of $\mathcal{U}$ is such that X'_M admits a $\mathbf{g}$-vector. This $\mathbf{g}$-vector $(g_1, \ldots, g_r)$ is given by $g_j = [\mathrm{ind}_T M : T_j]$, for $1 \leq j \leq r$.*

These considerations allow us to prove the following Theorem, whose parts (1), (3) and (4) were first shown in the same generality in [21] using decorated representations, and then in [65] using Donaldson–Thomas theory.

We say that a collection of vectors of $\mathbb{Z}^r$ are *sign-coherent* if the i-th coordinates of all the vectors of the collection are either all non-positive of all non-negative.

Theorem 4.3.7. *Let (Q, F) be any ice quiver without oriented cycles of length ≤ 2, and let $\mathcal{A}$ be the associated cluster algebra. Suppose that the matrix B associated with (Q, F) is of full rank r.*

1. *Conjecture 6.13 of [31] holds for $\mathcal{A}$, that is, the $\mathbf{g}$-vectors of the cluster variables of any given cluster are sign-coherent.*
2. *Conjecture 7.2 of [31] holds for $\mathcal{A}$, that is, the cluster monomials are linearly independent over $\mathbb{ZP}$, where $\mathbb{P}$ is the tropical semifield in the variables $x_{r+1}, \ldots, x_n$.*
3. *Conjecture 7.10 of [31] holds for $\mathcal{A}$, that is, different cluster monomials have different $\mathbf{g}$-vectors, and the $\mathbf{g}$-vectors of the cluster variables of any cluster form a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.*
4. *Conjecture 7.12 of [31] holds for $\mathcal{A}$, that is, if $\mathbf{g} = (g_1, \ldots, g_r)$ and $\mathbf{g}' = (g'_1, \ldots, g'_r)$ are the $\mathbf{g}$-vectors of one cluster monomial with respect to two clusters t and t' related by one mutation at the vertex i, then we have*

$$g'_j = \begin{cases} -g_i & \text{if } j = i \\ g_j + [b_{ji}]_+ g_i - b_{ji} \min(g_i, 0) & \text{if } j \neq i \end{cases}$$

where $B = (b_{j\ell})$ is the matrix associated with the seed t, and we set $[x]_+ = \max(x, 0)$ for any real number x.

Proof Choose a non-degenerate potential W on Q, and let $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category. Let $X'_?$ be the cluster character associated with Γ.

We first prove Conjecture 6.13. We reproduce the arguments of [18, Section 2.4]. To any cluster t of $\mathcal{A}$, we associate (using Theorem 3.4.1) a reachable rigid object T of $\mathcal{U}$, obtained by mutating at vertices not in F. Write T as the direct sum of the indecomposable objects $T_1, \ldots, T_n$. Then, for $1 \leq j \leq r$, we have that X'_{T_j} is a cluster variable lying in the cluster t. By Proposition 4.3.6, its $\mathbf{g}$-vector $(g^j_1, \ldots, g^j_r)$ is given by $g^j_\ell = [\mathrm{ind}_\Gamma T_j : \Gamma_\ell]$. Now, by Lemma 4.3.5, any minimal $\mathrm{add}\,\Gamma$-presentation of T

$$R_1 \longrightarrow R_0 \longrightarrow T \longrightarrow \Sigma R_1$$

is such that R_0 and R_1 have no direct factor in common. But this triangle is a direct sum of minimal presentations of $T_1, \ldots, T_n$. Therefore the indices of these objects must be sign-coherent. This proves Conjecture 6.13.

Next, we prove Conjecture 7.2. We prove it in the same way as in [32, Corollary 4.4 (b) and Theorem 6.3 (c)]. Using Theorem 3.4.1, we associate to any finite collection of clusters $(t_j)_{j \in J}$ of $\mathcal{A}$ a family of reachable rigid objects $(T^j)_{j \in J}$ of $\mathcal{U}$, obtained by mutating at vertices not in F (for the moment we do not know if this assignment is unique). Let $(M_j)_{j \in J}$ be a family of pairwise non-isomorphic objects, where each M_j lies in $\mathrm{add}\,T^j$ (in particular, these objects are rigid). Any $\mathbb{ZP}$-linear combination of cluster monomials can be written as a $\mathbb{Z}$-linear combination of some X'_{M_j}'s, where the M_j's are as above. Thus it is sufficient to show that the X'_{M_j}'s are linearly independent over $\mathbb{Z}$.

The key idea is to assign a degree to each x_j in such a way that each $\hat{y}_j$ is of degree 1. Such an assignment is obtained by putting $\deg(x_j) = k_j$, where the k_j's are rational numbers such that

$$(k_1, \dots, k_n)B = (1, \dots, 1).$$

This equation admits a solution, since the rank of B ir r. Thus the term of minimal degree in X'_M is $x^{\mathrm{ind}_\Gamma M}$, for any M in $\mathcal{U}$.

Now let $(c_j)_{j \in J}$ be a family of real numbers such that $\sum_{j \in J} c_j X'_{M_j} = 0$. The term of minimal degree of this polynomial has the form $\sum_{\ell \in L} c_\ell x^{\mathrm{ind}_\Gamma M_\ell}$ for some non-empty subset L of J, and this term must vanish. But according to Proposition 4.3.1, the indices of the M_ℓ's are pairwise distinct. Thus c_ℓ is zero for any $\ell \in L$. Repeating this argument, we get that c_j is zero for any $j \in J$. This proves the linear independence of cluster monomials.

The proof of Conjecture 7.10 goes as follows. Let $\{w_1, \dots, w_r\}$ be a cluster of $\mathcal{A}$, and let $w_1^{a_1} \dots w_r^{a_r}$ be a cluster monomial. Let $T = \bigoplus_{j=1}^r T_j \oplus \bigoplus_{j=r+1}^n \Gamma_n$ be the rigid object of $\mathcal{C}$ associated with that cluster. Then the cluster character

$$X'_M = x^{\mathrm{ind}_\Gamma M} \sum_e \Big(\chi(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M)) \Big) x^{-\iota(e)}$$

sends the object $\bigoplus_{j=1}^r T_j^{a_j}$ to the cluster monomial $w_1^{a_1} \dots w_r^{a_r}$. The $\mathbf{g}$-vector of this cluster monomial is the index of $\bigoplus_{j=1}^r T_j^{a_j}$ by Proposition 4.3.6, and by Proposition 4.3.1, this object is completely determined by its index. Therefore two different cluster monomials, being associated with different rigid objects of $\mathcal{C}$, have different $\mathbf{g}$-vectors.

Let us now prove that the $\mathbf{g}$-vectors of $w_1, \dots, w_r$ form a basis of $\mathbb{Z}^r$. For any object M of $\mathcal{D}$, denote by $(\mathrm{ind}_\Gamma M)_0$ the vector containing the first r components of $\mathrm{ind}_\Gamma M$. In view of Proposition 4.3.6, it is sufficient to prove that the vectors $(\mathrm{ind}_\Gamma T_1)_0, \dots, (\mathrm{ind}_\Gamma T_r)_0$ form a basis of $\mathbb{Z}^r$.

We prove this by induction. The statement is trivially true for Γ. Now suppose it is true for some reachable object T as above. Let $1 \leq \ell \leq r$ be a vertex of Q, and let $T' = \mu_\ell(T)$. We can write $T' = \bigoplus_{j=1}^n T'_j$, where $T'_j = T_j$ if $j \neq \ell$, and there are triangles

$$T_\ell \longrightarrow \bigoplus_{\substack{\alpha \in Q_1^T \\ s(\alpha)=\ell}} T_{t(\alpha)} \longrightarrow T'_\ell \longrightarrow \Sigma T_\ell \quad \text{and} \quad T'_\ell \longrightarrow \bigoplus_{\substack{\alpha \in Q_1^T \\ t(\alpha)=\ell}} T_{s(\alpha)} \longrightarrow T_\ell \longrightarrow \Sigma T'_\ell$$

thanks to [58]. Moreover, the space $\mathrm{Hom}_{\mathcal{C}}(T'_\ell, \Sigma T_\ell)$ is one-dimensional; by applying Lemma 3.3.8 (with the T of the Lemma being equal to our $\Sigma^{-1}\Gamma$), we get an isomorphism

$$(\Gamma)(T'_\ell, \Sigma T_\ell) \longrightarrow D\,\mathrm{Hom}_{\mathcal{C}}(T_\ell, \Sigma T'_\ell)/(\Gamma).$$

Therefore one of the two morphisms $T'_\ell \to \Sigma T_\ell$ and $T_\ell \to \Sigma T'_\ell$ in the triangles above is in (Γ). Depending on which one is in (Γ), and applying Lemma 3.3.4 (2), we get that either

$$\mathrm{ind}_\Gamma T'_j = \begin{cases} \mathrm{ind}_\Gamma T_j & \text{if } j \neq \ell \\ -\mathrm{ind}_\Gamma T_\ell + \sum_{\substack{\alpha \in Q_1^T \\ s(\alpha)=\ell}} \mathrm{ind}_\Gamma T_{t(\alpha)} & \text{if } j = \ell. \end{cases}$$

or

$$\mathrm{ind}_\Gamma T'_j = \begin{cases} \mathrm{ind}_\Gamma T_j & \text{if } j \neq \ell \\ -\mathrm{ind}_\Gamma T_\ell + \sum_{\substack{\alpha \in Q_1^T \\ t(\alpha)=\ell}} \mathrm{ind}_\Gamma T_{s(\alpha)} & \text{if } j = \ell. \end{cases}$$

holds. Therefore the $(\mathrm{ind}_\Gamma T'_j)_0$'s still form a basis of $\mathbb{Z}^r$. Conjecture 7.10 is proved.

Finally, let us now prove Conjecture 7.12. Let T and T' be reachable rigid objects related by a mutation at vertex ℓ, as above. Then we have two triangles

$$T_\ell \longrightarrow E \longrightarrow T'_\ell \longrightarrow \Sigma T_\ell \quad \text{and} \quad T'_\ell \longrightarrow E' \longrightarrow T_\ell \longrightarrow \Sigma T'_\ell,$$

where $E = \bigoplus_{\substack{\alpha\in Q_1^T\\ s(\alpha)=\ell}} T_{t(\alpha)}$ and $E' = \bigoplus_{\substack{\alpha\in Q_1^T\\ t(\alpha)=\ell}} T_{s(\alpha)}$. Moreover, the dimension of the space $\operatorname{Hom}_{\mathcal{C}}(T, \Sigma T')$ is one. Thus we can apply Proposition 3.2.7.

Let M be a rigid object in $\operatorname{pr}_{\mathcal{C}} T$, and let $T_1^M \to T_0^M \to M \to \Sigma T_1^M$ be a minimal $(\operatorname{add} T)$-presentation. Then, by Proposition 3.2.7, M is in $\operatorname{pr}_{\mathcal{C}} T'$. Moreover, if $T_0^M = \overline{T}_0^M \oplus T_\ell^a$ and $T_1^M = \overline{T}_1^M \oplus T_\ell^b$, where T_ℓ is not a direct summand of $\overline{T}_0^M \oplus \overline{T}_1^M$, then the end of the proof of that Proposition gives us a triangle

$$(E')^c \oplus E^{b-c} \oplus (T'_\ell)^{a-c} \oplus \overline{T}_1^M \longrightarrow (T'_\ell)^{b-c} \oplus \overline{T}_0^M \oplus (E')^a \longrightarrow M \longrightarrow \dots,$$

for some integer c. Notice that $[\operatorname{ind}_T M : T_\ell] = (a-b)$, and that since T_0^M and T_0^M have no direct factor in common by Lemma 4.3.5, one of a and b must vanish ; thus c also vanishes, since $c \leq \min(a,b)$. Notice further that $b = -\min([\operatorname{ind}_T M : T_\ell], 0)$. Thus

$$[\operatorname{ind}_{T'} M : T'_j] = \begin{cases} -[\operatorname{ind}_T M : T_\ell] \qquad (\text{if } j=\ell) \\ [\operatorname{ind}_T M : T_j] + [\operatorname{ind}_T M : T_\ell][b_{j\ell}^T]_+ - b_{j\ell}^T \min([\operatorname{ind}_T M : T_\ell], 0) \\ \qquad (\text{if } j \neq \ell). \end{cases}$$

This proves the desired result on **g**-vectors. □

Remark 4.3.8. Using the notations of the end of the proof of Theorem 4.3.7, we get that, if M is an object of $\mathcal{D}$ which is not necessarily rigid, then

$$[\operatorname{ind}_{T'} M : T'_j] = \begin{cases} -[\operatorname{ind}_T M : T_\ell] \qquad (\text{if } j=\ell) \\ [\operatorname{ind}_T M : T_j] + a[b_{j\ell}]_+ - b[-b_{j\ell}]_+ \qquad (\text{if } j \neq \ell). \end{cases}$$

Moreover, if the presentation $T_1^M \to T_0^M \to M \to \Sigma T_1^M$ is minimal, then the integer c vanishes. Indeed, in the proof of Proposition 3.2.7, c (or r in Proposition 3.2.7) is defined by means of the composition

$$T_1^M \longrightarrow \overline{T}_0^M \oplus T_\ell^a \longrightarrow \Sigma(T'_\ell)^a.$$

The minimality of the presentation implies that this composition vanishes, and thus that $c = 0$.

Using Theorem 4.3.7, we get a refinement of Theorem 3.4.1.

Corollary 4.3.9. *The cluster character $X'_?$ associated with Γ induces a bijection between the set of isomorphism classes of indecomposable reachable rigid objects of $\mathcal{C}$ and the set of cluster variables of $\mathcal{A}$.*

Proof It was proved in Theorem 3.4.1 that we have a surjection. We deduce from Theorem 4.3.7 that different indecomposable reachable rigid objects are sent to different cluster variables. Indeed, different such objects are sent to elements in $\mathcal{A}$ which are linearly independent, and thus different. □

We also get that the mutation of rigid objects governs the mutation of tropical Y-variables, as shown in [52, Corollary 6.9] in the Hom-finite case.

Corollary 4.3.10. *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let $\underline{i} = (i_1, \ldots, i_m)$ be an admissible sequence of vertices, and let T' be the object $\mu_{\underline{i}}(\Gamma)$. Let $(Q, \mathbf{y})$ be a Y-seed, with $\mathbf{y} = (y_1, \ldots, y_n)$.*

Then $\mu_{\underline{i}}(Q, \mathbf{y})$ is given by $(\mu_{\underline{i}}(Q), \mathbf{y}')$, where

$$y'_j = \prod_{s=1}^{n} y_s^{-[\mathrm{ind}_{\Sigma^{-1}T'} \Gamma_s : \Sigma^{-1} T'_j]}.$$

PROOF The result is proved by induction on m. It is trivially true for $m = 0$, that is, for empty sequences of mutations. Suppose it is true for any sequence of m mutations.

Let $\underline{i}' = (i_1, \ldots, i_m, \ell)$ be an admissible sequence of $m+1$ mutations. Let $T'' = \mu_{\underline{i}'}(\Gamma)$ and $(\mu_{\underline{i}'}(Q), \mathbf{y}'') = \mu_{\underline{i}'}(Q, \mathbf{y})$.

Using the mutation rule for Y-seeds (see section 4.2.1) and the induction hypothesis, we get that

$$y''_\ell = \prod_{s=1}^{n} y_s^{[\mathrm{ind}_{\Sigma^{-1}T'} \Gamma_s : \Sigma^{-1} T'_j]}$$

and that, for any vertex j different from ℓ,

$$y''_j = \prod_{s=1}^{n} y_s^{-[\mathrm{ind}_{\Sigma^{-1}T'} \Gamma_s : \Sigma^{-1} T'_j] - [\mathrm{ind}_{\Sigma^{-1}T'} \Gamma_s : \Sigma^{-1} T'_\ell][b^{T'}_{\ell j}]_+ - b^{T'}_{\ell j} \min(-[\mathrm{ind}_{\Sigma^{-1}T'} \Gamma_s : \Sigma^{-1} T'_\ell], 0)}$$

Now, recall from the end of the proof of Theorem 4.3.7 that for any object M of $\mathrm{pr}_{\mathcal{C}} T'$, we have an $(\mathrm{add}\, T'')$-presentation

$$(E')^c \oplus E^{b-c} \oplus (T''_\ell)^{a-c} \oplus \overline{T'}^M_1 \longrightarrow (T''_\ell)^{b-c} \oplus \overline{T'}^M_0 \oplus (E')^a \longrightarrow M \longrightarrow \ldots,$$

and that $[\mathrm{ind}_{T'} M : T'_\ell] = (a - b)$. Notice also that $a = -\min([-\mathrm{ind}_{T'} M : T'_\ell], 0)$. Thus

$$[\mathrm{ind}_{T''} M : T''_j] = \begin{cases} -[\mathrm{ind}_{T'} M : T'_\ell] & (\text{if } j = \ell) \\ [\mathrm{ind}_{T'} M : T'_j] + [\mathrm{ind}_{T'} M : T'_\ell][b^{T'}_{\ell j}]_+ + \\ \qquad + b^{T'}_{\ell j} \min(-[\mathrm{ind}_{T'} M : T'_\ell], 0) & (\text{if } j \neq \ell). \end{cases}$$

Replacing M by $\Sigma\Gamma_s$, and using the above computation of y''_j, we get exactly the desired equality. □

Remark 4.3.11. The opposite category $\mathcal{C}^{op}$ is triangulated with suspension functor $\Sigma_{op} = \Sigma^{-1}$. If T is a rigid object of $\mathcal{C}$, then it is rigid in $\mathcal{C}^{op}$, and any object X admitting an $(\mathrm{add}\, \Sigma^{-1} T)$-presentation in $\mathcal{C}$ admits an $(\mathrm{add}\, T)$-presentation in $\mathcal{C}^{op}$. If we denote by $\mathrm{ind}^{op}_T X$ the index of X with respect to T in $\mathcal{C}^{op}$, then we have the equality $\mathrm{ind}^{op}_T X = -\mathrm{ind}_{\Sigma^{-1}T} X$. Thus the equality of Corollary 4.3.10 can be written as

$$y'_j = \prod_{s=1}^{n} y_s^{[\mathrm{ind}^{op}_{T'} \Gamma_s : T'_j]}.$$

This corresponds to the notation and point of view adopted in [52, Corollary 6.9].

4.3.3 Cluster characters and F-polynomials

Let $\mathcal{A}$ be a cluster algebra with principal coefficients at a seed $\big((Q,F),\mathbf{x}\big)$. In particular, $n=2r$, and the matrix B associated with (Q,F) has full rank r.

Let W be a potential on Q, and let $\mathcal{C}=\mathcal{C}_{Q,W}$ be the cluster category associated with (Q,W). Let T be a rigid object of $\mathcal{C}$ reachable from Γ by mutation at an admissible sequence of vertices $(i_1,\ldots,i_s)$ not in F. Write T as $\bigoplus_{j=1}^{2r} T_j$, where $T_\ell=\Gamma_\ell$ for $r<\ell\leq 2r$.

For any vertex j not in F, X'_{T_j} is a cluster variable in $\mathcal{A}$. Specializing at $x_1=\ldots=x_r=1$, we obtain the corresponding F-polynomial (see Section 4.2.1), which we will denote by F_{T_j}.

We thus have the equality

$$F_{T_j}=\prod_{i=r+1}^{2r} x_i^{[\operatorname{ind}_\Gamma T_j:\Gamma_i]}\sum_e \chi\Big(\operatorname{Gr}_e\big(\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma,T_j)\big)\Big)\prod_{i=r+1}^{2r} x_i^{-\iota(e)_i},$$

where $\iota(e)$ was defined in section 3.3 and $\iota(e)_i$ is the i-th component of $\iota(e)$.

Remark 4.3.12. The element X'_{T_j} of $\mathcal{A}$ is the j-th cluster variable of the cluster obtained from the initial cluster at the sequence of vertices $(i_1,\ldots,i_s)$ by Theorem 3.4.1. Therefore, the polynomial F_{T_j} is the corresponding F-polynomial.

It follows from our computation in Section 4.3.2 that for $r<i\leq 2r$, there is an equality $-\iota(e)_i=\sum_{j=1}^r e_j b_{ij}$, and since our cluster algebra has principal coefficients, this number is e_{i-r}. Thus we get the equality

$$F_{T_j}=\prod_{i=r+1}^{2r} x_i^{[\operatorname{ind}_\Gamma T_j:\Gamma_i]}\sum_e \chi\Big(\operatorname{Gr}_e\big(\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma,T_j)\big)\Big)\prod_{i=r+1}^{2r} x_i^{e_{i-r}}.$$

From this we can prove the following theorem, using methods very similar to those found in [32], in which the theorem was proved in the Hom-finite case. Note that the theorem was shown in [21] using decorated representations and in [65] using Donaldson–Thomas theory.

Theorem 4.3.13. *Conjecture 5.6 of [31] holds for $\mathcal{A}$, that is, any F-polynomial has constant term* 1.

Proof It suffices to show that the polynomial F_{T_j} defined above has constant term 1. In order to do so, we will prove that, for any $r<i\leq 2r$, the number $[\operatorname{ind}_\Gamma T_j:\Gamma_i]$ vanishes.

We know that T_j lies in the subcategory $\mathcal{U}$ defined in Section 4.3.2, that is, for any $r<i\leq 2r$, the space $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i,T_j)$ vanishes. Using Proposition 3.2.16, we get that $\operatorname{Hom}_{\mathcal{C}}(T_j,\Sigma\Gamma_i)$ also vanishes.

Let $\overline{T}_1\to\overline{T}_0\to T_j\to\Sigma\overline{T}_1$ be a minimal $(\operatorname{add}\Gamma)$-presentation of T_j. Let $r<i\leq 2r$ be a vertex of Q. Suppose that Γ_i is a direct summand of $\overline{T}_1$. Since $\operatorname{Hom}_{\mathcal{C}}(T_j,\Sigma\Gamma_i)$ is zero, this implies that the presentation has the triangle

$$\Gamma_i\xrightarrow{1_{\Gamma_i}}\Gamma_i\longrightarrow 0\longrightarrow\Sigma\Gamma_i$$

as a direct summand, contradicting the minimality of the presentation. Thus Γ_i is not a direct summand of $\overline{T}_1$.

Suppose that Γ_i is a direct summand of $\overline{T}_0$. Since i is a sink in Q, and since Γ_i is not a direct summand of $\overline{T}_1$, we get that $\operatorname{Hom}_{\mathcal{C}}(\overline{T}_1,\Gamma_i)$ is zero. This implies that Γ_i is a direct summand of T_j, and since the latter is indecomposable, we get that it is isomorphic to the former. This is a contradiction, since T must be basic. $\square$

Definition 4.3.14. For any object M of $\mathcal{D}$, the *F-polynomial of M* is the polynomial

$$F_M = \sum_e \chi\Big(\mathrm{Gr}_e\big(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M)\big)\Big) \prod_{i=r+1}^{2r} x_i^{e_{i-r}}$$

in $\mathbb{Z}[x_{r+1}, \ldots, x_{2r}]$.

Thanks to Theorem 4.3.13, this definition is in accordance with the F_{T_i} used above. Note that we have the equality

$$X'_M\Big|_{x_1=\ldots=x_r=1} = \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma M:\Gamma_i]} F_M$$

We can deduce from the multiplication formula of Proposition 3.3.16 an equality for the polynomials F_M. This was first proved implicitly in [68, Section 5.1], see also [52, Theorem 6.12].

Proposition 4.3.15. *Let M and N be objects of $\mathcal{D}$ such that the space* $\mathrm{Hom}_{\mathcal{C}}(M, \Sigma N)$ *is one-dimensional. Let*

$$M \longrightarrow E \longrightarrow N \longrightarrow \Sigma M \quad \textit{and} \quad N \longrightarrow E' \longrightarrow M \longrightarrow \Sigma N$$

be non-split triangles. Then

$$F_M F_N = \prod_{i=r+1}^{2r} x_i^{d_{i-r}} F_E + \prod_{i=r+1}^{2r} x_i^{d'_{i-r}} F_{E'},$$

where $d = (d_1, \ldots, d_{2r})$ is the dimension vector of the kernel K of the induced morphism $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, E)$ *and $d' = (d'_1, \ldots, d'_{2r})$ is the dimension vector of the kernel K' of* $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, E')$.

PROOF We know from Proposition 3.3.16 that $X'_M X'_N = X'_E + X'_{E'}$. Specializing at $x_1 = \ldots = x_r = 1$, we get the equality

$$\prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma M:\Gamma_i]+[\mathrm{ind}_\Gamma N:\Gamma_i]} F_M F_N = \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma E:\Gamma_i]} F_E + \prod_{i=r+1}^{2r} x_i^{[\mathrm{ind}_\Gamma E':\Gamma_i]} F_{E'}.$$

It follows from Lemma 3.3.5 (applied to the above triangles shifted by Σ^{-1}, and with $T = \Sigma^{-1}\Gamma$) that

$$\begin{aligned} \mathrm{ind}_\Gamma M + \mathrm{ind}_\Gamma N &= \mathrm{ind}_\Gamma E + \mathrm{ind}_\Gamma K + \mathrm{ind}_\Gamma \Sigma K \\ &= \mathrm{ind}_\Gamma E' + \mathrm{ind}_\Gamma K' + \mathrm{ind}_\Gamma \Sigma K', \end{aligned}$$

where K and K' are as in the statement of the Proposition. But $\mathrm{ind}_\Gamma K + \mathrm{ind}_\Gamma \Sigma K = \iota(d)$, and using our computation of $\iota(e)$ of Section 4.3.2, we get that $-\iota(d)_i = d_{i-r}$ for $r < i \leq 2r$.

Similarly, we get that $\mathrm{ind}_\Gamma K' + \mathrm{ind}_\Gamma \Sigma K' = \iota(d')$, and that $-\iota(d')_i = d'_{i-r}$ for $r < i \leq 2r$. The desired equality follows.

$\square$

4.4 Link with decorated representations

In this section, an explicit link between cluster categories and the decorated representations of [22] is established. We show that the mutation of decorated representations of [22] corresponds to the derived-equivalence of [58], and we give an interpretation of the E-invariant of [21] as half the dimension of the space of selfextensions of an object in the cluster category.

4.4.1 Mutations

Let (Q, W) be a quiver with potential. Let $\Gamma = \Gamma_{Q,W}$ be the associated complete Ginzburg dg algebra, and $\mathcal{C} = \mathcal{C}_{Q,W}$ be the associated cluster category. Let $B = B_{Q,W}$ be the endomorphism algebra of Γ in $\mathcal{C}$. Recall from [58, Lemma 2.8] that B is the Jacobian algebra of (Q, W). Denote by F the functor $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$ from $\mathcal{C}$ to $\mathrm{Mod}\, B$. Let $\mathcal{D} = \mathcal{D}_{Q,W}$ be the full subcategory of $\mathrm{pr}_{\mathcal{C}}\Gamma \cap \mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma$ whose objects are those X such that FX is finite dimensional.

Consider the map $\Phi = \Phi_{Q,W}$ from the set of isomorphism classes of objects in $\mathcal{D}$ to the set of isomorphism classes of decorated representations of (Q, W) defined as follows. For any object X of $\mathcal{D}$, write $X = X' \oplus \bigoplus_{i \in Q_0} (e_i\Gamma)^{m_i}$, where X' has no direct summands in $\mathrm{add}\,\Gamma$. Such a decomposition of X is unique up to isomorphism, since $\mathrm{pr}_{\mathcal{C}}\Gamma$ is a Krull–Schmidt category, as shown in Chapter 3. Define $\Phi(X)$ to be the decorated representation $(F(X'), \bigoplus_{i \in Q_0} S_i^{m_i})$, where $(0, S_i)$ is the negative simple representation at the vertex i, for any i in Q_0.

Consider also the map $\Psi = \Psi_{Q,W}$ from the set of isomorphism classes of decorated representations of (Q, W) to the set of isomorphism classes of objects in $\mathcal{D}$ defined as follows. Recall from Chapter 3 that F induces an equivalence $\mathrm{pr}_{\mathcal{C}}\Sigma^{-1}\Gamma/(\Gamma) \to \mathrm{mod}\, B$, where $\mathrm{mod}\, B$ is the category of finitely presented B-modules. Let G be a quasi-inverse equivalence. For any decorated representation $(M, \bigoplus_{i \in Q_0} S_i^{m_i})$, choose a representative $\overline{M}$ of $G(M)$ in $\mathcal{D}$ which has no direct summands in $\mathrm{add}\,\Gamma$ (the representative can be chosen to be in $\mathcal{D}$ thanks to Lemma 3.3.2). Such a representative is unique up to (non-unique) isomorphism. The map Ψ then sends $(M, \bigoplus_{i \in Q_0} S_i^{m_i})$ to the object $\overline{M} \oplus \bigoplus_{i \in Q_0} (e_i\Gamma_i)^{m_i}$.

The diagram below summarizes the definitions of Φ and Ψ.

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{isoclasses of} \\ \text{objects of } \mathcal{D} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{isoclasses of decorated} \\ \text{representations of } (Q, W) \end{array} \right\} \\
X = X' \oplus \bigoplus_{i=1}^{n} (e_i\Gamma)^{m_i} & \longmapsto & \Phi(X) = (FX', \bigoplus_{i=1}^{n} (S_i)^{m_i}) \\
\Psi(\mathcal{M}) = \overline{M} \oplus \bigoplus_{i=1}^{n} (e_i\Gamma)^{m_i} & \longleftarrow\!\shortmid & \mathcal{M} = (M, \bigoplus_{i=1}^{n} S_i^{m_i})
\end{array}$$

The main result of this subsection states that the maps Φ and Ψ are mutually inverse bijections, on the one hand, and that, via these maps, the derived equivalences of [58] are compatible with the mutations of decorated representations of [22], on the other hand.

Proposition 4.4.1. *With the above notations, Φ and Ψ are mutual inverse maps. Moreover, if $i \in Q_0$ is not on any cycle of length ≤ 2, and if $(Q', W') = \widetilde{\mu}_i(Q, W)$, then for any object X of $\mathcal{D}$, we have that*

$$\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X)),$$

where the functor $\widetilde{\mu}_i^-$ is as defined after Theorem 3.2.6.

The rest of this section is devoted to the proof of the Proposition.

It is obvious from the definition of Φ and Ψ that the two maps are mutual inverses. Thus we only need to show that the two mutations agree.

Let Γ' be the complete Ginzburg dg algebra of (Q', W'). Note that $\mathrm{End}_{\mathcal{C}'}(\Gamma')$ is the Jacobian algebra $J(Q', W')$, by [58, Lemma 2.8]. Let $\mathcal{C}'$ be the cluster category associated with (Q', W'). We know from [21] that $\widetilde{\mu}_i(\Phi_{Q,W}(X))$ is a decorated representation of $(Q', W') = \widetilde{\mu}_i(Q, W)$. We need to show that it is isomorphic to $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X))$.

We can (and will) assume for the rest of the proof that X is indecomposable, as all the maps and functors considered commute with finite direct sums.

We first prove the proposition for some special cases.

Lemma 4.4.2. *Assume that X is an indecomposable object of $\mathcal{D}$ such that either*

– *X is of the form $e_j\Gamma$ for $j \neq i$, or*

– *X is the cone Γ_i^* of the morphism*

$$\Gamma_i \longrightarrow \bigoplus_\alpha \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α.

Then the equality $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X))$ holds.

PROOF Suppose that $X = e_j\Gamma$ for some vertex $i \neq j$. Then $\widetilde{\mu}_i(\Phi_{Q,W}(X)) = \widetilde{\mu}_i(0, S_j) = (0, S_j)$, and $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \Phi_{Q',W'}(e_j\Gamma') = (0, S_j)$, so the desired equality holds.

Suppose now that X is the cone Γ_i^* of the morphism

$$\Gamma_i \longrightarrow \bigoplus_\alpha \Gamma_{t(\alpha)}$$

whose components are given by left multiplication by α. In that case, $\widetilde{\mu}_i^-(X) = e_i\Gamma'$ and $\Phi(X) = (S_i, 0)$, so the desired equality is also satisfied. $\square$

Now suppose that X is not of the above form. Using the definition of $\widetilde{\mu}_i^-$, we get that $\Phi(\widetilde{\mu}_i^-(X))$ is equal to $\Phi(\mathcal{H}om_\Gamma(T, X))$, where T is as defined in section 4.2.4. Because of our assumptions on X, this decorated representation is $\Big(\mathrm{Hom}_{\mathcal{C}'}\left(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, X)\right), 0\Big)$.

We have the isomorphisms of $\mathrm{End}_{\mathcal{C}'}(\Gamma')$-modules

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}'}\left(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, X)\right) &= \mathrm{Hom}_{\mathcal{D}\Gamma'}\left(\Sigma^{-1}\Gamma', \mathcal{H}om_\Gamma(T, \overline{X})\right) \\ &= \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}\Gamma' \otimes^L_{\Gamma'} T, \overline{X}) \\ &= \mathrm{Hom}_{\mathcal{D}\Gamma}(\Sigma^{-1}T, \overline{X}) \\ &= \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X), \end{aligned}$$

where $\overline{X}$ is a lift of X in $\mathrm{pr}_{\mathcal{D}\Gamma}\Sigma^{-1}\Gamma$. Using this, we prove the Proposition for another special case.

Lemma 4.4.3. *If $X = e_i\Gamma$, then $\Phi_{Q',W'}(\widetilde{\mu}_i^-(X)) = \widetilde{\mu}_i(\Phi_{Q,W}(X))$.*

PROOF We have that $\widetilde{\mu}_i(\Phi_{Q,W}(e_i\Gamma)) = (S_i, 0)$. Moreover, the above calculation gives that $\Phi_{Q',W'}(\widetilde{\mu}_i^-(e_i\Gamma)) = \big(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma), 0\big)$.

For any vertex $j \neq i$, we have that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)e_j = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_jT), e_i\Gamma) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_j\Gamma), e_i\Gamma)$, and this last space is zero.

For the vertex i, we have isomorphisms $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)e_i = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_iT), e_i\Gamma) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, e_i\Gamma)$, and this space is one-dimensional.

Therefore $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, e_i\Gamma)$ is the simple module at the vertex i, and this proves the desired equality. $\Box$

We now treat the remaining cases, that is, those where X is not in $\mathrm{add}\,\Gamma$ and is not Γ_i^*. Then $\Phi(X) = (FX, 0)$, and $\widetilde{\mu}_i(\Phi_{Q,W}(X)) = \widetilde{\mu}_i(FX, 0) = (M', 0)$ is computed using section 4.2.2. We will show that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is isomorphic to M' as a $J(Q', W')$-module, using heavily the definition of T given in section 4.2.4.

Lemma 4.4.4. *For any vertex j, the vector spaces $M'e_j$ and $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_j$ are isomorphic.*

PROOF If j is a vertex different from i, then we have the isomorphisms of vector spaces $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_j = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_jT), X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_j\Gamma), X) = (FX)e_j = M'e_j$.

For the vertex i, we have isomorphisms $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)e_i = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}(e_iT), X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, X)$. Let us show that this space is isomorphic to $M'e_i$.

We have triangles in $\mathcal{C}$

$$e_i\Gamma \longrightarrow \bigoplus_{s(a)=i} e_{t(a)}\Gamma \longrightarrow \Gamma_i^* \longrightarrow \Sigma(e_i\Gamma) \quad \text{and}$$

$$\overline{\Gamma}_i^* \longrightarrow \bigoplus_{t(a)=i} e_{s(a)}\Gamma \longrightarrow e_i\Gamma \longrightarrow \Sigma\Gamma_i^*.$$

These triangles yield a diagram with exact rows

$$\begin{tikzcd}
(\Sigma^{-1}\overline{\Gamma}_i^*, X) \arrow[d, "\varphi_i^*"] & (\Sigma^{-1}\bigoplus_{t(a)=i} e_{s(a)}\Gamma, X) \arrow[l, "h"'] \arrow[d, "-\gamma"] & (\Sigma^{-1}(e_i\Gamma), X) \arrow[l, "\beta"'] \arrow[d, equal] & (\Gamma_i^*, X) \arrow[l] \\
(\Sigma^{-1}\Gamma_i^*, X) \arrow[r, "g"] & (\Sigma^{-1}\bigoplus_{s(a)=i} e_{t(a)}\Gamma, X) \arrow[r, "\alpha"] & (\Sigma^{-1}(e_i\Gamma), X) \arrow[r] & (\Sigma^{-2}\Gamma_i^*, X),
\end{tikzcd}$$

where we write (Y_1, Y_2) for $\mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2)$, where $-\gamma = g\varphi_i^* h$, and where φ_i was defined in section 4.2.4. Note that φ_i^* is an isomorphism.

Notice that, in the notations of section 4.2.2, we have that $(\Sigma^{-1}\bigoplus_{t(a)=i} e_{s(a)}\Gamma, X) = (FX)_{out}$ and $(\Sigma^{-1}\bigoplus_{s(a)=i} e_{t(a)}\Gamma, X) = (FX)_{in}$. Moreover, the maps α and β in the diagram above correspond to the maps α and β of section 4.2.2.

The map γ above also corresponds to the map γ defined in section 4.2.2. This follows from the computation we made in Remark 4.2.1.

Using the above diagram, we get isomorphisms

$$\begin{aligned}
(\Sigma^{-1}\Gamma_i^*, X) &\cong \mathrm{Im}\ g \oplus \mathrm{Ker}\, g \\
&\cong \mathrm{Ker}\,\alpha \oplus \mathrm{Ker}\, g
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ker}\,\gamma &\cong h^{-1}(\varphi_i^{*-1}(\mathrm{Ker}\, g)) \\
&\cong \mathrm{Ker}\, h \oplus \mathrm{Ker}\, g \\
&\cong \mathrm{Im}\ \beta \oplus \mathrm{Ker}\, g.
\end{aligned}$$

Thus $(\Sigma^{-1}\Gamma_i^*, X)$ is (non-canonically) isomorphic to $\mathrm{Ker}\,\alpha \oplus \frac{\mathrm{Ker}\,\gamma}{\mathrm{Im}\ \beta}$, which is in turn isomorphic to $\frac{\mathrm{Ker}\,\gamma}{\mathrm{Im}\ \beta} \oplus \mathrm{Im}\ \gamma \oplus \frac{\mathrm{Ker}\,\alpha}{\mathrm{Im}\ \gamma}$. But this is precisely $M'e_i$. $\Box$

It remains to be shown that the action of the arrows of Q' on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is the same as on M' in order to get the following Lemma.

Lemma 4.4.5. *As a $J(Q', W')$-module,* $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ *is isomorphic to* M'*.*

PROOF We know from Lemma 4.4.4 that the two modules considered are isomorphic as Λ-modules, where Λ is as in section 3.2.2.

Now let a be an arrow of Q not incident with i. Then a is an arrow of Q', and its action on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is obviously the same as its action on M'.

Consider now an arrow of Q' of the form $[ba]$, where $t(a) = i = s(b)$ in Q. By the definition of M' given in section 4.2.2, $[ba]$ acts as ba on M', that is, $M'_{[ba]} = (FX)_{ba}$.

According to the definition of T given in section 4.2.4, $[ba]$ acts on T as the map

$$\begin{array}{rcl} T_{s(a)} & \longrightarrow & T_{t(b)} \\ x & \longmapsto & bax. \end{array}$$

Hence the action of $[ba]$ on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ is also given by multiplication by ba. Thus the action of $[ba]$ on M' and on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ coincide.

There remains to be considered the action of the arrows incident with i.

Keep the notations introduced in the proof of Lemma 4.4.4. We assert that the maps $\varphi_i^* h$ and g encode the action of the arrows incident with i.

Recall that in $\mathcal{D}\Gamma$, the object Γ_i^* is isomorphic as a graded module to

$$\Sigma(e_i\Gamma) \oplus \bigoplus_{\substack{a\in Q_1 \\ s(a)=i}} e_{t(a)}\Gamma,$$

and that the map $\bigoplus_{\substack{a\in Q_1 \\ s(a)=i}} e_{t(a)}\Gamma \longrightarrow \Gamma_i^*$ is the canonical inclusion. Thus, its components are given by

$$\begin{array}{rcl} e_{t(a)}\Gamma & \longrightarrow & \Gamma_i^* \\ x & \longmapsto & e_a x. \end{array}$$

for any arrow a of Q such that $s(a) = i$. By the definition of T, this is multiplication by $a^\star$. Therefore g encodes the action of the arrows $a^\star$ of Q', where $s(a) = i$ in Q.

Similarly, recall that in $\mathcal{D}\Gamma$, the object $\overline{\Gamma}_i^*$ is isomorphic as a graded module to

$$(\bigoplus_{\substack{b\in Q_1 \\ t(b)=i}} e_{t(b)}\Gamma) \oplus \Sigma^{-1}(e_i\Gamma)$$

and that the map $\overline{\Gamma}_i^* \longrightarrow \bigoplus_{\substack{b\in Q_1 \\ t(b)=i}} e_{t(b)}\Gamma$ is given by the canonical projection. Thus its composition with φ_i^* is given by the matrix $\begin{pmatrix} -b^* & -\partial_{ab}W \end{pmatrix}$. Its components are the maps

$$\begin{array}{rcl} \Gamma_i^* & \longrightarrow & e_{s(b)}\Gamma \\ e_{\Sigma i}x_i + \sum_{s(a)=i} e_a x_a & \longmapsto & -b^* x_i + \sum_{s(a)=i} (\partial_{ab}W) x_a \end{array}$$

for any arrow b of Q such that $t(b) = i$. By the definition of T, this is multiplication by $b^\star$. Thus $\varphi_i^* h$ encodes the action of the arrows $b^\star$ of Q', where $t(b) = i$ in Q.

Finally, recall from Lemma 4.4.4 that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma_i^*, X)$ is isomorphic to $\frac{\mathrm{Ker}\,\gamma}{\mathrm{Im}\,\beta} \oplus \mathrm{Im}\,\gamma \oplus \frac{\mathrm{Ker}\,\alpha}{\mathrm{Im}\,\gamma}$. Recall that the summand $\frac{\mathrm{Ker}\,\gamma}{\mathrm{Im}\,\beta}$ corresponds to $\mathrm{Ker}\,g$, while the summand $\mathrm{Im}\,\gamma \oplus \frac{\mathrm{Ker}\,\alpha}{\mathrm{Im}\,\gamma}$ corresponds to $\mathrm{Im}\,g$.

We choose a spliting $\mathrm{Im}\ \gamma \oplus \frac{\mathrm{Ker}\ \alpha}{\mathrm{Im}\ \gamma}$ in such a way that $\mathrm{Im}\ \varphi_i^* h \cap \frac{\mathrm{Ker}\ \alpha}{\mathrm{Im}\ \gamma} = 0$. In that case, g is given in matrix form by $\begin{pmatrix} 0 & \iota & \iota\sigma \end{pmatrix}$ and $\varphi_i^* h$, by $\begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \end{pmatrix}$, in the notations of section 4.2.2. This proves that the action of the arrows of Q' on M' and on $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T, X)$ coincide, finishing the proof of the Lemma. $\square$

We have proved Proposition 4.4.1.

4.4.2 Interpretation of F-polynomials, g-vectors and h-vectors

In this section, we study the relation between the F-polynomials of objects of $\mathcal{D}$ and of decorated representations, and between the index of objects in $\mathcal{D}$ and the **g**-vectors of decorated representations. We also give an interpretation of the **h**-vector.

Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. We keep the notations of the previous section for the maps Φ and Ψ.

We first prove a result regarding F-polynomials.

Proposition 4.4.6. *Let X be an object of $\mathcal{D}$. Then we have the equality*

$$F_X(x_{r+1}, \ldots, x_n) = F_{\Phi(X)}(x_{r+1}, \ldots, x_n).$$

PROOF This is immediate from the definitions of F_X, Φ and $F_{\Phi(X)}$, given in Definition 4.3.14, Section 4.4.1 and Section 4.2.3, respectively. $\square$

We now prove that **g**-vectors of decorated representations and indices of objects in the cluster category are closely related. We will need the following Hom-infinite extension of [68, Lemma 7].

Lemma 4.4.7. *Let M be an indecomposable object of $\mathcal{D}$. Then*

$$[\mathrm{ind}_\Gamma M : e_i\Gamma] = \begin{cases} \delta_{ij} & \text{if } M \cong e_i\Gamma \\ \dim \mathrm{Ext}^1_B(S_i, FM) - \dim \mathrm{Hom}_B(S_i, FM) & \text{otherwise,} \end{cases}$$

where $B = \mathrm{End}_{\mathcal{C}}(\Gamma)$.

PROOF The result is obvious if M lies in $\mathrm{add}\,\Gamma$. Suppose it does not. Let $T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$ be an $(\mathrm{add}\,\Gamma)$-presentation of M.

The opposite category $\mathcal{C}^{op}$ is triangulated, with suspension functor $\Sigma_{op} = \Sigma^{-1}$. Thus, in $\mathcal{C}^{op}$, we have a triangle $\Sigma_{op}^{-1} T_0 \longrightarrow \Sigma_{op}^{-1} T_1 \longrightarrow M \longrightarrow T_0$. Applying the functor $F' = \mathrm{Hom}_{\mathcal{C}^{op}}(\Sigma_{op}^{-1}\Gamma, ?)$, we get a minimal projective resolution

$$F'\Sigma_{op}^{-1} T_0 \longrightarrow F'\Sigma_{op}^{-1} T_1 \longrightarrow F'M \longrightarrow 0$$

of $F'M$ as a B^{op}-module.

Letting S_i' be the simple B^{op}-module at the vertex i, we apply $\mathrm{Hom}_{B'}(?, S_i)$ to the above exact sequence and get a complex

$$0 \longrightarrow \mathrm{Hom}_{B^{op}}(F'\Sigma_{op}^{-1} T_1, S_i') \longrightarrow \mathrm{Hom}_{B^{op}}(F'\Sigma_{op}^{-1} T_0, S_i') \longrightarrow \ldots$$

whose differential vanishes, since the presentation is minimal.

Therefore we have the equalities

$$\begin{aligned}[\mathrm{ind}_\Gamma M : e_i\Gamma] &= \dim \mathrm{Ext}^1_{B^{op}}(F'M, S'_i) - \dim \mathrm{Hom}_{B^{op}}(F'M, S'_i)\\ &= \dim \mathrm{Ext}^1_B(S_i, DF'M) - \dim \mathrm{Hom}_B(S_i, DF'M),\end{aligned}$$

where S_i is the simple B-module at the vertex i.

Now, using Proposition 3.2.16, we get that

$$DF'M = D\,\mathrm{Hom}_{\mathcal{C}^{op}}(\Sigma_{op}^{-1}\Gamma, M) = D\,\mathrm{Hom}_{\mathcal{C}}(M, \Sigma\Gamma) \cong \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M) = FM.$$

Thus $DF'M$ is isomorphic to FM as a B-module. This proves the lemma. □

We now prove the result on $\mathbf{g}$-vectors of decorated representations.

Proposition 4.4.8. *Let (Q,W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X be an object of $\mathcal{D}$. Let $\mathbf{g}_{\Phi(X)} = (g_1, \ldots, g_n)$ be the $\mathbf{g}$-vector of the decorated representation $\Phi(X)$. Then we have the equality*

$$g_i = [\mathrm{ind}_\Gamma X : \Gamma_i]$$

for any vertex i of Q.

PROOF We can assume that X is indecomposable. If X lies in $\mathrm{add}\,\Gamma$, then the result is obviously true. Suppose that X does not lie in $\mathrm{add}\,\Gamma$.

Using the two triangles in $\mathcal{C}$

$$e_i\Gamma \longrightarrow \bigoplus_{s(a)=i} e_{t(a)}\Gamma \longrightarrow \Gamma_i^* \longrightarrow \Sigma(e_i\Gamma) \quad \text{and}$$

$$\overline{\Gamma}_i^* \longrightarrow \bigoplus_{t(a)=i} e_{s(a)}\Gamma \longrightarrow e_i\Gamma \longrightarrow \Sigma\Gamma_i^*.$$

and applying the functor $F = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$, we get a projective resolution of the simple B-module S_i at the vertex i:

$$P_i \longrightarrow \bigoplus_{s(a)=i} P_{t(a)} \longrightarrow \bigoplus_{t(a)=i} P_{s(a)} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0,$$

where P_j is the indecomposable projective B-module at the vertex j. Applying now the functor $\mathrm{Hom}_B(?, FM)$, we get the complex

$$0 \longrightarrow (FM)_i \xrightarrow{\beta_i} (FM)_{out} \xrightarrow{-\gamma_i} (FM)_{in} \xrightarrow{\alpha_i} (FM)_i.$$

From this complex, we see that $\mathrm{Hom}_B(S_i, M) = \mathrm{Ker}\,\beta_i$ and that $\mathrm{Ext}^1_B(S_i, M) = \mathrm{Ker}\,\gamma_i / \mathrm{Im}\ \beta_i$. We also deduce an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\beta_i \longrightarrow (FM)_i \xrightarrow{\beta_i} \mathrm{Ker}\,\gamma_i \longrightarrow \mathrm{Ker}\,\gamma_i/\mathrm{Im}\ \beta_i \longrightarrow 0.$$

Using the above arguments and Lemma 4.4.7, we get the equalities

$$\begin{aligned}[\mathrm{ind}_\Gamma X : e_i\Gamma] &= \dim \mathrm{Ext}^1_B(S_i, M) - \dim \mathrm{Hom}_B(S_i, M)\\ &= \dim(\mathrm{Ker}\,\gamma_i/\mathrm{Im}\ \beta_i) \dim \mathrm{Ker}\,\beta_i\\ &= \dim \mathrm{Ker}\,\gamma_i - \dim(FM)_i\\ &= g_i.\end{aligned}$$

This finishes the proof. □

As a corollary of the proof of the above Proposition, we get an interpretation of the $\mathbf{h}$-vector of a decorated representation.

Corollary 4.4.9. *For any decorated representation $\mathcal{M} = (M, V)$ of a quiver with potential (Q, W), we have the equality*

$$h_i = -\dim \operatorname{Hom}_{J(Q,W)}(S_i, M)$$

for any vertex i of Q.

This provides us with a way of "counting" the number of terms in a minimal presentation.

Corollary 4.4.10. *If $\mathbf{g} = (g_1, \ldots, g_n)$ and $\mathbf{h} = (h_1, \ldots, h_n)$ are the $\mathbf{g}$-vector and $\mathbf{h}$-vector of a decorated representation $\mathcal{M} = (M, V)$, $\mathbf{h}' = (h'_1, \ldots, h'_n)$ is the $\mathbf{h}$-vector of $\mu_i(\mathcal{M})$, and if*

$$T_1 \longrightarrow T_0 \longrightarrow \Psi(\mathcal{M}) \longrightarrow \Sigma T_1$$

is a minimal $(\operatorname{add}\Gamma)$-presentation of $\Psi(\mathcal{M})$ (see Proposition 4.4.1), then $-h_i$ and $-h'_i$ are the number of direct summands of T_1 and T_0 which are isomorphic to Γ_i, respectively.

Proof It follows from Corollary 4.4.9 that $-h_i = \dim \operatorname{Hom}_{J(Q,W)}(S_i, M)$.

Let T_i^* be an indecomposable object of $\mathcal{D}$ such that $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, T_i^*)$ is the simple S_i. Then, by Lemma 3.3.2, we have that

$$\operatorname{Hom}_{J(Q,W)}(S_i, M) \cong \operatorname{Hom}_{\mathcal{C}}(T_i^*, \Psi(M))/(\Gamma).$$

Applying $\operatorname{Hom}_{\mathcal{C}}(T_i^*, ?)$ to the presentation, we get a long exact sequence

$$(T_i^*, T_0) \xrightarrow{\psi_*} (T_i^*, \Psi(\mathcal{M})) \xrightarrow{\phi_*} (T_i^*, \Sigma T_1) \longrightarrow (T_i^*, \Sigma T_0).$$

We see that the image of ψ_* is $(\Gamma)(T_i^*, \Psi(M))$, so that $\operatorname{Hom}_{\mathcal{C}}(T_i^*, \Psi(M))/(\Gamma)$ is isomorphic to the image of ϕ_*. Thus $-h_i$ is the dimension of the image of ϕ_*.

Using Proposition 3.2.16, we get that the morphism $(T_i^*, \Sigma T_1) \longrightarrow (T_i^*, \Sigma T_0)$ is isomorphic to the morphism $D(\Sigma^{-1}T_1, T_i^*) \longrightarrow D(\Sigma^{-1}T_0, T_i^*)$, and this morphism is zero since the presentation is minimal. Thus ϕ_* is surjective.

Therefore $-h_i$ is equal to the dimension of $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}T_1, T_i^*)$, which is equal to the number of direct factors of T_1 isomorphic to Γ_i in any decomposition of T_1.

Furthermore, [21, Lemma 5.2] gives that $g_i = h_i - h'_i$, and by Proposition 4.4.8, $g_i = [\operatorname{ind}_\Gamma \Psi(\mathcal{M}) : \Gamma_i]$. This immediately implies that $-h'_i$ is equal to the number of direct factors of T_0 isomorphic to Γ_i, and finishes the proof. □

Remark 4.4.11. Corollary 4.4.10 allows us to reformulate Remark 4.3.8 in the following way. If M is any object of $\mathcal{D}$, and if $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{h}' = (h'_1, \ldots, h'_n)$ are the $\mathbf{h}$-vectors of $\Phi(M)$ and $\tilde{\mu}_i\Phi(M)$, respectively, then

$$[\operatorname{ind}_{T'} M : T'_j] = \begin{cases} -[\operatorname{ind}_T M : T_i] & \text{(if } i = j) \\ [\operatorname{ind}_T M : T_j] - h'_i[b_{ji}]_+ + h_i[-b_{ji}]_+ & \text{(if } i \neq j). \end{cases}$$

As a corollary, we get a proof of Conjecture 6.10 of [31].

Corollary 4.4.12. *Conjecture 6.10 of [31] is true, that is, if $\mathbf{g} = (g_1, \ldots, g_n)$ and $\mathbf{g}' = (g'_1, \ldots, g'_n)$ are the $\mathbf{g}$-vectors of one cluster variable with respect to two clusters t and t' related by one mutation at vertex i, and if $\mathbf{h} = (h_1, \ldots, h_n)$ and $\mathbf{h}' = (h'_1, \ldots, h'_n)$ are its $\mathbf{h}$-vectors with respect to those clusters, then we have that*

$$h'_i = -[g_i]_+ \quad \textit{and} \quad h_i = \min(0, g_i).$$

PROOF Let M be an indecomposable object of $\mathcal{D}$ such that X'_M is the cluster variable considered in the statement. In particular, M is reachable, and thus rigid. It follows from equation (5.5) of [21] that the **h**-vector of the cluster variable corresponds to the **h**-vector of the associated decorated representation.

Since M is rigid, Proposition 4.3.5 tells us that any minimal $(\operatorname{add}\Gamma)$-presentation of M has disjoint direct factors. The result follows directly from this and from Corollary 4.4.10. $\square$

Remark 4.4.13. Conjecture 6.10 of [31] also follows directly from Conjecture 7.12 (see Theorem 4.3.7(4) above) and equations (6.15) and (6.26) of [31]. We give the above proof because it is an application of the results developped in this thesis.

Finally, we get an interpretation of the substitution formula of [21, Lemma 5.2] in terms of cluster characters.

Corollary 4.4.14. *Let (Q, W) be a quiver with potential. Let i be an admissible vertex of Q, and let $\varphi_X : \mathbb{Q}(x'_1, \ldots, x'_n) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ be the field isomorphism sending x'_j to x_j if $i \neq j$ and to*

$$(x_i)^{-1}\Big(\prod_{\ell=1}^{n} x_\ell^{[b_{\ell i}]_+} + \prod_{\ell=1}^{n} x_\ell^{[-b_{\ell i}]_+}\Big)$$

if $i = j$. Let $\mathcal{C}$ and $\mathcal{C}'$ be the cluster categories of (Q, W) and $\widetilde{\mu}_i(Q, W)$, respectively, and let $\widetilde{\mu}_i^+ : \mathcal{C}' \longrightarrow \mathcal{C}$ be the associated functor (see [58, Theorem 3.2]).

Then for any object M of the subcategory $\mathcal{D}'$ of $\mathcal{C}'$, we have that

$$X'_{\widetilde{\mu}_i^+(M)} = \varphi_X(X'_M).$$

PROOF Consider the field isomorphism $\varphi_Y : \mathbb{Q}(y'_1, \ldots, y'_n) \longrightarrow \mathbb{Q}(y_1, \ldots, y_n)$ whose action on y'_j is given by

$$\varphi_Y(y'_j) = \begin{cases} y_i^{-1} & \text{if } i = j \\ y_j y_i^m (y_i + 1)^{-m} & \text{if there are } m \text{ arrows from } i \text{ to } j \\ y_j (y_i + 1)^m & \text{if there are } m \text{ arrows from } j \text{ to } i. \end{cases}$$

Consider also the morphism $(\hat{-}) : \mathbb{Q}(y_1, \ldots, y_n) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ sending each y_j to

$$\hat{y}_j = \prod_{\ell=1}^{n} x_\ell^{b_{\ell j}}.$$

Denote by the same symbol the corresponding map from the field $\mathbb{Q}(y'_1, \ldots, y'_n)$ to the field $\mathbb{Q}(x'_1, \ldots, x'_n)$. Then [31, Proposition 3.9] implies that $\varphi_X(\hat{z}) = \widehat{(\varphi_Y(z))}$ for any $z \in \mathbb{Q}(y'_1, \ldots, y'_n)$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{Q}(y_1, \ldots, y_n) \arrow[r, "(\hat{-})"] & \mathbb{Q}(x_1, \ldots, x_n) \\ \mathbb{Q}(y'_1, \ldots, y'_n) \arrow[r, "(\hat{-})"] \arrow[u, "\varphi_Y"] & \mathbb{Q}(x'_1, \ldots, x'_n). \arrow[u, "\varphi_X"'] \end{tikzcd}$$

Let us now compute $\varphi_X(X'_M)$. We have that

$$\begin{aligned} \varphi_X(X'_M) &= \varphi_X(x'^{\operatorname{ind}_{\Gamma'} M}) F_M(\hat{y}'_1, \ldots, \hat{y}'_n) \\ &= \varphi_X(x'^{\operatorname{ind}_{\Gamma'} M}) F_M(\widehat{(\varphi_Y(y'_1))}, \ldots, \widehat{(\varphi_Y(y'_n))}). \end{aligned}$$

Now, using [21, Lemma 5.2], we can express the right-hand side of the equation in terms of the $\hat{y}_j$. The equalities thus continue:

$$\begin{aligned}\varphi_X(X'_M) &= \varphi_X(x'^{\operatorname{ind}_{\Gamma'} M})\varphi_X(1+\hat{y}'_i)^{-h'_i}(1+\hat{y}_i)^{h_i}F_M(\hat{y}_1,\ldots,\hat{y}_n)\\ &= \varphi_X(x'^{\operatorname{ind}_{\Gamma'} M})\varphi_X(1+\hat{y}'_i)^{-h'_i}(1+\hat{y}_i)^{h_i}x^{-\operatorname{ind}_\Gamma \widetilde{\mu}_i^+(M)}X'_{\widetilde{\mu}_i^+(M)}.\end{aligned}$$

Thus, in order to prove the Corollary, we must show that

$$\varphi_X(x'^{\operatorname{ind}_{\Gamma'} M})\varphi_X(1+\hat{y}'_i)^{-h'_i}(1+\hat{y}_i)^{h_i}x^{-\operatorname{ind}_\Gamma \widetilde{\mu}_i^+(M)} = 1. \tag{4.1}$$

We do this in several steps. First, using the definition of φ_X and φ_Y, we get

$$\begin{aligned}\varphi_X(1+\hat{y}'_i)^{-h'_i}(1+\hat{y}_i)^{h_i} &= (1+\widehat{\varphi_Y(y'_i)})^{-h'_i}(1+\hat{y}_i)^{h_i}\\ &= (1+\hat{y}_i^{-1})^{-h'_i}(1+\hat{y}_i)^{h_i}\\ &= \hat{y}_i^{h'_i}(1+\hat{y}_i)^{h_i-h'_i}.\end{aligned}$$

Now, using Proposition 4.4.8, we get the equalities

$$\begin{aligned}\varphi_X\big((x')^{\operatorname{ind}_{\Gamma'} M}\big)x^{-\operatorname{ind}_\Gamma \widetilde{\mu}_i^+(M)} &= \varphi_X\big(\prod_{\ell=1}^n (x'_\ell)^{g'_\ell}\big)\prod_{\ell=1}^n x_\ell^{-g_\ell}\\ &= x_i^{g_i}\big(\prod_{\ell=1}^n x_\ell^{[b_{\ell i}]_+} + \prod_{\ell=1}^n x_\ell^{[-b_{\ell i}]_+}\big)^{-g_i}\big(\prod_{\ell\neq i} x_\ell^{g'_\ell-g_\ell}\big)x_i^{-g_i}\\ &= \big(\prod_{\ell=1}^n x_\ell^{[b_{\ell i}]_+} + \prod_{\ell=1}^n x_\ell^{[-b_{\ell i}]_+}\big)^{-g_i}\big(\prod_{\ell\neq i} x_\ell^{g'_\ell-g_\ell}\big).\end{aligned}$$

Thus we have, using the fact that $g_i = h_i - h'_i$ [21, Lemma 5.2], that the left-hand side of equation (4.1) is equal to

$$\hat{y}_i^{h'_i}(1+\hat{y}_i)^{g_i}\big(\prod_{\ell=1}^n x_\ell^{[b_{\ell i}]_+} + \prod_{\ell=1}^n x_\ell^{[-b_{\ell i}]_+}\big)^{-g_i}\big(\prod_{\ell\neq i} x_\ell^{g'_\ell-g_\ell}\big)$$

which is in turn equal to (using Remark 4.4.11)

$$\begin{aligned}\hat{y}_i^{h'_i}\big(\prod_{\ell=1}^n x_\ell^{-[-b_{\ell i}]_+}\big)^{g_i}\big(\prod_{\ell\neq i} x_\ell^{g'_\ell-g_\ell}\big) &= \hat{y}_i^{h'_i}\big(\prod_{\ell=1}^n x_\ell^{-[-b_{\ell i}]_+}\big)^{g_i}\big(\prod_{\ell\neq i} x_\ell^{h_i[-b_{\ell i}]_+-h'_i[b_{\ell i}]_+}\big)\\ &= \hat{y}_i^{h'_i}\big(\prod_{\ell\neq i} x_\ell^{h'_i[-b_{\ell i}]_+-h_i[-b_{\ell i}]_++h_i[-b_{\ell i}]_+-h'_i[b_{\ell i}]_+}\big)\\ &= \hat{y}_i^{h'_i}\big(\prod_{\ell\neq i} x_\ell^{-h'_i b_{\ell i}}\big)\\ &= \prod_{\ell\neq i} x_\ell^{h'_i b_{\ell i}-h'_i b_{\ell i}}\\ &= 1.\end{aligned}$$

This finishes the proof. □

4.4.3 Extensions and the E-invariant

In this section, we give an interpretation of the E-invariant of a decorated representation, as defined in [21] (its definition was recalled in section 4.2.3), as the dimension of a space of extensions, using the map Φ of section 4.4.1.

Proposition 4.4.15. *Let (Q, W) be a quiver with potential, and let $\mathcal{C}$ be the associated cluster category. Let X and Y be objects of $\mathcal{D}$. Then we have the following equalities:*

1. $E^{inj}(\Phi(X), \Phi(Y)) = \dim(\Sigma\Gamma)(X, \Sigma Y)$*;*
2. $E^{sym}(\Phi(X), \Phi(Y)) = \dim(\Sigma\Gamma)(X, \Sigma Y) + \dim(\Sigma\Gamma)(Y, \Sigma X)$*;*
3. $E(\Phi(X)) = (1/2) \dim \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X)$*,*

where $(\Sigma\Gamma)(X, Y)$ is the subspace of $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ containing all morphisms factoring through an object of $\operatorname{add}\Sigma\Gamma$.

PROOF The second equality follows immediately from the first one.

The third equality follows from the second one. Indeed, the second equality implies that $(\Sigma\Gamma)(X, \Sigma X)$ is finite-dimensional. It then follows from Lemma 3.3.8 that we have an isomorphism

$$(\Sigma\Gamma)(X, \Sigma X) \cong D \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X)/(\Sigma\Gamma).$$

Since $\dim \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X) = \dim(\Sigma\Gamma)(X, \Sigma X) + \dim \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X)/(\Sigma\Gamma)$, we get that

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{C}}(X, \Sigma X) &= 2\dim(\Sigma\Gamma)(X, \Sigma X) \\ &= E^{sym}(\Phi(X), \Phi(X)) \\ &= 2E(\Phi(X)). \end{aligned}$$

Let us now prove the first equality. Let

$$T_1^Y \longrightarrow T_0^Y \longrightarrow Y \longrightarrow \Sigma T_1^Y$$

be an $(\operatorname{add}\Gamma)$-presentation of Y. This triangle yields an exact sequence

$$(X, Y) \xrightarrow{u} (X, \Sigma T_1^Y) \longrightarrow (X, \Sigma T_0^Y) \longrightarrow (X, \Sigma Y) \xrightarrow{v} (X, \Sigma^2 T_1^Y),$$

which in turn gives an exact sequence

$$0 \longrightarrow \operatorname{Im} u \longrightarrow (X, \Sigma T_1^Y) \longrightarrow (X, \Sigma T_0^Y) \longrightarrow \operatorname{Ker} v \longrightarrow 0.$$

Since X is in $\mathcal{D}$, the two middle terms of this exact sequence are isomorphic to $(T_i^Y, \Sigma X)$ (for $i = 1, 2$) thanks to Proposition 3.2.16, and these spaces are finite-dimensional. Therefore all of the terms of the exact sequence are finite-dimensional.

Now, $\operatorname{Im} u$ is isomorphic to $(X, Y)/\operatorname{Ker} u$, and $\operatorname{Ker} u$ is exactly $(\Gamma)(X, Y)$. Therefore, by Lemma 3.3.2, $\operatorname{Im} u$ is isomorphic to the space $\operatorname{Hom}_{J(Q,W)}(FX, FY)$, where $F = \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$.

Moreover, $\operatorname{Ker} v$ is exactly $(\Sigma\Gamma)(X, \Sigma Y)$.

Thus, using the above exact sequence and Proposition 4.4.8, we have the equalities

$$\begin{aligned}
\dim(\Sigma\Gamma)(X,\Sigma Y) &= \dim \operatorname{Hom}_{J(Q,W)}(FX,FY) - \dim(X,\Sigma T_1^Y) + \dim(X,\Sigma T_0^Y) \\
&= \dim \operatorname{Hom}_{J(Q,W)}(FX,FY) - \dim(T_1^Y,\Sigma X) + \dim(T_0^Y,\Sigma X) \\
&= \dim \operatorname{Hom}_{J(Q,W)}(FX,FY) - \sum_{i=1}^{n}[T_1^Y : T_i](\dim(FX)_i) + \\
&\quad + \sum_{i=1}^{n}[T_0^X : T_i](\dim(FX)_i) \\
&= \dim \operatorname{Hom}_{J(Q,W)}(FX,FY) + \sum_{i=1}^{n}[\operatorname{ind}_\Gamma \Sigma Y : \Gamma_i](\dim(FX)_i) \\
&= \dim \operatorname{Hom}_{J(Q,W)}(FX,FY) + \sum_{i=1}^{n}\mathbf{g}_i(\Phi(Y))(\dim(FX)_i) \\
&= E^{inj}(\Phi(X),\Phi(Y)),
\end{aligned}$$

where $[T_j^Y : T_i]$ is the number of direct summands of T_j^Y isomorphic to T_i in any decomposition of T_j^Y into indecomposable objects, and where the $\mathbf{g}$-vector of $\Phi(Y)$ is given by $(\mathbf{g}_1(\Phi(Y)),\ldots,\mathbf{g}_n(\Phi(Y)))$. This finishes the proof. □

As a corollary, we get the following stronger version of [21, Lemma 9.2].

Corollary 4.4.16. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two decorated representations of a quiver with potential (Q,W). Assume that $E(\mathcal{M}')=0$. Then the following conditions are equivalent:*

1. *$\mathcal{M}$ and $\mathcal{M}'$ are isomorphic;*
2. *$E(\mathcal{M})=0$, and $\mathbf{g}_{\mathcal{M}}=\mathbf{g}_{\mathcal{M}'}$.*

Proof Condition (1) obviously implies condition (2). Now assume that condition (2) is satisfied. Then Proposition 4.4.15 implies that $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are rigid objects of $\mathcal{D}$. By Proposition 4.4.8, the indices of $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are given by $\mathbf{g}_{\mathcal{M}}$ and $\mathbf{g}_{\mathcal{M}'}$. By hypothesis, their indices are the same. Thus, by Proposition 4.3.1, $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}')$ are isomorphic, and so are $\mathcal{M}$ and $\mathcal{M}'$. □

Chapter 5

Indices and generic bases for cluster algebras

5.1 Introduction

One of the main motivations of S. Fomin and A. Zelevinsky for introducing cluster algebras in [29] was the search for a combinatorial framework in which one could study the canonical bases of M. Kashiwara [49] and G. Lusztig [63]. Recent results of C. Geiss, B. Leclerc and J. Schröer [35], who prove that coordinate rings of certain algebraic varieties have a natural cluster algebra structure and find a basis for them, give ample justification to this approach. The problem of finding "good" bases for cluster algebras is thus central in the theory. These bases should, as conjectured already in [29], contain the cluster monomials. We know from Theorem 4.3.7 that these are linearly independent when the defining matrix of the (skew-symmetric) cluster algebra is of full rank. Good bases for cluster algebras were previously constructed by G. Dupont [25] [24], by M. Ding, J. Xiao and F. Xu [23] and by G. Cerulli Irelli [15].

In their paper [35], C. Geiss, B. Leclerc and J. Schröer find bases for a certain class of (upper) cluster algebras and provide a candidate for a basis in general. In this chapter, inspired by their ideas, we use cluster categories to give another realization of this candidate set. We prove that its elements are linearly independent when the defining matrix is of full rank, and that it coincides with the basis of [35] when the cluster algebra arises from the setting studied therein.

The point of view that we adopt allows us to link a conjecture of V. Fock and A. Goncharov [27] to one of [35]. Our results apply to cluster algebras $\mathcal{A}_Q$ associated with a quiver Q on which there exists a non-degenerate potential W (in the sense of [22]) making (Q, W) Jacobi-finite.

More precisely, let (Q, W) be such a quiver with potential, and let $\mathcal{C}_{Q,W}$ be the associated cluster category (as defined by C. Amiot in [2]). Then Y. Palu's cluster character [68]

$$X'_M = x^{\mathrm{ind}_T M} \sum_e \Big(\chi\big(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, M))\big)\Big) x^{-\iota(e)},$$

can be applied to objects of $\mathcal{C}_{Q,W}$; the reachable indecomposable objects yield all the cluster variables of the cluster algebra $\mathcal{A}_Q$ in this way.

Each object M has an *index* $\mathrm{ind}_\Gamma M$ (as defined by Y. Palu in [68], see also [18]) which is an element of $K_0(\mathrm{add}\,\Gamma)$. The main theorem of this chapter states that a good candidate for a basis of $\mathcal{A}_Q$ is parametrized by the set of indices via the cluster character. Let $\mathcal{A}_Q^+$ be the upper cluster algebra of [5] associated with Q.

Theorem 5.1.1. *There exists a canonical map*

$$I : K_0(\operatorname{add}\Gamma) \longrightarrow \mathcal{A}_Q^+.$$

If the matrix of Q is of full rank, then the elements in the image of I are linearly independent over $\mathbb{Z}$. If (Q, W) arises from the setting of [35], then the image of I is the basis of the cluster algebra $\mathcal{A}_Q$ found in that paper.

The map I sends an element $[T_0] - [T_1]$ to the generic value taken by the cluster character on cones of morphisms in $\operatorname{Hom}_{\mathcal{C}}(T_1, T_0)$. It was first considered by G. Dupont in [24].

In their construction of a basis for cluster algebras, the authors of [35] consider *strongly reduced components* of the variety $\operatorname{rep}(A)$ of finite-dimensional representations of some finite-dimensional algebra A. It so happens that we can recover all such components from the set of indices $K_0(\operatorname{proj} A)$.

Theorem 5.1.2. *Let (Q, W) be a Jacobi-finite quiver with potential, and let A be its Jacobian algebra. Then there exists a canonical surjection*

$$\Psi : K_0(\operatorname{add} A) \longrightarrow \{\textit{strongly reduced components of } \operatorname{rep}(A)\}.$$

Two elements δ and δ' have the same image by Ψ if, and only if, their canonical decompositions (in the sense of H. Derksen and J. Fei [20], see section 5.3.2) can be written as

$$\delta = \delta_1 \oplus \overline{\delta} \quad \textit{and} \quad \delta_1' \oplus \overline{\delta},$$

with δ_1 and δ_1' non-negative.

Note that in the setting of the theorem, $K_0(\operatorname{add} A)$ is isomorphic to $K_0(\operatorname{add}\Gamma)$, so the notation is coherent with that of Theorem 5.1.1.

Elements of the cluster algebra can be mutated using the rules defined by S. Fomin and A. Zelevinsky [29]. Elements of $K_0(\operatorname{add}\Gamma)$ can also be mutated (in a way which we will make precise). The map I of Theorem 5.1.1 is well-behaved with respect to those different mutations, as conjectured in [24, Conjecture 9.2].

Theorem 5.1.3. *The map I commutes with mutation.*

As we shall see, this theorem allows us to link Conjecture 4.1 of [27] to one of [35]. This link, together with our results, also allows us to prove a part of Conjecture 4.1 of [27] for a certain class of cluster algebras.

The chapter is organized as follows. We define the map I in section 5.2. Next, we recall some notions on varieties of representations in section 5.3, and then prove Theorems 5.1.1, 5.1.2 and 5.1.3 in sections 5.6, 5.4 and 5.5, respectively. We end the chapter with an example of a Hom-finite cluster category for which the image of the map I is not contained in the cluster algebra, and in which there are cluster-tilting objects that are not related by a sequence of mutations.

5.2 Generic value of cluster characters

Let (Q, W) be a Jacobi-finite non-degenerate quiver with potential. Then C. Amiot's cluster category $\mathcal{C}_{Q,W}$ (see section 3.2.4) is Hom-finite, 2-Calabi–Yau and admits a cluster-tilting object Γ. In this setting, the cluster character $X_?'$ of Y. Palu (see section 3.3.3) is defined on the objects of $\mathcal{C}_{Q,W}$.

For two objects L and M of $\mathcal{C}_{Q,W}$, and for a morphism ε from L to ΣM, we denote by $mt(\varepsilon)$ any representative of the isomorphism class of "middle terms" U in triangles

$$M \longrightarrow U \longrightarrow L \longrightarrow \Sigma M.$$

This notation is borrowed from [67], as is the next result. In order to state it, we will need a bit of terminology (taken, for instance, from sections 2.3 to 2.5 of [43]). A *locally closed subset* of a variety is the intersection of an open subset with a closed subset. A *constructible subset* of a variety is a finite union of locally closed subsets. A function from an algebraic variety to any abelian group is *consctructible* if its image is finite and each fiber is a constructible subset of the variety.

Proposition 5.2.1 ([67]). *Let L and M be objects of $\mathcal{C}_{Q,W}$. Then the function*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(L, \Sigma M) & \longrightarrow & \mathbb{Q}(x_1, \ldots, x_n) \\ \varepsilon & \longmapsto & X'_{mt(\varepsilon)} \end{array}$$

is constructible.

PROOF This follows immediately from [67, Proposition 9]. □

Now, let T_0 and T_1 be objects in $\mathrm{add}\,\Gamma$. It follows from Proposition 5.2.1 that the function

$$\begin{array}{rcl} \eta_{T_0,T_1} : \mathrm{Hom}_{\mathcal{C}}(T_1, T_0) & \longrightarrow & \mathbb{Q}(x_1, \ldots, x_n) \\ \varepsilon & \longmapsto & X'_{mt(\Sigma\varepsilon)} \end{array}$$

is constructible. As in [24], we define the map I by using the fact that any constructible function on an irreducible variety admits a generic value (that is, there is a dense open subset of the domain of the function on which the function is constant).

Definition 5.2.2. We define the map

$$I : K_0(\mathrm{add}\,\Gamma) \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

by letting $I([T_0] - [T_1])$ be the generic value of the map η_{T_0,T_1} defined above.

Proposition 5.2.3. *If the matrix of Q is of full rank, then the elements in the image of I are lineraly independent over $\mathbb{Z}$.*

PROOF Similar to that of Theorem 4.3.7 (2). □

5.3 Varieties and projective presentations

In this section, we recall notions which we will need throughout the chapter.

5.3.1 Varieties of representations

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver, that is, an oriented graph with finitely many vertices and arrows. We denote by kQ its path algebra. Let I be an admissible ideal of kQ, and let $A = kQ/I$ be a finite-dimensional algebra (for general background on quivers and path algebras, we refer the reader to the book [3]).

Let $\mathbf{d}$ be a dimension vector for Q, that is, an element of $\mathbb{N}^{Q_0}$. The variety $\mathrm{rep}_{\mathbf{d}}(A)$ is the affine variety whose points are representations of Q with underlying space $\prod_{i\in Q_0} k^{d_i}$ satisfying the relations in I ; it is realized as a Zariski-closed subset of the affine space $\prod_{a\in Q_1} \mathrm{Hom}_k(k^{d_{s(a)}}, k^{d_{t(a)}})$.

We denote by $\mathrm{rep}(A)$ the disjoint union of all $\mathrm{rep}_{\mathbf{d}}(A)$ as $\mathbf{d}$ takes all possible values in $\mathbb{N}^{Q_0}$. For general background on varieties of representations, we refer the reader to [16].

The algebraic group $\mathrm{GL}_{\mathbf{d}}$ is defined to be $\prod_{i\in Q_0} \mathrm{GL}_{d_i}$. It acts on $\mathrm{rep}_{\mathbf{d}}(A)$ thus: for any $(g_i) \in \mathrm{GL}_{\mathbf{d}}$ and any $(\varphi_a) \in \mathrm{rep}_{\mathbf{d}}(A)$, $(g_i)(\varphi_a) = (g_{t(a)}\varphi_a(g_{s(a)})^{-1})$. The orbit of a representation M under the action of $GL_{\mathbf{d}}$ is the set of representations with underlying space $\prod_{i\in Q_0} k^{d_i}$ isomorphic to M.

We will need the following information on the dimension of morphism and extension spaces, and on minimal projective presentations.

Lemma 5.3.1 (Lemma 4.2 of [17]). *The functions*

$$\mathrm{rep}_{\mathbf{d}_1}(A) \times \mathrm{rep}_{\mathbf{d}_2}(A) \longrightarrow \mathbb{Z}$$

sending a pair (M_1, M_2) to the dimensions of the spaces $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Ext}^1_A(M_1, M_2)$ are upper semicontinuous.

Corollary 5.3.2. *Let $\mathcal{Z}$ be an irreducible component of $\mathrm{rep}_{\mathbf{d}}(A)$. There exist finitely generated projective A-modules P_1 and P_0 and a dense open subset $\mathcal{U}$ of $\mathcal{Z}$ such that any representation M in $\mathcal{U}$ admits a minimal projective presentation of the form*

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

PROOF Given any representation M and a minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, the multiplicities of an indecomposable projective Q in P_0 and P_1 are given by the dimensions of $\mathrm{Hom}_A(M, S)$ and $\mathrm{Ext}^1_A(M, S)$, respectively, where S is a simple module whose projective cover is Q.

Restrict the maps of Lemma 5.3.1 to $\mathcal{Z} \times \{S\}$. The restrictions are still upper semicontinuous. Therefore the subsets of $\mathcal{Z}$ on which these functions take their minimal values are (dense) open subsets of $\mathcal{Z}$. Their intersection is a dense open subset of $\mathcal{Z}$ on which the functions

$$\dim \mathrm{Hom}_A(?, S) \quad \text{and} \quad \dim \mathrm{Ext}^1_A(?, S)$$

are constant. This proves the result. □

We now introduce a slight modification of a definition of [35, Section 7.1]. Let $\mathcal{Z}$ be an irreducible component of $\mathrm{rep}(A)$. There is an open dense subset $\mathcal{U}$ of $\mathcal{Z}$ and positive integers $h(\mathcal{Z})$, $e(\mathcal{Z})$ and $c(\mathcal{Z})$ such that, for any M in $\mathcal{U}$,

1. $\dim \mathrm{Hom}_A(M, \tau M) = h(\mathcal{Z})$;
2. $\dim \mathrm{Ext}^1_A(M, M) = e(\mathcal{Z})$; and
3. $\mathrm{codim}_{\mathcal{Z}}(\mathrm{GL}_{\mathbf{d}} M) = c(\mathcal{Z})$,

where τ is the Auslander–Reiten translation (see, for example, [3, Chapter IV]). Moreover, we have that $c(\mathcal{Z}) \leq e(\mathcal{Z}) \leq h(\mathcal{Z})$.

Definition 5.3.3 (Section 7.1 of [35]). An irreducible component $\mathcal{Z}$ of $\operatorname{rep}(A)$ such that $c(\mathcal{Z}) = h(\mathcal{Z})$ is *strongly reduced.*

Remark 5.3.4. In the original definition of [35], the authors used the integer function $h'(\mathcal{Z}) = \dim \operatorname{Hom}_A(\tau^{-1}M, M)$ and defined $\mathcal{Z}$ to be strongly reduced if $c(\mathcal{Z}) = h'(\mathcal{Z})$.

In the case where A is the Jacobian algebra of a quiver with potential (see [22]), the two definitions coincide. Indeed, we have equalities

$$\begin{aligned}
\dim \operatorname{Hom}_A(M, \tau M) &= E^{proj}(M) \text{ (by [21, Corollary 10.9])}\\
&= E^{proj}(\tau^{-1}M) \text{ (by [20, Corollary 7.5])}\\
&= \dim \operatorname{Hom}_A(\tau^{-1}M, M) \text{ (by [21, Corollary 10.9]).}
\end{aligned}$$

This can also be seen by using the cluster category $\mathcal{C}$ of the quiver with potential, as defined in [2]. This category has a canonical cluster-tilting object Γ, and the functor $F = \operatorname{Hom}_{\mathcal{C}}(\Gamma, ?)$ induces an equivalence $\operatorname{Hom}_{\mathcal{C}}(\Gamma, ?)/(\Sigma\Gamma) \to \operatorname{mod} A$ ([57, Proposition 2.1(c)]), such that $F(\Sigma X) = \tau(FX)$ ([57, Section 3.5]). We then have

$$\begin{aligned}
\dim \operatorname{Hom}_A(M, \tau M) &= \dim \operatorname{Hom}_{\mathcal{C}}(\overline{M}, \Sigma\overline{M}) - \dim(\Sigma\Gamma)(\overline{M}, \Sigma\overline{M})\\
&= \dim \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\overline{M}, \overline{M}) - \dim(\Gamma)(\Sigma^{-1}\overline{M}, \overline{M})\\
&= \dim \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\overline{M}, \overline{M}) - \dim(\Sigma\Gamma)(\Sigma^{-1}\overline{M}, \overline{M})\\
&= \dim \operatorname{Hom}_A(\tau^{-1}M, M),
\end{aligned}$$

where $\overline{M}$ is a preimage of M by the functor F, and the second-to-last equality is a consequence of [68, Lemma 10].

5.3.2 Decomposition of projective presentations

Let A be a finite-dimensional algebra, and let P_1', P_0', P_1'' and P_0'' be finitely generated projective A-modules.

Definition 5.3.5 (Definition 3.1 of [20]). For any f' in $\operatorname{Hom}_A(P_1', P_0')$ and any f'' in $\operatorname{Hom}_A(P_1'', P_0'')$, define the space $E(f', f'')$ as

$$E(f', f'') = \operatorname{Hom}_{\mathrm{K}^b(\operatorname{proj} A)}(\Sigma^{-1}f', f''),$$

where f' and f'' are viewed as complexes in $\mathrm{K}^b(\operatorname{proj} A)$. Define $E(f')$ to be $E(f', f')$.

Lemma 5.3.6. *If* $P_1' \xrightarrow{f'} P_0' \longrightarrow M' \longrightarrow 0$ *is a projective presentation, then*

$$\dim E(f', f'') \geq \dim \operatorname{Hom}_A(M'', \tau M'),$$

where M'' is the cokernel of f''. Equality holds if the presentation is minimal.

PROOF Applying the right exact functor $D\operatorname{Hom}_A(?, A) = D(?)^t$ to the presentation, we get an exact sequence

$$0 \longrightarrow \tau M' \oplus I \longrightarrow D(P_1')^t \longrightarrow D(P_0')^t \longrightarrow D(M')^t \longrightarrow 0,$$

where I is a finite-dimensional injective A-module which vanishes if the presentation is minimal. We use the fact that the morphism of functors $D\operatorname{Hom}_A(X, ?) \to \operatorname{Hom}_A(?, DX^t)$

is an isomorphism whenever X is projective, and we get a commutative diagram with exact rows and vertical isomorphisms

$$\begin{tikzcd} & D\operatorname{Hom}_A(P_1', M'') \arrow[r, "D\phi"] \arrow[d, "\cong"] & D\operatorname{Hom}_A(P_0', M'') \arrow[d, "\cong"] \\ 0 \arrow[r] \operatorname{Hom}_A(M'', \tau M' \oplus I) \arrow[r] & \operatorname{Hom}_A(M'', D(P_1')^t) \arrow[r] & \operatorname{Hom}_A(M'', D(P_0')^t). \end{tikzcd}$$

Therefore $D\operatorname{Hom}_A(M'', \tau M' \oplus I)$ is isomorphic to the cokernel of ϕ, which is in turn isomorphic to $E(f', f'')$ by [20, Lemma 3.2]. This proves the inequality. If the presentation is minimal, then I vanishes and the equality holds.

□

We will need a result on the decomposition of general projective presentations, which follows from the work of H. Derksen and J. Fei on the one hand, and from that of W. Crawley-Boevey and J. Schröer on the other hand.

For any δ in $K_0(\operatorname{proj} A)$, let $\operatorname{PHom}_A(\delta)$ be the space $\operatorname{Hom}_A(P_-^\delta, P_+^\delta)$, where $\delta = [P_+^\delta] - [P_-^\delta]$, and P_+^δ and P_-^δ have no non-zero direct factors in common. If $[P_-^\delta] = 0$, then δ is called *non-negative.*

The vector δ is *indecomposable* if a general element of $\operatorname{PHom}_A(\delta)$ is indecomposable. Its *canonical decomposition* is $\delta_1 \oplus \ldots \oplus \delta_s$ if a general element of $\operatorname{PHom}_A(\delta)$ has the form $f_1 \oplus \ldots \oplus f_s$, with $f_i \in \operatorname{PHom}_A(\delta_i)$ and each δ_i is indecomposable [20, Definition 4.3].

Theorem 5.3.7 (Derksen–Fei)**.** *Any $\delta \in K_0(\operatorname{proj} A)$ admits a canonical decomposition $\delta_1 \oplus \ldots \oplus \delta_s$, where $\delta_1, \ldots, \delta_s \in K_0(\operatorname{proj} A)$ are unique up to reordering.*

PROOF Let $\mathbf{d} = \underline{\dim}\, P_-^\delta$ and $\mathbf{e} = \underline{\dim}\, P_+^\delta$. Then the orbit of P_-^δ (or P_+^δ) is a dense open subset of an irreducible component C_- of $\operatorname{rep}_{\mathbf{d}}(A)$ (or C_+ of $\operatorname{rep}_{\mathbf{e}}(A)$, respectively), since P_-^δ is projective and thus has no self-extensions (see [33, Corollary 1.2]). Let

$$\operatorname{rep}_{\mathbf{d},\mathbf{e}}(A\overrightarrow{A_2}) = \{(L, M, f) \mid L \in \operatorname{rep}_{\mathbf{d}}(A), M \in \operatorname{rep}_{\mathbf{e}}(A), f \in \operatorname{Hom}_A(L, M)\}.$$

Then we can view $\operatorname{PHom}_A(\delta) = \operatorname{Hom}_A(P_-^\delta, P_+^\delta)$ as an irreducible subvariety of the affine variety $\operatorname{rep}_{\mathbf{d},\mathbf{e}}(A\overrightarrow{A_2})$. Let C be an irreducible component of $\operatorname{rep}_{\mathbf{d},\mathbf{e}}(A\overrightarrow{A_2})$ which contains $\operatorname{PHom}_A(\delta)$. By [17, Theorem 1.1], there is a dense open subset $\mathcal{U}$ of C and indecomposable irreducible components $C_1, \ldots, C_s$ of $\operatorname{rep}(A\overrightarrow{A_2})$ such that $\mathcal{U} \subset C_1 \oplus \ldots \oplus C_s$. We have a diagram

$$\begin{tikzcd} & C \arrow[dl, "\pi_-"'] \arrow[dr, "\pi_+"] & \\ C_- & & C_+ \end{tikzcd}$$

where π_- and π_+ are the natural projections; their images intersect the orbits of P_-^δ and P_+^δ, respectively. Thus the preimages of these open orbits are dense open subsets of C, whose common intersection with $\mathcal{U}$ is a dense open subset $\mathcal{V}$ of C. Now $\mathcal{V} \cap \operatorname{PHom}_A(\delta)$ is non-empty, and is thus dense and open in $\operatorname{PHom}_A(\delta)$. The inclusion $\mathcal{V} \cap \operatorname{PHom}_A(\delta) \subset C_1 \oplus \ldots \oplus C_s$ induces the canonical decomposition $\delta_1 \oplus \ldots \oplus \delta_s$ of δ.

□

Corollary 5.3.8. *If the canonical decomposition of δ has no non-negative factors, then a general element in $\operatorname{PHom}_A(\delta)$ is a minimal projective presentation.*

5.4 Indices and strongly reduced components

5.4.1 Morphic cokernels

Let A be a finite-dimensional k-algebra as before, and let $\mathbf{d}$, $\mathbf{d}_1$ and $\mathbf{d}_0$ be dimension vectors. Define the affine varieties

$$\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2}) = \{(L, M, f)|\ L \in \mathrm{rep}_{\mathbf{d}_1}(A), M \in \mathrm{rep}_{\mathbf{d}_0}(A), f \in \mathrm{Hom}_A(L, M)\}$$

$$\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}} = \{(L, M, f) \in \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})|\ \underline{\dim}\,\mathrm{Coker}\, f = \mathbf{d}\}.$$

The latter is a locally closed subset of the former. The symbol $\overrightarrow{A_2}$ stands for the quiver $1 \to 2$; elements of the above sets are A-module-valued representations of $\overrightarrow{A_2}$.

Fix bases $\{u_1, u_2, \dots, u_\ell\}$ and $\{v_1, v_2, \dots, v_m\}$ of $\prod_{i\in Q_0} k^{d_{1,i}}$ and $\prod_{i\in Q_0} k^{d_{0,i}}$, respectively (these are the underlying vector spaces of L and M) ; choose the basis vectors so that they all lie in some $k^{d_{e,i}}$ for $e = 0, 1$ and $i \in Q_0$. For any subset $\mathbf{i}$ of $\{1, 2, \dots, m\}$, let $N_{\mathbf{i}}$ be the vector space generated by $\{v_i|\ i \in \mathbf{i}\}$, and let

$$E_{\mathbf{i}} = \{(L, M, f) \in \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})|\ M \cong N_{\mathbf{i}} \oplus \mathrm{Im}\ f \text{ as a vector space}\}.$$

Notice that $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})$ is the union of the $E_{\mathbf{i}}$, and that each $E_{\mathbf{i}}$ is contained in $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}}$ for some dimension vector $\mathbf{d}$. Notice also that $E_{\mathbf{i}}$ is the intersection of an open subset with a closed subset. Indeed, an element (L, M, f) of $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})$ lies in $E_{\mathbf{i}}$ if and only if the following two conditions are satisfied (here we write f as (a_{ij}) in matrix form with respect to the fixed bases) :

(a) There exists a subset $\mathbf{j}$ of $\{1, \dots, \dim P_1\}$ such that $|\mathbf{j}| = m - |\mathbf{i}|$ and the submatrix $(a_{ij})_{i\notin\mathbf{i}, j\in\mathbf{j}}$ has a non-zero determinant. This condition defines an open subset.

(b) For any $i_0 \in \mathbf{i}$, and any subset $\mathbf{j}$ of $\{1, \dots, \dim P_1\}$ such that $|\mathbf{j}| = m - |\mathbf{i}| + 1$, the submatrix (a_{ij}), where $j \in \mathbf{j}$ and i is either i_0 or not in $\mathbf{i}$, has vanishing determinant. This condition defines a closed subset.

In particular, if $E_{\mathbf{i}}$ is contained in $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}}$, then it is open inside it, since the second condition is then automatically satisfied. The next result is a slight generalization of a statement of [67, Lemma 4].

Lemma 5.4.1. *Assume that $E_{\mathbf{i}}$ is contained in $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}}$. Then there exists a morphism of varieties*

$$\Phi : E_{\mathbf{i}} \longrightarrow \mathrm{rep}_{\mathbf{d}}(A)$$

such that $\Phi(f)$ is isomorphic to Coker f *for any element f of $E_{\mathbf{i}}$.*

Proof Let (L, M, f) be an element of $E_{\mathbf{i}}$. We define $\Phi(L, M, f)$, as a vector space, to be the quotien of M by Im f, that is, $N_{\mathbf{i}}$. Let us define the A-module structure.

Let $\Omega_{\mathbf{j}}$ be the open subset of $E_{\mathbf{i}}$ consisting of maps satisfying condition (a) above for some fixed $\mathbf{j}$. Then the $\Omega_{\mathbf{j}}$ form an open cover of $E_{\mathbf{i}}$.

We define Φ on $\Omega_{\mathbf{j}}$ as follows. Assume that (L, M, f) lies in $\Omega_{\mathbf{j}}$. Let $D = (a_{ij})_{i\notin\mathbf{i}, j\in\mathbf{j}}$ and $C = (a_{ij})_{i\in\mathbf{i}, j\in\mathbf{j}}$; then D is invertible by condition (a).

Let b be an element of A. We will define the matrix of the action of b on $\Phi(L, M, f)$. We do this through the following diagram :

$$E_{\mathbf{i}} \xrightarrow{I_{\mathbf{i}}} M \xrightarrow{\rho_M(b)} M \xrightarrow{P_{\mathbf{i}}} E_{\mathbf{i}} \oplus E_{\mathbf{i}'} \xrightarrow{F} E_{\mathbf{i}} \oplus \mathrm{Im}\ f \xrightarrow{\pi} E_{\mathbf{i}}.$$

Here $I_{\mathbf{i}}$ is the natural inclusion; $\rho_M(b)$ is the action of b on M; $P_{\mathbf{i}}$ is the permutation matrix putting the basis vectors v_i, $i \in \mathbf{i}$, before the others; F is a base change matrix given by

$$\begin{pmatrix} 1 & -CD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

and π is the natural projection, given by $(1, 0)$. Thus the action of b on $\Phi(L, M, f)$ is given by the matrix

$$(1, -CD^{-1})P_{\mathbf{i}}(\rho_M(b))I_{\mathbf{i}}.$$

We have thus defined the action of Φ on $\Omega_{\mathbf{j}}$. This definition does not depend on $\mathbf{j}$; indeed, assume that (L, M, f) is also in $\Omega_{\mathbf{j}'}$. Let $D' = (a_{ij})_{i\notin\mathbf{i}, j\in\mathbf{j}'}$ and $C' = (a_{ij})_{i\in\mathbf{i}, j\in\mathbf{j}'}$. Then $CD^{-1} = C'(D')^{-1}$. To see this, notice that condition (b) above implies that any line of the matrix (a_{ij}) which is in $\mathbf{i}$ is a linear combination of the ones not in $\mathbf{i}$. Therefore there exists a matrix K such that $(a_{ij})_{i\in\mathbf{i}} = K(a_{ij})_{i\notin\mathbf{i}}$. Therefore $C = KD$ and $C' = KD'$, and we get the desired equality.

Therefore Φ is well-defined on an open cover of $E_{\mathbf{i}}$, and it is thus a morphism of varieties.

□

5.4.2 Codimensions of orbits

In the preceding section we have defined a morphism $\Phi : E_{\mathbf{i}} \longrightarrow \mathrm{rep}_{\mathbf{d}}(A)$. Recall that $\Phi(L, M, f)$ is isomorphic to Coker f, and that an open cover of $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(\overrightarrow{AA_2})_{\mathbf{d}}$ is formed by such $E_{\mathbf{i}}$'s.

Define $\mathrm{GL}_{\mathbf{d}_1,\mathbf{d}_0}$ as the algebraic group $\mathrm{GL}_{\mathbf{d}_1} \times \mathrm{GL}_{\mathbf{d}_0}$. Then the group $\mathrm{GL}_{\mathbf{d}_1,\mathbf{d}_0}$ acts on $\mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(\overrightarrow{AA_2})_{\mathbf{d}}$ thus : for any $(g_1, g_0) \in \mathrm{GL}_{\mathbf{d}_1,\mathbf{d}_0}$ and any $(L, M, f) \in \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(\overrightarrow{AA_2})_{\mathbf{d}}$, we have that $(g_1, g_0)(L, M, f) = (g_1 L, g_0 M, g_0 f g_1^{-1})$.

Lemma 5.4.2. *Let (L, M, f) be an element of $E_{\mathbf{i}}$. Then the orbit of $\Phi(L, M, f)$ in $\mathrm{rep}_{\mathbf{d}}(A)$ is equal to the image by Φ of the intersection of the orbit of (L, M, f) with $E_{\mathbf{i}}$. In short,*

$$\mathcal{O}_{\Phi(L,M,f)} = \Phi(E_{\mathbf{i}} \cap \mathcal{O}_{(L,M,f)}).$$

PROOF Let b be an element of A. Then we showed in lemma 5.4.1 that b acts on $\Phi(L, M, f)$ by the matrix

$$(1, -CD^{-1})P_{\mathbf{i}}(\rho_M(b))I_{\mathbf{i}}.$$

Let γ be an element of $\mathrm{GL}_{\mathbf{d}}$. Then the action of b on $\gamma\Phi(L, M, f)$ is

$$\gamma(1, -CD^{-1})P_{\mathbf{i}}(\rho_M(b))I_{\mathbf{i}}\gamma^{-1}.$$

Consider the element $G = (1, \overline{\gamma})$ of $\mathrm{GL}_{\mathbf{d}_1,\mathbf{d}_0}$, where

$$G = P_{\mathbf{i}}^{-1} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} P_{\mathbf{i}}.$$

Then $G(L, M, f) = (L, \overline{\gamma}M, \overline{\gamma}f)$. In matrix form, we have that

$$\overline{\gamma}f = P_{\mathbf{i}}^{-1} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} P_{\mathbf{i}} P_{\mathbf{i}}^{-1} \begin{pmatrix} C & A \\ D & B \end{pmatrix} P_{\mathbf{j}} = P_{\mathbf{i}}^{-1} \begin{pmatrix} \gamma C & \gamma A \\ D & B \end{pmatrix} P_{\mathbf{j}}.$$

Therefore $G(L,M,f)$ is still in $E_{\mathbf{i}}$, and b acts on $\Phi(G(L,M,f))$ by

$$\begin{aligned}
& (1,-\gamma CD^{-1})P_{\mathbf{i}}(\rho_{\overline{\gamma}M}(b))I_{\mathbf{i}} \\
= \ & (1,-\gamma CD^{-1})P_{\mathbf{i}}P_{\mathbf{i}}^{-1}\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} P_{\mathbf{i}}(\rho_M(b))P_{\mathbf{i}}^{-1}\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} P_{\mathbf{i}}I_{\mathbf{i}} \\
= \ & (\gamma,-\gamma CD^{-1})P_{\mathbf{i}}(\rho_M(b))P_{\mathbf{i}}^{-1}\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
= \ & \gamma(1,-CD^{-1})P_{\mathbf{i}}(\rho_M(b))P_{\mathbf{i}}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\gamma^{-1} \\
= \ & \gamma(1,-CD^{-1})P_{\mathbf{i}}(\rho_M(b))I_{\mathbf{i}}\gamma^{-1}.
\end{aligned}$$

Therefore $\Phi(G(L,M,f)) = \gamma\Phi(L,M,f)$. This proves that we have an inclusion $\mathcal{O}_{\Phi(L,M,f)} \subset \Phi(E_{\mathbf{i}} \cap \mathcal{O}_{(L,M,f)})$.

The other inclusion follows from the fact that if (L',M',f') lies in the orbit of (L,M,f), then the cokernels of f and f' are isomorphic. This proves the lemma.

□

In the course of this section, we will be relying heavily on the following theorem on dimensions, borrowed from the book [7].

Theorem 5.4.3 (Theorem AG.10.1 of [7]). *Let $a : X \to Y$ be a dominant morphism of irreducible varieties. Let W be an irreducible closed subvariety of Y and let Z be an irreducible component of $a^{-1}(W)$.*

There exists an open dense subset U of Y (depending only on a) such that
– $U \subset a(X)$, and
– if Z and $a^{-1}(U)$ have non-empty intersection, then $\mathrm{codim}_X Z = \mathrm{codim}_Y W$.

For the next lemma, we shall make the following identifications and definitions:

$$\begin{aligned}
\mathrm{Hom}_A(L_0,M_0) &= \{(L,M,f) \in \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2}) |\ L = L_0, M = M_0\}; \\
\mathrm{Hom}_A(L_0,M_0)_{\mathbf{d}} &= \mathrm{Hom}_A(L_0,M_0) \cap \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}}; \\
\mathrm{HOM}_A(L_0,M_0) &= \{(L,M,f) \in \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2}) |\ L \cong L_0, M \cong M_0\}; \\
\mathrm{HOM}_A(L_0,M_0)_{\mathbf{d}} &= \mathrm{HOM}_A(L_0,M_0) \cap \mathrm{rep}_{\mathbf{d}_1,\mathbf{d}_0}(A\overrightarrow{A_2})_{\mathbf{d}}; \\
\mathrm{GL}_{L_0,M_0} &= \mathrm{Aut}_A(L_0) \times \mathrm{Aut}_A(M_0).
\end{aligned}$$

We shall denote an element (L_0,M_0,f) of $\mathrm{Hom}_A(L_0,M_0)$ simply by the morphism f. Note that the first and the third varieties are irreducible; indeed, the first one is a vector space, and the third one is $GL_{\mathbf{d}_1,\mathbf{d}_0}\,\mathrm{Hom}_A(L_0,M_0)$, which is irreducible. Note that GL_{L_0,M_0} acts on $\mathrm{Hom}_A(L_0,M_0)$.

Notice that, inside $\mathrm{Hom}_A(L,M)$ and $\mathrm{HOM}_A(L,M)$, the subsets of the (L,M,f) such that f is of maximal rank are open subsets, and the cokernels of those f all have the same dimension vector. We denote those subsets by $\mathrm{Hom}_A(L,M)_{max}$ and $\mathrm{HOM}_A(L,M)_{max}$.

Lemma 5.4.4. *Fix $(L_0,M_0) \in \mathrm{rep}_{\mathbf{d}_1}(A) \times \mathrm{rep}_{\mathbf{d}_0}(A)$. Let $\mathbf{i}$ be such that $E_{\mathbf{i}}$ intersects $\mathrm{HOM}_A(L_0,M_0)_{\mathbf{d}}$, and consider the morphism $\Phi : E_{\mathbf{i}} \to \mathrm{rep}_{\mathbf{d}}(A)$ defined above. There exists an open subset $\mathcal{V}$ of $E_{\mathbf{i}} \cap \mathrm{HOM}_A(L_0,M_0)_{\mathbf{d}}$ such that for any (L,M,f) in $\mathcal{V}$, the following properties hold.*

1. *If $\mathcal{F}$ is an irreducible component of* $\mathrm{HOM}_A(L,M)_{\mathbf{d}}$ *which contains* $\mathcal{O}_{(L,M,f)}$, *then*

$$\mathrm{codim}_{\mathcal{F}}\mathcal{O}_{(L,M,f)} = \mathrm{codim}_{\Phi(E_{\mathbf{i}}\cap\mathcal{F})}\mathcal{O}_{\Phi(L,M,f)}.$$

   In particular, if $\mathrm{HOM}_A(L,M)_{\mathbf{d}} = \mathrm{HOM}_A(L,M)_{max}$, *then*

$$\mathrm{codim}_{\mathcal{Y}}\mathcal{O}_{(L,M,f)} = \mathrm{codim}_{\Phi(E_{\mathbf{i}}\cap\mathcal{Y})}\mathcal{O}_{\Phi(L,M,f)},$$

   where $\mathcal{Y} = \mathrm{HOM}_A(L,M)$.

2. *With the same notation as in (1), and letting* $\mathcal{X} = \mathrm{Hom}_A(L,M)$, *we have that*

$$\mathrm{codim}_{\mathcal{X}}\mathcal{O}_f = \mathrm{codim}_{\mathcal{Y}}\mathcal{O}_{(L,M,f)}.$$

PROOF We first prove (1). Consider the following commuting diagram :

$$\begin{tikzcd}
\overline{\mathcal{O}_{(L,M,f)}} \arrow[d, hook] & \overline{E_{\mathbf{i}}\cap\mathcal{O}_{(L,M,f)}} \arrow[l] \arrow[r, "\Phi"] \arrow[d, hook] & \overline{\mathcal{O}_{\Phi(L,M,f)}} \arrow[d, hook] \\
\mathcal{F} & E_{\mathbf{i}}\cap\mathcal{F} \arrow[l, hook'] \arrow[r, "\Phi"] & \Phi(E_{\mathbf{i}}\cap\mathcal{F}).
\end{tikzcd}$$

The three varieties in the lower row are irreducible. Since $E_{\mathbf{i}}\cap\mathcal{F}$ is a dense open subset of $\mathcal{F}$, the lower-left morphism is dominant. The lower-right morphism is also dominant (see, for instance, [7, AG.10.2]). So we can apply the dimension theorem 5.4.3; if $\mathcal{U}_1\subset\mathcal{F}$ and $\mathcal{U}_2\subset\Phi(E_{\mathbf{i}}\cap\mathcal{F})$ are the open subsets described by the theorem, let $\mathcal{V}$ be the intersection of their preimages in $E_{\mathbf{i}}\cap\mathcal{F}$.

Now, $\overline{E_{\mathbf{i}}\cap\mathcal{O}_{(L,M,f)}}$ is an irreducible component of $\Phi^{-1}(\overline{\mathcal{O}_{\Phi(L,M,f)}})$ thanks to Lemma 5.4.2. Moreover, $\overline{E_{\mathbf{i}}\cap\mathcal{O}_{(L,M,f)}}$ is the preimage of $\overline{\mathcal{O}_{(L,M,f)}}$ by the inclusion. Thus we can apply the dimension theorem and get

$$\begin{aligned}
\mathrm{codim}_{\mathcal{F}}\mathcal{O}_{(L,M,f)} &= \mathrm{codim}_{E_{\mathbf{i}}\cap\mathcal{F}}E_{\mathbf{i}}\cap\mathcal{O}_{(L,M,f)}\\
&= \mathrm{codim}_{\Phi(E_{\mathbf{i}}\cap\mathcal{F})}\mathcal{O}_{\Phi(L,M,f)}.
\end{aligned}$$

This proves the first result.

Let us now prove (2). Consider the diagram

$$\begin{tikzcd}
\mathcal{O}_f \arrow[r, hook] \arrow[d, hook] & \mathrm{Hom}_A(L,M) \arrow[r] \arrow[d, hook] & \{(L,M)\} \arrow[d, hook] \\
\mathcal{O}_{(L,M,f)} \arrow[r, hook] & \mathrm{HOM}_A(L,M) \arrow[r] & \mathcal{O}_L\times\mathcal{O}_M.
\end{tikzcd}$$

Two applications of the dimension theorem 5.4.3 yields equalities

$$\begin{aligned}
\mathrm{codim}_{\mathcal{O}_{(L,M,f)}}\mathcal{O}_f &= \dim\mathcal{O}_L\times\mathcal{O}_M\\
&= \mathrm{codim}_{\mathcal{Y}}\mathcal{X},
\end{aligned}$$

which in turn yields

$$\mathrm{codim}_{\mathcal{X}}\mathcal{O}_f = \mathrm{codim}_{\mathcal{Y}}\mathcal{O}_{(L,M,f)}.$$

□

For any $f\in\mathrm{Hom}_A(L,M)$, the action of $\mathrm{GL}_{L,M}$ induces a morphism

$$\begin{array}{rccc}
\pi: & \mathrm{GL}_{L,M} & \longrightarrow & \mathcal{O}_f\\
 & (g_1,g_0) & \longmapsto & g_0f(g_1)^{-1}
\end{array}$$

which, in turn, induces a linear map on tangent spaces

$$\begin{array}{rccc} d\pi : & \mathrm{End}_A L \oplus \mathrm{End}_A M & \longrightarrow & T_f(\mathcal{O}_f) \\ & (h_1, h_0) & \longmapsto & fh_1 - h_0 f. \end{array}$$

Here we view $T_f(\mathcal{O}_f)$ as a subspace of $T_f(\mathrm{Hom}_A(L, M))$, which we identify with the space $\mathrm{Hom}_A(L, M)$.

Lemma 5.4.5. *The map $d\pi$ is surjective.*

PROOF The morphism π is surjective by definition. In particular, it is dominant. Since we work over a field of characteristic zero, it is automatically separable. It then follows from [7, Proposition II.6.7 and AG.17.3] that $d\pi$ is surjective. □

5.4.3 Orbits and the E-invariant

As before, let A be a finite-dimensional k-algebra and let P_1 and P_0 be finitely generated A-modules.

Lemma 5.4.6. *Let f be any element of* $\mathrm{Hom}_A(P_1, P_0)$*. We have the equality*

$$\mathrm{codim}_{\mathcal{X}} \mathcal{O}_f = \dim E(f),$$

where $\mathcal{X}$ stands for $\mathrm{Hom}_A(P_1, P_0)$*.*

PROOF Consider the linear map

$$\begin{array}{rcl} \psi : \mathrm{Hom}_A(P_1, P_0) & \longrightarrow & E(f) \\ g & \longmapsto & \overline{g}, \end{array}$$

where $\overline{g}$ is the map from $\Sigma^{-1} f$ to f in $\mathrm{K}^b(\mathrm{proj}\, A)$ given by

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & P_1 \arrow[r, "f"] \arrow[d, "g"] & P_0 \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & P_1 \arrow[r, "f"] & P_0 \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

The map ψ is obviously surjective. Moreover, its kernel is exactly

$$T_f(\mathcal{O}_f) = \{h_0 f + f h_1 | h_i \in \mathrm{End}_A(P_i)\},$$

since this is the very definition of null-homotopic maps from $\Sigma^{-1} f$ to f (the above equality follows from Lemma 5.4.5). Therefore

$$\begin{array}{rcl} \mathrm{codim}_{\mathcal{X}} \mathcal{O}_f & = & \dim \left(\mathrm{Hom}_A(P_1, P_0) / T_f(\mathcal{O}_f) \right) \\ & = & \dim E(f). \end{array}$$

□

5.4.4 Proof of Theorem 5.1.2

We define the map

$$\Psi : K_0(\operatorname{add} A) \longrightarrow \{\text{strongly reduced components of } \operatorname{rep}(A)\}.$$

To do so, we first define a map

$$\Psi' : K_0(\operatorname{add} A^{op}) \longrightarrow \{\text{strongly reduced components of } \operatorname{rep}(A^{op})\}.$$

For any element of the form $\delta = [P_0] - [P_1]$, where P_1 and P_0 are two projective modules over A^{op} which share no non-zero direct factors, consider the morphism of varieties

$$\Phi : E_{\mathbf{i}} \cap \operatorname{HOM}_{A^{op}}(P_0, P_1)_{max} \longrightarrow \operatorname{rep}_{\mathbf{d}}(A^{op})$$

constructed in section 5.4.1.

By Lemma 5.4.4, there is a dense open subset of the set $\operatorname{HOM}_{A^{op}}(P_0, P_1)_{max}$ such that, for any (L, M, f) in that open subset,

$$\operatorname{codim}_{\mathcal{X}} \mathcal{O}_f = \operatorname{codim}_{\Phi(E_{\mathbf{i}} \cap \mathcal{Y})} \mathcal{O}_{\Phi(L,M,f)}.$$

Now, by Lemma 5.4.6, we have that

$$\operatorname{codim}_{\mathcal{X}} \mathcal{O}_f = \dim E(f).$$

Therefore,

$$\begin{array}{rcl} \operatorname{codim}_{\mathcal{Z}} \mathcal{O}_{\Phi(L,M,f)} & \geq & \operatorname{codim}_{\Phi(E_{\mathbf{i}} \cap \mathcal{Y})} \mathcal{O}_{\Phi(L,M,f)} \\ & = & \dim E(f) \\ & \geq & \dim \operatorname{Hom}_{A^{op}}(\Phi(L,M,f), \tau\Phi(L,M,f)) \\ & \geq & \operatorname{codim}_{\mathcal{Z}} \mathcal{O}_{\Phi(L,M,f)}, \end{array}$$

where the third inequality follows from Lemma 5.3.6. This implies that we have $\mathcal{Z} = \overline{\Phi(E_{\mathbf{i}} \cap \mathcal{Y})}$, and that $\mathcal{Z}$ is a strongly reduced component of $\operatorname{rep}(A^{op})$.

Define $\Psi'(\delta)$ to be this $\mathcal{Z}$.

Now the duality $D : \operatorname{mod}(A) \to \operatorname{mod}(A^{op})$ induces an isomorphism of varieties $\operatorname{rep}(A) \to \operatorname{rep}(A^{op})$ which preserves strongly reduced components; thus there is a strongly reduced component $\mathcal{Z}_0$ of $\operatorname{rep}(A)$ corresponding to $\mathcal{Z}$. Moreover, $K_0(\operatorname{add} A) \cong K_0(\operatorname{add} A^{op})$ in a natural way; δ thus corresponds to some $\delta_0 \in K_0(\operatorname{add} A)$. We define $\Psi(\delta_0)$ to be the strongly reduced component $\mathcal{Z}_0$.

From this definition, it follows immediately that two elements δ and δ' of $K_0(\operatorname{proj} A^{op})$ have the same image by Ψ' if, and only if, their canonical decompositions can be written as

$$\delta = \delta_1 \oplus \overline{\delta}, \quad \delta' = \delta'_1 \oplus \overline{\delta},$$

with δ_1, δ'_1 non-negative, for non-negative factors do not affect the cokernels.

Let us now prove that Ψ is surjective. It suffices to show that Ψ' is surjective.

Let $\mathcal{Z}'$ be a strongly reduced component of $\operatorname{rep}(A^{op})$. By Corollary 5.3.2, there is a dense open subset $\mathcal{U}$ of $\mathcal{Z}'$ and there are finitely generated projective modules P_1 and P_0 such that every representation M in $\mathcal{U}$ admits a minimal projective presentation

$$P_0 \longrightarrow P_1 \longrightarrow M \longrightarrow 0.$$

Consider the locally closed subset $\mathrm{HOM}_{A^{op}}(P_0, P_1)_{\mathbf{d}}$. There exists an irreducible component $\mathcal{F}$ of it and an $\mathbf{i}$ such that $\Phi(E_{\mathbf{i}} \cap \mathcal{F}) \cap \mathcal{U}$ is dense in $\mathcal{Z}'$. We get

$$\begin{aligned}
\mathrm{codim}_{\mathcal{F}}\mathcal{O}_{(P_1,P_0,f)} &= \mathrm{codim}_{\Phi(E_{\mathbf{i}}\cap\mathcal{F})}\mathcal{O}_{\Phi(P_0,P_1,f)} \text{ (Lemma 5.4.4)}\\
&= \mathrm{codim}_{\mathcal{Z}'}\mathcal{O}_{\Phi(P_0,P_1,f)}\\
&= \dim \mathrm{Hom}_{A^{op}}(\Phi(P_0,P_1,f), \tau\Phi(P_0,P_1,f)) \text{ ($\mathcal{Z}'$ str. reduced)}\\
&= \dim E(f) \text{ (Lemma 5.3.6)}\\
&= \mathrm{codim}_{\mathcal{X}}\mathcal{O}_f \text{ (Lemma 5.4.6)}\\
&= \mathrm{codim}_{\mathcal{Y}}\mathcal{O}_{(P_0,P_1,f)} \text{ (Lemma 5.4.4).}
\end{aligned}$$

Therefore $\mathcal{F}$ is of codimension zero in $\mathrm{HOM}_{A^{op}}(P_0, P_1)$. Thus we have the equality $\mathrm{HOM}_{A^{op}}(P_0, P_1)_{\mathbf{d}} = \mathrm{HOM}_{A^{op}}(P_0, P_1)_{max}$, and thus $\Psi'([P_0] - [P_1]) = \mathcal{Z}'$. This proves the surjectivity of the map Ψ'.

5.5 Invariance under mutation

We now define the mutation of indices. This notion comes from the mutation of Y-variables of [31], from the mutation of indices of [18] and from the mutation of $\mathcal{X}$-coordinates of [27].

Definition 5.5.1. Let (Q, W) be a non-degenerate quiver with potential. Let $(Q', W') = \mu_i(Q, W)$ be its mutation at a vertex i. Let Γ and Γ' be the corresponding Ginzburg dg algebras, considered as objects of the cluster categories $\mathcal{C}_{Q,W}$ and $\mathcal{C}_{Q',W'}$, respectively. The *mutation of indices* is given by the map

$$\mu_i : K_0(\mathrm{add}\,\Gamma) \longrightarrow K_0(\mathrm{add}\,\Gamma')$$

defined by $\mu_i(\sum_{j=1}^n y_j[\Gamma_j]) = \sum_{j=1}^n y'_j[\Gamma'_j]$, where

$$y'_j = \begin{cases} -y_i & \text{if } i = j;\\ y_j - m[-y_i]_+ & \text{if there are } m \text{ arrows from } i \text{ to } j;\\ y_j + m[y_i]_+ & \text{if there are } m \text{ arrows from } j \text{ to } i. \end{cases}$$

Remarks 5.5.2.
1. This definition comes from the mutation rule described in [27, Formula (13)], with elements taken in the tropicalization of the ring $\mathbb{Z}$. However, in order to get precisely the same mutation rule, one has to work over the opposite quiver.
2. Recall the triangle equivalence $\mu_i^- : \mathcal{C}_{Q,W} \to \mathcal{C}_{Q',W'}$ of section 3.2.8. If X is an object of $\mathcal{C}_{Q,W}$ whose index (with respect to Γ) is $\mathrm{ind}_\Gamma X$, then the index of $\mu_i^-(X)$ (with respect to Γ') is $\mu_i(\mathrm{ind}_\Gamma X)$. This is a consequence of Theorem 3.2.6 and of Proposition 3.2.7.

We reformulate Theorem 5.1.3 thus:

Theorem 5.5.3 (Reformulation of Theorem 5.1.3)**.** *We have a commutative diagram*

$$\begin{tikzcd}
K_0(\mathrm{add}\,\Gamma) \arrow[r, "\mu_i"] \arrow[d, "I"] & K_0(\mathrm{add}\,\Gamma') \arrow[d, "I"]\\
\mathbb{Q}(x_1, \ldots, x_n) \arrow[r, "\varphi_X^{-1}"] & \mathbb{Q}(x'_1, \ldots, x'_n),
\end{tikzcd}$$

where φ_X is as in Corollary 4.4.14.

During the proof of this result, we will need a lemma on the generic value of constructible functions.

Lemma 5.5.4. *Let W, X and Y be irreducible varieties, and let A be an abelian group. Assume that we have a commutative diagram*

$$\begin{tikzcd} & W \arrow[dl, "u"'] \arrow[dr, "v"] & \\ X \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & A & \end{tikzcd}$$

where u and v are dominant morphisms of varieties and φ and ψ are constructible functions. Then the generic values taken by φ and ψ are equal.

PROOF Let $x \in A$ be the generic value taken by the function φ, and let $y \in A$ be that taken by ψ. By definition, $\varphi^{-1}(x)$ is an open dense subset of X, and since X is irreducible and u is dominant, the intersection of $\varphi^{-1}(x)$ with the image of u contains a dense open subset of X. Thus $(\varphi \circ u)^{-1}(x)$ contains a dense open subset of W. For similar reasons, $(\psi \circ v)^{-1}(y)$ contains a dense open subset of W. Therefore $(\varphi \circ u)^{-1}(x)$ and $(\psi \circ v)^{-1}(y)$ have a non-empty intersection, and taking w in their intersection, we get

$$x = (\varphi \circ u)(w) = (\psi \circ v)(w) = y.$$

This proves the result. $\square$

We can now prove Theorem 5.5.3

PROOF (of Theorem 5.5.3) Let $[T_0] - [T_1]$ be an element of $K_0(\operatorname{add}\Gamma)$. We can assume that T_0 and T_1 have no direct factors in common. Then $I([T_0] - [T_1])$ is, by definition, the generic value taken by the constructible function

$$\begin{array}{rcl} \eta_{T_0,T_1} : \operatorname{Hom}_{\mathcal{C}}(T_1, T_0) & \longrightarrow & \mathbb{Q}(x_1, \ldots, x_n) \\ \varepsilon & \longmapsto & X'_{mt(\varepsilon)}. \end{array}$$

Let $\mu_i([T_0] - [T_1]) = [T'_0] - [T'_1]$ in $K_0(\operatorname{add}\Gamma')$. Then $I([T'_0] - [T'_1])$ is the generic value taken by the constructible function

$$\begin{array}{rcl} \eta_{T'_0,T'_1} : \operatorname{Hom}_{\mathcal{C}'}(T'_1, T'_0) & \longrightarrow & \mathbb{Q}(x'_1, \ldots, x'_n) \\ \varepsilon & \longmapsto & X'_{mt(\varepsilon)}. \end{array}$$

We want to show that $I([T'_0] - [T'_1])$ is the mutation of $I([T_0] - [T_1])$ at i; or, using our notation, that $I([T'_0] - [T'_1]) = \varphi_X^{-1}(I([T_0] - [T_1]))$. It is a consequence of Corollary 4.4.14 that we have a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(\mu_i^+(T'_1), \mu_i^+(T'_0)) \arrow[r, "\mu_i^-"] \arrow[d, "\eta"] & \operatorname{Hom}_{\mathcal{C}'}(T'_1, T'_0) \arrow[d, "\eta"] \\ \mathbb{Q}(x_1, \ldots, x_n) \arrow[r, "\varphi_X^{-1}"] & \mathbb{Q}(x'_1, \ldots, x'_n), \end{tikzcd}$$

where the two horizontal arrows are isomorphisms and where we omitted the indices of the maps η. Thus $I([T'_0] - [T'_1]) = \varphi_X^{-1}(I([\mu_i^+(T'_0)] - [\mu_i^+(T'_1)]))$, and to prove the theorem it is therefore sufficient to show that $I([\mu_i^+(T'_0)] - [\mu_i^+(T'_1)]) = I([T_0] - [T_1])$.

We consider two cases.

Step 1: Γ_i is not a direct summand of T_1. In that case, we can write $T_0 = \overline{T}_0 \oplus \Gamma_i^m$, where Γ_i is not a direct summand of $\overline{T}_0$. Recall that we have a (unique up to isomorphism) non-split triangle

$$\Gamma_i^* \xrightarrow{\alpha} E' \xrightarrow{\beta} \Gamma_i \xrightarrow{\gamma} \Sigma\Gamma_i^*$$

where $E' = \bigoplus_a \Gamma_{s(a)}$, the sum being taken over all arrows a ending in i, and the morphism α is given by multiplication by these arrows on each coordinate.

Then Proposition 3.2.7 (or rather, the triangle obtained at the end of its proof) allows us to write

$$\mu_i^+(T_0') = \overline{T}_0 \oplus (E')^m \quad \text{and} \quad \mu_i^+(T_1') = T_1 \oplus (\Gamma_i^*)^m.$$

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{C}}(\overline{T}_0 \oplus (E')^m) \times \mathrm{Hom}_{\mathcal{C}}(T_1, \overline{T}_0 \oplus (E')^m) \times \mathrm{Aut}_{\mathcal{C}}(T_1 \oplus (\Gamma_i^*)^m) \arrow[d, "u"] \arrow[dr, "v"] & \\
\mathrm{Hom}_{\mathcal{C}}(T_1 \oplus (\Gamma_i^*)^m, \overline{T}_0 \oplus (E')^m) \arrow[dr, "\eta"] & \mathrm{Hom}_{\mathcal{C}}(T_1, T_0) \arrow[d, "\eta"] \\
& \mathbb{Q}(x_1, \ldots, x_n).
\end{tikzcd}$$

where u and v are defined as follows. The morphism v takes a triple (g, f, g') and sends it to the composition $(id_{\overline{T}_0} \oplus \beta^{\oplus m}) \circ f$. The morphism u takes a triple (g, f, g') and sends it to the morphism given in matrix form by

$$g \begin{pmatrix} f_1 & 0 \\ f_2 & \alpha^{\oplus m} \end{pmatrix} g',$$

where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

If we could apply Lemma 5.5.4 to the above diagram, then the theorem would be proved. Let us show that the hypotheses of the Lemma are fulfilled. We easily see that the three varieties involved are irreducible, and we know from section 5.2 that the functions η are constructible. We must show that the square commutes and that u and v are dominant.

Substep 1 : the square commutes. To show that the square commutes, we first notice that, for any $(g, f, g') \in \mathrm{Aut}_{\mathcal{C}}(\overline{T}_0 \oplus (E')^m) \times \mathrm{Hom}_{\mathcal{C}}(T_1, \overline{T}_0 \oplus (E')^m) \times \mathrm{Aut}_{\mathcal{C}}(T_1 \oplus (\Gamma_i^*)^m)$, we have that $\eta v(g, f, g') = \eta v(id, f, id)$ (since g and g' do not occur in the definition of v) and that $\eta u(g, f, g') = \eta u(id, f, id)$ (since the map η takes the same value on orbits under the action of $\mathrm{Aut}_{\mathcal{C}}(\overline{T}_0 \oplus (E')^m) \times \mathrm{Aut}_{\mathcal{C}}(T_1 \oplus (\Gamma_i^*)^m)$). Thus it is sufficient to show that $\eta v(id, f, id) = \eta u(id, f, id)$. We invoke the octahedral axiom to get a diagram

$$\begin{tikzcd}
 & & Y''\oplus\Sigma(\Gamma_i^*)^m \arrow[ddl, "+\ast\oplus id"'] \arrow[d, dotted, "(a,b)"] & \\
Y' \arrow[urr] \arrow[dr, "+"] & & \Sigma(\Gamma_i^*)^m \arrow[ll, dotted, "+"] \arrow[dd, dotted, "+"] & \\
 & T_1\oplus(\Gamma_i^*)^m \arrow[rr, "(v(id,f,id),0)"] \arrow[dr, "u(id,f,id)"'] & & \overline{T}_0\oplus\Gamma_i^m \arrow[uul] \arrow[ul, dotted, "(0,\gamma^{\oplus m})"'] \\
 & & \overline{T}_0\oplus(E')^m \arrow[uull] \arrow[ur, "id\oplus\beta^{\oplus m}"'] &
\end{tikzcd}$$

where $*$ is an unknown morphism, Y' is the cone of $u(id,f,id)$ and Y'' is the cone of $v(id,f,id)$. We need to show that Y' and Y'' are isomorphic in order to show that the above square commutes, since $\eta u(id,f,id) = X'_{Y'}$ and $\eta v(id,f,id) = X'_{Y''}$. The octahedron yields a commutative square

$$\begin{tikzcd}
Y''\oplus\Sigma(\Gamma_i^*)^m \arrow[r, "(a,b)"] \arrow[d, "\ast\oplus id"] & \Sigma(\Gamma_i^*)^m \arrow[d, "(0,-\Sigma\alpha^{\oplus m})^t"] \\
\Sigma T_1\oplus\Sigma(\Gamma_i^*)^m \arrow[r, "-\Sigma u(id,f,id)"] & \Sigma\overline{T}_0\oplus\Sigma(E')^m
\end{tikzcd}$$

which, in turn, gives an equality of morphisms (in matrix form)

$$\begin{pmatrix} 0 & 0 \\ (-\Sigma\alpha^{\oplus m})\circ a & (-\Sigma\alpha^{\oplus m})\circ b \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & -\Sigma\alpha^{\oplus m} \end{pmatrix},$$

where again the stars are unknown morphisms. Thus $(-\Sigma\alpha^{\oplus m})\circ b = -\Sigma\alpha^{\oplus m}$, and using the fact that the triangle $\Gamma_i^* \xrightarrow{\alpha} E' \xrightarrow{\beta} \Gamma_i \xrightarrow{\gamma} \Sigma\Gamma_i^*$ is a minimal $(\operatorname{add}\Gamma)$-copresentation of Γ_i^*, we get that b is an isomorphism. Therefore, the triangle (in the octahedron)

$$Y' \longrightarrow Y''\oplus\Sigma(\Gamma_i^*)^m \xrightarrow{(a,b)} \Sigma(\Gamma_i^*)^m \longrightarrow \Sigma Y'$$

is isomorphic to a triangle

$$Y' \longrightarrow Y''\oplus\Sigma(\Gamma_i^*)^m \xrightarrow{(0,1)} \Sigma(\Gamma_i^*)^m \longrightarrow \Sigma Y'$$

which is a direct sum of two triangles, of the form $Y' \longrightarrow Y'' \longrightarrow 0 \longrightarrow \Sigma Y'$ and $0 \longrightarrow \Sigma(\Gamma_i^*)^m \longrightarrow \Sigma(\Gamma_i^*)^m \longrightarrow 0$. Thus Y' and Y'' are isomorphic, and substep 1 is proven, that is, the above square commutes.

Substep 2 : the morphism v is dominant. In fact, we show that v is surjective, and thus dominant. Indeed, let $f \in \operatorname{Hom}_{\mathcal{C}}(T_1, T_0)$. Since $T_0 = \overline{T}_0 \oplus \Gamma_i^m$, we can write f in matrix form as $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Now, since we have a triangle $\Gamma_i^* \xrightarrow{\alpha} E' \xrightarrow{\beta} \Gamma_i \xrightarrow{\gamma} \Sigma\Gamma_i^*$, and since the space $\operatorname{Hom}_{\mathcal{C}}(T_1, \Sigma\Gamma_i^*)$ vanishes (because Γ_i is not a direct summand of T_1), we have that any morphism from T_1 to Γ_i^m factors through $\beta^{\oplus m}$. Thus we can write $f_2 = (\beta^{\oplus m})f_2'$, and we have a preimage of f through v of the form

$$(id, \begin{pmatrix} f_1 \\ f_2' \end{pmatrix}, id).$$

Substep 3 : the morphism u is dominant. We will prove that the image of u contains the following dense open subset of $\mathrm{Hom}_{\mathcal{C}}(T_1 \oplus (\Gamma_i^*)^m, \overline{T}_0 \oplus (E')^m)$:

$$\{ \begin{pmatrix} f_1 & h \\ f_2 & g \circ \alpha^{\oplus m} \circ g' \end{pmatrix} \mid f_1, f_2, h \text{ are arbitrary, } g \in \mathrm{Aut}_{\mathcal{C}}((E')^m),\ g' \in \mathrm{Aut}_{\mathcal{C}}((\Gamma_i^*)^m)\}.$$

This subset is open because the subset $\{g \circ \alpha^{\oplus m} \circ g' \mid g \in \mathrm{Aut}_{\mathcal{C}}((E')^m),\ g' \in \mathrm{Aut}_{\mathcal{C}}((\Gamma_i^*)^m)\}$ of $\mathrm{Hom}_{\mathcal{C}}((\Gamma_i^*)^m, (E')^m)$ is open, thanks to the fact that Γ_i^m is rigid and to [18, Lemma 2.1]. Let us show that it is contained in the image of u. Let

$$\begin{pmatrix} f_1 & h \\ f_2 & g \circ \alpha^{\oplus m} \circ g' \end{pmatrix}$$

be an element of it. Then $h = h' \circ \alpha^{\oplus m}$ for some morphism h', and we have

$$\begin{pmatrix} f_1 & h \\ f_2 & g \circ \alpha^{\oplus m} \circ g' \end{pmatrix} = \begin{pmatrix} id & h' \\ 0 & g \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ g^{-1} f_2 & \alpha^{\oplus m} \end{pmatrix} \begin{pmatrix} id & o \\ 0 & g' \end{pmatrix},$$

which is in the image of u. Thus u is dominant.

Substep 4 We can now apply Lemma 5.5.4 to the above square, and as discussed earlier, this proves the theorem for the case considered in step 1.

Step 2 : Γ_i is not a direct summand of T_0. In that case, we can write $T_1 = \overline{T}_1 \oplus \Gamma_i^n$, where Γ_i is not a direct summand of $\overline{T}_1$. We can use arguments similar to those of step 1 to prove the theorem. We could also work in the opposite triangulated category $\mathcal{C}_{Q,W}^{op}$, and notice that

$$\mathrm{Hom}_{\mathcal{C}}(T_1, T_0) = \mathrm{Hom}_{\mathcal{C}^{op}}(T_0, T_1),$$

making step 2 in $\mathcal{C}_{Q,W}$ equivalent to step 1 in $\mathcal{C}_{Q,W}^{op}$. □

5.6 Proof of Theorem 5.1.1

Consider the map $I : K_0(\mathrm{add}\,\Gamma) \to \mathbb{Q}(x_1, \ldots, x_n)$ defined in Definition 5.2.2. We proved in Proposition 5.2.3 that the elements in the image of I are linearly independent over $\mathbb{Z}$.

The fact that the image of I is contained in the upper cluster algebra $\mathcal{A}_Q^+$ follows from Theorem 5.1.3. Indeed, let $(u_1, \ldots, u_n)$ be a cluster obtained from the initial seed by a sequence of mutations at vertices $i_1, \ldots, i_s$. Let w be an element of $\mathbb{Q}(x_1, \ldots, x_n)$, expressed in terms of the initial cluster. Then its expression with respect to the cluster $(u_1, \ldots, u_n)$ is given by $\varphi^{(s)} \circ \cdots \circ \varphi^{(1)}(w)$, where $\varphi^{(j)}$ is the isomorphism φ_X^{-1} of Corollary 4.4.14 (defined with respect to the vertex i_j). Theorem 5.1.3 implies that, for any $\delta \in K_0(\mathrm{add}\,\Gamma)$, we have

$$\varphi^{(s)} \circ \cdots \circ \varphi^{(1)}(I(\delta)) = I(\mu_{i_s} \circ \cdots \mu_{i_1}(\delta)),$$

which is a Laurent polynomial in the variables $u_1, \ldots, u_n$. This being true for any cluster, $I(\delta)$ belongs to the upper cluster algebra.

Finally, suppose that (Q, W) arises from the setting of [35], and let A be the Jacobian algebra of (Q, W). For any finite-dimensional representation M of A, define ψ_M thus: if

$$0 \longrightarrow M \longrightarrow I_1 \xrightarrow{f} I_0$$

is a minimal injective presentation of M, then the injective modules I_1 and I_0 lift in the cluster category $\mathcal{C}_{Q,W}$ to objects ΣT_1 and ΣT_0 of $\operatorname{add} \Sigma\Gamma$ through $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$ (see [57, Proposition 2.1]). Moreover, f lifts to a morphism $\overline{f} \in \operatorname{Hom}_{\mathcal{C}}(\Sigma T_1, \Sigma T_0)$. Then we put

$$\psi_M = X'_{mt(\overline{f})}.$$

For any irreducible component $\mathcal{Z}$ of $\operatorname{rep}_{\mathbf{d}}(A)$, let $\psi_{\mathcal{Z}}$ be the generic value taken by ψ_M in $\mathcal{Z}$, and let

$$null(\mathcal{Z}) = \{\mathbf{m} \in \mathbb{N}^{Q_0} |\ m_i = 0 \text{ if } d_i = 0\}.$$

Then it is proved in [35, Theorem 5] that the set

$$B = \{x^{\mathbf{m}}\psi_{\mathcal{Z}} |\ \mathcal{Z} \text{ is strongly reduced in } \operatorname{rep}(A), \mathbf{m} \in null(\mathcal{Z})\}$$

is a basis of the cluster algebra $\mathcal{A}_Q$. Let us prove that it is the image of the map I.

Assume that $\mathcal{Z}$ is a strongly reduced component of $\operatorname{rep}_{\mathbf{d}}(A)$. Then, by Theorem 5.1.2, we have $\mathcal{Z} = \Psi([T_0] - [T_1])$ for some $[T_0] - [T_1] \in K_0(\operatorname{add}\Gamma)$. By definition of Ψ (see section 5.4.4), $\mathcal{Z}$ is the dual component of some $\mathcal{Z}' = \Psi'([P_0] - [P_1])$, where $[P_0] - [P_1] \in K_0(\operatorname{add} A^{op})$. By definition of Ψ', there is a dense open subset $\mathcal{U}$ of $\operatorname{Hom}_{A^{op}}(P_0, P_1)$ such that the union of orbits of cokernels of morphisms in $\mathcal{U}$ contains a dense open subset of $\mathcal{Z}'$. Thus a generic representation in $\mathcal{Z}'$ is isomorphic to a cokernel of a generic morphism in $\operatorname{Hom}_{A^{op}}(P_0, P_1)$. Dualizing, we get that a generic representation in $\mathcal{Z}$ is isomorphic to a kernel of a generic morphism in $\operatorname{Hom}_A(DP_1, DP_0)$. Note that $\operatorname{Hom}_A(DP_1, DP_0)$ is isomorphic to $\operatorname{Hom}_{\mathcal{C}}(\Sigma T_1, \Sigma T_0)$ (since the DP_i are finite-dimensional injective A-modules, see [57, Proposition 2.1]).

Now, using Theorem 5.3.7, we get a canonical decomposition

$$[T_0] - [T_1] = \delta_1 \oplus \ldots \oplus \delta_s,$$

where the δ_i are indecomposable. Assume that there are no non-negative terms in this decomposition. This means that, generically in $\operatorname{Hom}_{\mathcal{C}}(\Sigma T_1, \Sigma T_0)$, $mt(\varepsilon)$ has no direct summand in $\operatorname{add}\Gamma$, so the generic value of $X'_{mt(\varepsilon)}$ is $\psi_{\mathcal{Z}}$; in other words,

$$I([T_0] - [T_1]) = \psi_{\mathcal{Z}} \in B.$$

Now, let $\mathbf{m} \in null(\mathcal{Z})$. Consider the non-negative element $[\bigoplus_{i=1}^n \Gamma_i^{m_i}] \in K_0(\operatorname{add}\Gamma)$. We will show that

$$[T_0] - [T_1] + [\bigoplus_{i=1}^n \Gamma_i^{m_i}] = \delta_1 \oplus \ldots \oplus \delta_s \oplus \bigoplus_{i=1}^n [\Gamma_i]^{\oplus m_i}$$

is a canonical decomposition. This will imply that

$$\begin{aligned} I([T_0] - [T_1] + [\bigoplus_{i=1}^n \Gamma_i^{m_i}]) &= I([\bigoplus_{i=1}^n \Gamma_i^{m_i}])I([T_0] - [T_1]) \\ &= x^{\mathbf{m}}\psi_{\mathcal{Z}}, \end{aligned}$$

and will thus prove that the set B is the image of the map I.

In order to do this, we work in the opposite category $\mathcal{C}_{Q,W}^{op}$. We use the functor $F = \mathrm{Hom}_{\mathcal{C}^{op}}((\Sigma^{op})^{-1}\Gamma, ?)$; in view of [20, Theorem 4.4], we only need to show that for generic morphisms

$$f' \in \mathrm{Hom}_{A^{op}}(F(\Sigma^{op})^{-1}T_0, F(\Sigma^{op})^{-1}T_1) \text{ and } f'' \in \mathrm{Hom}_{A^{op}}(\bigoplus_{i=1}^{n} F(\Sigma^{op})^{-1}\Gamma_i^{m_i}, 0),$$

the spaces $E(f', f'')$ and $E(f'', f')$ vanish. Note that, by the above, a generic f' has a cokernel M'' with dimension vector $\mathbf{d}$. Using [20, Lemma 3.2], we thus get

$$\begin{aligned} E(f', f'') &= \mathrm{Coker}\left(\mathrm{Hom}_{A^{op}}(F(\Sigma^{op})^{-1}T_1, 0) \to \mathrm{Hom}_{A^{op}}(F(\Sigma^{op})^{-1}T_0, 0)\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E(f'', f') &= \mathrm{Coker}\left(0, M''\right) \to \mathrm{Hom}_{A^{op}}(\bigoplus_{i=1}^{n} F(\Sigma^{op})^{-1}\Gamma_i^{m_i}, M'')) \\ &= \mathrm{Hom}_{A^{op}}(\bigoplus_{i=1}^{n} F(\Sigma^{op})^{-1}\Gamma_i^{m_i}, M'') \\ &= 0, \end{aligned}$$

since $\underline{\dim}\, M'' = \mathbf{d}$, and $\mathbf{m} \in null(\mathcal{Z})$. This finishes the proof of the theorem.

Remark 5.6.1. Our proof that the image of the map I is the set B is valid for any Jacobi-finite quiver with potential, and not only for those arising from the setting of [35].

5.6.1 Link with a conjecture of V. Fock and A. Goncharov

We now show how Theorems 5.1.1 and 5.1.3 are related to Conjecture 4.1 of [27]. Let Q be a quiver without oriented cycles of length ≤ 2, and let $\mathcal{A}_Q$ be the associated cluster algebra (without coefficients). The authors of [27] conjecture, in a slightly different language, that there exists a bijection

$$\mathbb{Z}^n \longrightarrow E(\mathcal{A}),$$

where $E(\mathcal{A})$ is the subset of the cluster algebra consisting of elements which are Laurent polynomials with positive coefficients in the cluster variables of every cluster, and which cannot be written as a sum of two or more such elements. This bijection should have the following properties:

1. It should commute with mutation (where the mutation in $\mathbb{Z}^n$ is as defined in Definition 5.5.1, when we identify the element $(a_1, \ldots, a_n)$ of $\mathbb{Z}^n$ with $\sum_{j=1}^{n} a_j[\Gamma_j]$ in $\mathrm{add}\,\Gamma$).
2. An element $(a_1, \ldots, a_n)$ of $\mathbb{Z}^n$ with non-negative coefficients should be sent to the element $\prod_{j=1}^{n} x_j^{a_j}$.
3. The set $E(\mathcal{A})$ should be a $\mathbb{Z}$-basis of the upper cluster algebra $\mathcal{A}_Q^+$.

Other conditions are described in [27, Conjecture 4.1], but we will not discuss them here. If we can equip the quiver Q with a non-degenerate potential W so that (Q, W) is Jacobi-finite, then we have a good candidate for such a map.

Theorem 5.6.2. *Let (Q, W) be a non-degenerate, Jacobi-finite quiver with potential. Then the map*

$$I : \mathbb{Z}^n \cong \operatorname{add}\Gamma \longrightarrow \mathcal{A}_Q^+$$

defined in Definition 5.2.2 satisfies conditions 1 and 2 above. If, moreover, (Q, W) arises from the setting of [35], then the image of I satisfies condition 3.

Proof Condition 1 is Theorem 5.1.3. Condition 2 follows from the definition of I. When we are in the setting of [35], condition 3 follows from Theorem 5.1.1 and from the fact [36, Theorem 3.3] that in that case, the cluster algebra and the upper cluster algebra coincide. $\square$

Note that the coefficients of the elements in the image of I need not be positive, as seen in [21, Example 3.6]. Thus the image of I is not contained in $E(\mathcal{A})$ in general.

The conjecture of [27] discussed above is linked to one of [35], where the authors conjecture that the set described in their Theorem 5 is a basis for the cluster algebra $\mathcal{A}_Q$, starting from an arbitrary non-degenerate quiver with potential (Q, W). Using Theorem 5.1.2, we know that, if (Q, W) is Jacobi-finite, then this set of [35] is exactly the image of the map I, and by Example 5.6.3 below, it is not necessarily contained in the cluster algebra, so that in the conjecture of [35], one should replace "cluster algebra" by "upper cluster algebra". If I is a good candidate for the above map, then this is compatible with [27, Conjecture 4.1].

Example 5.6.3. The quiver with potential described below arises from the work of D. Labardini-Fragoso [61][60]. We will show that the image of the map I for this example is not contained in the associated cluster algebra, and that its cluster-category (which is Hom-finite) has cluster-tilting objects which are not related by a finite sequence of mutations.

Consider the quiver

$$Q = \begin{tikzcd} & 2 \arrow[dr, shift left, "b_1{,}b_2"] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left, "a_1{,}a_2"] \arrow[ur, shift right] & & 3 \arrow[ll, shift left] \arrow[ll, shift right, "c_1{,}c_2"] \end{tikzcd}$$

with potential $W = c_1b_1a_1 + c_2b_2a_2 - c_1b_2a_1c_2b_1a_2$. As shown in [60, Example 8.2], this quiver with potential is Jacobi-finite and non-degenerate. Its Jacobian algebra A is the path algebra of Q, modulo the relations

$$\begin{array}{llll}
c_1b_1 = c_2b_1a_2c_1b_2; & c_1b_1a_2 = 0; & b_1a_1c_2 = 0; & a_1c_1b_2 = 0; \\
c_2b_2 = c_1b_2a_1c_2b_1; & c_2b_2a_1 = 0; & b_2a_2c_1 = 0; & a_2c_2b_1 = 0; \\
b_1a_1 = b_2a_1c_2b_1a_2; & c_1b_2a_2 = 0; & b_1a_2c_2 = 0; & a_1c_2b_2 = 0; \\
b_2a_2 = b_1a_2c_1b_2a_1; & c_2b_1a_1 = 0; & b_2a_1c_1 = 0; & a_2c_1b_1 = 0; \\
a_1c_1 = a_2c_1b_2a_1c_2; & & & \\
a_2c_2 = a_1c_2b_1a_2c_1; & & &
\end{array}$$

and all non-alternating paths of length 4 and all paths of length 7 are zero. These relations imply that $c_1b_1a_1 = c_2b_2a_2$; moreover, the Jacobian algebra is self-injective.

As a vector space, the indecomposable projective A-module $P_1 = e_1A$ has a basis given by

$$\{e_1, c_1, c_2, c_1b_1, c_1b_2, c_2b_1, c_2b_2, c_1b_1a_1, c_1b_2a_1, c_2b_1a_2, c_1b_2a_1c_2, c_2b_1a_2c_1\}.$$

Similar calculations can be done for P_2 and P_3. As a representation of the opposite quiver, we can draw P_1 as

$$
\begin{tikzcd}
 & & 1 \arrow[dll, "c_1"'] \arrow[drr, "c_2"] & & \\
3 \arrow[dr, "b_2"] \arrow[dddd, "b_1"'] & & & & 3 \arrow[dl, "b_1"'] \arrow[dddd, "b_2"] \\
 & 2 \arrow[d, "a_1"'] & & 2 \arrow[d, "a_2"] & \\
 & 1 \arrow[d, "c_2"'] & & 1 \arrow[d, "c_1"] & \\
 & 3 \arrow[drrr, "b_1"' near end] & & 3 \arrow[dlll, "b_2" near end] & \\
2 \arrow[drr, "a_1"'] & & & & 2 \arrow[dll, "a_2"] \\
 & & 1 & &
\end{tikzcd}
$$

Its socle is S_1; thus $P_1 = I_1$, the indecomposable injective at vertex 1. For similar reasons, $P_2 = I_2$ and $P_3 = I_3$.

Now, let $\mathcal{C}$ be the cluster category of (Q, W). It is Hom-finite, since (Q, W) is Jacobi-finite (by results of [2]). Moreover, the functor $F = \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}\Gamma, ?)$ sends $\Sigma^{-1}\Gamma_i$ to P_i and $\Sigma\Gamma_i$ to $I_i = P_i$, for $i = 1, 2, 3$.

Let us compute $I([\Gamma_1] - [\Gamma_3])$. By definition, it is the generic value of the cluster character $X'_?$ applied to cones of morphisms in $\operatorname{Hom}_{\mathcal{C}}(\Gamma_3, \Gamma_1)$. Equivalently, it is the value X'_M, where

$$\Gamma_1 \longrightarrow M \longrightarrow \Sigma\Gamma_3 \xrightarrow{f} \Sigma\Gamma_1$$

is a triangle and f is generic in $\operatorname{Hom}_{\mathcal{C}}(\Sigma\Gamma_3, \Sigma\Gamma_1)$. Applying the functor F, we get an injective presentation

$$0 \longrightarrow FM \longrightarrow P_3 \xrightarrow{Ff} P_1,$$

where Ff is generic. Now, a generic Ff in $\operatorname{Hom}_A(P_3, P_1)$ is one for which $\underline{\dim}\operatorname{Ker} Ff$ is minimal. We easily see that this minimal dimension vector is $(1, 0, 1)$ (for instance, one could take Ff to be the left multiplication by $c_1 + c_2$), so that

$$
FM = \begin{tikzcd}
 & 0 \arrow[dl, shift left] \arrow[dl, shift right] & \\
\mathbb{C} \arrow[rr, shift left, "\varphi_1, \varphi_2"] \arrow[rr, shift right] & & \mathbb{C} \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}
$$

with $\varphi_1\varphi_2 \neq 0$ as a representation of the opposite quiver. Thus FM has exactly 3 subrepresentations, of dimension vectors $(0, 0, 0)$, $(0, 0, 1)$ and $(1, 0, 1)$. Applying the cluster character, we get

$$\begin{aligned} X'_M &= x^{\mathrm{ind}_\Gamma M} \sum_e \chi(\mathrm{Gr}_e(FM)) x^{-\iota(e)} \\ &= x^{\mathrm{ind}_\Gamma M} \sum_e \chi(\mathrm{Gr}_e(FM)) \prod_{j=1}^{3} \hat{y}_j^{e_j} \\ &= \frac{x_1}{x_3}(1 + \hat{y}_3 + \hat{y}_1\hat{y}_3) \\ &= \frac{x_1}{x_3}(1 + x_1^{-2}x_2^2 + x_2^{-2}x_3^2x_1^{-2}x_2^2) \\ &= \frac{x_1^2 + x_2^2 + x_3^2}{x_1x_3}, \end{aligned}$$

where we use the notations of section 4.3.2 for the formula of the cluster character. This is the value of $I([\Gamma_1] - [\Gamma_3])$, and as shown in the proof of [5, Proposition 1.26], it does not lie in the cluster algebra $\mathcal{A}_Q$.

Now, the objects Γ and $\Sigma\Gamma$ are cluster-tilting objects in $\mathcal{C}$. Note that

$$\mathrm{ind}_\Gamma \Gamma = \sum_{j=1}^{3} [\Gamma_j] \quad \text{and} \quad \mathrm{ind}_\Gamma \Sigma\Gamma = -\sum_{j=1}^{3} [\Gamma_j].$$

Let Y be an object of $\mathcal{C}$ with index $\sum_{j=1}^{3} y_j[\Gamma_j]$. By Remark 5.5.2, the index of $\mu_i^-(Y)$ is given by $\sum_{j=1}^{3} y'_j[\Gamma'_j] = \mu_i(\mathrm{ind}_\Gamma Y)$, so that

$$y'_j = \begin{cases} -y_i & \text{if } i = j; \\ y_j + 2[y_i]_+ & \text{if there are arrows from } j \text{ to } i; \\ y_j - 2[-y_i]_+ & \text{if there are arrows from } i \text{ to } j. \end{cases}$$

Thus we have that $y'_i = -y_i$, $y'_{i+1} = y_{i+1} - 2[-y_i]_+$ and $y'_{i+2} = y_{i+2} + 2[y_i]_+$, where $i, i+1, i+2$ are considered modulo 3. Thus

$$y'_i + y'_{i+1} + y'_{i+2} = -y_i + y_{i+1} - 2[-y_i]_+ + y_{i+2} + 2[y_i]_+ = y_i + y_{i+1} + y_{i+2}.$$

This shows that the sum of the coefficients appearing in the index of Y is preserved under mutation of Y. Since this sum is 3 for Γ and -3 for $\Sigma\Gamma$, the two objects cannot be related by a sequence of mutations.

Remark 5.6.4. In the above example, the fact that the sum of the coefficients of the indices is invariant under mutation was proved in [26, Section 2.2] (in a slightly different language).

Bibliography

[1] Claire Amiot, *On generalized cluster categories*, arXiv:1101.3675v1 [math.RT].

[2] ———, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 6, 2525–2590.

[3] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006, Techniques of representation theory.

[4] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[5] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52.

[6] Mikhail V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, Journal of K-theory **6** (2010), no. 3, 387–504.

[7] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[8] Nicolas Bourbaki, *Éléments de mathématiques – topologie générale, chapitres 1 à 4*, Hermann, Paris, 1971.

[9] Tom Bridgeland, *An introduction to motivic Hall algebras*, arXiv:1002.4372v1 [math.AG].

[10] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[11] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1347–1364.

[12] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616.

[13] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras. II*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 983–1009.

[14] ———, *From triangulated categories to cluster algebras*, Invent. Math. **172** (2008), no. 1, 169–211.

[15] Giovanni Cerulli Irelli, *Canonically positive basis of cluster algebras of type $A_2^{(1)}$*, arXiv:0904.2543v2 [math.RT].

[16] William Crawley-Boevey, *Geometry of representations of algebras*, Lecture notes, 1993, available at the author's webpage.

[17] William Crawley-Boevey and Jan Schröer, *Irreducible components of varieties of modules*, J. Reine Angew. Math. **553** (2002), 201–220.

[18] Raika Dehy and Bernhard Keller, *On the combinatorics of rigid objects in 2-Calabi-Yau categories*, Int. Math. Res. Not. IMRN (2008), no. 11, Art. ID rnn029, 17.

[19] Laurent Demonet, *Categorification of skew-symmetrizable cluster algebras*, arXiv:0909.1633v1 [math.RT].

[20] Harm Derksen and Jiarui Fei, *General presentations of algebras*, arXiv:0911.4913v1 [math.RA].

[21] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations. II. Applications to cluster algebras*, arXiv:0904.0676v2 [math.RA].

[22] ______, *Quivers with potentials and their representations. I. Mutations*, Selecta Math. (N.S.) **14** (2008), no. 1, 59–119.

[23] Ming Ding, Jie Xiao, and Fan Xu, *Integral bases of cluster algebras and representations of tame quivers*, arXiv:0901.1937v1 [math.RT].

[24] Grégoire Dupont, *Generic cluster characters*, International Mathematics Research Notices IMRN (2011).

[25] ______, *Generic variables in acyclic cluster algebras*, J. Pure Appl. Algebra **215** (2011), no. 4, 628–641.

[26] Vladimir V. Fock and Alexander B. Goncharov, *Cluster X-varieties at infinity*, arXiv:1104.0407v1 [math.AG].

[27] ______, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 6, 865–930.

[28] Sergey Fomin, *Total positivity and cluster algebras*, Proceedings of the ICM 2010, Hyderabad.

[29] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic).

[30] ______, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121.

[31] ______, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164.

[32] Changjian Fu and Bernhard Keller, *On cluster algebras with coefficients and 2-Calabi-Yau categories*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 859–895.

[33] Peter Gabriel, *Finite representation type is open*, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 10 (Ottawa, Ont.), Carleton Univ., 1974, pp. 23 pp. Carleton Math. Lecture Notes, No. 9.

[34] Christof Geiss, Bernard Leclerc, and Jan Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, arXiv:math/0703039v4 [math.RT].

[35] ______, *Generic bases for cluster algebras and the Chamber Ansatz*, arXiv:1004.2781v2 [math.RT].

[36] ______, *Kac–Moody groups and cluster algebras*, arXiv:1001.3545v2 [math.RT].

[37] ______, *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 193–253.

[38] ———, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632.

[39] ———, *Semicanonical bases and preprojective algebras. II. A multiplication formula*, Compos. Math. **143** (2007), no. 5, 1313–1334.

[40] ———, *Preprojective algebras and cluster algebras*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 253–283.

[41] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3** (2003), no. 3, 899–934, 1199, {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.

[42] Victor Ginzburg, *Calabi–Yau algebras*, arXiv:math/0612139v3 [math.AG].

[43] Alexandre Grothendieck and Jean Dieudonné, *Éléments géométrie algébrique I*, Grundlehren der mathematischen Wissenschaften, vol. 166, Springer-Verlag, Berlin–Heidelberg, 1971.

[44] Rei Inoue, Osamu Iyama, Bernhard Keller, Atsuo Kuniba, and Tomoki Nakanishi, *Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: Type B_r*, arXiv:1001.1880v2 [math.QA].

[45] ———, *Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras II: Types C_r, F_4, and G_2*, arXiv:1001.1881v2 [math.QA].

[46] Osamu Iyama and Idun Reiten, *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*, Amer. J. Math. **130** (2008), no. 4, 1087–1149. MR 2427009 (2009h:16022)

[47] Osamu Iyama and Yuji Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), no. 1, 117–168.

[48] Dominic Joyce and Yinan Song, *A theory of generalized Donaldson-Thomas invariants*, arXiv:0810.5645v4 [math.AG].

[49] Masaki Kashiwara, *Bases cristallines*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 6, 277–280.

[50] Bernhard Keller, *Algèbres amassées et applications*, Séminaire Bourbaki, Exposé 1014, November 2009, 27 pages.

[51] ———, *Deformed Calabi–Yau completions*, arXiv:0908.3499v5 [math.RT].

[52] ———, *The periodicity conjecture for pairs of Dynkin diagrams*, arXiv:1001.1531v3 [math.RT].

[53] ———, *Quiver mutation in Java*, Java applet available at the author's homepage.

[54] ———, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[55] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[56] ———, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 467–489.

[57] Bernhard Keller and Idun Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

[58] Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, arXiv:0906.0761v2 [math.RT].

[59] Maxim Kontsevich and Yan Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435v1 [math.AG].

[60] Daniel Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces, Part II: Arc representations*, arXiv:0909.4100v2 [math.RT].

[61] ———, *Quivers with potentials associated to triangulated surfaces*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 3, 797–839.

[62] Bernard Leclerc, *Cluster algebras and representation theory*, Proceedings of the ICM 2010, Hyderabad.

[63] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[64] ———, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.

[65] Kentaro Nagao, *Donaldson-Thomas theory and cluster algebras*, arXiv:1002.4884v1 [math.AG].

[66] ———, *Electronic message to B. Keller*, October 2, 2009.

[67] Yann Palu, *Cluster characters II: A multiplication formula*, arXiv:0903.3281v3 [math.RT].

[68] ———, *Cluster characters for 2-Calabi-Yau triangulated categories*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 6, 2221–2248.

[69] Pierre-Guy Plamondon, *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, arXiv:1004.0830v2 [math.RT].

[70] ———, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Advances in Mathematics **227** (2011), no. 1, 1 – 39.

[71] Idun Reiten, *Cluster categories*, Proceedings of the ICM 2010, Hyderabad.

[72] Ralf Schiffler, *On cluster algebras arising from unpunctured surfaces. II*, Adv. Math. **223** (2010), no. 6, 1885–1923.

[73] Paul Sherman and Andrei Zelevinsky, *Positivity and canonical bases in rank 2 cluster algebras of finite and affine types*, Mosc. Math. J. **4** (2004), no. 4, 947–974, 982.

[74] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp.

[75] Andrei Zelevinsky, *What is ... a cluster algebra?*, Notices Amer. Math. Soc. **54** (2007), no. 11, 1494–1495.