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Rational points of bounded height on weighted projective stacks

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Résumé

Un champ projectif à poids est un quotient champêtre $\mathscr{P}(\mathbf{a}) = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$, où l'action de $\mathbb{G}_m$ est avec des poids $\mathbf{a} \in \mathbb{Z}^n_{>0}$. Des examples sont: le champ compactifié de modules de courbes elliptiques $\mathscr{P}(4,6)$ et le champs classifiant de μ_m-torseurs $B\mu_m = \mathscr{P}(m)$. Nous définissons des hauteurs sur ces champs. Les hauteurs généralisent la hauteur naïve d'une courbe et le discriminant absolu d'un torseur. Nous utilisons les hauteurs pour compter des points rationnels. Nous trouvons le comportement asymptotique pour le nombre de points rationnels de hauteur borne.

Mots clefs: Conjecture de Manin, Points rationnels, Champ projectif à poids, Formule de Poisson.

Abstract

A weighted projective stack is a stacky quotient $\mathscr{P}(\mathbf{a}) = (\mathbf{A}^n - \{0\})/\mathbb{G}_m$, where the action of $\mathbb{G}_m$ is with weights $\mathbf{a} \in \mathbb{Z}^n_{>0}$. Examples are: the compactified moduli stack of elliptic curves $\mathscr{P}(4,6)$ and the classifying stack of μ_m-torsors $B\mu_m = \mathscr{P}(m)$. We define heights on the weighted projective stacks. The heights generalize the naive height of an elliptic curve and the absolute discriminant of a torsor. We use the heights to count rational points. We find the asymptotic behaviour for the number of rational points of bounded heights.

Keywords: Manin conjecture, Rational points, Weighted projective stack, Poisson formula.

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CONTENTS

CHAPTER 1

INTRODUCTION EN FRANÇAIS

1.1. Notation

La notation suivante sera utilisée tout au long de la thèse. Par F nous désignerons un corps de nombres (que l'on peut fixer pour l'article entier). Soit M_F, M_F^0, M_F^∞, $M_F^\mathbb{R}$ et $M_F^\mathbb{C}$ l'ensemble des places, places finies, places infinies, places réeles et places complexes de F, respectivement. Pour $v \in M_F$, nous désignerons par F_v la completion v-adique de F. Pour $v \in M_F^0$, soit $\mathcal{O}_v$ l'anneau des entiers de F_v, fixons une uniformisante $\pi_v \in F_v$ et soit $|\cdot|_v$ la valeur absolue sur F_v normalisée par $|\pi_v|_v = [\mathcal{O}_v : \pi_v\mathcal{O}_v]^{-1}$. Pour $v \in M_F^\mathbb{R}$, on désigne par $|\cdot|_v$ la valeur absolue habituelle et pour $v \in M_F^\mathbb{C}$ par $|\cdot|_v$ le carré de la valeur absolue habituelle. Les normalisations sont choisies pour que la formule du produit soit valide, c'est-à-dire pour chaque $x \in F$, on a

$$\prod_{v \in M_F} |x|_v = 1.$$

On note $\mathcal{O}_F$ l'anneau des entiers de F et pour un sous-ensemble fini $S \subset M_F^0$, on note $\mathcal{O}_{F,S}$ l'anneau des S-entiers. Lorsque $v \in M_F^\infty$, on notera par n_v le degré $[F_v : \mathbb{R}]$. On note par $\mathbb{A}_F$ l'anneau des adèles de F et $\mathbb{A}_F^\times$ le groupe des idèles.

Pour un vecteur $\mathbf{x} \in \mathbb{R}^n$, on notera $|\mathbf{x}|$ la somme $x_1 + \cdots + x_n$.

1.2. Conjecture de Manin-Peyre

Rappelons une conjecture due à Manin et Peyre sur le comportement asymptotique du nombre de points rationnels de "taille" bornée.

1.2.1. — Une des questions fondamentales de la géométrie diophantienne est l'étude du nombre de solutions aux équations algébriques. La conjecture de Manin-Peyre traite d'une telle question. Il prédit le nombre de points rationnels sur les variétés algébriques de hauteur bornée, quand il y en a "beaucoup". Rappelons-le brièvement.

Soit X une variété de Fano sur un corps de nombres F et soit K_X^{-1} son fibré anticanonique. On croît que la condition de Fano, c'est-à-dire que K_X^{-1} est positif, rend, après éventuellement passage à une extension de F, les points rationnels de Zariski denses en X. Une métrique adélique sur K_X^{-1} est un choix de métriques pour chaque fibré en droites topologiques $K_X^{-1}(F_v) \to X(F_v)$ pour v dans l'ensemble des places M_F de F, avec une "condition de compatibilité". Le choix d'une métrique adelique sur K_X^{-1} produit deux choses. Premièrement, il donne une *hauteur*, c'est-à-dire une fonction $H : X(F) \to \mathbb{R}_{>0}$ qui vérifie la propriété de Northcott : pour chaque $B > 0$, l'ensemble $\{x \in X(F) | H(x) \leq B\}$ est fini. Ceci, en substance, généralise la notion classique de la hauteur sur l'espace projectif $\mathbb{P}^n$ lorsque $F = \mathbb{Q}$, qui est donné par $H(\mathbf{x}) = \max |x_j|$, où $\mathbf{x}$ sont des coordonnées entières qui satisfont $\gcd(\mathbf{x}) = 1$. La hauteur sert de "taille" d'un point rationnel. Deuxièmement, le choix de la métrique adélique produit une mesure ω_H sur l'espace adélique $X(\mathbb{A}_F) := \prod_{v \in M_F} X(\ Fv)$ (voir [47]). Soit τ_H la valeur $\omega_H(\overline{X(F)})$, où la fermeture $\overline{X(F)}$ est prise dans $X(\mathbb{A}_F)$. La question suivante est posée par Peyre dans [47] et raffine la question originale posée par Manin :

Conjecture 1.2.1.1. — *Supposons que les points rationnels $X(F)$ soient Zariski-denses dans X. Alors il existe une sous-variété fermée $Z \subsetneq X$, telle que l'on a*

$$|\{x \in (X - Z)(F) | H(x) \leq B\}| \sim_{B \to \infty} \alpha \tau_H B \log(B)^{\mathrm{rk}(\mathrm{Pic}(X)) - 1},$$

où $\alpha = \alpha(X)$ est une constante positive reliée à l'emplacement de K_X^{-1} dans le cône ample de X et $\mathrm{rk}(\mathrm{Pic}(X))$ *est le rang du Groupe Picard de X.*

On supprime une sous-variété fermée pour éviter les sous-variétés dites "accumulatrices", qui contiennent plus de points que le reste de la variété.

La conjecture a été réglée dans de nombreux cas différents. La preuve pour le cas de $\mathbb{P}^n$ est donnée par Schanuel dans [53], bien avant même que la conjecture ne soit formulée. D'autres cas connus importants de la conjecture sont les variétés toriques ([3]), les compactifications équivariantes de groupes de vecteurs ([18]), certaines familles de surfaces Châtelet

([23], [26]), etc. La version de 2.2.1.1 admet des contre-exemples (eg [4], [35]). Il existe une version pour laquelle il n'existe pas de contre-exemples connus : au lieu de supprimer les sous-variétés fermées, on supprime les ensembles “minces” (un ensemble mince est un sous-ensemble de l'image de l'ensemble des points rationnels $V(F)$ pour un morphisme de variétés $V \to X$, qui, au voisinage du point générique de V, est quasi-fini et n'admet aucune section). Pour une étude sur la conjecture de Manin-Peyre, nous renvoyons le lecteur à [16].

1.2.2. — Différentes méthodes sont disponibles pour aborder la question : torseurs universels, méthode du cercle, analyse harmonique, séries d'Eisenstein, etc. Nous rappelons brièvement la méthode d'analyse harmonique, d'abord utilisé dans [2] par Batyrev et Tschinkel pour prouver la conjecture de Manin-Peyre sur les compactifications de tores anisotropes et développé plus tard dans [3], [17], [18], [20] , etc. pour régler des exemples plus généraux et nouveaux. Soit X une variété torique et soit T son tore. Soit H la hauteur donnée par une métrique adélique sur le fibré du fibré anti-canonique. On compte les points rationnels de T (le diviseur à l'infini $X - T$ peut cependant accumuler des points). On a que $T(F)$ est discret dans le tore adélique $T(\mathbb{A}_F)$. On étend $H|_{T(F)}$ à une “hauteur” sur $T(\mathbb{A}_F)$. Soit $\widehat{H}(s, \chi)$ la transformée de Fourier de H^{-s} (où s est un nombre complexe) au caractère $\chi : T(\mathbb{A}_F) \to S^1$ qui s'annule à $T(F)$. La transformée globale est un produit d'Euler des transformées locales $\widehat{H}_v(s, \chi_v)$ pour $v \in M_F$. Les transformées locales sont des intégrales d'Igusa (voir [19]) et nous pouvons soit donner des formules exactes soit prouver certaines bornes. Ensuite, la transformée de Fourier globale $\widehat{H}(s, \chi)$ devient un produit de L-fonctions et une partie facile à analyser.

La formule de Poisson (7.1.1.4) donne $Z(s) = \int_{(T(\mathbb{A}_F)/T(F))^*} \widehat{H}(s, \chi) d\chi$, où $d\chi$ est une mesure de Haar convenablement normalisée sur le groupe des caractères $(T(\mathbb{A}_F)/T(F))^*$. Il existe des méthodes pour analyser les intégrales du côté droit, par ex. méthode des “fonctions M-contrôlées” de [17].

On obtient le pôle et une extension méromorphe de Z, qui, par des résultats taubériens, donne l'asymptotique recherchée pour le nombre de points rationnels de T de hauteur bornée.

1.3. Conjecture de Manin-Peyre pour les champs

Dans cette thèse, nous avons l'intention d'étendre la conjecture de Manin-Peyre aux champs algébriques.

Nous présentons deux motivations.

1.3.1. — Une *hauteur naïve* H_N d'une courbe elliptique $E/\mathbb{Q}$ est définie comme suit : écrivez l'équation de E sous la forme $Y^2 = X^3 + AX + B$, où $(A, B) \in \mathbb{Z}^2$ a la propriété que pour chaque nombre premier p on a $p^4|A \implies p^6 \nmid B$ et définit $H_N(E) := \max(|A^3|, |B^2|)$. Faltings, dans sa preuve de la conjecture de Mordell [**27**], définit différentes notions d'une hauteur d'une courbe elliptique appelée *une hauteur de Faltings instable* et *une hauteur de Faltings stable*. Pour la hauteur naïve et la hauteur instable de Faltings, il s'avère que si $B > 0$, il n'y a qu'un nombre fini de classes d'isomorphismes de courbes elliptiques de hauteur au plus B. Il n'est pas difficile de compter des courbes elliptiques sur $\mathbb{Q}$ de hauteur naïve bornée (et, comme nous le verrons plus tard, il est possible de le faire sur n'importe quel corps de nombres F). Pour le cas $F = \mathbb{Q}$, Hortsch dans [**31**] trouve le comportement asymptotique pour le nombre de classes d'isomorphismes de courbes elliptiques et de hauteur de Falting instable bornée. Les deux asymptotiques sont similaires aux asymptotiques apparaissant dans la conjecture de Manin-Peyre. Cependant, il y a une distinction : les courbes elliptiques sur un corps de nombres ne sont pas parametrisées par une variété, mais par un *champ algébrique.* Le champ est généralement désignée par $\mathcal{M}_{1,1}$.

1.3.2. — Nous présentons un autre exemple où l'on compte des points rationnels sur des champs algébriques. Malle dans [**37**] conjecture ce qui suit :

***Conjecture 1.3.2.1* (Malle, [37]).** — *Soit G un groupe de permutation transitif fini non trivial et soit F un corps de nombres. On dit que* $\mathrm{Gal}(K/F) = G$ *si K/F est une extension telle que le groupe de Galois de sa clôture de Galois est isomorphe à G en tant que groupe de permutation. Il existe $c(F, G) > 0$, tel que*

$$|\{K/F|\,\mathrm{Gal}(K/F) = G, \Delta(K/F) \le B\}| \sim c(F, G)B^{a(G)} \log(B)^{b(F,G)-1},$$

lorsque $B \to \infty$, où Δ est le discriminant absolu d'une extension, et $a(G)$ et $b(F, G)$ sont des invariants explicites de G et de F et G, respectivement.

La prédiction est prouvée pour certains cas comme le cas des groupes abéliens ([59]), d'autres familles de groupes (eg [57], [22]) et elle admet des contre-exemples ([33]). Un objet que l'on compte dans la question de Malle détermine un point sur le champ BG (c'est le champ algébrique qui parametrise les G-torseurs). Ainsi, la conjecture de Malle aussi, peut être étudiée comme comptant des points rationnels sur un champ algébrique. De plus, les prédictions des conjectures de Manin et Malle semblent similaires. Les similitudes ont déjà été observées par Yasuda dans [60] et par Ellenberg, Satriano et Zureick-Brown dans un ouvrage à paraître. La raison des similitudes des prédictions peut être cachée dans la géométrie du champ BG correspondant.

1.3.3. — Le but de notre travail est de formuler et d'étudier la conjecture de Manin-Peyre dans le contexte des champs algébriques. Plus précisément, nous allons le faire pour les *champs projectifs à poids*. Si $n \geq 1$ est un entier et $\mathbf{a} \in \mathbb{Z}^n_{>0}$, le champ projectif à poids $\mathscr{P}(\mathbf{a})$ est le quotient "champêtre" du schéma $\mathbb{A}^n - \{0\}$ par le schéma en groupes $\mathbb{G}_m$, où l'action est donnée par $t \cdot \mathbf{x} := (t^{a_j} x_j)_j$. Lorsque tous les poids a_j sont égaux à 1, alors $\mathscr{P}(\mathbf{a})$ est l'espace projectif $\mathbb{P}^{n-1}$. On a des coordonnées homogènes sur les champs projectifs à poids : un point rationnel sur $\mathscr{P}(\mathbf{a})$ est donné par n-tuple d'éléments de F et deux n-tuples $\mathbf{x}$ et $\mathbf{x}'$ représentent le même point s'il existe $t \in F^\times$ tel que $t^{a_j} x_j = x'_j$ pour $j = 1, \dots, n$.

Le champ de modules des courbes elliptiques $\mathcal{M}_{1,1}$ est un sous-champ ouvert du champ $\mathscr{P}(4,6)$ (le champ $\mathscr{P}(4,6)$ est le champ qui parametrise des courbes de genre 1 ayant au pire des singularités ordinaires). Un autre exemple est donné par le champ $B\mu_m$ (où $\mu_m = \operatorname{Spec}\big(F[X]/(X^m-1)\big)$ est le schéma de groupe de m-ièmes racines de l'unité), qui est précisément le champ projectif à poids $\mathscr{P}(m)$. Le champ $\mathscr{P}(\mathbf{a})$ est lisse, propre et *torique* : elle contient le tore champêtre $\mathscr{T}(\mathbf{a}) = \mathbb{G}^n_m/\mathbb{G}_m$. Sa similitude avec les variétés toriques en fait un excellent candidat pour étudier la conjecture de Manin-Peyre sur lui.

1.4. Principaux résultats

Nous énonçons les principaux résultats de notre thèse. Notre objectif est de fournir une théorie similaire à celle des points rationnels sur les variétés, plutôt de donner des preuves ad-hoc de certains cas. Le développement de la théorie occupe une partie importante de notre thèse.

Si X est un champ et R un anneau, afin de distinguer la catégorie $X(R)$ et l'ensemble des classes d'isomorphisme des objets de cette catégorie, on écrit $[X(R)]$ pour ce dernier. Soit F un corps de nombres.

1.4.1. — Expliquons d'abord que le comptage de points rationnels sur le champ projectif à poids $\mathscr{P}(\mathbf{a})$ est essentiellement différent du comptage de points rationnels sur l'espace projectif à poids $\mathbb{P}(\mathbf{a})$. Rappelons que l'espace projectif à poids $\mathbb{P}(\mathbf{a})$ est le quotient $(\mathbb{A}^n - \{0\})/\mathbb{G}_m$ dans la catégorie des schémas pour la même action que ci-dessus. Notons j le morphisme canonique $j : \mathscr{P}(\mathbf{a}) \to \mathbb{P}(\mathbf{a})$. Le schéma $\mathbb{P}(\mathbf{a})$ est une variété torique, et notons $T(\mathbf{a}) \cong \mathbb{G}_m^{n-1}$ son tore. Un isomorphisme de groupes $\mathbb{Z}^n/\mathbf{a}\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^{n-1} \times (\mathbb{Z}/\gcd(\mathbf{a})\mathbb{Z})$ induit un isomorphisme de tores champêtres

$$T(\mathbf{a})\times\mathscr{T}(\gcd(\mathbf{a})) \cong \mathbb{G}_m^{n-1}\times\mathscr{T}(\gcd(\mathbf{a})) = \mathscr{T}(\{1\}^{n-1}\times\{\gcd(\mathbf{a})\}) \xrightarrow{\sim} \mathscr{T}(\mathbf{a}) \\ = j^{-1}(T(\mathbf{a})).$$

Ainsi, un point rationnel $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$ est uniquement déterminé par la paire

$$(j(\mathbf{x}), \mathbf{x}') \in T(\mathbf{a})(F) \times [j^{-1}(\mathbf{x})(F)] \cong T(\mathbf{a})(F) \times [\mathscr{T}(\gcd(\mathbf{a}))(F)] \\ = T(\mathbf{a})(F) \times [\mathscr{P}(\gcd(\mathbf{a}))(F)].$$

Si $\gcd(\mathbf{a}) = 1$, alors $\mathscr{P}(\gcd(\mathbf{a}))$ est le schéma à un point. Il résulte de ce qui précède que le morphisme j induit une bijection $[\mathscr{T}(\mathbf{a})(F)] \xrightarrow{\sim} T(F)$. D'après [44, Proposition 6.1], l'homomorphisme de tiré en arrière $j^*_{\mathbb{Q}} : \operatorname{Pic}(\mathbb{P}(\mathbf{a}))_{\mathbb{Q}} \to \operatorname{Pic}(\mathscr{P}(\mathbf{a}))_{\mathbb{Q}}$ des groupes de Picard rationnels est un isomorphisme. Il s'ensuit que compter les points rationnels de $\mathscr{T}(\mathbf{a})$ correspond à compter les points rationnels de $\mathbb{P}(\mathbf{a})$ par rapport à une hauteur provenant d'un certain fibré un droites rationnel. Lorsque $\gcd(\mathbf{a}) > 1$, l'ensemble $[\mathscr{P}(\gcd(\mathbf{a}))(F)]$ est infini (Corollaire 4.6.2.2), et on voit que compter les points rationnels de (du tore champêtre de) $\mathscr{P}(\mathbf{a})$ n'est pas la même chose que de compter les points rationnels de (du tore de) $\mathbb{P}(\mathbf{a})$.

1.4.2. — Au chapitre 4, nous définissons une notion de *hauteur quasi-torique* sur l'ensemble des points rationnels $\mathscr{P}(\mathbf{a})$. C'est une fonction $H : [\mathscr{P}(\mathbf{a})(F)] \to \mathbb{R}_{\geq 0}$ et on établit un résultat de finitude sur le nombre de points rationnels de hauteur bornée ("propriété de Northcott faible"). Une hauteur dépend de la choix d'un fibré en droites sur le champ $\overline{\mathscr{P}(\mathbf{a})} = \mathbb{A}^n/\mathbb{G}_m$ (où l'action est canoniquement étendue) et d'une "métrique adélique". Pour $v \in M_F$, on définit des espaces topologiques $[\mathscr{P}(\mathbf{a})(F_v)] := (F_v^n - \{0\})/F_v^\times$, où l'action est induite à partir de l'action de $\mathbb{G}_m$ sur $\mathbb{A}^n - \{0\}$. L'espace produit $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ est un bon analogue de "l'espace adélique" d'une variété. Dans le chapitre 5, on définit une mesure ω_H sur l'espace produit $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ et on pose $\tau_H = \omega_H(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])$. Nous montrons que :

Thèorème **(Théorème 8.2.2.12, Proposition 8.3.2.3)**

Soit H une hauteur quasi-torique. On a ce

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| \sim_{B\to\infty} \frac{\tau_H}{|\mathbf{a}|} B.$$

Pour un type particulier de hauteur quasi-torique (que dans notre travail on appelle *hauteur torique*), le résultat a été établi dans [**12**]. Pour les autres hauteurs, le résultat est nouveau.

On établit en outre que les points rationnels de $\mathscr{P}(\mathbf{a})$ sont équidistribués dans $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ dans le sens suivant. Soit $i : [\mathscr{P}(\mathbf{a})(F)] \to \prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ l'application diagonale. Si $W \subset \prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ est un sous-ensemble ouvert tel que $\omega(\partial W) = 0$, dans le théorème 8.3.2.2, on prouver que

$$\lim_{B\to\infty} \frac{|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | i(\mathbf{x}) \in W \text{ et } H(\mathbf{x}) \leq B\}|}{|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}|} = \frac{\omega_H(W)}{\tau_H}.$$

1.4.3. — Énonçons le deuxième résultat principal de notre travail. On suppose que $n = 1$ et que $m \in \mathbb{Z}_{>1}$. On compte les μ_m-torseurs sur F (c'est-à-dire, les points rationnels de $\mathscr{P}(m)$) du discriminant borné. Au chapitre 9, nous définissons une notion de *hauteur quasi-discriminante*. C'est une fonction $H : [\mathscr{P}(m)(F)] \to \mathbb{R}_{\geq 0}$ qui satisfait la propriété faible de Northcott, et est essentiellement différente d'une hauteur quasi-torique (on ne peut pas la normaliser de telle façon que les quotients des deux hauteurs sont des fonctions bornées sur $[\mathscr{P}(m)(F)]$). Les normalisations sont prises de telle sorte que $H^{m(1-1/r)}$, où r est le plus petit nombre premier de m, est essentiellement le discriminant absolu Δ d'un μ_m- torseur (c'est-à-dire que les composantes locales de $H^{m(1-1/r)}$ ne sont différentes des composantes locales du discriminant absolu Δ qu'en un nombre fini de places, par conséquent $H^{m(1-1/r)}/\Delta$ est borné sur $[\mathscr{P}(m)(F)]$). Comme ci-dessus, un choix de la hauteur quasi-discriminante H définit une mesure ω_H sur $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ et on pose $\tau_H = \omega_H(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])$. Nous montrons que

***Thèorème* (Corollaire 9.2.5.9). —** *Soit H une hauteur quasi-discriminante. On a que*

$$|\{x \in [\mathscr{P}(m)(F)] | H(x) \leq B\}| \sim_{B\to\infty} \frac{1}{(r-2)!} \cdot \frac{\tau_H}{m} \cdot B \log(B)^{r-2}.$$

On montre à nouveau une propriété d'équidistribution des points rationnels dans l'espace $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ (Théorème 9.2.6.1). La propriété d'équidistribution est utilisée pour prouver qu'une proportion positive de μ_m-torseurs de hauteur quasi-discriminante bornée sont des corps et que, si m n'est pas un nombre premier, une proportion positive ne sont pas des corps. De plus, lorsque $4 \nmid m$ ou lorsque $i = \sqrt{-1} \in F$, on est en mesure de donner une formule pour la proportion de champs (Théorème 9.2.7.3).

Supposons que F contienne toutes les m-ième racines de l'unité. En particulier, on a que $4 \nmid m$ ou que $i \in F$. Nous avons que $\mu_m = \mathbb{Z}/m\mathbb{Z}$. Dans ce cas, l'asymptotique pour le nombre de $\mu_m = \mathbb{Z}/m\mathbb{Z}$-torseurs de discriminant borné qui sont des corps, a déjà été donnée par Wright dans [**59**]. L'avantage de notre méthode est que nous sommes en mesure de modifier le discriminant à un nombre fini de places.

1.5. Aperçu de la thèse

Donnons un aperçu de notre travail.

1.5.1. — Nous discutons d'une différence entre les hauteurs sur les variétés et sur les champs. Soit X une variété propre sur F et soit L un fibrés en droites sur X. Il est bien connu qu'un choix d'un $\mathcal{O}_{F,S}$-modèle de (X, L) (où S est un ensemble fini de places finies de F) munit L avec une métrique pour chaque place finie qui n'est pas dans S. Nous munissons le fibré en droites L aux places restants avec une métrique, et on obtient une métrique adélique sur L, et donc une hauteur sur $X(F)$.

Dans la construction de la métrique pour les places finies qui ne sont pas dans S, on utilise le *critère valuatif de proprété* qui donne que tout F_v-point de X s'étend à un $\mathcal{O}_v$-point du modèle. Cependant, ce n'est pas vrai pour les champs (par exemple, seuls les points de $\mathscr{P}(4, 6)$ correspondant à des courbes ayant de bonnes réductions à v s'étendent aux $\mathcal{O}_v$-points).

Le critère valuatif de proprété pour des champs donne seulement qu'un point F_v s'étend jusqu'à un A-point du modèle, où A est la normalisation de $\mathcal{O}_v$ dans une extension finie de F_v. Telles extensions intégrales donnent lieu à des hauteurs stables (c'est-à-dire, la hauteur d'un F-point reste la même lorsque le point est considéré comme un K-point, où K/F est une extension finie). Un desavantage des hauteurs stables est qu'elles ne satisfont pas à la « propriété faible de Northcott » (Définition 4.6.1.1), comme on le voit dans le cas de $\mathscr{P}(4,6)$. À savoir, la hauteur stable de deux F-courbes elliptiques qui ne sont pas isomorphes sur F est la même si elles ont le même j-invariant. Pour une courbe elliptique fixe E, il existe une infinité de telles courbes elliptiques (elles sont construites en effectuant des torsions quadratiques à E).

1.5.2. — Au chapitre 3, nous rappelons plusieurs résultats sur les champs, en mettant l'accent sur les stacks projectifs à poids. On introduit aussi le champ $\overline{\mathscr{P}(\mathbf{a})} = \mathbb{A}^n/\mathbb{G}_m$ (pour l'extension canonique de l'action de $\mathbb{G}_m$ sur $\mathbb{A}^n - \{0\}$). Ainsi, on a des immersions ouvertes $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a}) \subset \overline{\mathscr{P}(\mathbf{a})}$. Le champ $\overline{\mathscr{P}(\mathbf{a})}$ n'est pas séparé (Lemme 3.2.2.6), donc pas propre, mais elle satisfait la propriété que tous ses points rationnels s'étendent aux points entiers, et résout donc le problème du manque de points entiers de ci-dessus. Cette propriété sera utilisée au chapitre 4 pour produire des hauteurs *instables* sur les champs projectifs à poids.

Les travaux de Moret-Bailly de [**40**] fournissent une notion d'espace topologique associé à l'ensemble des (classes d'isomorphismes de) R-points d'un champs, lorsque R est un certain type d'anneau local. L'association est fonctoriale, c'est-à-dire que pour un morphisme $X \to Y$ de champs, l'application induite $[X(R)] \to [Y(R)]$ est continue. Une liste d'autres propriétés que la construction satisfait est donnée dans [**15**]. Nous prouvons la proposition suivante qui nous permet de comprendre cette topologie pour certains champs quotients :

Proposition 1.5.2.1. — *Supposons que X soit un champ quotient Y/G, avec G special (ses torseurs sont localement triviaux, par Hilbert 90, un exemple est fourni par $G = \mathbb{G}_m$). On a que $[X(R)]$ est le quotient topologique $Y(R)/G(R)$.*

Ainsi, on a par exemple que $[\mathscr{P}(\mathbf{a})(F_v)] = (F_v^n - \{0\})/F_v^\times$ et $[\mathscr{T}(\mathbf{a})(F_v)] = (F_v^\times)^n/F_v^\times$, où l'action de $F_v^\times = \mathbb{G}_m(F_v)$ est l'induite de l'action de $\mathbb{G}_m$ sur $\mathbb{A}^n - \{0\}$ et $\mathbb{G}_m^n$, respectivement. Dans la dernière partie du chapitre, nous parlons de l'espace adélique du tore $\mathscr{T}(\mathbf{a})$. Nous le définissons comme le produit restreint

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] := \prod_{v\in M_F}{}' [\mathscr{T}(\mathbf{a})(F_v)]$$

par rapport au sous-groupes compacts et ouverts $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \subset [\mathscr{T}(\mathbf{a})(F_v)]$. En utilisant les résultats de Česnavičius de [14] sur la cohomologie des adèles, nous montrons que $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ a des propriétés similaires au tore adélique $\mathbb{G}_m^n(\mathbb{A}_F)$ (par exemple l'image des points rationnels $[\mathscr{T}(\mathbf{a})(F)]$ pour l'application diagonale est discrète).

1.5.3. — Nous commençons le chapitre 4 en rappelant des faits sur les fibres en droites sur les champs, en particulier que les fibrés en droites sur le champ quotient Y/G correspondent à des fibrés en droites G-linearisés sur le schéma Y. Les groupes de Picard $\mathrm{Pic}(\mathscr{P}(\mathbf{a}))$ et $\mathrm{Pic}(\overline{\mathscr{P}(\mathbf{a})})$ sont calculés. Ensuite, nous définissons les métriques sur les fibrés en droites comme suit. Soit v une place de F, soit X un champ F_v-algébrique et soit L un fibré en droites sur X. Nous définissons une F_v-metric on L comme étant la donné fourniée par F_v-métriques "compatibles" sur y^*L pour chaque morphisme $y : Y \to X$ avec Y un F_v-schema (par une F_v-métrique sur un fibré en droites sur un schéma F_v, nous entendons un choix "continu" de normes sur toutes les F_v-fibrés). Notre métrique n'est pas forcement stable. Pour les champ quotients $X = Y/G$, lorsque G est supposé être un groupe algébrique spécial, nous relions le groupe des fibrés en droites F_v-métrisés $\widehat{\mathrm{Pic}}_v(Y/G)$ avec le groupe $\widehat{\mathrm{Pic}^G_v}(Y)$ des fibrés en droites F_v-métrisés sur X qui sont munis par une G-linéarisation et tels que la métrique est G-invariant :

***Proposition 1.5.3.1* (Proposition 4.3.4.5).** — *Soit G un schéma en groupes spécial de type fini sur F_v agissant sur localement de type fini F_v-schéma Y. L'homomorphisme canonique $\widehat{\mathrm{Pic}}_v(Y/G) \to \widehat{\mathrm{Pic}^G_v}(Y)$ est injectif, et est un isomorphisme si $\widehat{\mathrm{Pic}}_v(Y/G) \to \mathrm{Pic}(Y/G)$ est surjectif.*

Le champ $\mathscr{P}(\mathbf{a}) = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$ satisfait cette condition sur l'existence de F_v-métriques sur chacun de ses fibrés en droites (Lemme 4.3.6.3). Par conséquent, comme $\mathrm{Pic}(\mathbb{A}^n - \{0\})$ est trivial, on en déduit que pour définir une F_v-métrique sur un fibré en droites sur $\mathscr{P}(\mathbf{a})$, il suffit à définir une métrique $\mathbb{G}_m$-invariante sur la $\mathbb{G}_m$-linéarisation correspondante du fibré trivial sur $\mathbb{A}^n - \{0\}$. Une telle métrique est définie par la norme de la section 1 et la condition sur les linéarisations donne une condition d'"homogénéité" à la fonction $F_v^n - \{0\} \to \mathbb{R}_{>0}, \mathbf{x} \mapsto ||1||_{\mathbf{x}}$.

Supposons qu'un fibré en droites L sur $\mathscr{P}(\mathbf{a})$ soit muni d'une F_v-métrique pour chaque $v \in M_F$, sous réserve d'une condition de compatibilité qui permet aux normes d'une section être multiplié à n'importe quel $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$ (voir la condition dans la définition 4.4.1.1). Nous pouvons définir des hauteurs en multipliant les inverses de ces normes pour chaque v. La généralité, laisse la possibilité d'existence de hauteurs "essentiellement différentes" sur les mêmes fibres en droites, c'est-à-dire des hauteurs telles que leurs quotients ne sont pas des fonctions bornées sur l'ensemble $[\mathscr{P}(\mathbf{a})(F)]$. Des exemples sont : les *hauteurs stables* mentionnées, les *hauteurs quasi-toriques* (nous allons les expliquer maintenant) et dans le cas $n = 1$ les *hauteurs quasi-discriminantes* (elles seront expliquées dans le dernier chapitre). Les hauteurs quasi-toriques sont les hauteurs qui proviennent des familles de métriques qui se présentent de la manière suivante pour presque chaque place : étendre un F_v-point du champ $\overline{\mathscr{P}(\mathbf{a})}$ à un $\mathcal{O}_v$-point et utilisez la méthode classique (déjà discutée dans 2.5.1) pour obtenir une métrique (une petite modification, cependant, doit être faite, car $\overline{\mathscr{P}(\mathbf{a})}$ n'est pas séparé et donc le $\mathcal{O}_v$-extension d'un F_v-point n'est pas unique. Contrairement aux hauteurs stables, les hauteurs quasi-toriques satisfont à la propriété faible de Northcott :

***Theorem 1.5.3.2* (Théorème 4.6.8.2).** — *Soit H une hauteur quasi-torique sur $\mathscr{P}(\mathbf{a})$. Soit $\epsilon > 0$. On a qu'il existe $C > 0$ tel que*

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| \leq CB^{1+\epsilon}.$$

L'idée de la preuve est d'estimer séparément la hauteur finie et la hauteur infinie. La limite supérieure de la cardinalité dans le théorème est nécessaire pour assurer la convergence de la série zêta de hauteur correspondante. L'affirmation de 1.5.3.2 reste valide même lorsque les métriques à un nombre fini des places sont autorisées à avoir des singularités "logarithmiques" le long de diviseurs rationnels (voir le corollaire 4.7.1.3). La preuve de cette version découle immédiatement de 1.5.3.2, après avoir établi une estimation de la hauteur singulière de la forme : $H_{\mathrm{Sing}} \geq CH \log^{-\eta}(H)$, où $C, \eta > 0$, ce que nous faisons dans la proposition 4.7.1.2.

1.5.4. — Dans le chapitre 5, nous munissons les espaces topologiques associés aux R-points par des mesures. En particulier, nous définissons des mesures sur $[\mathscr{P}(\mathbf{a})(F_v)]$ (qui dépendent du choix des métriques) et sur $[\mathscr{T}(\mathbf{a})(F_v)]$ (qui ne dépendent pas du choix de métrique). Les mesures sont utilisées pour définir la constante de Peyre τ_H.

La dernière partie du chapitre est consacrée à la définition des mesures sur le “tore adélique” $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ et le nombre de Tamagawa du tore champêtre $\mathscr{T}(\mathbf{a})$. On établit que

***Proposition 1.5.4.1* (Proposition 5.4.4.4). —** *On a l'égalité* $\mathrm{Tam}(\mathscr{T}(\mathbf{a})) = 1$.

Lorsque $\mathbf{a} = \mathbf{1}$, c'est le résultat classique que le nombre de Tamagawa d'un tore scindé est 1. La preuve de la proposition 1.5.4.1 utilise les caractéristiques d'Euler-Poincaré d'Oesterlé de complexes de groupes abéliens localement compacts qui sont munis de mesures de Haar.

1.5.5. — Le chapitre 6 étudie characteurs du “tore adélique” $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. On introduit des normes “discrètes” et des normes “infinies” de ces caractères. On établit un résultat de finitude sur le nombre de caractères $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ qui s'annulent sur $[\mathscr{T}(\mathbf{a})(F)]$ et sur certains sous-groupes de bornée l'une ou l'autre de ces normes. Dans la dernière partie de ce chapitre nous rappelons les estimations de Rademacher sur L fonctions de caractères. Les résultats de ce chapitre seront utilisés au chapitre 7 pour prouver que la transformée de Fourier d'une fonction hauteur est intégrable.

1.5.6. — Au chapitre 7 nous adaptons la méthode d'analyse harmonique de Batyrev et Tschinkel de [**3**] à notre situation. Nous supposons que les métriques sont *lisses*. La première partie du chapitre est consacrée au calcul de la transformée de Fourier de la hauteur locale en une place finie v. Pour presque tous les v, nous pouvons donner la formule exacte qui s'avère être le produit de fonctions locales L de caractères et d'autres facteurs. Ensuite, pour une place infinie v, en utilisant les hypothèses de régularité, nous prouvons des estimés appropriés de la transformée de Fourier dans les deux normes de caractères. La preuve de cette affirmation est une adaptation de l'idée de Chambert-Loir et Tchinkel de [**18**] et [**20**], où les auteurs appliquent l'intégration par parties par rapport aux champs de vecteurs invariants. La transformée de Fourier globale s'écrit donc comme un produit de L fonctions et d'une partie sur laquelle nous avons un contrôle.

1.5.7. — Au chapitre 8 nous utilisons la théorie de [**17**] pour analyser la fonction zêta de hauteur.

L'accent est mis sur le tore champêtre $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a})$. De l'estimé du théorème 1.5.3.2, on déduit que la fonction hauteur zêta $Z(s) := \sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]} H(\mathbf{x})^{-s}$ converge et définit une fonction holomorphe de s dans le domaine $\Re(s) > 1$. La formule de Poisson donne que

$$Z(s) = \int_{([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(F)])^*} \widehat{H}(s,\chi)d\chi$$

chaque fois que les expressions des deux côtés convergent. Les estimés du chapitre 6 et du chapitre 7 et la “similarité” de la transformée de Fourier globale de la hauteur avec L fonctions, donnent la convergence de l'intégrale à droite pour $\Re(s) > 1$. De plus, la preuve impliquera que $s \mapsto Z(s)$ a une extension méromorphe dans un domaine $\Re(s) > 1-\delta$, pour un certain $\delta > 0$. Les estimés de Rademacher impliquent que Z satisfait les conditions de croissance nécessaires pour les théorèmes taubériens. Le résidu de Z à 1 est également calculé. En conséquence, les théorèmes taubériens donnent le comportement asymptotique du nombre de points rationnels de la hauteur au plus B :

***Theorem 1.5.7.1* (Théorème 8.2.2.12, Proposition 8.3.2.3)**
Soit H une hauteur quasi-torique. On a ce

$$\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\} \sim \frac{\tau_H}{|\mathbf{a}|}B,$$

quand B tend vers $+\infty$.

La constante $\frac{1}{|\mathbf{a}|}$ a la même interprétation que dans le cas des variétés (voir Remarque 8.3.2.7) et l'asymptotique reste la même quand on compte les points rationnels de $\overline{\mathscr{P}(\mathbf{a})}$ (car $[\overline{\mathscr{P}(\mathbf{a})}(F)] - [\mathscr{P}(\mathbf{a})(F)]$ est un ensemble à un point). Ainsi le théorème 1.5.7.1 peut être compris comme la conjecture de Manin-Peyre est vraie pour les champs projectifs à poids $\overline{\mathscr{P}(\mathbf{a})}$. La dernière partie du chapitre est consacrée à la compréhension *l'équidistribution* des points rationnels du champ $\mathscr{P}(\mathbf{a})$. L'idée est de trouver le comportement asymptotique du nombre de points rationnels de hauteur bornée qui sont demandés pour un nombre fini de places v à appartenir à certains sous-ensembles ouverts de l'espace v-adique du champ (par exemple, disons que la valuation 2-adic est paire). Une manière élégante de formuler cette question a été donnée par Peyre dans [47], en utilisant la mesure ω_H ci-dessus : si $W \subset \prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ est un sous-ensemble ouvert de frontière négligeable, on s'attend à ce que :

$$\lim_{B\to\infty}\frac{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|i(\ xxx)\in W \text{ et } H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|}=\frac{\omega_H(W)}{\tau_H},$$

où $i:[\mathscr{P}(\mathbf{a})(F)]\to\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ est l'application diagonale. Si cela est vrai pour chaque W, alors nous disons que les points rationnels sont équidistribués. Nous prouvons que :

***Théorème 1.5.7.2* (Théorème 8.3.2.2).** — *Les points rationnels de $\mathscr{P}(\mathbf{a})$ sont équidistribués dans l'espace $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$.*

1.5.8. — Au chapitre 9, nous utilisons nos méthodes pour étudier une question similaire à la conjecture de Malle. On trouve l'asymptotique pour le nombre de μ_m-torseurs sur F du discriminant absolu borné. Lorsque F contient toutes les m-ième racines de 1, cette question est la conjecture de Malle pour le groupe cyclique $\mathbb{Z}/m\mathbb{Z}$, mais il faut enlever les torseurs qui ne sont pas des corps et il peut y en avoir une proportion positive (le comptage des extensions cycliques a été couvert par le cas de l'abélienne groupes de [**59**]). Les μ_m-torseurs sur F sont parametrisés par le champ algébrique $B\mu_m=\mathscr{T}(m)=\mathscr{P}(m)$.

Nous utilisons le langage des hauteurs développé précédemment. On parle de hauteurs *quasi-discriminantes*, qui sont similaires aux discriminants, à la différence que les composantes locales de ces hauteurs aux nombres finis de places peuvent être différentes des composantes locales du discriminant. Notons qu'une hauteur quasi-discriminante n'est pas une hauteur quasi-torique, car les composantes locales des deux hauteurs sont différentes à presque tous les places. On va définir une mesure ω_H sur $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ et on va poser $\tau_H=\omega_H(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])$.

La méthode de la preuve est une adaptation de la méthode d'analyse harmonique ci-dessus. Il y a quelques simplifications, car les espaces locaux $[\mathscr{T}(m)(F_v)]$ sont finis, et des modifications dues à la différence avec les hauteurs quasi-toriques. Finalement, nous prouvons la convergence de la série zêta des hauteurs et utilisons la formule de Poisson comme précédemment. Dans le but d'avoir une formule plus élégante, nous indiquons ici l'asymptotique finale pour les hauteurs qui sont "essentiellement" $\Delta^{\frac{1}{m(1-1/r)}}$, où r est le plus petit nombre premier de m (c'est-à-dire que pour presque tous les places, la composante locale de H coïncide avec la composante locale de $\Delta^{\frac{1}{m(1-1/r)}}$).

***Théorème 1.5.8.1* (Corollaire 9.2.5.9). —** *Soit H une hauteur quasi-discriminante. On a que*

$$|\{x \in [\mathscr{P}(m)(F)] | H(x) \leq B\}| = \frac{\tau_H}{(r-2)!m} B \log(B)^{r-2}.$$

L'asymptotique rappelle beaucoup celle de la conjecture de Manin-Peyre. Comme dans le cas des hauteurs quasi-toriques, nous sommes capables de prouver une propriété d'équidistribution correspondante dans $\prod_{v \in M_F} [\mathscr{P}(m)(F_v)]$. Nous terminons le chapitre par la preuve qu'il existe une proportion positive de μ_m-torseurs qui sont des corps. Ceci est prouvé en trouvant un sous-ensemble ouvert $W \subset \prod_{v \in M_F} [\mathscr{P}(m)(F_v)]$ de volume positif tel que tous les μ_m-torseurs qu'il contient sont des corps. De plus, lorsque $4 \nmid m$ ou lorsque $i = \sqrt{-1} \in F$, nous donnons une formule exacte pour cette proportion.

1.6. Questions et remarques

Discutons de quelques questions qui découlent naturellement de notre travail.

1.6.1. — La conjecture de Manin-Peyre a été prouvée pour toutes les variétés toriques lisses. On aimerait savoir à quelle généralité la preuve s'applique à d'autres champs toriques (c.f. [**28**]). Nous ne savons pas ce qui se passe lorsque le "tore champêtre" n'est pas scindé, c'est-à-dire pas un quotient de deux tores scindés. La conjecture de Manin-Peyre pour les variétés toriques a également été prouvée pour le cas des corps de fonctions ([**11**]). Nous aimerions savoir quelle est la situation pour des champs toriques.

1.6.2. — Il serait intéressant de comprendre à quelles autres champs on peut développer une théorie des hauteurs et l'utiliser pour compter des points rationnels. Des exemples de tels champs pourraient être : le champ $\mathcal{M}_g$ qui parametrise les courbes du genre g, le champ de variétés abéliennes principalement polarisées $\mathcal{A}_g$, etc.

1.6.3. — On pourrait se demander s'il existe un champ X avec suffisamment de points entiers, tel que $B\mu_m \subset X$ et tel que le discriminant se présente comme une hauteur induite par un $\mathcal{O}_{F,S}$-modèle de X et un fibré en droites dessus (ici S est un ensemble fini de places). Nous voudrions savoir alors si le résultat du théorème 1.5.8.1 peut être réinterprété comme cette conjecture de Manin-Peyre est vraie pour X. On peut alors se demander comment la prédiction de la conjecture de Malle se compare à la prédiction de la conjecture de Manin-Peyre. La même question peut être posée pour n'importe quel (schéma en) groupe(s) fini G.

1.6.4. — Une notion différente de hauteur sur un champ, définie par des fibrés vectoriels, a été proposée par Ellenberg, Satriano et Zureick-Brown dans un travail à venir. Leur hauteur n'est pas additive dans les fibrés vectoriels. Nous aimerions savoir comment cette notion se compare à notre notion de la hauteur.

1.6.5. — Un contre-exemple à la conjecture de Malle a été construit par Klüners dans [**33**]. Les contre-exemples connus dans la conjecture de Manin-Peyre sont évités si l'on permet de supprimer les "ensembles minces". Nous aimerions savoir si la suppression des "ensembles minces" corrige la prédiction de Malle.

CHAPTER 2

INTRODUCTION

2.1. Notation

The following notation will be used throughout the thesis. By F we will denote a number field (which one may fix for the whole article). Let M_F, M_F^0, M_F^∞, $M_F^{\mathbb{R}}$ and $M_F^{\mathbb{C}}$ be the set of places, finite places, infinite places, real places and complex places of F, respectively. For $v \in M_F$ we let F_v be the v-adic completion of F. For $v \in M_F^0$, let $\mathcal{O}_v$ be the ring of integers of F_v, let us fix an uniformizer $\pi_v \in F_v$ and let $|\cdot|_v$ be the absolute value on F_v normalized by $|\pi_v|_v = [\mathcal{O}_v : \pi_v\mathcal{O}_v]^{-1}$. For $v \in M_F^{\mathbb{R}}$, we let $|\cdot|_v$ be the usual absolute value and for $v \in M_F^{\mathbb{C}}$ we let $|\cdot|_v$ be the square of the usual absolute value. The normalizations are chosen so that the product formula is valid i.e. for every $x \in F$, one has

$$\prod_{v \in M_F} |x|_v = 1.$$

By $\mathcal{O}_F$ we denote the ring of the integers of F and for a finite subset $S \subset M_F^0$, we denote by $\mathcal{O}_{F,S}$ the ring of S-integers. When $v \in M_F^\infty$, we will denote by n_v the degree $[F_v : \mathbb{R}]$. We denote by $\mathbb{A}_F$ the ring of the adeles of F and by $\mathbb{A}_F^\times$ the group of ideles.

For a vector $\mathbf{x} \in \mathbb{R}^n$, we will denote by $|\mathbf{x}|$ the sum $x_1 + \cdots + x_n$.

2.2. Manin-Peyre conjecture

Let us recall a conjecture due to Manin and Peyre on the asymptotic behaviour of the number of rational points of bounded “size”.

2.2.1. — One of the fundamental questions in Diophantine geometry is the study of the number of solutions to algebraic equations. The conjecture of Manin-Peyre deals with a such a question. It predicts the number of the rational points on algebraic varieties of bounded height, when there are “a lot” of them. Let us briefly recall it.

Let X be a Fano variety over a number field F and let K_X^{-1} be its anticanonical bundle. The Fano condition, i.e. that K_X^{-1} is positive, is believed to make, after possibly passing to an extension of F, rational points Zariski dense in X. An adelic metric on K_X^{-1} is a choice of metrics for every topological line bundle $K_X^{-1}(F_v) \to X(F_v)$ for v in the set of the places M_F of F, subject to certain compatibility conditions. A choice of an adelic metric on K_X^{-1} produces two things. Firstly, it gives a *height*, i.e. a function $H : X(F) \to \mathbb{R}_{>0}$ which satisfies the Northcott property: for every $B > 0$, the set $\{x \in X(F)|H(x) \leq B\}$ is finite. This in essence, generalizes the classical notion of the height on the projective space $\mathbb{P}^n$ when $F = \mathbb{Q}$, which is given by $H(\mathbf{x}) = \max |x_j|$, where $\mathbf{x}$ are integer coordinates which satisfy $\gcd(\mathbf{x}) = 1$. The height serves as a “size” of a rational point. Secondly, the choice of the adelic metric produces a measure ω_H on the adelic space $X(\mathbb{A}_F) := \prod_{v \in M_F} X(F_v)$ (see [**47**]). Let τ_H be the value $\omega_H(\overline{X(F)})$, where the closure $\overline{X(F)}$ is taken in $X(\mathbb{A}_F)$. The following question is asked by Peyre in [**47**] and refines the original question posed by Manin:

Conjecture 2.2.1.1. — *Suppose that the rational points $X(F)$ are Zariski-dense in X. Then there exist a closed subvariety $Z \subsetneq X$, such that one has*

$$|\{x \in (X - Z)(F)|H(x) \leq B\}| \sim_{B \to \infty} \alpha \tau_H B \log(B)^{\mathrm{rk}(\mathrm{Pic}(X))-1},$$

where $\alpha = \alpha(X)$ is a positive constant connected to the location of K_X^{-1} in the ample cone of X and $\mathrm{rk}(\mathrm{Pic}(X))$ *is the rank of the Picard group of X.*

One removes a closed subvariety to avoid so-called “accumulating” subvarieties, which contain more points than the rest of the variety.

The conjecture has been settled in many different cases. The proof for the case of $\mathbb{P}^n$ is given by Schanuel in [**53**], long before the conjecture was even formulated. Other important known cases of the conjecture are toric varieties ([**3**]), equivariant compactifications of vector groups ([**18**]), certain families of Châtelet surfaces ([**23**], [**26**]), etc. The version from 2.2.1.1 does admit counterexamples (e.g. [**4**], [**35**]). There exists a version for which no known counterexamples exist: instead of removing closed subvarieties, one removes "thin" sets (a thin set is a subset of the image of the set of rational points $V(F)$ for a morphism of varieties $V \to X$, which, in a neighbourhood of the generic point of V, is quasi-finite and admits no section). For a survey on Manin-Peyre conjecture, we refer the reader to [**16**].

2.2.2. — Different methods are available to tackle the question: universal torsors, circle method, harmonic analysis, Eisenstein series, etc. We briefly recall the harmonic analysis method, firstly used in [**2**] by Batyrev and Tschinkel to prove Manin-Peyre conjecture on compactifications of anisotropic tori and later developed in [**3**], [**17**], [**18**], [**20**], etc. to settle more general and new examples. Let X be a toric variety and let T be its torus. Let H be the height given by an adelic metric on the anti-canonical line bundle. We count the rational points of T (the divisor at the infinity $X - T$ may, however, accumulate points). We have that $T(F)$ is discrete in the adelic torus $T(\mathbb{A}_F)$. We extend $H|_{T(F)}$ to a "height" on $T(\mathbb{A}_F)$. We let $\widehat{H}(s, \chi)$ be the Fourier transform of H^{-s} (where s is a complex number) at the character $\chi : T(\mathbb{A}_F) \to S^1$ which vanishes at $T(F)$. The global transform is an Euler product of local transforms $\widehat{H}_v(s, \chi_v)$ for $v \in M_F$. The local transforms are Igusa integrals (see [**19**]) and we can either give exact formulas for them or prove certain bounds. Then global Fourier transform $\widehat{H}(s, \chi)$ turns out to be a product of L-functions and a part that is easy to analyse.

The Poisson formula (7.1.1.4) gives $Z(s) = \int_{(T(\mathbb{A}_F)/T(F))^*} \widehat{H}(s, \chi) d\chi$, where $d\chi$ is suitably normalized Haar measure on the group of the characters $(T(\mathbb{A}_F)/T(F))^*$. There are methods to analyse the integrals on the right hand side, e.g. method of "controlled M-functions" from [**17**].

One obtains the pole and a meromorphic extension of Z, which, by Tauberian results, gives the wanted asymptotic for the number of rational points of T of bounded height.

2.3. Manin-Peyre conjecture for stacks

In this thesis, we are intending to extend the conjecture of Manin-Peyre to the algebraic stacks.

We present two motivations.

2.3.1. — A *naive height* H_N of an elliptic curve $E/\mathbb{Q}$ is defined as follows: write the equation of E as $Y^2 = X^3 + AX + B$, where $(A, B) \in \mathbb{Z}^2$

has the property that for every prime p one has that $p^4|A \implies p^6 \nmid B$ and set $H_N(E) := \max(|A^3|, |B^2|)$. Faltings, in his proof of Mordell conjecture [**27**], defines different notions of a height of an elliptic curve called *unstable Faltings height* and *stable Faltings height*. For the naive height and the unstable Faltings height, it turns out that if $B > 0$, there are only finitely many isomorphism classes of elliptic curves of height at most B. It is not hard to count elliptic curves over $\mathbb{Q}$ of bounded naive height (and, as we will see later, it is possible to do so over any number field F). For the case $F = \mathbb{Q}$, Hortsch in [**31**] finds the asymptotic behaviour for the number of the isomorphism classes of elliptic curves and bounded unstable Faltings height. Both asymptotics are similar to the asymptotics appearing in Manin-Peyre conjecture. However, there is a distinction: the elliptic curves over a number field are not classified by a variety, but by an *algebraic stack*. The stack is usually denoted by $\mathcal{M}_{1,1}$.

2.3.2. — We present another example where one counts rational points on algebraic stacks. Malle in [**37**] conjectures the following:
Conjecture 2.3.2.1 **(Malle, [37]).** — *Let G be a non-trivial finite transitive permutation group and let F be a number field. We say that $\mathrm{Gal}(K/F) = G$ if K/F is an extension such that the Galois group of its Galois closure is isomorphic to G as a permutation group. There exists $c(F,G) > 0$, such that*

$$|\{K/F|\,\mathrm{Gal}(K/F) = G, \Delta(K/F) \leq B\}| \sim c(F,G)B^{a(G)}\log(B)^{b(F,G)-1},$$

when $B \to \infty$, where Δ is the absolute discriminant of an extension, and $a(G)$ and $b(F,G)$ are explicit invariants of G and of F and G, respectively.

The prediction is proved for some cases like the case of abelian groups ([**59**]), some other families of groups (e.g. [**57**], [**22**]) and it admits counter-examples ([**33**]). An object that one counts in Malle's question determines a point on the stack BG (this is the algebraic stack which classifies G-torsors). Thus, Malle conjecture too, can be studied as counting rational points on an algebraic stack. Moreover, the predictions of Manin and Malle conjectures appear similar. The similarities have already been observed by Yasuda in [**60**] and by Ellenberg, Satriano and Zureick-Brown in a forthcoming work. The reason for the similarities of the predictions may be hidden in the geometry of the corresponding BG-stack.

2.3.3. — The goal of our work is to formulate and investigate the conjecture of Manin-Peyre in the context of algebraic stacks. More precisely, we are going to so for the *weighted projective stacks*. If $n \geq 1$ is an integer and $\mathbf{a} \in \mathbb{Z}^n_{>0}$, the weighted projective stack $\mathscr{P}(\mathbf{a})$ is the "stacky" quotient of the scheme $\mathbb{A}^n - \{0\}$ by the group scheme $\mathbb{G}_m$, where the action is given by $t \cdot \mathbf{x} := (t^{a_j} x_j)_j$. When all the weights a_j are equal to 1, then $\mathscr{P}(\mathbf{a})$ is the projective space $\mathbb{P}^{n-1}$. One has homogenous coordinates on the weighted projective stacks: a rational point on $\mathscr{P}(\mathbf{a})$ is given by n-tuple of elements of F and two n-tuples $\mathbf{x}$ and $\mathbf{x}'$ represent the same point if there exists $t \in F^\times$ such that $t^{a_j} x_j = x'_j$ for $j = 1, \ldots, n$.

The moduli stack of elliptic curves $\mathcal{M}_{1,1}$ is an open substack of the stack $\mathscr{P}(4,6)$ (the stack $\mathscr{P}(4,6)$ itself is the classifying stack of the curves of genus 1 having at worst ordinary singularities). Another example is given by the stack $B\mu_m$ (where $\mu_m = \operatorname{Spec}\big(F[X]/(X^m - 1)\big)$ is the group scheme of m-th roots of unity), which is precisely the weighted projective stack $\mathscr{P}(m)$. The stack $\mathscr{P}(\mathbf{a})$ is smooth, proper and *toric*: it contains the stacky torus $\mathscr{T}(\mathbf{a}) = \mathbb{G}^n_m/\mathbb{G}_m$. Its similarity with toric varieties makes it a great candidate to study the Manin-Peyre conjecture on it.

2.4. Principal results

We state principal results of our thesis. Our goal is to provide a theory similar to the one for the rational points on varieties, rather to give ad-hoc proofs of certain cases. The development of the theory occupies a significant part of our thesis.

If X is a stack and R a ring, in order to distinguish between the category $X(R)$ and the set of isomorphism classes of objects of this category, we write $[X(R)]$ for the latter. Let F be a number field.

2.4.1. — Let us firstly explain when counting rational points on the weighted projective stack $\mathscr{P}(\mathbf{a})$ is essentially different from counting rational points on the weighed projective space $\mathbb{P}(\mathbf{a})$. Recall that the weighted projective space $\mathbb{P}(\mathbf{a})$ is the quotient $(\mathbb{A}^n - \{0\})/\mathbb{G}_m$ in the category of schemes for the same action as above. Let us denote by j the canonical morphism $j : \mathscr{P}(\mathbf{a}) \to \mathbb{P}(\mathbf{a})$. The scheme $\mathbb{P}(\mathbf{a})$ is a toric variety, and let us denote by $T(\mathbf{a}) \cong \mathbb{G}^{n-1}_m$ its torus. An isomorphism of groups $\mathbb{Z}^n/\mathbf{a}\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^{n-1} \times (\mathbb{Z}/\gcd(\mathbf{a})\mathbb{Z})$ induces an isomorphism of stacky tori

$$T(\mathbf{a})\times\mathscr{T}(\gcd(\mathbf{a})) \cong \mathbb{G}_m^{n-1}\times\mathscr{T}(\gcd(\mathbf{a})) = \mathscr{T}(\{1\}^{n-1}\times\{\gcd(\mathbf{a})\}) \xrightarrow{\sim} \mathscr{T}(\mathbf{a}) \\ = j^{-1}(T(\mathbf{a})).$$

Hence, a rational point $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$ is uniquely determined by the pair

$$(j(\mathbf{x}), \mathbf{x}') \in T(\mathbf{a})(F) \times [j^{-1}(\mathbf{x})(F)] \cong T(\mathbf{a})(F) \times [\mathscr{T}(\gcd(\mathbf{a}))(F)] \\ = T(\mathbf{a})(F) \times [\mathscr{P}(\gcd(\mathbf{a}))(F)].$$

If $\gcd(\mathbf{a}) = 1$, then $\mathscr{P}(\gcd(\mathbf{a}))$ is the one point scheme. It follows from above that the morphism j induces a bijection $[\mathscr{T}(\mathbf{a})(F)] \xrightarrow{\sim} T(F)$. According to [44, Proposition 6.1], the pullback homomorphism $j^*_{\mathbb{Q}}$: $\operatorname{Pic}(\mathbb{P}(\mathbf{a}))_{\mathbb{Q}} \to \operatorname{Pic}(\mathscr{P}(\mathbf{a}))_{\mathbb{Q}}$ of the rational Picard groups is an isomorphism. It follows that counting rational points of $\mathscr{T}(\mathbf{a})$ corresponds to counting the rational points of $\mathbb{P}(\mathbf{a})$ with respect to a height coming from a certain rational line bundle. When $\gcd(\mathbf{a}) > 1$, the set $[\mathscr{P}(\gcd(\mathbf{a}))(F)]$ is infinite (Corollary 4.6.2.2), and we see that counting the rational points of (the stacky torus of) $\mathscr{P}(\mathbf{a})$ is not the same as the counting the rational points of (the torus of) $\mathbb{P}(\mathbf{a})$.

2.4.2. — In Chapter 4, we define a notion of *quasi-toric height* on the set of rational points $\mathscr{P}(\mathbf{a})$. It is a function $H : [\mathscr{P}(\mathbf{a})(F)] \to \mathbb{R}_{\geq 0}$ and we establish a finiteness result on the number of rational points of bounded height ("weak Northcott property"). A height depends on the choice of a line bundle on the stack $\overline{\mathscr{P}(\mathbf{a})} = \mathbb{A}^n/\mathbb{G}_m$ (where the action is canonically extended) and an "adelic metric" on it. For $v \in M_F$, we define topological spaces $[\mathscr{P}(\mathbf{a})(F_v)] := (F_v^n - \{0\})/F_v^{\times}$, where the action is induced from the action of $\mathbb{G}_m$ on $\mathbb{A}^n - \{0\}$. The product space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is a good analogue of the "adelic space" of a variety. In Chapter 5, we define a measure ω_H on the product space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ and we set $\tau_H = \omega_H(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])$. We prove that:

***Theorem* (Theorem 8.2.2.12, Proposition 8.3.2.3)**

Let H be a quasi-toric height. One has that

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| \sim_{B\to\infty} \frac{\tau_H}{|\mathbf{a}|}B.$$

For a particular type of quasi-toric height (that in our work is called *toric height*), the result has been established in [**12**]. For the other heights, the result is new.

We establish furthermore that the rational points of $\mathscr{P}(\mathbf{a})$ are equidistributed in $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ in the following sense. Let $i:[\mathscr{P}(\mathbf{a})(F)]\to\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ be the diagonal map. If $W\subset\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is an open subset such that $\omega(\partial W)=0$, in Theorem 8.3.2.2, we prove that

$$\lim_{B\to\infty}\frac{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|i(\mathbf{x})\in W\text{ and }H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|}=\frac{\omega_H(W)}{\tau_H}.$$

2.4.3. — Let us state the second principal result of our work. We suppose that $n=1$ and that $m\in\mathbb{Z}_{>1}$. We count μ_m-torsors over F (i.e the rational points of $\mathscr{P}(m)$) of bounded discriminant. In Chapter 9, we define a notion of *quasi-discriminant height*. It is a function $H:[\mathscr{P}(m)(F)]\to\mathbb{R}_{\geq 0}$ which satisfies the weak Northcott property, and is essentially different from a quasi-toric height (one cannot normalize it so that the quotients of the two heights are bounded functions on $[\mathscr{P}(m)(F)]$). The normalizations are taken so that $H^{m(1-1/r)}$, where r is the least prime of m, is essentially the absolute discriminant Δ of a μ_m-torsor (i.e. the local components of $H^{m(1-1/r)}$ are different from the local components of the absolute discriminant Δ at only finitely many places, consequently $H^{m(1-1/r)}/\Delta$ is bounded on $[\mathscr{P}(m)(F)]$). As above, a choice of the quasi-discriminant height H defines a measure ω_H on $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ and we set $\tau_H=\omega_H(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])$. We prove that

***Theorem* (Corollary 9.2.5.9).** — *Let H be a quasi-discriminant height. One has that*

$$|\{x\in[\mathscr{P}(m)(F)]|H(x)\leq B\}|\sim_{B\to\infty}\frac{1}{(r-2)!}\cdot\frac{\tau_H}{m}\cdot B\log(B)^{r-2}.$$

Again we prove an equidistribution property of rational points in the space $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ (Theorem 9.2.6.1). The equidistribution property is used to prove that a positive proportion of μ_m-torsors of bounded quasi-discriminant height are fields and that, if m is not a prime, a positive proportion are not fields. Moreover, when $4\nmid m$ or when $i=\sqrt{-1}\in F$, we are able to give a formula for the proportion of fields (Theorem 9.2.7.3).

Suppose that F contains all m-th roots of unity. In particular, one has that $4\nmid m$ or that $i\in F$. We have that $\mu_m=\mathbb{Z}/m\mathbb{Z}$. In this case, the asymptotics for the number of $\mu_m = \mathbb{Z}/m\mathbb{Z}$-torsors of bounded discriminant which are fields, has already been given by Wright in [**59**]. The advantage of our method is that we are able to modify the discriminant at finitely many places.

2.5. Overview of the thesis

Let us make an overview of our work.

2.5.1. — Let us discuss a difference between heights on varieties and stacks. Let X be a proper F-variety and let L be a line bundle on X. It is well known that a choice of an $\mathcal{O}_{F,S}$-model of (X, L) (where S is a finite set of finite places of F) endows L with a metric for every finite place not in S. Endowing the line bundle L at the remaining places with a metric, gives an adelic metric on L, and hence a height on $X(F)$.

In the construction of the metric for the finite places not in S, one uses the *valuative criterion of properness* which gives that every F_v-point of X extends to an $\mathcal{O}_v$-point of the model. However, this is not true for stacks (e.g. only the points of $\mathscr{P}(4,6)$ corresponding to curves having good reductions at v do extend to $\mathcal{O}_v$-points).

The valuative criterion of properness for stacks gives only that an F_v-point extends to an A-point of the model, where A is the normalization of $\mathcal{O}_v$ in a finite extension of F_v. Such integral extensions give rise to stable heights (i.e. the height of an F-point stays the same when the point is looked as a K-point, where K/F is a finite extension). A drawback of the stable heights is that they do not satisfy the "weak Northcott property" (Definition 4.6.1.1), as one sees in the case of $\mathscr{P}(4,6)$. Namely, the stable height of two F-elliptic curves which are not isomorphic over F is the same if they have the same j-invariant. For a fixed elliptic curve E, there are infinitely many such elliptic curves (they are constructed by performing quadratic twists to E).

2.5.2. — In Chapter 3, we recall several results about stacks, with the focus on the weighted projective stacks. We also introduce the stack $\overline{\mathscr{P}(\mathbf{a})} = \mathbb{A}^n/\mathbb{G}_m$ (for the canonical extension of the action of $\mathbb{G}_m$ on $\mathbb{A}^n - \{0\}$). Thus, one has open immersions $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a}) \subset \overline{\mathscr{P}(\mathbf{a})}$. The stack $\overline{\mathscr{P}(\mathbf{a})}$ is not separated (Lemma 3.2.2.6), hence not proper, yet it exhibits the property that all of its rational points extend to integral points, and hence resolves the problem of lack of the integral points from above. This property will be used in Chapter 4 to produce *unstable* heights on the weighted projective stacks.

Work of Moret-Bailly from [**40**] provides a notion of a topological space associated to the set of (isomorphism classes of) R-points of stacks, when R is a certain kind of topological local ring. The association is functorial, that is, for a morphism $X \to Y$ of stacks, the induced map $[X(R)] \to [Y(R)]$ is continuous. A list of other properties that the construction satisfies is given in [**15**]. We prove the following proposition that enables us understand this topology for certain quotient stacks:

Proposition 2.5.2.1. — *Suppose that X is a quotient stack Y/G, with G special (its torsors are locally trivial, by Hilbert 90, an example is provided by $G = \mathbb{G}_m$). One has that $[X(R)]$ is the topological quotient $Y(R)/G(R)$.*

Thus, one has for example that $[\mathscr{P}(\mathbf{a})(F_v)] = (F_v^n - \{0\})/F_v^\times$ and $[\mathscr{T}(\mathbf{a})(F_v)] = (F_v^\times)^n/F_v^\times$, where the action of $F_v^\times = \mathbb{G}_m(F_v)$ is the induced from the action of $\mathbb{G}_m$ on $\mathbb{A}^n - \{0\}$ and $\mathbb{G}_m^n$, respectively. In the last part of the chapter, we speak about the adelic space of the torus $\mathscr{T}(\mathbf{a})$. We define it to be the restricted product

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] := \prod_{v \in M_F}{}' [\mathscr{T}(\mathbf{a})(F_v)]$$

with the respect to the compact and open subgroups $[\mathscr{T}(\mathbf{a})(\mathscr{O}_v)] \subset [\mathscr{T}(\mathbf{a})(F_v)]$. Using the results of Česnavičius from [**14**] on cohomology of the adeles, we prove that $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ has similar properties to the adelic torus $\mathbb{G}_m^n(\mathbb{A}_F)$ (e.g. the image of the rational points $[\mathscr{T}(\mathbf{a})(F)]$ for the diagonal map is discrete).

2.5.3. — We start Chapter 4 by recalling facts about line bundles on stacks, in particular that the line bundles on the quotient stack Y/G correspond to G-linearized line bundles on the scheme Y. The Picard groups $\operatorname{Pic}(\mathscr{P}(\mathbf{a}))$ and $\operatorname{Pic}(\overline{\mathscr{P}(\mathbf{a})})$ are calculated. Then, we define metrics on line bundles as follows. Let v be a place of F, let X be an F_v-algebraic stack and let L a line bundle on X. We define an F_v-metric on L to be the data given by "compatible" F_v-metrics on y^*L for every morphism $y : Y \to X$ with Y an F_v-scheme (by an F_v-metric on a line bundle over an F_v-scheme, we mean a "continuous" choice of norms on all F_v-fibers). Our metric does not need to be stable. For quotient stacks $X = Y/G$, when G is assumed to be a special algebraic group, we relate the group of F_v-metrized line bundles $\widehat{\mathrm{Pic}}_v(Y/G)$ with the group $\widehat{\mathrm{Pic}}_v^G(Y)$ of F_v-metrized line bundles on X which are endowed with a G-linearization and such that the metric is G-invariant:

***Proposition 2.5.3.1* (Proposition 4.3.4.5). —** *Let G be a special locally of finite type F_v-group scheme acting on locally of finite type F_v-scheme Y. The canonical homomorphism $\widehat{\mathrm{Pic}}_v(Y/G) \to \widehat{\mathrm{Pic}}_v^G(Y)$ is injective, and is an isomorphism if $\widehat{\mathrm{Pic}}_v(Y/G) \to \mathrm{Pic}(Y/G)$ is surjective.*

The stack $\mathscr{P}(\mathbf{a}) = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$ satisfies this condition on the existence of F_v-metrics on every of its line bundles (Lemma 4.3.6.3). Consequently, as $\mathrm{Pic}(\mathbb{A}^n - \{0\})$ is trivial, we deduce that to define an F_v-metric on a line bundle on $\mathscr{P}(\mathbf{a})$, it suffices to define a $\mathbb{G}_m$-invariant metric on the corresponding $\mathbb{G}_m$-linearization of the trivial line bundle on $\mathbb{A}^n - \{0\}$. Such metric is defined by the norm of the section 1 and condition on the linearizations gives a "homogeneity" condition to the function $F_v^n - \{0\} \to \mathbb{R}_{>0}, \mathbf{x} \mapsto ||1||_{\mathbf{x}}$.

Suppose that a line bundle L on $\mathscr{P}(\mathbf{a})$ is endowed with an F_v-metric for every $v \in M_F$, subject to a compatibility condition which allows that the norms of a section can be multiplied at any $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$ (see the condition in Definition 4.4.1.1). We can define heights by multiplying the inverses of these norms for every v. The generality, leaves possibility of existence of "essentially different" heights on the same line bundles, i.e. heights such that their quotients are not bounded functions on the set $[\mathscr{P}(\mathbf{a})(F)]$. Examples are: the mentioned *stable heights*, the *quasi-toric heights* (we are going to explain them now) and in the case $n = 1$ the *quasi-discriminant heights* (they will be explained in the last chapter). Quasi-toric heights are the heights, which come from the families of metrics which arise in the following way for almost every place: extend an F_v-point of the stack $\overline{\mathscr{P}(\mathbf{a})}$ to an $\mathcal{O}_v$-point and use the classical method (already discussed in 2.5.1) to get a metric (a smaller modification, however, must be done, because $\overline{\mathscr{P}(\mathbf{a})}$ is not separated and thus an $\mathcal{O}_v$-extension of an F_v-point is not unique). Contrary to the stable heights, the quasi-toric heights do satisfy the weak Northcott property:

***Theorem 2.5.3.2* (Theorem 4.6.8.2). —** *Let H be a quasi-toric height on $\mathscr{P}(\mathbf{a})$. Let $\epsilon > 0$. One has that there exists $C > 0$ such that*

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| \leq CB^{1+\epsilon}.$$

The idea of the proof of is to separately estimate the finite and the infinite height. The upper bound for the cardinality in the theorem is needed to provide convergence of the corresponding height zeta series. The claim of 2.5.3.2 stays valid even when metrics at finitely many places are allowed to have "logarithmic" singularities along rational divisors (see Corollary 4.7.1.3). The proof of that version follows immediately from 2.5.3.2, after establishing an estimate for the singular height of the form: $H_{\mathrm{Sing}} \geq CH\log^{-\eta}(H)$, where $C, \eta > 0$, which we do in Proposition 4.7.1.2.

2.5.4. — In Chapter 5, we endow the topological spaces associated to R-points with measures. In particular, we define measures on $[\mathscr{P}(\mathbf{a})(F_v)]$ (which depend on the choice of the metrics) and on $[\mathscr{T}(\mathbf{a})(F_v)]$ (which do not depend on the choice of metrics). The measures are used to define Peyre's constant τ_H.

The last part of the chapter is dedicated to the definition of the measures on the "adelic torus" $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and the Tamagawa number of the stacky torus $\mathscr{T}(\mathbf{a})$. We establish that

***Proposition 2.5.4.1* (Proposition 5.4.4.4).** — *One has that* $\mathrm{Tam}(\mathscr{T}(\mathbf{a})) = 1.$

When $\mathbf{a} = \mathbf{1}$, this is the classical result that the Tamagawa number of a split torus is 1. The proof of Proposition 2.5.4.1 uses Oesterlé's Euler-Poincaré characteristics of complexes of locally compact abelian groups which are endowed with Haar measures.

2.5.5. — Chapter 6 studies characters of the "adelic torus" $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. We introduce "discrete" norms and "infinity" norms of these characters. One establish a finiteness result on the number of the characters $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which vanish on $[\mathscr{T}(\mathbf{a})(F)]$ and on certain subgroups of bounded either of these norms. In the last part of this chapter we recall estimates of Rademacher on L functions of characters. The results of this chapter will be used in Chapter 7 to prove that the Fourier transform of a height function is integrable.

2.5.6. — In Chapter 7 we adapt the method of harmonic analysis of Batyrev and Tschinkel from [**3**] to our situation. We assume the metrics are *smooth*. The first part of the chapter is dedicated to the calculation of the Fourier transform of the local height at a finite place v. For almost all v, we can give the exact formula which turns out to be the product of local L functions of characters and other factors. Then, for infinite v, using the smoothness assumptions, we prove suitable decays of the Fourier transform in the two norms of characters. The proof for this claim is an adaptation of the idea of Chambert-Loir and Tchinkel from [18] and [20], where the authors apply integration by parts with the respect to invariant vector fields. The global Fourier transform, hence, writes as the product of L functions and a part over which we have a control.

2.5.7. — In Chapter 8 we use the theory of [17] to analyse height zeta function.

The accent is on the stacky torus $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a})$. From the estimate of Theorem 2.5.3.2, one deduces that the height zeta function $Z(s) := \sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]} H(\mathbf{x})^{-s}$ converges and defines a holomorphic function of s in the domain $\Re(s) > 1$. Poisson formula gives that

$$Z(s) = \int_{([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(F)])^*} \widehat{H}(s,\chi)d\chi$$

whenever the expressions on both hand side converge. The estimates from Chapter 6 and Chapter 7 and the "similarity" of the global Fourier transform of the height with L functions, give the convergence of the integral on the right hand side for $\Re(s) > 1$. Moreover, the proof will imply that $s \mapsto Z(s)$ has a meromorphic extension to a domain $\Re(s) > 1-\delta$, for some $\delta > 0$. The estimates of Rademacher imply that Z satisfies the growth conditions needed for Tauberian theorems. The residue of Z at 1 is also calculated. As a consequence, Tauberian theorems give the asymptotic behaviour of the number of rational points of the height at most B:

***Theorem 2.5.7.1* (Theorem 8.2.2.12, Proposition 8.3.2.3)**

Let H be a quasi-toric height. One has that

$$\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\} \sim \frac{\tau_H}{|\mathbf{a}|}B,$$

when B tends to $+\infty$.

The constant $\frac{1}{|\mathbf{a}|}$ has the same interpretation as in the case of varieties (see Remark 8.3.2.7) and the asymptotic stays the same when one counts rational points of $\overline{\mathscr{P}(\mathbf{a})}$ (because $[\overline{\mathscr{P}(\mathbf{a})}(F)] - [\mathscr{P}(\mathbf{a})(F)]$ is a one point set). Thus Theorem 2.5.7.1 can be understood as that Manin-Peyre's conjecture is true for weighted projective stacks $\overline{\mathscr{P}(\mathbf{a})}$. The last part of the chapter is dedicated to the understanding *equidistribution* of the rational points of the stack $\mathscr{P}(\mathbf{a})$. The idea is to find the asymptotic behaviour for the number of rational points of bounded height which are required for finitely many places v to belong to certain open subsets of the v-adic space of the stack (e.g. say that the 2-adic valuation is even). An elegant way to phrase this question has been given by Peyre in [47], using the measure ω_H from above: if $W \subset \prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is an open subset of negligible boundary, one expects that:

$$\lim_{B\to\infty}\frac{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|i(\mathbf{x})\in W \text{ and } H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|}=\frac{\omega_H(W)}{\tau_H},$$

where $i:[\mathscr{P}(\mathbf{a})(F)]\to\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is the diagonal map. If this is true for every such W, then we say that the rational points are equidistributed. We prove that:

***Theorem 2.5.7.2* (Theorem 8.3.2.2). —** *The rational points of $\mathscr{P}(\mathbf{a})$ are equidistributed in the space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$.*

2.5.8. — In Chapter 9, we use our methods to study a question similar to Malle conjecture. We find the asymptotic for the number of μ_m-torsors over F of bounded absolute discriminant. When F contains all m-th roots of 1, this question is the Malle conjecture for the cyclic group $\mathbb{Z}/m\mathbb{Z}$, but one needs to remove the torsors which are not fields and there may be a positive proportion of them (the counting of the cyclic extensions has been covered by the case of the abelian groups from [59]). The μ_m-torsors over F are classified by the algebraic stack $B\mu_m=\mathscr{T}(m)=\mathscr{P}(m)$.

We use the language of heights developed earlier. We speak about *quasi-discriminant* heights, which are similar to the discriminants, with the difference that the local components of these heights at the finitely many places may be different from the local components of the discriminant. Let us note that a quasi-discriminant height is not a quasi-toric height, as the local components of the two heights are different at almost every place. We will define a measure ω_H on $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ and we will set $\tau_H=\omega_H(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])$.

The method of the proof is an adaption of the above harmonic analysis method. There are some simplifications, as the local spaces $[\mathscr{T}(m)(F_v)]$ are finite, and modifications because of the difference with quasi-toric heights. Eventually, we prove the convergence of the height zeta series and use Poisson's formula as before. For the purpose of having more elegant formula, here we state the final asymptotic for heights that are "essentially" $\Delta^{\frac{1}{m(1-1/r)}}$, where r is the least prime of m (that is for almost all places the local component of H coincides with the local component of $\Delta^{\frac{1}{m(1-1/r)}}$).

***Theorem 2.5.8.1* (Corollary 9.2.5.9). —** *Let H be a quasi-discriminant height. One has that*

$$|\{x \in [\mathscr{P}(m)(F)] | H(x) \leq B\}| = \frac{\tau_H}{(r-2)!m} B \log(B)^{r-2}.$$

The asymptotic is very reminiscent of the one from Manin-Peyre conjecture. As in the case of quasi-toric heights, we are able to prove a corresponding equidistribution property in $\prod_{v \in M_F} [\mathscr{P}(m)(F_v)]$. We end the chapter by the proof that there is a positive proportion of μ_m-torsors which are fields. This is proven by finding an open subset $W \subset \prod_{v \in M_F} [\mathscr{P}(m)(F_v)]$ of positive volume such that all of μ_m-torsors contained in it are fields. Moreover, when $4 \nmid m$ or when $i = \sqrt{-1} \in F$, we give an exact formula for this proportion.

2.6. Questions and Remarks

Let us discuss some questions that arise naturally from our work.

2.6.1. — The conjecture of Manin-Peyre has been proved for all smooth toric varieties. We would like to know to what generality the proof applies to other toric stacks (c.f. [28]). We do not know what happens when the "stacky torus" is not split, i.e. not a quotient of two split tori. Manin-Peyre conjecture for toric varieties has also been proved for the case of function fields ([11]). We would like to know what is the situation for toric stacks.

2.6.2. — It would be interesting to understand to what other stacks one can develop a theory of heights and use it to count rational points. Examples of such stacks could be: the stack $\mathcal{M}_g$ which classifies the curves of genus g, the stack of principally polarized abelian varieties $\mathcal{A}_g$, etc.

2.6.3. — One could ask whether there exists a stack X with enough integral points, such that $B\mu_m \subset X$ and such that the discriminant arises as a height induced by an $\mathcal{O}_{F,S}$-model of X and a line bundle on it (here S is a finite set of places). We would like to know then whether the result of Theorem 2.5.8.1 can be reinterpreted as that Manin-Peyre conjecture is true for X. We may then ask how the prediction of Malle conjecture compares with the prediction of Manin-Peyre conjecture. The same question can be asked for any finite group (scheme) G.

2.6.4. — A different notion of a height on stack, defined by vector bundles, has been proposed Ellenberg, Satriano and Zureick-Brown in a forthcoming work. Their height is not additive in the vector bundles. We would like to know how this notion compares with our notion of the height.

2.6.5. — A counterexample to Malle conjecture has been constructed by Klüners in [**33**]. The known counterexamples in the conjecture of Manin-Peyre are avoided if one allows removing "thin sets". We would like to know whether removing "thin sets" fixes the prediction of Malle.

CHAPTER 3

WEIGHTED PROJECTIVE STACKS

A weighted projective stack is a stacky quotient $\mathscr{P}(\mathbf{a}) := (\mathbb{A}^n - \{0\})/\mathbb{G}_m$, where the action of $\mathbb{G}_m$ is weighted with the weights $a_1, \ldots, a_n$, where $a_1, \ldots, a_n$ are positive integers. In the first part of this chapter we will recall several properties of such stacks. It turns out that the weighted projective stacks are proper, however, not all of its rational point extends to an integral point. This is a fundamental feature that enables one to define heights. The stack $\overline{\mathscr{P}(\mathbf{a})} := \mathbb{A}^n/\mathbb{G}_m$ (that by the abuse of the terminology we may also call a weighted projective stack) has enough of integral points in this sense. The second part of the chapter is dedicated to the topological spaces associated to weighted projective stacks.

3.1. Weighted projective stacks

In this section we recall several facts about stacks and weighted projective stacks.

3.1.1. — In this paragraph we recall some generalities on quotient stacks. We follow [56].

Let Z be a scheme. Let X be a Z-scheme and let $a : G \times_Z X \to X$ be a left Z-action of locally of finite presentation flat Z-algebraic group G on X. Denote by p_2 the projection to the second coordinate $G \times_Z X \to X$.

One has a commutative diagram

$$
\begin{tikzcd}
G \times_Z X \arrow[r, "u"] \arrow[dr, "a"'] & G \times_Z X \arrow[d, "p_2"] \\
& X,
\end{tikzcd}
\tag{3.1.1.1}
$$

where the morphism u is given by $(g,x) \mapsto (g, a(g,x))$. The morphism u is an automorphism as its inverse is provided by $(g,x) \mapsto (g, a(g^{-1},x))$, hence, the morphism a is surjective, flat and locally of finite presentation.

We write X/G for the quotient stack ([**56**, Tag 044O]). Recall that if V is a Z-scheme a 1-morphism $x : V \to X/G$ is given by a G_V-equivariant morphism $\widetilde{x} : T \to X_V$, where T is a G_V-torsor. A 2-morphism $\theta : x \to y$ of 1-morphisms $x : V \to X/G$ and $y : V \to X/G$ corresponding to G_V-equivariant morphisms from G_V-torsors $\widetilde{x} : T \to X$ and $\widetilde{y} : R \to X$, respectively, is given by a morphism $\theta : R \to T$ of G_V-torsors such that $\widetilde{x} = \widetilde{y} \circ \theta$.

In the following proposition we recall some of the properties of a quotient stack. Let $q : X \to X/G$ be the quotient 1-morphism, i.e. the one given by the trivial G_X-torsor $G \times_Z X \xrightarrow{p_2} X$ and the G_X-equivariant morphism $G_X = G \times_Z X \xrightarrow{a} X$.

Proposition 3.1.1.2. — *1. For every* 1*-morphism* $x : V \to X/G$ *over* Z *with* V *a scheme, which is given by* G_V*-equivariant morphism* $\widetilde{x} : T \to X$*, where* T *is a* G_V*-torsor, the diagram*

$$
\begin{tikzcd}
T \arrow[r, "\widetilde{x}"] \arrow[d] & X \arrow[d, "q"] \\
V \arrow[r, "x"] & X/G.
\end{tikzcd}
$$

is 2*-commutative* 2*-cartesian (*[**56**, Section 04UV]*).*

2. *The morphism* $q : X \to X/G$ *is surjective, flat, representable and locally of finite presentation* [**56**, Lemma 06FH].
3. *The stack* X/G *is algebraic (*[**56**, Theorem 06FI]*).*
4. *The stack* X/G *is smooth over* Z *if* X *is smooth over* Z *(*[**56**, Lemma 0DLS]*).*
5. *If* Y *is a* Z*-scheme, we let* X_Y/G_Y *be the quotient stack for the induced* Y*-action* $a_Y : G_Y \times_Y X_Y \to X_Y$*. The canonical* 1*-morphism* $X_Y/G_Y \to (X/G) \times_Z Y$ *is an equivalence (*[**56**, Lemma 04WX]*).*

Let $\Delta_{X/G}$ be the diagonal morphism of $X/G \to Z$. The following lemma will be quoted several times:

Lemma 3.1.1.3. — *The diagram*

$$\begin{tikzcd} G \times_Z X \arrow[r, "a \times p_2"] \arrow[d, "q \circ p_2"] & X \times_Z X \arrow[d, "q \times_Z q"] \\ X/G \arrow[r, "\Delta_{X/G}"] & (X/G) \times_Z (X/G). \end{tikzcd} \tag{3.1.1.4}$$

is 2*-commutative* 2*-cartesian.*

Proof. — In this proof p_1 and p_2 denote the obvious projections, while Id stands for the identity 1-morphism. The diagram

$$\begin{tikzcd} G \times_Z X \arrow[r, "(a,p_2)"] \arrow[d, "p_2"'] & X \times_Z X \arrow[r, "p_2"] \arrow[d, "(\mathrm{Id}_X, q)"] & X \arrow[d, "q"] \\ X \arrow[r, "\Gamma_q"'] & X \times_Z (X/G) \arrow[r, "p_2"'] & X/G \end{tikzcd}$$

is 2-commutative. By (1) of Proposition 3.1.1.2, its big subdiagram is 2-cartesian. The diagram

$$\begin{tikzcd} X \times_Z X \arrow[r, "p_2"] \arrow[d, "(\mathrm{Id}_X, q)"] & X \arrow[d, "q"] \\ X \times_Z (X/G) \arrow[r, "p_2"'] & X/G \end{tikzcd}$$

is 2-commutative and 2-cartesian. It follows that

$$\begin{tikzcd} G \times_Z X \arrow[r, "(a,p_2)"] \arrow[d, "p_2"'] & X \times_Z X \arrow[d, "(\mathrm{Id}_X, q)"] \\ X \arrow[r, "\Gamma_q"'] & X \times_Z (X/G) \end{tikzcd} \tag{3.1.1.5}$$

is 2-commutative and 2-cartesian. The diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma_q"'] \arrow[d, "q"] & X \times_Z (X/G) \arrow[r, "p_1"'] \arrow[d, "(q, \mathrm{Id}_{X/G})"'] & X \arrow[d, "q"] \\ X/G \arrow[r, "\Delta_{X/G}"'] & (X/G) \times_Z (X/G) \arrow[r, "p_2"'] & X/G \end{tikzcd}$$

is 2-commutative. Its big subdiagram is 2-cartesian, because the horizontal maps are the identity 1-morphisms. The diagram

$$\begin{tikzcd} X \times_Z (X/G) \arrow[r, "p_1"'] \arrow[d, "(q,\mathrm{Id}_{X/G})"'] & X \arrow[d, "q"] \\ (X/G) \times_Z (X/G) \arrow[r, "p_2"'] & X/G \end{tikzcd}$$

is 2-commutative and 2-cartesian, hence the diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma_q"'] \arrow[d, "q"] & X \times_Z (X/G) \arrow[d, "(q,\mathrm{Id}_{X/G})"'] \\ X/G \arrow[r, "\Delta_{X/G}"'] & (X/G) \times_Z (X/G) \end{tikzcd} \tag{3.1.1.6}$$

is 2-commutative and 2-cartesian. The diagram in the statement is the big subdiagram of the diagram that one gets by merging 2-commutative and 2-cartesian diagrams (3.1.1.5) and (3.1.1.6), hence itself is 2-commutative and 2-cartesian. The statement is proven. □

We say that affine algebraic group G is *special* (Serre, Section 4.1 in [**54**]) if every G-torsor $Y \to W$, with W and Y schemes, is locally trivial for the Zariski topology on W. Hilbert 90 theorem states that the general linear groups GL_d, for $d \geq 1$ are special (see e.g. [**38**, Lemma 4.10, Chapter III]).

Lemma 3.1.1.7. — *Suppose G is a flat, locally of finite presentation special algebraic group. Let R be a local Z-ring. For every 1-morphism of Z-stacks $x : \operatorname{Spec} R \to X/G$, there exists a 2-commutative 2-cartesian square*

$$\begin{tikzcd} G_R \arrow[r, "\widetilde{x}"] \arrow[d] & X \arrow[d, "q"] \\ \operatorname{Spec}(R) \arrow[r, "x"] & X/G, \end{tikzcd} \tag{3.1.1.8}$$

with $\widetilde{x}$ being G-equivariant morphism.

Proof. — As R is local and G special, every G-torsor over $\operatorname{Spec} R$ is isomorphic to the trivial one. Now the claim follows from part (1) of Proposition 3.1.1.2. □

One can also see that in the situation of Lemma 3.1.1.7, the category $(X/G)(R)$ is equivalent to the following category: its objects are G_R-equivariant morphisms $G_R \to X_R$ and a morphism $t : (x : G_R \to X_R) \to (y : G_R \to X_R)$ is an element $t \in G_R(R)$ such that $x = y \circ t$, when t is seen as a morphism $t : G_R \to G_R$ by multiplication to the left. We will often by the abuse of notation write $(X/G)(R)$ for the latter category.

3.1.2. — In this paragraph we work over $\mathrm{Spec}(\mathbb{Z})$. Let $n \geq 1$ be an integer. Let $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$. The smooth group scheme $\mathbb{G}_m$ acts on $\mathbb{A}^n$ via the formula:

$$a : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n \qquad (t, \mathbf{x}) = (t^{a_j} x_j)_j. \tag{3.1.2.1}$$

We will often write $t \cdot \mathbf{x}$ instead of $a(t, \mathbf{x})$. Note that $\mathbb{A}^n - \{0\} \subset \mathbb{A}^n$ and $\mathbb{G}_m^n = (\mathbb{A}^1 - \{0\})^n \subset \mathbb{A}^n$ are $\mathbb{G}_m$-invariant open subschemes for this action. We have, hence, induced actions of $\mathbb{G}_m$ on $\mathbb{G}_m^n$ and on $\mathbb{A}^n - \{0\}$.

Lemma 3.1.2.2. — *1. The morphism*

$$(a, p_2) : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$$

is of finite presentation and affine (hence separated and quasi-compact by [**56**, Lemma 01S7]*).*

2. The morphisms

$$(a|_{\mathbb{G}_m \times (\mathbb{A}^n - \{0\})}, p_2) : \mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\})$$

and

$$(a|_{\mathbb{G}_m \times \mathbb{G}_m^n}, p_2) : \mathbb{G}_m \times \mathbb{G}_m^n \to \mathbb{G}_m^n \times \mathbb{G}_m^n$$

induced from $\mathbb{G}_m$-invariant open subschemes $\mathbb{A}^n - \{0\} \subset \mathbb{A}^n$ and $\mathbb{G}_m^n \subset \mathbb{A}^n$, respectively, are finite.

Proof. — 1. The morphism (a, p_2) is of finite presentation as both a and p_2 are of finite presentation (see Diagram (3.1.1.1) in Proposition 3.1.1.2). The morphism (a, p_2) is affine because it is a morphism of affine schemes.

2. Let us verify that $(a|_{\mathbb{G}_m \times (\mathbb{A}^n - \{0\})}, p_2)$ is proper. It is affine, hence separated, and of finite type, as it is the base change of the affine and finite type morphism $\mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ along the open immersion $(\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\}) \to \mathbb{A}^n \times \mathbb{A}^n$. We use the valuative criterion for finite type morphism with Noetherian target to be universally closed [**56**, Lemma 0CM5]. Let R be a discrete valuation ring, v_R

its valuation and K its fraction field. Consider a diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "(t,\mathbf{z})"] \arrow[d] & \mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "(\mathbf{x},\mathbf{y})"] & (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\}). \end{tikzcd}$$

It follows that $\mathbf{z} = \mathbf{y} \in (\mathbb{A}^n - \{0\})(R)$ and that $t \cdot \mathbf{z} = \mathbf{x}$. There exists i such that $v_R(z_i) = v_R(y_i) = 0$. We have that $0 \leq v_R(x_i) = v_R(t^{a_i} z_i) = a_i v_R(t)$ and hence $v_R(t) \geq 0$. There exists k such that $v_R(x_k) = 0$. We have $0 = v_R(x_k) = v_R(t^{a_k} z_k) = a_k v_R(t) + v_R(z_k) \geq a_k v_R(t)$ and hence $v_R(t) \leq 0$. We deduce $v_R(t) = 0$ i.e. $t \in \mathbb{G}_m(R)$. We deduce $(t, \mathbf{z}) \in (\mathbb{G}_m \times (\mathbb{A}^n - \{0\}))(R)$ and the valuative criterion is verified. It follows that $(a|_{\mathbb{G}_m \times (\mathbb{A}^n - \{0\})}, p_2)$ is universally closed and we deduce that it is proper. It is affine, and we deduce that it is also finite. Now, the morphism $(a|_{\mathbb{G}_m \times \mathbb{G}_m^n}, p_2)$ is the base change of the finite morphism $(a|_{\mathbb{G}_m \times (\mathbb{A}^n - \{0\})}, p_2)$ along the open immersion $\mathbb{G}_m^n \times \mathbb{G}_m^n \to (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\})$, hence is finite.

□

Definition 3.1.2.3. — *We define quotient stacks for the actions from above:*

$$\begin{aligned} \overline{\mathscr{P}(\mathbf{a})} &:= \mathbb{A}^n / \mathbb{G}_m, \\ \mathscr{P}(\mathbf{a}) &:= (\mathbb{A}^n - \{0\}) / \mathbb{G}_m, \\ \mathscr{T}(\mathbf{a}) &:= \mathbb{G}_m^n / \mathbb{G}_m. \end{aligned}$$

The first two stacks we may call weighted projective stacks.

The $\mathbb{G}_m$-equivariant open immersions $\mathbb{G}_m^n \subset \mathbb{A}^n - \{0\}$, $\mathbb{A}^n - \{0\} \subset \mathbb{A}^n$ induce 1-morphisms of stacks $\mathscr{T}(\mathbf{a}) \to \mathscr{P}(\mathbf{a})$ and $\mathscr{P}(\mathbf{a}) \to \overline{\mathscr{P}(\mathbf{a})}$ by [**56**, Lemma 046Q], which are open immersions by [**56**, Lemma 04YN].

Lemma 3.1.2.4. — *The stacks $\mathscr{T}(\mathbf{a})$, $\mathscr{P}(\mathbf{a})$ and $\overline{\mathscr{P}(\mathbf{a})}$ satisfy the following:*

1. *They are smooth algebraic stacks.*
2. *They are quasi-compact.*
3. *Their diagonals for the canonical morphisms to* $\mathrm{Spec}(\mathbb{Z})$ *are representable and affine (hence, separated and quasi-compact). The diagonals of $\mathscr{T}(\mathbf{a})$ and $\mathscr{P}(\mathbf{a})$ are further finite.*
4. *They are of finite presentation.*

Proof. — 1. This claim follows from parts (3) and (4) of Proposition 3.1.1.2.

2. The quotient 1-morphisms $q^{\mathbf{a}} : \mathbb{A}^n \to \mathbb{A}^n/\mathbb{G}_m = \overline{\mathscr{P}(\mathbf{a})}$, $q^{\mathbf{a}}|_{\mathbb{A}^n-\{0\}} : (\mathbb{A}^n - \{0\}) \to \mathscr{P}(\mathbf{a})$ and $q^{\mathbf{a}}|_{\mathbb{G}_m^n} : \mathbb{G}_m^n \to \mathscr{T}(\mathbf{a})$ are surjective by Proposition 3.1.1.2 and as $\mathbb{A}^n, \mathbb{A}^n - \{0\}$ and $\mathbb{G}_m^n$ are quasi-compact, we deduce by [**56**, Lemma 04YC] that $\overline{\mathscr{P}(\mathbf{a})}$, $\mathscr{P}(\mathbf{a})$ and $\mathscr{T}(\mathbf{a})$ are quasi-compact.

3. All three diagonals are representable, because all three stacks are algebraic. Let us prove that the diagonal morphism $\Delta_{\overline{\mathscr{P}(\mathbf{a})}} : \overline{\mathscr{P}(\mathbf{a})} \to \overline{\mathscr{P}(\mathbf{a})} \times \overline{\mathscr{P}(\mathbf{a})}$ is affine. By Lemma 3.1.1.3, we have a 2-commutative 2-cartesian square

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{A}^n \arrow[r, "(a,p_2)"] \arrow[d, "q\circ p_2"'] & \mathbb{A}^n \times \mathbb{A}^n \arrow[d, "(q,q)"] \\
\overline{\mathscr{P}(\mathbf{a})} \arrow[r, "\Delta_{\overline{\mathscr{P}(\mathbf{a})}}"'] & \overline{\mathscr{P}(\mathbf{a})} \times \overline{\mathscr{P}(\mathbf{a})}.
\end{tikzcd}$$

The 1-morphism $(q, q) : \mathbb{A}^n \times \mathbb{A}^n \to \overline{\mathscr{P}(\mathbf{a})} \times \overline{\mathscr{P}(\mathbf{a})}$ is surjective, flat and locally of finite presentation and the morphism $(a, p_2) : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ is affine by Lemma 3.1.2.2. It follows from [**56**, Lemma 06TY] that $\Delta_{\overline{\mathscr{P}(\mathbf{a})}}$ is affine. Let us prove that the diagonal $\Delta_{\mathscr{P}(\mathbf{a})} : \mathscr{P}(\mathbf{a}) \to \mathscr{P}(\mathbf{a}) \times \mathscr{P}(\mathbf{a})$ is finite. By Lemma 3.1.1.3, we have a 2-commutative 2-cartesian square

$$\begin{tikzcd}
\mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \arrow[r, "(a,p_2)"] \arrow[d, "q\circ p_2"'] & (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\}) \arrow[d, "(q,q)"] \\
\mathscr{P}(\mathbf{a}) \arrow[r, "\Delta_{\mathscr{P}(\mathbf{a})}"'] & \mathscr{P}(\mathbf{a}) \times \mathscr{P}(\mathbf{a}).
\end{tikzcd}$$

The 1-morphism

$$(q|_{\mathbb{A}^n-\{0\}}, q|_{\mathbb{A}^n-\{0\}}) : (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\}) \to \mathscr{P}(\mathbf{a}) \times \mathscr{P}(\mathbf{a})$$

is surjective, flat and locally of finite presentation and the morphism $(a, p_2) : \mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\})$ is finite by 3.1.2.2. It follows from [**56**, Lemma 06TY] that $\Delta_{\mathscr{P}(\mathbf{a})}$ is finite. Now, one has that the diagonal $\Delta_{\mathscr{T}(\mathbf{a})}$ is just the base changes of $\Delta_{\mathscr{P}(\mathbf{a})}$ along the open immersion $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a})$, hence is finite by [**56**, Lemma 045C].

4. Recall that of finite presentation means quasi-compact, quasi-separated and locally of finite presentation. We have seen in (1) and (2) that $\mathscr{T}(\mathbf{a})$, $\mathscr{P}(\mathbf{a})$ and $\overline{\mathscr{P}(\mathbf{a})}$ are smooth, thus locally of finite presentation, and quasi-compact. By (3), the diagonals $\Delta_{\mathscr{T}(\mathbf{a})}, \Delta_{\mathscr{P}(\mathbf{a})}$ and $\Delta_{\overline{\mathscr{P}(\mathbf{a})}}$ are quasi-compact and separated, thus quasi-separated by [**56**, Lemma 050E], i.e. $\mathscr{T}(\mathbf{a}), \mathscr{P}(\mathbf{a})$ and $\overline{\mathscr{P}(\mathbf{a})}$ are quasi-separated. We deduce that $\mathscr{T}(\mathbf{a})$, $\mathscr{P}(\mathbf{a})$ and $\overline{\mathscr{P}(\mathbf{a})}$ are all of finite presentation.

□

Proposition 3.1.2.5. — *The stack $\mathscr{P}(\mathbf{a})$ is proper.*

Proof. — Recall that a proper 1-morphism is a 1-morphism which is of finite type, separated and universally closed. In Lemma 3.1.2.4, we have verified that $\mathscr{P}(\mathbf{a})$ is of finite presentation, hence of finite type [**56**, Lemma 06Q5], and of finite diagonal, hence separated.

We will now apply the valuative criterion for separated 1-morphisms with target locally Noetherian algebraic stacks to be universally closed [**56**, Lemma 0CQM], to the 1-morphism $\mathscr{P}(\mathbf{a}) \to \operatorname{Spec}\mathbb{Z}$. Let again R be a discrete valuation ring, let K be its field of fractions, let v_R be its valuation and π_R its uniformizer. We pick an object $\mathbf{x}$ in the groupoid $\mathscr{P}(\mathbf{a})(K)$ and we prove that there exists a finite extension K' of K and a valuation ring $R' \subset K'$ such that $\mathfrak{m}_R = \mathfrak{m}_{R'} \cap K$, where $\mathfrak{m}_R$ and $\mathfrak{m}_{R'}$ are maximal ideals of R and R' respectively, and such that the restriction $\mathbf{x}_{K'} \in \mathscr{P}(\mathbf{a})(K')$, is in the essential image of the functor

$$\mathscr{P}(\mathbf{a})(R') \to \mathscr{P}(\mathbf{a})(K').$$

Let $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_K \to (\mathbb{A}^n - \{0\})_K$ be the $(\mathbb{G}_m)_K$-equivariant morphism given by $\mathbf{x}$. We set $\ell := \operatorname{lcm}(\mathbf{a})$. Let us set $K' = K(\pi_R^{1/\ell})$ and let R' be the integral closure of R in K'. By [**56**, Lemma 09EV], one has that $R' = R[\pi_R^{1/\ell}]$, that R' is a discrete valuation ring, and that $\pi_R^{1/\ell}$ is a uniformizer of R'. Thus the maximal ideal of R' is given by $(\pi_R^{1/\ell})$ and its intersection with R is precisely the maximal ideal (π_R) of R. We extend canonically v_R to R' and K'. We set $k = -\min_j \frac{v_R(\widetilde{x}_j(1))}{a_j}$, where $\widetilde{x}_j(1)$ is the j-th coordinate of $\widetilde{\mathbf{x}}(1)$, so that $\ell k \in \mathbb{Z}$. Note that one has that

$$\pi_R^k \cdot \widetilde{\mathbf{x}}(1) = (\pi_R^{a_j k}\widetilde{x}_j(1))_j \in (\mathbb{A}^n - \{0\})(R'),$$

because for every index i one has

$$v_R(\pi_R^{a_i k}\widetilde{x}_i(1)) = -a_i \min_j \left(\frac{v_R(\widetilde{x}_j(1))}{a_j}\right) + a_i v_R(\widetilde{x}_i(1)) \geq 0$$

and because for the index i such that $\frac{\widetilde{x}_i(1)}{a_i}$ is minimal one has that

$$v_R(\pi_R^{a_i k}\widetilde{x}_i(1)) = 0.$$

We define a $(\mathbb{G}_m)_{R'}$-equivariant morphism by

$$\widetilde{\mathbf{z}} : (\mathbb{G}_m)_{R'} \to (\mathbb{A}^n - \{0\})_{R'} \qquad 1 \mapsto \pi_R^k \cdot \widetilde{\mathbf{x}}(1),$$

and $\widetilde{\mathbf{z}}$ defines a morphism $\mathbf{z} : \operatorname{Spec}(R') \to \mathscr{P}(\mathbf{a})$. One has that $\pi_R^k = (\pi_R^{1/\ell})^{k\ell} \in K'$ and thus π_R^k defines by the multiplication a morphism $(\mathbb{G}_m)_{K'} \to (\mathbb{G}_m)_{K'}$ which satisfies $\widetilde{\mathbf{z}} = \widetilde{\mathbf{x}} \circ \pi_R^k$. It follows that $\mathbf{z}_{K'}$ and $\mathbf{x}_{K'}$ are isomorphic. The valuative criterion is verified and $\mathscr{P}(\mathbf{a}) \to \operatorname{Spec}(\mathbb{Z})$ is universally closed. It follows that the algebraic stack $\mathscr{P}(\mathbf{a})$ is proper. $\square$

3.2. Models with enough integral points

In this section, we will define models of stacks which admit "enough integral points" in order to define unstable heights on stacks.

3.2.1. — In this paragraph we define models of stacks.

Definition 3.2.1.1. — *Let X be a finite presentation algebraic stack over a number field F and let $\mathcal{O}_F \subset A \subset F$ be a ring. A model of X over* $\operatorname{Spec}(A)$ *is a finite presentation A-algebraic stack $\mathscr{X}$ endowed with a 1-isomorphism $x : \mathscr{X}_F \xrightarrow{\sim} X$.*

A base change of a 1-morphism of finite presentation is of finite presentation [**56**, Lemma 06Q4]. We deduce that if $(\mathscr{X}, x : \mathscr{X}_F \xrightarrow{\sim} X)$ is a model of X over $\operatorname{Spec}(A)$, for some $\mathcal{O}_F \subset A \subset F$, then for every A' such that $A \subset A' \subset F$, one has that $(\mathscr{X}_{A'}, x : \mathscr{X}_F \xrightarrow{\sim} X)$ is a model of X. The model is unique in the following sense.

Lemma 3.2.1.2. — *Let X be a finite presentation F-algebraic stack. Let S_1 and S_2 be finite sets of finite places of F. Let $(\mathscr{Y}, y : \mathscr{Y}_F \xrightarrow{\sim} X)$ and $(\mathscr{Z}, z : \mathscr{Z}_F \xrightarrow{\sim} X)$ be models of X over $\mathcal{O}_{F,S_1}$ and $\mathcal{O}_{F,S_2}$, respectively. There exists a finite set $S \supset S_1 \cup S_2$ of finite places of F, a 1-isomorphism of stacks $f : \mathscr{Y}_{\mathcal{O}_{F,S}} \xrightarrow{\sim} \mathscr{Z}_{\mathcal{O}_{F,S}}$ and a 2-isomorphism $y \xrightarrow{\sim} z \circ f_F$.*

Proof. — Fix 1-inverses $y^{-1}: X \to \mathscr{Y}_F$ and $z^{-1}: X \to \mathscr{Z}_F$. We set $S_0 = S_1 \cup S_2$ and $T_0 = \operatorname{Spec}(\mathcal{O}_{F,S_0})$. For every finite subset $\Lambda \supset S_0$ of the set of finite places of F, we set $T_\Lambda = \operatorname{Spec}(\mathcal{O}_{F,\Lambda})$. The schemes T_Λ form an inverse system and

$$\varprojlim_{\Lambda} T_\Lambda = \operatorname{Spec}(\varinjlim_{\Lambda} \mathcal{O}_{F,\Lambda}) = \operatorname{Spec}(F).$$

Set $Y_0 := \mathscr{Y}_{\mathcal{O}_{F,S_0}}$ and $Z_0 := \mathscr{Z}_{\mathcal{O}_{F,S_0}}$, and for finite subset $\Lambda \supset S_0$ of finite places of F, we set $Y_\Lambda := Y_0 \times_{T_0} T_\Lambda$ and $Z_\Lambda := Z_0 \times_{T_0} T_\Lambda$. Note that by the definition of the model and by the fact that the base change of finite presentation 1-morphism is of finite presentation, the stack Y_0 is quasi-compact and quasi-separated and the stack Z_0 is locally of finite presentation. We have verified the conditions of [51, Proposition B2]. It follows that there exists finite subset $S' \supset S_0$ of the set of finite places of F, a 1-morphism of stacks $f' : Y_{S'} \to Z_{S'}$ and a 2-isomorphism $f'_F \xrightarrow{\sim} z^{-1} \circ y$. Hence, there exists a 2-isomorphism $y \xrightarrow{\sim} z \circ f'_F$. For every finite subset $\Lambda \supset S_0$, we set $f'_\Lambda : \mathscr{Y}_{\mathcal{O}_{F,\Lambda}} = Y_\Lambda \to Z_\Lambda = \mathscr{Z}_{\mathcal{O}_{F,\Lambda}}$ for the base change morphism $f' \times_{\mathcal{O}_{F,S'}} \mathcal{O}_{F,\Lambda}$. For every finite subset $\Lambda \supset S_0$ of the set of finite places of F, the stacks Y_Λ and Z_Λ are of finite presentation, thus by [51, Proposition B3], there exists Λ big enough such that f'_Λ is a 1-isomorphism. We set $S = \Lambda$ and $f = f'_\Lambda$. One clearly has that $f'_F = f_F$, thus there exists a 2-isomorphism $y \xrightarrow{\sim} z \circ f_F$. The statement follows. □

Example 3.2.1.3. — The pair $(\mathscr{P}(\mathbf{a})_{\mathcal{O}_F}, \mathrm{Id}_{\mathscr{P}(\mathbf{a})_F})$ is a model over $\operatorname{Spec}(\mathcal{O}_F)$ of the stack $\mathscr{P}(\mathbf{a})_F = (\mathbb{A}^n - \{0\})_F/(\mathbb{G}_m)_F$. Indeed, it follows from 3.1 that

$$\mathscr{P}(\mathbf{a})_F = (\mathbb{A}^n - \{0\})_F/(\mathbb{G}_m)_F = ((\mathbb{A}^n - \{0\})_{\mathcal{O}_F}/(\mathbb{G}_m)_{\mathcal{O}_F})_F = \mathscr{P}(\mathbf{a})_F$$

and from Lemma 3.1.2.4 and from [56, Lemma 06Q4] that $\mathscr{P}(\mathbf{a})_{\mathcal{O}_F} = \mathscr{P}(\mathbf{a}) \times_{\mathbb{Z}} \mathcal{O}_F$ is of finite presentation. An analogous argument shows that $\overline{\mathscr{P}(\mathbf{a})}_F = \mathbb{A}^n_F/(\mathbb{G}_m)_F$ admits a model $(\overline{\mathscr{P}(\mathbf{a})}, \mathrm{Id}_{\overline{\mathscr{P}(\mathbf{a})}_F})$ over $\operatorname{Spec}(\mathbb{Z})$.

3.2.2. — We propose the following definitions to have sufficiently $\mathcal{O}_v$-integral points to define unstable heights on stacks.

Definition 3.2.2.1. — *Let v be a finite place of F and let $\mathcal{O}_F \subset A \subset \mathcal{O}_v$ be a ring. Let X be a finite presentation A-algebraic stack. We say that X has enough $\mathcal{O}_v$-integral points if the canonical functor $X(\mathcal{O}_v) \to X(F_v)$ is essentially surjective.*

Definition 3.2.2.2. — *Let X be an F-algebraic stack of finite presentation. Let S be a finite set of finite places of F and $(\mathscr{X}, x)$ be a model of X. We say that $(\mathscr{X}, x)$ has enough integral points, if for every finite place v of F which is not in S, the stack $\mathscr{X}_{\mathcal{O}_v}$ has enough $\mathcal{O}_v$-integral points.*

Note that in the situation of Definition 3.2.2.2, the property "has enough integral points" is in fact a property of $\mathscr{X}$. It follows that if $(\mathscr{Y}, y)$ is another model of X such that there exists an equivalence $\mathscr{X} \xrightarrow{\sim} \mathscr{Y}$, then $(\mathscr{Y}, y)$ admits has enough integral points. For v not in S, every F_v-point "extends" to an $\mathcal{O}_v$-point of $\mathscr{X}$ in the following sense: the functor

$$\mathscr{X}(\mathcal{O}_v) = \mathscr{X}_{\mathcal{O}_v}(\mathcal{O}_v) \to \mathscr{X}(F_v) = \mathscr{X}_{F_v}(F_v) = \mathscr{X}_F(F_v) \xrightarrow{x(F_v)} X_{F_v}(F_v) = X(F_v)$$

is essentially surjective (this follows from the fact that $x(F_v) : \mathscr{X}_F(F_v) \to X_{F_v}(F_v)$ is an equivalence).

Lemma 3.2.2.3. — *Suppose for some index i, one has $a_i > 1$. Let v be a finite place of F. The $\mathcal{O}_v$-stack $\mathscr{P}(\mathbf{a})_{\mathcal{O}_v}$ does not have enough $\mathcal{O}_v$-integral points.*

Proof. — We prove that the point $\mathbf{x} : q^{\mathbf{a}}_{F_v}(F_v)(\pi_v, \dots, \pi_v) \in \mathscr{P}(\mathbf{a})_{F_v}(F_v)$ is not in the essential image of the canonical functor $\mathscr{P}(\mathbf{a})_{\mathcal{O}_v}(\mathcal{O}_v) \to \mathscr{P}(\mathbf{a})_{F_v}(F_v)$. The group scheme $\mathbb{G}_m$ is special and let $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_{F_v} : (\mathbb{G}_m)_{F_v} \to (\mathbb{A}^n - \{0\})_{F_v}$ be the $(\mathbb{G}_m)_{F_v}$-equivariant morphism defined by $\mathbf{x}$. If $\mathbf{y} \in \mathscr{P}(\mathbf{a})_{F_v}(F_v)$, an isomorphism $\mathbf{x} \xrightarrow{\sim} \mathbf{y}$ is given by an element $t \in \mathbb{G}_m(F_v)$ such that $\widetilde{\mathbf{x}} = \widetilde{\mathbf{y}} \circ t$, where $\widetilde{\mathbf{y}} : (\mathbb{G}_m)_{F_v} \to (\mathbb{A}^n - \{0\})_{F_v}$ is the $(\mathbb{G}_m)_{F_v}$-equivariant morphism given by $\mathbf{y}$ and t is seen as a morphism $(\mathbb{G}_m)_{F_v} \to (\mathbb{G}_m)_{F_v}$ by multiplication. It follows that if $\mathbf{y}$ is isomorphic to $\mathbf{x}$, then

$$\begin{aligned}\widetilde{\mathbf{y}}(1) \in \{\widetilde{\mathbf{x}}(t) | t \in \mathbb{G}_m(F_v)\} &= \{t \cdot \widetilde{\mathbf{x}}(1) | t \in \mathbb{G}_m(F_v)\} \\ &= \{(t^{a_j}\pi_v)_j | t \in \mathbb{G}_m(F_v)\}.\end{aligned}$$

On the other side, if $\mathbf{y}$ is the image of an $\mathcal{O}_v$-point for the canonical morphism $\mathscr{P}(\mathbf{a})_{\mathcal{O}_v}(\mathcal{O}_v) \to \mathscr{P}(\mathbf{a})_{F_v}(F_v)$, it follows that $\widetilde{\mathbf{y}}$ extends to a $(\mathbb{G}_m)_{\mathcal{O}_v}$-equivariant morphism $(\mathbb{G}_m)_{\mathcal{O}_v} \to (\mathbb{A}^n - \{0\})_{\mathcal{O}_v}$ and in particular that $\widetilde{\mathbf{y}}(1) \in (\mathbb{A}^n - \{0\})_{\mathcal{O}_v}(\mathcal{O}_v)$. We will show that the sets

$$(\mathbb{A}^n - \{0\})_{\mathcal{O}_v}(\mathcal{O}_v) = \{(z_j)_j \in \mathcal{O}_v^n | \exists j : v(z_j) = 0\}$$

and

$$\{(t^{a_j}\pi_v)_j | t \in \mathbb{G}_m(F_v)\}$$

are disjoint. Suppose that $(t^{a_1}\pi_v, \dots, t^{a_n}\pi_v) \in (\mathbb{A}^n - \{0\})_{\mathcal{O}_v}(\mathcal{O}_v)$ for some $t \in \mathbb{G}_m(F_v)$. One has $v(t^{a_i}\pi_v) = a_i v(t) + 1 \geq 0$ and as $a_i > 1$, we deduce $v(t) \geq 0$. Now for every index j one has $v(t^{a_j}\pi_v) = a_j v(t) + 1 > 0$, a contradiction with the assumption that $(t^{a_1}\pi_v, \dots, t^{a_n}\pi_v) \in (\mathbb{A}^n - \{0\})(\mathcal{O}_v)$. We deduce that $\mathbf{x}$ is not in the essential image of $\mathscr{P}(\mathbf{a})_{\mathcal{O}_v}(\mathcal{O}_v) \to \mathscr{P}(\mathbf{a})_{F_v}(F_v)$ and consequently $\mathscr{P}(\mathbf{a})_{\mathcal{O}_v}$ does not have enough $\mathcal{O}_v$-points. □

Corollary 3.2.2.4. — *Suppose for some index i, one has $a_i > 1$. For any finite subset S of the set of the finite places of F, there exists no model $(\mathscr{X}, x)$ of $\mathscr{P}(\mathbf{a})_F$ over $\operatorname{Spec}(\mathcal{O}_{F,S})$ such that the following condition is satisfied: there exists a finite place $v \notin S$ such that the stack $\mathscr{X}$ has enough $\mathcal{O}_v$-points.*

Proof. — Suppose there exists finite set S of finite places of F and a model $(\mathscr{X}, x)$ of $\mathscr{P}(\mathbf{a})_F$ such that for every finite place $v \notin S$ one has that $\mathscr{X}$ has enough $\mathcal{O}_v$-integral points. By Lemma 3.2.1.2, we can increase S if needed and find a 1-isomorphism $f : \mathscr{X} \xrightarrow{\sim} \mathscr{P}(\mathbf{a})_{\mathcal{O}_{F,S}}$ and a 2-isomorphism $x \xrightarrow{\sim} \mathrm{Id}_{\mathscr{P}(\mathbf{a})_F} \circ f_F = f_F$. Let $f^{-1} : \mathscr{P}(\mathbf{a})_{\mathcal{O}_{F,S}} \to \mathscr{X}$ be an inverse to f. By Lemma 3.2.2.3, for every finite $v \notin S$, one has that

$$\mathscr{X}(\mathcal{O}_v) \xrightarrow{f(\mathcal{O}_v)} \mathscr{P}(\mathbf{a})_{\mathcal{O}_{F,S}}(\mathcal{O}_v) = \mathscr{P}(\mathbf{a})_{\mathcal{O}_v}(\mathcal{O}_v) \to \mathscr{P}(\mathbf{a})_F(F) \xrightarrow{f_F^{-1}(F)} \mathscr{X}_F(F)$$

is not essentially surjective. We obtain a contradiction and the claim follows □

Proposition 3.2.2.5. — *The stack $\overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_F} = \mathbb{A}^n_{\mathcal{O}_F}/(\mathbb{G}_m)_{\mathcal{O}_F}$ is a model of $\overline{\mathscr{P}(\mathbf{a})_F}$ which for every $v \in M_F^0$ has enough $\mathcal{O}_v$-integral points.*

Proof. — Let $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}_{F_v}(F_v)$ and let $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_{F_v} \to \mathbb{A}^n_{F_v}$ be the $(\mathbb{G}_m)_{F_v}$-equivariant morphism defined by $\mathbf{x}$. Let $v \in M_F^0$. By the fact that all a_j are positive, there exists $k \in \mathbb{Z}$ such that for every $j = 1, \dots, n$ one has that $v(\widetilde{x}_j(1) + a_j k) > 0$. The $(\mathbb{G}_m)_{F_v}$-equivariant morphism given by

$$(\mathbb{G}_m)_{F_v} \to \mathbb{A}^n_{F_v} \qquad 1 \mapsto \pi_v^k \cdot \widetilde{\mathbf{x}}(1)$$

is isomorphic to $\widetilde{\mathbf{x}}$ and is the base change of the $(\mathbb{G}_m)_{\mathcal{O}_v}$-equivariant morphism

$$\widetilde{\mathbf{x}}_{\mathcal{O}_v} : (\mathbb{G}_m)_{\mathcal{O}_v} \to \mathbb{A}^n_{\mathcal{O}_v} \qquad 1 \mapsto \pi_v^k \cdot \widetilde{\mathbf{x}}(1)$$

along $\operatorname{Spec} F_v \to \operatorname{Spec} \mathcal{O}_v$. The $(\mathbb{G}_m)_{\mathcal{O}_v}$-equivariant morphism defines a morphism $\mathbf{x}_{\mathcal{O}_v} : \operatorname{Spec} \mathcal{O}_v \to \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}$. By construction we have $\mathbf{x}_{\mathcal{O}_v} \times_{\mathbb{Z}} F_v \cong \mathbf{x}$. $\square$

The stacks $\overline{\mathscr{P}(\mathbf{a})}$ are not proper because they are not separated as the following lemma shows.

Lemma 3.2.2.6. — *Let v be a finite place of F. Let $\mathcal{O}_F \subset A \subset \mathcal{O}_v$ be a ring. The canonical morphism $\overline{\mathscr{P}(\mathbf{a})}_A \to \operatorname{Spec}(A)$ is not separated.*

Proof. — By the fact that property of being separated is stable for a base changes, one can assume that $A = \mathcal{O}_v$. We will verify that the diagonal $\Delta_{\overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}} : \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v} \to \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v} \times_{\mathcal{O}_v} \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}$ is not proper. The diagram

$$\begin{tikzcd}
(\mathbb{G}_m)_{\mathcal{O}_v} \times \mathbb{A}^n_{\mathcal{O}_v} \arrow[r, "(a,p_2)_{\mathcal{O}_v}"] \arrow[d, "(q\circ p_2)_{\mathcal{O}_v}"'] & \mathbb{A}^n_{\mathcal{O}_v} \times \mathbb{A}^n_{\mathcal{O}_v} \arrow[d, "(q,q)_{\mathcal{O}_v}"'] \\
\overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v} \arrow[r, "\Delta_{\overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}}"'] & \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v} \times_{\mathcal{O}_v} \overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}.
\end{tikzcd}$$

is 2-commutative 2-cartesian. By the fact that being proper is stable for a base change, it suffices to see that $(a, p_2)_{\mathcal{O}_v} : (\mathbb{G}_m)_{\mathcal{O}_v} \times \mathbb{A}^n_{\mathcal{O}_v} \to \mathbb{A}^n_{\mathcal{O}_v} \times \mathbb{A}^n_{\mathcal{O}_v}$ is not proper. We verify that the valuative criterion of properness [**56**, Lemma 0BX5] is not satisfied for the finite type and quasi-separated morphism $(a, p_2)_{\mathcal{O}_v}$. The diagram

$$\begin{tikzcd}
\operatorname{Spec}(F_v) \arrow[r, "(\pi_v,(1)_j)"] \arrow[d] & (\mathbb{G}_m)_{\mathcal{O}_v} \times_{\mathcal{O}_v} \mathbb{A}^n_{\mathcal{O}_v} \arrow[d, "(a,p_2)_{\mathcal{O}_v}"] \\
\operatorname{Spec}(\mathcal{O}_v) \arrow[r, "((\pi_v^{a_j})_j,(1)_j)"'] & \mathbb{A}^n_{\mathcal{O}_v} \times_{\mathcal{O}_v} \mathbb{A}^n_{\mathcal{O}_v}
\end{tikzcd}$$

does not admit admit an arrow $\operatorname{Spec}(\mathcal{O}_v) \to (\mathbb{G}_m)_{\mathcal{O}_v} \times_{\mathcal{O}_v} \mathbb{A}^n_{\mathcal{O}_v}$ so that the diagram commutes. Indeed if $(t, \mathbf{x}) : \operatorname{Spec}(\mathcal{O}_v) \to (\mathbb{G}_m)_{\mathcal{O}_v} \times_{\mathcal{O}_v} \mathbb{A}^n_{\mathcal{O}_v}$ was a such an arrow, then $\mathbf{x} = (1)_j$ and $v(t) = 0$. One has that $a \circ (t, (1)_j) = (\pi_v^{a_j})_j$, thus $v(t^{a_1}) = a_1 v(t) = 0 \neq a_1 = v(\pi_v^{a_1})$, a contradiction. It follows that $(a, p_2)_{\mathcal{O}_v}$ is not proper, and hence that $\Delta_{\overline{\mathscr{P}(\mathbf{a})}_{\mathcal{O}_v}}$ is not proper, i.e. that $\overline{\mathscr{P}(\mathbf{a})}$ is not separated. $\square$

3.3. Topology on R-points of stacks

We recall a definition, originally due to Moret-Bailly in [**40**], of a topology that one can put on R-points of a stack. Let R be a local topological ring that satisfies the following conditions:

(a) the group of units $\mathbb{G}_m(R)$ is open in R,

(b) the inverse map $\mathbb{G}_m(R) \xrightarrow{x \mapsto x^{-1}} \mathbb{G}_m(R)$ is continuous, when $\mathbb{G}_m(R) \subset R$ is endowed with the subspace topology.

We will call such a ring "topologically suitable". The principal examples are $R = F_v$ for $v \in M_F$ or $R = \mathcal{O}_v$ for $v \in M_F^0$.

3.3.1. — The following proposition is given in [**21**]. We consider schemes that are locally of finite type over a suitable ring R.

Proposition 3.3.1.1 **(Conrad,** [**21**, Proposition 3.1]**)**

Let R be a topologically suitable ring. There exists a unique way to topologize $Y(R)$ for every scheme Y locally of finite type over R subject to the requirements of functoriality, carrying closed (open) immersions into embeddings (open embeddings) of topological spaces, compatibility with fiber products, and giving $Y(R)$ the usual topology when Y is the affine line over R. One also has that if Y is separated and R is Hausdorff, then $Y(R)$ is Hausdorff and that if R is Hausdorff and locally compact, then $Y(R)$ is locally compact.

The following suffices to make $Y(R)$ compact.

Lemma 3.3.1.2. — *Suppose R is a topologically suitable ring.*

1. *Suppose R is compact. If Y is an R-scheme of finite type, then $Y(R)$ is compact.*
2. (Conrad, [**21**, Corollary 5.7]) *Suppose R is a local field. If Y is a proper R-scheme, then $Y(R)$ is compact.*

Proof. — We prove (1). Take a finite Zariski open covering $\{U_i\}_i$ of Y with U_i affine. Every affine scheme U_i is a closed subscheme of an affine space $\mathbb{A}^{n_i}$. We deduce that $U_i(R)$ is a closed subset of a compact set $\mathbb{A}^{n_i}(R) = R^{n_i}$, hence is compact. Now, the sets $\{U_i(R)\}_i$ cover $Y(R)$, because R is local, and thus $Y(R)$ is compact. □

A direct consequence of Proposition 3.3.1.1 is the following corollary.

Corollary 3.3.1.3. — *Let R be a topologically suitable locally compact Hausdorff ring. Suppose G is a locally of finite type algebraic group. Then $G(R)$ is locally compact group. If G is commutative, then $G(R)$ is commutative. If $a : G \times Y \to Y$ is an action to the left of G on a* *locally of finite type scheme Y, then $a(R) : G(R) \times Y(R) \to Y(R)$ is a continuous action of $G(R)$ on $Y(R)$.*

3.3.2. — As in the case of schemes, we work only with stacks that are locally of finite type. If X is an algebraic stack and R a ring by $[X(R)]$ we denote the set of isomorphism classes of objects in the groupoid $X(R)$. For $x \in X(R)$ we denote by $[x]$ its image in $[X(R)]$. The following definition is firstly given by Moret-Bailly in [**40**, Definition 2.2] for stacks with separated and quasi-compact diagonals. Česnavičius gives it for stacks without such hypothesis.

Definition 3.3.2.1 **(Česnavičius,**[**15**, Section 2.4]**)**

Let R be a topologically suitable ring. Let X be a locally of finite type R-algebraic stack of separated diagonal. We endow $[X(R)]$ with the finest topology such that for every 1-morphism $f : Y \to X$, with Y a locally of finite type R-scheme, the maps $[f(R)]$ are continuous.

The following lemma follows from properties of the finest topology.

Lemma 3.3.2.2. — *A subset $U \subset [X(R)]$ is open if and only if for every 1-morphism $f : Y \to X$ of algebraic stacks, with Y locally of finite type R-scheme, the preimage $[f(R)]^{-1}(U)$ is open in $Y(R)$. Let T be a topological space. A map $h : X(R) \to T$ is continuous if and only if for every 1-morphism $g : Y \to X$ of algebraic stacks, with Y locally of finite type R-scheme, the composite map $h \circ g(R)$ is continuous.*

Proof. — Those are consequences of [**7**, Chapter I, §2, n^0 4, Proposition 4] and [**7**, Chapter I, §2, n^0 4, Proposition 6]. □

We recall some of properties which are proven in [**15**] and which we are going to use.

Proposition 3.3.2.3 **(Česnavičius,**[**15**, Corollary 2.7]**)**

Let $f : X \to W$ be a 1-morphism of R-stacks that are locally of finite type, where R is a topologically suitable ring.

1. *The induced map on R-points $[f(R)] : [X(R)] \to [W(R)]$ is continuous.*
2. *Suppose f is an open immersion. Then $[f(R)] : [X(R)] \to [W(R)]$ is an open immersion.*
3. *Suppose R is Hausdorff and f is a closed immersion. Then the map $[f(R)] : [X(R)] \to [W(R)]$ is a closed immersion.*
4. *Let R' another topologically suitable ring. Let $h : R \to R'$ a continuous ring homomorphism. The canonical map $[X(R)] \to [X(R')]$ is continuous.*

3.3.3. — Let us study the topological spaces $[(Y/G)(R)]$, when the algebraic group G is special (see 3.1).

Proposition 3.3.3.1. — *Let R be a topologically suitable ring. Let $G=(G,m,e)$ be a flat locally of finite presentation R-algebraic group. Let Y be a locally of finite type R-scheme endowed with an action of G and let $\pi : Y \to Y/G$ be the quotient morphism.*

1. *The map $[\pi(R)] : Y(R) \to [(Y/G)(R)]$ is $G(R)$-invariant and continuous.*
2. *Assume G is special. The map $[\pi(R)]$ is surjective and open. The canonical continuous map*

$$\overline{[\pi(R)]} : Y(R)/G(R) \to [(Y/G)(R)]$$

induced from $G(R)$-invariant map $[\pi(R)]$ is a homeomorphism.

Proof. — 1. We prove that $[\pi(R)]$ is continuous and $G(R)$-invariant.

– The fact that the map $[\pi(R)]$ is continuous follows from the functoriality (Claim (1) in Proposition 3.3.2.3).

– By Proposition 3.1.1.2, the diagram

$$\begin{tikzcd} G \times_R Y \arrow[r, "a"] \arrow[d, "p_2"] & Y \arrow[d, "\pi"'] \\ Y \arrow[r, "\pi"] & (Y/G), \end{tikzcd}$$

where a is the action, is 2-commutative. It follows that if $(g,x) \in (G \times_R X)(R)$, then

$$[\pi(R)](a(R)(g,x)) = [\pi(R)](p_2(R)(g,x)) = [\pi(R)(x)].$$

Thus $[\pi(R)]$ is $G(R)$-invariant.

2. We assume that G is special.

– We establish that $[\pi(R)]$ is surjective. Let $x : \operatorname{Spec} R \to Y/G$ be a 1-morphism of algebraic stacks. As G is special, by Lemma 3.1.1.7, one has the following 2-commutative diagram

$$\begin{tikzcd} G_R \arrow[r, "\widetilde{x}"] \arrow[d] & Y \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r, "x"] & Y/G. \end{tikzcd}$$

By 2-commutativity, it follows that $\operatorname{Spec} R \xrightarrow{\pi\circ\widetilde{x}\circ e_R} Y/G$ and $\operatorname{Spec} R \xrightarrow{x} Y/G$ are 2-isomorphic. We deduce that $[x]$ is the image of $[\pi(R)(\widetilde{x}(e_R))]$ and it follows that $[\pi(R)]$ is surjective.

- Let us verify that $\overline{[\pi(R)]}$ is a bijection. Denote by q the quotient map $Y(R) \to Y(R)/G(R)$. One has that $[\pi(R)] = \overline{[\pi(R)]} \circ q$. The map $\overline{[\pi(R)]}$ is surjective because $[\pi(R)]$ is surjective. Let us verify that $\overline{[\pi(R)]}$ is injective. Suppose x, y are such that $[\overline{\pi(R)}(x)] = [\overline{\pi(R)}(y)]$. Let $x', y' \in Y(R)$ be lifts of x, y, respectively. We have that $[\pi(R)](x') = [\pi(R)](y')$, hence, $\pi(R)(x')$ and $\pi(R)(y')$ are isomorphic in the groupoid $(Y/G)(R)$. This means precisely that there exists $g \in G(R)$ such that the G_R-equivariant morphisms $\widetilde{x} : G_R \to Y, e_R \mapsto x'$ and $\widetilde{y} : G_R \to Y, e_R \mapsto y'$ satisfy $\widetilde{x} = \widetilde{y} \circ g$, where $g \in G(R)$ is seen as a morphism $G_R \to G_R$ via the left multiplication. As $\widetilde{y}$ is G_R-equivariant, we deduce $x' = \widetilde{x}(e_R) = g \cdot \widetilde{y}(e_R) = g \cdot y'$. This means that $x = q(x') = q(g \cdot y') = q(y') = y$. It follows that $\overline{[\pi(R)]}$ is injective and hence bijective.
- We establish that the map $[\pi(R)]$ is open. Let $V \subset Y(R)$ be an open subset, we are going to prove that $[\pi(R)](V)$ is open in $[(Y/G)(R)]$. By Definition 3.3.2.1, we need to establish that if $s : W \to Y/G$ is a 1-morphism of stacks with W a scheme, then $[s(R)]^{-1}([\pi(R)](V))$ is open in $W(R)$. Set $\widetilde{W} := W \times_{Y/G} Y$ and set $\widetilde{s} : \widetilde{W} \to Y$ to be the base change morphism. The following diagram is commutative:

$$\begin{tikzcd}
\widetilde{W}(R) \arrow[r, "\widetilde{s}(R)"] \arrow[d, "\pi_W(R)"'] & Y(R) \arrow[r, "q"] \arrow[d, "{[\pi(R)]}"] & Y(R)/G(R) \arrow[dl, "\overline{[\pi(R)]}"] \\
W(R) \arrow[r, "s(R)"'] & {[(Y/G)(R)]} &
\end{tikzcd}$$

The morphism $\pi_Y : \widetilde{W} \to W$ is a G-torsor, hence, as G is special, it is locally Zariski trivial on W. Let $\cup_{i\in I} U_i$ be an open covering of W, such that for all i, the morphism $\pi_W|_{\pi_W^{-1}(U_i)}$ is a trivial G-torsor. For all i, the map $\pi_i : \pi(W)^{-1}(U_i)(R) \to U_i(R)$ decomposes as $\pi(W)^{-1}(U_i)(R) \xrightarrow{\sim} U_i(R) \times G(R) \to U_i(R)$, where the first morphism comes from an isomorphism of G-torsors $\pi(W)^{-1}(U_i) \xrightarrow{\sim} U_i \times G$ and the second map is the projection, and, hence, π_i is open and surjective. As $\cup_{i\in I}U_i(R)$ is a covering of $W(R)$, the map $\pi_W(R)$ is open and surjective. We have that

$$\begin{aligned}\pi_W(R)^{-1}(s(R)^{-1}([\pi(R)](V))) &= \widetilde{s}(R)^{-1}([\pi(R)]^{-1}([\pi(R)](V))) \\ &= \widetilde{s}(R)^{-1}(q^{-1}(q(V))),\end{aligned}$$

where the last equality follows from the fact that $\overline{[\pi(R)]}$ is a bijection. It follows that $\pi_W(R)^{-1}(s(R)^{-1}([\pi(R)](V)))$ is open in $\widetilde{W}(R)$, as q is open and continuous and $\widetilde{s}(R)$ is continuous. Finally, we get that

$$\pi_W(R)(\pi_W(R)^{-1}(s(R)^{-1}([\pi(R)](V)))) = [s(R)]^{-1}([\pi(R)])$$

is open in $W(R)$. We deduce that $[\pi(R)](V)$ and hence $[\pi(R)]$ are open. As $\underline{[\pi(R)]}$ is open and continuous bijection, it follows that $\overline{[\pi(R)]}$ is a homeomorphism. The statement is now proven.

□

By Hilbert 90 theorem, the algebraic group $\mathbb{G}_m$ is special. We can establish that:

Corollary 3.3.3.2. — *Let R be a topologically suitable ring.*

1. *The map $(\mathbb{A}^n - \{0\})(R) \to [\mathscr{P}(\mathbf{a})(R)]$ is $\mathbb{G}_m(R)$-invariant, continuous and open, and the induced map*

$$(\mathbb{A}^n - \{0\})(R)/\mathbb{G}_m(R) \xrightarrow{\sim} [\mathscr{P}(\mathbf{a})(R)]$$

is a homeomorphism.

2. *The map*

$$\mathbb{G}_m^n(R) = (\mathbb{A}^1 - \{0\})^n(R) \to [\mathscr{T}(\mathbf{a})(R)]$$

is a $\mathbb{G}_m(R)$-invariant, continuous and open map and the induced map

$$\mathbb{G}_m^n(R)/\mathbb{G}_m(R) \to [\mathscr{T}(\mathbf{a})(R)] \tag{3.3.3.3}$$

is a homeomorphism.

3. *The inclusion $[\mathscr{T}(\mathbf{a})(R)] \subset [\mathscr{P}(\mathbf{a})(R)]$ is an open embedding.*

Proof. — The first two claims are direct consequences of Proposition 3.3.3.1. The last claim is a consequence of Proposition 3.3.2.3. □

3.3.4. — To say more about spaces $[\mathscr{T}(\mathbf{a})(R)]$ and $[\mathscr{P}(R)]$, we will need additional assumptions on R. In [**15**, Section 2.12], Česnavičius defines a notion of proper-closed ring: it is a topologically suitable ring R such that for every proper morphism $f : X \to Y$ of finite type R-schemes, the induced continuous map $f(R) : X(R) \to Y(R)$ is closed. The following examples are presented: a local field or the ring of integers $\mathcal{O}_v$ in the completion F_v for a finite place v.

Proposition 3.3.4.1. — *Let R be a proper-closed integral domain. (e.g. the completion F_v for some $v \in M_F$ or the ring of integers $\mathcal{O}_v$ in the completion F_v for some $v \in M_F^0$).*

1. *The topological actions $\mathbb{G}_m(R) \times (\mathbb{A}^n - \{0\})(R) \to (\mathbb{A}^n - \{0\})(R)$ and $\mathbb{G}_m(R) \times \mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)$, deduced from the actions $\mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to (\mathbb{A}^n - \{0\})$ and $\mathbb{G}_m \times \mathbb{G}_m^n \to \mathbb{G}_m^n$ by Corollary 3.3.1.3, are proper.*
2. *The map $\mathbb{G}_m(R) \to \mathbb{G}_m^n(R), t \mapsto (t^{a_j})_j$ is proper, and the subgroup $\mathbb{G}_m(R)_{\mathbf{a}} = \{(t^{a_j})_j | t \in \mathbb{G}_m(R)\}$ is a closed subgroup of $\mathbb{G}_m^n(R)$. The canonical map $\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}} = [\mathscr{T}(\mathbf{a})(R)]$ is $\mathbb{G}_m(R)$-invariant, continuous, open and surjective, and the induced map*
$$\mathbb{G}_m^n(R)/\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}} = [\mathscr{T}(\mathbf{a})(R)] \tag{3.3.4.2}$$
is a homeomorphism.
3. *Suppose R is locally compact (e.g. the completion F_v for some $v \in M_F$ or the ring of integers $\mathcal{O}_v$ in the completion F_v for some $v \in M_F^0$), then $[\mathscr{P}(\mathbf{a})(R)]$ and $[\mathscr{T}(\mathbf{a})(R)]$ are locally compact and Hausdorff.*

Proof. — 1. As R is proper-closed and as $\mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to (\mathbb{A}^n - \{0\}) \times (\mathbb{A}^n - \{0\})$ is proper (Lemma 3.1.2.2), it follows that the map
$$\mathbb{G}_m(R) \times (\mathbb{A}^n - \{0\})(R) \to (\mathbb{A}^n - \{0\})(R) \times (\mathbb{A}^n - \{0\})(R) \tag{3.3.4.3}$$
is closed. Let us verify that it's fibers are finite. Suppose that $(t, \mathbf{x})$ is a preimage of $(\mathbf{y}, \mathbf{z})$. This means that $\mathbf{x} = \mathbf{z}$ and that $t \cdot \mathbf{x} = \mathbf{y}$. Let i be an index such that $x_i \neq 0$. As R is an integral domain, there are only finitely many elements $t \in R$ for which $t^{a_i} x_i = z_i$. We deduce that the map (3.3.4.3) has finite fibers. Now, it follows from [**7**, Chapter III, §10, n^0 2, Theorem 1], that the map (3.3.4.3) is proper. We deduce from [**7**, Chapter III, §4, n^0 1, Example 2], that the restriction of the action of $\mathbb{G}_m(R)$ to the $\mathbb{G}_m(R)$-invariant subset $\mathbb{G}_m^n(R) \subset (\mathbb{A}^n - \{0\})(R)$ is proper. The claim is proven.

2. By (1) the action of $\mathbb{G}_m(R)$ on $\mathbb{G}_m^n(R)$ is proper. Thus by [7, Chapter III, §4, n^0 2, Proposition 4], the induced map $t \mapsto t \cdot (1)_j = (t^{a_j})_j$ is proper and its image $\mathbb{G}_m(R)_{\mathbf{a}}$ is closed in $\mathbb{G}_m^n(R)$. The map is $\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$ is continuous, open and surjective, because it is a quotient map. It follows that the induced map $[\mathscr{T}(\mathbf{a})(R)] = \mathbb{G}_m^n(R)/\mathbb{G}_m(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$ is continuous, open and surjective. Moreover, note that for $t \in \mathbb{G}_m(R)$ and $\mathbf{x} \in \mathbb{G}_m^n(R)$ one has that the image of $t \cdot \mathbf{x} = (t^{a_j})_j \mathbf{x}$ in $\mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$ coincides with the image of $\mathbf{x}$ in $\mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$. Observe that if $\mathbf{x}, \mathbf{y} \in \mathbb{G}_m^n(R)$ have the same image in $\mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$, then there exists $(t^{a_j})_j \in \mathbb{G}_m(R)_{\mathbf{a}}$ such that $(t^{a_j})_j \mathbf{x} = \mathbf{y}$, hence $t \cdot \mathbf{x} = \mathbf{y}$. It follows that the induced map $\mathbb{G}_m^n(R)/\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$ is injective. We deduce that it is a homeomorphism.
3. The action of $\mathbb{G}_m(R)$ on $(\mathbb{A}^n - \{0\})(R)$ and $\mathbb{G}_m^n(R)$ is proper, hence the spaces $[\mathscr{P}(\mathbf{a})(R)]$ and $[\mathscr{T}(\mathbf{a})(R)]$ are Hausdorff by [7, Chapter III, §4, n^0 2, Proposition 3]. It follows from Proposition 3.3.1.1 that the spaces $(\mathbb{A}^n - \{0\})(R)$ and $\mathbb{G}_m^n(R)$ are locally compact. Now, by [7, Chapter III, §4, n^0 5, Proposition 9] imply that $[\mathscr{P}(\mathbf{a})(R)]$ and $[\mathscr{T}(\mathbf{a})(R)]$ are locally compact.

$\square$

We finish the paragraph by establishing that $[\mathscr{P}(\mathbf{a})(\mathcal{O}_v)]$, $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ and $[\mathscr{P}(F_v)]$ are compact and that $[\mathscr{T}(\mathbf{a})(F_v)]$ is paracompact. First, we prove the following lemma.

Lemma 3.3.4.4. — *Let v be a place of F. If v is finite, we define*

$$\mathcal{D}_v^{\mathbf{a}} := (\mathcal{O}_v)^n - (\pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v)$$

and if v is infinite we define

$$\mathcal{D}_v^{\mathbf{a}} := \{\mathbf{x} \in F_v^n |\ ||\mathbf{x}||_{\max} = 1\},$$

where $||\mathbf{x}||_{\max} = \max_j(|x_j|_v)$.

1. *Suppose v is finite. The set $\mathcal{D}_v^{\mathbf{a}}$ is an open, a closed and a compact subset of $F_v^n - \{0\}$.*
2. *Suppose v is finite and let $\mathbf{x} \in F_v^n - \{0\}$. The set $\{k \in \mathbb{Z} | \pi_v^k \cdot \mathbf{x} \in \mathcal{O}_v^n\}$ is non-empty and we define*

$$r_v(\mathbf{x}) := \inf\{k \in \mathbb{Z} | \pi_v^k \cdot \mathbf{x} \in \mathcal{O}_v^n\}.$$

One has that $\pi_v^{r_v(\mathbf{x})} \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}$.
3. *Suppose v is infinite. The set $\mathcal{D}_v^{\mathbf{a}}$ is a compact subset of $F_v^n - \{0\}$.*

4. *Suppose v is infinite and let $\mathbf{x} \in F_v^n - \{0\}$. There exists $t \in F_v^\times$ such that $||t \cdot \mathbf{x}||_{v,\max} = 1$.*

Proof. — 1. The subset $\mathcal{D}_v^{\mathbf{a}}$ writes as

$$\mathcal{D}_v^{\mathbf{a}} = (\mathcal{O}_v)^n \cap (\pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v)^c.$$

As $(\mathcal{O}_v)^n$ and $(\pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v)^c$ are a ball and a complement of a ball in F_v^n, they are both open and closed subsets of F_v^n. Hence, the subset $\mathcal{D}_v^{\mathbf{a}}$ is both open and closed in F_v^n and, as $F_v^n - \{0\}$ is open in F_v^n, also in $F_v^n - \{0\}$. Moreover, $\mathcal{D}_v^{\mathbf{a}}$ is a closed subset of $(\mathcal{O}_v)^n$, hence $\mathcal{D}_v^{\mathbf{a}}$ is compact.

2. As all a_j are strictly positive, there exists a positive integer ℓ such that for all j one has $a_j\ell > -v(x_j)$. For such ℓ one has $\pi_v^\ell \cdot \mathbf{x} \in \mathcal{O}_v^n$, thus $\{k \in \mathbb{Z} | \pi_v^k \cdot \mathbf{x} \in \mathcal{O}_v^n\}$ is non-empty. Suppose that $\pi_v^{r_v(\mathbf{x})} \cdot \mathbf{x} \in (\pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v)$. One has that

$$\pi_v^{r_v(\mathbf{x})-1} \cdot \mathbf{x} = \pi_v^{-1} \cdot (\pi_v^{r_v(\mathbf{x})} \cdot \mathbf{x}) \in \pi_v^{-1} \cdot (\pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v) = \mathcal{O}_v^n,$$

a contradiction. The claim follows.

3. The set $\mathcal{D}_v^{\mathbf{a}}$ is a sphere for the norm $||\cdot||_{v,\max}$ in the finite dimensional F_v-vector space F_v^n, hence is compact.

4. The function

$$F_v^\times \to \mathbb{R}_{>0} \qquad t \mapsto ||t \cdot \mathbf{x}||_{v,\max}$$

is continuous. From the fact that all a_j are strictly positive it follows that

$$\lim_{|t|_v \to 0} ||t \cdot \mathbf{x}||_{v,\max} = 0$$

and that

$$\lim_{|t|_v \to \infty} ||t \cdot \mathbf{x}||_{v,\max} = +\infty.$$

We deduce that there exists $t \in F_v^\times$ such that $t \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}$.

□

When $R = F_v$, where v is a place of F, one can say the following.

Proposition 3.3.4.5. — *Let $v \in M_F$. One has that:*

1. *the space $[\mathscr{P}(\mathbf{a})(F_v)]$ is compact;*
2. *the space $[\mathscr{T}(\mathbf{a})(F_v)]$ is paracompact;*
3. *the spaces $[\mathscr{P}(\mathbf{a})(\mathcal{O}_v)]$ and $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ are compact.*

Proof. — 1. Suppose firstly that $v \in M_F^0$. It follows from Lemma 3.3.4.4 that the restriction of the quotient map $[q^{\mathbf{a}}(F_v)]$ to $\mathcal{D}_v^{\mathbf{a}}$ is surjective and that $\mathcal{D}_v^{\mathbf{a}}$ is compact. We deduce that $[\mathscr{P}(\mathbf{a})(F_v)]$ is compact.

2. The group $\mathbb{G}_m^n(R)$ is locally compact and $(\mathbb{G}_m^n(R))_{\mathbf{a}}$ is its closed subgroup. The quotient $[\mathscr{T}(\mathbf{a})(R)] = \mathbb{G}_m^n(R)/(\mathbb{G}_m(R))_{\mathbf{a}}$ is paracompact by [7, Chapter III, §4, n^0 6, Proposition 13].
3. The spaces $(\mathbb{A}^n - \{0\})(\mathcal{O}_v)$ and $\mathbb{G}_m^n(\mathcal{O}_v)$ are compact by Lemma 3.3.1.2. Therefore the corresponding quotients by $\mathbb{G}_m(\mathcal{O}_v)$ are compact.

$\square$

3.3.5. — The last paragraph of this section is dedicated to the group structure of $[\mathscr{T}(\mathbf{a})(R)]$.

If R is a proper-closed integral domain, by Proposition 3.3.4.1, one has a homeomorphism of topological spaces $[\mathscr{T}(\mathbf{a})(R)] = \mathbb{G}_m^n(R)/\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$ (induced from $\mathbb{G}_m(R)$-invariant homomorphism $\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R)/\mathbb{G}_m(R)_{\mathbf{a}}$). We will transfer the structure of an abelian group to $[\mathscr{T}(\mathbf{a})(R)]$ using the inverse of this isomorphism and we may write $[\mathscr{T}(\mathbf{a})(R)] = \mathbb{G}_m^n(R)/(\mathbb{G}_m(R)_{\mathbf{a}})$. If, furthermore, R is assumed to be locally compact, then $[\mathscr{T}(\mathbf{a})(R)]$ is a locally compact abelian group.

Lemma 3.3.5.1. — *1. Suppose that $h : R \to R'$ is a morphism of rings. The canonical map $[\mathscr{T}(\mathbf{a})(h)] : [\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is a homomorphism.*

2. Suppose that $R \to R'$ is an injective morphism of rings. The canonical map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is injective if and only if $\mathbb{G}_m(R)_{\mathbf{a}} = \mathbb{G}_m(R)^n \cap \mathbb{G}_m(R')_{\mathbf{a}}$.

3. Suppose that $h : R \to R'$ is a continuous map of topologically suitable rings. The canonical map $[\mathscr{T}(\mathbf{a})(h)] : [\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is a continuous homomorphism.

4. Suppose that $R \to R'$ is an open embedding of proper-closed integral domains, then the canonical map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is open.

Proof. — 1. The map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is the induced map from $\mathbb{G}_m(R)_{\mathbf{a}}$-invariant homomorphism

$$\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R') \to [\mathscr{T}(\mathbf{a})(R')], \tag{3.3.5.2}$$

hence is a homomorphism.

2. The kernel of the map (3.3.5.2) is given by $\mathbb{G}_m(R')_{\mathbf{a}} \cap \mathbb{G}_m^n(R)$ and it contains $\mathbb{G}_m(R)_{\mathbf{a}}$. The induced map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ from $\mathbb{G}_m(R)_{\mathbf{a}}$-invariant map (3.3.5.2) is injective if and only if $\mathbb{G}_m(R')_{\mathbf{a}} \cap \mathbb{G}_m^n(R) = \mathbb{G}_m(R)_{\mathbf{a}}$.

3. The map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is continuous by Proposition 3.3.2.3 is a homomorphism by (1).
4. By [**15**, Section 2.2, parts (vii) and (x)], the inclusion $\mathbb{G}_m^n(R) \to \mathbb{G}_m^n(R')$ is continuous and open. The map $[\mathscr{T}(\mathbf{a})(R)] \to [\mathscr{T}(\mathbf{a})(R')]$ is the induced map from the $\mathbb{G}_m(R)_{\mathbf{a}}$-invariant, continuous and open map (3.3.5.2), thus is open.

□

Lemma 3.3.5.3. — *Let $v \in M_F^0$. One has that*

$$\mathbb{G}_m(F_v)_{\mathbf{a}} \cap \mathbb{G}_m^n(\mathcal{O}_v) = (\mathbb{G}_m(\mathcal{O}_v))_{\mathbf{a}}.$$

Proof. — It is obvious that $\mathbb{G}_m(\mathcal{O}_v)_{\mathbf{a}} \subset \mathbb{G}_m(F_v)_{\mathbf{a}} \cap \mathbb{G}_m^n(\mathcal{O}_v)$ and let us prove the inverse inclusion. Let $\mathbf{x} \in \mathbb{G}_m(F_v)_{\mathbf{a}} \cap \mathbb{G}_m^n(\mathcal{O}_v)$. This means that there exists $t \in \mathbb{G}_m(F_v)$ such that for every j one has $t^{a_j} = x_j$ and that $x_j \in \mathbb{G}_m(\mathcal{O}_v)$. We deduce $v(t^{a_j}) = a_j v(t) = x_j = 0$, and as $a_j > 0$ we get $v(t) = 0$. It follows that $t \in \mathbb{G}_m(\mathcal{O}_v)$ and hence $\mathbf{x} \in \mathbb{G}_m(\mathcal{O}_v)_{\mathbf{a}}$. □

We are ready to prove that:

Proposition 3.3.5.4. — *Let $v \in M_F^0$. The map $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \to [\mathscr{T}(\mathbf{a})(F_v)]$, induced from $(\mathcal{O}_v)_{\mathbf{a}}$-invariant map $q_v^{\mathbf{a}}|_{(\mathcal{O}_v^\times)^n}$, is continuous, injective and open homomorphism and induces an identification of $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ with an open and compact subgroup of $[\mathscr{T}(\mathbf{a})(F_v)]$.*

Proof. — By applying (1), (3) and (4) of Lemma 3.3.5.1 to the inclusion $\mathcal{O}_v \to F_v$, we obtain that $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \to [\mathscr{T}(\mathbf{a})(F_v)]$ is a continuous and open homomorphism. Lemma 3.3.5.3 gives that $\mathbb{G}_m(F_v)_{\mathbf{a}} \cap \mathbb{G}_m^n(\mathcal{O}_v) = (\mathbb{G}_m(\mathcal{O}_v))_{\mathbf{a}}$ and thus by (2) of Lemma 3.3.5.1, the map $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \to [\mathscr{T}(\mathbf{a})(F_v)]$ is injective. Moreover, by Proposition 3.3.4.5, the topological group $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ is compact by Proposition 3.3.4.5. The claim is proven.

□

Lemma 3.3.5.5. — *Let $\mathbf{a}\mathbb{Z}$ denotes the subgroup $\{(a_j x)_j | x \in \mathbb{Z}\}$ of $\mathbb{Z}^n$. The homomorphism*

$$(F_v^\times)^n \to \mathbb{Z}^n \to \mathbb{Z}^n/(\mathbf{a}\mathbb{Z}), \tag{3.3.5.6}$$

where the first homomorphism is given by $\mathbf{x} \mapsto (v(x_j))_j$ and the second homomorphism is the quotient one, is $(F_v^\times)_{\mathbf{a}}$-invariant. The kernel of the induced homomorphism $[\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{Z}^n/(\mathbf{a}\mathbb{Z})$ is $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$.

Proof. — Note that the image of $(t^{a_j})_j \in (F_v^\times)_{\mathbf{a}}$ in $\mathbb{Z}^n$ under the map $(F_v^\times)^n \to \mathbb{Z}^n$ from above is $(v(t^{a_j}))_j = (a_j v(t))_j$, and the image of $(a_j v(t))_j$ in $\mathbb{Z}^n/(\mathbf{a}\mathbb{Z})$ under the quotient homomorphism is 0. Thus the homomorphism (3.3.5.6) is $F_v^\times$-invariant. The kernel of $(F_v^\times)^n \to \mathbb{Z}^n$ is the subgroup $(\mathcal{O}_v^\times)^n \subset (F_v^\times)^n$. The kernel of the induced homomorphism $[\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{Z}^n/(\mathbf{a}\mathbb{Z})$ is the image of $(\mathcal{O}_v^\times)^n$ (as $(\mathcal{O}_v^\times)^n$ is the kernel of $(F_v^\times)^n \to \mathbb{Z}^n$) under the quotient map. We have the following "snake diagram":

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
1 \arrow[r] & (\mathcal{O}_v^\times)_{\mathbf{a}} \arrow[r] \arrow[d] & (F_v^\times)_{\mathbf{a}} \arrow[r] \arrow[d] & \mathbf{a}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & (\mathcal{O}_v^\times)^n \arrow[r] \arrow[d] & (F_v^\times)^n \arrow[r] \arrow[d] & \mathbb{Z}^n \arrow[r] \arrow[d] & 0 \\
 & {[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} \arrow[r] & {[\mathscr{T}(\mathbf{a})(F_v)]} \arrow[r] & \mathbb{Z}^n/\mathbf{a}\mathbb{Z}. &
\end{tikzcd}$$

Snake lemma gives that

$$1 \to [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \to [\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{Z}^n/\mathbf{a}(\mathbb{Z}) \to 0$$

is exact. The statement follows □

We add another fact that will be used later. When $n = 1$, the spaces $[\mathscr{T}(\mathbf{a})(F_v)]$ is finite and discrete (finiteness and the fact that $(F_v^\times)_m \subset F_v^\times$ is closed imply that $(F_v^\times)_m$ is open, hence discreteness follows):

Lemma 3.3.5.7 ([**42**, Corollary 5.8, Chapter II])

Suppose $n = 1$ and that $a = a_1 \in \mathbb{Z}_{\geq 1}$. The space $[\mathscr{T}(a)(F_v)] = F_v^\times/(F_v^\times)_a$ is discrete and of the cardinality is $\frac{a}{|a|_v}|\mu_a(F_v)|$, where $|\mu_a(F_v)|$ is the number of a-th roots of 1 in F_v.

3.4. Adelic situation

We define "adelic space" $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ of the stack $\mathscr{T}(\mathbf{a})$. It is defined as a restricted product of $[\mathscr{T}(\mathbf{a})(F_v)]$ with the respect to open subgroups $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \subset [\mathscr{T}(\mathbf{a})(F_v)]$ for $v \in M_F^0$. It turns out that $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is a locally compact abelian group.

3.4.1. — Let us recall some facts on restricted product homomorphisms.

Lemma 3.4.1.1. — *Let I be a set. Suppose for every $i \in I$ we are given locally compact abelian groups G_i and H_i and for every $i \in I'$, where $I' \subset I$ is a subset of finite complement, we are given open and* *compact subgroups $G'_i \subset G_i$ and $H'_i \subset H_i$. Suppose for $i \in I$ we are given continuous homomorphism $\phi_i : G_i \to H_i$ such that for every $i \in I'$ one has $\phi_i(G'_i) \subset H'_i$. Let us set $G := \prod'_{i\in I} G_i$, with the respect to the open and compact subgroups $(G'_i \subset G_i)_{i\in I'}$ and let us set $H := \prod'_{i\in I} H_i$, with the respect to the open and compact subgroups $(H'_i \subset H_i)_{i\in I'}$*

1. *The topological groups G and H are locally compact.*
2. *The canonical inclusion $G \subset \prod_{i\in I} G_i$ is continuous.*
3. *The image of $G \subset \prod_{i\in I} G_i$ under $\prod_{i\in I} \phi_i$ lies in H. We let $\phi : G \to H$ be the homomorphism induced by $\big(\prod_{i\in I} \phi_i\big)$.*
4. *The homomorphism $\phi : G \to H$ is continuous.*
5. *One has that $\ker(\phi) = \big(\prod_i \ker(\phi_i)\big) \cap G$.*
6. *Suppose for every $i \in I'$ one has $\phi_i(G'_i) = H'_i$. Suppose further that for every $i \in I$, the homomorphism ϕ_i is surjective (respectively, open, isomorphism of topological groups). Then the homomorphism ϕ is surjective (respectively, open, isomorphism of topological groups).*

Proof. — 1. Let $(x_i)_i \in G$. There exists a subset $I_1 \subset I'$ such that $I' - I_1$ is finite and such that for all $i \in I_1$ one has $x_i \in G'_i$. For $i \in I' - I_1$, one can pick a compact neighbourhood U_i of x_i in G_i, because G_i is locally compact. Then $\prod_{i\in I'-I_1} U_i \times \prod_{i\in I_1} G'_i$ is a compact neighbourhood of $(x_i)_i$ in G. It follows that G is locally compact, and the same is true for H.

2. A basis open subset of $\prod_{i\in I} G_i$ writes as $\prod_{i\in S} U_i \times \prod_{i\in I-S} G_i$, where S is finite. Its preimage in G is given by

$$\bigcup_{\substack{T \text{ is finite} \\ T\supset S}} \prod_{i\in S} U_i \times \prod_{i\in T-S} G_i \times \prod_{i\in I-T} G'_i,$$

hence is open. It follows that the canonical inclusion is continuous.

3. Suppose $(x_i)_i \in G$. There exists a subset $I_1 \subset I'$ such that $I' - I_1$ is finite and such that for all $i \in I_1$ one has $x_i \in G'_i$. For $i \in I_1$, we have $\phi_i(x_i) \in H'_i$, hence $(\phi_i(x_i))_i \in H$.

4. It suffices to verify that the preimage under ϕ of a basis open subset of H is open in G. A basis open subset of H is given by $\prod_{i\in J} U_i \times \prod_{i\in I-J} H'_i$, for some finite subset $J \subset I$ containing $I - I'$ and some open subsets $U_i \subset H_i$ for $i \in J$. We establish that every point $(x_i)_i \in \phi^{-1}\big(\prod_{i\in J} U_i \times \prod_{i\in I-J} H'_i\big)$ admits an open neighbourhood contained in $\phi^{-1}\big(\prod_{i\in J} U_i \times \prod_{i\in I-J} H'_i\big)$. There exists a subset $I_1 \subset$

I' such that $I' - I_1$ is finite and such that for all $i \in I_1$ one has $x_i \in G'_i$. We note that

$$\prod_{i\in(J\cap I)-I_1}\phi_i^{-1}(U_i)\times\prod_{i\in I-I_1-J}\phi_i^{-1}(H'_i)\times\prod_{i\in I_1}G'_i$$

is an open neighbourhood of $(x_i)_i$ contained in $\phi^{-1}\big(\prod_{i\in J}U_i\times\prod_{i\in I-J}H'_i\big)$. The set $\phi^{-1}\big(\prod_{i\in J}U_i\times\prod_{i\in I-J}H'_i\big)$ is thus open and it follows that ϕ is continuous.

5. Let $(x_i)_i \in G$. One has that $0 = \phi((x_i)_i) = (\phi_i(x_i))_i$ if and only if $x_i \in \ker(\phi_i)$ for every $i \in I$, i.e. if and only if $(x_i)_i \in \prod_i \ker(\phi_i)$. It follows that $\ker(\phi) = \big(\prod_i \ker(\phi_i)\big)\cap G$.
6. Let us suppose that for every $i \in I'$ one has $\phi_i(G'_i) = H'_i$.
   (a) Suppose ϕ_i are surjective and let $(x_i)_i \in H$. There exists $I_1 \subset I'$ such that $I' - I_1$ is finite and such that $x_i \in H'_i$ for every $i \in I_1$. As $\phi_i(G'_i) = H'_i$ for every $i \in I_1$, we can pick $y_i \in G'_i$ such that $\phi_i(y_i) = x_i$. As maps ϕ_i are surjective for every $i \in I - I_1$, we can pick $y_i \in G_i$ such that $\phi_i(y_i) = x_i$ for every $i \in I - I_1$. It follows that $(y_i)_i$ is an element of G such that $\phi((y_i)_i) = (x_i)_i$. Hence, ϕ is surjective.
   (b) Suppose ϕ_i are open. It suffices to prove that the image of a basis open subset of G is open in H. A basis open subset of G is given by $\prod_{i\in J}U_i\times\prod_{i\in I-J}G'_i$, for some finite subset $J \subset I$ such that $J \supset I - I'$ and some open subsets $U_i \subset G_i$. We have that

$$\begin{aligned}\phi\Big(\prod_{i\in J}U_i\times\prod_{i\in I-J}G'_i\Big)&=\prod_{i\in J}\phi_i(U_i)\times\prod_{i\in I-J}\phi_i(G'_i)\\&=\prod_{i\in J}\phi_i(U_i)\times\prod_{i\in I-J}H'_i\end{aligned}$$

   is open in H. It follows that ϕ is open.
   (c) Suppose ϕ_i are isomorphisms of topological groups. It follows from above that ϕ is injective, surjective, continuous and open. Thus ϕ is an isomorphism of topological groups.

□

3.4.2. — Let $n \geq 1$ be an integer and let $\mathbf{a} \in \mathbb{Z}^n_{>0}$. We define an "adelic space" of the stack $\mathscr{T}(\mathbf{a})$.

Let $\mathbb{A}_F^\times$ be the group of ideles of F, that is, it is the restricted product

$$\prod_{v\in M_F}{}' F_v^\times,$$

with the respect to the family of open and compact subgroups

$$(\mathscr{O}_v^\times\subset F_v^\times)_{v\in M_F^0}.$$

It follows from Lemma 3.4.1.1 that the group $\mathbb{A}_F^\times$ is locally compact.

Definition 3.4.2.1. — *We define*

$$[\mathscr{T}^{\mathbf{a}}(\mathbb{A}_F)]:=\prod_{v\in M_F}{}'[\mathscr{T}^{\mathbf{a}}(F_v)],$$

where the restricted product is taken with the respect to the family of compact and open subgroups

$$([\mathscr{T}^{\mathbf{a}}(\mathscr{O}_v)]\subset[\mathscr{T}^{\mathbf{a}}(F_v)])_{v\in M_F^0}.$$

For $v\in M_F$, let $q_v^{\mathbf{a}}:(F_v^\times)^n\to(F_v^\times)^n/(F_v^\times)_{\mathbf{a}}=[\mathscr{T}(\mathbf{a})(F_v)]$ be the quotient morphism. By Proposition 3.3.5.4, for every $v\in M_F^0$, one has that $q_v^{\mathbf{a}}((\mathscr{O}_v^\times)^n)=[\mathscr{T}(\mathbf{a})(\mathscr{O}_v)]$ is an open and compact subgroup of $[\mathscr{T}(\mathbf{a})(F_v)]$. Lemma 3.4.1.1 provides a homomorphism

$$q^{\mathbf{a}}_{\mathbb{A}_F}=\Big(\prod_{v\in M_F}q_v^{\mathbf{a}}\Big):(\mathbb{A}_F^\times)^n\to[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)].$$

We apply Lemma 3.4.1.1 to our situation and we get the following lemma.

Lemma 3.4.2.2. — *The abelian topological group $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is locally compact. The map $q^{\mathbf{a}}_{\mathbb{A}_F}=\big(\prod_{v\in M_F}q_v^{\mathbf{a}}\big):(\mathbb{A}_F^\times)^n\to[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is continuous, open and surjective. The kernel of $q^{\mathbf{a}}_{\mathbb{A}_F}$ is the group $(\mathbb{A}_F^\times)_{\mathbf{a}}=\{(\mathbf{x}^{a_j})_j|\mathbf{x}\in\mathbb{A}_F^\times\}$ (in particular $(\mathbb{A}_F^\times)_{\mathbf{a}}\subset(\mathbb{A}_F^\times)^n$ is closed).*

Proof. — The abelian topological group $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is locally compact by Lemma 3.4.1.1. For every $v\in M_F$, the quotient map $q_v^{\mathbf{a}}$ is continuous, open and surjective. By Lemma 3.4.1.1, one has that $q^{\mathbf{a}}_{\mathbb{A}_F}$ is continuous, open and surjective. For $v\in M_F$, one has that $\ker(q_v^{\mathbf{a}})=(F_v^\times)_{\mathbf{a}}$ and Lemma 3.4.1.1 gives that the kernel of $q^{\mathbf{a}}_{\mathbb{A}_F}$ is given by

$$\ker(q^{\mathbf{a}}_{\mathbb{A}_F})=\Big(\prod_v\ker(q_v^{\mathbf{a}})\Big)\cap(\mathbb{A}_F^\times)^n=\Big(\prod_v(F_v^\times)_{\mathbf{a}}\Big)\cap(\mathbb{A}_F^\times)^n$$

By Lemma 3.3.5.3, for $x\in F_v^\times$ one has that $(x^{a_j})_j\in(\mathscr{O}_v^\times)_{\mathbf{a}}$ if and only if $x\in\mathscr{O}_v^\times$. It follows that for $\mathbf{x}\in\big(\prod_v F_v^\times\big)$, one has that $(\mathbf{x}^{a_j})_j\in(\mathbb{A}_F^\times)_{\mathbf{a}}$

if and only if $\mathbf{x} \in \mathbb{A}_F^\times$. We deduce that

$$\ker(q^{\mathbf{a}}_{\mathbb{A}_F}) = \Big(\prod_v (F_v^\times)_{\mathbf{a}}\Big) \cap (\mathbb{A}_F^\times)^n = (\mathbb{A}_F^\times)_{\mathbf{a}}.$$

□

For $v \in M_F$, let $i_v : F^\times \to F_v^\times$ be the canonical inclusion. The homomorphism

$$(F^\times)^n \xrightarrow{i_v^n} (F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)]$$

is $F^\times$-invariant, and one deduces a homomorphism $[\mathscr{T}(\mathbf{a})(i_v)] : [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(F_v)]$. Let $i : F^\times \to \mathbb{A}_F^\times$ be the product map $i = \prod_{v\in M_F} i_v$. It is well known that the image of $F^\times$ is discrete in $\mathbb{A}_F^\times$ (see e.g. [**49**, Theorem 5-11]). The homomorphism

$$(F^\times)^n \xrightarrow{i} (\mathbb{A}_F^\times)^n \xrightarrow{q^{\mathbf{a}}_{\mathbb{A}_F}} [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$$

is $(F^\times)$-invariant and we deduce homomorphism $[\mathscr{T}(\mathbf{a})(i)] : [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. For every v, the homomorphism

$$(F_v^\times)^n \xrightarrow{\mathbf{x}\mapsto((\mathbf{x})_v,(\mathbf{1})_{w\in M_F-\{v\}})} (\mathbb{A}_F^\times)^n \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$$

is $F_v^\times$-invariant, and we deduce a homomorphism $[\mathscr{T}(\mathbf{a})(F_v)] \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$.

Lemma 3.4.2.3. — *The map $[\mathscr{T}(\mathbf{a})(i)]$ coincides with the product map $\prod_v[\mathscr{T}(\mathbf{a})(i_v)] : [\mathscr{T}(\mathbf{a})(F)] \to \prod_v[\mathscr{T}(\mathbf{a})(F_v)]$.*

Proof. — For every $v \in M_F$, the following diagram is commutative

$$\begin{tikzcd}
(F^\times)^n \arrow[r, "i_v^n"] \arrow[d] & (F_v^\times)^n \arrow[r] \arrow[d] & (\mathbb{A}_F^\times)^n \arrow[d] \\
{[\mathscr{T}(\mathbf{a})(F)]} \arrow[r, "{[\mathscr{T}(\mathbf{a})(i_v)]}"] & {[\mathscr{T}(\mathbf{a})(F_v)]} \arrow[r] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]}
\end{tikzcd}$$

for every $v \in M_F$. It follows that the map $[\mathscr{T}(\mathbf{a})(i)]$ coincides with the product map $\prod_v[\mathscr{T}(\mathbf{a})(i_v)] : [\mathscr{T}(\mathbf{a})(F)] \to \prod_v[\mathscr{T}(\mathbf{a})(F_v)]$. □

Definition 3.4.2.4. — *We define $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ to be the image of $(\mathbb{A}_F^1)^n$ in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ under the map $q^{\mathbf{a}}_{\mathbb{A}_F}$.*

3.4.3. — In this paragraph we suppose that $n = 1$ and that $a = a_1 \geq 2$. The following proposition is due to Česnavičius:

Proposition 3.4.3.1. — *One has that:*

1. [**14**, Proposition 4.12] *The image*
$$[\mathscr{T}(a)(i)]([\mathscr{T}(a)(F)]) \subset [\mathscr{T}(a)(\mathbb{A}_F)]$$
is discrete, closed and cocompact.
2. [**14**, Lemma 4.4] *The group* $\text{Ш}^1(F, \mu_a) := \ker([\mathscr{T}(a)(i)])$ *is finite.*

Proof. — Let R be a local ring. Kummer exact sequence
$$1 \to \mu_a \to \mathbb{G}_m \xrightarrow{t \mapsto t^a} \mathbb{G}_m \to 1$$
provides a long exact sequence
$$H^0_{\text{fppf}}(R, \mathbb{G}_m) \xrightarrow{t \mapsto t^a} H^0_{\text{fppf}}(R, \mathbb{G}_m) \to H^1_{\text{fppf}}(R, \mu_a) \to H^1_{\text{fppf}}(R, \mathbb{G}_m).$$
By Hilbert 90 theorem, the group $H^1_{\text{fppf}}(R, \mathbb{G}_m)$ is trivial. We deduce an identification of abelian groups $H^1_{\text{fppf}}(R, \mu_a)$ and $R^\times/(R^\times)_{\mathbf{a}} = [\mathscr{T}(a)(R)]$. It follows that the space $\mathscr{T}(a)(\mathbb{A}_F)$ is precisely the space $H^1(\mathbb{A}_F^{\in M_F}, \mu_a)$ from [**14**, Section 3]. Moreover, with these identifications, the map $[\mathscr{T}(a)(i)]$ becomes the map $\operatorname{loc}^1(\mu_a)$ in the notation of [**14**]. We are thus in the situations of the mentioned statements of [**14**]. □

Lemma 3.4.3.2. — *One has that*
$$[\mathscr{T}(a)(\mathbb{A}_F)]_1 = [\mathscr{T}(a)(\mathbb{A}_F)].$$

Proof. — We will establish that any $(x_v)_v \in [\mathscr{T}(a)(\mathbb{A}_F)]$ admits a lift in $\mathbb{A}^1_F$ for the map $q^a_{\mathbb{A}_F} = \prod_v q^a_v : \mathbb{A}^\times_F \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. It follows from Lemma 3.4.2.2 that there exists a lift $(y_v)_v \in \mathbb{A}^\times_F$ of $(x_v)_v$. We set $A := \prod_{v \in M_F} |y_v|_v \in \mathbb{R}_{>0}$. Let $w \in M^\infty_F$. Set $z = A^{1/n_w} \in F^\times_w$, so that $|z|_w = A$. One has that $((y_v)_{v \neq w}, (y_w z^{-1})_w) \in \mathbb{A}^1_F$, because
$$|y_w z^{-1}|_w \prod_{v \neq w} |y_v| = |z^{-1}|_w \prod_v |y_v|_v = A^{-1} \cdot A = 1.$$
We note that $z^{-1} = A^{-1/n_v} = (A^{-1/an_v})^a$, hence $z^{-1} \in (F^\times_w)_a$. Now one has that
$$\begin{aligned} q^a_{\mathbb{A}_F}((y_v)_{v \neq w}, y_w z^{-1}) &= ((q^a_v(y_v))_{v \neq w}, (q^a_w(y_w z^{-1}))_w) \\ &= ((x_v)_{v \neq w}, (q^a_w(y_w))_w) \\ &= (x_v)_v. \end{aligned}$$

Thus $((y_v)_{v\neq w},(y_w z^{-1})_w)$ is a lift of $(x_v)_v$ lying in $\mathbb{A}^1_F$. It follows that $[\mathscr{T}(a)(\mathbb{A}_F)] = q^a_{\mathbb{A}_F}(\mathbb{A}^1_F) = [\mathscr{T}(a)(\mathbb{A}_F)]_1$. □

3.4.4. — If A is an abelian group written additively, we will write $\mathbf{a}A$ for the subgroup $\{(a_j m)_j | m \in A\}$ of A^n. If the group is written multiplicatively, then we will use notation $A_{\mathbf{a}}$ to be consistent with earlier notation. In this paragraph we construct an isomorphism $A/dA \times A^{n-1} \xrightarrow{\sim} A^n/\mathbf{a}A$, where $d := \gcd(\mathbf{a})$.

Let $n \geq 1$ be an integer and let $\mathbf{a} \in \mathbb{Z}^n_{>0}$. For $j = 1,\dots,n$, we set

$$b_{j1} := \frac{a_j}{d}.$$

There exists a matrix $E = (b_{ji})_{ji} \in SL(n,\mathbb{Z})$ which has $(b_{j1})^t_j$ for the first column (see e.g. [**50**]).

Lemma 3.4.4.1. — *1. The kernel of the homomorphism*

$$\mathbb{Z}^n \xrightarrow{E} \mathbb{Z}^n \to \mathbb{Z}^n/\mathbf{a}\mathbb{Z}, \tag{3.4.4.2}$$

where the second homomorphism is the quotient homomorphism, is the subgroup $d\mathbb{Z} \times \{0\}^{n-1} \subset \mathbb{Z}^n$.

2. The homomorphism

$$\overline{E} : (\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}^{n-1} = \mathbb{Z}^n/(d\mathbb{Z} \times \{0\}^{n-1}) \xrightarrow{\sim} \mathbb{Z}^n/\mathbf{a}\mathbb{Z},$$

induced from $\mathbf{a}\mathbb{Z}$-invariant homomorphism 3.4.4.2, is an isomorphism. Let A be an abelian group. Let us write E_A and $\overline{E}_A$ for the tensor products $E \otimes_{\mathbb{Z}} A$ and $\overline{E} \otimes_{\mathbb{Z}} A$, respectively. The following diagram $C(A)$ is commutative, its horizontal sequences are exact and its vertical arrows are isomorphisms of abelian groups:

$$\begin{array}{ccccccc} dA \times \{0\}^{n-1} & \longrightarrow & A^n & \longrightarrow & (A/dA) \times A^{n-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle E_A} & & \downarrow{\scriptstyle E_A} & & \downarrow{\scriptstyle \overline{E}_A} & & \\ \mathbf{a}A & \longrightarrow & A^n & \longrightarrow & A^n/\mathbf{a}A & \longrightarrow & 0. \end{array} \tag{3.4.4.3}$$

Moreover, if $A \to B$ is a homomorphism of abelian groups, the canonical homomorphisms provide a morphism of diagrams $C(A) \to C(B)$.

Proof. — 1. Obviously, the kernel of (3.4.4.2) coincides with $E^{-1}(\mathbf{a}\mathbb{Z})$, thus the kernel is a free abelian group of the rank 1. Moreover, it contains the vector $(d,0,\dots,0)^t$. We deduce that the generator of the kernel of the homomorphism (3.4.4.2) is given by $(k, 0, \ldots, , 0)^t$ for some $k|d$. One has that

$$E \cdot (k, 0, \ldots, , 0)^t = (kb_{j1})_j^t = (ka_j/d)_j^t \in \mathbf{a}\mathbb{Z},$$

hence $d|k$, hence $d = \pm k$. We deduce that the kernel of the homomorphism (3.4.4.2) is precisely the subgroup $d\mathbb{Z} \times \{0\}^{n-1} \subset \mathbb{Z}^n$.

2. The homomorphism $\overline{E}$ is evidently an isomorphism. By (1), the following diagram is commutative, its horizontal sequences are exact and its vertical arrows are isomorphisms:

$$\begin{tikzcd} d\mathbb{Z} \times \{0\}^{n-1} \arrow[r] \arrow[d, "E"'] & \mathbb{Z}^n \arrow[r] \arrow[d, "E"'] & (\mathbb{Z}/d\mathbb{Z}) \times \mathbb{Z}^{n-1} \arrow[r] \arrow[d, "\overline{E}"'] & 0 \\ \mathbf{a}\mathbb{Z} \arrow[r] & \mathbb{Z}^n \arrow[r] & \mathbb{Z}^n/\mathbf{a}\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{3.4.4.4}$$

The diagram (3.4.4.3) is obtained by tensoring the diagram (3.4.4.4) by A. Thus the diagram (3.4.4.3) is commutative, its horizontal sequences are exact and its vertical arrows are isomorphisms. Moreover, the morphism of diagrams $C(A) \to C(B)$ is deduced by functoriality of the tensor product.

□

3.4.5. — In this paragraph we prove that the isomorphisms $E_{F_v^\times}$ and $\overline{E}_{F_v^\times}$ are continuous and that they preserve the compact open subgroups from before.

Lemma 3.4.5.1. — *The following claims are valid:*

1. *Let $v \in M_F$. The homomorphism $E_{F_v^\times} : (F_v^\times)^n \to (F_v^\times)^n$ is an isomorphism of abelian topological groups. The homomorphism $\overline{E}_{F_v^\times} : [\mathscr{T}(d)(F_v)] \times (F_v^\times)^{n-1} \to [\mathscr{T}(d)(F_v)]$ is an isomorphism of abelian topological groups.*
2. *Let $v \in M_F^0$. One has that*

$$E_{F_v^\times}((\mathcal{O}_v^\times)^n) = (\mathcal{O}_v^\times)^n$$

and that

$$\overline{E}_{F_v^\times}([\mathscr{T}(d)(\mathcal{O}_v)] \times (\mathcal{O}_v^\times)^{n-1}) = [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)].$$

Proof. — 1. We have seen in Lemma 3.4.4.1 that the homomorphisms $E_{F_v^\times}$ and $\overline{E}_{F_v^\times}$ are isomorphisms of abelian groups. The map $E_{F_v^\times}$:

$(F_v^\times)^n \to (F_v^\times)^n$ is given by

$$\mathbf{x} \mapsto \Big(\prod_{i=1}^n x_i^{b_{ji}}\Big)_j$$

and is continuous and open. It follows that $E_{F_v^\times}$ is an isomorphism of topological groups. Moreover, by Lemma 3.4.4.1, the map $\overline{E}_{F_v^\times}$ is the induced map from $(F_v^\times)_d \times \{1\}^{n-1}$-invariant map

$$(F_v^\times)^n \xrightarrow{E_{F_v^\times}} (F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)],$$

thus is continuous and open. It follows that $\overline{E}_{F_v^\times}$ is an isomorphism of topological groups.

2. Let $v \in M_F^0$ and let $\mathbf{x} \in (\mathcal{O}_v^\times)^n$. The j-th coordinate of $E_{F_v^\times}(\mathbf{x})$ is equal to $\prod_{i=1}^n x_i^{b_{ji}}$ and is an element of $\mathcal{O}_v^\times$, thus $E_{F_v^\times}(\mathbf{x})$ is an element of $(\mathcal{O}_v^\times)^n$. We have established $E_{F_v^\times}((\mathcal{O}_v^\times)^n) \subset (\mathcal{O}_v^\times)^n$. Let $E^{-1} = (c_{ji})_{ji}$. For $\mathbf{y} \in (F_v^\times)^n$, one has that $E^{-1}_{F_v^\times}(\mathbf{y}) = \prod_{j=1}^n y_i^{b_{ji}}$ and it follows that $E^{-1}(\mathbf{y}) \in (\mathcal{O}_v^\times)^n$, hence $E^{-1}((\mathcal{O}_v^\times)^n) \subset (\mathcal{O}_v^\times)^n$. We deduce $E_{F_v^\times}((\mathcal{O}_v^\times)^n) = (\mathcal{O}_v^\times)^n$. Now one has that

$$\begin{aligned}\overline{E}_{F_v^\times}([\mathscr{T}(d)(\mathcal{O}_v)] \times (\mathcal{O}_v^\times)^{n-1}) &= \overline{E}_{F_v^\times}(q_v^d \times \mathrm{Id}_{(F_v^\times)^{n-1}}((\mathcal{O}_v^\times)^n)) \\ &= q_v^{\mathbf{a}}(E_{F_v^\times}((\mathcal{O}_v^\times)^{n-1})) \\ &= q_v^{\mathbf{a}}((\mathcal{O}_v^\times)^n) \\ &= [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)].\end{aligned}$$

□

3.4.6. — We will now use properties given in Lemma 3.4.5.1 to define maps $(\mathbb{A}_F^\times)^n \to (\mathbb{A}_F^\times)^n$ and $[\mathscr{T}(d)(\mathbb{A}_F)] \times (\mathbb{A}_F^\times)^{n-1} \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$.

Lemma 3.4.6.1. — *The following claims are valid*

1. *The map* $\big(\prod_{v\in M_F} E_{F_v^\times}\big) : (\prod_v (F_v^\times)^n) \to \prod_v (F_v^\times)^n$ *is precisely the map* $E \otimes_{\mathbb{Z}} (\prod_v F_v^\times)$. *The map* $\big(\prod_{v\in M_F} \overline{E}_{F_v^\times}\big) : (\prod_v([\mathscr{T}(d)(F_v)] \times (F_v^\times)^{n-1})) \to \prod_v [\mathscr{T}(\mathbf{a})(F_v)]$ *is precisely the map* $\overline{E} \otimes_{\mathbb{Z}} (\prod_v F_v^\times)$.
2. *One has that* $\big(\prod_{v\in M_F} E_{F_v^\times}\big)((\mathbb{A}_F^\times)^n) \subset (\mathbb{A}_F^\times)^n$. *Moreover, the induced homomorphism*

$$E_{\mathbb{A}_F^\times} : (\mathbb{A}_F^\times)^n \to (\mathbb{A}_F^\times)^n$$

is an isomorphism of topological abelian groups.

3. *One has that* $\big(\prod_{v\in M_F}\overline{E}_{F_v^\times}\big)([\mathscr{T}(d)(\mathbb{A}_F)]\times(\mathbb{A}_F^\times)^{n-1})\subset[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. *Moreover, the induced homomorphism*

$$\overline{E}_{\mathbb{A}_F^\times}:([\mathscr{T}(d)(\mathbb{A}_F)]\times(\mathbb{A}_F^\times)^{n-1})\to[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$$

is an isomorphism of abelian topological groups.

Proof. — 1. One has that E and $\overline{E}$ are homomorphisms of finitely presented $\mathbb{Z}$-modules, thus tensoring by E and $\overline{E}$ commutes with the direct products. The claim follows

2. For every $v\in M_F^0$, Lemma 3.4.5.1 gives that $E_{F_v^\times}$ is a continuous homomorphism which satisfies that $E_{F_v^\times}((\mathcal{O}_v^\times)^n)=(\mathcal{O}_v^\times)^n$, and hence by Lemma 3.4.1.1, it follows that

$$\big(\prod_{v\in M_F}E_{F_v^\times}\big)((\mathbb{A}_F^\times)^n)\subset(\mathbb{A}_F^\times)^n.$$

Moreover, by Lemma 3.4.5.1 the maps $E_{F_v^\times}$ are isomorphisms of abelian topological groups, thus by Lemma 3.4.1.1 the homomorphism $\big(\prod_{v\in M_F}E_{F_v^\times}\big):(\mathbb{A}_F^\times)^n\to(\mathbb{A}_F^\times)^n$ is an isomorphism of abelian topological groups.

3. For every $v\in M_F^0$, Lemma 3.4.5.1 gives that $\overline{E}_{F_v^\times}$ is a continuous homomorphism which satisfies that $\overline{E}_{F_v^\times}([\mathscr{T}(d)(\mathcal{O}_v)]\times(\mathcal{O}_v^\times)^{n-1})=[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$, and hence by Lemma 3.4.1.1, it follows that

$$\big(\prod_{v\in M_F}\overline{E}_{F_v^\times}\big)([\mathscr{T}(d)(\mathbb{A}_F)]\times(\mathbb{A}_F^\times)^{n-1}\big)\subset[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)].$$

Moreover, by Lemma 3.4.5.1 the maps $\overline{E}_{F_v^\times}$ are isomorphisms of abelian topological groups, thus by Lemma 3.4.1.1 the homomorphism $\big(\prod_{v\in M_F}\overline{E}_{F_v^\times}\big):[\mathscr{T}(d)(\mathbb{A}_F)]\times(\mathbb{A}_F^\times)^{n-1}\to[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is an isomorphism of abelian topological groups.

□

Lemma 3.4.6.2. — *The following claims are valid:*

1. *One has that* $E_{\mathbb{A}_F^\times}((\mathbb{A}_F^1)^n)=(\mathbb{A}_F^1)^n$.
2. *One has that* $\overline{E}_{\mathbb{A}_F^\times}([\mathscr{T}(d)(\mathbb{A}_F)]_1\times(\mathbb{A}_F^1)^{n-1})=[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$.

Proof. — 1. Let $(x_{jv})_v\in(\mathbb{A}_F^1)^n$. The j-th coordinate of its image under $E_{\mathbb{A}_F^\times}=\prod_v E_{F_v^\times}$ is $(\prod_{k=1}^n x_{kv}^{b_{jk}})_v$ and one has

$$\prod_v\big|\prod_{k=1}^n x_{kv}^{b_{jk}}\big|_v=\prod_{k=1}^n\prod_{v\in M_F}|x_{kv}|_v^{b_{jk}}=1.$$

It follows that $E_{\mathbb{A}_F^\times}((\mathbb{A}_F^1)^n) \subset (\mathbb{A}_F^1)^n$. Let $E^{-1} = (c_{ji})_{ji} \in SL_n(\mathbb{Z})$. For $j = 1, \dots, n$, the j-th coordinate of $E_{\mathbb{A}_F^\times}^{-1}(x_{jv})_v = (E_{F_v^\times}^{-1}(x_{jv}))_v$ is equal to $\prod_{k=1}^n x_{kv}^{c_{jk}}$ and one has

$$\prod_{v \in M_F} \Big|\prod_{k=1}^n x_{kv}^{c_{jk}}\Big|_v = \prod_{k=1}^n \prod_{v \in M_F} |x_{kv}|_v^{c_{jk}} = 1$$

It follows that $E_{\mathbb{A}_F^\times}^{-1}(\mathbb{A}_F^1)^n) \subset (\mathbb{A}_F^1)^n$. We deduce that $E_{\mathbb{A}_F^\times}((\mathbb{A}_F^1)^n) = (\mathbb{A}_F^1)^n$.

2. The diagram

$$\begin{tikzcd} (\mathbb{A}_F^\times)^n \arrow[rr, "E_{\mathbb{A}_F^\times}"'] \arrow[d, "q^d_{\mathbb{A}_F} \times \mathrm{Id}_{(\mathbb{A}_F^\times)^{n-1}}"'] & & (\mathbb{A}_F^\times)^n \arrow[d, "q^{\mathbf{a}}_{\mathbb{A}_F}"'] \\ {[\mathscr{T}(d)(\mathbb{A}_F)] \times (\mathbb{A}_F^\times)^{n-1}} \arrow[rr, "\overline{E}_{\mathbb{A}_F^\times}"'] & & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} \end{tikzcd}$$

is commutative by Lemma 3.4.4.1. One has that $[\mathscr{T}^d(\mathbb{A}_F)]_1 \times (\mathbb{A}_F^1)^{n-1}$ and $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ are images of $(\mathbb{A}_F^1)^n$ in $[\mathscr{T}(d)(\mathbb{A}_F)] \times (\mathbb{A}_F^\times)^{n-1}$ and in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ under the surjective maps $q^d_{\mathbb{A}_F} \times \mathrm{Id}_{(\mathbb{A}_F^\times)^{n-1}}$ and $q^{\mathbf{a}}_{\mathbb{A}_F}$, respectively. The claim now follows from (1).

□

3.4.7. — We will now study the kernel and the image of the map $[\mathscr{T}(\mathbf{a})(i)]$ when $n \geq 2$ and $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$.

Lemma 3.4.7.1. — *One has that*

$$\begin{aligned} \overline{E}_{F^\times}\big(\text{Ш}^1(F, \mu_d) \times \{0\}^{n-1}\big) &= \overline{E}_{F^\times}\Big(\ker([\mathscr{T}(d)(i)]) \times \{0\}^{n-1}\Big) \\ &= \ker([\mathscr{T}(\mathbf{a})(i)]) \end{aligned}$$

is finite.

Proof. — By Lemma 3.4.4.1, the following diagram is commutative and the vertical arrows are isomorphisms:

$$\begin{tikzcd} {[\mathscr{T}(d)(F)] \times (F^\times)^{n-1}} \arrow[r, "{[\mathscr{T}(d)(i)] \times i^{n-1}}"'] \arrow[d, "\overline{E}_{F^\times}"'] & {[\mathscr{T}(d)(\mathbb{A}_F)] \times (\mathbb{A}_F^\times)^{n-1}} \arrow[d, "E_{\mathbb{A}_F^\times}"] \\ {[\mathscr{T}(\mathbf{a})(F)]} \arrow[r, "{[\mathscr{T}(\mathbf{a})(i)]}"] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)].} \end{tikzcd}$$

We deduce that

$$\overline{E}_{F^\times}(\ker([\mathscr{T}(d)(i)] \times i^{n-1}]) = \ker([\mathscr{T}(\mathbf{a})(i)]).$$

The map $i^{n-1} : (F^\times)^{n-1} \to (\mathbb{A}_F^1)^{n-1}$ given by the diagonal inclusion is injective, hence the kernel of $[\mathscr{T}(d)(i)] \times i^{n-1}$ is precisely the group $\ker([\mathscr{T}(d)(i)]) \times \{0\}^{n-1} = Ш^1(F, \mu_d) \times \{0\}^{n-1}$. Finiteness of the Tate-Shafarevich group has been established in [52, Lemma 1.2]. The claim is proven $\square$

Proposition 3.4.7.2. — *The image $[\mathscr{T}(\mathbf{a})(F)]$ under $[\mathscr{T}(\mathbf{a})(i)]$ lies in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$. Moreover, the subgroup $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]) \subset [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ is discrete, closed and cocompact.*

Proof. — The following diagram is commutative by the definition of $[\mathscr{T}(\mathbf{a})(i)(F)]$:

$$\begin{tikzcd}
(F^\times)^n \arrow[r, "i^n"] \arrow[d, "{[q^{\mathbf{a}}(F)]}"'] & (\mathbb{A}_F^\times)^n \arrow[d, "q^{\mathbf{a}}_{\mathbb{A}_F}"] \\
{[\mathscr{T}(\mathbf{a})(F)]} \arrow[r, "{[\mathscr{T}(\mathbf{a})(i)]}"'] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]}
\end{tikzcd}$$

The image under i^n of $F^{\times n}$ is contained in $(\mathbb{A}_F^1)^n$. We deduce that

$$[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]) = (q^{\mathbf{a}}_{\mathbb{A}_F} \circ i^n)(F^{\times n}) \subset q^{\mathbf{a}}_{\mathbb{A}_F}(((\mathbb{A}_F)^1)^n) = [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1.$$

The map $\overline{E}_{\mathbb{A}_F^\times}$ is an isomorphism of abelian topological groups. Thus the subgroup $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]) = \overline{E}^{-1}_{\mathbb{A}_F^\times}((([\mathscr{T}(d)(i)] \times i^{n-1})([\mathscr{T}(d)(F)] \times (F^\times)^{n-1}))$ is discrete, closed and cocompact in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 = \overline{E}^{-1}_{\mathbb{A}_F^\times}([\mathscr{T}(d)(\mathbb{A}_F)]_1 \times (\mathbb{A}_F^1)^{n-1})$ if and only the subgroup

$$[\mathscr{T}(d)(i)] \times i^{n-1})([\mathscr{T}(d)(F)] \times (F^\times)^{n-1}) \subset [\mathscr{T}(d)(\mathbb{A}_F)]_1 \times (\mathbb{A}_F^1)^{n-1}$$

is discrete, closed and cocompact. The subgroup $(F^\times)^n \subset (\mathbb{A}_F^1)^n$ is discrete, closed and cocompact by [49, Theorem 5-15]. Recall that by Lemma 3.4.3.2 one has that $[\mathscr{T}(d)(\mathbb{A}_F)]_1 = [\mathscr{T}(d)(\mathbb{A}_F)]$. Now, by Proposition 3.4.3.1, one has that $[\mathscr{T}(d)(i)]([\mathscr{T}(d)(F)]) \subset [\mathscr{T}(d)(\mathbb{A}_F)]_1 = [\mathscr{T}(d)(\mathbb{A}_F)]$ is discrete, closed and cocompact. We deduce that

$$[\mathscr{T}(d)(F)] \times (F^\times)^{n-1} \subset [\mathscr{T}(d)(\mathbb{A}_F)]_1 \times ((\mathbb{A}_F)^1)^n$$

is discrete, closed and compact. The statement follows. $\square$

3.4.8. — In this paragraph we establish some more basic properties of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$.

Lemma 3.4.8.1. — *For $v\in M_F$, the groups $F_v^\times$ are countable at infinity (i.e. are countable unions of compact subsets). The groups $\mathbb{A}_F^\times$, $\mathbb{A}_F^1$, $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ are countable at infinity.*

Proof. — For $v\in M_F - v\in M_F^{\mathbb{C}}$, one has that $F_v^\times$ is the countable union of of compact balls:

$$F_v^\times = \bigcup_{q\in\mathbb{Q}_{\neq 0}} B\big(q, \frac{|q|_v}{2}\big)$$

and for $v\in M_F^{\mathbb{C}}$ one has that $F_v^\times = \mathbb{C}-\{0\}$ is the countable union of compact balls

$$F_v^\times = \bigcup_{q\in\mathbb{Q}_{\neq 0}} B\big(q, \frac{|q|}{2}\big).$$

For finite subset $S\subset M_F$, it follows that $\prod_{v\in S}F_v^\times$ is countable at infinity. It follows that the group $\mathbb{A}_F^\times$, which writes as the countable union

$$\mathbb{A}_F^\times = \bigcup_{\substack{M_F^\infty\subset S\subset M_F\\ S\text{ finite}}} \prod_{v\in S} F_v^\times \times \prod_{v\in M_F - S}\mathcal{O}_v^\times,$$

is countable at infinity. The group $\mathbb{A}_F^1$ is countable at infinity as it is a closed subgroup of $\mathbb{A}_F^\times$. The group $(\mathbb{A}_F^\times)^n$ and the group $(\mathbb{A}_F^1)^n$ are countable at infinity, as finite products of groups which are countable at infinity. Finally, the groups $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ are countable at infinity, as they admit surjective continuous maps from groups which are countable at infinity. □

Lemma 3.4.8.2. — *The maps $\mathbb{A}_F^\times\to(\mathbb{A}_F^\times)^n$, $\mathbb{A}_F^1\to(\mathbb{A}_F^1)^n$ and $\mathbb{R}_{>0}\to\mathbb{R}_{>0}^n$ given by $t\mapsto(t^{a_j})_j$ are proper.*

Proof. — The morphism

$$\mathbb{G}_m\to\mathbb{G}_m^n \qquad t\mapsto(t^{a_j})_j,$$

is the base change along $\mathbb{G}_m^n\times\{\mathbf{1}\}\to\mathbb{G}_m^n\times\mathbb{G}_m^n$ of the morphism $\mathbb{G}_m\times\mathbb{G}_m^n\xrightarrow{(a,p_2)}\mathbb{G}_m^n\times\mathbb{G}_m^n$ which is proper by Lemma 3.1.2.2, hence itself is proper. By [**21**, Proposition 4.4], for a proper morphism of separated schemes $X\to Y$, the induced topological map $X(\mathbb{A}_F)\to Y(\mathbb{A}_F)$ is proper. We deduce that the map $\mathbb{A}_F^\times=\mathbb{G}_m(\mathbb{A}_F)\to\mathbb{G}_m^n(\mathbb{A}_F)=(\mathbb{A}_F^\times)^n$ is proper. One has that the preimage of $(\mathbb{A}_F^1)^n$ under the map $\mathbb{A}_F^\times\to(\mathbb{A}_F^\times)^n$ is $\mathbb{A}_F^1$, thus by [7, Chapter I, §10, n^0 1, Proposition 3] the map $\mathbb{A}_F^1\to$

$(\mathbb{A}_F^1)^n$ is proper. Note that under the identification $\log : \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}$, the homomorphism $t \mapsto (t^{a_j})_j$ becomes $t \mapsto (a_j t)_j$. As for every j one has $a_j > 0$, the latter morphism is proper. The statement is proven. □

Let us write $|\cdot|$ for the map $\mathbb{A}_F^\times \to \mathbb{R}_{>0}$ given by $|(x_v)_v| = \prod_{v \in M_F} |x_v|_v$, and by $|\cdot|^n$ the product map $(\mathbb{A}_F^\times)^n \to \mathbb{R}_{>0}^n$. We recall that in Lemma 3.4.2.2 we have established that the map

$$q^{\mathbf{a}}_{\mathbb{A}_F} : (\mathbb{A}_F^\times)^n \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \qquad (\mathbf{x}_v)_v \mapsto (q_v^{\mathbf{a}}(\mathbf{x}_v))_v$$

is open, continuous and surjective. The image of $(\mathbb{A}_F^1)^n$ under $q^{\mathbf{a}}_{\mathbb{A}_F}$ we have denoted by $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$. Let $q^{\mathbf{a}}_{\mathbb{R}_{>0}} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}$ be the quotient map. The map

$$q^{\mathbf{a}}_{\mathbb{R}_{>0}} \circ |\cdot|^n : (\mathbb{A}_F^\times)^n \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}$$

is $(\mathbb{A}_F^\times)_{\mathbf{a}}$-invariant, and let

$$|\cdot|_{\mathbf{a}} : [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}} \tag{3.4.8.3}$$

be the induced map.

Lemma 3.4.8.4. — *In the commutative diagram*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & (\mathbb{A}_F^1)_{\mathbf{a}} \arrow[r] \arrow[d] & (\mathbb{A}_F^\times)_{\mathbf{a}} \arrow[r, "|\cdot|^n"] \arrow[d] & (\mathbb{R}^n_{>0})_{\mathbf{a}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (\mathbb{A}_F^1)^n \arrow[r] \arrow[d, "q^{\mathbf{a}}_{\mathbb{A}_F}"] & (\mathbb{A}_F^\times)^n \arrow[r, "|\cdot|^n"] \arrow[d, "q^{\mathbf{a}}_{\mathbb{A}_F}"] & \mathbb{R}^n_{>0} \arrow[r] \arrow[d, "q^{\mathbf{a}}_{\mathbb{R}_{>0}}"] & 1, \\
1 \arrow[r] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1} \arrow[r] \arrow[d] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} \arrow[r, "|\cdot|_{\mathbf{a}}"] \arrow[d] & \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where the maps that are not named are either canonical inclusions or the canonical maps to singletons, all horizontal and all vertical sequences are exact.

Proof. — – One has an exact sequence $1 \to \mathbb{A}_F^1 \to \mathbb{A}_F^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0} \to 1$ and its n-th product with itself is the second horizontal exact sequence, which is therefore exact.

– We establish that the first horizontal sequence is exact. We establish that the map $|\cdot|^n|_{(\mathbb{A}_F^\times)_{\mathbf{a}}} : (\mathbb{A}_F^\times)_{\mathbf{a}} \to (\mathbb{R}_{>0})_{\mathbf{a}}$ is surjective. An element in $(\mathbb{R}_{>0})_{\mathbf{a}}$ is of the form $(r^{a_j})_j$. One can find $(x_v)_v \in \mathbb{A}_F^\times$ such that $\prod_v |x_v|_v = r$. Then one has $((x_v)_v^{a_j}) \in (\mathbb{A}_F^\times)_{\mathbf{a}}$ satisfies that its image under $|\cdot|^n$ is precisely $(r^{a_j})_j$. To establish the exactness at $(\mathbb{A}_F^\times)_{\mathbf{a}}$, let $((y_v)_v^{a_j}) \in \ker(|\cdot|^n)$ and we observe that one must have for $j = 1, \dots, n$ that $\prod_v |y_v^{a_j}|_v = \big(\prod_v |y_v|_v\big)^{a_j} = 1$. As $a_j > 0$, we deduce $(y_v)_v \in \mathbb{A}_F^1$. We conclude the first horizontal sequence is exact.
– The third vertical sequence is exact by the definition.
– We establish that the second vertical sequence is exact. That the map $q^{\mathbf{a}}_{\mathbb{A}_F}$ is surjective is proven in Lemma 3.4.2.2. We prove that its kernel is $(\mathbb{A}_F^\times)_{\mathbf{a}}$. Suppose $\mathbf{x}_v \in \ker q^{\mathbf{a}}_{\mathbb{A}_F} = \ker\big(\prod_{v\in M_F} q^{\mathbf{a}}_v\big)$. Then for every $v \in M_F$ one has that $\mathbf{x}_v \in \ker q^{\mathbf{a}}_v$ and for almost every v, one has $\mathbf{x}_v \in (\mathscr{O}_v^\times)^n$. Hence, for every $v \in M_F$, there exists $y_v \in F_v^\times$ such that for almost every v one has that $y_v \in \mathscr{O}_v^\times$ and such that for every v one has $(y_v^{a_j}) = \mathbf{x}_v$. We deduce $\mathbf{x} = (y_v^{a_j})_j \in (\mathbb{A}_F^\times)_{\mathbf{a}}$. Suppose now $\mathbf{z} \in (\mathbb{A}_F^\times)_{\mathbf{a}}$ and pick $\mathbf{y} \in \mathbb{A}_F^\times$ such that $(\mathbf{y}^{a_j})_j = \mathbf{z}$. We have for every $v \in M_F$ that $q^{\mathbf{a}}_v(\mathbf{z}_v) = q^{\mathbf{a}}_v(\mathbf{y}_v^{a_j}) = 1$ and hence $q^{\mathbf{a}}_{\mathbb{A}_F}(\mathbf{z}) = 1$. We have established that the kernel of $q^{\mathbf{a}}_{\mathbb{A}_F}$ is $(\mathbb{A}_F^\times)_{\mathbf{a}}$ and we deduce the exactness of the second vertical sequence.
– The map $q^{\mathbf{a}}_{\mathbb{A}_F}|_{(\mathbb{A}_F^1)^n} : (\mathbb{A}_F^1)^n \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ is surjective and its kernel is $\ker q^{\mathbf{a}}_{\mathbb{A}_F}|_{(\mathbb{A}_F^1)^n} = \ker q^{\mathbf{a}}_{\mathbb{A}_F} \cap (\mathbb{A}_F^1)^n = (\mathbb{A}_F^1)_{\mathbf{a}}$. We deduce that the first vertical sequence is exact.
– The long exact sequence deduced from applying Five lemma on the first two horizontal sequences contains the third horizontal sequence and the statement is proven.

□

3.4.9. — We end the chapter, by observing that as in the classical situation, the short exact sequence

$$\{1\} \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \xrightarrow{|\cdot|_{\mathbf{a}}} (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}) \to \{1\}$$

splits and we give its section.

The exact sequence

$$1 \to \mathbb{A}_F^1 \to \mathbb{A}_F^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0} \to 1$$

admits for a section the map

$$\sigma : \mathbb{R}_{>0} \to \mathbb{A}_F^\times \qquad x \mapsto ((\rho_v(x)^{\frac{1}{r_1+r_2}})_{v\in M_F^\infty}, (1)_{v\in M_F^0}),$$

where, for $v \in M_F^\infty$, we define $\rho_v : \mathbb{R}_{>0} \to F_v^\times$ by $x \mapsto x^{1/n_v}$, and r_1 and r_2 are the number of real and complex places of F, respectively. We deduce an isomorphism :

$$\mathbb{A}_F^1 \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{A}_F^\times \qquad (\mathbf{x}, r) \mapsto \mathbf{x}\sigma(r). \tag{3.4.9.1}$$

The map

$$|\cdot|^n : (\mathbb{A}_F^\times)^n \to (\mathbb{R}_{>0})^n \qquad (x_j)_j \mapsto (|x_j|)_j$$

admits a section

$$\sigma^n : \mathbb{R}_{>0}^n \to (\mathbb{A}_F^\times)^n \qquad (x_j)_j \mapsto (\sigma(x_j))_j.$$

Note that $\sigma^n((\mathbb{R}_{>0})_\mathbf{a}) \subset (\mathbb{A}_F^\times)_\mathbf{a}$ and let $\sigma^\mathbf{a} : \mathbb{R}_{>0}^n/(\mathbb{R}_{>0})_\mathbf{a}$ be the map induced from $(\mathbb{R}_{>0})_\mathbf{a}$-invariant map $q^\mathbf{a}_{\mathbb{A}_F} \circ \sigma^n : \mathbb{R}_{>0}^n \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. The map $\sigma^\mathbf{a}$ is a section to the map $|\cdot|_\mathbf{a} : [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \to \mathbb{R}_{>0}^n/(\mathbb{R}_{>0})_\mathbf{a}$ and we deduce an isomorphism

(3.4.9.2)

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 \times \mathbb{R}_{>0}^n/(\mathbb{R}_{>0})_\mathbf{a} \xrightarrow{\sim} [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \qquad (\mathbf{x}, \mathbf{r}) \mapsto \mathbf{x}\sigma^\mathbf{a}(\mathbf{r}).$$

The image $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ is contained in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ by Proposition 3.4.7.2. The isomorphism (3.4.9.2) induces, hence, an identification that may be used implicitly

$$\begin{aligned}\big([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])\big) \times \big(\mathbb{R}^n/(\mathbb{R}_{>0})_\mathbf{a}\big)& \\ \xrightarrow{\sim} [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]).&\end{aligned} \tag{3.4.9.3}$$

CHAPTER 4

QUASI-TORIC HEIGHTS

In this chapter we define heights on weighted projective stacks. A height on a stack can be stable or unstable. Stable means that any two rational points of stack that are $\mathbb{C}$-isomorphic have same heights. Stable heights feature a drawback: they are not "weak Northcott heights", i.e. there may exist $B > 0$ such that there are infinitely many rational points of the stack having the height less than B (see Corollary 4.6.2.2 for an example). Hence, such heights cannot be used to count rational points.

We start the chapter by recalling several facts on the line bundles on the weighted projective stacks. We make a formalism of unstable metrics and unstable heights in sections 4.3 and 4.4. Examples of unstable heights are *quasi-toric* heights and *quasi-discriminant heights* (the latter one appears in the last chapter of this article).

A quasi-toric height is a height arising from a model with enough integral points (see 4.5) of the stack $\overline{\mathscr{P}(\mathbf{a})}$. We prove in 4.6, that they are weak Northcott heights. These heights will be used in the following chapters to make an estimate of Manin-Peyre for the number of rational points. The last section of the chapter is dedicated to the proof that quasi-toric heights admitting logarithmic singularities are weak Northcott heights.

4.1. Line bundles on a quotient stack

In the next paragraphs, we recall that line bundles on quotient stacks correspond to G-linearizations of line bundles on presentations. We use this to determine the line bundles on $\overline{\mathscr{P}(\mathbf{a})}$ and on $\mathscr{P}(\mathbf{a})$. Let Z be a scheme.

4.1.1. — Let X be a Z-algebraic stack. By a line bundle on X we mean a quasi-coherent $\mathcal{O}_X$-module L for which there exists a faithfully flat 1-morphism of finite presentation $f : X' \to X$ and an isomorphism $f^*L \cong \mathcal{O}_{X'}$.

We give another presentation of line bundles. The category of X-schemes is the category the objects of which are pairs (T, t), where T is a Z-scheme and $t : T \to X$ is a 1-morphism over Z. A morphism of X-schemes $(T', t') \to (T, t)$ is a pair $(f, f^{\#})$, where $f : T' \to T$ is a Z-morphism and $f^{\#} : t' \xrightarrow{\sim} t \circ f$. The composition of $(g, g^{\#}) : (T'', t'') \to (T', t')$ and $(f, f^{\#}) : (T', t') \to (T, t)$ is defined to be the pair consisting of $g \circ f$ and the 2-morphism:

$$t'' \xrightarrow{g^{\#}} t' \circ g \xrightarrow{g(f^{\#})} t \circ f \circ g = t \circ (f \circ g).$$

The big fppf site of X (see [**45**, Exercise 9.F]) is defined to be the category of X-schemes endowed with the Grothendieck topology defined by coverings:

$$\{(T_i, t_i) \to Y\}_i \text{ is a covering of Y, if } \coprod_i T_i \to Y \text{ is surjective and fppf.}$$

We write X_{fppf} for this site. Then a line bundle L on X is simply an fppf-locally trivial quasi-coherent sheaf on X_{fppf}.

4.1.2. — We recall the definition of a linearization of a module on a scheme (see e.g. [**41**, Definition 1.6, Section 3, Chapter 3] for the case of line bundle).

Definition 4.1.2.1 **(Stacks Project, [56, Definition 03LF])**

Let Y be a Z-scheme and let $a : G \times Y \to Y$ be a Z-algebraic action of a Z-group scheme $G = (G, m_G, e_G)$ to the left on Y. Denote by $p_2 : G \times Y \to Y$ the second projection.

1. *A G-linearized quasi-coherent $\mathcal{O}_Y$-module is a pair (M, ψ) where M is a quasi-coherent $\mathcal{O}_Y$-module and $\psi : p_2^*M \xrightarrow{\sim} a^*M$ is an isomorphism of quasi-coherent $\mathcal{O}_Y$-modules which satisfies the following cocycle condition*

$$\begin{array}{ccc} (p_2 \circ (m_G \times \mathrm{Id}_Y))^*M = (p_2 \circ p_{23})^*M & \xrightarrow{p_{23}^*\psi} & (a \circ (p_{23}))^*M = (p_2 \circ (\mathrm{Id}_G \times a))^*M \\ \downarrow{\scriptstyle (m_G \times \mathrm{Id}_Y)^*\psi} & & \downarrow{\scriptstyle (\mathrm{Id}_G \times a)^*\psi} \\ (a \circ (m_G \times \mathrm{Id}_Y))^*M & \xrightarrow{\quad = \quad} & (a \circ (\mathrm{Id}_G \times a))^*M, \end{array}$$

where $p_{23} : G\times G\times Y \to G\times Y$ is the projection obtained by forgetting the first coordinate. We also say that ψ is a G-linearization of M.

2. *A morphism $\ell : (M_1, \psi_1) \to (M_2, \psi_2)$ of G-linearized quasi-coherent $\mathcal{O}_Y$-modules is an isomorphism of $\mathcal{O}_Y$-modules $\ell : M_1 \to M_2$ such that the following diagram is commutative*

$$\begin{tikzcd} a^*M_1 \arrow[r, "\psi_1"] \arrow[d, "a^*\ell"'] & p_2^*M_1 \arrow[d, "p_2^*\ell"] \\ a^*M_2 \arrow[r, "\psi_2"'] & p_2^*M_2. \end{tikzcd}$$

3. *The tensor product of G-linearized quasi-coherent $\mathcal{O}_Y$-modules is defined by*

$$(M, \psi) \otimes (M', \psi') := (M \otimes M', \psi \otimes \psi').$$

4. *The trivial G-linearized quasi-coherent $\mathcal{O}_Y$-modules is the one given by the pair $(\mathcal{O}_Y, \mathrm{Id}_{\mathcal{O}_{G\times_Z Y}})$.*

The inverse of a morphism $\ell : (M, \psi) \to (M', \psi')$ is the morphism given by the isomorphism $\ell^{-1} : M' \to M$. One verifies easily that G-linearized quasi-coherent $\mathcal{O}_Y$-modules and their morphisms form a category, which is a Picard category (see [**24**, Exposé XVIII, Definition 1.4.2]) with respect to $\otimes$.

4.1.3. — In this paragraph we are going to discuss linearizations of the trivial line bundle. Let Y be a Z-scheme endowed with a left Z-action of a Z-group scheme G. If ψ is a G-linearization of the trivial line bundle on Y, using that $a^*\mathcal{O}_Y = \mathcal{O}_{G\times_Z Y}$ and that $p_2^*\mathcal{O}_Y = \mathcal{O}_{G\times_Z Y}$, we deduce that $\psi \in \mathcal{O}^*_{G\times_Z Y}$. The cocycle condition translates as

$$\psi(g'g, x) = \psi(g', g\cdot x)\psi(g, x) \qquad g, g' \in G, x \in Y. \tag{4.1.3.1}$$

The following things are immediate for G-linearizations of the trivial line bundle on Y.

- A morphism $\ell : (\mathcal{O}_Y, \psi_1) \to (\mathcal{O}_Y, \psi_2)$ of G-linearizations is given by an element $\ell \in \mathcal{O}_Y^*$ such that

$$\ell(g\cdot x)\psi_1(g, x) = \ell(x)\psi_2(g, x) \qquad g \in G, x \in Y.$$

- The trivial G-linearization of the trivial line bundle is the G-linearization given by $(g, x) \mapsto 1$.

– If $\ell_1 : (\mathcal{O}_Y, \psi_1) \to (\mathcal{O}_Y, \psi_2)$ and $\ell_2 : (\mathcal{O}_Y, \psi_2) \to (\mathcal{O}_Y, \psi_3)$ are morphisms, the morphism $\ell_2 \circ \ell_1 : (\mathcal{O}_Y, \psi_1) \to (\mathcal{O}_Y, \psi_3)$ is given by $l_2 l_1 \in \mathcal{O}_Y^*$.
– If ψ_1 are ψ_2 are two G-linearizations of the trivial line bundle, the G-linearization of the trivial line bundle $\psi_1 \otimes \psi_2$ is given by

$$(g, x) \mapsto \psi_1(g, x)\psi_2(g, x).$$

– The inverse of the isomorphism $\ell : (\mathcal{O}_Y, \psi_1) \to (\mathcal{O}_Y, \psi_2)$ is given by the element $\ell^{-1} \in \mathcal{O}_Y^*$.

By abuse of the terminology, we will speak about morphisms or tensor products of G-linearizations of the trivial line bundle, when if fact we mean the morphisms or tensor products of corresponding G-linearized line bundles. We have that G-linearizations of $\mathcal{O}_Y$ form a Picard category $\mathscr{P}ic^G(\mathcal{O}_Y)$. We let $\mathrm{Pic}^G(\mathcal{O}_Y)$ be the abelian group formed by the isomorphism classes of objects of $\mathscr{P}ic^G(\mathcal{O}_Y)$. Let $G^\vee$ be the group of characters of G. One has a homomorphism

$$G^\vee \to \mathrm{Pic}^G(\mathcal{O}_Y)$$

given by associating to $\chi \in G^\vee$ the isomorphism class of the $\mathbb{G}_m$-linearization of $\mathcal{O}_Y$

$$(g, x) \mapsto \chi(g). \tag{4.1.3.2}$$

Lemma 4.1.3.3. — *1. Suppose that* $\mathrm{Pic}(Y) = 0$. *One has that* $\mathrm{Pic}^G(Y) = \mathrm{Pic}^G(\mathcal{O}_Y)$.

2. Suppose that there are no non-constant morphisms $Y \to \mathbb{G}_m$. *Any two isomorphic* G*-linearizations of* $\mathcal{O}_Y$ *are identical. The homomorphism (4.1.3.2) is an isomorphism.*

Proof. — 1. It suffices to prove that $(L, \psi) \in \mathscr{P}ic^G(Y)$ is isomorphic to an element $(\mathcal{O}_Y, \psi) \in \mathscr{P}ic^G(\mathcal{O}_Y)$. As the Picard group of Y is trivial, one can find an isomorphism of line bundles $\ell : \mathcal{O}_Y \xrightarrow{\sim} L$. One readily verifies that $\theta : (a^*\ell)^{-1} \circ \psi \circ (p_2^*\ell)$ is a linearization of $\mathcal{O}_Y$. Now, we have an isomorphism $(\ell, (a^*\ell)^{-1} \circ \psi \circ (p_2^*\ell)) : (L, \psi) \to (\mathcal{O}_Y, \theta)$. It follows that $\mathrm{Pic}^G(Y) = \mathrm{Pic}^G(\mathcal{O}_Y)$.

2. An isomorphism $\ell : (\mathcal{O}_Y, \psi_1) \xrightarrow{\sim} (\mathcal{O}_Y, \psi_2)$ is given by an element $\ell \in \mathcal{O}_Y^*$ such that

$$\ell(gx)\psi_1(g, x) = \ell(x)\psi_2(g, \mathbf{x}) \qquad \forall g \in G, \forall x \in Y.$$

As ℓ is a constant morphism and $\ell(x) = \ell(gx) \neq 0$ for every g and x, we get that $\psi_1 = \psi_2$. We also get that the homomorphism

(4.1.3.2) is injective. Let us prove the second claim. As there are no non-constant morphisms $Y \to \mathbb{G}_m$, for fixed g, the morphism $\psi(g,-) : Y \to \mathbb{G}_m$ is constant. Set $\chi(g) := \psi(g,x)$ for some $x \in Y$. The cocycle condition (4.1.3.1) gives that $\chi(g'g) = \chi(g')\chi(g)$ for every $g, g' \in G$, i.e. χ is a character of G. It follows that (4.1.3.2) is surjective, and hence is an isomorphism.

□

4.1.4. — Let now Y be a Z-scheme and G a flat, locally of finite presentation Z-algebraic group scheme. Suppose we are given an action $a : G \times Y \to Y$. We are going to recall that the category of line bundles on the quotient stack Y/G is equivalent to the category of line bundles on Y endowed with a G-linearization.

Let $q : Y \to Y/G$ be the quotient 1-morphism. Let M be a $\mathcal{O}_{Y/G}$-module. Note that the pullback module q^*M is G-linearized as follows. Let $t : q \circ a \xrightarrow{\sim} q \circ pr_2$ be a 2-morphism making the diagram

$$\begin{array}{ccc} G \times_Z Y & \xrightarrow{a} & Y \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[q]{} & Y/G \end{array}$$

2-commutative. We have an isomorphism

$$pr_2^*(q^*M) = (q \circ pr_2)^*M \xrightarrow{t^*M} (q \circ a)^*M = a^*(q^*M),$$

where t^*M is the isomorphism given by [**56**, Lemma 06WK]. We omit the proof that the cocycle condition from 4.1.2.1 is satisfied. If $\ell : M \to M'$ is a morphism of $\mathcal{O}_{Y/G}$-modules, then a morphism of G-linearized $\mathcal{O}_Y$-modules is provided by $q^*\ell$. We have, hence, a functor $\mathcal{F}$ from the category of $\mathcal{O}_{Y/G}$-modules to the category of G-linearized $\mathcal{O}_Y$-modules given by $\mathcal{F} : M \mapsto (M, t^*M)$. One can verify that $t^*(M \otimes M') = t^*M \otimes t^*M'$ and we deduce that the functor is an additive functor. According to [**56**, Proposition 06WT], $\mathcal{F}$ is an equivalence of categories. We consider the restriction of $\mathcal{F}$ on the category of the line bundles on Y. It is immediate that if L is a line bundle on Y/G then $\mathcal{F}(Y)$ is a G-linearized line bundle on Y.

Proposition 4.1.4.1. — *The above functor $\mathcal{F}$ induces an additive (*[**24**, Exposé XVIII, Definition 1.4.2]*) equivalence of the Picard category $\mathscr{P}ic(Y/G)$ of the line bundles on Y/G and the Picard category $\mathscr{P}ic^G(Y)$ of G-linearized line bundles on Y.*

Proof. — By [**56**, Proposition 06WT] the functor $\mathcal{F}$ is fully faithful, thus the restricted functor $\mathcal{F}|_{\mathscr{P}ic(Y/G)}$ is an equivalence to its image in $\mathscr{P}ic^G(Y)$. It suffices therefore to verify that an object $(L,\psi) \in \mathscr{P}ic^G(Y/G)$ is isomorphic to an object in the image of $F|_{\mathscr{P}ic(Y/G)}$. It follows from [**56**, Proposition 06WT] that there exists a $\mathcal{O}_{Y/G}$-module M such that $\mathcal{F}(M) \cong (L,\psi)$. Let us prove M is locally fppf trivial. The pullback π^*M is a line bundle, thus, there exists Zariski open covering $y : Y' \to Y$ such that $y^*\pi^*M$ is locally trivial. It follows that for the fppf covering $\pi \circ y : Y' \to Y$ one has $(\pi \circ y)^*M \cong \mathcal{O}_{Y'}$. □

If X is an algebraic stack, we denote by $\mathrm{Pic}(X)$ the Picard group of X. We denote by $\mathrm{Pic}^G(Y)$ the abelian group the elements of which are isomorphism classes of objects in $\mathscr{P}ic^G(Y)$, and the addition of which is defined by the tensor product in $\mathscr{P}ic^G(Y)$. Functor $\mathcal{F}$ induces a homomorphism $\mathrm{Pic}(Y/G) \to \mathrm{Pic}^G(Y)$. Proposition 4.1.4.1 gives that it is an isomorphism.

4.1.5. — Let $\phi : G_1 \to G_2$ be a homomorphism of flat, locally of finite type Z-group schemes. Let $a_X : G_1 \times_Z X \to X$ and $a_Y : G_2 \times_Z Y \to Y$ be Z-actions on Z-schemes X and Y. Suppose $f : X \to Y$ is a morphism of Z-schemes and suppose furthermore that the following diagram is commutative:

$$\begin{tikzcd} G_1 \times_Z X \arrow[r, "(\phi,f)"] \arrow[d, "a_X"'] & G_2 \times_Z Y \arrow[d, "a_Y"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

We say that f is ϕ-equivariant. We construct an additive functor of Picard categories $\mathscr{P}ic_{G_2}(Y) \to \mathscr{P}ic_{G_1}(X)$ as follows. Let $(L,\psi) \in \mathscr{P}ic_{G_2}(Y)$. Denote by $pr_X : G_1 \times_Z X \to X$ and $pr_Y : G_2 \times_Z Y \to Y$ the projections to the second coordinate. The linearization on f^*L is provided by the isomorphism

$$(\phi,f)^*\psi \ : \ (pr_X)^*(f^*L) = (\phi,f)^*(pr_Y)^*L \xrightarrow{(\phi,f)^*\psi} (\phi,f)^*a_Y^*L = a_X^*(f^*L).$$

If $\ell : (L,\psi) \to (L',\psi')$ is a morphism in $\mathscr{P}ic_{G_2}(Y)$, then $f^*\ell$ is a morphism of $(f^*L, (\phi,f)^*\psi) \to (f^*L', (\phi,f)^*\psi)$. It is a straightforward verification that this construction provides a functor $\mathscr{P}ic_{G_2}(Y) \to \mathscr{P}ic_{G_1}(X)$, and it is moreover an additive functor. The induced map $\mathrm{Pic}_{G_2}(Y) \to \mathrm{Pic}_{G_1}(X)$ is thus a homomorphism.

Lemma 4.1.5.1. — *Let $\overline{f} : X/G_1 \to Y/G_2$ be the morphism of quotient stacks given by* [**56**, Lemma 046Q]. *The following diagram is 2-commutative:*

$$\begin{array}{ccc} \mathscr{P}ic^{G_2}(Y) & \longrightarrow & \mathscr{P}ic^{G_1}(X) \\ \uparrow & & \uparrow \\ \mathscr{P}ic(Y/G_2) & \longrightarrow & \mathscr{P}ic(X/G_1). \end{array} \tag{4.1.5.2}$$

The diagram

$$\begin{array}{ccc} \mathrm{Pic}^{G_2}(Y) & \longrightarrow & \mathrm{Pic}^{G_1}(X) \\ \uparrow & & \uparrow \\ \mathrm{Pic}(Y/G_2) & \longrightarrow & \mathrm{Pic}(X/G_1). \end{array} \tag{4.1.5.3}$$

is commutative.

Proof. — Fix 2-isomorphisms $t_X : q_X \circ pr_X \xrightarrow{\sim} q_X \circ a_X$, $t_Y : q_Y \circ pr_Y \xrightarrow{\sim} q_Y \circ a_Y$ and $t_f : q_Y \circ f \xrightarrow{\sim} \overline{f} \circ q_X$ that make corresponding diagrams 2-commutative. Let $L \in \mathscr{P}ic(Y/G_2)$. The image of L in $\mathscr{P}ic^{G_2}(Y)$ is (q_Y^*L, t_Y^*L) and the image of (q_Y^*L, t_Y^*L) in $\mathscr{P}ic^{G_1}(X)$ is $(f^*q_Y^*L, (\phi, f)^*t_Y^*L)$. The image of L in $\mathscr{P}ic(X/G_1)$ is $\overline{f}^*L$, and the image of $\overline{f}^*L$ in $\mathscr{P}ic^{G_1}(X)$ is $(q_X^*\overline{f}^*L, t_X^*(\overline{f}^*L))$. It suffices to verify that the two images under the composite functors of L in $\mathscr{P}ic^{G_1}(X)$ are isomorphic. In fact, we will verify that

$$t_f^*L : f^*q_Y^*L = (q_Y \circ f)^*L \xrightarrow{\sim} (\overline{f} \circ q_X)^*L = q_X^*\overline{f}^*L$$

is an isomorphism $(f^*q_Y^*L, (\phi, f)^*t_Y^*L) \xrightarrow{\sim} (q_X^*\overline{f}^*L, t_X^*\overline{f}^*L)$. For that we need to verify the commutativity of the following diagram:

$$\begin{array}{ccc} pr_X^*f^*q_Y^*L & \xrightarrow{(\phi,f)^*t_Y^*L} & a_X^*f^*q_Y^*L \\ \Big\downarrow{\scriptstyle pr_X^*t_f^*L} & & {\scriptstyle a_X^*t_f^*L}\Big\downarrow \\ pr_X^*q_X^*\overline{f}^*L & \xrightarrow[t_X^*\overline{f}^*L]{} & a_X^*q_X^*\overline{f}^*L \end{array}$$

This is true as in fact

$$(t_f * a_X) \circ (t_Y * (\phi, f)) = (\overline{f} * t_X) \circ (t_f * pr_X),$$

as one verifies in a straightforward way, where $*$ is the operation given by the structures of 2-categories (if a is 1-morphism compozable with 1-morphisms i and j and $b : i \to j$ a 2-morphism, then $b*a$ is a 2-morphism $i \circ a \to j \circ a$, and analogously for the composability on the other side). Therefore the diagram (4.1.5.2) is 2-commutative, and hence the diagram (4.1.5.3) is commutative. □

4.2. Picard group of a weighted projective stack

We are going to calculate Picard groups of weighted projective stacks.

Let Z be a scheme. All the schemes and stacks are understood to be over Z. Let $n \geq 1$ and let $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$.

4.2.1. — We calculate the Picard group of $\overline{\mathscr{P}(\mathbf{a})}$ and of $\mathscr{P}(\mathbf{a})$. Proposition 4.1.4.1 gives that the canonical homomorphisms

$$\operatorname{Pic}(\overline{\mathscr{P}(\mathbf{a})}) \to \operatorname{Pic}^{\mathbb{G}_m}(\mathbb{A}^n)$$

and

$$\operatorname{Pic}(\mathscr{P}(\mathbf{a})) \to \operatorname{Pic}^{\mathbb{G}_m}(\mathbb{A}^n - \{0\})$$

are isomorphisms.

Proposition 4.2.1.1. — *Let $n \geq 1$ be an integer and let $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$.*

1. *One has that*

$$\operatorname{Pic}(\overline{\mathscr{P}(\mathbf{a})}) \cong \mathbb{Z}$$

and a generator of $\operatorname{Pic}(\overline{\mathscr{P}(\mathbf{a})})$ is given by the isomorphism class of line bundles defined by the $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{A}^n}$:

$$\psi : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{G}_m \qquad (t, \mathbf{x}) \mapsto t.$$

2. *Suppose that $n \geq 2$. One has that*

$$\operatorname{Pic}(\mathscr{P}(\mathbf{a})) \cong \mathbb{Z}$$

and a generator of $\operatorname{Pic}(\mathscr{P}(\mathbf{a}))$ is given by the isomorphism class of line bundles defined by the $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{A}^n - \{0\}}$:

$$\psi : \mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to \mathbb{G}_m \qquad (t, \mathbf{x}) \mapsto t.$$

3. *Suppose that $n = 1$. One has that*

$$\operatorname{Pic}(\mathscr{P}(a)) \cong \mathbb{Z}/a\mathbb{Z}$$

and a generator of $\mathrm{Pic}(\mathscr{P}(a))$ *is given by the isomorphism class of line bundles defined by the* $\mathbb{G}_m$*-linearization of* $\mathcal{O}_{\mathbb{A}^1-\{0\}}$:

$$\psi : \mathbb{G}_m \times (\mathbb{A}^1 - \{0\}) \to \mathbb{G}_m \qquad (t, x) \mapsto t.$$

Proof. — Let us prove the first two statements. We have that $\mathrm{Pic}(\mathbb{A}^n) = \mathrm{Pic}(\mathbb{A}^n - \{0\}) = 0$ and thus by Lemma 4.1.3.3, one has equalities of abelian groups $\mathrm{Pic}^{\mathbb{G}_m}(\mathbb{A}^n) = \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^n})$ and $\mathrm{Pic}^{\mathbb{G}_m}(\mathbb{A}^n - \{0\}) = \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^n-\{0\}})$. There are no non-constant morphisms $\mathbb{A}^n \to \mathbb{G}_m$ (respectively, non-constant morphisms $\mathbb{A}^n - \{0\} \to \mathbb{G}_m$) and thus by Lemma 4.1.3.3 any two $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{A}^n}$ (respectively, of $\mathcal{O}_{\mathbb{A}^n-\{0\}}$) are identical. Moreover, by the same lemma, the homomorphisms $\mathbb{G}_m^\vee \to \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^n}) = \mathrm{Pic}^{\mathbb{G}_m}(\mathbb{A}^n)$ and $\mathbb{G}_m^\vee \to \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^n-\{0\}}) = \mathrm{Pic}^{\mathbb{G}_m}(\mathbb{A}^n - \{0\})$ given by

$$\chi \mapsto ((t, \mathbf{x}) \mapsto \chi(t))$$

are isomorphisms. The group $\mathbb{G}_m^\vee$ is the infinite cyclic group with generator $t \mapsto t$, therefore the isomorphism class of the linearization $(t, \mathbf{x}) \mapsto t$ of the trivial line bundle is a generator of the group of the $\mathbb{G}_m$-linearizations of the trivial line bundle (in both cases). The first two claim follow.

Let us prove the third claim. One has that $\mathrm{Pic}(\mathbb{A}^1 - \{0\}) = 0$, thus by Lemma 4.1.3.3, one has an equality of abelian groups $\mathrm{Pic}^{\mathbb{G}_m}(\mathbb{A}^1 - \{0\}) = \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1-\{0\}})$. We study the latter group. Let $\widetilde{\mathrm{Pic}}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1-\{0\}})$ be the group of $\mathbb{G}_m$-linearizations of $\mathcal{O}_{\mathbb{A}^1-\{0\}}$. We will split the proof into two parts. First, we prove that $\widetilde{\mathrm{Pic}^{\mathbb{G}_m}}(\mathcal{O}_{\mathbb{A}^1-\{0\}})$ is isomorphic to $\mathbb{Z}$ and that its generator is given by $(t, x) \mapsto t$. Then we will prove that the canonical surjective homomorphism $\widetilde{\mathrm{Pic}}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1)-\{0\}} \to \mathrm{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1-\{0\}})$ has for the kernel the subgroup generated by the $\mathbb{G}_m$-linearization $(t, x) \mapsto t^a$.

The cocycle condition $\psi(t't, x) = \psi(t', t \cdot x)\psi(t, x)$ implies that the degree of ψ in x must be zero i.e. $\psi(t, -) : (\mathbb{A}^1 - \{0\}) \to \mathbb{G}_m$ is a constant morphism. Set $\chi(t) := \psi(t, x)$ for some $x \in \mathbb{A}^1 - \{0\}$. The cocycle condition gives $\chi(t't) = \chi(t')\chi(t)$, i.e. χ is a character of $\mathbb{G}_m$. It follows that the homomorphism

$$\mathbb{G}_m^\vee \to \widetilde{\mathrm{Pic}^{\mathbb{G}_m}}(\mathcal{O}_{\mathbb{A}^1-\{0\}}) \qquad \chi \mapsto \big((t, x) \mapsto \chi(t)\big)$$

is surjective. This homomorphism is evidently injective, hence is an isomorphism. The group $\mathbb{G}_m^\vee$ is isomorphic to $\mathbb{Z}$ and its generator is given by $t \mapsto t$, thus $\widetilde{\mathrm{Pic}^{\mathbb{G}_m}}(\mathcal{O}_{\mathbb{A}^1-\{0\}})$ is isomorphic to $\mathbb{Z}$ and its generator is given by $(t,x)\mapsto t$. Let us now prove the second claim. Let $\ell:(\mathcal{O}_{\mathbb{A}^1-\{0\}},(t,x)\mapsto t^k)\to(\mathcal{O}_{\mathbb{A}^1},(t,x)\mapsto 1)$ be an isomorphism of $\mathbb{G}_m$-linearizations of $\mathcal{O}_{\mathbb{A}^1-\{0\}}$. This means that $\ell:\mathbb{A}^1-\{0\}\to\mathbb{G}_m$ is a morphism such that for every $t\in\mathbb{G}_m$ and every $x\in\mathbb{A}^1-\{0\}$ one has that $\ell(t\cdot x)t^k=\ell(x)$ i.e. $\ell(t^ax)=\ell(x)t^{-k}$. Thus the degree of the rational function ℓ must be $(-k)/a$. It follows that if $(t,x)\mapsto t^k$ and $(t,x)\mapsto 1$ are isomorphic, then $k\equiv 0 \pmod a$. Let us see that if $a|k$, then the $\mathbb{G}_m$-linearizations $(t,x)\mapsto t^k$ and $(t,x)\mapsto 1$ are isomorphic. Indeed if one sets $\ell(x)=x^{-k/a}$, we have that

$$\ell(t\cdot x)t^k=(t\cdot x)^{(-k)/a}t^k=(t^ax)^{(-k)/a}t^{n_1}=x^{(-k)/a}t^{n_2}=\ell(x).$$

The second claim follows. We deduce that $\operatorname{Pic}^{\mathbb{G}_m}(\mathbb{A}^1-\{0\})=\operatorname{Pic}^{\mathbb{G}_m}(\mathcal{O}_{\mathbb{A}^1-\{0\}})\cong\mathbb{Z}/n\mathbb{Z}$ that its generator is given by the isomorphism class of the $\mathbb{G}_m$-linearization of the trivial line bundle $(t,x)\mapsto t$. As $\operatorname{Pic}(\mathscr{P}(a))\xrightarrow{\sim}\operatorname{Pic}^{\mathbb{G}_m}(\mathbb{A}^1-\{0\})$ is an isomorphism by Proposition 4.1.4.1, the statement follows. □

Definition 4.2.1.2. — *Let $\mathcal{O}(1)$ be the isomorphism class of line bundles on $\overline{\mathscr{P}(\mathbf{a})}$ given by the linearization of the trivial line bundle*

$$\psi:\mathbb{G}_m\times\mathbb{A}^n\to\mathbb{G}_m\qquad (t,\mathbf{x})\mapsto t.$$

For $k\in\mathbb{Z}$, we write $\mathcal{O}(k)$ for $\mathcal{O}(1)^{\otimes k}$. By abuse of notation, we may also write $\mathcal{O}(k)$ for a line bundle in the corresponding isomorphism class of line bundles.

Definition 4.2.1.3. — *Let $\mathcal{O}(1)$ be the isomorphism class of line bundles on $\mathscr{P}(\mathbf{a})$ given by the linearization of the trivial line bundle*

$$\psi:\mathbb{G}_m\times(\mathbb{A}^n-\{0\})\to\mathbb{G}_m\qquad (t,\mathbf{x})\mapsto t.$$

For $k\in\mathbb{Z}$, we write $\mathcal{O}(k)$ for $\mathcal{O}(1)^{\otimes k}$. By abuse of notation, we may also write $\mathcal{O}(k)$ for a line bundle in the corresponding isomorphism class of line bundles.

When $n=1$ one has that $\mathcal{O}(a)=\mathcal{O}(0)=\mathcal{O}_{\mathscr{P}(\mathbf{a})}$. For $\mathbf{b}\in\mathbb{Z}^n$, we denote $|\mathbf{b}|:=b_1+\cdots+b_n$.

Definition 4.2.1.4. — *We say that a line bundle on $\overline{\mathscr{P}(\mathbf{a})}$ with isomorphism class $\mathcal{O}(|\mathbf{a}|)$ is anti-canonical. We may write $(K_{\overline{\mathscr{P}(\mathbf{a})}})^{-1}=\mathcal{O}(|\mathbf{a}|)$.*

4.3. Metric on a line bundle on a stack

In this section we define metrics on line bundles on algebraic stacks.

4.3.1. — Let v be a place of F. We present a definition of an F_v-metric on a line bundle on a stack.

Definition 4.3.1.1. — *Let X be a locally of finite type F_v-algebraic stack and let L be a line bundle on X. An F_v-metric $||\cdot||$ is the following data:*

- *for every 1-morphism of stacks $x : \mathrm{Spec}(F_v) \to X$ we give a norm $||\cdot||_x$ on $L(x) := x^*L$.*
- *for every 2-morphism $x \xrightarrow{\sim} y$ of 1-morphisms $x : \mathrm{Spec}(F_v) \to X$ and $y : \mathrm{Spec}(F_v) \to X$, the canonical morphism $L(x) \to L(y)$ is an isometric isomorphism.*
- *for every 1-morphism $f : U \to X$ over F_v, with U locally of finite type F_v-scheme and every section s of f^*L over U, the map*

$$U(F_v) \to \mathbb{R}_{\geq 0} \qquad z \mapsto ||s(z)||_{f \circ z}$$

is continuous.

An F_v-metrized line bundle is a pair $(L, ||\cdot||)$ of a line bundle L and an F_v-metric on L.

Let v be a place of F and let X be a locally of finite type F_v-algebraic stack.

- A morphism $r : (L, ||\cdot||) \to (L', ||\cdot||')$ of F_v-metrized line bundles on X is an isomorphism of line bundles $r : L \to L'$ which is isometric i.e. for every $x \in X(F_v)$, the morphism $r(x) : L(x) \to L'(x)$ is isometric.
- The trivial line bundle can be endowed with the following metric: set $||1||_x = 1$ for each 1-morphism $x : \mathrm{Spec}(F_v) \to X$. The corresponding F_v-metrized line bundle will be called the trivial F_v-metrized line bundle.
- One defines the tensor product of F_v-metrized line bundles on X as follows: let $(L, ||\cdot||) \otimes (L', ||\cdot||')$ be F_v-metrized line bundles on X, endow $L \otimes L'$ with the metric $||\cdot|| \otimes ||\cdot||'$ defined by $||\cdot||_x \otimes ||\cdot||'_x$ for every 1-morphism $x : \mathrm{Spec}(F_v) \to X$ (it is immediate that one indeed gets a metric on $L \otimes L'$).

Lemma 4.3.1.2. — *Let X be an algebraic stack and let L be a line bundle on X.*

1. *Suppose $||\cdot||$ is an F_v-metric on L. For every $x \in X(F_v)$, let $||\cdot||_x^{-1}$ be the metric on $L^{-1}(x)$ defined by*
$$||\cdot||_x^{-1}(\lambda) := ||\ell||_x^{-1},$$
for $0 \neq \ell \in L(x)$ and $\lambda \in L^{-1}(x)$ such that $\lambda(\ell) = 1$. The metrics $||\cdot||_x^{-1}$ on $L^{-1}(x)$ for $x \in X(F_v)$ define an F_v-metric $||\cdot||^{-1}$ on L^{-1}.
2. *Suppose $||\cdot||$ is an F_v-metric on $L^{\otimes m}$ where m is a positive integer. For every $x \in X(F_v)$ and every $\ell \in L(x)$ let $\sqrt[m]{||\ell||}_x$ be the metric on $L(x)$ defined by*
$$\sqrt[m]{||\cdot||}_x(\ell) := \sqrt[m]{||\ell^m||_x}.$$
The metrics $\sqrt[m]{||\cdot||}_x$ on $L(x)$ for $x \in X(F_v)$ define an F_v-metric $\sqrt[m]{||\cdot||}$ on L.

Proof. — 1. Suppose that $x \xrightarrow{\sim} y$ is a 2-morphism, where $x, y \in X(F_v)$. The induced linear map $L(x) \to L(y)$ is isometric, and it follows that $L^{-1}(x) \to L^{-1}(y)$ is isometric. Let now $g : U \to X$ be a 1-morphism, with U is a locally of finite type F_v-scheme and let $s \in g^*L(U)$. The map $U(F_v) \to \mathbb{R}_{>0}$ given by $z \mapsto ||s(z)||^{-1}$ is precisely the composition of the map
$$U(F_v) \to \mathbb{R}_{>0} \qquad z \mapsto ||s(z)||_{g\circ x}$$
and the map $\mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto x^{-1}$ and is thus continuous. It follows that $||\cdot||^{-1}$ is an F_v-metric on L^{-1}.

2. Suppose that $x \xrightarrow{\sim} y$ is a 2-morphism, where $x, y \in X(F_v)$. The induced map $L^{\otimes m}(x) \to L^{\otimes m}(y)$ is isometric, and it follows that $L(x) \to L(y)$ is isometric. Let now $g : U \to X$ be a 1-morphism, with U is a locally of finite type F_v-scheme and let $s \in g^*L(U)$. The map $U(F_v) \to \mathbb{R}_{>0}$ given by $z \mapsto \sqrt[m]{||s(z)||}$ is precisely the composition of the map
$$U(F_v) \to \mathbb{R}_{>0} \qquad z \mapsto ||\ell^m||_{g\circ x}$$
and the map $\sqrt[m]{\cdot} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and is thus continuous. It follows that $\sqrt[m]{||\cdot||}$ is an F_v-metric on L.

□

One sees that the category $\widehat{\mathscr{P}ic}_v(X)$ of F_v-linearized line bundles is a Picard category. The abelian group of isomorphism classes of objects in this category will be denoted by $\widehat{\mathrm{Pic}}_v(X)$. By abuse of terminology, we may call an element of $\widehat{\mathrm{Pic}}_v(X)$ an F_v-metrized line bundle.

For topological spaces A, B, let us denote by $\mathscr{C}^0(A,B)$ the set of continuous functions from A to B. If B is a topological abelian group, then $\mathscr{C}^0(A,B)$ carries a structure of an abelian group.

Lemma 4.3.1.3. — *Let v be a place of F and let X be a locally of finite type F_v-algebraic stack. If $f : [X(F_v)] \to \mathbb{R}_{>0}$ is a continuous function, then setting $||\ell||_x^f := |\ell(x)|_v f([x])$, for $\ell \in \mathcal{O}_X(x) = F_v$, defines an F_v-metric $||\cdot||^f$ on $\mathcal{O}_X$. The sequence*

$$0 \to \mathscr{C}^0([X(F_v)], \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}}_v(X) \to \mathrm{Pic}(X)$$

where the first homomorphism is given by $f \mapsto (\mathcal{O}_X, (||\cdot||^f)$ and where the second homomorphism is the one that forgets the metric, is exact.

Proof. — Let us verify that $||\cdot||^f$ is a metric on $\mathcal{O}_X$. Let us verify that for any 2-morphism $x \xrightarrow{\sim} y$, where $x, y : \mathrm{Spec}(F_v) \to X$ are 1-morphisms of algebraic stacks, the induced linear map $\mathcal{O}_X(x) \xrightarrow{=} F_v \xrightarrow{=} \mathcal{O}_X(y)$ is an isometry. This is true because $||1||_x^f = f([x]) = f([y]) = ||1||_y^f$. We now verify the last condition of Definition 4.3.1.1. Let $r : V \to X$ be a 1-morphism with V a locally of finite type F_v-scheme and let s be a section over V of $r^*\mathcal{O}_X = \mathcal{O}_V$ i.e. a morphism $s : V \to \mathbb{A}^1$. The map

$$V(F_v) \to \mathbb{R}_{>0} \qquad z \mapsto ||s(z)||_{r\circ z}$$

coincides with the map

$$V(F_v) \to \mathbb{R}_{>0} \qquad z \mapsto |s(z)|_v f([r(z)])$$

and is thus is continuous. We have verified that $||\cdot||^f$ is a metric on $\mathcal{O}_X$.

It is evident that the composite homomorphism

$$\mathscr{C}^0([X(F_v)], \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}}_v(X) \to \mathrm{Pic}(X)$$

is the zero homomorphism. Let $(L, ||\cdot||)$ be in the kernel of $\widehat{\mathrm{Pic}}_v(X) \to \mathrm{Pic}(X)$. We verify that $(L, ||\cdot||)$ is in the image of $\mathscr{C}^0([X(F_v)], \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}}_v(X)$. Obviously, $L = \mathcal{O}_X$ in $\mathrm{Pic}(X)$. By the fact that a 2-morphism $x \xrightarrow{\sim} y$ induces an isometry $\mathcal{O}_X(x) \xrightarrow{=} F_v \xrightarrow{=} \mathcal{O}_X(y)$, it follows that

$$[X(F_v)] \to \mathbb{R}_{>0} \qquad [x] \mapsto ||1||_x$$

is a well defined function. We verify it is continuous. For every 1-morphism $g : U \to X$, with U locally of finite type F_v-scheme, the function

$$U(F_v) \xrightarrow{[g(F_v)]} [X(F_v)] \xrightarrow{[x]\mapsto ||1||_x} \mathbb{R}_{>0}$$

is continuous as it coincides with the map

$$U(F_v) \to \mathbb{R}_{>0} \qquad x \mapsto ||1||_{g(x)},$$

which is continuous by the definition of an F_v-metric. We deduce from Lemma 3.3.2.2 that the map $[x] \mapsto ||1||_x$ is a continuous function. It follows that $(L, ||\cdot||) = (\mathscr{O}_X, ||\cdot||)$ is in the image of $\mathscr{C}^0([X(F_v)], \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}}_v(X)$. The statement is proven. $\square$

4.3.2. — Let v be a place of F. Suppose $g : Y \to X$ is a 1-morphism of locally of finite type F_v-algebraic stacks. Let $(L, ||\cdot||_L)$ be an F_v-metrized line bundle on X. We define the pullback metric on g^*L. Suppose $x : \mathrm{Spec}(F_v) \to Y$ is an F_v-point of Y. For a section $\ell \in (g^*L)(x) = x^*(g^*L) = (g\circ x)^*L = L(g(x))$, we set $g^*||\ell||_x := ||\ell||_{g\circ x}$.

Lemma 4.3.2.1. — *Let $g : Y \to X$ be a 1-morphism of locally of finite type F_v-algebraic stacks.*

1. *If $(L, ||\cdot||)$ is an F_v-metrized line bundle on X. The pair $(g^*L, g^*||\cdot||)$ is an F_v-metrized line bundle on Y.*
2. *If $(L, ||\cdot||)$ is the trivial F_v-metrized line bundle on X, then $(g^*L, g^*||\cdot||)$ is the trivial F_v-metrized line bundle on Y.*
3. *If*
$$r : (L, ||\cdot||) \to (L', ||\cdot||')$$
is a morphism of F_v-metrized line bundles on X, then g^*r is isometric.
4. *The functor $g^* : \widehat{\mathscr{P}ic}_v(X) \to \widehat{\mathscr{P}ic}_v(Y)$ given by*
$$(L, ||\cdot||) \mapsto (g^*L, g^*||\cdot||)$$
$$r : (L, ||\cdot||) \to (L', ||\cdot||') \mapsto g^*r$$
is an additive functor.
5. *Let $f : Y \to X$ be another 1-morphism and let $t : g \xrightarrow{\sim} f$ be a 2-isomorphism. Let $(L, ||\cdot||)$ be an F_v-metrized line bundle on X. The canonical isomorphism $t^*L : g^*L \xrightarrow{\sim} f^*L$ is an isometry.*

Proof. — 1. Let us verify that $g^*||\cdot||$ is an F_v-metric on g^*L. Let $y : \mathrm{Spec}(F_v) \to Y$ be such that there exists a 2-morphism $x \xrightarrow{\sim} y$. The canonical morphism
$$(g^*L)(x) = x^*(g^*L) = (g\circ x)^*L = L(g(x)) \to L(g(y)) = (g^*L)(y)$$
is an isometric isomorphism, as such is $L(g(x)) \to L(g(y))$. Let $h : U \to Y$ be a 1-morphism of algebraic stacks with U locally of finite type F_v-scheme. Pick $s \in (h^*(g^*L))(U) = ((g\circ h)^*L)(U)$. The map
$$U(F_v) \to \mathbb{R}_{\geq 0} \qquad z \mapsto g^*||s(z)||_{h\circ z}$$

is continuous, because it coincides with the continuous map

$$U(F_v) \to \mathbb{R}_{\geq 0} \qquad z \mapsto ||s(z)||_{g\circ(h\circ z)}.$$

2. One has that $g^*\mathcal{O}_X = \mathcal{O}_Y$. Let $x \in Y(F_v)$. One has that

$$g^*||1||_x = ||1||_{g(x)} = 1.$$

The claim follows.
3. Let $x \in Y(F_v)$. The linear map $(g^*r)(x) : (g^*L')(x) \to (g^*L')(g(x))$ is isometric as it coincides with the isometric linear map $L(g(x)) \to L'(g(x))$. It follows that g^*r is isometric.
4. Let $x \in Y(F_v)$ and pick $\ell \in (g^*L)(x) = L(g(x))$ and $\ell' \in (g^*L')(x) = L'(g(x))$. We have that

$$g^*||\ell||_x \cdot g^*||\ell'||'_x = ||\ell||_{g(x)} \cdot ||\ell'||'_{g(x)}.$$

It follows that $g^*(||\cdot||) \otimes g^*(||\cdot||') = g^*(||\cdot|| \otimes ||\cdot||')$. It follows that g is an additive functor.
5. Let y be an F_v-point of Y. The isomorphism $t : g \xrightarrow{\sim} f$ induces the isomorphism $(t * y) : g \circ y \xrightarrow{\sim} f \circ y$. The linear map t^*L is precisely the linear map $g^*L(y) = L(g(y)) \xrightarrow{(t*y)^*L} L(f(y)) = f^*L(y)$ and is thus an isometry by the definition of an F_v-metric. The claim is proven.

$\square$

The lemma provides a group homomorphism $\widehat{\mathrm{Pic}}_v(X) \to \widehat{\mathrm{Pic}}_v(Y)$ that we also denote by g^*.

4.3.3. — In this paragraph we study F_v-metrics which are invariant for an action of an algebraic group.

Let v be a place of F. Let G be a locally of finite type F_v-algebraic group acting on a locally of finite type F_v-scheme X. By Corollary 3.3.1.3, one gets a topological action $G(F_v) \times X(F_v) \to X(F_v)$.

Definition 4.3.3.1. — *Let L be a G-linearized line bundle on X. An F_v-metric $||\cdot||$ on L is said to be G-invariant if for every $t \in G(F_v)$ and every $x \in X(F_v)$, one has that the linear map $L(x) \to L(t\cdot x)$ given by the linearization is an isometric isomorphism. The F_v-metrized line bundle $(L, ||\cdot||)$ will be said to be G-invariant.*

Let us introduce the category of G-invariant F_v-metrized line bundles.

- A morphism $\ell : (L, ||\cdot||) \to (L', ||\cdot||')$ of G-invariant F_v-metrized line bundles is a morphism of G-linearized line bundles $\ell : L \to L'$ which is isometric.

- The trivial G-invariant F_v-metrized line bundle is the trivial G-linearized line bundle endowed with the metric: $||1||_x = 1$ for every $x : \mathrm{Spec}(F_v) \to X$ (it is immediate the metric is G-invariant).
- A morphism of G-invariant F_v-metrized line bundles is a morphism of corresponding G-linearized line bundles which is an isometry.
- The tensor product of two G-invariant F_v-metrized line bundles is a G-invariant F_v-metrized line bundle.

The G-invariant F_v-metrized line bundles form a Picard category that we denote by $\widehat{\mathscr{P}ic^G_v}(X)$. Let $\widehat{\mathrm{Pic}^G_v}(X)$ be the abelian group given by the isomorphism classes of objects of $\widehat{\mathscr{P}ic^G_v}(X)$. One has a homomorphism of abelian groups

$$\widehat{\mathrm{Pic}^G_v}(X) \to \mathrm{Pic}^G(X)$$

which forgets the structure of F_v-metrized line bundle. One also has a canonical morphism $\widehat{\mathrm{Pic}^G_v}(X) \to \widehat{\mathrm{Pic}}_v(X)$ by simply forgetting that F_v-metrized line bundle is G-invariant.

If E is a topological group acting on a topological space A, we denote by $\mathscr{C}^0_E(A, B)$ the set of E-invariant continuous functions $A \to B$. If B has a structure of a topological abelian group, then $\mathscr{C}^0_E(A, B)$ carries a structure of an abelian group. The map $\mathscr{C}^0(A/E, B) \to \mathscr{C}^0(A, B)$ induces an isomorphism

$$\mathscr{C}^0(A/E, B) \xrightarrow{\sim} \mathscr{C}^0_E(A, B).$$

Lemma 4.3.3.2. — *If $f \in \mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0})$ then setting*

$$||\ell||^f_x := |\ell(x)|_v f(x)$$

for every $x \in X(F_v)$ and every $\ell \in \mathcal{O}_X(x) = F_v$, defines an F_v-metric $||\cdot||^f$ on $\mathcal{O}_X$ which is G-invariant for the trivial G-linearization of $\mathcal{O}_X$. The sequence

$$0 \to \mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}^G_v}(X) \to \mathrm{Pic}^G(X),$$

where $\mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}^G_v}(X)$ is given by $f \mapsto (\mathcal{O}_X, \mathrm{Id}_{\mathcal{O}_{G\times X}}, ||\cdot||^f)$, is exact.

Proof. — Lemma 4.3.1.3 gives that $||\cdot||^f$ is a metric on $\mathcal{O}_X$. Moreover, for every $x \in X(F_v)$ and every $t \in G(F_v)$, the linear map $\mathcal{O}_X(x) \xrightarrow{=} F_v \xrightarrow{=} \mathcal{O}_X(t \cdot x)$ given by the trivial G-linearization, is an isometry, because $||1||^f_x = f(x) = f(t \cdot x) = ||1||^f_{t\cdot x}$. It follows that the F_v-metric $||\cdot||^f$ is G-invariant.

The composite homomorphism $\mathscr{C}^0_{G(F_v)}(X(F_v),\mathbb{R}_{>0}) \to \widehat{\mathrm{Pic}^G_v}(X) \to \widehat{\mathrm{Pic}}(X)$ is the zero homomorphism. To complete proof, it suffices to verify that if $(L,||\cdot||)$ is in the kernel of $\widehat{\mathrm{Pic}^G_v}(X)\to\widehat{\mathrm{Pic}}(X)$, then $(L,||\cdot||)$ is in the image of $\mathscr{C}^0_{G(F_v)}(X(F_v),\mathbb{R}_{>0})\to\widehat{\mathrm{Pic}^G_v}(X)$. Obviously, L is the trivial G-linearized line bundle $(\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_{G\times X}})$. For every $x\in X(F_v)$ and every $t\in\mathbb{G}_m(F_v)$, the canonical linear map $F_v=\mathcal{O}_X(x)\xrightarrow{=}\mathcal{O}_X(t\cdot x)=F_v$, given by the trivial G-linearization, is isometry and maps $1\in\mathcal{O}_X(x)$ to $1\in\mathcal{O}_X(t\cdot x)$, thus the function $x\mapsto ||1||_x$ is G-invariant. It follows that $(L,||\cdot||)=(\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_{G\times X}})$ is the image of $x\mapsto||1||_x$ for the homomorphism $\mathscr{C}^0_{G(F_v)}(X(F_v),\mathbb{R}_{>0})\to\widehat{\mathrm{Pic}^G_v}(X)$ from above. The statement is proven. □

In the rest of paragraph we study G-invariant F_v-metrics on the trivial line bundle. Let us denote by $\widehat{\mathscr{P}ic^G_v}(\mathcal{O}_X)$ the full subcategory of $\widehat{\mathscr{P}ic^G_v}(X)$ given by G-invariant F_v-metrized line bundles $(L,\psi,||\cdot||)$ such that $L=\mathcal{O}_X$. It is immediate that $\widehat{\mathscr{P}ic^G_v}(\mathcal{O}_X)$ is a Picard category. Let $\widehat{\mathrm{Pic}^G_v}(\mathcal{O}_X)$ be the abelian group formed by isomorphism classes of objects of $\widehat{\mathscr{P}ic^G_v}(\mathcal{O}_X)$. The canonical inclusion

$$\widehat{\mathrm{Pic}^G_v}(\mathcal{O}_X)\to\widehat{\mathrm{Pic}^G_v}(X) \tag{4.3.3.3}$$

is an equality, if $\mathrm{Pic}(X)=0$.

Lemma 4.3.3.4. — *Let $\psi:G\times X\to\mathbb{G}_m$ be a G-linearization of the trivial line bundle on X. An G-invariant F_v-metric $||\cdot||$ satisfies that $x\mapsto(||1||_x)^{-1}$ is a continuous function and that for every $x\in X(F_v)$ and every $t\in G(F_v)$ one has*

$$(||1||_{t\cdot x})^{-1}=|\psi(t,x)|_v(||1||_x)^{-1}.$$

Conversely, let $f:X(F_v)\to\mathbb{R}_{>0}$ be a continuous function such that for every $x\in X(F_v)$ and every $t\in G(F_v)$ one has $f(t\cdot x)=|\psi(t,x)|_vf(x)$. Then setting $||1||_x:=(f(x))^{-1}$ for every $x\in X(F_v)$ defines a G-invariant F_v-metric on the G-linearized line bundle $(\mathcal{O}_X,\psi)$.

Proof. — The function $x\mapsto(||1||_x)^{-1}$ is continuous as $||\cdot||$ is an F_v-metric. For $x\in X(F_v)$ and $t\in G(F_v)$, the linear map $F_v=\mathcal{O}_X(x)\to\mathcal{O}_X(t\cdot x)=F_v$, induced from the G-linearization ψ, is given by multiplication by $\psi(t,x)$. Now, the fact that $||\cdot||$ is G-invariant gives that for every $x \in X(F_v)$ and every $t \in G(F_v)$ one has that

$$||1||_x = ||\psi(t,x)||_{t\cdot x} = |\psi(t,x)|_v ||1||_{t\cdot x},$$

i.e.

$$(||1||_{t\cdot x})^{-1} = |\psi(t,x)|_v (||1||_x)^{-1}.$$

Suppose now $f : X(F_v) \to \mathbb{R}_{>0}$ is a continuous function such that for every $x \in X(F_v)$ and every $t \in G(F_v)$ one has $f(t \cdot x) = |\psi(t,x)|_v f(x)$. For every $x \in X(F_v)$, set $||1||_x = (f(x))^{-1}$. The function $x \mapsto ||1||_x$ is continuous and hence defines an F_v-metric on $\mathcal{O}_X$. We have that $F_v = \mathcal{O}_X(x) \to \mathcal{O}_X(t\cdot x) = F_v$ maps 1 to $\psi(t,x)$ and thus

$$(||1||_{t\cdot x})^{-1} = f(t\cdot x) = |\psi(t,x)|_v f(x) = |\psi(t,x)|_v (||1||_x)^{-1}.$$

□

4.3.4. — In this paragraph we compare the F_v-metrized line bundles on the quotient X/G and G-linearized F_v-metrized line bundles on the scheme X.

Let v be a place of F and let X be a locally of finite type F_v-scheme. Let G be a locally of finite presentation F_v-group scheme acting on X. The quotient stack X/G is an algebraic stack by Proposition 3.1.1.2 and locally of finite type by [**56**, Lemma 06FM].

Lemma 4.3.4.1. — *Let $q : X \to X/G$ be the quotient morphism. Let $\mathcal{F} : \mathscr{P}ic(X/G) \to \mathscr{P}ic^G(X)$ be the equivalence defined in 4.1.4.*

1. *Let $||\cdot||$ be an F_v-metric on $L \in \mathscr{P}ic(X/G)$. The F_v-metric $q^*||\cdot||$ on q^*L makes $\mathcal{F}(L)$ a G-invariant F_v-metrized line bundle that we denote $\widehat{\mathcal{F}}(L)$.*
2. *If $\ell : (L, ||\cdot||) \to (L', ||\cdot||)$ is a morphism of F_v-metrized line bundles, then $\mathcal{F}(\ell) : \widehat{\mathcal{F}}(L) \to \widehat{\mathcal{F}}(L')$ is an isometry, thus a morphism of G-invariant F_v-metrized line bundles. In this case we set $\widehat{\mathcal{F}}(\ell) := \mathcal{F}(\ell)$.*
3. *If L is the trivial F_v-metrized line bundle on X/G, then $\widehat{\mathcal{F}}(L)$ is the trivial G-invariant F_v-metrized line bundle.*
4. *The functor $\widehat{\mathcal{F}} : \widehat{\mathscr{P}ic}(X/G) \to \widehat{\mathscr{P}ic}{}^G_v(X)$ is an additive functor.*

Proof. — 1. Let $x \in X(F_v)$ and $t \in G(F_v)$. The map $(q^*L)(x) \to (q^*L)(t\cdot x)$ given by the G-linearization defining $\mathcal{F}(L)$ coincides with the linear map $(q^*L)(x) = L(q(x)) \to L(q(t\cdot x)) = (q^*L)(t\cdot x)$ given by the isomorphism $q(x) \xrightarrow{\sim} q(t\cdot x)$. The map $L(q(x)) \to$

$L(q(t\cdot x))$ is an isometry by the definition of an F_v-metric, hence is $(q^*L)(x) \to (q^*L)(t\cdot x)$ is an isometry. The claim follows.

2. The morphism $\mathcal{F}(\ell)$ identifies with the morphism $q^*\ell$, which is isometric by the third part of Lemma 4.3.2.1.
3. The G-linearized line bundle defining $\widehat{\mathcal{F}}(\mathcal{O}_{X/G})$ is the trivial G-linearized line bundle. Moreover, the F_v-metrized line bundle defining $\widehat{\mathcal{F}}(\mathcal{O}_{X/G})$ is precisely the trivial F_v-metrized line bundle by Lemma 4.3.2.1. It follows that $\widehat{\mathcal{F}}(\mathcal{O}_{X/G})$ is the trivial G-invariant F_v-metrized line bundle.
4. Let $(L, ||\cdot||)$ and $(L', ||\cdot||')$ be two F_v-metrized line bundles on X/G. The G-linearized line bundle defining $\widehat{\mathcal{F}}((L, ||\cdot||)\otimes(L', ||\cdot||'))$ is the G-linearized line bundle $\mathcal{F}(L\otimes L') = \mathcal{F}(L)\otimes\mathcal{F}(L')$. The F_v-metrized line bundle defining $\widehat{\mathcal{F}}((L, ||\cdot||)\otimes(L', ||\cdot||'))$ is precisely the F_v-metrized line bundle $(q^*(L\otimes L'), q^*(||\cdot||\otimes||\cdot||')) = (q^*L\otimes q^*L', q^*||\cdot||\otimes q^*||\cdot||')$. It follows that $\widehat{\mathcal{F}}$ is an additive functor.

□

If follows from Lemma 4.3.4.1, that we have a homomorphism of abelian groups

$$\widehat{\operatorname{Pic}}_v(X/G) \to \widehat{\operatorname{Pic}}^G_v(X). \tag{4.3.4.2}$$

The map $X(F_v) \to [(X/G)(F_v)]$ is $G(F_v)$-invariant by Proposition 3.3.3.1 and thus if $f \in \mathscr{C}^0([(X/G)(F_v)], \mathbb{R}_{>0})$ its pullback along $[q(F_v)]$ is an element of $\mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0})$.

Lemma 4.3.4.3. — *The following diagram is commutative:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{C}^0([(X/G)(F_v)], \mathbb{R}_{>0}) & \longrightarrow & \widehat{\operatorname{Pic}}_v(X/G) & \longrightarrow & \operatorname{Pic}(X/G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0}) & \longrightarrow & \widehat{\operatorname{Pic}}^G_v(X) & \longrightarrow & \operatorname{Pic}^G(X).
\end{array}$$

Proof. — The diagram

$$\begin{array}{ccc}
\widehat{\operatorname{Pic}}_v(X/G) & \longrightarrow & \operatorname{Pic}(X/G) \\
\downarrow & & \downarrow \\
\widehat{\operatorname{Pic}}^G_v(X) & \longrightarrow & \operatorname{Pic}^G(X)
\end{array}$$

is commutative by the construction of (4.3.4.2). Let $q : X \to X/G$ be the quotient 1-morphism. We prove the commutativity of

$$\begin{tikzcd} \mathscr{C}^0([(X/G)(F_v)], \mathbb{R}_{>0}) \arrow[r] \arrow[d] & \widehat{\mathrm{Pic}}_v(X/G) \arrow[d] \\ \mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0}) \arrow[r] & \widehat{\mathrm{Pic}}{}^G_v(X) \end{tikzcd} \tag{4.3.4.4}$$

Let $f \in \mathscr{C}^0([(X/G)(F_v)], \mathbb{R}_{>0})$. Its image in $\mathscr{C}^0_{G(F_v)}(X(F_v), \mathbb{R}_{>0})$ is $f \circ [q(F_v)]$. The image of $f \circ [q(F_v)]$ in $\widehat{\mathrm{Pic}}{}^G_v(X)$ is $(\mathcal{O}_X, \mathrm{Id}_{\mathcal{O}_{G\times X}}, ||\cdot||^{f\circ[q(F_v)]})$, where $||\cdot||^{f\circ[q(F_v)]}$ is defined by $||1||_x^{f\circ[q(F_v)]} = f([q(F_v)](x))$ for $x \in X(F_v)$. The image of f in $\widehat{\mathrm{Pic}}_v(X/G)$ is the F_v-metrized line bundle $(\mathcal{O}_X, ||\cdot||^f)$, where $||\cdot||^f$ is defined by $||1||_y^f = f(y)$ for $y \in [X/G](F_v)$. The image of $(\mathcal{O}_X, ||\cdot||^f)$ in $\widehat{\mathrm{Pic}}{}^G_v(X)$ is the G-invariant F_v-metrized line bundle $(\mathcal{O}_X, \mathrm{Id}_{\mathcal{O}_{G\times X}}, q^*||\cdot||^f)$. Note that $q^*||1||_x^f = ||1||^f_{[q(F_v)](x)} = f([q(F_v)](x))$, i.e. $q^*||\cdot||^f = ||\cdot||^{f\circ[q(Fv)]}$. We deduce the commutativity of the diagram 4.3.4.4. The statement is proven. $\square$

We give a conditions for the homomorphism (4.3.4.2) to be injective or an isomorphism.

Proposition 4.3.4.5. — *Let v be a place of F. Let G be a special locally of finite type F_v-group scheme acting on locally of finite type F_v-scheme X. The homomorphism (4.3.4.2) is injective. If moreover $\widehat{\mathrm{Pic}}_v(X/G) \to \mathrm{Pic}(X/G)$ is surjective, then the homomorphism (4.3.4.2) is an isomorphism.*

Proof. — Consider the diagram from Lemma 4.3.4.3. Both horizontal sequences are exact by Lemma 4.3.3.2 and Lemma 4.3.1.3. Obviously, the first vertical homomorphism is an isomorphism. As G is special, one can identify $[(X/G)(F_v)]$ with the topological quotient $X(F_v)/G(F_v)$ using Proposition 3.3.3.1. Thus, the second vertical homomorphism, given by pulling back continuous functions on $[X/G(F_v)] = X(F_v)/G(F_v)$ to $X(F_v)$, is an isomorphism. The fourth vertical homomorphism is an isomorphism by Proposition 4.1.4.1. By 4-lemma, we deduce that $\widehat{\mathrm{Pic}}_v(X/G) \to \widehat{\mathrm{Pic}}{}^G_v(X)$ is injective. Suppose that $\widehat{\mathrm{Pic}}_v(X/G) \to \mathrm{Pic}(X/G)$ is surjective. The following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}^0([(X/G)(F_v)],\mathbb{R}_{>0}) & \longrightarrow & \widehat{\mathrm{Pic}}_v(X/G) & \longrightarrow & \mathrm{Pic}(X/G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{C}^0_{G(F_v)}(X(F_v),\mathbb{R}_{>0}) & \longrightarrow & \widehat{\mathrm{Pic}^G_v}(X) & \longrightarrow & \mathrm{Pic}^G(X) & \longrightarrow & E, \end{array}$$

where E is the quotient of $\mathrm{Pic}^G(X)$ by the image of $\widehat{\mathrm{Pic}^G_v}(X)$. Obviously, the first and the fifth vertical homomorphisms are an isomorphism and a monomorphism, respectively. Again, the second and the fourth vertical homomorphisms are isomorphisms. By 5-lemma, we deduce that $\widehat{\mathrm{Pic}}_v(X/G) \to \widehat{\mathrm{Pic}^G_v}(X)$ is an isomorphism. □

4.3.5. — We give some conditions that will make $\widehat{\mathrm{Pic}}_v(X) \to \mathrm{Pic}(X)$ surjective for an algebraic stack X. Let v be a place of F. We are going to construct F_v-metrics on line bundles on algebraic stacks, by pulling back metrics on line bundles on schemes.

Lemma 4.3.5.1. — *Let X be a locally of finite type F_v-algebraic stack satisfying the following condition: for every line bundle L on X, there exist positive integer m, a 1-morphism to a locally of finite type F_v-scheme $g : X \to Z$ and a line bundle L' on Z which is in the image of the canonical homomorphism $\widehat{\mathrm{Pic}}_v(Z) \to \mathrm{Pic}(Z)$, such that $L^{\otimes m} = g^*(L')$. Then $\widehat{\mathrm{Pic}}_v(X) \to \mathrm{Pic}(X)$ is surjective.*

Proof. — The line bundle L' admits an F_v-metric, hence $L^{\otimes m} = g^*(L')$ admits an F_v-metric by Lemma 4.3.2.1. It follows that L admits an F_v-metric by Lemma 4.3.1.2. The statement follows. □

Remark 4.3.5.2. — In the case Z is a separated scheme, then every line bundle on Z admits an F_v-metric. Indeed $Z(F_v)$ is locally compact and Hausdorff topological space. Moreover, it is a finite union of paracompact spaces (if U is an open affine subset of Z, then $U(F_v)$ is a closed subspace of F_v^n, thus paracompact). By [7, Chapter I, §9, n^0 10, Proposition 18], we deduce that $Z(F_v)$ is paracompact. By [8, Chapter IX, §4, n^0 4, Proposition 4], the space $Z(F_v)$ is normal. By [8, Chapter IX, §4, n^0 3, Theorem 3], one can find a partitions of unity subordinated to every locally finite open covering and one can use them to construct metrics in a usual way.

Remark 4.3.5.3. — The following is true. A locally of finite type F_v-stack X of finite diagonal, that admits a separated good moduli space $p : X \to \mathscr{X}$ (see [1, Definition 1.2]), satisfies that $\widehat{\mathrm{Pic}}(X) \to \mathrm{Pic}(X)$ is surjective. Indeed, by the fact that the diagonal of X is finite and by [1, Section 2], every line bundle L on X admits a multiple $L^{\otimes m}$ which is a pullback of a line bundle on $\mathscr{X}$. We can endow $L^{\otimes m}$ with the pullback of a metric on the line bundle on $\mathscr{X}$, and we can endow L with the "m-th root" metric by Lemma 4.3.1.2.

4.3.6. — We will prove that every line bundle on $\mathscr{P}(\mathbf{a})_{F_v}$ admits an F_v-metric without using remarks from 4.3.5. Let us firstly prove the following lemma.

Lemma 4.3.6.1. — *Let ℓ be a positive integer divisible by* $\mathrm{lcm}(\mathbf{a})$. *Consider the morphism*

$$\begin{aligned} J(\ell) : \mathbb{A}^n - \{0\} &\to \mathbb{A}^n - \{0\} \\ \mathbf{x} &\mapsto (x_j^{\ell/a_j})_j. \end{aligned}$$

Endow the first $\mathbb{A}^n - \{0\}$ with the $\mathbb{G}_m$-action with the weights $a_1, \dots, a_n$, that is $t \cdot_{\mathbf{a}} \mathbf{x} = (t^{a_j} x_j)_j$, and the second $\mathbb{A}^n - \{0\}$ with the $\mathbb{G}_m$-action with the weights $1, \dots, 1$, that is $t \cdot_{\mathbf{1}} \mathbf{x} = (t x_j)_j$.

1. *The morphism $J(\ell)$ is $t \mapsto t^\ell$-equivariant, that is*

$$J(\ell)(t \cdot_{\mathbf{a}} \mathbf{x}) = t^\ell \cdot_{\mathbf{1}} (J(\ell)(\mathbf{x})) \qquad \forall t \in \mathbb{G}_m, \forall \mathbf{x} \in \mathbb{A}^n - \{0\}.$$

The diagram

$$\begin{tikzcd} \mathbb{A}^n - \{0\} \arrow[r, "J(\ell)"'] \arrow[d, "q^{\mathbf{a}}"'] & \mathbb{A}^n - \{0\} \arrow[d, "q^{\mathbf{1}}"] \\ \mathscr{P}(\mathbf{a}) \arrow[r, "\overline{J(\ell)}"] & \mathbb{P}^{n-1}. \end{tikzcd} \tag{4.3.6.2}$$

is 2-commutative, where $\overline{J(\ell)}$ is given by [56, Lemma 046Q].

2. *The pullback $\overline{J(\ell)}^*(\mathcal{O}(k))$ for $k \in \mathbb{Z}$ is the line bundle $\mathcal{O}(\ell k)$.*

Proof. — 1. One has that

$$\begin{aligned} J(\ell)(t \cdot_{\mathbf{a}} \mathbf{x}) = J(\ell)((t^{a_j} x_j)_j) &= (t^\ell x_j^{\ell/a_j})_j \\ &= t^\ell \cdot_{\mathbf{1}} J(\ell)(\mathbf{x}) \quad \forall t \in \mathbb{G}_m, \forall \mathbf{x} \in \mathbb{A}^n - \{0\}. \end{aligned}$$

The 2-commutativity of the diagram (4.3.6.2) follows from the universal property of $\overline{J(\ell)}$, see [56, Lemma 0436].

2. By Lemma 4.1.5.1, the pullback $\overline{J(\ell)}^*\mathcal{O}(k)$ is the line bundle determined by the pullback linearization $J(\ell)^*\psi$ of the trivial line bundle, where $\psi : (t, \mathbf{x}) \mapsto t^k$ is the $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{A}^n-\{0\}}$ that defines $\mathcal{O}(k)$. One has that

$$(J(\ell))^*\psi = \psi(\phi(t), J(\ell)(\mathbf{x})) = \phi(t)^k = t^{\ell k}.$$

It follows that $\overline{J(\ell)}^*(\mathcal{O}(k)) = \mathcal{O}(\ell k)$.

□

Lemma 4.3.6.3. — *The canonical morphism* $\widehat{\operatorname{Pic}}_v(\mathscr{P}(\mathbf{a})_{F_v}) \to \operatorname{Pic}(\mathscr{P}(\mathbf{a})_{F_v})$ *is surjective. The canonical homomorphism*

$$\widehat{\operatorname{Pic}}_v(\mathscr{P}(\mathbf{a})_{F_v}) \to \widehat{\operatorname{Pic}}^{\mathbb{G}_m}_{F_v}((\mathbb{A}^n - \{0\})_{F_v}) \tag{4.3.6.4}$$

is an isomorphism.

Proof. — Let ℓ be an integer divisible by $\operatorname{lcm}(\mathbf{a})$. By Lemma 4.3.6.1, one has that $\overline{J(\ell)}^*(\mathcal{O}(1)) = \mathcal{O}(\ell k)$. We deduce that every line bundle on $\mathscr{P}(\mathbf{a})$ admits a non-zero power which is a pullback of a line bundle on $\mathbb{P}^{n-1}$. The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}$ admits a metric, e.g. one can endow it with the Fubiny-Study metric. It follows from Lemma 4.3.5.1, that $\widehat{\operatorname{Pic}}_v(\mathscr{P}(\mathbf{a})) \to \operatorname{Pic}(\mathscr{P}(\mathbf{a}))$ is surjective. Proposition 4.3.4.5 now gives that the homomorphism (4.3.6.4) is an isomorphism. □

Let us dedicate the end of this paragraph to a dictionary given by Lemma 4.3.3.4 for the case of weighted projective stacks.

Definition 4.3.6.5. — *Let $v \in M_F$, let $d \in \mathbb{C}$ and let $f : F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$ be a function. We say that f is $\mathbf{a}$-homogenous of weighted degree d if for every $\mathbf{x} \in F_v^n - \{0\}$ and every $t \in F_v^\times$ one has*

$$f(t \cdot \mathbf{x}) = |t|_v^d f(\mathbf{x}).$$

Lemma 4.3.6.6. — *Let v be a place of F and let k be an integer. If $||\cdot||$ is an F_v-metric on the line bundle $\mathcal{O}(k)$ on $\mathscr{P}(\mathbf{a})_{F_v}$, the pullback metric $(q^{\mathbf{a}}_{F_v})^*||\cdot||$ on $\mathcal{O}_{\mathbb{A}^n-\{0\}}$ is $\mathbb{G}_m$-invariant and the function $f_{||\cdot||} : \mathbf{x} \mapsto (q^{\mathbf{a}}_{F_v})^*||1(\mathbf{x})||$ is an $\mathbf{a}$-homogenous continuous function $F_v^n - \{0\} \to \mathbb{R}_{>0}$ of weighted degree k. Conversely, let $f : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be an $\mathbf{a}$-homogenous continuous function of weighted degree k. Then setting $||1(\mathbf{x})||' := (f(\mathbf{x}))^{-1}$ defines a $\mathbb{G}_m$-invariant F_v-metric on the $\mathbb{G}_m$-linearized line bundle $(\mathcal{O}_{\mathbb{A}^n-\{0\}}, (t, \mathbf{x}) \mapsto t^k)$. Furthermore, there exists a metric $||\cdot||_f$ on $\mathcal{O}(k)$ such that $(q^{\mathbf{a}}_{F_v})^*||\cdot|| = ||\cdot||'$.*

Proof. — The line bundle $\mathcal{O}(k)$ is given by the linearization $\psi : (t, \mathbf{x}) \mapsto t^k$ of $\mathcal{O}_{\mathbb{A}^n-\{0\}}$. Let $||\cdot||$ be an F_v-metric on $\mathcal{O}(k)$. The pullback metric $(q^{\mathbf{a}}_{F_v})^*||\cdot||$ is $\mathbb{G}_m$-invariant by Lemma 4.3.4.1. By Lemma 4.3.3.4, the function $\mathbf{x} \mapsto (q^{\mathbf{a}}_{F_v})^*||1(\mathbf{x})||$ is continuous and satisfies that

$$((q^{\mathbf{a}}_{F_v})^*||1(t\cdot\mathbf{x})||)^{-1} = |t^k|_v((q^{\mathbf{a}}_{F_v})^*||1(\mathbf{x})||)^{-1},$$

i.e. it is $\mathbf{a}$-homogenous continuous of weighted degree k. Let us prove the converse claim. Note that Lemma 4.3.3.4 gives that $||\cdot||'$ is an $\mathbb{G}_m(F_v)$-invariant metric on $(\mathcal{O}_{\mathbb{A}^n-\{0\}}, (t, \mathbf{x}) \mapsto t^k)$. By Lemma 4.3.6.3, there exists an F_v-metric $||\cdot||$ on $\mathcal{O}(k)$ such that $(q^{\mathbf{a}}_{F_v})^*||\cdot|| = ||\cdot||'$. The statement is proven. □

Remark 4.3.6.7. — By allowing that f takes value 0, we allow "singular" metrics.

4.4. Heights on $\mathscr{P}(\mathbf{a})(F)$

In this section we will define heights on $\mathscr{P}(\mathbf{a})(F)$. An obvious approach is to define heights to be pullbacks of heights for some morphism to a scheme. However, such heights exhibit a drawback, they do not satisfy the weak Northcott property and hence are not suitable for countings. For the purpose of counting, we define quasi-toric heights, for which in 4.6 we establish that they satisfy the weak Northcott property.

4.4.1. — In this paragraph we define heights on stacks. We give a condition that will enable us to define heights. It is not a "very restrictive" condition, as we discuss in Remark 4.4.1.5.

Definition 4.4.1.1. — *Let $k \in \mathbb{Z}$. For $v \in M_F$, let $f_v : F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$ be an $\mathbf{a}$-homogenous function of weighted degree d and let us set*

$$E_v := \{\mathbf{x} \in F_v^n - \{0\} | \forall j\ |x_j|_v = 1 \text{ or } x_j = 0.\}$$

We say that a family $(f_v)_v$ is generalized adelic if for almost every $v \in M_F^0$, one has that

$$f|_{E_v} = 1.$$

We use the language of $\mathbf{a}$-homogenous continuous functions to define heights.

Lemma 4.4.1.2. — *Let $(f_v)_{v\in M_F}$ be a generalized adelic family of $\mathbf{a}$-homogenous functions $f_v : F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$ of weighted degree $d \in \mathbb{Z}$. For $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$, let us denote by $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_{F_v} \to (\mathbb{A}^n - \{0\})_{F_v}$ the induced*

$(\mathbb{G}_m)_{F_v}$-equivariant morphism defined by $\mathbf{x}$. For every $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$, the product

$$H((f_v)_v)(\mathbf{x}) := \prod_{v \in M_F} f_v(\widetilde{\mathbf{x}}(1)) \tag{4.4.1.3}$$

is a finite product. Moreover, if $\mathbf{x} \xrightarrow{\sim} \mathbf{y}$ is a 2-isomorphism, then $H((f_v)_v)(\mathbf{x}) = H((f_v)_v)(\mathbf{y})$.

Proof. — Let $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$. If $\widetilde{x}_j(1) \neq 0$, then for almost every $v \in M_F^0$ one has $|\widetilde{x}_j(1)|_v = 1$. We conclude that for almost all $v \in M_F^0$, one has $\widetilde{\mathbf{x}}(1) \in E_v$. As $(f_v)_v$ is generalized adelic, we deduce that the product (4.4.1.3) is indeed finite. Let now $\mathbf{x} \xrightarrow{\sim} \mathbf{y}$ be a 2-isomorphism, it is given by an element $t \in \mathbb{G}_m(F_v)$ such that $t \cdot \widetilde{\mathbf{x}}(1) = \widetilde{\mathbf{y}}(1)$. By the product formula, one has that

$$\prod_{v \in M_F} f_v(\widetilde{\mathbf{y}}(1)) = \prod_{v \in M_F} f_v(t \cdot \widetilde{\mathbf{x}}(1)) = \prod_{v \in M_F} |t|_v^k f_v(\widetilde{\mathbf{x}}(1)) = \prod_{v \in M_F} f_v(\widetilde{\mathbf{x}}(1)).$$

□

Definition 4.4.1.4. — *Let $(f_v)_v$ be a generalized adelic family of $\mathbf{a}$-homogenous continuous functions $F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$ of weighted degree k.*

1. *The function that associates to $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ the value of the product (4.4.1.3), we call the resulting height defined by the family $(f_v)_v$ and we denote it by $H((f_v)_v)$.*
2. *For $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$, we define $H((f_v)_v)(\mathbf{x})$ by setting it to be $H((f_v)_v)(\mathbf{y})$ where $\mathbf{y} \in \mathscr{P}(\mathbf{a})(F)$ is such that the isomorphism class of $\mathbf{y}$ is $\mathbf{x}$.*

When there is no confusion, we call it simply the height.

Remark 4.4.1.5. — The condition that $f_v|_{E_v} = 1$ for almost all v is not a strong one. For $\mathbf{x} \in F_v^n - \{0\}$, it can happen that there is no $t \in F_v^\times$ such that $t \cdot \mathbf{x} \in E_v$. This means that the condition does not impose anything on the value of f_v at $\mathbf{x}$. The weakness of the condition enable us to have essentially different heights. Soon we will give much more restricting conditions on f_v to produce so called "quasi-toric heights". In Chapter 9 another restrictive condition is given and the heights we produce are "essentially" discriminants (of F-algebras).

When two families consist of functions which are non-vanishing at every place and which coincide at almost ever places, the resulting heights can be compared as follows.

Lemma 4.4.1.6. — *Let $d \in \mathbb{Z}$, and let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ and $(f'_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be two generalized adelic degree d families of* $\mathbf{a}$*-homogenous functions. Suppose that for almost all $v \in M_F$, one has that $f_v = f'_v$. There exist $C_1, C_2 > 0$ such that for every $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ one has that*

$$C_1 < \frac{H((f_v)_v)(\mathbf{x})}{H((f'_v)_v)(\mathbf{x})} < C_2.$$

Proof. — For $v \in M_F$, the function

$$F_v^n - \{0\} \to \mathbb{R}_{>0} \qquad \mathbf{y} \mapsto \frac{f_v(\mathbf{y})}{f'_v(\mathbf{y})}$$

is $\mathbb{G}_m(F_v)$-invariant, thus descends to a positive valued function on $[\mathscr{P}(\mathbf{a})(F_v)]$, which is compact by Proposition 3.3.4.5. We deduce that for every $v \in M_F$, there exist $C_{v,1}, C_{v,2} > 0$ such that for every $\mathbf{y} \in F_v^n - \{0\}$ one has that

$$C_{v,1} < \frac{f_v(\mathbf{y})}{f'_v(\mathbf{y})} < C_{v,2}.$$

Let us write H for $H((f_v)_v)$ and H' for $H((f'_v)_v)$. Let $\mathbf{y} \in F_v^n - \{0\}$ be a lift of $\mathbf{x}$. Let S be the finite set of places, for which $f_v \neq f'_v$. For every $v \notin S$, one has that $f_v = f'_v$. We deduce that

$$\frac{H(\mathbf{x})}{H'(\mathbf{x})} = \prod_{v \in S} \frac{f_v(\widetilde{\mathbf{x}})}{f'_v(\widetilde{\mathbf{x}})} < \prod_{v \in S} C_{v,2}$$

and that

$$\prod_{v \in S} C_{v,1} < \prod_{v \in S} \frac{f_v(\widetilde{\mathbf{x}})}{f'_v(\widetilde{\mathbf{x}})} = \frac{H(\mathbf{x})}{H'(\mathbf{x})}.$$

The statement is proven. □

4.4.2. — In this paragraph, we define stable heights on $\mathscr{P}(\mathbf{a})(F)$, where $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$. These heights are pullbacks of the heights on varieties. Such height H satisfies that if $\mathbf{x}, \mathbf{y} \in \mathscr{P}(\mathbf{a})(F)$ are such that $\mathbf{x}_K \cong \mathbf{y}_K$ for some extension K/F, then $H(\mathbf{x}) = H(\mathbf{y})$. The definition of the height can hence be naturally carried to any $\overline{F}$-point of $\mathscr{P}(\mathbf{a})$. They are called "stable" because the height of an F-point stays invariant when regarding this point as a K-point of $[\mathscr{P}(\mathbf{a})]$. We will see later in 4.6, that such heights do not always satisfy the weak Northcott property.

If $v \in M_F$ and $||\cdot||$ is an F_v-metric on the line bundle $\mathcal{O}(k)$ on $\mathscr{P}(\mathbf{a})_{F_v}$, for some $k \in \mathbb{Z}$, we will denote by $f_{||\cdot||} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ the $\mathbf{a}$-homogenous function of weighted degree k given by Lemma 4.3.6.6.

If $r \geq 0$ is an integer, let us denote by $||\cdot||_{v,\max}$ the metric on $\mathcal{O}(1)$ on $\mathbb{P}^r$ given by the $\mathbf{1}$-homogenous continuous function $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ of weighted degree 1

$$f_v^{\#} : \mathbf{x} \mapsto \max_j(|x_j|_v).$$

By a stable F_v-metric on a line bundle L on locally of finite type F_v-scheme Z, we mean a metric which is the restriction of a metric on the analytic line bundle L^{an} on the analytic space $Z_{F_v}^{\mathrm{an}}$. The metric $||\cdot||_{v,\max}$ on the line bundle $\mathcal{O}(1)$ on the projective space is stable.

Definition 4.4.2.1. — *Let X be a locally of finite type F_v-algebraic stack and let L be a line bundle on X. An F_v-metric $||\cdot||$ on L is said to be stable, if there exist an integer $m \neq 0$, a locally of finite type F_v-scheme Z, an F_v-metrized line bundle $(L', ||\cdot||')$ on Z with $||\cdot||'$ stable and a 1-morphism of algebraic stacks $g : X \to Z$ such that $(L^{\otimes m}, ||\cdot||^{\otimes m}) = g^*(L', ||\cdot||')$.*

Recall that an adelic metric on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}$ is a collection of metrics $(||\cdot||_v)_v$, where each $||\cdot||_v$ is a stable metric on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}_{F_v}$, and for almost all v, one has $||\cdot||_v = ||\cdot||_{v,\max}$.

Definition 4.4.2.2. — *Let k be an integer. Let $(f_v)_v$ be a generalized adelic family of $\mathbf{a}$-homogenous continuous functions of weighted degree k. We say that $(f_v)_v$ is stable if there exists an integer $\ell \neq 0$, an integer $r \geq 0$, an adelic metric $(||\cdot||_v)_v$ on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^r$ and a 1-morphism of algebraic stacks $h : \mathscr{P}(\mathbf{a}) \to \mathbb{P}^r$ such that the line bundle $\mathcal{O}(\ell k)$ is the pullback line bundle $h^*\mathcal{O}(1)$ and such that for every v one has that $f_v^{\ell} = f_{h^*||\cdot||_v}$. In this case, we also say that the height $H = H((f_v)_v)$ is stable.*

A fundamental property of the stable heights is that the height of two points, which become isomorphic after passing to an extension, is the same.

Lemma 4.4.2.3. — *Let k be an integer, let $(f_v)_v$ be a stable generalized adelic family of $\mathbf{a}$-homogenous continuous functions of weighted degree k. Let $H = H((f_v)_v)$ be the corresponding height. Let $\mathbf{x}, \mathbf{y} \in \mathscr{P}(\mathbf{a})(F)$ be such that there exists an extension K/F and an isomorphism $\mathbf{x}|_K \xrightarrow{\sim} \mathbf{y}_K$ in $\mathscr{P}(\mathbf{a})(K)$. Then $H(\mathbf{x}) = H(\mathbf{y})$.*

Proof. — There exists an $\ell \neq 0$, an integer $r \geq 0$, an adelic metric $(||\cdot||_v)_v$ on the line bundle $\mathcal{O}(1)$, a 1-morphism of algebraic stacks $g : \mathscr{P}(\mathbf{a}) \to \mathbb{P}^r$

such that $\mathcal{O}(\ell k) = g^*\mathcal{O}(1)$ and such that for each v one has $f_v^\ell = f_{g^*||\cdot||_v}$. Let $H_{\mathbb{P}^{n-1}}$ be the height on $\mathbb{P}^{n-1}(F)$ given by $(||\cdot||_v)_v$. For $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ that

$$\begin{aligned} H(\mathbf{x})^\ell = \prod_v f_v(\widetilde{\mathbf{x}}(1))^\ell = \prod_v f_{g^*||\cdot||_v}(\widetilde{\mathbf{x}}(1)) = \prod_v (g^*||1(\mathbf{x})||_v)^{-1} \\ = \prod_v ||1(g(\mathbf{x})||_v^{-1} = H_{\mathbb{P}^{n-1}}(g(\mathbf{x})). \end{aligned}$$

The image of $\mathbf{x}_K$ in $\mathbb{P}^{n-1}(K)$ is precisely $g(\mathbf{x})_K$ and the image of $\mathbf{y}_K$ in $\mathbb{P}^{n-1}(K)$ is precisely $g(\mathbf{x})_K$. The existence of an isomorphism $\mathbf{x}_K \xrightarrow{\sim} \mathbf{y}_K$ gives that $g(\mathbf{x})_K = g(\mathbf{y})_K$. It follows that $g(\mathbf{x}) = g(\mathbf{y})$, and hence

$$H(\mathbf{x}) = H_{\mathbb{P}^{n-1}}(g(\mathbf{x})) = H_{\mathbb{P}^{n-1}}(g(\mathbf{y})) = H(\mathbf{y}).$$

□

We give an example of a stable family.

Lemma 4.4.2.4. — *For $v \in M_F$, the functions $f_v : F_v^n - \{0\} \to \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \max(|x_j|_v^{1/a_j})$ are continuous and $\mathbf{a}$-homogenous of weighted degree 1. The family $(f_v)_v$ is generalized adelic and stable. Moreover, the resulting height $H^{\max} = H((f_v))$ satisfies for every $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ that $H(\mathbf{x}) \geq 1$.*

Proof. — It is evident that f_v is continuous and that $f_v(t \cdot \mathbf{x}) = \max_j(|t^{a_j}x_j|_v^{1/a_j}) = |t|_v \max_j(|x_j|_v^{1/a_j}) = |t|_v f_v(\mathbf{x})$, and thus for every $v \in M_F$ one has that f_v is a continuous $\mathbf{a}$-homogenous function of weighted degree 1. Moreover, for every $v \in M_F$, one has that if $\mathbf{y} \in E_v = \{\mathbf{x} \in F_v^n - \{0\} | \, |x_j|_v = 1 \text{ or } x_j = 0\}$, then $f_v(\mathbf{y}) = 1$. Thus, the family $(f_v)_v$ is generalized adelic.

We will verify that $(f_v)_v$ is stable. Let $v \in M_F$, an F_v-metric on $\mathcal{O}_{\mathbb{A}^n - \{0\}}$ identifies with a continuous function $g : F_v^n - \{0\} \to \mathbb{R}_{>0}$ by setting $g(\mathbf{x}) = ||1(\mathbf{x})||^{-1}$, where $\mathbf{x} \in F_v^n - \{0\}$ and $1(\mathbf{x})$ is the value of $1 \in \Gamma(\mathbb{A}^n - \{0\}, \mathcal{O}_{\mathbb{A}^n - \{0\}})$ at $\mathbf{x}$. Let us set $\ell = \operatorname{lcm}(\mathbf{a})$. Consider the morphism $J(\ell) : \mathbb{A}^n - \{0\} \to \mathbb{A}^n - \{0\}$ given by

$$J(\ell) : \mathbf{x} \mapsto (x_j^{\ell/a_j})_j.$$

In Lemma 4.3.6.1, we have established that $J(\ell)$ is $(t \mapsto t^\ell)$-invariant and that $J(\ell)^*\mathcal{O}(1) = \mathcal{O}(\ell)$. Moreover, the following diagram is 2-commutative:

$$\begin{tikzcd} \mathbb{A}^n - \{0\} \arrow[r, "J(\ell)"'] \arrow[d, "q^{\mathbf{a}}"'] & \mathbb{A}^n - \{0\} \arrow[d, "q^{\mathbf{1}}"] \\ \mathscr{P}(\mathbf{a}) \arrow[r, "\overline{J(\ell)}"] & \mathbb{P}^{n-1}. \end{tikzcd}$$

By Lemma 4.3.2.1, it follows that the diagram

$$\begin{tikzcd} \widehat{\mathrm{Pic}}_v(\mathbb{A}^n - \{0\}) & \widehat{\mathrm{Pic}}_v(\mathbb{A}^n - \{0\}) \arrow[l, "J(\ell)^*"'] \\ \widehat{\mathrm{Pic}}_v(\mathscr{P}(\mathbf{a})) \arrow[u, "(q^{\mathbf{a}}_{F_v})^*"'] & \widehat{\mathrm{Pic}}_v(\mathbb{P}^{n-1}) \arrow[l, "\overline{J(\ell)}^*"'] \arrow[u, "(q^{\mathbf{1}}_{F_v})^*"] \end{tikzcd}$$

is commutative. Hence, the image of the F_v-metrized line bundle $(\mathcal{O}(k), ||\cdot||_{v,\max})$ under $(q^{\mathbf{a}}_{F_v})^* \circ \overline{J(\ell)}^*$ identifies with

$$\begin{aligned} J(\ell)^*(q^{\mathbf{1}}_{F_v})^*(\mathcal{O}(k), ||\cdot||_{v,\max}) &= (\mathcal{O}_{\mathbb{A}^n-\{0\}}, J(\ell)^*(q^{\mathbf{1}}_{F_v})^*||\cdot||_{v,\max}) \\ &= (\mathcal{O}_{\mathbb{A}^n-\{0\}}, \mathbf{x} \mapsto J(\ell)^*(\mathbf{x} \mapsto \max_j |x_j|_v)) \\ &= (\mathcal{O}_{\mathbb{A}^n-\{0\}}, \mathbf{x} \mapsto \max_j(|x_j|_v^{\ell/a_j})) \\ &= (\mathcal{O}_{\mathbb{A}^n-\{0\}}, \mathbf{x} \mapsto f_v(\mathbf{x})^\ell). \end{aligned}$$

The $\mathbb{G}_m$-invariant F_v-linearized line bundle $(\mathcal{O}_{\mathbb{A}^n-\{0\}}, \mathbf{x} \mapsto f_v(\mathbf{x})^\ell)$ is precisely the image of $\overline{J(\ell)}^*(\mathcal{O}(1), ||\cdot||_{v,\max}) = (\mathcal{O}(k), \overline{J(\ell)}^*||\cdot||_{v,\max})$ under the pullback $(q^{\mathbf{a}}_{F_v})^*$, so that $f_v^\ell = f_{J(\ell)^*||\cdot||_{v,\max}}$. It follows that the family $(f_v)_v$ is stable.

Finally, let us prove the estimate $H(\mathbf{x}) \geq 1$. Let $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ and let $\widetilde{\mathbf{x}}$ be the $\mathbb{G}_m$-equivariant F-morphism $\mathbb{G}_m \to \mathbb{A}^n - \{0\}$ given by $\mathbf{x}$. Let i be an index such that $\widetilde{\mathbf{x}}(1)_i \neq 0$. Let $K = F\big(\sqrt[a_i]{\widetilde{x}_i(1)}\big)$. For $v \in M_F$, the absolute value $|\cdot|_v$ on F admits a unique extension to K.

By the product formula, we get that

$$\begin{aligned}
H(\mathbf{x}) &= \prod_{v\in M_F} \max_j(|\widetilde{x}_j(1)|_v^{1/a_j}) \\
&= \prod_{v\in M_F} |\sqrt[a_i]{\widetilde{x}_i(1)})|_v \max_j \left(\left|\frac{\widetilde{x}_j(1)}{(\sqrt[a_i]{\widetilde{x}_j(1)})^{a_j}}\right|_v^{1/a_j}\right) \\
&= \prod_{v\in M_F} \max_j \left(\left|\frac{\widetilde{x}_j(1)}{(\sqrt[a_i]{\widetilde{x}_j(1)})^{a_j}}\right|_v^{1/a_j}\right) \\
&\geq 1.
\end{aligned}$$

□

4.4.3. — In this paragraph we present an example of a height on $\mathscr{P}(\mathbf{a})$ which is intrinsic to stacks. We will call them quasi-toric heights. In 4.6, we are going to show that they satisfy the weak Northcott property (i.e. for every $B>0$, there are only finitely many points in $[\mathscr{P}(\mathbf{a})(F)]$ which have the height less than B). In Lemma 3.3.4.4, we have introduced the set $\mathcal{D}_v^{\mathbf{a}} = \mathcal{O}_v^n - (\pi_v^{a_1}\mathcal{O}_v)\times\cdots\times(\pi_v^{a_n}\mathcal{O}_v)$. For $v\in M_F^0$, we have defined in Lemma 3.3.4.4 a function $r_v : F_v^n - \{0\}\to\mathbb{Z}$ by

$$r_v(\mathbf{x}) = \inf\{k\in\mathbb{Z}|\pi_v^k\cdot\mathbf{x}\in\mathcal{O}_v^n\}$$

and we have established that for every $\mathbf{x}\in F_v^n-\{0\}$ that

$$\pi_v^{r_v(\mathbf{x})}\cdot\mathbf{x}\in\mathcal{D}_v^{\mathbf{a}}.$$

Lemma 4.4.3.1. — *Let $v\in M_F^0$.*

1. *For every $\mathbf{x}\in F_v^n-\{0\}$, one has that*
$$r_v(\mathbf{x}) = \sup_{\substack{j=1,\dots,n\\ x_j\neq 0}} \left\lceil -\frac{v(x_j)}{a_j}\right\rceil.$$
2. *For every $t\in F_v^\times$ and every $\mathbf{x}\in F_v^n-\{0\}$, one has that*
$$r_v(t\cdot\mathbf{x}) = r_v(\mathbf{x}) - v(t).$$
3. *For every $k\in\mathbb{Z}$ one has that*
$$\{\mathbf{x}\in F_v^n-\{0\}|r_v(\mathbf{x})=k\} = \pi_v^{-k}\cdot\mathcal{D}_v^{\mathbf{a}}.$$
4. *Let $k\in\mathbb{Z}$. Suppose that for $u\in\mathbb{G}_m(F_v)$ one has $u\cdot(\pi_v^k\cdot\mathcal{D}_v^{\mathbf{a}})\cap(\pi_v^k\cdot\mathcal{D}_v^{\mathbf{a}})\neq\emptyset$, then $v(u)=0$.*

5. *The function*

$$f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0} \qquad \mathbf{x} \mapsto |\pi_v|_v^{-r_v(\mathbf{x})}$$

is $\mathbf{a}$*-homogenous of weighted degree* 1 *and is locally constant.*

Proof. — 1. For every index j such that $x_j \neq 0$, one has that $\pi_v^{a_j k} x_j \in \mathcal{O}_v$ if and only if $k \geq -\frac{v(x_j)}{a_j}$. We conclude

$$r_v(\mathbf{x}) = \left\lceil \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \frac{-v(x_j)}{a_j} \right\rceil = \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \left\lceil -\frac{v(x_j)}{a_j} \right\rceil.$$

2. We observe that

$$r_v(t \cdot \mathbf{x}) = \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \left\lceil - \frac{v(t^{a_j} x_j)}{a_j} \right\rceil = \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \left\lceil - \frac{v(x_j) + a_j v(t)}{a_j} \right\rceil = r_v(\mathbf{x}) - v(t).$$

3. We verify the claim for $k = 0$. Pick $\mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}$. For every index j one has $v(x_j) \geq 0$ and there exists an index i such that $x_i \notin \pi_v^{a_i} \mathcal{O}_v$, i.e. such that $v(x_i) < a_i$. This implies that

$$0 \geq r_v(\mathbf{x}) = \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \left\lceil - \frac{v(x_j)}{a_j} \right\rceil \geq 0.$$

Pick now $\mathbf{x} \in F_v^n - \{0\}$ such that $r_v(\mathbf{x}) = 0$. This means that for every j such that $x_j \neq 0$, one has that

$$\frac{-v(x_j)}{a_j} \leq 0,$$

hence that $v(x_j) \geq 0$, and that there exists an index i such that

$$-1 < \frac{-v(x_i)}{a_i} \leq 0,$$

i.e. such that $0 \leq v(x_i) < a_i$. We deduce that $\mathbf{x} \in \mathcal{O}_v^n$ and that $\mathbf{x} \notin \pi_v^{a_1} \mathcal{O}_v \times \cdots \times \pi_v^{a_n} \mathcal{O}_v$, i.e. one has $\mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}$. Let now $k \neq 0$ be an integer. It follows from (1) and the case $k = 0$ that

$$\begin{aligned}
\{\mathbf{x} \in F_v^n - \{0\} | r_v(\mathbf{x}) = k\} &= \{\mathbf{x} \in F_v^n - \{0\} | r_v(\pi_v^k \cdot \mathbf{x}) = 0\} \\
&= \{\mathbf{x} \in F_v^n - \{0\} | \pi_v^k \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}\} \\
&= \pi_v^{-k} \cdot \mathcal{D}_v^{\mathbf{a}}.
\end{aligned}$$

4. We have that
$$\emptyset \neq \pi_v^{-k} \cdot (u \cdot (\pi_v^k \cdot \mathcal{D}_v^{\mathbf{a}}) \cap (\pi_v^k \cdot \mathcal{D}_v^{\mathbf{a}})) = (u \cdot \mathcal{D}_v^{\mathbf{a}}) \cap \mathcal{D}_v^{\mathbf{a}}.$$
If $v(u) > 0$, then $(u \cdot \mathcal{D}_v^{\mathbf{a}}) \subset \pi_v^{a_1}\mathcal{O}_v \times \cdots \times \pi_v^{a_n}\mathcal{O}_v$ and hence $(u \cdot \mathcal{D}_v^{\mathbf{a}}) \cap \mathcal{D}_v^{\mathbf{a}} = \emptyset$, a contradiction. Suppose $v(u) < 0$, then $(u \cdot \mathcal{D}_v^{\mathbf{a}}) \subset \pi_v^{-a_1}\mathcal{O}_v \times \cdots \times \pi_v^{-a_n}\mathcal{O}_v - \mathcal{O}_v^n$, and hence $u \cdot \mathcal{D}_v^{\mathbf{a}} \cap \mathcal{D}_v^{\mathbf{a}} = \emptyset$, a contradiction. We deduce $v(u) = 0$.
5. Let $t \in F_v^{\times}$. One has that
$$f_v^{\#}(t \cdot \mathbf{x}) = |\pi_v|_v^{-r_v(t\cdot\mathbf{x})} = |\pi_v|_v^{-r_v(\mathbf{x})+1} = |\pi_v|_v \cdot f_v^{\#}(\mathbf{x}),$$
hence $f_v^{\#}$ is $\mathbf{a}$-homogenous. We have seen in Lemma 3.3.4.4 that $\mathcal{D}_v^{\mathbf{a}}$ is open and closed in F_v^n, and hence in $F_v^n - \{0\}$. Hence, the sets $\pi_v^k \cdot \mathcal{D}_v^{\mathbf{a}}$ are open and closed in $F_v^n - \{0\}$ for every $k \in \mathbb{Z}$. It follows that r_v is locally constant, so is $f_v^{\#}$.

□

Definition 4.4.3.2. — *Let $v \in M_F^0$. We call the $\mathbf{a}$-homogenous function $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ of weighted degree $d \in \mathbb{Z}$ given by*
$$f_v : \mathbf{x} \mapsto \begin{cases} |\pi_v|_v^{-dr_v(\mathbf{x})} & \text{if } v \in M_F^0, \\ (\max_j(|x_j|_v^{1/a_j}))^d & \text{if } v \in M_F^{\infty}, \end{cases}$$
the toric $\mathbf{a}$-homogenous function of weighted degree d.

We remark that when $\mathbf{a} = \mathbf{1}$, for every $v \in M_F$, the toric $\mathbf{1}$-homogenous function of weighted degree 1 is given by $f_v^{\#} : \mathbf{x} \mapsto \max_j(|x_j|_v)$.

Definition 4.4.3.3. — *Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{\geq 0})_v$ be a family of $\mathbf{a}$-homogenous continuous functions of weighted degree d. We say that the degree of the family $(f_v)_v$ is d.*

1. *We say that $(f_v)_v$ is quasi-toric if for almost all $v \in M_F^0$, the function f_v is toric.*
2. *We say that $(f_v)_v$ is toric if for every v one has that $f_v = f_v^{\#}$, where $f_v^{\#}$ is the toric $\mathbf{a}$-homogenous function of weighted degree d.*

For every $v \in M_F^0$, as $E_v \subset \mathcal{D}_v^{\mathbf{a}}$, one has that $f_v^{\#}|_{E_v} = 1$. Thus every quasi-toric family $(f_v)_v$ is automatically generalized adelic.

Definition 4.4.3.4. — *Let $(f_v)_v$ be a quasi-toric family of $\mathbf{a}$-homogenous continuous functions of a fixed weighted degree. We say that the resulting height $H((f_v)_v)$ is quasi-toric. If $(f_v)_v$ is furthermore assumed to be toric, we say that H is toric and may be denoted by $H^{\#}$.*

Example 4.4.3.5. — We present a formula for the toric height in the case $F = \mathbb{Q}$. Let d be a strictly positive integer. Let $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(\mathbb{Q})]$

and let $\widetilde{\mathbf{x}} \in \mathbb{Z}^n$ be a lift of $\mathbf{x}$ which satisfies $\mathbf{x} \in \mathcal{D}_p^{\mathbf{a}}$ for every prime p. Then for every prime p, one has that $f_p^{\#}(\widetilde{\mathbf{x}}) = 1$. It follows that the toric height defined by the degree d toric family satisfies:

$$H^{\#}(\mathbf{x}) = (\max(|\widetilde{x}_j|^{1/a_j}))^d.$$

When $n = 2$ and $\mathbf{a} = (4, 6)$, the toric height defined by the degree 12 toric family is sometimes called *naive height* (e.g. [**13**]).

Remark 4.4.3.6. — Quasi-toric heights are not stable. In fact, we will verify in 4.6, that quasi-toric heights satisfy the weak Northcott property, while the stable heights do not .

Lemma 4.4.3.7. — *For $v \in M_F$, let $f_v^{\#}$ be the toric $\mathbf{a}$-homogenous function of weighted degree* 1*.*

1. *For every $v \in M_F$ and every $\mathbf{y} \in F_v^n - \{0\}$, one has that $f_v^{\#}(\mathbf{y}) \geq \max(|y_j|_v^{1/a_j})$.*
2. *For every $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ one has that $H^{\#}(\mathbf{x}) \geq H^{\max}(\mathbf{x}) \geq 1$, where $H^{\max}$ is the height given by $(\mathbf{y} \mapsto \max(|y_j|_v^{1/a_j}))_v$.*

Proof. — For $v \in M_F^{\infty}$, we recall that $f_v^{\#}(\mathbf{y}) = \max(|y_j|_v^{1/a_j})$. Let $v \in M_F^0$. For every $\mathbf{y} \in \mathcal{D}_v^{\mathbf{a}}$ one has that

$$r_v(\mathbf{y}) \geq \sup_{\substack{j=1,\dots,n \\ x_j \neq 0}} \Big(\frac{-v(y_j)}{a_j}\Big) =: r_v^{\max}(\mathbf{y})$$

and thus

$$f_v(\mathbf{y}) = |\pi_v|_v^{-r_v(\mathbf{y})} \geq |\pi_v|_v^{-r_v^{\max}(\mathbf{y})} \geq (\max(|y_j|_v^{1/a_j}).$$

The function $\mathbf{y} \mapsto f_v^{\#}(\mathbf{y})(\max(|y_j|_v^{1/a_j}))^{-1}$ is $\mathbb{G}_m(F_v)$-invariant. As by Lemma 4.4.3.1 any $\mathbf{z} \in F_v^n - \{0\}$ writes us $t \cdot \mathbf{y}$, where $\mathbf{y} \in \mathcal{D}_v^{\mathbf{a}}$ it follows that $f_v^{\#}(\mathbf{z}) \geq (\max(|z_j|_v^{1/a_j})$ for every $\mathbf{z} \in F_v^n - \{0\}$. The first claim is proven.

Now let $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F)$ and let $\widetilde{\mathbf{x}} : \mathbb{G}_m \to \mathbb{A}^n - \{0\}$ be the $\mathbb{G}_m$-equivariant morphism over F defined by $\mathbf{x}$. We have that

$$H^{\#}(\mathbf{x}) = \prod_{v \in M_F} f_v^{\#}(\widetilde{\mathbf{x}}(1)) \geq \prod_{v \in M_F} \max_j(|\widetilde{\mathbf{x}}(1)|_v^{1/a_j}) = H^{\max}(\mathbf{x}) \geq 1,$$

by Lemma 4.4.2.4. The second claim is proven. □

4.4.4. — We dedicate this paragraph to state and prove a lemma that will be used latter for the proof of Proposition 4.7.1.2. It is motivated by [30, Theorem B.2.5].

Let K be a field and let $\mathbf{a} \in \mathbb{Z}^n_{>0}$. Let us define weighted degrees of polynomials in $K[X_1, \dots, X_n]$. For $j = 1, \dots, n$, we define $\mathbf{a}$-weighted degree (or simply weighted degree) of polynomial X_j by setting $\deg_{\mathbf{a}}(X_j) = a_j$. For a monomial $cX_1^{d_1} \cdots X_n^{d_n}$, where $c \in R$, we define $\deg_{\mathbf{a}}(cX_1^{d_1} \cdots X_n^{d_n}) = \mathbf{a} \cdot \mathbf{d}$. Finally, if $P = \sum_i Q_i$, with Q_i monomials, we define $\deg_{\mathbf{a}}(P) = \max_i(\deg_{\mathbf{a}}(Q_i))$.

Definition 4.4.4.1. — *We say that a polynomial $P \in K[X_1, \dots, X_n]$ is $\mathbf{a}$-homogenous if it is a sum of monomials of the same $\mathbf{a}$-weighted degree.*

Lemma 4.4.4.2. — *Let P be an $\mathbf{a}$-homogenous polynomial of weighted degree $d \geq 1$.*

1. *For $t \in \mathbb{G}_m$ and $\mathbf{x} \in \mathbb{A}^n$, one has that $P(t \cdot \mathbf{x}) = t^d P(\mathbf{x})$.*
2. *Let Q be an $\mathbf{a}$-homogenous polynomial of weighted degree $k \geq 0$. The polynomial PQ is an $\mathbf{a}$-homogenous polynomial of weighted degree $d + k$.*
3. *Let $\{P_1, \dots, P_m\} \in F_v[X_1, \dots, X_n]$ be a set of non-constant $\mathbf{a}$-homogenous polynomials. The closed subscheme $Z(\{P_1, \dots, P_m\}) \subset \mathbb{A}^n$, given by the common zero set of $P_1, \dots, P_m$ is $\mathbb{G}_m$-invariant for the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ with weights $a_1, \dots, a_n$. The open subscheme $D(P_1) \subset \mathbb{A}^n$, given by the locus where P_1 does not vanish is $\mathbb{G}_m$-invariant for the same action of $\mathbb{G}_m$ on $\mathbb{A}^n$.*

Proof. —
1. Let $t \in \mathbb{G}_m$ and let $CX_1^{m_1} \cdots X_n^{m_n}$ be a monomial. For $\mathbf{x} \in \mathbb{A}^n$, we have that
$$CX_1^{m_1} \cdots X_n^{m_n}(t \cdot \mathbf{x}) = Ct_1^{a_1 m_1} x_1^{m_1} \cdots t_n^{a_n m_n} x_n^{m_n} = t^{\mathbf{a} \cdot \mathbf{m}} X_1^{m_1} \cdots X_n^{m_n}(\mathbf{x}).$$
Now, P can be written as sum $\sum_i C_i X_1^{m_{1,i}} \cdots X_n^{m_{n,i}}$ where for each i one has $\mathbf{a} \cdot \mathbf{m}_i = d$ and we deduce that $P(t \cdot \mathbf{x}) = t^d P(\mathbf{x})$.
2. The product PQ is a sum of monomials $CX_1^{m_1} \cdots X_n^{m_n} \cdot DX_1^{r_1} \cdots X_n^{r_n}$, with $\mathbf{a} \cdot \mathbf{m} = d$ and $\mathbf{a} \cdot \mathbf{r} = k$. It follows that PQ is $\mathbf{a}$-homogenous of weighted degree $\mathbf{a}(\mathbf{m} + \mathbf{r}) = k + d$.
3. For every $\mathbf{x} \in \mathbb{A}^n$ and every $i \in \{1, \dots, m\}$, we have that $P_i(t \cdot \mathbf{x}) = t^d P_i(\mathbf{x}) = 0$ if and only if $P_i(\mathbf{x}) = 0$. The claim follows.

□

Definition 4.4.4.3. — *Let $\{P_1, \dots, P_m\} \subset F[X_1, \dots, X_n]$ be a set of non-constant $\mathbf{a}$-homogenous polynomials. We define $\mathcal{Z}(\{P_1, \dots, P_m\})$ to* *be the substack of $\mathscr{P}(\mathbf{a})$ defined by the $\mathbb{G}_m$-invariant closed subscheme $Z(P_1,\dots,P_m)-\{0\}\subset\mathbb{A}^n-\{0\}$. We define $\mathcal{D}(P_1)$ to be the substack of $\mathscr{P}(\mathbf{a})$ defined by the $\mathbb{G}_m$-invariant open subscheme $D(P_1)\subset\mathbb{A}^n-\{0\}$.*

It follows from [**56**, Lemma 04YN] that $\mathcal{Z}(\{P_1,\dots,P_k\})$ is a closed substack of $\mathscr{P}(\mathbf{a})$ and that $\mathcal{D}(P_1\cdots P_m)$ is an open substack of $\mathscr{P}(\mathbf{a})$.

Let $Q_1,\dots,Q_{r+1}$ be $\mathbf{a}$-homogenous polynomials of the same weighted degree d. The morphism

$$J(Q_1,\dots,Q_{r+1}):(\mathbb{A}^n-Z(\{Q_1,\dots,Q_{r+1}\}))\to\mathbb{A}^{r+1}-\{0\}\qquad \mathbf{x}\mapsto(Q_i(\mathbf{x}))_i$$

is $t\mapsto t^d$-equivariant (see 4.1.5 for the terminology), when the left scheme is endowed with the action $t\cdot_{\mathbf{a}}\mathbf{x}=(t^{a_j}x_j)_j$ of $\mathbb{G}_m$ and the right scheme is endowed with the action $t\cdot_{\mathbf{1}}\mathbf{x}=(tx_j)_j$. Let us denote by

$$\overline{J(Q_1,\dots,Q_{r+1})}:\mathscr{P}(\mathbf{a})\to\mathbb{P}^r \tag{4.4.4.4}$$

the 1-morphism of stacks induced by [**56**, Lemma 046Q].

We state our lemma:

Lemma 4.4.4.5. — *Let $P_1,\dots,P_{r+1}\in F[X_1,\dots,X_n]$ be $\mathbf{a}$-homogenous polynomials of the same weighted degree d. Let $\overline{J}(P_1,\dots,P_{r+1}):(\mathscr{P}(\mathbf{a})-\mathcal{Z}(P_1,\dots,P_{r+1}))\to\mathbb{P}^r$ be the 1-morphism given by the $t\mapsto t^d$-equivariant morphism $(P_1,\dots,P_{r+1}):(\mathbb{A}^n-\{0\})\to(\mathbb{A}^n-\{0\})$ (see 4.1.5). Let $H^{\#}$ be the toric height on $[\mathscr{P}(\mathbf{a})(F)]$ defined by the toric degree d family. There exists $C>0$ such that for all $\mathbf{x}\in[(\mathscr{P}(\mathbf{a})-Z(P_1,\dots,P_{r+1}))(F)]$ one has that*

$$CH^{\#}(\mathbf{x})\geq H_{\mathbb{P}^r}(\phi(\mathbf{x})),$$

where $H_{\mathbb{P}^r}$ is the toric height defined by the toric degree 1 family on $\mathbb{P}^r$.

Proof. — The strategy from the proof of [**30**, Theorem B.2.5] applies here. Let us denote by $X^{\mathbf{m}}$ the monomial $X_1^{m_1},\dots,X_n^{m_n}$, where $\mathbf{m}\in\mathbb{Z}^n_{\geq 0}$. We denote by $w(\mathbf{a},d)$ the number of $\mathbf{m}\in\mathbb{Z}^n_{\geq0}$ for which $\mathbf{a}\cdot\mathbf{m}=d$, this is precisely the number of monomials which have $\mathbf{a}$-weighted degree equal to d. For $i\in\{1,\dots,r+1\}$, we write

$$P_i=\sum_{\mathbf{a}\cdot\mathbf{m}=d}A_{i,\mathbf{m}}X^{\mathbf{m}},$$

where the sum runs over $\mathbf{m}\in\mathbb{Z}^n_{\geq0}$ for which $\mathbf{a}\cdot\mathbf{m}=\sum_{j=1}^n a_jm_j=d$. For $v\in M_F$, we denote by $|P_i|_v=\max_{\mathbf{m}}|A_{i,\mathbf{m}}|_v$. For almost all v, one has that $|P_i|_v = 1$. For $k \in \mathbb{Z}$, we set

$$\varepsilon_v(k) := \begin{cases} k & \text{if } v \in M_F^\infty, \\ 1 & \text{if } v \in M_F^0. \end{cases}$$

We note that for any $v \in M_F$, any $k \geq 1$ and any $z_1, \ldots, z_k \in F$, one has that

$$|z_1 + \cdots + z_k|_v \leq \varepsilon_v(r) \max(|z_1|_v, \ldots, |z_k|_v).$$

For $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$, let $\widetilde{\mathbf{x}} \in F_v^n - \{0\}$ be a lift of $\mathbf{x}$. For $i \in \{1, \ldots, r+1\}$ and $v \in M_F$, we deduce that

$$\begin{aligned} |P_i(\widetilde{\mathbf{x}})|_v &= \big|\sum_{\mathbf{a}\cdot\mathbf{m}} A_{i,\mathbf{m}} \widetilde{x}_1^{m_1} \cdots \widetilde{x}_n^{m_n}\big|_v \\ &\leq \varepsilon(w(\mathbf{a}, d))(\max_{\mathbf{m}} |A_{i,\mathbf{m}}|_v)(\max_{\mathbf{m}} |\widetilde{x}_1^{m_1} \cdots \widetilde{x}_n^{m_n}|_v) \\ &\leq \varepsilon(w(\mathbf{a}, d))|P_i|_v \max_{\mathbf{m}} \big(\prod_{\ell=1}^n \max_j (|\widetilde{x}_j|_v^{1/a_j})^{a_\ell m_\ell}\big) \\ &= \varepsilon(w(\mathbf{a}, d))|P_i|_v \max_j (|\widetilde{x}_j|_v^{1/a_j})^{\mathbf{a}\cdot\mathbf{m}} \\ &= \varepsilon(w(\mathbf{a}, d))|P_i|_v \max_j (|\widetilde{x}_j|_v^{1/a_j})^d. \end{aligned}$$

Set $C_v = \varepsilon(w(\mathbf{a}, d)) \max_i |P_i|_v$. For almost all v, one has that $C_v = 1$ and we set $C = \prod_v C_v$. Let us define

$$r_v : F_v^n - \{0\} \to \mathbb{Z} \qquad \mathbf{y} \mapsto \sup_{\substack{j=1,\ldots,n \\ x_j \neq 0}} \left\lceil -\frac{v(y_j)}{a_j} \right\rceil$$

and

$$r_v^{\max} : F_v^n - \{0\} \to \mathbb{Q} \qquad \mathbf{y} \mapsto \sup_{\substack{j=1,\ldots,n \\ y_j \neq 0}} \big(\frac{v(y_j)}{a_j}\big).$$

One has that

$$f_v^{\#}(\mathbf{y}) = |\pi_v|_v^{-dr_v(\mathbf{y})} \geq |\pi_v|_v^{-dr_v^{\max}(\mathbf{y})} = \max_j (|y_j|_v^{1/a_j}|_v)^d.$$

We deduce that

$$\max_i (|P_i(\widetilde{\mathbf{x}})|_v) \leq C_v \max_j (|y_j|_v^{1/a_j})^d \leq C_v f_v^{\#}(\widetilde{\mathbf{x}}).$$

By multiplying this inequality for all v we obtain that for every $\mathbf{x} \in [(\mathscr{P}(\mathbf{a}) - Z(P_1, \dots, P_n))(F)]$ one has that

$$H_{\mathbb{P}^r}(\overline{J}(P_1, \dots, P_{r+1})(\mathbf{x})) = \prod_{v \in M_F} (\max_i(|P_i(\widetilde{\mathbf{x}})|_v)) \leq \prod_v C_v f_v^{\#}(\widetilde{\mathbf{x}})$$
$$= CH^{\#}(\mathbf{x}).$$

The claim is proven. □

4.4.5. — We establish several facts on "local heights" that will be needed in 4.6.

Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a generalized adelic family of $\mathbf{a}$-homogenous continuous function of weighted degree $|\mathbf{a}|$. Let $H = H((f_v)_v)$ be the corresponding height on $[\mathscr{P}(\mathbf{a})(F)]$. For $v \in M_F$, $t \in F_v^\times$ and $\mathbf{x} \in (F_v^\times)^n$ one has that

$$f_v(t \cdot \mathbf{x}) \prod_{j=1}^n |t^{a_j} x_j|_v^{-1} = |t|_v^{|\mathbf{a}|} f_v(\mathbf{x}) \prod_{j=1}^n |t|_v^{-a_j} |x_j|_v^{-1} = f_v(\mathbf{x}) \prod_{j=1}^n |x_j|_v^{-1}$$

i.e. for $v \in M_F$, the continuous function

$$(F_v^\times)^n \to \mathbb{R}_{>0}, \qquad \mathbf{x} \mapsto f_v(\mathbf{x}) \prod_{j=1}^n |x_j|_v^{-1} \tag{4.4.5.1}$$

is $(F_v^\times)_{\mathbf{a}}$-invariant. Let $H_v : [\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{R}_{>0}$ be the function induced from $(F_v)_{\mathbf{a}}$-invariant function (4.4.5.1). For $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$, we write $H_v(\mathbf{x})$ for what is technically $H_v([\mathscr{T}(\mathbf{a})(i_v)](\mathbf{x}))$, where $[\mathscr{T}(\mathbf{a})(i_v)] : [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(F_v)]$ is the induced homomorphism from $(F^\times)_{\mathbf{a}}$-invariant homomorphism

$$(F^\times)^n \hookrightarrow (F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)].$$

Lemma 4.4.5.2. — *Let $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$. One has that $H(\mathbf{x}) = \prod_v H_v(\mathbf{x})$.*

Proof. — Let $\widetilde{\mathbf{x}} \in (F^\times)^n$ be a lift of $\mathbf{x}$. By using the product formula, one gets that

$$\begin{aligned}\prod_{v\in M_F} H_v(\mathbf{x}) &= \prod_{v\in M_F} \big(f_v(\widetilde{\mathbf{x}})\prod_{j=1}^n |\widetilde{x}_j|_v^{-1}\big)\\ &= \big(\prod_{v\in M_F} f_v(\widetilde{\mathbf{x}})\big)\prod_{v\in M_F}\prod_{j=1}^n |\widetilde{x}_j|_v^{-1}\\ &= H(\mathbf{x})\end{aligned}$$

□

Lemma 4.4.5.3. — *Let $v\in M_F$ and suppose that $f_v = f_v^{\#}$ is the toric* $\mathbf{a}$*-homogenous function of weighted degree* 1*. Let $\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]$. One has that $H_v^{\#}(\mathbf{x})\geq 1$.*

Proof. — Suppose $v\in M_F^0$. It follows from Lemma 4.4.3.1 that there exists a lift $\widetilde{\mathbf{x}}$ of $[\mathscr{T}(\mathbf{a})(i_v)](\mathbf{x})$ lying in $\mathcal{D}_v^{\mathbf{a}}$. By using that $f_v^{\#}|_{\mathcal{D}_v^{\mathbf{a}}} = 1$ and that $\mathcal{D}_v^{\mathbf{a}}\subset(\mathcal{O}_v)^n$, we obtain

$$H_v^{\#}(\mathbf{x}) = f_v^{\#}(\widetilde{\mathbf{x}})\prod_{j=1}^n |\widetilde{x}_j|_v^{-1_j} = \prod_{j=1}^n |\widetilde{x}_j|_v^{-1_j}\geq 1.$$

Suppose now $v\in M_F^\infty$. Let $\widetilde{\mathbf{x}}\in(F_v^\times)^n$ be a lift of $\mathbf{x}$. One has that

$$\begin{aligned}H_v^{\#}(\mathbf{x}) &= f_v^{\#}(\widetilde{\mathbf{x}})\prod_{j=1}^n |\widetilde{x}_j|_v^{-1}\\ &= \big(\max_k(|\widetilde{x}_k|_v^{1/a_k})\big)^{|\mathbf{a}|}\prod_{j=1}^n |\widetilde{x}_j|_v^{-1}\\ &= \prod_{j=1}^n\big(\max_{k=1,\dots,n}(|\widetilde{x}_k|_v^{1/a_k})\big)^{a_j}\big(|\widetilde{x}_j|_v^{1/a_j}\big)^{-a_j}\\ &\geq \prod_{j=1}^n\big(\max_k(|\widetilde{x}_k|_v^{1/a_k})\big)^{a_j}\big(\max_k(|\widetilde{x}_k|_v^{1/a_k})\big)^{-a_j}\\ &= 1.\end{aligned}$$

□

4.5. Metrics induced by models

We use models with enough integral points to define metrics. We establish that the toric metric comes from models of weighted projective stacks from 3.2.

4.5.1. — We use $\mathcal{O}_v$-points of $\overline{\mathscr{P}(\mathbf{a})}$ to define F_v-metrics.

Let $v \in M_F^0$. By an $\mathcal{O}_v$-extension of $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}(F_v)$ we mean a pair $(\mathbf{y}, t)$ where $\mathbf{y} \in \overline{\mathscr{P}(\mathbf{a})}(\mathcal{O}_v)$ and $t : \mathbf{y}_{F_v} \xrightarrow{\sim} \mathbf{x}$ is a 2-isomorphism. Let $S_{\mathbf{x}}$ be the set of $\mathcal{O}_v$-extensions of $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}(F_v)$. Proposition 3.2.2.5 gives that the set $S_{\mathbf{x}}$ is non-empty for any $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}(F_v)$.

For $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}(F_v)$ (respectively, $\mathbf{x} \in \overline{\mathscr{P}(\mathbf{a})}(\mathcal{O}_v)$), we will denote by $\widetilde{\mathbf{x}}$ the canonical $(\mathbb{G}_m)_{F_v}$-equivariant morphism $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_{F_v} \to \mathbb{A}^n_{F_v}$ (respectively, the canonical $(\mathbb{G}_m)_{\mathcal{O}_v}$-equivariant morphism $\widetilde{\mathbf{x}} : (\mathbb{G}_m)_{\mathcal{O}_v} \to \mathbb{A}^n_{\mathcal{O}_v}$) induced by $\mathbf{x}$. One has that $q^{\mathbf{a}} \circ \widetilde{\mathbf{x}}(1) = \mathbf{x}$.

If L is a line bundle on $\overline{\mathscr{P}(\mathbf{a})}$, a 2-isomorphism $t : \mathbf{x} \xrightarrow{\sim} \mathbf{x}'$, where $\mathbf{x}, \mathbf{x}' \in \overline{\mathscr{P}(\mathbf{a})}(F_v)$, induces a linear map $L(t) : L(\mathbf{x}) \xrightarrow{\sim} L(\mathbf{x}')$.

Definition 4.5.1.1. — *Let L be a line bundle on $\overline{\mathscr{P}(\mathbf{a})}$. Let $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F_v)$ and let $\ell \in L(\mathbf{x})$. We define*

$$||\ell||_{\mathbf{x}} := \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^{\times} : \ell \in a(L(t)(\mathbf{y}^*L))\}\}.$$

We calculate these metrics.

Lemma 4.5.1.2. — *Let $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F_v)$. Let k be an integer and let $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the $\mathbf{a}$-homogenous toric function of weighted degree k. Let $\ell \in \mathcal{O}(k)(\mathbf{x}) = F_v$. One has that*

$$||\ell||_{\mathbf{x}} := f_v^{\#}(\widetilde{\mathbf{x}}(1))^{-1}|\ell|_v.$$

Proof. — If $t : \mathbf{x}' \to \mathbf{x}''$ is a 2-isomorphism in $\overline{\mathscr{P}(\mathbf{a})}(F_v)$, the induced linear map $\mathcal{O}(k)(t) : F_v = \mathcal{O}(k)(\mathbf{x}') \xrightarrow{\sim} \mathcal{O}(k)(\mathbf{x}'') = F_v$ is the linear map

$x \mapsto t^k x$. It follows that:

$$\begin{aligned}||\ell||_{\mathbf{x}} &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^\times : \ell \in at^k(\mathbf{y}^*\mathcal{O}(k))\}\} \\ &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^\times : \ell \in at^k((\widetilde{\mathbf{y}}(1))^*(q^{\mathbf{a}}_{\mathcal{O}_v})^*\mathcal{O}(k))\}\} \\ &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^\times : \ell \in at^k((\widetilde{\mathbf{y}}(1))^*\mathcal{O}_{\mathbb{A}^n})\}\} \\ &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^\times : \ell \in at^k\mathcal{O}_v\}\} \\ &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{\inf\{|a|_v \mid a \in F_v^\times : 1 \in a\ell^{-1}t^k\mathcal{O}_v\}\} \\ &= \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{|t|_v^{-k}|\ell|_v\} \\ &= |\ell|_v \sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{|t|_v^{-k}\}.\end{aligned}$$

Note that if $(\mathbf{y}, t) \in S_{\mathbf{x}}$, then as $t^{-1} \cdot \widetilde{\mathbf{x}}(1) = \widetilde{\mathbf{y}}(1) \in \mathcal{O}_v^n$, it follows from Lemma 4.4.3.1 that $v(t^{-1}) \geq r_v(\widetilde{\mathbf{x}}(1))$, with the equality if and only if $\widetilde{\mathbf{y}}(1) \in \mathcal{D}_v^{\mathbf{a}}$. We deduce that

$$\sup_{(\mathbf{y},t)\in S_{\mathbf{x}}} \{|t|_v^{-k}\} = |\pi_v|_v^{-kr_v(\widetilde{\mathbf{x}}(1))} = f_v^{\#}(\widetilde{\mathbf{x}}(1))^{-1}.$$

The claim follows. □

We can deduce that:

Corollary 4.5.1.3. — *Let v be a finite place of F and let k be an integer. The metrics $||\cdot||_{\mathbf{x}}$ from Definition 4.5.1.1 on $\mathcal{O}(k)(\mathbf{x})$ for $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F_v)$ define an F_v-metric $||\cdot||$ on $\mathcal{O}(k)|_{\mathscr{P}(\mathbf{a})_{F_v}}$. The F_v-metric $||\cdot||$ is the induced F_v-metric from the function $x \mapsto f_v^{\#}(\mathbf{x})$ by Lemma 4.3.3.4.*

Proof. — For $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F_v)$, let us pick $\ell = 1 \in \mathcal{O}(k)(\mathbf{x})$. By Lemma 4.5.1.2, we obtain that $||1||_{\mathbf{x}} = f_v^{\#}(\widetilde{\mathbf{x}})^{-1}$ for every $\mathbf{x} \in \mathscr{P}(\mathbf{a})(F_v)$. As $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ is continuous (Lemma 4.4.3.1) and satisfies that $f_v^{\#}(t \cdot \mathbf{x}) = |t|_v^k f_v^{\#}(\mathbf{x})$, we deduce that $||\cdot||$ is the induced F_v-metric from the function $f_v^{\#}$ using Lemma 4.3.6.6. □

4.6. Finiteness property of quasi-toric heights

We say that a height is a weak Northcott height if on some non empty open substack $\mathscr{U} \subset \mathscr{P}(\mathbf{a})$ one has that for every $B > 0$ there are only finitely many points in $[\mathscr{U}(F)]$ having the height less than B. We establish that quasi-toric heights are not weak Northcott heights. We improve it further for heights that are degenerate if the singularities of f_v at certain places v are "logarithmic" along a rational divisor. Let $\mathbf{a}\in\mathbb{Z}^n_{\geq 1}$.

4.6.1. — We define weak Northcott heights.

Definition 4.6.1.1. — *Suppose $(f_v:F_v^n-\{0\}\to\mathbb{R}_{\geq 0})_v$ is a generalized adelic family of continuous $\mathbf{a}$-homogenous functions. We say that the corresponding height $H((f_v)_v)$ is a weak Northcott height if there exists a non empty open substack $\mathscr{U}\subset\mathscr{P}(\mathbf{a})$ for every $B>0$, the set*

$$\{\mathbf{x}\in[\mathscr{U}(F)]|H((f_v)_v)(\mathbf{x})<B\}$$

is finite.

Remark 4.6.1.2. — For varieties, it is possible to define heights of its algebraic points and to normalize them so that an L-point (where L/F is a finite extension) has the same height viewed as an K-point (where K/L is a finite extension). Then Northcott theorem at its full strength guarantees a finiteness on a open subvariety of the number of algebraic points of bounded degree and bounded height. However, for stacks the heights which are possible to be normalized in the above way (we will call them stable heights) fail to satisfy even the above "weak" Northcott property as we will see shortly. All the "useful" heights are not stable (i.e. the height of an F-point may changes when the same point is looked as a K-point). For that reason the "strong" Northcott theorem does not have a meaningful analogy in the setting of stacks.

4.6.2. — We give heights which are not weak Northcott heights.

Lemma 4.6.2.1. — *Suppose $n=1$ and $a\in\mathbb{Z}_{\geq 2}$. The set $[\mathscr{P}(a)(F)]$ is infinite.*

Proof. — One has that $[\mathscr{P}(a)(F)]=[\mathscr{T}(\mathbf{a})(F)]=F^\times/(F^\times)_a$, where $(F^\times)_a$ is the subgroup given by the non-zero a-th powers in $F^\times$. There are infinitely many non-zero principal prime ideals in $\mathcal{O}_F$ (because there are only finitely many prime ideals in $\mathbb{Z}$ which ramify in $\mathcal{O}_F$). For any of those principal prime ideals $\mathfrak{p}$, let $b_{\mathfrak{p}}\in F^\times$ be a generator. Note that for any principal prime ideals $\mathfrak{p}_1,\mathfrak{p}_2$ one has that $b_{\mathfrak{p}_1}b_{\mathfrak{p}_2}^{-1}\not\in(F^\times)_a$, because $a\nmid v_{\mathfrak{p}_1}(b_{\mathfrak{p}_1}b_{\mathfrak{p}_2}^{-1})=1$, where $v_{\mathfrak{p}_1}$ is the valuation corresponding to $\mathfrak{p}_1$. It follows that the images in $F^\times/(F^\times)_a$ of different $b_{\mathfrak{p}}$ are different. It follows that $[\mathscr{P}(a)(F)]=F^\times/(F^\times)_a$ is infinite. $\square$

Corollary 4.6.2.2. — *Suppose that $n=1$ and $a\in\mathbb{Z}_{\geq 2}$.*

1. *There are no non empty open substacks of the stack $\mathscr{P}(a)$.*
2. *Let $(f_v)_v$ be a stable generalized adelic family of a-homogenous functions $F_v - \{0\} \to \mathbb{R}_{>0}$. For every $x, y \in \mathscr{P}(a)(F)$ one has that $H((f_v)_v)(\mathbf{x}) = H((f_v)_v)(\mathbf{y})$. The height $H = H((f_v)_v)$ is not a weak Northcott height.*

Proof. — 1. Let $\mathscr{U} \subset \mathscr{P}(a)$ be a non empty open substack. Its preimage $(q^a)^{-1}(\mathscr{U})$ for the $\mathbb{G}_m$-invariant quotient 1-morphism $q^a : \mathbb{G}_m \to \mathscr{P}(a)$ is a non empty $\mathbb{G}_m$-invariant open subscheme of $\mathbb{G}_m$. Let $r \in (q^a)^{-1}(\mathscr{U})(F)$. Suppose that $\mathbb{G}_m - (q^a)^{-1}(\mathscr{U})$ is non empty and let z be a geometric point of $\mathbb{G}_m - (q^a)^{-1}(\mathscr{U})(F)$. There exists a finite extension K of F such that z and all a-th roots of zr^{-1} are defined over K. Then, if $t \in K - \{0\}$ is an a-th root of zr^{-1}, one has that $t \cdot r = z$, a contradiction with the fact that $(q^a)^{-1}(\mathscr{U})$ is $\mathbb{G}_m$-invariant. It follows that $(q^a)^{-1}(\mathscr{U}) = \mathbb{G}_m$ and hence that $\mathscr{U} = \mathscr{P}(a)$. The statement follows.

2. Let $x, y \in \mathscr{P}(a)(F)$ and let $\widetilde{x}, \widetilde{y} : (\mathbb{G}_m)_{F_v} \to (\mathbb{A}^1 - \{0\})_{F_v}$ be the two $(\mathbb{G}_m)_{F_v}$-equivariant morphisms defined by x and y. Let $K = F(\sqrt[a]{\widetilde{x}(1)\widetilde{y}(1)^{-1}})$. Note that for $t := \sqrt[a]{\widetilde{x}(1)\widetilde{y}(1)^{-1}}$ one has $t^a x = y$, thus $x_K \cong y_K$. By Lemma 4.4.2.3, one has that for every $x, y \in \mathscr{P}(a)(F)$ one has $H(x) = H(y)$. Thus for every $z, w \in [\mathscr{P}(\mathbf{a})(F)]$ one has $H(z) = H(w)$. Let $B > H(w)$, where $w \in [\mathscr{P}(a)(F)]$. By Lemma 4.6.2.1, the set $\{z | z \in [\mathscr{P}(a)(F)] \text{ and } H(z) < B\}$ is infinite. Thus H is not a weak Northcott height.

□

4.6.3. — We dedicate the next paragraphs to the proof that the toric heights are weak Northcott heights. By the property of the boundedness of the quotients (Lemma 4.4.1.6), it is then immediate that any quasi-toric height is a weak Northcott height.

Let $\operatorname{Div}(F)$ be the group of fractional ideals of F. We define

$$\operatorname{Div}(F)_{\mathbf{a}} := \{(x^{a_j})_j |\, x \in \operatorname{Div}(F)\}.$$

In this paragraph we will give an estimate to the number of elements of the abelian group $\operatorname{Div}(F)^n / \operatorname{Div}(F)_{\mathbf{a}}$ of bounded "height". This will be useful, as in the next paragraph, we relate the finite part of the height on $\mathscr{T}(\mathbf{a})(F)$ with the "height" on $\operatorname{Div}(F)^n / \operatorname{Div}(F)_{\mathbf{a}}$. If $v \in M_F^0$, for a fractional ideal x of $\mathcal{O}_F$ we define $v(x)$ by setting it to be the exponent of the prime ideal corresponding to v, in the prime factorization of x.

We define set

$$\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}} := \{\mathbf{x}\in \mathrm{Div}(F)^n | \mathbf{x}\subset \mathcal{O}_F^n \text{ and } \forall v\in M_F^0, \exists j : v(x_j) < a_j\}.$$

Lemma 4.6.3.1. — *1. Let us define*

$$r_v : \mathrm{Div}(F)^n \to \mathbb{Z} \qquad \mathbf{x}\mapsto \sup_{\substack{j=1,\dots,n\\ x_j\neq 0}} \left\lceil \frac{-v(x_j)}{a_j}\right\rceil.$$

Let $\mathbf{x}\in \mathrm{Div}(F)^n$. The fractional ideal

$$y = y(\mathbf{x}) := \prod_{v\in M_F^0} \mathfrak{m}_v^{r_v(\mathbf{x})},$$

where $\mathfrak{m}_v$ is the maximal ideal corresponding to $v\in M_F^0$, satisfies that

$$(x_j y(\mathbf{x})^{a_j})_j \in \mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}.$$

2. The restriction of the quotient map

$$q_{\mathrm{Div}(F),\mathbf{a}} : \mathrm{Div}(F)^n \to \mathrm{Div}(F)^n/\,\mathrm{Div}(F)_{\mathbf{a}}$$

to $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$ is a bijection $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}} \xrightarrow{\sim} \mathrm{Div}(F)^n/\,\mathrm{Div}(F)_{\mathbf{a}}$.

Proof. — 1. Let $v\in M_F^0$. For every j one has that

$$v(x_jy^{a_j}) = v(x_j) + a_j r_v(\mathbf{x}) \geq 0.$$

Let i be the index such that $\frac{-v(x_i)}{a_i}$ is maximal, then $r_v(\mathbf{x}) < 1 + \frac{-v(x_i)}{a_i}$. We deduce that

$$v(x_iy^{a_i}) = v(x_i) + a_i r_v(\mathbf{x}) < a_i.$$

Hence, $(x_jy^{a_j})_j$ is an element of $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$.

2. We prove the surjectivity of $q_{\mathrm{Div}(F),\mathbf{a}}$. Let $\mathbf{z}\in \mathrm{Div}(F)^n/\,\mathrm{Div}(F)_{\mathbf{a}}$ and let $\widetilde{\mathbf{z}}\in\mathrm{Div}(F)^n$ be a lift of $\mathbf{z}$. Then $(\widetilde{z}_j y(\mathbf{z})^{a_j})_j$ is a lift of $\mathbf{z}$ belonging to $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$. Let us prove the injectivity. Suppose that the elements $\mathbf{x},\mathbf{r}\in \mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$ are lying above the same element in $\mathrm{Div}(F)^n/\,\mathrm{Div}(F)_{\mathbf{a}}$. Then there exists $t\in\mathrm{Div}(F)$ such that $x_j = t^{a_j}r_j$ for $j=1,\dots,n$. We need to establish that $t=(1)$. Suppose on the contrary $t\neq (1)$. One can choose v such that $v(t)\neq 0$. If $v(t)>0$, then for every j one has

$$v(x_j) = a_jv(t) + v(r_j) \geq a_j\cdot 1 + 0 = a_j,$$

which is a contradiction with the fact that $\mathbf{x}$ is primitive. If $v(t) < 0$, then there exists j such that $a_j > v(r_j)$, which gives that

$$v(x_j) = a_j v(t) + v(r_j) < a_j - a_j = 0,$$

which is a contradiction with the fact that $\mathbf{x}$ is primitive. We deduce that $t = (1)$, and hence $\mathbf{x} = \mathbf{r}$. Therefore $q_{\mathrm{Div}(F),\mathbf{a}}|_{\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}}$ is injective. The claim is proven.

□

For every $\mathbf{x} \in \mathrm{Div}(F)^n / \mathrm{Div}(F)_{\mathbf{a}}$, let us denote by $\widetilde{\mathbf{x}}$ the unique lift of $\mathbf{x}$ lying in $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$. For $\mathbf{x} \in \mathrm{Div}(F)^n / \mathrm{Div}(F)_{\mathbf{a}}$, we set

$$H_{\mathrm{Ideal}}(\mathbf{x}) := \prod_{j=1}^{n} N(\widetilde{x}_j)$$

where N is the ideal norm.

Lemma 4.6.3.2. — *For $m \in \mathbb{Z}_{\geq 1}$, there exists $C_m > 0$ such that for every $B > 0$ one has that*

$$\{\mathbf{y} \in \mathrm{Div}(F)^m_{\geq 0} | \prod_{j=1}^{n} N(y_j) \leq B\} \leq C_m B \log(1+B)^{m-1},$$

where $\mathrm{Div}_{\geq 0}(F) = \{y \in \mathrm{Div}(F) | y \subset \mathcal{O}_F\}$.

Proof. — We will use the following result: there exists $C_1 > 0$ such that

$$|\{y \in \mathrm{Div}(F) | y \subset \mathcal{O}_F, N(y) \leq B\}| \leq C_1 B.$$

This is proven in [**32**, Pages 145-150]. We use induction on m. Let $m \geq 2$. For $k \geq 1$, we set

$$g(k) = \left|\left\{\mathbf{x} \in \mathrm{Div}_{\geq 0}(F)^{m-1} | \prod_{j=1}^{m-1} N(x_j) = k\right\}\right|.$$

For $B \geq 1$, using Abel's summation formula, we get that there exists $C'_m, C_m > 0$ such that

$$
\begin{aligned}
&|\{\mathbf{y} \in \mathrm{Div}_{\geq 0}(F)^m | \prod_{j=1}^{m} N(y_j) \leq B\}| \\
&= \sum_{\substack{\mathbf{y} \in \mathrm{Div}_{\geq 0}(F)^m \\ N(y_m) \leq B / \prod_{j=1}^{m-1} N(y_j)}} 1 \\
&\leq \sum_{\substack{(y_j)_{j=1}^{m-1} \in \mathrm{Div}_{\geq 0}(F)^{m-1} \\ \prod_{j=1}^{m-1} N(y_j) \leq B}} \frac{C_1 B}{\prod_{j=1}^{m-1} N(y_i)} \\
&= C_1 B \sum_{k=1}^{\lfloor B \rfloor} \frac{g(k)}{k} \\
&= C_1 B \Big(\frac{1}{\lfloor B \rfloor} \sum_{j=1}^{\lfloor B \rfloor} g(k) + \sum_{j=1}^{\lfloor B \rfloor - 1} \big(\frac{1}{j} - \frac{1}{j+1} \big) \sum_{k=1}^{j} g(k) \Big) \\
&\leq 2 C_1 C_{m-1} \log(\lfloor B \rfloor + 1)^{m-2} + C_1 B \sum_{j=1}^{\lfloor B \rfloor - 1} \big(\frac{1}{j} - \frac{1}{j+1} \big) C_{m-1} j \log(j+1)^{m-2} \\
&\leq 2 C_1 C_{m-1} \log(B+1)^{m-2} + C_1 C_{m-1} B \sum_{j=1}^{\lfloor B \rfloor - 1} \frac{\log(j+1)^{m-2}}{j} \\
&\leq C'_m \log(B+1)^{m-2} + C'_m B \log(\lfloor B \rfloor)^{m-2} \sum_{j=1}^{\lfloor B \rfloor - 1} \frac{1}{j+1} \\
&\leq C'_m \log(B+1)^{m-2} + C'_m B \log(\lfloor B \rfloor) \log(\lfloor B \rfloor + 1)^{m-1} \\
&\leq C_m B \log(B+1)^{m-1}.
\end{aligned}
$$

The statement follows. □

We can estimate the number of elements of $\mathrm{Div}(F)^n / \mathbf{a}\,\mathrm{Div}(F)$ having H_{Ideal} less than B.

Corollary 4.6.3.3. — *There exists $C > 0$ such that for any $B > 0$, one has*

$$
|\{\mathbf{x} \in \mathrm{Div}(F)^n / \mathbf{a}\,\mathrm{Div}(F) |\, H_{\mathrm{Ideal}}(\mathbf{x}) \leq B\}| \leq C B \log(1+B)^{n-1}.
$$

Proof. — By Lemma 4.6.3.2, there exists $C_n > 0$ such that:

$$\begin{aligned}
&|\{\mathbf{x} \in \operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}} |\, H_{\mathrm{Ideal}}(\mathbf{x}) < B\}| \\
&\qquad = |\{\mathbf{y} \in \operatorname{Div}_{\mathbf{a}-\mathrm{prim}}(F)| \prod_{j=1}^n N(y_j) < B\}| \\
&\qquad \leq |\{\mathbf{y} \in \operatorname{Div}(F)^n | \mathbf{y} \subset \mathcal{O}_F^n, \prod_{j=1}^n N(y_j) < B\}| \\
&\qquad \leq C_n B \log(1+B)^{n-1}.
\end{aligned}$$

The statement is proven. □

4.6.4. — For $v \in M_F$, we let $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the toric **a**-homogenous function of weighted degree $|\mathbf{a}|$. Let $H^{\#}$ be the corresponding toric height on $[\mathscr{P}(\mathbf{a})(F)]$. For $v \in M_F$, we let $H_v^{\#}$ to be the "local heights" from 4.4.5 i.e. the functions $[\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{R}_{>0}$ induced from $F_v^{\times}$-invariant maps

$$(F_v^{\times})^n \to \mathbb{R}_{>0}, \qquad \mathbf{x} \mapsto f_v^{\#}(\mathbf{x}) \prod_{j=1}^n |x_j|_v^{-1}.$$

In this paragraph we relate $\prod_{v \in M_F^0} H_v^{\#}$ and H_{Ideal}. For $x \in F^{\times}$, let us denote by $\mathfrak{I}(x) \in \operatorname{Div}(F)$ the fractional ideal $x\mathcal{O}_F$. We denote by $\mathfrak{I}^n$ the product homomorphism $(F^{\times})^n \to \operatorname{Div}(F)^n$. Note that if $\mathbf{x} \in (F^{\times})_{\mathbf{a}} = \{(x^{a_j})_j | x \in F^{\times}\}$, then $\mathfrak{I}^n(\mathbf{x}) \in \operatorname{Div}(F)_{\mathbf{a}}$. We denote by $\overline{\mathfrak{I}}$ the induced homomorphism

$$\overline{\mathfrak{I}} : [\mathscr{T}(\mathbf{a})(F)] = (F^{\times})^n/(F^{\times})_{\mathbf{a}} \to \operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}} \tag{4.6.4.1}$$

from $(F^{\times})_{\mathbf{a}}$-invariant map

$$F^{\times n} \xrightarrow{\mathbf{x} \mapsto \mathfrak{I}^n(\mathbf{x})} \operatorname{Div}(F)^n \to \operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}},$$

where the second homomorphism is the quotient homomorphism.
Lemma 4.6.4.2. — *Let* $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$. *One has that*

$$H_{\mathrm{Ideal}}(\overline{\mathfrak{I}}(\mathbf{x})) = \prod_{v \in M_F^0} H_v^{\#}(\mathbf{x}).$$

Proof. — Let $\widetilde{\mathbf{x}} \in (F^\times)^n$ be a lift of $\mathbf{x}$. It follows from Lemma 4.6.3.1 that

$$\mathfrak{I}^n(\widetilde{\mathbf{x}})(y(\mathfrak{I}^n(\widetilde{\mathbf{x}}))^{a_j})_j = (\mathfrak{I}(\widetilde{x}_j)\prod_{v\in M_F^0}(\mathfrak{m}_v^{a_j r_v(\mathfrak{I}^n(\widetilde{\mathbf{x}}))})_j,$$

where $\mathfrak{m}_v$ is the maximal ideal corresponding to $v \in M_F^0$, is the unique lift of $\overline{\mathfrak{I}}(\mathbf{x})$ lying in $\mathrm{Div}(F)_{\mathbf{a}-\mathrm{prim}}$. We can calculate (if $0 \neq I \in \mathrm{Div}(F)$, we write $v(I)$ for the exponent of the prime ideal corresponding to v in the prime factorization of I):

$$\begin{aligned}
H_{\mathrm{Ideal}}(\overline{\mathfrak{I}}(\mathbf{x})) &= \prod_{j=1}^n N\Big(\mathfrak{I}(\widetilde{x}_j)\prod_{v\in M_F^0}(\pi_v^{a_j r_v(\mathfrak{I}^n(\widetilde{\mathbf{x}}))}\mathcal{O}_v\cap\mathcal{O}_F)\Big)\\
&= \prod_{j=1}^n N\Big(\prod_{v\in M_F^0}\pi_v^{v(\mathfrak{I}(\widetilde{x}_j))+a_j r_v(\mathfrak{I}^n(\widetilde{\mathbf{x}}))}\mathcal{O}_v\cap\mathcal{O}_F\Big)\\
&= \prod_{j=1}^n\prod_{v\in M_F^0}|\pi_v|_v^{-(v(\mathfrak{I}(\widetilde{x}_j))+a_j r_v(\mathfrak{I}^n(\widetilde{\mathbf{x}})))}\\
&= \prod_{v\in M_F^0}\prod_{j=1}^n|\pi_v|_v^{-(v(\widetilde{x}_j)+a_j r_v(\widetilde{\mathbf{x}}))}\\
&= \prod_{v\in M_F^0}|\pi_v|_v^{-|\mathbf{a}|r_v(\widetilde{\mathbf{x}})}\prod_{j=1}^n|\pi_v|_v^{-v(\widetilde{x}_j)}\\
&= \prod_{v\in M_F^0}f_v(\widetilde{\mathbf{x}})\prod_{j=1}^n|\widetilde{x}_j|_v^{-1}\\
&= \prod_{v\in M_F^0}H_v^{\#}(\mathbf{x}).
\end{aligned}$$

□

4.6.5. — In this paragraph we study the kernel of the homomorphism $\overline{\mathfrak{I}}$ defined in 4.6.4. Let U be the group of units of F and let $U_{\mathbf{a}} := \{(u^{a_j})_j | u \in U\}$. As $U_{\mathbf{a}} = (F^\times)_{\mathbf{a}} \cap U^n$, we have a canonical identification of $U^n/U_{\mathbf{a}}$ with a subgroup of $[\mathscr{T}(\mathbf{a})(F)]$. Note that $U \subset \ker(\mathfrak{I})$ and thus $U^n \subset \ker(\mathfrak{I}^n)$. It follows that

$$U^n/U_{\mathbf{a}} \subset \ker(\overline{\mathfrak{I}}).$$

We prove the following fact:

Lemma 4.6.5.1. — *Let $d = \gcd(\mathbf{a})$. One has that*

$$(\ker(\overline{\mathfrak{I}}) : (U^n/U_{\mathbf{a}})) = |\mathrm{Cl}(F)[d]|,$$

where $\mathrm{Cl}(F)[d]$ is the d-torsion subgroup of the class group $\mathrm{Cl}(F)$.

Proof. — The following "snake diagram" is commutative

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
 & U \arrow[r, "t\mapsto(t^{a_j})_j"] \arrow[d] & U^n \arrow[r] \arrow[d] & \ker(\overline{\mathfrak{I}}) \arrow[d] & \\
 & F^\times \arrow[r, "t\mapsto(t^{a_j})_j"] \arrow[d, "\mathfrak{I}"] & (F^\times)^n \arrow[r] \arrow[d, "\mathfrak{I}^n"] & {[\mathscr{T}(\mathbf{a})(F)]} \arrow[r] \arrow[d, "\overline{\mathfrak{I}}"'] & 1 \\
0 \arrow[r] & \mathrm{Div}(F) \arrow[r, "t\mapsto(a_jt)_j"] \arrow[d] & \mathrm{Div}(F)^n \arrow[r] \arrow[d] & \mathrm{Div}(F)^n/\mathrm{Div}(F)_{\mathbf{a}} \arrow[d] & \\
 & \mathrm{Cl}(F) \arrow[r, "t\mapsto(a_jt)_j"] \arrow[d] & \mathrm{Cl}(F)^n \arrow[r] \arrow[d] & \mathrm{coker}(\overline{\mathfrak{I}}) \arrow[d] & \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where $\mathrm{Cl}(F)$ is the class group of F. All vertical and horizontal sequences are exact. The snake lemma provides an exact sequence

$$U \xrightarrow{t\mapsto(t^{a_j})_j} U^n \to \ker(\overline{\mathfrak{I}}) \xrightarrow{\sigma} \mathrm{Cl}(F) \xrightarrow{t\mapsto(a_jt)_j} \mathrm{Cl}(F)^n \to \mathrm{coker}(\overline{\mathfrak{I}}).$$

It follows that $\ker(\sigma) = \mathrm{Im}\left(U^n \to \ker(\overline{\mathfrak{I}})\right) = U^n/U_{\mathbf{a}}$. The kernel of

$$\mathrm{Cl}(F) \to \mathrm{Cl}(F)^n \qquad t \mapsto (a_jt)_j$$

is given by the subgroup $\mathrm{Cl}(F)[d]$. We deduce that

$$|\,\mathrm{Cl}(F)[d]| = |\,\mathrm{Im}(\sigma)| = |\ker(\overline{\mathfrak{I}})/\ker(\sigma)| = (\ker(\overline{\mathfrak{I}}) : (U^n/U_{\mathbf{a}})).$$

$\square$

4.6.6. — In this paragraph for fixed $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$, we bound the number of elements $\mathbf{u} \in U^n/U_{\mathbf{a}}$ for which $\prod_{v \in M_F^\infty} H_v^{\#}(\mathbf{ux}) < B$.

We define an auxiliary function $h_\infty : (\mathbb{R}^{M_F^\infty})^n \to \mathbb{R}$ by

$$h_\infty : (\mathbf{y}_v)_v \mapsto \sum_{v \in M_F^\infty} \mathbf{a} \cdot \big(\max_{k=1,\dots,n} \left(\frac{y_{v,k}}{a_k} - \frac{y_{v,j}}{a_j} \right) \big)_j.$$

We define a homomorphism

$$\rho : F^\times \to \mathbb{R}^{M_F^\infty} \qquad x \mapsto \log(|x|_v),$$

and a homomorphism

$$\rho^n : (F^\times)^n \to (\mathbb{R}^{M_F^\infty})^n \qquad \mathbf{x} \mapsto (\rho(x_j))_j.$$

Lemma 4.6.6.1. — *Let $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$ and let $\widetilde{\mathbf{x}} \in (F^\times)^n$ be a lift of $\mathbf{x}$. We have that*

$$\log \left(\prod_{v \in M_F^\infty} H_v^{\#}(\mathbf{x}) \right) = h_\infty(\rho^n(\widetilde{\mathbf{x}})).$$

Proof. — For $v \in M_F^\infty$, we have that

$$\begin{aligned}
\log(H_v^{\#}(\mathbf{x})) &= \log \big(\max_k (|\widetilde{x}_k|_v^{1/a_k})^{|\mathbf{a}|} \prod_{j=1}^n |\widetilde{x}_j|_v^{-1} \big) \\
&= \log \big(\prod_{j=1}^n \max_k (|\widetilde{x}_k|_v^{1/a_k})^{a_j} |\widetilde{x}_j|_v^{-1} \big) \\
&= \log \big(\prod_{j=1}^n \max_k (|\widetilde{x}_k|_v^{a_j/a_k}) |\widetilde{x}_j|_v^{-1} \big) \\
&= \log \big(\prod_{j=1}^n \max_k (|\widetilde{x}_k|_v^{a_j/a_k} |\widetilde{x}_j|_v^{-1}) \big) \\
&= \sum_{j=1}^n \log \big(\max_k \left(\frac{|\widetilde{x}_k|_v^{1/a_k}}{|\widetilde{x}_j|_v^{1/a_j}} \right)^{a_j} \big) \\
&= \sum_{j=1}^n a_j \max_k \left(\frac{\log(|\widetilde{x}_k|_v)}{a_k} - \frac{\log(|\widetilde{x}_j|_v)}{a_j} \right) \\
&= \mathbf{a} \cdot \big(\max_k \left(\frac{\log(|\widetilde{x}_k|_v)}{a_k} - \frac{\log(|\widetilde{x}_j|_v)}{a_j} \right) \big)_j.
\end{aligned}$$

We deduce:

$$\log\big(\prod_{v\in M_F^\infty} H_v^{\#}(\mathbf{x})\big) = \sum_{v\in M_F^\infty} \mathbf{a}\cdot\big(\max_k\Big(\frac{\log(|\widetilde{x}_k|_v)}{a_k} - \frac{\log(|\widetilde{x}_j|_v)}{a_j}\Big)\big)_j$$
$$= h_\infty(\rho(\widetilde{\mathbf{x}})).$$

□

Lemma 4.6.6.2. — *For $j = 1,\dots,n$, let us define a homomorphism $\alpha_j : (\mathbb{R}^{M_F^\infty})^n \to (\mathbb{R}^{M_F^\infty})^n$ by*

$$\alpha_j : (\mathbf{y}_i)_i \mapsto \big(\frac{\mathbf{y}_i}{a_i} - \frac{\mathbf{y}_j}{a_j}\big)_i.$$

Set $\alpha = \prod_{j=1}^n \alpha_j : (\mathbb{R}^{M_F^\infty})^n \to (\mathbb{R}^{M_F^\infty})^{n^2}$.

1. *One has that*
$$\ker(\alpha) = \big\{(a_j\mathbf{y})_j | \mathbf{y}\in\mathbb{R}^{M_F^\infty}\big\}.$$
2. *Let $K > 0$ and let $(\mathbf{y}_j)_j \in (\mathbb{R}^{M_F^\infty})^n$ be such that $h_\infty(\mathbf{y}) < K$. One has that*
$$\alpha(\mathbf{y}) \in \prod_{i,j}\left[\frac{-K}{a_i}, \frac{K}{a_j}\right]^{M_F^\infty}.$$
3. *Suppose L is a lattice of $\mathbb{R}^{M_F^\infty}$. Then $\alpha(L^n)$ is contained in a lattice of $(\mathbb{R}^{M_F^\infty})^{n^2}$.*

Proof. — 1. For $k,\ell\in\{1,\dots,n\}$, one has that

$$\ker(\alpha_\ell) = \{\big(\frac{a_i\mathbf{y}_\ell}{a_\ell}\big)_i | \mathbf{y}_\ell\in\mathbb{R}^{M_F^\infty}\} = \{(a_i\mathbf{y})_i | \mathbf{y}\in\mathbb{R}^{M_F^\infty}\}$$
$$= \{\big(\frac{a_i\mathbf{y}_k}{a_k}\big)_j | \mathbf{y}_k\in\mathbb{R}^{M_F^\infty}\} = \ker(\alpha_k).$$

Therefore

$$\ker(\alpha) = \bigcap_{j=1}^n \ker(\alpha_j) = \ker(\alpha_1) = \{(a_j\mathbf{y})_j | \mathbf{y}\in\mathbb{R}^{M_F^\infty}\}.$$

2. Let $\ell\in\{1,\dots,n\}$. Note that from

$$h_\infty(\mathbf{y}) = \sum_{v\in M_F^\infty}\mathbf{a}\cdot\big(\max_k\Big(\frac{y_{v,k}}{a_k} - \frac{y_{v,j}}{a_j}\Big)\big)_j < K,$$

it follows that for every $i \in \{1, \dots, n\}$ and every $w \in M_F^\infty$, one has that

$$a_\ell\Big(\frac{y_{w,i}}{a_i} - \frac{y_{w,\ell}}{a_\ell}\Big) \leq a_\ell \max_k \Big(\frac{y_{v,k}}{a_k} - \frac{y_{v,\ell}}{a_\ell}\Big) \leq \sum_{v \in M_F^\infty} \mathbf{a} \cdot \big(\max_k \Big(\frac{y_{v,k}}{a_k} - \frac{y_{v,j}}{a_j}\Big)\big)_j < K,$$

and hence that

$$\frac{y_{w,i}}{a_i} - \frac{y_{w,\ell}}{a_\ell} < \frac{K}{a_\ell}.$$

Similarly, for every $i \in \{1, \dots, n\}$ and every $w \in M_F^\infty$, one has that

$$\frac{y_{w,\ell}}{a_\ell} - \frac{y_{w,i}}{a_i} < \frac{K}{a_i}.$$

We deduce that

$$\frac{\mathbf{y}_i}{a_i} - \frac{\mathbf{y}_j}{a_j} \in \Big[-\frac{K}{a_i}, \frac{K}{a_\ell} \Big]_{v \in M_F^\infty}$$

and hence that

$$\alpha_\ell((\mathbf{y}_i)_i) = \big(\frac{\mathbf{y}_i}{a_i} - \frac{\mathbf{y}_\ell}{a_\ell}\big)_i \in \Big[-\frac{K}{a_i}, \frac{K}{a_\ell} \Big]_{v,i}.$$

Finally, it follows that

$$\alpha((\mathbf{y}_i)_i) = (\alpha_j((\mathbf{y}_i)_i))_j \in \prod_{i,j} \Big[-\frac{K}{a_i}, \frac{K}{a_j} \Big]^{M_F^\infty}.$$

3. Let $(\mathbf{y}_j)_j \in L^n$. Then for every i, j one has

$$\frac{\mathbf{y}_i}{a_i} - \frac{\mathbf{y}_j}{a_j} \in \frac{L}{\prod_{j=1}^n a_j}.$$

We deduce

$$\alpha_j(\mathbf{y}) = \big(\frac{\mathbf{y}_i}{a_i} - \frac{\mathbf{y}_j}{a_j}\big)_i \in \Big(\frac{L}{\prod_{j=1}^n a_j}\Big)_{i=1}^n.$$

It follows that

$$\alpha(L^n) = (\alpha_j(L^n))_j \subset \Big(\frac{L}{\prod_{j=1}^n a_j}\Big)_{\ell=1}^{n^2}.$$

The claim is proven.

$\square$

Let us set

$$W := \big\{(w_v)_v \in \mathbb{R}^{M_F^\infty} \,|\, \sum_v w_v = 0\big\}$$

For $u \in U$, we have that

$$\rho(u) = (\log |u|_v)_v \in W,$$

and hence for $\mathbf{u} \in U^n$, we have that $\rho^n(\mathbf{u}) = (\rho(u_j))_j \in W^n$.

Lemma 4.6.6.3. — *The following claims are valid:*

1. *The homomorphism ρ^n is of finite kernel and its image is a lattice of W^n.*
2. *One has that $W^n \cap \ker(\alpha) = \{(a_j\mathbf{w})_j | \mathbf{w} \in W\}$.*
3. *One has that $\mathrm{rk}(\ker(\alpha \circ (\rho^n|_{U^n}))) \leq r_1 + r_2 - 1$, where r_1 is the number of the real and r_2 is the number of complex places of F.*
4. *One has that $U_\mathbf{a} \subset \ker(\alpha \circ (\rho^n|_{U^n}))$ and the map $\beta : U^n/U_\mathbf{a} \to (\mathbb{R}^{M_F^\infty})^{n^2}$, induced from $U_\mathbf{a}$-invariant map $\alpha \circ (\rho^n|_{U^n})$, is of finite kernel.*
5. *The image $\beta(U^n/U_\mathbf{a})$ is contained in a lattice of $(\mathbb{R}^{M_F^\infty})^{n^2}$.*

Proof. — 1. By Dirichlet's unit theorem, one has that $\ker(\rho|_U)$ is finite and that $\rho(U)$ is a lattice of W. It follows that $\ker(\rho^n|_{U^n}) = \ker(\rho|_U)^n$ is finite and that $\rho^n(U^n) = \rho(U)^n$ is a lattice of W^n.

2. By Lemma 4.6.6.2 one has that $\ker(\alpha) = \{(a_j\mathbf{y})_j | \mathbf{y} \in \mathbb{R}^{M_F^\infty}\}$ and thus

$$W^n \cap \ker(\alpha) = W^n \cap \{(a_j\mathbf{y})_j | \mathbf{y} \in \mathbb{R}^{M_F^\infty}\} = \{(a_j\mathbf{w})_j | \mathbf{w} \in W\}.$$

3. By (1), the kernel of $\rho^n|_{\ker(\alpha\circ\rho^n|_{U^n})} : \ker(\alpha \circ \rho^n|_{U^n}) \to W^n$ is finite. It suffices therefore to show that the image $\rho^n(\ker(\alpha \circ \rho^n|_{U^n}))$ is of rank no more than $r_1 + r_2 - 1$. The image $\rho^n(\ker(\alpha \circ \rho^n|_{U^n}))$ is contained in $\ker(\alpha)$ and by (1), it is also contained in the lattice $\rho^n(U^n)$ of W^n, thus

$$\rho^n(\ker(\alpha \circ \rho^n|_{U^n})) \subset \rho^n(U^n) \cap \ker(\alpha) = \rho^n(U^n) \cap W^n \cap \ker(\alpha).$$

The rank of the intersection of the lattice $\rho^n(U^n)$ of W^n and the vector subspace $\ker(\alpha) \cap W^n$ of W^n cannot be more than $\dim(W^n \cap \ker(\alpha)) = \dim(\{(a_j\mathbf{w})_j | \mathbf{w} \in W\}) = r_1 + r_2 - 1$. The claim follows.

4. Let $(u^{a_j})_j \in U_\mathbf{a}$, where $u \in U$. We have that

$$\rho^n((u^{a_j})_j) = (a_j\rho(u))_j \in \ker(\alpha).$$

We deduce $U_{\mathbf{a}} \subset \ker(\alpha \circ (\rho^n|_{U^n}))$. Let us establish that $U_{\mathbf{a}}$ is of finite index in $\ker(\alpha \circ (\rho^n|_{U^n}))$. The map

$$U \to U^n \qquad u \mapsto (u^{a_j})_j$$

is of finite kernel and thus

$$\mathrm{rk}(U_{\mathbf{a}}) = \mathrm{rk}(U) = r_1 + r_2 - 1 \geq \mathrm{rk}(\ker(\alpha \circ (\rho^n|_{U^n}))).$$

We conclude that $\mathrm{rk}(U_{\mathbf{a}}) = r_1 + r_2 - 1 = \mathrm{rk}(\ker(\alpha \circ (\rho^n|_{U^n})))$, and hence is $U_{\mathbf{a}}$ of finite index in $\ker(\alpha \circ (\rho^n|_{U^n}))$. Now, it follows that the homomorphism β, which is precisely the composite homomorphism

$$\beta : U^n/U_{\mathbf{a}} \to U^n/\ker(\alpha \circ (\rho^n|_{U^n})) \hookrightarrow (\mathbb{R}^{M_F^\infty})^{n^2},$$

is of finite kernel. The claim is proven.

5. One has that $\beta(U^n/U_{\mathbf{a}}) = \alpha(\rho(U^n))$. By Lemma 4.6.6.2 one has that $\alpha(\rho^n(U^n))$ is contained in a lattice of $(\mathbb{R}^{M_F^\infty})^{n^2}$. The statement follows.

□

We now prove the principal result of the paragraph.

Lemma 4.6.6.4. — *There exists $C > 0$ such that for every $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$ and every $B > 2$, one has that*

$$\{\mathbf{u} \in U^n/U_{\mathbf{a}} | H^{\#}_{\infty}(\mathbf{u}\mathbf{x}) < B\} \leq C\big(\log(B)\big)^{n^2(r_1+r_2)}.$$

Proof. — For $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$, let $\widetilde{\mathbf{x}} \in (F^\times)^n$ be a lift of $\mathbf{x}$ and for $\mathbf{u} \in U^n/U_{\mathbf{a}}$, let $\widetilde{\mathbf{u}} \in U^n$ be a lift of $\mathbf{u}$. For $B > 1$, by Lemma 4.6.6.1, one has

(4.6.6.5)
$$\begin{aligned}\{\mathbf{u} \in U^n/U_{\mathbf{a}} | H^{\#}_{\infty}(\mathbf{u}\mathbf{x}) < B\} &= \{\mathbf{u} \in U^n/U_{\mathbf{a}} | h_\infty(\rho^n(\widetilde{\mathbf{u}}\widetilde{\mathbf{x}})) < B\} \\ &= \{\mathbf{u} \in U^n/U_{\mathbf{a}} | h_\infty(\rho^n(\widetilde{\mathbf{x}}) + \rho^n(\widetilde{\mathbf{u}})) < \log(B)\}.\end{aligned}$$

Further, by using Lemma 4.6.6.2 and the fact that $\ker(\beta)$ is finite (Lemma 4.6.6.3), we obtain for every $B>1$ that

(4.6.6.6)

$$
\begin{aligned}
&|\{\mathbf{u}\in U^n/U_{\mathbf{a}}|h_\infty(\rho^n(\widetilde{\mathbf{x}})+\rho(\widetilde{\mathbf{u}}))<\log(B)\}|\\
&\quad=\big|\big\{\mathbf{u}\in U^n/U_{\mathbf{a}}|\alpha(\rho^n(\widetilde{\mathbf{x}})+\rho^n(\widetilde{\mathbf{u}}))\in\big[\frac{-\log(B)}{a_i},\frac{\log(B)}{a_i}\big]_{v,i,j}\big\}\big|\\
&\quad=\big|\big\{\mathbf{u}\in U^n/U_{\mathbf{a}}|\alpha(\rho^n(\widetilde{\mathbf{u}}))\in-\alpha(\rho^n(\widetilde{\mathbf{x}}))+\big[\frac{-\log(B)}{a_i},\frac{\log(B)}{a_i}\big]_{v,i,j}\big\}\big|\\
&\quad=\big|\big\{\mathbf{u}\in U^n/U_{\mathbf{a}}|\beta(\mathbf{u})\in-\alpha(\rho^n(\widetilde{\mathbf{x}}))+\big[\frac{-\log(B)}{a_i},\frac{\log(B)}{a_j}\big]_{v,i,j}\big\}\big|\\
&\quad=|\ker(\beta)|\cdot\big|\big\{\beta(U^n/U_{\mathbf{a}})\cap\big(-\alpha(\rho^n(\widetilde{\mathbf{x}}))+\big[\frac{-\log(B)}{a_i},\frac{\log(B)}{a_j}\big]_{v,i,j}\big)\big\}\big|.
\end{aligned}
$$

As $\beta(U^n/U_{\mathbf{a}})$ is a subgroup (Lemma 4.6.6.3) of a lattice of $(\mathbb{R}^{M_F^\infty})^{n^2}$, there exists $C'>0$ such that

(4.6.6.7)

$$
\begin{aligned}
&\Big|\beta(U^n/U_{\mathbf{a}})\cap\big(-\alpha(\rho(\widetilde{\mathbf{x}}))+\big[\frac{-\log(B)}{a_i},\frac{\log(B)}{a_j}\big]_{v,i,j}\big)\big\}\Big|\\
&\qquad\le C'\prod_{i,j}\Big(\frac{\log(B)}{a_i}+\frac{\log(B)}{a_j}\Big)^{r_1+r_2}\\
&\qquad\le C'\prod_{i,j}\big(\log(B^{1/a_i+1/a_j})\big)^{r_1+r_2}\\
&\qquad=C'\big(1/a_i+1/a_j\big)^{n^2(r_1+r_2)}\big(\log(B)\big)^{n^2(r_1+r_2)}.
\end{aligned}
$$

for every $B>2$. By combining estimates (4.6.6.5), (4.6.6.6) and (4.6.6.7) we deduce that there exists $C>0$ such that for every $B>2$ one has that

$$\{\mathbf{u}\in U^n/U_{\mathbf{a}}|H^{\#}_\infty(\mathbf{ux})<B\}\le C\big(\log(B)\big)^{n^2(r_1+r_2)}.$$

□

4.6.7. — In this paragraph we prove that quasi-toric heights are weak Northcott heights. We will establish that on the open substack $\mathscr{T}(\mathbf{a})$ there are only finitely many isomorphism classes of rational points of bounded height. We also give an upper estimate for the number of rational points.

Proposition 4.6.7.1. — *Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a quasi-toric family of* $\mathbf{a}$*-homogenous functions of weighted degree* $|\mathbf{a}|$*. The resulting height $H = H((f_v)_v)$ is a weak Northcott height. There exists $C > 0$ such that for every $B > 0$ one has*

$$\left|\left\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H(\mathbf{x}) < B\right\}\right| < CB\log(2+B)^{n^2(r_1+r_2)+n-1}.$$

Proof. — Let $(f_v^{\#})_v$ be the family of toric $\mathbf{a}$-homogenous continuous functions of weighted degree $|\mathbf{a}|$. We firstly prove the statement for the family $(f_v = f_v^{\#})_v$. Recall that $\overline{\mathfrak{I}} : [\mathscr{T}(\mathbf{a})(F)] \to \operatorname{Div}(F)^n / \operatorname{Div}(F)_{\mathbf{a}}$ is the induced homomorphism from $\operatorname{Div}(F)_{\mathbf{a}}$-invariant homomorphism $(F^{\times})^n \to \operatorname{Div}(F)^n \to \operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}}$. We set

$$H_0^{\#}(\mathbf{x}) := \prod_{v\in M_F^0} H_v^{\#}(\mathbf{x})$$
$$H_\infty^{\#}(\mathbf{x}) := \prod_{v\in M_F^\infty} H_v^{\#}(\mathbf{x}).$$

Recall that for every $v \in M_F$ we have $H_v^{\#}(\mathbf{x}) \geq 1$. Using this fact and Lemma 4.6.4.2, for $B > 0$ we deduce that

(4.6.7.2)
$$\begin{aligned}
&\left|\left\{\mathbf{x} \in [\mathscr{T}^{\mathbf{a}}(F)] | H^{\#}(\mathbf{x}) < B\right\}\right| \\
&\qquad \leq \left|\left\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H_0^{\#}(\mathbf{x}) < B, H_\infty^{\#}(\mathbf{x}) < B\right\}\right| \\
&\qquad \leq \left|\left\{\mathbf{x} \in \mathscr{T}(\mathbf{a})(F) | H_{\mathrm{Ideal}}(\overline{\mathfrak{I}}(\mathbf{x})) < B, H_\infty^{\#}(\mathbf{x}) < B\right\}\right|.
\end{aligned}$$

For $\mathbf{y} \in \operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}}$, let $\widetilde{\mathbf{y}} \in \operatorname{Div}(F)^n$ be a lift of $\mathbf{y}$. We have that

$$\begin{aligned}
&\left|\left\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H_{\mathrm{Ideal}}(\overline{\mathfrak{I}}(\mathbf{x})) < B, H_\infty^{\#}(\mathbf{x}) < B\right\}\right| \\
&= \left|\{(\mathbf{y},\mathbf{z}) \in (\operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}}) \times \ker(\overline{\mathfrak{I}}) | H_{\mathrm{Ideal}}(\mathbf{y}) < B, H_\infty^{\#}(\mathbf{z}\widetilde{\mathbf{y}}) < B\}\right|
\end{aligned}$$

and thus by the estimate (4.6.7.2) we get that

(4.6.7.3)
$$\begin{aligned}
&\left|\left\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H(\mathbf{x}) < B\right\}\right| \\
&\leq \left|\left\{(\mathbf{y},\mathbf{z}) \in (\operatorname{Div}(F)^n/\operatorname{Div}(F)_{\mathbf{a}}) \times \ker(\overline{\mathfrak{I}}) | H_{\mathrm{Ideal}}(\mathbf{y}) < B, H_\infty^{\#}(\mathbf{z}\widetilde{\mathbf{y}}) < B\right\}\right|.
\end{aligned}$$

Recall from 4.6.5, that $U^n/U_{\mathbf{a}} \subset \ker(\overline{\mathfrak{I}})$. Let δ be a set theoretical section to the quotient map $\ker(\overline{\mathfrak{I}}) \to \ker(\overline{\mathfrak{I}})/(U^n/U_{\mathbf{a}})$. By Lemma 4.6.5.1, the group $\ker(\overline{\mathfrak{I}})/(U^n/U_{\mathbf{a}})$ is finite, thus is $\delta(\ker(\overline{\mathfrak{I}})/(U^n/U_{\mathbf{a}}))$ finite. By using the estimate from Lemma 4.6.6.4, we deduce that there exists $C_0 > 0$ such that for every $\mathbf{y} \in \mathrm{Div}(F)^n / \mathrm{Div}(F)_{\mathbf{a}}$ and every $B > 0$ one has

(4.6.7.4)
$$\begin{aligned}
&|\{\mathbf{z} \in \ker(\overline{\mathfrak{I}}) | H^{\#}_{\infty}(\mathbf{z}\widetilde{\mathbf{y}}) < B\}| \\
&\qquad \leq \big|\big\{(\mathbf{u}, \mathbf{d}) \in (U^n/U_{\mathbf{a}}) \times \delta(\ker(\overline{\mathfrak{I}})/(U^n/U_{\mathbf{a}}))\big| H^{\#}_{\infty}(\mathbf{u}\mathbf{d}\widetilde{\mathbf{y}}) < \log(B)\big\}\big| \\
&\qquad \leq \sum_{\mathbf{d} \in \delta(\ker(\overline{\mathfrak{I}})/(U^n/U_{\mathbf{a}}))} |\{\mathbf{u} \in U^n/U_{\mathbf{a}} | H^{\#}_{\infty}(\mathbf{u}\mathbf{d}\widetilde{\mathbf{y}}) < \log(B)\}| \\
&\qquad \leq C_0(\log(B+2))^{n^2(r_1+r_2)}.
\end{aligned}$$

Corollary 4.6.3.3 gives that there exists $C_1 > 0$ such that for every $B > 0$ one has that

$$|\{\mathbf{y} \in \mathrm{Div}(F)^n / \mathbf{a}\,\mathrm{Div}(F)|\ H_{\mathrm{Ideal}}(\mathbf{y}) < B\}| \leq C_1 B \log(1+B)^{n-1}.$$

We deduce that for every $B > 0$ one has

$$\begin{aligned}
&\big|\big\{\mathbf{x} \in \mathscr{T}^{\mathbf{a}}(F) | H(\mathbf{x}) < B\big\}\big| \\
&\quad \leq \big|\big\{(\mathbf{y}, \mathbf{z}) | (\mathbf{y}, \mathbf{z}) \in (\mathrm{Div}(F)^n / \mathrm{Div}(F)_{\mathbf{a}}) \times \ker(\overline{\mathfrak{I}}); \\
&\qquad\qquad\qquad\qquad H_{\mathrm{Ideal}}(\mathbf{y}) < B, H^{\#}_{\infty}(\mathbf{z}\widetilde{\mathbf{y}}) < B\big\}\big| \\
&\quad \leq \big|\big\{\mathbf{y} \in \mathrm{Div}(F)^n / \mathrm{Div}(F)_{\mathbf{a}} | H_{\mathrm{Ideal}}(\mathbf{y}) < B\big\}\big| \big(\log(B)\big)^{n^2(r_1+r_2)} \\
&\quad \leq C_0 C_1 B \log(1+B)^{n-1} \log(2+B)^{n^2(r_1+r_2)} \\
&\quad \leq C B \log(2+B)^{n^2(r_1+r_2)+n-1},
\end{aligned}$$

for some $C > 0$.

Let $(f_v)_v$ now be any quasi-toric family of $\mathbf{a}$-homogenous continuous functions of weighted degree $|\mathbf{a}|$. By Lemma 4.4.1.6, there exists $C' > 0$ such that

$$H^{\#}(\mathbf{x}) < C' H(\mathbf{x}) \qquad \forall \mathbf{x} \in F_v^n - \{0\}.$$

We deduce that

$$\begin{aligned}
|\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H(\mathbf{x}) < B\}| &\leq |\{\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)] | H^{\#}(\mathbf{x}) < C'B\}| \\
&\leq C(C'B) \log(2 + C'B)^{n^2(r_1+r_2)+n-1} \\
&= C'' B \log(2+B)^{n^2(r_1+r_2)+n-1}
\end{aligned}$$

for $C'' \gg 0$ and every $B > 0$. The statement is proven. $\square$

4.6.8. — In this paragraph we prove that the finiteness result is in fact valid on the whole of $\mathscr{P}(\mathbf{a})$ and not only on the open substack $\mathscr{T}(\mathbf{a}) \subset \mathscr{P}(\mathbf{a})$. Again, we give an upper estimate on the number of rational points of bounded height. For $j \in \{1,\dots,n\}$, we will denote by $\{j\}^c$, the set $\{1,\dots,n\} - \{j\}$. For $j \in \{1,\dots,n\}$, we denote by $p^j : \mathbb{A}^n \to \prod_{i\in\{j\}^c} \mathbb{A}^1$ the canonical projection and by d_j the closed immersion

$$d_j : \prod_{i\in\{j\}^c} \mathbb{A}^1 \to \mathbb{A}^n \qquad (x_i)_i \mapsto ((x_i)_{i\neq j}, (0)_j).$$

By abuse of notation, we will shorten $p^j(\mathbb{Z})$ and write simply p^j. The morphisms d_j are $\mathbb{G}_m$-equivariant when $\prod_{i\in\{j\}^c} \mathbb{A}^1$ is endowed with the action of $\mathbb{G}_m$ given by

$$t\cdot(x_i)_i = (t^{a_i}x_i)_i.$$

For every j, one has that $d_j^{-1}(\mathbb{A}^n - \{0\}) = \left(\prod_{i\in\{j\}^c} \mathbb{A}^1\right) - \{0\}$. We deduce $\mathbb{G}_m$-equivariant morphisms

$$d_j|_{(\prod_{i\in\{j\}^c}\mathbb{A}^1)-\{0\}} : \Big(\prod_{i\in\{j\}^c} \mathbb{A}^1\Big) - \{0\} \to \mathbb{A}^n - \{0\}.$$

Let $\overline{d_j}$ denotes the induced closed immersion of stacks $\mathscr{P}(p^j(\mathbf{a})) \to \mathscr{P}(\mathbf{a})$ from the $\mathbb{G}_m$-equivariant morphism $d_j|_{(\prod_{i\in\{j\}^c}\mathbb{A}^1)-\{0\}}$.

Lemma 4.6.8.1. — *Let $j \in \{1,\dots,n\}$.*

1. *Let $v \in M_F$ and let $f : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be an $\mathbf{a}$-homogenous continuous function of weighted degree $b \in \mathbb{Z}_{\geq 0}$. The function $f\circ(d_j(F_v)) : (\prod_{i\in\{j\}^c} F_v) - \{0\} \to \mathbb{C}$ is $p^j(\mathbf{a})$-homogenous, continuous and of weighted degree b. Moreover, if f is the toric $\mathbf{a}$-homogenous function of weighted degree b, then $f\circ(d_j(F_v))$ is the toric $\mathbf{a}$-homogenous of weighted degree b.*
2. *Let $(g_v : F_v^n - \{0\} \to \mathbb{R}_{\geq 0})_v$ be a quasi-toric family of $\mathbf{a}$-homogenous continuous functions of weighted degree b and let $H = H((g_v)_v)$ be the corresponding height. The family $g_v\circ d_j(F_v)$ is a degree b quasi-toric family of $p_I(\mathbb{Z})(\mathbf{a})$-homogenous continuous functions and the corresponding height $H^j = H((g_v\circ d_I(F_v))_v)$ satisfies that*

$$H^j = H\circ(\overline{d_j}(F)).$$

3. Let $\mathfrak{i} : \mathscr{T}(\mathbf{a}) \to \mathscr{P}(\mathbf{a})$ *be the inclusion induced by the* $\mathbb{G}_m$*-invariant subscheme* $\mathbb{G}_m^n \subset \mathbb{A}^n - \{0\}$*. The map*

$$\Big(\coprod_{j=1}^n [\overline{d_j}(F)] \coprod [\mathfrak{i}(F)]\Big) : \coprod_{j=1}^n [\mathscr{P}(p^j(\mathbb{Z})(\mathbf{a}))(F)] \coprod [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{P}(\mathbf{a})(F)]$$

is surjective.

Proof. — 1. The function $f \circ d_j(F_v)$ is continuous, as it coincides with the restriction $f|_{(\prod_{i\in\{j\}^c} F_v)-\{0\}}$ and f is continuous. Let $t \in F_v^\times$ and let $(x_i)_{i\in\{j\}^c} \in (\prod_{i\in\{j\}^c} F_v) - \{0\}$. We have that

$$\begin{aligned} f(d_j(F_v)(t\cdot(x_i)_{i\in\{j\}^c})) &= f(d_j(F_v)((t^{a_i}x_i)_{i\in\{j\}^c})) \\ &= f((t^{a_i}x_i)_{i\in\{j\}^c}, (0)_j) \\ &= |t|_v^b f((x_i)_{i\in\{j\}^c}, (0)_j) \\ &= |t|_v^b (f\circ d_j(F_v))((x_i)_{i\in\{j\}^c}), \end{aligned}$$

thus $f\circ d_j(F_v)$ is $p^j(\mathbf{a})$ homogenous of weighted degree b. Suppose f is the toric $\mathbf{a}$-homogenous of weighted degree b, that is

$$f(\mathbf{y}) = |\pi_v|_v^{-b\cdot r_v^{\mathbf{a}}(\mathbf{y})}, \text{ where } r_v^{\mathbf{a}}(\mathbf{y}) = \sup_{\substack{j=1,\dots,n\\ y_j\neq 0}} \left\lceil \frac{-v(y_j)}{a_j} \right\rceil,$$

if v is finite and

$$f(\mathbf{y}) = (\max_{j=1,\dots,n}(|y_j|_v^{1/a_j}))^{-b}$$

if v is infinite. Suppose v is finite. Let $\mathbf{x} \in (\prod_{i\in\{j\}^c} F_v) - \{0\}$. One has that $d_j(F_v)(\mathbf{x}) = ((x_i)_{i\in\{j\}^c}, (0)_j)$ and that

$$(r_v^{\mathbf{a}} \circ (d_j(F_v)))(\mathbf{x}) = \sup_{\substack{i\in\{j\}^c\\ x_i\neq 0}} \left\lceil \frac{-v(x_i)}{a_i} \right\rceil =: r^{p^j(\mathbf{a})}(\mathbf{x}),$$

because the j-th coordinate of $d_j(F_v)(\mathbf{x})$ is zero. It follows that the function $f\circ(d_j(F_v))$ given by

$$\mathbf{x} \mapsto f(d_j(F_v)(\mathbf{x})) = |\pi_v|_v^{-b\cdot r_v^{\mathbf{a}}(d_j(F_v)(\mathbf{x}))} = |\pi_v|_v^{-b\cdot r_v^{p^j(\mathbf{a})}(\mathbf{x})}$$

is toric.

Suppose v is infinite. The toric $p^j(\mathbf{a})$-homogenous function of weighted degree b is the function

$$f' : (\prod_{i\in\{j\}^c} F_v) - \{0\} \to \mathbb{R}_{>0} \qquad \mathbf{x} \mapsto \max_{i\in\{j\}^c}(|x_i|_v^{1/a_i}).$$

For $\mathbf{x} \in (\prod_{i\in\{j\}^c} F_v) - \{0\}$, we have that

$$f(d_j(F_v)(\mathbf{x})) = \max_{i\in\{j\}^c}\left(|x_i|_v^{1/a_i}\right) = f'(\mathbf{x}),$$

because the j-th coordinate of $d_j(F_v)(\mathbf{x})$ is zero. The claim follows.

2. By (1), for every $v \in M_F$, the function $g_v \circ d_I(F_v)$ is a continuous $p_I(\mathbb{Z})(\mathbf{a})$-homogenous of weighted degree b and if g_v is toric, then $g_v \circ d^j(F_v)$ is toric of the same weighted degree. It follows that the family $(g_v \circ d^j(F_v))_v$ is quasi-toric of the degree b. Let $\mathbf{y} \in [\mathscr{P}(p^j(\mathbf{a}))(F_v)]$ and let $\widetilde{\mathbf{y}} \in (\prod_{i\in\{j\}^c} F) - \{0\}$ be a lift. Then $d^j(F)((\widetilde{y}_i)_{i\in\{j\}^c})$ is a lift of $\overline{d^j}(F)(\mathbf{y})$. We have that

$$H^j(\mathbf{y}) = \prod_{v\in M_F}(g_v\circ d^j(F_v))(\widetilde{\mathbf{y}}) = \prod_{v\in M_F} g_v(d^j(F)(\widetilde{\mathbf{y}})) = H((g_v)_v)(\mathbf{y}) = H(\mathbf{y}).$$

The claim follows.

3. Let $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$ and let $\widetilde{\mathbf{x}} \in F^n - \{0\}$ be a lift of $\mathbf{x}$. If $\widetilde{\mathbf{x}} \in (F^\times)^n$, then $\mathbf{x} \cong [\mathfrak{i}(F)(q^{\mathbf{a}}(F)(\widetilde{\mathbf{x}}))]$, where $q^{\mathbf{a}}$ is the quotient 1-morphism $\mathbb{G}_m^n \to \mathscr{T}(\mathbf{a})$. Suppose at least one coordinate, say the j-th coordinate, of $\widetilde{\mathbf{x}}$ is equal to zero. Then,

$$\widetilde{\mathbf{x}} = d_j(F)((\widetilde{x}_i)_{i\in\{j\}^c})$$

and so

$$\mathbf{x} \cong \overline{d_j}(q^{p^j(\mathbf{a})}((\widetilde{x}_i)_{i\in\{j\}^c})),$$

where $q^{p^j(\mathbf{a})} : \big((\prod_{i\in\{j\}^c} \mathbb{A}^1) - \{0\}\big) \to \mathscr{P}(p^j(\mathbf{a}))$, is the quotient 1-morphism. The claim follows.

□

Theorem 4.6.8.2. — *Let $n \geq 1$ be an integer and let $\mathbf{a} \in \mathbb{Z}^n_{>0}$. Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a quasi-toric family of $\mathbf{a}$-homogenous functions of weighted degree $|\mathbf{a}|$ and let $H = H((f_v)_v)$.*

1. *There exists $C > 0$ such that for every $B > 0$ one has that*

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] - [\mathscr{T}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| < CB^{\frac{|\mathbf{a}|-\min_i a_i}{|\mathbf{a}|}} \log(2 + B^{\frac{|\mathbf{a}|-\min_i a_i}{|\mathbf{a}|}})^{n^2(r_1+r_2)+n-1}.$$

2. There exists $C>0$ such that for every $B>0$ one has that

$$|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|<CB\log(2+B)^{n^2(r_1+r_2)+n-1}.$$

Proof. — We prove the both statements simultaneously. We apply the induction on n. When $n=1$, one has that $\mathscr{P}(a)=\mathscr{T}(a)$. Thus the first claim is trivial and the second is proven in Proposition 4.6.7.1.

Suppose that the both claims are true for some $n-1\geq 0$ and let us prove them for n. The map

$$\Big(\coprod_{j=1}^n[\overline{d_j}(F)]\coprod[\mathrm{i}(F)]\Big):\coprod_{j=1}^n[\mathscr{P}(p^j(\mathbb{Z})(\mathbf{a}))(F)]\coprod[\mathscr{T}(\mathbf{a})(F)]\to[\mathscr{P}(\mathbf{a})(F)]$$

is surjective by Lemma 4.6.8.1. For $j=1,\dots,n$, by Lemma 4.6.8.1, one has that

$$H^j=H((f_v\circ(d_j(F_v)))_v)=H\circ\overline{d}_j,\qquad\text{where } H^j:=H((f_v\circ(d_j(F_v)))_v).$$

We deduce that

$$|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}| \tag{4.6.8.3}$$

$$\leq|\{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|+\sum_{j=1}^n|\{\mathbf{x}\in[\mathscr{P}(p^j(\mathbf{a}))(F)]|H^j(\mathbf{x})\leq B\}|.$$

For $j=1,\dots,n$, by Lemma 4.6.8.1, the height H^j is defined by degree $|\mathbf{a}|$ quasi-toric family. The family

$$(g_v\circ(d_j(F_v))^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}})_v$$

is quasi-toric of degree $|p^j(\mathbf{a})|$ and the resulting height $H'=H((g_v\circ(d_j(F_v))^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}})_v)$ is related to H^j as follows

$$H'=(H^j)^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}.$$

By the induction hypothesis, we deduce that for every $j=1,\dots,n$ there exists $C_j>0$ such that for every $B>1$ one has that

$$\begin{aligned}|\{\mathbf{x}\in[\mathscr{P}(p^j(\mathbf{a}))(F)]|H^j(\mathbf{x})\leq B\}|&\\&=|\{\mathbf{x}\in[\mathscr{P}(p^j(\mathbf{a}))(F)]|H^j(\mathbf{x})^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}\leq B^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}\}|\\&=|\{\mathbf{x}\in[\mathscr{P}(p^j(\mathbf{a}))(F)]|H'(\mathbf{x})\leq B^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}\}|\\&\leq C_jB^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}\log(2+B^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}})^{n^2(r_1+r_2)+n-1}.\end{aligned}$$

It follows that

$$\begin{aligned}
&|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|\\
&=|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|-|\{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|\\
&\leq\sum_{j=1}^{n}|\{\mathbf{x}\in[\mathscr{P}(p^j(\mathbf{a}))(F)]|H^j(\mathbf{x})\leq B\}|\\
&\leq\sum_{j=1}^{n}C_jB^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}}\log(2+B^{\frac{|p^j(\mathbf{a})|}{|\mathbf{a}|}})\\
&\leq C''B^{\frac{|\mathbf{a}|-\min_i a_i}{|\mathbf{a}|}}\log(2+B^{\frac{|\mathbf{a}|-\min_i a_i}{|\mathbf{a}|}})
\end{aligned}$$

for $C''\gg 0$. Thus the first claim is proven for n. By Proposition 4.6.7.1, there exists $C'>0$ such that

$$|\{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|<C'B\log(2+B)^{n^2(r_1+r_2)+n-1}. \tag{4.6.8.4}$$

By combining the estimates (4.6.8.3), (4.6.8.4) and the first claim, we get that there exists $C>0$ such that for every $B>1$ one has that

$$|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|\leq CB\log(2+B)^{n^2(r_1+r_2)+n-1}.$$

The statement is proven. □

Remark 4.6.8.5. — In Chapter 8, we establish that

$$|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|\sim CB,$$

for some $C>0$.

4.7. Weak Northcott property for singular heights

In this section we allow finitely many f_v to take values in 0 but we require that f_v^{-1} admits a "logarithmic singularity" over a rational divisor. We establish that corresponding heights are weak Northcott heights.

4.7.1. — In this paragraph we study heights that are obtained when v-adic metric is singular. We require the singularities to be "logarithmic" over a rational divisor. We establish a finiteness result on the number of points of bounded height outside of the divisor.

Definition 4.7.1.1. — *Let $P\in F[X_1,\dots,X_n]$ be a non-constant* **a**-*homogenous polynomial and let S be a finite set of places. A collection of continuous $\mathbb{G}_m(F_v)$-invariant functions $g_v:D(P)(F_v)\to\mathbb{R}_{>0}$ for $v\in S$,* *will be said to be logarithmically suitable if the following condition is satisfied: there exists a set $\{Q_i\}_i$ of non-constant* **a***-homogenous polynomials $Q_i \in F[X_1,\dots,X_n]$ such that Q_i and P have no common factors of degree at least 1, such that $Z(\{Q_i\}_i)=\{0\}$ and such that for every $v\in S$ and every i there exist $\alpha_{v,i},\beta_{v,i}>0$ such that*

$$g_v(\mathbf{x})\leq \alpha_{v,i}\max\bigg(-\log\bigg(\frac{|P(\mathbf{x})|_v^{\deg_{\mathbf{a}}(Q)/\deg_{\mathbf{a}}(P)}}{|Q_i(\mathbf{x})|_v}\bigg)^{\beta_{v,i}},1\bigg)$$

for every $\mathbf{x}\in D(PQ_i)(F_v)$.

By the quasi-compactness of the scheme $\mathbb{A}^n$, one can always ask for the set $\{Q_i\}_i$ to be finite.

For $v\in M_F$, let $f_v^{\#}:F_v^n-\{0\}\to\mathbb{R}_{>0}$ be the v-adic toric **a**-homogenous continuous function of weighted degree $d\geq 1$. Let $P\in F[X_1,\dots,X_n]$ be a non-constant **a**-homogenous polynomial. Let S be a finite set of places and let $(g_v)_v$ be a logarithmically suitable family of continuous $\mathbb{G}_m(F_v)$-invariant functions. For $v\in S$ we define $f_v:F_v^n-\{0\}\to\mathbb{R}_{\geq 0}$ by

$$f_v(\mathbf{x}):=\begin{cases} f_v^{\#}g_v^{-1}(\mathbf{x}) & \text{if } \mathbf{x}\in D(P)(F_v),\\ 0 & \text{if } \mathbf{x}\in Z(P)(F_v).\end{cases}$$

and for $v\in M_F-S$ we let $f_v=f_v^{\#}$. For every $v\in M_F$, the function $f_v:F_v^n-\{0\}\to\mathbb{R}_{\geq 0}$ is an **a**-homogenous continuous function of weighted degree $d\geq 1$. We define height $H=H((f_v)_v)$ and $H^{\#}=H((f_v^{\#}))$ on $\mathscr{P}(\mathbf{a})(F)$. We recall that for every $\mathbf{x}\in\mathscr{P}(\mathbf{a})(F)$, one has that $H^{\#}(\mathbf{x})\geq 1$ by Lemma 4.4.3.7.

Motivated by [**55**, Proposition 2.1], we establish that:

Proposition 4.7.1.2. — *There exist $C,\beta>0$ such that for all $\mathbf{x}\in\mathcal{D}(P)(F)$ one has*

$$H(\mathbf{x})\geq CH^{\#}(\mathbf{x})\log(1+H^{\#}(\mathbf{x}))^{-\beta}.$$

Proof. — For $\mathbf{x}\in\mathcal{D}(P)(F)$, let $\widetilde{\mathbf{x}}:\mathbb{G}_m\to\mathbb{A}^n-\{0\}$ be the $\mathbb{G}_m$-equivariant morphism over F defined by $\mathbf{x}$. For simplicity we will write $\widetilde{\mathbf{x}}$ for $\widetilde{\mathbf{x}}(1)$.

There exists a finite set of non-constant **a**-homogenous polynomials $\{Q_i\}_i$ which have no common factors with P_v of degree at least 1 such that $Z(\{Q_i\}_i)=\{0\}$, and such that for every $v\in S$ and every i, there exist $\alpha_{i,v},\beta_{i,v}$ such that

$$g_v(\mathbf{x})\leq \alpha_{i,v}\max\bigg(-\log\bigg(\frac{|P(\mathbf{x})|_v^{\deg_{\mathbf{a}}(Q)/\deg_{\mathbf{a}}(P)}}{|Q_i(\mathbf{x})|_v}\bigg)^{\beta_{i,v}},1\bigg)$$

for every $\mathbf{x} \in D(PQ_i)(F_v)$. Fix an index i. For $\mathbf{x} \in \mathcal{D}(Q_iP)(F)$, using the fact that $f_v = f_v^{\#}$ for $v \notin S$ we deduce that

$$\begin{aligned}
H(\mathbf{x}) &= \prod_{v \in M_F} f_v(\widetilde{\mathbf{x}}) \\
&= \prod_{v \in M_F} f_v^{\#}(\widetilde{\mathbf{x}}) \prod_{v \in S} g_v(\widetilde{\mathbf{x}})^{-1} \\
&\geq H^{\#}(\mathbf{x}) \prod_{v \in S} \alpha_{v,i}^{-1} \max\left(-\log\left(\frac{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}}{|Q_i(\widetilde{\mathbf{x}})|_v} \right), 1 \right)^{-\beta_{v,i}} \\
&\geq H^{\#}(\mathbf{x}) \prod_{v \in S} \alpha_{v,i}^{-1} \max\left(\log\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}} \right), 1 \right)^{-\beta_{v,i}} \\
&= H^{\#}(\mathbf{x}) \prod_{v \in S} \alpha_{v,i}^{-1} \log\left(\max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}} \right), e \right)^{-\beta_{v,i}} \\
&\geq \alpha_i H^{\#}(\mathbf{x}) \prod_{v \in S} \log\left(\max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}} \right), e \right)^{-\beta_{v,i}},
\end{aligned}$$

where $\alpha = \prod_{v \in S} \alpha_{v,i}^{-1}$. For two polynomials $A, B \in F[X_1, \dots, X_n]$ of the same weighted degree, we denote by $\overline{J}(A,B) : \mathscr{P}(\mathbf{a}) \to \mathbb{P}^1$, the 1-morphism of stacks given by Lemma 4.4.4.4. Lemma 4.4.4.5 gives that for every $v \in S$, there exists $\gamma_v > 1$ such that for every $\mathbf{x} \in \mathcal{D}(Q_iP)(F)$ one has that :

$$\begin{aligned}
\max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}}, e \right) &\leq e \max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)/\deg_{\mathbf{a}}(P)}}, 1 \right) \\
&\leq e \max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(P)}}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)}}, 1 \right)^{1/\deg_{\mathbf{a}}(P)} \\
&\leq e \prod_{v \in M_F} \max\left(\frac{|Q_i(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(P)}}{|P(\widetilde{\mathbf{x}})|_v^{\deg_{\mathbf{a}}(Q_i)}}, 1 \right)^{1/\deg_{\mathbf{a}}(P)} \\
&= e H_{\mathbb{P}^1}(\overline{J}(Q_i^{\deg_{\mathbf{a}}(P)}, P^{\deg_{\mathbf{a}}(Q_i)})(\mathbf{x}))^{1/\deg_{\mathbf{a}}(P)} \\
&\leq \gamma_v H^{\#}(\mathbf{x})^{\deg_{\mathbf{a}}(Q_i)}.
\end{aligned}$$

For $v \in S$ let us set $\delta_v = \max(\log(\gamma_v), \deg_{\mathbf{a}}(Q_i))$. We have for every $\mathbf{x} \in \mathcal{D}(Q_i P)(F)$ that

$$\begin{aligned}
H(\mathbf{x}) &\geq \alpha_i H^{\#}(\mathbf{x}) \prod_{v\in S} \log(\gamma_v H^{\#}(\mathbf{x})^{\deg_{\mathbf{a}}(Q_i)})^{-\beta_{v,i}} \\
&= \alpha_i H^{\#}(\mathbf{x}) \prod_{v\in S} \big(\log(\gamma_v) + \deg_{\mathbf{a}}(Q_i)\log(H^{\#}(\mathbf{x}))\big)^{-\beta_{v,i}} \\
&\geq \alpha_i H^{\#}(\mathbf{x}) \prod_{v\in S} \delta_v^{-\beta_{v,i}}\big(1+\log(H^{\#}(\mathbf{x}))\big)^{-\beta_{v,i}} \\
&= C_i H^{\#}(\mathbf{x}) \log(1+H^{\#}(\mathbf{x}))^{-\beta_i},
\end{aligned}$$

where we have set $C_i = \alpha_i \prod_{v\in S} \delta_v^{-\beta_{v,i}}$ and $\beta_i = \sum_{v\in S}\beta_{v,i}$. We have that $\bigcup_i D(Q_i P) = D(P)$. Thus for $C = \min_i C_i$ and $\beta = \max \beta_i$ we get

$$H(\mathbf{x}) \geq C H^{\#} \log(1+H^{\#}(\mathbf{x}))^{-\beta},$$

for every $\mathbf{x} \in D(P)(F)$. □

Corollary 4.7.1.3. — *Let $(f_v)_v$ be as above and let H be the corresponding height. The height H is a weak Northcott height. Moreover, for every $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that for every $B > 1$ one has that*

$$|\{\mathbf{x} \in [\mathcal{D}(P)(F)] | H(\mathbf{x}) \leq B\}| \leq CB^{1+\epsilon}$$

and that

$$\begin{aligned}
&|\{\mathbf{x} \in [\mathcal{D}(P)(F)] \cap ([\mathscr{P}(\mathbf{a})(F)] - [\mathscr{T}(\mathbf{a})(F)]) | H(\mathbf{x}) \leq B\}| \\
&\qquad\qquad\qquad\qquad \leq CB^{\frac{(1+\epsilon)(|\mathbf{a}|-\min_j a_j)}{|\mathbf{a}|}}.
\end{aligned}$$

Proof. — By Proposition 4.7.1.2, there exist $C', N > 0$, such that for every $\mathbf{x} \in [\mathcal{D}(P)(F)]$ one has that

$$H(\mathbf{x}) \geq C' H^{\#}(\mathbf{x}) \log(1+H^{\#}(\mathbf{x}))^{-N}.$$

and thus there exists $A > 0$ and $\frac{\epsilon}{1+\epsilon} > \delta > 0$ such that

$$H(\mathbf{x}) \geq AH^{\#}(\mathbf{x})^{1-\delta} \qquad \forall \mathbf{x} \in [\mathcal{D}(P)(F)].$$

Using Theorem 4.6.8.2, one has hence that there exists $C>0$ such that

$$\begin{aligned}
&|\{\mathbf{x}\in[\mathcal{D}(P)(F)]|H(\mathbf{x})\leq B\}|\\
&\qquad\leq|\{\mathbf{x}\in[\mathcal{D}(P)(F)]|AH^{\#}(\mathbf{x})^{1-\delta}\leq B\}|\\
&\qquad=|\{\mathbf{x}\in[\mathcal{D}((P)(F))]|H^{\#}(\mathbf{x})\leq A^{-1/(1-\delta)}B^{1/(1-\delta)}\}|\\
&\qquad\leq A^{-1/(1-\delta)}B^{1/(1-\delta)}\log(2+A^{-1/(1-\delta)}B^{1/(1-\delta)})^{n^2(r_1+r_2)+n-1}\\
&\qquad\leq CB^{1+\epsilon}
\end{aligned}$$

and such that

$$\begin{aligned}
&|\{\mathbf{x}\in[\mathcal{D}(P)(F)]\cap([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|\\
&\quad\leq|\{\mathbf{x}\in[\mathcal{D}(P)(F)]|AH^{\#}(\mathbf{x})^{1-\delta}\leq B\}|\\
&\quad=|\{\mathbf{x}\in[\mathcal{D}((P)(F))]\cap([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])|H^{\#}(\mathbf{x})\leq A^{-\frac{1}{1-\delta}}B^{\frac{1}{1-\delta}}\}|\\
&\quad\leq(A^{-1}B)^{\frac{(|\mathbf{a}|-\min_j a_j)}{(1-\delta)|\mathbf{a}|}}\log(2+(A^{-1}B)^{\frac{|\mathbf{a}|-\min_j a_j}{(1-\delta)|\mathbf{a}|}})^{n^2(r_1+r_2)+n-1}\\
&\quad\leq CB^{\frac{(1+\epsilon)(|\mathbf{a}|-\min_j a_j)}{|\mathbf{a}|}}.
\end{aligned}$$

The statement is proven. $\square$

CHAPTER 5

MEASURES

Let $n \geq 1$ be an integer. Let $\mathbf{a} \in \mathbb{Z}_{>0}^n$. Recall that $\mathscr{P}(\mathbf{a})$ is the quotient stack for the action

$$\mathbb{G}_m \times (\mathbb{A}^n - \{0\}) \to \mathbb{A}^n - \{0\} \qquad t \cdot \mathbf{x} = (t^{a_j} x_j)_j.$$

The open subscheme $\mathbb{G}_m^n \subset \mathbb{A}^n - \{0\}$ is $\mathbb{G}_m$-invariant for this action and $\mathscr{T}(\mathbf{a})$ is defined to be the quotient $\mathbb{G}_m^n/\mathbb{G}_m$. In this chapter we define measures on $[\mathscr{P}(\mathbf{a})(F_v)]$ and $[\mathscr{T}(\mathbf{a})(F_v)]$ for $v \in M_F$. We use these measures to define Peyre's constant as in [**47**]. Later in the section we define a Haar measure on the adelic space $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and we define and calculate the corresponding Tamagawa number.

5.1. Quotient measures

5.1.1. — We make several conventions on measures that will be used throughout the chapter. Let X be a locally compact topological space. Let $\mathscr{C}_c^0(X, \mathbb{C})$ be the set of continuous compactly supported functions on X. We endow $\mathscr{C}_c^0(X, \mathbb{C})$ with the uniform convergence topology. By a *measure* on X [**9**, Chapter III, §1, n^0 3, Definition 2], we mean a continuous linear functional $\mu : \mathscr{C}_c^0(X, \mathbb{C}) \to \mathbb{C}$.

Let μ be a measure on X. Let $L^1(X, \mu)$ be the Banach space of absolutely μ-integrable complex valued functions modulo negligible functions [**9**, Chapter IV, §3, n^0 4, Definition 2]. By abuse of the terminology, we may call an element $f \in L^1(X, \mu)$ a function and for a function $g : X \to \mathbb{C}$ which is μ-absolutely integrable we may write $g \in L^1(X, \mu)$. For $f \in L^1(X, \mu)$, we denote by $\int_X f\mu$ the integral of f

against μ [9, Chapter IV, §4, n^0 1, Definition 1]. If $U \subset X$ is a subset, such that $\mathbf{1}_{U,X} \in L^1(X, \mu)$, we write $\mu(U)$ for $\mu(\mathbf{1}_{U,X})$ (where $\mathbf{1}_{U,X}$ stands for the characteristic function of U in X, written sometimes as $\mathbf{1}_U$). Such U will be said to be μ-measurable.

5.1.2. — We recall some facts about quotient measures from [10, Chapter VII, §2]. Let X be a locally compact Hausdorff topological space. Let G be a locally compact topological group acting on the right on X continuously and properly (that is the action $X \times G \to X$ is continuous and proper). The quotient topological space X/G is separated [7, Chapter III, §2, n^0 2, Proposition 3] and locally compact [7, Chapter I, §10, n^0 4, Proposition 10].

Let dg be a left Haar measure on G. Let $\Delta_G : G \to \mathbb{R}_{>0}$ be the modular function of G (we recall the definition of the modular function: according to [10, Chapter VII, §1, n^0 1, Formula (11)], for every $h \in G$, the association $A \mapsto dg(Ah)$, for dg-measurable subset A of G, is a left Haar measure on G, hence, by the unicity of the Haar measure there exists a unique positive real number $\Delta_G(h)$ such that for every dg-measurable subset A of G, one has that $dg(Ag) = \Delta_G(g)dg(A)$; further, it does not depend on the choice of the Haar measure dg).

Proposition 5.1.2.1 **(**[10, Chapter VII, §2, n^0 2, Proposition 1]**)**
For $x \in X/G$, let $\widetilde{x} \in X$ be a lift of x. Let $\phi : X \to \mathbb{C}$ be a compactly supported continuous function. For every $y \in X$, one has that $g \mapsto \phi(yg) \in L^1(G, \mathbb{C})$. Moreover, for $x \in X/G$, the value of $\int_G \phi(\widetilde{x}g)dg$ does not depend on the choice of $\widetilde{x}$. The function $\phi^* : x \mapsto \int_G \phi(\widetilde{x}g)dg$ is continuous and compactly supported.

Proposition 5.1.2.2 **(**[10, Chapter VII, §2, n^0 2, Proposition 4]**)**

1. *Let μ be a measure on X such that for every μ-measurable U one has that $\mu(Ug) = \Delta_G(g)\mu(U)$ (such measures will be called G-invariant measures). There exists a unique measure μ/dg on X/G such that for every compactly supported function $\phi : X \to \mathbb{C}$ one has that*
$$\int_X \phi\mu = \int_{X/G} (\phi^*)(\mu/dg).$$
2. *Let μ' be a measure on X/G. There exists a unique G-invariant measure μ on X such that $\mu/dg = \mu'$.*

Let $\pi : X \to X/G$ be the quotient map. We quote some more propositions from [10].

Proposition 5.1.2.3 **(**[10, Chapter VII, §2, n^0 4, Proposition 8]**)**
Suppose that X/G is paracompact. Let $\lambda > 0$. There exists a continuous function $k : X \to \mathbb{R}_{\geq 0}$, the support of which has a compact intersection with the preimage under π of any compact of X/G and such that for any $x \in X$ one has that

$$\int_G k(xg)dg = \lambda.$$

Of course, when X/G is compact, the condition on the support of k becomes that it is compact.

Lemma 5.1.2.4. — *Let $f : X \to \mathbb{C}$ be a continuous G-invariant function. Let $\overline{f} : X/G \to \mathbb{C}$ be the function induced from f. One has that $(f\mu)/dg = (\overline{f})(\mu/dg)$.*

Proof. — Let $\phi : X \to \mathbb{C}$ be a compactly supported continuous function. The function ϕf is compactly supported. Let $x \in X/G$ and let $\widetilde{x} \in X$ be its lift. One has that

$$\begin{aligned}(\phi \cdot f)^*(x) = \int_G \phi f(\widetilde{x}g)dg = \int_G \phi(\widetilde{x}g)f(\widetilde{x}g)dg = \overline{f}(x)\int_G \phi(\widetilde{x}g)dg \\ = ((\phi^*) \cdot \overline{f})(x).\end{aligned}$$

It follows that: $(\phi \cdot f)^* = \phi^* \cdot \overline{f}$. We deduce that:

$$\begin{aligned}\int_{X/G}(\phi^*)(f\mu/dg) &= \int_X (\phi)(f\mu)\\ &= \int_X (\phi f)\mu\\ &= \int_{X/G}((\phi\cdot f)^*)(\mu/dg)\\ &= \int_{X/G}(\phi^*\overline{f})(\mu/dg)\\ &= \int_{X/G}(\phi^*)((\overline{f})(\mu/dg)).\end{aligned}$$

It follows that $(f\mu/dg) = (\overline{f})(\mu/dg)$. □

Proposition 5.1.2.5 **(**[10, Chapter VII, §2, n^0 4, Proposition 9]**)**
Let us suppose that X/G is paracompact. Let $k : X \to \mathbb{R}_{\geq 0}$ be a continuous function, the support of which has compact intersection with

the preimage under π of any compact of X/G and such that for every $x \in X$ one has

$$\int_G k(xg)dg = 1.$$

Then for any function $h : X/G \to \mathbb{C}$ one has that $h \in L^1(X/G, \mu/dg)$ if and only if $k \cdot (h \circ \pi) \in L^1(X, \mu)$ and if $h \in L^1(X/G, \mu/dg)$ then

$$\int_{X/G} (h)(\mu/dg) = \int_X k \cdot (h \circ \pi)\mu.$$

5.1.3. — We recall the notion of the quotient measure when we have inclusion of locally compact abelian groups.

Let G is an abelian locally compact Hausdorff topological group. The action of $G \times G \to G$ of G on G given by the multiplication is proper (because the induced map $G \times G \xrightarrow{(m_G, p_2)} G \times G$ is a homeomorphism, where m_G is the multiplication map and p_2 the projection to the second coordinate). Thus if A a closed subgroup, the action of A on G, given by the restriction of the action of G, is proper by [**7**, Chapter III, §4, n^0 1, Example 1]. Let dg be a Haar measure on G and let da be a Haar measure on A. By [**10**, Chapter VII, §2, n^0 7, Proposition 10], the quotient measure dg/da is a Haar measure on G/A.

5.1.4. — In this paragraph we recall the theory of [**43**] on the Euler characteristics of complexes of locally compact abelian groups endowed with Haar measures.

A homomorphism of topological groups is said to be *strict* (see [**7**, Chapter III, §2, n^0 8, Definition 1]), if the induced homomorphism to its image is open. By [**8**, Chapter IX, §5, n^0 3, Corollary of Proposition 6], any morphism $\rho : A \to B$ of locally compact abelian groups which are countable at infinity is strict if and only if the induced homomorphism $A/\ker(\rho) \to \mathrm{Im}(\rho)$ is an isomorphism. Consider a complex $C_\bullet$ of locally compact abelian groups which are countable at infinity

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \cdots$$

such that the following conditions are satisfied:

- the homomorphisms d_n are continuous for every $n \in \mathbb{Z}$,
- the complex is bounded and of finite homology (i.e. $\ker(d_n)/\mathrm{Im}(d_{n+1})$ is a finite group for every $n \in \mathbb{Z}$.)

Using this conditions, Oesterlé establishes that the homomorphisms d_n are strict. Suppose that for every $n \in \mathbb{Z}$, we are given a Haar measure λ_n on C_n and that for almost all n the measure λ_n is normalized by $\lambda_n(C_n) = 1$. For every $n \in \mathbb{Z}$, let us choose Haar measures ν_n and θ_n on $\ker(d_n)$ and $\operatorname{Im}(d_n)$, respectively. Let α_n be the volume of the finite set $\ker(d_n)/\operatorname{Im}(d_{n+1})$ for the quotient measure ν_n/θ_{n+1}. Let $\beta_n > 0$ be the unique real number such that $\lambda_n/\nu_n = \beta_n\theta_n$ (after identifying $C_n/\ker(d_n)$ and $\operatorname{Im}(d_n)$ via the isomorphism induced from d_n). Oesterlé defines the number

$$\chi(C_\bullet) = \chi(C_\bullet, (\lambda_n)_n) := \prod_{n\in\mathbb{Z}} (\alpha_n\beta_n)^{(-1)^n},$$

which does not depend on the choice of ν_n and θ_n. We may say that $C_\bullet$ is of trivial measure Euler-Poincaré characteristic if $\chi(C_\bullet) = 1$.

***Lemma 5.1.4.1* (Oesterlé,** [43], Examples 2 and 3]**)**

The following claims are valid:

1. *Suppose that every C_n is compact. Then*
$$\chi(C_\bullet) = \prod_{n\in\mathbb{Z}} \lambda_n(C_n)^{(-1)^n}.$$
2. *Suppose that every C_n is discrete and that λ_n are counting measures. One has that*
$$\chi(C_\bullet) = \prod_{n\in\mathbb{Z}} [\ker(d_n) : \operatorname{Im}(d_{n+1})]^{(-1)^n}.$$

***Proposition 5.1.4.2* (Oesterlé,** [43], Proposition 1, n^0 4]**)**

Consider a bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C_{n,m-1} & \xrightarrow{d'_{n,m-1}} & C_{n-1,m-1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d''_{n,m}} & & \uparrow{\scriptstyle d''_{n-1,m}} & & \\
\cdots & \longrightarrow & C_{n,m} & \xrightarrow{d'_{n,m}} & C_{n-1,m} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

of locally compact abelian groups endowed with Haar measures, such that $C_{n,m}$ is the trivial group for $|n| + |m|$ big enough and such that for every

$n \in \mathbb{Z}$ and every $m \in \mathbb{Z}$, the complexes $C_{n,\bullet}$ and $C_{\bullet,m}$ satisfy the above conditions. One has that

$$\prod_{n\in\mathbb{Z}} \chi(C_{n,\bullet})^{(-1)^n} = \prod_{m\in\mathbb{Z}} \chi(C_{\bullet,m})^{(-1)^m}.$$

We end the paragraph by a lemma that will be used on several occasions.

Lemma 5.1.4.3. — *Let $\epsilon : H \to G$ be a proper continuous homomorphism of locally compact Hausdorff abelian topological groups. Let dg and dh be Haar measures on G and H. Consider the left action of H on G*

$$H \times G \to G \qquad h \cdot g = \epsilon(h)g.$$

1. *The action is continuous and proper. The quotient G/H identifies with the quotient group $G/\epsilon(H)$.*
2. *The group $K := \ker(\epsilon)$ is compact and let dk be the probability Haar measure on K. We have an equality of measures on $\epsilon(H)$:*

$$\epsilon_*(dh) = dh/dk.$$

3. *The subgroup $\epsilon(H)$ is closed in G. The measure dg/dh is the quotient Haar measure $dg/\epsilon_*(dh)$.*
4. *Suppose that H and G are countable at infinity. The exact sequence $1 \to K \to H \to G \to G/H \to 1$ is of trivial measure Euler-Poincaré characteristics, when the measures on K, H, G and G/H are dk, dh, dg and dg/dh.*

Proof. — 1. The continuity of the group action follows from the continuity of the multiplication by an element in G. The action is proper, as one has a Cartesian diagram

$$\begin{tikzcd} G \times H \arrow[r, "(g,h) \mapsto (g, g\epsilon(h))"] \arrow[d, "p_2"'] & G \times G \arrow[d] \\ H \arrow[r, "\epsilon"'] & G, \end{tikzcd}$$

where the right vertical homomorphism is given by $(g_1, g_2) \mapsto g_1^{-1}g_2$. The canonical homomorphism $G \to G/\epsilon(H)$ is continuous and open and induces a continuous and open map $G/H \to G/\epsilon(H)$. The map is bijective because one has an equality of sets $G/H = G/\epsilon(H)$.

We deduce a topological identification $G/H = G/\epsilon(H)$. The claim follows.

2. The map ϵ is proper and it follows that K is a compact subgroup of H. The quotient space $\epsilon(H) = H/\ker(\epsilon)$ is paracompact [**9**, Chapter III, §4, n^0 6, Proposition 13]. Note that the constant function $H \to \mathbb{C}$ given by $h \mapsto 1$, satisfies the condition that its support (i.e. the whole of H) has compact intersection with the preimage $\epsilon^{-1}(A)$ for every compact $A \subset \epsilon(H)$, because ϵ is proper. Now, by Proposition 5.1.2.5, for every $r : \epsilon(H) \to \mathbb{C}$ with $r \in L^1(\epsilon(H), dh/dk)$ we have
$$\int_{\epsilon(H)} (r)(dh/dk) = \int_H (r \circ \epsilon) \cdot 1 dh = \int_H (r \circ \epsilon) dh,$$
which means precisely that $\epsilon_*(dh) = dh/dk$.
3. The subgroup $\epsilon(H)$ is closed as ϵ is a proper map. Let $\phi : G/H \to \mathbb{C}$ be a compactly supported function. By [**10**, Chapter VII, §2, n^0 1, Proposition 2], there exists a compactly supported continuous function $\Phi : G \to \mathbb{C}$, such that for any $x \in G/H$ and any lift $\widetilde{x} \in G$ of x, we have
$$\phi(x) = \int_H \Phi(h\widetilde{x}) dh.$$
Note that
$$\phi(x) = \int_{\epsilon(H)} \Phi(h\widetilde{x})\epsilon_*(dh).$$
We have that
$$\int_{G/H} (\phi)(dg/dh) = \int_G \Phi \, dg = \int_{G/\epsilon(H)} (\phi)(dg/\epsilon_*(dh)).$$
It follows that $dg/dh = dg/\epsilon_*(dh)$ as claimed.
4. The short exact sequences
$$1 \longrightarrow K \longrightarrow H \longrightarrow \epsilon(H) \longrightarrow 1$$
$$1 \longrightarrow \epsilon(H) \longrightarrow G \longrightarrow G/H \longrightarrow 1$$
are of trivial measure Euler-Poincaré characteristics by (2) and (3). Thus $1 \to K \to H \to G \to G/H \to 1$ is of trivial measure Euler-Poincaré characteristics.

$\square$

5.2. Measures on $[\mathscr{P}(\mathbf{a})(F_v)]$

For $v \in M_F$, the abelian locally compact group $F_v^\times$ acts on $F_v^n - \{0\}$ by $t\cdot\mathbf{x} = (t^{a_j}x_j)_j$. This action is proper by Proposition 3.3.4.1. By Corollary 3.3.3.2, the quotient $(F_v^n - \{0\})/F_v^\times$ identifies with $[\mathscr{P}(\mathbf{a})(F_v)]$. The goal of this section is to define measures on $[\mathscr{P}(\mathbf{a})_{F_v}(F_v)] = [\mathscr{P}(\mathbf{a})(F_v)]$ for $v \in M_F$. The topological spaces $[\mathscr{P}(\mathbf{a})(F_v)]$ are Hausdorff and compact by Proposition 3.3.4.5. Let us denote by $q_v^{\mathbf{a}}$ the quotient maps $F_v^n - \{0\} \to [\mathscr{P}(\mathbf{a})(F_v)]$ for $v \in M_F$.

5.2.1. — In this paragraph, we define measure on F_v.

For $v \in M_F$, let dx_v be the Haar measure on F_v normalized by

- $dx_v(\mathcal{O}_v) = 1$ if v is finite,
- dx_v is the Lebesgue measure on $F_v \cong \mathbb{R}$ if v is real,
- $dx_v = 2dxdy$ on $F_v \cong \mathbb{C} \cong \mathbb{R}^2$ if v is complex.

For $v \in M_F$, let d^*x_v be the Haar measure $\frac{dx_v}{|x_v|_v}$ on $F_v^\times$. For $v \in M_F^0$, it satisfies that

$$d^*x_v(\mathcal{O}_v^\times) = \int_{\mathcal{O}_v^\times} 1d^*x_v = \int_{\mathcal{O}_v^\times} 1dx_v = 1 - |\pi_v|_v.$$

When v is real, the measure d^*x_v identifies with the measure

$$d^*x := \frac{dx}{|x|}$$

on $\mathbb{R}^\times$ and when v is complex, the measure d^*x_v identifies with the measure

$$\frac{2dxdy}{x^2+y^2}$$

on $\mathbb{C}^2 - \{0\} \cong \mathbb{R}^2 - \{0\}$. Let us set

$$F_{v,1} := \{x \in F_v |\ |x|_v = 1\},$$

so that $F_{v,1} = \{\pm 1\}$ when v is real and

$$F_{v,1} = S^1 = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 = 1\}$$

when v is complex. Recall that we have set $n_v = 1$ if v is a real place and let $n_v = 2$ if v is complex. The exact sequence

$$\{1\} \to F_{v,1} \xrightarrow{i_{F_{v,1}}} F_v^\times \xrightarrow{x \mapsto |x|_v} \mathbb{R}_{>0} \to \{1\},$$

where $i_{F_{v,1}}$ is the inclusion map, admits a section

$$\rho_v : \mathbb{R}_{>0} \to F_v^\times \qquad r \mapsto r^{1/n_v}.$$

The section induces a continuous isomorphism

$$\widetilde{\rho}_v : \mathbb{R}_{>0} \times F_{v,1} \to F_v^\times \qquad (r, z) \mapsto \rho_v(r)z. \tag{5.2.1.1}$$

The inverse of this homomorphism is given by

$$F_v^\times \to \mathbb{R}_{>0} \times F_{v,1} \qquad x \mapsto (|x|_v, x\rho_v(|x|_v)^{-1}) \tag{5.2.1.2}$$

and is also continuous. We deduce that $\widetilde{\rho}_v$ is an isomorphism of topological groups. Let us set $\lambda_{v,1}$ to be the counting measure on $\{\pm 1\}$ when v is real and let us set $\lambda_{v,1}$ to be the Haar measure on $F_{v,1}$ normalized by $\lambda_{v,1}(F_{v,1}) = 2\pi$ when v is complex.

Lemma 5.2.1.3. — *Let $v \in M_F^\infty$. One has that and that $(\widetilde{\rho}_v)_*\big(dr \times \lambda_{v,1}\big) = dx_v|_{F_v^\times}$ and that $(\widetilde{\rho}_v)_*\big(d^*r \times \lambda_{v,1}\big) = d^*x_v$.*

Proof. — We will firstly verify that $(\widetilde{\rho}_v)_*\big(d^*r \times \lambda_{v,1}\big) = d^*x_v$. Suppose v is real. Both measures are Haar measures on $F_v^\times$, so it suffices to check their equality on a single Borel subset of $F_v^\times \cong \mathbb{R}^\times$, and we will verify it on $[1, 2]$. We have that $[1, 2] = \widetilde{\rho}_v(\{1\} \times [1, 2])$ and

$$\begin{aligned} d^*x_v([1, 2]) = d^*x([1, 2]) &= 1 \cdot d^*x([1, 2]) \\ &= \lambda_{v,1}(\{1\}) \cdot d^*r([1, 2]) \\ &= \big(\lambda_{v,1} \times d^*r\big)(\{1\} \times [1, 2]) \\ &= \big((\widetilde{\rho}_v)_*\big(d^*r \times \lambda_{v,1}\big)\big)(\widetilde{\rho}_v(\{1\} \times [1, 2])). \end{aligned}$$

Suppose v is complex. Note that $\lambda_{v,1}$ is the pushforward measure for the map $[0, 2\pi[\to S^1$ given by

$$\theta \mapsto e^{i\theta} = (\cos(\theta), \sin(\theta)).$$

Hence, the map

$$S^1 \times \mathbb{R}_{>0} \to \mathbb{R}^2 - \{0\} \qquad (z, r) \mapsto zr^{1/2}$$

is measure preserving if and only if the map

$$[0, 2\pi[\times \mathbb{R}_{>0} \to \mathbb{R}^2 - \{0\} \qquad (\theta, r) \mapsto (r^{1/2}\cos(\theta), r^{1/2}\sin(\theta))$$

is measure preserving. The corresponding Jacobian matrix is

$$\begin{pmatrix} -r^{1/2}\sin(\theta) & \frac{1}{2}r^{-1/2}\cos(\theta) \\ r^{1/2}\cos(\theta) & \frac{1}{2}r^{-1/2}\sin(\theta) \end{pmatrix}$$

and its determinant equals $-\frac{1}{2}$. It follows that

$$dxdy = (\widetilde{\rho}_v)_*(|-1/2|drd\theta) = (\widetilde{\rho}_v)_*((1/2)drd\theta),$$

and hence that

$$d^*x_v = \frac{2dxdy}{x^2+y^2} = (\widetilde{\rho}_v)_*\Big(\frac{2\cdot(1/2)drd\theta}{r}\Big) = (\widetilde{\rho}_v)_*(d^*rd\theta).$$

We have proven that $(\widetilde{\rho}_v)_*(d^*r\times\lambda_{v,1}) = d^*x_v$.

Let us now verify that $(\widetilde{\rho}_v)_*(dr\times\lambda_{v,1}) = dx_v|_{F_v^\times}$. One has that $|\widetilde{\rho}_v(r,z)|_v = r$ for every $(r,z)\in\mathbb{R}_{>0}\times F_{v,1}$. For a Borel subset $U\subset F_v^\times$, one has that

$$\begin{aligned} dx_v|_{F_v^\times}(U) = \int_U |x_v|_v d^*x_v &= \int_U |\widetilde{\rho}_v(r,z)|_v(\rho_v)_*(d^*r\times\lambda_{v,1}) \\ &= \int_{\widetilde{\rho_v}^{-1}(U)} r\,(d^*r\times\lambda_{v,1}) \\ &= \int_{\widetilde{\rho_v}^{-1}(U)} dr\times\lambda_{v,1} \\ &= (dr\times\lambda_{v,1})(U). \end{aligned}$$

It follows that $dx_v|_{F_v^\times} = (\widetilde{\rho}_v)_*(dr\times\lambda_{v,1})$. The statement is proven. $\square$

We will often write dx and d^*x for dx_v and d^*x_v, respectively.

5.2.2. — In this paragraph we will define compactly supported continuous functions $k_v^{\mathbf{a}}: F_v^n - \{0\}\to\mathbb{R}_{\geq 0}$ which satisfy that their integrals in every orbit for the weighted action of $\mathbb{G}_m(F_v)$ on $F_v^n-\{0\}$ are equal to 1. These functions will enable us to use Proposition 5.1.2.5.

For every $v\in M_F^0$, recall that

$$\mathcal{D}_v^{\mathbf{a}} = \{\mathbf{y}\in(\mathcal{O}_v)^n | \exists j : v(y_j) < a_j\} = (\mathcal{O}_v)^n - \pi_v^{a_1}\mathcal{O}_v\times\cdots\times\pi_v^{a_n}\mathcal{O}_v$$

and set

$$k_v^{\mathbf{a}} := \frac{\mathbf{1}_{\mathcal{D}_v^{\mathbf{a}}, F_v^n-\{0\}}}{1-|\pi_v|_v}.$$

Lemma 5.2.2.1. — *Let $v\in M_F^0$ and let $\mathbf{x}\in F_v^n-\{0\}$. The function $k_v^{\mathbf{a}}$ is compactly supported, locally constant and one has that*

$$\int_{F_v^\times} k_v^{\mathbf{a}}(t\cdot\mathbf{x})d^*t = 1.$$

Proof. — We have seen in Lemma 3.3.4.4 that the subset $\mathcal{D}_v^{\mathbf{a}}$ is open, closed and compact subset of $F_v^n - \{0\}$. We conclude that $\mathbf{1}_{\mathcal{D}_v^{\mathbf{a}}}$ is locally constant and compactly supported, hence is such $k_v^{\mathbf{a}}$. In Lemma 4.4.3.1 we have defined $r_v : F_v^n - \{0\} \to \mathbb{Z}$ by

$$r_v(\mathbf{y}) = \sup_{\substack{j=1,\dots,n \\ y_j \neq 0}} \left\lceil -\frac{v(y_j)}{a_j} \right\rceil.$$

One has that

$$t \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}} \iff \mathbf{x} \in t^{-1} \cdot \mathcal{D}_v^{\mathbf{a}} \iff \mathbf{x} \in t^{-1}\pi_v^{r_v(\mathbf{x})} \cdot (\pi_v^{-r_v(\mathbf{x})} \cdot \mathcal{D}_v^{\mathbf{a}}).$$

Lemma 4.4.3.1 gives that $\mathbf{x} \in \pi_v^{-r_v(\mathbf{x})} \cdot \mathcal{D}_v^{\mathbf{a}}$ and that if $(u \cdot (\pi_v^{-r_v(\mathbf{x})} \cdot \mathcal{D}_v^{\mathbf{a}})) \cap (\pi_v^{-r_v(\mathbf{x})} \cdot \mathcal{D}_v^{\mathbf{a}}) \neq \emptyset$ then $v(u) = 0$. We conclude $v(t^{-1}\pi_v^{r_v(\mathbf{x})}) = 0$ and hence

$$\{t | t \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}\} = \{t | v(t) = r_v(\mathbf{x})\}.$$

We deduce

$$d^*t\{t \in F_v^\times | t \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}\} = d^*t(\pi_v^{r_v(\mathbf{x})}\mathcal{O}_v^\times) = 1 - |\pi_v|_v.$$

One has further that

$$\int_{F_v^\times} k_v^{\mathbf{a}}(t \cdot \mathbf{x}) d^*t = \int_{F_v^\times} \frac{\mathbf{1}_{\mathcal{D}_v^{\mathbf{a}}}(t \cdot \mathbf{x})}{1 - |\pi_v|_v} d^*t = \frac{d^*t(\{t \in F_v^\times | t \cdot \mathbf{x} \in \mathcal{D}_v^{\mathbf{a}}\})}{1 - |\pi_v|_v} = 1.$$

The claim is proven. □

The following auxiliary lemma will be used in the definition of $k_v^{\mathbf{a}}$ for v infinite.

Lemma 5.2.2.2. — *Let a_{G_1} and a_{G_2} be continuous actions of topological groups G_1 and G_2 on a topological space X. Suppose the actions are permutable, that is for every $g_1 \in G_1$, every $g_2 \in G_2$ and every $x \in X$ one has $g_1 g_2 x = g_2 g_1 x$.*

1. *The map*

$$a_{G_1 \times G_2} : G_1 \times G_2 \times X \to X$$
$$(g_1, g_2, x) \mapsto g_1(g_2 x)$$

defines a continuous action of $G_1 \times G_2$ on X.

2. *There exists a continuous action of G_2 on X/G_1 such that*

$$g_2 \cdot [x]_{G_1} = [g_2 x]_{G_1}$$

for every $x \in X$, where $[x]_{G_1}$ is the image of x in X/G_1 for the quotient map.

3. *The canonical map $X/G_1 \to X/(G_1 \times G_2)$ is open, continuous, surjective and G_2-invariant. The induced map $(X/G_1)/G_2 \to X/(G_1 \times G_2)$ is a homeomorphism.*

Proof. — 1. The map $a_{G_1\times G_2}$ factorizes as $a_{G_2} \circ (\mathrm{Id}_{G_1} \times a_{G_1})$ thus is continuous. If e_{G_1}, e_{G_2} are neutral elements of G_1 and G_2, respectively, by definition one has $(e_{G_1}, e_{G_2}) \cdot x = e_{G_1}e_{G_2}x = x$. Moreover, if (g_1, g_2) and (g_1', g_2') are elements of $G_1 \times G_2$, then for $x \in X$ one has

$$(g_1g_1', g_2g_2')x = (g_1g_1')(g_2g_2'x) = g_1((g_1'(g_2g_2'x))) = g_1(g_2(g_1'g_2'x)) \\ = (g_1, g_2)(g_1', g_2')x.$$

We deduce that $a_{G_1\times G_2}$ is a continuous action of $G_1 \times G_2$ on X.

2. This is proven in [7, Chapter III, §4, n^0 4, Remark to Proposition 11].
3. The map $X/G_1 \to X/(G_1 \times G_2)$ is the induced map from G_1-invariant continuous, open and surjective map $X \to X/(G_1 \times G_2)$, thus itself is continuous, open and surjective. If $g_2 \in G_2$ and $x \in X$, one has that

$$[g_2 \cdot [x]_{G_1}]_{G_1\times G_2} = [[g_2x]_{G_1}]_{G_1\times G_2} = [g_2x]_{G_1\times G_2} = [x]_{G_1\times G_2},$$

where $[\cdot]_{G_1\times G_2}$ is the image in $X/(G_1 \times G_2)$. Hence, $X/G_1 \to X/(G_1 \times G_2)$ is continuous. Suppose now elements $[x]_{G_1}$ and $[y]_{G_1}$ have the same image in $X/(G_1 \times G_2)$. This means precisely that there exists $(g_1, g_2) \in G_1 \times G_2$ such that $(g_1, g_2) \cdot x = g_1g_2x = y$. We deduce that

$$[y]_{G_1} = [g_2g_1x]_{G_1} = [g_2x]_{G_1} = g_2[x]_{G_1},$$

i.e. $[x]_{G_1}$ and $[y]_{G_2}$ are in the same orbit of G_2. We deduce that the induced map $((X/G_1)/G_2) \to (X/(G_1 \times G_2))$ is bijective. It follows that it is a homeomorphism.

□

Lemma 5.2.2.3. — *Let $v \in M_F^\infty$. There exists a continuous compactly supported function $k_v^{\mathbf{a}} : F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$, which is $F_{v,1}$-invariant and such that for every $\mathbf{x} \in F_v^n - \{0\}$ one has that*

$$\int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(\rho_v(t) \cdot \mathbf{x})d^*t = \frac{1}{\lambda_{v,1}(F_{v,1})},$$

where $\rho_v : \mathbb{R}_{>0} \to F_v^\times$ *is given by* $\rho_v : r \mapsto r^{1/n_v}$. *Such function satisfies furthermore for every* $\mathbf{x} \in F_v^n - \{0\}$ *that*

$$\int_{F_v^\times} k_v^{\mathbf{a}}(t \cdot \mathbf{x}) d^*t = 1.$$

Proof. — The isomorphism (5.2.1.1)

$$\widetilde{\rho}_v : \mathbb{R}_{>0} \times F_{v,1} \xrightarrow{\sim} F_v^\times \qquad (r, x) \mapsto \rho_v(r)x$$

satisfies $(\widetilde{\rho}_v)_*(d^*r \times \lambda_{v,1}) = d^*x_v$ by Lemma 5.2.1.3. It induces a topological action of $\mathbb{R}_{>0} \times F_{v,1}$ on $F_v^n - \{0\}$. Moreover, as $F_{v,1} = \{1\} \times F_{v,1}$ is a closed subgroup of $\mathbb{R}_{>0} \times F_{v,1}$, by [7, Chapter III, §4, n^0 1, Example 1] the restriction of this action to $F_{v,1}$ is continuous and proper. Let $q_{v,1} : F_v^n - \{0\} \to (F_v^n - \{0\})/F_{v,1}$ be the corresponding quotient map, by [7, Chapter III, §4, n^0 1, Proposition 2], the map $q_{v,1}$ is proper. Let $\mathbb{R}_{>0}$ acts on $F_v^n - \{0\}$ via the identification $\mathbb{R}_{>0} = \mathbb{R}_{>0} \times \{1\}$. Lemma 5.2.2.2 provides an action of $\mathbb{R}_{>0}$ on $(F_v^n - \{0\})/F_{v,1}$ which satisfies

$$r \cdot q_{v,1}(\mathbf{x}) = q_{v,1}(r \cdot \mathbf{x}) = q_{v,1}(\rho_v(r) \cdot \mathbf{x}),$$

for $r \in \mathbb{R}_{>0}$ and $\mathbf{x} \in F_v^n - \{0\}$. Moreover, the canonical map $(F_v^n - \{0\})/F_{v,1} \to [\mathscr{P}(\mathbf{a})(F_v)]$ induces an identification $((F_v^n - \{0\})/F_{v,1})/\mathbb{R}_{>0} \xrightarrow{\sim} [\mathscr{P}(\mathbf{a})(F_v)]$. As $[\mathscr{P}(\mathbf{a})(F_v)]$ is compact, hence paracompact, Proposition 5.1.2.3 gives that there exists a continuous compactly supported function $k' : (F_v^n - \{0\})/F_{v,1} \to \mathbb{R}_{\geq 0}$ such that for every $\mathbf{y} \in (F_v^n - \{0\})/F_{v,1}$, we have that

$$\int_{\mathbb{R}_{>0}} k'(r \cdot \mathbf{y}) d^*r = \frac{1}{\lambda_{v,1}(F_{v,1})}.$$

Let us set $k_v^{\mathbf{a}} = k' \circ q_{v,1}$. As $q_{v,1}$ is proper and $F_{v,1}$-invariant, the map $k_v^{\mathbf{a}}$ is compactly supported and $F_{v,1}$-invariant. Let $\mathbf{x} \in F_v^n - \{0\}$, we have that

$$\begin{aligned}
\int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(\rho_v(r) \cdot \mathbf{x}) d^*r &= \int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(r \cdot \mathbf{x}) d^*r \\
&= \int_{\mathbb{R}_{>0}} k'(q_{v,1}(r \cdot \mathbf{x})) d^*r \\
&= \int_{\mathbb{R}_{>0}} k'(r \cdot q_{v,1}(\mathbf{x})) d^*r \\
&= \frac{1}{\lambda_{v,1}(F_{v,1})},
\end{aligned}$$

and that

$$\begin{aligned}\int_{F_v^\times} k_v^{\mathbf{a}}(t\cdot\mathbf{x})d^*t &= \int_{\mathbb{R}_{>0}\times F_{v,1}} k_v^{\mathbf{a}}((r,z)\cdot\mathbf{x})d^*r d\lambda_{v,1}(z)\\ &= \int_{F_{v,1}} d\lambda_{v,1}(z)\int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(r\cdot(z_j^{a_j}x_j)_j)d^*r\\ &= \int_{F_{v,1}} \frac{1}{\lambda_{v,1}(F_{v,1})}d\lambda_{v,1}\\ &= 1.\end{aligned}$$

The statement is proven. □

5.2.3. — The goal of this paragraph is to give several equivalent conditions that make $f^{-1}dx_1\dots dx_n$ a measure on $F_v^n-\{0\}$, where f is an $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$.

Let $v\in M_F$. We will use the following conventions $a\cdot\infty=\infty$ for $a\in\mathbb{R}_{>0}$ and $\infty^{-1}=0$. If $t\in\mathbb{G}_m(F_v)$, we will see it as a function $t:F_v^n-\{0\}\to F_v^n-\{0\}$ given by its action on $F_v^n-\{0\}$. We say that a function $f:F_v^n-\{0\}\to\mathbb{C}\cup\{\infty\}$ is $\mathbf{a}$-homogenous of weighted degree $|\mathbf{a}|$ if $f(t\cdot\mathbf{x})=|t|_v^{|\mathbf{a}|}f(\mathbf{x})$ for every $t\in F_v^\times$ and every $\mathbf{x}\in F_v^n-\{0\}$.

Lemma 5.2.3.1. — *Let $f:F_v^n-\{0\}\to\mathbb{C}\cup\{\infty\}$ be an $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ such that*

$$dx_1\dots dx_n(\{\mathbf{x}\in F_v^n-\{0\}|f(\mathbf{x})=0\})=0.$$

Let $\phi\in\mathscr{C}_c^0(F_v^n-\{0\},\mathbb{C})$. One has that $\phi f^{-1}\in L^1(F_v^n-\{0\},dx_1\dots dx_n)$ if and only if for every $t\in F_v^\times$ one has that $(\phi\circ t^{-1})f^{-1}\in L^1(F_v^n-\{0\},dx_1\dots dx_n)$ and if $\phi f^{-1}\in L^1(F_v^n-\{0\},dx_1\dots dx_n)$ then

$$\int_{F_v^n-\{0\}}\phi f^{-1}dx_1\dots dx_n=\int_{F_v^n-\{0\}}(\phi\circ t^{-1})f^{-1}dx_1\dots dx_n$$

for every $t\in F_v^\times$.

Proof. — Let $t\in\mathbb{G}_m(F_v)$. The action of t on $F_v^n-\{0\}$ is given by the multiplication to the left by the diagonal matrix D_t which has the diagonal vector $(t^{a_1},\dots,t^{a_n})$. By using the formula for the change of the coordinates, we get that

$$\begin{aligned}
\int_{F_v^n-\{0\}} (\phi\circ t^{-1})f^{-1}dx_1\dots dx_n \\
&= \int_{F_v^n-\{0\}} |\det(\mathrm{Jac}(D_t))|_v(\phi\circ t^{-1})(t\cdot\mathbf{x})f_v(t\cdot\mathbf{x})^{-1}dx_1\dots dx_n \\
&= \int_{F_v^n-\{0\}} |t^{|\mathbf{a}|}|_v\phi(\mathbf{x})\cdot|t|_v^{-|\mathbf{a}|}f(\mathbf{x})^{-1}dx_1\dots dx_n \\
&= \int_{F_v^n-\{0\}} \phi f^{-1}dx_1\dots dx_n,
\end{aligned}$$

if one, and hence every, integral converges absolutely. The statement follows. □

The following lemma will be needed to treat the case $v\in M_F^\infty$.

Lemma 5.2.3.2. — *Suppose $v\in M_F^\infty$. Let $\rho_v:\mathbb{R}_{>0}\to F_v^\times$ be the map $r\mapsto r^{1/n_v}$.*

1. *Consider the map $A:(F_v^\times)^n\to(F_v^\times)^n$ given by*

$$A:\mathbf{x}\mapsto((\rho_v(|x_n|_v^{a_j/a_n})x_j)_{j=1}^{n-1},x_n).$$

One has that

$$|x_n|^{\frac{|\mathbf{a}|}{a_n}-1}A_*(dx_1\dots dx_n)=dx_1\dots dx_n.$$

2. *Let $f:F_v^n-\{0\}\to\mathbb{C}\cup\{\infty\}$ be an $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ such that*

$$dx_1\dots dx_n(\{\mathbf{x}\in F_v^n-\{0\}|f(\mathbf{x})=0\})=0.$$

Let $\phi:F_v^n-\{0\}\to\mathbb{C}$ be a compactly supported continuous function. One has that $\phi f^{-1}\in L^1(F_v^n-\{0\},dx_1\dots dx_n)$ if and only if $f^{-1}|_{(F_v^\times)^{n-1}\times F_{v,1}}\in L^1((F_v^\times)^{n-1}\times F_{v,1},dx_1\dots dx_n\times\lambda_{v,1})$ and if $\phi f^{-1}\in L(F_v^n-\{0\},dx_1\dots dx_n)$ then

(5.2.3.3)
$$\begin{aligned}
\int_{F_v^n-\{0\}}\phi f^{-1}dx_1\dots dx_n = a_n\int_{F_{v,1}}d\lambda_{v,1}(z)\int_{\mathbb{R}_{>0}}\phi\big(\rho_v(t)\cdot((x_j)_{j=1}^{n-1},z)\big)d^*t\times \\
\times\int_{(F_v^\times)^{n-1}}f(x_1,\dots,x_{n-1},z)^{-1}dx_1\dots dx_{n-1}.
\end{aligned}$$

Proof. — 1. Suppose that v is real. The function defined by A is smooth. The Jacobian of A is given by the diagonal matrix having for the diagonal vector $((\rho_v(|x_n|_v^{a_j/a_n}))_{j=1}^{n-1}, 1)$. The v-adic absolute value of the determinant of the Jacobian is equal to

$$\Big|\prod_{j=1}^{n-1}\rho_v(|x_n|_v^{a_j/a_n})\Big|_v=\prod_{j=1}^{n-1}\big|\rho_v(|x_n|_v^{a_j/a_n})\big|_v=\prod_{j=1}^{n-1}|x_n|_v^{a_j/a_n}=|x_n|_v^{\frac{|\mathbf{a}|}{a_n}-1}.$$

Thus, by the formula for the change of variables we get that

$$|x_n|_v^{\frac{|\mathbf{a}|}{a_n}-1}A_*(dx_1\dots dx_n)=dx_1\dots dx_n.$$

Suppose now that v is complex. We use identification $F_v=\mathbb{R}^2$ as in paragraph 5.2.1 and let y and r be coordinates of $\mathbb{R}^2$. The measure dx_v becomes $2dydr$. The function $A:(\mathbb{R}^2-\{0\})^n\to(\mathbb{R}^2-\{0\})^n$ in the new coordinates is given by

$$A(\mathbf{y},\mathbf{r})=\big((y_j\cdot(r_n^2+y_n^2)^{a_j/(2a_n)})_{j=1}^{n-1},y_n,(r_j\cdot(r_n^2+y_n^2)^{a_j/(2a_n)})_{j=1}^{n-1},r_n\big)$$

and is smooth (because the locus $r_n=y_n=0$ is outside of the domain of the definition). Its Jacobian is an upper triangular matrix having for the diagonal vector

$$\big(((r_n^2+y_n^2)^{a_j/(2a_n)})_{j=1}^{n-1},1,((r_n^2+y_n^2)^{a_j/(2a_n)})_{j=1}^{n-1},1\big).$$

Its determinant is

$$\begin{aligned}\prod_{j=1}^{n-1}((r_n^2+y_n^2)^{a_j/(2a_n)})\cdot\prod_{j=1}^{n-1}((r_n^2+y_n^2)^{a_j/(2a_n)})&=\prod_{j=1}^{n-1}(r_n^2+y_n^2)^{a_j/a_n}\\&=(r_n^2+y_n^2)^{(|\mathbf{a}|-a_n)/a_n}.\end{aligned}$$

By the formula for the change of variables, we get that

$$(r_n^2+y_n^2)^{(|\mathbf{a}|-a_n)/a_n}A_*(dy_1\cdots dy_ndr_1\cdots dr_n)=dy_1\cdots dy_ndr_1\cdots dr_n.$$

Multiplying both hand sides by 2^n and using that $r_n^2+y_n^2=|x_n|_v$ gives that

$$|x_n|_v^{\frac{|\mathbf{a}|}{a_n}-1}A_*(dx_1\dots dx_n)=dx_1\dots dx_n,$$

as claimed

2. Let us define $B,C:(F_v^\times)^n\to(F_v^\times)^n$ by

$$\begin{aligned}B&:\mathbf{x}\mapsto(x_j\rho_v(|x_n|_v^{-a_j/a_n}))_j,\\C&:\mathbf{x}\mapsto((x_j)_{j=1}^{n-1},x_n\rho_v(|x_n|_v)^{-1}).\end{aligned}$$

Note that for $\mathbf{x}\in(F_v^\times)^n$, one has that

$$B(A(\mathbf{x}))=B((x_j\rho_v(|x_n|_v^{a_j/a_n}))_{j=1}^{n-1},x_n)=((x_j)_{j=1}^{n-1},x_n\rho_v(|x_n|^{-1}))=C(\mathbf{x})$$

and that

$$A(\mathbf{x})=((\rho_v(|x_n|_v^{\frac{a_j}{a_n}})x_j)_{j=1}^{n-1},x_n)=\rho_v(|x_n|_v^{\frac{1}{a_j}})\cdot((x_j)_{j=1}^{n-1},x_n\rho_v(|x_n|_v)^{-1})$$
$$=\rho_v(|x_n|_v^{\frac{1}{a_n}})\cdot C(\mathbf{x}).$$

For every $\mathbf{x}\in F_v^n-\{0\}$, we have that

$$f_v(\mathbf{x})=|x_n|_v^{|\mathbf{a}|/a_n}f_v\big(\big(x_j\rho_v(|x_n|_v^{-a_j/a_n})\big)_j\big)=f_v(B(\mathbf{x})),$$

by the fact that f_v is $\mathbf{a}$-homogenous of weighted degree $|\mathbf{a}|$. Let $\phi\in\mathscr{C}^0_c(F_v^n-\{0\},\mathbb{C})$. By using that $dx_1\dots dx_n(F_v^n-\{0\}-(F_v^\times)^n)=0$, because $F_v^n-\{0\}-(F_v^\times)^n$ is contained in a finite union of hyperplanes of $(F_v)^n$, and the part (1), we get that:

$$\begin{aligned}
&\int_{F_v^n-\{0\}}\phi f_v^{-1}dx_1\dots dx_n\\
&\qquad=\int_{(F_v^\times)^n}\phi f_v^{-1}dx_1\dots dx_n\\
&\qquad=\int_{(F_v^\times)^n}|x_n|_v^{-|\mathbf{a}|/a_n}\phi\cdot(f_v^{-1}\circ B)dx_1\dots dx_n\\
&\qquad=\int_{(F_v^\times)^n}|x_n|_v^{-1}\phi\cdot(f_v^{-1}\circ B)A_*(dx_1\dots dx_n)\\
&\qquad=\int_{(F_v^\times)^n}|x_n\circ A|_v^{-1}(\phi\circ A)(f_v^{-1}\circ(B\circ A))dx_1\dots dx_n\\
&\qquad=\int_{(F_v^\times)^n}|x_n|_v^{-1}(\phi\circ A)f_v((x_j)_{j=1}^{n-1},x_n\rho_v(|x_n|_v^{-1}))^{-1}dx_1\dots dx_n\\
&\qquad=\int_{(F_v^\times)^n}(\phi\circ A)f_v((x_j)_{j=1}^{n-1},x_n\rho_v(|x_n|_v^{-1}))^{-1}dx_1\dots dx_{n-1}d^*x_n\\
&\qquad=\int_{(F_v^\times)^n}\phi(\rho_v(|x_n|_v^{1/a_n})\cdot C(\mathbf{x}))f_v(C(\mathbf{x}))^{-1}dx_1\dots dx_{n-1}d^*x_n.
\end{aligned}$$

if one (and hence every) integral converges absolutely. It follows from Lemma 5.2.1.3 that the map

$$\widetilde{\rho}_v^{-1}:F_v^\times\to\mathbb{R}_{>0}\times F_{v,1}\qquad x\mapsto(|x|_v,x\rho_v(|x|_v)^{-1})$$

satisfies that $(\widetilde{\rho}_v^{-1})_* d^*x = d^*r \times \lambda_{v,1}$. The last integral is hence equal to

$$\int_{F_{v,1}\times\mathbb{R}_{>0}}\int_{(F_v^\times)^{n-1}} \phi(\rho_v(r^{1/a_j})\cdot((x_j)_{j=1}^{n-1},z))\times \\ \times f_v((x_j)_{j=1}^{n-1},z)^{-1}dx_1..dx_{n-1}d^*rd\lambda_{v,1}(z).$$

Note that setting $u^{a_n} = r$ gives $d^*r = a_n d^*u$ and thus the last integral becomes

(5.2.3.4)
$$a_n\int_{F_{v,1}\times\mathbb{R}_{>0}}\int_{(F_v^\times)^{n-1}} \frac{\phi(\rho_v(u)\cdot((x_j)_{j=1}^{n-1},z))}{f_v((x_j)_{j=1}^{n-1},z)}dx_1\dots dx_{n-1}d^*ud\lambda_{v,1}(z).$$

By Fubini theorem, we get that the integral (5.2.3.4) is equal to

$$(5.2.3.5)\quad a_n\int_{F_{v,1}} d\lambda_{v,1}(z)\int_{\mathbb{R}_{>0}}\phi(\rho_v(u\cdot((x_j)_{j=1}^{n-1},z)))d^*u\times \\ \times\int_{(F_v^\times)^{n-1}} f_v((x_j)_{j=1}^{n-1},z)^{-1}dx_1\dots dx_{n-1}.$$

For every $((x_j)_j,z)\in(F_v^\times)^{n-1}\times F_{v,1}$, the map $\mathbb{R}_{>0}\to\mathbb{C}$ given by $u\mapsto\phi(\rho_v(u)\cdot((x_j)_j,z))$ is compactly supported, because ϕ is compactly supported, the map $y\mapsto y\cdot((x_j)_j,z)$ is proper (the action of $F_v^\times$ on $(F_v^\times)^n$ is proper by Proposition 3.3.4.1, thus by [7, Chapter III, §4, n^0 2, Proposition 4], the map $y\mapsto y\cdot((x_j)_j,z)$ is proper) and $\rho_v : r\mapsto r^{1/n_v}$ is proper. It follows that

$$\int_{\mathbb{R}_{>0}}\phi(\rho_v(u)\cdot((x_j)_j,z))d^*u$$

converges absolutely. It follows that the integral (5.2.3.5) converges absolutely if and only if $\int_{F_{v,1}} d\lambda_{v,1}(z)\int_{(F_v^\times)^n} f_v((x_j)_{j=1}^{n-1},z)^{-1}dx_1\dots dx_{n-1}$ converges absolutely. The statement follows.

□

In the following proposition, we give equivalent conditions to the condition that $f^{-1}dx_1\dots dx_n$ is a measure on $F_v^n-\{0\}$.

Proposition 5.2.3.6. — *Let $f : F_v^n-\{0\}\to\mathbb{C}\cup\{\infty\}$ be an* **a**-*homogenous function of weighted degree $|\mathbf{a}|$ such that*

$$dx_1\dots dx_n(\{\mathbf{x}\in F_v^n-\{0\}|f(\mathbf{x})=0\})=0.$$

The following are equivalent:

1. *For every* $\phi \in \mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C})$ *one has that* $\phi f^{-1} \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$ *and that*
$$\mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C}) \to \mathbb{C} \qquad \phi \mapsto \int_{F_v^n - \{0\}} \phi \cdot f^{-1} dx_1 \dots dx_n$$
is a measure on $F_v^n - \{0\}$.
2. *For every compactly supported function* $\phi : F_v^n - \{0\} \to \mathbb{C}$ *one has that* $\phi f^{-1} \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$ *and that there exists a unique measure* ω_v *on* $[\mathscr{P}(\mathbf{a})(F_v)]$ *such that*
$$\int_{F_v^n - \{0\}} \phi f^{-1} dx_1 \dots dx_n = \int_{[\mathscr{P}(\mathbf{a})(F_v)]} d\omega_v(\mathbf{y}) \int_{F_v^\times} \phi(t \cdot \widetilde{\mathbf{y}}) d^*t, \tag{5.2.3.7}$$
where $\widetilde{\mathbf{y}}$ *is a lift on an element* $\mathbf{y} \in [\mathscr{P}(\mathbf{a})(F_v)]$.
3. *One has that* $k_v^{\mathbf{a}} \cdot f^{-1} \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$, *where the function* $k_v^{\mathbf{a}}$ *is defined in* 5.2.2.
4. *If* $v \in M_F^0$, *one has that* $f^{-1}|_{\mathcal{D}_v^{\mathbf{a}}} \in L^1(\mathcal{D}_v^{\mathbf{a}}, dx_1 \dots dx_n)$. *If* $v \in M_F^\infty$, *one has that* $f^{-1}|_{(F_v^\times)^{n-1} \times F_{v,1}} \in L^1((F_v^\times)^{n-1} \times F_{v,1}, dx_1 \dots dx_{n-1} \times \lambda_{v,1})$.

If any of the conditions is satisfied, one has that
$$\omega_v([\mathscr{P}(\mathbf{a})(F_v)]) = \int_{F_v^n - \{0\}} k_v^{\mathbf{a}} f^{-1} dx_1 \dots dx_n \tag{5.2.3.8}$$
Moreover, if $v \in M_F^0$, *both quantities in the equality* (5.2.3.8) *are equal to*
$$\frac{1}{1 - |\pi_v|_v} \int_{\mathcal{D}_v^{\mathbf{a}}} f^{-1} dx_1 \dots dx_n$$
and are equal to
$$\frac{a_n}{\lambda_{v,1}(F_{v,1})} \int_{(F_v^\times)^{n-1} \times F_{v,1}} f^{-1} dx_1 \dots dx_{n-1} \times \lambda_{v,1}$$
if $v \in M_F^\infty$.

Proof. — Note that the implication (1) $\implies$ (2) follows from Proposition 5.1.2.2. Facts that $[\mathscr{P}(\mathbf{a})(F_v)]$ is compact, hence paracompact, and that for every $\mathbf{x} \in F_v^n - \{0\}$ one has that the equality $\int_{F_v^\times} k_v^{\mathbf{a}}(t \cdot \mathbf{x}) dt^* = 1$ is valid (Lemma 5.2.2.1 and Lemma 5.2.2.3), enable us to apply Proposition 5.1.2.5 and we get that (2) $\implies$ (3) and that if the condition (2) is satisfied, then
$$\omega_v^{\mathbf{a}}([\mathscr{P}(\mathbf{a})(F_v)]) = \int_{F_v^n - \{0\}} k_v^{\mathbf{a}} f^{-1} dx_1 \dots dx_n.$$

(3) $\implies$ (4). Suppose v is a finite place. As $k_v^{\mathbf{a}} f^{-1} = \frac{\mathbf{1}_{\mathcal{D}_v^{\mathbf{a}}}\cdot f^{-1}}{1-|\pi_v|_v}$, we deduce that the restriction of $\mathbf{x}\mapsto f(\mathbf{x})^{-1}$ to $\mathcal{D}_v^{\mathbf{a}}$ is $dx_1\dots dx_n$-absolutely integrable and that

$$\int_{F_v^n-\{0\}} k_v^{\mathbf{a}} f^{-1} dx_1\dots dx_n = \frac{1}{1-|\pi_v|_v}\int_{\mathcal{D}_v^{\mathbf{a}}} f^{-1} dx_1\dots dx_n.$$

Suppose v is infinite. By Lemma 5.2.3.2 one has that

$$f^{-1}|_{(F_v^{\times})^{n-1}\times F_{v,1}} \in L^1((F_v^{\times})^{n-1}\times F_{v,1}, dx_1\dots dx_{n-1}\times\lambda_{v,1})$$

and that

$$\begin{aligned}\int_{F_v^n-\{0\}} & k_v^{\mathbf{a}} f^{-1} dx_1\dots dx_n \\ &= a_n\int_{F_{v,1}} d\lambda_{v,1}(z)\int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(\rho_v(u)\cdot((x_j)_j,z))d^*u\times \\ &\qquad\times\int_{(F_v^{\times})^{n-1}} f(x_1,\dots,x_{n-1},z)^{-1}dx_1\dots dx_{n-1}.\end{aligned}$$

We have by Lemma 5.2.3.2 that

$$\int_{\mathbb{R}_{>0}} k_v^{\mathbf{a}}(\rho_v(u)\cdot((x_j)_j,z))d^*u = \frac{1}{\lambda_{v,1}(F_{v,1})},$$

for every $((x_j)_j,z)\in(F_v^{\times})^{n-1}\times F_{v,1}$. It follows that

$$\int_{F_v^n-\{0\}} k_v^{\mathbf{a}} f^{-1} dx_1\dots dx_n = \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^{\times})^{n-1}\times F_{v,1}} f^{-1}dx_1\dots dx_{n-1}\times\lambda_{v,1}.$$

The implication is proven.

(4) $\implies$ (1). If $\phi : F_v^n-\{0\}\to\mathbb{C}$ is compactly supported, we set $||\phi||_{\sup} := \sup_{\mathbf{x}\in F_v^n-\{0\}}|\phi(\mathbf{x})|$. Suppose firstly $v\in M_F^0$ and let $\phi\in\mathscr{C}_c^0(F_v^n-\{0\},\mathbb{C})$. It follows from Lemma 4.4.3.1 that $\cup_{t\in\mathbb{G}_m(F_v)}(t\cdot\mathcal{D}_v^{\mathbf{a}}) = F_v^n-\{0\}$. As $\operatorname{supp}(\phi)$ is compact, there exists a finite set $\{t_1,\dots,t_m\}$ such that $(\cup_{i=1}^m(t_i\cdot\mathcal{D}_v^{\mathbf{a}}))\supset\operatorname{supp}(\phi)$. For $\mathbf{x}\in\mathcal{D}_v^{\mathbf{a}}$ and $t\in\mathbb{G}_m(F_v)$ one has that $f(t\cdot\mathbf{x})^{-1} = |t|_v^{-|\mathbf{a}|}f(\mathbf{x})^{-1}$. Thus from the fact that $f^{-1}|_{\mathcal{D}_v^{\mathbf{a}}}\in L^1(\mathcal{D}_v^{\mathbf{a}}, dx_1\dots dx_n)$, it follows that $f^{-1}|_{t\cdot\mathcal{D}_v^{\mathbf{a}}}\in L^1(t\cdot\mathcal{D}_v^{\mathbf{a}}, dx_1\dots dx_n)$ and hence that

$$f^{-1}|_{\cup_{i=1}^m(t_i\cdot\mathcal{D}_v^{\mathbf{a}})}\in L^1(\cup_{i=1}^m(t_i\cdot\mathcal{D}_v^{\mathbf{a}}), dx_1\dots dx_n).$$

We deduce that $\phi f^{-1} \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$ and that

$$\left| \int_{F_v^n - \{0\}} \phi f^{-1} dx_1 \dots dx_n \right| \leq \int_{\sup(\phi)} ||\phi||_{\sup} f^{-1} dx_1 \dots dx_n$$
$$\leq ||\phi||_{\sup} \int_{(\cup_{i=1}^m (t_i \cdot \mathcal{D}_v^{\mathbf{a}}))} f^{-1} dx_1 \dots dx_n.$$

It follows furthermore that $\phi \mapsto \int_{F_v^n - \{0\}} \phi f^{-1} dx_1 \dots dx_n$ is a bounded, hence a continuous operator, i.e. $f^{-1} dx_1 \dots dx_n$ is a measure on $F_v^n - \{0\}$.

Suppose now $v \in M_F^{\infty}$. For any $\phi \in \mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C})$, one has by Lemma 5.2.3.2 that $\phi f^{-1} \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$. We will use [**10**, Chapter III, §1, n^0 3, Proposition 6] to prove the continuity of the operator $\phi \mapsto \int_{F_v^n - \{0\}} \phi f^{-1} dx_1 \dots dx_n$. By the mentioned proposition, it suffices to pick a sequence of compacts $\{K_\alpha\}_\alpha$, the interiors of which cover $F_v^n - \{0\}$ and to establish that for every α there exists $M_\alpha > 0$ such that

$$\left| \int_{F_v^n - \{0\}} \phi f^{-1} dx_1 \dots dx_n \right| \leq M_\alpha ||\phi||_{\sup}$$

for $\phi \in \mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C})$ with $\operatorname{supp}(\phi) \subset K_\alpha$. We set

$$K_\alpha := \{\mathbf{x} \in F_v^n - \{0\} | \forall j : \ \alpha^{-1} \leq |x_j|_v \leq \alpha\} \qquad \text{for } \alpha \in \mathbb{Z}_{\geq 2}.$$

Let $\xi_\alpha \in \mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{R}_{\geq 0})$ such that $\xi_\alpha(\mathbf{x}) \geq 1$ for every $\mathbf{x} \in K_\alpha$. For every $\mathbf{x} \in F_v^n - \{0\}$, the map $\mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ given by $r \mapsto \xi_\alpha(\rho_v(r) \cdot \mathbf{x})$ is compactly supported, because ξ_α is compactly supported, the map $y \mapsto y \cdot \mathbf{x}$ is proper (the action of $F_v^\times$ on $(F_v^\times)^n$ is proper by Proposition 3.3.4.1, thus by [**7**, Chapter III, §4, n^0 2, Proposition 4], the map $y \mapsto y \cdot \mathbf{x}$ is proper) and $\rho_v : r \mapsto r^{1/n_v}$ is proper. Let $\phi \in \mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C})$ with $\operatorname{supp}(\phi) \subset K_\alpha$. We have that

$$\left| \int_{\mathbb{R}_{>0}} \phi(\rho_v(r) \cdot ((u_j)_{j=1}^{n-1}, z)) d^* r \right|$$
$$\leq \int_{\mathbb{R}_{>0}} ||\phi||_{\sup} \cdot \xi_\alpha(\rho_v(r) \cdot ((u_j)_{j=1}^{n-1}, z)) d^* r$$
$$\leq ||\phi||_{\sup} \int_{\mathbb{R}_{>0}} \xi_\alpha(\rho_v(t) \cdot ((u_j)_{j=1}^{n-1}, z)) d^* t$$
$$\leq ||\phi||_{\sup} ||\xi_\alpha||_{\sup} \int_{\operatorname{supp}(r \mapsto \xi_\alpha(\rho_v(r) \cdot \mathbf{x}))} 1 d^* t$$
$$= ||\phi||_{\sup} C(\alpha),$$

for every $\mathbf{x} \in F_v^n - \{0\}$. We deduce from Lemma 5.2.3.2 that

$$\begin{aligned}\int_{F_v^n-\{0\}} & \phi f^{-1} dx_1 \dots dx_n \\ &= a_n \int_{F_{v,1}} d\lambda_{v,1}(z) \int_{\mathbb{R}_{>0}} \phi\big(\rho_v(t)\cdot((x_j)_{j=1}^{n-1}, z)\big) d^*t \times \\ &\qquad \times \int_{(F_v^\times)^{n-1}} f(x_1,\dots,x_{n-1},z)^{-1} dx_1 \dots dx_{n-1} \\ &\leq ||\phi||_{\sup} C(\alpha) \int_{(F_v^\times)^{n-1}\times F_{v,1}} f^{-1} dx_1 \dots dx_{n-1} \lambda_{v,1}.\end{aligned}$$

By [9, Chapter III, §1, n^0 3, Proposition 6] the operator

$$\mathscr{C}_c^0(F_v^n - \{0\}, \mathbb{C}) \to \mathbb{C} \qquad \phi \mapsto \int_{F_v^n-\{0\}} \phi f^{-1} dx_1 \dots dx_n$$

is continuous. We conclude that $f^{-1}dx_1 \dots dx_n$ is a measure. The statement is proven. □

When f satisfies the equivalent conditions of Proposition 5.2.3.6, the measure $f^{-1}dx_1 \dots dx_n$ is $F_v^\times$-invariant by Lemma 5.2.3.1.

Example 5.2.3.9. — Suppose f is continuous and $f(\mathbf{x}) \in \mathbb{C} - \{0\}$ for every $\mathbf{x} \in F_v^n - \{0\}$. From the fact that a product of a measure and a continuous function is a measure [9, Chapter III, §1, n^0 4], it follows that $f^{-1}dx_1 \dots dx_n$ is a measure and hence f satisfies the equivalent conditions of Proposition 5.2.3.6.

5.2.4. — Proposition 5.2.3.6 provides a measure ω_v on $[\mathscr{P}(\mathbf{a})(F_v)]$.

Definition 5.2.4.1. — *Let $v \in M_F$ and let $f_v : F_v^n - \{0\} \to \mathbb{C} \cup \{\infty\}$ be an $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ such that*

$$dx_1 \dots dx_n(\{\mathbf{x} | f_v(\mathbf{x}) = 0\}) = 0$$

and such that $f_v^{-1}dx_1 \dots dx_n$ is a measure on $F_v^n - \{0\}$. We define ω_v to be the quotient measure

$$\omega_v := (f_v^{-1} dx_1 \dots dx_n)/d^*x$$

on $[\mathscr{P}(\mathbf{a})(F_v)]$.

Recall from Corollary 3.3.3.2 that $[\mathscr{T}(\mathbf{a})(F_v)]$ is the open subset of $[\mathscr{P}(\mathbf{a})(F_v)]$ given by the image of $q_v^{\mathbf{a}}((F_v^\times)^n)$. We prove that the complement of the open subset $[\mathscr{T}(\mathbf{a})(F_v)] \subset [\mathscr{P}(\mathbf{a})(F_v)]$ is ω_v-negligible.

Lemma 5.2.4.2. — *Let f_v be as in Definition 5.2.4.1. One has that*

$$\omega_v([\mathscr{P}(\mathbf{a})(F_v)] - [\mathscr{T}(\mathbf{a})(F_v)]) = 0.$$

Proof. — The preimage $(q_v^{\mathbf{a}})^{-1}([\mathscr{P}(\mathbf{a})(F_v)] - [\mathscr{T}(\mathbf{a})(F_v)])$ is $dx_1 \dots dx_n$-negligible. Now, [9, Chapter II, §2, $\mathrm{n}^0\,3$, Proposition 6] gives that $\omega_v([\mathscr{P}(\mathbf{a})(F_v)] - [\mathscr{T}(\mathbf{a})(F_v)]) = 0$. □

We explain how to do the integration against ω_v.

Lemma 5.2.4.3. — *Let $v \in M_F$. Let f_v be as in Definition 5.2.4.1. Let $h : [\mathscr{P}(\mathbf{a})(F_v)] \to \mathbb{C}$ be a function. Suppose $v \in M_F^0$. Then $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$ if and only if $(h \circ q_v^{\mathbf{a}}) \cdot f_v^{-1}|_{\mathcal{D}_v^{\mathbf{a}}} \in L^1(\mathcal{D}_v^{\mathbf{a}}, dx_1 \dots dx_n)$ and if $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$, then*

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} h\omega_v = \frac{1}{1 - |\pi_v|_v} \int_{\mathcal{D}_v^{\mathbf{a}}} (h \circ q_v^{\mathbf{a}}) f_v^{-1} dx_1 \dots dx_n.$$

Suppose $v \in M_F^\infty$. Then $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$ if and only if

$$(h \circ q_v^{\mathbf{a}}) \cdot f_v^{-1}|_{(F_v^\times)^{n-1} \times F_{v,1}} \in L^1((F_v^\times)^{n-1} \times F_{v,1}, dx_1 \dots dx_{n-1} \times \lambda_{v,1})$$

and if $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$, then

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} h\omega_v = \frac{a_n}{\lambda_{v,1}(F_{v,1})} \int_{(F_v^\times)^{n-1} \times F_{v,1}} (h \circ q_v^{\mathbf{a}}) f_v^{-1} dx_1 \dots dx_{n-1} \times \lambda_{v,1}.$$

Proof. — As $[\mathscr{P}(\mathbf{a})(F_v)]$ is compact and hence paracompact, by Proposition 5.1.2.5, one has that $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$ if and only if $k_v^{\mathbf{a}} \cdot (h \circ q_v^{\mathbf{a}}) \in L^1(F_v^n - \{0\}, f_v^{-1} dx_1 \dots dx_n)$, and if $h \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v)$ then

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} h\omega_v = \int_{F_v^n - \{0\}} k_v^{\mathbf{a}} \cdot (h \circ q_v^{\mathbf{a}}) f_v^{-1} dx_1 \dots dx_n.$$

The function $q_v^{\mathbf{a}}$ is $F_v^\times$-invariant, hence is such $h^{-1} \circ q_v^{\mathbf{a}}$. It follows that $(h^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v$ is an $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. The set where it vanishes coincides with the set where f_v vanishes, thus this set is $dx_1 \dots dx_n$-negligible. We apply Proposition 5.2.3.6 for the function $(h^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v$. It follows that $k_v^{\mathbf{a}} \cdot ((h \circ q_v^{\mathbf{a}}) f_v^{-1}) \in L^1(F_v^n - \{0\}, dx_1 \dots dx_n)$ if and only if

$$(h \circ q_v^{\mathbf{a}}) \cdot f_v^{-1}|_{\mathcal{D}_v^{\mathbf{a}}} \in L^1(\mathcal{D}_v^{\mathbf{a}}, dx_1 \dots dx_n)$$

if v is finite, and if and only if

$$(h \circ q_v^{\mathbf{a}}) \cdot f_v^{-1}|_{(F_v^\times)^{n-1} \times F_{v,1}} \in L^1((F_v^\times)^{n-1} \times F_{v,1}, dx_1 \dots dx_{n-1} \times \lambda_{v,1})$$

if v is infinite. Moreover, Proposition 5.2.3.6 gives that if

$$k_v^{\mathbf{a}}\cdot(h\circ q_v^{\mathbf{a}})f_v^{-1}\in L^1(F_v^n-\{0\},dx_1\dots dx_n),$$

then

$$\int_{F_v^n-\{0\}}k_v^{\mathbf{a}}(h\circ q_v^{\mathbf{a}})f_v^{-1}dx_1\dots dx_n=\frac{1}{1-|\pi_v|_v}\int_{\mathcal{D}_v^{\mathbf{a}}}(h\circ q_v^{\mathbf{a}})f_v^{-1}dx_1\dots dx_n.$$

if v is finite, and

$$\frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^{\times})^{n-1}\times F_{v,1}}(h\circ q_v^{\mathbf{a}})f_v^{-1}dx_1\dots dx_{n-1}\times\lambda_{v,1}.$$

if v is infinite. The statement follows. $\square$

5.3. Peyre's constant

In this section from a quasi-toric family of $\mathbf{a}$-homogenous functions of weighted degree $|\mathbf{a}|$, we will define a measure on the product space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$, and Peyre's constant of the stacks $\mathscr{P}(\mathbf{a})$ and $\overline{\mathscr{P}(\mathbf{a})}$.

5.3.1. — In this paragraph, we calculate the volume ω_v for $v\in M_F^0$ from Definition 5.2.4.1 when the function f_v is toric.

Let $v\in M_F^0$. As in 4.4.3, we set $r_v:F_v^n-\{0\}\to\mathbb{R}_{>0}$ for the function

$$r_v(\mathbf{x})=\sup_{\substack{j=1,\dots,n\\x_j\neq 0}}\left\lceil-\frac{v(x_j)}{a_j}\right\rceil.$$

Let $f_v^{\#}:F_v^n-\{0\}\to\mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. Recall that this means

$$f_v^{\#}(\mathbf{x})=|\pi_v|_v^{-|\mathbf{a}|r_v(\mathbf{x})}.$$

In Lemma 4.4.3.1, we have established that $r_v|_{\mathcal{D}_v^{\mathbf{a}}}=0$, thus $f_v^{\#}|_{\mathcal{D}_v^{\mathbf{a}}}=1$, where $\mathcal{D}_v^{\mathbf{a}}=(\mathcal{O}_v)^n-(\pi_v^{a_1}\mathcal{O}_v\times\dots\times\pi_v^{a_n}\mathcal{O}_v)$. For $v\in M_F^0$ and $s\in\mathbb{C}$, we denote

$$\zeta_v(s)=\frac{1}{1-|\pi_v|_v^s}.$$

Lemma 5.3.1.1. — *Let $v\in M_F^0$ and let $f_v^{\#}:F_v^n-\{0\}\to\mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. Let $\omega_v^{\#}$ be the measure on $[\mathscr{P}(\mathbf{a})(F_v)]$ that is given by*

$$((f_v^{\#})^{-1}dx_1\dots dx_n)/d^*x.$$

One has that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = \frac{\zeta_v(1)}{\zeta_v(|\mathbf{a}|)}.$$

Proof. — By applying Lemma 5.2.4.3 and using the fact that $f_v^{\#}|_{\mathcal{D}_v^{\mathbf{a}}} = 1$, we get that

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} 1\omega_v^{\#} = \frac{1}{1-|\pi_v|_v}\int_{\mathcal{D}_v^{\mathbf{a}}} (f_v^{\#})^{-1} dx_1 \dots dx_n = \zeta_v(1)\int_{\mathcal{D}_v^{\mathbf{a}}} dx_1 \dots dx_n.$$

In turn one has that

$$dx_1 \dots dx_n(\mathcal{D}_v^{\mathbf{a}}) = 1 - \prod_{j=1}^{n} dx(\pi_v^{a_j}\mathcal{O}_v) = 1 - |\pi_v|_v^{|\mathbf{a}|} = \zeta_v(|\mathbf{a}|)^{-1},$$

and the claim follows. □

Remark 5.3.1.2. — We will generalize the calculation of Lemma 5.3.1.1 in Lemma 7.2.1.1, when will be calculating the Fourier transform of a local toric height at a non-archimedean place.

5.3.2. — Let us calculate the volume $\omega_v([\mathscr{P}(\mathbf{a})(F_v)])$ when $v \in M_F^{\infty}$ and the function $f_v = f_v^{\#}$ is the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$.

Recall that the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ is the function

$$f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0} \qquad \mathbf{x} \mapsto \max_{j=1,\dots,n} (|x_j|_v^{1/a_j})^{|\mathbf{a}|}.$$

Lemma 5.3.2.1. — *Let $v \in M_F^{\infty}$ and let $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. Let $\omega_v^{\#}$ be the measure on $[\mathscr{P}(\mathbf{a})(F_v)]$ that is given by*

$$((f_v^{\#})^{-1} dx_1 \dots dx_n)/d^*x.$$

One has that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = 2^{n-1}|\mathbf{a}|$$

if v is real, and that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = (2\pi)^{n-1}|\mathbf{a}|$$

if v is complex.

Proof. — Lemma 5.2.4.3 gives that:

$$\begin{aligned}
&\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)])\\
&= \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} (f_v^{\#})^{-1}dx_1\dots dx_{n-1}\times\lambda_{v,1}\\
&= \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} \max_{j=1,\dots,n}(|x_j|_v^{1/a_j})^{-|\mathbf{a}|}dx_1\dots dx_{n-1}\times\lambda_{v,1}\\
&= \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} \max(\max_{j=1,\dots,n-1}(|x_j|_v^{1/a_j}),1)^{-|\mathbf{a}|}dx_1\dots dx_{n-1}\times\lambda_{v,1}\\
&= \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{F_{v,1}}\lambda_{v,1}\int_{(F_v^\times)^{n-1}} \max(\max_{j=1,\dots,n-1}(|x_j|_v^{1/a_j}),1)^{-|\mathbf{a}|}dx_1\dots dx_{n-1}\\
&= a_n\int_{(F_v^\times)^{n-1}} \max(\max_{j=1,\dots,n-1}(|x_j|_v^{1/a_j}),1)^{-|\mathbf{a}|}dx_1\dots dx_{n-1}.
\end{aligned}$$

Lemma 5.2.1.3 gives that the homomorphism

$$\widetilde{\rho}_v:\mathbb{R}_{>0}\times F_{v,1}\to F_v^\times\qquad (r,z)\mapsto\rho_v(r)z,$$

where $\rho_v(r)=r^{1/[F_v:\mathbb{R}]}$, satisfies that $(\widetilde{\rho}_v)_*(dr\times\lambda_{v,1})=dx|_{F_v^\times}$. One has that $|\widetilde{\rho}_v(r,z)|_v=r$ and we deduce that

$$\begin{aligned}
&\omega_v^{\#}(\mathscr{P}(\mathbf{a})(F_v))\\
&= a_n\int_{(F_v^\times)^{n-1}} \max(\max_{j=1,\dots,n-1}(|x_j|_v^{1/a_j}),1)^{-|\mathbf{a}|}dx_1\dots dx_{n-1}\\
&= a_n\int_{(\mathbb{R}_{>0}\times F_{v,1})^{n-1}} \max(\max_{j=1,\dots,n-1}(|\widetilde{\rho}_v(r_jz)|_v^{1/a_j}),1)^{-|\mathbf{a}|}dr_1\dots dr_{n-1}\otimes\lambda_{v,1}^{\otimes(n-1)}\\
&= a_n\int_{F_{v,1}^{n-1}}\lambda_{v,1}^{\otimes(n-1)}\int_{\mathbb{R}_{>0}^{n-1}} \max(\max_{j=1,\dots,n-1}(r_j^{1/a_j}),1)^{-|\mathbf{a}|}dr_1\dots dr_{n-1}\\
&= a_n(\lambda_{v,1}(F_{v,1}))^{n-1}\int_{\mathbb{R}_{>0}^{n-1}} \max(\max_{j=1,\dots,n-1}(r_j^{1/a_j}),1)^{-|\mathbf{a}|}dr_1\dots dr_{n-1}.
\end{aligned}$$

Let us evaluate the last integral. Define $V_0:=\{\mathbf{x}\in\mathbb{R}_{>0}^{n-1}|\forall i: x_i\leq 1\}$. For $i=1,\dots,n-1$, define

$$V_i:=\{\mathbf{x}\in\mathbb{R}_{>0}^{n-1}|x_i^{1/a_i}=\max_j(x_j^{1/a_j})\}.$$

For every $i,j \in \{0,\dots,n-1\}$ with $i \neq j$, one has that $V_i \cap V_j$ is $dr_1 \dots dr_{n-1}$-negligible. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}_{>0}} \max(\max_j(r_j^{1/a_j}),1)^{-|\mathbf{a}|} dr_1 \dots dr_{n-1} \\
&\qquad = \sum_{i=0}^{n-1} \int_{V_i} \max(\max_j(r_j^{1/a_j}),1)^{-|\mathbf{a}|} dr_1 \dots dr_{n-1} \\
&\qquad = \int_{V_0} 1 dr_1 \dots dr_{n-1} + \sum_{i=1}^{n-1} \int_{V_i} r_i^{-|\mathbf{a}|/a_i} dr_1 \dots dr_{n-1} \\
&\qquad = 1 + \sum_{i=1}^{n-1} \int_1^{\infty} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n-1} \int_0^{r_i^{a_j/a_i}} 1 dr_j \right) \cdot r_i^{-|\mathbf{a}|/a_i} dr_i \\
&\qquad = 1 + \sum_{i=1}^{n-1} \int_1^{\infty} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n-1} r_i^{a_j/a_i} \right) r_i^{-|\mathbf{a}|/a_i} dr_i \\
&\qquad = 1 + \int_1^{\infty} r_i^{-1-a_n/a_i} dr_i \\
&\qquad = 1 + \sum_{i=1}^{n-1} \frac{a_i}{a_n} \\
&\qquad = \frac{|\mathbf{a}|}{a_n},
\end{aligned}$$

where the third equality follows from Fubini theorem. We deduce that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = a_n \lambda_{v,1}(F_{v,1})^{n-1} \frac{|\mathbf{a}|}{a_n} = \lambda_{v,1}(F_{v,1})^{n-1} |\mathbf{a}|.$$

Thus if v is real one has that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = 2^{n-1}|\mathbf{a}|$$

and if v is complex, one has that

$$\omega_v^{\#}([\mathscr{P}(\mathbf{a})(F_v)]) = (2\pi)^{n-1}|\mathbf{a}|.$$

□

5.3.3. — We define Peyre's constant for stack $\mathscr{P}(\mathbf{a})$ for quasi-toric families $(f_v)_v$.

We will define a measure on the product space

$$\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)].$$

The space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is compact and Hausdorff, as for every $v\in M_F$ by Proposition 3.3.4.5 and by Proposition 3.3.4.1, the spaces $[\mathscr{P}(\mathbf{a})(F_v)]$ are compact and Hausdorff. Let $(f_v:F_v^n-\{0\}\to\mathbb{R}_{\geq 0})_{v\in M_F}$ be a quasi-toric family of $|\mathbf{a}|$-homogenous functions of weighted degree $|\mathbf{a}|$ such that for every v, one has that the set $\{\mathbf{x}\in F_v^n-\{0\}|f_v(\mathbf{x})=0\}$ is $dx_1\dots dx_n$-negligible and that $f_v^{-1}dx_1\dots dx_n$ is a measure on $F_v^n-\{0\}$. For every $v\in M_F$, we set $\omega_v=(f_v^{-1}dx_1\dots dx_n)/d^*x$. Using measures ω_v, we define a product measure on $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ (by [9, Chapter III, §4, n^0 6, Proposition 9] we indeed get a measure on the product).

Definition 5.3.3.1. — *We define a measure $\omega=\omega((f_v)_v)$ on $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ by*

$$|\mu_{\gcd(\mathbf{a})}(F)|\Delta(F)^{-\frac{n-1}{2}}\operatorname{Res}(\zeta_F,1)\bigotimes_{v\in M_F^0}\big(\zeta_v(1)^{-1}\omega_v\big)\bigotimes_{v\in M_F^\infty}\omega_v.$$

We set

$$\tau=\tau((f_v)_v)=\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big),$$

where $\Delta(F)$ is the absolute discriminant of F.

We explain how ω changes, when the quasi-toric family is changed.

Lemma 5.3.3.2. — *Let S be a finite set of places and for $v\in S$, let $h_v:[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{R}_{>0}$ be a continuous function. For $v\in M_F-S$, we set $h_v=1$. Let us denote by $h:\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{R}_{>0}$ the function $\otimes_{v\in M_F}h_v$. One has that*

$$\omega((h_vf_v)_v)=h^{-1}\omega((f_v)_v).$$

Proof. — For $v\in M_F$, it follows directly from Lemma 5.1.2.4 that

$$(((h_v\circ q_v^{\mathbf{a}})\cdot f_v)^{-1}dx_1\dots dx_n)/d^*x=(h_v^{-1})((f_v^{-1}dx_1\dots dx_n)/d^*x)=h_v^{-1}\omega_v.$$

It follows that the measure $\omega(((h_{v,\eta}\circ q_v^{\mathbf{a}})\cdot f_v)_v)$ on $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ defined by the quasi-toric family $((h_v\circ q_v^{\mathbf{a}})\cdot f_v)_v$ satisfies that

$$\begin{aligned}\omega(((h_v\circ q_v^{\mathbf{a}})\cdot f_v)_v)&\\ &=\frac{|\mu_{\gcd(\mathbf{a})}(F)|}{\Delta(F)^{\frac{n-1}{2}}}\operatorname{Res}(\zeta_F,1)\prod_{v\in M_F^0}\left(\zeta_v(1)^{-1}h_v^{-1}\omega_v\right)\prod_{v\in M_F^\infty}h_v^{-1}\omega_v\\ &=h^{-1}\omega.\end{aligned}$$

□

We give another expression for τ.

Lemma 5.3.3.3. — *Let S be the finite set of places v for which f_v is not toric. One has that*

$$\begin{aligned}\tau((f_v)_v)&\\ &=\frac{\operatorname{Res}(\zeta_F,1)|\mu_{\gcd(\mathbf{a})}(F)|}{\Delta(F)^{\frac{n-1}{2}}\zeta_F(|\mathbf{a}|)}\prod_{v\in S\cap M_F^0}\frac{\zeta_v(|\mathbf{a}|)\omega_v([\mathscr{P}(\mathbf{a})(F_v)])}{\zeta_v(1)}\times\\ &\qquad\times\prod_{v\in M_F^\infty}\omega_v([\mathscr{P}(\mathbf{a})(F_v)]).\end{aligned}$$

Proof. — Lemma 5.3.1.1 gives that for every $v\in M_F^0-S$ one has

$$\omega_v([\mathscr{P}(\mathbf{a})(F_v)])\zeta_v(1)^{-1}=\zeta_v(|\mathbf{a}|)^{-1}$$

and thus:

$$\begin{aligned}&\tau((f_v)_v)|\mu_{\gcd(\mathbf{a})}(F)|^{-1}\Delta(F)^{\frac{n-1}{2}}\\ &\quad=\operatorname{Res}(\zeta_F,1)\prod_{v\in S\cap M_F^0}\frac{\omega_v([\mathscr{P}(\mathbf{a})(F_v)])}{\zeta_v(1)}\times\\ &\qquad\qquad\times\prod_{v\in M_F^0-S}\zeta_v(|\mathbf{a}|)^{-1}\times\prod_{v\in M_F^\infty}\omega_v([\mathscr{P}(\mathbf{a})(F_v)])\\ &\quad=\frac{\operatorname{Res}(\zeta_F,1)}{\zeta(|\mathbf{a}|)}\prod_{v\in S\cap M_F^0}\left(\frac{\zeta_v(|\mathbf{a}|)\omega_v([\mathscr{P}(\mathbf{a})(F_v)])}{\zeta_v(1)}\right)\times\prod_{v\in M_F^\infty}\omega_v([\mathscr{P}(\mathbf{a})(F_v)]).\end{aligned}$$

□

5.4. Haar measure on $[\mathscr{T}(\mathbf{a})(F_v)]$

Let $F_v^\times$ acts on $(F_v^\times)^n$ by $t\cdot\mathbf{x} = (t^{a_j}x_j)_j$. This action is proper by Proposition 3.3.4.1. The quotient for this action is $[\mathscr{T}(\mathbf{a})(F_v)]$ by Corollary 3.3.3.2 and is locally compact by Proposition 3.3.4.1. By Corollary 3.3.3.2, one has that the map $[\mathscr{T}(\mathbf{a})(F_v)]\to[\mathscr{P}(\mathbf{a})(F_v)]$, induced from $F_v^\times$-invariant map $(F_v^\times)^n\hookrightarrow F_v^n-\{0\}\to[\mathscr{P}(\mathbf{a})(F_v)]$ is an open embedding. By Proposition 3.3.4.1, the map

$$\epsilon : F_v^\times \to (F_v^\times)^n \qquad t\mapsto (t^{a_j})_j$$

is proper, its image $(F_v^\times)_{\mathbf{a}} := \epsilon(F_v^\times) = \{(t^{a_j})_j | t\in F_v^\times\}$, is a closed subgroup of $(F_v^\times)^n$ and one has an identification $[\mathscr{T}(\mathbf{a})(F_v)] = (F_v^\times)^n/(F_v^\times)_{\mathbf{a}}$. Using this identification, we endow $[\mathscr{T}(\mathbf{a})(F_v)]$ with a structure of a topological group (which is necessary abelian). The goal of this section is to define a Haar measure on $[\mathscr{T}(\mathbf{a})(F_v)]$ and relate it with the measure ω_v on $[\mathscr{P}(\mathbf{a})(F_v)]$.

5.4.1. — Let $v\in M_F$. We are going to define a Haar measure on $[\mathscr{T}(\mathbf{a})(F_v)]$.

By the fact that a product of a continuous function and a measure is a measure [9, Chapter III, §1, n^0 4], one has that

$$\mathscr{C}_c^0((F_v^\times)^n,\mathbb{C})\to\mathbb{C} \qquad \phi\mapsto\int_{(F_v^\times)^n}\phi\cdot\prod_{j=1}^n|x_j|_v^{-1}dx_1\dots dx_n$$

is a measure on $(F_v^\times)^n$. The function

$$F_v^n-\{0\}\to\mathbb{R}_{\geq 0} \qquad \mathbf{x}\mapsto\prod_{j=1}^n|x_j|_v$$

is $\mathbf{a}$-homogenous of weighted degree $|\mathbf{a}|$ and the set where it vanishes is given by $\{\mathbf{x}\in F_v^n-\{0\}|\exists j : x_j=0\}$, thus this set is $dx_1\dots dx_n$-negligible (because it is contained in a finite union of hyperplanes in $(F_v)^n$.) It follows from Lemma 5.2.3.1 that

$$\prod_{j=1}^n|x_j|_v^{-1}dx_1\dots dx_n = d^*x_1\dots d^*x_n$$

is $F_v^\times$-invariant measure on $F_v^\times$.

Definition 5.4.1.1. — *We define a measure μ_v on $[\mathscr{T}(\mathbf{a})(F_v)]$ by*

$$\mu_v := (d^*x_1\dots d^*x_n)/d^*x_v.$$

By Lemma 5.1.4.3 the measure μ_v is a Haar measure on $[\mathscr{T}(\mathbf{a})(F_v)]$.

Lemma 5.4.1.2. — *Let $v\in M_F$. Let $f_v:F_v^n-\{0\}\to\mathbb{R}_{\geq 0}$ be a continuous $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ such that*

$$f_v^{-1}dx_1\dots dx_n$$

is a measure on $F_v^n-\{0\}$ (this in particular implies that $dx_1\dots dx_n(\{\mathbf{x}|f_v(\mathbf{x})=0\})=0$). Let $H_v:[\mathscr{T}(\mathbf{a})(F_v)]\to\mathbb{R}_{\geq 0}$ be the function given by the continuous $F_v^\times$-invariant function

$$\mathbf{x}\mapsto f_v(\mathbf{x})\prod_{j=1}^n|x_j|_v^{-1}.$$

1. *One has that $\mu_v(\{\mathbf{y}|H_v(\mathbf{y})=0\})=0$.*
2. *One has an equality of the measures $\mu_v=(H_v)(\omega_v|_{[\mathscr{T}(\mathbf{a})(F_v)]})$.*

Proof. — 1. By the definition of H_v one has that $\{\mathbf{y}|H_v(\mathbf{y})=0\}=q_v^{\mathbf{a}}(\{\mathbf{x}\in(F_v^\times)^n|f_v(\mathbf{x})=0\})$. Note that as $dx_1\dots dx_n(\{\mathbf{x}|f_v(\mathbf{x})=0\})=0$, it follows that the set $\{\mathbf{x}\in(F_v^\times)^n|f_v(\mathbf{x})=0\}$ is $dx_1\dots dx_n$-negligible and thus $d^*x_1\dots d^*x_n$-negligible. Hence, by [10, Chapter VII, §2, n^0 3, Proposition 6], one has that $\mu_v(q_v^{\mathbf{a}}(\{\mathbf{x}\in(F_v^\times)^n|f(\mathbf{x})=0\}))=0$.

2. Observe that

$$\begin{aligned}d^*x_1\dots d^*x_n=\prod_{j=1}^n|x_j|_v^{-1}dx_1\dots dx_n&=f_v(\mathbf{x})\prod_{j=1}^n|x_j|_v^{-1}f_v(\mathbf{x})^{-1}dx_1\dots dx_n\\&=(H_v\circ q_v^{\mathbf{a}})f_v^{-1}dx_1\dots dx_n.\end{aligned}$$

Now Lemma 5.1.2.4 gives precisely that

$$\mu_v=(d^*x_1\dots d^*x_n)/d^*x=H_v(f_v^{-1}dx_1\dots dx_n)/d^*x=H_v\omega_v.$$

□

Lemma 5.4.1.3. — *Let $h:[\mathscr{T}(\mathbf{a})(F_v)]\to\mathbb{C}$ be a function. Suppose $v\in M_F^0$. One has that $h\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\mu_v)$ if and only if $(h\circ q_v^{\mathbf{a}})\in L^1((F_v^\times)^n\cap\mathcal{D}_v^{\mathbf{a}},d^*x_1\dots d^*x_n)$, and if $h\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\mu_v)$, one has that:*

$$\int_{[\mathscr{T}(\mathbf{a})(F_v)]}h\mu_v=\frac{1}{1-|\pi_v|_v}\int_{(F_v^\times)^n\cap\mathcal{D}_v^{\mathbf{a}}}(h\circ q_v^{\mathbf{a}})d^*x_1\dots d^*x_n.$$

Suppose $v\in M_F^\infty$. One has that $h\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\mu_v)$ if and only if $(h\circ q_v^{\mathbf{a}})\in L^1((F_v^\times)^{n-1}\times F_{v,1},d^*x_1\dots d^*x_{n-1}\times\lambda_{v,1})$, and if

$h \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \mu_v)$, *one has that:*

$$\frac{a_n}{\lambda_{v,1}(F_{v,1})} \int_{(F_v^\times)^{n-1} \times F_{v,1}} (h \circ q_v^{\mathbf{a}}) d^* x_1 \dots d^* x_{n-1} \times \lambda_{v,1}.$$

Proof. — Let $f_v^\# : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. Let $H_v^\# : [\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{R}_{>0}$ be the induced function from $F_v^\times$-invariant function $(F_v^\times)^n \to \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto f_v^\#(\mathbf{x}) \prod_{j=1}^n |x_j|_v$. It follows from Lemma 5.4.1.2, that one has an equality of the measures $\mu_v = H_v^\# \omega_v^\# |_{[\mathscr{T}(\mathbf{a})(F_v)]}$. We deduce that $h \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \mu_v)$ if and only if $hH_v^\# \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \omega_v^\#)$, and as by Lemma 5.2.4.2 one has $\omega_v^\#([\mathscr{P}(\mathbf{a})(F_v)] - [\mathscr{T}(\mathbf{a})(F_v)]) = 0$, if and only if $hH_v^\# \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v^\#)$. Moreover, it follows that if $h \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \mu_v)$, then

$$\int_{[\mathscr{T}(\mathbf{a})(F_v)]} h\mu_v = \int_{[\mathscr{P}(\mathbf{a})(F_v)]} (hH_v^\#)\omega_v^\#.$$

Suppose $v \in M_F^0$. Recall that by Lemma 4.4.3.1 one has that $f_v^\# |_{\mathcal{D}_v^{\mathbf{a}}} = 1$. By Lemma 5.2.4.3, one has that $hH_v^\# \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v^\#)$ if and only if

$$(((hH_v^\#) \circ q_v^{\mathbf{a}})(f_v^\#)^{-1})|_{\mathcal{D}_v^{\mathbf{a}}} = (h \circ q_v^{\mathbf{a}})(H_v^\# \circ q_v^{\mathbf{a}})|_{\mathcal{D}_v^{\mathbf{a}}} = (h \circ q_v^{\mathbf{a}})(|x_1|_v^{-1} \cdots |x_n|_v^{-1})|_{\mathcal{D}_v^{\mathbf{a}}}$$

is an element of $L^1(\mathcal{D}_v^{\mathbf{a}}, dx_1 \dots dx_n)$. Moreover, Lemma 5.2.4.3 gives that if $hH_v^\# \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v^\#)$ then

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} hH_v^\# \omega_v^\# = \frac{1}{1 - |\pi_v|_v} \int_{\mathcal{D}_v^{\mathbf{a}}} (h \circ q_v^{\mathbf{a}}) |x_1|_v^{-1} \cdots |x_n|_v^{-1} dx_1 \dots dx_n.$$

As $dx_1 \dots dx_n(\mathcal{D}_v^{\mathbf{a}} - (\mathcal{D}_v^{\mathbf{a}} \cap (F_v^\times)^n)) = 0$, (because $\mathcal{D}_v^{\mathbf{a}} - (\mathcal{D}_v^{\mathbf{a}} \cap (F_v^\times)^n)$ is contained in a finite union of hyperplanes of F_v^n,) the last integral is equal to

$$\frac{1}{1 - |\pi_v|_v} \int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} (h \circ q_v^{\mathbf{a}}) d^* x_1 \dots d^* x_n.$$

It follows that one has $h \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \mu_v)$ if and only if $(h \circ q_v^{\mathbf{a}}) \in L^1((F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}, d^* x_1 \dots d^* x_n)$ and if $h \in L^1([\mathscr{T}(\mathbf{a})(F_v)], \mu_v)$, then

$$\int_{[\mathscr{T}(\mathbf{a})(F_v)]} h\mu_v = \frac{1}{1 - |\pi_v|_v} \int_{(F_v^\times)^n} (h \circ q_v^{\mathbf{a}}) d^* x_1 \dots d^* x_n.$$

Suppose that $v \in M_F^\infty$. By Lemma 5.2.4.3, one has that $hH_v^\# \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v^\#)$ if and only if

$$((hH_v^\#) \circ q_v^{\mathbf{a}})(f_v^\#)^{-1} = ((h \circ q_v^{\mathbf{a}}) \cdot (H^\# \circ q_v^{\mathbf{a}}))(f_v^\#)^{-1} = (h \circ q_v^{\mathbf{a}}) \prod_{j=1}^{n-1} |x_j|_v^{-1}$$

is an element of $L^1((F_v^\times)^{n-1} \times F_{v,1}, dx_1 \dots dx_{n-1} \times \lambda_{v,1})$ i.e. if and only if

$$(h \circ q_v^{\mathbf{a}}) \in L^1((F_v^\times)^{n-1} \times F_{v,1}, d^*x_1 \dots d^*x_{n-1}\lambda_{v,1}).$$

Moreover, Lemma 5.2.4.3 gives that if $hH_v^\# \in L^1([\mathscr{P}(\mathbf{a})(F_v)], \omega_v^\#)$ then

$$\begin{aligned}
&\int_{[\mathscr{P}(\mathbf{a})(F_v)]} hH_v^\# \omega_v^\# \\
&\quad = \frac{a_n}{\lambda_{v,1}(F_{v,1})} \int_{(F_v^\times)^{n-1} \times F_{v,1}} (h \circ q_v^{\mathbf{a}}) \prod_{j=1}^{n-1} |x_j|_v^{-1} dx_1 \dots dx_{n-1} \times \lambda_{v,1} \\
&\quad = \frac{a_n}{\lambda_{v,1}(F_{v,1})} \int_{(F_v^\times)^{n-1} \times F_{v,1}} (h \circ q_v^{\mathbf{a}}) d^*x_1 \dots d^*x_{n-1} \times \lambda_{v,1}.
\end{aligned}$$

□

5.4.2. — Recall that for $v \in M_F^0$ by Proposition 3.3.5.4, the group $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ identifies with the image of $\mathcal{O}_v^{\times n}$ in $[\mathscr{T}(\mathbf{a})(F_v)]$ for the quotient homomorphism $(F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)]$ and that with this identification, becomes open and compact subgroup of $[\mathscr{T}(\mathbf{a})(F_v)]$. We calculate the volume of $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ against μ_v.

Lemma 5.4.2.1. — *Let $v \in M_F^0$. The Haar measure μ_v is normalized by*

$$\mu_v([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]) = (1 - |\pi_v|_v)^{n-1} = \zeta_v(1)^{-(n-1)}$$

.

Proof. — Let us firstly establish that $(q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]) \cap \mathcal{D}_v^{\mathbf{a}} = (\mathcal{O}_v^\times)^n$. One has that

$$(q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])^{-1} = \bigcup_{t \in \mathbb{G}_m(F_v)} t \cdot (\mathcal{O}_v^\times)^n.$$

Note that if $\mathbf{u} \in (\mathcal{O}_v^\times)^n$, then for every index j one has $v(t^{a_j}u_j) = a_j v(t) + v(u_j) = a_j v(t)$. Now if $v(t) > 0$ it follows that $t \cdot \mathbf{u} \in \prod_{j=1}^n (\pi_v^{a_j}\mathcal{O}_v)$, thus

$t \cdot \mathbf{u} \notin \mathcal{D}_v^{\mathbf{a}}$ and if $v(t) < 0$ then $t \cdot \mathbf{u} \notin \mathcal{O}_v^n$, thus $t \cdot \mathbf{u} \notin \mathcal{D}_v^{\mathbf{a}}$. We have obtained if $v(t) \neq 0$, then $(t \cdot \mathcal{O}_v^{\times n}) \cap \mathcal{D}_v^{\mathbf{a}} = \emptyset$ and hence,

$$\mathcal{D}_v^{\mathbf{a}} \cap (q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]) = \mathcal{D}_v^{\mathbf{a}} \cap \bigcup_{t \in \mathbb{G}_m(F_v)} t \cdot \mathcal{O}_v^{\times n} = \mathcal{D}_v^{\mathbf{a}} \cap \mathcal{O}_v^{\times n} = \mathcal{O}_v^{\times n}.$$

We deduce that one has equality of functions

$$(\mathbf{1}_{(q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])})|_{(F_v^{\times})^n \cap \mathcal{D}_v^{\mathbf{a}}} = \mathbf{1}_{(\mathcal{O}_v^{\times})^n}.$$

Now, we can calculate $\mu_v([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])$. One has by Lemma 5.4.1.3 that

$$\begin{aligned}
\int_{[\mathscr{T}(\mathbf{a})(F_v)]} \mathbf{1}_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} \mu_v &= \frac{1}{1 - |\pi_v|_v} \int_{(F_v^{\times})^n \cap \mathcal{D}_v^{\mathbf{a}}} \mathbf{1}_{(q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])} d^*x_1 \dots d^*x_n \\
&= \frac{1}{1 - |\pi_v|_v} \int_{(F_v^{\times})^n} \mathbf{1}_{(\mathcal{O}_v^{\times})^n} d^*x_1 \dots d^*x_n \\
&= \frac{1}{1 - |\pi_v|_v} \left(\int_{F_v^{\times}} \mathbf{1}_{\mathcal{O}_v^{\times}} d^*x \right)^n \\
&= (1 - |\pi_v|_v)^{n-1}.
\end{aligned}$$

The statement follows. □

5.4.3. — Let us dedicate this paragraph to the definition of a Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. Let $n \in \mathbb{Z}_{>0}$ and let $\mathbf{a} \in \mathbb{Z}_{>0}^n$. We set $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ to be the "adelic space" of $\mathscr{T}(\mathbf{a})$, i.e.

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] = \prod_{v \in M_F}{}' [\mathscr{T}(\mathbf{a})(F_v)],$$

where the restricted product is taken with respect to the sequence of the open and compact subgroups for $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)] \subset [\mathscr{T}(\mathbf{a})(F_v)]$ for $v \in M_F^0$.

Proposition 5.4.3.1. — *Let I be a set and let I' be a subset such that $I - I'$ is finite. For $i \in I$, let G_i be a locally compact abelian group endowed with a Haar measure dg_i. For $i \in I'$, let H_i be an open and compact subgroup of G_i such that $dg_i(H_i) = 1$. Set G to be the restricted product $\prod'_{i \in I} G_i$ with respect to the subgroups $H_i \subset G_i$ for $i \in I'$.*

1. [**10**, Chapter VII, §1, n^0 5, Proposition 5] *Let S be a finite subset of I containing $I - I'$. Set*

$$G_S := \prod_{i \in S} G_i \times \prod_{i \in I - S} H_i.$$

The measure

$$\bigotimes_{i\in S} dg_i \bigotimes_{i\in I-S}(dg_i|_{H_i})$$

is a Haar measure on G_S.

2. [**49**, Proposition 5.5] *There exists a unique Haar measure dg on G such that for every finite subset $S\subset I$ which contains $I-I'$ one has that*

$$dg|_{G_S}=\bigotimes_{i\in S} dg_i \bigotimes_{i\in I-S}(dg_i|_{H_i}).$$

The measure dg will be called the restricted product Haar measure and, by the abuse of the notation, may be denoted as $dg=\otimes dg_i$. We present a way how to calculate the integral of a function.

Proposition 5.4.3.2 **(**[**49**, Proposition 5-6]**).** — *In the situation of Proposition 5.4.3.1, let $f_i\in L^1(G_i,dg_i)$ be a continuous complex valued function such that there exists a finite subset $I''\subset I'$ with $I'-I''$ is finite such that $f_i|H_i=1$ for every $i\in I''$. The function*

$$f:(x_i)_i\mapsto\prod_i f_i(x_i)$$

is continuous. Suppose that

$$\prod_i\int_{G_i} f_i dg_i$$

converges. Then $f\in L^1(G)$ and

$$\int_G f dg=\prod_i\int_{G_i} f_i dg_i.$$

For $v\in M_F^0$, we have established in Lemma 5.4.2.1 that $\mu_v([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])=\zeta_v(1)^{-(n-1)}$. We will apply Proposition 5.4.3.1 to define a Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$.

Definition 5.4.3.3. — *Let $\mu_{\mathbb{A}_F}$ be the restricted product measure*

$$\mu_{\mathbb{A}_F}=\bigotimes_{v\in M_F^0}\zeta_v(1)^{(n-1)}\mu_v\otimes\bigotimes_{v\in M_F^\infty}\mu_v.$$

The group $\mathbb{A}_F^\times$ acts on $(\mathbb{A}_F^\times)^n$ via the proper homomorphism $(x_v)_v\mapsto((x_v^{a_j})_j)_v$ (Lemma 3.4.8.2) and one has an identification $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]=$

$(\mathbb{A}_F^\times)^n/(\mathbb{A}_F^\times)$ (Lemma 3.4.8.4 together with Lemma 5.1.4.3). Endow $\mathbb{A}_F^\times$ with the Haar measure

$$d^*x_{\mathbb{A}_F} := \bigotimes_{v\in M_F^0} \zeta_v(1) d^*x_v \otimes \bigotimes_{v\in M_F^\infty} d^*x_v.$$

Let $d^*\mathbf{x}_{\mathbb{A}_F} := d^*x_{\mathbb{A}_F}^{\otimes n}$ be the product Haar measure on $(\mathbb{A}_F^\times)^n$.

Lemma 5.4.3.4. — *One has the following equality of the measures on* $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] = (\mathbb{A}_F^\times)^n/\mathbb{A}_F^\times$ *:*

$$\mu_{\mathbb{A}_F} = d^*\mathbf{x}_{\mathbb{A}_F}/d^*x_{\mathbb{A}_F}.$$

Proof. — The quotient measure $d^*\mathbf{x}_{\mathbb{A}_F}/d^*x_{\mathbb{A}_F}$ is a Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ by Lemma 5.1.4.3. Therefore, it suffices to verify the equality on a single non-trivial compactly supported function on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ which takes non-negative values. For $v \in M_F^0$ we set $\phi_v = \mathbf{1}_{(\mathcal{O}_v^\times)^n} : (F_v^\times)^n \to \mathbb{C}$ and for $v \in M_F^\infty$, we let $\phi_v : (F_v^\times)^n \to \mathbb{R}_{\geq 0}$ be a non-trivial continuous function with compact support. The function $\phi = \bigotimes_v \phi_v$ is continuous by Proposition 5.4.3.2 and compactly supported (its support is the set $\prod_{v\in M_F^0}(\mathcal{O}_v^\times)^n \times \prod_{v\in M_F^\infty} \operatorname{supp}(\phi_v))$. For $v \in M_F$ and $\mathbf{y} \in [\mathscr{T}(\mathbf{a})(F_v)]$, let $\widetilde{\mathbf{y}} \in (F_v^\times)^n$ be a lift of $\mathbf{y}$ and for $\mathbf{y} \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$, let $\widetilde{\mathbf{y}} \in (\mathbb{A}_F^\times)^n$ be a lift of $\mathbf{y}$. For $v \in M_F^0$ we define

$$\phi_v^* : [\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{R}_{\geq 0} \qquad \mathbf{y} \mapsto \zeta_v(1)\int_{F_v^\times} \phi_v(x\cdot\widetilde{\mathbf{y}})d^*x$$

and for $v \in M_F^\infty$ we define

$$\phi_v^* : [\mathscr{T}(\mathbf{a})(F_v)] \to \mathbb{R}_{\geq 0} \qquad \mathbf{y} \mapsto \int_{F_v^\times} \phi_v(x\cdot\widetilde{\mathbf{y}})d^*x,$$

the functions are well defined, continuous compactly supported and of non-negative values by Proposition 5.1.2.1. By the definition we have that $(d^*x)^{\otimes n}/d^*x = \mu_v$ and thus $(d^*x)^{\otimes n}/(\zeta_v(1)d^*x) = \zeta_v(1)^{-1}\mu_v$. Now by Proposition 5.1.2.2, one has that

$$\begin{aligned}\int_{(F_v^\times)^n}(\phi)(d^*x)^{\otimes n} &= \int_{[\mathscr{T}(\mathbf{a})(F_v)]}(\mu_v/\zeta_v(1))\left(\mathbf{y}\mapsto\int_{F_v^\times}\phi(x\cdot\widetilde{\mathbf{y}})\zeta_v(1)d^*x\right)\\ &= \int_{[\mathscr{T}(\mathbf{a})(F_v)]}(\phi^*)(\mu_v/\zeta_v(1))\\ &= \zeta_v(1)^{-1}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}\phi^*\mu_v.\end{aligned}$$

If $v \in M_F^0$, let us prove that

$$\phi_v^* = \mathbf{1}_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]}.$$

Indeed, if $\mathbf{z} \notin (q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])$, then $\phi_v(x \cdot \mathbf{z}) = 0$ for every $x \in F_v^\times$, thus $\phi_v^*(q_v^{\mathbf{a}}(\mathbf{z})) = 0$. If $\mathbf{z} \in (q_v^{\mathbf{a}})^{-1}([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$, then $x \cdot \mathbf{z} \in (\mathcal{O}_v^\times)^n$ if and only if $v(x) = v(z_1)/a_1$, and thus $d^*x(\{x \in F_v^\times | x \cdot \mathbf{z} \in (\mathcal{O}_v^\times)^n\}) = \zeta_v(1)^{-1}$, and hence $\zeta_v(1) \int_{F_v^\times} \phi_v(x \cdot \mathbf{z}) d^*x = 1$. It follows that $\phi_v^* = \mathbf{1}_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]}$.

We also define

$$\phi^* : [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \to \mathbb{C} \qquad \mathbf{y} \mapsto \int_{\mathbb{A}_F^\times} \phi(x_{\mathbb{A}_F} \cdot \widetilde{\mathbf{y}}) d^* x_{\mathbb{A}_F},$$

by Proposition 5.1.2.1 it is well defined and continuous function. We are going to verify that the two measures coincide on ϕ^*. For every $\mathbf{y} \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$, one has that

$$\begin{aligned} \phi^*(\mathbf{y}) &= \int_{\mathbb{A}_F^\times} \phi(x_{\mathbb{A}_F} \cdot \widetilde{\mathbf{y}}) d^* x_{\mathbb{A}_F} \\ &= \prod_{v \in M_F^0} \zeta_v(1) \int_{[\mathscr{T}(\mathbf{a})(F_v)]} \phi_v(x \cdot \widetilde{\mathbf{y}}_v) d^*x \times \prod_{v \in M_F^\infty} \int_{[\mathscr{T}(\mathbf{a})(F_v)]} \phi_v(x \cdot \widetilde{\mathbf{y}}_v) d^*x \\ &= \prod_{v \in M_F} \phi_v^*(\mathbf{y}_v), \end{aligned}$$

i.e. $\phi^* = \otimes_{v \in M_F} \phi_v^*$. By Proposition 5.1.2.2, we have that

$$\int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} (\phi^*)(d^*\mathbf{x}_{\mathbb{A}_F}/d^*x_{\mathbb{A}_F}) = \int_{(\mathbb{A}_F^\times)^n} \phi d^* \mathbf{x}_{\mathbb{A}_F}.$$

On the other side, one has that

$$\begin{aligned}
&\int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]}\phi^*\mu_{\mathbb{A}_F}\\
&\qquad=\int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]}\big(\otimes_v\phi_v^*\big)\mu_{\mathbb{A}_F}\\
&\qquad=\prod_{v\in M_F^0}\zeta_v(1)^{n-1}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}\phi_v^*\mu_v\times\prod_{v\in M_F^\infty}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}\phi_v^*\mu_v\\
&\qquad=\prod_{v\in M_F^0}\zeta_v(1)^{n}\int_{(F_v^\times)^n}\phi_v d^*x_1\dots d^*x_n\times\prod_{v\in M_F^\infty}\int_{(F_v^\times)^n}\phi_v d^*x_1\dots d^*x_n\\
&\qquad=\int_{(\mathbb{A}_F^\times)^n}(\otimes_v\phi_v)d^*\mathbf{x}_{\mathbb{A}_F}\\
&\qquad=\int_{(\mathbb{A}_F^\times)^n}\phi d^*\mathbf{x}_{\mathbb{A}_F}.
\end{aligned}$$

We have verified that the Haar measures $d^*\mathbf{x}_{\mathbb{A}_F}/d^*x$ and $\mu_{\mathbb{A}_F}$ satisfy that

$$(d^*\mathbf{x}_{\mathbb{A}_F}/d^*x)(\phi^*)=\mu_{\mathbb{A}_F}(\phi^*)>0,$$

thus $d^*\mathbf{x}_{\mathbb{A}_F}/d^*x=\mu_{\mathbb{A}_F}$. The statement is proven. □

5.4.4. — In this paragraph we define and calculate the Tamagawa number of the "stacky torus" $\mathscr{T}(\mathbf{a})$.

Denote by $\mu_{\mathbb{R}}$ the quotient measure $(d^*x)^{\otimes n}/d^*x$ on the quotient $\mathbb{R}^n_{>0}/\mathbb{R}_{>0}$ for the action of $\mathbb{R}_{>0}$ via the proper map $t\mapsto(t^{a_j})_j$. By 5.1.4.3 the quotient identifies with the quotient group $\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}$ and $\mu_{\mathbb{R}}$ is a Haar measure on it.

Definition 5.4.4.1. — *We define μ_1 to be the Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)_1]$ normalized by the condition $\mu_1\otimes\mu_{\mathbb{R}}=\mu_{\mathbb{A}_F}$ for the identification*

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1\times(\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})=[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$$

given by the isomorphism (3.4.9.2).

We can write μ_1 as a quotient measure:

Lemma 5.4.4.2. — *The measure μ_1 identifies with the measure $d^*\mathbf{x}^1_{\mathbb{A}_F}/d^*x^1_{\mathbb{A}_F}$ on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1=(\mathbb{A}^1_F)^n/\mathbb{A}^1_F$, where the action is given by* *the proper morphism*

$$\mathbb{A}_F^1 \to (\mathbb{A}_F^1)^n \qquad (x_v) \mapsto ((x_v^{a_j})_j)_v.$$

Proof. — We have the following bicomplex (we have written the corresponding measures next to the groups)

$$\begin{array}{ccccc}
K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{A}_F^1, dx^1_{\mathbb{A}_F}) & \longrightarrow & (\mathbb{A}_F^\times, d^*x_{\mathbb{A}_F}) & \longrightarrow & (\mathbb{R}_{>0}, d^*x) \\
\downarrow & & \downarrow & & \downarrow \\
((\mathbb{A}_F^1)^n, (dx^1_{\mathbb{A}_F})^{\otimes n}) & \longrightarrow & ((\mathbb{A}_F^\times)^n, d^*\mathbf{x}_{\mathbb{A}_F}) & \longrightarrow & (\mathbb{R}^n_{>0}, (d^*x)^{\otimes n}) \\
\downarrow & & \downarrow & & \downarrow \\
([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1, \mu_1) & \longrightarrow & ([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)], \mu_{\mathbb{A}_F}) & \longrightarrow & ((\mathbb{R}^n/(\mathbb{R}_{>0})_{\mathbf{a}}, (d^*x)^{\otimes n}/d^*x),
\end{array}$$

where K_1, K_2 are K_3 are the corresponding kernels, which are compact by Lemma 3.4.8.2, endowed with the probability Haar measures, all terms that are not drawn and corresponding measures are assumed to be the trivial groups endowed with the probability Haar measures. All horizontal sequences and the last vertical sequence are of trivial measure Euler-Poincaré characteristic. By Lemma 5.4.3.4, the second vertical sequence is of trivial measure Euler-Poincaré characteristic. By Proposition 5.1.4.2, it follows that the first vertical sequence is of trivial measure Euler-Poincaré characteristic. The statement follows. □

For every space X, we denote by count_X the counting measure on X. Proposition 3.4.7.2 gives that the image of $[\mathscr{T}(\mathbf{a})(F)]$ under the map $[\mathscr{T}(\mathbf{a})(i)]$ (which is induced map from the canonical inclusion $(F^\times)^n \to (\mathbb{A}_F^\times)^n$) is contained in $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ and that, moreover, it is discrete, closed and cocompact subgroup of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$.

The kernel of the map of discrete groups

$$F^\times \to (F^\times)^n \qquad t \mapsto (t^{a_j})_j$$

is the finite group $\mu_{\gcd(\mathbf{a})}(F)$. We endow $[\mathscr{T}(\mathbf{a})(F)] = (F^\times)^n/F^\times$ with the unique Haar measure which makes the complex

$$1 \to \mu_{\gcd(\mathbf{a})}(F) \to F^\times \to (F^\times)^n \to [\mathscr{T}(\mathbf{a})(F)] \to 1$$

to have trivial measure Euler-Poincaré characteristics. This measure is precisely $\frac{1}{|\mu_{\gcd(\mathbf{a})}(F)|}\mathrm{count}_{[\mathscr{T}(\mathbf{a})(F)]}$. The kernel of the homomorphism $[\mathscr{T}(\mathbf{a})(i)] : [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ is the finite group $|Ш^1(F, \mu_{\gcd(\mathbf{a})})|$. We endow the discrete group $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ with the pushforward measure of the measure $\frac{1}{|\mu_{\gcd(\mathbf{a})}(F)|}\mathrm{count}_{[\mathscr{T}(\mathbf{a})(F)]}$ on $[\mathscr{T}(\mathbf{a})(F)]$. This measure is precisely the measure

$$\frac{|Ш^1(F, \mu_{\gcd(\mathbf{a})})|}{|\mu_{\gcd(\mathbf{a})}(F)|}\mathrm{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])}.$$

Definition 5.4.4.3. — *We define a Haar measure on the quotient* $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ *by*

$$\overline{\mu_1} := \mu_1 / \left(\frac{|Ш^1(F, \mu_{\gcd(\mathbf{a})})|}{|\mu_{\gcd(\mathbf{a})}(F)|}\mathrm{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])} \right).$$

We define

$$\begin{aligned}&\mathrm{Tam}(\mathscr{T}(\mathbf{a}))\\ &:= \mathrm{Res}(\zeta_F, 1)^{-(n-1)}\Delta(F)^{-\frac{n-1}{2}}\overline{\mu_1}([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])),\end{aligned}$$

where $\Delta(F)$ is the absolute discriminant of F.

Proposition 5.4.4.4. — *One has that*

$$\mathrm{Tam}(\mathscr{T}(\mathbf{a})) = 1.$$

Proof. — Consider the following bicomplex (where $\lambda_{\mathbf{a}}$ stands for the map $t \mapsto (t^{a_j})_j$, whatever the domain is; K_1, K_2 and K_3 are the corresponding kernels and E is the corresponding quotient; and every term that is not drawn is assumed to be the trivial group):

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \mu_{\gcd(\mathbf{a})}(F) \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & K_3 \arrow[d] \\
1 \arrow[r] \arrow[d] & F^\times \arrow[r] \arrow[d, "\lambda_{\mathbf{a}}"] & \mathbb{A}^1_F \arrow[r] \arrow[d, "\lambda_{\mathbf{a}}"] & (\mathbb{A}^1_F/F^\times) \arrow[d, "\lambda_{\mathbf{a}}"] \\
1 \arrow[r] \arrow[d] & (F^\times)^n \arrow[r] \arrow[d] & (\mathbb{A}^1_F)^n \arrow[r] \arrow[d] & (\mathbb{A}^1_F)^n/(F^\times)^n \arrow[d] \\
Ш^1(F, \mu_{\gcd(\mathbf{a})}) \arrow[r] & [\mathscr{T}(\mathbf{a})(F)] \arrow[r] & [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 \arrow[r] & E,
\end{tikzcd}$$

Endow every finite group in the bicomplex with the probability Haar measure. The group K_2 is compact by Lemma 3.4.8.2, we endow it with the probability Haar measure. The group K_3 is compact, as it is a closed subgroup of the compact group $\mathbb{A}^1_F/F^\times$ and we endow it with the probability Haar measure. Endow the discrete groups $F^\times$ and $(F^\times)^n$ with the counting measures. Endow $\mathbb{A}^1_F$, $(\mathbb{A}^1_F)^n$ and $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ with the measure $d^*x^1_{\mathbb{A}_F}$, $(d^*x^1_{\mathbb{A}_F})^{\otimes n}$ and μ_1, respectively. Finally, endow $(\mathbb{A}^1_F/F^\times)$, $(\mathbb{A}^1_F)^n/(F^\times)^n$ and E with the unique Haar measures so that the corresponding rows are of trivial measure Euler-Poincaré characteristics, that is $(\mathbb{A}^1_F/F^\times)$ and $(\mathbb{A}^1_F)^n/(F^\times)^n$ are endowed with the corresponding quotient measures, while E is endowed with the measure $\overline{\mu_1}$. We apply Proposition 5.1.4.2. The measure Euler-Poincaré characteristics of every row is 1. The measure Euler-Poincaré characteristics of the first three columns is 1. Proposition 5.1.4.2 gives that the measure Euler-Poincaré characteristics of the fourth column is 1. It follows from Part (2) of Lemma 5.1.4.1 that

$$\begin{aligned}\overline{\mu_1}(E) &= \overline{\mu_1}([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]))\\ &= \frac{d^*x^1_{\mathbb{A}_F}(\mathbb{A}^1_F/F^\times)^n}{d^*x^1_{\mathbb{A}_F}(\mathbb{A}^1_F/F^\times)}\\ &= d^*x^1_{\mathbb{A}_F}(\mathbb{A}^1_F/F^\times)^{n-1}\\ &= (\operatorname{Res}(\zeta_F,1)\Delta(F)^{\frac12})^{n-1},\end{aligned}$$

where we have used that $d^*x^1_{\mathbb{A}_F}(\mathbb{A}^1_F/F^\times) = \operatorname{Res}(\zeta_F,1)\Delta(F)^{\frac12}$ (see e.g. [**58**, Page 116]). We obtain that

$$\operatorname{Tam}(\mathscr{T}(\mathbf{a})) = 1.$$

□

Remark 5.4.4.5. — When $\mathbf{a} = \mathbf{1}$, the result is a classical result that the Tamagawa number of a split torus is 1 ([**46**, Theorem 3.5.1]).

Remark 5.4.4.6. — When $n = 1$, Oesterlé has calculated in [**43**, Proposition 2] the volume of the fundamental domain for the action of the subgroup $[\mathscr{T}(a)(i)([\mathscr{T}(a)(F)])]$ on $[\mathscr{T}(a)(\mathbb{A}_F)]$. The volume of the fundamental domain is not 1, because Oesterlé has used a different normalization of the Haar measure.

CHAPTER 6

ANALYSIS OF CHARACTERS OF $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$

In this chapter, we will study the characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. Later in the chapter, we recall several facts on the estimates of L-functions.

6.1. Characters of $\mathbb{A}_F^\times$

We are going to define two "norms" for Hecke characters and we are going to compare them for the characters vanishing on certain compact subgroups. Later we establish that there are only finitely many characters vanishing on such subgroup of bounded "norm". The analogues for the characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ are stated and proven in 6.2.

6.1.1. — In this paragraph we recall several facts about characters of locally compact topological groups.

If G is a locally compact topological group, by a character of G we mean a continuous homomorphism $G \to S^1$. Let G^* be the group of characters of G ([5, Chapter II, §1, n^0 1, Definition 2]). The group G^* is locally compact by [5, Chapter II, §1, n^0 1, Corollary 2]. A morphism of topological groups $\phi : G \to G'$ induces a continuous homomorphism $\phi^* : (G')^* \to G^*, \phi^*(\chi) = \chi \circ \phi$ (see [5, Chapter II, §1, n^0 7]). If A is a subset of G, by $A^\perp$ we denote the subgroup of G^* given by the characters vanishing on A.

Proposition 6.1.1.1 ([5, Chapter II, §1, n^0 7, Theorem 4])

Let G be a commutative Hausdorff locally compact group. Let $i : G_1 \to G$ be the inclusion of a closed subgroup G_1 into G. Let $G_2 = G/G_1$ and let $p : G \to G_2$ be the quotient map. In the sequence

$$G_2^* \xrightarrow{p^*} G^* \xrightarrow{i^*} G_1^*,$$

the homomorphism p^* is an isomorphism of G_2^* onto $G_1^\perp$ and i^* is a strict homomorphism $G^* \to G_1^*$ of kernel $G_1^\perp$ (it follows that $G_1^\perp$ is closed).

Occasionally, we may identify G_2^* with its image $G_1^\perp$ under the homomorphism p^*. Recall also that if $G = H_1 \times \cdots H_n$, the canonical homomorphism $G^* \to (H^1)^* \times \cdots \times (H^n)^*$ is an isomorphism of topological groups ([**5**, Chapter II, §1, n^0 7, Corollary 5]) and we may identify G^* with $(H^1)^* \times \cdots \times (H^n)^*$ using this isomorphism.

6.1.2. — We define two "norms" of Hecke characters and we compare them. For $v \in M_F^\infty$, in 5.2.1 we have defined

$$F_{v,1} = \{x | x \in F_v^\times : |x|_v = 1\}.$$

We have furthermore established that

$$\widetilde{\rho}_v : \mathbb{R}_{>0} \times F_{v,1} \xrightarrow{\sim} F_v^\times \qquad (r,z) \mapsto \rho_v(r)z,$$

where $\rho_v : \mathbb{R}_{>0} \to F_v^\times$ is defined by $\rho_v(r) = r^{1/n_v}$, is an isomorphism of abelian topological groups. For a character $\chi_v \in (F_v^\times)^*$, we set $m(\chi_v)$ to be the unique real number m such that the character $\chi_v \circ \widetilde{\rho}_v : \mathbb{R}_{>0} \to S^1$ is given by $r \mapsto r^{im}$. If v is a real place, we set $\ell(\chi_v)$ to be $0 \in \mathbb{Z}$ if the character $\chi_v \circ \widetilde{\rho}_v : F_{v,1} \to S^1$ is the trivial character, otherwise we set $\ell(\chi_v) = 1 \in \mathbb{Z}$ (the only reason why we let ℓ have values in $\mathbb{Z}$ for v real is to speak of norms of vectors). If v is a complex place, we set $\ell(\chi_v)$ to be the unique integer ℓ such that $\chi_v \circ \widetilde{\rho}_v : F_{v,1} \to S^1$ is given by $z \mapsto z^\ell$.

Let $\mathbb{A}_F^\times$ be the group of ideles of F and let $\mathbb{A}_F^1$ be the subgroup of $\mathbb{A}_F^\times$ given by $(x_v)_v$ which satisfy that $\prod_v |x_v|_v = 1$. For a character $\chi \in (\mathbb{A}_F^\times)^*$, let us define

$$||\chi||_{\text{discrete}} := \max_{v \in M_F^\infty} (||\ell(\chi_v)||),$$
$$||\chi||_\infty := \max_{v \in M_F^\infty} (||m(\chi_v)||).$$

Let $K_{\max}^0$ be the topological group $\prod_{v \in M_F^0} \mathcal{O}_v^\times$. For an open subgroup $K \subset K_{\max}^0$, we let $\mathfrak{A}_K$ be the subgroup of $(\mathbb{A}_F^1)^*$ given by the characters vanishing on $F^\times$ (technically we mean $i(F^\times)$) and on the compact subgroup $K \times \prod_{v \in M_F^\infty} \{1\} \subset (\mathbb{A}_F^1)^*$. By the abuse of notation, we may write sometimes K for what is technically $K \times \prod_{v \in M_F^\infty} \{1\}$. The group $K_{\max}^0$ is compact, therefore, the subgroup K is of finite index in $K_{\max}^0$.

The following lemma will be used on several occasions:

Lemma 6.1.2.1. — *Let G be an abelian group and let A and B be two subgroups such that $A \subset B$.*

1. *Let $H \subset G$ be a subgroup. The homomorphism $B/A \to (B+H)/(A+H)$ induced from A-invariant homomorphism $B \to (B+H)/(A+H)$ is surjective.*
2. *Suppose G is a topological group and that $(B : A)$ is finite. Then $(A^\perp : B^\perp)$ is finite and $(A^\perp : B^\perp) \le (B : A)$.*

Proof. — 1. We have the following commutative diagram, with first two horizontal and all three vertical sequences exact:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A+H \arrow[r] \arrow[d] & B+H \arrow[r] \arrow[d] & (B+H)/(A+H) \arrow[r] \arrow[d] & 0 \\
& (A+H)/A \arrow[r] & (B+H)/B \arrow[r] & E \arrow[r] & 0,
\end{tikzcd}$$

where $E = \operatorname{coker}((B/A) \to (B+H)/(A+H))$. By snake lemma, the third horizontal sequence is exact. By the second isomorphism theorem, the homomorphism $(A+H)/A \to (B+H)/B$ identifies with the homomorphism $H/(H \cap A) \to H/(H \cap B)$ induced from the inclusion $(H \cap A) \subset (H \cap B)$, hence is surjective and thus $E = 0$. It follows that $B/A \to (B+H)/(A+H)$ is surjective.

2. The kernel of the homomorphism

$$A^\perp \to \operatorname{Hom}_{\mathbb{Z}}(B, S^1) \qquad \chi \mapsto \chi|_B$$

is the subgroup $B^\perp$. We deduce an injective homomorphism

$$A^\perp / B^\perp \to \operatorname{Hom}_{\mathbb{Z}}(B, S^1). \tag{6.1.2.2}$$

The image of the homomorphism (6.1.2.2) is contained in the subgroup of $\operatorname{Hom}_{\mathbb{Z}}(B, S^1)$ given by the homomorphisms which vanish on A, i.e. in the image of the canonical homomorphism $\operatorname{Hom}_{\mathbb{Z}}(B/A, S^1) \to \operatorname{Hom}_{\mathbb{Z}}(B, S^1)$. It follows that

$$(A^\perp : B^\perp) \le |\operatorname{Hom}_{\mathbb{Z}}(B/A, S^1)| = (B : A).$$

□

Corollary 6.1.2.3. — *One has that*

$$(K^0_{\max} : K) \ge ((F^\times K^0_{\max}) : (F^\times K)) \ge (\mathfrak{A}_K : \mathfrak{A}_{K^0_{\max}}).$$

Proof. — To obtain the first inequality, we apply Lemma 6.1.2.1 for $G = \mathbb{A}^1_F$, $A = K$, $B = K^0_{\max}$ and $H = F^\times$. It follows that the homomorphism from Lemma 6.1.2.1

$$K^0_{\max}/K \to (F^\times K^0_{\max}/F^\times K)$$

is surjective, thus $(K^0_{\max} : K) \geq ((F^\times K^0_{\max}) : (F^\times K))$. The second inequality is the case of Lemma 6.1.2.1 for $G = \mathbb{A}^1_F$, $A = F^\times K$ and $B = F^\times K^0_{\max}$. □

Lemma 6.1.2.4. — *Let $K \subset K^0_{\max}$ be an open subgroup. The group $\mathfrak{A}_K$ is finitely generated and its rank is at most r_2, where r_2 is the number of complex places of F.*

Proof. — The group $\mathfrak{A}_{K^0_{\max}}$ is the group of Hecke characters $(\mathbb{A}^1_F/F^\times) \to S^1$ which are unramified at the finite places of F and we deduce that the kernel of the homomorphism

$$\phi : \mathfrak{A}_{K^0_{\max}} \to \prod_{v \in M^\infty_F} (F_{v,1})^* \qquad\qquad \chi \mapsto \prod_{v \in M^\infty_F} \chi_v|_{F_{v,1}}$$

is given by the unramified Hecke characters $(\mathbb{A}^1_F/F^\times) \to S^1$, hence is finite (they are canonically identified with the characters of $\mathbb{A}^1_F/(F^\times K_{\max})$, where the subgroup $K_{\max} \subset \mathbb{A}^1_F$ is given by the norm 1 elements at every place; the group $\mathbb{A}^1_F/(F^\times K_{\max})$ is finite by [**49**, Theorem 5-18] and hence its character group is finite). The group $(\prod_{v \in M^\infty_F} (F_{v,1})^*)$ is finitely generated and of rank r_2, because $F^*_{v,1}$ is of order 2 if v is real and is an infinite cyclic group if v is complex. It follows that

$$\operatorname{rk}(\mathfrak{A}_{K^0_{\max}}) = \operatorname{rk}(\operatorname{Im}(\phi)) \leq r_2.$$

By Corollary 6.1.2.3, one has that $\mathfrak{A}_{K^0_{\max}}$ is of finite index in $\mathfrak{A}_K$, thus one has that $\operatorname{rk}(\mathfrak{A}_K) = \operatorname{rk}(\mathfrak{A}_{K^0_{\max}}) \leq r_2$. The statement is proven. □

The following lemma will be used in the proof of Proposition 6.1.2.6.

Lemma 6.1.2.5. — *Let K be an open subgroup of $K^0_{\max}$. Consider the homomorphism*

$$\ell^{\mathbb{C}} : (\mathbb{A}^1_F/F^\times)^* \to \mathbb{Z}^{M^{\mathbb{C}}_F} \qquad\qquad \chi \mapsto (\ell(\chi_v))_{v \in M^{\mathbb{C}}_F}$$

The group $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K$ is finite.

Proof. — Firstly, let us establish that $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}}$ is finite. The group $\ker(\ell^{\mathbb{C}})$ (respectively, the group $\mathfrak{A}_{K^0_{\max}}$) is the group of Hecke characters

$(\mathbb{A}_F^1/F^\times) \to S^1$ which are unramified at the complex (respectively, at the finite) places of F. Hence, the kernel of the map

$$\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}} \to \prod_{v \in M_F^{\mathbb{R}}} (F_{v,1})^* \qquad\qquad \chi \mapsto \prod_{v \in M_F^{\mathbb{R}}} \chi_v|_{F_{v,1}}$$

is given by the unramified Hecke characters $(\mathbb{A}_F^1/F^\times) \to S^1$, therefore is finite. As $F_{v,1}^*$ is a cyclic group of order 2, the group $\prod_{v \in M_F^{\mathbb{R}}} (F_{v,1})^*$ is finite and we conclude that $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}}$ is finite. Now let us establish that $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}}$ is of finite index in $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K$. By applying snake lemma to the "snake" diagram:

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & \ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}} \arrow[r] \arrow[d] & \mathfrak{A}_{K^0_{\max}} \arrow[r, "\ell^{\mathbb{C}}"] \arrow[d] & l^{\mathbb{C}}(\mathfrak{A}_{K^0_{\max}}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K \arrow[r] \arrow[d] & \mathfrak{A}_K \arrow[r, "\ell^{\mathbb{C}}"] \arrow[d] & \ell^{\mathbb{C}}(\mathfrak{A}_K) \arrow[r] & 1 \\
 & E \arrow[r] & \mathfrak{A}_K/\mathfrak{A}_{K^0_{\max}}, & &
\end{tikzcd}$$

where $E = (\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K)/(\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}})$ we get an exact sequence

$$1 \to E \to \mathfrak{A}_K/\mathfrak{A}_{K^0_{\max}}.$$

By Corollary 6.1.2.3, one has that $\mathfrak{A}_K/\mathfrak{A}_{K^0_{\max}}$ is finite, thus $E = (\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K)/(\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}})$ is finite. Using the fact that $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_{K^0_{\max}}$ is finite, we deduce that $\ker(\ell^{\mathbb{C}}) \cap \mathfrak{A}_K$ is finite. $\square$

The main proposition of this paragraph is the following one.

Proposition 6.1.2.6. — *For every open subgroup $K \subset K^0_{\max}$, there exists a constant $C = C(K) > 0$ such that*

$$||\chi||_\infty \leq C||\chi||_{\text{discrete}}$$

for all $\chi \in \mathfrak{A}_K$.

Proof. — Let K be an open subgroup of $K^0_{\max}$. Let $\ell^{\mathbb{C}} : (\mathbb{A}_F^1/F^\times)^* \to \mathbb{Z}^{M_F^{\mathbb{C}}} = \mathbb{Z}^{r_2}$ be as in Lemma 6.1.2.5. To simplify notation, in the rest of the proof we will write $\ell^{\mathbb{C}}$ for $\ell^{\mathbb{C}}|_{\mathfrak{A}_K}$. The abelian group $\ell^{\mathbb{C}}(\mathfrak{A}_K)$ is finitely generated and free, let us pick $\chi_1, \dots, \chi_k \in \mathfrak{A}_K$ such that $\ell^{\mathbb{C}}(\chi_1), \dots, \ell^{\mathbb{C}}(\chi_k)$

is a basis of $\ell^{\mathbb{C}}(\mathfrak{A}^K)$. Obviously, $k\leq|M_F^{\mathbb{C}}|=r_2$. One has an isomorphism

$$\mathbb{Z}^k\to\ell^{\mathbb{C}}(\mathfrak{A}_K)\qquad\qquad (d_1,\dots,d_k)\mapsto d_1\ell^{\mathbb{C}}(\chi_1)+\dots+d_k\ell^{\mathbb{C}}(\chi_k).$$

One can pick a section to the surjective homomorphism $\ell^{\mathbb{C}}:\mathfrak{A}_K\to\ell^{\mathbb{C}}(\mathfrak{A}_K)$, we obtain an induced splitting

$$\ell^{\mathbb{C}}(\mathfrak{A}_K)\oplus\ker(\ell^{\mathbb{C}})\xrightarrow{\sim}\mathfrak{A}_K.$$

We deduce an isomorphism

$$\mathbb{Z}^k\oplus\ker(\ell^{\mathbb{C}})\xrightarrow{\sim}\mathfrak{A}_K. \tag{6.1.2.7}$$

A character $\chi\in\mathfrak{A}_K$ writes as $\chi=p_2(\chi)\chi_1^{q_1(p_1(\chi))}\cdots\chi_k^{q_k(p_1(\chi))}$. Let us firstly estimate $||\chi||_\infty$ for $\chi\in\mathfrak{A}_K$. The group $\ker(\ell^{\mathbb{C}})$ is finite by Lemma 6.1.2.5 and let r be its order. For every $\chi\in\ker(\ell^{\mathbb{C}})$ one has that

$$\mathbf{0}=(m(1_v))_v=(m(\chi_v^r))_v=(rm(\chi_v))_v,$$

hence $m(\chi_v)=0$ for every $v\in M_F^\infty$. Now, we can make the following estimate for every character $\chi\in\mathfrak{A}_K$:

$$\begin{aligned}||\chi||_\infty&=\max_{v\in M_F^\infty}|m(\chi_v)|\\&=\max_{v\in M_F^\infty}|m((p_2(\chi))_v)+m(\chi_{1v})q_1(p_1(\chi))+\cdots m(\chi_{kv})q_k(p_1(\chi))|\\&=\max_{v\in M_F^\infty}|m(\chi_{1v})q_1(p_1(\chi))+\cdots m(\chi_{kv})q_k(p_1(\chi))|\\&\leq\max_{v\in M_F^\infty,i=1,\dots,k}|m(\chi_{iv})|\max_{i=1,\dots,k}|q_i(p_1(\chi))|\\&=C_0\max_{i=1,\dots,k}|q_i(p_1(\chi))|,\end{aligned}$$

where we have shortened $C_0=\max_{v\in M_F^\infty,i=1,\dots,k}|m(\chi_{iv})|$. Let us now estimate $||\chi||_\infty$ for $\chi\in\mathfrak{A}_K$. For every $\chi\in\mathfrak{A}_K$ we have that:

$$\begin{aligned}||\chi||_{\text{discrete}}&=\max_{v\in M_F^\infty}|\ell(\chi_v)|\\&\geq\max_{v\in M_F^{\mathbb{C}}}|\ell(\chi_v)|\\&=\max_{v\in M_F^{\mathbb{C}}}|\ell\big((p_2(\chi))_v\chi_{1v}^{q_1(p_1(\chi))}\cdots\chi_{kv}^{q_k(p_1(\chi))}\big)|\\&=\max_{v\in M_F^{\mathbb{C}}}|\ell((p_2(\chi))_v)+q_1(p_1(\chi))\ell(\chi_{1v})+\cdots+q_k(p_1(\chi))\ell(\chi_{kv})|\\&=\max_{v\in M_F^{\mathbb{C}}}|q_1(p_1(\chi))\ell(\chi_{1v})+\cdots+q_k(p_1(\chi))\ell(\chi_{kv})|.\end{aligned}$$

Tensoring the injective map $\ell^{\mathbb{C}}|_{\mathbb{Z}^k}$ with the flat $\mathbb{Z}$-module $\mathbb{R}$ gives an injective map $\mathbb{R}^k \to \mathbb{R}^{r_2}$. The pullback of the norm $(x_v)_{v\in M_F^{\mathbb{C}}} \mapsto \max_v |x_v|$ on $\mathbb{R}^{r_2}$ along this map is given by

$$\mathbf{x} \mapsto \max_{v\in M_F^{\mathbb{C}}} |x_1\ell(\chi_{1v}) + \cdots x_k\ell(\chi_{k,v})|,$$

and is a norm on $\mathbb{R}^k$. One can find, hence, a constant $C_1 > 0$ such that

$$C_1 \max_{v\in M_F^{\mathbb{C}}} |x_1\ell(\chi_{1v}) + \cdots + x_k\ell(\chi_{k,v})| \geq \max_{i=1,\dots,k} |x_k|$$

for every $\mathbf{x} \in \mathbb{R}^k$. Thus for every $\chi \in \mathfrak{A}_K$ one has that

$$C_1||\chi||_{\text{discrete}} \geq \max_{i=1,\dots,k} |q_k(p_1(\chi))|.$$

Let us now prove the wanted inequality. We set $C = C_1C_0$. For every $\chi \in \mathfrak{A}_K$, we have that

$$C||\chi||_{\text{discrete}} \geq C_0C_1||\chi||_{\text{discrete}} \geq C_0 \max_{i=1,\dots,k} |q_i(p_1(\chi))| \geq ||\chi||_\infty.$$

The statement is proven. □

6.1.3. — In this paragraph, we bound the number of characters χ for which $||\chi||_{\text{discrete}}$ is bounded.

Lemma 6.1.3.1. — *Let $h_F = |\operatorname{Cl}(F)|$ be the class number of F. For every open subgroup $K \subset K^0_{\max}$ and any $C > 0$, there are no more than*

$$h_F(K^0_{\max} : K)2^{r_1}(2C+1)^{r_2}$$

characters $\chi \in \mathfrak{A}_K$ for which $||\chi||_{\text{discrete}} \leq C$ (recall that r_1 and r_2 are the numbers of real and complex places of F, respectively).

Proof. — Let $K \subset K^0_{\max}$ be an open subgroup. For $v \in M_F^{\mathbb{C}}$, we define $\widetilde{\ell}(\chi_v) := \ell(\chi_v)$, and for $v \in M_F^{\mathbb{R}}$ we define $\widetilde{\ell}(\chi_v) \in \mathbb{Z}/2\mathbb{Z}$ by

$$\widetilde{\ell}(\chi_v) := \begin{cases} 0 & \text{if } \chi_v|_{F_{v,1}} \text{ is the trivial character on } F_{v,1} \\ 1 & \text{otherwise.} \end{cases} \tag{6.1.3.2}$$

(The difference between ℓ and $\widetilde{\ell}$ is of technical nature, recall that we have defined $\ell(\chi_v) \in \mathbb{Z}$ for $v \in M_F^{\mathbb{R}}$.) For a character $\chi \in \mathfrak{A}_K$, the r_1+r_2-tuple given by $(\ell(\chi_v))_{v\in M_F^\infty} \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2}$ will be called the signature of χ.

Let us estimate the number of characters with fixed signature. Let $(\ell_v)_{v\in M_F^\infty} \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2}$ be a signature of some character $\delta \in \mathfrak{A}_K$. Then the characters in $\mathfrak{A}_K$ having $(\ell_v)_{v\in M_F^\infty}$ for the signature are in a bijection with the characters in $\mathfrak{A}_K$ having $(0)_{v\in M_F^\infty}$ for the signature.

Indeed, a bijection between the two sets is given by $\chi \mapsto \chi\delta^{-1}$. The group of $\chi \in \mathfrak{A}_K$ having $(0)_{v\in M_F^\infty}$ for signature is given by the subgroup $(F^\times(K \times \prod_{v\in M_F^\infty} F_{v,1}))^\perp$ of $(\mathbb{A}_F^1)^*$. Lemma 6.1.2.1 gives that

$$\begin{aligned}(K^0_{\max} : K) &\geq \Big(\big(F^\times(K^0_{\max} \times \prod_{v\in M_F^\infty} F_{v,1})\big) : \big(F^\times(K \times \prod_{v\in M_F^\infty} F_{v,1})\big)\Big)\\ &\geq \Big((F^\times(K \times \prod_{v\in M_F^\infty} F_{v,1}))^\perp : (F^\times(K^0_{\max} \times \prod_{v\in M_F^\infty} F_{v,1}))^\perp\Big).\end{aligned}$$

The group $(F^\times(K^0_{\max} \times \prod_{v\in M_F^\infty} F_{v,1}))^\perp$ is the group of the unramified Hecke characters $(\mathbb{A}_F^1/F^\times) \to S^1$ and its order is h_F. It follows that the number of characters $\chi \in \mathfrak{A}_K$ having $(0)_{v\in M_F^\infty}$ for the signature is

$$|(F^\times(K \times \prod_{v\in M_F^\infty} F_{v,1}))^\perp| \leq h_F(K^0_{\max} : K).$$

Let us estimate the number of signatures when $||\cdot||_{\text{discrete}}$ is bounded. Note that if for $C > 0$ and a character $\chi \in \mathfrak{A}_K$ one has $||\chi||_{\text{discrete}} < C$, then the signature of χ lies in $(\mathbb{Z}/2\mathbb{Z})^{r_1} \times ([-C, C]^{r_2} \cap \mathbb{Z}^{r_2})$. Thus the number of signatures can be bounded by $2^{r_1}(2C+1)^{r_2}$.

It follows that the number of characters $\chi \in \mathfrak{A}_K$ such that $||\chi||_{\text{discrete}} \leq C$ is bounded by $h_F(K^0_{\max} : K)2^{r_1}(2C+1)^{r_2}$. The statement is proven. □

6.2. Results for the characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$

Let $n \geq 1$ be an integer and let $\mathbf{a} \in \mathbb{Z}^n_{\geq 1}$. We make analogous estimates to those in 6.1 for characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. The results are simple consequences of the corresponding results in 6.1

6.2.1. — In this paragraph we explain our notation and define “norms” for the characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and compare them for characters vanishing on certain compact subgroups of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. In this paragraph, we discuss local characters.

Recall that the subgroups $(F_v^\times)_{\mathbf{a}} \subset (F_v^\times)^n$ and $(\mathbb{A}_F^\times)_{\mathbf{a}} \subset (\mathbb{A}_F^\times)^n$ are closed by Proposition 3.3.4.1 and Lemma 3.4.2.2, respectively. Recall that $[\mathscr{T}(\mathbf{a})(F_v)]$ identifies with the quotient group $(F_v^\times)^n/(F_v^\times)_{\mathbf{a}}$ by Proposition 3.3.4.1. Recall that $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ identifies with the quotient group $(\mathbb{A}_F^\times)^n/(\mathbb{A}_F^\times)_{\mathbf{a}}$ by Lemma 3.4.2.2.

Lemma 6.2.1.1. — *The subgroup $(F_v^\times)^\perp_{\mathbf{a}}$ is the subgroup*

$$\{(\chi_j)_j|\prod_j \chi_j^{a_j}=1\}$$

of $((F_v^\times)^*)^n$. Moreover, $(F_v^\times)^\perp_{\mathbf{a}}$ is closed in $((F_v^\times)^*)^n$, and it is the image of the pullback homomorphism $[\mathscr{T}(\mathbf{a})(F_v)]^*\to((F_v^\times)^n)^*$. The subgroup $(\mathbb{A}_F^\times)^\perp_{\mathbf{a}}$ is the subgroup $\{(\chi_j)_j|\prod_j\chi_j^{a_j}=1\}$ of $((\mathbb{A}_F^\times)^*)^n$. Moreover, $(\mathbb{A}_F^\times)^\perp_{\mathbf{a}}$ is closed in $((\mathbb{A}_F^\times)^*)^n$, and it is the image of the pullback homomorphism $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*\to((\mathbb{A}_F^\times)^*)^n$.

Proof. — Note that if $\chi=(\chi_1,\dots,\chi_n)$ is a character of $((F_v^\times)^*)^n$ which vanishes on $(F_v^\times)_{\mathbf{a}}=\{(t^{a_j})_j|t\in F_v^\times\}$, then for every $t\in F_v^\times$, one has that

$$1=\chi_1(t^{a_1})\cdots\chi_n(t^{a_n})=\chi_1(t)^{a_1}\cdots\chi_n(t)^{a_n}.$$

The first claim follows. The facts that $(F_v^\times)^\perp_{\mathbf{a}}$ is closed and that it is the image of the pullback homomorphism follow from Proposition 6.1.1.1. Analogous argument shows the claims about $(\mathbb{A}_F^\times)^\perp_{\mathbf{a}}$. □

Let v be a place of F. For $j\in\{1,\dots,n\}$, we let $\delta_v^j:F_v^\times\to(F_v^\times)^n$ be the inclusion

$$x\mapsto((x)_j,(1)_{i\neq j}).$$

If $\chi\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ is a character we let $\chi^{(j)}$ be the character $\chi\circ q_v^{\mathbf{a}}\circ\delta^j:F_v^\times\to S^1$. The lemma from above gives that $\prod_{j=1}^n(\chi^{(j)})^{a_j}=1$ for every $\chi\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. We define $\mathbf{m}(\chi):=(m(\chi^{(j)}))_j$ and $\boldsymbol{\ell}(\chi)=(\ell(\chi^{(j)}))_j$.

If χ is a character of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$, for $j\in\{1,\dots,n\}$, we will denote by $\delta^j_{\mathbb{A}_F}$ the inclusion $x\mapsto((x)_j,(1)_{i\neq j})$ and by $\chi^{(j)}$ the character $\chi\circ q^{\mathbf{a}}_{\mathbb{A}_F}\circ\delta^j_{\mathbb{A}_F}:\mathbb{A}_F^\times\to S^1$. The lemma from above gives that $\prod_{j=1}^n(\chi^{(j)})^{a_j}=1$ for every $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$. Moreover, it is immediate that

$$\chi|_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])}=1\implies \text{ for } j=1,\dots,n \text{ one has } \chi^{(j)}|_{i(F^\times)}=1.$$

For a character $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, we denote by χ_v the restriction $\chi|_{[\mathscr{T}(\mathbf{a})(F_v)]}$.

Remark 6.2.1.2. — Let χ be a character of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. A priori the notation $\chi_v^{(j)}$ can make confusion as it may design $(\chi^{(j)})|_{F_v^\times}$ and $(\chi|_{[\mathscr{T}(\mathbf{a})(F_v)]})^{(j)}$. The commutativity of the following diagram shows that no confusion arises:

$$\begin{tikzcd}
F_v^\times \arrow[r] \arrow[d, "\delta_v^j"'] & \mathbb{A}_F^\times \arrow[d, "\delta^j_{\mathbb{A}_F}"] \\
(F_v^\times)^n \arrow[r] \arrow[d, "q_v^{\mathbf{a}}"'] & (\mathbb{A}_F^\times)^n \arrow[d, "q^{\mathbf{a}}_{\mathbb{A}_F}"] \\
{[\mathscr{T}(\mathbf{a})(F_v)]} \arrow[r] & {[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)].}
\end{tikzcd}$$

Namely, for a character χ, the character $(\chi^{(j)})_v$ is the pullback of χ for the composite by the two vertical homomorphisms on the right and the most upper horizontal homomorphism, while $(\chi_v)^{(j)}$ is the pullback for the lowest horizontal and then two left vertical homomorphisms.

For a character $\chi \in [\mathscr{T}^{\mathbf{a}}(\mathbb{A}_F)]^*$, let us define

$$\begin{aligned}
||\chi||_{\text{discrete}} &:= \max_{v\in M_F^\infty}(||\boldsymbol{\ell}(\chi_v)||),\\
||\chi||_\infty &:= \max_{v\in M_F^\infty}(||\mathbf{m}(\chi_v)||).
\end{aligned}$$

Note that

$$\begin{aligned}
||\chi||_{\text{discrete}} = \max_{v\in M_F^\infty}(||\boldsymbol{\ell}(\chi_v)||) &= \max_{v\in M_F^\infty}\left(\max_{j=1,\dots,n}(|\ell(\chi_v^{(j)})|)\right)\\
&= \max_{j=1,\dots,n}\left(\max_{v\in M_F^\infty}|\ell(\chi_v^{(j)})|\right)\\
&= \max_{j=1,\dots,n}||\chi^{(j)}||_{\text{discrete}};
\end{aligned}$$

and analogously

$$||\chi||_\infty = \max_{j=1,\dots,n}||\chi^{(j)}||_\infty.$$

Lemma 6.2.1.3. — *Suppose that $n=1$ and $a_1 = a \in \mathbb{Z}_{\geq 1}$. For every $\chi \in [\mathscr{T}(a)(\mathbb{A}_F)]^*$, one has that $||\chi||_{\text{discrete}} \leq 1$ and that $||\chi||_\infty = 0$.*

Proof. — Let $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$. To simplify notation, in this proof we write $\widetilde{\chi}$ for $\chi^{(1)}$. For every $\chi \in [\mathscr{T}(a)(\mathbb{A}_F)]^*$, one has that $\chi^a = 1$. It follows that

$$(0)_{v\in M_F^\infty} = (m(1))_{v\in M_F^\infty} = (m((\widetilde{\chi}_v)^a))_{v\in M_F^\infty} = (am((\widetilde{\chi}_v)))_{v\in M_F^\infty},$$

hence $(m(\widetilde{\chi}_v^a))_{v\in M_F^\infty} = (0)_{v\in M_F^\infty}$, and thus $||\widetilde{\chi}||_\infty = 0$. We obtain that

$$||\chi||_\infty = ||\widetilde{\chi}||_\infty = 0.$$

We have that
$$||\chi||_{\text{discrete}} = ||\widetilde{\chi}||_{\text{discrete}} \leq \max(1, \max_{v\in M_F^{\mathbb{C}}}(\ell(\chi_v))).$$
For every $v \in M_F^{\mathbb{C}}$, one has that $0 = \ell(1) = \ell(\widetilde{\chi}^a) = a\ell(\widetilde{\chi}_v)$, hence $\ell(\widetilde{\chi}_v) = 0$. We deduce that $||\chi||_{\text{discrete}} \leq 1$. □

6.2.2. — In this paragraph we present analogous results to those of 6.1 for the characters of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$.

Let $K^{\mathbf{a}}_{\max}$ be the topological group given by
$$K^{\mathbf{a}}_{\max} := \prod_{v\in M_F^0}[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)].$$
For an open subgroup K of $K^{\mathbf{a}}_{\max}$, we let $\mathfrak{A}_K$ be the subgroup of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)_1]^*$ given by the characters vanishing on $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ and on $K\times\prod_{v\in M_F^\infty}\{1\}$ (we may simply write K for what is technically $K\times\prod_{v\in M_F^\infty}\{1\}$). The group $K^{\mathbf{a}}_{\max}$ is compact, therefore, the subgroup K is of finite index in $K^{\mathbf{a}}_{\max}$. We present another corollary of Lemma 6.1.2.1.

Corollary 6.2.2.1. — *Let $K\subset K^{\mathbf{a}}_{\max}$ be an open subgroup. One has that*
$$\begin{aligned}(K^{\mathbf{a}}_{\max}:K) &\geq ([\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])K^{\mathbf{a}}_{\max} : [\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])K)\\ &\geq (\mathfrak{A}_K : \mathfrak{A}_{K^0_{\max}}).\end{aligned}$$

Proof. — The left inequality follows from Lemma 6.1.2.1 for $G = [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$, $A = K$, $B = K^{\mathbf{a}}_{\max}$ and $H = [\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$. The right inequality follows from Lemma 6.1.2.1 for $G = [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$, $A = [\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])K$ and $B = [\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])K^{\mathbf{a}}_{\max}$. □

Note that
$$q^{\mathbf{a}}_{\mathbb{A}_F}\big((K^0_{\max})^n\big) = (q^{\mathbf{a}}_v((\mathcal{O}_v^\times)^n))_v = ([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)])_v = K^{\mathbf{a}}_{\max}.$$
If $K\subset K^{\mathbf{a}}_{\max}$ is an open subgroup, let us set
$$\widetilde{K} = \bigcap_{j=1}^n (\delta^j_{\mathbb{A}_F})^{-1}(q^{\mathbf{a}}_{\mathbb{A}_F}|_{(K^0_{\max})^n})^{-1}(K),$$
it is an open subgroup of $K^0_{\max}$. For a character $\chi\in\mathfrak{A}_K$, one has that
$$\chi^{(j)}|_{\widetilde{K}} = (\chi\circ q^{\mathbf{a}}_{\mathbb{A}_F}\circ\delta^j_{\mathbb{A}_F})|_{\widetilde{K}} = 1.$$
The following statements are simple corollaries of corresponding statements for characters of $\mathbb{A}^1_F$.

Corollary 6.2.2.2. — *The group $\mathfrak{A}_K$ is finitely generated and of rank at most nr_2.*

Proof. — Recall that by Lemma 6.1.2.4, the abelian group $\mathfrak{A}_{\widetilde{K}}$ is finitely generated and of rank at most r_2. The image of $\mathfrak{A}_K$ under the injective homomorphism

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1^* \to ((\mathbb{A}_F^1)^n)^* = ((\mathbb{A}_F^1)^*)^n \qquad \chi \mapsto \chi \circ q^{\mathbf{a}}_{\mathbb{A}_F} = (\chi^{(j)})_j$$

lies in $(\mathfrak{A}_{\widetilde{K}})^n$. It follows that $\mathfrak{A}_K$ is finitely generated and of rank at most nr_2. □

Corollary 6.2.2.3. — *For every compact group $K \subset K^{\mathbf{a}}_{\max}$, there exists a constant $C = C(K) > 0$ such that for all $\chi \in \mathfrak{A}_K$ one has that*

$$||\chi||_\infty \leq C||\chi||_{\text{discrete}}.$$

Proof. — Proposition 6.1.2.6 gives that there exists constant $C = C(\widetilde{K}) > 0$ such that for every $\chi \in \mathfrak{A}_{\widetilde{K}}$ one has that

$$||\chi||_\infty \leq C||\chi||_{\text{discrete}}.$$

For $\chi \in \mathfrak{A}_K$ we deduce that:

$$||\chi||_\infty = \max_{j=1,\dots,n} ||\chi^{(j)}||_\infty \leq C \max_{j=1,\dots,n} ||\chi^{(j)}||_0 = C||\chi||_{\text{discrete}}.$$

It follows that $C(K) = C(\widetilde{K})$ is the wanted constant. □

Corollary 6.2.2.4. — *For every open subgroup $K \subset K^{\mathbf{a}}_{\max}$ and every $C > 0$, there are no more than*

$$(h_F(K^0_{\max} : \widetilde{K})2^{r_1}(2C+1)^{r_2})^n$$

characters $\chi \in \mathfrak{A}_K$ for which $||\chi||_{\text{discrete}} \leq C$.

Proof. — Let $K \subset K^{\mathbf{a}}_{\max}$ be an open subgroup. Lemma 6.1.3.1 gives that for every $C > 0$, there are no more than

$$h_F(K^0_{\max} : \widetilde{K})2^{r_1}(2C+1)^{r_2}$$

characters in $\mathfrak{A}_{\widetilde{K}}$ having $||\chi||_{\text{discrete}} \leq C$. A character $\chi \in \mathfrak{A}_K$ is completely determined by the characters $\chi^{(j)} \in \mathfrak{A}_{\widetilde{K}}$ for $j = 1, \dots, n$. It follows that there are no more than

$$(h_F(K^0_{\max} : \widetilde{K})2^{r_1}(2C+1)^{r_2})^n$$

characters $\chi \in \mathfrak{A}_K$ having $||\chi||_{\text{discrete}} = \max_j(||\chi^{(j)}||_{\text{discrete}}) \leq C$. □

By Lemma 6.2.1.3, we have that when $n = 1$ and $a = a_1 \in \mathbb{Z}_{\geq 1}$, we have that $||\chi||_{\text{discrete}} \leq 1$. We deduce that

Corollary 6.2.2.5. — *Suppose that* $n = 1$ *and that* $a = a_1 \in \mathbb{Z}_{\geq 1}$*. For every open subgroup* $K \subset K^a_{\max}$*, the group* $\mathfrak{A}_K$ *is finite.*

6.3. Estimates of Rademacher

In this section we recall some bounds on the growth of L functions of Hecke characters in vertical strips. If $v \in M^0_F$, for $s \in \mathbb{C}$ and $\chi_v \in (F^\times_v)^*$, one defines

$$L_v(s, \chi_v) := \frac{1}{1 - |\pi_v|^s_v \chi_v(\pi_v)},$$

and one writes $\zeta_v(s)$ for $L_v(s, 1)$. For a character $\chi = (\chi_v)_v : \mathbb{A}^\times_F \to S^1$, we set

$$L(s, \chi) := \prod_{v \in M^0_F} L_v(s, \chi_v),$$

and we denote

$$\zeta_F(s) = L(s, 1).$$

6.3.1. — In this paragraph we restrict ourselves to the characters of $\mathbb{A}^1_F$. The corresponding estimates for the characters of $\mathbb{A}^\times_F$ will be established in 6.3.2.

Rademacher in [**48**, Theorem 5] establishes that the growth of the L-function of a character in vertical strip is moderate (i.e. bounded by a polynomial). As the notation there is cumbersome, let us quote the variant given as [**39**, Theorem 14.A, Chapter III] as part (1) of the next theorem. Part (2) is [**48**, Theorem 3].

Theorem 6.3.1.1. — *Let* $0 < \eta \leq \frac{1}{2}$ *and let* $\chi = \prod_v \chi_v : (\mathbb{A}^1_F / F^\times) \to S^1$ *be a Hecke character. Let* $\operatorname{cond}(\chi)$ *be the conductor ideal of* χ*. We set* $d_F = N_{F/\mathbb{Q}}(\operatorname{disc}_{F/\mathbb{Q}})$*.*

1. (Moreno, [**39**, Theorem 14.A, Chapter III]) *Suppose* χ *is not the trivial character. One has that*

$$|L(s, \chi)| \leq \zeta_F(1+\eta) \left(\frac{d_F N_{F/\mathbb{Q}}(\operatorname{cond}(\chi))}{(2\pi)^{[F:\mathbb{Q}]}} \prod_{v \in M^\infty_F} |1 + s + m(\chi_v)|^{n_v} \right)^{\frac{1+\eta-\Re(s)}{2}}$$

   in the strip $-\eta \leq \Re(s) \leq 1 + \eta$*.*

2. (Rademacher, [**48**, Theorem 3]) *One has that*

$$\left| \frac{\zeta_F(s)(1-s)}{1+s} \right| \leq 3\zeta_F(1+\eta)^{[F:\mathbb{Q}]} \left(d_F \left(\frac{|1+s|}{2\pi} \right)^{[F:\mathbb{Q}]} \right)^{\frac{1+\eta-\Re(s)}{2}}$$

in the strip $-\eta \leq \Re(s) \leq 1+\eta$.

The following proposition is a corollary of Theorem 6.3.1.1. A similar version, but only for unramified Hecke characters, has been invoked in the analysis of height zeta functions in [**2**, Theorem 3.2.3], [**17**, Corollary 4.2.3], etc. As our metrics at infinite places are not invariant for the maximal compact subgroups, we present the following version.

Proposition 6.3.1.2. — *Let* $K \subset K^0_{\max} = \prod_{v \in M^0_F} \mathcal{O}_v^\times$ *be an open subgroup. For every* $\epsilon > 0$, *there exist* $C = C(\epsilon) > 0$ *and* $\delta = \delta(\epsilon) > 0$ *such that the following conditions are satisfied whenever* $\Re(s) \geq 1-\delta$:

1. *for every non trivial Hecke character* $\chi : (\mathbb{A}^1_F/F^\times) \to S^1$ *which vanishes on* $K \subset K^0_{\max}$ *one has*

$$|L(s,\chi)| \leq C\big((1+|\Im(s)|)(1+||\chi||_\infty)\big)^\epsilon; \tag{6.3.1.3}$$

2. *one has*

$$\left|\frac{(s-1)\zeta_F(s)}{s}\right| \leq C(1+|\Im(s)|)^\epsilon. \tag{6.3.1.4}$$

Proof. — We may assume that $\frac{1}{6} > \epsilon > 0$. We set

$$C = 27\zeta_F\big(1+\frac{\epsilon}{[F:\mathbb{Q}]}\big)^{[F:\mathbb{Q}]} d_F^{\frac{1}{[F:\mathbb{Q}]}} (K^0_{\max}:K)^{\frac{1}{[F:\mathbb{Q}]}}.$$

We are going to verify that C and $\delta = \frac{\epsilon}{2[F:\mathbb{Q}]}$ verify the above conditions.

Let us prove the estimates (6.3.1.3) and (6.3.1.4) in the domain $\Re(s) > \frac{4}{3}$. For a non-trivial character χ one can estimate

$$|L(s,\chi)| \leq \zeta_F(\Re(s)) \leq \zeta_F\big(\frac{4}{3}\big) \leq \zeta_F\big(1+\frac{\epsilon}{[F:\mathbb{Q}]}\big) \leq C.$$

One also has

$$\left|\frac{(s-1)L(s,1)}{s}\right| \leq \left|1-\frac{1}{s}\right|\zeta_F\left(\frac{4}{3}\right) \leq \frac{7}{4}\zeta_F\left(\frac{4}{3}\right) \leq \frac{7}{4}\zeta_F(1+\frac{\epsilon}{[F:\mathbb{Q}]}) \leq C.$$

It follows that the estimates (6.3.2.2) and (6.3.2.3) are satisfied in the domain $\Re(s) > \frac{4}{3}$.

Now we prove the estimates (6.3.1.3) and (6.3.1.4) in the domain $1 - \frac{1}{2[F:\mathbb{Q}]}\epsilon < \Re(s) < \frac{4}{3}$. Let us set $\eta(s) = \frac{1}{[F:\mathbb{Q}]}\epsilon + \Re(s) - 1$, we have that

$$0 < \frac{\epsilon}{2[F:\mathbb{Q}]} < \eta(s) < \frac{1}{6[F:\mathbb{Q}]} + \frac{4}{3} - 1 \leq \frac{1}{2}$$

and that $-\eta(s) \leq 0 \leq \Re(s) < 1+\eta(s)$. We will apply Theorem 6.3.1.1 for s and $\eta = \eta(s)$. The following estimate will be used: for every s in the domain $1 - \frac{1}{2[F:\mathbb{Q}]} < \Re(s) < \frac{4}{3}$ one has

$$1 + |s| \leq 1 + |\Re(s)| + |\Im(s)| < 3(1 + |\Im(s)|). \tag{6.3.1.5}$$

Let us firstly prove the estimate for the non-trivial characters. Using the first part of Theorem 6.3.1.1, we deduce that for every $\chi \neq 1$ in the domain $1 - \frac{\epsilon}{2[F:\mathbb{Q}]} < \Re(s) < \frac{4}{3}$ one has

$$\begin{aligned}
&|L(s,\chi)| \\
&\leq |\zeta_F(1+\eta(s))|^{[F:\mathbb{Q}]}\Bigg(\frac{d_F N_{F/\mathbb{Q}}(\mathrm{cond}(\chi))}{(2\pi)^{[F:\mathbb{Q}]}}\prod_{v\in M_F^\infty}|1+s+m(\chi_v)|^{n_v}\Bigg)^{\frac{\epsilon}{2[F:\mathbb{Q}]}} \\
&\leq \zeta_F(1+\frac{\epsilon}{2[F:\mathbb{Q}]})^{[F:\mathbb{Q}]}\times \\
&\qquad \times\Bigg(d_F N_{F/\mathbb{Q}}(\mathrm{cond}(\chi))\prod_{v\in M_F^\infty}(1+|s|)^2(1+|m(\chi_v)|)^2\Bigg)^{\frac{\epsilon}{2[F:\mathbb{Q}]}} \\
&\leq \zeta_F(1+\frac{\epsilon}{2[F:\mathbb{Q}]})^{[F:\mathbb{Q}]}\times \\
&\qquad \times\Bigg(d_F^{1/2} N_{F/\mathbb{Q}}(\mathrm{cond}(\chi))^{1/2}\prod_{v\in M_F^\infty}3(1+|\Im(s)|)(1+|m(\chi_v)|)\Bigg)^{\frac{\epsilon}{[F:\mathbb{Q}]}} \\
&\leq \zeta_F(1+\frac{\epsilon}{2[F:\mathbb{Q}]})^{[F:\mathbb{Q}]}\times \\
&\qquad \times\big((d_F N_{F/\mathbb{Q}}(\mathrm{cond}(\chi)))^{\frac{1}{2[F:\mathbb{Q}]}}3((1+|\Im(s)|)(1+||\chi||_\infty))\big)^{\epsilon}.
\end{aligned}$$

Moreover, as χ_v vanishes at K_v, we have

$$N_{F/\mathbb{Q}}(\mathrm{cond}(\chi)) \leq \prod_{v\in M_F^0}(\mathcal{O}_v^\times : K_v) \leq (K^0_{\max} : K),$$

where K_v is the image in $\mathcal{O}_v^\times$ of the v-adic projection of K. The inequality (6.3.1.3) now follows from the observation that

$$\begin{aligned}
&\zeta_F(1+\frac{\epsilon}{2[F:\mathbb{Q}]})^{[F:\mathbb{Q}]}\big((d_F\cdot(K:K^0_{\max}))^{\frac{1}{2[F:\mathbb{Q}]}}3\big)^{\epsilon} \\
&\qquad\qquad \leq 27\zeta_F\big(1+\frac{\epsilon}{[F:\mathbb{Q}]}\big)^{[F:\mathbb{Q}]}d_F^{\frac{1}{[F:\mathbb{Q}]}}(K^0_{\max}:K)^{\frac{1}{[F:\mathbb{Q}]}} = C.
\end{aligned}$$

Let us now consider the trivial character. When $\Re(s) > 1 - \frac{\epsilon}{2[F:\mathbb{Q}]} > \frac{1}{2}$, one has that

$$\left|\frac{3s}{s-1}\right| \geq \frac{|s|+1}{|s-1|} \geq \left|\frac{s+1}{s-1}\right|. \tag{6.3.1.6}$$

Using the second part of Theorem 6.3.1.1 and the inequality (6.3.1.6), we deduce that

$$\begin{aligned}\left|\frac{(s-1)\zeta_F(s)}{s}\right| &\leq 9\zeta_F(1+\frac{\epsilon}{2[F:\mathbb{Q}]})^{[F:\mathbb{Q}]}\frac{d_F^{\epsilon/[F:\mathbb{Q}]}3^{\epsilon}(1+||\Im(s)||)^{\epsilon}}{(2\pi)^{\epsilon}}\\ &\leq C(1+|\Im(s)|)^{\epsilon}.\end{aligned}$$

The proposition is proven. □

6.3.2. — We will now present a bound on the growth in the vertical strips of the L-function of a general Hecke character $\chi : (\mathbb{A}_F^{\times}/F^{\times}) \to S^1$. In the equality (3.4.9.1), we have established an identification:

$$\mathbb{A}_F^1 \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{A}_F^{\times}.$$

For a character $\chi \in (\mathbb{R}_{>0})^*$ we denote by $m(\chi)$ the unique real number m such that $\chi(x) = x^{im}$ for every $x \in \mathbb{R}_{>0}$. For a character $\chi \in (\mathbb{A}_F^{\times})^*$ we denote by χ_0 the restriction $\chi|_{\mathbb{A}_F^1}$ and we write $m(\chi)$ for $m(\chi|_{\mathbb{R}_{>0}})$ so that $\chi = \chi_0|\cdot|^{im(\chi)}$.

For a character $\chi : \mathbb{A}_F^{\times} \to S^1$, one has that

$$\begin{aligned}L(s,\chi) = L(s,\chi_0|\cdot|^{im(\chi)}) &= \prod_{v\in M_F^0} L_v(s,(\chi_0)_v|\cdot|_v^{im(\chi)})\\ &= \prod_{v\in M_F^0}\frac{1}{1-|\pi_v|_v^s(\chi_0)_v(\pi_v)|\pi_v|_v^{im(\chi)}}\\ &= \prod_{v\in M_F^0}\frac{1}{1-|\pi_v|_v^{s+im(\chi)}(\chi_0)_v(\pi_v)}\\ &= L(s+im(\chi),\chi_0).\end{aligned}$$

The following proposition deduces easily from Proposition 6.3.1.2.

Corollary 6.3.2.1. — *Let $K \subset K^0_{\max}$ be an open subgroup. For every $\epsilon > 0$, there exist $C = C(\epsilon) > 0$ and $\delta = \delta(\epsilon) > 0$ such that the following conditions are satisfied if provided that $\Re(s) \geq 1-\delta$:*

1. *for every non trivial Hecke character* $\chi : (\mathbb{A}_F^\times/F^\times) \to S^1$ *with* $\chi_0 \neq 1$ *which vanishes on* $K \subset K^0_{\max}$ *one has*

$$|L(s,\chi)| \leq C\big((1+|\Im(s)|)(1+||\chi_0||_\infty)(1+|m(\chi)|)\big)^\epsilon; \tag{6.3.2.2}$$

2. *for every Hecke character* $\chi : (\mathbb{A}_F^\times/F^\times) \to S^1$ *with* $\chi_0 = 1$ *one has that*

$$\left|\frac{(s+im(\chi)-1)L(s,\chi)}{s+im(\chi)}\right| \leq C((1+|\Im(s)|)(1+|m(\chi)|)^\epsilon. \tag{6.3.2.3}$$

Proof. — Let $\epsilon > 0$ and let $C = C(\epsilon)$ and $\delta = \delta(\epsilon) > 0$ be given by Proposition 6.3.1.2.

1. Let $\chi : \mathbb{A}_F^\times \to S^1$ be a Hecke character which vanishes on K such that $\chi_0 \neq 1$. Then χ_0 vanishes on K and Proposition 6.3.1.2 gives that

$$\begin{aligned} L(s,\chi) = L(s+im(\chi),\chi_0) &\leq C((1+|\Im(s)+m(\chi)|)(1+||\chi_0||_\infty))^\epsilon \\ &\leq C((1+|\Im(s)|)(1+|m(\chi)|)(1+||\chi_0||_\infty))^\epsilon. \end{aligned}$$

2. Let $\chi : \mathbb{A}_F^\times \to S^1$ be a Hecke character with $\chi_0 = 1$. Proposition 6.3.1.2 gives that

$$\begin{aligned} \left|\frac{(s+im(\chi)-1)L(s,\chi)}{s+im(\chi)}\right| &= \left|\frac{(s+im(\chi)-1)L(s+im(\chi),1)}{s+im(\chi)}\right| \\ &= \left|\frac{(s+im(\chi)-1)\zeta_F(s+im(\chi))}{s+im(\chi)}\right| \\ &\leq C((1+|\Im(s)+m(\chi)|))^\epsilon \\ &\leq C((1+|\Im(s)|)(1+|m(\chi)|))^\epsilon. \end{aligned}$$

□

CHAPTER 7

FOURIER TRANSFORM OF THE HEIGHT FUNCTION

In this chapter we analyse the Fourier transform of the height function, when the functions f_v are smooth. Let n be a positive integer and let $\mathbf{a}\in\mathbb{Z}^n_{>0}$ if $n\geq 2$ and $a=a_1\in\mathbb{Z}_{>1}$ if $n=1$. As before, we use notation $f_v^{\#}$ for the toric $\mathbf{a}$-homogenous function $F_v^n-\{0\}\to\mathbb{R}_{>0}$ of weighted degree $|\mathbf{a}|$. For $v\in M_F^0$, we have established in Lemma 4.4.3.1 that $f_v^{\#}$ is locally constant i.e. smooth. Let $(f_v:F_v^n-\{0\}\to\mathbb{R}_{>0})_v$ be a degree $|\mathbf{a}|$ quasi-toric $\mathbf{a}$-homogenous family of *smooth* functions. Let S be the union of the set consisting of the finite places v at which f_v is not toric and the set of the infinite places. Let $H=H((f_v)_v)$ be the corresponding height on $[\mathscr{P}(\mathbf{a})(F)]$. If $v\in M_F$, for a character $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ and $j\in\{1,\dots,n\}$, we denote by $\chi_v^{(j)}$ the character $F_v^{\times}\to S^1$ given by

$$x\mapsto \chi_v(q_v^{\mathbf{a}}((1)_{\substack{k=1,\dots,n,\\ k\neq j}},(x)_k)).$$

7.1. Local transform

In this section we study local Fourier transform.

7.1.1. — In the first paragraph we recall several facts from abstract harmonic analysis.

Let G be an abelian locally compact group and let dg be a Haar measure on G. For $f\in L^1(G)$ and $\chi\in G^*$, we denote the Fourier transform of f by

$$\widehat{f}(\chi):=\int_G (f\chi)(dg)^*.$$

By $(dg)^*$ or by dg^* we denote the dual measure on the character group G^* (see [5, Chapter II, §1, n^0 3, Definition 4]). It is characterized by the following property: it is the unique Haar measure on G^* such that *Fourier inversion formula* ([5, Chapter II, §1, n^0 4, Proposition 4]) is valid

$$f(x) = \int_{G^*} \overline{\chi(g)}\widehat{f}(\chi)(dg^*(\chi)) \tag{7.1.1.1}$$

for every $x \in G$.

Lemma 7.1.1.2 **(**[5, Chapter II, §1, n^0 8, Proposition 9]**)**

Let H be a closed subgroup of G and let dh be a Haar measure on H. The dual measure $(dg/dh)^*$ on $(G/H)^* = H^\perp$ of the measure dg/dh on B/A satisfies that

$$(dg)^*/((dg/dh)^*) = (dh)^*.$$

Lemma 7.1.1.3 **(**[5, Chapter II, §1, n^0 9, Proposition 11]**)**

The group G^* is compact if and only if G is discrete. If dg is the counting measure on G, then $(dg)^*$ is normalized by $(dg)^*(G^*) = 1$.

Proposition 7.1.1.4 **(**[5, Chapter II, §1, n^0 8, Proposition 8]**)**

Let H be a closed subgroup of an abelian locally compact group G. Let dh and dg be Haar measures on H and G, respectively. Let $f \in L^1(G)$. We suppose that

1. *the restriction of the Fourier transform $\widehat{f}|_{H^\perp}$ is an element of $L^1(H^\perp) = L^1((G/H)^*)$,*
2. *for every $x \in G$, one has that $(h \mapsto f(xh)) \in L^1(H)$,*
3. *the function $x \mapsto \int_H f(xh)dh$ is a continuous function $G \to \mathbb{C}$.*

Then Poisson formula is valid:

$$\int_H f(h)dh = \int_{H^\perp} (\widehat{f}\,)(dg/dh)^*.$$

7.1.2. — In this paragraph we define height pairing.

For $v \in M_F$, $\mathbf{s} \in \mathbb{C}^n$, $t \in F_v^\times$ and $\mathbf{x} \in (F_v^\times)^n$ one has that

$$f_v(t\cdot\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |t^{a_j}x_j|_v^{-s_j} = |t|_v^{\mathbf{a}\cdot\mathbf{s}} f_v(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |t|_v^{-a_js_j}|x_j|_v^{-s_j}$$
$$= f_v(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{-s_j}$$

i.e. for $v \in M_F$ and $\mathbf{s} \in \mathbb{C}^n$, the continuous function

$$(F_v^\times)^n \to \mathbb{C}, \qquad \mathbf{x} \mapsto f_v(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{-s_j} \tag{7.1.2.1}$$

is $(F_v^\times)_{\mathbf{a}}$-invariant. Let $H_v(\mathbf{s}, -) : [\mathscr{T}(\mathbf{a})(F)] \to \mathbb{C}$ be the function induced from $(F_v)_{\mathbf{a}}$-invariant function (7.1.2.1). For $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$, we write $H_v(\mathbf{s}, \mathbf{x})$ for what is technically $H_v(\mathbf{s}, [\mathscr{T}(\mathbf{a})(i_v)](\mathbf{x}))$, where $[\mathscr{T}(\mathbf{a})(i_v)] : [\mathscr{T}(\mathbf{a})(i)(F)] \to [\mathscr{T}(\mathbf{a})(i)(F_v)]$ is the induced homomorphism from $(F^\times)_{\mathbf{a}}$-invariant homomorphism

$$(F^\times)^n \hookrightarrow (F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)].$$

Lemma 7.1.2.2. — *Let $\mathbf{s} \in \mathbb{C}^n$.*

1. *Suppose that $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$. One has that $H_v^{\#}(\mathbf{s}, \mathbf{x}) = 1$.*
2. *Let $(\mathbf{x}_v)_v \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$. The product*

$$H(\mathbf{s}, \mathbf{x}) := \prod_{v \in M_F} H_v(\mathbf{s}, \mathbf{x})$$

   is a finite product.

3. *Let $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F)]$. One has that*

$$H(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} = H(\mathbf{s}, [\mathscr{T}(\mathbf{a})(i)](\mathbf{x}))$$

   (recall that $[\mathscr{T}(\mathbf{a})(i)] : [\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is the map induced from $(F^\times)_{\mathbf{a}}$-invariant map

$$F^{\times n} \to (\mathbb{A}_F^\times)^n \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)],$$

   where the first map is the diagonal inclusion and the second map is the quotient map).

Proof. — 1. Let $\widetilde{\mathbf{x}} \in (\mathcal{O}_v^\times)^n$ be a lift of $\mathbf{x}$. One has that $(\mathcal{O}_v^\times)^n \subset (\mathcal{O}_v^n - \prod_{j=1}^n \pi_v^{a_j}\mathcal{O}_v)$ and thus by Lemma 4.4.3.1, one has $f_v^{\#}|_{(\mathcal{O}_v^\times)^n} = 1$. We deduce that

$$H_v^{\#}(\mathbf{x}) = f_v^{\#}(\widetilde{\mathbf{x}})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{x}_j|_v^{-s_j} = 1.$$

2. By definition of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ for almost every $v \in M_F$ one has that $\mathbf{x}_v \in [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$. For almost every v, hence, one has that $H_v(\mathbf{s}, \mathbf{x}_v) = H_v^{\#}(\mathbf{s}, \mathbf{x}_v) = 1$. Thus the product defining $H(\mathbf{s}, \mathbf{x})$ is a finite product.

3. Let $\widetilde{\mathbf{x}} \in F^{\times n}$ be a lift of $\mathbf{x}$. Recall from Lemma 3.4.2.3 that v-th coordinate of $[\mathscr{T}(\mathbf{a})(i)](\mathbf{x})$ is $[\mathscr{T}(\mathbf{a})(i_v)](\mathbf{x})$. Using the product formula, we get that

$$\begin{aligned} H(\mathbf{s}, [\mathscr{T}(\mathbf{a})(i)](\mathbf{x})) &= \prod_v H_v(\mathbf{s}, [\mathscr{T}(\mathbf{a})(i_v)](\mathbf{x})) \\ &= \prod_v \left(f_v(\widetilde{\mathbf{x}})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{x}_j|_v^{-s_j} \right) \\ &= \left(\prod_v f_v(\widetilde{\mathbf{x}})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \right) \prod_v \prod_{j=1}^n |\widetilde{x}_j|_v^{-s_j} \\ &= \prod_v f_v^{\#}(\widetilde{\mathbf{x}})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \\ &= H(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}. \end{aligned}$$

□

We may write $H(\mathbf{s}, \mathbf{x})$ for what is technically $H(\mathbf{s}, [\mathscr{T}(\mathbf{a})(i)](\mathbf{x}))$.

7.1.3. — In this paragraph we establish that the functions $H_v^{-1}(\mathbf{s}, -)$ are integrable and that their Fourier transforms are holomorphic and bounded in $\mathbf{s}$.

Let $v \in M_F$. In Definition 5.4.1.1, we have defined a Haar measure on $[\mathscr{T}(\mathbf{a})(F_v)] = (F_v^\times)^n/(F_v^\times)_\mathbf{a}$ by $(d^*x_1 \dots d^*x_n)/d^*x$. If $v \in M_F^0$, we have established in Lemma 5.4.2.1 that $\mu_v([\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]) = \zeta_v(1)^{-n+1}$. For $\mathbf{s} \in \mathbb{C}^n$ and a character $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ we define formally

$$\widehat{H}_v(\mathbf{s}, \chi_v) := \begin{cases} \zeta_v(1)^{n-1} \int_{[\mathscr{T}(\mathbf{a})(F_v)]} H_v(\mathbf{s}, -)^{-1}\chi_v\mu_v & \text{if } v \in M_F^0, \\ \int_{[\mathscr{T}(\mathbf{a})(F_v)]} H_v(\mathbf{s}, -)^{-1}\chi_v\mu_v & \text{if } v \in M_F^\infty. \end{cases}$$

In this paragraph, we are going to prove that this integral converges absolutely when $\mathbf{s} \in \Omega_{>0}$ and that it is a holomorphic function of $\mathbf{s}$ in this domain. The following result, in a bit weaker form, has been given as Lemma 8.3 in [18]

Lemma 7.1.3.1. — *Let $B > 0$. For every $\epsilon > 0$, the integral*

$$\int_{\{x \in F_v \mid |x|_v \le B\}} |x|_v^{s-1} dx_v \tag{7.1.3.2}$$

converges absolutely and uniformly in the domain $s \in \mathbb{R}_{>\epsilon} + i\mathbb{R}$. The function that associates to s the value of the integral (7.1.3.2) is holomorphic in the domain $\mathbb{R}_{>0} + i\mathbb{R}$.

Proof. — Suppose $v \in M_F^0$. Let r be the largest integer satisfying that $(|\pi_v|_v^{-1})^r \leq B$. For every $\epsilon > 0$ and every $s \in \mathbb{R}_{>\epsilon} + i\mathbb{R}$, we have that

$$\begin{aligned}
\int_{\{x \in F_v |\ |x|_v \leq B\}} |x|_v^{s-1} dx_v &= \sum_{k=-\infty}^{r} \int_{|x|_v = (|\pi_v|_v^{-1})^k} |x|_v^{s-1} dx \\
&= \sum_{k=-\infty}^{r} \int_{|x|_v = (|\pi_v|_v^{-1})^k} |\pi_v|_v^{-k(s-1)} dx \\
&= \sum_{k=-\infty}^{r} |\pi_v|_v^{-k(s-1)} dx(\pi_v^{-k} \mathcal{O}_v) \\
&= \sum_{k=-\infty}^{r} |\pi_v|_v^{-ks} \\
&= \sum_{k=-r}^{\infty} |\pi_v|_v^{ks} \\
&= |\pi_v|_v^{-rs} \sum_{k=0}^{\infty} |\pi_v|_v^{ks}.
\end{aligned}$$

The last series converges absolutely and uniformly in the domain $\mathbb{R}_{>\epsilon} + i\mathbb{R}$. Moreover, $s \mapsto |\pi_v|_v^{-rs} \sum_{k=0}^{\infty} |\pi_v|_v^{ks} = \frac{|\pi_v|_v^{-rs}}{1 - |\pi_v|_v^s}$ is a holomorphic function in the domain $\mathbb{R}_{>0} + i\mathbb{R}$. Suppose v is a real place. For every $\epsilon > 0$, we have that

$$\int_{|x|_v \leq B} |x|_v^{s-1} dx_v = \int_{|x| \leq B} |x|^{s-1} dx = 2 \int_0^B x^{s-1} dx$$

converges absolutely and uniformly for $s \in \mathbb{R}_{>\epsilon} + i\mathbb{R}$. Moreover,

$$s \mapsto 2 \int_0^B x^{s-1} dx = 2 \frac{x^s}{s} \bigg|_{x=0}^{x=B} = 2 \frac{B^s}{s}$$

is a holomorphic function in s in the domain $\mathbb{R}_{>0}+i\mathbb{R}$. Suppose v is a complex place. For every $\epsilon>0$, we have that

$$\begin{aligned}\int_{|x|_v\leq B}|x|_v^{s-1}dx_v &= \int_{x^2+y^2\leq B}(x^2+y^2)^{s-1}2dxdy\\ &= \int_0^{2\pi}\int_{r^2\leq B} r^{2(s-1)}2rdrd\phi\\ &= 4\pi\int_0^{\sqrt{B}} r^{2s-1}dr\end{aligned}$$

converges absolutely and uniformly for $s\in\mathbb{R}_{>\epsilon}+i\mathbb{R}$. Moreover,

$$s\mapsto 4\pi\int_0^{\sqrt{B}}r^{2s-1}dr=4\pi\frac{r^{2s}}{2s}\Big|_{r=0}^{\sqrt{B}}=\frac{2\pi B^s}{s}$$

is a holomorphic function in s in the domain $\mathbb{R}_{>0}+i\mathbb{R}$. The statement is proven. □

For $\mathbf{y}\in\mathbb{R}^n$, we define

$$\Omega_{>\mathbf{y}}:=\{\mathbf{s}\in\mathbb{C}^n|\Re(\mathbf{s})>\mathbf{y}\}.$$

Proposition 7.1.3.3. — *For every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$, the integral defining $\widehat{H}_v(\mathbf{s},\chi_v)$ converges absolutely in the domain $\mathbf{s}\in\Omega_{>0}$. Moreover, for every compact $\mathcal{K}\subset\Omega_{>0}$, there exists $C(\mathcal{K})>0$ such that for every $\mathbf{s}\in\mathcal{K}$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$, one has that*

$$|\widehat{H}_v(\mathbf{s},\chi_v)|\leq C(\mathcal{K}).$$

Proof. — As our characters are assumed unitary (that is with the values in S^1), by the triangle inequality, it suffices to prove the statement when $\chi_v=1$. Let $\mathcal{K}\subset\Omega_{>0}$ be a compact. Let ω_v be the quotient measure $f_v^{-1}dx_1\dots dx_n/d^*x$ on $(F_v^n-\{0\})/F_v^\times=[\mathscr{P}(\mathbf{a})(F_v)]$ (see Definition 5.2.4.1). By Lemma 5.4.1.2, one has an inequality of the measures $H_v(\mathbf{1},-)\omega_v|_{[\mathscr{T}(\mathbf{a})(F_v)]}=\mu_v$. We deduce that $H_v(\mathbf{s},-)^{-1}\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\mu_v)$ if and only if $H_v(\mathbf{s},-)^{-1}\in L^1([\mathscr{T}(\mathbf{a})(F_v)],H_v(\mathbf{1},-)\omega_v)$, i.e. if and only if

$$H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\omega_v).$$

Moreover, if $H_v(\mathbf{s},-)^{-1}\in L^1([\mathscr{T}(\mathbf{a})(F_v)],\mu_v)$, then

$$\int_{[\mathscr{T}(\mathbf{a})(F_v)]} H_v(\mathbf{s},-)^{-1}\mu_v = \int_{[\mathscr{T}(\mathbf{a})(F_v)]} H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\omega_v$$
$$= \int_{[\mathscr{P}(\mathbf{a})(F_v)]} H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\omega_v,$$

where the last equality follows from the fact that $\omega_v([\mathscr{P}(\mathbf{a})(F_v)]-[\mathscr{T}(\mathbf{a})(F_v)])=0$, which we have established in Lemma 5.2.4.2. In 5.2.2, we have defined a compactly supported function $k_v^{\mathbf{a}}:F_v^n-\{0\}\to\mathbb{R}_{\geq 0}$ which satisfies that for every $\mathbf{x}\in F_v^n-\{0\}$ one has that $\int_{F_v^\times}k_v^{\mathbf{a}}(t\cdot\mathbf{x})d^*t=1$. Proposition 5.1.2.5 gives that $H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\in L^1([\mathscr{P}(\mathbf{a})(F_v)],\omega_v)$ if and only if

$$\begin{aligned}&((H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-))\circ q_v^{\mathbf{a}})\cdot k_v^{\mathbf{a}}\\ &\qquad= \Big(f_v(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}\prod_{j=1}^n|x_j|_v^{s_j}\cdot f_v(\mathbf{x})\prod_{j=1}^n|x_j|^{-1}\Big)\cdot k_v^{\mathbf{a}}\\ &\qquad= f_v(\mathbf{x})^{1-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}\prod_{j=1}^n|x_j|_v^{s_j-1}\cdot k_v^{\mathbf{a}}\\ &\qquad\in L^1(F_v^n-\{0\},dx_1\dots dx_n),\end{aligned}$$

and that if $H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\in L^1([\mathscr{P}(\mathbf{a})(F_v)],\omega_v)$, then

$$\begin{aligned}&\int_{[\mathscr{P}(\mathbf{a})(F_v)]}H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\omega_v\\ &\qquad=\int_{F_v^n-\{0\}}f_v(\mathbf{x})^{1-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}\prod_{j=1}^n|x_j|_v^{s_j-1}\cdot k_v^{\mathbf{a}}\cdot f_v^{-1}dx_1\dots dx_n\\ &\qquad=\int_{\operatorname{supp}(k_v^{\mathbf{a}})}f_v(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}\prod_{j=1}^n|x_j|_v^{s_j-1}\cdot k_v^{\mathbf{a}}dx_1\dots dx_n.\end{aligned}$$

For every $\mathbf{s}\in\mathcal{K}$, the function $\mathbf{x}\mapsto f_v(\mathbf{x})^{-\frac{\mathbf{a}\cdot\Re(\mathbf{s})}{|\mathbf{a}|}}k_v^{\mathbf{a}}(\mathbf{x})$ is non vanishing and continuous, moreover it can be uniformly bounded for $\mathbf{s}\in\mathcal{K}$ and $\mathbf{x}\in\operatorname{supp}(k_v^{\mathbf{a}})$. Moreover, as $k_v^{\mathbf{a}}$ is compactly supported, there exists $B>0$, such that

$$\operatorname{supp}(k_v^{\mathbf{a}})\subset\{\forall j:|x_j|_v\leq B\}.$$

It follows from Lemma 7.1.3.1 that the integral

$$\int_{|x|_v\leq B}\prod_{j=1}^{n}|x_j|_v^{s_j-1}dx_1\dots dx_n=\prod_{j=1}^{n}\left(\int_{|x|_v\leq B}|x|_v^{s_j-1}dx\right)$$

converges absolutely and uniformly for $\Re(\mathbf{s})\in\mathcal{K}$. Hence,

$$\begin{aligned}\int_{\mathrm{supp}(k_v^{\mathbf{a}})} f_v(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}\prod_{j=1}^{n}|x_j|_v^{s_j-1}\cdot k_v^{\mathbf{a}}dx_1\dots dx_n \\ =\int_{[\mathscr{P}(\mathbf{a})(F_v)]}H_v(\mathbf{s},-)^{-1}H_v(\mathbf{1},-)\omega_v \\ =\int_{[\mathscr{T}(\mathbf{a})(F_v)]}H_v(\mathbf{s},-)^{-1}\mu_v\end{aligned}$$

converges absolutely and uniformly for $\Re(\mathbf{s})\in\mathcal{K}$. The statement is proven □

Corollary 7.1.3.4. — *The function $\mathbf{s}\mapsto\widehat{H}_v(\mathbf{s},\chi_v)$ is holomorphic.*

Proof. — We apply the Morera's criterion. Let $\Delta\subset\Omega_{>0}$ be a triangle. By Proposition 7.1.3.3, the function $H_v(\mathbf{s},-)^{-1}$ is absolutely μ_v-integrable and the function $\widehat{H}_v(-,-)$ can be uniformly bounded on $\Delta\times[\mathscr{T}(\mathbf{a})(F_v)]^*$, we deduce that

$$\int_{\Delta}\widehat{H}_v(\mathbf{s},\chi_v)d\mathbf{s}=\int_{\Delta}\zeta_v(1)^{n-1}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}H_v(\mathbf{s},-)^{-1}\chi_v\mu_v d\mathbf{s},$$

where if $v\in M_F^{\infty}$ one sets $\zeta_v(1)=1$, converges absolutely. By using Fubini's theorem, we get that

$$\int_{\Delta}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}H_v(\mathbf{s},-)^{-1}\chi_v\mu_v d\mathbf{s}=\int_{[\mathscr{T}(\mathbf{a})(F_v)]}\int_{\Delta}H_v(\mathbf{s},-)^{-1}\chi_v d\mathbf{s}\mu_v=0.$$

By Morera's criterion, $\mathbf{s}\mapsto\widehat{H}_v(\mathbf{s},\chi_v)$ is holomorphic. □

7.2. Calculations in non-archimedean case

We establish some properties of the local transform in the non-archimedean case. Firstly we treat the case when f_v is the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ and give the exact value of the integral in Lemma 7.2.1.1.

7.2.1. — In this paragraph we calculate the Fourier transform at the finite places v when $f_v = f_v^{\#}$ is toric.

Let $v \in M_F^0$. Let $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$ (see Definition 4.4.3.2). Recall from Lemma 4.4.3.1, that $f_v^{\#}|_{\mathcal{D}_v^{\mathbf{a}}} = 1$, where $\mathcal{D}_v^{\mathbf{a}} = \mathcal{O}_v^n - \prod_{j=1}^n \pi_v^{a_j}\mathcal{O}_v$.

When $v \in M_F^0$, we observe that

$$\chi_v^{(j)}|_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} = 1 \implies \chi_v^{(j)}|_{\mathcal{O}_v^\times} = 1 \text{ for } j = 1, \ldots, n.$$

Lemma 7.2.1.1. — *Let $v \in M_F^0$ and let $f_v^{\#} : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be the toric $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. Let $\mathbf{s} \in \Omega_{>0}$ and let $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ be a character. We have that*

$$\widehat{H}_v^{\#}(\mathbf{s}, \chi_v) := \begin{cases} \dfrac{\prod_{j=1}^n L_v(s_j, \chi_v^{(j)})}{\zeta_v(\mathbf{a}\cdot\mathbf{s})} & \text{if } \chi_v|_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.1.2}$$

Proof. — By applying Lemma 5.4.1.3, we have that

$$\begin{aligned} &\int_{[\mathscr{T}(\mathbf{a})(F_v)]} H_v^{\#}(\mathbf{s}, -)^{-1}\chi\mu_v \\ &\qquad = \zeta_v(1)\int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} f_v^{\#}(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{s_j}\chi_v^{(j)}(x_j)\, d^*x_1 \ldots d^*x_n \\ &\qquad = \zeta_v(1)\int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} \prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)\, dx_1 \ldots dx_n. \end{aligned}$$

We calculate the last integral as the difference of the integrals of $\prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)$ over $(\mathcal{O}_v)^n \cap (F_v^\times)^n = (\mathcal{O}_v - \{0\})^n$ and $\big(\prod_{j=1}^n \pi_v^{a_j}\mathcal{O}_v\big) \cap (F_v^\times)^n = \prod_{j=1}^n(\pi_v^{a_j}\mathcal{O}_v - \{0\})$:

$$\begin{aligned} &\int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} \prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)\, dx_1 \ldots dx_n \\ &\qquad = \int_{(\mathcal{O}_v - \{0\})^n} \prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j) dx_1 \ldots dx_n \\ &\qquad\quad - \int_{\prod_{j=1}^n(\pi_v^{a_j}\mathcal{O}_v - \{0\})} \Big(\prod_{j=1}^n |x_j|_v^{s_j}\chi_v^{(j)}(x_j)\Big) dx_1 \ldots dx_n. \end{aligned}$$

Let us integrate over $(\mathcal{O}_v - \{0\})^n$. We have that
(7.2.1.3)

$$\int_{(\mathcal{O}_v-\{0\})^n} \prod_{j=1}^{n} |x_j|_v^{s_j-1} \chi_v^{(j)}(x_j) dx_1 \dots dx_n = \prod_{j=1}^{n} \int_{\mathcal{O}_v-\{0\}} \chi_v^{(j)}(x)|x|_v^{s_j-1} dx.$$

When $\Re(s) > 0$, we have that

$$\begin{aligned} \int_{\mathcal{O}_v-\{0\}} \chi_v^{(j)}(x)|x|^s d^*x &= \sum_{r=0}^{\infty} \int_{\pi_v^r \mathcal{O}_v^\times} \chi_v^{(j)}(x) \cdot |x|^s d^*x \\ &= \sum_{r=0}^{\infty} \int_{\mathcal{O}_v^\times} \chi_v^{(j)}(\pi_v^r x) \cdot |\pi_v^r x|_v^s d^*x \\ &= \sum_{r=0}^{\infty} \chi_v^{(j)}(\pi_v)^r |\pi_v|_v^{rs} \int_{\mathcal{O}_v^\times} \chi_v^{(j)}(x) d^*x. \end{aligned}$$

The integral of a non trivial character of compact group for a Haar measure on the group is 0, while it is the volume of the group if the character is trivial. We deduce that, in the case $\chi_v^{(j)}|_{\mathcal{O}_v^\times} \neq 1$ one has that

$$\int_{\mathcal{O}_v-\{0\}} \chi_v^{(j)}(x)|x|^s d^*x = 0,$$

otherwise

$$\begin{aligned} \int_{\mathcal{O}_v-\{0\}} \chi_v^{(j)}(x)|x|_v^s d^*x &= \sum_{r=0}^{\infty} \int_{\pi_v^r \mathcal{O}_v^\times} \chi_v^{(j)}(x)|x|_v^s d^*x \\ &= \sum_{r=0}^{\infty} \chi_v^{(j)}(\pi_v^r)|\pi_v|_v^{rs} d^*x(\pi_v^r \mathcal{O}_v^\times) \\ &= d^*x(\mathcal{O}_v^\times) \sum_{r=0}^{\infty} (\chi^{(j)}(\pi_v)|\pi_v|_v^s)^r \\ &= \frac{1 - |\pi_v|_v}{1 - \chi_v^{(j)}(\pi_v)|\pi_v|_v^s} \\ &= \zeta_v(1)^{-1} L_v(s, \chi^{(j)}). \end{aligned}$$

The integral over $(\mathcal{O}_v - \{0\})^n$ is hence

$$\prod_{j=1}^{n} (1 - |\pi_v|_v) L_v(s, \chi_v^{(j)}) = \zeta_v(1)^{-n} \prod_{j=1}^{n} L_v(s, \chi^{(j)}).$$

Let us calculate the integral of $\prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)$ over $\prod_{j=1}^n(\pi_v^{a_j}\mathcal{O}_v - \{0\})$. The v-adic absolute value of the determinant of the Jacobian of the map

$$(\mathcal{O}_v - \{0\})^n \to (\pi_v^{a_j}\mathcal{O}_v - \{0\})_j \qquad \mathbf{x} \mapsto (\pi_v^{a_j}x_j)_j$$

is equal to $|\pi_v|_v^{|\mathbf{a}|}$. Using the formula for the change of variables and the fact that $\prod_{j=1}^n(\chi_v^{(j)})^{a_j} = 1$, we get that

$$\begin{aligned}
&\int_{\prod_{j=1}^n(\pi_v^{a_j}\mathcal{O}_v - \{0\})} \big(\prod_{j=1}^n |x_j|_v^{s_j}\chi_v^{(j)}(x_j)\big)dx_1\dots dx_n \\
&= \int_{(\mathcal{O}_v-\{0\})^n} |\pi_v|_v^{|\mathbf{a}|}\big(\prod_{j=1}^n |\pi_v^{a_j}x_j|_v^{s_j-1}\chi_v^{(j)}(\pi_v^{a_j}x_j)\big)dx_1\dots dx_n \\
&= |\pi_v|_v^{|\mathbf{a}|+\mathbf{a}\cdot(\mathbf{s}-\mathbf{1})}\prod_{j=1}^n \chi_v^{(j)}(\pi_v)^{a_j}\int_{(\mathcal{O}_v-\{0\})^n}\big(\prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)\big)dx_1\dots dx_n \\
&= |\pi_v|_v^{\mathbf{a}\cdot\mathbf{s}}\int_{(\mathcal{O}_v-\{0\})^n}\big(\prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)\big)dx_1\dots dx_n \\
&= |\pi_v|_v^{\mathbf{a}\cdot\mathbf{s}}\prod_{j=1}^n\int_{\mathcal{O}_v-\{0\}} |x|_v^{s_j-1}\chi_v^{(j)}(x)dx \\
&= |\pi_v|_v^{\mathbf{a}\cdot\mathbf{s}}\zeta_v(1)^{-n}\prod_{j=1}^n L_v(s_j,\chi_v^{(j)}).
\end{aligned}$$

We deduce that

$$\int_{(F_v^\times)^n\cap\mathcal{D}_v^{\mathbf{a}}}\prod_{j=1}^n |x_j|_v^{s_j-1}\chi_v^{(j)}(x_j)\,dx_1\dots dx_n = 0$$

if $\chi|_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} \neq 1$ and

$$\int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} \prod_{j=1}^n |x_j|_v^{s_j-1} \chi_v^{(j)}(x_j)\, dx_1 \dots dx_n$$
$$= \zeta_v(1)^{-n} \prod_{j=1}^n L(s_j, \chi_v^{(j)}) - \zeta_v(1)^{-n} |\pi_v|_v^{\mathbf{a}\cdot\mathbf{s}} \prod_{j=1}^n L_v(s_j, \chi_v^{(j)})$$
$$= \zeta_v(1)^{-n} \zeta_v(\mathbf{a}\cdot\mathbf{s})^{-1} \prod_{j=1}^n L_v(s_j, \chi_v^{(j)}),$$

if $\chi|_{[\mathscr{T}(\mathbf{a})(F_v)]} = 1$. Finally, it follows that

$$\widehat{H}_v^{\#}(\mathbf{s}, \chi) = \zeta_v(1)^{(n-1)} \cdot \zeta_v(1) \int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} \prod_{j=1}^n |x_j|_v^{s_j-1} \chi_v^{(j)}(x_j)\, dx_1 \dots dx_n = 0$$

if $\chi|_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} \neq 1$ and

$$\widehat{H}_v^{\#}(\mathbf{s}, \chi) = \zeta_v(1)^{n-1} \cdot \zeta_v(1)^1 \int_{(F_v^\times)^n \cap \mathcal{D}_v^{\mathbf{a}}} \prod_{j=1}^n |x_j|_v^{s_j-1} \chi_v^{(j)}(x_j)\, dx_1 \dots dx_n$$
$$= \zeta_v(\mathbf{a}\cdot\mathbf{s}) \prod_{j=1}^n L_v(s_j, \widetilde{\chi}_v^{(j)})$$

if $\chi_v|_{[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]} = 1$. The statement is proven. □

7.2.2. — Let $v \in M_F^0$. When f_v is assumed to be smooth, we establish in Lemma 7.2.2.2 that, whenever $\chi_v \notin [\mathscr{T}(\mathbf{a})(F_v)]^*$, there exists a compact and open subgroup $K_v \subset [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ such that $\widehat{H}_v(\mathbf{s}, \chi_v) = 0$.

The following lemma will be used.

Lemma 7.2.2.1. — *Let $v \in M_F^0$ and let $f_v : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be a locally constant $\mathbf{a}$-homogenous function of weighted degree $b \geq 0$. There exists an open and compact subgroup $\Lambda_v \subset (F_v^\times)^n$ such that for every $(\lambda_j)_j \in \Lambda_v$ and every $(x_j)_j \in F_v^n - \{0\}$ one has that*

$$f_v((\lambda_j x_j)_j) = f_v(\mathbf{x}).$$

Proof. — We set as before $\mathcal{D}_v^{\mathbf{a}} := (\mathcal{O}_v^n) - (\pi_v^{a_1}\mathcal{O}_v) \times \dots \times (\pi_v^{a_n}\mathcal{O}_v)$. By Lemma 3.3.4.4, the set $\mathcal{D}_v^{\mathbf{a}} \subset F_v^n - \{0\}$ is open and compact subset of $F_v^n - \{0\}$. There exists a finite set $\{B(\mathbf{x}_i, d_i)\}_i \subset \mathcal{D}_v^{\mathbf{a}}$ of open balls in $F_v^n - \{0\}$, where $d_i > 0$, which cover $\mathcal{D}_v^{\mathbf{a}}$ and such that for every i, the restriction $f_v|_{B(\mathbf{x}_i, d_i)}$ is constant. The open balls in $F_v^n - \{0\}$ are also closed and hence compact. For every i and every $\mathbf{y} \in B(\mathbf{x}_i, d_i)$, the set of $(\lambda_j)_j \in (F_v^\times)^n$ such that $(\lambda_j y_j)_j \in B(\mathbf{x}_i, d_i)$ is an open neighbourhood of $\mathbf{1} \in (F_v^\times)^n$ and thus contains an open subgroup $\Lambda^i_{\mathbf{y}} \subset (F_v^\times)^n$. For every i, the open sets $\{\Lambda^i_{\mathbf{y}} \cdot \mathbf{y}\}_{\mathbf{y}\in B(\mathbf{x}_i,d_i)}$ form an open covering of $B(\mathbf{x}_i, d_i)$ and there exists a finite set of points $\mathbf{y}^i_1, \dots, \mathbf{y}^i_{m_i} \in B(\mathbf{x}_i, d_i)$ such that $\{\Lambda^i_{\mathbf{y}^i_e} \cdot \mathbf{y}^i_e\}_{e=1,\dots,m_i}$ is an open covering of $B(\mathbf{x}_i, d_i)$. Now $\Lambda_v := \cap_i \cap_{e=1}^{m_i} \Lambda^i_{\mathbf{y}^i_e}$ is an open subgroup of $(F_v^\times)^n$ satisfying that for any $\mathbf{x} \in \mathscr{D}^{\mathbf{a}}_v$ and any $(\lambda_j)_j \in \Lambda_v$, one has $f_v((\lambda_j x_j)_j) = f_v(\mathbf{x})$. Let $\mathbf{y} \in F_v^n - \{0\}$ and $(\lambda_j)_j \in \Lambda_v$. By Lemma 3.3.4.4, there exists $t \in F_v^\times$ such that $t \cdot \mathbf{y} \in \mathscr{D}^{\mathbf{a}}_v$. We have

$$|t|_v^{|\mathbf{a}|} f_v((\lambda_j y_j)_j) = f_v((t^{a_j}\lambda_j y_j)_j) = f_v((\lambda_j t^{a_j} y_j)) = f_v((t^{a_j} y_j)_j) = |t|_v^{|\mathbf{a}|} f_v(\mathbf{y}),$$

and the statement follows. □

Lemma 7.2.2.2. — *Let* $\mathbf{s} \in \Omega_{>0}$. *Let* $v \in M_F^0$ *and let* $f_v : F_v^n - \{0\} \to \mathbb{R}_{\geq 0}$ *be locally constant* $\mathbf{a}$*-homogenous function of weighted degree* $|\mathbf{a}|$. *There exists a compact open subgroup* K_v *of* $[\mathscr{T}(\mathbf{a})(F_v)]$ *such that* $\widehat{H}_v(\mathbf{s}, \chi) = 0$, *for any character* $\chi \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ *not vanishing on* K_v. *Moreover, if* $f_v = f_v^{\#}$ *is toric, one can choose* $K_v = [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$.

Proof. — By Lemma 7.2.2.1, there exists an open and compact subgroup $\Lambda_v \subset (F_v^\times)^n$ such that f_v is Λ_v-invariant. Let us set $K_v := q^{\mathbf{a}}_v(\Lambda_v \cap \mathcal{O}_v^{\times n}) \subset [\mathscr{T}(\mathbf{a})(F_v)]$. It is open and compact subgroup of $[\mathscr{T}(\mathbf{a})(F_v)]$. Let $\mathbf{x} \in [\mathscr{T}(\mathbf{a})(F_v)]$ and let $\widetilde{\mathbf{x}} \in (F_v^\times)^n$ be a lift of $\mathbf{x}$. Let $\mathbf{y} \in K_v$ and let $\widetilde{\mathbf{y}} \in \Lambda \cap \mathcal{O}_v^{\times n}$ be a lift of $\mathbf{y}$. We have that

$$H_v(\mathbf{s}, \mathbf{y}\mathbf{x}) = f_v((\widetilde{y}_j\widetilde{x}_j)_j)^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{y}_j\widetilde{x}_j|_v^{-s_j} = f_v(\widetilde{\mathbf{x}})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{x}_j|^{s_j} = H_v(\mathbf{s}, \mathbf{x}).$$

Therefore $\mathbf{x} \mapsto H_v(\mathbf{s}, \mathbf{x})$, and hence $\mathbf{x} \mapsto H_v(\mathbf{s}, \mathbf{x})^{-1}$ are invariant for the open and compact subgroup $q^{\mathbf{a}}_v(\Lambda_v \cap \mathcal{O}_v^{\times n}) \subset [\mathscr{T}(\mathbf{a})(F_v)]$. We deduce that for any $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ which does not vanish on K_v one has $\widehat{H}_v(\mathbf{s}, \chi_v) = 0$. Moreover, in the case f_v is toric, by Lemma 7.2.1.1, one has that $\widehat{H}_v(\mathbf{s}, \chi_v) = 0$ for every χ_v not vanishing on $[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$. □

7.3. Product of transforms over finite places

Using the results from local analysis, we establish some growth properties on the product of transforms over all finite places.

The assumption on $\mathbf{a}$ is that $\mathbf{a} \in \mathbb{Z}^n_{>0}$ if $n \geq 2$, and $\mathbf{a} = a \in \mathbb{Z}_{>1}$ if $n = 1$. As before f_v are assumed to be locally constant for $v \in M_F^0$.

7.3.1. — For a character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ and any $\mathbf{s} \in \Omega_{>0}$, we set

$$\widehat{H}_{\mathrm{fin}}(\mathbf{s}, \chi) := \prod_{v \in M_F^0} \widehat{H}_v(\mathbf{s}, \chi_v).$$

One has following proposition.

Proposition 7.3.1.1. — *Let $\mathbf{a} \in \mathbb{Z}^n_{>0}$ if $n \geq 2$ and let $\mathbf{a} = a_1 \in \mathbb{Z}_{>1}$ if $n = 1$. For every character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, the infinite product $\widehat{H}_{\mathrm{fin}}(\mathbf{s}, \chi)$ converges for $\mathbf{s} \in \Omega_{>1}$ and defines a holomorphic function in the domain $\Omega_{>1}$. There exists a unique holomorphic function $\phi_{\mathrm{fin}}(-, \chi)$ on $\Omega_{>\frac{2}{3}}$ such that one has an equality of meromorphic functions in the domain $\Omega_{>\frac{2}{3}}$:*

$$\widehat{H}_{\mathrm{fin}}(\mathbf{s}, \chi) = \phi_{\mathrm{fin}}(\mathbf{s}, \chi) \prod_{j=1}^{n} L(s_j, \chi^{(j)}).$$

Moreover, for every compact $\mathcal{K} \subset \mathbb{R}^n_{>\frac{2}{3}}$, there exists $C(\mathcal{K})$ such that $|\phi_{\mathrm{fin}}(\mathbf{s}, \chi)| \leq C(\mathcal{K})$ for every character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, provided that $\Re(\mathbf{s}) \in \mathcal{K}$.

Proof. — Let S be the union of the set of finite places v for which f_v is not toric and the set of the infinite places. By Corollary 7.1.3.4, for every character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, the function

$$\Omega_{>0} \to \mathbb{C} \qquad \mathbf{s} \mapsto \prod_{v \in S \cap M_F^0} \widehat{H}_v(\mathbf{s}, \chi_v)$$

is holomorphic. For every character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ and every $v \in M_F^0 - S$, the functions

$$\mathbf{s} \mapsto \prod_{j=1}^{n} L_v(s_j, \chi_v^{(j)}) = \prod_{j=1}^{n} \frac{1}{1 - |\pi_v|_v^{s_j} \chi_v^{(j)}(\pi_v)}$$

and

$$\mathbf{s} \mapsto \zeta_v(\mathbf{a} \cdot \mathbf{s}) = \frac{1}{1 - |\pi_v|_v^{\mathbf{a} \cdot \mathbf{s}}}$$

are holomorphic and non vanishing in the domain $\Omega_{>0}$. We deduce that for every $\mathbf{s} \in \Omega_{>1}$ and every $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, the product

$$\prod_{v \in M_F^0 - S} \widehat{H}_v(\mathbf{s}, \chi_v) = \prod_{v \in M_F^0 - S} \widehat{H}_v^{\#}(\mathbf{s}, \chi_v)$$
$$= \prod_{v \in M_F^0 - S} \frac{\prod_{j=1}^n L_v(s_j, \chi_v^{(j)})}{\zeta_v(\mathbf{a} \cdot \mathbf{s})}$$

converges to

$$\frac{\prod_{j=1}^n L(s_j, \chi_v^{(j)})}{\zeta_F(\mathbf{a} \cdot \mathbf{s}) \prod_{v \in S \cap M_F^0} \frac{\prod_{j=1}^n L_v(s_j, \chi_v^{(j)})}{\zeta_v(\mathbf{a} \cdot \mathbf{s})}}.$$

We conclude that for $\mathbf{s} \in \Omega_{>1}$ and $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, the product $\widehat{H}_{\mathrm{fin}}(\mathbf{s}, \chi)$ converges to

$$\left(\prod_{v \in S \cap M_F^0} \widehat{H}_v(\mathbf{s}, \chi_v) \right) \cdot \frac{\prod_{j=1}^n L(s_j, \chi_v^{(j)})}{\zeta_F(\mathbf{a} \cdot \mathbf{s}) \prod_{v \in S \cap M_F^0} \frac{\prod_{j=1}^n L_v(s_j, \chi_v^{(j)})}{\zeta_v(\mathbf{a} \cdot \mathbf{s})}}$$

and using holomorphicity of $\mathbf{s} \mapsto \widehat{H}_v(\mathbf{s}, \chi_v)$ for $\mathbf{s} \in \Omega_{>0}$, that the resulting function is holomorphic in $\mathbf{s}$ in the domain $\Omega_{>1}$. Let us establish the meromorphic extension to the domain $\Omega_{>\frac{2}{3}}$. We set

$$\phi_{\mathrm{fin}}(\mathbf{s}, \chi) := \frac{1}{\zeta_F(\mathbf{a} \cdot \mathbf{s})} \prod_{v \in S \cap M_F^0} \frac{\widehat{H}_v(\mathbf{s}, \chi_v) \zeta_v(\mathbf{a} \cdot \mathbf{s})}{\prod_{j=1}^n L_v(s_j, \chi^{(j)})}.$$

If $\mathbf{s} \in \Omega_{>\frac{2}{3}}$, then $\Re(\mathbf{a} \cdot \mathbf{s}) > \frac{4}{3}$ (because if $n \geq 2$ then $\mathbf{a} \in \mathbb{Z}_{>0}^n$ and if $n = 1$ then $\mathbf{a} \in \mathbb{Z}_{>1}$). Using the fact that the function ζ_F is holomorphic and without zeros in the domain $\Omega_{>\frac{4}{3}}$, we deduce that the function $\mathbf{s} \mapsto \zeta_F(\mathbf{a} \cdot \mathbf{s})^{-1}$ is holomorphic in the domain $\Omega_{>\frac{2}{3}}$. We have already seen that for $v \in M_F^0 \cap S$, the function $\mathbf{s} \mapsto \left(\prod_{j=1}^n L_v(s_j, \chi_v^{(j)}) \right)^{-1}$, the function $\mathbf{s} \mapsto \zeta_v(\mathbf{a} \cdot \mathbf{s})$ and $\mathbf{s} \mapsto \widehat{H}_v(\mathbf{s}, \chi_v)$ are holomorphic in the domain $\mathbf{s} \in \Omega_{>0}$. It follows that $\phi_{\mathrm{fin}}(-, \chi)$ is holomorphic in the domain $\Omega_{>\frac{2}{3}}$ and is the unique holomorphic function which satisfies that $\widehat{H}_{\mathrm{fin}}(-, \chi) = \phi_{\mathrm{fin}}(-, \chi) \prod_{j=1}^n L(s_j, \chi^{(j)})$ in this domain.

Let $\mathcal{K} \subset \mathbb{R}^n_{>\frac{2}{3}}$ be a compact. By Proposition 7.1.3.3, for every $v \in S \cap M_F^0$, there exists $C_1 > 0$ such that for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every

$\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, one has $|\widehat{H}_v(\mathbf{s},\chi_v)| \leq C_1$. One has that

$$\left|\frac{1}{\prod_{j=1}^n L_v(s_j,\chi_v^{(j)})}\right| = \prod_{j=1}^n \left|(1-\chi_v^{(j)}(\pi_v)|\pi_v|_v^{s_j})\right| \leq \prod_{j=1}^n 2 = 2^n \tag{7.3.1.2}$$

for every $\mathbf{s} \in \Omega_{>\frac{2}{3}}$. Note that for $v \in S \cap M_F^0$, we have

$$\left|\zeta_v(\mathbf{a}\cdot\mathbf{s})\right| = \left|\frac{1}{1-|\pi_v|_v^{\mathbf{a}\cdot\mathbf{s}}}\right| \leq \frac{1}{1-|\pi_v|_v} = \zeta_v(1),$$

whenever $\mathbf{s} \in \Omega_{>\frac{2}{3}}$. Finally, for $\mathbf{s} \in \Omega_{>\frac{2}{3}}$, one has $\Re(\mathbf{a}\cdot\mathbf{s}) > \frac{4}{3}$, and we have

$$\left|\frac{1}{\zeta_F(\mathbf{a}\cdot\mathbf{s})}\right| \leq \frac{1}{\zeta_F(\Re(\mathbf{a}\cdot\mathbf{s}))} \leq \frac{1}{\zeta_F(\frac{4}{3})}. \tag{7.3.1.3}$$

We conclude that for $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ one has

$$\begin{aligned}|\phi_{\mathrm{fin}}(\mathbf{s},\chi)| &= \frac{1}{\zeta_F(\mathbf{a}\cdot\mathbf{s})}\prod_{v\in S\cap M_F^0}\frac{\widehat{H}_v(\mathbf{s},\chi_v)\zeta_v(\mathbf{a}\cdot\mathbf{s})}{\prod_{j=1}^n L_v(s_j,\chi^{(j)})}\\ &\leq \frac{(2^nC_1\zeta_v(1))^{|S\cap M_F^0|}}{\zeta_F\left(\frac{4}{3}\right)}.\end{aligned}$$

We have proven that ϕ_{fin} is uniformly bounded for $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ when $\Re(\mathbf{s}) \in \mathcal{K}$. The proof is completed. □

7.4. Calculations in archimedean case

The goal of this section is to analyse the Fourier transforms of local heights at the infinite places.

7.4.1. — In this paragraph we recall some facts about integration by parts. Let $v \in M_F^\infty$.

Consider the vector field

$$\frac{\partial}{\partial x_v} := \begin{cases}\frac{\partial}{\partial x}, & \text{if } v \text{ is real}\\ \frac{\partial}{\partial z}, & \text{if } v \text{ is complex}\end{cases} \tag{7.4.1.1}$$

defined on F_v.

Lemma 7.4.1.2. — *Let $v \in M_F^\infty$. Let $f, g : F_v^\times \to \mathbb{C}$ be smooth functions. We suppose that $\lim_{|x_v|_v \to \infty} fg(x_v) = 0$ and that the functions $F_v^\times \to \mathbb{C}$ given by $x_v \mapsto \frac{\partial f(x_v)}{\partial x_v} g(x_v)$ and $x_v \mapsto f(x_v)\frac{\partial g(x_v)}{\partial x_v}$ are absolutely dx_v-integrable. If v is real, we suppose further that $\lim_{x_v \to 0} fg(x_v)$ exists and is a finite real number. One has that*

$$\int_{F_v} \frac{\partial f}{\partial x_v} g dx_v = -\int_{F_v} f \frac{\partial g}{\partial x_v} dx_v.$$

Proof. — Suppose v is real. By applying the integration by parts, we get that

$$\begin{aligned}
&\int_{F_v} \frac{\partial f}{\partial x_v} g dx_v \\
&\qquad = \int_{\mathbb{R}} \frac{\partial f}{\partial x} g dx \\
&\qquad = \int_{\mathbb{R}_{>0}} \frac{\partial f}{\partial x} g dx + \int_{\mathbb{R}_{<0}} \frac{\partial f}{\partial x} g dx \\
&\qquad = 0 - \lim_{x \to 0} fg(x) - \int_{\mathbb{R}_{>0}} f \frac{\partial g}{\partial x} dx + \lim_{x \to 0} fg(x) - 0 - \int_{\mathbb{R}_{<0}} f \frac{\partial g}{\partial x} dx \\
&\qquad = -\int_{\mathbb{R}} f \frac{\partial g}{\partial x} dx.
\end{aligned}$$

Suppose v is complex. By Fubini's theorem we have

(7.4.1.3)

$$\begin{aligned}
&\int_{F_v} \frac{\partial f}{\partial x_v} g dx_v \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{\partial f(x+iy)}{\partial x} - i \frac{\partial f(x+iy)}{\partial y} \right) g(x+iy) dx dy \\
&\quad = \int_{\mathbb{R}} dy \int_{\mathbb{R}} \frac{\partial f(x+iy)}{\partial x} g(x+iy) dx - \int_{\mathbb{R}} dx \int_{\mathbb{R}} i \frac{\partial f(x+iy)}{\partial y} g(x+iy) dy.
\end{aligned}$$

For every $y \in \mathbb{R}$, by the conditions of our lemma, one has that

$$\lim_{x \to \pm\infty} \frac{\partial f(x+iy)}{\partial x} g(x+iy) = \lim_{x \to \pm\infty} \frac{\partial g(x+iy)}{\partial x} f(x+iy) = 0.$$

Using that $z \mapsto \frac{\partial f(z)}{\partial z} g(z)$ and of $z \mapsto f(z)\frac{\partial g(z)}{\partial x_v}$ are absolutely $-idzd\overline{z} = 2dxdy$-integrable, we deduce that $x \mapsto \frac{\partial f(x+iy)}{\partial x} g(x+iy)$ and $x \mapsto f(x+$

$iy)\frac{\partial g(x+iy)}{\partial x}$ are absolutely integrable for almost every y. For such y, one gets

$$\int_{\mathbb{R}} \frac{\partial f(x+iy)}{\partial x} g(x+iy)dx = -\int_{\mathbb{R}} f(x+iy)\frac{\partial g(x+iy)}{\partial x}dx.$$

Similarly, for almost every $x \in \mathbb{R}$ one has that

$$\int_{\mathbb{R}} \frac{\partial f(x+iy)}{\partial y} g(x+iy)dy = -\int_{\mathbb{R}} f(x+iy)\frac{\partial g(x+iy)}{\partial y}dy.$$

We deduce that the last integral of the equality (7.4.1.3) is equal to

$$\begin{aligned}
&= -\int_{\mathbb{R}} dy \int_{\mathbb{R}} f(x+iy)\frac{\partial g(x+iy)}{\partial x}dx + \int_{\mathbb{R}} dx \int_{\mathbb{R}} if(x+iy)\frac{\partial g(x+iy)}{\partial y}dy \\
&= -\int_{\mathbb{R}^2} f\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right)dxdy \\
&= -\int_{F_v} f\frac{\partial g}{\partial x_v}dx_v.
\end{aligned}$$

□

Note that

$$\frac{\partial \frac{x_v}{|x_v|_v}}{\partial x_v} = 0 \qquad (7.4.1.4)$$

whenever $x_v \neq 0$ (indeed, when v is real, one has that $\frac{x_v}{|x_v|}$ is a piecewise constant function and when v is complex, one has that $\frac{\partial \frac{x_v}{|x_v|_v}}{\partial x_v} = \frac{\partial(\overline{z}^{-1})}{\partial z} = 0$).

Let ∇ be the vector field on F_v given by $\frac{x_v \partial}{\partial x_v}$.

Corollary 7.4.1.5. — *Let $v \in M_F^\infty$. Suppose that $f, g : F_v \to \mathbb{C}$ are continuous functions the restrictions of which to $F_v^\times$ are smooth. Suppose further that $\lim_{|x_v|_v \to \infty} fg(x_v) = \lim_{|x_v|_v \to 0} fg(x_v) = 0$ and that the functions $\nabla(f)g, f\nabla(g) : F_v^\times \to \mathbb{C}$ are absolutely d^*x_v-integrable. One has that*

$$\int_{F_v^\times} \nabla(f) g d^*x_v = -\int_{F_v^\times} f\nabla(g)d^*x_v.$$

Proof. — We will apply the previous lemma for f and $\frac{x_v}{|x_v|_v}g$. We note that if v is real, then

$$\lim_{x_v \to 0} fg(x_v)\frac{x_v}{|x_v|} = \lim_{x \to 0} fg(x)\frac{x}{|x|} = 0.$$

By applying the fact (7.4.1.4), we get that $\frac{\partial(\frac{x_v}{|x_v|_v}g)}{\partial x_v}=\frac{x_v}{|x_v|_v}\frac{\partial g}{\partial x_v}$. It follows from the conditions of the lemma that the functions $F_v^\times\to\mathbb{C}$ given by $x_v\mapsto\frac{\partial f(x_v)}{\partial x_v}\frac{x_vg(x_v)}{|x_v|_v}$ and $x_v\mapsto\frac{x_vf(x_v)}{|x_v|_v}\frac{\partial g(x_v)}{\partial x_v}$ are absolutely dx_v-integrable, and that $\lim_{|x_v|_v\to\infty}\frac{x_v\cdot fg(x_v)}{|x_v|_v}=0$. Using Lemma 7.4.1.2, we get that

$$\begin{aligned}\int_{F_v^\times}\nabla(f)gd^*x_v&=\int_{F_v^\times}\frac{\partial f}{\partial x_v}\frac{x_v}{|x_v|_v}gdx_v\\&=-\int_{F_v^\times}f\frac{\partial(\frac{x_v}{|x_v|_v}\cdot g)}{\partial x_v}dx_v\\&=-\int_{F_v^\times}f\frac{x_v}{|x_v|_v}\cdot\frac{\partial g}{\partial x_v}dx_v\\&=-\int_{F_v^\times}f\nabla(g)d^*x_v.\end{aligned}$$

$\square$

7.4.2. — In this paragraph we define and prove properties of auxiliary functions h_j, which will used in 7.4.3 to perform desired integration by parts. For $k\in\{1,\dots,n\}$, let ∇_k be the vector field on F_v^n given by $\frac{x_{kv}\partial}{\partial x_{kv}}$.

We start with the following lemma.

Lemma 7.4.2.1. — *Let $U\subset F_v^n-\{0\}$ be an open and $F_v^\times$-invariant subset. Let $g:U\to\mathbb{C}$ be a smooth, $\mathbf{a}$-homogenous function of weighted degree $s\in\mathbb{C}$ (that is whenever $t\in F_v^\times$, one has $g(t\cdot\mathbf{w})=|t|_v^sg(\mathbf{w})$ for every $\mathbf{w}\in U$). Let $k\in\{1,\dots,n\}$. The function $\nabla_k(g):U\to\mathbb{C}$ is a smooth $\mathbf{a}$-homogenous function of weighted degree s.*

Proof. — Let $\mathbf{w}\in U$ and let $t\in F_v^\times$. Suppose v is real. We have

$$\begin{aligned}\frac{x_{kv}\partial g}{\partial x_{kv}}(t\cdot\mathbf{w})&=\frac{x_k\partial g}{\partial x_k}(t\cdot\mathbf{w})\\&=t^{a_i}w_k\lim_{\epsilon\to0}\frac{g((t^{a_k}w_k+\epsilon)_k,(t^{a_j}\omega_j)_{j\neq k})-g((t^{a_j}w_j)_j)}{\epsilon}\\&=t^{a_k}w_k\lim_{\epsilon\to0}\frac{|t|^sg((w_k+\epsilon/t^{a_k})_k,(w_j)_{j\neq k})-|t|^sg((w_j)_j)}{\epsilon}\\&=t^{a_k}w_k\frac{|t|^s\frac{\partial g}{\partial x_k}(\mathbf{w})}{t^{a_k}}\\&=|t|_v^s\nabla_k(g)(\mathbf{w}).\end{aligned}$$

It follows that for v real, the function $\nabla_k(g)$ is $\mathbf{a}$-homogenous and of weighted degree s. Moreover, it is smooth. Suppose that v is complex. We have that:

$$\frac{x_{kv}\partial g}{\partial x_{kv}}((t^{a_j}w_j)_j) = \frac{z_k\partial g}{\partial z_k}((t^{a_j}w_j)_j) = \frac{t^{a_k}w_k}{2}\left(\frac{\partial g}{\partial x_k} - i\frac{\partial g}{\partial y_k}\right)(t\cdot\mathbf{w}).$$

We have that

$$\begin{aligned}\frac{\partial g((t^{a_j}w_j)_j)}{\partial x_k} &= \lim_{\epsilon\to 0}\frac{g((t^{a_k}w_k+\epsilon)_k,(t^{a_j}w_j)_{j\neq k}) - g(t\cdot\mathbf{w})}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{|t|^{2s}\big(g((w_k+\epsilon/t^{a_k})_k,(w_j)_{j\neq k}) - g(\mathbf{w})\big)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{|t|^{2s}|t|^{-a_k}\big(g((w_k+\epsilon/t^{a_k})_k,(w_j)_{j\neq k}) - g(\mathbf{w})\big)}{(\epsilon/t^{a_k})}\\ &= t^{-a_k}|t|^{2s}\frac{\partial g}{\partial x_k}(\mathbf{w}).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial g((t^{a_j}w_j)_j)}{\partial y_k} &= \lim_{\epsilon\to 0}\frac{g((t^{a_k}w_k+i\epsilon)_k,(t^{a_j}w_j)_{j\neq k}) - g(t\cdot\mathbf{w})}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{|t|^{2s}\big(g((w_k+i\epsilon/t^{a_k})_k,(w_j)_{j\neq k}) - g(\mathbf{w})\big)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{|t|^{2s}t^{-a_k}\big(g((w_k+i\epsilon/t^{a_k})_k,(w_j)_{j\neq k}) - g(\mathbf{w})\big)}{(\epsilon/t^{a_k})}\\ &= t^{-a_k}|t|^{2s}\frac{\partial g}{\partial x_k}(\mathbf{w}).\end{aligned}$$

We deduce that:

$$\begin{aligned}\frac{x_{kv}\partial g}{\partial x_{kv}}(t\cdot\mathbf{w}) &= \frac{t^{a_k}w_k}{2}\left(\frac{\partial g}{\partial x_k} - i\frac{\partial g}{\partial y_k}\right)(t\cdot\mathbf{w})\\ &= \frac{w_k|t|^{2s}}{2}\left(\frac{\partial g}{\partial x_k} - i\frac{\partial g}{\partial y_k}\right)(\mathbf{w})\\ &= \frac{w_k|t|_v^{s}}{2}\left(\frac{\partial g}{\partial x_k} - i\frac{\partial g}{\partial y_k}\right)(\mathbf{w})\\ &= |t|_v^s\nabla_k(g)(\mathbf{w}).\end{aligned}$$

It follows that for v complex, the function $\nabla_k(g)$ is $\mathbf{a}$-homogenous and of weighted degree s. Moreover, it is smooth. The statement is proven. $\square$

Recall that $f_v : F_v^n - \{0\} \to \mathbb{R}_{>0}$ is smooth $\mathbf{a}$-homogenous function of weighted degree $|\mathbf{a}|$. For $j = 1, \dots, n$, let $h_j : F_v^n - \{x_j = 0\} \to \mathbb{R}$ be given by

$$\mathbf{x} \mapsto -\log\big(|x_j|_v f_v(\mathbf{x})^{-a_j/|\mathbf{a}|}\big).$$

Note that

$$h_j(t\cdot\mathbf{x}) = -\log(|t^{a_j}|_v |x_j|_v f_v(t\cdot\mathbf{x})^{-a_j/|\mathbf{a}|}) = -\log(|x_j|_v f_v(\mathbf{x})^{-a_j/|\mathbf{a}|}) = h_j(\mathbf{x})$$

for every $t \in F_v^{\times}$ and every $\mathbf{x} \in F_v^n - \{x_j = 0\}$.

Lemma 7.4.2.2. — *Let $k, j \in \{1, \dots, n\}$. The function $\nabla_k(h_j)$ extends to a smooth $F_v^{\times}$-invariant function $F_v^n - \{0\} \to \mathbb{R}$.*

Proof. — By Lemma 7.4.2.1, $\nabla_k(h_j)$ is smooth and $F_v^{\times}$-invariant on the domain $(F_v)^n - \{x_j = 0\}$. When $x_j \neq 0$, we have that

$$\nabla_k(h_j) = \nabla_k(\log(f_v^{a_j/|\mathbf{a}|})) - \nabla_k(\log(|x_j|_v))$$

As f_v is smooth and non-vanishing, the function $\nabla_k(\log(f_v^{a_j/|\mathbf{a}|}))$ is a smooth function defined of $F_v^n - \{0\}$. By Lemma 7.4.2.1, we have for $t \in F_v^{\times}$ and $\mathbf{y} \in F_v^n - \{0\}$ that

$$\begin{aligned}\nabla_k(\log(f_v^{a_j/|\mathbf{a}|}))(t\cdot\mathbf{y}) = \frac{\nabla_k(f_v^{a_j/|\mathbf{a}|})(t\cdot\mathbf{y})}{f_v^{a_j/|\mathbf{a}|}(t\cdot\mathbf{y})} &= \frac{|t|_v^{a_j}\nabla_k(f_v^{a_j/|\mathbf{a}|})(\mathbf{y})}{|t|_v^{a_j} f_v^{a_j/|\mathbf{a}|}(\mathbf{y})}\\ &= \nabla_k(\log(f_v^{a_j/|\mathbf{a}|}))(\mathbf{y}),\end{aligned}$$

i.e. $\nabla_k(\log(f_v^{a_j/|\mathbf{a}|}))$ is $F_v^{\times}$-invariant. For a real place v, we have that

$$\nabla_k(\log(|x_k|_v)) = \frac{x\partial(\log(|x|))}{\partial x} = 1,$$

and for a complex place v, we have that

$$\nabla_k(\log(|x_k|_v)) = \frac{z\partial\log(|z|^2)}{\partial z} = 1.$$

We deduce that

$$\nabla_k(\log(|x_j|_v)) = \begin{cases} 1, & \text{if } k = j\\ 0, & \text{otherwise.}\end{cases} \tag{7.4.2.3}$$

Therefore $\nabla_k(\log|x_j|_v)$ extends to a smooth $F_v^{\times}$-invariant function $F_v^n - \{0\} \to \mathbb{R}$. Now, we deduce that the function

$$\nabla_k(h_j) = \nabla_k(\log(f_v^{a_j/|\mathbf{a}|})) - \nabla_k(\log(|x_j|_v))$$

extends to $F_v^{\times}$-invariant and smooth function $F_v^n - \{0\} \to \mathbb{R}_{>0}$. The statement is proven. $\square$

Continuous $F_v^\times$-invariant functions $F_v^n - \{0\} \to \mathbb{R}$ descend to continuous functions on the compact $[\mathscr{P}(\mathbf{a})(F_v)]$. We deduce that:
Corollary 7.4.2.4. — *Let $k \in \{1, \dots, n\}$ and let $N \geq 1$ be an integer. The functions $\nabla_k^N(h_j)$ are bounded.*

7.4.3. — In this paragraph we derivate the pullback of the function $H_v(\mathbf{s}, -)$ for the quotient map $(F_v^\times)^n \to [\mathscr{T}(\mathbf{a})(F_v)]$ using the vector fields ∇_k. For $\mathbf{s} \in \mathbb{C}^n$, let us set $\widetilde{H}_v(\mathbf{s}, -) = H_v(\mathbf{s}, -) \circ q_v^{\mathbf{a}} : (F_v^\times)^n \to \mathbb{C}$. We have that

$$\widetilde{H}_v(\mathbf{s}, \mathbf{x}) = f_v(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{-s_j} = \prod_{j=1}^n \exp(s_j h_j(\mathbf{x}))$$

for $\mathbf{x} \in (F_v^\times)^{n-1}$ and $\mathbf{s} \in \mathbb{C}^n$.
Lemma 7.4.3.1. — *Let $k \in \{1, \dots, n-1\}$. For every $N \in \mathbb{Z}_{>0}$, there exists an isobaric polynomial $P_N \in \mathbb{R}[\{X_{j,d}\}_{\substack{1 \leq j \leq n \\ 1 \leq d \leq N}}]$ which is of weighted degree N (where the degree of $X_{j,d}$ is d) such that*

$$\nabla_k^N(\widetilde{H}_v(\mathbf{s}, -)^{-1}) = \widetilde{H}_v(\mathbf{s}, -)^{-1} P_N((s_j \nabla_k^d(h_j))_{\substack{1 \leq j \leq n \\ 1 \leq d \leq N}})$$

for every $\mathbf{s} \in \mathbb{C}^n$.
Proof. — Let $\mathbf{s} \in \mathbb{C}^n$. For every $\mathbf{x} \in (F_v^\times)^n$, we have that

$$\widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1} = \exp\bigg(- \sum_{j=1}^n s_j h_j(\mathbf{x})\bigg),$$

and hence that

$$\begin{aligned}\nabla_k\big(\widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1}\big) &= \exp\big(- \sum_j s_j h_j(\mathbf{x})\big) \sum_{j=1}^n \frac{x_k \partial h_j}{\partial x_k} \\ &= \widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1} \sum_{j=1}^n s_j \nabla_k(h_j)(\mathbf{x}).\end{aligned}$$

We deduce that when $N = 1$, we can take $P_1((X_{j,d})_{j,d}) = \sum_{j=1}^n X_{j,1}$. Suppose the statement is true for some N and let us verify it for $N+1$.

We have

$$\begin{aligned}
&\nabla_k^{N+1}(\widetilde{H}_v(\mathbf{s},-)^{-1})\\
&= \nabla_k\big(\widetilde{H}_v(\mathbf{s},-)^{-1}P_N((s_j\nabla_k^d h_j)_{j,d})\big)\\
&= \widetilde{H}_v(\mathbf{s},-)^{-1}\cdot\Big(\sum_{j=1}^n s_j\nabla_k(h_j)\Big)P_N((s_j\nabla_k^d h_j)_{j,d})\\
&\qquad\qquad + \widetilde{H}_v(\mathbf{s},-)^{-1}\nabla_k(P_N((s_j\nabla_k^d h_j)_{j,d})\\
&= \widetilde{H}_v(\mathbf{s},-)^{-1}\cdot\Big(\big(\sum_{j=1}^n s_j\nabla_k(h_j)\big)P_N((s_j\nabla_k^d h_j)_{j,d}) + \nabla_k\big(P_N((s_j\nabla_k^d h_j)_{j,d})\big)\Big).
\end{aligned}$$

Let $\delta : \mathbb{R}[\{X_{j,d}\}_{\substack{1\le j\le n\\ 1\le d\le N}}] \to \mathbb{R}[\{X_{j,d}\}_{\substack{1\le j\le n\\ 1\le d\le N+1}}]$ be the $\mathbb{R}$-linear map given by

$$X_{j_1,d_1}^{q_1}\cdots X_{j_r,d_r}^{q_r} \mapsto \sum_{e=1}^r q_e\frac{X_{j_e,d_e+1}}{X_{j_e,d_e}}\big(X_{j_1,d_1}^{q_1}\cdots X_{j_r,d_r}^{q_r}\big) \qquad q_e\in\mathbb{Z}_{\ge 0}.$$

Note that if $Q \in \mathbb{R}[\{X_{j,d}\}_{\substack{1\le j\le n\\ 1\le d\le N}}]$ is isobaric of weighted degree N, then $\delta(Q)$ is isobaric of weighted degree $N+1$. As the polynomial $\big(\sum_{j=1}^n X_{j,1}\big)P_N((X_{j,d})_{j,d})$ is isobaric of weighted degree $N+1$, the polynomial

$$P_{N+1} = \big(\sum_{j=1}^n X_{j,1}\big)P_N + \delta(P_N)$$

is isobaric of weighted degree $N+1$ and from above one has that:

$$\widetilde{H}_v(\mathbf{s},-)^{-1} = \widetilde{H}_v(\mathbf{s},-)^{-1}P_{N+1}((s_j\nabla_k^d h_j)_{\substack{1\le j\le n\\ 1\le d\le N+1}}).$$

The statement is proven. □

7.4.4. — In this paragraph we calculate several limits that will enable us to perform integration by parts as in Corollary 7.4.1.5 in paragraph 7.4.6. The developed theory, along with the theory of 7.4.5 and 7.4.6, is ultimately used to prove Proposition 7.4.6.5 on the decay of Fourier transforms in the "discrete" and the "infinite" norms of a character. Throughout the paragraph one assumes that $n \ge 2$, because the following lemma is not valid when $n = 1$. Proposition 7.4.6.5 will, however, also be valid in the case $n = 1$ and will follow independently from the rest of the theory.

Lemma 7.4.4.1. — *Suppose that $n \geq 2$. Let us fix $(x_j)_{\substack{j=1\\ j\neq k}}^n \in (F_v^\times)^{n-1}$ and let $\mathbf{s} \in \Omega_{>0}$. One has that*

$$\lim_{x_k \to 0} \widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1} = 0,$$
$$\lim_{|x_k|_v \to \infty} \widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1} = 0.$$

Proof. — We have

$$\begin{aligned}\lim_{x_k \to 0} \widetilde{H}_v(\mathbf{s}, \mathbf{x})^{-1} &= \lim_{x_k \to 0} \prod_{j=1}^n |x_j|_v^{s_j} f_v(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \\ &= \lim_{x_k \to 0} \prod_{j=1}^n |x_j|_v^{s_j} f_v((x_j)_{j\neq k}, (0)_k)^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} = 0.\end{aligned}$$

Let us calculate the other limit. For every $x_k \in F_v^\times$, we have that

$$\begin{aligned}f_v(\mathbf{x}) &= f_v(|x_k|_v^{1/(n_v a_k)} \cdot (x_j |x_k|_v^{-a_j/(n_v a_k)})_j) \\ &= |x_k|_v^{|\mathbf{a}|/(a_k)} f_v((x_j |x_k|_v^{-a_j/(n_v a_k)})_j)\end{aligned}$$

and hence that

$$f_v(\mathbf{x})^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} = |x_k|_v^{\frac{\mathbf{a}\cdot\mathbf{s}}{a_k}} f_v\big((x_j |x_k|_v^{-a_j/(n_v a_k)})_j\big)^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}.$$

Thus

$$\begin{aligned}&\lim_{|x_k|_v \to \infty} \widetilde{H}_v(\mathbf{x})^{-1} \\ &\quad= \lim_{|x_k|_v \to \infty} \left(f_v(\mathbf{x})^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{s_j} \right) \\ &\quad= \lim_{|x_k|_v \to \infty} \left(|x_k|_v^{\frac{-\mathbf{a}\cdot\mathbf{s}}{a_k}} f_v\big((x_j |x_k|_v^{-a_j/(n_v a_k)})_j\big)^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |x_j|_v^{s_j} \right) \\ &\quad= \lim_{|x_k|_v \to \infty} \left(|x_k|_v^{\frac{a_k s_k - \mathbf{a}\cdot\mathbf{s}}{a_k}} f_v\big((x_j |x_k|_v^{-a_j/(n_v a_k)})_j\big)^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{\substack{j=1\\ j\neq k}}^n |x_j|_v^{s_j} \right).\end{aligned}$$

Note that $\lim_{|x_k|_v \to \infty} |x_k|_v^{\frac{a_k s_k - \mathbf{a}\cdot\mathbf{s}}{a_k}} = 0$. Let us define

$$\mathcal{B}_k := \{\mathbf{y} \in F_v^n - \{0\} | \forall j |y_j|_v \leq 1 \text{ and } |y_k|_v = 1\}.$$

The set $\mathcal{B}_k$ is compact. As f_v is strictly positive, there exists $\epsilon_1 > 0$ such that $f_v(\mathbf{y}) > \epsilon_1$ for every $\mathbf{y} \in \mathcal{B}_k$. We deduce that $f_v^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}$ is bounded above by $\epsilon_1^{-\mathbf{a}\cdot\Re(\mathbf{s})/|\mathbf{a}|}$ on $\mathcal{B}_k$. For $|x_k|_v \gg 0$, one has $(x_j|x_k|_v^{-a_j/(n_v a_k)})_j \in \mathcal{B}_k$. We conclude that

$$\lim_{|x_k|_v \to 0} \widetilde{H}_v(\mathbf{s},\mathbf{x})^{-1} = 0.$$

□

By using the formula given in Lemma 7.4.3.1 and the fact that the functions $\nabla_k^d(h_j)$ are bounded, we obtain immediately the following corollary .

Corollary 7.4.4.2. — *Suppose that $n \geq 2$. Let $\mathbf{s} \in \Omega_{>0}$, let $N \geq 0$ and let $k \in \{1,\dots,n\}$. Let us fix $(x_j)_{\substack{j=1\\ j\neq k}}^n \in (F_v^\times)^{n-1}$. One has that*

$$\lim_{|x_k|_v \to \infty} \nabla_k^N(\widetilde{H}_v(\mathbf{s},\mathbf{x})^{-1}) = 0,$$

$$\lim_{x_k \to 0} \nabla_k^N(\widetilde{H}_v(\mathbf{s},\mathbf{x})^{-1}) = 0.$$

7.4.5. — In this paragraph we present several formulas for the derivation with ∇_k, that will be used in 7.4.6. Let $v \in M_F^\infty$.

Lemma 7.4.5.1. — *If s is a complex number, one has that $\nabla(x_v \mapsto |x_v|_v^s) = s|x_v|^s$ in the domain $x_v \in F_v^\times$.*

Proof. — If v is real then

$$\nabla(|x_v|_v^s) = \frac{x\partial|x|^s}{\partial x} = xs|x|^{s-1}\mathrm{sgn}(x) = s|x|^s = s|x_v|_v^s.$$

If v is complex, then

$$\nabla(|x_v|_v^s) = \frac{z\partial(|z|^{2s})}{\partial z} = sz|z|^{2(s-1)}\overline{z} = s|z|^{2s} = s|x_v|_v^s.$$

□

We set $F_{v,1} := \{x|\,|x|_v = 1\}$. We have established in 5.2.1 an identification

$$\widetilde{\rho}_v : \mathbb{R}_{>0} \times F_{v,1} \xrightarrow{\sim} F_v^\times \qquad (r,z) \mapsto \rho_v(r)z,$$

where $\rho_v : \mathbb{R}_{>0} \to F_v^\times$ is defined by $\rho_v(r) = r^{1/n_v}$. For a character $\chi_v \in (F_v^\times)^*$, we have set $m(\chi_v)$ to be the unique real number m such that the character $\chi_v|_{\mathbb{R}_{>0}}$ is given by $r \mapsto r^{im}$. If v is a real place, we set $\ell(\chi_v)$ to be 0 if the character $\chi_{vF_{v,1}}$ is the trivial character, otherwise we set $\ell(\chi_v) = 1$. If v is a complex place, we have set $\ell(\chi_v)$ to be the unique integer ℓ such that $\chi_v|_{F_{v,1}}$ is given by $z\mapsto z^{\ell}$. Let $\chi\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ be a character. We set $\widetilde{\chi}_v:=\chi_v\circ q_v^{\mathbf{a}}$. We note that the function $\widetilde{\chi}_v:(F_v^{\times})^n\to\mathbb{C}$ is given by

$$\mathbf{x}\mapsto\prod_{j=1}^n|x_j|_v^{im(\chi_v^{(j)})/n_v}(x_j|x_j|_v^{-1/n_v})^{\ell(\chi_v^{(j)})},$$

If $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ is a character, we set

$$\mathbf{m}(\chi_v):=(m(\chi_v^{(j)}))_j\in\mathbb{R}^n \tag{7.4.5.2}$$

and

$$\boldsymbol{\ell}(\chi_v):=(\ell(\chi_v^{(j)}))_j\in\mathbb{Z}^n, \tag{7.4.5.3}$$

where $\chi_v^{(j)}$ is given by

$$x\mapsto\chi_v(q_v^{(\mathbf{a})}((1)_{\substack{k=1,\dots,n,\\ k\neq j}},(x)_{k=j})).$$

It follows from the definition that

$$\mathbf{m}(\chi_v)\in M:=\big\{\mathbf{x}\in\mathbb{R}^n|\sum_{j=1}^n a_jx_j=0\big\}.$$

Lemma 7.4.5.4. — *Let $k\in\{1,\dots,n\}$. Suppose $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ is a character. We set $d(k,\chi_v)=(1-\frac{1}{n_v})\ell(\chi_v^{(k)})+im(\chi_v^{(k)})$. One has that*

$$\nabla_k(\widetilde{\chi}_v)=d(k,\chi_v)\cdot\widetilde{\chi}_v.$$

Proof. — If $k\neq j$, then we have that

$$\nabla_k(|x_j|_v^{im(\chi_v^{(j)})})=0$$

and that

$$\nabla_k((x_j|x_j|_v^{-1/n_v})^{\ell(\chi_v^{(j)})})=0.$$

By using this and the product rule, we obtain that:

$$\begin{aligned}&\nabla_k\bigg(\prod_{j=1}^n|x_j|_v^{im(\chi_v^{(j)})}\big(x_j|x_j|_v^{-1/n_v}\big)^{\ell(\chi_v^{(j)})}\bigg)\\ &\quad=\nabla_k(x_k^{\ell(\chi_v^{(k)})}|x_k|_v^{im(\chi_v^{(k)})-\ell(\chi_v^{(k)})/n_v})\prod_{\substack{j=1\\ j\neq k}}^n|x_j|_v^{im(\chi_v^{(j)})}\big(x_j|x_j|_v^{-1/n_v}\big)^{\ell(\chi_v^{(j)})}.\end{aligned}$$

One has that

$$\begin{aligned}
&\nabla_k\big(x_k^{\ell(\chi_v^{(k)})}|x_k|_v^{im(\chi_v^{(k)})-\ell(\chi_v^{(k)})/n_v}\big)\\
&\quad= \ell(\chi_v^{(k)})x_k^{\ell(\chi_v^{(k)})}|x_k|_v^{im(\chi_v^{(k)})-\frac{\ell(\chi_v^{(k)})}{n_v}}\\
&\qquad+\big(im(\chi_v^{(k)})-\frac{\ell(\chi_v^{(k)})}{n_v}\big)x_k^{\ell(\chi_v^{(k)})}|x_k|_v^{im(\chi_v^{(k)})-\frac{\ell(\chi_v^{(k)})}{n_v}}\\
&\quad=\Big((1-\frac{1}{n_v})\ell(\chi_v^{(k)})+im(\chi_v^{(k)})\Big)x_k^{\ell(\chi_v^{(k)})}|x_k|_v^{im(\chi_v^{(k)})-\ell(\chi_v^{(k)})/n_v}\\
&\quad= d(k,\chi_v)\cdot\widetilde{\chi}_v.
\end{aligned}$$

It follows that

$$\begin{aligned}
\nabla_k(\widetilde{\chi}_v) &= \nabla_k\bigg(\prod_{j=1}^n |x_j|_v^{im(\chi_v^{(j)})}\big(x_j|x_j|_v^{-1/n_v}\big)^{\ell(\chi_v^{(j)})}\bigg)\\
&= d(k,\chi_v)\cdot\prod_{j=1}^n |x_j|_v^{im(\chi_v^{(j)})}\big(x_j|x_j|_v^{-1/n_v}\big)^{\ell(\chi_v^{(j)})}\\
&= d(k,\chi_v)\cdot\widetilde{\chi}_v.
\end{aligned}$$

The statement is proven. □

7.4.6. — In this paragraph we make the wanted estimates on the absolute value of the Fourier transform. We use the integration by parts with respect to the vector fields ∇_k.

Lemma 7.4.6.1. — *Suppose that $n\geq 2$. Let $k\in\{1,\dots,n-1\}$, let $\mathbf{s}\in\Omega_{>0}$ and let N be a non-negative integer. The function $\nabla_k^N(\widetilde{H}_v(\mathbf{s},-)^{-1}):(F_v^\times)^{n-1}\times F_{v,1}\to\mathbb{C}$ is absolutely $dx_1\dots dx_{n-1}\times\lambda_{v,1}$-integrable. Moreover, if $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ is a character, one has that*

$$\begin{aligned}
&\widehat{H}_v(\mathbf{s},\chi_v)\cdot(-d(k,\chi_v))^N\\
&\qquad=\frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}}\nabla_k^N(\widetilde{H}_v(\mathbf{s},-)^{-1})\widetilde{\chi}_v d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.
\end{aligned}$$

Proof. — Without loss of the generality, we can suppose that $k=1$. Suppose $N=0$. Proposition 7.1.3.3 gives that the integral defining the Fourier transform $\widehat{H}_v(\mathbf{s},\chi_v)$ converges absolutely. Now, it follows from Lemma 5.4.1.3 that (where $q_v^{\mathbf{a}}:(F_v^\times)^n\to[\mathscr{T}(\mathbf{a})(F_v)]$ is the quotient map)

$$(H_v(\mathbf{s},-)^{-1}\chi_v)\circ q_v^{\mathbf{a}} = \widetilde{H}_v(\mathbf{s},-)^{-1}\widetilde{\chi}_v \\ \in L^1((F_v^{\times})^{n-1}\times F_{v,1}, d^*x_1\dots d^*x_{n-1}\lambda_{v,1})$$

and that

$$\widehat{H}(\mathbf{s},\chi_v) = \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^{\times})^{n-1}\times F_{v,1}} \widetilde{H}_v(\mathbf{s},-)^{-1}\widetilde{\chi}_v d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.$$

The statement is thus true when $N=0$ and we suppose it is true for $N-1$, where $N\geq 1$. By Proposition 7.4.3.1, we have that

$$\nabla_1^N(\widetilde{H}_v(\mathbf{s},-)^{-1}) = \widetilde{H}_v(\mathbf{s},-)^{-1}P_N((s_j\nabla_1^d h_j)_{j,d}).$$

By Corollary 7.4.2.4, the functions $\nabla_k^r(h_j)$ are bounded. It follows that

$$\nabla_1^N(\widetilde{H}_v(\mathbf{s},-)^{-1})\in L^1((F_v^{\times})^{n-1}\times F_{v,1}, d^*x_1\dots d^*x_{n-1}\lambda_{v,1}).$$

Using the induction hypothesis and the fact that $\nabla_1(\widetilde{\chi}) = d(1,\chi_v)\cdot\widetilde{\chi}_v$ from Lemma 7.4.5.4, we obtain that:

$$\begin{aligned}
&\widehat{H}_v(\mathbf{s},\chi_v)\cdot(-d(1,\chi_v))^N \\
&\quad= \widehat{H}_v(\mathbf{s},\chi_v)\cdot(-d(1,\chi_v))^{N-1}(-d(1,\chi_v)) \\
&\quad= \frac{-d(1,\chi_v)a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^{\times})^{n-1}\times F_{v,1}} \nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-)^{-1})\widetilde{\chi}_v\, d^*x_1\dots d^*x_{n-1}\lambda_{v,1} \\
&\quad= \frac{-a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^{\times})^{n-1}\times F_{v,1}} \nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-)^{-1})\nabla_1(\widetilde{\chi}_v)\, d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.
\end{aligned}$$

The last integral, by Fubini theorem, writes as

$$\int_{(F_v^{\times})^{n-2}\times F_{v,1}} \otimes_{j=2}^{n-1}d^*x_j\otimes d\lambda_{v,1}(u)\times \\ \times\int_{F_v^{\times}} \nabla_1^{N-1}\big(\widetilde{H}_v(\mathbf{s},(x_j)_{j=1}^{n-1},u)^{-1}\big)\nabla_1(\widetilde{\chi}_v((x_j)_{j=1}^{n-1},u))d^*x_1.$$

As $\nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-)^{-1}):(F_v^{\times})^{n-1}\times F_{v,1}\to\mathbb{C}$ is absolutely $d^*x_1\dots d^*x_{n-1}\lambda_{v,1}$-integrable, we deduce that for almost every $((x_j)_{j=2}^{n-1},u)\in(F_v^{\times})^{n-2}\times F_{v,1}$, we have $\nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-,(x_j)_{j=2}^{n-1},u)^{-1})$ is absolutely d^*x-integrable. Now, for such $((x_j)_{j=2}^{n-1},u)\in(F_v^{\times})^{n-2}\times F_{v,1}$, Lemma 7.4.4.2 and Lemma 7.4.5.4

give that the functions $x_1 \mapsto \nabla_1^{N-1}\widetilde{H}_v(\mathbf{s},(x_j)_j)^{-1}$ and $x_1 \mapsto \widetilde{\chi}_v((x_j)_j)$ satisfy the conditions of Corollary 7.4.1.5 and we can apply the integration by parts with the respect to ∇_1. We get that:

$$\int_{F_v^\times} \nabla_1^{N-1}\big(\widetilde{H}_v(\mathbf{s},(x_j)_{j=1}^{n-1},u)^{-1}\big)\nabla_1\big(\widetilde{\chi}_v((x_j)_{j=1}^{n-1},u)\big)d^*x_1$$
$$= -\int_{F_v^\times} \nabla_1^N\big(\widetilde{H}_v(\mathbf{s},(x_j)_{j=1}^{n-1},u)^{-1}\big)\widetilde{\chi}_v((x_j)_{j=1}^{n-1},u)d^*x_1,$$

and hence that

$$\int_{(F_v^\times)^{n-1}\times F_{v,1}} \nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-)^{-1})\nabla_1(\widetilde{\chi}_v)\, d^*x_1\dots d^*x_{n-1}\lambda_{v,1}$$
$$= -\int_{(F_v^\times)^{n-1}\times F_{v,1}} \nabla_1^{N}(\widetilde{H}_v(\mathbf{s},-)^{-1})\widetilde{\chi}_v d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.$$

Finally, we deduce that

$$\begin{aligned}&\widehat{H}_v(\mathbf{s},\chi_v)\cdot(-d(1,\chi_v))^N\\ &\quad= \frac{-a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} \nabla_1^{N-1}(\widetilde{H}_v(\mathbf{s},-)^{-1})\nabla_1(\widetilde{\chi}_v)\, d^*x_1\dots d^*x_{n-1}\lambda_{v,1}\\ &\quad= \frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} \nabla_1^{N}(\widetilde{H}_v(\mathbf{s},-)^{-1})\widetilde{\chi}_v d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.\end{aligned}$$

The statement is proven. □

The integration by parts from the last lemma enables us to make the following estimate:

Lemma 7.4.6.2. — *Let $v\in M_F^\infty$. Suppose that $n\geq 2$. Let $k\in\{1,\dots,n-1\}$ and let N be a positive integer. Let $\mathcal{K}\subset\mathbb{R}^n_{>0}$ be a compact. There exists $C=C(k,N,\mathcal{K})>0$ such that for every character $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ and every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$, one has that*

$$|d(k,\chi_v)|^N\cdot|\widehat{H}_v(\mathbf{s},\chi_v)|\leq C(1+||\Im(\mathbf{s})||)^N.$$

Proof. — It follows from Lemma 7.4.6.1 that

$$|d(k,\chi_v)|^N|\widehat{H}_v(\mathbf{s},\chi_v)|\leq$$
$$\frac{a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}} |\nabla_k^N(\widetilde{H}_v(\mathbf{s},-)^{-1})|d^*x_1\dots d^*x_{n-1}\lambda_{v,1}.$$

By Lemma 7.4.3.1 there exists an isobaric polynomial P_N of weighted degree N such that
$$\nabla_k^N(\widetilde{H}_v(\mathbf{s},-)^{-1}) = \widetilde{H}_v(\mathbf{s},-)^{-1}P_N((s_j\nabla_k^d(h_j))_{j,d}).$$
Moreover, by Corollary 7.4.2.4 the functions $\nabla_k^d(h_j)$ are bounded, and we deduce that there exists $C'>0$ such that
$$|P_N((s_j\nabla_k^d h_j(\mathbf{x}))_{j,k}) \leq C'(1+||\Im(\mathbf{s})||)^N$$
for every $\mathbf{x}\in F_v^n-\{0\}$ and every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$. It follows that
$$\begin{aligned}|d(k,\chi_v)|^N|\widehat{H}_v(\mathbf{s},\chi_v)| &\leq \frac{C'a_n}{\lambda_{v,1}(F_{v,1})}\int_{(F_v^\times)^{n-1}\times F_{v,1}}\left|\widetilde{H}_v(\mathbf{s},-)^{-1}\right|d^*x_1\cdots d^*x_{n-1}\lambda_{v,1}\\ &\leq (1+||\Im(\mathbf{s})||)^N\frac{C'a_n}{\lambda_{v,1}(F_{v,1})}\int_{[\mathscr{T}(\mathbf{a})(F)]}|H(\mathbf{s},-)^{-1}|\mu_v\\ &\leq (1+||\Im(\mathbf{s})||)^N\frac{C'a_n}{\lambda_{v,1}(F_{v,1})}\int_{[\mathscr{T}(\mathbf{a})(F)]}|H(\mathbf{s},-)^{-1}|\mu_v\\ &= (1+||\Im(\mathbf{s})||)^N\frac{C'a_n}{\lambda_{v,1}(F_{v,1})}\widehat{H}(\Re(\mathbf{s}),1)\end{aligned}$$
for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. Proposition 7.1.3.3 gives that there exists $A>0$, such that $|\widehat{H}_v(\mathbf{s},\chi_v)|\leq A$ for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. We set $C=\frac{C'a_nA}{\lambda_{v,1}(F_{v,1})}$. It follows that for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ one has that
$$|d(k,\chi_v)|^N|\widehat{H}_v(\mathbf{s},\chi_v)|\leq C(1+||\Im(\mathbf{s})||)^N.$$
□

Lemma 7.4.6.3. — *Let $v\in M_F^\infty$. Suppose that $n\geq 2$. Let $k\in\{1,\dots,n-1\}$ and let N be a positive integer. Let $\mathcal{K}\subset\mathbb{R}^n_{>0}$ be a compact. There exists $C=C(k,N,\mathcal{K})>0$ such that for every character $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$ and every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$, one has that*
$$(|\ell(\chi_v^{(k)})|+|m(\chi_v^{(k)})|)^N|\widehat{H}_v(\mathbf{s},\chi_v)|\leq C(1+||\Im(\mathbf{s})||)^N.$$
Proof. — Lemma 7.4.6.2 gives that there exists $C'>0$ such that
$$|d(k,\chi_v)|^N\cdot|\widehat{H}_v(\mathbf{s},\chi_v)|\leq C'(1+||\Im(\mathbf{s})||)^N \tag{7.4.6.4}$$
for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. Proposition 7.1.3.3 gives that there exists $A>0$, such that $|\widehat{H}_v(\mathbf{s},\chi_v)|\leq A$ for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. We prove the claim of the lemma with $C=4^N\max(C',A)$. Let $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and let $\chi_v\in[\mathscr{T}(\mathbf{a})(F_v)]^*$. We

suppose first that v is complex or that v is real and that $\ell(\chi_v^{(k)}) = 0$. This condition implies that

$$\left|(1-\frac{1}{n_v})\ell(\chi_v^{(k)})\right| \geq \frac{1}{2}|\ell(\chi_v^{(k)})|.$$

We obtain that

$$\begin{aligned}|d(k,\chi_v)| &= \left|(1-\frac{1}{n_v})\ell(\chi_v^{(k)}) + im(\chi_v^{(k)})\right| \\ &\geq \frac{1}{2}|\ell(\chi_v^{(k)}) + im(\chi_v^{(k)})| \\ &\geq \frac{|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|}{4}.\end{aligned}$$

From the inequality (7.4.6.4), we deduce that

$$\begin{aligned}\frac{(|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|)^N}{4^N}|\widehat{H}_v(\mathbf{s},\chi_v)| &\leq |d(k,\chi_v)| \cdot |\widehat{H}_v(\mathbf{s},\chi_v)| \\ &\leq C'(1+||\Im(\mathbf{s})||)^N,\end{aligned}$$

and hence that

$$(|\ell(\chi_v^{(k)})|+|m(\chi_v^{(k)})|)^N|\widehat{H}_v(\mathbf{s},\chi_v)| \leq 4^N C''(1+||\Im(\mathbf{s})||)^N \leq C(1+||\Im(\mathbf{s})||)^N.$$

The claim is proven in the case v is complex or v is real and $\ell(\chi_v^{(k)}) = 0$. We suppose now that v is real and that $\ell(\chi_v^{(k)}) = 1$. One has that $d(k,\chi) = im(\chi_v^{(k)})$ and rewriting the inequality (7.4.6.4) gives that

$$|m(\chi_v^{(k)})|^N \cdot |\widehat{H}_v(\mathbf{s},\chi_v)| \leq C'(1+||\Im(\mathbf{s})||)^N.$$

Suppose that $|m(\chi_v^{(k)})| \leq 1$. Then one has that

$$\begin{aligned}(|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|)^N \cdot |\widehat{H}_v(\mathbf{s},\chi_v)| &= (1+|m(\chi_v^{(k)})|)^N \cdot |\widehat{H}_v(\mathbf{s},\chi_v)| \\ &\leq 2^N \cdot |\widehat{H}_v(\mathbf{s},\chi_v)| \\ &\leq 2^N A \\ &\leq C \\ &\leq C(1+||\Im(\mathbf{s})||)^N.\end{aligned}$$

Suppose that $|m(\chi_v^{(k)})| > 1$. Then one has that

$$\begin{aligned}(|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|)^N \cdot |\widehat{H}_v(\mathbf{s}, \chi_v)| &= (1 + |m(\chi_v^{(k)})|)^N \cdot |\widehat{H}_v(\mathbf{s}, \chi_v)| \\ &\leq 2^N |m(\chi_v^{(k)})|^N \cdot |\widehat{H}_v(\mathbf{s}, \chi_v)| \\ &\leq 2^N C'(1 + ||\Im(\mathbf{s})||)^N \\ &\leq C(1 + ||\Im(\mathbf{s})||)^N.\end{aligned}$$

The claim is thus verified also for the case v is real and $\ell(\chi_v^{(k)}) = 1$. The statement of the lemma is proven. □

We are ready to prove:

Proposition 7.4.6.5. — *Let $v \in M_F^\infty$ and let $\mathcal{K} \subset \mathbb{R}^n_{>0}$ be a compact. For every integer $N > 1$, there exists $C = C(N) > 0$ such that for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ one has*

$$|\widehat{H}_v(\mathbf{s}, \chi_v)| \leq \frac{C(1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\mathbf{m}(\chi_v)||)(1 + ||\boldsymbol{\ell}(\chi_v)||))^{N/(2(n-1))}}.$$

Proof. — Let us first suppose that $n \geq 2$. We have already seen in Proposition 7.1.3.3 that there exists $A > 0$ such that

$$|\widehat{H}_v(\mathbf{s}, \chi_v)| \leq A$$

for every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]$ and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$. Using this and Lemma 7.4.6.3 we get that there exists $M > 0$ such that

$$\begin{aligned}(7.4.6.6) \quad & \prod_{\substack{k \in \{1,..,n-1\} \\ |\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})| = 0}} |\widehat{H}_v(\mathbf{s}, \chi_v)| \times \\ & \times \prod_{\substack{k \in \{1,..,n-1\} \\ |\ell(\chi_v^{(k))})| + |m(\chi_v^{(k)})| \neq 0}} (|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|)^N |\widehat{H}_v(\mathbf{s}, \chi_v)| \\ & \leq M(1 + ||\Im(\mathbf{s})||)^{N(n-1)}\end{aligned}$$

for every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]$ and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$. Using the fact that

$$a_1 m(\chi_v^{(1)}) + \cdots + a_n m(\chi_v^{(n)}) = 0$$

we deduce that

$$||\mathbf{m}(\chi_v)|| \leq \max\big(\max_{j=1,\ldots,n-1} |m(\chi_v^{(j)})|, a_1|m(\chi_v^{(1)})| + \cdots + a_{n-1}|m(\chi_v^{(n-1)})|\big)$$

and hence there exists an index $o(\chi_v) \in \{1, \dots, n-1\}$ such that

$$(n-1) \cdot \max_j a_j \cdot |m(\chi_v^{(o(\chi_v))})| \geq ||\mathbf{m}(\chi_v)||. \tag{7.4.6.7}$$

In an analogous way, we conclude that there exists index $r(\chi_v) \in \{1, \dots, n-1\}$ such that

$$(n-1) \cdot \max_j a_j \cdot |\ell(\chi_v^{(r(\chi_v))})| \geq ||\boldsymbol{\ell}(\chi_v)||. \tag{7.4.6.8}$$

Using the arithmetic-geometric inequality and the estimates (7.4.6.7) and (7.4.6.8), we conclude that there exists $D > 0$ such that

$$\begin{aligned} D \prod_{|l(\chi_v^{(k)})|+|m(\chi_v^{(k)})| \neq 0} & (|\ell(\chi_v^{(k)})| + |m(\chi_v^{(k)})|)^N \\ & \geq \max(||\boldsymbol{\ell}(\chi_v)||^{N/2} ||\mathbf{m}(\chi_v)||^{N/2}, ||\mathbf{m}(\chi_v)||^{N/2}, ||\boldsymbol{\ell}(\chi_v)||^{N/2}). \end{aligned} \tag{7.4.6.9}$$

Combining the estimates (7.4.6.9) and (7.4.6.6) and taking the $n-1$-th root gives

$$\begin{aligned} |\widehat{H}_v(\mathbf{s}, \chi_v)| \max(||\boldsymbol{\ell}(\chi_v)|| \cdot ||\mathbf{m}(\chi_v)||, ||\mathbf{m}(\chi_v)||, ||\boldsymbol{\ell}(\chi_v)||)^{N/(2(n-1))} & \\ \leq (MD)^{1/(n-1)} (1 + ||\Im(\mathbf{s})||)^N & \end{aligned} \tag{7.4.6.10}$$

for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$.

For every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ with $\boldsymbol{\ell}(\chi_v) = (0)_j$ and $||\mathbf{m}(\chi_v)|| \leq 1$ and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$, one has that

$$\begin{aligned} |\widehat{H}_v(\mathbf{s}, \chi_v)| & \leq \frac{2^{N/(2(n-1))} A}{(1 + ||\mathbf{m}(\chi_v)||)^{N/(2(n-1)}} \\ & \leq \frac{2^{N/(2(n-1))} A (1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\boldsymbol{\ell}(\chi_v)||)(1 + ||\mathbf{m}(\chi_v)||))^{N/(2(n-1))}}. \end{aligned}$$

For every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ for which $\boldsymbol{\ell}(\chi_v) \neq (0)_j$ and for which $||\mathbf{m}(\chi_v)|| \leq 1$ and for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ one has

$$\begin{aligned} |\widehat{H}_v(\mathbf{s}, \chi_v)| & \leq \frac{(MD)^{1/(n-1)} (1 + ||\Im(\mathbf{s})||)^N}{||\boldsymbol{\ell}(\chi_v)||^{N/(2(n-1))}} \\ & \leq \frac{2^{2N/(2(n-1))} (MD)^{1/(n-1)} (1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\boldsymbol{\ell}(\chi_v)||)(1 + ||\mathbf{m}(\chi_v)||))^{N/(2(n-1))}}. \end{aligned}$$

For every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ with $\boldsymbol{\ell}(\chi_v) = (0)_j$ and $||\mathbf{m}(\chi_v)|| > 1$ and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$, one has that

$$\begin{aligned}|\widehat{H}_v(\mathbf{s}, \chi_v)| &\leq \frac{(DM)^{1/(n-1)}(1 + ||\Im(\mathbf{s})||)^N}{||\mathbf{m}(\chi_v)||^{N/(2(n-1))}} \\ &\leq \frac{2^{N/(2(n-1))}(MD)^{1/(n-1)}(1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\boldsymbol{\ell}(\chi_v)||)(1 + ||\mathbf{m}(\chi_v)||))^{N/(2(n-1))}}.\end{aligned}$$

For every $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ with $\boldsymbol{\ell}(\chi_v) \neq (0)_j$ and $||\mathbf{m}(\chi_v)|| > 1$ and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$, one has that

$$\begin{aligned}|\widehat{H}_v(\mathbf{s}, \chi_v)| &\leq \frac{(MD)^{1/(n-1)}(1 + ||\Im(\mathbf{s})||)^N}{||\boldsymbol{\ell}(\chi_v)||^{N/(2(n-1))}||\mathbf{m}(\chi_v)||^{N/(2(n-1))}} \\ &\leq \frac{2^{2N/(2(n-1))}(MD)^{1/(n-1)}(1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\mathbf{m}(\chi_v)||)(1 + ||\boldsymbol{\ell}(\chi_v)||))^{N/(2(n-1))}}\end{aligned}$$

Therefore $C = 2^{2N/(2(n-1))} \max((MD)^{1/(n-1)}, A)$ satisfies the wanted condition. Hence, the claim is valid when $n \geq 2$.

Suppose now that $n = 1$. In this case, one has that $m(\chi_v) = 0$ and that $||\ell(\chi_v)|| \geq 1$. The claim follows from the boundness of the Fourier transform (Proposition 7.1.3.3). □

7.4.7. — In 6.1.2 and in 6.2.1, we have defined norms of the characters of $\mathbb{A}_F^\times$ and of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, respectively. In this paragraph we give a corollary of Proposition 7.4.6.5 and we write it in terms on the norms.

For a character $\chi \in (\mathbb{A}_F^\times)^*$, we have defined:

$$\begin{aligned}||\chi||_{\text{discrete}} &= \max_{v \in M_F^\infty}(||\ell(\chi_v)||), \\ ||\chi||_\infty &= \max_{v \in M_F^\infty}(||m(\chi_v)||).\end{aligned}$$

For a character $\chi \in [\mathscr{T}^{\mathbf{a}}(\mathbb{A}_F)]^*$, we have defined

$$\begin{aligned}||\chi||_{\text{discrete}} &= \max_{v \in M_F^\infty}(||\boldsymbol{\ell}(\chi_v)||), \\ ||\chi||_\infty &= \max_{v \in M_F^\infty}(||\mathbf{m}(\chi_v)||).\end{aligned}$$

Proposition 7.4.6.5 implies that:

Corollary 7.4.7.1. — *Let $\mathcal{K} \subset \mathbb{R}^n_{>0}$ be a compact and let N be a positive integer. There exists $C > 0$ such that for every $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$*

and every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ *one has*

$$\prod_{v \in M_F^\infty} |\widehat{H}_v(\mathbf{s}, \chi_v)| \leq \frac{C(1 + ||\Im(\mathbf{s})||)^{N(r_1+r_2)}}{((1 + ||\chi||_{\text{discrete}})(1 + ||\chi||_\infty))^{N/(2(n-1))}}.$$

Proof. — We have that

$$(1 + ||\chi||_{\text{discrete}}) \leq \prod_{v \in M_F^\infty} (1 + ||\boldsymbol{\ell}(\chi_v)||)$$

and that

$$(1 + ||\chi||_\infty) \leq \prod_{v \in M_F^\infty} (1 + ||\mathbf{m}(\chi_v)||).$$

It follows from Proposition 7.4.6.5 that there exists $C_1 > 0$ such that for every $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ one has that

$$\begin{aligned}\prod_{v \in M_F^\infty} |\widehat{H}_v(\mathbf{s}, \chi_v)| &\leq \prod_{v \in M_F^\infty} \frac{C_1(1 + ||\Im(\mathbf{s})||)^N}{((1 + ||\mathbf{m}(\chi_v)||)(1 + |\boldsymbol{\ell}(\chi_v)||))^{N/(2(n-1))}} \\ &\leq \frac{C_1^{r_1+r_2}(1 + ||\Im(\mathbf{s})||)^{N(r_1+r_2)}}{\big((1 + ||\chi||_{\text{discrete}})(1 + ||\chi||_\infty)\big)^{N/(2(n-1)}}.\end{aligned}$$

The statement is proven. □

7.5. Global transform

Using the results of 7.3, 7.4 and 6.3, we obtain estimates for the global Fourier transform.

Let $\mu_{\mathbb{A}_F}$ be the Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ given by Definition 5.4.3.3

$$\mu_{\mathbb{A}_F} = \bigotimes_{v \in M_F^0} \zeta_v(1)^{n-1}\mu_v \otimes \bigotimes_{v \in M_F^\infty} \mu_v.$$

7.5.1. — In this paragraph we are going to present a Haar measure on the dual group $(\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$. Let us set

$$M = M(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a} \cdot \mathbf{x} = 0\},$$

$$\lambda_{\mathbf{a}} : \mathbb{R}_{>0} \to (\mathbb{R}_{>0})_{\mathbf{a}} \qquad x \mapsto (x^{a_j})_j.$$

The subspace M is the kernel of the surjective linear map $\mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{a} \cdot \mathbf{x}$ and we will identify $\mathbb{R}^n/M$ with $\mathbb{R}$ (for this identification the quotient map $\mathbb{R}^n \to \mathbb{R}^n/M$ becomes $\mathbf{x} \mapsto \mathbf{a} \cdot \mathbf{x}$).

Lemma 7.5.1.1. — *Let* $d\mathbf{m}$ *be the unique Lebesgue measure on* M *such that*

$$(dx_1 \dots dx_n)/d\mathbf{m} = dx.$$

1. *For* $A > 0$, *let us define*
$$\theta^n_A : \mathbb{R}^n \to \mathbb{R}^n \qquad \mathbf{x} \mapsto (Ax_j)_j.$$
We have that $(\theta^n_A|M)_*(d\mathbf{m}) = A^{-(n-1)}d\mathbf{m}$, *i.e. for any* $d\mathbf{m}$-*integrable function* f, *one has that*
$$\int_M f d\mathbf{m} = \frac{1}{A^{n-1}} \int_M f \circ \theta^n_A|_M d\mathbf{m}.$$
2. *The homomorphism*
$$\xi : \mathbb{R} \to (\mathbb{R}_{>0})^* \qquad t \mapsto |\cdot|^{2i\pi t}$$
is an isomorphism and one has $\xi_*(dx) = d^*x$. *The homomorphism*
$$\xi^n : \mathbb{R}^n \to (\mathbb{R}^n_{>0})^* \qquad \mathbf{x} \mapsto \prod_{j=1}^n |pr_j(\cdot)|^{2i\pi x_j},$$
where $pr_j : \mathbb{R}^n \to \mathbb{R}$ *is the projection to the* j*-th coordinate, is an isomorphism and one has* $\xi^n_*(dx_1 \dots dx_n) = d^*x_1 \dots d^*x_n$.
3. *Let* $(\mathbb{R}_{>0})^\perp_{\mathbf{a}}$ *be the group of characters of* $(\mathbb{R}^n_{>0})^*$ *which are trivial on* $(\mathbb{R}_{>0})_{\mathbf{a}}$. *One has that* $\xi^n(M) = (\mathbb{R}_{>0})^\perp_{\mathbf{a}} = (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$.
4. *The isomorphism* $\xi^n|_M : M \xrightarrow{\sim} (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$ *from (2) satisfies*
$$(\xi^n|_M)_*\big(d\mathbf{m}\big) = \big(d^*x_1 \dots d^*x_n/(\lambda_{\mathbf{a}})_*(d^*x)\big)^*.$$

Proof. — For $a, b > 0$, we have that $d^*x([a,b]) = \log b - \log a$. The homomorphism $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is an isomorphism, for which, hence, one has that $(\exp)_*dx = d^*x$. In the proof, using the isomorphism exp, we identify $\mathbb{R}$ and $\mathbb{R}_{>0}$ and the corresponding measures dx and d^*x.

1. For $A > 0$, let $\theta^1_A : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto Ax$. We observe that $(\theta^n_A)_*(dx_1 \cdots dx_n) = A^{-n}dx_1 \dots dx_n$ and that $(\theta^1_A)_*(dx) = A^{-1}dx$. Now, in the commutative diagram
$$\begin{tikzcd}
\{0\} \arrow[r] & M \arrow[r] \arrow[d, "\theta^n_A|_M"] & \mathbb{R}^n \arrow[r, "\mathbf{x}\mapsto\mathbf{a}\cdot\mathbf{x}"] \arrow[d, "\theta^n"] & \mathbb{R} \arrow[r] \arrow[d, "\theta^1_A"] & \{0\} \\
\{0\} \arrow[r] & M \arrow[r] & \mathbb{R}^n \arrow[r, "x\mapsto\mathbf{a}\cdot\mathbf{x}"] & \mathbb{R} \arrow[r] & \{0\}
\end{tikzcd}$$

the vertical maps are isomorphisms, and we deduce

$$\begin{aligned} A^{-1}dx = (\theta_A^1)_*(dx) &= (\theta_A^n)_*(dx_1 \dots dx_n)/(\theta_A^n|_M)_* d\mathbf{m} \\ &= A^{-n}dx_1 \dots dx_n/(\theta_A^n|_M)_* d\mathbf{m}. \end{aligned}$$

This gives $(\theta_A^n|_M)_* d\mathbf{m} = A^{-(n-1)} d\mathbf{m}$. For every $d\mathbf{m}$-integrable function f one has that

$$\int_M f d\mathbf{m} = \int_M (f \circ (\theta_A^n|_M))(\theta_A^n|_M)_* d\mathbf{m} = A^{-n+1} \int_M f \circ (\theta_A^n|M) d\mathbf{m}.$$

2. With this identification, the first claim is given in [**5**, Chapter II, §1, n^0 9, Corollary 1 of Proposition 11]. The second claim we deduce by the fact that the dual of a finite product of Haar measures is the product of the duals.
3. The isomorphism $\exp^n : \mathbb{R}^n \to (\mathbb{R}_{>0})^n$ identifies $\mathbf{a}(\mathbb{R}) = \{(a_j x)_j | x \in \mathbb{R}\}$ with $(\mathbb{R}_{>0})_{\mathbf{a}}$. Let $0 \neq \mathbf{x} \in \mathbf{a}(\mathbb{R})$ and let $0 \neq x \in \mathbb{R}$ such that $a_j x = x_j$ for every j. For every $\mathbf{y} \in M$, we have that

$$\xi^n(\mathbf{y})(\mathbf{x}) = \exp\left(\sum_{j=1}^n 2i\pi y_j x_j\right) = \exp\left(2i\pi x \sum_{j=1}^n a_j y_j\right).$$

Hence, $\xi^n(\mathbf{y}) \in (\mathbf{a}(\mathbb{R}))^\perp$ if and only if $\mathbf{y} \in M$ and the claim follows.
4. The dual sequence of the short exact sequence

$$0 \to \mathbb{R}_{>0} \xrightarrow{i_{(\mathbb{R}_{>0})_{\mathbf{a}}} \circ \lambda_{\mathbf{a}}} \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}} \to 0$$

is the short exact sequence sequence

$$0 \to (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^* \to \mathbb{R}^n \xrightarrow{\mathbf{x} \mapsto \mathbf{a}\cdot\mathbf{x}} \mathbb{R} \to 0.$$

Lemma 7.1.1.2 gives

$$dx_1 \dots dx_n / \big(d^*x_1 \dots d^*x_n/(\lambda_{\mathbf{a}})_*(d^*x)\big)^* = dx.$$

By definition $dx_1 \dots dx_n / d\mathbf{m} = dx$. Now, the commutativity of the diagram

$$\begin{tikzcd} \{0\} \arrow[r] & M \arrow[r] \arrow[d, "\xi^n_M"] & \mathbb{R}^n \arrow[r, "\mathbf{x}\mapsto\mathbf{a}\cdot\mathbf{x}"] \arrow[d, "\xi^n"] & \mathbb{R} \arrow[r] \arrow[d, "\xi"] & \{0\} \\ \{1\} \arrow[r] & (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^* \arrow[r] & \mathbb{R}^n \arrow[r] & \mathbb{R} \arrow[r] & \{0\}, \end{tikzcd}$$

gives that

$$(\xi^n|_M)_*(d\mathbf{m}) = (d^*x_1 \dots d^*x_n/(\lambda_{\mathbf{a}})_*(d^*x))^*.$$

□

For a character $\chi \in (\mathbb{R}_{>0})^*$, we denote by $m(\chi)$ the unique real number m such that χ is given by $x \mapsto x^{im}$. For a character $\chi \in (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$ we write

$$\mathbf{m}(\chi) := (m(\chi^{(j)}))_j \in M,$$

where $\chi^{(j)}$ is the pullback character $\mathbb{R}_{>0}$ given by the composite homomorphism

$$\mathbb{R}_{>0} \xrightarrow{x\mapsto((1)_{i\neq j},x)} \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}.$$

It follows from Lemma 7.5.1.1, that

$$(\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^* \to M \qquad \chi \mapsto \mathbf{m}(\chi)$$

is an isomorphism and we write $|\cdot|^{i\mathbf{m}}$ for the unique character of $(\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$ such that its image under the isomorphism is $\mathbf{m}$. Let $\widetilde{\mathbf{x}} \in \mathbb{R}^n_{>0}$ be a lift of $\mathbf{x} \in (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})$. We observe that

$$\xi^n(\mathbf{m})(\mathbf{x}) = \prod_{j=1}^n \widetilde{x}_j^{i2\pi m_j}$$

and that

$$|\mathbf{x}|^{i\mathbf{m}} = \prod_{j=1}^n \widetilde{x}_j^{i m_j}.$$

In other words

(7.5.1.2) $$\xi^n(\mathbf{m}) = |\cdot|^{i2\pi\mathbf{m}}.$$

7.5.2. — In this paragraph we estimate the global Fourier transform of the height.

In the equality 3.4.9.1, we have established an identification

$$\mathbb{A}_F^\times \xrightarrow{\sim} \mathbb{A}_F^1 \times \mathbb{R}_{>0}.$$

For a character $\chi \in (\mathbb{A}_F^\times)^*$ we write $m(\chi)$ for $m(\chi|_{\mathbb{R}_{>0}})$. In the equality 3.4.9.2, we have established an identification

$$[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)] \xrightarrow{\sim} [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 \times (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}).$$

For $(\mathbf{x}_v)_v \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$, let $(\widetilde{\mathbf{x}}_v)_v \in (\mathbb{A}_F^\times)^n$ be its lift. The morphism to the second coordinate is given by

$$\mathbf{x} \mapsto q_{\mathbb{R}_{>0}}\Big(\big(\prod_{v\in M_F} |\widetilde{x}_{jv}|_v\big)_j\Big),$$

where $q_{\mathbb{R}_{>0}} : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}$ is the quotient map. For a character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, we write $\mathbf{m}(\chi)$ for $\mathbf{m}(\chi|_{\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}})$. We write $|\cdot|^{i\cdot\mathbf{m}}$ for the unique character $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which satisfies $\chi|_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1} = 1$ and $\chi|_{\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}} = |\cdot|^{i\mathbf{m}}$. For $(\mathbf{x}_v)_v \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$, one has that

$$|(\mathbf{x}_v)_v|^{i\mathbf{m}} = \prod_{j=1}^n \Big| \prod_{v\in M_F} |\widetilde{x}_{jv}|_v \Big|^{im_j} = \prod_{j=1}^n \prod_{v\in M_F} |\widetilde{x}_{jv}|_v^{im_j}.$$

Lemma 7.5.2.1. — *1. For every $\mathbf{m} \in M$, one has that*

$$H(\mathbf{s},\cdot)|\cdot|^{i\cdot\mathbf{m}} = H(\mathbf{s}+i\mathbf{m},\cdot).$$

2. Let $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$. Whenever the quantities on the both hand sides converge, one has that

$$\widehat{H}(\mathbf{s},\chi) = \widehat{H}(\mathbf{s}+i\mathbf{m},\chi_0).$$

Proof. — 1. Let $(\mathbf{x}_v)_v \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ and let $(\widetilde{\mathbf{x}}_v)_v \in (\mathbb{A}_F^\times)^n$ be its lift. By using that $\mathbf{m} \in M$, we get that

$$\begin{aligned}
H(\mathbf{s},(\mathbf{x}_v)_v)|(\mathbf{x}_v)_v|^{i\mathbf{m}} &= \prod_{v\in M_F} H_v(\mathbf{s},\mathbf{x}_v) \prod_{j=1}^n |\widetilde{x}_{jv}|^{im_j} \\
&= \prod_{v\in M_F} \left(\left(f_v(\widetilde{\mathbf{x}}_v)^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{x}_j|_v^{s_j} \right) \prod_{j=1}^n |\widetilde{x}_{jv}|^{im_j} \right) \\
&= \prod_{v\in M_F} \left(f_v(\widetilde{\mathbf{x}}_v)^{\frac{-\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} \prod_{j=1}^n |\widetilde{x}_j|_v^{s_j+im_j} \right) \\
&= H(\mathbf{s}+i\mathbf{m},(\mathbf{x}_v)_v),
\end{aligned}$$

as claimed.

2. It follows from (1) that

$$H(\mathbf{s},\cdot)\chi = H(\mathbf{s}+i\mathbf{m},\cdot)\chi_0|\cdot|^{i\mathbf{m}(\chi)} = H(\mathbf{s}+i\mathbf{m},\cdot)\chi_0.$$

Now, one has that

$$\begin{aligned}\widehat{H}(\mathbf{s},\chi) &= \int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} H(\mathbf{s},\cdot)\chi\mu_{\mathbb{A}_F}\\ &= \int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} H(\mathbf{s},\cdot)\chi_0|\cdot|^{i\mathbf{m}(\chi)}\mu_{\mathbb{A}_F}\\ &= \int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]} H(\mathbf{s}+i\mathbf{m}(\chi),\chi_0)\mu_{\mathbb{A}_F}\\ &= \widehat{H}(\mathbf{s}+i\mathbf{m}(\chi),\chi_0),\end{aligned}$$

whenever every quantity converges. The claim is proven.

□

Given a character $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, let us denote by χ_0 the character $\chi|_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1}$. For every $\mathbf{s}\in\mathbb{C}^n$, Lemma 7.5.2.1 gives that

Lemma 7.5.2.2. — *1. For every $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, the product*

$$\widehat{H}(\mathbf{s},\chi):=\prod_{v\in M_F}\widehat{H}_v(\mathbf{s},\chi_v)$$

converges when $\mathbf{s}\in\Omega_{>1}$.

2. Let $K\subset K^{\mathbf{a}}_{\max}:=\prod_{v\in M_F^0}[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ be an open subgroup. For every $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ vanishing on K, the function

$$\mathbf{s}\mapsto\widehat{H}(\mathbf{s},\chi)\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}$$

extends to a holomorphic function in the domain $\Omega_{>\frac{2}{3}}$. Moreover, there exists $\frac{1}{3}>\delta>0$ such that for any compact in the domain $\mathcal{K}\subset\mathbb{R}^n_{>1-\delta}$ and any positive integer N, there exists $C(\mathcal{K},N)>0$ such that for any $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which vanishes on K and any $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ one has that

$$\left|\widehat{H}(\mathbf{s},\chi)\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\right| \leq\frac{C(\mathcal{K},N)(1+||\Im(\mathbf{s})||)^{N(r_1+r_2)+1}(1+||\mathbf{m}(\chi)||)}{(1+||\chi||_{\text{discrete}})^{N/2(n-1)-1}((1+||\chi||_\infty))^{N/2(n-1)-1}}. \tag{7.5.2.3}$$

Proof. — 1. Suppose $\mathbf{s}\in\Omega_{>1}$. For every $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$, by Proposition 7.3.1.1, one has that the product $\prod_{v\in M_F^0}\widehat{H}_v(\mathbf{s},\chi_v)$

converges absolutely and by Corollary 7.4.7.1, one has that $\mathbf{s} \mapsto \prod_{v \in M_F^\infty} \widehat{H}_v(\mathbf{s}, \chi_v)$ is holomorphic in the domain $\Omega_{>0}$. The claim follows.

2. By Proposition 7.3.1.1, there exists a unique holomorphic function $\phi(\cdot, \chi) : \Omega_{>\frac{2}{3}} \to \mathbb{C}$ such that

$$\begin{aligned}
&\widehat{H}(\mathbf{s}, \chi) \\
&\quad = \widehat{H}(\mathbf{s} + i\mathbf{m}(\chi), \chi_0) \\
&\quad = \phi_{\mathrm{fin}}(\mathbf{s} + i\mathbf{m}(\chi), \chi_0) \prod_{j=1}^{n} L(s_j + im(\chi^{(j)}), \chi_0^{(j)}) \prod_{v \in M_F^\infty} \widehat{H}_v(\mathbf{s} + i\mathbf{m}(\chi), \chi_{0v}).
\end{aligned}$$

The function $\mathbf{s} \mapsto \prod_{v \in M_F^\infty} \widehat{H}_v(\mathbf{s} + i\mathbf{m}(\chi), \chi_{0v})$ is holomorphic in the domain $\Omega_{>0}$ by Corollary 7.4.7.1. Recall that $L(\cdot, \chi_0)$ is an entire function for every $0 \neq \chi_0 \in \mathbb{A}_F^1$, while $L(\cdot, 1)$ is a meromorphic function with the single pole at 1 and no other poles. Therefore, the function

$$\mathbf{s} \mapsto \prod_{\chi_0^{(j)}=1} \frac{s_j + im(\chi^{(j)}) - 1}{s_j + im(\chi^{(j)})} L(s_j + im(\chi^{(j)}), \chi_0^{(j)}) \prod_{\chi_0^{(j)} \neq 1} L(s_j + im(\chi^{(j)}), \chi_0^{(j)})$$

extends to a holomorphic function in the domain $\Omega_{>\frac{2}{3}}$. Hence

$$\mathbf{s} \mapsto \widehat{H}(\mathbf{s} + i\mathbf{m}(\chi), \chi_0) \prod_{\chi_0^{(j)}=1} \frac{s_j + im(\chi^{(j)}) - 1}{s_j + im(\chi^{(j)})}$$

extends to a holomorphic function in the domain $\Omega_{>1-\delta}$. By Proposition 6.3.1.2, there exists $\frac{1}{3} > \delta > 0$ and $C_2 > 0$ such that

$$\begin{aligned}
&\Big| \prod_{\chi_0^{(j)}=1} \frac{s_j + im(\chi^{(j)}) - 1}{s_j + im(\chi^{(j)})} L(s_j + im(\chi^{(j)}), \chi_0^{(j)}) \prod_{\chi_0^{(j)} \neq 1} \cdot L(s_j + im(\chi^{(j)}), \chi_0^{(j)}) \Big| \\
&\qquad \leq C_2 \Big(\prod_{j=1}^{n} (1 + |\Im(s_j)|)(1 + ||\chi_0^{(j)}||_\infty)(1 + |m(\chi^{(j)})|) \Big)^{1/n} \\
&\qquad \leq C_2 (1 + ||\Im(\mathbf{s})||)(1 + ||\chi_0||_\infty)(1 + ||\mathbf{m}(\chi)||)
\end{aligned}$$

provided that $\Re(s_j) > 1 - \delta$ for $j = 1, \ldots, n$. Let N be an integer and let $\mathcal{K} \subset \mathbb{R}^n_{>1-\delta}$ be a compact. Proposition 7.3.1.1 gives that there exists $C_1 > 0$ such that $|\phi_{\text{fin}}(\mathbf{s},\chi)| \leq C_1$ for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$. By Lemma 7.4.7.1, there exists $C_3 > 0$ such that

$$\prod_{v\in M_F^\infty} |\widehat{H}_v(\mathbf{s}+i\mathbf{m}(\chi),\chi_{0v})| \leq \frac{C_3(1+||\Im(\mathbf{s})||)^{N(r_1+r_2)}}{(1+||\chi||_{\text{discrete}})^{N/(2(n-1))}(1+||\chi||_\infty)^{N/(2(n-1))}}$$

for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which vanishes on K. By Corollary 6.2.2.3, there exists $C_4 > 1$ such that

$$\frac{1+||\chi_0||_\infty}{1+||\chi_0||_{\text{discrete}}} \leq C_4.$$

We deduce that for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which vanishes on K, one has

$$\left|\widehat{H}(\mathbf{s}+i\mathbf{m}(\chi),\chi_0)\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\right| = \Bigg|\phi_{\text{fin}}(\mathbf{s}+i\mathbf{m}(\chi),\chi_0)\times$$

$$\times\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\prod_{j=1}^n L(s_j+im(\chi^{(j)}),\chi_0^{(j)})\prod_{v\in M_F^\infty}\widehat{H}_v(\mathbf{s}+i\mathbf{m}(\chi),\chi_{0v})\Bigg|$$

$$\leq \frac{C_1C_2C_3(1+||\Im(\mathbf{s})||)^{N(r_1+r_2)+1}(1+||\chi_0||_\infty)(1+||\mathbf{m}(\chi)||)}{((1+||\chi_0||_{\text{discrete}})(1+||\chi||_\infty))^{N/(2(n-1))}}$$

$$\leq \frac{C_1C_2C_3C_4(1+||\Im(\mathbf{s})||)^{N(r_1+r_2)+1}(1+||\mathbf{m}(\chi)||)}{(1+||\chi_0||_{\text{discrete}})^{N/(2(n-1))-1}(1+||\chi_0||_\infty)^{N/(2(n-1))}}.$$

The claim follows.

□

Note that if $\chi_1,\chi_2 \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ are two characters then

$$\begin{aligned}||\chi_1\chi_2||_\infty &= \max_{v\in M_F^\infty}||\mathbf{m}(\chi_{1v}\chi_{2v})||\\ &= \max_{v\in M_F^\infty}||\mathbf{m}(\chi_{1v})+\mathbf{m}(\chi_{2v})||\\ &\leq \max_{v\in M_F^\infty}||\mathbf{m}(\chi_{1v})|| + \max_{v\in M_F^\infty}||\mathbf{m}(\chi_{2v})||\\ &= ||\chi_1||_\infty + ||\chi_2||_\infty.\end{aligned}$$

Now, if $\chi_0 = \chi|\cdot|^{-i\mathbf{m}}$ for $\chi \in [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ and $\mathbf{m} \in M$, using the inequality

$$\frac{1}{1+|x+y|} \leq \frac{1+|x|}{1+|y|} \qquad x,y\in\mathbb{C},$$

we can deduce that

$$\frac{1}{1+||\chi||_\infty}=\frac{1}{1+||\chi_0|\cdot|^{-i\mathbf{m}}||_\infty}\leq\frac{1+||\chi_0||_\infty}{1+||\mathbf{m}||}. \tag{7.5.2.4}$$

We can establish the following corollary:

Corollary 7.5.2.5. — *Let $K\subset K^{\mathbf{a}}_{\max}$ be an open subgroup. For every $\alpha>0$ there exist $\beta=\beta(\alpha)>0$ and $\delta=\delta(\alpha)>0$ such that for every compact $\mathcal{K}\subset\mathbb{R}^n_{>1-\delta}$, one has that there exists $C=C(\alpha,\mathcal{K})>0$ such that for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\chi\in\mathscr{T}(\mathbf{a})(\mathbb{A}_F)^*$ which vanishes on K one has*

$$\left|\widehat{H}(\mathbf{s},\chi)\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\right|\leq\frac{C(1+||\Im(\mathbf{s})||)^\beta}{((1+||\chi_0||_{\text{discrete}})(1+||\mathbf{m}(\chi)||))^\alpha}.$$

Proof. — Firstly, let $C_1>0$ be such that for every $\chi_0\in\mathfrak{A}_K$ one has

$$\frac{1+||\chi_0||_\infty}{1+||\chi_0||_{\text{discrete}}}\leq C_1,$$

such C_1 exists by Corollary 6.2.2.3. Let N be an integer bigger than $2(n-1)(2\alpha+2)$ and let $\beta>N(r_1+r_2)+1$. Let δ be given by Lemma 7.5.2.2. It follows from this lemma and from the estimate (7.5.2.4) that for every compact $\mathcal{K}\subset\mathbb{R}^n_{>1-\delta}$, there exists $C(\mathcal{K},N)$ such that for every $\chi\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]^*$ which vanishes on K and every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ one has

$$\begin{aligned}
&\left|\widehat{H}(\mathbf{s},\chi)\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\right|\\
&\qquad\leq\frac{C(\mathcal{K},N)(1+||\Im(\mathbf{s})||)^{N(r_1+r_2)+1}(1+||\mathbf{m}(\chi)||)}{(1+||\chi_0||_{\text{discrete}})^{\frac{N}{2(n-1)}-1}(1+||\chi||_\infty)^{\frac{N}{2(n-1)}}}\\
&\qquad\leq\frac{C(\mathcal{K},N)(1+||\Im(\mathbf{s})||)^{\beta}(1+||\mathbf{m}(\chi)||)}{((1+||\chi_0||_{\text{discrete}})(1+||\chi||_\infty))^{2\alpha+1}}.\\
&\qquad\leq\frac{C(1+||\Im(\mathbf{s})||)^{\beta}(1+||\chi_0||_\infty)^{\alpha+1}}{(1+||\chi_0||_{\text{discrete}})^{2\alpha+1}(1+||\mathbf{m}(\chi)||)^\alpha}\\
&\qquad\leq\frac{CC_1^{\alpha+1}(1+||\Im(\mathbf{s})||)^{\beta}}{((1+||\chi_0||)(1+||\mathbf{m}(\chi)||))^\alpha}.
\end{aligned}$$

The claim follows. □

7.5.3. — For an open subgroup $K \subset K^{\mathbf{a}}_{\max} = \prod_{v\in M_F^0}[\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$, we denote by $\mathfrak{A}_K$ the subgroup of $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1^*$ given by the characters that vanish on $[\mathscr{T}(\mathbf{a})(i)(F)]K$. In this paragraph we explain that one can sum transforms over the characters of $\mathfrak{A}_K$. The following lemma will be used.

Lemma 7.5.3.1. — *Suppose X is a discrete set. Let $f : X \to \mathbb{R}_{\geq 0}$ be such that there exists $A > 0$ and $d > 0$ such that for every $B > 0$ one has*

$$|\{x \in X | f(x) \leq B\}| \leq AB^d.$$

Let $\epsilon > 0$. The series

$$\sum_{x\in X} \frac{1}{(1+f(x))^N}$$

and the series

$$\sum_{\substack{x\in X\\ f(x)>0}} \frac{1}{f(x)^N}$$

converge for $N > d + \epsilon$.

Proof. — For every $i \in \mathbb{Z}_{\geq 1}$, let us set

$$w(i) = |\{x \in X | i-1 \leq f(x) < i\}|.$$

Let $B \geq 1$ be an integer. By Abel's summation formula, for $N > d + \epsilon$ we have that

$$\begin{aligned}
\sum_{\substack{x\in X\\ f(x)\leq B}} \frac{1}{(1+f(x))^N} &\leq \sum_{i=1}^{B+1} \frac{w(i)}{i^N} \\
&= \frac{1}{(1+B)^N}\sum_{r=1}^{B+1} w(r) + \sum_{i=1}^{B}\Big(\frac{1}{i^N} - \frac{1}{(i+1)^N}\Big)\sum_{j=1}^{i} w(j) \\
&\leq \frac{AB^d}{(1+B)^N} + \sum_{i=1}^{B}\Big(\frac{1}{i^N} - \frac{1}{(1+i)^N}\Big)Ai^d \\
&\leq A + \sum_{i=1}^{B} Ai^d \frac{2^N i^{N-1}}{i^N(i+1)^N} \\
&\leq A + \sum_{i=1}^{B} \frac{2^N A}{i^{N+1-d}} \\
&\leq A + 2^N A\zeta(N+1-d).
\end{aligned}$$

(we have used that $(1+i)^N - i^N \leq 2^N i^{N-1}$). It follows that $\sum_{x\in X}\frac{1}{(1+f(x))^N}$ converges uniformly for $N > d+\epsilon$. Moreover, for $N > d+\epsilon$, one has that

$$\begin{aligned}\sum_{\substack{x\in X\\ f(x)>0}}\frac{1}{f(x)^N} &= \sum_{\substack{x\in X\\ 1>f(x)>0}}\frac{1}{f(x)^N}+\sum_{\substack{x\in X\\ f(x)>1}}\frac{1}{f(x)^N}\\ &\leq \sum_{\substack{x\in X\\ 1>f(x)>0}}\frac{1}{f(x)^N}+\sum_{\substack{x\in X\\ f(x)>1}}\frac{2}{(1+f(x))^N}.\end{aligned}$$

As the sum $\sum_{\substack{x\in X\\ 1>f(x)>0}}\frac{1}{f(x)^N}$ is a finite sum and as the sum $\sum_{\substack{x\in X\\ f(x)>1}}\frac{2}{(1+f(x))^N}$ converges uniformly for $N > d+\epsilon$, we deduce that $\sum_{\substack{x\in X\\ f(x)>0}}\frac{1}{f(x)^N}$ converges uniformly for $N > d+\epsilon$. The statement is proven. □

We set

$$\widehat{H}^*(\mathbf{s},\chi) := \widehat{H}(\mathbf{s},\chi)\prod_{j=1}^{n}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}.$$

Definition 7.5.3.2. — *We define formally*

$$g_K(\mathbf{s}) := \sum_{\chi_0\in\mathfrak{A}_K}\widehat{H}(\mathbf{s},\chi_0)$$

and

$$g_K^*(\mathbf{s}) := \sum_{\chi_0\in\mathfrak{A}_K}\widehat{H}^*(\mathbf{s},\chi_0).$$

Proposition 7.5.3.3. — *Let $K\subset K^{\mathbf{a}}_{\max}$ be an open subgroup and let $\alpha > 0$. There exist $\delta=\delta(\alpha)>0$ and $\beta=\beta(\alpha)>0$ such that the following conditions are satisfied:*

1. *The series*

$$g_K^*(\mathbf{s}) := \sum_{\chi_0\in\mathfrak{A}_K}\widehat{H}^*(\mathbf{s},\chi_0) \tag{7.5.3.4}$$

   converges absolutely and uniformly on compacts in the domain $\Omega_{>1-\delta}$ and the function $\mathbf{s}\mapsto g_K^*(\mathbf{s})$ is holomorphic in the domain $\Omega_{>1-\delta}$.
2. *For every compact $\mathcal{K}\subset\Omega_{>1-\delta}$ one has that there exists $C = C(\alpha,\mathcal{K})>0$ such that for every $\mathbf{s}\in\mathcal{K}+i\mathbb{R}^n$ and every $\mathbf{m}\in M$ one*

has that

$$|g_K^*(\mathbf{s}+i\mathbf{m})| \leq \frac{C(1+||\Im(\mathbf{s})||)^\beta}{(1+||\mathbf{m}||)^\alpha}.$$

Proof. — Let $\alpha > 0$. By Corollary 7.5.2.5, there exist $\frac{1}{3} > \delta > 0$ and $\beta > 0$ such that for any compact $\mathcal{K} \subset \mathbb{R}^n_{>1-\delta}$ there exists $C > 0$ such that for every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$, every $\chi_0 \in \mathfrak{A}_K$ and every $\mathbf{m} \in M$ one has for $\chi = \chi_0|\cdot|^{i\mathbf{m}}$ that:

(7.5.3.5)

$$\begin{aligned}\left|\widehat{H}(\mathbf{s}+i\mathbf{m},\chi_0)\prod_{\chi_0^{(j)}=1}\frac{s_j+im_j-1}{s_j+im_j}\right| &= \left|\widehat{H}(\mathbf{s},\chi_0|\cdot|^{i\mathbf{m}})\prod_{\chi_0^{(j)}=1}\frac{s_j+im(\chi^{(j)})-1}{s_j+im(\chi^{(j)})}\right| \\ &\leq \frac{C(1+||\Im(\mathbf{s})||)^\beta}{((1+||\chi_0||_{\text{discrete}})(1+||\mathbf{m}||))^\alpha}.\end{aligned}$$

We prove that the series (7.5.3.4) converges absolutely and uniformly on compacts of $\Omega_{>1-\delta}$. Let $\mathcal{G} \subset \Omega_{>1-\delta}$ be a compact set and let $C(\mathcal{G}) > 0$ be such that for every $\mathbf{s} \in \mathcal{G}$ one has $||\Im(\mathbf{s})|| < C(\mathcal{G})$. Let $\mathcal{K} \subset \mathbb{R}^n_{>1/2}$ be a compact such that $\mathcal{G} \subset \mathcal{K} + i\mathbb{R}^n$. Note that for $j = 1,\dots,n$ and $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ one has that

$$\left|\frac{s_j-1}{s_j}\right| \leq 3.$$

Given $\alpha > nr_2 + 1$, it follows from the estimate (7.5.3.5) that for every $\chi_0 \in \mathfrak{A}_K$ and every $\mathbf{s} \in \mathcal{G}$ one has that

$$\begin{aligned}\left|\widehat{H}(\mathbf{s},\chi_0)\prod_{j=1}^n\frac{s_j-1}{s_j}\right| &\leq \left|\frac{C(\alpha,\mathcal{K})(1+||\Im(\mathbf{s})||)^\beta}{(1+||\chi_0||_{\text{discrete}})^\alpha}\prod_{\chi^{(j)}\neq 1}3\right| \\ &\leq \frac{(1+C(\mathcal{G}))^\beta C(\alpha,\mathcal{K})3^n}{(1+||\chi_0||_{\text{discrete}})^\alpha}.\end{aligned} \tag{7.5.3.6}$$

Now, by Corollary 6.2.2.4, there exists $B > 0$ such that for every $A > 0$ one has that $|\{\chi_0 \in \mathfrak{A}_K|\ ||\chi_0||_{\text{discrete}} < A\}| \leq BA^{nr_2}$. The set $\mathfrak{A}_K$ is discrete (Lemma 6.1.2.4), and therefore the estimate (7.5.3.6) and Lemma 7.5.3.1 give that the series $\sum_{\chi_0\in\mathfrak{A}_K}\frac{1}{(1+||\chi_0||)^\alpha}$ converges. We deduce that the series (7.5.3.4) converges absolutely for every $\mathbf{s} \in \mathcal{G}$. Moreover, for $M > 0$ one has that if $||\chi_0||_{\text{discrete}} > M$, then

$$|\widehat{H}(\mathbf{s},\chi_0)| \leq \frac{C(\mathcal{G})^\beta C(\alpha,\mathcal{K})3^n}{(1+M)^\alpha}$$

for every $\mathbf{s} \in \mathcal{G}$, and, hence, the convergence is uniform on compacts. As for every $\chi_0 \in \mathfrak{A}_K$, the function $\mathbf{s} \mapsto \widehat{H}(\mathbf{s}, \chi_0) \prod_{j=1}^n \frac{s_j - 1}{s_j}$ is holomorphic in the domain $\Omega_{>1-\delta}$, we deduce that $\mathbf{s} \mapsto g_K^*(\mathbf{s})$ is holomorphic in the domain $\Omega_{>1-\delta}$. Let $\mathcal{K} \subset \mathbb{R}^n_{>1-\delta}$ be a compact. For every $\mathbf{s} \in \mathcal{K} + i\mathbb{R}^n$ and every $\mathbf{m} \in M$, it follows from the estimate (7.5.3.5) that

$$\begin{aligned} |g_K^*(\mathbf{s} + i\mathbf{m})| &= \left| \sum_{\chi_0 \in \mathfrak{A}_K} \widehat{H}(\mathbf{s} + i\mathbf{m}, \chi_0) \prod_{j=1}^n \frac{s_j + m_j - 1}{s_j + m_j} \right| \\ &\leq \frac{C(1 + ||\Im(\mathbf{s})||)^\beta}{(1 + ||\mathbf{m}||)^\alpha} \sum_{\chi_0 \in \mathfrak{A}_K} \frac{1}{(1 + ||\chi_0||_{\mathrm{discrete}})^\alpha}. \end{aligned}$$

The statement follows. □

The function $\mathbf{s} \mapsto \prod_{j=1}^n \frac{s_j - 1}{s_j}$ does not vanish in the domain $\Omega_{>1}$. We deduce that in this domain the series defining

$$g_K(\mathbf{s}) = g_K^*(\mathbf{s}) \prod_{j=1}^n \frac{s_j}{s_j - 1}$$

converges absolutely. Therefore $\mathbf{s} \mapsto g_K(\mathbf{s})$ is a holomorphic function in this domain.

CHAPTER 8

ANALYSIS OF HEIGHT ZETA FUNCTIONS

8.1. Analysis of M-controlled functions

In this section we adapt the analysis of [17] to our needs. We recall the definition of M-controlled functions and establish properties of their integrals.

8.1.1. — Let $d \geq 1$ be an integer and $U \subset \mathbb{R}^d$ an open subset. For a vector subspace $M \subset \mathbb{R}^d$, we say that a function $f : U + i\mathbb{R}^d \to \mathbb{C}$ is M-controlled if for every $\alpha > 0$, there exists $\beta > 0$ such that for any compact $\mathcal{K} \subset U$, there exists $C(\mathcal{K}) > 0$ such that for every $\mathbf{m} \in M$ and every $\mathbf{s} \in \mathbb{C}^n$ with $\Re(\mathbf{s}) \in \mathcal{K}$, one has

$$|f(\mathbf{s} + i \cdot \mathbf{m})| \leq \frac{C(\mathcal{K})(1 + ||\Im(\mathbf{s})||)^{\beta}}{(1 + ||\mathbf{m}||)^{\alpha}}. \tag{8.1.1.1}$$

Remark 8.1.1.2. — Note that if f is M-controlled, then f is M-controlled in the sense of [17, Section 4.3]. There, the condition is that there exists a family of linear forms $(\ell_j)_j$ in V^* such that the $\ell_j|M$ form a basis of M and such that there exists $\beta' > 0$ and $1 > \epsilon > 0$ for which for any compact $\mathcal{K} \subset U$ there exists $C'(\mathcal{K}) > 0$ such that for every $\mathbf{m} \in M$ one has

$$|f(\mathbf{s} + i \cdot \mathbf{m})| \leq \frac{C'(\mathcal{K})(1 + ||\Im(\mathbf{s})||)^{\beta'}}{(1 + ||\mathbf{m}||)^{1-\epsilon}} \frac{1}{\prod(1 + |\ell_j(\mathbf{s} + \mathbf{m})|)} \tag{8.1.1.3}$$

if provided $\Re(\mathbf{s}) \in \mathcal{K}$. We verify that our condition is stronger. The inequality

$$\frac{1}{1 + |x + y|} \leq \frac{1 + |x|}{1 + |y|} \qquad x, y \in \mathbb{C}$$

gives that
$$\frac{1+|\ell_j(\mathbf{s})|}{1+|\ell_j(\mathbf{s}+\mathbf{m})|} \geq \frac{1}{1+|\ell_j(\mathbf{m})|}.$$
Now there exist $C_0, C_1 > 0$ such that
$$\prod_{\ell_j}(1+|\ell_j(\mathbf{s})|) \leq C_0(1+||\Im(\mathbf{s})||)^{\dim M}$$
and
$$\frac{1}{\prod_{\ell_j}(1+|\ell_j(\mathbf{m})|)} \geq \frac{C_1}{(1+||\mathbf{m}||)^{\dim M}},$$
provided that $\Re(\mathbf{s}) \in \mathcal{K}$. Hence, if f satisfies our condition it satisfies the estimate (8.1.1.3).

The following result is given as Lemma 3.1.6 in [**17**].

Lemma 8.1.1.4 **(Chambert-Loir, Tschinkel,** [**17**, Lemma 3.1.6]**)**

Let $q : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective map and let $M \subset \mathbb{R}^n$ be its kernel. For $\mathbf{s} \in \mathbb{C}^n$, let $\widetilde{\mathbf{s}} \in \mathbb{C}^n$ be such that $q_{\mathbb{C}}(\widetilde{\mathbf{s}}) = \mathbf{s}$. Endow M with the unique Lebesgue measure $d\mathbf{m}$ such that $(dx_1 \dots dx_n)/d\mathbf{m} = dx_1 \dots dx_k$. Suppose that $f : U + i\mathbb{R}^k \to \mathbb{C}$ is an M-controlled holomorphic function. The integral
$$\frac{1}{(2\pi)^{n-k}} \int_M f(\widetilde{\mathbf{s}} + i\mathbf{m})d\mathbf{m}$$
converges for every $\mathbf{s} \in U + i\mathbb{R}^n$ and the value of
$$\mathscr{S}_M(f) : \mathbf{s} \mapsto \frac{1}{(2\pi)^{n-k}} \int_M f(\widetilde{\mathbf{s}} + i\mathbf{m})d\mathbf{m}$$
does not depend on the choice of $\widetilde{\mathbf{s}}$ and the resulting map $\mathscr{S}_M(f) : q(U) + i\mathbb{R}^k \to \mathbb{C}$ is holomorphic and $\{0\}$-controlled.

Remark 8.1.1.5. — The original Lemma 3.1.6 has been simplified by assuming $M' = M$ (with the notation as in 3.1.6 in [**17**]).

The following result is Theorem 3.1.14 in [**17**].

Theorem 8.1.1.6 **(Chambert-Loir, Tschinkel,** [**17**, Theorem 3.1.14]**)**

Let $q : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map such that $q(\mathbb{R}^n_{\geq 0}) = \mathbb{R}^k_{\geq 0}$ and let $M = \ker q$. Let $f : \Omega_{>0} \to \mathbb{C}$ be a holomorphic function such that there exists an open ball $B \subset \mathbb{R}^n$ centred at 0 such that $\mathbf{s} \mapsto f(\mathbf{s}) \prod_{j=1}^n \frac{s_j}{s_j+1}$ extends to an M-controlled holomorphic function on $(B + i\mathbb{R}^n) \cup \Omega_{>0}$. Then there exists an open neighbourhood B' of 0 in $\mathbb{R}^k$ such that $\mathscr{S}_M(f) \prod_{j=1}^k \frac{s_j}{s_j+1}$ extends to a holomorphic $\{0\}$-controlled *function in the domain* $(B' + i\mathbb{R}^k) \cup (\mathbb{R}^k_{>0} + i\mathbb{R}^k)$. *Moreover, if one has for every* $\mathbf{x} \in \mathbb{R}^n_{>0}$ *that*

$$\lim_{s\to 0^+} \left(s^n \prod_{j=1}^{n} x_j\right) f(s\mathbf{x}) = a,$$

for some $0 \neq a \in \mathbb{R}$, *then one has for every* $\mathbf{x}' \in \mathbb{R}^k_{>0}$ *that*

$$\lim_{s\to 0^+} \left(s^k \prod_{j=1}^{k} x'_j\right) \mathscr{S}_M(f)(s\mathbf{x}') = a.$$

Remark 8.1.1.7. — The original statement of Theorem 8.1.1.6 is somewhat simplified here. With notation as in Theorem 3.1.14 of [**17**], we have supposed that $M = M'$ and $C = \Lambda = \mathbb{R}^n_{\geq 0}$. We have also added the condition $q(\mathbb{R}^n_{\geq 0}) = \mathbb{R}^k_{\geq 0}$ in order to make calculations of the characteristic functions of cones ([**17**, Section 3.1.7]) simple. For every $k \in \mathbb{Z}_{\geq 1}$, the cone $\mathbb{R}^k_{>0}$ is simplicial and its characteristic function (according to [**17**, Section 3.1.7]) is given by

$$\Omega_{>0} \to \mathbb{C} \qquad \mathbf{s} \mapsto \frac{1}{\prod_{j=1}^{n} s_k}.$$

8.2. Height zeta function

We define and prove holomorphicity of a height zeta function. Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a quasi-toric family of weighted degree d.

8.2.1. — We define height zeta functions.

Definition 8.2.1.1. — *For* $\mathbf{s} \in \mathbb{C}^n$ *we formally define series*

$$\mathring{Z}((f_v)_v)(\mathbf{s}) = \mathring{Z}(\mathbf{s}) := \sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]} H(\mathbf{s},\mathbf{x})^{-1}$$

and

$$Z((f_v)_v)(\mathbf{s}) = Z(\mathbf{s}) := \sum_{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]} H(\mathbf{s},\mathbf{x})^{-1}$$

Proposition 8.2.1.2. — *Let* $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ *be a degree* $|\mathbf{a}|$ *quasi-toric family of* $\mathbf{a}$*-homogenous functions. For* $\mathbf{s} \in \Omega_{>0}$ *we set* $H(\mathbf{s}, -) := H^{\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}$. *Let* $\frac{1}{2} > \epsilon > 0$. *The height zeta function series defining* $Z(\mathbf{s})$ *and* $\mathring{Z}(\mathbf{s})$ *converge absolutely and uniformly in the domain* $\Omega_{>1+\epsilon}$, *and defines a holomorphic function in this domain.*

Proof. — By Theorem 4.6.8.2, one has that there exists $C > 0$ such that

$$|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x}) \leq B\}| \leq CB^{1+\epsilon/2}$$

for every $B > 0$. Note that if $\mathbf{s} \in \Omega_{>1+\epsilon}$ then one has that $\frac{\mathbf{a}\cdot\Re(\mathbf{s})}{|\mathbf{a}|} > 1+\epsilon > 1+\epsilon/2$. Thus by Lemma 7.5.3.1, we have that the series

$$\sum_{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]} H(\mathbf{s},\mathbf{x})^{-1} = \sum_{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]} H(\mathbf{x})^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}$$

converges absolutely and uniformly in the domain $\Omega_{>1+\epsilon}$. It follows that the function Z is holomorphic in this domain. One has that

$$Z(\mathbf{s}) = \mathring{Z}(\mathbf{s}) + \sum_{\mathbf{x}\in([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])} H(\mathbf{s},\mathbf{x}).$$

Let $\delta > 0$ be such that $(1+\delta)(|\mathbf{a}| - \min_j a_j)/|\mathbf{a}| \leq \max(1/2, 1-\epsilon)$. Theorem 4.6.8.2 gives that there exists $C'(\delta) > 0$ such that

$$\begin{aligned}|\{\mathbf{x} \in ([\mathscr{P}(\mathbf{a})(F)] - [\mathscr{T}(\mathbf{a})(F)]) | H(\mathbf{x}) \leq B\}|& \\ \leq C'(\delta)B^{(1+\delta)(|\mathbf{a}|-\min_j a_j)/|\mathbf{a}|}&.\end{aligned}$$

Note that if $\mathbf{s} \in \Omega_{>\max(1/2,1-\epsilon)}$, then $\frac{\mathbf{a}\cdot\Re(\mathbf{s})}{|\mathbf{a}|} > \max(1/2, 1-\epsilon)$. Lemma 7.5.3.1 gives that

$$\sum_{\mathbf{x}\in([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])} H(\mathbf{s},\mathbf{x}) = \sum_{\mathbf{x}\in([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])} H(\mathbf{x})^{-\frac{\mathbf{a}\cdot\Re(\mathbf{s})}{|\mathbf{a}|}}$$

converges absolutely and uniformly in the domain $\mathbf{s} \in \Omega_{>\max(1/2,1-\epsilon)}$. It follows that the function defined by the series is holomorphic. Consequently, the series defining $\mathring{Z}$ also converges absolutely and uniformly in the domain $\mathbf{s} \in \Omega_{>1+\epsilon}$ and defines a holomorphic function in this domain. The statement is proven. $\square$

8.2.2. — The goal of this paragraph is to apply Poisson formula to understand the analytic behaviour of the height zeta series.

We suppose $n \geq 1$ is an integer and $\mathbf{a} \in \mathbb{Z}^n_{>0}$ if $n \geq 2$ and $\mathbf{a} = a \in \mathbb{Z}_{>1}$ if $n = 1$. Recall that in Definition 8.2.1.1 for $\mathbf{s} \in \mathbb{C}^n$, we have defined formally

$$\mathring{Z}(\mathbf{s}) = \sum_{\mathbf{x}\in\mathscr{T}(\mathbf{a})(F)} H(\mathbf{x})^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}}$$

and in Proposition 8.2.1.2, we have established that there exists $\gamma > 0$ such that the series converges absolutely and uniformly for any $\mathbf{s} \in \Omega_{>\gamma}$ and defines a holomorphic function in this domain.

For $v \in S$ (that is the finite places v for which f_v is not toric), by Lemma 7.2.2.2, one can take an open subgroup $K_v \subset [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$, such that if $\chi_v \in [\mathscr{T}(\mathbf{a})(F_v)]^*$ does not vanish at K_v, then $\widehat{H}_v(\mathbf{s}, \chi_v) = 0$ for every $\mathbf{s} \in \Omega_{>0}$. We set $K_v = [\mathscr{T}(\mathbf{a})(\mathcal{O}_v)]$ for $v \in M_F^0 \cap S$. Let us set $K = \prod_{v \in M_F^0} K_v$ and let $\mathfrak{A}_K$ be the group of characters $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1 \to S^1$ which vanish on $[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])$ and on K. By Corollary 6.2.2.2, the group $\mathfrak{A}_K$ is a finitely generated abelian group.

In Definition 7.5.3.2, we have defined $g_K(\mathbf{s}) = \sum_{\chi_0 \in \mathfrak{A}_K} \widehat{H}(\mathbf{s}, \chi_0)$. Recall that $M = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = 0\}$. By Proposition 7.5.3.3 one has that $g_K(\mathbf{1} + \mathbf{s})$ converges absolutely and uniformly in the domain $\Omega_{>0}$, that $\mathbf{s} \mapsto g_K(\mathbf{1} + \mathbf{s})$ is M-controlled and holomorphic function in the domain $\Omega_{>0}$ and that there exists $\delta > 0$ such that $\mathbf{s} \mapsto g_K(\mathbf{1} + \mathbf{s}) \prod_{j=1}^n \frac{s_j}{s_j+1}$ extends to a holomorphic M-controlled function in the domain $\Omega_{>-\delta}$.

The following lemma will be used to determine the exact constant in Poisson formula.

Lemma 8.2.2.1. — *The measure $(\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])})^*$ on*

$$([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]))^* = \big([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])\big)^* \times (\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$$

(the identification follows from the identification (3.4.9.3) and Lemma 7.1.1.2) satisfies that

$$(\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])})^* = \frac{1}{E(\mathbf{a})}\operatorname{count}_{([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]))^*} \times (d^*\mathbf{r}_{\mathbf{a}})^*,$$

where

$$E(\mathbf{a}) := \frac{|\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})| \operatorname{Res}(\zeta_F, 1)^{n-1} \Delta(F)^{\frac{n-1}{2}}}{|\mu_{\gcd(\mathbf{a})}(F)|}.$$

and where we write $d^*\mathbf{r}_{\mathbf{a}}$ for the measure $(d^*r_1 \dots d^*r_n)/(\lambda_{\mathbf{a}})_*(d^*r)$ on $\mathbb{R}^n_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}}$.

Proof. — Let μ_1 be the Haar measure on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1$ normalized by $\mu_{\mathbb{A}_F}/\mu_1 = d^*\mathbf{r}_{\mathbf{a}}$. For the above identification, one has that

$$\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])} = \mu_1/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])} \times d^*\mathbf{r}_{\mathbf{a}}.$$

We denote $\widetilde{\mu_1} := \mu_1 / \operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])}$. This measure satisfies

$$\widetilde{\mu_1} = \frac{|\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})|}{|\mu_{\gcd(\mathbf{a})}(F)|}\overline{\mu_1},$$

where $\overline{\mu_1}$ is the measure from Definition 5.4.4.3. Proposition 5.4.4.4 gives that

$$\begin{aligned}
&\widetilde{\mu_1}([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])) \\
&\qquad = \frac{|\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})|}{|\mu_{\gcd(\mathbf{a})}(F)|}\overline{\mu_1}([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])) \\
&\qquad = \frac{|\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})| \operatorname{Res}(\zeta_F, 1)^{n-1}\Delta(F)^{\frac{n-1}{2}}}{|\mu_{\gcd(\mathbf{a})}(F)|} \\
&\qquad = E(\mathbf{a}).
\end{aligned}$$

Using Lemma 7.1.1.3, we obtain that

$$\widetilde{\mu_1}^* = \frac{1}{E(\mathbf{a})} \operatorname{count}_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])^*}.$$

The statement follows. $\square$

Proposition 8.2.2.2. — *For every $\mathbf{s} \in \Omega_{>0}$, both sides of*

$$\mathring{Z}(\mathbf{1}+\mathbf{s}) = \frac{|\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})} \int_M g_K(\mathbf{1}+\mathbf{s}+i\cdot\mathbf{m})d\mathbf{m}$$

converge and the equality is valid.

Proof. — By Lemma 3.4.7.1, the kernel of the map $[\mathscr{T}(\mathbf{a})(F)] \to [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ is isomorphic to the finite group $\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})$. Using this and the fact that

$$H(\mathbf{x})^{-\frac{\mathbf{a}\cdot\mathbf{s}}{|\mathbf{a}|}} = H(\mathbf{s}, \mathscr{T}(\mathbf{a})(i)(\mathbf{x}))$$

(Lemma 7.1.2.2), we deduce that

$$\mathring{Z}(\mathbf{1}+\mathbf{s}) = |\text{Ш}^1(F, \mu_{\gcd(\mathbf{a})})| \sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])} H(\mathbf{1}+\mathbf{s}, \mathbf{x})^{-1}.$$

Poisson formula (Proposition 7.1.1.4) applied to the inclusion

$$[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]) \subset [\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$$

gives that

$$\sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])} H(\mathbf{1}+\mathbf{s},\mathbf{x})^{-1}$$
$$= \int_{([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]))^*} \widehat{H}(\mathbf{1}+\mathbf{s},-)(\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])})^*,$$

for every $\mathbf{s}$ for which the both sides converge and, hence,

$$\mathring{Z}(\mathbf{1}+\mathbf{s}) = |\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|\times$$
$$\times\int_{[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])^*} \widehat{H}(\mathbf{1}+\mathbf{s},-)(\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])})^*,$$

for every $\mathbf{s}$ that the both sides converge. By Lemma 7.5.1.1, the homomorphism

$$\xi^n:\mathbb{R}^n\to(\mathbb{R}^n_{>0})^* \qquad \mathbf{x}\mapsto(\mathbf{r}\mapsto\prod_{j=1}^n r_j^{2i\pi x_j})$$

induces an isomorphism $\xi^n|_M : M \to (\mathbb{R}_{>0}/(\mathbb{R}_{>0})_{\mathbf{a}})^*$, which satisfies $(\xi^n|_M)_*d\mathbf{m} = (d^*\mathbf{r}_{\mathbf{a}})^*$. Now, Lemma 8.2.2.1 and Fubini theorem give that

$$\mathring{Z}(\mathbf{s})$$
$$= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})}\int_M \sum_{\chi_0\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])^*} \widehat{H}(\mathbf{s},\chi_0\xi^n(\mathbf{m}))d\mathbf{m}.$$

Lemma 7.5.2.1 gives $\widehat{H}(1+\mathbf{s},\chi_0\xi^n(\mathbf{m})) = \widehat{H}(1+\mathbf{s}+2i\pi\mathbf{m},\chi_0)$. Moreover, it follows from Lemma 7.2.2.2 that $\widehat{H}(1+\mathbf{s}+2i\pi,\chi_0) = 0$, for every $\chi_0\in[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(F)]^* - \mathfrak{A}_K$. We deduce that

$$\begin{aligned}\sum_{\chi_0\in([\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]_1/[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)]))^*} \widehat{H}(\mathbf{1}+\mathbf{s},\chi_0\xi^n(\mathbf{m}))\\ = \sum_{\chi_0\in\mathfrak{A}_K}\widehat{H}(\mathbf{1}+\mathbf{s}+2i\pi\mathbf{m},\chi_0)\\ = g_K(\mathbf{1}+\mathbf{s}+2i\pi\mathbf{m}).\end{aligned}$$

Therefore,

$$\mathring{Z}(\mathbf{1}+\mathbf{s}) = \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})}\int_M g_K(\mathbf{1}+\mathbf{s}+2i\pi\mathbf{m})d\mathbf{m}.$$

We introduce

$$G_K(\mathbf{1}+\mathbf{s}) := \int_M g_K(\mathbf{1}+\mathbf{s}+2i\pi\mathbf{m})d\mathbf{m},$$

which, if the integral converges, by Part (1) of Lemma 7.5.1.1, is the same as

$$G_K(\mathbf{1}+\mathbf{s}) = \frac{1}{(2\pi)^{n-1}}\int_M g_K(\mathbf{1}+\mathbf{s}+i\cdot\mathbf{m})d\mathbf{m}$$

The function $\mathbf{s}\mapsto g_K(\mathbf{1}+\mathbf{s})$ is M-controlled and so the integral defining G_K converges for $\mathbf{s}\in\Omega_{>0}$ by Lemma 8.1.1.4. By Proposition 8.2.1.2, the series defining $\mathring{Z}(\mathbf{1}+\mathbf{s})$ converges for $\mathbf{s}\in\Omega_{>0}$. We get

$$\begin{aligned}\mathring{Z}(\mathbf{1}+\mathbf{s}) &= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})}G_K(\mathbf{1}+\mathbf{s})\\ &= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})}\int_M g_K(\mathbf{1}+\mathbf{s}+i\cdot\mathbf{m})d\mathbf{m}\end{aligned}$$

in the domain $\Omega_{>0}$. The claim is proven. □

Proposition 8.2.2.3. — *Let $n\geq 1$. Let $\mathbf{a}\in\mathbb{Z}^n_{\geq 1}$ if $n\geq 2$ and let $\mathbf{a}=a\geq 2$ if $n=1$. For every $\mathbf{x}\in\mathbb{R}^n_{>0}$ one has that*

$$\lim_{s\to 0^+}\Big(\prod_{j=1}^n sx_j\Big)\cdot g_K(\mathbf{1}+s\mathbf{x}) = \frac{\Delta(F)^{\frac{n-1}{2}}\operatorname{Res}(\zeta_F,1)^{n-1}\tau}{|\mu_{\gcd(\mathbf{a})}(F)|}.$$

Proof. — Let $\mathbf{x}\in\mathbb{R}^n_{>0}$. The series defining

$$g_K(\mathbf{1}+s\mathbf{x}) = \sum_{\chi_0\in\mathfrak{A}_K}\widehat{H}(\mathbf{1}+s\mathbf{x},\chi_0)$$

converges absolutely and uniformly when $s>0$, so we can exchange the limit and the sum, we get:

(8.2.2.4)

$$\lim_{s\to 0^+}\Big(\prod_{j=1}^n sx_j\Big)\cdot g_K(\mathbf{1}+s\mathbf{x}) = \sum_{\chi_0\in\mathfrak{A}_K}\lim_{s\to 0^+}\Big(\prod_{j=1}^n sx_j\Big)\widehat{H}((1+sx_j)_j,\chi_0).$$

By Proposition 7.3.1.1, we have that for every $\chi_0\in\mathfrak{A}_K$, there exists a holomorphic function $\phi(-,\chi_0):\Omega_{>\frac{2}{3}}\to\mathbb{C}$ such that one has an equality of the meromorphic functions

$$\widehat{H}(\mathbf{s},\chi_0) = \phi(\mathbf{s},\chi_0)\prod_{j=1}^n L(s_j,\chi_0^{(j)})\prod_{v\in M_F^\infty}\widehat{H}_v(\mathbf{s},\chi_{0v})$$

for $\mathbf{s} \in \Omega_{>1}$. Suppose $\chi_0 \neq 1$. There exists index k such that $\chi_0^{(k)} \neq 1$, and the function $s \mapsto L(s, \chi_0^{(k)})$ is entire. For $j \neq k$, we have that

$$\lim_{s \to 0^+} sx_j L(1 + sx_j, \chi_0^{(j)}) = \lim_{s \to 0^+} sx_j \zeta_F(1 + sx_j)$$

exists. We conclude that

$$\begin{aligned}
&\lim_{s \to 0^+} \big(\prod_{j=1}^n sx_j\big) \widehat{H}((1 + sx_j)_j, \chi_0) \\
&= \lim_{s \to 0^+} \phi((1 + sx_j)_j, \chi_0) \prod_{j=1}^n sx_j L(1 + sx_j, \chi_0^{(j)}) \prod_{v \in M_F^\infty} \widehat{H}_v((1 + sx_j)_j, \chi_{0v}) \\
&= 0.
\end{aligned}$$

Therefore, the only surviving term in the sum on the right hand side of the equality (8.2.2.4) is, hence, for $\chi_0 = 1$. Lemma 7.2.1.1 gives for every $v \in M_F^0 - S$ (that is for every finite v such that f_v is toric) that

$$\widehat{H}_v((1 + sx_j)_j, 1) = \zeta_v(\mathbf{a} \cdot (1 + sx_j)_j)^{-1} \prod_{j=1}^n \zeta_v(1 + sx_j).$$

Hence we have

$$\begin{aligned}
&\widehat{H}((1 + sx_j)_j, 1) \prod_{j=1}^n \zeta_F(1 + sx_j)^{-1} \\
&= \prod_{v \in M_F^0} \left(\widehat{H}_v((1 + sx_j)_j, 1) \prod_{j=1}^n \zeta_v(1 + sx_j)^{-1}\right) \times \prod_{v \in M_F^\infty} \widehat{H}_v((1 + sx_j)_j, 1) \\
&= \prod_{v \in S \cap M_F^0} \left(\widehat{H}_v((1+sx_j)_j, 1) \prod_{j=1}^n \zeta_v(1+sx_j)^{-1}\right) \prod_{v \in M_F^0 - S} \zeta_v(\mathbf{a} \cdot (1+sx_j)_j)^{-1} \times \\
&\qquad \times \prod_{v \in M_F^\infty} \widehat{H}_v((1 + sx_j)_j, 1).
\end{aligned}$$

When $s > 0$, one has $\Re(\mathbf{a} \cdot (1 + sx_j)_j) > 1$ (because $\mathbf{a} \in \mathbb{Z}_{\geq 1}^n$ if $n \geq 2$ and $\mathbf{a} = a \in \mathbb{Z}_{\geq 2}^n$ if $n = 1$), and thus the product $\prod_{v \in M_F^0 - S} \zeta_v(\mathbf{a} \cdot (1 + sx_j)_j)^{-1}$ converges to

$$\zeta_F(\mathbf{a} \cdot (1 + sx_j)_j)^{-1} \prod_{v \in S \cap M_F^0} \zeta_v(\mathbf{a} \cdot (1 + sx_j)_j).$$

We deduce

$$(8.2.2.5)\quad \widehat{H}((1+sx_j)_j,1)$$
$$= \Big(\prod_{j=1}^{n}\zeta_F(1+sx_j)\Big)\prod_{v\in S\cap M_F^0}\Big(\widehat{H}_v((1+sx_j)_j,1)\prod_{j=1}^{n}\zeta_v((1+sx_j)_j)^{-1}\Big)\times$$
$$\times\,\zeta_F(\mathbf{a}\cdot(1+sx_j)_j)^{-1}\Big(\prod_{v\in S\cap M_F^0}\zeta_v(\mathbf{a}\cdot(1+sx_j)_j)\Big)\prod_{v\in M_F^\infty}\widehat{H}_v((1+sx_j)_j,1)$$

We will calculate the limit of the last product multiplied by $\prod_{j=1}^n sx_j$, when s goes to zero. One has that

$$(8.2.2.6)\qquad \lim_{s\to 0^+}\big(\prod_{j=1}^{n}sx_j\big)\zeta_F(1+sx_j)=\operatorname{Res}(\zeta_F,1)^n.$$

Now we calculate:

(8.2.2.7)

$$\lim_{s\to 0^+}\prod_{v\in S\cap M_F^0}\Big(\widehat{H}_v((1+sx_j)_j,1)\prod_{j=1}^{n}\zeta_v(1+sx_j)^{-1}\Big)\times\prod_{v\in M_F^\infty}\widehat{H}_v((1+sx_j)_j,1)$$
$$=\prod_{v\in S\cap M_F^0}\Big(\widehat{H}_v(\mathbf{1},1)\zeta_v(1)^{-n}\Big)\times\prod_{v\in M_F^\infty}\widehat{H}_v(\mathbf{1},1).$$

By Lemma 5.4.1.2, for $v\in M_F^0$ we have that

$$\begin{aligned}\widehat{H}_v(\mathbf{1},1)&=\zeta_v(1)^{n-1}\int_{[\mathscr{T}(\mathbf{a})(F_v)]}H_v(\mathbf{1},-)^{-1}\mu_v\\&=\zeta_v(1)^{n-1}\omega_v([\mathscr{T}(\mathbf{a})(F_v)])\\&=\zeta_v(1)^{n-1}\omega_v([\mathscr{P}(\mathbf{a})(F_v)]),\end{aligned}$$

where we have used that $\omega([\mathscr{P}(\mathbf{a})(F_v)]-[\mathscr{T}(\mathbf{a})(F_v)])=0$ which we have established in Lemma 5.2.4.2. For $v\in M_F^\infty$ we have that

$$\widehat{H}_v(\mathbf{1},1)=\int_{[\mathscr{T}(\mathbf{a})(F_v)]}H_v(\mathbf{1},-)^{-1}\mu_v=\omega_v([\mathscr{P}(\mathbf{a})(F_v)]).$$

We conclude that the product on the right hand side of the equality (8.2.2.7) is equal to

$$(8.2.2.8)\qquad \prod_{v\in S\cap M_F^0}\zeta_v(1)^{-1}\omega_v([\mathscr{P}(\mathbf{a})(F_v)])\times\prod_{v\in M_F^\infty}\omega_v([\mathscr{P}(\mathbf{a})(F_v)]).$$

Finally, we can calculate

(8.2.2.9)
$$\lim_{s\to 0^+}\zeta_F(\mathbf{a}\cdot(1+sx_j)_j)^{-1}\prod_{v\in S\cap M_F^0}\zeta_v(\mathbf{a}\cdot(1+sx_j)_j)=\zeta_F(|\mathbf{a}|)^{-1}\prod_{v\in S\cap M_F^0}\zeta_v(|\mathbf{a}|).$$

Using the facts (8.2.2.6), (8.2.2.8) and (8.2.2.9), we conclude

$$\lim_{s\to 0^+}\Big(\prod_{j=1}^n sx_j\Big)\widehat{H}((1+sx_j)_j,1)$$
$$=\frac{\mathrm{Res}(\zeta_F,1)^n}{\zeta_F(|\mathbf{a}|)}\prod_{v\in S\cap M_F^0}\frac{\zeta_v(|\mathbf{a}|)\omega_v(\mathscr{P}(\mathbf{a})(F_v))}{\zeta_v(1)}\times\prod_{v\in M_F^\infty}\omega_v(\mathscr{P}(\mathbf{a})(F_v)).$$

Using the formula given in Lemma 5.3.3.3 for τ we deduce that the last number is
$$\mathrm{Res}(\zeta_F,1)^{n-1}\tau|\mu_{\mathrm{gcd}(\mathbf{a})}(F)|^{-1}\Delta(F)^{\frac{n-1}{2}},$$
as claimed. □

Theorem 8.2.2.10. — *There exists $\gamma>0$ such that the series series defining*

(8.2.2.11)
$$\mathring{Z}((s)_j)=\sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]}H(\mathbf{x})^{-s}=|\text{Ш}^1(F,\mu_{\mathrm{gcd}(\mathbf{a})})|\sum_{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(i)]([\mathscr{T}(\mathbf{a})(F)])}H((s)_j,\mathbf{x})^{-1}$$

converges for $s\in\mathbb{R}_{>\gamma}+i\mathbb{R}$ and the function that associates to s the value of $\mathring{Z}((s)_j)$ is holomorphic in the domain $\Re(s)>\gamma$. There exists $1>\delta>0$ such that the function $s\mapsto\mathring{Z}((s)_j)$ extends to a meromorphic function in the domain $\Re(s)>1-\delta$ with the only pole at $s=1$ which is simple, and such that for every compact $\mathcal{K}\subset\mathbb{R}_{>1-\delta}$ there exists $C(\mathcal{K})>0$ and $\beta(\mathcal{K})>0$ such that
$$\left|\frac{s-1}{s}Z((s)_j)\right|\leq C(\mathcal{K})(1+|\Im(s)|)^{\beta(\mathcal{K})}$$
if provided $\Re(s)\in\mathcal{K}$. We have further that
$$\lim_{s\to 1}(s-1)\mathring{Z}((s)_j)=\frac{\tau}{|\mathbf{a}|}.$$

Proof. — Let us firstly establish the convergence and holomorphicity. We have seen in Proposition 8.2.1.2 that there exists $\gamma>0$ such that the series defining $\mathring{Z}(\mathbf{s})$ converges absolutely for $\mathbf{s}\in\Omega_{>\gamma}$ and defines a holomorphic function in this domain. We deduce that the series defining $\mathring{Z}((s)_j)$ converges in the domain $\mathbb{R}_{>\gamma}+i\mathbb{R}$. The function $s\mapsto Z((s_j)_j)$ is holomorphic as it this the composition of the holomorphic map $\mathbb{R}_{>\gamma}+i\mathbb{R}\to\Omega_{>\gamma}$ which is given by $s\mapsto (s)_j$ and the holomorphic map $\Omega_{>\gamma}\to\mathbb{C}$ which is given by $\mathbf{s}\mapsto\mathring{Z}(\mathbf{s})$.

Let us now prove the meromorphic extension and the bound. The facts that $\mathbf{s}\mapsto g_K(1+\mathbf{s})$ is holomorphic for $\mathbf{s}\in\Omega_{>0}$ and that $\mathbf{s}\mapsto g_K(\mathbf{1}+\mathbf{s})\prod_{j=1}^n\frac{s_j}{s_j+1}$ is M-controlled in the domain $\Omega_{>-\delta'}$ for some $\delta'>0$ (Proposition 7.5.3.3), enable us to apply Theorem 8.1.1.6. We apply this theorem for the map $\mathbb{R}^n\to\mathbb{R},\mathbf{x}\mapsto\mathbf{a}\cdot\mathbf{x}$ and the function

$$\mathbf{s}\mapsto\frac{|\text{Ш}^1(F,\mu_{\gcd(a_j)_j})|}{E(\mathbf{a})}g_K(\mathbf{1}+\mathbf{s}+i\mathbf{m}).$$

We get that

$$\begin{aligned}s\mapsto\mathring{Z}((1+s)_j)&=\frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})(2\pi)^{n-1}}\int_M g_K((1+s+m_j)_j)d\mathbf{m}\\&=\frac{|\text{Ш}^1(F,\mu_{\gcd(a_j)_j})|}{E(\mathbf{a})}\mathscr{S}_M(g_K)(|\mathbf{a}|+|\mathbf{a}|\cdot s)\end{aligned}$$

is holomorphic in the domain $\Re(s)>0$ and that there exists $1>\delta>0$ such that $\frac{s}{s+1}\mathring{Z}((1+s)_j)$ extends to a holomorphic functions for $\Re(s)>-\delta$ and is $\{0\}$-controlled in this domain. This means that for every compact $\mathcal{K}\subset\mathbb{R}_{>-\delta}$, there exist $C(\mathcal{K}),\beta(\mathcal{K})>0$ such that

$$\left|\frac{s\mathring{Z}((1+s)_j)}{s+1}\right|\leq C(\mathcal{K})(1+|\Im(s)|)^{\beta(\mathcal{K})}$$

provided that $\Re(s)\in\mathcal{K}$. Pick a compact $\mathcal{K}\subset\mathbb{R}_{>1-\delta}$. For every s with $\Re(s)\in\mathcal{K}$, one has that

$$\left|\frac{s-1}{s}\mathring{Z}(s)\right|=\left|\frac{(s-1)\mathring{Z}((1+(s-1))_j)}{(s-1)+1}\right|\leq C(\mathcal{K}-1)(1+|\Im(s)|)^{\beta(\mathcal{K}-1)},$$

where $\mathcal{K}-1=\{x-1|x\in\mathcal{K}\}$.

Let us calculate the limit. By Proposition 8.2.2.3, for every $\mathbf{x}\in\mathbb{R}^n_{>0}$ one has

$$\lim_{s\to0^+}s^n g_K(1+s\mathbf{x})\prod_{j=1}^n x_j=\operatorname{Res}(\zeta_F,1)^{n-1}|\mu_{\gcd(\mathbf{a})}(F)|^{-1}\Delta(F)^{\frac{n-1}{2}}\tau.$$

Now, the second part of Theorem 8.1.1.6 gives that for every $x > 0$ one has that

$$\begin{aligned}
\lim_{s\to 0^+} s\mathring{Z}((1+sx)_j) &= \lim_{s\to 0^+} \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})} sx\mathscr{S}_M(g_K)(|\mathbf{a}|+|\mathbf{a}|sx)\\
&= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{E(\mathbf{a})} \lim_{s\to 0^+} sx\mathscr{S}_M(g_K)(|\mathbf{a}|+|\mathbf{a}|\cdot sx)\\
&= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{|\mathbf{a}|E(\mathbf{a})} \lim_{s\to 0^+} |\mathbf{a}|sx\mathscr{S}_M(g_K)(|\mathbf{a}|+|\mathbf{a}|\cdot sx)\\
&= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|}{|\mathbf{a}|E(\mathbf{a})} \lim_{s\to 0^+} s^n g_K(1+s\mathbf{x})\prod_{j=1}^n x_j\\
&= \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|\operatorname{Res}(\zeta_F,1)^{n-1}\Delta(F)^{\frac{n-1}{2}}\tau}{|\mathbf{a}|\cdot E(\mathbf{a})\cdot|\mu_{\gcd(\mathbf{a})}(F)|}\\
&= \frac{\tau}{|\mathbf{a}|},
\end{aligned}$$

where the last equality follows directly from the definition of $E(\mathbf{a}) = \frac{|\text{Ш}^1(F,\mu_{\gcd(\mathbf{a})})|\operatorname{Res}(\zeta_F,1)^{n-1}\Delta(F)^{\frac{n-1}{2}}}{|\mu_{\gcd(\mathbf{a})}(F)|}$ (see Lemma 8.2.2.1). We deduce that the function $s\mapsto \mathring{Z}((1+s)_j)$ admits a pole of order 1 at 0, which is simple and

$$\operatorname{Res}(s\mapsto \mathring{Z}((1+s)_j),1) = \frac{\tau}{|\mathbf{a}|}.$$

The statement follows. □

By using the Tauberian result given as [17, Theorem A1] we deduce the following theorem.

Theorem 8.2.2.12. — *Let $(f_v)_v$ be a quasi-toric degree $|\mathbf{a}|$ family of $\mathbf{a}$-homogenous smooth functions. One has that*

$$\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)] | H(\mathbf{x})\leq B\} \sim \frac{\tau}{|\mathbf{a}|}B,$$

when B tends to $+\infty$.

Proof. — We first establish the following fact

$$\{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)] | H(\mathbf{x})\leq B\} \sim \frac{\tau}{|\mathbf{a}|}B.$$

Let $W\subset[\mathscr{T}(\mathbf{a})(F)]$ be the set of points for which $H(\mathbf{x})<1$, it is finite by Corollary 4.7.1.3. We define $\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}((s)_j) = \sum_{\mathbf{x}\in\mathscr{T}(\mathbf{a})(F)-W} H(\mathbf{x})^{-s}$.

It follows from Theorem 8.2.2.10 that the series defining $\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}((s)_j)$ converges absolutely in the domain $\mathbb{R}_{>\gamma}+i\mathbb{R}$ and that there exists $\delta>0$ such that the function $s\mapsto \mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}$ extends meromorphically to the domain $\Re(s)>1-\delta$ and has one and only one pole in this domain which is moreover simple with the residue

$$\operatorname{Res}(\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W},1)=\frac{\tau}{|\mathbf{a}|}.$$

By the absolute convergence of the defining series and the fact that we are summing only for those $\mathbf{x}$ for which $H(\mathbf{x})\geq 1$, the function $s\mapsto \mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}((s)_j)$ is decreasing on $]\gamma,+\infty[$. Let us pick $\mathcal{K}=[1-\delta/2,\gamma+1]$. It follows from Theorem 8.2.2.10 and the triangle inequality, that there exists $C(\mathcal{K}),\beta(\mathcal{K})>0$ such that for $\Re(s)\in\mathcal{K}$ one has that

$$\left|\frac{(s-1)\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}((s)_j)}{s}\right|\leq\left|\frac{s-1}{s}\right|\sum_{\mathbf{x}\in W}\left|H(\mathbf{x})^{-s}\right|+C(\mathcal{K})(1+|\Im(s)|)^{\beta(\mathcal{K})}.$$

The function $s\mapsto\left|\frac{s-1}{s}\right|\sum_{\mathbf{x}\in W}\left|H(\mathbf{x})^{-s}\right|$ is bounded for $s\in\mathcal{K}+i\mathbb{R}$, say by $A>0$. By the fact that $\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}$ is decreasing on $]\gamma,+\infty[$ we deduce that for $\Re(s)>1-\delta/2$ one has that

$$\left|\frac{(s-1)\mathring{Z}(s)(\gamma+1)}{s}\right|\leq(A+C(\mathcal{K})+\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}(\gamma+1))(1+|\Im(s)|)^{\beta}.$$

Therefore, $\mathring{Z}_{\mathscr{T}(\mathbf{a})(F)-W}$ satisfies the conditions we need for the Tauberian theorem. Our claim for the rational points of $\mathscr{T}(\mathbf{a})$ follows from the direct application of the theorem.

Let us now prove the statement of the theorem. By Theorem 4.6.8.2, we have that there exists $A>0$ such that for every $B>0$ one has that

$$\begin{aligned}\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}\\ \leq AB^{|\mathbf{a}|-\min_j a_j}\log(2+B^{|\mathbf{a}|-\min_j a_j}|\mathbf{a}|)^{n^2(r_1+r_2)+n-1}.\end{aligned}$$

Thus the statement follows. □

8.3. Equidistribution of rational points

We study the "equidistribution" of the set of rational points of a weighted projective stack in its adelic space.

8.3.1. — In this paragraph we recall what do we mean by the equidistribution. Let X be a compact topological space and let μ be a measure on X. Let U be a subset of X. Let $H : U \to \mathbb{R}_{>0}$ be a function such that for every $B > 0$, one has that $\{x \in U | H(x) \leq B\}$ is finite. We say that U is equidistributed in (X, μ), (or simply in X) with respect to H if for any open μ-measurable subset $W \subset X$ such that $\mu(\partial W) = 0$ one has that

$$\lim_{B\to\infty} \frac{|\{x \in U \cap W | H(x) \leq B\}|}{|\{x \in U | H(x) \leq B\}|} \to \frac{\mu(W)}{\mu(X)}.$$

8.3.2. — Let $n \in \mathbb{Z}_{\geq 1}$ and let $\mathbf{a} \in \mathbb{Z}^n_{>0}$. We establish that the rational points of $\mathscr{P}(\mathbf{a})$ are equidistributed in the space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$.

If $v \in M_F$, we say that a function $f : [\mathscr{P}(\mathbf{a})(F_v)] \to \mathbb{C}$ is smooth if its pullback $F_v^n - \{0\} \to \mathbb{C}$ is smooth. If $A \subset \mathbb{C}$ is open, the sets of smooth functions $[\mathscr{P}(\mathbf{a})(F_v)] \to A$ will be denoted by $\mathscr{C}^\infty([\mathscr{P}(\mathbf{a})(F_v)], A)$.

Lemma 8.3.2.1. — *Let $v \in M_F$.*

1. *There exists a unique structure of a v-adic manifold on $[\mathscr{P}(\mathbf{a})(F_v)]$ such that for an open subset $A \subset [\mathscr{P}(\mathbf{a})(F_v)]$ and $f \in \mathscr{C}^0(A, \mathbb{C})$ one has that $f \in \mathscr{C}^\infty(A, \mathbb{C})$ if and only if f is smooth in the usual sense (i.e. f is locally constant if $v \in M_F^0$ and f is an infinitely differentiable function if $v \in M_F^\infty$).*
2. *Let $f_v : F_v^n - \{0\} \to \mathbb{R}_{>0}$ be an $\mathbf{a}$-homogenous continuous function and let $\omega_v := (f_v^{-1} dx_1 \dots dx_n)/d^*x$ be the induced measure on $[\mathscr{P}(\mathbf{a})(F_v)]$. Let $W \subset [\mathscr{P}(\mathbf{a})(F_v)]$ be an open subset such that $\omega_v(\partial W) = 0$. Let $\epsilon > 0$. There exist smooth functions $h, g : [\mathscr{P}(\mathbf{a})(F_v)] \to \mathbb{R}_{>0}$ such that*

$$0 \leq h \leq \mathbf{1}_W \leq g \text{ and } \int_{[\mathscr{P}(\mathbf{a})(F_v)]} (g - h)\omega < \epsilon.$$

Proof. — 1. We have seen in Proposition 3.3.4.1, that the action

$$F_v^\times \times (F_v^n - \{0\}) \to (F_v^n - \{0\}) \qquad (t, \mathbf{x}) \mapsto (t^{a_j} x_j)_j$$

is proper, i.e. the canonical morphism $F_v^\times \times (F_v^n - \{0\}) \to (F_v^n - \{0\}) \times (F_v^n - \{0\})$ is proper. Let $e : F_v^\times \to (F_v^\times)_{\mathbf{a}}$ be given by $e(t) = (t^{a_j})_j$. The group $(F_v)_{\mathbf{a}} = \{(t^{a_j})_j | t \in F_v^\times\}$ acts on $F_v^n - \{0\}$ by the component-wise multiplication. The two actions are compatible in the following sense: $t \cdot \mathbf{x} = e(t)\mathbf{x}$. Now the proper morphism $F_v^\times \times (F_v^n - \{0\}) \to (F_v^n - \{0\}) \times (F_v^n - \{0\})$ factorizes as

$$F_v^\times \times (F_v^n - \{0\}) \xrightarrow{(e, \mathrm{Id}_{F_v^n - \{0\}})} (F_v^\times)_{\mathbf{a}} \times (F_v^n - \{0\}) \to (F_v^n - \{0\}) \times (F_v^n - \{0\}).$$

The first morphism is surjective, hence by [7, Chapter I, §10, n^0 1Proposition 5], the second morphism is proper, i.e. the action of $(F_v^\times)_{\mathbf{a}}$ on $F_v^n - \{0\}$ is proper. By [6, Paragraph 6.2.3] we deduce that the quotient $[\mathscr{P}(\mathbf{a})(F_v)] = (F_v^n - \{0\})/F_v^\times = (F_v^n - \{0\})/(F_v^\times)_{\mathbf{a}}$ carries a unique structure of a compact v-adic manifold and with this structure, the smooth functions in our sense are precisely the smooth functions on the v-adic manifold $[\mathscr{P}(\mathbf{a})(F_v)]$.

2. To prove the statement, we use the existence of *bump* functions on v-adic manifolds. First, we recall the proof for this statement when $v \in M_F^0$ and when $v \in M_F^\infty$ we refer the reader to [36, Lemma 2.15]. Let $\mathcal{K}$ be a compact of $[\mathscr{P}(\mathbf{a})(F_v)]$. If $U \supset \mathcal{K}$ is open, for every $\mathbf{x} \in \mathcal{K}$, there exists an open and closed neighbourhood $N_{\mathbf{x}}$ of $\mathbf{x}$ contained in U (because every point in the spaces F_v^r for $r \geq 0$ has a basis of its neighbourhoods given by open and closed balls). By the compacity of $\mathcal{K}$, there exist $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ such that $N := \bigcup N_{\mathbf{x}_\ell} \supset \mathcal{K}$. Moreover, $N \subset U$ is open and closed, thus $\mathbf{1}_N$ is a bump function which extends the function $\mathbf{1}_{\mathcal{K}}$.

   Now let us prove the claim. For every open $U \supset \overline{W}$, there exists a smooth function $g_U : [\mathscr{P}(\mathbf{a})(F_v)] \to [0,1]$ such that $g_U|_{\overline{W}} = 1$ and $\operatorname{supp}(g_U) \subset U$. For every compact $K \subset W$ there exists a smooth function $h_K : K \to [0,1]$ such that $h_K|_K = 1$ and $\operatorname{supp}(h_K) \subset W$. For every $\epsilon > 0$, by the regularity of ω and the fact that $\omega(\partial W) = 0$, there exists open $U \supset \overline{W}$ and a compact $K \subset W$ such that $\omega(U) - \omega(K) < \epsilon$. It follows that

$$\int_{[\mathscr{P}(\mathbf{a})(F_v)]} (g_U - h_K)\omega \leq \int_{[\mathscr{P}(\mathbf{a})(F_v)]} \mathbf{1}_{U-K}\omega = \omega(U) - \omega(K) < \epsilon.$$

□

Let $(f_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a quasi-toric degree $|\mathbf{a}|$-family of $\mathbf{a}$-homogenous *smooth* functions. Let $H = H((f_v)_v)$ be the resulting height on $[\mathscr{P}(\mathbf{a})(F)]$ and let $\omega = \omega((f_v)_v)$ be the resulting measure on $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. The goal of the rest of the paragraph is to establish that the set $[\mathscr{P}(\mathbf{a})(i)]([\mathscr{P}(\mathbf{a})(F)])$ is equidistributed in $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. We will write i for the map $[\mathscr{P}(\mathbf{a})(i)]$ The following theorem is motivated by [47, Proposition 3.3].

Theorem 8.3.2.2 **("Equidistribution of rational points")**

The set $i([\mathscr{P}(\mathbf{a})(F)])$ is equidistributed in $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ with respect to H.

Proof. — In the case $n=1$ and $a_1=1$, the statement is trivially true. In the rest of the proof we suppose that $n\geq 1$ and that $\mathbf{a}\in\mathbb{Z}^n_{>0}$ if $n\geq 2$ and that $\mathbf{a}=a_1\in\mathbb{Z}_{>1}$ if $n=1$. The proof is adaptation of the proof of [47, Proposition 3.3]. We split it in several parts.

1. In the first part we establish that the asymptotic for counting points of $[\mathscr{P}(\mathbf{a})(F)]$ equals to $|Ш^1(F,\mu_{\gcd(\mathbf{a})})|$ times the asymptotic for counting the points of $i([\mathscr{P}(\mathbf{a})(F)])$. By Theorem 8.2.2.12, one has that
$$|\{\mathbf{x}\in[\mathscr{T}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|\sim_{B\to\infty}\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])}{|\mathbf{a}|}B.$$
It follows from Proposition 3.4.3.1 that
$$|\{\mathbf{x}\in i([\mathscr{T}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|\sim_{B\to\infty}\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])}{|Ш^1(F,\mu_{\gcd(\mathbf{a})})|\cdot|\mathbf{a}|}B.$$
It follows from Theorem 4.6.8.2, that for every $B>0$ there exists $C'>0$ such that
$$\begin{aligned}|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)]-[\mathscr{T}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|\\ \leq C'B^{\frac{|\mathbf{a}|-\min_j a_j}{|\mathbf{a}|}}\log(2+B)^{n^2(r_1+r_2)+n-1}.\end{aligned}$$
We deduce that
$$|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|\sim_{B\to\infty}\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])}{|Ш^1(F,\mu_{\gcd(\mathbf{a})})|\cdot|\mathbf{a}|}B.$$
2. We are going to prove the claim for the open subsets $W\subset\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ of the form
$$W=\prod_{v\in S_W}W_v\times\prod_{v\in M_F-S_W}[\mathscr{P}(\mathbf{a})(F_v)],$$
where S is a finite set and for $v\in S$ the set $W_v\subset[\mathscr{P}(\mathbf{a})(F_v)]$ is open satisfying that $\omega_v(\partial W_v)=0$ (we will call such open subset *elementary*). For $v\in S_W$, by Lemma 8.3.2.1 there exist smooth functions $g_v,h_v:W_v\to\mathbb{R}_{>0}$ such that
$$0\leq h_v\leq\mathbf{1}_{W_v}\leq g_v\leq 1\text{ and }\int_{W_v}(g_v-h_v)\omega_v\leq\frac{\epsilon\omega_v([\mathscr{P}(\mathbf{a})(F_v)])}{8|S_W|}.$$
Let us set $\eta=\epsilon/4$. For $v\in S$, we define $h_{v,\eta}=(1-\eta)h_v+\eta$ and $g_{v,\eta}=(1-\eta)g_v+\eta$ and for $v\in M_F-S$, we define $h_{v,\eta}=g_{v,\eta}=1$. We

define $h_\eta = \prod_{v\in M_F} h_{v,\eta}$ and $g_\eta = \prod_{v\in M_F} g_{v,\eta}$. For $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$, let $\widetilde{\mathbf{x}} \in F^n - \{0\}$ be a lift of $\mathbf{x}$. Observe that

$$\begin{aligned}
H(((h_{v,\eta}^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v)(\mathbf{x}) &= \prod_{v\in M_F} ((h_{v,\eta}^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)(\widetilde{\mathbf{x}}) \\
&= \prod_{v\in M_F} h_{v,\eta}^{-1}(q_v^{\mathbf{a}}(\widetilde{\mathbf{x}}))f_v(\widetilde{\mathbf{x}}) \\
&= \Big(\prod_{v\in M_F} h_{v,\eta}^{-1}(\mathbf{x})\Big)H(\mathbf{x}) \\
&= h_\eta(i(\mathbf{x}))^{-1} \cdot H(\mathbf{x})
\end{aligned}$$

for $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$. By Lemma 5.3.3.2, one has that $\omega(((h_{v,\eta}^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v) = h_\eta\omega$. Similarly $H(((g_{v,\eta}^{-1}\circ q_v^{\mathbf{a}})\cdot f_v)_v) = g_v([\mathscr{P}(\mathbf{a})(i_v)](\mathbf{x}))^{-1}H(\mathbf{x})$ for $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]$ and

$$\omega(((g_v^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v) = g_\eta\omega.$$

Now, it follows from (1) that for the quasi-toric degree $|\mathbf{a}|$ families of smooth functions $((h_{v,\eta}^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v$ and $((g_{v,\eta}^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v$ we have that

$$\begin{aligned}
&|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq h_\eta(\mathbf{x}) \cdot B\}| \\
&\qquad = |\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(((h_v^{-1} \circ q_v^{\mathbf{a}}) \cdot f_v)_v)(\mathbf{x}) \leq B\}| \\
&\qquad\qquad \sim_{B\to\infty} \frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} h_\eta\omega}{|Ш^1(F,\mu_{\mathrm{gcd}(\mathbf{a})})| \cdot |\mathbf{a}|}B.
\end{aligned}$$

and that

$$\begin{aligned}
&|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq g_\eta(\mathbf{x})B\}| \\
&\qquad\qquad \sim_{B\to\infty} \frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} g_\eta\omega}{|Ш^1(F,\mu_{\mathrm{gcd}(\mathbf{a})})| \cdot |\mathbf{a}|}B.
\end{aligned}$$

Using (1), we deduce that

$$\frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq h_\eta(\mathbf{x})B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|} \sim_{B\to\infty} \frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} h_\eta\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}$$

and analogously for g_η. Thus, there exists $B_0 > 0$ such that if $B > B_0$, one has that

$$\begin{aligned}
&\frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} h_\eta\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)} - \frac{\epsilon}{8} \\
&\qquad \le \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\prod_{v\in S_W}((1-\eta)\mathbf{1}_{W_v}+\eta)\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} \\
&\qquad \le \frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} g_\eta\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)} + \frac{\epsilon}{8}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])} - \frac{\epsilon}{8} \\
&\qquad \le \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\prod_{v\in S_W}((1-\eta)\mathbf{1}_{W_v}+\eta)\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} \\
&\qquad \le \frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])} + \frac{\epsilon}{4}
\end{aligned}$$

One has that

$$\lim_{B\to\infty} \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le \eta B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} = \eta = \frac{\epsilon}{4}.$$

For $B \gg 0$, we deduce that

$$\begin{aligned}
-\frac{\epsilon}{8} &\le \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\prod_{v\in S_W}((1-\eta)\mathbf{1}_{W_v}+\eta)\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} \\
&\qquad - \frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])} \\
&\le \frac{|\{\mathbf{x}\in W|H(\mathbf{x})\le B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} + \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le \eta B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} \\
&\qquad - \frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])} \\
&\le \frac{|\{\mathbf{x}\in W|H(\mathbf{x})\le B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} - \frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])} + \frac{\epsilon}{2} \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{2}.
\end{aligned}$$

Thus the claim follows for elementary open subsets $W \subset \prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$.

3. We prove claim for every W which is a finite union of elementary open subsets of $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. The collection of the elementary open subsets of $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$ is stable for finite intersections. Suppose the claim is valid for every union of k elementary open sets. Let $V_1,\dots,V_{k+1}$ be elementary open sets, we have that

$$\begin{aligned}
&\frac{|\{\mathbf{x}\in\bigcup_{j=1}^{k}V_j\cup V_{k+1}|H(\mathbf{x})\le B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|}\\
&\qquad\qquad=\frac{|\{\mathbf{x}\in\bigcup_{j=1}^{k}V_j|H(\mathbf{x})\le B\}|+|\{\mathbf{x}\in V_{k+1}|H(\mathbf{x})\le B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|}\\
&\qquad\qquad\quad-\frac{|\{\mathbf{x}\in\bigcup_{j=1}^{k}(V_j\cap V_{k+1})|H(\mathbf{x})\le B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|}\\
&\qquad\qquad\sim_B\frac{\omega\big(\bigcup_{j=1}^{k}V_j\big)+\omega(V_{k+1})-\omega\big(\bigcup_{j=1}^{k}(V_j\cap V_{k+1})\big)}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}\\
&\qquad\qquad=\frac{\omega\big(\bigcup_{j=1}^{k+1}V_j\big)}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.
\end{aligned}$$

It follows by the induction, that the claim is valid for W which is a union of finitely many elementary open subsets of $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$.

4. Let us now prove the claim for a general open subset W with $\omega(\partial W)=0$. We shall first establish that for every $\epsilon>0$, there exist W' and W'' which are finite unions of the elementary open subsets of $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$, such that $W'\subset W\subset W''$ and such that $\omega(W''-W')<\epsilon$. For $v\in M_F$, open sets of $[\mathscr{P}(\mathbf{a})(F_v)]$ with negligible boundary form a basis of the topologies of $[\mathscr{P}(\mathbf{a})(F_v)]$ (the collection of such open sets contains the images of open balls in $F_v^n-\{0\}$, and the open balls in $F_v^n-\{0\}$ form a basis of the topologies and have negligible boundaries). It follows that the elementary open subsets form a basis of the topology of $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. Let $\epsilon>0$. The space $\overline{W}$ is a compact, thus can be covered by the finitely many elementary open sets of volume no more than ϵ_2. We let W'' be the union of these sets. By the inner regularity of ω (e.g. [29, Theorem 2.5.13]), there exists a compact set $K\subset W$ such that

$\omega(W)-\omega(K)<\epsilon/2$. Cover the set K by finitely many elementary open subsets lying completely in W. We let W' be the union of this sets. Clearly, $W'\subset W\subset W''$. By using that $\omega(\partial W)=0$ we get that

$$\begin{aligned}\omega(W''-W')&\leq \omega(W''-\overline{W})+\omega(\overline{W}-W')\\ &\leq \omega(W''-(W\cup\partial W))+\omega((W\cup\partial W)-W')\\ &\leq \omega(W''-W)+\omega(W-W')\\ &<\epsilon.\end{aligned}$$

Now, for $\delta>0$, one has that there exists $B_1>0$ such that if $B>B_1$, then:

$$\begin{aligned}\frac{\omega(W')}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}-\delta&\leq \frac{|\{\mathbf{x}\in W'|H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}\\ &\leq \frac{|\{\mathbf{x}\in W|H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}.\end{aligned}$$

and that

$$\begin{aligned}\frac{|\{\mathbf{x}\in W|H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}&\leq \frac{|\{\mathbf{x}\in W''|H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}\\ &\leq \frac{\omega(W'')}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}+\delta.\end{aligned}$$

It follows that

$$\frac{|\{\mathbf{x}\in W|H(\mathbf{x})\leq B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}\sim_{B\to\infty}\frac{\omega(W)}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.$$

The statement is proven.

□

We deduce the following proposition (it is analogous to parts (a) and (b) of [**47**, Proposition 3.3])

Proposition 8.3.2.3. — *The following claims are valid:*

1. *Let* $f:\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{C}$ *be a step function (the sum is assumed to be finite)* $\sum\lambda_i\mathbf{1}_{W_i}$, *where* W_i *are open sets with negligible*

boundaries. One has that

$$\lim_{B\to\infty}\frac{\sum_{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]} f(\mathbf{x})}{|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x})\leq B\}|}$$
$$=\lim_{B\to\infty}\frac{\sum_{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])} f(\mathbf{x})}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\leq B\}|}$$
$$=\frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} f\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.$$

2. *For every continuous function $f:\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{C}$, the equality from (1) is valid*
3. *The asymptotic formula from Theorem 8.2.2.12 is valid for any quasi-toric degree $|\mathbf{a}|$ family of $\mathbf{a}$-homogenous functions $(g_v:F_v^n-\{0\}\to\mathbb{R}_{>0})_v$.*

Proof. — For the first two claims, the proofs of the corresponding claims in [47, Proposition 3.3] work here. For the third one, we make minor modifications.

1. By Theorem 8.3.2.2, the equality is valid for the characteristic functions of open sets with negligible boundaries. Clearly, the equality stays valid for any step function (the sum is assumed to be finite) $\sum_i\lambda_i\mathbf{1}_{W_i}$, where W_i are open sets with negligible boundaries, assumed to be pairwise disjoint. We verify that the same equality stays valid for a step function $\sum_i\lambda_i\mathbf{1}_{W_i}$, where W_i are open sets with negligible boundaries (not assumed pairwise disjoint). For every point $\mathbf{x}\in\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$, let $A(\mathbf{x})$ be the set of the indices i for which $\mathbf{x}\in W_i$. We let $W_{A(\mathbf{x})}=\bigcap_{i\in A(\mathbf{x})}W_i$. The function $\sum_i\lambda_i\mathbf{1}_{W_i}$ coincides with the function
$$\sum_{A(\mathbf{x})}\bigg(\sum_{i\in A(x)}\lambda_i\bigg)\mathbf{1}_{W_{A(\mathbf{x})}},$$
where the sum is taken over all subsets that appear as $A(\mathbf{x})$ for some $\mathbf{x}\in\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. A finite intersection of open sets with negligible boundary is an open set with negligible boundary (because the boundary of an intersection is contained in the union of the boundaries), thus the sets $W_{A(\mathbf{x})}$ are open sets with negligible boundary. Hence, the equality stays valid for described step functions.

2. The open sets with negligible boundaries form a basis of the compact topological space $\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]$. Now, any continuous function $f:\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{C}$ can be approached uniformly by step functions $\sum_i \lambda_i \mathbf{1}_{W_i}$, where W_i are open sets with negligible boundaries. The claim follows.
3. Let $(g_v : F_v^n - \{0\} \to \mathbb{R}_{>0})_v$ be a quasi-toric degree $|\mathbf{a}|$ family of $\mathbf{a}$-homogenous functions and let H^g be the resulting height. Let S' be the finite set of places for which $f_v \neq g_v$. For $v \in S'$ and for $\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F_v)]$, let $\widetilde{\mathbf{x}} \in F_v^n - \{0\}$ be a lift of $\mathbf{x}$. The function

$$h:\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\to\mathbb{R}_{>0}\qquad (\mathbf{x}_v)_v\mapsto\prod_{v\in S'}\frac{f_v(\widetilde{\mathbf{x}}_v)}{g_v(\widetilde{\mathbf{x}}_v)},$$

does not depend on the choices of $\widetilde{\mathbf{x}}_v$, is a continuous function (because f_v and g_v are of the same weighted degree and are continuous functions $F_v^n - \{0\} \to \mathbb{R}_{>0}$). Note that when $\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])$, one has that

$$h(\mathbf{x})=\frac{H(\mathbf{x})}{H^g(\mathbf{x})},$$

because for every v, one can take $\widetilde{\mathbf{x}}_v$ to be a fixed element in $(F^\times)^n$. By Theorem 8.3.2.2, for every open W with negligible boundary, one has that

$$\lim_{B\to\infty}\frac{|\{\mathbf{x}\in i(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])|H(\mathbf{x})\le \mathbf{1}_W B\}|}{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x})\le B\}|} = \frac{\int_{\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]}\mathbf{1}_W\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}. \tag{8.3.2.4}$$

The same equality is valid when $\mathbf{1}_W$ is replaced by a step function $\sum_i \lambda_i \mathbf{1}_{W_i}$, where W_i are open with the negligible boundaries. Let $\epsilon > 0$, there exists a step function $\sum_i \lambda_i \mathbf{1}_{W_i}$, with W_i open with the negligible boundaries, such that $0 \le h - \sum_i \lambda_i \mathbf{1}_{W_i} \le \epsilon$. It follows that

$$\begin{aligned}
\frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H^g(\mathbf{x}) \leq B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|}
&= \frac{|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x}) \leq h(\mathbf{x})B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|} \\
&\leq \frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq (\epsilon + \sum_i \lambda_i \mathbf{1}_{W_i})B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F_v)])|H(\mathbf{x}) \leq B\}|} \\
&=_{B\to\infty} \frac{\int_{\prod_{v \in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} (\epsilon + \sum_i \lambda_i \mathbf{1}_{W_i})\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)} \\
&= \epsilon + \frac{\int_{\prod_{v \in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} (\sum_i \lambda_i \mathbf{1}_{W_i})\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H^g(\mathbf{x}) \leq B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|}
&= \frac{|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x}) \leq h(\mathbf{x})B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|} \\
&\geq \frac{|\{\mathbf{x} \in [\mathscr{P}(\mathbf{a})(F)]|H(\mathbf{x}) \leq (\sum_i \lambda_i \mathbf{1}_{W_i})B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|} \\
&=_{B\to\infty} \frac{\int_{\prod_{v \in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} (\sum_i \lambda_i \mathbf{1}_{W_i})\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.
\end{aligned}$$

By decreasing ϵ, we deduce that

$$\lim_{B\to\infty} \frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H^g(\mathbf{x}) \leq B\}|}{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|} = \frac{\int_{\prod_{v \in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} h\omega}{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}.$$

It follows from Lemma 5.3.3.2 that

$$\int_{\prod_{v \in M_F}[\mathscr{P}(\mathbf{a})(F_v)]} h\omega = \omega((g_v)_v)\Big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\Big).$$

Finally, Theorem 8.2.2.12 gives that

$$\lim_{B\to\infty} \frac{|\{\mathbf{x} \in i([\mathscr{P}(\mathbf{a})(F)])|H(\mathbf{x}) \leq B\}|}{B} = \frac{\omega\big(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\big)}{|\mathbf{a}|},$$

thus

$$\lim_{B\to\infty} \frac{|\{\mathbf{x}\in i([\mathscr{P}(\mathbf{a})(F)])|H^g(\mathbf{x})\le B\}|}{B} = \frac{\omega((g_v)_v)\left(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)]\right)}{|\mathbf{a}|}$$

as claimed.

$\square$

Remark 8.3.2.5. — Suppose that $(f_v^{\#})_v$ is the toric degree $|\mathbf{a}|$ family of $\mathbf{a}$-homogenous functions. Using the expression for τ from Lemma 5.3.3.3 and the formula for the volumes $\omega_v([\mathscr{P}(\mathbf{a})(F_v)])$ for $v\in M_F^\infty$ from Lemma 5.3.2.1, we get that

$$\begin{aligned}\tau &= \frac{\operatorname{Res}(\zeta_F,1)|\mu_{\gcd(\mathbf{a})}(F)|(2^{n-1}|\mathbf{a}|)^{r_1}((2\pi)^{n-1}|\mathbf{a}|)^{r_2}}{\Delta(F)^{\frac{n-1}{2}}\zeta_F(|\mathbf{a}|)}\\ &= \frac{\operatorname{Reg}(F)h_F 2^{r_1}(2\pi)^{r_2}}{\Delta(F)^{\frac{1}{2}}}\frac{|\mu_{\gcd(\mathbf{a})(F)}|2^{(n-1)r_1}(2\pi)^{(n-1)r_2}|\mathbf{a}|^{r_1+r_2}}{\Delta(F)^{\frac{n-1}{2}}w_F\zeta_F(|\mathbf{a}|)}\\ &= \frac{h_F}{\zeta_F(|\mathbf{a}|)}\left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\Delta(F)}}\right)^n|\mathbf{a}|^{r_1+r_2}\frac{\operatorname{Reg}(F)|\mu_{\gcd(\mathbf{a})}(F)|}{w_F}.\end{aligned}$$

Here, $\operatorname{Reg}(F)$ is the regulator of F and w_F the number of roots of unity in F. Hence,

$$\begin{aligned}&|\{\mathbf{x}\in[\mathscr{P}(\mathbf{a})(F)]|H^{\#}(\mathbf{x})\le B\}|\\ &\qquad\sim_{B\to\infty}\frac{h_F}{\zeta_F(|\mathbf{a}|)}\left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{\Delta(F)}}\right)^n|\mathbf{a}|^{r_1+r_2-1}\frac{\operatorname{Reg}(F)|\mu_{\gcd(\mathbf{a})}(F)|}{w_F}B.\end{aligned}$$

The counting result in this case has been obtained by Bruin and Najman ([**12**, Theorem 3.7]). They used the method of Deng from [**25**], which is similar to the original method of Schanuel from [**53**] for the case of rational points of the projective space.

Remark 8.3.2.6. — Theorem 8.2.2.10 and Theorem 8.2.2.12 give that the closed substack $\mathscr{Z}(\{X_1\cdots X_n\})\subset\mathscr{P}(\mathbf{a})$ given by the $\mathbb{G}_m$-invariant closed subscheme $Z(X_1\cdots X_n)\subset\mathbb{A}^n-\{0\}$, is not an "accumulating" substack (see [**47**, Definition 1.3] for the terminology).

The same estimate as in Theorem 8.2.2.12 is true for the rational points of the stack $\overline{\mathscr{P}(\mathbf{a})}$ (because the stack $\overline{\mathscr{P}(\mathbf{a})}-\mathscr{P}(\mathbf{a})$ has only one rational point).

Remark 8.3.2.7. — For $L \in \mathrm{Pic}(\overline{\mathscr{P}(\mathbf{a})}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Pic}(\overline{\mathscr{P}(\mathbf{a})})_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$, we define a measure θ_L on the set

$$\mathcal{H}_L(\lambda) = \{y \in \mathrm{Pic}(\overline{\mathscr{P}(\mathbf{a})})^*_{\mathbb{R}} | y(L) = \lambda\} = \begin{cases} \mathbb{R} & \text{if } L = 0 \text{ and } \lambda = 0, \\ \left\{\frac{\lambda}{L}\right\} & \text{if } L \neq 0 \\ \emptyset & \text{if } L = 0 \text{ and } \lambda \neq 0. \end{cases},$$

by setting it to be the Lebesgue measure, the Haar measure which is normalized by $\theta_L\big(\left\{\frac{\lambda}{L}\right\}\big) = \frac{1}{L}$ and by $\theta_0(\emptyset) = 1$, respectively. We define

$$\alpha = \alpha(\overline{\mathscr{P}(\mathbf{a})}) = \theta_{|\mathbf{a}|}(\{y > 0\} \cap \mathcal{H}_{|\mathbf{a}|}(1)) = \theta_{|\mathbf{a}|}\big(\frac{1}{|\mathbf{a}|}\big) = \frac{1}{|\mathbf{a}|}.$$

(This definition is [**47**, Definition 2.4] for the case rk(Pic) = 1). Thus the leading constant in the asymptotics of Theorem 8.2.2.12 and in Part (3) of Proposition 8.3.2.3, writes as $\alpha\tau$, as predicted by Peyre in [**47**] for Fano varieties.

CHAPTER 9

NUMBER OF μ_m-TORSORS OF BOUNDED DISCRIMINANT

In this chapter F will be a number field and $\mathcal{O}_F$ its ring of integers. The goal of this chapter is to give the asymptotic behaviour for the number of μ_m-torsors over F of bounded discriminant, where $m \geq 2$ is an integer. We are going to use the language of heights on weighted projective stacks from 4.

9.1. Calculations of the discriminant

In this section we will be calculating the discriminants of μ_m-torsors over F_v where v is a finite place of F, where $m \geq 2$ is an integer.

9.1.1. — We use [**42**] as principal reference for the definition and basic properties of discriminants. Let R be a Dedekind domain and let K be its field of fractions. Let L be a finite extension of K. Let R' be the integral closure of R in L. If $x_1, \ldots, x_n$ is a basis of L over K, we set

$$\Delta(x_1, \ldots, x_n) := \det((\mathrm{Tr}(x_i x_j))_{ij}),$$

where $\mathrm{Tr} : L \to K$ is the trace map. We say that Δ is the discriminant of the basis $x_1, \ldots, x_n$. We define $\Delta(R', R)$ to be the ideal of R generated by all $\Delta(x_1, \ldots, x_n)$ when $x_1, \ldots, x_n$ range over all bases of L/K which are contained R'. By the abuse of notation, we may write $\Delta(L/K)$ for $\Delta(R'/R)$ if R and R' are understood from the context.

Proposition 9.1.1.1 **(**[**42**], Corollary 2.10, Chapter III]**)**
For a tower of fields $K \subset L \subset M$ one has that:

$$\Delta(M/K) = \Delta(L/K)^{[M:L]} N_{L/K}(\Delta(M/L)).$$

Proposition 9.1.1.2 **(**[**42**, Corollary 2.12, Chapter III]**)**
Suppose that L/K is unramified. One has that $\Delta(L/K) = (1)$.

Let $\mathfrak{D}(L/K)$ denotes the different ideal ([**42**, Definition 2.1, Chapter III]). The following statement is true for any *tamely ramified* primes ([**42**, Definition 7.6, Chapter II]), but for our needs the following version suffices (clearly, it makes primes automatically tamely ramified):

Proposition 9.1.1.3 **(**[**42**, Theorem 2.6]**).** — *Suppose that $\mathfrak{p}$ is a non-zero prime ideal of R such that $R/\mathfrak{p}$ is a finite field of characteristic coprime to $[L:K]$ and let $\mathfrak{q}$ be a prime of R' lying over $\mathfrak{p}$. One has that $v_{\mathfrak{q}}(\mathfrak{D}(L/K)) = e_{\mathfrak{q}|\mathfrak{p}} - 1$ where $v_{\mathfrak{q}}(\mathfrak{D}(L/K))$ is the exponent of $\mathfrak{q}$ in the prime factorisation of the ideal $\mathfrak{D}(L/K)$ and $e_{\mathfrak{q}|\mathfrak{p}}$ is the degree of the ramification of the prime ideal $\mathfrak{p}$ in $\mathfrak{q}$.*

If $A = K_1 \times \cdots \times K_r$ be a product of finite separable extensions K_i/K, we define $\Delta(A/K) := \prod_i \Delta(K_i/K)$. The following proposition is given in [**42**, Corollary 2.11, Chapter III] when A is a field, nonetheless, it is true when A is a finite product of finite extensions of F.

Proposition 9.1.1.4. — *Let A/F be a finite product of finite extensions of F. Let $v \in M_F^0$. One has that*

$$v(\Delta(A/F)) = v\bigg(\Delta\big((A \otimes_F F_v)/F_v\big)\bigg).$$

Proof. — Let $A = K_1 \times \cdots \times K_r$, with K_i/F finite extensions of F. For every i, by [**42**, Corollary 2.11, Chapter III], one has that

$$v(\Delta(K_i/F)) = v\bigg(\prod_{w^i|v} \Delta(K_{w^i}/F_v)\bigg),$$

where w^i are places of K_i lying above v. For every i, by [**42**, Proposition 8.3, Chapter II], one has that

$$\prod_{w^i|v} K_{w^i} = K_i \otimes_F F_v$$

and hence

$$\prod_{w^i|v} \Delta(K_{w^i}/F_v) = \Delta(K_i \otimes_F F_v).$$

We deduce that

$$v(\Delta(K_i/F)) = v\bigg(\Delta(K_i \otimes_F F_v/F_v)\bigg).$$

Hence,

$$v(\Delta(A/F)) = v\Big(\prod_{i=1}^{r} \Delta(K_i/F)\Big) = v\Big(\prod_{i=1}^{r} \Delta\big((K_i \otimes_F F_v)/F_v\big)\Big)$$
$$= v\Big(\Delta\big((A \otimes_F F_v)/F_v\big)\Big).$$

The statement is proven. □

9.1.2. — Let again $m \geq 2$ be an integer and let $v \in M_F^0$ be such that $v(m) = 0$. In this paragraph we calculate the discriminant of a μ_m-torsor over F_v. We have not found a reference for these calculations.

Lemma 9.1.2.1. — *Let n be an integer such that $v(n) = 0$. Let $\sqrt[n]{\pi_v}$ be a formal n-th root of the uniformizer π_v. One has that*

$$\Delta\big(F_v(\sqrt[n]{\pi_v})/F_v\big) = \pi_v^{n-1}\mathcal{O}_v.$$

Proof. — Let us write K for $F_v(\sqrt[n]{\pi_v})$. By Eisenstein's criterion, the polynomial $X^n - \pi_v$ is irreducible. It follows that the degree of the extension K/F_v is equal to n and that the elements $\sqrt[n]{\pi_v}, \sqrt[n]{\pi_v^2}, \ldots, \sqrt[n]{\pi_v^{n-1}}$ do not belong to $\mathcal{O}_v$. We deduce that $(\sqrt[n]{\pi_v}) \cap \mathcal{O}_v = (\pi_v)$. Thus the degree of the ramification of π_v in K is at least n and hence is equal to n (because the degree of the extension is n). Proposition 9.1.1.3 gives that $(\mathfrak{D}(K/F_v)) = (\sqrt[n]{\pi_v})^{n-1}$. We deduce that

$$\begin{aligned}\Delta(K/F_v) &= N_{K/F_v}(\mathfrak{D}(F_v(\sqrt[n]{\pi_v})/F_v)) \\ &= N_{K/F_v}((\sqrt[n]{\pi_v})^{n-1}) \\ &= N_{K/F_v}((\sqrt[n]{\pi_v}))^{n-1}\end{aligned}$$

As $N_{K/F_v}(\pi_v) = (\pi_v^n)$, it follows that $N_{K/F_v}((\sqrt[n]{\pi_v})) = (\pi_v)$, and hence

$$\Delta(F_v(\sqrt[n]{\pi_v})/F_v) = \Delta(K/F_v) = N_{K/F_v}((\sqrt[n]{\pi_v}))^{r-1} = (\pi_v)^{r-1}.$$

The statement is proven. □

Lemma 9.1.2.2. — *Let $a \in F_v^{\times}$ and let n be an integer such that $v(n) = 0$. We set $d = \gcd(v(a), n)$. Let $\sqrt[n]{a}$ be an n-th root of a (lying in an algebraic closure of F_v). One has that*

$$\Delta(F_v(\sqrt[n]{a})/F_v) = \pi_v^{[F_v(\sqrt[n]{a}):F_v](1-d/n)}\mathcal{O}_v.$$

Proof. — Let us set $r = v(a)$ and $d = \gcd(v(a), n)$. We write $a = \pi_v^r u$ for some $u \in \mathcal{O}_v^\times$. There exist integers b, c such that $br + cn = d$ so that $(\pi_v^r u)^b(\pi_v^c)^n = \pi_v^d u'$ for some $u' \in \mathcal{O}_v^\times$. Let $\pi_v^{1/(n/d)}$ and $u^{1/n}$ be formal n/d-th and $1/n$-th roots of π_v and u, respectively. We have the following towers of extensions:

$$\begin{tikzcd}
 & F_v({\pi_v}^{1/(n/d)}, u^{1/n}) \arrow[dl, no head] \arrow[dr, no head] & \\
F_v(a^{1/n}) \arrow[dr, no head] & & F_v({\pi_v}^{1/(n/d)}) \arrow[dl, no head] \\
 & F_v &
\end{tikzcd}$$

Let us set $M = F_v({\pi_v}^{1/(n/d)}, u^{1/n})$, $K = F_v(a^{1/n})$ and $L = F_v(\pi_v^{1/(n/d)})$. The extensions M/K and M/L are unramified because $v(n) = 0$. Now, Proposition 9.1.1.1 gives that

$$\Delta(K/F_v)^{[M:K]} = \Delta(L/F_v)^{[M:L]}.$$

One has that

$$[K : F_v] \cdot [M : K] = [L : F_v] \cdot [M : L] = (n/d)[M : L],$$

thus

$$\Delta(K/F_v) = \Delta(L/F_v)^{[M:L]/[M:K]} = \Delta(L/F_v)^{(d/n)[K:F_v]}.$$

Recall that by Lemma 9.1.2.1, one has that

$$\Delta(L/F_v) = \Delta(F_v(\pi_v^{1/(n/d)})/F_v) = \pi_v^{(n/d)-1}\mathcal{O}_v,$$

thus

$$\begin{aligned}\Delta(K/F_v) = \pi_v^{[K:F_v]\cdot(d/n)((n/d)-1)}\mathcal{O}_v &= \pi_v^{[K:F_v](1-d/n)}\mathcal{O}_v \\ &= \pi_v^{[F(\sqrt[n]{a}):F_v](1-d/n)}\mathcal{O}_v.\end{aligned}$$

□

Proposition 9.1.2.3. — *Let $a \in F_v^\times$. Let $m \geq 2$ be an integer such that $v(m) = 0$. Let us set $d = \gcd(v(a), m)$. One has that*

$$\Delta\big((F_v[X]/(X^m - a))/F_v\big) = \pi_v^{d-m}\mathcal{O}_v.$$

Proof. — Let $X^m - a = \prod_{j=1}^{\ell} b_j(X)$, be the composition of $X^m - a$ into a product of irreducible unitary polynomials (repetitions are allowed). For every $j = 1, \ldots, \ell$, let η_j be a root of $b_j(X)$ so that $F_v(b_j)$ are fields and the homomorphism

$$F_v[X]/(X^m - a) \to \big(\prod_{j=1}^{\ell} F_v(\eta_j)\big)$$

induced from the homomorphism

$$F[X] \to \prod_{j=1}^{\ell} F_v(b_j) \qquad X \mapsto (b_j)_j,$$

is an isomorphism. For every j, one has that $\eta_j^m = a$ and by 9.1.2.2, we have that

$$\Delta(F_v(\eta_j)/F_v) = \pi_v^{\deg(b_j)(1-(d/m))}\mathcal{O}_v.$$

We deduce that

$$\begin{aligned}\Delta((F_v[X]/(X^m - a))/F_v) = \prod_{i=1}^{\ell}(\pi_v^{\deg(b_j)(1-d/m)}\mathcal{O}_v) = \pi_v^{m(d/m-1)}\mathcal{O}_v \\ = \pi_v^{d-m}\mathcal{O}_v.\end{aligned}$$

□

9.1.3. — In this paragraph, we define the heights that will be used for the counting. The following notation will be used in the rest of the Chapter: $m \geq 2$ will be a fixed integer, r will be the smallest prime of m and $\alpha(m) := m^2 - m^2/r$. We will use the terminology from 4.4.

Lemma 9.1.3.1. — *For $v \in M_F^0$, let us define $f_v^\Delta : F_v^\times \to \mathbb{R}_{>0}$ by*

$$f_v^\Delta(y) = |y|_v^{1/m}\bigg(N\big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\big)^{1/\alpha(m)}\bigg),$$

where N stands for the ideal norm. For $v \in M_F^\infty$, we set $f_v^\Delta(y) = |y|_v^{1/m}$.

1. *For every $v \in M_F$, the function f_v^Δ is m-homogenous and of weighted degree 1.*
2. *For every $v \in M_F^0$, the function f_v^Δ is locally constant.*
3. *Let $v \in M_F^0$ such that $v(m) = 0$. For every $y \in F_v^\times$, one has that*

$$f_v^\Delta(y) = |y|_v^{1/m}|\pi_v|_v^{(\gcd(v(y),m)-m)/\alpha(m)} = |\pi_v|_v^{v(y)/m+(\gcd(v(y),m)-m)/\alpha(m)}.$$

For every $y \in \mathcal{O}_v^\times$, one has that $f_v^\Delta(y) = 1$.

Proof. — 1. The function $y \mapsto |y|_v^{1/m}$ is m-homogenous of weighted degree 1. It follows that for $v \in M_F^\infty$ one has that f_v is m-homogenous of weighted degree 1. Let $v \in M_F^0$ and let $t \in F_v^\times$. The image of the ideal $(X^m - y)$ under the isomorphism

$$F_v[X] \to F_v[X] \qquad X \mapsto t^{-1}X$$

is the ideal $(t^{-m}X^m - y) = (X^m - t^m y)$. It follows that $F_v[X]/(X^m - y)$ and $F_v[X]/(X^m - t^m y)$ are isomorphic, hence the norms of the corresponding discriminants are the same. It follows that

$$y \mapsto \big(N\big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\big)\big)^{1/\alpha(m)}$$

is $F_v^\times$-invariant. We deduce that f_v is m-homogenous of weighted degree 1. The claim is proven.

2. The function $y \mapsto \big(N\big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\big)\big)^{1/\alpha(m)}$ is $(F_v^\times)_m$-invariant by (1). The subgroup $(F_v^\times)_m \subset F_v^\times$ is of the finite index in $F_v^\times$ by Lemma 3.3.5.7, thus open in $F_v^\times$ by [**42**, Exercice 4, Chapter II]. The function $y \mapsto |y|_v$ is invariant for the open subgroup $\mathcal{O}_v^\times \subset F_v^\times$. We deduce that f_v is invariant for the open subgroup $(F_v^\times)_m \cap \mathcal{O}_v^\times$ of $F_v^\times$. It follows that f_v is locally constant.
3. As $v(m) = 0$, Proposition 9.1.2.3 gives that

$$\Delta\big((F_v[X]/(X^m - y))/F_v\big) = \pi_v^{m - \gcd(v(y), m)}\mathcal{O}_v.$$

We deduce that for every $y \in F_v^\times$ one has that

$$\begin{aligned} f_v^\Delta(y) &= |y|_v^{1/m}\left(N\big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\big)\right)^{1/\alpha(m)} \\ &= |y|_v^{1/m}|\pi_v|_v^{(\gcd(v(y),m)-m)/\alpha(m)} \\ &= |\pi_v|_v^{v(y)/m + (\gcd(v(y),m)-m)/\alpha(m)}. \end{aligned}$$

If $y \in \mathcal{O}_v^\times$, one has that

$$f_v^\Delta(y) = |y|_v^{1/m}|\pi_v|_v^{(\gcd(v(y),m)-m)/\alpha(m)} = 1.$$

The claim is proven.

□

Definition 9.1.3.2. — *Let $v \in M_F$ and let $k \in \mathbb{Z}$. Let f_v^Δ be as in Lemma 9.1.3.1.*

- *The function $x \mapsto (f_v^\Delta(x))^k$ will be called the discriminant m-homogenous function of weighted degree k.*

- *A degree k family $(f_v : F_v^\times \to \mathbb{R}_{\geq 0})_v$ of m-homogenous continuous functions, will be said to be quasi-discriminant if for almost all v, one has that $f_v = (x \mapsto (f_v^\Delta(x))^k)$. It follows from Lemma 9.1.3.1 that quasi-discriminant families are generalized adelic (see Definition 4.4.1.1) and the resulting height $H = H((f_v)_v)$ on $\mathscr{P}(m)(F)$ will be said to be quasi-discriminant height.*
- *If for every $v \in M_F$ one has that $f_v = (x \mapsto (f_v^\Delta(x))^k)$, then the family $(f_v)_v$ will be said to be the discriminant degree k family. The resulting height $H^\Delta = H((f_v)_v)$ will be said to be the discriminant height.*

As usual, we will write H for the resulting heights on the set of the isomorphism classes $[\mathscr{P}(m)(F)]$.

Remark 9.1.3.3. — Note that by Lemma 9.1.3.1 and by Lemma 4.4.3.1, a "quasi-discriminant" and a "quasi-toric" family are different notions.

Remark 9.1.3.4. — The calculations from Proposition 9.1.2.3 and Lemma 9.1.3.1 may be well known, however we have not found an adequate reference.

Lemma 9.1.3.5. — *Let $(f_v^\Delta)_v$ be the discriminant degree 1 family of m-homogenous continuous functions and let H^Δ be the resulting height. Let $y \in F^\times$. One has that*

$$H^\Delta(q^m(y)) = N\Big(\Delta\big(F[X]/(X^m - y))/F\big)\Big)^{1/\alpha(m)},$$

where $q^m : (\mathbb{A}^1 - \{0\}) \to \mathscr{P}(m)$ is the quotient 1-morphism.

Proof. — By the product formula, one has that

$$\begin{aligned}
H^\Delta(y) &= \prod_{v \in M_F} f_v^\Delta(y) \\
&= \Big(\prod_{v \in M_F^\infty} |y|_v^{1/m}\Big) \prod_{v \in M_F^0} |y|_v^{1/m} N\Big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\Big)^{\frac{1}{\alpha(m)}} \\
&= \prod_{v \in M_F^0} N\Big(\Delta\big((F_v[X]/(X^m - y))/F_v\big)\Big)^{\frac{1}{\alpha(m)}}.
\end{aligned}$$

Proposition 9.1.1.4 gives that

$$\prod_{v\in M_F^0} N\Big(\Delta\big((F_v[X]/(X^m-y))/F_v\big)\Big)^{\frac{1}{\alpha(m)}} = N\Big(\Delta\big((F[X]/(X^m-y))/F\big)\Big)^{\frac{1}{\alpha(m)}}.$$

We deduce that

$$H^{\Delta}(y) = N\Big(\Delta\big((F[X]/(X^m-y))/F\big)\Big)^{1/\alpha(m)}.$$

□

9.1.4. — In this paragraph, we prove the weak Northcott property. Our proof does not involve Hermite-Minkowski theorem ([**42**, Theorem 2.13]) and relies on a comparison with toric heights that we establish in Lemma 9.1.4.3.

Let $(f_v : F_v^\times \to \mathbb{R}_{>0})_v$ be a degree 1 quasi-discriminant family of m-homogenous functions and let $H = H((f_v)_v)$. For $v \in M_F$, let H_v be the function $H_v : [\mathscr{T}(m)(F_v)] \to \mathbb{R}_{>0}$ induced from $F_v^\times$-invariant function $F_v^\times \to \mathbb{R}_{>0}, y \mapsto |y|_v^{-1/m} f_v(y)$ (we have studied such functions in 4.4.5 for a general generalized adelic family). If $f_v = f_v^{\Delta}$, we may write H_v^{Δ} for H_v. By Lemma 4.4.5.2, one has that if $x \in [\mathscr{T}(m)(F)]$, then

$$H(x) = \prod_{v\in M_F} H_v([\mathscr{T}(m)(i_v)](x)), \tag{9.1.4.1}$$

where for $v \in M_F$, the map $[\mathscr{T}(m)(i_v)] : [\mathscr{T}(m)(F)] \to [\mathscr{T}(m)(F_v)]$ is the induced map from the $F_v^\times$-invariant inclusion $i_v : (F^\times)^n \to (F_v^\times)^n$.

Lemma 9.1.4.2. — *The following claims are valid:*

1. *Suppose that $v(m) = 0$. For $y \in F_v^\times$, one has that*
$$H_v^{\Delta}(q_v^m(y)) = |\pi_v|_v^{(\gcd(v(y),m)-m)/\alpha(m)}.$$
The function $H_v^{\Delta} : [\mathscr{T}(m)(F_v)] \to \mathbb{R}_{>0}$ is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. If $x \in [\mathscr{T}(m)(\mathcal{O}_v)]$, then $H_v^{\Delta}(x) = 1$.
2. *Suppose that $v \in M_F^\infty$. One has that $H_v^{\Delta} = 1$.*

Proof. — 1. As $v(m) = 0$, by Lemma 9.1.3.1, one has that

$$H_v^{\Delta}(q_v^m(y)) = |y|_v^{-1/m} f_v^{\Delta}(y) = |\pi_v|_v^{(\gcd(v(y),m)-m)/\alpha(m)}$$

for $y \in F_v^\times$. Let us prove that H_v^Δ is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. Let $x \in [\mathscr{T}(m)(F_v)]$ and let $u \in [\mathscr{T}(m)(\mathcal{O}_v)]$. Let $\widetilde{x}$ and $\widetilde{u}$ be its lifts in $F_v^\times$ and $\mathcal{O}_v^\times$, respectively. We have that

$$H_v^\Delta(x) = |\pi_v|_v^{(\gcd(v(\widetilde{x}),m)-m)/\alpha(m)} = |\pi_v|_v^{(\gcd(v(\widetilde{u}\widetilde{x}),m)-m)/\alpha(m)} = H_v^\Delta(xu).$$

It follows that H_v^Δ is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. Suppose that $x \in [\mathscr{T}(m)(\mathcal{O}_v)] = q^m(\mathcal{O}_v^\times)$. Then $\widetilde{x}$ can be taken in $\mathcal{O}_v^\times$. Thus

$$H_v^\Delta(x) = |\pi_v|_v^{(\gcd(v(\widetilde{x}),m)-m)/\alpha(m)} = 1.$$

2. The function f_v^Δ is the function $x \mapsto |x|_v^{1/m}$. The function H_v^Δ is the induced function from the constant function $x \mapsto |x|_v^{-1/m} f_v^\Delta(x) = 1$, hence $H_v^\Delta = 1$.

□

Lemma 9.1.4.3. — *There exists $C > 0$ such that for every $x \in [\mathscr{T}(m)(F)]$ one has that*

$$CH^\Delta(x) \geq H^\#(x)^{\frac{1}{\alpha(m)}},$$

where $H^\#$ is the height defined by the degree 1 toric family $(f_v^\#)_v$ of m-homogenous functions.

Proof. — For $v \in M_F$, let H_v^Δ be the function induced from $F_v^\times$-invariant function $y \mapsto |y|_v^{-1/m} f_v^\Delta(y)$ and let $H_v^\#$ be the function induced from $F_v^\times$-invariant function $y \mapsto |y|_v^{-1/m} f_v^\#(y)$. Recall that for $v \in M_F^\infty$, by the definitions of f_v^Δ and $f_v^\#$ (see Lemma 9.1.3.1 and Definition 4.4.3.2), one has $f_v^\Delta = f_v^\#$, thus by Lemma 9.1.4.2, one has that $H_v^\Delta = H_v^\# = 1$. Using this and using Lemma 4.4.5.2, for $x \in [\mathscr{T}(m)(F)]$, we get that

$$H^\Delta(x) = \prod_{v \in M_F^0} H_v^\Delta([\mathscr{T}(m)(i_v)](x))$$
$$H^\#(x) = \prod_{v \in M_F^0} H_v^\#([\mathscr{T}(m)(i_v)](x)),$$

where the maps $[\mathscr{T}(m)(i_v)] : [\mathscr{T}(m)(F)] \to [\mathscr{T}(m)(F_v)]$ are the induced maps from $(F^\times)_m$-invariant inclusions $i_v : (F^\times)^n \to (F_v^\times)^n$. For every finite v such that $v(m) = 0$, by the finiteness of the space $[\mathscr{T}(m)(F_v)]$ one has that there exists $C_v > 0$ such that for every $x \in [\mathscr{T}(m)(F_v)]$ one has that

$$H_v^\Delta(x) \geq C_v H_v^\#(x)^{1/\alpha(m)}.$$

Let $v \in M_F^0$ be such that $v(m) = 0$. By Lemma 9.1.3.1, one has for $y \in F_v^\times$ that

$$H_v^\Delta(q_v^m(y)) = |\pi_v|_v^{-(m-\gcd(m,v(y)))/\alpha(m)}.$$

On the other side, by Lemma 4.4.3.1, one has for $y \in F_v^\times$ that

$$H_v^\#(q_v^m(y)) = |\pi_v|_v^{-(\frac{v(y)}{m} - \lfloor\frac{v(y)}{m}\rfloor)}.$$

For every $k \in \mathbb{Z}$ one has that

$$m - \gcd(k,m) \geq \frac{k}{m} - \left\lfloor \frac{k}{m} \right\rfloor$$

(if k is divisible by m, then the quantities on both hand sides are equal to zero, and if k is not divisible by m, then the quantity on the left hand side is at least 1, hence is bigger than the quantity on the right hand side). We deduce that

$$H_v^\Delta \geq (H_v^\#)^{1/\alpha(m)}.$$

It follows that

$$H^\Delta \geq \Big(\prod_{v(m)\neq 0} C_v\Big)(H^\#)^{1/\alpha(m)}.$$

The statement is proven. □

Proposition 9.1.4.4. — *Let $(f_v : F_v^\times \to \mathbb{R}_{>0})_v$ be a degree 1 quasi-discriminant family of m-homogenous functions. The height $H = H((f_v)_v)$ is a weak Northcott height. Moreover, for every $B > 0$, there exists $C > 0$ such that*

$$|\{x \in [\mathscr{P}(m)(F)] | H(x) < B\}| < CB^{m\alpha(m)}.$$

Proof. — Let $(f_v^\Delta : F_v^\times \to \mathbb{R}_{>0})_v$ be the discriminant degree 1 family of m-homogenous functions. The families $(f_v^\Delta)_v$ and $(f_v)_v$ are degree 1 families of m-homogenous continuous functions and for almost all v one has $f_v^\Delta = f_v$ (because $(f_v)_v$ is quasi-discriminant), thus by Lemma 4.4.1.6, there exists a constant $C_1 > 0$ such that $C_1 H^\Delta \leq H(y)$ for every $y \in [\mathscr{P}(m)(F)]$. Let $(f_v^\# : F_v^\times \to \mathbb{R}_{>0})_v$ be the toric family of m-homogenous functions of weighted degree 1 and let $H^\# = H((f_v^\#)_v)$ be the resulting height. Lemma 9.1.4.3 gives that there exists $C_2 > 0$ such that

$$H^\Delta(y) \geq C_2 H^\#(y)^{1/\alpha(m)},$$

for every $y \in [\mathscr{P}(m)(F)]$. Now, for every $B > 0$, one has that

$$\begin{aligned}|\{x \in [\mathscr{P}(m)(F)] | H(x) < B\}| \\ &\leq |\{x \in [\mathscr{P}(m)(F)] | C_1 H^{\Delta}(x) < B\}| \\ &\leq |\{x \in [\mathscr{P}(m)(F)] | C_2 C_1 (H^{\#}(x))^{1/\alpha(m)} < B\}| \\ &= |\{x \in [\mathscr{P}(m)(F)] | H^{\#}(x) < C_2^{-1} C_1^{-1} B^{\alpha(m)}\}|.\end{aligned}$$

Theorem 8.2.2.12 implies that there exists $C_3 > 0$ such that

$$|\{x \in [\mathscr{P}(m)(F)] | H^{\#}(x) < C_0^{-1} C_1^{-1} B^{\alpha(m)}\}| < C_3 C_2^{-1} C_1^{-m} B^{m\alpha(m)}$$

for every $B > 0$ (recall that in 8.2.2.12, the degree of the toric family is m and our toric family is of the degree 1). Thus for every $B > 0$, we have that

$$|\{x \in [\mathscr{P}(m)(F)] | H(x) < B\}| \leq C_3 C_2^{-1} C_1^{-m} B^{m\alpha(m)}.$$

The statement is proven. □

Remark 9.1.4.5. — In the next sections we establish the precise asymptotic behaviour of $|\{x \in [\mathscr{P}(m)(F)] | H(x) < B\}|$ when $B \to \infty$.

9.2. Analysis of height zeta function

The goal of this section is to establish the asymptotic behaviour of $|\{x \in [\mathscr{P}(m)(F)] | H(x) \leq B\}$. Using the "Tauberian dictionary", the task translates into the study the convergence of the height zeta series. For that purpose we use Fourier analysis.

Let $(f_v : F_v^{\times} \to \mathbb{R}_{>0})_v$ be a quasi-discriminant **degree** m family of m-homogenous functions. For $v \in M_F$, we will denote by f_v^{Δ} the discriminant m-homogenous function of weighted degree m. In the entire section we will denote by S the finite set

$$S := \{v \in M_F^0 | f_v \neq f_v^{\Delta} \text{ or } v(m) \neq 0\}.$$

9.2.1. — In this paragraph we study the local Fourier transform of a local height.

Let $v \in M_F$. Let $q_v^m : F_v^{\times} \to (F_v^{\times})/(F_v^{\times})_m = [\mathscr{P}(m)(F_v)] = [\mathscr{T}(m)(F_v)]$ be the quotient map. By Lemma 5.2.3.1, the measure d^*x on $F_v^{\times}$ is $F_v^{\times}$-invariant for the action $t \cdot y = t^m y$ of $F_v^{\times}$ on $F_v^{\times}$. We set μ_v to be the quotient Haar measure d^*x/d^*x on $[\mathscr{T}(m)(F_v)]$ (see Definition 5.4.1.1). Recall that the sets $[\mathscr{T}(m)(F_v)]$ are finite by Lemma 3.3.5.7, hence $\mu_v([\mathscr{T}(m)(F_v)])$ are finite positive numbers.

Lemma 9.2.1.1. — *The Haar measure μ_v is normalized by*

$$\mu_v([\mathscr{T}(m)(F_v)]) = m.$$

Proof. — Suppose that $v \in M_F^0$. Recall from Proposition 3.3.5.4 that $[\mathscr{T}(m)(\mathcal{O}_v)]$ identifies with the open and compact subgroup $q_v^m(\mathcal{O}_v^\times)$ of $[\mathscr{T}(m)(F_v)]$ and is of index m by Lemma 3.3.5.5. By Lemma 5.4.2.1, the measure μ_v is normalized by $\mu_v([\mathscr{T}(m)(\mathcal{O}_v)]) = 1$ i.e. by $\mu_v([\mathscr{T}(m)(F_v)]) = m$. Suppose that $v \in M_F^\infty$. One has by Lemma 5.4.1.3 that

$$\int_{[\mathscr{T}(m)(F_v)]} 1\mu_v = \frac{m}{\lambda_{v,1}(F_v)} \int_{F_{v,1}} 1\lambda_{v,1} = m.$$

□

If $\chi \in [\mathscr{T}(m)(F_v)]^*$ is a character, we denote by $\widetilde{\chi}$ the pullback character $(q_v^m)^*(\chi) : F_v^\times \to S^1$. Lemma 6.2.1.1 gives that $\widetilde{\chi}^m = 1$ and that $\chi \mapsto \widetilde{\chi}$ is an isomorphism of $[\mathscr{T}(m)(F_v)]^*$ to the closed subgroup $(F_v)_m^\perp \subset (F_v^\times)^*$. For a complex number s and a character $\chi \in [\mathscr{T}(m)(F_v)]^*$ we define formally

$$\widehat{H}_v(s,\chi) := \int_{[\mathscr{T}(m)(F_v)]} H_v^{-s}\chi\mu_v.$$

Lemma 9.2.1.2. — *Let $v \in M_F$.*

1. *For every $s \in \mathbb{C}$ one has that $H_v^{-s} \in L^1([\mathscr{T}(m)(F_v)], \mu_v)$. For every $\chi \in [\mathscr{T}(m)(F_v)]^*$, one has that $s \mapsto \widehat{H}_v(s,\chi)$ is an entire function. Moreover, for every compact $\mathcal{K} \subset \mathbb{R}$, there exists $C(\mathcal{K}) > 0$ such that for every $s \in \mathcal{K} + i\mathbb{R}$, one has that $\widehat{H}_v(s,\chi) \leq C$.*
2. *Suppose that $v \in M_F^0 - S$. Let $s \in \mathbb{C}$ and let $\chi \in [\mathscr{T}(m)(F_v)]^*$ be a character. One has that*

(9.2.1.3)
$$\widehat{H}_v^{\Delta}(s,\chi) := \begin{cases} \sum_{j=0}^{m-1} |\pi_v|_v^{(s(m^2 - m\gcd(j,m)))/\alpha(m)}\widetilde{\chi}(\pi_v^j) & \text{if } \chi_v|_{[\mathscr{T}^m(\mathcal{O}_v)]} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. *Suppose that $v \in M_F^\infty$. For every $s \in \mathbb{C}$ and every $\chi \in [\mathscr{T}(m)(F_v)]^*$, one has that*

$$\widehat{H}_v(s,\chi) := \begin{cases} m & \text{if } \chi_v = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2.1.4}$$

Proof. — 1. The group $[\mathscr{T}(m)(F_v)]$ is finite by Lemma 3.3.5.7. Moreover, μ_v is a Haar measure on $[\mathscr{T}(m)(F_v)]$, hence a non-zero multiple of the counting measure. It follows that $H_v^{-s}\in L^1([\mathscr{T}(m)(F_v)],\mu_v)$. Let $\chi\in[\mathscr{T}(m)(F_v)]^*$ be a character. For every $x\in[\mathscr{T}(m)(F_v)]$, one has that $s\mapsto H_v(x)^{-s}\chi(x)$ is an entire function. We deduce that $s\mapsto\widehat{H}_v(s,\chi)$ is an entire function. Moreover, for every $s\in\mathbb{C}$, by the triangle inequality, one has that $|\widehat{H}_v(s,\chi)|\leq\widehat{H}_v(\Re(s),1)$. For every compact $\mathcal{K}\subset\mathbb{R}$, by the fact that μ_v is a multiple of the counting measure, there exists $C'(\mathcal{K})>0$ such that if $\Re(s)\in\mathcal{K}$ then

$$|\widehat{H}_v(s,\chi)|\leq\widehat{H}(\Re(s),1)\leq C'(\mathcal{K})\sup_{y\in\mathcal{K}}\sum_{x\in[\mathscr{T}(m)(F_v)]}H_v(x)^{-y}.$$

The claim is proven.

2. It follows from Lemma 9.1.4.2 that

$$[\mathscr{T}(m)(\mathcal{O}_v)]\to\mathbb{C}\qquad x\mapsto(H_v^{\Delta}(x))^{-s}$$

is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. We deduce that if $\chi|_{[\mathscr{T}(m)(\mathcal{O}_v)]}\neq1$, then

$$\widehat{H}_v^{\Delta}(s,\chi)=\int_{[\mathscr{T}(m)(F_v)]}(H_v^{\Delta})^{-s}\chi\mu_v=0.$$

Suppose that $\chi|_{[\mathscr{T}(m)(\mathcal{O}_v)]}=1$. It follows that $\widetilde{\chi}|_{\mathcal{O}_v^{\times}}=1$. The function $(H_v^{\Delta})^{-s}$ is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant, hence $((H_v^{\Delta})^{-s}\circ q_v^{\mathbf{a}})=x\mapsto(|x|_v^{-1}f_v^{\Delta}(x))^{-s}$ is $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. Using Lemma 5.4.1.3, we get that

$$\begin{aligned}
\widehat{H}_v^{\Delta}(s,\chi) &= \int_{[\mathscr{T}(m)(F_v)]} (H_v^{\Delta})^{-s}\chi\mu_v \\
&= \zeta_v(1)\int_{F_v^{\times}\cap\mathcal{D}_v^m} ((H_v^{\Delta})^{-s}\circ q_v^{\mathbf{a}})\cdot(\chi\circ q_v^{\mathbf{a}})d^*x \\
&= \zeta_v(1)\int_{F_v^{\times}\cap\mathcal{D}_v^m} (|x|_v^{-1}f_v^{\Delta}(x))^{-s}\widetilde{\chi}(x)d^*x \\
&= \zeta_v(1)\int_{F_v^{\times}\cap\mathcal{D}_v^m} (|\pi_v|_v^{(m\gcd(v(x),m)-m^2)/\alpha(m)})^{-s}\widetilde{\chi}(x)d^*x \\
&= \zeta_v(1)\sum_{j=0}^{m-1}\int_{\pi_v^j\mathcal{O}_v^{\times}} |\pi_v|_v^{s(m^2-m\gcd(v(x),m))/\alpha(m)}\widetilde{\chi}(x)d^*x \\
&= \zeta_v(1)\sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2-m\gcd(j,m))/\alpha(m)}\widetilde{\chi}(\pi_v^j)\cdot d^*x(\pi_v^j\mathcal{O}_v^{\times}) \\
&= \sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2-m\gcd(j,m))/\alpha(m)}\widetilde{\chi}(\pi_v^j).
\end{aligned}$$

The claim is proven.

3. For every $s\in\mathbb{C}$, the function $(H_v^{\Delta})^{-s}$ is the constant function 1. Thus for $\chi\in[\mathscr{T}(m)(F_v)]^*$ such that $\chi_v\neq 1$ one has that $\widehat{H}_v(s,\chi)=0$. Suppose that $\chi=1$. We have that

$$\int_{[\mathscr{T}(m)(F_v)]}(H_v^{\Delta})^{-s}\mu_v = \int_{[\mathscr{T}(m)(F_v)]} 1\mu_v = m$$

by Lemma 9.2.1.1.

□

9.2.2. — We will compare the Fourier transform of the local "discriminant" height with a product of certain local L-functions.

Lemma 9.2.2.1. — *Let $v\in M_F^0-S$. Let $s\in\mathbb{C}$ with $\Re(s)>0$ and let $\chi\in[\mathscr{T}(m)(F_v)]^*$. One has that*

$$\left|\frac{\widehat{H}_v^{\Delta}(s,\chi)}{\prod_{j=1}^{r-1}L_v(s,\widetilde{\chi}^{mj/r})}\right| \leq \left(\frac{\zeta_v\big(\Re(s)(1+1/\alpha(m))\big)}{\zeta_v\big(2\Re(s)(1+1/\alpha(m))\big)}\right)^{2^{r-1}rm^3}.$$

Proof. — Suppose first $\chi|_{[\mathscr{T}(m)(\mathcal{O}_v)]} \neq 1$. Lemma 9.2.1.2 gives that $\widehat{H}_v^{\Delta}(s,\chi) = 0$, hence the inequality is trivially verified. Suppose now that $\chi|_{[\mathscr{T}(m)(\mathcal{O}_v)]} = 1$. As $v \in M_F^0 - S$, Lemma 9.2.1.2 gives that

$$\widehat{H}_v^{\Delta}(s,\chi) = \sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2 - m\gcd(j,m))/\alpha(m)} \widetilde{\chi}(\pi_v^j).$$

We have that:

(9.2.2.2)

$$\begin{aligned}
&\left| \frac{\widehat{H}_v^{\Delta}(s,\chi)}{\prod_{j=1}^{r-1} L_v(s, \widetilde{\chi}^{mj/r})} \right| \\
&\quad = \left| \sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2 - m\gcd(j,m))/\alpha(m)} \widetilde{\chi}(\pi_v^j) \right| \cdot \left| \prod_{j=1}^{r-1} (1 - |\pi_v|_v^s \widetilde{\chi}^{mj/r}(\pi_v)) \right| \\
&\quad = \left| \left(\sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2 - m\gcd(j,m))/\alpha(m)} \widetilde{\chi}(\pi_v^j) \right) \prod_{j=1}^{r-1} (1 - |\pi_v|_v^s \widetilde{\chi}(\pi_v^{mj/r})) \right|.
\end{aligned}$$

Whenever $\gcd(m,j) = m/r$, we have $m^2 - m\gcd(m,j) = m^2 - m\gcd(m,j) = \alpha(m)$, and we rewrite the last sum as:

$$\begin{aligned}
&\sum_{j=0}^{m-1} |\pi_v|_v^{s(m^2 - m\gcd(j,m))/\alpha(m)} \widetilde{\chi}(\pi_v^j) \\
&= 1 + \sum_{\gcd(j,m) = \frac{m}{r}} |\pi_v|_v^s \widetilde{\chi}(\pi_v^j) + \sum_{\gcd(m,j) < \frac{m}{r}} |\pi_v|_v^{s(m^2 - m\gcd(m,j))/\alpha(m)} \widetilde{\chi}(\pi_v^j) \\
&= 1 + |\pi_v|_v^s \left(\sum_{j=1}^{r-1} \widetilde{\chi}(\pi_v^{\frac{mj}{r}}) \right) + \sum_{\gcd(m,j) < \frac{m}{r}} |\pi_v|_v^{s(m^2 - m\gcd(j,m))/\alpha(m)} \widetilde{\chi}(\pi_v^j).
\end{aligned}$$

Now, we expand the last product of the equality (9.2.2.2) and collect the terms:

$$\left| \frac{\widehat{H}_v^{\Delta}(s,\chi)}{\prod_{j=1}^{r-1} L_v(s, \widetilde{\chi}^{mj/r})} \right| = \left| 1 + \sum_{k=1}^{m^2 r} A_k(\widetilde{\chi}) |\pi_v|_v^{sk/\alpha(m)} \right|.$$

It is clear that

$$A_1(\widetilde{\chi}) = A_2(\widetilde{\chi}) = \cdots = A_{\alpha(m)-1}(\widetilde{\chi}) = 0,$$

and one also has that

$$A_{\alpha(m)}(\widetilde{\chi}) = 0,$$

because every $\widetilde{\chi}(\pi_v^{\frac{mj}{r}})$ appears exactly twice in the expanded product with the different signs. Moreover, each $A_k(\widetilde{\chi})$ is a sum of no more than $2^{r-1}m$ numbers of the absolute value 1, thus for every $k=1,\dots,m^2r$, one can estimate that $|A_k(\widetilde{\chi})|\leq 2^{r-1}m$. Now, using the triangle inequality and the fact that $|\pi_v|_v<1$ we deduce that

$$\begin{aligned}\left|\frac{\widehat{H}^{\Delta}_v(s,\chi)}{\prod_{j=1}^{r-1}L_v(s,\widetilde{\chi}^{mj/r})}\right| &= \left|1+\sum_{k=1}^{m^2r}A_k(\chi)|\pi_v|_v^{sk/\alpha(m)}\right|\\ &\leq 1+2^{r-1}m\sum_{k=\alpha(m)+1}^{m^2r}|\pi_v|_v^{\Re(s)k/\alpha(m)}\\ &\leq 1+2^{r-1}m(m^2r-\alpha(m)-1)|\pi_v|_v^{\Re(s)(\alpha(m)+1)/\alpha(m)}\\ &\leq 1+2^{r-1}m^3r|\pi_v|_v^{\Re(s)(1+(1/\alpha(m)))}\\ &\leq (1+|\pi_v|_v^{\Re(s)(1+(1/\alpha(m)))})^{2^{r-1}m^3r}\\ &= \left(\frac{\zeta_v\big(\Re(s)(1+(1/\alpha(m)))\big)}{\zeta_v\big(2\Re(s)(1+(1/\alpha(m)))\big)}\right)^{2^{r-1}m^3}.\end{aligned}$$

The statement is proven. □

9.2.3. — In this paragraph we study the global Fourier transform.

Let $[\mathscr{T}(m)(\mathbb{A}_F)]$ be the restricted product

$$[\mathscr{T}(m)(\mathbb{A}_F)]=\prod_{v\in M_F}{}'[\mathscr{T}(m)(F_v)],$$

where the restricted product is taken with the respect to the open and compact subgroups $[\mathscr{T}(m)(\mathcal{O}_v)]\subset[\mathscr{T}(m)(F_v)]$ for $v\in M_F^0$. Let $[\mathscr{T}(m)(i)]:[\mathscr{T}(m)(F)]\to[\mathscr{T}(m)(\mathbb{A}_F)]$ be the diagonal map. If $(x_v)_v\in[\mathscr{T}(m)(\mathbb{A}_F)]$, by Lemma 9.1.4.2, the product

$$H(x):=\prod_{v\in M_F}H_v(x)$$

is finite. By Proposition 5.4.3.2, the function $H:[\mathscr{T}(m)(\mathbb{A}_F)]\to\mathbb{R}_{>0}$ is continuous. By the equality (9.1.4.1), for $x\in[\mathscr{T}(m)(F)]$, one has that

$$H(x)=H([\mathscr{T}(m)(i)](x)). \tag{9.2.3.1}$$

For $v \in M_F$, let K_v be the maximal subgroup of the finite group $[\mathscr{T}(m)(F_v)]$ such that H_v is K_v-invariant. By Lemma 9.1.4.2, for every $v \in M_F^0 - S$, one has that $K_v \supset [\mathscr{T}(m)(\mathcal{O}_v)]$.

Lemma 9.2.3.2. — *Let $x \in [\mathscr{T}(m)(\mathbb{A}_F)]$. We denote by S_x the finite set of places of F given by the union of the set M_F^∞, of the set of places v for which $v(x) \neq 0$ and of the set of the places v for which H_v is not $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. Let us set*

$$C(x) := \prod_{v\in S_x} \frac{\max_{z\in[\mathscr{T}(F_v)]} H_v(z)}{\min_{z\in[\mathscr{T}(F_v)]} H_v(z)}.$$

For every $y \in [\mathscr{T}(m)(\mathbb{A}_F)]$, one has that

$$C(x)^{-1}H(y) \leq H(xy) \leq C(x)H(y).$$

Proof. — For $v \in M_F - S_x$, one has that $H_v(x) = 1$. For $v \in S_x$, let us set

$$C_v(x) := \frac{\max_{z\in[\mathscr{T}(F_v)]} H_v(z)}{\min_{z\in[\mathscr{T}(F_v)]} H_v(z)}$$

so that $C(x) = \prod_{v\in S_x} C_v(x)$. For every $y \in [\mathscr{T}(m)(\mathbb{A}_F)]$, one has that

$$\begin{aligned} H(xy) &= \prod_{v\in M_F} H_v(xy) \\ &= \prod_{v\in S_x} H_v(xy) \cdot \prod_{v\in M_F - S_x} H_v(xy) \\ &\leq \prod_{v\in S_x} C_v(x)H_v(y) \cdot \prod_{M_F - S_x} H_v(y) \\ &= C(x)H(y). \end{aligned}$$

Analogously, one verifies that $C(x)^{-1}H(xy) \leq H(y)$.

For $z \in \mathbb{R}$, recall that $\Omega_{>z}$ is the "tube":

$$\Omega_{>z} := \mathbb{R}_{>z} + i\mathbb{R} \subset \mathbb{C}.$$

□

Lemma 9.2.3.3. — *Let $\chi \in ([\mathscr{T}(i)]([\mathscr{T}(m)(F)]))^\perp$.*

1. *Suppose χ does not vanish on K. Then for every $s \in \mathbb{C}$, one has that $\widehat{H}(s, \chi) = 0$.*

2. *Suppose χ vanishes on K. For $s \in \Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$, the product*

$$\prod_{v \in M_F^0} \frac{\widehat{H}_v(s, \chi_v)}{\prod_{j=1}^{r-1} L_v(s, \widetilde{\chi}^{mj/r})}$$

converges uniformly on compacts in the domain $\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$. The function

$$\gamma(-, \chi) : s \mapsto \prod_{v \in M_F^0} \frac{\widehat{H}_v(s, \chi_v)}{\prod_{j=1}^{r-1} L_v(s, \widetilde{\chi}^{mj/r})},$$

is a holomorphic function $\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}} \to \mathbb{C}$ which satisfies that for every compact $\mathcal{K} \subset \mathbb{R}_{>\frac{\alpha(m)}{\alpha(m)+1}}$ there exists $C = C(\mathcal{K}) > 0$ such that

$$|\gamma(s, \chi)| \leq C,$$

for $s \in \mathcal{K} + i\mathbb{R}$.

3. *One has that $\gamma(1, 1) > 0$.*

Proof. — 1. Let $v \in M_F^0$ such that $\chi_v|_{K_v} \neq 1$. By Lemma 9.2.1.2, one has that $\widehat{H}_v(s, \chi_v) = 0$ for every $s \in \mathbb{C}$. It follows that $\widehat{H}(s, \chi) = 0$.

2. For $v \in M_F^0$, let us denote by

$$\gamma_v(s, \chi_v) := \frac{\widehat{H}_v(s, \chi_v)}{\prod_{j=1}^{r-1} L_v(s, \widetilde{\chi}^{mj/r})}.$$

For every $v \in M_F^0$, the function $\gamma_v(-, \chi_v)$ is an entire function, because by Lemma 9.2.1.2, the function $\widehat{H}_v(s, \chi_v)$ is an entire function and because for $j = 1, \ldots, r-1$, the function $(L_v(s, \widetilde{\chi}^{mj/r}))^{-1} = (1 - |\pi_v|_v^s \widetilde{\chi}^{mj/r}(|\pi_v|_v))$ is an entire function. Moreover, by Lemma 9.2.2.1, for $v \in M_F^0 - S$, there exists a positive integer A such that

$$|\gamma_v(s, \chi)| \leq \left(\frac{\zeta_v\big(\Re(s)(1 + \frac{1}{\alpha(m)})\big)}{\zeta_v\big(2\Re(s)(1 + \frac{1}{\alpha(m)})\big)} \right)^A. \tag{9.2.3.4}$$

For every $y > \alpha(m)/(\alpha(m) + 1)$, the product

$$\prod_{v \in M_F^0 - S} \left(\frac{\zeta_v\big(y(1 + \frac{1}{\alpha(m)})\big)}{\zeta_v\big(2y(1 + \frac{1}{\alpha(m)})\big)} \right)^A$$

converges uniformly in the domain $\mathbb{R}_{>\frac{\alpha(m)}{(\alpha(m)+1)}}$ to

$$\left(\frac{\zeta\big(y\big(1+\frac{1}{\alpha(m)}\big)\big)}{\zeta\big(2y(1+\frac{1}{\alpha(m)})\big)}\times\prod_{v\in S}\frac{\zeta_v\big(2y(1+\frac{1}{\alpha(m)})\big)}{\zeta_v\big(y(1+\frac{1}{\alpha(m)})\big)}\right)^A.$$

It follows that the product

$$\prod_{v\in M_F^0-S}\gamma_v(s,\chi_v)$$

converges absolutely and uniformly in the domain $s\in\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$. We deduce that the product $\gamma(s,\chi)=\prod_{v\in M_F^0}\gamma_v(s,\chi_v)$ converges absolutely and uniformly in the domain $s\in\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$. Moreover, $\gamma(-,\chi):\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}\to\mathbb{C}$ is a holomorphic function and it satisfies that

$$\begin{aligned}&|\gamma(s,\chi)|\\&\leq\left(\frac{\zeta\big(\Re(s)(1+\frac{1}{\alpha(m)})\big)}{\zeta\big(2\Re(s)(1+\frac{1}{\alpha(m)})\big)}\times\prod_{v\in S}\frac{\zeta_v\big(2\Re(s)(1+\frac{1}{\alpha(m)})\big)}{\zeta_v\big(\Re(s)(1+\frac{1}{\alpha(m)})\big)}\right)^A\times\prod_{v\in S}|\gamma_v(s,\chi)|.\end{aligned}$$

We deduce that for a compact $\mathcal{K}\subset\mathbb{R}_{>1/(\alpha(m)+1)}$ one has that

$$|\gamma(s,\chi)|\leq\sup_{y\in\mathcal{K}}\left(\left(\frac{\zeta\big(y(1+\frac{1}{\alpha(m)})\big)}{\zeta\big(2y(1+\frac{1}{\alpha(m)})\big)}\times\prod_{v\in S}\frac{\zeta_v\big(2y(1+\frac{1}{\alpha(m)})\big)}{\zeta_v\big(y(1+\frac{1}{\alpha(m)})\big)}\right)^A\prod_{v\in S}\gamma_v(y,\chi)\right).$$

The statement is proven.

3. For $v\in M_F^0-S$, one has that

$$\begin{aligned}\gamma_v(1,1)&=\frac{\widehat{H}_v(1,1)}{\zeta_v(1)^{r-1}}\\&=\Big(\sum_{j=0}^{m-1}|\pi_v|_v^{(m^2-m\gcd(j,m))/\alpha(m)}\Big)(1-|\pi_v|_v)^{r-1}\\&=\Big(1+(r-1)|\pi_v|_v+O(|\pi_v|_v^{\frac{m^2-m^2/r+1}{\alpha(m)}})\Big)\times\\&\quad\times\Big(1-(r-1)|\pi_v|_v+O(|\pi_v|_v^2)\Big)\\&=\Big(1-(r-1)^2|\pi_v|_v^2+O(|\pi_v|_v^{\frac{m^2-m^2/r+1}{\alpha(m)}})\Big),\end{aligned}$$

where by $O(|\pi_v|_v^k)$ is meant a quantity that for all v is bounded by a constant (independent of v) times $|\pi_v|_v^k$. For every $v\in M_F^0$, one has that $\widehat{H}_v(1,1)=\int_{[\mathscr{T}(m)(F_v)]}H_v^{-1}\mu_v>0$, because H_v^{-1} is strictly positive and μ_v is a Haar measure on a finite group, hence

$$\gamma_v(1,1)=\frac{\widehat{H}_v(1,1)}{\zeta_v(1)^{r-1}}>0.$$

It follows that

$$\gamma(1,1)=\prod_{v\in M_F^0}\gamma_v(1,1)>0.$$

□

9.2.4. — In this paragraph we estimate the global Fourier transform.

In the equality 3.4.9.1, we have provided an identification $\mathbb{A}_F^1\times\mathbb{R}_{>0}\xrightarrow{\sim}\mathbb{A}_F^\times$. If χ is a character of $[\mathscr{T}(m)(\mathbb{A}_F)]=\mathbb{A}_F^\times/(\mathbb{A}_F^\times)_m$, then $\widetilde{\chi}$ is of order dividing m, thus $\widetilde{\chi}|\mathbb{R}_{>0}=0$. Recall that by Lemma 3.4.3.2, one has that $[\mathscr{T}(m)(\mathbb{A}_F)]=[\mathscr{T}(m)(\mathbb{A}_F)]_1$.

Let $K^0_{\max}:=\prod_{v\in M_F^0}[\mathscr{T}(m)(\mathscr{O}_v)]$. The group $K^0_{\max}$, as well as its any open subgroup K, is compact. By Corollary 6.2.2.5, for an open subgroup $K\subset K^0_{\max}:=\prod_{v\in M_F^0}[\mathscr{T}(\mathbf{a})(\mathscr{O}_v)]$, one has that

$$\mathfrak{A}_K:=(K[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)]))^\perp$$

is finite. We are ready to estimate the global Fourier transform. Let $\mu_{\mathbb{A}_F}$ be the restricted product measure

$$\mu_{\mathbb{A}_F} = \bigotimes_{v \in M_F} \mu_v$$

on $[\mathscr{T}(\mathbf{a})(\mathbb{A}_F)]$ (as in Definition 5.4.3.3). For $s \in \mathbb{C}$ and a character $\chi \in [\mathscr{T}(m)(\mathbb{A}_F)]^*$, we define formally

$$\widehat{H}(s,\chi) = \int_{[\mathscr{T}(m)(\mathbb{A}_F)]} H^{-s}\chi\mu_{\mathbb{A}_F}.$$

Lemma 5.4.3.2 gives that if the product $\prod_{v\in M_F}\widehat{H}_v(s,\chi_v)$ converges, then

$$\widehat{H}(s,\chi) = \prod_{v\in M_F}\widehat{H}_v(s,\chi_v).$$

Lemma 9.2.4.1. — *For $\chi \in \mathfrak{A}_K$ let us define $d(\chi) = 0$ if $\chi^{m/r} \neq 1$ and $d(\chi) = r-1$, otherwise. For every $\chi \in \mathfrak{A}_K$, the function $\widehat{H}(s,\chi)$ is holomorphic in the domain $s \in \Omega_{>1}$. Moreover, there exists $\delta > 0$ such that*

$$\left(\frac{s-1}{s}\right)^{d(\chi)}\widehat{H}(s,\chi)$$

extends to a holomorphic function in the domain $\Omega_{>1-\delta}$ and such that for every compact $\mathcal{K} \subset \mathbb{R}_{>1-\delta}$ there exists $C = C(\mathcal{K}) > 0$, such that

$$\left|\left(\frac{s-1}{s}\right)^{d(\chi)}\widehat{H}(s,\chi)\right| \leq C(\mathcal{K})(1+|\Im(s)|)$$

for every $\chi \in \mathfrak{A}_K$.

Proof. — Set $\gamma_v(s,\chi_v) = \frac{\widehat{H}_v(s,\chi_v)}{\prod_{j=1}^{r-1}L_v(s,\widetilde{\chi}^{mj/r})}$ for $v \in M_F^0$. By Lemma 9.2.3.3, for every $\chi \in \mathfrak{A}_K$, the product $\prod_{v\in M_F^0}\gamma_v(s,\chi_v)$ converges absolutely and uniformly on the compacts in the domain $\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$ to a holomorphic function $\gamma(-,\chi) : \Omega_{>\frac{\alpha(m)}{\alpha(m)+1}} \to \mathbb{C}$ and, moreover as $\mathfrak{A}_K$ is finite, for every compact $\mathcal{K} \subset \mathbb{R}_{>\frac{\alpha(m)}{\alpha(m)+1}}$, there exists $C_0 > 0$ such that $|\gamma(s,\chi)| \leq C_0$ for every $\chi \in \mathfrak{A}_K$ and every $s \in \mathcal{K} + i\mathbb{R}$.

Let us define $\widetilde{K} := (q^m|_{K^0_{\max}})^{-1}(K)$. It is an open subgroup of finite index of the compact group $K^0_{\max}$, hence $\widetilde{K}$ is of the finite index in $K^0_{\max}$. Note that if $\chi|_K = 1$, then $\widetilde{\chi}|_{\widetilde{K}} = 1$. Now, by Proposition 6.3.1.2, we get that there exist $\frac{1}{\alpha(m)+1} > \delta > 0$ and $C > 0$, such that for every $\chi \in \mathfrak{A}_K$ one has that

$$\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}=1}}^{r-1}\frac{s-1}{s}\Big)\Big(\prod_{j=1}^{r-1}L(s,\widetilde{\chi}^{mj/r})\Big)$$
$$=\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}=1}}^{r-1}\frac{(s-1)L(s,\widetilde{\chi}^{mj/r})}{s}\Big)\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}\neq 1}}^{r-1}L(s,\widetilde{\chi}^{mj/r})\Big)$$

extends to a holomorphic function in the domain $\Omega_{>1-\delta}$ and satisfies the inequality

$$\Big|\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}=1}}^{r-1}\frac{s-1}{s}\Big)\Big(\prod_{j=1}^{r-1}L(s,\widetilde{\chi}^{mj/r})\Big)\Big|\leq C(1+|\Im(s)|)$$

in this domain. We deduce that in the domain $\Omega_{>1-\delta}$ one has that

$$\widehat{H}(s,\chi)=\prod_{v\in M_F^0}\widehat{H}_v(s,\chi_v)\times\prod_{v\in M_F^\infty}\widehat{H}_v(s,\chi_v)$$
$$=\Big(\prod_{v\in M_F^0}\gamma_v(s,\chi)\prod_{j=1}^{r-1}L_v(s,\widetilde{\chi}_v^{mj/r})\Big)\times\prod_{v\in M_F^\infty}\widehat{H}_v(s,\chi)$$

converges to $\gamma(s,\chi)\prod_{j=1}^{r-1}L(s,\widetilde{\chi}^{mj/r})\prod_{v\in M_F^\infty}\widehat{H}_v(s,\chi)$. Moreover, using Lemma 9.2.1.2, if $\mathcal{K}\subset\mathbb{R}_{>1-\delta}$ is a compact, we deduce that there exists $C>0$ such that

$$\Big|\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj}=1}}^{r-1}\frac{s-1}{s}\Big)\widehat{H}(s,\chi)\Big|$$
$$=\Big|\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj}=1}}^{r-1}\frac{s-1}{s}\Big)\gamma(s,\chi)\Big(\prod_{j=1}^{r-1}L(s,\widetilde{\chi}^{mj/r})\Big)\times\prod_{v\in M_F^\infty}\widehat{H}_v(s,\chi)\Big|$$
$$=|\gamma(s,\chi)|\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}=1}}^{r-1}\frac{s-1}{s}L(s,1)\Big)\Big(\prod_{\substack{j=1\\ \widetilde{\chi}^{mj/r}\neq 1}}^{r-1}|L(s,\widetilde{\chi}^{mj/r})|\Big)\Big|\prod_{v\in M_F^\infty}\widehat{H}_v(s,\chi)\Big|$$
$$\leq C(1+|\Im(s)|).$$

To complete the proof, it suffices to see that for every $\chi \in \mathfrak{A}_K$, one has that

$$|\{j|\, 0 \leq j \leq r-1 \text{ and } \widetilde{\chi}^{mj/r} = 1\}| = d(\chi). \tag{9.2.4.2}$$

Suppose that $\chi^{m/r} = 1$. For every $j \in \{1, \dots, r-1\}$ one has that $\widetilde{\chi}^{mj/r} = ((q^m_{\mathbb{A}_F})^*\chi)^{mj/r} = (q^m_{\mathbb{A}_F})^*(\chi^{mj/r}) = 1$, hence

$$|\{j|\, 0 \leq j \leq r-1 \text{ and } \widetilde{\chi}^{mj/r} = 1\}| = r-1 = d(\chi).$$

Suppose that $\chi^{m/r} \neq 1$. Suppose that for some $j \in \{1, \dots, r-1\}$ one has that

$$\widetilde{\chi}^{mj/r} = ((q^m_{\mathbb{A}_F})^*\chi^{mj/r}) = 1.$$

Let $k \in \{1, \dots, r-1\}$ be such that $kj = 1 + \ell r$ for some $\ell \in \mathbb{Z}_{>0}$. Then,

$$1 = \widetilde{\chi}^{mjk/r} = \widetilde{\chi}^{(m\ell r+1)/r} = \widetilde{\chi}^{m\ell}\widetilde{\chi}^{m/r} = \widetilde{\chi}^{m/r},$$

because $\widetilde{\chi}^{m\ell} = (q^m_{\mathbb{A}_F})^*(\chi^\ell) = 1$ (because $[\mathscr{T}(m)(\mathbb{A}_F)]$ is an m-torsion group).This is a contradiction. We deduce that

$$|\{j|\, 0 \leq j \leq r-1 \text{ and } \widetilde{\chi}^{mj/r} = 1\}| = 0 = d(\chi).$$

In either of the two cases, we obtain that the equality (9.2.4.2) is valid. We deduce that

$$\left|\left(\frac{s-1}{s}\right)^{d(\chi)} \widehat{H}(s,\chi)\right| \leq C(1 + |\Im(s)|)$$

The statement is proven. □

9.2.5. — In this paragraph we define the height zeta function, establish its convergence and the meromorphic extension of the function it defines. For $s \in \mathbb{C}$, we define formally

$$Z(s) := \sum_{x \in [\mathscr{T}(m)(F)]} H(x)^{-s}.$$

The following lemma verifies some of the conditions that are needed to be satisfied in order to apply Poisson formula. We will write $i([\mathscr{T}(m)(F)])$ for $[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])$.

Lemma 9.2.5.1. — *The following claims are valid.*

1. *Let $\epsilon > 0$. For $s \in \Omega_{>\alpha(m)+\epsilon}$, the series defining $Z(s)$ converges absolutely and uniformly. The function $s \mapsto Z(s)$ is holomorphic in the domain $\Omega_{>\alpha(m)}$.*

2. *Let $s \in \Omega_{>\alpha(m)}$. For every $x \in [\mathscr{T}(m)(\mathbb{A}_F)]$, the series*

$$\sum_{y \in i([\mathscr{T}(m)(F)])} H(xy)^{-s}$$

converges absolutely. The function

$$x \mapsto \sum_{y \in i([\mathscr{T}(m)(F)])} H(xy)^{-s}$$

is continuous.

Proof. — 1. It follows from Proposition 9.1.4.4 that there exists $C_1 > 0$ such that for every $B > 0$ one has that

$$|\{y \in [\mathscr{P}(m)(F)] | H(y) < B\}| < C_1 B^{\alpha(m)}$$

(recall that in Proposition 9.1.4.4 we had a quasi-discriminant degree 1 family, while now it is a degree m quasi-discriminant degree m family). Now, it follows from Lemma 7.5.3.1, that the series defining $Z(s)$ converges absolutely and uniformly for $s \in \Omega_{>\alpha(m)+\epsilon}$. Thus Z is holomorphic in the domain $\Omega_{>\alpha(m)+\epsilon}$, and by decreasing ϵ we deduce that Z is holomorphic in the domain $\Omega_{>\alpha(m)}$.

2. For $x \in [\mathscr{T}(m)(\mathbb{A}_F)]$, we denote by S_x the finite set of places of F given by the union of the set M_F^∞, of the set of places v for which $v(x) \neq 0$ and of the set of the places v for which H_v is not $[\mathscr{T}(m)(\mathcal{O}_v)]$-invariant. Let us set $U_{x,v} := \{x_v\} \subset [\mathscr{T}(m)(F_v)]$ (where x_v is the v-adic component of x). We define

$$U_x = \prod_{v \in S_x} U_{x,v} \times \prod_{v \in M_F^0 - S_x} [\mathscr{T}(m)(\mathcal{O}_v)].$$

We are going to prove that the series converges absolutely and uniformly on U_x. For every $x' \in U_x$ and every $y \in i([\mathscr{T}(m)(F)])$, it follows from Lemma 9.2.3.2, that there exists $C(x) > 0$ such that

$$\left|H(x'y)^{-s}\right| = H(x'y)^{-\Re(s)} \leq C(x)^{-\Re(s)} H(y)^{-\Re(s)}.$$

The kernel of the homomorphism $[\mathscr{T}(m)(F)] \to [\mathscr{T}(m)(\mathbb{A}_F)]$ is $Ш^1(F, \mu_m)$ by Proposition 3.4.3.1. As $\sum_{y \in i([\mathscr{T}(m)(F)])} H(y)^{-s} =$

$\frac{Z(s)}{|\text{Ш}^1(F,\mu_m)|}$ converges absolutely, it follows that

$$\sum_{y\in i([\mathscr{T}(m)(F)])} \left|H(x'y)^{-s}\right| = \sum_{y\in i([\mathscr{T}(m)(F)])} H(x'y)^{-\Re(s)} \leq C(x)^{-\Re(s)} \frac{Z(\Re(s))}{|\text{Ш}^1(F,\mu_m)|},$$

and hence the series $\sum_{y\in[\mathscr{T}(m)(F)]} H(x'y)^{-s}$ converges absolutely and uniformly in the domain $x' \in U_x$. It follows that $x' \mapsto \sum_{y\in i([\mathscr{T}(m)(F)])} H(x'y)^{-s}$ is continuous on U_x. We deduce that $x \mapsto \sum_{y\in i([\mathscr{T}(m)(F)])} H(xy)^{-s}$ is continuous on $[\mathscr{T}(m)(\mathbb{A}_F)]$.

□

Let $\Xi \subset [\mathscr{T}(m)(\mathbb{A}_F)]^*$ be the group of the characters χ which satisfy that $\chi^{m/r} = 1$. Note that $\Xi^\perp$ is given by

$$[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r} = \{x^{m/r} | x \in [\mathscr{T}(m)(\mathbb{A}_F)]\}.$$

For an open subgroup $K \subset K^0_{\max} = \prod_{v\in M^0_F}[\mathscr{T}(m)(\mathcal{O}_v)]$, we denote by $\Xi_K = \mathfrak{A}_K \cap \Xi$. By Corollary 6.2.2.5, the subgroups $\mathfrak{A}_K$ are finite, thus discrete, hence the groups Ξ_K are finite and discrete. To simplify notation, in the rest of the paragraph we may write $[\mathscr{T}(m)(F)]$ for what is technically $[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])$.

Lemma 9.2.5.2. — *The following claims are valid:*

1. *For every open subgroup $K \subset K^0_{\max}$, the group $\Xi_K^\perp$ is the kernel of the homomorphism $[\mathscr{T}(m)(\mathbb{A}_F)] \to \Xi_K^*$, which is induced from the inclusion $\Xi_K \subset [\mathscr{T}(m)(\mathbb{A}_F)]^*$. Moreover, it is open, closed and of index $|\Xi_K|$ in $[\mathscr{T}(m)(\mathbb{A}_F)]$. One has that $\Xi_K^\perp = [\mathscr{T}(m)(F)]K[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}$.*
2. *The group $\Xi_\infty^\perp := \bigcap \Xi_K^\perp$, where the intersection is over all open subgroups K of $K^0_{\max}$, identifies with the closure*
$$\overline{[\mathscr{T}(m)(F)][\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}} \subset [\mathscr{T}(m)(\mathbb{A}_F)].$$
3. *Let $K \subset K^0_{\max}$ be an open subgroup and let f be a K-invariant continuous complex valued function lying in $L^1([\mathscr{T}(m)(\mathbb{A}_F)], \mu_{\mathbb{A}_F})$. One has that*
$$\frac{1}{|\Xi_K|}\sum_{\chi\in\Xi_K} \widehat{f}(\chi) = \int_{\Xi_K^\perp} f\mu_{\mathbb{A}_F}.$$
4. *There exists a unique Haar measure $\mu_\infty^\perp$ on $\Xi_\infty^\perp$ such that for every open subgroup $K \subset K^0_{\max}$, any K-invariant continuous function*

$f : [\mathscr{T}(m)(\mathbb{A}_F)] \to \mathbb{C}$ one has that $f \in L^1(\Xi_K^\perp, |\Xi_K|\mu_{\mathbb{A}_F})$ if and only if $f \in L^1(\Xi_\infty^\perp, \mu_\infty^\perp)$ and if $f \in L^1(\Xi_K^\perp, |\Xi_K|\mu_{\mathbb{A}_F})$, then

$$\int_{\Xi_\infty^\perp} f\mu_\infty^\perp = |\Xi_K| \int_{\Xi_K^\perp} f\mu_{\mathbb{A}_F}.$$

Proof. — 1. The claim that $\Xi_K^\perp$ is the kernel of the homomorphism $[\mathscr{T}(m)(\mathbb{A}_F)] \to \Xi_K^*$ follows from Proposition 6.1.1.1. The same proposition gives that the homomorphism $[\mathscr{T}(m)(\mathbb{A}_F)] \to \Xi_K^*$ is surjective. The group Ξ_K^* is finite and discrete, of order $|\Xi_K|$, thus $\Xi_K^\perp$ is an open, closed and of index $|\Xi_K|$, as claimed. Clearly,

$$\Xi_K^\perp = ([\mathscr{T}(m)(F)]^\perp K^\perp \cap [\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}^\perp)^\perp = \overline{[\mathscr{T}(m)(F)]K[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}}.$$

The subgroup $[\mathscr{T}(m)(F)]K \subset [\mathscr{T}(m)(\mathbb{A}_F)]$ is closed (because it is a product of a discrete subgroup $[\mathscr{T}(m)(F)]$ and a compact subgroup K in $[\mathscr{T}(m)(\mathbb{A}_F)]$), hence, is equal to $\mathfrak{A}_K^\perp = (([\mathscr{T}(m)(F)]K)^\perp)^\perp = \overline{[\mathscr{T}(m)(F)]K}$, which is of the finite index in $[\mathscr{T}(m)(\mathbb{A}_F)]$. It follows that the subgroup

$$[\mathscr{T}(m)(F)]K[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r} \subset [\mathscr{T}(m)(\mathbb{A}_F)]$$

is closed, as it contains $\mathfrak{A}_K^\perp$ as a finite index subgroup. The claim follows.

2. First, let us observe that $\overline{[\mathscr{T}(m)(F)][\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}} \subset \Xi_\infty^\perp$, because $\Xi_\infty^\perp$ is a closed subgroup of $[\mathscr{T}(m)(\mathbb{A}_F)]$ and because for every compact and open $K \subset [\mathscr{T}(m)(\mathbb{A}_F)]$ one has that

$$[\mathscr{T}(m)(F)][\mathscr{T}(m)(\mathbb{A}_F)]_{m/r} \subset [\mathscr{T}(m)(F)][\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}K = \Xi_K^\perp.$$

Let now

$$y \in [\mathscr{T}(m)(\mathbb{A}_F)] - \overline{([\mathscr{T}(m)(F)])[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}}.$$

The open subgroups of $K^0_{\max}$ form a basis of neighbourhoods of $1 \in [\mathscr{T}(m)(\mathbb{A}_F)]$, thus there exists an open subgroup $K \subset K^0_{\max}$ such that $yK \cap \overline{([\mathscr{T}(m)(F)])[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}} = \emptyset$. Hence,

$$y \notin ([\mathscr{T}(m)(F)][\mathscr{T}(m)(\mathbb{A}_F)]_{m/r})K = \Xi_K^\perp.$$

It follows that $y \notin \Xi_\infty^\perp$ and the claim is proven.

3. We apply the Poisson formula for the inclusion $\Xi_K \subset [\mathscr{T}(m)(\mathbb{A}_F)]^*$, where Ξ_K is endowed with the counting measure and $[\mathscr{T}(m)(\mathbb{A}_F)]^*$, with the dual measure $\mu^*_{\mathbb{A}_F}$ of the measure $\mu_{\mathbb{A}_F}$. Every $x \in [\mathscr{T}(m)(\mathbb{A}_F)]$ will be regarded as a character of $[\mathscr{T}(m)(\mathbb{A}_F)]^*$ by the evaluation map. By (1), the group $\Xi_K^\perp$ identifies with the kernel of the homomorphism $[\mathscr{T}(m)(\mathbb{A}_F)] \to \Xi_K^*$, given by $x \mapsto x|_{\Xi_K}$, and is an open subgroup of index $|\Xi_K|$ in $[\mathscr{T}(m)(\mathbb{A}_F)]$. The dual measure of the measure count_{Ξ_K} on the dual group Ξ_K^* is given by $\frac{1}{|\Xi_K|} \cdot \mathrm{count}_{\Xi_K^*}$, thus we have an equality of measures on $([\mathscr{T}(m)(\mathbb{A}_F)]^*/\Xi_K)^* = \Xi_K^\perp$:
$$(\mu^*_{\mathbb{A}_F}/\,\mathrm{count}_{\Xi_K})^* = |\Xi_K| \cdot \mu_{\mathbb{A}_F}|_{\Xi_K^\perp}.$$
The Fourier transform of $\chi \mapsto \widehat{f}(\chi)$ at the character x, by the Fourier inversion formula (7.1.1.1), is equal to f. By the finiteness of Ξ_K and the continuity of $\chi \mapsto \widehat{f}(\chi)$ ([**5**, Chapter II, §1, n^0 2, Proposition 2]), the conditions (2) and (3) of Poisson formula (Proposition 7.1.1.4) are satisfied, and applying it gives that
$$\sum_{\chi \in \Xi_K} \widehat{f}(\chi) = |\Xi_K| \int_{\Xi_K^\perp} f \mu_{\mathbb{A}_F},$$
as claimed.
4. For an open subgroup $K \subset K^0_{\max}$, let dk be the probability Haar measure on K. For every K, using the fact that the product subset of a closed subset and a compact subgroup is closed ([**7**, Chapter III, §4, n^0 1, Corollary 1 of Proposition 1]) and that $\Xi_K^\perp$ is closed (1), we get that
$$\begin{aligned}\Xi_\infty^\perp K &= \overline{([\mathscr{T}(m)(F)])[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}}K \\ &= \overline{([\mathscr{T}(m)(F)])[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}K} \\ &= ([\mathscr{T}(m)(F)])[\mathscr{T}(m)(\mathbb{A}_F)]_{m/r}K \\ &= \Xi_K^\perp.\end{aligned}$$
. Let us denote by g_K the canonical morphism
$$g_K : \Xi_\infty^\perp \times K \to \Xi_\infty^\perp K = \Xi_K^\perp \qquad (y,k) \mapsto yk.$$

The group $[\mathscr{T}(m)(\mathbb{A}_F)]$ is countable at infinity by Lemma 3.4.8.1. For every open $K \subset K^0_{\max}$, it follows from [**10**, Chapter VII, §2, n^0 9, Corollary of Proposition 13] that there exists a unique Haar measure $\mu_\infty^{\perp K}$ on $\Xi_\infty^\perp$ such that for every continuous positive valued function f on $\Xi_K^\perp$ one has that $f \in L^1(\Xi_K^\perp, |\Xi_K|\mu_{\mathbb{A}_F})$ if and only if $f \circ g_K \in L^1(\Xi_\infty^\perp \times K, \mu_\infty^{\perp K} \times dk)$ and if $f \in L^1(\Xi_K^\perp, |\Xi_K|\mu_{\mathbb{A}_F})$ then

$$|\Xi_K| \int_{\Xi_K^\perp} f\mu_{\mathbb{A}_F} = b(K) \int_{\Xi_\infty^\perp \times K} (f \circ g_K)\mu_\infty^{\perp K} dk,$$

for some $b(K) > 0$. We deduce that if $f \in L^1(\Xi_K^\perp, |\Xi_K|\mu_{\mathbb{A}_F})$ is moreover assumed to be K-invariant, then

$$|\Xi_K| \int_{\Xi_K^\perp} f\mu_{\mathbb{A}_F} = b(K) \int_{\Xi_\infty^\perp} f\mu_\infty^{\perp K}.$$

We now prove that the measures $b(K)\mu_\infty^{\perp K}$ are independent of the choice of K. First, we prove it for open subgroups of $K' \subset K$ (such subgroups are compact and of finite index in K). Let $h : [\mathscr{T}(m)(\mathbb{A}_F)] \to \mathbb{R}_{\geq 0}$ be a K-invariant function in $L^1([\mathscr{T}(m)(\mathbb{A}_F)], \mu_{\mathbb{A}_F})$. Using (3) and the fact that for a character $\chi \in \Xi_{K'}^\perp$ which is not trivial on K, one has that $\widehat{h}(\chi) = 0$ (because h is K'-invariant), we deduce that

$$\begin{aligned} b(K) \int_{\Xi_\infty^\perp} h\mu_\infty^{\perp K} = |\Xi_K| \int_{\Xi_K^\perp} h\mu_{\mathbb{A}_F} = \sum_{\chi \in \Xi_K} \widehat{h}(\chi) = \sum_{\chi \in \Xi_{K'}} \widehat{h}(\chi) \\ = |\Xi_{K'}| \int_{\Xi_{K'}^\perp} h\mu_{\mathbb{A}_F} = b(K') \int_{\Xi_\infty^\perp} h\mu_\infty^{\perp K'}. \end{aligned}$$

As $b(K)\mu_\infty^{\perp K}$ and $b(K')\mu_\infty^{\perp K'}$ are Haar measures on $\Xi_\infty^\perp$, it follows that $b(K)\mu_\infty^{\perp K} = b(K')\mu_\infty^{\perp K'}$. Now we prove the claim for general K'. As for any open subgroup $K' \subset K^0_{\max}$, we have that $K \cap K'$ is open in K and K', we deduce

$$b(K)\mu_\infty^{\perp K} = b(K \cap K')\mu_\infty^{\perp K \cap K'} = b(K')\mu_\infty^{\perp K'}.$$

Thus $\mu_\infty^\perp := b(K)\mu_\infty^{\perp K}$ is the wanted measure.

□

Denote by $j : \Xi_\infty^\perp \to \prod_{v \in M_F}[\mathscr{P}(m)(F_v)]$ the canonical inclusion. Note that j is the composite of the closed embedding $\Xi_\infty^\perp \hookrightarrow [\mathscr{T}(m)(\mathbb{A}_F)]$ and the canonical inclusion $[\mathscr{T}(m)(\mathbb{A}_F)] \hookrightarrow \prod_{v \in M_F}[\mathscr{T}(m)(F_v)] = \prod_{v \in M_F}[\mathscr{P}(m)(F_v)]$. The later map is continuous by Lemma 3.4.1.1, hence j is continuous.

For any compactly supported continuous $\phi : \prod_{v\in M_F}[\mathscr{P}(m)(F_v)] \to \mathbb{R}_{\geq 0}$, it follows from Lemma 9.2.4.1 that the limit

$$\lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}(\phi\circ j)H^{-s}\mu_\infty = \lim_{s\to 1^+}(s-1)^{r-1}|\Xi_K|\int_{\Xi_K^\perp}(\phi\circ j)H^{-s}\mu_{\mathbb{A}_F},$$

where $K\subset K^0_{\max}$ is open, exists and is a non-negative number. It follows that

$$\mathscr{C}^0_c\Big(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)],\mathbb{R}_{\geq 0}\Big)\to\mathbb{R}_{\geq 0}\quad \phi\mapsto \lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}(\phi\circ j)H^{-s}\mu_\infty$$

is a non-negative linear form, hence by [**9**, Chapter III, §1, n^0 6, Theorem 1] extends to a measure on $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$

Definition 9.2.5.3. — *Let $(f_v : F_v^\times\to\mathbb{R}_{>0})_v$ be a quasi-discriminant degree m family of m-homogenous functions. We define a measure ω on $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ by*

$$\omega=\omega((f_v)_v)=|\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}j_*(H^{-s}\mu_\infty).$$

We set

$$\tau=\tau((f_v)_v)=\omega\Big(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]\Big).$$

Lemma 9.2.5.4. — *Assuming the conditions of Definition 9.2.5.3, the following claims are valid*

1. *One has that*

$$\lim_{s\to 1^+}(s-1)^{r-1}\widehat{H}(s,1)=\lim_{s\to 1^+}(s-1)^{r-1}\int_{[\mathscr{T}(m)(\mathbb{A}_F)]}H^{-s}\mu_{\mathbb{A}_F}>0.$$

2. *One has that $\tau>0$.*

Proof. — 1. Recall that $\widehat{H}_v(s,1)=m$ for every $v\in M_F^\infty$ by Lemma 9.2.1.2. Now, by Lemma 9.2.3.3, one has that

$$\widehat{H}(s,1)=\gamma(s,1)m^{r_1+r_2}\prod_{j=1}^{r-1}L(s,1)=\gamma(s,1)m^{r_1+r_2}\zeta(s)^{r-1},$$

where $\gamma(-,1)$ is a holomorphic function in the domain $\Omega_{>\frac{\alpha(m)}{\alpha(m)+1}}$ and $\gamma(1,1)>0$. Thus

$$\begin{aligned}\lim_{s\to 1^+}(s-1)^{r-1}\widehat{H}(s,1) &= \gamma(1,1)m^{r_1+r_2}\lim_{s\to 1^+}(s-1)^{r-1}\zeta(s)^{r-1}\\ &= \gamma(1,1)m^{r_1+r_2}\operatorname{Res}(\zeta,1)^{r-1}\\ &> 0.\end{aligned}$$

2. Let $K\subset K^0_{\max}$ be an open subgroup such that $H=H((f_v)_v)$ is K-invariant. By Lemma 9.2.5.2 one has that

$$\begin{aligned}\omega\Big(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]\Big) &= |\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi^\perp_\infty}H^{-s}\mu_\infty\\ &= |\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}|\Xi_K|\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F}.\end{aligned}$$

Let $\{x_1\dots x_{|\Xi_K|}\}$ be a set of elements of $[\mathscr{T}(m)(\mathbb{A}_F)]$ such that for any $i\neq j\in\{1,\dots,|\Xi_K|\}$, one has that $x_ix_j^{-1}\notin\Xi_K^\perp$. Using Lemma 9.2.3.2, we obtain that for $s>1$, one has that

$$\begin{aligned}\int_{[\mathscr{T}(m)(\mathbb{A}_F)]}H^{-s}\mu_{\mathbb{A}_F} &= \sum_i\int_{\Xi^\perp_K}H(x_iy)^{-s}d\mu_{\mathbb{A}_F}(y)\\ &\leq \sum_i C(x_i)^{-s}\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F},\end{aligned}$$

for certain $C(x_i)>0$. It follows that

$$\begin{aligned}0 &< \lim_{s\to 1^+}(s-1)^{r-1}\int_{[\mathscr{T}(m)(\mathbb{A}_F)]}H^{-s}\mu_{\mathbb{A}_F}\\ &= \lim_{s\to 1^+}(s-1)^{r-1}\sum_i C(x_i)^{-s}\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F}\\ &\leq \Big(\sum_i C(x_i)^{-1}\Big)\lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F},\end{aligned}$$

and hence that $\lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F}>0$. We deduce that

$$\tau = |\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}|\Xi_K|\int_{\Xi^\perp_K}H^{-s}\mu_{\mathbb{A}_F}>0,$$

as claimed.

□

Theorem 9.2.5.5. — *There exists $\delta > 0$, such that Z extends to a meromorphic function in the domain $s \in \Omega_{>1-\delta}$ with the only pole in this domain at 1 which is of order $r-1$ and such that for every compact $\mathcal{K} \subset \mathbb{R}_{>1-\delta}$ one has that there exists $C(\mathcal{K}) > 0$ such that*

$$\left| \left(\frac{s-1}{s} \right)^{r-1} Z(s) \right| \leq C(\mathcal{K})(1 + |\Im(s)|)^{r-1}$$

if provided $\Re(s) \in \mathbb{R}_{>1-\delta}$. The principal value at the pole $s = 1$ is equal to

$$\frac{\tau}{m}.$$

Proof. — By the equality (9.2.3.1), one has that $H(x) = H([\mathscr{T}(m)(i)](x))$ and by Proposition 3.4.3.1 the group $\ker([\mathscr{T}(m)(i)]) = Ш^1(F, \mu_m)$ is finite. It follows that formally one has that

$$Z(s) = |Ш^1(F, \mu_m)| \sum_{x \in [\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])} H(x)^{-s}. \tag{9.2.5.6}$$

By Lemma 9.2.5.1 for every $\epsilon > 0$, the series defining $Z(s)$ converges absolutely and uniformly to a holomorphic function in the domain $\Omega_{>\alpha(m)+\epsilon}$ and it follows from 9.2.4.1 that for $s \in \Omega_{>1}$ the sum on the right hand side converges and is a holomorphic function in s in this domain. Thus, the equality (9.2.5.6) is valid in $\Omega_{>1}$ as an equality of holomorphic functions. We apply Poisson formula (Proposition 7.1.1.4) to the inclusion

$$[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)]) \subset [\mathscr{T}(m)(\mathbb{A}_F)]$$

(we have already verified the conditions (2) and (3) of Proposition 7.1.1.4 in Lemma 9.2.5.1). We have formally

$$\sum_{x \in [\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])} H(x)^{-s}$$
$$= \int_{([\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)]))^{\perp}} \widehat{H}(s, \chi)(\mu_{\mathbb{A}_F} / \operatorname{count}_{[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])})^*.$$

We use Lemma 8.2.2.1 to understand the measure

$$(\mu_{\mathbb{A}_F} / \operatorname{count}_{[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])})^*.$$

A volume of a subset of $(\mathbb{R}_{>0})_m = \mathbb{R}_{>0}$ when $(\mathbb{R}_{>0})_m$ is endowed with the pushforward measure of the measure d^*r for the map $\mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto mx$ is $1/m$ times it was for the measure d^*r. Thus the Haar measure d^*r_m from Lemma 8.2.2.1 is normalized by $(d^*r_m)(\mathbb{R}_{>0}/(\mathbb{R}_{>0})_m) = m$. Hence, the dual measure of d^*r_m satisfies that $(d^*r_m)^*((\mathbb{R}_{>0}/(\mathbb{R}_{>0})_m)^*) = \frac{1}{m}$. Now, Lemma 8.2.2.1 gives that

$$(\mu_{\mathbb{A}_F}/\operatorname{count}_{[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)])})^* \\ = \frac{|\mu_m(F)|}{m|Ш^1(F,\mu_m)|}\operatorname{count}_{([\mathscr{T}(m)(\mathbb{A}_F)]_1/[\mathscr{T}(m)(i)]([\mathscr{T}(m)(F)]))^*}.$$

Whenever $\chi \notin \mathfrak{A}_K$, we have by Lemma 9.2.3.3 that $\widehat{H}(s,\chi) = 0$. We deduce that formally one has

(9.2.5.7)

$$Z(s) = \frac{|Ш^1(F,\mu_m)|\cdot|\mu_m(F)|}{m|Ш^1(F,\mu_m)|}\sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi) = \frac{|\mu_m(F)|}{m}\sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi).$$

For every $\chi \in \mathfrak{A}_K$, by Lemma 9.2.4.1 one has that $s \mapsto \widehat{H}(s,\chi)$ is a holomorphic function in the domain $\Omega_{>1}$. As the group $\mathfrak{A}_K$ is finite, we deduce that $s \mapsto \sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi)$ is a holomorphic function in the domain $\Omega_{>1}$. It follows that the equality (9.2.5.7) is valid as an equality of holomorphic functions in the domain $\Omega_{>1}$. Moreover, the equality (9.2.5.7) is valid as an equality of the maximal meromorphic extensions of the functions from the both hand sides. Moreover, Lemma 9.2.4.1 gives that there exists $\delta > 0$ such that

$$\left(\frac{s-1}{s}\right)^{d(\chi)}\widehat{H}(s,\chi)$$

extends to a holomorphic function in the domain $\Omega_{>1-\delta}$ and that for every compact $\mathcal{K} \subset \mathbb{R}_{>1-\delta}$ there exists $C'(\mathcal{K}) > 0$, such that

$$\left|\left(\frac{s-1}{s}\right)^{d(\chi)}\widehat{H}(s,\chi)\right| \leq C'(\mathcal{K})(1+|\Im(s)|)$$

for every $\chi \in \mathfrak{A}_K$ (here $d(\chi) = r-1$ if $\chi^{m/r} = 1$ otherwise $d(\chi) = 0$). By the finiteness of $\mathfrak{A}_K$ and the fact that $d(\chi) \leq r-1$, we deduce that

$$\sum_{\chi\in\mathfrak{A}_K}\left(\frac{s-1}{s}\right)^{r-1}\widehat{H}(s,\chi) = \left(\frac{s-1}{s}\right)^{r-1}\sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi)$$

extends to a holomorphic function in the domain $\Omega_{>1-\delta}$ and such that for every compact $\mathcal{K} \subset \mathbb{R}_{>1-\delta}$ there exists $C(\mathcal{K}) > 0$, such that

$$\left|\left(\frac{s-1}{s}\right)^{r-1}\sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi)\right| \leq C(\mathcal{K})(1+|\Im(s)|).$$

We deduce that Z extends to a meromorphic function in the domain $s \in \Omega_{>1-\delta}$ with the only possible pole at $s = 1$ in this domain which is of order at most $r - 1$. Moreover, for every compact $\mathcal{K} \subset \mathbb{R}_{>1-\delta}$ one has that

$$\left|\left(\frac{s-1}{s}\right)^{r-1} Z(s)\right| \leq C(\mathcal{K})(1 + |\Im(s)|)^{r-1}$$

if provided $\Re(s) \in \mathcal{K}$.

The last part of the proof we dedicate to the proving that Z indeed has a pole at 1, that this pole is of order exactly $r - 1$ and to the calculation of the principal value. We calculate the limit

$$\lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \sum_{\chi\in\mathfrak{A}_K} \widehat{H}(s,\chi) = \sum_{\chi\in\mathfrak{A}_K} \lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \widehat{H}(s,\chi).$$

Recall that by Ξ_K we have denoted the subgroup of $\chi \in \mathfrak{A}_K$ such that $\chi^{m/r} = 1$. If $\chi \in \mathfrak{A}_K - \Xi_K$, then by Lemma 9.2.4.1, one has that $s \mapsto \widehat{H}(s,\chi)$ is holomorphic in the domain $\Omega_{>1-\delta}$, and thus

$$\lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \widehat{H}(s,\chi) = 0.$$

We deduce that

$$\lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \sum_{\chi\in\mathfrak{A}_K} \widehat{H}(s,\chi) = \sum_{\chi\in\Xi_K} \lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \widehat{H}(s,\chi)$$

Using Lemma 9.2.5.2, we have that

$$\sum_{\chi\in\Xi_K} \widehat{H}(s,\chi) = |\Xi_K| \int_{\Xi_K^\perp} H^{-s}\mu_{\mathbb{A}_F} = \int_{\Xi_\infty^\perp} H^{-s}\mu_\infty^\perp \tag{9.2.5.8}$$

whenever the quantities on both hand sides converge. By the fact that H^{-s} is absolutely integrable over $[\mathscr{T}(m)(\mathbb{A}_F)]$ for $s \in \Omega_{>1}$ and by Lemma 9.2.5.2, we deduce that the equality (9.2.5.8) is valid in the domain $\Omega_{>1}$. Moreover, the equality (9.2.5.8) is valid, as an equality of the meromorphic functions on the domain $\Omega_{>1-\delta}$. We deduce that

$$\lim_{s\to 1^+} \left(\frac{s-1}{s}\right)^{r-1} \int_{\Xi_\infty^\perp} H^{-s}\mu_\infty^\perp.$$

We recognize this quantity from Definition 9.2.5.3 as $\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])}{|\mu_m(F)|}$, and hence

$$\begin{aligned}\lim_{s\to 1^+}\left(\frac{s-1}{s}\right)^r Z(s) &= \lim_{s\to 1^+}(s-1)^{r-1}Z(s)\\ &= \frac{|\mu_m(F)|}{m}\sum_{\chi\in\mathfrak{A}_K}\widehat{H}(s,\chi)\\ &= \frac{|\mu_m(F)|\cdot\omega(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])}{|\mu_m(F)|m}\\ &= \frac{\omega(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])}{m}\\ &= \frac{\tau}{m}.\end{aligned}$$

The statement is proven. □

In Theorem 9.2.5.5, we have verified the conditions of the Tauberian result [**17**, Theorem A1]. We deduce that:

Corollary 9.2.5.9. — *Let $(f_v : F_v^\times \to \mathbb{R}_{>0})_v$ be a quasi-discriminant degree m family of m-homogenous functions and let H be the resulting height on $[\mathscr{P}(m)(F)]$. One has that*

$$|\{x\in[\mathscr{P}(m)(F)]|H(x)\leq B\}|\sim\frac{\tau}{(r-2)!\cdot m}B\log(B)^{r-2},$$

when $B\to\infty$.

If for every v one has that $f_v = f_v^\Delta$, where f_v^Δ is the discriminant m-homogenous functions of weighted degree m, then by 9.1.3.5, we get for $y\in[\mathscr{P}(m)(F)]$ that

$$H^\Delta(y) = N\bigg(\Delta\big(F[X]/(X^m-\widetilde{y})\big)/F\big)\bigg)^{m/\alpha(m)},$$

where $\widetilde{y}$ is a lift of y. Let us write $|\Delta|(y)$ for $H^\Delta(y)^{\alpha(m)/m}$. It is precisely the norm of the discriminant of a torsor corresponding to y. We deduce that:

Corollary 9.2.5.10. — *One has that*

$$\begin{aligned}&|\{x\in[\mathscr{P}(m)(F)]|\,|\Delta|(x)\leq B\}|\\ &\qquad\sim_{B\to\infty}\frac{r^{r-2}\tau^\Delta}{m^{r-1}(r-1)^{r-2}\cdot(r-2)!}B^{\frac{m}{\alpha(m)}}\log(B)^{r-2},\end{aligned}$$

where $\tau^{\Delta} = \tau((f_v^{\Delta})_v)$.

9.2.6. — In this paragraph we explain the equidistribution of rational points in $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$. We will write i for the map $[\mathscr{P}(m)(i)]$.

Theorem 9.2.6.1. — *The set* $i([\mathscr{P}(m)(F)])$ *is equidistributed in* $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$ *with respect to* H.

Proof. — The proof follows the proof of Theorem 8.3.2.2 with some modifications and simplifications. Corollary 9.2.5.9, together with the fact that $\ker(i) = Ш^1(F,\mu_m)$ from Proposition 3.4.3.1, gives that

$$|\{x\in i([\mathscr{P}(m)(F)])|H(x)\leq B\}| \sim_{B\to\infty} \frac{\omega(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])}{(r-2)!m|Ш^1(F,\mu_m)|}B\log(B)^{r-2}.$$

We say that an open subset W is elementary if W writes as $W = \prod_{v\in M_F} W_v$, where $W_v\subset[\mathscr{P}(m)(F)]$ is open, and for almost all v, one has that $W_v = [\mathscr{P}(m)(F_v)]$. As for every v, the spaces $[\mathscr{P}(m)(F_v)]$ are finite and discrete by Lemma 3.3.5.7, we deduce that W is open and closed, hence $\partial W = \emptyset$. We prove that for every elementary W, one has that

$$\frac{|x\in W|H(x)\leq B\}|}{|x\in i([\mathscr{P}(m)(F)]|H(x)\leq B\}|}\sim_{B\to\infty}\frac{\omega(W)}{\omega(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)])}.$$

For $v\in M_F$, we define $g_v = \mathbf{1}_{W_v} = h_v$. Let $\epsilon>0$ and set $\eta=\epsilon/4$. For $v\in M_F$, we set $g_{\eta,v} = (1-\eta)\mathbf{1}_{W_v}+\eta$ and $h_{\eta,v} = g_{\eta,v}$. The proof that $H((f_v\cdot(g_v^{-1}\circ q_v^m))_v) = g_\eta^{-1}H$ is identical to the proof of the corresponding claim in Part (2) proof of Theorem 8.3.2.2. Let us establish that

$$\omega((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v) = g_\eta^{-1}\omega.$$

For a compactly supported continuous function $\phi:\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]\to\mathbb{C}$ we say that it is decomposable, if it can be written as $\otimes_{v\in M_F}\phi_v$, where for almost all v one has that $\phi_v = 1$. Let $\phi:[\mathscr{P}(m)(F_v)]\to\mathbb{R}_{\geq 0}$ be decomposable. One has that

$$\begin{aligned}
&\omega((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v)(\phi)\\
&\qquad\qquad = \lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}(\phi)H(((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v))^{-s}\mu_\infty^\perp\\
&\qquad\qquad = \lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}(\phi g_\eta^{-s})H^{-s}\mu_\infty^\perp\\
&\qquad\qquad \geq \lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}(\phi g_\eta^{-1})H^{-s}\mu_\infty^\perp\\
&\qquad\qquad = \omega(\phi g_\eta^{-1}),
\end{aligned}$$

where the only inequality follows from the fact that $g_\eta^{-s}\geq g_\eta^{-1}$ (which is true because g_η takes values in the interval $]0,1[$). On the other side, for every $\delta>1$, by taking in the limit only those s contained in the domain $]1,\delta[$, we deduce that

$$\begin{aligned}
\omega((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v)(\phi) &\leq \lim_{s\to 1^+}(s-1)^{r-1}\int_{\Xi_\infty^\perp}\phi g_\eta^{-\delta}H^{-s}\mu_\infty^\perp\\
&= \omega(\phi g_\eta^{-\delta}).
\end{aligned}$$

From the fact that ω is a measure, it follows that $\lim_{\delta\to 1^+}\omega(\phi g^{-\delta}) = \omega(\phi g^{-1})$, and hence that

$$\omega((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v)(\phi)=\omega(\phi g_\eta^{-1}) \tag{9.2.6.2}$$

for any non-negative decomposable function ϕ. Clearly, any real valued decomposable function is a difference of two non-negative decomposable functions, and the equality (9.2.6.2) is hence valid for all real valued decomposable functions. Any decomposable ϕ writes as $\phi_1+i\phi_2$, for some decomposable real valued functions ϕ_1 and ϕ_2. The equality (9.2.6.2) is thus valid for any decomposable ϕ. Moreover, the equality (9.2.6.2) is valid for finite sums of decomposable functions. The finite sums of decomposable continuous compactly supported functions are dense in the set $\mathscr{C}_c^0(\prod_{v\in M_F}[\mathscr{P}(m)(F_v)],\mathbb{C})$ by [**9**, Chapter III, §4, n^0 5, Lemma 3], hence $\omega((f_v\cdot(g_{\eta,v}^{-1}\circ q_v^m))_v)=g_\eta^{-1}\omega$. To prove the claim for the elementary open subset W, we use the same steps as in the part (2) of Theorem 8.3.2.2, with the only change in the function of B (that is replace $\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])}{|Ш^1(F,\mu_{\gcd(\mathbf{a})})|\cdot|\mathbf{a}|}B$ with $\frac{\omega(\prod_{v\in M_F}[\mathscr{P}(\mathbf{a})(F_v)])}{(r-2)!m|Ш^1(F,\mu_m)|}B\log(B)^{r-2}$).

We have hence established the claim for every elementary subset of $[\mathscr{P}(m)(F)]$. The rest of the proof is identical to the proofs of parts (3) and (4) in the proof of Theorem 8.3.2.2. $\square$

9.2.7. — In the last paragraph, we prove that μ_m-torsors which are fields are of positive proportion among all μ_m torsors of bounded quasi-discriminant height.

Proposition 9.2.7.1. — *There exists $A(m,(f_v)_v)>0$ such that for $B\gg 0$, one has that*

$$|\{x\in[\mathscr{P}(m)(F)]|\ x \text{ is a field}, H(x)\leq B\}|\geq A(m,(f_v)_v)B\log(B)^{r-2}.$$

Proof. — Let w be a finite place of F which does not extend the place 2 of $\mathbb{Q}$. Let us denote by w_m the canonical map

$$[\mathscr{T}(m)(F_v)]\to\mathbb{Z}/m\mathbb{Z},$$

given in Lemma 3.3.5.5. Recall that if $\widetilde{y}\in F_w^\times$ is a lift of $y\in[\mathscr{T}(m)(F_w)]$, then the image of $w(\widetilde{y})$ under the quotient map $\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}$ is $w_m(y)$. We let

$$W=\{y\in[\mathscr{P}(m)(F_w)]:w_m(y)=1\}\times\prod_{v\in M_F-\{w\}}[\mathscr{P}(m)(F_v)].$$

The set $\{y\in[\mathscr{P}(m)(F_v)]|w_m(y)=1\}$ is open and closed in the finite discrete set $[\mathscr{P}(m)(F_v)]$, thus W is open and closed in $\prod_{v\in M_F}[\mathscr{P}(m)(F_v)]$. Hence, $\partial W=\emptyset$. We prove two claims that will imply the statement of the proposition.

1. Let us prove that if $x\in[\mathscr{P}(m)(F)]$ satisfies that $i(x)\in W$, then x is a field. One has that x is field if and only if $X^m-\widetilde{x}$ is irreducible, where $\widetilde{x}\in F^\times$ is a lift of x. By [34, Theorem 9.1, Chapter VI], this is true if and only if $\widetilde{x}$ is not an element of $(F^\times)_p$ for p prime divisor of m and if $4|m$, then also $\widetilde{x}\notin -4(F^\times)_4$. Let us write $i_v:[\mathscr{P}(m)(F)]=[\mathscr{T}(m)(F)]\to[\mathscr{T}(m)(F_v)]=[\mathscr{P}(m)(F_v)]$ for the map induced from $(F_v^\times)_m$-invariant map $F^\times\to F_v^\times\xrightarrow{q^m}[\mathscr{T}(m)(F_v)]$. For every prime $p|m$, one has that $p\nmid 1=w_m(i_v(x))=w(\widetilde{x})$, hence $\widetilde{x}\notin(F^\times)_p$. One has that $2\nmid 1=w_m(i_v(x))=w(\widetilde{x})$, thus $\widetilde{x}\notin(-4F^\times)_4$. It follows that x is a field.

2. Let us prove that W has a strictly positive volume. By Lemma 9.2.1.2, one has that
$$\widehat{H}_w(s,1)=\sum_{j=0}^{m-1}|\pi_v|_v^{\frac{s(m^2-m\gcd(j,m))}{\alpha(m)}},$$
for $s>0$. Hence, $\widehat{H}_w(-,1)$ does not vanish for $s>0$. Using Lemma 9.2.5.4, we deduce that
$$\begin{aligned}\lim_{s\to 1^+}(s-1)^{r-1}\int_{\substack{[\mathscr{T}(m)(\mathbb{A}_F)]\\ w_m=1}}H_w^{-s}\mu_{\mathbb{A}_F}&=\lim_{s\to 1^+}(s-1)^{r-1}\widehat{H}(s,1)\frac{\int_{w_m=1}H_w^{-s}\mu_w}{\widehat{H}_w(s,1)}\\ &=\lim_{s\to 1^+}(s-1)^{r-1}\widehat{H}(s,1)\frac{\int_{w_m=1}H^{-1}\mu_w}{\widehat{H}_w(s,1)}\end{aligned}$$
is positive. By Lemma 9.2.5.2 one has that
$$\begin{aligned}\omega(W)&=|\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}\int_{\substack{\Xi_\infty^\perp\\ w_m=1}}H^{-s}\mu_\infty\\ &=|\mu_m(F)|\lim_{s\to 1^+}(s-1)^{r-1}|\Xi_K|\int_{\substack{\Xi_K^\perp\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F}.\end{aligned}$$
Let $\{x_1\dots x_{|\Xi_K|}\}$ be a set of representatives of classes of Ξ_K in $[\mathscr{T}(m)(\mathbb{A}_F)]$. Using Lemma 9.2.3.2, we obtain that for $s>1$, one has that
$$\begin{aligned}\int_{\substack{[\mathscr{T}(m)(\mathbb{A}_F)]\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F}&=\sum_i\int_{\substack{\Xi_K^\perp\\ w_m=1}}H(x_iy)^{-s}d\mu_{\mathbb{A}_F}(y)\\ &\leq\sum_i C(x_i)^{-s}\int_{\substack{\Xi_K^\perp\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F},\end{aligned}$$
for certain $C(x_i)>0$. It follows that
$$\begin{aligned}0&<\lim_{s\to 1^+}(s-1)^{r-1}\int_{\substack{[\mathscr{T}(m)(\mathbb{A}_F)]\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F}\\ &\leq\lim_{s\to 1^+}(s-1)^{r-1}\sum_i C(x_i)^{-s}\int_{\substack{\Xi_K^\perp\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F}\\ &=\Big(\sum_i C(x_i)^{-1}\Big)\lim_{s\to 1^+}(s-1)^{r-1}\int_{\substack{\Xi_K^\perp\\ w_m=1}}H^{-s}\mu_{\mathbb{A}_F},\end{aligned}$$

and hence that $\lim_{s\to 1^+}(s-1)^{r-1}\int_{\substack{\Xi_K^\perp\\ w_m=1}} H^{-s}\mu_{\mathbb{A}_F} > 0$. We deduce that

$$\omega(W) = |\mu_m(F)| \lim_{s\to 1^+}(s-1)^{r-1}|\Xi_K| \int_{\substack{\Xi_K^\perp\\ w_m=1}} H^{-s}\mu_{\mathbb{A}_F} > 0,$$

as claimed.

Now, as $\omega(\partial W)=0$, by Theorem 9.2.6.1, when $B\to\infty$, one has that

$$\begin{aligned}|\{x\in[\mathscr{P}(m)(F)]|\ x \text{ is a field}, H(x)\leq B\}|&\\ &\geq |\{x\in[\mathscr{P}(m)(F)]|i(x)\in W\}|\\ &\sim_{B\to\infty}\frac{\omega(W)}{m}B\log(B)^{r-2}.\end{aligned}$$

The statement follows. □

Let us give the precise asymptotics for the number of fields of bounded height under the condition that $4\nmid m$. For a divisor $d|m$, we denote by j^d the homomorphism

$$j^d:[\mathscr{T}(d)(F)]\to[\mathscr{T}(m)(F)]$$

induced from $(F^\times)_d$-invariant homomorphism

$$F^\times\to F^\times\to[\mathscr{T}(m)(F)] \qquad y\mapsto[q^m(F)](y^{m/d}),$$

where $[q^m(F)]:F^\times\to[\mathscr{T}(m)(F)]$ is the quotient map. The map j^d induces an isomorphism $[\mathscr{T}(d)(F)]\to[\mathscr{T}(m)(F)]_{d/m}$, where, as usual $[\mathscr{T}(m)(F)]_d$ denotes the subgroup given by d-th powers of the elements of $[\mathscr{T}(m)(F)]$. For $v\in M_F$ and $d|m$ we set

$$f_v^{*d}=\Big(\big(y\mapsto|y|_v^{1-\frac{m}{d}}\big)\cdot(f_v\circ(x\mapsto x^{\frac{m}{d}}))\Big)^{\left(1-\frac{1}{r}\right)/\left(1-\frac{1}{q}\right)},$$

where q is the smallest prime of d.

Lemma 9.2.7.2. — *Let d be a divisor of m and let q be the smallest prime of d. The following claims are valid:*

1. *The family $(f_v^{*d})_v$ is a quasi-discriminant degree d family of d-homogenous functions.*
2. *One has equality of functions $[\mathscr{T}(d)(F)]\to\mathbb{R}_{\geq 0}$:*

$$H((f_v)_v)\circ j^d=\big(H((f_v^{*d})_v)\big)^{(1-\frac{1}{q})/(1-\frac{1}{r})}.$$

Proof. — 1. Let $t, x \in F_v^\times$. For $v \in M_F$, one has that

$$f_v^{*d}(t\cdot x) = f_v^{*d}(t^d x) = |t^d x|_v^{1-\frac{m}{d}} f_v(t^m x^{\frac{m}{d}}) = |t^d|_v^{1-\frac{m}{d}} |x|_v^{1-\frac{m}{d}} |t|_v^m f_v(x^{\frac{m}{d}})$$
$$= |t|_v^d f_v^{*d}(x).$$

It follows that f_v^{*d} is d-homogenous of weighted degree d. For almost all v, the function f_v is the discriminant m-homogenous function of the weighted degree m. Recall from Lemma 9.1.3.1 that for v finite such that $v(m) = 0$, this means that

$$f_v(y) = |y|_v |\pi_v|_v^{\frac{m\gcd(v(y),m)-m^2}{\alpha(m)}}$$

for every $y \in F_v^\times$. A direct calculation gives that

$$\begin{aligned}\left(f_v^{*d}(y)\right)^{(1-\frac{1}{q})/(1-\frac{1}{r})} = |y|_v^{1-\frac{m}{d}} f_v(y^{\frac{m}{d}}) &= |y|_v^{1-\frac{m}{d}}\cdot |y|_v^{\frac{m}{d}} |\pi_v|_v^{\frac{m\gcd((m/d)v(y),m)-m^2}{\alpha(m)}} \\ &= |y|_v |\pi_v|_v^{\frac{(m/d)m\gcd(v(y),d)-m^2}{m^2(1-\frac{1}{r})}} \\ &= |y|_v |\pi_v|_v^{\frac{(1/d)\gcd(v(y),d)-1}{1-\frac{1}{r}}} \\ &= |y|_v |\pi_v|_v^{\frac{d\gcd(v(y),d)-d^2}{d^2(1-\frac{1}{r})}} \\ &= |y|_v |\pi_v|_v^{\frac{d\gcd(v(y),d)-d^2}{\alpha(d)}\cdot\frac{1-\frac{1}{q}}{1-\frac{1}{r}}},\end{aligned}$$

i.e.

$$f_v^{*d}(y) = |y|_v |\pi_v|_v^{\frac{d\gcd(v(y),d)-d^2}{\alpha(d)}}.$$

In other words f_v^{*d} is the discriminant d-homogenous function of the weighted degree d. It follows that $(f_v^{*d})_v$ is a quasi-discriminant degree d family of d-homogenous functions.

2. Let $x \in [\mathscr{T}(d)(F)]$ and let $\widetilde{x} \in F^\times$ be its lift. One has that $\widetilde{x}^{\frac{m}{d}} \in F^\times$ is a lift of $j^d(x) \in [\mathscr{T}(m)(F)]$. Hence,

$$\begin{aligned} H((f_v)_v)(j^d(x)) &= \prod_{v\in M_F} f_v(\widetilde{x}^{\frac{m}{d}}) \\ &= \left(\prod_{v\in M_F} |\widetilde{x}|_v^{1-\frac{m}{d}}\right)\cdot \prod_{v\in M_F} f_v(\widetilde{x}^{\frac{m}{d}}) \\ &= \prod_{v\in M_F} |\widetilde{x}|_v^{1-\frac{m}{d}} f_v(\widetilde{x}^{\frac{m}{d}}) \\ &= \prod_{v\in M_F} (f_v^{*d}(\widetilde{x}))^{(1-\frac{1}{q})/(1-\frac{1}{r})} \\ &= (H((f_v^{*d}))(x))^{(1-\frac{1}{q})/(1-\frac{1}{r})}. \end{aligned}$$

We deduce that

$$H((f_v)_v)\circ j^d = \big(H((f_v^{*d})_v)\big)^{(1-\frac{1}{q})/(1-\frac{1}{r})},$$

as claimed.

$\square$

Theorem 9.2.7.3. — *Suppose that $4\nmid m$ or that $i=\sqrt{-1}\in F$. One has that*

$$|\{x\in[\mathscr{P}(m)(F)]|\ x \text{ is a field},\ H(x)\leq B\}| \sim_{B\to\infty} C(m,(f_v)_v)B\log(B)^{r-2},$$

where

$$C(m,((f_v)_v)) = \frac{\left(\sum_{\substack{d|m\\ r|d}} d\cdot\mu(d)\cdot\tau((f_v^{*(m/d)})_v)\right)}{(r-2)!m},$$

and μ stands for the Möbius function (here, the sum is taken over divisors d of m, which are divisible by the prime r).

Proof. — We introduce notation

$$[\mathscr{T}(m)(F)]^0 = \{x\in[\mathscr{T}(m)(F)]|x \text{ is a field}\}.$$

For $x\in[\mathscr{T}(m)(F)]$, let $\widetilde{x}\in F^\times$ be its lift. Suppose for instant that $4\nmid m$. It follows from [**34**, Theorem 9.1, Chapter VI] that $x\in[\mathscr{T}(m)(F)]$ is a field if and only if $\widetilde{x}$ is not an element of $(F^\times)_p = \{y^p|y\in F^\times\}$ for primes $p|m$. Suppose now that $4|m$, by the hypothesis $i\in F$, hence $-4(F^\times)_4\subset(F^\times)_2$. Therefore the same conclusion of [**34**, Theorem 9.1, Chapter VI] applies.

We deduce that x is a field if and only if $x \notin [\mathscr{T}(m)(F)]_p = \{y^p | y \in [\mathscr{T}(m)(F)]\}$ for primes $p|m$, i.e. we can write

$$[\mathscr{T}(m)(F)]^0 = [\mathscr{T}(m)(F)] - \bigcup_{\substack{p \text{ prime} \\ p|m}} [\mathscr{T}(m)(F)]_p.$$

We are going to use the inclusion-exclusion principle. For that purpose, we verify that for positive integers k, ℓ such that $\gcd(k, \ell) = 1$, one has that

$$[\mathscr{T}(m)(F)]_k \cap [\mathscr{T}(m)(F)]_\ell = [\mathscr{T}(m)(F)]_{k\ell}.$$

Indeed, the inclusion "$\supset$" is clear and let us prove the reverse inclusion. Write $u_1k + u_2\ell = 1$. If $y \in [\mathscr{T}(m)(F)]_k \cap [\mathscr{T}(m)(F)]_\ell$, then there exists y' and y'' such that $y = (y')^k$ and $y = (y'')^\ell$. Thus

$$y = y^{u_1k+u_2\ell} = y^{u_1k}y^{u_2\ell} = (y'')^{\ell u_1 k}(y')^{ku_2\ell} = ((y'')^{u_1}(y')^{u_2})^{k\ell},$$

and the claim is verified. We deduce that for $B > 0$ one has that:

$$\begin{aligned}
&\Big|\Big\{x \in \bigcup_{\substack{p \text{ prime} \\ p|m}} [\mathscr{T}(m)(F)]_p | H(x) \leq B\Big\}\Big| \\
&\qquad = \sum_{\substack{j \geq 1 \\ p_1 < \cdots < p_j \text{ primes of } m}} (-1)^{j+1} |\{x \in [\mathscr{T}(m)(F)]_{p_1 \cdots p_j} | H(x) \leq B\}|,
\end{aligned}$$

and thus that

$$\begin{aligned}
&\Big|\Big\{x \in [\mathscr{T}(m)(F)]^0 | H(x) \leq B\Big\}\Big| \\
&\quad = \Big|\Big\{x \in [\mathscr{T}(m)(F)] | H(x) \leq B\Big\}\Big| \\
&\qquad - \sum_{\substack{j \geq 1 \\ p_1 < \cdots < p_j \text{ primes of } m}} (-1)^{j+1} |\{x \in [\mathscr{T}(m)(F)]_{p_1 \cdots p_j} | H(x) \leq B\}|.
\end{aligned}$$

Using the Möbius function μ, we write the last equality as

$$\begin{aligned}
&\Big|\Big\{x \in [\mathscr{T}(m)(F)]^0 | H(x) \leq B\Big\}\Big| \\
&\qquad\qquad = \sum_{d|m} \mu(d) \big|\{x \in [\mathscr{T}(m)(F)]_d | H(x) \leq B\}\big|.
\end{aligned}$$

We write $r(k)$ for the smallest prime of an integer k. Lemma 9.2.7.2 for $d|m$ gives that

$$\begin{aligned}\{x \in [\mathscr{T}(m)(F)]_d | H(x) \leq B\} \\ &= \{x \in [\mathscr{T}(m/d)(F)] | (H((f_v^{*(m/d)})_v)(j^{\frac{m}{d}}(x)))^{(1-\frac{1}{r(m/d)})/(1-\frac{1}{r})} \leq B\} \\ &= \{x \in [\mathscr{T}(m/d)(F)] | (H((f_v^{*(m/d)})_v)(j^{\frac{m}{d}}(x)))) \leq B^{(1-\frac{1}{r})/(1-\frac{1}{r(m/d)})}\}\end{aligned}$$

Now, it follows from Corollary 9.2.5.9 that

$$\begin{aligned}&|\{x \in [\mathscr{T}(m)(F)]_d | H((f_v^{*(m/d)})_v)(x) \leq B\}| \\ &\sim_{B\to\infty} \frac{\tau((f_v^{*(m/d)})_v)}{(r(m/d)-2)!(m/d)} B^{(1-\frac{1}{r})/(1-\frac{1}{r(m/d)})} \log(B^{(1-\frac{1}{r})/(1-\frac{1}{r(m/d)})})^{r(m/d)-2},\end{aligned}$$

where $\tau((f_v^{*(m/d)})_v) = \omega((f_v^{*(m/d)})_v)(\prod_{v\in M_F}[\mathscr{P}(m/d)(F_v)])$. Thus, for a divisor $d|m$ for which $r(m/d) > r$, the term $|\{x \in [\mathscr{T}(m)(F)]_d | H(x) \leq B\}|$ does not influence the leading constant. We deduce that

$$\begin{aligned}&|\{x \in [\mathscr{T}(m)(F)]^0 | H(x) \leq B\}| \\ &\qquad = \frac{\left(\sum_{\substack{d|m \\ r|d}} d \cdot \mu(d) \cdot \tau((f_v^{*(m/d)})_v)\right)}{(r-2)!m} B \log(B)^{r-2}.\end{aligned}$$

The theorem has been proven. □

Remark 9.2.7.4. — When m is not a prime the proof of Theorem 9.2.7.3 gives that there exists a positive proportion of μ_m-torsors, which are not fields. Indeed, let $d > 1$ be a divisor of m such that $r|(m/d)$. We have that any $x \in [\mathscr{P}(m)(F)]_d$ is not a field. It follows from above that

$$\begin{aligned}&|\{x \in [\mathscr{P}(m)(F)]_d | H(x) \leq B\}| \\ &\qquad = |\{x \in [\mathscr{P}(m/d)(F)] | H((f_v^{*(m/d)})_v)(x) \leq B\}| \\ &\qquad \sim_{B\to\infty} \frac{d \cdot \tau((f_v^{*(m/d)})_v}{(r-2)!m} B \log(B)^{r-2}.\end{aligned}$$

Remark 9.2.7.5. — Suppose that F contains all m-th roots of 1 (in particular if $4|m$ then $i \in F$, so Theorem 9.2.7.3 applies and gives the leading constant). One has that $\mu_m = \mathbb{Z}/m\mathbb{Z}$. The result of Theorem 9.2.7.3 has been established by Wright in [59]: he finds the asymptotic behaviour for the number of abelian extensions. The proof there also gives the precise leading constant, however, we find it is difficult to compare it with our constant.

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