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Théorie de Chern-Weil sous les groupes quantiques

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Résumé

Dans ce travail, nous étudions trois sujets liés à l'opérateur de Yang-Baxter: algèbres d'endormorphismes et la q-trace, constructions d'algèbres de Yang-Baxter et de cogèbres de Yang-Baxter, algèbres $\mathbf{B}_\infty$ quantiques et algèbres de quasi-battage quantiques. Ce sont des quantifications d'objets familiers correspondants au sens où le flip classique est remplacé par un tressage.

Ce travail est divisé en trois chapitres.

Chapitre 1: Soit (V, σ) un espace avec un tressage σ de type de Hecke et tel que $\dim S_\sigma^i(V) = 1$ pour certain suffisamment grand i. Nous étudions l'algèbre d'endomorphismes $\oplus_{k=1}^i \mathrm{End} S_\sigma^k(V)$. Après avoir défini trois produits associatifs sur cet espace, nous construisons une q-analogue de la trace classique, appelé q-trace, de tout endomorphisme de $S_\sigma^k(V)$. Cette nouvelle trace est un morphisme de l'algèbre si on considère le troisième produit. Et nous montrons que cette q-trace est proportionnele à la trace quantique.

Chapitre 2: Nous présentons des méthodes pour construire des algèbres de Yang-Baxter et des cogèbres de Yang-Baxter. Ils comprennent: modules de Yetter-Drinfel'd avec conditions de compatibilité supplémentaires, algèbres de battage quantiques et algèbres $\mathbf{B}_\infty$ quantiques. L'algèbre $\mathbf{B}_\infty$ quantique est une généralisation de l'algèbre de Yang-Baxter et de l'algèbre $\mathbf{B}_\infty$. Nous également introduisons l'algèbre de 2-YB qui est motivée par les travaux de Loday et Ronco. Ils fournissent des algèbres $\mathbf{B}_\infty$ quantiques.

Chapitre 3: Nous définissons l'algèbre de quasi-battage quantique par algèbres $\mathbf{B}_\infty$ quantiques, dans l'esprit de l'algèbre de battage quantique intruduite par Rosso. Nous étudions des propriétés de ces algèbres de quasi-battage quantiques. Par exemple, la propriété universelle, la commutativité, etc.

Mots-clefs

Q-traces, algèbres de Yang-Baxter, algèbres de battage quantiques, algèbres $\mathbf{B}_\infty$ quantiques, algèbres de quasi-battage quantiques.

Chern-Weil Theory on quantum groups

Abstract

In this work, we study three topics related to the Yang-Baxter operator: endomorphism algebras and the q-trace, constructions of Yang-Baxter algebras and Yang-Baxter coalgebras, quantum $\mathbf{B}_\infty$-algebras, and quantum quasi-shuffle algebras. They are the quantizations of the corresponding objects in the sense that the usual flip is replaced by a braiding.

This work is divided into three chapters.

Chapter 1: Let (V, σ) be a braided space with a braiding σ of Hecke type and such that $\dim S_\sigma^i(V) = 1$ for some sufficiently large i. We study the endomorphism algebra $\oplus_{k=1}^i \mathrm{End} S_\sigma^k(V)$. After defining three associative products on this space, we construct a q-analogue of the usual trace, called q-trace, for any endomorphism of $S_\sigma^k(V)$. This new trace is an algebra morphism with respect to the third product. And we show that this q-trace is just the quantum trace up to some scalar.

Chapter 2: We introduce several methods to construct Yang-Baxter (or short for YB) algebras and Yang-Baxter coalgebras. They include: Yetter-Drinfel'd modules with extra compatible conditions, quantum-shuffle algebras and quantum $\mathbf{B}_\infty$-algebras. Quantum $\mathbf{B}_\infty$-algebras are generalizations of both YB algebras and $\mathbf{B}_\infty$-algebras. We also introduce 2-YB algebras, which are motivated by the work of Loday and Ronco, to provide quantum $\mathbf{B}_\infty$-algebras.

Chapter 3: Using the tool of quantum $\mathbf{B}_\infty$-algebras, we quantize quasi-shuffle algebras in the spirit of Rosso's quantum shuffle algebras. We study various properties of these quantum quasi-shuffle algebras. For instance, the universal property, the commutativity and so on.

Keywords

Q-traces, Yang-Baxter algebras, quantum shuffle algebras, $\mathbf{B}_\infty$-algebras, quantum quasi-shuffle algebras.

Contents

Notations

Let $(H, \triangle, \varepsilon, S)$ be a Hopf algebra. We adopt Sweedler's notations for coalgebras and comodules: for any $h \in H$,

$$\triangle(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)},$$

and for a left H-comodule (M, ρ) and any $m \in M$,

$$\rho(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}.$$

The symmetric group of n letters $\{1, 2, \ldots, n\}$ is written by $\mathfrak{S}_n$. An $(i_1, \ldots, i_l)$-shuffle is an element $w \in \mathfrak{S}_{i_1+\cdots+i_l}$ such that $w(1) < \cdots < w(i_1), w(i_1+1) < \cdots < w(i_1+i_2), \ldots, w(i_1+\cdots+i_{l-1}+1) < \cdots < w(i_1+\cdots+i_l)$. We denote by $\mathfrak{S}_{i_1,\ldots,i_l}$ the set of all $(i_1, \ldots, i_l)$-shuffles.

A braiding σ on a vector space V is an invertible linear map in $\mathrm{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation on $V^{\otimes 3}$:

$$(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes \sigma)(\sigma \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma).$$

A braided vector space (V, σ) is a vector space V equipped with a braiding σ. For any $n \in \mathbb{N}$ and $1 \leq i \leq n-1$, we denote by σ_i the operator $\mathrm{id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{id}_V^{\otimes(n-i-1)} \in \mathrm{End}(V^{\otimes n})$. For any $w \in \mathfrak{S}_n$, we denote by T_w the corresponding lift of w in the braid group B_n, defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of w, where $s_i = (i, i+1)$, then $T_w = \sigma_{i_1} \cdots \sigma_{i_1}$. Sometimes we also use T_w^σ to indicate the action of σ.

The usual flip switching two factors is denoted by τ. For a vector space V, we denote by $\otimes$ the tensor product within $T(V)$, and by $\underline{\otimes}$ the one between $T(V)$ and $T(V)$ respectively.

Let q be a nonzero number in $\mathbb{C}$. For $q \neq 1$ and any $n = 0, 1, 2, \ldots$, we denote $(n)_q = \frac{1-q^n}{1-q}$, and

$$(n)_q! = \begin{cases} 1, & n = 0, \\ \frac{(1-q)\cdots(1-q^n)}{(1-q)^n}, & n \geq 1. \end{cases}$$

Chapter 1

Endomorphism algebras and q-traces

1.1 Introduction

More than two decades ago, Osborn investigated the space $\oplus_{i\geq 0}\mathrm{End}\bigwedge^i(V)$ of endomorphisms of the exterior algebra in order to give an algebraic description of Chern-Weil theory (see [22] and [23]). He defined three associative products on this space. The first one is just the composition of endomorphisms. Since the exterior algebra is also a coalgebra, he defined the second one to be the convolution product. And then he combined the first two ones to make out the third product. Assuming that $\dim\bigwedge^i(V) = 1$ for sufficiently large i, he constructed a trace function by using the second product. This trace gives the usual one when it is restricted on $\mathrm{End}(V)$. And when one considers the third product, it is an algebra morphism.

On the other side, after the birth of quantum groups, mathematicians use Yang-Baxter operators to quantize many algebra structures. In his paper [7], Gurevich studied Yang-Baxter operators of Hecke type, which he called Hecke symmetries. And then he defined symmetric algebras and exterior algebras with respect to these operators, which are analogue to the usual ones. Later, Hashimoto and Hayashi ([9]), Wambst ([28]) discussed these algebras from different aspects. In [26], a remarkable property of the quantized symmetric algebra is discovered. For some special Yang-Baxter operators, the symmetric one, as Hopf algebra, is isomorphic to the "upper triangular part" of the quantized enveloping algebra associated with a symmetrizable Cartan matrix.

It is interesting to extend Osborn's trace to the quantum case. Let (V,σ) be a braided vector space with braiding σ of Hecke type, and $S_\sigma^i(V)$ be the i-th component of the quantum symmetric algebra $S_\sigma(V)$ built on (V,σ). We assume that $\dim S_\sigma^N(V) = 1$ for some N and $\dim S_\sigma^k(V) = 0$ for $k > N$. Then, on the vector space $\oplus_{p=0}^N \mathrm{End} S_\sigma^p(V)$, the convolution product, the third product, and the trace can be constructed step-by-step following the ones in [23]. And this trace, called q-trace, is an algebra morphism with respect to the third product. Specially, let V be the fundamental representation of $\mathcal{U}_q\mathfrak{sl}_{N+1}$

and σ the braiding given by the R-matrix of $\mathcal{U}_q\mathfrak{sl}_{N+1}$. Then σ is of Hecke type and $S_\sigma^i(V)$ vanishes when i is sufficiently large. To our surprise, the q-trace in this case has already existed for more than one decade.

In the theory of quantum groups, there is an important invariant which generalizes the usual trace of endomorphisms. It is the so-called quantum trace. If $\mathcal{C}$ is a ribbon category with unit I, V is an object of $\mathcal{C}$ and f is an endomorphism of V, then the quantum trace is an element of the monoid $\mathrm{End}(I)$. It coincides with the usual trace when $\mathcal{C} = Vect(k)$ (see [15]). When we take $\mathcal{C}$ to be the category of finite dimensional representations of u_ϵ (see [16]), the quantum trace is given by the usual trace and the group-like elements K_i's. This is a functorial approach. After an easy computation, we can show that it is a proportion of the q-trace. So we give a more elementary approach of the quantum trace of A type.

This chapter is organized as follows. In Section 2 we give some essential properties of braidings of Hecke type which we will use in the rest of this chapter. We define three products on $\oplus_{p=0}^N \mathrm{End}S_\sigma^p(V)$ for a braided vector space V with a braiding σ of Hecke type in Section 3. In Section 4 we construct the q-trace of $\oplus_{p=0}^N \mathrm{End}S_\sigma^p(V)$ and prove that it is an algebra morphism with respect to the third product. And then in Section 5, we apply our constructions to the the special braided vector space (V, σ), where V is the fundamental representation of $\mathcal{U}_q\mathfrak{sl}_{N+1}$ and σ is the braiding given by the R-matrix of this $\mathcal{U}_q\mathfrak{sl}_{N+1}$-module. Section 6 contains an other approach of the usual quantum trace of $\mathcal{U}_q\mathfrak{sl}_{N+1}$. Finally, we use the quantum alternating multilinear form to obtain the quantum determinant of a matrix.

1.2 Braidings of Hecke type

We first recall some notions and properties of braidings of Hecke type and quantum symmetric algebras. For more details, one can see [7], [26] and [6].

A braiding σ is said to be of *Hecke type* if it satisfies the following *Iwahori's quadratic equation*:

$$(\sigma + \mathrm{id}_{V\otimes V})(\sigma - \nu \mathrm{id}_{V\otimes V}) = 0, \tag{1.1}$$

where ν is a nonzero scalar in $\mathbb{C}$.

In the following, we fix σ a braiding of Hecke type with parameter ν, and use $\mathtt{I}_k$ to denote $\mathrm{id}_{V^{\otimes k}}$.

For $k \geq 1$, we define the following three operators $\Pi_i, B^{(k)}, A^{(k)} \in \mathrm{End}(V^{\otimes k})$ by:

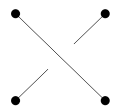
$$\Pi_i = \begin{cases} \mathrm{I}_k, & i = k, \\ \sigma_i\sigma_{i+1}\cdots\sigma_{k-1}, & i = 1, 2, \ldots k-1, \end{cases}$$

$$B^{(k)} = \sum_{l=1}^{k} \Pi_l,$$

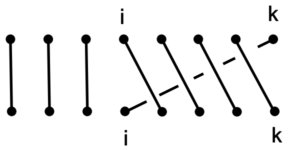
and

$$A^{(k)} = \begin{cases} \mathrm{I}_1, & k = 1, \\ B^{(k)}(A^{(k-1)} \otimes \mathrm{I}_1), & k \geq 2. \end{cases}$$

It is convenient to use the following figure to denote σ:



Then Π_i is:



Now we give another approach to $A^{(k)}$. The following lemma is well-known (see [10]).

Lemma 1.1. *Set*

$$\begin{aligned} S_1 &= \{1, s_1\}, \\ S_2 &= \{1, s_2, s_2 s_1\}, \\ &\vdots \\ S_{k+1} &= \{1\} \cup s_{k+1} A_k, \\ &\vdots \\ S_{p-1} &= \{1, s_{p-1}, \cdots, s_{p-1} s_{p-2} \cdots s_1\}. \end{aligned}$$

Then any $\sigma \in \mathfrak{S}_p$ can be written in a unique form $\sigma = u_1 \cdots u_{p-1}$ with $u_i \in S_i$.

Lemma 1.2. *For any $u_i \in S_i$, we have*

$$l(u_1 \cdots u_{p-1}) = l(u_1) + \cdots + l(u_{p-1}).$$

Proof. First we notice that for $k \leq q - 1$,

$$s_{p-1}\cdots s_k(t)=\begin{cases} p, & t=k,\\ t, & t<k,\\ t-1, & t>k.\end{cases}$$

So we get that if $u_1\cdots u_{p-1}(k)=p$ then $u_{p-1}=s_{p-1}\cdots s_k$.

$$l(u_{p-1})=p-k=\#\{(k,j)|k<j\le p, u_1\cdots u_{p-1}(k)>u_1\cdots u_{p-1}(j)\}.$$

Now we want to prove that

$$\begin{aligned}&\#\{(i,j)|1\le i<j\le p-1, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\}\\ =\;&\#\{(i,j)|1\le i<j\le p, i\neq k, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(j)\}.\end{aligned}$$

Notice that

$$\begin{aligned}&\#\{(i,j)|k<i<j\le p, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(j)\}\\ =\;&\#\{(i,j)|k<i<j\le p, u_1\cdots u_{p-2}(i-1)>u_1\cdots u_{p-2}(j-1)\}\\ =\;&\#\{(i,j)|k\le i<j\le p-1, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\},\end{aligned}$$

$$\begin{aligned}&\#\{(i,j)|1\le i<j\le p, i<k, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(j)\}\\ =\;&\#\{(i,j)|1\le i<k<j\le p, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(j)\}\\ &+\#\{(i,j)|1\le i<k=j, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(k)\}\\ &+\#\{(i,j)|1\le i<j<k, u_1\cdots u_{p-1}(i)>u_1\cdots u_{p-1}(j)\}\\ =\;&\#\{(i,j)|1\le i<k<j\le p, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j-1)\}\\ &+\#\{(i,j)|1\le i<j<k, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\}\\ =\;&\#\{(i,j)|1\le i<k\le j\le p-1, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\}\\ &+\#\{(i,j)|1\le i<j<k, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\}\\ =\;&\#\{(i,j)|1\le i<j\le p-1, i<k, u_1\cdots u_{p-2}(i)>u_1\cdots u_{p-2}(j)\}.\end{aligned}$$

By combining the above computations, we get

$$l(u_1\cdots u_{p-1})=l(u_1\cdots u_{p-2})+l(u_{p-1}).$$

So the conclusion follows from the induction on p. □

Proposition 1.3. *For $p\ge 1$, we have*

$$\sum_{w\in\mathfrak{S}_p}T_w=A^{(p)}.$$

Proof. We use induction on p.
The case of $p = 1$ is trivial. We assume that the result holds for k.

$$\begin{aligned} A^{(k+1)} &= B^{(k+1)}(A^{(k)} \otimes \mathrm{I}_1) \\ &= (\sum_{l=1}^{k+1} \Pi_l) \circ (\sum_{w \in \mathfrak{S}_k \times 1_{\mathfrak{S}_1}} T_w) \\ &= (\sum_{l=1}^{k+1} \sigma_l \sigma_{l+1} \cdots \sigma_k) \circ (\sum_{w \in \mathfrak{S}_k \times 1_{\mathfrak{S}_1}} T_w) \\ &= (\sum_{l=1}^{k+1} T_{s_l s_{l+1} \cdots s_k}) \circ (\sum_{w \in \mathfrak{S}_k \times 1_{\mathfrak{S}_1}} T_w) \\ &= \sum_{l=1}^{k+1} \sum_{w \in \mathfrak{S}_k \times 1_{\mathfrak{S}_1}} T_{s_l s_{l+1} \cdots s_k} \circ T_w \\ &= \sum_{w \in \mathfrak{S}_{k+1}} T_w, \end{aligned}$$

where the fifth equality follows from the above two lemmas. □

The following proposition plays an important role in the construction of q-trace. It is due to Gurevich ([7], Proposition 2.4):

Proposition 1.4. *For $k \geq 1$ we have*

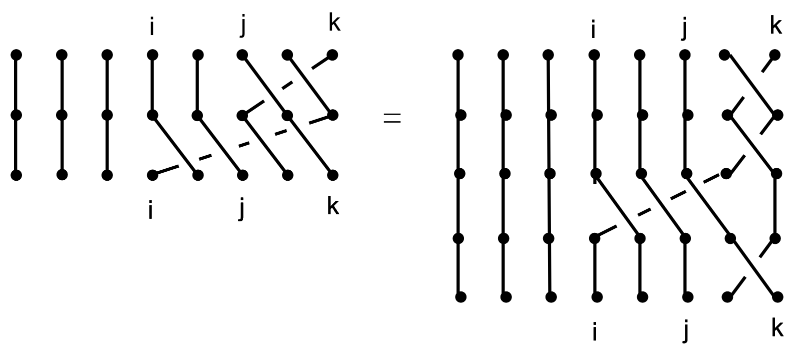
$$(A^{(k)})^2 = (k)_\nu! A^{(k)}.$$

Proof. In $\mathrm{End}(V^{\otimes k})$, we have

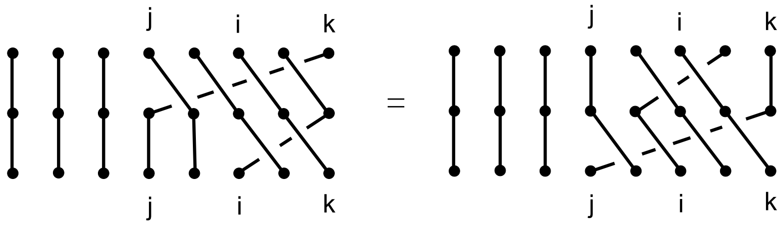
$$\begin{cases} \Pi_i \Pi_j = \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1)\sigma_{k-1}^2, \ 1 \leq i \leq j \leq k-1, \\ \Pi_i \Pi_j = \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1), \ 1 \leq j < i \leq k-1. \end{cases}$$

We illustrate the verification by the following figures:

1) $1 \leq i \leq j \leq k-1$:



. 2) $1 \le j < i \le k-1$:



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$$
\begin{aligned}
& B^{(k)}(B^{(k-1)} \otimes \mathrm{I}_1)\sigma_{k-1} \\
= \;& (\sum_{i=1}^{k} \Pi_i)(\sum_{j=1}^{k-1} \Pi_j \otimes \mathrm{I}_1)\sigma_{k-1} \\
= \;& (\sum_{i=1}^{k} \Pi_i)(\sum_{j=1}^{k-1} \Pi_j) \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le i\le j\le k-1} \Pi_i\Pi_j + \sum_{1\le j<i\le k-1} \Pi_i\Pi_j \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le i\le j\le k-1} \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1)\sigma_{k-1}^2 + \sum_{1\le j<i\le k-1} \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1) \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le j<i\le k-1} \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1) \\
& +(\nu-1) \sum_{1\le i\le j\le k-1} \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1)\sigma_{k-1} + \nu \sum_{1\le i\le j\le k-1} \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1) \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le j<i\le k-1} \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1) \\
& +(\nu-1) \sum_{1\le i\le j\le k-1} \Pi_{j+1}\Pi_i + \nu \sum_{1\le i\le j\le k-1} \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1) \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le j<i\le k-1} \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1) + (\nu-1)(\sum_{i=1}^{k-1} \Pi_i + \sum_{1\le i<j\le k-1} \Pi_j\Pi_i) \\
& +\nu \sum_{1\le i\le j\le k-1} \Pi_{j+1}(\Pi_i \otimes \mathrm{I}_1) \\
= \;& \sum_{j=1}^{k-1} \Pi_j + \sum_{1\le j<i\le k-1} \Pi_j(\Pi_{i-1} \otimes \mathrm{I}_1)
\end{aligned}
$$

$$
\begin{aligned}
& +(\nu-1)\big(\sum_{i=1}^{k-1}\Pi_i + \sum_{1\leq i<j\leq k-1}\Pi_i(\Pi_{j-1}\otimes \mathrm{I}_1)\big) + \nu \sum_{1\leq i\leq j\leq k-1}\Pi_{j+1}(\Pi_i\otimes \mathrm{I}_1) \\
= {} & \nu\big(\sum_{i=1}^{k-1}\Pi_i + \sum_{1\leq i<j\leq k-1}\Pi_i(\Pi_{j-1}\otimes \mathrm{I}_1) + \nu \sum_{1\leq i\leq j\leq k-1}\Pi_{j+1}(\Pi_i\otimes \mathrm{I}_1)\big) \\
= {} & \nu B^{(k)}(B^{(k-1)}\otimes \mathrm{I}_1).
\end{aligned}
$$

And for any $k \geq 2$ and $1 \leq i \leq k-1$, we have

$$
\begin{aligned}
& A^{(k)}\sigma_i \\
= {} & B^{(k)}(B^{(k-1)}\otimes \mathrm{I}_1)\cdots(B^{(1)}\otimes \mathrm{I}_{k-1})\sigma_i \\
= {} & B^{(k)}(B^{(k-1)}\otimes \mathrm{I}_1)\cdots((B^{(i+1)}\otimes \mathrm{I}_{k-i-1})(B^{(i)}\otimes \mathrm{I}_{k-i})\sigma_i)\cdots(B^{(1)}\otimes \mathrm{I}_{k-1}) \\
= {} & \nu A^{(k)}.
\end{aligned}
$$

So

$$
\begin{aligned}
A^{(k)}A^{(k)} &= A^{(k)}B_i^{(k)}(A^{(k-1)}\otimes \mathrm{I}_1) \\
&= A^{(k)}(\sum_{l=1}^{k}\Pi_l)(A^{(k-1)}\otimes \mathrm{I}_1) \\
&= (\sum_{l=1}^{k}A^{(k)}\sigma_l\sigma_{l+1}\cdots\sigma_{k-1})(A^{(k-1)}\otimes \mathrm{I}_1) \\
&= (1+\nu+\nu^2+\cdots+\nu^{k-1})A^{(k)}(A^{(k-1)}\otimes \mathrm{I}_1) \\
&= (k)_\nu A^{(k)}(B^{(k-1)}(A^{(k-2)}\otimes \mathrm{I}_2)\otimes \mathrm{I}_1) \\
&= \cdots \\
&= (k)_\nu! A^{(k)}.
\end{aligned}
$$

□

1.3 The endomorphism algebra of $S_\sigma(V)$

Let (V,σ) be a braided vector space. Using the braiding σ, one can generalize the usual shuffle algebra structure on $T(V)$ to the so-called quantum shuffle algebra structure (for more detail, one can see [26] and [6]). The *quantum shuffle algebra* is $T(V)$ equipped with the following associative product sh: for any $v_1,\ldots,v_{i+j}\in V$,

$$
sh(v_1\otimes\cdots\otimes v_i \underline{\otimes} v_{i+1}\otimes\cdots\otimes v_{i+j}) = \sum_{w\in\mathfrak{S}_{i,j}} T_w(v_1\otimes\cdots\otimes v_{i+j}). \tag{1.2}
$$

We denote by $T_\sigma(V)$ the quantum shuffle algebra. The subalgebra $S_\sigma(V)$ of $T_\sigma(V)$ generated by V with respect to the quantum shuffle product is called the *quantum symmetric algebra*. In fact, $S_\sigma(V) = \oplus_{i\geq 0}\mathrm{Im}(\sum_{w\in\mathfrak{S}_i} T_w)$. We also denote $S^i_\sigma(V) = \mathrm{Im}(\sum_{w\in\mathfrak{S}_i} T_w)$.

$T_\sigma(V)$ is a coalgebra with the deconcatenation coproduct δ:

$$\delta(v_1\otimes\cdots\otimes v_n) = \sum_{i=0}^{n} v_1\otimes\cdots\otimes v_i \underline{\otimes} v_{i+1}\otimes\cdots\otimes v_n. \tag{1.3}$$

We denote by δ_{ij} the composition of δ with the projection $T(V)\underline{\otimes}T(V)\to V^{\otimes i}\otimes V^{\otimes j}$.

We endow $T_\sigma(V)\otimes T_\sigma(V)$ with the associative product $\star$, which is given by $(sh\otimes sh)(\mathrm{id}_V^{\otimes i}\otimes T_{\chi_{jk}}\otimes \mathrm{id}_V^{\otimes l})$ on the component $V^{\otimes i}\underline{\otimes}V^{\otimes j}\underline{\otimes}V^{\otimes k}\underline{\otimes}V^{\otimes l}$. Here

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix}.$$

Then δ is an algebra morphism from $T_\sigma(V)$ to $T_\sigma(V)\otimes T_\sigma(V)$. So $(S_\sigma(V),\delta)$ is also a coalgebra.

Let σ be a braiding of Hecke type on V such that $\dim S^N_\sigma(V) = 1$ for some N and $\dim S^k_\sigma(V) = 0$ for $k > N$.

For $\mathbf{A}\in\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V)$, we write $\mathbf{A} = (A_0, A_1, \ldots, A_N)$, where $A_p\in\mathrm{End}S^p_\sigma(V)$ is the p-th component of $\mathbf{A}$.

For $\mathbf{A},\mathbf{B}\in\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V)$, we define the *composition product* $\mathbf{A}\circ\mathbf{B}\in\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V)$ by $(\mathbf{A}\circ\mathbf{B})_p = A_p\circ B_p$ with the usual composition. Obviously, $\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V)$ is an associative algebra with a two-sided unit element $\mathbf{I} = (\mathtt{I}_0, \mathtt{I}_1, \ldots \mathtt{I}_N)$, where $\mathtt{I}_p$ is the identity map of $S^p_\sigma(V)$.

We can also define the *convolution product* $\mathbf{A}*\mathbf{B}\in\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V)$ by

$$(\mathbf{A}*\mathbf{B})_p = \sum_{l=0}^{p} A_l * B_{p-l},$$

where $A_i * B_j = sh\circ(A_i\otimes B_j)\circ\delta_{i,j}\in\mathrm{End}S^{i+j}_\sigma(V)$. It is well-known that the convolution product of endomorphisms is associative. It follows immediately that $(\oplus_{p=0}^N \mathrm{End}S^p_\sigma(V), *)$ is an associative algebra with the two-sided unit element $\mathrm{I}_0 = (\mathrm{I}_0, 0, \ldots, 0)$.

Proposition 1.5. *For $0\leq p\leq N$, we have*

$$\mathrm{I}_1^{*p} = (p)_\nu!\mathrm{I}_p.$$

Proof. It follows from that

$$\mathrm{I}_1^{*p}\circ A^{(p)} \;=\; A^{(p)}\circ \mathrm{I}_1^{\otimes p}\circ A^{(p)}$$

$$\begin{aligned} &= (A^{(p)})^2 \\ &= (p)_\nu! A^{(p)}. \end{aligned}$$

□

Corollary 1.6. *For $0 \leq i, j \leq N$ with $i + j \leq N$, we have*

$$\mathrm{I}_i * \mathrm{I}_j = \binom{i+j}{i}_\nu \mathrm{I}_{i+j},$$

where $\binom{i+j}{i}_\nu = \frac{(i+j)_\nu!}{(i)_\nu!(j)_\nu!}$.

For any $A \in \mathrm{End} S^1_\sigma(V) = \mathrm{End}(V)$ and ν is not a root of 1, we define

$$e_\nu^{*A} = (\mathtt{I}_0, \frac{1}{(1)_\nu!}A, \frac{1}{(2)_\nu!}A^{*2}, \ldots, \frac{1}{(N)_\nu!}A^{*N}).$$

In particular, $e_\nu^{*\mathtt{I}_1} = (\mathtt{I}_0, \mathtt{I}_1, \ldots, \mathtt{I}_N)$.

If we write

$$(e_\nu^{*A})^{-1} = (\mathtt{I}_0, \frac{-1}{(1)_\nu!}A, \frac{\nu}{(2)_\nu!}A^{*2}, \ldots, \frac{(-1)^N \nu^{N(N-1)/2}}{(N)_\nu!}A^{*N}),$$

then we have

$$(e_\nu^{*A})^{-1} * e_\nu^{*A} = e_\nu^{*A} * (e_\nu^{*A})^{-1} = \mathtt{I}_0.$$

We define

$$\begin{aligned} \alpha : \oplus_{p=0}^N \mathrm{End} S^p_\sigma(V) &\rightarrow \oplus_{p=0}^N \mathrm{End} S^p_\sigma(V), \\ \mathbf{A} &\mapsto \mathbf{A} * e_\nu^{*\mathtt{I}_1}. \end{aligned}$$

Consequently α has an inverse defined by $\alpha^{-1}(\mathbf{A}) = \mathbf{A} * (e_\nu^{*\mathtt{I}_1})^{-1}$.

Definition 1.7. *For any $\boldsymbol{A}, \boldsymbol{B} \in \oplus_{p=0}^N \mathrm{End} S^p_\sigma(V)$, we define the* third product $\boldsymbol{A} \times \boldsymbol{B}$ *of $\boldsymbol{A}$ and $\boldsymbol{B}$ by*

$$\boldsymbol{A} \times \boldsymbol{B} = \alpha^{-1}((\alpha \boldsymbol{A}) \circ (\alpha \boldsymbol{B})) = \big((\boldsymbol{A} * e_\nu^{*\mathrm{I}_1}) \circ (\boldsymbol{B} * e_\nu^{*\mathrm{I}_1})\big) * (e_\nu^{*\mathrm{I}_1})^{-1}.$$

Remark 1.8. *$\oplus_{p=0}^N \mathrm{End} S^p_\sigma(V)$ is an associative algebra with two-sided unit element $\mathtt{I}_0$ with respect to the third product.*

Indeed,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \alpha^{-1}\Big(\big(\alpha(\mathbf{A} \times \mathbf{B})\big) \circ (\alpha \mathbf{C})\Big)$$

$$\begin{aligned} &= \alpha^{-1}\Big(\big(\alpha \circ \alpha^{-1}((\alpha\mathbf{A}) \circ (\alpha\mathbf{B}))\big) \circ (\alpha\mathbf{C})\Big) \\ &= \alpha^{-1}\big((\alpha\mathbf{A}) \circ (\alpha\mathbf{B}) \circ (\alpha\mathbf{C})\big) \\ &= \mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \end{aligned}$$

And

$$\begin{aligned} \mathtt{I}_0 \times \mathbf{A} &= \alpha^{-1}\big((\alpha\mathtt{I}_0) \circ (\alpha\mathbf{A})\big) \\ &= \alpha^{-1}\big((\mathtt{I}_0 * e_\nu^{*\mathtt{I}_1}) \circ (\alpha\mathbf{A})\big) \\ &= \alpha^{-1}\big(e_\nu^{*\mathtt{I}_1} \circ (\alpha\mathbf{A})\big) \\ &= \alpha^{-1}(\alpha\mathbf{A}) \\ &= \mathbf{A}. \end{aligned}$$

Proposition 1.9. *For $0 \le r \le N$, $A_i \in \mathrm{End}S^i_\sigma(V)$ and $B_j \in \mathrm{End}S^j_\sigma(V)$, we have*

$$(A_i \times B_j)_r = \sum_{s=0}^{r} \frac{\nu^{s(s-1)/2}}{(s)_\nu!}((A_i * \mathrm{I}_{r-s-i}) \circ (B_j * \mathrm{I}_{r-s-j})) * \mathrm{I}_1^{*s},$$

where $\mathrm{I}_t = 0$ for $t < 0$.

Proof. The formula follows from the definition of the third product. □

Corollary 1.10. *$(A_i \times B_j)_r = 0$ for $r < \max(i, j)$ and $(A_r \times B_r)_r = A_r \circ B_r$.*

1.4 The q-trace

Definition 1.11. *The* q-trace *of any $\boldsymbol{A} \in \oplus_{p=0}^{N}\mathrm{End}S^p_\sigma(V)$ is the unique element $\mathrm{Tr}_q\boldsymbol{A} \in \mathbb{C}$ such that $(\alpha\boldsymbol{A})_N = (\mathrm{Tr}_q\boldsymbol{A})\mathrm{I}_N \in \mathrm{End}S^N_\sigma(V)$.*

Theorem 1.12. *The q-trace is an algebra morphism with respect to the third product. Precisely, for $\boldsymbol{A}, \boldsymbol{B} \in \oplus_{p=0}^{N}\mathrm{End}S^p_\sigma(V)$, we have*

1. $\mathrm{Tr}_q(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{Tr}_q\boldsymbol{A} + \mathrm{Tr}_q\boldsymbol{B}$,
2. $\mathrm{Tr}_q(\boldsymbol{A} \times \boldsymbol{B}) = (\mathrm{Tr}_q\boldsymbol{A})(\mathrm{Tr}_q\boldsymbol{B})$,
3. $\mathrm{Tr}_q(\boldsymbol{A} \times \boldsymbol{B}) = \mathrm{Tr}_q(\boldsymbol{B} \times \boldsymbol{A})$.

Proof. 1.

$$\begin{aligned} (\alpha(\mathbf{A} + \mathbf{B}))_N &= \sum_{k=0}^{N}(A_k + B_k) * \mathtt{I}_{N-k} \\ &= (\alpha\mathbf{A})_N + (\alpha\mathbf{B})_N. \end{aligned}$$

So

$$\begin{aligned}\mathrm{Tr}_q(\mathbf{A}+\mathbf{B})\mathtt{I}_N &= (\mathrm{Tr}_q\mathbf{A})\mathtt{I}_N + (\mathrm{Tr}_q\mathbf{B})\mathtt{I}_N \\ &= (\mathrm{Tr}_q\mathbf{A} + \mathrm{Tr}_q\mathbf{B})\mathtt{I}_N.\end{aligned}$$

Therefore $\mathrm{Tr}_q(\mathbf{A}+\mathbf{B}) = \mathrm{Tr}_q\mathbf{A} + \mathrm{Tr}_q\mathbf{B}$.

2. By the definition, $\mathbf{A}\times\mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})\circ(\alpha\mathbf{B}))$, we have $\alpha(\mathbf{A}\times\mathbf{B}) = (\alpha\mathbf{A})\circ(\alpha\mathbf{B})$. So

$$(\alpha(\mathbf{A}\times\mathbf{B}))_N = (\alpha\mathbf{A})_N \circ (\alpha\mathbf{B})_N,$$

which implies

$$\begin{aligned}\mathrm{Tr}_q(\mathbf{A}\times\mathbf{B})\mathtt{I}_N &= (\mathrm{Tr}_q\mathbf{A})\mathtt{I}_N \circ (\mathrm{Tr}_q\mathbf{B})\mathtt{I}_N \\ &= (\mathrm{Tr}_q\mathbf{A})(\mathrm{Tr}_q\mathbf{B})\mathtt{I}_N.\end{aligned}$$

So we have $\mathrm{Tr}_q(\mathbf{A}\times\mathbf{B}) = (\mathrm{Tr}_q\mathbf{A})(\mathrm{Tr}_q\mathbf{B})$.

3. It follows from the identity stated in 2 immediately. □

1.5 Fundamental representation of $\mathcal{U}_q\mathfrak{sl}_{N+1}$

We first recall the quantum group $\mathcal{U}_q\mathfrak{sl}_{N+1}$ and fix some notations.

Let $(a_{ij})_{1\le i,j\le N+1}$ be the Cartan matrix of $\mathfrak{sl}_{N+1}$, i.e.,

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ 0 & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}_{N\times N}.$$

$\mathcal{U}_q\mathfrak{sl}_{N+1}$ is the $\mathbb{C}$-algebra with generators $E_i, F_i, K_i^{\pm1}$, where $1\le i\le N$, and relations

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j,$$

$$K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}},$$

and the quantum Serre relations:

$$\begin{gathered} E_iE_j = E_jE_i,\ |i-j| \geq 2, \\ E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0,\ |i-j| = 1, \\ F_iF_j = F_jF_i,\ |i-j| \geq 2, \\ F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0,\ |i-j| = 1. \end{gathered}$$

$\mathcal{U}_q\mathfrak{sl}_{N+1}$ is a Hopf algebra with the structures:

$$\begin{cases} \triangle K_i^{\pm 1} & = & K_i^{\pm 1} \otimes K_i^{\pm 1}, \\ \triangle E_i & = & 1 \otimes E_i + E_i \otimes K_i, \\ \triangle F_i & = & K_i^{-1} \otimes F_i + F_i \otimes 1, \end{cases}$$

$$\begin{cases} \varepsilon(K_i^{\pm 1}) & = & 1, \\ \varepsilon(E_i) & = & 0, \\ \varepsilon(F_i) & = & 0, \end{cases}$$

$$\begin{cases} SK_i^{\pm 1} & = & K_i^{\mp 1}, \\ SE_i & = & -E_iK_i^{-1}, \\ SF_i & = & -K_iF_i. \end{cases}$$

Let $V = \mathbb{C}^{N+1}$ and E_{ij} be the matrix with entry 1 in the position (i,j) and entries 0 elsewhere. We define the *fundamental representation* of $\mathcal{U}_q\mathfrak{sl}_{N+1}$ to be

$$\begin{array}{rcl} \rho : \mathcal{U}_q\mathfrak{sl}_{N+1} & \to & \mathrm{End}V, \\ E_i & \mapsto & E_{i,i+1}, \\ F_i & \mapsto & E_{i+1,i}, \\ K_i & \mapsto & \sum_{l \neq i,i+1} E_{ll} + qE_{ii} + q^{-1}E_{i+1,i+1}. \end{array}$$

Then the action of the R-matrix on $V \otimes V$is

$$R_\rho = q\sum_{i=1}^{N+1} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q-q^{-1})\sum_{i<j} E_{jj} \otimes E_{ii}.$$

Let $c = q^{-1}R_\rho \in \mathrm{GL}(V \otimes V)$. If we denote by $e_i = (0,\ldots,0,1,0,\ldots,0)^t \in V$ the unit column vector whose components are zero except the i-th component is 1. then we have:

$$c(e_i \otimes e_j) = \begin{cases} e_i \otimes e_i, & i = j, \\ q^{-1}e_j \otimes e_i, & i < j, \\ q^{-1}e_j \otimes e_i + (1-q^{-2})e_i \otimes e_j, & i > j. \end{cases}$$

c is a braiding on V and satisfies the *Iwahori's quadratic equation*:

$$(c - \mathrm{id}_{V\otimes V})(c + q^{-2}\mathrm{id}_{V\otimes V}) = 0. \tag{1.4}$$

We define $c^\vee = (c^{-1})^t$, where t means the transpose of the operator. Then $c^\vee \in \mathrm{GL}(V^* \otimes V^*)$. Let $\{f_i\}$ be the dual basis of $\{e_i\}$. We have

$$c^\vee(f_i \otimes f_j) = \begin{cases} f_i \otimes f_i, & i = j, \\ qf_j \otimes f_i + (1-q^2)f_i \otimes f_j, & i < j \\ qf_j \otimes f_i, & i > j. \end{cases}$$

Obviously $c^\vee$ is a braiding on V^* and satisfies the Iwahori's equation:

$$(c^\vee - \mathrm{id}_{V^*\otimes V^*})(c^\vee + q^2\mathrm{id}_{V^*\otimes V^*}) = 0.$$

Let $\mathfrak{I}$ be the two-sided ideal generated by $\mathrm{Ker}(\mathrm{id}_{V\otimes V} - c)$ in $\mathrm{T}(V)$. Since $\mathfrak{I}$ is generated by homogeneous elements of degree 2, we could give $\mathfrak{I}$ a natural grading:

$$\mathfrak{I}^k = \mathfrak{I} \cap \mathrm{T}^k(V).$$

Here $\mathfrak{I} \cap \mathrm{T}^k(V)$ means the ideal generated by $\mathrm{Ker}(\mathrm{id}_{V\otimes V} - c)$ in $\mathrm{T}^k(V)$.

Definition 1.13. *We define the* quantum exterior algebra $\bigwedge_c(V)$ *with respect to c by:*

$$\bigwedge\nolimits_c^k(V) = \mathrm{T}^k(V)/\mathfrak{I}^k,$$

$$\bigwedge\nolimits_c(V) = \mathrm{T}(V)/\mathfrak{I} = \bigoplus_{k\geq 0} \bigwedge\nolimits_c^k(V).$$

Let $\pi : \mathrm{T}(V) \to \bigwedge_c(V)$ be the canonical projection. For any $e_{i_1} \otimes \cdots \otimes e_{i_p} \in \mathrm{T}^p(V)$, we will write $e_{i_1} \wedge \cdots \wedge e_{i_p} = \pi(e_{i_1} \otimes \cdots \otimes e_{i_p})$.

From easy computation, we have

$$\mathrm{Ker}(\mathrm{id}_{V\otimes V} - c) = \mathrm{Span}_{\mathbb{C}}\{e_i \otimes e_i, q^{-1}e_i \otimes e_j + e_j \otimes e_i (i < j)\}.$$

So it follows immediately that:

1. $\bigwedge_c(V)$ is a graded algebra generated by $\{e_1, \ldots, e_{N+1}\}$ with relations:

$$e_i \wedge e_i = 0,$$

and

$$e_j \wedge e_i = -q^{-1}e_i \wedge e_j \ (i < j).$$

2. $\bigwedge_c^0(V) = \mathbb{C}$, $\bigwedge_c^1(V) = V$ and $\bigwedge_c^p(V) = 0$ for $p > N + 1$.

3. For $1 \leq p \leq N + 1$, $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq N + 1\}$ is a basis of the linear space $\bigwedge_c^p(V)$.

4. $\bigwedge_c(V)$ is a noncommutative local ring with the maximal ideal $\{0\} \cup \bigoplus_{k\geq 1} \bigwedge_c^k(V)$.

Let $A^{(p)} = \sum_{w\in\mathfrak{S}_p} (-1)^{l(w)} T_w^c = \sum_{w\in\mathfrak{S}_p} T_w^{-c}$.

Proposition 1.14 ([7], Proposition 2.13). *For $k \geq 1$, we have the following linear isomorphism:*

$$\mathrm{Im}A^{(k)} \cong \wedge_c^k(V).$$

So $\bigwedge_c(V)$ is just the quantum symmetric algebra $S_{-c}(V)$. We can identify the wedge product $\wedge$ on $\bigwedge_c(V)$ with the quantum shuffle product sh on $S_{-c}(V)$. The map $e_i \mapsto f_i$ induces an isomorphism of algebras:

$$i : \bigwedge\nolimits_c(V) \;\; \rightarrow \;\; \bigwedge\nolimits_{c^\vee}(V^*).$$

Indeed, we have

$$\mathrm{Ker}(\mathrm{id}_{V^*\otimes V^*} - c^\vee) = \mathrm{Span}_{\mathbb{C}}\{f_i \otimes f_i, f_i \otimes f_j + q^{-1} f_j \otimes f_i (i > j)\}.$$

So in $\wedge_{c^\vee}(V^*)$, we have

$$f_i \wedge f_i = 0, \;\; f_j \wedge f_i = -q^{-1} f_i \wedge f_j \;\; (i < j).$$

Now we give a more explicit description of the action of $A^{(p)}$, which will be used to compute the q-trace.

We first introduce a notation which will be used frequently in the rest of this chapter. Given $A, B, C \subset \mathfrak{S}_p$, $A \circ B = C$ means that the image of the composition map $\circ$ on $A \times B$ lies in C and it is bijective.

Proposition 1.15. *1. For $1 \leq i_1 < i_2 < \cdots < i_p \leq N+1$, we have*

$$\begin{aligned} A^{(p)}(e_{i_1} \otimes \cdots \otimes e_{i_p}) &= \sum_{w \in \mathfrak{S}_p} (-q)^{-l(w)} w(e_{i_1} \otimes \cdots \otimes e_{i_p}) \\ &= \sum_{w \in \mathfrak{S}_p} (-q)^{-l(w)} e_{i_{w^{-1}(1)}} \otimes \cdots \otimes e_{i_{w^{-1}(p)}} \end{aligned}$$

2. For $1 \leq i_1 < i_2 \cdots < i_p \leq N+1$, and $1 \leq t \leq p$, we have

$$\begin{aligned} & A^{(p)}(e_{i_1} \otimes \cdots \otimes e_{i_p}) \\ &= \sum_{w \in \mathfrak{S}_{t,p-t}} (-q)^{-l(w)} A^{(t)}(e_{i_{w(1)}} \otimes \cdots \otimes e_{i_{w(t)}}) \otimes A^{(p-t)}(e_{i_{w(t+1)}} \otimes \cdots \otimes e_{i_{w(p)}}). \end{aligned}$$

In particular,

$$A^{(p)}(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum_{l=1}^{p} (-q)^{1-l} e_{i_l} \otimes A^{(p-1)}(e_{i_1} \otimes \cdots \otimes \widehat{e_{i_l}} \otimes \cdots \otimes e_{i_p}),$$

where the symbol $\widehat{\ }$ means that the term is omitted.

Proof. 1.

$$\begin{aligned} A^{(p)}(e_{i_1}\otimes\cdots\otimes e_{i_p}) &= \sum_{w\in\mathfrak{S}_p}(-1)^{l(w)}T_w(e_{i_1}\otimes\cdots\otimes e_{i_p}) \\ &= \sum_{w\in\mathfrak{S}_p}(-1)^{l(w)}q^{-l(w)}w(e_{i_1}\otimes\cdots\otimes e_{i_p}). \end{aligned}$$

The second equality follows from the fact that every permutation $w\in\mathfrak{S}_p$ can be written in a unique reduced form $w=w_1\cdots w_{p-1}$, where $w_i\in\{1,s_i,s_is_{i-1},\dots,s_is_{i-1}\cdots s_1\}$.

2. Without loss of generality, we assume $i_1=1,\dots,i_p=p$. For any $w\in\mathfrak{S}_{t,p-t}$, we have

$$\begin{aligned} &(-q)^{-l(w)}A^{(p)}(e_{w(1)}\otimes\cdots\otimes e_{w(t)})\otimes A^{(p-t)}(e_{w(t+1)}\otimes\cdots\otimes e_{w(p)}) \\ &= (-q)^{-l(w)}(\sum_{\sigma\in\mathfrak{S}_t}(-q)^{-l(\sigma)}\sigma(e_{w(1)}\otimes\cdots\otimes e_{w(t)})) \\ &\quad\otimes(\sum_{\tau\in\mathfrak{S}_{p-t}}(-q)^{-l(\tau)}\tau(e_{w(t+1)}\otimes\cdots\otimes e_{w(p)})) \\ &= \sum_{\sigma\in\mathfrak{S}_t,\tau\in\mathfrak{S}_{p-t}}(-q)^{-l(w)-l(\sigma)-l(\tau)}\sigma(e_{w(1)}\otimes\cdots\otimes e_{w(t)})) \\ &\quad\otimes\tau(e_{w(t+1)}\otimes\cdots\otimes e_{w(p)})). \end{aligned}$$

Since $\mathfrak{S}_{t,p-t}\circ(\mathfrak{S}_t\times\mathfrak{S}_{p-t})=\mathfrak{S}_p$ and all the expressions are reduced, we have the formula. □

For $1\le t\le p\le N+1$ and $1\le i_1<i_2<\cdots<i_p\le N+1$, we define

$$\triangle_{t,p-t}(e_{i_1}\wedge\cdots\wedge e_{i_p})=\sum_{w\in\mathfrak{S}_{t,p-t}}(-q)^{-l(w)}e_{i_{w(1)}}\wedge\cdots\wedge e_{i_{w(t)}}\otimes e_{i_{w(t+1)}}\wedge\cdots\wedge e_{i_{w(p)}}.$$

More explicitly,

$$\begin{aligned} &\triangle_{t,p-t}(e_{i_1}\wedge\cdots\wedge e_{i_p}) \\ &= \triangle_{t,p-t}\circ A^{(p)}(e_{i_1}\otimes\cdots\otimes e_{i_p}) \\ &= \sum_{w\in\mathfrak{S}_{t,p-t}}(-q)^{-l(w)}(A^{(t)}\otimes A^{(p-t)})\circ w^{-1}(e_{i_1}\otimes\cdots\otimes e_{i_p}) \\ &= \sum_{w\in\mathfrak{S}_{t,p-t}}(-q)^{-l(w)}((\sum_{\alpha\in\mathfrak{S}_t}(-q)^{-l(\alpha)}\alpha)\otimes(\sum_{\beta\in\mathfrak{S}_{p-t}}(-q)^{-l(\beta)}\beta)) \\ &\quad\circ w^{-1}(e_{i_1}\otimes\cdots\otimes e_{i_p}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{w\in\mathfrak{S}_{t,p-t}}\sum_{\alpha\in\mathfrak{S}_t}\sum_{\beta\in\mathfrak{S}_{p-t}}(-q)^{-l(w)-l(\alpha)-l(\beta)}(\alpha\otimes\beta)\circ w^{-1}(e_{i_1}\otimes\cdots\otimes e_{i_p})\\
&= \sum_{w\in\mathfrak{S}_p}(-q)^{-l(w)}(e_{i_{w^{-1}(1)}}\otimes\cdots e_{i_{w^{-1}(t)}})\otimes(e_{i_{w^{-1}(t+1)}}\otimes\cdots\otimes e_{i_{w^{-1}(p)}}).
\end{aligned}$$

So $\triangle$ is just the deconcatenation coproduct on $S_{-c}(V)$.

If we define a symmetric nondegenerated bilinear form $(,)$ on V by requiring that the basis $\{e_i\}$ is orthonormal and extend it to $T(V)$, then we have:

Proposition 1.16. *$(\bigwedge_c(V), \wedge, \triangle, \epsilon)$ is self-dual with respect to the above bilinear form.*

Proof. For $s+t \le N+1$ and any multi-indices sets $\underline{i} = \{i_1, \ldots, i_s\}$ with $1 \le i_1 < \cdots < i_s \le N+1$, $\underline{j} = \{j_1, \ldots, j_t\}$ with $1 \le j_1 < \cdots < j_t \le N+1$ and $\underline{k} = \{k_1, \ldots, k_{s+t}\}$ with $1 \le k_1 < \cdots < k_{s+t} \le N+1$,

1) if $\underline{i} \cup \underline{j} \neq \underline{k}$, then obviously we have

$$< (e_{i_1}\wedge\cdots\wedge e_{i_s})\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t}), e_{k_1}\wedge\cdots\wedge e_{k_{s+t}} >= 0,$$

and

$$< e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes e_{j_1}\wedge\cdots\wedge e_{j_t}, \triangle_{s,t}(e_{k_1}\wedge\cdots\wedge e_{k_{s+t}}) >= 0.$$

2) if $\underline{i} \cup \underline{j} = \underline{k}$, then

$$\begin{aligned}
&< e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes e_{j_1}\wedge\cdots\wedge e_{j_t}, \triangle_{s,t}(e_{k_1}\wedge\cdots\wedge e_{k_{s+t}}) >\\
&= \sum_{\tau\in S_{s,t}}(-q)^{-l(\tau)} < e_{i_1}\wedge\cdots\wedge e_{i_s}, e_{k_{\tau(1)}}\wedge\cdots\wedge e_{k_{\tau(t)}} >\\
&\quad \times < e_{j_1}\wedge\cdots\wedge e_{j_t}, e_{k_{\tau(t+1)}}\wedge\cdots\wedge e_{k_{\tau(p)}} >\\
&= (-q)^{l(i_1,\ldots,i_s,j_1,\ldots,j_t)}\\
&= < (e_{i_1}\wedge\cdots\wedge e_{i_s})\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t}), e_{k_1}\wedge\cdots\wedge e_{k_{s+t}} > .
\end{aligned}$$

$$< 1, x >= \begin{cases} x, & \text{if } x \in \mathbb{C},\\ 0, & \text{if } x \notin \mathbb{C}. \end{cases}$$

In both cases, we have

$$< 1, x >= \varepsilon(x).$$

□

1.6 The relation between q-traces and quantum traces

In Section 2, we have defined three products on $\oplus_{i\ge 0}\mathrm{End}(\bigwedge_c^i V)$. Now we describe the convolution product more precisely in this case.

For $s < t$,

$$\begin{aligned} E_{ij} * E_{kl}(e_s \wedge e_t) &= sh_{1,1} \circ (E_{ij} \otimes E_{kl}) \circ \triangle_{1,1}(e_s \wedge e_t) \\ &= sh_{1,1} \circ (E_{ij} \otimes E_{kl})(e_s \otimes e_t - q^{-1} e_t \otimes e_s) \\ &= \delta_{js}\delta_{lt} e_i \wedge e_k - q^{-1}\delta_{jt}\delta_{ls} e_i \wedge e_k \\ &= (\delta_{js}\delta_{lt} - q^{-1}\delta_{jt}\delta_{ls}) e_i \wedge e_k. \end{aligned}$$

Similarly, $E_{kl} * E_{ij}(e_s \wedge e_t) = (\delta_{ls}\delta_{jt} - q^{-1}\delta_{lt}\delta_{js}) e_k \wedge e_i$. So we get that

$$\begin{cases} E_{ij} * E_{ik} = E_{ij} * E_{kj} = 0, & \forall i, j, k, \\ E_{kj} * E_{il} = -q^{-1} E_{ij} * E_{kl}, & \text{if} i < k, \forall j, l, \\ E_{il} * E_{kj} = -q^{-1} E_{ij} * E_{kl}, & \text{if} j < l, \forall i, k. \end{cases} \tag{3}$$

Generally, for $1 \le l_1 < \cdots < l_p \le N+1$ and $1 \le i_1 < \cdots < i_p \le N+1$, we have

$$E_{i_1 j_1} * \cdots * E_{i_p j_p}(e_{l_1} \wedge \cdots \wedge e_{l_p}) = \begin{cases} (-q)^{-l(j_1, \cdots, j_p)} e_{i_1} \wedge \cdots \wedge e_{i_p}, & \text{if} \underline{j} = \underline{l}, \\ 0, & \text{otherwise.} \end{cases}$$

So $\{E_{i_1 j_1} * \cdots * E_{i_p j_p} | 1 \le i_1 < \cdots < i_p \le N+1, 1 \le j_1 < \cdots < j_p \le N+1\}$ is a $\mathbb{C}$-basis of $\mathrm{End} \bigwedge_c^p(V)$, which implies that $\oplus_{p=0}^{N+1} \mathrm{End} \bigwedge_c^p(V)$ is a $\mathbb{C}$-algebra generated by $\{E_{ij}\}$.

From easy computations we get that if $1 \le i_1 < \cdots < i_p \le N+1$, $1 \le j_1 < \cdots < j_p \le N+1$, $1 \le k_1 < \cdots < k_p \le N+1$ and $1 \le l_1 < \cdots < l_p \le N+1$, then

$$(E_{i_1 j_1} * \cdots * E_{i_p j_p}) \circ (E_{k_1 l_1} * \cdots * E_{k_p l_p}) = \delta_{j_1 k_1} \cdots \delta_{j_p k_p} E_{i_1 l_1} * \cdots * E_{i_p l_p}.$$

For given $A_m = \sum_{\substack{1 \le i_1 < \cdots < i_m \le N+1 \\ 1 \le j_1 < \cdots < j_m \le N+1}} a^{i_1 \cdots i_m}_{j_1 \cdots j_m} E_{i_1 j_1} * \cdots * E_{i_m j_m} \in \mathrm{End} \bigwedge_c^m(V)$ and $B_n = \sum_{\substack{1 \le k_1 < \cdots < k_n \le N+1 \\ 1 \le l_1 < \cdots < l_n \le N+1}} b^{k_1 \cdots k_n}_{l_1 \cdots l_n} E_{k_1 l_1} * \cdots * E_{k_n l_n} \in \mathrm{End} \bigwedge_c^n(V)$, we have

$$\begin{aligned} A_m * B_n &= \sum_{i,j,k,l} a^{i_1 \cdots i_m}_{j_1 \cdots j_m} b^{k_1 \cdots k_n}_{l_1 \cdots l_n} E_{i_1 j_1} * \cdots * E_{i_m j_m} * E_{k_1 l_1} * \cdots * E_{k_n l_n} \\ &= \sum_{\sigma, \tau \in \mathfrak{S}_{m+n, N+1-m-n}} \sum_{\alpha, \beta \in S_{m,n}} a^{\alpha\sigma(1) \cdots \alpha\sigma(m)}_{\beta\tau(1) \cdots \beta\tau(m)} b^{\alpha\sigma(m+1) \cdots \alpha\sigma(m+n)}_{\beta\tau(m+1) \cdots \beta\tau(m+n)} \\ &\quad E_{\alpha\sigma(1), \beta\tau(1)} * \cdots * E_{\alpha\sigma(m+n), \beta\tau(m+n)} \\ &= \sum_{\sigma, \tau \in \mathfrak{S}_{m+n, N+1-m-n}} \sum_{\alpha, \beta \in S_{m,n}} a^{\alpha\sigma(1) \cdots \alpha\sigma(m)}_{\beta\tau(1) \cdots \beta\tau(m)} b^{\alpha\sigma(m+1) \cdots \alpha\sigma(m+n)}_{\beta\tau(m+1) \cdots \beta\tau(m+n)} \\ &\quad \times (-q)^{-l(\alpha) - l(\beta)} E_{\sigma(1), \tau(1)} * \cdots * E_{\sigma(m+n), \tau(m+n)}. \end{aligned}$$

For $1 \le p \le N+1$, we let $\{F_{i_1 \cdots i_p} | 1 \le i_1 < \cdots < i_p \le N+1\}$ be the dual basis of $\{e_{j_1} \wedge \cdots \wedge e_{j_p} | 1 \le j_1 < \cdots < j_p \le N+1\}$, i.e.,

$$F_{i_1 \cdots i_p}(e_{j_1} \wedge \cdots \wedge e_{j_p}) = \delta_{i_1 j_1} \cdots \delta_{i_p j_p}.$$

We define

$$\begin{array}{rcl}\varphi: \quad {\bigwedge}_{c^\vee}^p(V^*) & \to & ({\bigwedge}_c^p(V))^*, \\ f_{i_1}\wedge\cdots\wedge f_{i_p} & \mapsto & F_{i_1\cdots i_p}.\end{array}$$

It is a linear isomorphism. Then we have the following linear isomorphism:

$$\begin{array}{rcl}\iota_p: \quad \mathrm{End}\,{\bigwedge}_c^p(V) & \to & {\bigwedge}_c^p(V)\otimes{\bigwedge}_{c^\vee}^p(V^*), \\ E_{i_1j_1}*\cdots*E_{i_pj_p} & \mapsto & e_{i_1}\wedge\cdots\wedge e_{i_p}\otimes f_{j_1}\wedge\cdots\wedge f_{j_p}.\end{array}$$

If we endow ${\bigwedge}_c(V)\otimes{\bigwedge}_{c^\vee}(V^*)$ with the tensor algebra structure, then $\oplus_{p=0}^{N+1}{\bigwedge}_c^p(V)\otimes{\bigwedge}_{c^\vee}^p(V^*)$ is a subalgebra.

Proposition 1.17. $\iota=\oplus_{p=0}^{N+1}\iota_p:(\oplus_{p=0}^{N+1}\mathrm{End}\,{\bigwedge}_c^p(V),*)\to\oplus_{p=0}^{N+1}{\bigwedge}_c^p(V)\otimes{\bigwedge}_{c^\vee}^p(V^*)$ *is an isomorphism of algebras.*

Proof. In $\oplus_{p=0}^{N+1}{\bigwedge}_c^p(V)\otimes{\bigwedge}_{c^\vee}^p(V^*)$ we have

$$\begin{cases}(e_i\otimes f_j)(e_i\otimes f_k) &= e_i\wedge e_i\otimes f_j\wedge f_k=0, & \forall i,j,\\ (e_i\otimes f_j)(e_k\otimes f_j) &= e_i\wedge e_k\otimes f_j\wedge f_j=0, & \forall i,j,\\ (e_k\otimes f_j)(e_i\otimes f_l) &= e_k\wedge e_i\otimes f_j\wedge f_l=-q^{-1}e_i\wedge e_k\otimes f_j\wedge f_l & \\ &= -q^{-1}(e_i\otimes f_j)(e_k\otimes f_l), & \text{if} i<k,\forall j,l,\\ (e_i\otimes f_l)(e_k\otimes f_j) &= e_i\wedge e_k\otimes f_l\wedge f_j=-q^{-1}e_i\wedge e_k\otimes f_j\wedge f_l & \\ &= -q^{-1}(e_i\otimes f_j)(e_k\otimes f_l), & \text{if} j<l,\forall i,k.\end{cases}$$

It shares the same multiplication rule in (3). So we get the result since $\oplus_{p=0}^{N+1}{\bigwedge}_c^p(V)\otimes{\bigwedge}_{c^\vee}^p(V^*)$ is generated by $e_i\otimes f_j$ as an algebra. □

The *quantum matrix algebra* of rank $N+1$, denoted by $\mathrm{M}_q(N+1)$, is the algebra generated by $\{x_{ij}|1\le i,j\le N+1\}$ with relations:

$$\begin{cases}x_{jt}x_{it} &= qx_{it}x_{jt}, & \forall i<j,\ \forall t,\\ x_{it}x_{is} &= qx_{is}x_{it}, & \forall s<t,\ \forall i,\\ x_{it}x_{js} &= x_{js}x_{it}, & \forall i<j,\ \forall s<t,\\ x_{jt}x_{is}-x_{is}x_{jt} &= (q-q^{-1})x_{it}x_{js}, & ,\forall i<j,\ \forall s<t.\end{cases} \tag{4}$$

The following properties of quantum matrix algebras is well-known (see [27]):

1. $\mathrm{M}_q(N+1)$ is a bialgebra with coproduct

$$\begin{array}{rcl}\triangle:\mathrm{M}_q(N+1) & \to & \mathrm{M}_q(N+1)\otimes\mathrm{M}_q(N+1),\\ x_{ij} & \mapsto & \sum_{k=1}^{N+1}x_{ik}\otimes xkj,\end{array}$$

and counit

$$\varepsilon:\mathrm{M}_q(N+1) \ \to \ \mathbb{C},$$

$$x_{ij} \mapsto \delta_{ij}.$$

2. For any $k \in \mathbb{N}$, $V^{\otimes k}$ is a right $\mathrm{M}_q(N+1)$-comodule:

$$\begin{array}{rlcl}\delta: & V^{\otimes k} & \to & V^{\otimes k} \otimes \mathrm{M}_q(N+1), \\ & e_{i_1} \otimes \cdots \otimes e_{i_k} & \mapsto & \sum_{1 \leq j_1, \cdots, j_k \leq N+1} e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes x_{j_1 i_1} \cdots e_{j_k i_k}.\end{array}$$

Moreover c is an $\mathrm{M}_q(N+1)$-comodule map. So both $\bigwedge_c(V)$ and $S_c(V)$ inherit $\mathrm{M}_q(N+1)$-comodule structures.

So $\oplus_{i=0}^{N+1}\mathrm{End} \bigwedge_c^p(V)$ is a $\mathrm{M}_q(N+1)$-comodule with the structure map:

$$\begin{array}{rlcl}\overline{\delta}: & \mathrm{End} \bigwedge_c^p(V) & \to & \mathrm{End} \bigwedge_c^p(V) \otimes \mathrm{M}_q(N+1) \\ & E_{i_1 j_1} * \cdots * E_{i_p j_p} & \mapsto & \sum_{k,l} E_{i_1 j_1} * \cdots * E_{i_p j_p} \otimes x_{k_1 i_1} \cdots x_{k_p i_p} x_{l_1 j_1} \cdots x_{l_p j_p}.\end{array}$$

If $A \in \mathrm{End} \bigwedge_c^1(V) = \mathrm{End}(V)$ with $Ae_i = \sum_{j=1}^{N+1} a_i^j e_j$, then

$$\begin{aligned}
&(\alpha A)_{N+1}(e_1 \wedge \cdots \wedge e_{N+1}) \\
&= (A * e_{q^2}^{*\mathtt{I}_1})_{N+1}(e_1 \wedge \cdots \wedge e_n) \\
&= (A * \mathtt{I}_N)(e_1 \wedge \cdots \wedge e_{N+1}) \\
&= sh_{1,N} \circ (A \otimes \mathtt{I}_N) \circ \triangle_{1,N}(e_1 \wedge \cdots \wedge e_{N+1}) \\
&= sh_{1,N} \circ (A \otimes \mathtt{I}_N)(\sum_{i=1}^{N+1} (-q)^{1-i} e_i \otimes e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{N+1}) \\
&= sh_{1,N}(\sum_{i=1}^{N} (-q)^{1-i} Ae_i \otimes e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_{N+1}) \\
&= (\sum_{i=1}^{N+1} q^{-2(i-1)} a_i^i) e_1 \wedge \cdots \wedge e_{N+1}.
\end{aligned}$$

So

$$\mathrm{Tr}_q A = \sum_{i=1}^{N+1} q^{-2(i-1)} a_i^i.$$

Now let us recall the definition of the quantum trace. We know the positive roots of $\mathfrak{sl}_{N+1}(\mathbb{C})$ are

$$\begin{array}{lllll}\alpha_1, & \alpha_1+\alpha_2, & \alpha_1+\alpha_2+\alpha_3, & \ldots, & \alpha_1+\cdots+\alpha_N, \\ \alpha_2, & \alpha_2+\alpha_3, & \alpha_2+\alpha_3+\alpha_4, & \ldots, & \alpha_2+\cdots+\alpha_N, \\ \ldots, & & & & \\ \alpha_N. & & & & \end{array}.$$

The sum of all positive roots is

$$\sum_{i=1}^{N} i(N+1-i)\alpha_i.$$

Set

$$K = K_1^N K_2^{2(N-1)} \cdots K_N^N.$$

For any $A \in \mathrm{End}(V)$, we call

$$\mathrm{tr}_q(A) = \mathrm{Tr}(\rho(K)A)$$

the *quantum trace* of A, where Tr is the usual trace of endomorphism. From direct computation, one gets that if $A \in \mathrm{End}(V)$ with $Ae_i = \sum_{j=1}^{N+1} a_i^j e_j$, then

$$\mathrm{tr}_q(A) = \sum_{i=1}^{N+1} q^{N-2(i-1)} a_i^i.$$

Hence, the q-trace is the quantum trace, just up to some scalar.

Proposition 1.18. *For any $A \in \mathrm{End}(V)$, we have*

$$\mathrm{Tr}_q A = q^{-N}\mathrm{tr}_q(A).$$

In general, the quantum trace of $\mathrm{tr}_q A$ for $A \in \mathrm{End}\bigwedge_c^p(V)$ is defined by:

$$\mathrm{tr}_q A = \mathrm{tr}(\rho^p(K)A),$$

where $\rho^p : \mathcal{U}_q\mathfrak{sl}_{N+1} \to \mathrm{End}\bigwedge_c^p(V)$ is the representation of $\mathcal{U}_q\mathfrak{sl}_{N+1}$ on $\mathrm{End}\bigwedge_c^p(V)$ induced by the fundamental representation ρ.

We have the generalization of the above proposition:

Proposition 1.19. *For any $A \in \mathrm{End}\bigwedge_c^p(V)$, we have*

$$\mathrm{Tr}_q A = q^{-p(N+1-p)}\mathrm{tr}_q A.$$

Proof. For any $A \in \mathrm{End}\bigwedge_c^p(V)$ with $A = \sum_{\substack{1\le i_1<\cdots<i_p\le N+1\\ 1\le j_1<\cdots<j_p\le N+1}} a_{j_1\cdots j_p}^{i_1\cdots i_p} E_{i_1j_1} * \cdots * E_{i_pj_p}$, we have

$$\begin{aligned} &A * \mathrm{I}_{N+1-p} \\ &= \Big(\sum_{\substack{1\le i_1<\cdots<i_p\le N+1\\ 1\le j_1<\cdots<j_p\le N+1}} a_{j_1\cdots j_p}^{i_1\cdots i_p} E_{i_1j_1} * \cdots * E_{i_pj_p}\Big) \end{aligned}$$

$$
\begin{aligned}
&*\Big(\sum_{1\le k_1<\cdots<k_{N+1-p}\le N+1} E_{k_1k_1} * \cdots * E_{k_{N+1-p}k_{N+1-p}}\Big)\\
=&\sum_{\underline{i},\ \underline{j},\ \underline{k}} a^{i_1\cdots i_p}_{j_1\cdots j_p} E_{i_1j_1} * \cdots * E_{i_pj_p} * E_{k_1k_1} * \cdots * E_{k_{N+1-p}k_{N+1-p}}\\
=&\sum_{\sigma\in S_{p,\ N+1-p}} a^{\sigma(1)\cdots\sigma(p)}_{\sigma(1)\cdots\sigma(p)} E_{\sigma(1)\sigma(1)} * \cdots * E_{\sigma(N+1)\sigma(N+1)}\\
=&\sum_{\sigma\in S_{p,\ N+1-p}} (-q)^{-2l(\sigma)} a^{\sigma(1)\cdots\sigma(p)}_{\sigma(1)\cdots\sigma(p)} E_{11} * \cdots * E_{N+1,N+1}.
\end{aligned}
$$

So we get

$$
\mathrm{Tr}_q A = \sum_{\sigma\in\mathfrak{S}_{p,\ N+1-p}} (-q)^{-2l(\sigma)} a^{\sigma(1)\cdots\sigma(p)}_{\sigma(1)\cdots\sigma(p)}.
$$

For $1 \le j_1 < \cdots < j_p \le N+1$, we have

$$
\begin{aligned}
&\rho^p(K)A(e_{j_1}\wedge\cdots\wedge e_{j_p})\\
&= \rho^p(K)\Big(\sum_{1\le i_1<\cdots<i_p\le N+1} a^{i_1\cdots i_p}_{j_1\cdots j_p} e_{i_1}\wedge\cdots\wedge e_{i_p}\Big)\\
&= \sum_{1\le i_1<\cdots<i_p\le N+1} a^{i_1\cdots i_p}_{j_1\cdots j_p} Ke_{i_1}\wedge\cdots\wedge Ke_{i_p}\\
&= \sum_{1\le i_1<\cdots<i_p\le N+1} a^{i_1\cdots i_p}_{j_1\cdots j_p} q^{p(N+2)-2(i_1+\cdots+i_p)} e_{i_1}\wedge\cdots\wedge e_{i_p},
\end{aligned}
$$

since $K = \mathrm{diag}(q^N, q^{N-2}, \cdots, q^{-N})$. So

$$
\mathrm{tr}_q A = \sum_{1\le i_1<\cdots<i_p\le N+1} q^{p(N+2)-2(i_1+\cdots+i_p)} a^{i_1\cdots i_p}_{i_1\cdots i_p}.
$$

For an easy observation, we know that for any $\sigma \in S_{p,N+1-p}$ with $\sigma(1) = i_1, \cdots, \sigma(p) = i_p$ we have $l(\sigma) = (i_1 - 1) + \cdots + (i_p - p) = (i_1 + \cdots + i_p) - \frac{(1+p)p}{2}$. Hence the proposition holds. □

Proposition 1.20. *For any $A \in \mathrm{End}(V)$ with $Ae_i = \sum_{j=1}^{N+1} a^j_i e_j$, and $0 \le p \le N+1$, we have*

$$
\mathrm{Tr}_q A^p = \sum_{i=1}^{n} q^{-2(i-1)} \sum_{j_1,\cdots,j_p} a^i_{j_1} a^{j_1}_{j_2} \cdots a^{j_p}_i,
$$

and

$$
\mathrm{Tr}_q A^{*p} = \sum_{\sigma,\tau\in\mathfrak{S}_p} \sum_{w\in S_{p,N+1-p}} (-q)^{-(2l(w)+l(\sigma)+l(\tau))} a^{\tau w(1)}_{\sigma w(1)} \cdots a^{\tau w(p)}_{\sigma w(p)}.
$$

In particular,

$$\mathrm{Tr}_q A^{*N+1} = \sum_{\sigma,\tau\in\mathfrak{S}_{N+1}} (-q)^{-l(\sigma)-l(\tau)} a_{\sigma(1)}^{\tau(1)} \cdots a_{\sigma(N+1)}^{\tau(N+1)}.$$

Proof. The first formula is trivial.

$$\begin{aligned}
&(\alpha A^{*p})_{N+1} \\
&= A^{*p} * \mathrm{I}_{N+1-p} \\
&= (\sum_{i,j} a_i^j E_{ji})^{*p} * (\sum_{w\in\mathfrak{S}_{p,N+1-p}} E_{w(p+1)w(p+1)} * \cdots * E_{w(N+1)w(N+1)}) \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_p} \sum_{w\in\mathfrak{S}_{N+1-p,p}} a_{\sigma w(1)}^{\tau w(1)} \cdots a_{\sigma w(p)}^{\tau w(p)} E_{\tau w(1),\sigma w(1)} * \cdots * E_{\tau w(p),\sigma w(p)} \\
&\quad *(E_{w(1)w(1)} * \cdots * E_{w(N+1-p)w(N+1-p)}) \\
&= \sum_{\sigma,\tau\in\mathfrak{S}_p} \sum_{w\in\mathfrak{S}_{p,N+1-p}} (-q)^{-(2l(w)+l(\sigma)+l(\tau))} a_{\sigma w(1)}^{\tau w(1)} \cdots a_{\sigma w(p)}^{\tau w(p)} \\
&\quad \times E_{11} * \cdots * E_{N+1,N+1}.
\end{aligned}$$

□

Specially, if $A \in \mathrm{End}V$ with $Ae_i = a_i^i e_i$, then

1) $\mathrm{Tr}_q A^{*N+1} = (N+1)_{q^{-2}}! a_1^1 \cdots a_{N+1}^{N+1}$,

2) $\mathrm{Tr}_q A^{*p} = (p)_{q^{-2}}! \sum_{w\in\mathfrak{S}_{p,N+1-p}} (-q)^{-2l(w)} a_{w(1)}^{w(1)} \cdots a_{w(p)}^{w(p)}$,

3) $\mathrm{Tr}_q A^p = \sum_{i=0}^{N+1} (-q)^{-2(i-1)} (a_i^i)^p$.

Let $\mathbb{C} = V(1) \subset V(2) \subset \cdots \subset V(i) \subset \cdots$ be a sequence of vector spaces with $V(i) = Span_{\mathbb{C}}\{e_1, \ldots, e_i\}$. We still use c to denote the action of c restricted on $V(i)$ for all i. For any $1 \leq p \leq N$, we define

$$\begin{aligned}
(\mathrm{Tr}_q)_{p+1} :\ \mathrm{End}\textstyle\bigwedge_c^{p+1}(V_{N+1}) &\to \mathrm{End}\textstyle\bigwedge_c^{p}(V_N), \\
E_{i_1 j_1} * \cdots * E_{i_{p+1} j_{p+1}} &\mapsto E_{i_1 j_1} * \cdots * E_{i_p j_p} \mathrm{Tr}_q E_{i_{p+1} j_{p+1}},
\end{aligned}$$

where $1 \leq i_1 < \cdots < i_{p+1} \leq N+1$ and $1 \leq j_1 < \cdots < j_{p+1} \leq N+1$.

Proposition 1.21. *For any $A \in \mathrm{End}\bigwedge_c^p(V)$, we have*

$$\mathrm{Tr}_q A = (-q)^{p(p-1)} (\mathrm{Tr}_q)_1 (\mathrm{Tr}_q)_2 \cdots (\mathrm{Tr}_q)_p A.$$

Proof. We set $A = \sum_{\substack{1\leq i_1<\cdots<i_p\leq N+1 \\ 1\leq j_1<\cdots<j_p\leq N+1}} a_{j_1\cdots j_p}^{i_1\cdots i_p} E_{i_1 j_1} * \cdots * E_{i_p j_p}$. Then

$$(\mathrm{Tr}_q)_1 (\mathrm{Tr}_q)_2 \cdots (\mathrm{Tr}_q)_p A$$

$$
\begin{aligned}
&= \sum_{\substack{1\le i_1<\cdots<i_p\le N+1\\ 1\le j_1<\cdots<j_p\le N+1}} a^{i_1\cdots i_p}_{j_1\cdots j_p}\mathrm{Tr}_q E_{i_1j_1}\cdot\cdots\cdot \mathrm{Tr}_q E_{i_pj_p}\\
&= \sum_{\substack{1\le i_1<\cdots<i_p\le N+1\\ 1\le j_1<\cdots<j_p\le N+1}} a^{i_1\cdots i_p}_{j_1\cdots j_p}\delta_{i_1j_1}(-q)^{-2(i_1-1)}\cdot\cdots\cdot \delta_{i_pj_p}(-q)^{-2(i_1-1)}\\
&= \sum_{1\le i_1<\cdots<i_p\le N+1} a^{i_1\cdots i_p}_{i_1\cdots i_p}(-q)^{-2(i_1+\cdots+i_p-p)}\\
&= (-q)^{p(1-p)}\mathrm{Tr}_q A.
\end{aligned}
$$

□

1.7 Quantum forms and quantum determinants

Set

$$
q_{ij} = \begin{cases} 0, & i=j,\\ -q, & i<j,\\ -q^{-1}, & i>j. \end{cases}
$$

Then $e_i\wedge e_j = q_{ij}e_j\wedge e_i$ and $q_{ij}q_{ji}=1$ if $i\neq j$.

Definition 1.22. *The elements of* $A^p_c(V) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge^p_c(V),\mathbb{C})$ *are called* quantum p-forms. *Sometimes we denote* $A^p_c = A^p_c(V)$ *simply and* $A_c = \bigoplus_{p\ge 0} A^p_c$.

For any $\omega\in A^p_c$, we could view it as a map

$$
\underbrace{V\times\cdots\times V}_{\text{p times}} \to \mathbb{C}
$$

with the following properties: for any $w\in\mathfrak{S}_p$,

$$
\omega(e_{i_1},\cdots,e_{i_p}) = (-q)^{-l(i_1,\cdots,i_p)+l(i_{w(1)},\cdots,i_{w(p)})}\omega(e_{i_{w(1)}},\cdots,e_{i_{w(p)}}).
$$

Specially, for any $1\le k\neq l\le p$,

$$
\begin{aligned}
&\omega(e_{i_1},\cdots,e_{i_k},\cdots,e_{i_l},\cdots,e_{i_p})\\
&\quad = q_{i_{k-1}i_k}\cdots q_{i_ji_k}q_{i_ji_{j+1}}\cdots q_{i_ji_{k-1}}\omega(e_{i_1},\cdots,e_{i_l},\cdots,e_{i_k},\cdots,e_{i_p}).
\end{aligned}
$$

As an algebra, A_c is generated by $A^1_c = V^*$ with respect to the convolution product: for any $\omega_1\in A^i_c$, $\omega\in A^j_c$ and any $1\le k_1<\cdots<k_{i+j}\le N+1$,

$$
\begin{aligned}
&\omega_1 * \omega_2(e_{k_1}\wedge\cdots\wedge e_{k_{i+j}})\\
&\quad = \cdot\circ(\omega_1\otimes\omega_2)\circ\triangle_{i,j}(e_{k_1}\wedge\cdots\wedge e_{k_{i+j}})
\end{aligned}
$$

$$= \sum_{w\in\mathfrak{S}_{i,j}} (-q)^{-l(w)}\omega_1(e_{k_{w(1)}}\wedge\cdots\wedge e_{k_{w(i)}})\omega_2(e_{k_{w(i+1)}}\wedge\cdots\wedge e_{k_{w(i+j)}}).$$

For any matrix $A = (a^i_j) \in \mathrm{M}_n(\mathbb{C})$, we denote

$$\det{}_q A = \sum_{w\in\mathfrak{S}_n} (-q)^{-l(w)} a^1_{w(1)}\cdots a^n_{w(n)},$$

and call it the *quantum determinant* of A.

It is easy to see that:

Proposition 1.23. *For any $\omega_1, \cdots, \omega_p \in A^1_c$ with $\omega_i(e_j) = a^i_j$ and any $1 \le k_1 < \cdots < k_p \le N+1$, we have*

$$\begin{aligned}(\omega_1 * \cdots * \omega_p)(e_{k_1}\wedge\cdots\wedge e_{k_p}) &= \sum_{w\in\mathfrak{S}_p} (-q)^{-l(w)} a^1_{k_{w(1)}}\cdots a^p_{k_{w(p)}} \\ &= \det{}_q(a^i_{k_j}).\end{aligned}$$

Chapter 2

Constructions of YB algebras and YB coalgebras

2.1 Introduction

For any algebra (resp. coalgebra) A, there is a natural algebra (resp. coalgebra) structure on $A^{\otimes n}$ defined by using the multiplication (resp. comultiplication) and the usual flip. Hashimoto and Hayashi [9] showed that if a Yang-Baxter operator is compatible with the multiplication (resp. comultiplication) in some sense, then one can also provide a new algebra (resp. coalgebra) structure on $A^{\otimes n}$ by using the Yang-Baxter operator instead of the flip. They called the algebra (resp. coalgebra) with the compatible Yang-Baxter operator a Yang-Baxter algebra (resp. Yang-Baxter coalgebra). These algebras and coalgebras play an important role in braided categories. For example, Baez [2] used them to study the Hochschild homology. But there are few examples, and the question is how to provide some. We provide a first series of examples as follows. For any Hopf algebra H, Woronowicz [30] constructed two Yang-Baxter operators T_H and T'_H on H. By direct computation one can show that these give Yang-Baxter algebra and Yang-Baxter coalgebra structures on H. So one may wonder whether there is a general machinery behind this. One may observe that Woronowicz's braidings come from some special Yetter-Drinfel'd module structures on H. Moreover the multiplication and comultiplication of H are compatible with the Yetter-Drinfel'd module structures in some sense. This provides a systematic way to construct Yang-Baxter algebras and Yang-Baxter coalgebras in the category of Yetter-Drinfel'd modules.

We also observe that the quantum shuffle algebras discussed in [26] are certainly interesting such examples. They are the quantization of the usual shuffle algebras on $T(V)$. They are obtained by replacing the flip by the braiding on V to construct the multiplication. This leads to consider the following question: what are the possible Yang-Baxter algebra structures on the tensor space $T(V)$ of a braided vector space compatible with the natural braiding on $T(V)$.

In [19], Loday and Ronco proved a classification theorem for connected cofree bialgebras which are the analogues of the Poincaré-Birkhoff-Witt theorem and of the Cartier-Milnor-Moore theorem for non-cocommutative Hopf algebras. The main tool used is the notion of B_∞-algebra. This enables one to investigate all associative algebra structures on $T(V)$ compatible with the deconcatenation coproduct. The point is that $T(V)$ is a connected coalgebra in the sense of Quillen [24]. So by using the universal property of $T(V)$ with respect to the connected coalgebra structure, the product can be rebuilt from the data of some linear maps $M_{pq} : V^{\otimes p} \otimes V^{\otimes q} \to V$ for $p, q \geq 0$. Conversely, one can construct associative algebra structure for such given maps under some conditions. Furthermore, with this algebra structure and the deconcatenation coproduct, $T(V)$ becomes a bialgebra. We extend this to the braided framework, where we use the "quantized" coproduct instead of the tensor deconcatenation coproduct of $T(V) \otimes T(V)$. $T(V)$ will become a "twisted" Hopf algebra in the sense of [26]. The underlying associative algebra structure is provided by the quantum B_∞-algebra structure. This new object is not just the generalization of B_∞-algebras, but also of Yang-Baxter algebras.

Works on multiple zeta values led naturally to quasi-shuffle algebras. Mainly, the underlying vector space used to construct the shuffle algebra has also an algebra structure. These algebras were already studied by [20], and there were some attempts to quantize them, for examples, [1] and [11]. The quantum B_∞-algebras provide a good framework to "deform" quasi-shuffle algebras in the spirit of quantum shuffle algebras, where the usual flip is replaced by a braiding and we have to impose compatibility between the braiding and the algebra structure on V. Then Yang-Baxter algebras or Yang-Baxter coalgebras appear to be the relevant structures.

This chapter is organized as follows. In Section 2, we show that an algebra (resp. coalgebra) with compatible Yetter-Drinfel'd module structure is a Yang-Baxter algebra (resp. coalgebra). Specially, modules over a quasi-triangular Hopf algebra make sense. And we use Woronowicz's braidings to illustrate our constructions. Section 3 contains the interesting example of Yang-Baxter algebra, which is the so-called quantum shuffle algebra (introduced in [26]). We also prove that the cotensor algebra $T_H^c(M)$ over a Hopf algebra H and a H-Hopf bimodule M is both a Yang-Baxter algebra and a Yang-Baxter coalgebra. As a consequence, the "upper triangular part" U_q^+ of the quantized enveloping algebra with a symmetrizable Cartan matrix is a Yang-Baxter algebra. In Section 4, we define quantum $\mathbf{B}_\infty$-algebra and prove that its tensor space has a Yang-Baxter algebra structure. Quantum shuffle algebras and quantum quasi-shuffle algebras are special quantum $\mathbf{B}_\infty$-algebras. Finally, in Section 5, we introduce the notion of 2-YB algebras and use them to construct quantum $\mathbf{B}_\infty$-algebras.

2.2 Machineries arising from Yetter-Drinfel'd Modules

We will first recall the definitions of Yang-Baxter algebra and Yang-Baxter coalgebra which were introduced in [9].

Definition 2.1. *1. Let $A = (A, m, \eta)$ be an algebra with product m and unit η. Let σ be a braiding on A. We call (A, σ) a* Yang-Baxter algebra, *or short for YB algebra, if the following diagram is commutative:*

$$\begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{\sigma_1\sigma_2} & A^{\otimes 3} & \xrightarrow{\sigma_2\sigma_1} & A^{\otimes 3} \\ \downarrow{\scriptstyle m\otimes \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_A\otimes m} & & \downarrow{\scriptstyle m\otimes \mathrm{id}_A} \\ A^{\otimes 2} & \xrightarrow{\sigma} & A^{\otimes 2} & \xrightarrow{\sigma} & A^{\otimes 2} \\ \uparrow{\scriptstyle \eta\otimes \mathrm{id}_A} & & \uparrow{\scriptstyle \mathrm{id}_A\otimes \eta} & & \uparrow{\scriptstyle \eta\otimes \mathrm{id}_A} \\ K\otimes A & \xrightarrow{\simeq} & A\otimes K & \xrightarrow{\simeq} & K\otimes A. \end{array}$$

2. Let $C = (C, \triangle, \varepsilon)$ be a coalgebra with coproduct $\triangle$ and counit ε. Let σ be a braiding on C. We call (C, σ) a Yang-Baxter coalgebra, *or short for YB coalgebra, if the following diagram is commutative:*

$$\begin{array}{ccccc} C^{\otimes 3} & \xrightarrow{\sigma_1\sigma_2} & C^{\otimes 3} & \xrightarrow{\sigma_2\sigma_1} & C^{\otimes 3} \\ \uparrow{\scriptstyle \triangle\otimes \mathrm{id}_C} & & \uparrow{\scriptstyle \mathrm{id}_C\otimes \triangle} & & \uparrow{\scriptstyle \triangle\otimes \mathrm{id}_C} \\ C^{\otimes 2} & \xrightarrow{\sigma} & C^{\otimes 2} & \xrightarrow{\sigma} & C^{\otimes 2} \\ \downarrow{\scriptstyle \varepsilon\otimes \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}_C\otimes \varepsilon} & & \downarrow{\scriptstyle \varepsilon\otimes \mathrm{id}_C} \\ K\otimes C & \xrightarrow{\simeq} & C\otimes K & \xrightarrow{\simeq} & K\otimes C. \end{array}$$

These definitions give a right way to extend the usual algebra (resp. coalgebra) structure on the tensor products of algebras (resp. coalgebras).

Proposition 2.2 ([9], Proposition 4.2). *1. For a YB algebra (A, σ) and any $i \in \mathbb{N}$, the YB pair $(A^{\otimes i}, T^{\sigma}_{\chi_{ii}})$ becomes a YB algebra with product $m_{\sigma,i} = m^{\otimes i} \circ T^{\sigma}_{w_i}$ and unit $\eta^{\otimes i} : K \simeq K^{\otimes i} \to A^{\otimes i}$, where $\chi_{ii}, w_i \in \mathfrak{S}_{2i}$ are given by*

$$\chi_{ii} = \left(\begin{array}{cccccccc} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & 2i \\ i+1 & i+2 & \cdots & 2i & 1 & 2 & \cdots & i \end{array} \right),$$

and

$$w_i = \left(\begin{array}{ccccccccc} 1 & 2 & 3 & \cdots & i & i+1 & i+2 & \cdots & 2i \\ 1 & 3 & 5 & \cdots & 2i-1 & 2 & 4 & \cdots & 2i \end{array} \right).$$

2. For a YB coalgebra (C, σ), the YB pair $(C^{\otimes i}, T^{\sigma}_{\chi_{ii}})$ becomes a YB coalgebra with coproduct $\triangle_{\sigma,i} = T^{\sigma}_{w_i^{-1}} \circ \triangle^{\otimes i}$ and counit $\varepsilon^{\otimes i} : C^{\otimes i} \to K^{\otimes i} \simeq K$.

Remark 2.3. *1. Any algebra (resp. coalgebra) is a YB algebra (resp. coalgebra) with the usual flip.*

2. Obviously, if (A, σ) is a YB algebra, then so is (A, σ^{-1}). And if (C, σ) is a YB coalgebra, then so is (C, σ^{-1}).

3. Assume there is a nondegenerate bilinear form $(,)$ between two vector spaces A and B. It can be extended to $(,) : A^{\otimes i} \times B^{\otimes i} \to K$ in the usual way. For any $f \in \mathrm{End}(A^{\otimes i})$, the adjoint operator $adj(f) \in \mathrm{End}(B^{\otimes i})$ of f is defined to be the one such that $(x, adj(f)(y)) = (f(x), y)$ for any $x \in A^{\otimes i}$ and $y \in B^{\otimes i}$. If (A, m, η, σ) is a YB algebra, then its adjoint $(B, \mathrm{adj}(m), \mathrm{adj}(\eta), \mathrm{adj}(\sigma))$ is a YB coalgebra.

The YB algebra and YB coalgebra structures given by Remark 2.3.1 are trivial. We will give non trivial examples by using braided vector spaces as follows.

Let (V, σ) be a braided vector space. For any $i, j \geq 1$, we denote

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix},$$

and define $\beta : T(V)\underline{\otimes}T(V) \to T(V)\underline{\otimes}T(V)$ by requiring that $\beta_{ij} = T^{\sigma}_{\chi_{ij}}$ on $V^{\otimes i}\underline{\otimes}V^{\otimes j}$. For convenience, we denote by β_{0i} and β_{i0} the usual flip maps.

It is easy to see that β is a braiding on $T(V)$ and $(T(V), m, \beta)$ is a YB algebra, where m is the concatenation product. And $(T(V), m, \beta)$ has a sort of universal property in the category of YB algebras.

Definition 2.4. *Let (A_1, α_1) and (A_2, α_2) be two YB algebras. A linear map $f : A_1 \to A_2$ is called a* morphism of YB algebras *if*

1. f is an algebra map, i.e., $f(ab) = f(a)f(b)$ for any $a, b \in A_1$,

2. $(f \otimes f)\alpha_1 = \alpha_2(f \otimes f)$, i.e., f is a morphism of braided vector spaces.

Definition 2.5. *Let (V, σ) be braided vector space. A* free YB algebra *over V is a YB algebra $(F_{\mathcal{YB}}(V), \overline{\sigma})$ with the following universal property:*

1. there is an injective map $i : V \to F_{\mathcal{YB}}(V)$ such that $(i \otimes i)\sigma = \overline{\sigma}(i \otimes i)$,

2. for any YB algebra (A, α) and linear map $\varphi : V \to A$ such that $(\varphi\otimes\varphi)\sigma = \alpha(\varphi\otimes\varphi)$, there is a unique morphism of YB algebras $\overline{\varphi} : F_{\mathcal{YB}}(V) \to A$ such that $\overline{\varphi} \circ i = \varphi$.

Proposition 2.6. *The free YB algebra exists and it is unique up to isomorphism.*

Proof. Let $\varphi : V \to A$ be a linear map such that $(\varphi \otimes \varphi)\sigma = \alpha(\varphi \otimes \varphi)$. By the universal property of tensor algebra, there is a unique algebra map $\overline{\varphi} : T(V) \to A$ which extends φ. Let $i : V \to T(V)$ be the inclusion map. Consider the YB algebra $(T(V), m, \beta)$ where m is the concatenation product. Then $(T(V), i, \beta)$ is a free YB algebra over V. We only need to check that $(\overline{\varphi} \otimes \overline{\varphi})\beta_{kl} = \alpha(\overline{\varphi} \otimes \overline{\varphi})$ on $V^{\otimes k}\underline{\otimes}V^{\otimes l}$. We use induction on $k + l$ for $k, l \geq 1$.

When $i=j=1$, $(\overline{\varphi}\otimes\overline{\varphi})\beta_{11}=(\varphi\otimes\varphi)\sigma=\alpha(\varphi\otimes\varphi)=\alpha(\overline{\varphi}\otimes\overline{\varphi})$.

$$\begin{array}{ll}\alpha(\overline{\varphi}\otimes\overline{\varphi})(u_1\otimes\cdots\otimes u_k\underline{\otimes}v_1\otimes\cdots\otimes v_l) & \\ = & \alpha\Big(\varphi(u_1)\cdots\varphi(u_k)\underline{\otimes}\varphi(v_1)\cdots\varphi(v_l)\Big)\\ = & \alpha(\mathrm{id}_A\otimes m_A)\Big(\varphi(u_1)\cdots\varphi(u_k)\underline{\otimes}\varphi(v_1)\cdots\varphi(v_{l-1})\underline{\otimes}\varphi(v_l)\Big)\\ = & (m_A\otimes\mathrm{id}_A)\alpha_2\alpha_1\Big(\varphi(u_1)\cdots\varphi(u_k)\underline{\otimes}\varphi(v_1)\cdots\varphi(v_{l-1})\underline{\otimes}\varphi(v_l)\Big)\\ = & (m_A\otimes\mathrm{id}_A)\alpha_2\Big((\overline{\varphi}\otimes\overline{\varphi})\beta_{k,l-1}\otimes\overline{\varphi}\Big)(u_1\otimes\cdots v_l)\\ = & (m_A\otimes\mathrm{id}_A)\big(\overline{\varphi}\otimes\alpha(\overline{\varphi}\otimes\overline{\varphi})\big)(\beta_{k,l-1}\otimes\mathrm{id}_V)(u_1\otimes\cdots v_l)\\ = & (m_A\otimes\mathrm{id}_A)(\overline{\varphi}\otimes\overline{\varphi}\otimes\overline{\varphi})(\mathrm{id}_V^{\otimes l-1}\otimes\beta_{k1})(\beta_{k,l-1}\otimes\mathrm{id}_V)(u_1\otimes\cdots v_l)\\ = & (\overline{\varphi}\otimes\overline{\varphi})\beta_{kl}(u_1\otimes\cdots v_l).\end{array}$$

□

The following property above YB algebras is useful for the further discussion.

Lemma 2.7. *Let $(V,*,\sigma)$ be a YB algebra. Then for any $k,l\geq 1$, we have*

$$\begin{cases}\beta_{1l}(*^k\otimes\mathrm{id}_V^{\otimes l}) &=& (\mathrm{id}_V^{\otimes l}\otimes *^k)\beta_{k+1,l},\\ \beta_{l1}(\mathrm{id}_V^{\otimes l}\otimes *^k) &=& (*^k\otimes\mathrm{id}_V^{\otimes l})\beta_{l,k+1},\end{cases}$$

where $*^k=*^{\otimes k}:V^{\otimes k+1}\to V$ given by $v_1\otimes\cdots\otimes v_{k+1}\mapsto v_1*\cdots*v_{k+1}$.

Proof. We use induction on $k+l$. We prove the first equality. The second one can be proved similarly.

When $k=l=1$, it is just the condition of YB algebra.

$$\begin{array}{rcl}\beta_{1l}(*^k\otimes\mathrm{id}_V^{\otimes l}) &=& \beta_{1l}(*\otimes\mathrm{id}_V^{\otimes l})(\mathrm{id}_V\otimes *^{k-1}\otimes\mathrm{id}_V^{\otimes l})\\ &=& (\mathrm{id}_V^{\otimes l}\otimes *)\beta_{2l}(\mathrm{id}_V\otimes *^{k-1}\otimes\mathrm{id}_V^{\otimes l})\\ &=& (\mathrm{id}_V^{\otimes l}\otimes *)(\beta_{1l}\otimes\mathrm{id}_V)(\mathrm{id}_V\otimes\beta_{1l})(\mathrm{id}_V\otimes *^{k-1}\otimes\mathrm{id}_V^{\otimes l})\\ &=& (\mathrm{id}_V^{\otimes l}\otimes *)(\beta_{1l}\otimes\mathrm{id}_V)(\mathrm{id}_V^{\otimes l+1}\otimes *^{k-1})(\mathrm{id}_V\otimes\beta_{kl})\\ &=& (\mathrm{id}_V^{\otimes l}\otimes *)(\mathrm{id}_V^{\otimes l+1}\otimes *^{k-1})(\beta_{1l}\otimes\mathrm{id}_V)(\mathrm{id}_V\otimes\beta_{kl})\\ &=& (\mathrm{id}_V^{\otimes l}\otimes *^k)\beta_{k+1,l}.\end{array}$$

□

We define δ to be the deconcatenation on $T(V)$, i.e.,

$$\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \underline{\otimes} (v_{i+1} \otimes \cdots \otimes v_n).$$

We denoted by $T^c(V)$ the coalgebra $(T(V), \delta)$.

$T^c(V)$ is the dual construction of $(T(V), m)$. So $(T^c(V), \beta)$ is a YB coalgebra. It is valuable to present the following short proof of this statement, from which we can see how it works more clearly.

Since

$$\begin{cases} \chi_{i+j,k} &= (\chi_{ik} \times 1_{\mathfrak{G}_j})(1_{\mathfrak{G}_i} \times \chi_{jk}), \\ \chi_{i+j,k} &= (1_{\mathfrak{G}_j} \times \chi_{ik})(\chi_{ij} \times 1_{\mathfrak{G}_k}), \end{cases}$$

and all the expressions are reduced, we have

$$\begin{cases} \beta_{i+j,k} &= (\beta_{ik} \otimes \mathrm{id}_V^{\otimes j})(\mathrm{id}_V^{\otimes i} \otimes \beta_{jk}), \\ \beta_{i+j,k} &= (\mathrm{id}_V^{\otimes j} \otimes \beta_{ik})(\beta_{ij} \otimes \mathrm{id}_V^{\otimes k}). \end{cases}$$

On $V^{\otimes k}$, we have $\delta = \oplus_{i+j=k} \delta_{ij}$ with $\delta_{ij}(v_1 \otimes \cdots \otimes v_{i+j}) = v_1 \otimes \cdots \otimes v_i \underline{\otimes} v_{i+1} \otimes \cdots \otimes v_{i+j}$. We identify $\delta_{ij} = \mathrm{id}_V^{\otimes i} \underline{\otimes} \mathrm{id}_V^{\otimes j}$. So on $V^{\otimes i+j} \underline{\otimes} V^{\otimes k}$, we have

$$\begin{aligned} (\beta_{ik} \otimes \mathrm{id}_V^{\otimes j})(\mathrm{id}_V^{\otimes i} \otimes \beta_{jk})(\delta_{ij} \otimes \mathrm{id}_V^{\otimes k}) &= \beta_{i+j,k}(\mathrm{id}_V^{\otimes i} \underline{\otimes} \mathrm{id}_V^{\otimes j} \otimes \mathrm{id}_V^{\otimes k}) \\ &= (\mathrm{id}_V^{\otimes k} \otimes \mathrm{id}_V^{\otimes i} \underline{\otimes} \mathrm{id}_V^{\otimes j})\beta_{i+j,k} \\ &= (\mathrm{id}_V^{\otimes k} \otimes \delta_{ij})\beta_{i+j,k}. \end{aligned}$$

The other compatibility condition for δ and β can be proved similarly, and other conditions for the YB coalgebra follow from the definitions of ε and β.

Before giving general constructions of YB algebras and YB coalgebras, we recall some terminologies first.

Definition 2.8. *Let H be a Hopf algebra. A triple $(V, \cdot, \rho)$ is called a* (left) Yetter-Drinfel'd module *over H if*

1. *$(V, \cdot)$ is a left H-module,*
2. *(V, ρ) is a left H-comodule,*
3. *for any $h \in H$ and $v \in V$,*

$$\sum h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = \sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)}.$$

For any Yetter-Drinfel'd module V, there is a natural braiding $\sigma_V : V \otimes V \to V \otimes V$ defined by $\sigma_V(v \otimes w) = \sum v_{(-1)} \cdot w \otimes v_{(0)}$. In the following, we will always adopt this σ_V as the braiding of a Yetter-Drinfel'd module.

Definition 2.9. *Let $(H, \triangle, \varepsilon)$ be a Hopf algebra. An algebra A is called a* module-algebra *over H if*

1. *A is an H-module, and*
2. *for any $h \in H$ and $a, b \in A$,*

$$\begin{aligned} h \cdot (ab) &= \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b), \\ h \cdot 1 &= \varepsilon(h)1. \end{aligned}$$

Definition 2.10. *Let H be a Hopf algebra. An algebra A is called a* comodule-algebra *over H if*

1. *A is an H-comodule with structure map $\rho : A \to H \otimes A$,*
2. *ρ is an algebra morphism, i.e., for any $a, b \in A$,*

$$\sum_{(ab)} (ab)_{(-1)} \otimes (ab)_{(0)} = \sum_{(a),(b)} a_{(-1)} b_{(-1)} \otimes a_{(0)} b_{(0)},$$

$$\rho(1_A) = 1_H \otimes 1_A.$$

Theorem 2.11. *Let $(V, \cdot, \rho)$ be a Yetter-Drinfel'd module over H. If V is both a comodule-algebra and module-algebra, then (V, σ_V) is a YB algebra.*

Proof. For any $x, y, z \in V$, we have

$$\begin{aligned} &(\mathrm{id}_V \otimes m)(\sigma_V \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma_V)(x \otimes y \otimes z) \\ &= (\mathrm{id}_V \otimes m)(\sigma_V \otimes \mathrm{id}_V)(\sum_{(y)} x \otimes y_{(-1)} \cdot z \otimes y_{(0)}) \\ &= (\mathrm{id}_V \otimes m)(\sum_{(x),(y)} x_{(-1)} \cdot (y_{(-1)} \cdot z) \otimes x_{(0)} \otimes y_{(0)}) \\ &= \sum_{(x),(y)} (x_{(-1)} y_{(-1)}) \cdot z \otimes x_{(0)} y_{(0)} \\ &= \sum_{(xy)} (xy)_{(-1)} \cdot z \otimes (xy)_{(0)} \\ &= \sigma_V(m \otimes \mathrm{id}_V)(x \otimes y \otimes z). \end{aligned}$$

Also,

$$\begin{aligned} &(m \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma_V)(\sigma_V \otimes \mathrm{id}_V)(x \otimes y \otimes z) \\ &= (m \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \sigma_V)(\sum_{(x)} x_{(-1)} \cdot y \otimes x_{(0)} \otimes z) \end{aligned}$$

$$\begin{aligned}
&= (m \otimes \mathrm{id}_V)(\sum_{(x)} x_{(-2)} \cdot y \otimes x_{(-1)} \cdot z \otimes x_{(0)} \\
&= \sum_{(x)} (x_{(-2)} \cdot y)(x_{(-1)} \cdot z) \otimes x_{(0)} \\
&= \sum_{(x)} x_{(-1)} \cdot (yz) \otimes x_{(0)} \\
&= \sigma_V(\mathrm{id}_V \otimes m)(x \otimes y \otimes z).
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_V(x \otimes 1_V) &= \sum_{(x)} x_{(-1)} \cdot 1_V \otimes x_{(0)} \\
&= \sum_{(x)} \varepsilon(x_{(-1)}) 1_M \otimes x_{(0)} \\
&= 1_V \otimes x,
\end{aligned}$$

and,

$$\begin{aligned}
\sigma_V(1_V \otimes x) &= 1_H \cdot x \otimes 1_V \\
&= x \otimes 1_V.
\end{aligned}$$

□

Corollary 2.12. *Under the above assumption, $V^{\otimes 2}$ is an associative algebra with the product: for any $w, x, y, z \in V$,*

$$(w \otimes x)(y \otimes z) = \sum_{(x)} w(x_{(-1)} \cdot y) \otimes x_{(0)} z.$$

Proof. It follows from the above theorem and Proposition 2.2. □

There is a dual description of coalgebras. We have a correspondence between algebras and coalgebras, modules and comodules, etc. So the dual notions of module-algebras and comodule-algebras are the following.

Definition 2.13. *Let H be a Hopf algebra. A coalgebra $(C, \triangle)$ is called a* module-coalgebra *over H if*

1. *C is an H-module,*
2. *for any $h \in H$ $c \in C$, we have*

$$\begin{aligned}
\triangle(h \cdot c) &= \sum_{(c),(h)} (h_{(1)} \cdot c_{(1)}) \otimes (h_{(2)} \cdot c_{(2)}), \\
\varepsilon(h \cdot c) &= \varepsilon_H(h) \varepsilon_C(c).
\end{aligned}$$

Definition 2.14. *Let H be a Hopf algebra. A coalgebra $(C,\triangle)$ is called a* comodule-coalgebra *over H if*

1. C is an H-comodule with structure map $\rho: C \to H\otimes C$, and we denote for any $c\in C$, $\rho(c)=\sum_{(c)} c_{(-1)}\otimes c_{(0)}$,

2. for any $c\in C$, we denote $\triangle(c)=\sum_{(c)} c^{(1)}\otimes c^{(2)}$. And we have

$$\sum_{(c)} c_{(-1)}\otimes (c_{(0)})^{(1)}\otimes (c_{(0)})^{(2)} = \sum_{(c)} (c^{(1)})_{(-1)}(c^{(2)})_{(-1)}\otimes (c^{(1)})_{(0)}\otimes (c^{(2)})_{(0)},$$

$$\sum_{(c)} c_{(-1)}\varepsilon(c_{(0)}) = \varepsilon(c)1_H.$$

Theorem 2.15. *Let $(V,\cdot,\rho)$ be a Yetter-Drinfel'd module over H. If V is both a comodule-coalgebra and module-coalgebra, then (V,σ_V) is a YB coalgebra.*

Proof. For any $x,y\in V$, we have

$$\begin{aligned}
&(\sigma_V\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes\sigma_V)(\triangle\otimes\mathrm{id}_V)(x\otimes y)\\
&= (\sigma_V\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes\sigma_V)(\sum_{(x)} x^{(1)}\otimes x^{(2)}\otimes y)\\
&= (\sigma_V\otimes \mathrm{id}_V)(\sum_{(x)} x^{(1)}\otimes (x^{(2)})_{(-1)}\cdot y\otimes (x^{(2)})_{(0)})\\
&= \sum_{(x)} (x^{(1)})_{(-1)}\cdot((x^{(2)})_{(-1)}\cdot y)\otimes (x^{(1)})_{(0)}\otimes (x^{(2)})_{(0)}\\
&= \sum_{(x)} ((x^{(1)})_{(-1)}(x^{(2)})_{(-1)})\cdot y\otimes (x^{(1)})_{(0)}\otimes (x^{(2)})_{(0)}\\
&= \sum_{(x)} x_{(-1)}\cdot y\otimes (x_{(0)})^{(1)}\otimes (x_{(0)})^{(2)}\\
&= (\mathrm{id}_V\otimes\triangle)\sigma_V(x\otimes y),
\end{aligned}$$

where the fifth equality follows from the Condition 2 in the definition of comodule-coalgebra.

Also,

$$\begin{aligned}
&(\mathrm{id}_V\otimes\sigma_V)(\sigma_V\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes\triangle)(x\otimes y)\\
&= (\mathrm{id}_V\otimes\sigma_V)(\sigma_V\otimes \mathrm{id}_V)(\sum_{(y)} x\otimes y^{(1)}\otimes y^{(2)})\\
&= (\mathrm{id}_V\otimes\sigma_V)(\sum_{(x),(y)} x_{(-1)}\cdot y^{(1)}\otimes x_{(0)}\otimes y^{(2)})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{(x),(y)} x_{(-2)} \cdot y^{(1)} \otimes x_{(-1)} \cdot y^{(2)} \otimes x_{(0)} \\
&= \sum_{(x),(y)} (x_{(-1)} \cdot y)^{(1)} \otimes (x_{(-1)} \cdot y)^{(2)} \otimes x_{(0)} \\
&= (\triangle \otimes \mathrm{id}_V)\sigma(x \otimes y).
\end{aligned}$$

Finally,

$$\begin{aligned}
(\mathrm{id} \otimes \varepsilon_V)\sigma_V(x \otimes y) &= \sum_{(x)} x_{(-1)} \cdot y \otimes \varepsilon_V(x_{(0)}) \\
&= \sum_{(x)} x_{(-1)}\varepsilon_V(x_{(0)}) \cdot y \\
&= \varepsilon(x)1_H \cdot y \\
&= \varepsilon(x)y,
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon_V \otimes \mathrm{id})\sigma_V(x \otimes y) &= \sum_{(x)} \varepsilon_V(x_{(-1)} \cdot y) \otimes x_{(0)} \\
&= \sum_{(x)} \varepsilon(x_{(-1)})\varepsilon_V(y) \otimes x_{(0)} \\
&= \varepsilon(y)x.
\end{aligned}$$

□

Corollary 2.16. *Under the above assumption, $V^{\otimes 2}$ is a coalgebra with the coproduct: for any $x, y \in V$,*

$$\triangle(x \otimes y) = \sum_{(x),(y)} x^{(1)} \otimes (x^{(2)})_{(-1)} \cdot y^{(1)} \otimes (x^{(2)})_{(0)} \otimes y^{(2)}.$$

Proof. It follows from the above theorem and Proposition 2.2. □

Examples (Woronowicz's braidings).

For any Hopf algebra $(H, m, \eta, \triangle, \varepsilon, S)$, Woronowicz [30] constructed two braidings on H: for any $a, b \in H$,

$$\begin{aligned}
T_H(a \otimes b) &= \sum_{(b)} b_{(2)} \otimes aS(b_{(1)})b_{(3)}, \\
T'_H(a \otimes b) &= \sum_{(b)} b_{(1)} \otimes S(b_{(2)})ab_{(3)}.
\end{aligned}$$

They are invertible with inverses respectively:

$$\begin{aligned} T_H^{-1}(a\otimes b) &= \sum_{(a)} bS^{-1}(a_{(3)})a_{(1)}\otimes a_{(2)},\\ (T'_H)^{-1}(a\otimes b) &= \sum_{(a)} a_{(3)}bS^{-1}(a_{(2)})\otimes a_{(1)}. \end{aligned}$$

We consider $H^{op}=(H, m\circ\tau, \eta, \triangle, \varepsilon, S^{-1})$ and $H^{cop}=(H, m, \eta, \tau\circ\triangle, \varepsilon, S^{-1})$. Denote $F_H=T_{H^{op}}^{-1}$ and $F'_H=(T'_{H^{cop}})^{-1}$. Precisely,

$$\begin{aligned} F_H(a\otimes b) &= \sum_{(a)} a_{(1)}S(a_{(3)})b\otimes a_{(2)},\\ F'_H(a\otimes b) &= \sum_{(a)} a_{(1)}bS(a_{(2)})\otimes a_{(3)}. \end{aligned}$$

It is well-known that H is a Yetter-Drinfel'd module over itself with the following structures: for any $x,h\in H$,

$$\begin{cases} x\cdot h &= \sum_{(x)} x_{(1)}hS(x_{(2)}),\\ \rho(h) &= \sum_{(h)} h_{(1)}\otimes h_{(2)}. \end{cases}$$

It is easy to check that H is a module-algebra and comodule-algebra with these structures. The action of σ_H is $\sigma_H(x\otimes y)=\sum_{(x)} x_{(1)}yS(x_{(2)})\otimes x_{(3)}$. It is just the braiding F'. So by Theorem 2.11, (H,F') is a YB algebra.

H has also the following Yetter-Drinfel'd module structure: for any $x,h\in H$,

$$\begin{cases} x\cdot h &= xh,\\ \rho(h) &= \sum_{(h)} h_{(1)}S(h_{(3)})\otimes h_{(2)}. \end{cases}$$

It is easy to check that H is a module-coalgebra and comodule-coalgebra with these structures. The action of σ_H is $\sigma_H(x\otimes y)=\sum_{(x)} x_{(1)}S(x_{(3)})y\otimes x_{(3)}$. It is just the braiding F. Then by Theorem 2.15, (H,F) is a YB coalgebra.

The product and coproduct introduced in Corollary 2.12 and 2.16 are the generalizations of the smash product and smash coproduct respectively. This is related to some work of Lambe and Radford ([17], p. 115-p. 119) who gave slightly a more general result, but without considering YB algebras. Let V and W be Yetter-Drinfel'd modules. They proved that:

1. If V and W are both module-algebras and comodule-algebras. Then $V\otimes W$ has an associative algebra structure given by: for any $v,v'\in V$ and $w,w'\in W$,

$$(v\otimes w)\star(v'\otimes w')=\sum v(w_{(-1)}\cdot v')\otimes w_{(0)}w'.$$

2. If V and W are both module-coalgebras and comodule-coalgebras. Then $V \otimes W$ has an coassociative coalgebra structure given by: for any $v, v' \in V$,

$$\triangle(v \otimes w) = \sum_{(v),(w)} v^{(1)} \otimes (v^{(2)})_{(-1)} \cdot w^{(1)} \otimes (v^{(2)})_{(0)} \otimes w^{(2)}.$$

Notice that the category of Yetter-Drinfel'd modules is a braided monoidal category (for the definition, one can see [16]). So $V \otimes W$ is again a Yetter-Drinfel'd module with the usual tensor product module and comodule structure. The natural braiding of $V \otimes W$ is just $\Sigma = (\mathrm{id}_V \otimes \theta' \otimes \mathrm{id}_W)(\sigma_V \otimes \sigma_W)(\mathrm{id}_V \otimes \theta \otimes \mathrm{id}_W)$, where θ and θ' are given by, for any $v \in V$ and $w \in W$, $\theta(v \otimes w) = \sum v_{(-1)} \cdot w \otimes v_{(0)}$ and $\theta'(w \otimes v) = \sum w_{(-1)} \cdot v \otimes w_{(0)}$ respectively. If V and W are both module-algebras (resp. coalgebras) and comodule-algebras (resp. coalgebras), then so does $V \otimes W$ (cf. [17]). Therefore we have immediately that:

Proposition 2.17. *Let V and W be Yetter-Drinfel'd modules.*

1 If V and W are both module-algebras and comodule-algebras. Then $(V \otimes W, \Sigma)$ is a YB algebra with the product introduced above.

1 If V and W are both module-coalgebras and comodule-coalgebras. Then $(V \otimes W, \Sigma)$ is a YB coalgebra with the coproduct introduced above.

In the following, we will focus on some special cases of the machinery introduced above.

Definition 2.18. *A Hopf algebra H is said to be* quasi-triangular *if there exists an invertible element $\mathcal{R} \in H \otimes H$ such that for all $x \in H$,*

$$\mathcal{R} \triangle (x) \mathcal{R}^{-1} = \triangle^{op}(x), \tag{2.1}$$

$$(\triangle \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{2.2}$$

$$(\mathrm{id} \otimes \triangle)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.3}$$

where $\triangle^{op} = \tau \circ \triangle$, $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$ and $\mathcal{R}_{13} = (\tau \otimes \mathrm{id})(1 \otimes \mathcal{R})$.

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra with $\mathcal{R} = \sum_i s_i \otimes t_i \in H \otimes H$. We have that $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{R})$. For any H-module M, we define $\rho : M \to H \otimes M$ by $\rho(m) = \sum_i t_i \otimes s_i \cdot m$. Then $(M, \cdot, \rho)$ is a Yetter-Drinfel'd module over H and the braiding σ_M is just the action of the R-matrix of H (e.g., see [3]).

Theorem 2.19. *Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra and (A, m) be a module-algebra over H. Then (A, σ_A) is a YB algebra.*

Proof. We only need to check that A is also a comodule-algebra. We denote $\mathcal{R} = \sum_i s_i \otimes t_i$. Then from equation (2) we have

$$\sum_i \triangle(s_i) \otimes t_i = \sum_{k,l} s_k \otimes s_l \otimes t_k t_l.$$

Hence

$$\sum_i \sum_{(s_i)} t_i \otimes (s_i)_{(1)} \otimes (s_i)_{(2)} = \sum_{k,l} t_k t_l \otimes s_k \otimes s_l.$$

For any $a, b \in A$, we have

$$\begin{aligned}
\sum_{(ab)} (ab)_{(-1)} \otimes (ab)_{(0)} &= \sum_i t_i \otimes s_i \cdot (ab) \\
&= \sum_{i,(s_i)} t_i \otimes ((s_i)_{(1)} \cdot a)((s_i)_{(2)} \cdot b) \\
&= \sum_{k,l} t_k t_l \otimes (s_k \cdot a)(s_l \cdot b) \\
&= \sum_{(a),(b)} a_{(-1)} b_{(-1)} \otimes a_{(0)} b_{(0)}.
\end{aligned}$$

And

$$\begin{aligned}
\rho(1_A) &= \sum_i t_i \otimes s_i \cdot 1_A \\
&= \sum_i \varepsilon(s_i) t_i \otimes 1_A \\
&= 1_H \otimes 1_A.
\end{aligned}$$

□

Theorem 2.20. *Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra and $(C, \triangle)$ be a module-coalgebra over H. Then (C, σ_C) is a YB coalgebra.*

Proof. We only need to check that C is also a comodule-coalgebra. For any $c \in C$

$$\begin{aligned}
&\sum_{(c)} c_{(-1)} \otimes (c_{(0)})^{(1)} \otimes (c_{(0)})^{(2)} \\
&= \sum_{i,(s_i \cdot c)} t_i \otimes (s_i \cdot c)_{(1)} \otimes (s_i \cdot c)_{(2)} \\
&= \sum_{i,(s_i),(c)} t_i \otimes (s_i)_{(1)} \cdot c_{(1)} \otimes (s_i)_{(2)} \cdot c_{(2)} \\
&= \sum_{k,l,(c)} t_k t_l \otimes s_k \cdot c_{(1)} \otimes s_l \cdot c_{(2)} \\
&= \sum_{(c)} (c^{(1)})_{(-1)} (c^{(2)})_{(-1)} \otimes (c^{(1)})_{(0)} \otimes (c^{(2)})_{(0)}.
\end{aligned}$$

And

$$\sum_{(c)} c_{(-1)} \varepsilon(c_{(0)}) = \sum_i t_i \varepsilon(s_i \cdot c)$$

$$\begin{aligned} &= \sum_i t_i \varepsilon_H(s_i)\varepsilon_C(c) \\ &= \varepsilon_C(c)1_H. \end{aligned}$$

□

2.3 Examples related to quantum shuffle algebras

For a Yetter-Drinfel'd module V which is both a module-algebra and a comodule-algebra, $V^{\otimes i}$ is a YB algebra by Proposition 2.2.1. One can have an interesting YB algebra structure on $T(V)$ as follows, which will be generalized for any braided vector space later.

We first review some terminologies. An $(i_1, \ldots, i_l)$-shuffle is an element $w \in \mathfrak{S}_{i_1+\cdots+i_l}$ such that $w(1) < \cdots < w(i_1), w(i_1+1) < \cdots < w(i_1+i_2), \ldots, w(i_1+\cdots+i_{l-1}+1) < \cdots < w(i_1+\cdots+i_l)$. We denote by $\mathfrak{S}_{i_1,\ldots,i_l}$ the set of all $(i_1, \ldots, i_l)$-shuffles.

Let V be a Yetter-Drinfel'd module over a Hopf algebra H with the natural braiding σ. In [26], the following associative product on $T(V)$ was constructed (in fact, the construction works for any braided vector space): for any $x_1, \ldots, x_{i+j} \in V$,

$$(x_1 \otimes \cdots \otimes x_i) ш_\sigma (x_{i+1} \otimes \cdots \otimes x_{i+j}) = \sum_{w \in \mathfrak{S}_{i,j}} T_w(x_1 \otimes \cdots \otimes x_{i+j}). \tag{2.4}$$

$T(V)$ equipped with $ш_\sigma$ is called the *quantum shuffle algebra* and denoted by $T_\sigma(V)$. Moreover, the Yetter-Drinfel'd module $T_\sigma(V)$ is a module-algebra and a comodule-algebra with the diagonal action and coaction respectively (cf. [26], Proposition 9). So $T_\sigma(V)$ is a YB algebra. In fact, the result holds for any braided vector space.

Theorem 2.21. *Let (V,σ) be a braided vector space. Then $(T_\sigma(V), \beta)$ is a YB algebra. The subalgebra $S_\sigma(V)$ of $T_\sigma(V)$ generated by V is also a YB algebra with braiding β.*

Proof. For any triple (i,j,k) of positive integers and any $w \in \mathfrak{S}_{i,j}$, we have that

$$(1_{\mathfrak{S}_k} \times w)(\chi_{ik} \times 1_{\mathfrak{S}_j})(1_{\mathfrak{S}_i} \times \chi_{jk}) = \chi_{i+j,k}(w \times 1_{\mathfrak{S}_k}).$$

And all the expressions are reduced. This gives us that

$$(\mathrm{id}_V^{\otimes k} \otimes ш_\sigma)(\beta_{ik} \otimes \mathrm{id}_V^{\otimes j})(\mathrm{id}_V^{\otimes i} \otimes \beta_{jk}) = \beta_{i+j,k}(ш_\sigma \otimes \mathrm{id}_V^{\otimes k}).$$

The other conditions can be proved similarly. Hence $(T_\sigma(V), \beta)$ is a YB algebra.

From the definition, $S_\sigma(V) = \oplus_{i\geq 0}\mathrm{Im}(\sum_{w\in\mathfrak{S}_i} T_w^\sigma)$. By observing that $\chi_{ij}(\mathfrak{S}_i \times \mathfrak{S}_j) = (\mathfrak{S}_j \times \mathfrak{S}_i)\chi_{ij}$ and all the expressions are reduced, we see β is a braiding on $S_\sigma(V)$. Certainly it is a YB algebra since it is a subalgebra of $T_\sigma(V)$. □

Remark 2.22. *By using the dual construction, we know* $(T(V),\beta)$ *is a YB coalgebra with the following coproduct* $\triangle$*: for any* $x_1,\ldots,x_n\in V$*, the component of* $\triangle(x_1\otimes\cdots\otimes x_n)$ *in* $V^{\otimes p}\underline{\otimes}V^{\otimes n-p}$ *is*

$$\triangle(x_1\otimes\cdots\otimes x_n)=\sum_{w\in\mathfrak{S}_{p,n-p}}T_{w^{-1}}(x_1\otimes\cdots\otimes x_n).$$

Example (Quantum exterior algebras). Let V be a vector space over $\mathbb{C}$ with basis $\{e_1,\ldots,e_N\}$. Take a nonzero scalar $q\in\mathbb{C}$. We define a braiding σ on V by

$$\sigma(e_i\otimes e_j)=\begin{cases} e_i\otimes e_i, & i=j,\\ q^{-1}e_j\otimes e_i, & i<j,\\ q^{-1}e_j\otimes e_i+(1-q^{-2})e_i\otimes e_j, & i>j.\end{cases}$$

Then σ satisfies the Iwahori's quadratic equation $(\sigma-\mathrm{id}_{V\otimes V})(\sigma+q^{-2}\mathrm{id}_{V\otimes V})=0$. In fact, this σ is given by the R-matrix in the fundamental representation of $U_q\mathfrak{sl}_N$. By a result of Gurevich (cf. [7], Proposition 2.13), we know that $T(V)/I\cong\oplus_{i\geq 0}\mathrm{Im}(\sum_{w\in\mathfrak{S}_i}(-1)^{l(w)}T_w)$ as algebras, where $l(w)$ is the length of w and I is the ideal of $T(V)$ generated by $\mathrm{Ker}(\mathrm{id}_{V^{\otimes 2}}-\sigma)$. So by easy computation, we get that $\mathrm{Ker}(\mathrm{id}_{V\otimes V}-c)=\mathrm{Span}_{\mathbb{C}}\{e_i\otimes e_i, q^{-1}e_i\otimes e_j+e_j\otimes e_i(i<j)\}$. We denote by $e_{i_1}\wedge\cdots\wedge e_{i_s}$ the image of $e_{i_1}\otimes\cdots\otimes e_{i_s}$ in $S_\sigma(V)$. So $S_\sigma(V)$ is an algebra generated by (e_i) and the relations $e_i^2=0$ and $e_j\wedge e_i=-q^{-1}e_i\wedge e_j$ if $i<j$. This $S_\sigma(V)$ is called the *quantum exterior algebra* over V. It is a finite dimensional YB algebra with the braiding β.

The quantum exterior algebra has another YB algebra structure as follows. We denote the increasing set $(i_1,\ldots,i_s)$ by $\underline{i}$ and so on. For $1\leq i_1<\cdots<i_s\leq N$ and $1\leq j_1<\cdots<j_t\leq N$, we denote

$$(i_1,\cdots,i_s|j_1,\cdots,j_t)=\begin{cases} 0, & \text{, if } \underline{i}\cap\underline{j}\neq\emptyset,\\ 2\sharp\{(i_k,j_l)|i_k>j_l\}-st, & \text{otherwise.}\end{cases}$$

Using the above notation, it is easy to see that

$$e_{i_1}\wedge\cdots\wedge e_{i_s}\wedge e_{j_1}\wedge\cdots\wedge e_{j_t}=(-q)^{-(i_1,\cdots,i_s|j_1,\cdots,j_t)}e_{j_1}\wedge\cdots\wedge e_{j_t}\wedge e_{i_1}\wedge\cdots\wedge e_{i_s}.$$

Definition 2.23. *The* q-flip $\mathscr{T}=\bigoplus_{s,t}\mathscr{T}_{s,t}$*:* $S_\sigma(V)\otimes S_\sigma(V)\to S_\sigma(V)\otimes S_\sigma(V)$ *is defined by: for* $1\leq i_1<\cdots<i_s\leq N$ *and* $1\leq j_1<\cdots<j_t\leq N$*,*

$$\mathscr{T}_{s,t}(e_{i_1}\wedge\cdots\wedge e_{i_s}\otimes e_{j_1}\wedge\cdots\wedge e_{j_t})=(-q)^{(i_1,\cdots,i_s|j_1,\cdots,j_t)}e_{j_1}\wedge\cdots\wedge e_{j_t}\otimes e_{i_1}\wedge\cdots\wedge e_{i_s}.$$

Obviously, $\mathscr{T}$ is a braiding. And $\mathscr{T}$ induces a representation of the symmetric group, i.e., $\mathscr{T}^2=\mathrm{id}$. Indeed, for $1\leq i_1<\cdots<i_s\leq N+1$ and $1\leq j_1<\cdots<j_t\leq N+1$, the result is trivial if $\underline{i}\cap\underline{j}\neq\emptyset$. Otherwise,

$$(i_1,\cdots,i_s|j_1,\cdots,j_t)+(j_1,\cdots,j_t|i_1,\cdots,i_s)$$

$$\begin{aligned} &= 2\sharp\{(i_k, j_l)|i_k > j_l\} - st + 2\sharp\{(j_u, i_v)|j_u > i_v\} - st \\ &= 2st - 2st \\ &= 0. \end{aligned}$$

For any sequence $I = (i_1, \ldots, i_n)$, we denote by $l(I)$ the number of pairs (i_a, i_b) such that $a < b$ but $i_a > i_b$, and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_n}$. $(S_\sigma(V), \wedge, \mathscr{T})$ is a YB algebra and $(S_\sigma(V), \delta, \mathscr{T})$ is a YB coalgebra. Given any increasing multi-indices I, J and K. If $J \cap K \neq \emptyset$, the result is trivial. Assume that $J \cap K = \emptyset$.

$$\begin{aligned} &(\wedge \otimes \mathrm{id}_{S_\sigma(V)})\mathscr{T}_2\mathscr{T}_1(e_I \otimes e_J \otimes e_K) \\ &\quad= (-q)^{(I|J)+(I|K)} e_J \wedge e_K \otimes e_I \\ &\quad= (-q)^{(I|J\uplus K)} e_J \wedge e_K \otimes e_I \\ &\quad= (-q)^{-l(J,\ K)+(I|J\uplus K)} e_{J\uplus K} \otimes e_I \\ &\quad= \mathscr{T}((-q)^{-l(J,K)} e_I \otimes e_{J\uplus K}) \\ &\quad= \mathscr{T}(e_I \otimes e_J \wedge e_K) \\ &\quad= \mathscr{T}(\mathrm{id}_{S_\sigma(V)} \otimes \wedge)(e_I \otimes e_J \otimes e_K), \end{aligned}$$

where $J \uplus K$ is the the rearrangement of the disjoint union $J \sqcup K$ with the increasing order.

For $s + t = |J|$,

$$\begin{aligned} &\mathscr{T}_2\mathscr{T}_1(\mathrm{id}_{S_\sigma(V)} \otimes \delta)(e_I \otimes e_J) \\ &\quad= \mathscr{T}_2\mathscr{T}_1(e_I \otimes \sum_{w\in S_{s,t}} (-q)^{-l(w)} e_{J_1(w)} \otimes e_{J_2(w)}) \\ &\quad= \sum_{w\in S_{s,t}} (-q)^{-l(w)+(I|J_1(w))+(I|J_2(w))} e_{J_1(w)} \otimes e_{J_2(w)} \otimes e_I \\ &\quad= (\mathrm{id}_{S_\sigma(V)} \otimes \delta)((-q)^{(I|J)} e_J \otimes e_I) \\ &\quad= (\mathrm{id}_{S_\sigma(V)} \otimes \delta)\mathscr{T}(e_I \otimes e_J), \end{aligned}$$

where $J_1(w) = (j_{w^{-1}(1)}, \ldots, j_{w^{-1}(s)})$ and $J_2(w) = (j_{w^{-1}(s+1)}, \ldots, j_{w^{-1}(s+t)})$. The other conditions can be verified similarly.

Originally, quantum shuffle algebras were discovered from the cotensor algebras (cf. [26]). Combining the discussions in the previous section, it is not hard to see that the cotensor algebra is both a YB algebra and YB coalgebra. Here, we give a more general description of this phenomenon in the framework of bialgebras with a projection onto a Hopf algebra which is due to Radford [25].

Definition 2.24 (cf. [21], [29]). *Let H be a Hopf algebra. A* Hopf bimodule *over H is a vector space M given with an H-bimodule structure, an H-bicomodule structure with left and right coactions $\delta_L : M \to H \otimes M$, $\delta_R : M \to M \otimes H$ which commute in the* *following sense:* $(\delta_L \otimes \mathrm{id}_M)\delta_R = (\mathrm{id}_M \otimes \delta_R)\delta_L$*, and such that* δ_L *and* δ_R *are morphisms of* H*-bimodules.*

We denote by V the subspace of right coinvariants $M^R = \{m \in M | \delta_R(m) = m \otimes 1\}$. Then V is a left Yetter-Drinfel'd module with coaction δ and the left adjoint action given by: for any $h \in H$ and $m \in M$,

$$h \cdot m = \sum h_{(1)} m S(h_{(2)}).$$

Let H be a Hopf algebra with antipode S and A be a bialgebra. Suppose there are two bialgebra maps $i : H \to A$ and $\pi : A \to H$ such that $\pi \circ i = \mathrm{id}_H$. Set $\Pi = \mathrm{id}_A \star (i \circ S \circ \pi)$, where $\star$ is the convolution product on $\mathrm{End}(A)$, and $B = \Pi(A)$. The following statements are easy to verify (cf. [25]).

1. A is a Hopf bimodule over H with actions $h \cdot a = i(h)a$ and $a \cdot h = ai(h)$, coactions $\delta_L(a) = \sum \pi(a_{(1)}) \otimes a_{(2)}$ and $\delta_R(a) = \sum a_{(1)} \otimes \pi(a_{(2)})$ for any $h \in H$ and $a \in A$. Obviously, by the projection formula from a Hopf bimodule to its right coinvariant subspace, $A^R = B$. So B is a left Yetter-Drinfel'd module over H with the left adjoint action.

2. B is a subalgebra of A. Furthermore it is both a module-algebra and a comodule-algebra. B has a coalgebra structure such that Π is a coalgebra map. And with this coalgebra structure, B is both a module-coalgebra and comodule-coalgebra.

3. The map $B \otimes H \to A$ given by $b \otimes h \mapsto bi(h)$ is a bialgebra isomorphism, where $B \otimes H$ is with the smash product and smash coproduct.

So combining Woronowicz's examples and Proposition 2.16, the bialgebra A is both a YB algebra and a YB coalgebra. If A is moreover a Hopf algebra, then it is again a YB algebra and YB coalgebra with Woronowicz's braidings. Obviously, these two YB algebra (resp. coalgebra) structures are different.

Now we restrict our attention on cotensor algebras, which will give us YB algebras related to quantum groups. For a Hopf bimodule M over H, one can construct the cotensor algebra over $T_H^c(M)$. More precisely, we define $M \square M = \mathrm{Ker}(\delta_R \otimes \mathrm{id}_M - \mathrm{id}_M \otimes \delta_L)$ and $M^{\square k} = M^{\square k-1} \square M$ for $k \geq 3$. And the cotensor algebra built over H and M is $T_H^c(M) = H \oplus M \oplus \oplus_{k \geq 2} M^{\square k}$. It is again a Hopf bimodule over H. From the universal property of cotensor algebra, one can construct a Hopf algebra structure with a complicated multiplication on $T_H^c(M)$. We denote by $S_H(M)$ the subalgebra of $T_H^c(M)$ generated by H and M. Then $S_H(M)$ is a sub-Hopf algebra. For more details, one can see [21]. Obviously, $T^c(V)$ defined above is the cotensor algebra over the trivial Hopf algebra K. Here V is a Hopf bimodule with scalar multiplication and the coactions defined by $\delta_L(v) = 1 \otimes v$ and $\delta_R(v) = v \otimes 1$ for any $v \in V$.

Since the inclusion $H \to T_H^c(M)$ and the projection $T_H^c(M) \to H$ are bialgebra maps, we get:

Theorem 2.25. *Let* M *be a Hopf bimodule over* H*. Then both* $T_H^c(M)$ *and* $S_H(M)$ *are YB algebras and YB coalgebras.*

As an application of the above theorem, we consider the following special case. Let $G = \mathbb{Z}^r \times \mathbb{Z}/l_1 \times \mathbb{Z}/l_2 \times \cdots \mathbb{Z}/l_p$ and $H = K[G]$ be the group algebra of G. We fix generators $K_1, \ldots, K_N$ of G ($N = r + p$). Let V be a vector space over $\mathbb{C}$ with basis $\{e_1, \ldots, e_N\}$. We know V is a Yetter-Drinfel'd module over H with action and coaction given by $\delta_L(e_i) = K_i \otimes e_i$ and $K_i \cdot e_j = q_{ij} e_j$ with some nonzero scalar $q_{ij} \in \mathbb{C}$ respectively. The braiding coming from the Yetter-Drinfel'd module structure is given by $\sigma(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$. Now we choose special q_{ij} to construct meaningful examples. Let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a symetrizable generalized Cartan matrix, $(d_1, \ldots, d_N)$ positive integers relatively prime such that $(d_i a_{ij})$ is symmetric. Let $q \in \mathbb{C}$ and define $q_{ij} = q^{d_i a_{ij}}$. By Theorem 15 in [26], $S_H(M)$ is isomorphic, as a Hopf algebra, to the sub Hopf algebra U_q^+ of the quantized universal enveloping algebra associated with A when $G = \mathbb{Z}^N$ and q is not a root of unity; $S_H(M)$ is isomorphic, as a Hopf algebra, to the quotient of the restricted quantized enveloping algebra u_q^+ by the two-sided Hopf ideal generated by the elements $(K_i^l - 1)$, $i = 1, \ldots, N$ when $G = (\mathbb{Z}/l)^N$ and q is a primitive l-th root of unity. Then we have:

Corollary 2.26. *Both U_q^+ and u_q^+ are YB algebras and YB coalgebras.*

We use the above special $S_\sigma(V) \otimes H$ to illustrate the difference between the braiding coming from Woronowicz's construction and the one from the tensor product of two Yetter-Drinfel'd modules.

We use the following notation: for any $g = K_1^{i_1} \cdots K_N^{i_N} \in G$, $q_{gj} = q_{1j}^{i_1} \cdots q_{Nj}^{i_N}$, i.e., $g \cdot e_j = q_{gj} e_j$. For any $g, h \in G$, Woronowicz's braiding F' has the following action on $S_\sigma(V) \otimes H$:

$$
\begin{aligned}
& F'\Big((e_i \otimes g) \otimes (e_j \otimes h)\Big) \\
= \ & \sum (e_i \otimes g)_{(1)} (e_j \otimes h) S\big((e_i \otimes g)_{(2)}\big) \otimes (e_i \otimes g)_{(3)} \\
= \ & (K_i g)(e_j \otimes h) S(K_i g) \otimes (e_i \otimes g) \\
& + (K_i g)(e_j \otimes h) S(e_i \otimes g) \otimes g \\
& + (e_i \otimes g)(e_j \otimes h) S(g) \otimes g \\
= \ & q_{ij} q_{gj} (e_j \otimes h) \otimes (e_i \otimes g) \\
& - q_{ij} q_{gj} \big(e_j \otimes K_i g h\big)\big((K_i^{-1} g^{-1}) \cdot e_i \otimes K_i^{-1} g^{-1}\big)\big) \otimes g \\
& + (e_i \otimes g)(e_j \otimes h g^{-1}) \otimes g \\
= \ & q_{ij} q_{gj} (e_j \otimes h) \otimes (e_i \otimes g) \\
& - q_{ij} q_{gj} q_{hi} (e_j e_i \otimes h) \otimes g \\
& + q_{gj} (e_i e_j \otimes h) \otimes g,
\end{aligned}
$$

where the second and third equalities follow from the formulas $\triangle(e_i \otimes g) = K_i g \otimes (e_i \otimes g) + (e_i \otimes g) \otimes g$ and $S(e_i \otimes g) = -(K_i^{-1} g^{-1})(e_i \otimes g)$.

And the braiding in the category of Yetter-Drinfel'd modules is :

$$\Sigma\Big((e_i \otimes g) \otimes (e_j \otimes h)\Big) = q_{ij}(e_j \otimes h) \otimes (e_i \otimes g).$$

2.4 Quantum B_∞-algebras

Let $(C, \triangle, \varepsilon)$ be a coalgebra with a preferred group-like element $1_C \in C$ and denote $\overline{\triangle}(x) = \triangle(x) - x \otimes 1_C - 1_C \otimes x$ for any $x \in C$. And $\overline{\triangle}$ is called the *reduced coproduct*. We also denote $\overline{C} = \mathrm{Ker}\varepsilon$. $C = K1_C \oplus \overline{C}$ since $x - \varepsilon(x)1_C \in \overline{C}$ for any $x \in C$.

Definition 2.27 (cf. [19]). *$(C, \triangle)$ is said to be* connected *if $C = \cup_{r\geq 0} F_r C$, where*

$$\begin{aligned} F_0 C &= K1_C, \\ F_r C &= \{x \in C | \overline{\triangle}(x) \in F_{r-1}C \otimes F_{r-1}C\}, \quad \text{for r} \geq 1. \end{aligned}$$

Now we collect some properties of the reduced coproduct and of connected coalgebras. They are certainly well-known. We provide a proof because we could not find one in the literature.

Proposition 2.28. *Let $(C, \triangle, \varepsilon)$ be a coalgebra.*

1. The reduced coproduct is coassociative, i.e., $(\overline{\triangle} \otimes \mathrm{id}_C)\overline{\triangle} = (\mathrm{id}_C \otimes \overline{\triangle})\overline{\triangle}$. So we can adopt the following notations: $\overline{\triangle}^{(0)} = \mathrm{id}$, $\overline{\triangle}^{(1)} = \overline{\triangle}$, and $\overline{\triangle}^{(n)} = (\overline{\triangle} \otimes \mathrm{id}_C^{\otimes n-1})\overline{\triangle}^{(n-1)}$ for $n \geq 2$.

2. $\overline{\triangle}(\overline{C}) \subset \overline{C} \otimes \overline{C}$.

3. If C is connected, then $\overline{\triangle}^{(r)}(\overline{C} \cap F_r C) = 0$ for any $r \geq 0$.

Proof. 1. For any $x \in C$,

$$\begin{aligned} &(\overline{\triangle} \otimes \mathrm{id}_C)\overline{\triangle}(x) \\ &= (\overline{\triangle} \otimes \mathrm{id}_C)(x_{(1)} \otimes x_{(2)} - 1_C \otimes x - x \otimes 1_C) \\ &= \overline{\triangle}(x_{(1)}) \otimes x_{(2)} - \overline{\triangle}(1_C) \otimes x - \overline{\triangle}(x) \otimes 1_C \\ &= (x_{(1)} \otimes x_{(2)} - 1_C \otimes x_{(1)} - \otimes x_{(1)} \otimes 1_C) \otimes x_{(3)} \\ &\quad -(1_C \otimes 1_C - 1_C \otimes 1_C - 1_C \otimes 1_C) \otimes x \\ &\quad -(x_{(1)} \otimes x_{(2)} - 1_C \otimes x - x \otimes 1_C) \otimes 1_C \\ &= x_{(1)} \otimes x_{(2)} \otimes x_{(3)} - 1_C \otimes x_{(1)} \otimes x_{(2)} - x_{(1)} \otimes 1_C \otimes x_{(2)} \\ &\quad +1_C \otimes 1_C \otimes x - x_{(1)} \otimes x_{(2)} \otimes 1_C + 1_C \otimes x \otimes 1_C + x \otimes 1_C \otimes 1_C \\ &= x_{(1)} \otimes (x_{(2)} \otimes x_{(3)} - 1_C \otimes x_{(2)} - x_{(2)} \otimes 1_C) \\ &\quad -1_C \otimes (x_{(1)} \otimes x_{(2)} - 1_C \otimes x - x \otimes 1_C) + x \otimes 1_C \otimes 1_C \end{aligned}$$

$$\begin{aligned} &= x_{(1)} \otimes \overline{\triangle}(x_{(2)}) - 1_C \otimes \overline{\triangle}(x) - x \otimes \overline{\triangle}(1_C) \\ &= (\mathrm{id}_C \otimes \overline{\triangle})\overline{\triangle}(x). \end{aligned}$$

2. For any $x \in \overline{C}$,

$$\begin{aligned} (\mathrm{id}_C \otimes \varepsilon)\overline{\triangle}(x) &= (\mathrm{id}_C \otimes \varepsilon)(\triangle(x) - 1_C \otimes x - x \otimes 1_C) \\ &= x - 0 - x \\ &= 0. \end{aligned}$$

So $\overline{\triangle}(x) \in \mathrm{Ker}(\mathrm{id}_C \otimes \varepsilon) = H \otimes \mathrm{Ker}\varepsilon = H \otimes \overline{C}$. By applying $\varepsilon \otimes \mathrm{id}_C$ on $\overline{\triangle}(x)$, we get $\overline{\triangle}(x) \in \overline{C} \otimes H$. Hence we have $\overline{\triangle}(x) \in (H \otimes \overline{C}) \cap (\overline{C} \otimes H) = \overline{C} \otimes \overline{C}$.

3. We use induction on r.

The case $r = 0$. $\overline{\triangle}^{(0)}(\overline{C} \cap F_0C) = \mathrm{id}_C(\overline{C} \cap K1_C) = 0$.

The case $r = 1$. Given any $x \in \overline{C} \cap F_1C$. Since $x \in F_1C$, we have $\overline{\triangle}(x) \in F_1C \otimes F_1C = K1_C \otimes 1_C$. So we assume $\overline{\triangle}(x) = \alpha 1_C \otimes 1_C$ for some $\alpha \in K$.

$$\begin{aligned} (\varepsilon \otimes \varepsilon)\overline{\triangle}(x) &= (\varepsilon \otimes \varepsilon)(\triangle(x) - x \otimes 1_C - 1_C \otimes x) \\ &= \varepsilon(x) - \varepsilon(1_C)\varepsilon(x) - \varepsilon(x)\varepsilon(1_C) \\ &= 0. \end{aligned}$$

But we also have $(\varepsilon \otimes \varepsilon)\overline{\triangle}(x) = (\varepsilon \otimes \varepsilon)(\alpha 1_C \otimes 1_C) = \alpha$, which implies $\alpha = 0$. So $\overline{\triangle}^{(1)}(\overline{C} \cap F_1C) = 0$.

We assume the result holds for r. For any $x \in \overline{C} \cap F_rC$, $\overline{\triangle}^{(r+1)}(x) = (\mathrm{id}_C \otimes \overline{\triangle}^{(r)})\overline{\triangle}(x) \in F_rC \otimes \overline{\triangle}^{(r)}(F_r \cap \overline{C}) = 0$. □

There is a well-known universal property for $T^c(V)$:

Proposition 2.29. *Given a connected coalgebra $(C, \triangle, \varepsilon)$ and a linear map $\phi : C \to V$ such that $\phi(1_C) = 0$, there is a unique coalgebra morphism $\overline{\phi} : C \to T^c(V)$ which extends ϕ, i.e., $P_V \circ \overline{\phi} = \phi$, where $P_V : T^c(V) \to V$ is the projection onto V. Explicitly, $\overline{\phi} = \varepsilon + \sum_{n \geq 1} \phi^{\otimes n} \circ \overline{\triangle}^{(n-1)}$.*

Corollary 2.30. *Let C be a connected coalgebra. If $\Phi, \Psi : C \to T^c(V)$ are coalgebra maps such that $P_V \circ \Phi = P_V \circ \Psi$ and $P_V \circ \Phi(1_C) = 0 = P_V \circ \Psi(1_C)$, then $\Phi = \Psi$.*

Proof. C is connected and $P_V \circ \Phi : C \to V$ is a linear map with $P_V \circ \Phi(1_C) = 0$. Then by the universal property we know there is a unique coalgebra map $\overline{P_V \circ \Phi} : C \to T^c(V)$ such that $P_V \circ \overline{P_V \circ \Phi} = P_V \circ \Phi$. But obviously Φ is such a map. So $\overline{P_V \circ \Phi} = \Phi$. From the same reason we have $\overline{P_V \circ \Psi} = \Psi$. Hence we have $\Phi = \Psi$. □

Using Proposition 2.2 and the fact, which is mentioned in Section 2, that $(T^c(V), \beta)$ is a YB coalgebra, we know there is a coalgebra structure on $T^c(V)^{\otimes i}$ by combining β and δ:

$$\triangle_{\beta,i} = T^{\beta}_{w_i^{-1}} \circ \delta^{\otimes i},$$

and the counit is $\varepsilon^{\otimes i}$.

Proposition 2.31. *Let (V, σ) be a braided vector space. Then for any $n \geq 1$, $(T^c(V)^{\otimes n}, \triangle_{\beta,n})$ is connected.*

Proof. Obviously, $1^{\underline{\otimes} n}$ is a group-like element of $T^c(V)^{\otimes n}$. For any $r \geq 0$, we have that

$$F_r = F_r(T^c(V)^{\otimes n}) = \bigoplus_{0 \leq i_1 + \cdots + i_n \leq r} V^{\otimes i_1} \underline{\otimes} \cdots \underline{\otimes} V^{\otimes i_n}.$$

□

From now on, we use $\triangle_\beta$ to denote $\triangle_{\beta,2}$ for $n = 2$. Since $w_2^{-1} = s_2 \in \mathfrak{S}_4$, $\triangle_\beta = (\mathrm{id}_{T^c(V)} \otimes \beta \otimes \mathrm{id}_{T^c(V)}) \circ (\delta \otimes \delta)$.

Let $M = \oplus M_{pq} : T^c(V) \underline{\otimes} T^c(V) \to V$ be a linear map such that $M_{pq} : V^{\otimes p} \underline{\otimes} V^{\otimes q} \to V$, and

$$\begin{cases} M_{00} &= 0, \\ M_{10} &= \mathrm{id}_V = M_{01}, \\ M_{n0} &= 0 = M_{0n}, \text{ for } n \geq 2. \end{cases}$$

Since $M(1 \underline{\otimes} 1) = 0$, there is a unique coalgebra map $* : T^c(V) \underline{\otimes} T^c(V) \to T^c(V)$ by the universal property of $T^c(V)$. Explicitly,

$$* = (\varepsilon \otimes \varepsilon) + \sum_{n \geq 1} M^{\otimes n} \circ \overline{\triangle_\beta}^{(n-1)}.$$

We shall investigate conditions under which $*$ is an associative product. Here we start by giving another form of $*$ by using the map M and the deconcatenation δ.

Proposition 2.32. *For $n \geq 0$, we have that*

$$\triangle_\beta^{(n)} = T^{\beta}_{w_{n+1}} \circ (\delta^{(n)})^{\otimes 2}.$$

Proof. We use induction on n.

When $n = 0$, it is trivial since $w_1 = 1_{\mathfrak{S}_2}$.

When $n = 1$, $\triangle_\beta^{(1)} = \triangle_\beta = \beta_2(\delta \otimes \delta) = T^{\beta}_{w_2} \circ (\delta^{(1)})^{\otimes 2}$ since $w_2 = s_2$.

When $n = 2$,

$$\triangle_\beta^{(2)} = (\triangle_\beta \otimes \mathrm{id}_{T^c(V)} \otimes \mathrm{id}_{T^c(V)})\, \triangle_\beta$$

$$\begin{aligned}
&= \beta_2(\delta\otimes\delta\otimes \mathrm{id}_{T^c(V)}\otimes \mathrm{id}_{T^c(V)})\beta_2(\delta\otimes\delta)\\
&= \beta_2\big(\delta\otimes(\delta\otimes \mathrm{id}_{T^c(V)})\beta_2\otimes \mathrm{id}_{T^c(V)}\big)\circ(\delta\otimes\delta)\\
&= \beta_2\big(\delta\otimes\beta_2\beta_1(\mathrm{id}_{T^c(V)}\otimes\delta)\otimes \mathrm{id}_{T^c(V)}\big)\circ(\delta\otimes\delta)\\
&= \beta_2\beta_4\beta_3(\delta\otimes \mathrm{id}_{T^c(V)}\otimes\delta\otimes \mathrm{id}_{T^c(V)})\circ(\delta\otimes\delta)\\
&= T^\beta_{w_3}\circ(\delta^{(2)})^{\otimes 2}.
\end{aligned}$$

For $n\geq 3$,

$$\begin{aligned}
\triangle_\beta^{(n+1)} &= (\triangle_\beta\otimes \mathrm{id}_{T^c(V)}^{\otimes 2n})\,\triangle_\beta^{(n)}\\
&= \beta_2(\delta\otimes\delta\otimes \mathrm{id}_{T^c(V)}^{\otimes 2n})T^\beta_{w_{n+1}}\circ(\delta^{(n)})^{\otimes 2}\\
&= \beta_2(\delta\otimes\delta\otimes \mathrm{id}_{T^c(V)}^{\otimes 2n})(\mathrm{id}_{T^c(V)}^{\otimes 2}\otimes T^\beta_{w_n})\beta_1\cdots\beta_{n+1}\circ(\delta^{(n)})^{\otimes 2}\\
&= \beta_2(\mathrm{id}_{T^c(V)}^{\otimes 2}\otimes T^\beta_{w_n})(\delta\otimes\delta\otimes \mathrm{id}_{T^c(V)}^{\otimes 2n})\beta_1\cdots\beta_{n+1}\circ(\delta^{(n)})^{\otimes 2}\\
&= \beta_2(\mathrm{id}_{T^c(V)}^{\otimes 2}\otimes T^\beta_{w_n})\beta_3\beta_3\beta_5\beta_4\cdots\beta_{n+3}\beta_{n+2}\\
&\quad \circ(\delta\otimes\otimes \mathrm{id}_{T^c(V)}^{\otimes n}\otimes\delta\otimes \mathrm{id}_{T^c(V)}^{\otimes n})\circ(\delta^{(n)})^{\otimes 2}\\
&= T^\beta_{w_{n+2}}\circ(\delta^{(n+1)})^{\otimes 2}.
\end{aligned}$$

The third and last equalities follow from the fact that $w_{n+1}=(1_{\mathfrak{S}_2}\times w_n)s_2\cdots s_{n+1}$ for $n\geq 1$, $w_{n+2}=s_2(1_{\mathfrak{S}_4}\times w_n)s_4s_3s_5s_4\cdots s_{n+3}s_{n+2}$ for $n\geq 3$ and both expressions are reduced. □

Lemma 2.33. *For $n\geq 1$, we have $M^{\otimes n}\,\triangle_\beta^{(n-1)}\,(1\underline{\otimes}1)=0$.*

Proof. It follows from the fact that $\triangle_\beta^{(n-1)}(1\underline{\otimes}1)=(1\underline{\otimes}1)^{\otimes n}$ and $M_{00}=0$. □

Proposition 2.34. *For $n\geq 1$, we have $M^{\otimes n}\overline{\triangle_\beta}^{(n-1)}=M^{\otimes n}\triangle_\beta^{(n-1)}$*

Proof. We use induction on n.

When $n=1$, it is trivial.

For $n\geq 2$ any $u,v\in T^c(V)$,

$$\begin{aligned}
&M^{\otimes n}\overline{\triangle_\beta}^{(n-1)}\\
&\quad= \big((M^{\otimes n-1}\overline{\triangle_\beta}^{(n-2)})\otimes M\big)\overline{\triangle_\beta}(u\underline{\otimes}v)\\
&\quad= \big((M^{\otimes n-1}\triangle_\beta^{(n-2)})\otimes M\big)\Big(\triangle_\beta\,(u\underline{\otimes}v)-(1\underline{\otimes}1)\underline{\otimes}(u\underline{\otimes}v)-(u\underline{\otimes}v)\underline{\otimes}(1\underline{\otimes}1)\Big)\\
&\quad= M^{\otimes n}\,\triangle_\beta^{(n-1)}\,(u\underline{\otimes}v)-\big(M^{\otimes n-1}\,\triangle_\beta^{(n-2)}\,(1\underline{\otimes}1)\big)\underline{\otimes}M_{11}(u\underline{\otimes}v)\\
&\quad\quad -\big(M^{\otimes n-1}\,\triangle_\beta^{(n-2)}\,(u\underline{\otimes}v)\big)\underline{\otimes}M_{00}(1\underline{\otimes}1)
\end{aligned}$$

$$= \quad M^{\otimes n} \bigtriangleup_\beta^{(n-1)} (u \underline{\otimes} v).$$

□

Corollary 2.35. *The $*$ defined by the M_{pq}'s can be rewritten as*

$$* = \varepsilon \otimes \varepsilon + \sum_{n \geq 1} M^{\otimes n} \circ T^\beta_{w_n} \circ (\delta^{(n-1)})^{\otimes 2}.$$

But this $*$ is not an associative product on $T^c(V)$ in general. Now we will generalize the definition of YB algebra by giving some compatibility conditions between M_{pq}'s and the braiding, and prove that under these conditions the new object makes $*$ to be associative automatically and $T^c(V)$ become a YB algebra with $*$.

Definition 2.36. *A* quantum $\mathbf{B}_\infty$-algebra (V, M, σ) *is a braided vector space* (V, σ) *equipped with a operation* $M = \oplus M_{pq}$, *where*

$$M_{pq} : V^{\otimes p} \otimes V^{\otimes q} \to V, \quad p \geq 0, \;\; q \geq 0,$$

satisfying

1.
$$\begin{cases} M_{00} &= 0, \\ M_{10} &= \mathrm{id}_V &= M_{01}, \\ M_{n0} &= 0 &= M_{0n}, \text{ for } n \geq 2, \end{cases}$$

2. *Yang-Baxter conditions: for any $i, j, k \geq 1$,*
$$\begin{cases} \beta_{1k}(M_{ij} \otimes \mathrm{id}_V^{\otimes k}) &= (\mathrm{id}_V^{\otimes k} \otimes M_{ij})\beta_{i+j,k}, \\ \beta_{i1}(\mathrm{id}_V^{\otimes i} \otimes M_{jk}) &= (M_{jk} \otimes \mathrm{id}_V^{\otimes i})\beta_{i,j+k}, \end{cases}$$

3. *Associativity condition: for any triple (i, j, k) of positive integers,*
$$\begin{aligned} &\sum_{r=1}^{i+j} M_{rk} \circ \big((M^{\otimes r} \circ \overline{\bigtriangleup_\beta}^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\big) \\ &= \sum_{l=1}^{j+k} M_{il} \circ \big(\mathrm{id}_V^{\otimes i} \otimes (M^{\otimes l} \circ \overline{\bigtriangleup_\beta}^{(l-1)})\big). \end{aligned} \tag{2.5}$$

Remark 2.37. *For any vector space V, (V, τ) is always a braided vector space with the usual flip τ. And the Yang-Baxter conditions in the above definition hold automatically. In this case, the quantum $\boldsymbol{B}_\infty$-algebra returns to the classical $\boldsymbol{B}_\infty$-algebra (for the definition of $\boldsymbol{B}_\infty$-algebras, one can see [19]).*

Examples. 1. A braided vector space (V,σ) is a quantum $\mathbf{B}_\infty$-algebra with $M_{ij}=0$ except for the pairs $(1,0)$ and $(0,1)$.

2. A YB algebra (A,m,σ) is a quantum $\mathbf{B}_\infty$-algebra with $M_{11}=m$ and $M_{ij}=0$ except for the pairs $(1,0)$, $(0,1)$ and $(1,1)$.

In the following, we adopt the notation $M_{(i_1,j_1,\dots,i_k,j_k)}=M_{i_1j_1}\otimes\cdots\otimes M_{i_kj_k}$.

Lemma 2.38. *Let (V,M,σ) be a quantum $\boldsymbol{B}_\infty$-algebra. Then for any $k,l\geq 1$, we have*

$$\begin{cases} \beta_{kl}(M_{(i_1,j_1,\dots,i_k,j_k)}\otimes \mathrm{id}_V^{\otimes l}) &= (\mathrm{id}_V^{\otimes l}\otimes M_{(i_1,j_1,\dots,i_k,j_k)})\beta_{i_1+j_1+\cdots+i_k+j_k,l},\\ \beta_{lk}(\mathrm{id}_V^{\otimes l}\otimes M_{(i_1,j_1,\dots,i_k,j_k)}) &= (M_{(i_1,j_1,\dots,i_k,j_k)}\otimes \mathrm{id}_V^{\otimes l})\beta_{l,i_1+j_1+\cdots+i_k+j_k}.\end{cases}$$

Proof. We use induction on k.

The case $k=1$ is trivial.

$$\begin{aligned}
&\beta_{k+1,l}(M_{(i_1,j_1,\dots,i_{k+1},j_{k+1})}\otimes \mathrm{id}_V^{\otimes l})\\
&= (\beta_{kl}\otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes k}\otimes\beta_{1l})(M_{(i_1,j_1,\dots,i_{k+1},j_{k+1})}\otimes \mathrm{id}_V^{\otimes l})\\
&= (\beta_{kl}\otimes \mathrm{id}_V)\Big(M_{(i_1,j_1,\dots,i_k,j_k)}\otimes\beta_{1l}(M_{i_{k+1}j_{k+1}}\otimes \mathrm{id}_V^{\otimes l})\Big)\\
&= (\beta_{kl}\otimes \mathrm{id}_V)\Big(M_{(i_1,j_1,\dots,i_k,j_k)}\otimes(\mathrm{id}_V^{\otimes l}\otimes M_{i_{k+1}j_{k+1}})\beta_{i_{k+1}+j_{k+1},l}\Big)\\
&= \Big(\beta_{kl}(M_{(i_1,j_1,\dots,i_k,j_k)}\otimes \mathrm{id}_V^{\otimes l})\otimes \mathrm{id}_V\Big)\\
&\quad\circ(\mathrm{id}_V^{\otimes i_1+\cdots+i_k+l}\otimes M_{i_{k+1}j_{k+1}})(\mathrm{id}_V^{\otimes i_1+\cdots+i_k}\otimes\beta_{i_{k+1}+j_{k+1},l})\\
&= \Big((\mathrm{id}_V^{\otimes l}\otimes M_{(i_1,j_1,\dots,i_k,j_k)})\beta_{i_1+j_1+\cdots+i_k+j_k,l}\otimes \mathrm{id}_V\Big)\\
&\quad\circ(\mathrm{id}_V^{\otimes i_1+\cdots+i_k+l}\otimes M_{i_{k+1}j_{k+1}})(\mathrm{id}_V^{\otimes i_1+\cdots+i_k}\otimes\beta_{i_{k+1}+j_{k+1},l})\\
&= (\mathrm{id}_V^{\otimes l}\otimes M_{(i_1,j_1,\dots,i_{k+1},j_{k+1})})\\
&\quad\circ(\beta_{i_1+j_1+\cdots+i_k+j_k,l}\otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes i_1+\cdots+i_k}\otimes\beta_{i_{k+1}+j_{k+1},l})\\
&= (\mathrm{id}_V^{\otimes l}\otimes M_{(i_1,j_1,\dots,i_k,j_k)})\beta_{i_1+j_1+\cdots+i_k+j_k,l}.
\end{aligned}$$

The another equality is proved similarly. □

Lemma 2.39. *Let $(C,\triangle,\sigma)$ be a YB coalgebra and 1_C be a group-like element of C. If $\sigma(1_C\otimes x)=x\otimes 1_C$ and $\sigma(x\otimes 1_C)=1_C\otimes x$ for any $x\in C$, then we have*

$$\begin{cases} (\mathrm{id}_C\otimes\overline{\triangle})\sigma &= \sigma_1\sigma_2(\overline{\triangle}\otimes \mathrm{id}_C),\\ (\overline{\triangle}\otimes \mathrm{id}_C)\sigma &= \sigma_2\sigma_1(\mathrm{id}_C\otimes\overline{\triangle}).\end{cases}$$

Proof. For any $x, y \in C$, we denote $\sigma(x \otimes y) = y' \otimes x'$. Then

$$\begin{aligned}
&(\mathrm{id}_C \otimes \overline{\triangle})\sigma(x \otimes y)\\
&= (\mathrm{id}_C \otimes \overline{\triangle})(y' \otimes x')\\
&= y' \otimes \overline{\triangle}(x')\\
&= y' \otimes \triangle(x') - y' \otimes 1_C \otimes x' - y' \otimes x' 1_C\\
&= (\mathrm{id}_C \otimes \triangle)\sigma(x \otimes y) - \sigma_1\sigma_2(1_C \otimes x \otimes y) - \sigma_1\sigma_2(x \otimes 1_C \otimes y)\\
&= \sigma_1\sigma_2\big((\triangle(x) - 1_C \otimes x - x \otimes 1_C) \otimes y\big)\\
&= \sigma_1\sigma_2(\overline{\triangle} \otimes \mathrm{id}_C)(x \otimes y).
\end{aligned}$$

The another equality is proved similarly. □

The following notation is adopted to shorten the length of identities. We denote by $\overline{\triangle}_{\beta(i_1,j_1,i_2,j_2)}$ the composition of $\overline{\triangle_\beta} : V^{\otimes i_1+i_2} \underline{\otimes} V^{\otimes j_1+j_2} \to (T(V)\underline{\otimes}T(V))\underline{\otimes}(T(V)\underline{\otimes}T(V))$ with the projection $(T(V)\underline{\otimes}T(V))\underline{\otimes}(T(V)\underline{\otimes}T(V)) \to V^{\otimes i_1}\underline{\otimes}V^{\otimes j_1}\underline{\otimes}V^{\otimes i_2}\underline{\otimes}V^{\otimes j_2}$. And we denote

$$\overline{\triangle}^{(k-1)}_{\beta(i_1,j_1,\ldots,i_k,j_k)} = (\overline{\triangle}_{\beta(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+i_k+j_k}) \circ \overline{\triangle}^{(k-2)}_{\beta(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_k,j_k)},$$

the map from $V^{\otimes i_1+\cdots+i_k}\underline{\otimes}V^{\otimes j_1+\cdots+j_k}$ to $V^{\otimes i_1}\underline{\otimes}V^{\otimes j_1}\underline{\otimes}\cdots\underline{\otimes}V^{\otimes i_k}\underline{\otimes}V^{\otimes j_k}$.

Lemma 2.40. *For any $k, l \geq 1$, we have*

$$\begin{cases}
\beta_{i_1+j_1+\cdots+i_k+j_k,l}(\overline{\triangle}^{(k-1)}_{\beta(i_1,j_1,\ldots,i_k,j_k)} \otimes \mathrm{id}_V^{\otimes l})\\
\quad = (\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle}^{(k-1)}_{\beta(i_1,j_1,\ldots,i_k,j_k)})\beta_{i_1+j_1+\cdots+i_k+j_k,l},\\
\beta_{l,i_1+j_1+\cdots+i_k+j_k}(\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle}^{(k-1)}_{\beta(i_1,j_1,\ldots,i_k,j_k)})\\
\quad = (\overline{\triangle}^{(k-1)}_{\beta(i_1,j_1,\ldots,i_k,j_k)} \otimes \mathrm{id}_V^{\otimes l})\beta_{l,i_1+j_1+\cdots+i_k+j_k}.
\end{cases}$$

Proof. Since $(T^c(V)^{\otimes 2}, \triangle_\beta, T^\beta_{\chi_{22}})$ is a YB coalgebra, by the above lemma, we have

$$\begin{cases}
(\mathrm{id}_{T^c(V)^{\otimes 2}} \otimes \overline{\triangle_\beta})T^\beta_{\chi_{22}}\\
\quad = (T^\beta_{\chi_{22}} \otimes \mathrm{id}_{T^c(V)^{\otimes 2}})(\mathrm{id}_{T^c(V)^{\otimes 2}} \otimes T^\beta_{\chi_{22}})(\overline{\triangle_\beta} \otimes \mathrm{id}_C),\\
(\overline{\triangle_\beta} \otimes \mathrm{id}_{T^c(V)^{\otimes 2}})T^\beta_{\chi_{22}}\\
\quad = (\mathrm{id}_{T^c(V)^{\otimes 2}} \otimes T^\beta_{\chi_{22}})(T^\beta_{\chi_{22}} \otimes \mathrm{id}_{T^c(V)^{\otimes 2}})(\mathrm{id}_C \otimes \overline{\triangle_\beta}).
\end{cases}$$

On $V^{\otimes i}\underline{\otimes}V^{\otimes j}\underline{\otimes}V^{\otimes k}\underline{\otimes}V^{\otimes l}$, $T^\beta_{\chi_{22}} = T^\sigma_{\chi_{i+j,k+l}} = \beta_{i+j,k+l}$.

So on $V^{\otimes i_1+i_2}\underline{\otimes}V^{\otimes j_1+j_2}\underline{\otimes}V^{\otimes r}\underline{\otimes}V^{\otimes s}$,

$$(\mathrm{id}_V^{\otimes r+s} \otimes \overline{\triangle}_{\beta(i_1,j_1,i_2,j_2)}\beta_{i_1+j_1+i_2+j_2,r+s})$$

$$= \quad (\beta_{i_1+j_1,r+s} \otimes \mathrm{id}_V^{\otimes i_2+j_2})(\mathrm{id}_V^{\otimes i_1+j_1} \otimes \beta_{i_2+j_2,r+s})(\overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes r+s}),$$

and on $V^{\otimes i} \underline{\otimes} V^{\otimes j} \underline{\otimes} V^{\otimes k} \underline{\otimes} V^{\otimes l}$,

$$\begin{aligned}
&(\overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes r+s})\beta_{r+s,i_1+j_1+i_2+j_2} \\
&= \quad (\mathrm{id}_V^{\otimes i_1+j_1} \otimes \beta_{r+s,i_2+j_2})(\beta_{r+s,i_1+j_1} \otimes \mathrm{id}_V^{\otimes i_2+j_2})(\mathrm{id}_V^{\otimes r+s} \otimes \overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)}).
\end{aligned}$$

In order to prove our lemma, we use induction on k and the above formulas for $r = l$ and $s = 0$.

The cases $k = 1$ and $k = 2$ are trivial.

$$\begin{aligned}
&\beta_{i_1+j_1+\cdots+i_{k+1}+j_{k+1},l}(\overline{\triangle_\beta}^{(k)}_{(i_1,j_1,\ldots,i_k,j_k)} \otimes \mathrm{id}_V^{\otimes l}) \\
&= (\beta_{i_1+i_2+j_1+j_2,l} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}})(\mathrm{id}_V^{\otimes i_1+j_1+i_2+j_2} \otimes \beta_{i_3+j_3+\cdots+j_{k+1},l}) \\
&\quad \circ(\overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}+l})(\overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})} \otimes \mathrm{id}_V^{\otimes l}) \\
&= (\beta_{i_1+i_2+j_1+j_2,l} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}})(\overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}+l}) \\
&\quad \circ(\mathrm{id}_V^{\otimes i_1+j_1+i_2+j_2} \otimes \beta_{i_3+j_3+\cdots+j_{k+1},l})(\overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})} \otimes \mathrm{id}_V^{\otimes l}) \\
&= (\beta_{i_1+j_1+i_2+j_2,l}(\overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes l}) \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}}) \\
&\quad \circ(\mathrm{id}_V^{\otimes i_1+j_1+i_2+j_2} \otimes \beta_{i_3+j_3+\cdots+j_{k+1},l})(\overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})} \otimes \mathrm{id}_V^{\otimes l}) \\
&= ((\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)})\beta_{i_1+j_1+i_2+j_2,l} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}}) \\
&\quad \circ(\mathrm{id}_V^{\otimes i_1+j_1+i_2+j_2} \otimes \beta_{i_3+j_3+\cdots+j_{k+1},l})(\overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})} \otimes \mathrm{id}_V^{\otimes l}) \\
&= (\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}}) \\
&\quad \circ\beta_{i_1+j_1+\cdots+j_{k+1},l}(\overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})} \otimes \mathrm{id}_V^{\otimes l}) \\
&= (\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle_\beta}_{(i_1,j_1,i_2,j_2)} \otimes \mathrm{id}_V^{\otimes i_3+j_3+\cdots+j_{k+1}}) \\
&\quad \circ(\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle_\beta}^{(k-1)}_{(i_1+i_2,j_1+j_2,i_3,j_3,\ldots,i_{k+1},j_{k+1})})\beta_{i_1+j_1+\cdots+j_{k+1},l} \\
&= (\mathrm{id}_V^{\otimes l} \otimes \overline{\triangle_\beta}^{(k)}_{(i_1,j_1,\ldots,i_{k+1},j_{k+1})})\beta_{i_1+j_1+\cdots+i_{k+1}+j_{k+1},l}.
\end{aligned}$$

The another equality can be proved similarly. □

Proposition 2.41. *Let (V, M, σ) be a quantum $\boldsymbol{B}_\infty$-algebra. Then we have*

$$\begin{cases} \beta(* \otimes \mathrm{id}_{T^c(V)}) &= (\mathrm{id}_{T^c(V)} \otimes *)\beta_1\beta_2, \\ \beta(\mathrm{id}_{T^c(V)} \otimes *) &= (* \otimes \mathrm{id}_{T^c(V)})\beta_2\beta_1, \end{cases}$$

where $* = \varepsilon \otimes \varepsilon + \sum_{r\geq 1} M^{\otimes r} \circ \overline{\triangle_\beta}^{(r-1)}$.

Proof. We only need to verify that for all $k, l \geq 1$,

$$\begin{cases} \beta_{kl}\big((M_{(i_1,j_1,\dots,i_k,j_k)} \circ \overline{\triangle_{\beta}}^{(k-1)}_{(i_1,j_1,\dots,i_k,j_k)}) \otimes \mathrm{id}_V^{\otimes l}\big) \\ \quad = \big(\mathrm{id}_V^{\otimes l} \otimes (M_{(i_1,j_1,\dots,i_k,j_k)} \circ \overline{\triangle_{\beta}}^{(k-1)}_{(i_1,j_1,\dots,i_k,j_k)})\big)\beta_{i_1+j_1+\dots+i_k+j_k,l}, \\ \beta_{lk}\big(\mathrm{id}_V^{\otimes l} \otimes (M_{(i_1,j_1,\dots,i_k,j_k)} \circ \overline{\triangle_{\beta}}^{(k-1)}_{(i_1,j_1,\dots,i_k,j_k)})\big) \\ \quad = \big((M_{(i_1,j_1,\dots,i_k,j_k)} \circ \overline{\triangle_{\beta}}^{(k-1)}_{(i_1,j_1,\dots,i_k,j_k)}) \otimes \mathrm{id}_V^{\otimes l}\big)\beta_{l,i_1+j_1+\dots+i_k+j_k}. \end{cases}$$

They follow from the above lemmas immediately. □

Theorem 2.42. *Let (V, M, σ) be a quantum $\boldsymbol{B}_\infty$-algebra. Then $(T(V), *, \beta)$ is a YB algebra.*

Proof. We only need to show that $*$ is associative. First we show that $*(* \otimes \mathrm{id}_{T^c(V)})$ and $*(\mathrm{id}_{T^c(V)} \otimes *)$ are coalgebra maps from $(T^c(V)^{\otimes 3}, \triangle_{\beta,3})$ to $T^c(V)$.

$$\begin{aligned} &\delta \circ *(* \otimes \mathrm{id}_{T^c(V)}) \\ &= (* \otimes *) \circ \triangle_\beta \circ (* \otimes \mathrm{id}_{T^c(V)}) \\ &= (* \otimes *) \circ \beta_2 \circ \delta^{\otimes 2} \circ (* \otimes \mathrm{id}_{T^c(V)}) \\ &= (* \otimes *) \circ \beta_2 \circ (\delta * \otimes \delta) \\ &= (* \otimes *) \circ \beta_2 \circ ((* \otimes *) \circ \triangle_\beta \otimes \delta) \\ &= (* \otimes *) \circ \beta_2 \circ (* \otimes * \otimes \mathrm{id}_{T^c(V)} \otimes \mathrm{id}_{T^c(V)}) \circ \beta_2 \circ \beta^{\otimes 3} \\ &= (* \otimes *) \circ (* \otimes \beta(* \otimes \mathrm{id}_{T^c(V)}) \otimes \mathrm{id}_{T^c(V)}) \circ \beta_2 \circ \delta^{\otimes 3} \\ &= (* \otimes *) \circ (* \otimes (\mathrm{id}_{T^c(V)} \otimes *)\beta_1\beta_2 \otimes \mathrm{id}_{T^c(V)}) \circ \beta_2 \circ \delta^{\otimes 3} \\ &= (* \otimes *) \circ (* \otimes \mathrm{id}_{T^c(V)} \otimes * \otimes \mathrm{id}_{T^c(V)}) \circ \beta_3\beta_4\beta_2 \circ \delta^{\otimes 3} \\ &= (* \otimes *) \circ (* \otimes \mathrm{id}_{T^c(V)} \otimes * \otimes \mathrm{id}_{T^c(V)}) \circ T^\beta_{w_3^{-1}} \circ \delta^{\otimes 3} \\ &= (*(* \otimes \mathrm{id}_{T^c(V)}) \otimes *(* \otimes \mathrm{id}_{T^c(V)}))\, \triangle_{\beta,3}\,. \end{aligned}$$

The first and third equalities follow from the fact that $* : T^C(V)\underline{\otimes} T^c(V) \to T^c(V)$ is a coalgebra map.

Similarly, we can prove that $*(\mathrm{id}_{T^c(V)} \otimes *)$ is also a coalgebra map.

Now we show that $P_V \circ *(* \otimes \mathrm{id}_{T^c(V)}) = P_V \circ *(\mathrm{id}_{T^c(V)} \otimes *)$. On $V^{\otimes i}\underline{\otimes} V^{\otimes j}\underline{\otimes} V^{\otimes k}$,

$$\begin{aligned} &P_V \circ \big(* (* \otimes \mathrm{id}_{T^c(V)})\big) \\ &= P_V\Big(\sum_{s=1}^{i+j+k} M^{\otimes s} \circ \overline{\triangle_\beta}^{(s-1)} \circ \big(\sum_{r=1}^{i+j} (M^{\otimes r} \circ \overline{\triangle_\beta}^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=1}^{i+j} M_{rk} \circ \big((M^{\otimes r} \circ \overline{\triangle_\beta}^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k} \big) \\
&= \sum_{l=1}^{j+k} M_{il} \circ \big(\mathrm{id}_V^{\otimes i} \otimes (M^{\otimes l} \circ \overline{\triangle_\beta}^{(l-1)}) \big) \\
&= P_V \Big(\sum_{s=1}^{i+j+k} M^{\otimes s} \circ \overline{\triangle_\beta}^{(s-1)} \circ \big(\sum_{l=1}^{j+k} \mathrm{id}_V^{\otimes i} \otimes (M^{\otimes l} \circ \overline{\triangle_\beta}^{(l-1)}) \big) \Big) \\
&= P_V \circ *(\mathrm{id}_{T^c(V)} \otimes *).
\end{aligned}
$$

Then by the Corollary 4.4, we have that $*(* \otimes \mathrm{id}_{T^c(V)}) = *(\mathrm{id}_{T^c(V)} \otimes *)$. The compatibility conditions for the unit and braiding are trivial. □

Remark 2.43. *By using the dual construction stated in Remark 2.3.3, we can easily define* quantum $\mathbf{B}_\infty$-coalgebras *and prove that they provide YB coalgebras.*

Example (Reconstruction of quantum shuffle algebras). Let (V, σ) be a braided vector space. Then (V, M, σ) is a $\mathbf{B}_\infty$ -algebra with $M_{10} = \mathrm{id}_V = M_{01}$ and $M_{pq} = 0$ for other cases. The resulting algebra $T(V)$ in the above theorem is just the quantum shuffle algebra, i.e., $* = ш_\sigma$.

Example (Quantum quasi-shuffle algebras). Let (V, m, σ) be a YB algebra. Then (V, M, σ) is a $\mathbf{B}_\infty$ -algebra with $M_{10} = \mathrm{id}_V = M_{01}$, $M_{11} = m$ and $M_{pq} = 0$ for other cases. The resulting algebra $T(V)$ in the above theorem is called the quantum quasi-shuffle algebra. We denote by $\Join_\sigma$ the quantum quasi-shuffle product. This new product has the following inductive relation: for any $u_1, \ldots, u_i, v_1, \ldots, v_j \in A$,

$$
\begin{aligned}
&(u_1 \otimes \cdots \otimes u_i) \Join_\sigma (v_1 \otimes \cdots \otimes v_j) \\
&= \big((u_1 \otimes \cdots \otimes u_i) \Join_\sigma (v_1 \otimes \cdots \otimes v_{j-1}) \big) \otimes v_j \\
&\quad + (\Join_\sigma \otimes \mathrm{id}_A) \sigma_{i+j-1} \cdots \sigma_i (u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j) \\
&\quad + (\Join_\sigma \otimes m) \sigma_{i+j-2} \cdots \sigma_i (u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j). \qquad (2.6)
\end{aligned}
$$

It is the generalization of quantum shuffle algebra and the quantization of the classical quasi-shuffle algebra. We will discuss systematically the quantum quasi-shuffle algebra in next chapter.

Proposition 2.44. *Let V be a Yetter-Drinfel'd module over a Hopf algebra H which is both a module-algebra and comodule-algebra with multiplication m_V. Then the quantum quasi-shuffle algebra built on V is a module-algebra with the diagonal action and a comodule-algebra with the diagonal coaction.*

Proof. We use induction to prove the statement. On $V \underline{\otimes} V$, $\Join_\sigma = m_V + ш_\sigma$. Since $T_\sigma(V)$ is both a module-algebra and comodule-algebra with the diagonal action and coaction respectively, and m_V is both a module map and comodule map, the result holds. Using the inductive relation (6) to reduce the degree, the rest of the proof follows from that m_V is both a module map and comodule map. □

Remark 2.45. *We make the assumptions above. Using the inductive relation (6), we can define a map $\bowtie_\sigma$: $T(V)\otimes T(V)\to T(V)$. It is not difficult to prove by induction that this $\bowtie_\sigma$ defines an associative product on $T(V)$. Noticing that the natural braiding of the Yetter-Drinfel'd module $T(V)$ is just β, $T(V)$ satisfies all conditions of Theorem 2.11. Hence we can reprove that $(T(V),\bowtie_\sigma,\beta)$ is a YB algebra in this special case.*

Let (V,M,σ) be a quantum $\mathbf{B}_\infty$-algebra and $*$ be the product constructed by M and σ as before. We denote by $Q_\sigma(V)$ the subalgebra of $(T(V),*)$ generated by V. If we define $*^n: V^{\otimes n+1}\to T(V)$ by $v_1\otimes\cdots\otimes v_{n+1}\mapsto v_1*\cdots*v_{n+1}$, and $*^0=\mathrm{id}_V$ for convenience, then $Q_\sigma(V)=K\oplus\oplus_{n\geq 0}\mathrm{Im}*^n$. This algebra is a generalization of the quantum symmetric algebra over V.

Proposition 2.46. *$(Q_\sigma(V),\beta)$ is a YB algebra.*

Proof. In order to prove the statement, we only need to verify that β is a braiding on $Q_\sigma(V)$. In fact, we have that $\beta(*^k\otimes *^l)=(*^l\otimes *^k)\beta_{k+1,l+1}$. We use induction on $k+l$.

The case $k=l=0$ is trivial since $\sigma(\mathrm{id}_V\otimes\mathrm{id}_V)=(\mathrm{id}_V\otimes\mathrm{id}_V)\sigma$.

When $k+l\geq 1$,

$$\begin{aligned}
\beta(*^k\otimes *^l) &= \beta(*\otimes\mathrm{id}_{T(V)})(\mathrm{id}_V\otimes *^{k-1}\otimes *^l)\\
&= (\mathrm{id}_{T(V)}\otimes *)\beta_1\beta_2(\mathrm{id}_V\otimes *^{k-1}\otimes *^l)\\
&= (\mathrm{id}_{T(V)}\otimes *)\beta_1\big(\mathrm{id}_V\otimes\beta(*^{k-1}\otimes *^l)\big)\\
&= (\mathrm{id}_{T(V)}\otimes *)\beta_1(\mathrm{id}_V\otimes *^l\otimes *^{k-1})(\mathrm{id}_V\otimes\beta_{k,l+1})\\
&= (\mathrm{id}_{T(V)}\otimes *)(\beta(\mathrm{id}_V\otimes *^l)\otimes *^{k-1})(\mathrm{id}_V\otimes\beta_{k,l+1})\\
&= (*^l\otimes *^k)(\beta_{1,l+1}\otimes\mathrm{id}_V^{\otimes k})(\mathrm{id}_V\otimes\beta_{k,l+1})\\
&= (*^l\otimes *^k)\beta_{k+1,l+1}.
\end{aligned}$$

□

For any quantum $\mathbf{B}_\infty$-algebra (V,M,σ), if we endow $T(V)$ with the usual grading, then the algebra $(T(V),*)$ is not graded in general. But with this grading, we have:

Proposition 2.47. *The term of highest degree in the product $*$ is the quantum shuffle product.*

Proof. We need to verify that for any $i,j\geq 1$, $M^{\otimes i+j}\overline{\triangle_\beta}^{(i+j-1)}=\sum_{w\in\mathfrak{S}_{ij}}T^\sigma_w$. We use induction on $i+j$. When $i=j=1$, $M^{\otimes 2}\overline{\triangle_\beta}(u\otimes v)=u\otimes v+\sigma(u\otimes v)=u ш_\sigma v$.

$$M^{\otimes i+j}\overline{\triangle_\beta}^{(i+j-1)}(u_1\otimes\cdots\otimes u_i\underline{\otimes}v_1\otimes\cdots\otimes v_j)$$

$$
\begin{aligned}
&= \Big((M^{\otimes i+j-1}\overline{\triangle_\beta}^{(i+j-2)}) \otimes M\Big)\overline{\triangle_\beta}(u_1 \otimes \cdots \otimes u_i \underline{\otimes} v_1 \otimes \cdots \otimes v_j) \\
&= \Big((M^{\otimes i+j-1}\overline{\triangle_\beta}^{(i+j-2)}) \otimes M\Big)(u_1 \otimes \cdots \otimes u_i \underline{\otimes} v_1 \otimes \cdots \otimes v_{j-1} \underline{\otimes} 1 \underline{\otimes} v_j \\
&\quad + u_1 \otimes \cdots \otimes u_{i-1} \underline{\otimes} \beta_{1j}(u_i \underline{\otimes} v_1 \otimes \cdots \otimes v_j) \underline{\otimes} 1) \\
&= \Big(\sum_{w \in \mathfrak{S}_{i,j-1}} T_w^\sigma \otimes \mathrm{id}_V + \sum_{w' \in \mathfrak{S}_{i-1,j}} (T_w^\sigma \otimes \mathrm{id}_V)\sigma_{i+j-1} \cdots \sigma_i\Big)(u_1 \otimes \cdots \otimes v_j) \\
&= (u_1 \otimes \cdots \otimes u_i) ш_\sigma (v_1 \otimes \cdots \otimes v_j).
\end{aligned}
$$

The third equality follows from the fact that $w \in \mathfrak{S}_{i,j}$ implies either $w(i+j) = i+j$ or $w(i) = i+j$. □

From the classical theory (cf. [19]), we also know that $(T^c(V), *)$ has an antipode S given by $S(1) = 1$ and $S(x) = \sum_{n \geq 0} (-1)^{n+1} *^{\otimes n} \circ \overline{\delta}^{(n)}(x)$ for any $x \in \mathrm{Ker}\varepsilon$.

2.5 Construction of quantum B_∞-algebras

We now introduce a new notion motivated by [19].

Definition 2.48. *A* unital 2-YB algebra *is a braided vector space (V, σ) equipped with two associative algebra structure $*$ and $\cdot$, which share the same unit, such that both $(V, *, \sigma)$ and $(V, \cdot, \sigma)$ are YB algebras. We denote a 2-YB algebra by $(V, *, \cdot, \sigma)$.*

Examples. 1. Let (A, m, α) be a YB algebra. Then (A, m, m, α) is a trivial unital 2-YB algebra.

2. Let (V, σ) be a braided vector space. Then $(T(V), m, ш_\sigma, \beta)$ is a unital 2-YB algebra, where m is the concatenation product.

Let $(V, *, \cdot, \sigma)$ be a unital 2-YB algebra. We adopt the notation $\cdot^k = \cdot^{\otimes k} : V^{\otimes k+1} \to V$. We define $M_{pq} : V^{\otimes p} \otimes V^{\otimes q} \to V$ for $p, q \geq 0$ inductively as follows:

$$
\begin{cases}
M_{00} = 0, \\
M_{10} = \mathrm{id}_V = M_{01}, \\
M_{n0} = 0 = M_{0n}, \text{ for } n \geq 2,
\end{cases}
$$

and

$$
\begin{aligned}
&M_{pq}(u_1 \otimes \cdots \otimes u_p \underline{\otimes} v_1 \otimes \cdots \otimes v_q) \\
&= (u_1 \cdot \cdots \cdot u_p) * (v_1 \cdot \cdots \cdot v_q)
\end{aligned}
$$

$$-\sum_{k=2}^{p+q}\sum_{I_k,J_k} \cdot^{k-1} M_{(i_i,j_1,\dots,i_k,j_k)} \circ \overline{\triangle}^{(k-1)}_{\beta_{(i_1,j_1,\dots,i_k,j_k)}}(u_1\otimes\cdots\otimes u_p\underline{\otimes} v_1\otimes\cdots\otimes v_q),$$

where $I_k=(i_1,\dots,i_k)$ and $J_k=(j_1,\dots,j_k)$ run through all the partitions of length k of p and q respectively.

For instance,

$$\begin{aligned} M_{11}(u\underline{\otimes} v) &= u*v \\ &\quad -\cdot(M_{01}\otimes M_{10})(1\otimes\sigma(u\otimes v)\otimes 1) \\ &\quad -\cdot(M_{10}\otimes M_{01})(u\otimes\sigma(1\otimes 1)\otimes v) \\ &= u*v-\cdot\sigma(u\otimes v)-u\cdot v, \end{aligned}$$

$$\begin{aligned} M_{21}(u\otimes v\underline{\otimes} w) &= (u\cdot v)*w \\ &\quad -u\cdot M_{11}(v\underline{\otimes} w)-\cdot(M_{11}\otimes \mathrm{id}_V)(u\otimes\sigma(v\otimes w)) \\ &\quad -\cdot^2\big(u\otimes v\otimes w+\sigma_2(u\otimes v\otimes w)+\sigma_1\sigma_2(u\otimes v\otimes w)\big) \\ &= (u\cdot v)*w-u\cdot(v*w) \\ &\quad +\cdot^2\sigma_2(u\otimes v\otimes w)-\cdot(*\otimes\mathrm{id}_V)\sigma_2(u\otimes v\otimes w), \end{aligned}$$

and

$$\begin{aligned} M_{12}(u\underline{\otimes} v\otimes w) &= u*(v\cdot w)-(u*v)\cdot w \\ &\quad +\cdot^2\sigma_1(u\otimes v\otimes w)-\cdot(\mathrm{id}_V\otimes *)\sigma_1(u\otimes v\otimes w). \end{aligned}$$

Proposition 2.49. *Let $(V,*,\cdot,\sigma)$ be a unital 2-YB algebra and $M=(M_{pq})$ be the maps defined above. Then (V,M,σ) is a quantum $\boldsymbol{B}_\infty$-algebra.*

Proof. First we verify the Yang-Baxter conditions. We use induction on $i+j+k$.

When $i=j=k=1$,

$$\begin{aligned} \beta_{11}(M_{11}\otimes \mathrm{id}_V) &= \sigma(*\otimes\mathrm{id}_V-(\cdot\otimes\mathrm{id}_V)\sigma_1-\cdot\otimes\mathrm{id}_V) \\ &= (\mathrm{id}_V\otimes *)\sigma_1\sigma_2-(\mathrm{id}_V\otimes\cdot)\sigma_1\sigma_2\sigma_1-(\mathrm{id}_V\otimes\cdot)\sigma_1\sigma_2 \\ &= (\mathrm{id}_V\otimes *)\sigma_1\sigma_2-(\mathrm{id}_V\otimes\cdot)\sigma_2\sigma_1\sigma_2-(\mathrm{id}_V\otimes\cdot)\sigma_1\sigma_2 \\ &= \big(\mathrm{id}_V\otimes(*-\cdot\sigma-\cdot)\big)\sigma_1\sigma_2 \\ &= (\mathrm{id}_V\otimes M_{11})\beta_{21}. \end{aligned}$$

$$\begin{aligned} &\beta_{1k}(M_{pq}\otimes\mathrm{id}_V^{\otimes k}) \\ &\quad= \beta_{1k}(*\otimes\mathrm{id}_V^{\otimes k})(\cdot^{p-1}\otimes\cdot^{q-1}\otimes\mathrm{id}_V^{\otimes k}) \\ &\qquad -\sum\beta_{1k}(\cdot^{r-1}\otimes\mathrm{id}^{\otimes l})\Big((M_{(i_i,j_1,\dots,i_r,j_r)}\circ\overline{\triangle}_{\beta_{(i_1,j_1,\dots,i_r,j_r)}})\otimes\mathrm{id}_V^{\otimes l}\Big) \end{aligned}$$

$$
\begin{aligned}
&= (\mathrm{id}_V^{\otimes k} \otimes *)(\beta_{1k} \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes \beta_{1k})(\cdot^{p-1} \otimes \cdot^{q-1} \otimes \mathrm{id}_V^{\otimes k}) \\
&\quad - \sum (\mathrm{id}^{\otimes l} \otimes \cdot^{r-1})\beta_{rk}\Big((M_{(i_i,j_1,\ldots,i_r,j_r)} \circ \overline{\triangle_\beta}_{(i_1,j_1,\ldots,i_r,j_r)}) \otimes \mathrm{id}^{\otimes l}\Big) \\
&= (\mathrm{id}_V^{\otimes k} \otimes *)(\mathrm{id}_V^{\otimes k} \otimes \cdot^{p-1} \otimes \cdot^{q-1})\beta_{p+q,k} \\
&\quad - \sum (\mathrm{id}^{\otimes l} \otimes \cdot^{r-1})\Big((M_{(i_i,j_1,\ldots,i_r,j_r)} \circ \overline{\triangle_\beta}_{(i_1,j_1,\ldots,i_r,j_r)}) \otimes \mathrm{id}_V^{\otimes l}\Big)\beta_{p+q,k} \\
&= (\mathrm{id}_V^{\otimes k} \otimes M_{pq})\beta_{p+q,k}.
\end{aligned}
$$

The third equality follows from the induction hypothesis and a similar method used in the proof of Proposition 4.12.

The condition $\beta_{i1}(\mathrm{id}_V^{\otimes i} \otimes M_{jk}) = (M_{jk} \otimes \mathrm{id}_V^{\otimes i})\beta_{i,j+k}$ can be verified similarly.

Now we want to prove that $M = (M_{pq})$ also satisfy the associativity condition. We also use induction on $i + j + k$.

When $i = j = k = 1$, the associativity condition is just $M_{11}(M_{11} \otimes \mathrm{id}_V) + M_{21} + M_{21}\sigma_1 = M_{11}(\mathrm{id}_V \otimes M_{11}) + M_{12} + M_{12}\sigma_2$

$$
\begin{aligned}
&M_{11}(M_{11} \otimes \mathrm{id}_V) + M_{21} + M_{21}\sigma_1 \\
&= *^2 - \cdot\sigma(* \otimes \mathrm{id}_V) - \cdot(* \otimes \mathrm{id}_V) \\
&\quad - *(\cdot \otimes \mathrm{id}_V)\sigma_1 + \cdot\sigma(\cdot \otimes \mathrm{id}_V)\sigma_1 + \cdot^2\sigma_1 \\
&\quad - *(\cdot \otimes \mathrm{id}_V) + \cdot\sigma(\cdot \otimes \mathrm{id}_V) + \cdot^2 \\
&\quad + *(\cdot \otimes \mathrm{id}_V) - \cdot(\mathrm{id}_V \otimes *) + \cdot^2\sigma_2 - \cdot(* \otimes \mathrm{id}_V)\sigma_2 \\
&\quad + *(\cdot \otimes \mathrm{id}_V)\sigma_1 - \cdot(\mathrm{id}_V \otimes *)\sigma_1 + \cdot^2\sigma_2\sigma_1 - \cdot(* \otimes \mathrm{id}_V)\sigma_2\sigma_1 \\
&= *^2 - \cdot(\mathrm{id}_V \otimes *)\sigma_1\sigma_2 - \cdot(* \otimes \mathrm{id}_V) \\
&\quad + \cdot^2\,\sigma_1\sigma_2\sigma_1 + \cdot^2\sigma_1 + \cdot^2\sigma_1\sigma_2 + \cdot^2 \\
&\quad - \cdot\,(\mathrm{id}_V \otimes *) + \cdot^2\sigma_2 - \cdot(* \otimes \mathrm{id}_V)\sigma_2 \\
&\quad - \cdot\,(\mathrm{id}_V \otimes *)\sigma_1 + \cdot^2\sigma_2\sigma_1 - \cdot\sigma(\mathrm{id}_V \otimes *) \\
&= *^2 - \cdot\sigma(\mathrm{id}_V \otimes *) - \cdot(\mathrm{id}_V \otimes *) \\
&\quad - *(\mathrm{id}_V \otimes \cdot)\sigma_2 + \cdot\sigma(\mathrm{id}_V \otimes \cdot)\sigma_2 + \cdot^2\sigma_2 \\
&\quad - *(\mathrm{id}_V \otimes \cdot) + \cdot\sigma(\mathrm{id}_V \otimes \cdot) + \cdot(\mathrm{id}_V \otimes \cdot) \\
&\quad + *(\mathrm{id}_V \otimes \cdot) - \cdot(* \otimes \mathrm{id}_V) + \cdot^2\sigma_1 - \cdot(\mathrm{id}_V \otimes *)\sigma_1 \\
&\quad + *(\mathrm{id}_V \otimes \cdot)\sigma_2 - \cdot(* \otimes \mathrm{id}_V)\sigma_2 + \cdot^2\sigma_1\sigma_2 - \cdot(\mathrm{id}_V \otimes *)\sigma_1\sigma_2 \\
&= M_{11}(\mathrm{id}_V \otimes M_{11}) + M_{12} + M_{12}\sigma_2.
\end{aligned}
$$

For $i+j+k \geq 2$, we notice that $*(\cdot^{i-1} \otimes \cdot^{j-1}) = \sum_{r\geq 1} \cdot^{r-1} M^{\otimes r}\overline{\triangle_\beta}^{(r-1)} = \sum_{r\geq 1} \cdot^{r-1} M^{\otimes r}\triangle_\beta^{(r-1)}$.

$$
\begin{aligned}
&\sum_{r=1}^{i+j} M_{rk} \circ \big((M^{\otimes r} \circ \overline{\triangle_\beta}^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\big) \\
&= \sum_{r\geq 1} \big(*(\cdot^{r-1} \otimes \cdot^{r-1}) - \sum_{l\geq 2} \cdot^{l-1} M^{\otimes l} \circ \triangle_\beta^{(l-1)})\big) \circ \big((M^{\otimes r} \circ \triangle_\beta^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= *\Big((\sum_{r\geq 1} \cdot^{r-1} M^{\otimes r} \circ \triangle_\beta^{(r-1)}) \otimes \cdot^{k-1}\Big) \\
&\quad - \sum_{r\geq 1}\sum_{l\geq 2} \cdot^{l-1} M^{\otimes l} \circ \triangle_\beta^{(l-1)} \circ \Big((M^{\otimes r} \circ \triangle_\beta^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\Big) \\
&= *\Big(* (\cdot^{i-1} \otimes \cdot^{j-1}) \otimes \cdot^{k-1}\Big) \\
&\quad - \sum_{r\geq 1}\sum_{l\geq 2} \cdot\Big((\cdot^{l-2} M^{\otimes l-1} \circ \triangle_\beta^{(l-2)}) \otimes M\Big) \triangle_\beta \circ\Big((M^{\otimes r} \circ \triangle_\beta^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\Big) \\
&= *(* \otimes \mathrm{id}_V)(\cdot^{i-1} \otimes \cdot^{j-1} \otimes \cdot^{k-1}) \\
&\quad - \sum_{r\geq 1}\sum_{\underline{2}} \cdot\Big(* (\cdot^{p_1-1} \otimes \cdot^{q_1-1}) \otimes M_{p_2,q_2}\Big) \\
&\qquad\qquad \circ \triangle_{\beta\ (p_1,q_1,p_2,q_2)} \circ\Big((M^{\otimes r} \circ \triangle_\beta^{(r-1)}) \otimes \mathrm{id}_V^{\otimes k}\Big) \\
&= *(* \otimes \mathrm{id}_V)(\cdot^{i-1} \otimes \cdot^{j-1} \otimes \cdot^{k-1}) \\
&\quad - \sum_{r\geq 1}\sum_{\underline{2}} \cdot\Big(* (\cdot^{p_1-1} \otimes \cdot^{q_1-1}) \otimes M_{p_2,q_2}\Big) \circ (\mathrm{id}_V^{\otimes p_1} \otimes \beta_{p_2,q_1} \otimes \mathrm{id}_V^{\otimes q_2}) \\
&\qquad \circ(\sum M_{(r_1,s_1,\ldots,r_{p_1},s_{p_1})} \triangle^{(p_1-1)}_{\beta\ (r_1,s_1,\ldots,r_{p_1},s_{q_1})} \\
&\qquad\quad \otimes M_{(r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \triangle^{(p_2-1)}_{\beta\ (r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \\
&\qquad\quad \otimes \mathrm{id}_V^{\otimes q_1} \otimes \mathrm{id}_V^{\otimes q_2}) \\
&\qquad \circ(\triangle_{\beta\ (r_1+\cdots+r_{p_1},s_1+\cdots+s_{p_1},r_{p_1+1}+\cdots+r_{p_1+p_2},s_{p_1+1}+\cdots+s_{p_1+p_2})} \otimes \mathrm{id}_V^{\otimes k}) \\
&= *(* \otimes \mathrm{id}_V)(\cdot^{i-1} \otimes \cdot^{j-1} \otimes \cdot^{k-1}) \\
&\quad - \sum_{r\geq 1}\sum_{\underline{2}} \cdot\Big(* (\cdot^{p_1-1} \otimes \cdot^{q_1-1}) \otimes M_{p_2,q_2}\Big) \\
&\qquad \circ(\sum M_{(r_1,s_1,\ldots,r_{p_1},s_{p_1})} \triangle^{(p_1-1)}_{\beta\ (r_1,s_1,\ldots,r_{p_1},s_{q_1})} \otimes \mathrm{id}_V^{\otimes q_1} \\
&\qquad \otimes M_{(r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \triangle^{(p_2-1)}_{\beta\ (r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \otimes \mathrm{id}_V^{\otimes q_2}) \\
&\qquad \circ(\mathrm{id}_V^{\otimes r_1+\cdots+s_{p_1}} \otimes \beta_{r_{p_1+1}+\cdots+s_{p_1+p_2},q_1} \otimes \mathrm{id}_V^{\otimes q_2}) \\
&\qquad \circ(\triangle_{\beta\ (r_1+\cdots+r_{p_1},s_1+\cdots+s_{p_1},r_{p_1+1}+\cdots+r_{p_1+p_2},s_{p_1+1}+\cdots+s_{p_1+p_2})} \otimes \mathrm{id}_V^{\otimes k}) \\
&= *(* \otimes \mathrm{id}_V)(\cdot^{i-1} \otimes \cdot^{j-1} \otimes \cdot^{k-1}) \\
&\quad - \sum_{r\geq 1}\sum_{\underline{2}} \cdot\Big(* (\cdot^{p_1-1} \otimes \cdot^{q_1-1}) \otimes M_{p_2,q_2}\Big) \\
&\qquad \circ(\sum M_{(r_1,s_1,\ldots,r_{p_1},s_{p_1})} \triangle^{(p_1-1)}_{\beta\ (r_1,s_1,\ldots,r_{p_1},s_{q_1})} \otimes \mathrm{id}_V^{\otimes q_1} \\
&\qquad\quad \otimes M_{(r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \triangle^{(p_2-1)}_{\beta\ (r_{p_1+1},s_{p_1+1},\ldots,r_{p_1+p_2},s_{p_1+p_2})} \otimes \mathrm{id}_V^{\otimes q_2})
\end{aligned}
$$

$$
\begin{aligned}
&\circ \triangle_{\beta,3,(r_1+\cdots+r_{p_1},s_1+\cdots+s_{p_1},q_1,r_{p_1+1}+\cdots+r_{p_1+p_2},s_{p_1+1}+\cdots+s_{p_1+p_2},q_2)} \\
= \;& *(*\otimes \mathrm{id}_V)(\cdot^{i-1}\otimes \cdot^{j-1}\otimes \cdot^{k-1}) \\
&- \cdot \sum_{p+q+r<i+j+k} \big(*\,(*\otimes \mathrm{id}_V)(\cdot^{i-p-1}\otimes \cdot^{j-q-1}\otimes \cdot^{k-r-1}) \\
&\qquad \otimes \sum_{s\geq 1} M_{sr}\circ \big((M^{\otimes s}\circ \triangle_\beta^{(s-1)})\otimes \mathrm{id}_V^{\otimes s}\big)\circ \triangle_{\beta,3,(i-p,j-q,k-r,p,q,r)} \\
= \;& *(\mathrm{id}_V\otimes *)(\cdot^{i-1}\otimes \cdot^{j-1}\otimes \cdot^{k-1}) \\
&- \cdot \sum_{p+q+r<i+j+k} \big(*\,(\mathrm{id}_V\otimes *)(\cdot^{i-p-1}\otimes \cdot^{j-q-1}\otimes \cdot^{k-r-1}) \\
&\qquad \otimes \sum_{s\geq 1} M_{ps}\circ \big(\mathrm{id}_V^{\otimes p}\otimes (M^{\otimes s}\circ \triangle_\beta^{(s-1)})\big)\circ \triangle_{\beta,3,(i-p,j-q,k-r,p,q,r)} \\
= \;& \sum_{l=1}^{j+k} M_{il}\circ \big(\mathrm{id}_V^{\otimes i}\otimes (M^{\otimes l}\circ \overline{\triangle_\beta}^{(l-1)})\big).
\end{aligned}
$$

The third equality follows from the induction hypothesis and the associativity of $*$. And here $\triangle_{\beta,3,(i,j,k,l,m,n)}$ means the composition of $\triangle_{\beta,3} : V^{\otimes i+k}\underline{\otimes}V^{\otimes j+m}\underline{\otimes}V^{\otimes l+n} \to T(V)^{\otimes 6}$ with the projection from $T(V)^{\otimes 6}$ to $V^{\otimes i}\underline{\otimes}V^{\otimes j}\underline{\otimes}V^{\otimes k}\underline{\otimes}V^{\otimes l}\underline{\otimes}V^{\otimes m}\underline{\otimes}V^{\otimes n}$. □

Let $A_{2\mathcal{YB}}$ be the category of unital 2-YB algebras and $A_{\mathcal{QB}_\infty}$ be the category of quantum $\mathbf{B}_\infty$-algebras. By the above proposition, we get a functor

$$(-)_{Q\mathbf{B}_\infty} : A_{2\mathcal{YB}} \;\to\; A_{\mathcal{QB}_\infty},$$

by $(V)_{Q\mathbf{B}_\infty} = (V, M, \sigma)$ for any $(V, *, \cdot, \sigma) \in A_{2\mathcal{YB}}$.

By the above proposition, we have immediately that:

Corollary 2.50. *Let (V, M, σ) be a quantum $\boldsymbol{B}_\infty$-algebra and $(T(V), *, m, \beta)$ be the 2-YB algebra with product $* = \varepsilon\otimes\varepsilon + \sum_{n\geq 1} M^{\otimes n}\overline{\triangle_\beta}^{(n-1)}$ and m the concatenation. Then the inclusion $i : V \to T(V)$ is a quantum $\boldsymbol{B}_\infty$-algebra morphism, i.e., $i\circ M_{pq} = \overline{M}_{pq}\circ (i^{\otimes p}\otimes i^{\otimes q})$, for any $p, q \geq 0$. Here $\overline{M}_{pq}$ is the quantum $\boldsymbol{B}_\infty$-algebra structure on $T(V)$ defined above.*

Chapter 3

Quantum quasi-shuffle algebras

3.1 Introduction

Quasi-shuffle algebras are the generalization of shuffle algebras. As we know, they are first constructed by Newman and Radford ([20]) for the study of the cofree irreducible Hopf algebra built on an associative algebra. For an algebra U, Newman and Radford defined an associative algebra structure on $T(U)$ by combining the multiplication of U and the shuffle product of $T(U)$. These algebras have their particular interest in many branches of algebras and a number of applications have been found in the past decade. For example, they can be applied to commutative TriDendriform algebras [18], Rota-Baxter algebras [5], multiple zeta values [11].

After the birth of quantum groups, many algebraic objects had better understandings in a more genenral framework, the braided category. For instance, shuffle algebras, special examples of quasi-shuffle algebras, had been quantized in [26] ten years ago, and led a more intrinsic understanding of quantum enveloping algebras. The next task is to find a suitable way to quantize the quasi-shuffle algebra. There were some attempts, for example, [1] and [11]. In Chapter 2, the quasi-shuffle algebra structure is quantized in the spirit of quantum shuffle algebras ([26]), by replacing the usual flip by a braiding. The resulting algebras, called quantum quasi-shuffle algebras, are the generalization of quantum shuffle algebras and provide YB algebras. Since there are many good properties for quasi-shuffle algebras, we hope that the quantum one can also share some of them or some "q-analogues". The aim of this chapter is to provide some interesting properties of these new algebras.

3.2 Properties of quantum quasi-shuffle algebras

We first illustrate the quantum quasi-shuffle algebra by some examples.

1. Let (H, F') be the Woronowicz's YB algebra. Then the first three products of the quantum quasi-shuffle algebra $(T(H), \bowtie_{F'} \beta)$ are: for any $a, b, c \in H$,

$$a \bowtie_{F'} b = ab + a \otimes b + \sum a_{(1)} b S(a_{(2)}) \otimes a_{(3)},$$

$$\begin{aligned}(a \otimes b) \bowtie_{F'} c &= a \otimes bc + \sum ab_{(1)} c S(b_{(2)}) \otimes b_{(3)} \\ &\quad + a \otimes b \otimes c + \sum a \otimes b_{(1)} c S(b_{(2)}) \otimes b_{(3)} \\ &\quad + \sum a_{(1)} b_{(1)} c S(a_{(2)} b_{(2)}) \otimes a_{(3)} \otimes b_{(3)},\end{aligned}$$

and

$$\begin{aligned}a \bowtie_{F'} (b \otimes c) &= ab \otimes c + \sum a_{(1)} b S(a_{(2)}) \otimes a_{(3)} c \\ &\quad + a \otimes b \otimes c + \sum a_{(1)} b S(a_{(2)}) \otimes a_{(3)} \otimes c \\ &\quad + \sum a_{(1)} b S(a_{(2)}) \otimes a_{(3)} c S(a_{(4)}) \otimes a_{(5)}.\end{aligned}$$

2. Let V be a vector space with basis $\{e_1, \ldots, e_n\}$. We define a braiding σ on by $\sigma(e_i \otimes e_j) = q_{ij} e_j \otimes e_i$ for some non zero scalars q_{ij}. For $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$, set $q_{IJ} = \prod_{r \in \{1,\ldots,k\}, s \in \{1,\ldots,l\}} q_{i_r, j_s}$. We use $e_{i_1} \cdots e_{i_k}$ to denote $e_{i_1} ш_\sigma \cdots ш_\sigma e_{i_k}$. Since $\chi_{rs}(\mathfrak{S}_r \times \mathfrak{S}_s) = (\mathfrak{S}_s \times \mathfrak{S}_r)\chi_{rs}$ and all expressions are reduced, we have $\beta((e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}})) = q_{IJ}(e_{i_{k+1}} \cdots e_{i_{k+l}}) \otimes (e_{i_1} \cdots e_{i_k})$. Then

$$\begin{aligned}&(e_{i_1} \cdots e_{i_k}) \bowtie_\sigma (e_{i_{k+1}} \cdots e_{i_{k+l}}) \\ &= (e_{i_1} \cdots e_{i_{k+l}}) \\ &\quad + (e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}}) + q_{IJ}(e_{i_{k+1}} \cdots e_{i_{k+l}}) \otimes (e_{i_1} \cdots e_{i_k}),\end{aligned}$$

$$\begin{aligned}&((e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}})) \bowtie_\sigma (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \\ &= (e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l+m}}) \\ &\quad + q_{JK}(e_{i_1} \cdots e_{i_k} e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \otimes (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \\ &\quad + (e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}}) \otimes (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \\ &\quad + q_{JK}(e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}}) \\ &\quad + q_{IK} q_{JK}(e_{i_{k+l+1}} \cdots e_{i_{k+l+m}}) \otimes (e_{i_1} \cdots e_{i_k}) \otimes (e_{i_{k+1}} \cdots e_{i_{k+l}}).\end{aligned}$$

and

$$\begin{aligned}&(e_{i_1} \cdots e_{i_k}) \bowtie_\sigma ((e_{i_{k+1}} \cdots e_{i_{k+l}}) \otimes (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}})) \\ &= (e_{i_1} \cdots e_{i_{k+l}}) \otimes (e_{i_{k+l+1}} \cdots e_{i_{k+l+m}})\end{aligned}$$

$$
\begin{aligned}
&+q_{IJ}(e_{i_{k+1}}\cdots e_{i_{k+l}})\otimes(e_{i_1}\cdots e_{i_k}e_{i_{k+l+1}}\cdots e_{i_{k+l+m}})\\
&+(e_{i_1}\cdots e_{i_k})\otimes(e_{i_{k+1}}\cdots e_{i_{k+l}})\otimes(e_{i_{k+l+1}}\cdots e_{i_{k+l+m}})\\
&+q_{IJ}(e_{i_{k+1}}\cdots e_{i_{k+l}})\otimes(e_{i_1}\cdots e_{i_k})\otimes(e_{i_{k+l+1}}\cdots e_{i_{k+l+m}})\\
&+q_{IJ}q_{IK}(e_{i_{k+1}}\cdots e_{i_{k+l}})\otimes(e_{i_{k+l+1}}\cdots e_{i_{k+l+m}})\otimes(e_{i_1}\cdots e_{i_k}).
\end{aligned}
$$

We recall the following inductive relation: for any $u_1,\ldots,u_i,v_1,\ldots,v_j\in A$,

$$
\begin{aligned}
&(u_1\otimes\cdots\otimes u_i)\bowtie_\sigma(v_1\otimes\cdots\otimes v_j)\\
&\quad= ((u_1\otimes\cdots\otimes u_i)\bowtie_\sigma(v_1\otimes\cdots\otimes v_{j-1})\otimes v_j)\\
&\qquad+(\bowtie_{\sigma i-1,j}\otimes \mathrm{id}_A)\sigma_{i+j-1}\cdots\sigma_i(u_1\otimes\cdots\otimes u_i\otimes v_1\otimes\cdots\otimes v_j)\\
&\qquad+(\bowtie_{\sigma i-1,j-1}\otimes m)\sigma_{i+j-2}\cdots\sigma_i(u_1\otimes\cdots\otimes u_i\otimes v_1\otimes\cdots\otimes v_j)
\end{aligned}
\tag{3.1}
$$

where $\bowtie_{\sigma k,l}$ means the restriction of $\bowtie_\sigma$ on $V^{\otimes k}\underline{\otimes}V^{\otimes l}$.

Let (V,σ) be a braided vector space. $M=\oplus M_{pq}:T^c(V)^{\otimes 2}\to V$ is a linear map such that $M_{10}=\mathrm{id}_V=M_{01}$ and $M_{ij}=0$ except for the pairs $(1,0)$, $(0,1)$ and $(1,1)$. Define $*=\varepsilon\otimes\varepsilon+\sum_{n\geq 1}M^{\otimes n}\circ\overline{\triangle_\beta}^{(n-1)}$. Then we want to find out the necessary and sufficient conditions for making $(T^c(V),*)$ into a YB algebra.

Theorem 3.1. *For a braided vector space (V,σ) and the above $(T^c(V),*,\beta)$ is a YB algebra if and only if (V,M_{11},σ) is a YB algebra.*

Proof. Obviously, if (V,M_{11},σ) is a YB algebra, then the result holds.

Conversely, if $*$ is associative, then for any $u,v,w\in V$,

$$
\begin{aligned}
u*v &= (\varepsilon\otimes\varepsilon+M\circ\overline{\triangle_\beta}^{(0)}+M^{\otimes 2}\circ\overline{\triangle_\beta}^{(1)})(u\underline{\otimes}v)\\
&= M_{11}(u\underline{\otimes}v)\\
&\quad +M^{\otimes 2}(1\underline{\otimes}\beta_{10}(u\underline{\otimes}1)\underline{\otimes}v+1\underline{\otimes}\beta_{11}(u\underline{\otimes}v)\underline{\otimes}1\\
&\quad +u\underline{\otimes}\beta_{00}(1\underline{\otimes}1)\underline{\otimes}v+u\underline{\otimes}\beta_{01}(1\underline{\otimes}v)\underline{\otimes}1\\
&\quad -(1\underline{\otimes}1)\underline{\otimes}(u\underline{\otimes}v)-(u\underline{\otimes}v)\underline{\otimes}(1\underline{\otimes}1))\\
&= M_{11}(u\underline{\otimes}v)+(M_{01}\otimes M_{10})(1\underline{\otimes}\sigma(u\underline{\otimes}v)\underline{\otimes}1)\\
&\quad +(M_{10}\otimes M_{01})((u\underline{\otimes}1)\underline{\otimes}(1\underline{\otimes}v))\\
&= M_{11}(u\underline{\otimes}v)+u\underline{\otimes}v+\sigma(u\underline{\otimes}v)\\
&= M_{11}(u\underline{\otimes}v)+u\text{ш}_\sigma v.
\end{aligned}
$$

$$(u\otimes v)*w$$

$$\begin{aligned}
&= (\varepsilon\otimes\varepsilon + M\circ\overline{\triangle_\beta}^{(0)} + M^{\otimes 2}\circ\overline{\triangle_\beta}^{(1)} + M^{\otimes 3}\circ\overline{\triangle_\beta}^{(2)})(u\otimes v\underline{\otimes}w)\\
&= M^{\otimes 2}[1\underline{\otimes}\beta_{20}(u\otimes v\underline{\otimes}1)\underline{\otimes}w + u\underline{\otimes}\beta_{10}(v\underline{\otimes}1)\underline{\otimes}w\\
&\quad +(u\otimes v)\underline{\otimes}\beta_{00}(1\underline{\otimes}1)\underline{\otimes}w + 1\underline{\otimes}\beta_{21}(u\otimes v\underline{\otimes}w)\underline{\otimes}1\\
&\quad +u\underline{\otimes}\beta_{11}(v\underline{\otimes}w)\underline{\otimes}1 + (u\otimes v)\underline{\otimes}\beta_{01}(1\underline{\otimes}w)\underline{\otimes}1\\
&\quad -(1\underline{\otimes}1)\underline{\otimes}(u\otimes v\underline{\otimes}w) - (u\otimes v\underline{\otimes}w)\underline{\otimes}(1\underline{\otimes}1)]\\
&\quad +M^{\otimes 3}\big(\overline{\triangle_\beta}(u\underline{\otimes}1)\underline{\otimes}(v\underline{\otimes}w) + \overline{\triangle_\beta}(u\otimes v\underline{\otimes}1)\underline{\otimes}(1\underline{\otimes}w)\\
&\quad +(\overline{\triangle_\beta}\otimes M_{20})(1\underline{\otimes}\beta_{21}(u\otimes v\underline{\otimes}w)\underline{\otimes}1)\\
&\quad +(\overline{\triangle_\beta}\otimes M_{11})(u\underline{\otimes}\beta_{11}(v\underline{\otimes}w)\underline{\otimes}1)\big)\\
&= u\otimes M_{11}(v\underline{\otimes}w) + (M_{11}\otimes M_{10})(u\underline{\otimes}\sigma(v\underline{\otimes}w)\underline{\otimes}1)\\
&\quad +M^{\otimes 3}\big(1\underline{\otimes}\beta_{20}(u\otimes v\underline{\otimes}1)\underline{\otimes}1\underline{\otimes}(1\underline{\otimes}w) + u\underline{\otimes}\beta_{10}(v\underline{\otimes}1)\underline{\otimes}1\underline{\otimes}(1\underline{\otimes}w)\\
&\quad +(u\otimes v)\underline{\otimes}\beta_{00}(1\underline{\otimes}1)\underline{\otimes}1\underline{\otimes}(1\underline{\otimes}w) - (1\underline{\otimes}1)\underline{\otimes}(u\otimes v\underline{\otimes}1)\underline{\otimes}(1\underline{\otimes}w)\\
&\quad -(u\otimes v\underline{\otimes}1)\underline{\otimes}(1\underline{\otimes}1)\underline{\otimes}(1\underline{\otimes}w) + (\overline{\triangle_\beta}\otimes M_{10})(u\underline{\otimes}\beta_{11}(v\underline{\otimes}w)\underline{\otimes}1)\big)\\
&= u\otimes M_{11}(v\underline{\otimes}w) + (M_{11}\otimes \mathrm{id}_V)(u\otimes\sigma(v\otimes w))\\
&\quad +u\otimes v\otimes w + \sigma_2(u\otimes v\otimes w) + \sigma_1\sigma_2(u\otimes v\otimes w)\\
&= u\otimes M_{11}(v\underline{\otimes}w) + (M_{11}\otimes \mathrm{id}_V)(u\otimes\sigma(v\otimes w)) + (u\otimes v)ш_\sigma w.
\end{aligned}$$

And

$$u*(v\otimes w) = M_{11}(u\underline{\otimes}v)\otimes w + (\mathrm{id}_V\otimes M_{11})(\sigma(u\otimes v)\otimes w) + uш_\sigma(v\otimes w).$$

$$\begin{aligned}
(u*v)*w &= (M_{11}(u\underline{\otimes}v) + uш_\sigma v)*w\\
&= M_{11}(M_{11}(u\underline{\otimes}v)\underline{\otimes}w) + M_{11}(u\underline{\otimes}v)ш_\sigma w\\
&\quad +(\mathrm{id}_V\otimes M_{11})(uш_\sigma v\underline{\otimes}w) + (M_{11}\otimes\mathrm{id})(\mathrm{id}_V\otimes\sigma)(uш_\sigma v\underline{\otimes}w))\\
&\quad +uш_\sigma vш_\sigma w\\
&= [M_{11}(M_{11}\otimes\mathrm{id}_V) + (\mathrm{id}_V+\sigma)(M_{11}\otimes\mathrm{id}_V)\\
&\quad +(\mathrm{id}_V\otimes M_{11})((\mathrm{id}_V+\sigma)\otimes\mathrm{id}_V)\\
&\quad +(M_{11}\otimes\mathrm{id}_V)(\sigma_2+\sigma_2\sigma_1)](u\underline{\otimes}v\underline{\otimes}w)\\
&\quad +uш_\sigma vш_\sigma w.
\end{aligned}$$

$$\begin{aligned}
u*(v*w) &= u*(M_{11}(v\underline{\otimes}w) + vш_\sigma w)\\
&= M_{11}(u\underline{\otimes}M_{11}(v\underline{\otimes}w)) + uш_\sigma M_{11}(v\underline{\otimes}w)
\end{aligned}$$

$$
\begin{aligned}
&+(M_{11}\otimes \mathrm{id}_V)(u\underline{\otimes}v ш_\sigma w)+(\mathrm{id}_V\otimes M_{11})(\sigma\otimes \mathrm{id}_V)(u\underline{\otimes}v ш_\sigma w))\\
&+u ш_\sigma v ш_\sigma w\\
=\ &[M_{11}(\mathrm{id}_V\otimes M_{11})+(\mathrm{id}_V+\sigma)(\mathrm{id}_V\otimes M_{11})\\
&+(M_{11}\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes(\mathrm{id}_V+\sigma))\\
&+(\mathrm{id}_V\otimes M_{11})(\sigma_1+\sigma_1\sigma_2)](u\underline{\otimes}v\underline{\otimes}w)\\
&+u ш_\sigma v ш_\sigma w.
\end{aligned}
$$

$(u*v)*w=u*(v*w)$ if and only if

$$
\begin{aligned}
&M_{11}(M_{11}\otimes \mathrm{id}_V)+M_{11}\otimes \mathrm{id}_V+\sigma(M_{11}\otimes \mathrm{id}_V)\\
&+\mathrm{id}_V\otimes M_{11}+(\mathrm{id}_V\otimes M_{11})\sigma_1+(M_{11}\otimes \mathrm{id}_V)\sigma_2+(M_{11}\otimes \mathrm{id}_V)\sigma_2\sigma_1\\
=\ &M_{11}(\mathrm{id}_V\otimes M_{11})+\mathrm{id}_V\otimes M_{11}+\sigma(\mathrm{id}_V\otimes M_{11})\\
&+M_{11}\otimes \mathrm{id}_V+(M_{11}\otimes \mathrm{id}_V)\sigma_2+(\mathrm{id}_V\otimes M_{11})\sigma_1+(\mathrm{id}_V\otimes M_{11})\sigma_1\sigma_2,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&M_{11}(M_{11}\otimes \mathrm{id}_V)+\sigma(M_{11}\otimes \mathrm{id}_V)+(M_{11}\otimes \mathrm{id}_V)\sigma_2\sigma_1\\
=\ &M_{11}(\mathrm{id}_V\otimes M_{11})+\sigma(\mathrm{id}_V\otimes M_{11})+(\mathrm{id}_V\otimes M_{11})\sigma_1\sigma_2.
\end{aligned}
$$

By comparing the degree of the result tensor vectors, we must have $M_{11}(M_{11}\otimes \mathrm{id}_V)=M_{11}(\mathrm{id}_V\otimes M_{11})$.

On $V\underline{\otimes}V$, $(\mathrm{id}_V\otimes *)\sigma_1\sigma_2=\sigma(*\otimes \mathrm{id}_V)$. It implies that $(\mathrm{id}_V\otimes M_{11})\sigma_1\sigma_2+(\mathrm{id}_V\otimes ш_\sigma)\sigma_1\sigma_2=\sigma(M_{11}\otimes \mathrm{id}_V)+\sigma(ш_\sigma\otimes \mathrm{id}_V)$. Comparing the degree, we get $(\mathrm{id}_V\otimes M_{11})\sigma_1\sigma_2=\sigma(M_{11}\otimes \mathrm{id}_V)$. Similarly, we have $(M_{11}\otimes \mathrm{id}_V)\sigma_2\sigma_1=\sigma(\mathrm{id}_V\otimes M_{11})$. □

Definition 3.2. *A quadruple* $(H,\cdot,\triangle,\sigma)$ *is called a* twisted YB bialgebra *if*

1. $(H,\cdot,\sigma)$ *is a YB algebra,*

2. $(H,\triangle,\sigma)$ *is a YB coalgebra,*

3. $\cdot: H\otimes H\to H$ *is a coalgebra map, where* $H\otimes H$ *is equipped with the twisted coalgebra structure. Or equivalently,* $\triangle: H\to H\otimes H$ *is an algebra map, where* $H\otimes H$ *is equipped with the twisted algebra structure.*

From the condition 3 above, we have that $\triangle(1)=1\otimes 1$.

Examples. 1. Let (V,σ) be a braided vector space. Then the quantum shuffle algebra $(T_\sigma(V),\delta,\beta)$ is a twisted YB bialgebra (see [26]).

2. Let (V,m,σ) be a YB algebra. Then the quantum quasi-shuffle algebra $(T^c(V),\bowtie_\sigma,\beta)$ is a twisted YB bialgebra with the deconcatenation coproduct δ.

We denote by $CB_{\mathcal{YB}}$ the category of connected twisted YB bialgebras. It consists of the following data:

1. the objects of $CB_{\mathcal{YB}}$ are the twisted YB bialgebras $(H, \cdot, \triangle, \sigma)$ such that both H and $H \otimes H$ are connected, where $H \otimes H$ is equipped with the twisted coalgebra structure;

2. a morphism f from object (H_1, σ_1) to object (H_2, σ_2) is both an algebra map and a coalgebra map and satisfies that $(f \otimes f)\sigma_1 = \sigma_2(f \otimes f)$.

It is easy to see that both $(T(V), ш_\sigma, \delta, \beta)$ and $(T^c(V), \bowtie_\sigma, \delta, \beta)$ are in $CB_{\mathcal{YB}}$.

Lemma 3.3. *Let (V_1, σ_1) and (V_2, σ_2) be two braided vector spaces and $f : V_1 \to V_2$ be a linear map such that $\sigma_2(f \otimes f) = (f \otimes f)\sigma_1$. Then for any $i, j \geq 1$, $T^{\sigma_2}_{\chi_{ij}}(f^{\otimes i} \otimes f^{\otimes j}) = (f^{\otimes j} \otimes f^{\otimes i})T^{\sigma_1}_{\chi_{ij}}$.*

Proof. We use induction on $i + j$.

When $i = j = 1$, it is trivial.

For $i + j \geq 3$,

$$\begin{aligned}
T^{\sigma_2}_{\chi_{ij}}(f^{\otimes i} \otimes f^{\otimes j}) &= (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \mathrm{id}_{V_2})(\mathrm{id}_{V_2}^{\otimes i-1} \otimes T^{\sigma_2}_{\chi_{1,j}})(f^{\otimes i} \otimes f^{\otimes j}) \\
&= (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \mathrm{id}_{V_2})(f^{\otimes i-1} \otimes T^{\sigma_2}_{\chi_{1,j}}(f \otimes f^{\otimes j})) \\
&= (T^{\sigma_2}_{\chi_{i-1,j}} \otimes \mathrm{id}_{V_2})(f^{\otimes i-1} \otimes f^{\otimes j} \otimes f)(\mathrm{id}_{V_1}^{\otimes i-1} \otimes T^{\sigma_1}_{\chi_{1,j}}) \\
&= (f^{\otimes j} \otimes f^{\otimes i})(T^{\sigma_1}_{\chi_{i-1,j}} \otimes \mathrm{id}_{V_1})(\mathrm{id}_{V_1}^{\otimes i-1} \otimes T^{\sigma_1}_{\chi_{1,j}}) \\
&= (f^{\otimes j} \otimes f^{\otimes i})T^{\sigma_1}_{\chi_{ij}}.
\end{aligned}$$

□

Let (V, m, σ) be a YB algebra. We have the following universal property in $CB_{\mathcal{YB}}$:

Proposition 3.4. *For any $(H, \cdot, \triangle, \alpha) \in CB_{\mathcal{YB}}$ and a linear map $f : H \to V$ such that $m \circ (f \otimes f) = f \circ \cdot$, $f(1) = 0$ and $(f \otimes f)\alpha = \sigma(f \otimes f)$, there exists a unique morphism $\overline{f} : H \to (T(V), \bowtie_\sigma, \delta, \beta)$ which extends f.*

Proof. Since $f(1) = 0$ and H is connected, there is a unique coalgebra map $\overline{f} : H \to T^c(V)$ which extends f. More precisely, $\overline{f} = \varepsilon_H + \sum_{n\geq 1} f^{\otimes n} \circ \overline{\triangle_H}^{(n-1)}$.

We first prove that $\beta(\overline{f} \otimes \overline{f}) = (\overline{f} \otimes \overline{f})\alpha$. We only need to verify it on $\overline{H} \otimes \overline{H}$.

$$\begin{aligned}
\beta(\overline{f} \otimes \overline{f}) &= \beta(\sum_{i,j\geq 1}(f^{\otimes i} \otimes f^{\otimes j})(\overline{\triangle_H}^{(i-1)} \otimes \overline{\triangle_H}^{(j-1)})) \\
&= \sum_{i,j\geq 1} T^{\sigma}_{\chi_{ij}}(f^{\otimes i} \otimes f^{\otimes j})(\overline{\triangle_H}^{(i-1)} \otimes \overline{\triangle_H}^{(j-1)}) \\
&= \sum_{i,j\geq 1}(f^{\otimes j} \otimes f^{\otimes i})T^{\alpha}_{\chi_{ij}}(\overline{\triangle_H}^{(i-1)} \otimes \overline{\triangle_H}^{(j-1)})
\end{aligned}$$

$$\begin{aligned} &= \sum_{i,j\geq 1}(f^{\otimes j}\otimes f^{\otimes i})(\overline{\triangle_H}^{(j-1)}\otimes\overline{\triangle_H}^{(i-1)})\alpha \\ &= (\overline{f}\otimes\overline{f})\alpha. \end{aligned}$$

The third and the forth equalities follow from the above lemma and Lemma 2.39 respectively.

The next step is to prove that $\overline{f}$ is an algebra map. We define two maps:

$$\begin{aligned} F_1 : H\otimes H &\to T(V), \\ h\otimes g &\mapsto \overline{f}(h)\Join_\sigma\overline{f}(g), \end{aligned}$$

and

$$\begin{aligned} F_2 : H\otimes H &\to T(V), \\ h\otimes g &\mapsto \overline{f}(hg). \end{aligned}$$

We claim that both F_1 and F_2 are coalgebra maps, where $H\otimes H$ is equipped the twisted coalgebra structure.

Indeed,

$$\begin{aligned} \delta\circ F_1 &= \delta\circ\Join_\sigma(\overline{f}\otimes\overline{f}) \\ &= (\Join_\sigma\otimes\Join_\sigma)\triangle_\beta(\overline{f}\otimes\overline{f}) \\ &= (\Join_\sigma\otimes\Join_\sigma)(\mathrm{id}_{T(V)}\otimes\beta\otimes\mathrm{id}_{T(V)})(\delta\otimes\delta)(\overline{f}\otimes\overline{f}) \\ &= (\Join_\sigma\otimes\Join_\sigma)(\mathrm{id}_{T(V)}\otimes\beta\otimes\mathrm{id}_{T(V)})(\delta\circ\overline{f}\otimes\delta\circ\overline{f}) \\ &= (\Join_\sigma\otimes\Join_\sigma)(\mathrm{id}_{T(V)}\otimes\beta\otimes\mathrm{id}_{T(V)})(\overline{f}\otimes\overline{f}\otimes\overline{f}\otimes\overline{f})(\triangle_H\otimes\triangle_H) \\ &= (\Join_\sigma\otimes\Join_\sigma)(\overline{f}\otimes\beta(\overline{f}\otimes\overline{f})\otimes\overline{f})(\triangle_H\otimes\triangle_H) \\ &= (F_1\otimes F_1)(\mathrm{id}_H\otimes\alpha\otimes\mathrm{id}_H)(\triangle_H\otimes\triangle_H) \\ &= (F_1\otimes F_1)\triangle_\alpha . \end{aligned}$$

And

$$\begin{aligned} \delta\circ F_2 &= \delta\circ\overline{f}\circ\cdot \\ &= (\overline{f}\otimes\overline{f})\circ\triangle_H\circ\cdot \\ &= (\overline{f}\otimes\overline{f})(\cdot\otimes\cdot)(\mathrm{id}_H\otimes\alpha\otimes\mathrm{id}_H)(\triangle_H\otimes\triangle_H) \\ &= (F_2\otimes F_2)\triangle_\alpha . \end{aligned}$$

For any $h,g\in H$, we have

$$\begin{aligned} Pr_V\circ F_1(h\otimes g) &= Pr_V(\overline{f}(h)\Join_\sigma\overline{f}(g)) \\ &= Pr_V\big(\sum_{n\geq 1}M^{\otimes n}\overline{\triangle_\beta}^{(n-1)}(\overline{f}(h)\otimes\overline{f}(g))\big) \end{aligned}$$

$$
\begin{aligned}
&= M(\overline{f}(h)\otimes \overline{f}(g))\\
&= \sum_{i,j\geq 1} M_{ij}((f^{\otimes i}\otimes f^{\otimes j})(\overline{\triangle_H}^{(i-1)}(h)\otimes \overline{\triangle_H}^{(j-1)}(g))\\
&= M_{11}(f\otimes f)(h\otimes g)\\
&= f\circ\cdot(h\otimes g)\\
&= Pr_V\circ F_2(h\otimes g).
\end{aligned}
$$

Since $H\otimes H$ is connected with the twisted coalgebra structure, $F_1=F_2$ follows from the Corollary 2.30. □

Definition 3.5. *A YB algebra (A,m,σ) is called* twisted commutative *if $m\circ\sigma=m$.*

Examples. 1. Let (A,m) be an algebra. Then the trivial YB algebra structure (A,m,τ) is twisted commutative if and only if A is commutative.

2. Then quantum exterior algebra $(S_\sigma(V),\wedge,\mathscr{T})$ is a YB algebra. Moreover it is twisted commutative.

Lemma 3.6. *Let σ be a braiding on V such that $\sigma^2=\mathrm{id}^{\otimes 2}$. Then the braiding β on $T(V)$ also satisfies that $\beta^2=\mathrm{id}_{T(V)}^{\otimes 2}$.*

Proof. We prove the statement for β_{ij} by using induction on $i+j$.

When $i=j=1$, it is trivial since $\beta_{11}=\sigma$.

For $i+j\geq 3$,

$$
\begin{aligned}
\beta_{ji}\circ\beta_{ij} &= (\beta_{j-1,i}\otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes j-1}\otimes\beta_{1i})(\mathrm{id}_V^{\otimes j-1}\otimes\beta_{i1})(\beta_{i,j-1}\otimes\mathrm{id}_V)\\
&= \mathrm{id}_{T(V)}^{\otimes 2}.
\end{aligned}
$$

□

If $\sigma=\pm\tau$, then $\sigma^2=\mathrm{id}_V^{\otimes 2}$. The first nontrivial example is the q-flip $\mathscr{T}$.

Theorem 3.7. *Let (V,m,σ) be a YB algebra. Then he quantum quasi-shuffle algebra $(T^c(V),\Join_\sigma,\beta)$ is twisted commutative if and only if (V,m,σ) is twisted commutative and $\sigma^2=\mathrm{id}_V^{\otimes 2}$.*

Proof. If $(T^c(V),\Join_\sigma,\beta)$ is twisted commutative, then on $V\underline{\otimes}V$ we have

$$
\begin{aligned}
m+\mathrm{id}_V^{\otimes 2}+\sigma &= m+ш_\sigma\\
&= \Join_{\sigma 1,1}\\
&= \Join_{\sigma 1,1}\circ\sigma\\
&= m\circ\sigma+\sigma+\sigma^2.
\end{aligned}
$$

Comparing the degree, we have that $m = m \circ \sigma$ and $\sigma^2 = \mathrm{id}_V^{\otimes 2}$.

Conversely, we use induction on $i + j$ where i and j are the powers of $V^{\otimes i} \underline{\otimes} V^{\otimes j}$.

When $i = j = 1$, it is trivial.

For $i + j \geq 3$, we use the inductive relation (3.1).

$$
\begin{aligned}
&\bowtie_{\sigma j,i} \circ \beta_{ij} \\
&= (\bowtie_{\sigma j,i-1} \otimes \mathrm{id}_V)(\beta_{i-1,j} \otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes i-1} \otimes \beta_{1,j}) \\
&\quad + (\bowtie_{\sigma j-1,i} \otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes j-1} \otimes \beta_{1,i})(\mathrm{id}_V^{\otimes j-1} \otimes \beta_{i,1})(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&\quad + (\bowtie_{\sigma j-1,i-1} \otimes m)(\mathrm{id}_V^{\otimes j-1} \otimes \beta_{1,i-1} \otimes \mathrm{id}_V) \\
&\qquad \circ (\mathrm{id}_V^{\otimes j-1} \otimes \beta_{i-1,1} \otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes i+j-2} \otimes \beta_{1,1})(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&= \big((\bowtie_{\sigma j,i-1} \circ \beta_{i-1,j}) \otimes \mathrm{id}_V\big)(\mathrm{id}_V^{\otimes i-1} \otimes \beta_{1,j}) \\
&\quad + (\bowtie_{\sigma j-1,i} \otimes \mathrm{id}_V)\big(\mathrm{id}_V^{\otimes j-1} \otimes (\beta_{1,i}\beta_{i,1})\big)(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&\quad + (\bowtie_{\sigma j-1,i-1} \otimes m)(\mathrm{id}_V^{\otimes j-1} \otimes (\beta_{1,i-1}\beta_{i-1,1}) \otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes i+j-2} \otimes \beta_{1,1})(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&= (\bowtie_{\sigma i-1,j} \otimes \mathrm{id}_V)(\mathrm{id}_V^{\otimes i-1} \otimes \beta_{1,j}) \\
&\quad + (\bowtie_{\sigma j-1,i} \otimes \mathrm{id}_V)(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&\quad + (\bowtie_{\sigma j-1,i-1} \otimes m)(\beta_{i,j-1} \otimes \mathrm{id}_V) \\
&= \bowtie_{\sigma i,j} .
\end{aligned}
$$

□

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