Thèse de doctorat

Discipline : Mathématiques

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Poisson and coisotropic structures in derived algebraic geometry

(Structures de Poisson et coïsotropes en géométrie algébrique dérivée)

effectuée sous la direction de Gabriele Vezzosi et Grégory GINOT
Soutenue à Paris le 30 Septembre 2016 devant le jury composé de :

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Abstract

In this thesis, we define and study Poisson and coisotropic structures on derived stacks in the framework of derived algebraic geometry. We consider two possible presentations of Poisson structures of different flavour: the first one is purely algebraic, while the second is more geometric. We show that the two approaches are in fact equivalent. We also introduce the notion of coisotropic structure on a morphism between derived stacks, once again presenting two equivalent definitions: one of them involves an appropriate generalization of the Swiss Cheese operad of Voronov, while the other is expressed in terms of relative polyvector fields. In particular, we show that the identity morphism carries a unique coisotropic structure; in turn, this gives rise to a non-trivial forgetful map from n-shifted Poisson structures to (n-1)-shifted Poisson structures. We also prove that the intersection of two coisotropic morphisms inside a n-shifted Poisson stack is naturally equipped with a canonical (n-1)-shifted Poisson structure. Moreover, we provide an equivalence between the space of non-degenerate coisotropic structures and the space of Lagrangian structures in derived geometry, as introduced in the work of Pantev-Toën-Vaquié-Vezzosi.

Résumé

Dans cette thèse, on définit et on étudie les notions de structure de Poisson et coïsotrope sur un champ dérivé, dans le contexte de la géométrie algébrique dérivée. On considère deux présentations différentes de structure de Poisson : la première est purement algébrique, alors que la deuxième est plus géométrique. On montre que les deux approches sont en fait équivalentes. On introduit aussi la notion de structure coïsotrope sur un morphisme de champs dérivés, encore une fois en présentant deux définitions équivalentes : la première est basée sur une généralisation appropriée de l'opérade Swiss-Cheese de Voronov, tandis que la deuxième est formulée en termes de champs de multivecteurs rélatifs. En particulier, on montre que le morphisme identité admet une unique structure coïsotrope ; cela produit une application d'oubli des structures de Poisson n-décalées aux structures de Poisson (n-1)-décalées. On montre aussi que l'intersection de deux morphismes coïsotropes dans un champ de Poisson n-décalé est naturellement équipée d'une structure de Poisson (n-1)-décalée canonique. En outre, on fournit une équivalence entre l'espace de structures coïsotropes non-dégénérées et l'espace des structures Lagrangiennes en géométrie dérivée, introduites dans les travaux de Pantev-Toën-Vaquié-Vezzosi.

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Introduction

This thesis is concerned with generalizing classical constructions and results from ordinary Poisson geometry to the broader context of derived algebraic geometry. In particular, we develop the theory of Poisson and coisotropic structures for derived Artin stacks, which are the main geometric objects of study of derived algebraic geometry. We study differences and similarities of derived Poisson and coisotropic structures, with respect to their classical versions. The comparison with the existent literature about derived symplectic geometry is also discussed. The results contained in this text should hopefully open the way to a possible deformation quantization of derived coisotropic moduli spaces.

Derived algebraic geometry was introduced as a homotopical generalization of classical algebraic geometry. The basic idea behind it is to change the category of affine pieces one uses to construct geometric objects (schemes, algebraic spaces, stacks). Classically, the building blocks of algebraic geometry are k-algebras, where k is an ordinary ring; gluing together those elementary pieces, one produces global geometric objects.

The starting point of derived algebraic geometry is to take as affine pieces *simplicial* k-algebras, that is to say simplicial objects in the category of k-algebras. When the characteristic of k is zero, these can be thought as non positively graded commutative differential graded algebras. In any case, the category of new affines has some sort of inner homotopical nature: the obvious notion of equivalence between two derived affines is not an isomorphism, but rather a weak homotopy equivalence (or a quasi-isomorphism, depending on the model we are working with). It is thus natural, when gluing such objects, to do it up to weak equivalences.

This vague ideas are formalized using the technology of ∞-categories. Namely, one needs to present the category of derived affines as an ∞-category, to specify some Grothendieck topology on it, and to explain how to glue affines with respect to this topology (that is to say, construct an ∞-category of sheaves). This approach has been carried out by Toën and Vezzosi in [HAG-I] and [HAG-II].

This makes this homotopical version of classical algebraic geometry indeed possible. Motivations for developing such a theory come from different fields of mathematics, and in particular algebraic geometry and topology. For example, the so-called *hidden smoothness philosophy* affirms that singular moduli spaces are in fact truncations of the "true" higher moduli spaces, which are smooth.

One easy example is the moduli stack $\underline{\mathrm{Vect}}_n(S)$ of rank n vector bundles of a smooth projective surface S: in this case the dimension of the (stacky) tangent space is not locally constant, and thus $\underline{\mathrm{Vect}}_n(S)$ cannot be smooth.

This lack of smoothness is philosophically due to the fact that often times the right moduli space to consider is the derived one: this is now a geometrically treatable object, whose classical part is what one saw in classical algebraic geometry.

During the last decade, derived algebraic geometry has become one of the most rapidly expanding topics in modern mathematics. In particular, the introduction of the notions of symplectic and Poisson structures on derived stacks provided a formalism in which many previous results and intuitions found a rigorous and unified treatment. The construction of symplectic and Poisson structures on moduli spaces has been a central interest in mathematics for many years, and most of the classically observed symplectic and Poisson structures on moduli spaces have their derived analogs.

Moreover, derived Poisson geometry paves the way to deformation quantization of derived moduli spaces: we refer to [CPTVV] and [PV] for more details.

The study of derived symplectic and Poisson geometry is still at its early stages, but it has already had interesting consequence on the underived level. To mention a few moduli theoretic results, we mention that (-1)-shifted symplectic structures naturally induce obstruction theories in the sense of [BF]. Moreover, the results in [BBJ] implies that the Donaldson-Thomas moduli space, which is (-1)-symplectic, is Zariski locally isomorphic to a critical locus of a potential. More recently, Shende and Takeda showed in [ST] that there exist derived Lagrangian structures on a variety of moduli space, such as wild character varieties, certain cluster varieties, multiplicative Nakajima varieties, and the augmentation variety of knot contact homology of character varieties.

In classical symplectic geometry, the most relevant sub-manifolds of a symplectic manifold are the Lagrangians. In their paper [PTVV], Pantev, Toën, Vaquié and Vezzosi defined what it means for a morphisms of derived stacks $f : X \to Y$ to have a Lagrangian structures, provided Y is (shifted) symplectic.

In the ordinary categorical approach to symplectic geometry, Lagrangian submanifolds are the objects of the *symplectic category* of [We1]. In classical Poisson geometry, the notion of Lagrangian submanifold does not make sense, and the role of Lagrangians is now played by coisotropic submanifolds. The importance of coisotropic submanifolds is also evident from the point of view of deformation quantization: if a Poisson algebra A is quantized to an associative algebra B, then we have a correspondences

$$\left\{ \text{ Coisotropic sub-algebras of } A \ \right\} \leftrightarrows \left\{ \text{ One sided ideals of } B \ \right\}.$$

This is the so-called *Poisson creed* of Lu (see [Lu]), which is the Poisson analogue of the *symplectic creed* of Weinstein in [We1].

However, contrary to the Lagrangian and symplectic case of [PTVV], the treatment of coisotropic structures given in [CPTVV] is somewhat unsatisfactory: the definition relies on a conjectural result (the additivity property of the Poisson operad), and its high level of abstraction implies that it would be difficult to give concrete examples of coisotropic structures.

The goal of this thesis is precisely to study in detail Poisson and coisotropic structures in derived algebraic geometry. In a first part (Chapter 1), we give two definitions of derived Poisson structures on derived affine schemes, prove that they are equivalent, and then extend these notions and their equivalence to general derived stacks (Chapter 2). We also propose a definition of coisotropic structures on a morphism; again we start with the case of derived affine schemes (Chapter 3), where we give two definitions of derived coisotropic structures and prove that they are in fact equivalent. Then in Chapter 4 we transplant these equivalent definitions in the general context of derived Artin stacks, and prove derived extensions of many classical results (see for instance [We2]) on coisotropics.

Shifted symplectic structures were introduced in [PTVV], while Poisson structures were studied in [CPTVV], building on our work [Me]. While classically defining Poisson structures is in no way different that defining symplectic structures, things change dramatically as soon as we move in the derived context. A rapid glimpse at the two papers will convince the reader that there is a remarkable change of point of view between the two approaches. This is justified by some intrinsic technical problems in the very definition of shifted Poisson structures. Namely, polyvectors are not as functorial as differential forms are: given any map of algebras $A \to B$, there is no sensible way to define a map between multi-derivations (the algebraic incarnation of polyvectors) on A and on B. On the other hand, there is a natural notion of pullback of differential forms, which induces a natural map $\Omega^1_A \to \Omega^1_B$, which can be used to define a functorial de Rham algebra of differential forms. Geometrically, this is nothing more than observing that for smooth manifold the construction of the tangent bundle is functorial, while the cotangent bundle is not, or at least not in a trivial way.

Another crucial problem is that for a smooth manifold, one defines a Poisson structure on a smooth manifold X to be a bivector field π such that $[\pi, \pi] = 0$, where the bracket is the Schouten-Nijenhuis bracket of polyvectors fields on X (see for instance [Va]). Now, if X is taken to be a derived stack, there is no easy or elementary way to endow its algebra of polyvectors field with a natural bracket. Said in another way, the canonical Poisson structure that should exist on the cotangent bundle of a derived stack X is not easy to define. One reason is the fact that we are working in a ∞-categorical context, and thus there is little hope to construct algebraic structures explicitly: most of the time we have to deal with algebraic relations that only hold up to homotopy.

In order to overcome such problems, in [CPTVV] the authors had to introduce new and broader techniques, with the goal of developing differential calculus in very general symmetric monoidal model categories. Most importantly, they developed a new tool, which they call *formal localization*, that will be probably prove to be extremely useful in further developing of derived algebraic geometry. The results presented in this thesis are among other things another proof of the usefulness and the flexibility of such techniques, since we use them extensively, and often in a relative context (that is to say, we work with maps instead of working with objects, like in the case of coisotropic structures on a morphism).

Let us now briefly go through the content of this text.

In chapter 0, we review the notions of formal derived geometry and formal localization, as introduced and studied in [CPTVV]. We keep the exposition to a minimum, and we do not give any proof. The interested reader will find details and proofs in the original paper, while an excellent review is the very recent [PV]. The main object of study is a derived Artin stack X, together with the natural projection $X \to X_{DR}$ to its de Rham stack X_{DR}. The de Rham stack has the same reduced points of X, and the fundamental property of the projection $X \to X_{DR}$ is that its fiber taken at any closed point $x : \mathrm{Spec}\ k \to X_{DR}$ is the formal completion of X at x, which is denoted $\widehat{X}_x$. This allows us to define an algebra $\mathcal{P}_X(\infty)$ inside a properly defined symmetric monoidal ∞-category $\mathcal{M}_X$ of prestacks over X_{DR}, which knows a lot about the original stack X. In particular we have an equivalence

$$\mathrm{Perf}(X) \simeq \mathcal{P}_X(\infty) - \mathrm{mod}^{\mathrm{perf}},$$

where $\mathrm{Perf}(X)$ is the ∞-category of perfect complexes on X, and $\mathcal{P}_X(\infty) - \mathrm{mod}^{\mathrm{perf}}$ is a suitably defined sub-category of $P_X(\infty)$-modules. Notice that this is already a very strong result, since in particular it allows to do differential calculus on X treating it as it was just an algebra inside the category $\mathcal{M}_X$. In particular, we have

$$\mathrm{Symp}(X, n) \simeq \mathrm{Symp}(\mathcal{P}_X(\infty), n),$$

where $\mathrm{Symp}(-,n)$ is the space of n-shifted symplectic structures of [PTVV]. In other terms, the geometrically defined shifted symplectic structures on X are equivalent to algebraic symplectic structures on $\mathcal{P}_X(\infty)$.

It seems now reasonable to define n-shifted Poisson structures on X as $\mathbb{P}_{n+1}$-structures on the algebra $\mathcal{P}_X(\infty)$, where $\mathbb{P}_{n+1}$ is the operad encoding dg Poisson algebras with brackets of degree $-n$. On the other hand, we are now given an algebra $\mathcal{P}_X(\infty)$, and we could therefore construct its graded algebra of (shifted) multi-derivations. By the general differential calculus formalism of [CPTVV], this algebra naturally carries a (shifted) Lie bracket, making it into a shifted Poisson algebra. One is led to ask whether these two natural definition coincide. Notice that at the classical level, this is the same of asking if, starting a smooth manifold X, to give a Poisson bracket on the algebra of functions $C^\infty(X)$ is the same as to give a bivector π satisfying $[\pi,\pi]=0$. This is course trivially true for underived objects, but as explained before the derived nature of the question requires one to handle it with care.

In chapter 1, we address this precise question. Working in the simpler language of commutative differential graded algebras, we give two possible (and sensible) definitions of shifted Poisson structure on an algebra A. Algebraically speaking, one could consider extending the commutative structure to get a Poisson structure on A. The natural appearance of moduli space of algebraic structures brings us in the context of operad theory, by which we explicitly study what it means for A to have a shifted Lie bracket compatible with the given multiplication. Namely, one can give the following definition.

Definition 0.0.1. *Let A be a commutative dg algebra. The* space of $\mathbb{P}_{n+1}$-structures on A *is defined as the pullback of the following diagram*

$$\begin{tikzcd} P_{n+1}(A) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A) \end{tikzcd}$$

of spaces, where the bottom map is the given multiplicative structure on A.

Here the fact that we are working in a derived, homotopical context implies that we constantly have to work with up-to-homotopy algebraic structures and up-to-homotopy compatibilities between those. This is encoded in the fact that in the above definition, we are using mapping spaces in the model category of dg operads, which can be computed by first resolving the source. In some way, $\mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A)$ has to be considered a moduli space of homotopy $\mathbb{P}_{n+1}$-structures.

On the other hand, A is implicitly thought as a geometric object, namely a derived affine scheme. As such, under mild assumptions it has a nicely behaved tangent complex $\mathbb{T}_A$ (which is a dg A-module), a derived analogue for vector fields. Let us now also introduce a derived analogue for the algebra of polyvector fields: we denote with $\mathrm{Pol}(A,n)$ the graded complex $\mathrm{Sym}_A(\mathbb{T}_A[-n])$, and we will call it the algebra of *n-shifted polyvectors*. This algebra can be endowed with an explicit Poisson structure, which is induced (as in the classical case) by the Lie bracket of vector fields. In particular, $\mathrm{Pol}(A,n)[n]$ has a natural structure of graded dg Lie algebra. We now give the second possible definition of Poisson structures on A, more in line with the ordinary geometric approach of Poisson geometry.

Let us denote by dgMod^{gr} the category of graded cochain complexes. Notice that objects of dgMod^{gr} have two natural gradings, to which we will refer as *cohomological grading* and *weight* *grading.* Inside dgMod^{gr}, consider the category dgLie^{gr} of graded complexes endowed with a Lie bracket of cohomological degree 0 and weight degree -1.

Definition 0.0.2. *Let A be a commutative dg algebra. The* space of n-shifted Poisson structures *on A is*

$$\mathrm{Pois}(A, n) := \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(A, n+1)[n+1])$$

where $k[-1](2)$ is the trivial Lie algebra k sitting in cohomological degree 1 and weight 2.

Note that if A is a classical (not dg) algebra and $n = 0$, then $\mathrm{Pol}(A, 1)$ is the ordinary Gerstenhaber algebra of polyvectors fields, described for example in [Va]. If moreover we replace the mapping space above with the set of strict morphisms of graded dg Lie algebras, then we get precisely the set of bivectors π on A such that $[\pi, \pi] = 0$. It is thus clear that the definition above is an homotopical generalization of the ordinary definition of Poisson structures in terms of bivector fields.

As already mentioned, the homotopical character of the question is such that writing down explicit equations becomes an hard task, which we however make unnecessary by using a construction of Kapranov and Manin [KM], and using it to give a relatively clean and equation-free description of the dg Lie algebra controlling algebraic Poisson structures on A.

The main result of the chapter states that the dg Lie algebra controlling algebraic Poisson structures on A is in fact equivalent to the polyvectors algebra on A, giving us the expected equivalence between the two definitions of Poisson structures.

Theorem 0.0.3 ([Me], see also Theorem 1.3.2)**.** *Let A be a commutative dg algebra, and suppose moreover that its cotangent complex $\mathbb{L}_A$ is a perfect A-module. Then there is a natural isomorphism*

$$\mathrm{Pois}(A, n) \xrightarrow{\sim} \mathbb{P}_{n+1}(A)$$

in the homotopy category of simplicial sets.

At the end of the chapter we also give an alternative way of proving the result, which makes use of the canonical explicit resolution of the Lie operad.

Theorem 0.0.3 is the fundamental starting point of the theory of derived Poisson geometry, as developed in [CPTVV]. Using the above result, we are now able to finally define shifted Poisson structures on a derived Artin stack X. This is done in chapter 2, where we follow the exposition of [CPTVV] and [PV]. As mentioned, Theorem 0.0.3 plays a fundamental role here, as it guarantees that one can safely pass from bivectors to algebraic structures. In particular, having an algebraic description of shifted Poisson structures on derived Artin stacks opens the possibility to discuss shifted deformation quantization, as in the last section of [CPTVV].

More specifically, thanks to formal localization one can define a polyvectors algebra $\mathrm{Pol}(X, n)$. Then, having in mind to follow the bivector approach, we can define n-shifted Poisson structures on X using $\mathrm{Pol}(X, n)$, exactly as in the affine case.

Definition 0.0.4. *Let X be a derived Artin stack, locally of finite presentation. The* space of n-shifted Poisson structures on X *is*

$$\mathrm{Pois}(X, n) := \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(X, n+1)[n+1])$$

again dgLie^{gr} is the ∞-category of graded dg Lie algebras over k.

Alternatively, the formal localization machinery also encodes pretty much all of the geometric information of X in the prestack $\mathcal{P}_X(\infty)$, together with its $\mathbb{D}_{X_{DR}}(\infty)$-linear structure. Another natural thing to do is to define Poisson structures in terms of additional algebraic structures on the commutative algebra $\mathcal{P}_X(\infty)$.

Definition 0.0.5. *Let X be a derived Artin stack, locally of finite presentation. The* space of n-shifted Poisson structures $\mathrm{Pois}'(X,n)$ on X *is the space of lifts of the given commutative algebra structure on $\mathcal{P}_X(\infty)$ to a compatible $\mathbb{D}_{X_{DR}}(\infty)$-linear $\mathbb{P}_{n+1}$-structure. Explicitly,* $\mathrm{Pois}'(X,n)$ *is the fiber product*

$$\begin{tikzcd} \mathrm{Pois}'(X,n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \end{tikzcd}$$

where $\mathcal{P}_X(\infty)$ is viewed as an object inside the symmetric monoidal ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules.

The equivalence of the two definitions above is proven using Theorem 0.0.3. More specifically, we need a slightly modified version of it: in fact, we observe in chapter 2 that even if we drop the hypothesis on the cotangent complex being perfect, the theorem remains valid, provided we define the polyvector algebra only using the cotangent complex. Apart from this caveat, the arguments given in the proof of Theorem 0.0.3 work exactly in the same way for the general case, so that we get the following result.

Theorem 0.0.6 (see Theorem 2.2.3)**.** *With notations as above, there is a natural equivalence of spaces*

$$\mathrm{Pois}(X,n) \simeq \mathrm{Pois}'(X,n).$$

Now that the definition of Poisson structure is behind us, we move on in chapter 3 to coisotropic structures. Given a map $f : X \to Y$ between derived Artin stacks, we follow the same path used for Poisson structures in chapters 1 and 2: thanks to formal localization, we can first start studying coisotropic structures on f where X and Y are affine, and then apply these definitions to the general case of derived Artin stacks by passing to the associated algebras $\mathcal{P}_X(\infty)$ and $\mathcal{P}_Y(\infty)$.

Contrary to the Poisson case, for a classical coisotropic it is not entirely clear which is the underlying algebraic structure encoding the geometric information: this thesis solves this problem, and we will show that the algebraic structure one can see on a classical coisotropic is a particular case of much more general homotopical algebraic structure.

Let us first of all recall the classical geometric definition of a coisotropic submanifold.

Definition 0.0.7. *Let X be a smooth, underived manifold, and let $C \to X$ be a sub-manifold. Let $\pi \in \Gamma(\Lambda^2 TX)$ be a Poisson bivector on X. Then C is said to be* coisotropic *if the restriction of the induced map*

$$\pi^* : T^*X \longrightarrow TX$$

to the conormal bundle N^*C lands in TC. Diagrammatically, C if coisotropic if it exists a dotted arrow

$$\begin{tikzcd} N^*C \arrow[r, hook] \arrow[d, dotted] & T^*X \arrow[d] \\ TC \arrow[r, hook] & TX \end{tikzcd}$$

making the diagram commute, where the top and the bottom arrows are the natural inclusions.

Remark that we can rephrase this notion by saying that the image of the bivector π in $\Gamma(\Lambda^2(NC))$ under the map induced by $TX \to NC$ is zero. Having this definition in mind, we can construct a derived version of it. Let $f : A \to B$ a morphism of commutative dg algebras; it induces [HAG-II] a natural fiber sequence of B-modules

$$\mathbb{T}_f \to \mathbb{T}_B \to \mathbb{T}_A \otimes_A B.$$

Notice that, following the analogy with ordinary differential geometry, $\mathbb{T}_A \otimes_A B$ is the pullback of the tangent bundle of A (in the case of a sub-manifold, this is just the restriction), and thus $\mathbb{T}_f[1]$ plays the role of the normal bundle. In particular, we can use the previous sequence to get a natural map of A-modules $\mathbb{T}_A \to \mathbb{T}_A \otimes_A B \to \mathbb{T}_f[1]$. This in turn induces a morphism of algebras

$$\mathrm{Sym}_A(\mathbb{T}_A[-n-1]) \longrightarrow \mathrm{Sym}_B(\mathbb{T}_f[-n])$$

for every n. Let us denote the homotopy fiber of this map $\mathrm{Pol}(f, n+1)$, and call it the *relative polyvectors algebra of f*. Shifting back by $n+1$, we get

$$\mathrm{Pol}(A, n+1)[n+1] \longrightarrow \mathrm{Sym}_B(\mathbb{T}_f[-n])[n+1].$$

Notice that the source of the map is naturally a graded dg Lie, while the target is not, for trivial degree reasons. Thus a priori there is no reasons to $\mathrm{Pol}(f, n+1)[n+1]$ to be a Lie algebra, since it is not a limit of Lie algebras. Nevertheless, we prove in chapter 3 that $\mathrm{Pol}(f, n+1)[n+1]$ does have a Lie structure, and it moreover fits in the following fiber sequence of graded Lie algebras

$$\mathrm{Sym}_B(\mathbb{T}_f[-n])[n] \to \mathrm{Pol}(f, n+1)[n+1] \to \mathrm{Pol}(A, n+1)[n+1].$$

The relative polyvectors by definition control coisotropic structures.

Definition 0.0.8. *Let $f : A \to B$ a map of commutative dg algebras. Suppose moreover that we are given an n-shifted Poisson structure π on A. Then the* space $\mathrm{Cois}(f, \pi)$ of coisotropic structures on f relative to π *is the space of dotted maps of Lie algebras*

$$\begin{tikzcd}
\mathrm{Sym}_B(\mathbb{T}_f[-n])[n] \arrow[r] & \mathrm{Pol}(f, n+1)[n+1] \arrow[r] & \mathrm{Pol}(A, n+1)[n+1] \\
& k[-1](2) \arrow[u, dotted] \arrow[ur, "\pi"] &
\end{tikzcd}$$

making the diagram commute. More precisely, $\mathrm{Cois}(f, \pi)$ *is the homotopy fiber of the map of spaces*

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(A, n+1)[n+1])$$

taken at the point corresponding to π.

By the previous discussion, it is clear that in the case of ordinary affine schemes and $n = 0$ this notion is equivalent the classical coisotropic definition.

The natural question is now to find an equivalent algebraic structure on a morphism of algebras: as we said, this is not an easy goal. One proposal for such a derived algebraic structure is given in [Sa]; in order to recall this notion, let us fix some notations. If B is a $\mathbb{P}_n$-algebra, then we can naturally produce a $\mathbb{P}_{n+1}$-algebra $Z(B)$, called the *Poisson center* of B. As a commutative algebra, we have

$$Z(B) = \widehat{\mathrm{Sym}}_B(\mathbb{T}[-n])$$

where the Lie bracket is the standard Lie bracket of multi-derivations, while the differential has two components: one given by the internal differential of the B-module $\mathbb{T}_B$ and another given by $[\pi_B, -]$. Notice that there is a natural projections of commutative dg algebras $Z(B) \to B$.

Let $\mathbb{P}_{[n+1,n]}$ be the two-colored operad whose algebras are couples of objects (V, W) together with the following additional structure:

- a $\mathbb{P}_{n+1}$-structure on V;
- a $\mathbb{P}_n$-structure on W;
- a map of $\mathbb{P}_{n+1}$-algebras $V \to Z(W)$.

In order to explicitly construct the operad $\mathbb{P}_{[n+1,n]}$, in Chapter 3 we will define generalized Swiss cheese operads, inspired by the classical Swiss cheese operad of Voronov (see [Vo]). Our $\mathbb{P}_{[n+1,n]}$ will then be a particular case of this construction.

Notice that by composing with the projection $Z(W) \to W$, we get a natural forgetful functor

$$\mathbb{P}_{[n+1,n]}-\mathrm{alg} \longrightarrow \mathrm{Comm}_{\Delta^1}-\mathrm{alg},$$

where Comm_{Δ^1} is the two-colored operad encoding morphisms of commutative algebras.

Definition 0.0.9. *Let $f : A \to B$ a map of commutative dg algebra. Then the moduli space $\mathbb{P}_{[n+1,n]}(f)$ of $\mathbb{P}_{[n+1,n]}$-structures on f is the fiber product of simplicial sets*

$$\begin{tikzcd}
\mathbb{P}_{[n+1,n]}(f) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[d] \\
* \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}_{\Delta^1}, \mathrm{End}_{A,B})
\end{tikzcd}$$

where the bottom arrow is given by f itself. If moreover we were given a $\mathbb{P}_{n+1}$-structure π_A on A, Then we define the $\mathrm{Cois}'(f, \pi_A)$ to be the following fiber product on simplicial sets

$$\begin{tikzcd}
\mathrm{Cois}'(f, \pi_A) \arrow[r] \arrow[d] & \mathbb{P}_{[n+1,n]}(f) \arrow[d] \\
* \arrow[r] & \mathbb{P}_{n+1}(A)
\end{tikzcd}$$

where the bottom map is the given $\mathbb{P}_{n+1}$-structure π_A.

Notice that the algebraic definition is particularly badly suited for an immediate interpretation as a derived version of the classical definition of coisotropic: for example, if X is a derived Artin stack endowed with a n-shifted Poisson structure, then if $C \to X$ is a coisotropic it follows by definition that C is itself $(n-1)$-shifted Poisson. This structure cannot be seen at the classical level, as smooth ordinary objects can only carry 0-shifted Poisson structures, and moreover those coincide with classical Poisson structures.

One of the main results in chapter 3 proves that the algebraic notion is in fact equivalent with the previous (derived) geometric definition.

Theorem 0.0.10 (see Theorem 3.4.11). *Let again $f : A \to B$ be a map of cdgas, and let π_A be a n-shifted Poisson structure on A. Then the space* $\mathrm{Cois}(f, \pi_A)$ *of coisotropic structures on f in the sense of definition 0.0.8 is equivalent to the space* $\mathrm{Cois}'(f, \pi_A)$ *of coisotropic structures in the sense of definition 0.0.9.*

This theorem has of course a reassuring component: the (perhaps) strangely looking algebraic definition is in fact equivalent to a more down-to-earth one. Nevertheless, this means that definition 0.0.9 is justified, in the sense that it does have in fact a precise geometric interpretation.

The procedure adopted for the proof of this theorem is philosophically similar to what happened in chapter 1 for Poisson structure: we develop a general operadic formalism (building on ideas of Calaque and Willwacher) to get a description of the dg Lie algebra encoding $\mathbb{P}_{[n+1,n]}$-structures on f, and we then prove that this Lie algebra is in fact quasi-isomorphic to $\mathrm{Pol}(f, n+1)[n+1]$.

In chapter 4 we complete the picture, using the results of chapter 3 to give a sensible definition of coisotropic structures on morphisms $f : L \to X$ between derived Artin stacks. Here we use the full power of the formalism of formal localization, as we are able to work with the objects $\mathcal{P}_L(\infty)$ and $\mathcal{P}_X(\infty)$, which are algebra in the symmetric monoidal ∞-categories $\mathcal{M}_L$ and $\mathcal{M}_X$ respectively. More specifically, the morphism f naturally induces a monoidal pullback functor $f^* : \mathcal{M}_X \to \mathcal{M}_L$, together with a canonical map

$$f^*_{\mathcal{P}} \ : \ f^*\mathcal{P}_X(\infty) \longrightarrow \mathcal{P}_L(\infty)$$

of algebras in the category $\mathcal{M}_L$. We can now use the algebraic description of coisotropic structures given before to define general coisotropics.

Definition 0.0.11. *Let $L \to X$ be a morphisms of derived Artin stacks, locally of finite presentation. Let π_X be a n-shifted Poisson structure on X. The space* $\mathrm{Cois}(f, \pi_X)$ *of coisotropic structures on f is the fiber product in simplicial sets*

$$\begin{tikzcd} \mathrm{Cois}(f, \pi_X) \arrow[r] \arrow[d] & \mathbb{P}_{[n+1,n]}(f^*_{\mathcal{P}}) \arrow[d] \\ * \arrow[r] & \mathbb{P}_{n+1}(f^*\mathcal{P}_X(\infty)) \end{tikzcd}$$

where the bottom map is the $\mathbb{P}_{n+1}$-structure on $f^*\mathcal{P}_X(\infty)$ coming from π_X.

Again, this is a reasonable definition, but it fails to produce immediate examples. To this purpose, we can use Theorem 0.0.10 to give an alternative definition.

Theorem 0.0.12 (see Theorem 4.1.5). *Let $f : L \to X$ be a map of derived Artin stacks, locally of finite presentation. Suppose π_X is a n-shifted Poisson structure on X. Then the space* $\mathrm{Cois}(f, \pi_X)$ *can also be described as the homotopy fiber of the morphism*

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f^*_{\mathcal{P}}, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f^*\mathcal{P}_X(\infty), n+1)[n+1])$$

taken at the point given by the Poisson structure π_X.

This definition can be now more easily unzipped: by results contained in [CPTVV], the polyvectors algebra on $\mathcal{P}_X(\infty)$ is a model for the geometric polyvectors on the derived Artin stack X. This means in particular that we are now able to construct a series of expected examples of derived coisotropic structures:

- Suppose L is now a smooth sub-scheme of a smooth schemes X, and π_X is a (0-shifted) Poisson structure on X is the ordinary sense. Let us denote by $f : L \to X$ the inclusion. Then the space of coisotropic structures on f is either empty or contractible, and it is non-empty precisely when L is a coisotropic subscheme in the usual sense.

- Let X be a derived Artin stack, endowed with a n-shifted Poisson structure π_X. Then the identity morphism $X \to X$ has a canonical coisotropic structure.

- Let $f : X \to Y$ be a morphisms of derived Artin stacks. Suppose both X and Y are endowed with n-shifted Poisson structures, denoted π_X and π_Y respectively. Then f is a Poisson map (in the sense that it preserves Poisson structures) if and only if its graph $g : X \to X \times Y$ is coisotropic, where $X \times Y$ is taken with the n-shifted Poisson structure $\pi_X - \pi_Y$.

  More precisely, there is a canonical equivalence of spaces

  $$\mathrm{Pois}(f) \simeq \mathrm{Cois}(g, \pi_X - \pi_Y)$$

  where with $\mathrm{Pois}(f)$ we denoted the space of lifts of f to a Poisson map.

Notice that many of this points were observed in the seminal paper of Weinstein about coisotropic calculus [We2]. We thus proved that many of its classical results stay true in the derived world, therefore confirming that the chosen definitions are the "right" ones.

On the other hand, we also discovered some strictly derived phenomena. For example, let X be a derived stack, and consider the canonical map $p_x : X \to \mathrm{Spec}k$. Consider $\mathrm{Spec}k$ as being equipped with the trivial n-shifted Poisson structure. Then, giving a coisotropic structure on p_X is equivalent to giving an $(n-1)$-shifted Poisson structure on X.

Let us now explain some of the advantages of having a theory of coisotropic structures in the derived context. Let us turn back to the classical theory of coisotropic calculus, as constructed in [We2]. One of the main results of that paper is about what the author calls *coisotropic relations*. By definition, a coisotropic relation from a Poisson manifold (X, π_X) to another Poisson manifold (Y, π_Y) is a coisotropic submanifold in $(X \times Y, \pi_X - \pi_Y)$. Weinstein then proves that under quite strong transversality assumption, these relations can be composed: given a relation from X to Y and another one from Y to Z, these can be combined in order to obtain a relation from X to Z. In practice, the transversality condition is generally not satisfied, which limits the scope of Weinstein's theorem. Notice that this result has no hope of being true in the classical setting without the transversality hypothesis. In particular, this prevents the Poisson relations to be the morphisms of an ordinary category of Poisson manifolds.

But in the context of derived algebraic geometry, all issues about transverse intersections magically disappear. In particular, we prove in chapter 4 the following theorem, which has to be thought as a first step towards a complete derived generalization of Weinsten's composition of Poisson relations, and therefore possibly to the construction to a (suitably higher) category of (shifted) Poisson manifolds.

Theorem 0.0.13 (see Theorem 4.2.2)**.** *Let X, L_1 and L_2 be derived Artin stacks, locally of finite presentation, and let $\pi_X \in \mathrm{Pois}(X, n)$ be a n-shifted Poisson structure on X. Let $f_1 : L_1 \to X$ and $f_2 : L_2 \to X$ be morphisms of stacks, and suppose both f_1 and f_2 are equipped with coisotropic* *structures. Then the derived intersection Y, defined as the fiber product*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & L_1 \arrow[d, "f_1"] \\ L_2 \arrow[r, "f_2"] & X \end{tikzcd}$$

has a canonical $(n-1)$-shifted Poisson structures. This structure is such that the map $Y \to L_1 \times L_2$ is a morphism of $(n-1)$-shifted Poisson stacks, where L_2 is taken with the opposite Poisson structure.

Notice that a similar result was proven (at the cohomological underived level) by Baranovsky and Ginzburg in [BG]. On the other hand, a parallel statement in the context of derived symplectic geometry was one of the main theorems of [PTVV], where of course coisotropics are now substituted with Lagrangians. Our result vastly generalizes the theorem in [BG], which becomes a consequence of Theorem 0.0.13.

The end of chapter 4 is devoted to the study of the relation existing between coisotropic and Lagrangian structures. In [CPTVV], the authors proved that a n-symplectic structure on a derived Artin stack X naturally induce a n-Poisson structure, which is moreover non-degenerate in an appropriate sense. This could look like a trivial statement, but it is important to remark this is far from being easy. One needs to start by once again using formal localization techniques to reduce the question to a similar question on the algebra $\mathcal{P}_X$. Even here, the homotopical nature of the structures involved creates complications: the non-degeneracy condition imposed to symplectic structures only ask for a quasi-isomorphism between the tangent and the cotangent complex of X. While this is the natural thing to ask in derived geometry, it prevents all classical methods to work.

We follow the same path for our relative case: we define a notion of non-degeneracy for coisotropic structures, and we get to prove the following theorem, which was conjectured in both [JS] and [CPTVV].

Theorem 0.0.14 (see Theorem 4.3.7)**.** *Let $f : L \to X$ a map of derived Artin stacks, locally of finite presentation. Let π_X be a non-degenerate n-shifted Poisson structure on X, in the sense of [CPTVV]. Then on X there is an induced n-shifted symplectic structure ω_X. We have a natural equivalence of spaces*

$$\mathrm{Cois}^{nd}(f, \pi_X) \xrightarrow{\sim} \mathrm{Lagr}(f, \omega_X),$$

where $\mathrm{Cois}^{nd}(f, \pi_X)$ is the space of non-degenerate coisotropic structures on f, and $\mathrm{Lagr}(f, \omega_X)$ is the space of Lagriangan structures on f (in the sense of [PTVV]).

Apart from its foundational value, the above theorem also has concrete interesting consequences. As already mentioned, there are many natural moduli spaces that appear as derived Lagrangians inside shifted symplectic stacks (see [ST] for a detailed list). The above results allows us to consider these moduli spaces as having a natural coisotropic structure, which can now hopefully be quantized, extending ideas and results of ordinary Poisson geometry (see for example [CF] and [OP]). This process should then produce interesting quantizations of moduli spaces coming from topology and representation theory.

We remark also that Theorem 0.0.14 allows us to find back results on Lagrangian structures using the theory of coisotropic structures developed in this thesis. For example, one can check that Theorem 2.9 in [PTVV] is now a direct consequence of Theorem 0.0.13 together with Theorem 0.0.14.

The final chapter is devoted to the problem of comparing our definition of coisotropic structure with the one proposed in section 3.4 of [CPTVV]. We propose two promising approaches, but we refer to the paper [MS] for more details on this particular subject.

Background

Throughout this thesis, we will assume the reader has some familiarity with derived algebraic geometry, for which useful reviews are [To1], and the more recent [To2], while the foundational works are Toën-Vezzosi [HAG-II], J. Lurie's DAG series [Lur1], and also the recent [Lur4]. We will use both the old but very often still useful language of model categories (see e.g. [Ho] or [Hir]), and the more modern language and theory of ∞-categories ([Lur2]). We try to recall most of the operadic notions we use, but good references are [Hi], [BM], [LV], [DR].

Acknoledgments

Even though this Ph.D. thesis is supposed to be a personal work, I am much indebted to a whole lot of people, and it is a pleasure to thank all the persons that helped me during the last three years.

My advisors, Gabriele Vezzosi and Grégory Ginot, offered constant support for the all the difficulties I had, related (or un-related) to the subject of this thesis. They were able to teach me wonderful mathematics, and my scientific debt to them is enormous. At the same time, the human aspect of our relation is certainly what helped me the most, and I owe them much more than what they probably know.

I would like to thank Benoit Fresse and Dominic Joyce for having accepted to write a report for this thesis, and Francesco Bottacin, Damien Calaque, Domenico Fiorenza, Muriel Livernet, Paolo Salvatore for having accepted to be part of my jury. I felt really honored when they showed interest in my work.

I also want to thank Damien Calaque, Tony Pantev, Bertrand Toën and Michel Vaquié, for many stimulating and interesting conversations, and for forgiving me for the triviality of many of my doubts. They also showed me that it is indeed possible to work in a joyful atmosphere.

A very special thought is for my PhD brother Mauro Porta. I had the luck to share these three years with him, and it was absolutely great. He is of course an outstanding mathematician, but that is almost irrelevant. Simply put, Mauro is one of best persons I've ever met.

The second half of this thesis is part of a joint work with Pavel Safronov. I would like to express my gratitude to him: without its resiliency in attacking difficult problems, we would never have managed to solve them.

During my PhD I had the occasion to encounter many mathematicians and colleagues, and I wish to thank them for all the enlightening discussions we had. The list is of course vastly incomplete, but I recall fruitful mathematical discussions with Mathieu Anel, Samuel Bach, Dario Beraldo, Anthony Blanc, Francesco Bonechi, John Calabrese, Martino Cantadore, Cyrus Cohier-Chevaux, Domenico Fiorenza, Dragos Fratila, Mattia Galeotti, Ezra Getzler, Benjamin Hennion, Rémi Jaoui, Mikhail Kapranov, Damien Lejay, Nicolàs Matte Bon, François Petit, Rafael Renaudin-Avino, Marco Robalo, Nick Rozenblyum, Claudia Scheimbauer, Vivek Shende, Davide Stefani, Alex Takeda, Bruno Vallette, Pietro Vertechi, Ping Xu, Sinan Yalin, Tony Yue Yu, Vito Felice Zenobi.

I firmly believe that one of the main ingredients of a good mathematical research is the happiness of the underlying researcher. If there is some good in this thesis, I should therefore thank many people; I already named some of them (disguised as mathematicians), but there are of course many others. Thanks.

Résumé substantiel

Cette thèse veut généraliser des constructions et des résultats classiques de la géométrie de Poisson ordinaire au contexte plus vaste de la géométrie algébrique dérivée. En particulier, on développe la théorie des structures de Poisson et des structures coisotropes pour des champs d'Artin dérivés, qui sont les principaux objets géométriques étudiés en géométrie algébrique dérivée. On étudie différences et similarités entre les structures de Poisson et coisotrope dérivées et leurs versions classiques. On s'occupe aussi de la comparaison avec la littérature déjà présente à propos de la géométrie symplectique dérivée. Les résultats contenus dans cette thèse devraient ouvrir la voie à une quantification par déformation des champs dérivés de modules coisotropes.

Dans le chapitre 0, on rappelle les notions de géométrie dérivée formelle et de localisation formelle, qui ont étés introduites et étudiées dans [CPTVV]. On ne s'agissant que de préliminaires, on ne donne aucune preuve. Le lecteur intéressé trouvera tous les détails et les preuves dans l'article [CPTVV], ainsi que dans l'excellente *review* [PV]. Le principal objet qu'on étudie est un champ d'Artin dérivé X, avec la projection naturelle $X \to X_{DR}$, où X_{DR} est le champ de de Rham associé à X. Le champ X_{DR} a les mêmes points réduits de X, et la propriété fondamentale de la projection $X \to X_{DR}$ est que sa fibre en un point fermé $x : \mathrm{Spec} k \to X_{DR}$ est le complété formel de X en x, dénoté $\widehat{X}_x$. Cela permet de définir une algèbre $\mathcal{P}_X(\infty)$ dans une ∞-catégorie symétrique monoidale $\mathcal{M}_X$ de pré-champs sur X_{DR}, qui connait beaucoup du champ de départ X. En particulier, on a une équivalence

$$\mathrm{Perf}(X) \simeq \mathcal{P}_X(\infty) - \mathrm{mod}^{\mathrm{perf}},$$

où $\mathrm{Perf}(X)$ est l'∞-catégorie des complexes parfaits sur X, et $\mathcal{P}_X(\infty) - \mathrm{mod}^{\mathrm{perf}}$ est une sous-catégorie appropriée de $P_X(\infty)$-modules. On remarque qu'il s'agit déjà d'un résultat très fort, car il nous permets de faire du calcul différentiel sur X en le traitant comme s'il était juste une algèbre dans la catégorie $\mathcal{M}_X$. En particulier, on obtient

$$\mathrm{Symp}(X, n) \simeq \mathrm{Symp}(\mathcal{P}_X(\infty), n),$$

où $\mathrm{Symp}(-, n)$ est l'espace des structures symplectiques n-décalées de [PTVV]. Autrement dit, les structures symplectiques décalées de X définies géométriquement sont équivalentes aux structures symplectiques algébriques de l'algèbre $\mathcal{P}_X(\infty)$.

On est donc emmenés à définir les structures de Poisson n-décalées sur un champ X comme étant les structures $\mathbb{P}_{n+1}$ sur l'algèbre $\mathcal{P}_X(\infty)$, où $\mathbb{P}_{n+1}$ est l'opérade des dg algèbres de Poisson avec crochet de degré $-n$. Au même temps, on dispose maintenant d'une algèbre $\mathcal{P}_X(\infty)$: on peut donc construire l'algèbre graduée de ses multi-dérivations (décalées), et l'utiliser pour donner une définition alternative de structure de Poisson sur X. Il est naturel à ce point de comparer ces deux possibles définitions. Au niveau classique, cela revient á se demander si, étant donnée une variété lisse X, la donnée d'un crochet de Poisson sur l'algèbre des fonctions $C^\infty(X)$ est équivalente à la donnée d'un bivecteur π tel que $[\pi,\pi]=0$. La question est triviale pour des objets non-dérivés, mais la nature homotopique du contexte dérivé la rend beaucoup plus délicate dans le cas de l'algèbre $\mathcal{P}_X(\infty)$.

Dans le chapitre 1, on attaque précisement cette question. En travaillant dans le langage plus simples des dg algèbres commutatives, on donne deux définitions possibles de structure de Poisson décalée sur une algèbre A. D'un point de vue algébrique, on peut considérer d'étendre la structure commutative sur A pour obtenir une structure de Poisson. Cela nous conduit dans le contexte de la théorie des opérades, avec laquelle on donne la définition suivante.

Definition 0.0.15. *Soit A une dg algèbre commutative. L'*espace des structures $\mathbb{P}_{n+1}$ sur A *est défini avec le diagramme cartésien suivant*

$$\begin{tikzcd} P_{n+1}(A) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A) \end{tikzcd}$$

d'espaces, où le morphisme en bas est la structure multiplicative donnée sur A.

D'autre part A est toujours implicitement considérée comme un objet géométrique, c'est-à-dire un schéma affine dérivé. Sous des hypothèses assez faibles, il a donc un complexe tangent $\mathbb{T}_A$ qui est un dg A-module dualisable. Le module $\mathbb{T}_A$ est un analogue dérivé des champs de vecteurs sur A. On introduit aussi un analogue dérivé de l'algèbre des champs de poly-vecteurs : on dénote $\mathrm{Pol}(A,n)$ le complexe gradué $\mathrm{Sym}_A(\mathbb{T}_A[-n])$, et on l'appelle l'algèbre des *poly-vecteurs n-décalés*. Cette algèbre peut être munie d'une structure de Poisson explicite, qui est induite (tout comme dans le cas classique) par le crochet de Lie des champs de vecteurs. En particulier, $\mathrm{Pol}(A,n)[n]$ a une structure naturelle de dg algèbre de Lie graduée. Cela nous permet de donner une deuxième définition possible de structure de Poisson sur A, plus en ligne avec l'approche géométrique classique.

Soit dgMod^{gr} la catégorie des complexes gradués. Les objets de dgMod^{gr} ont deux graduations naturelles, qu'on va appeler le *degré cohomologique* et le *poids*. Dans dgMod^{gr}, on considère la catégorie dgLie^{gr} des complexes gradués munis d'un crochet de Lie de degré cohomologique 0 et de poids -1.

Definition 0.0.16. *Soit A une dg algèbre commutative. L'*espace des structures de Poisson n-décalées sur A *est*

$$\mathrm{Pois}(A,n) := \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(A,n+1)[n+1])$$

où $k[-1](2)$ est l'algèbre de Lie triviale k en degré cohomologique 1 et poids 2.

Le résultat principal du chapitre est l'équivalence des deux défnitions de structure de Poisson.

Theorem 0.0.17 ([Me], see also Theorem 1.3.2)**.** *Soit A une dg algèbre commutative telle que son complexe cotangent $\mathbb{L}_A$ est un A-module parfait. Alors on a un isomorphisme*

$$\mathrm{Pois}(A,n) \xrightarrow{\sim} \mathbb{P}_{n+1}(A)$$

dans la catégorie homotopique des ensembles simpliciaux.

Le Théorème 0.0.17 est un des points de départ de la géométrie de Poisson dérivée, développée dans [CPTVV]. En utilisant le résultat ci-dessus, on est maintenant capables de définir une structure de Poisson décalée sur un champ d'Artin dérivé X. Cela est fait dans le chapitre 2, où on suit l'exposition de [CPTVV] et [PV]. Le Théorème 0.0.3 joue ici un rôle fondamentale : en particulier, le fait d'avoir une description opéradique des structures de Poisson permet de pouvoir parler de quantification par déformation de champs d'Artin dérivés (voir la dernière section de [CPTVV]).

Plus précisément, la localisation formelle permet de définir une algèbre des poly-vecteurs $\mathrm{Pol}(X, n)$. On peut définir les structures de Poisson décalées dans la manière suivante :

Definition 0.0.18. *Soit X un champ d'Artin dérivé, localement de présentation finie. L'*espace des structures de Poisson n-décalées sur X *est*

$$\mathrm{Pois}(X, n) := \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(X, n+1)[n+1])$$

où dgLie^{gr} est toujours l'∞-catégorie des algèbres de Lie graduées sur k.

Alternativement, la localisation formelle nous dit qu'une bonne partie de l'information géométrique de X se retrouve dans le pré-champ $\mathcal{P}_X(\infty)$ avec sa structure $\mathbb{D}_{X_{DR}}(\infty)$-linéaire.

Definition 0.0.19. *Soit X un champ d'Artin dérivé, localement de présentation finie. L'*espace $\mathrm{Pois}'(X, n)$ des structures de Poisson n-décalées sur X *est le produit fibré*

$$\begin{tikzcd} \mathrm{Pois}'(X,n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \end{tikzcd}$$

où $\mathcal{P}_X(\infty)$ est considéré comme un objet dans l'∞-catégorie monoidale symétrique des $\mathbb{D}_{X_{DR}}(\infty)$-modules.

L'équivalence des deux définitions ci-dessus est encore montrée avec le Théorème 0.0.3.

Theorem 0.0.20 (see Theorem 2.2.3)**.** *Avec les mêmes notations, on a une équivalence*

$$\mathrm{Pois}(X, n) \simeq \mathrm{Pois}'(X, n).$$

Dans le chapitre 3, on s'occupe des structures coisotropes. Si $X \to Y$ est un morphisme de champs dérivés, on définit d'abord une structure coisotrope dans le cas où X et Y sont affines, et après on applique ces définitions au cas général en passant aux algèbres associées $\mathcal{P}_X(\infty)$ et $\mathcal{P}_Y(\infty)$.

Tout d'abord, on rappelle la définition classique de sous-variété coisotrope.

Definition 0.0.21. *Soit X une variété lisse, et soit $C \to X$ une sous-variété. Soit $\pi \in \Gamma(\Lambda^2 TX)$ un bivecteur de Poisson sur X. Alors on dit que C est* coisotrope *si l'image de la restriction de*

$$\pi^* : T^*X \longrightarrow TX$$

au fibré conormale N^*C est contenue TC. Alternativement, C est coisotrope s'il existe un morphisme

$$\begin{tikzcd} N^*C \arrow[r, hook] \arrow[d, dotted] & T^*X \arrow[d] \\ TC \arrow[r, hook] & TX \end{tikzcd}$$

qui fait commuter le diagramme, où les flèches horizontales sont les inclusions naturelles.

On remarque que cette notion peut se traduire aussi en disant que l'image du bivecteur π dans $\Gamma(\Lambda^2(NC))$ est zéro. On peut construire une version dérivée de cette définition de la manière suivante. Si $f : A \to B$ est un morphisme de dg algèbres commutatives, il induit une séquence exacte naturelle de B-modules

$$\mathbb{T}_f \to \mathbb{T}_B \to \mathbb{T}_A \otimes_A B,$$

où $\mathbb{T}_f[1]$ joue le rôle du fibré normal. En particulier, on obtient un morphisme d'algèbres graduées

$$\mathrm{Sym}_A(\mathbb{T}_A[-n-1]) \longrightarrow \mathrm{Sym}_B(\mathbb{T}_f[-n])$$

pour tout n. Soit $\mathrm{Pol}(f, n+1)$ la fibre homotopique de ce morphisme, qui sera appelée l'algèbre des *poly-vecteurs relatifs de f*. En appliquant un décalage de $n+1$, on a

$$\mathrm{Pol}(A, n+1)[n+1] \longrightarrow \mathrm{Sym}_B(\mathbb{T}_f[-n])[n+1].$$

Notons que la source de ce morphisme est une algèbre de Lie graduée, alors que le but ne l'est pas. Donc a priori il n'y a aucune raison pour avoir un crochet de Lie sur $\mathrm{Pol}(f, n+1)[n+1]$, car il n'est pas une limite d'algèbres de Lie. Cependant, on montre dans le chapitre 3 que $\mathrm{Pol}(f, n+1)[n+1]$ est en fait une algèbre de Lie, et qu'on a une séquence exacte d'algèbres de Lie graduées

$$\mathrm{Sym}_B(\mathbb{T}_f[-n])[n] \to \mathrm{Pol}(f, n+1)[n+1] \to \mathrm{Pol}(A, n+1)[n+1].$$

Definition 0.0.22. *Soit $f : A \to B$ un morphisme de dg algèbres commutatives. Supposons qu'on a une structure de Poisson n-décalée π sur A. Alors l'*espace $\mathrm{Cois}(f, \pi)$ des structures coisotropes sur f relatives à π *est l'espace des morphismes d'algèbres de Lie*

$$\begin{tikzcd}
\mathrm{Sym}_B(\mathbb{T}_f[-n])[n] \arrow[r] & \mathrm{Pol}(f, n+1)[n+1] \arrow[r] & \mathrm{Pol}(A, n+1)[n+1] \\
& k[-1](2) \arrow[u, dotted] \arrow[ur, "\pi"] &
\end{tikzcd}$$

qui font commuter le diagramme. Plus précisément, $\mathrm{Cois}(f, \pi)$ *est la fibre homotopique du morphisme d'espaces*

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(A, n+1)[n+1])$$

prise au point qui correspond à π.

La question est maintenant de trouver une structure algébrique équivalente sur un morphisme d'algèbres. Une proposition a été donnée dans [Sa]. Si B est une $\mathbb{P}_n$-algèbre, on peut produire naturellement une $\mathbb{P}_{n+1}$-algèbre $Z(B)$, dite le *centre de Poisson* de B. En tant qu'algèbre commutative, on a

$$Z(B) = \widehat{\mathrm{Sym}}_B(\mathbb{T}[-n])$$

où le crochet de Lie est le crochet standard des multi-dérivations, et la différentielle a deux composantes: une induite par la différentielle interne du B-module $\mathbb{T}_B$ et une autre donnée par $[\pi_B, -]$. On a immédiatement une projection naturelle d'algèbres commutatives $Z(B) \to B$.

Soit $\mathbb{P}_{[n+1,n]}$ L'opérade colorée dont les algèbres sont les couples d'objets (V, W) avec les structures additionnelles suivantes:

- une structure $\mathbb{P}_{n+1}$ sur V;
- une structure $\mathbb{P}_n$ sur W;
- un morphisme de $\mathbb{P}_{n+1}$-algèbres $V \to Z(W)$.

On a un foncteur d'oubli naturel

$$\mathbb{P}_{[n+1,n]}-\mathrm{alg} \longrightarrow \mathrm{Comm}_{\Delta^1}-\mathrm{alg},$$

où Comm_{Δ^1} est l'opérade colorée des morphismes d'algèbres commutatives.

Definition 0.0.23. *Soit $f : A \to B$ un morphisme d'algèbres commutatives. Alors l'espace des modules $\mathbb{P}_{[n+1,n]}(f)$ de structures $\mathbb{P}_{[n+1,n]}$ sur f est le produit fibré*

$$\begin{tikzcd} \mathbb{P}_{[n+1,n]}(f) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}_{\Delta^1}, \mathrm{End}_{A,B}) \end{tikzcd}$$

où le morphisme en bas est donné par f. Si de plus on avait une structure donnée π_A on A, on définit l'espace $\mathrm{Cois}'(f, \pi_A)$ comme le produit fibré

$$\begin{tikzcd} \mathrm{Cois}'(f, \pi_A) \arrow[r] \arrow[d] & \mathbb{P}_{[n+1,n]}(f) \arrow[d] \\ * \arrow[r] & \mathbb{P}_{n+1}(A) \end{tikzcd}$$

où le morphisme en bas est π_A.

Un des résultats principaux du chapitre 3 est le suivant.

Theorem 0.0.24 (see Theorem 3.4.11)**.** *Soit $f : A \to B$ un morphisme de dg algèbres commutatives, et soit π_A une structure de Poisson n-décalée sur A. Alors l'espace $\mathrm{Cois}(f, \pi_A)$ des structures coisotropes sur f au sens de la Définition 0.0.22 est équivalent à l'espace $\mathrm{Cois}'(f, \pi_A)$ des structures coisotropes au sens de la Définition 0.0.23.*

Dans le chapitre 4 on utilise les résultats du chapitre 3 pour donner une définition de structure coisotrope sur un morphisme $f : L \to X$ de champs d'Artin dérivés. On utilise ici la puissance du formalisme de la localisation formelle, et on est donc capable de travailler avec les objets $\mathcal{P}_L(\infty)$ et $\mathcal{P}_X(\infty)$, qui sont des algèbres dans les ∞-catégories monoidales symétriques $\mathcal{M}_L$ et $\mathcal{M}_X$ respectivement. Plus spécifiquement, le morphisme f induit un foncteur monoidal de pullback $f^* : \mathcal{M}_X \to \mathcal{M}_L$, avec une flèche naturelle

$$f^*_{\mathcal{P}} \ : \ f^*\mathcal{P}_X(\infty) \longrightarrow \mathcal{P}_L(\infty)$$

d'algèbres dans la catégorie $\mathcal{M}_L$.

Definition 0.0.25. *Soit $L \to X$ un morphisme de champs d'Artin dérivés, localement de présentation finie. Soit π_X une structure de Poisson n-décalée sur X. L'espace* $\mathrm{Cois}(f, \pi_X)$ *de structures coisotropes sur f est le produit fibré*

$$\begin{tikzcd} \mathrm{Cois}(f, \pi_X) \arrow[r] \arrow[d] & \mathbb{P}_{[n+1,n]}(f^*_{\mathcal{P}}) \arrow[d] \\ * \arrow[r] & \mathbb{P}_{n+1}(f^*\mathcal{P}_X(\infty)) \end{tikzcd}$$

où le morphisme en bas est la structure $\mathbb{P}_{n+1}$ sur $f^*\mathcal{P}_X(\infty)$ venant de π_X.

Encore une fois, on montre qu'il est possible de donner une caractérisation différente des structures coisotropes: on utilise le Théorème 0.0.24 pour montrer l'énoncé suivant.

Theorem 0.0.26 (see Theorem 4.1.5)**.** *Soit $f : L \to X$ un morphisme de champs d'Artin dérivés, localement de présentation finie. Supposons que π_X est une structure de Poisson n-décalée sur X. Alors l'espace* $\mathrm{Cois}(f, \pi_X)$ *est aussi équivalent à la fibre homotopique du morphisme*

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f^*_{\mathcal{P}}, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f^*\mathcal{P}_X(\infty), n+1)[n+1])$$

prise au point correspondant à la structure de Poisson π_X.

Le fait d'avoir une théorie des structures coisotropes dans le contexte dérivé a ses avantages. Par exemple, un des résultats principaux de Weinstein dans [We2] est à propos de ce qu'il appelle *relations coisotropes*. Par définition, une relation coisotrope (ou *correspondance coisotrope*) d'une variété de Poisson (X, π_X) vers une autre variété de Poisson (Y, π_Y) est une sous-variété de Poisson dans $(X \times Y, \pi_X - \pi_Y)$. Weinstein montre alors que sous des hypothèses assez fortes de transversalité, ces relations peuvent être composées : si on dispose d'une relation de X vers Y et d'une autre relation de Y vers Z, on peut en obtenir une de X vers Z. Dans la pratique, la condition de transversalité n'est pas toujours satisfaite, ce qui rend moins fort le théorème de Weinstein. D'autre part le résultat n'a aucun espoir d'être vrai dans le contexte classique sans l'hypothèse de transversalité. En particulier, cela empêche les correspondances coisotropes d'être les morphismes d'une catégorie des variétés de Poisson.

Mais dans le contexte de la géométrie dérivée, tous les problèmes d'intersection transverse disparaissent de façon magique. En particulier, on montre dans le chapitre 4 le théorème suivant, qui doit être considéré comme un premier pas vers une généralisation dérivée de la composition de correspondances de Weinstein, et donc vers la construction d'une ∞-catégorie de champs dérivés de Poisson.

Theorem 0.0.27 (see Theorem 4.2.2)**.** *Soient X, L_1 et L_2 des champs d'Artin dérivés, localement de présentation finie, et soit $\pi_X \in$* $\mathrm{Pois}(X, n)$ *une structure de Poisson n-décalée sur X. Soient $f_1 : L_1 \to X$ et $f_2 : L_2 \to X$ des morphismes de champs, et supposons que f_1 et f_2 soient équipés avec des structures coisotropes. Alors l'intersection dérivée Y, définie comme étant le produit fibré*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & L_1 \arrow[d, "f_1"] \\ L_2 \arrow[r, "f_2"] & X \end{tikzcd}$$

a une structure de Poisson $(n-1)$-décalée canonique. Cette structure est telle que la flèche $Y \to L_1 \times L_2$ est un morphisme de champs de Poisson $(n-1)$-décalés, où L_2 est considéré avec la structure de Poisson opposée.

Un résultat similaire avait été montré (au niveau cohomologique non-dérivé) par Baranovsky et Ginzburg dans [BG]. De l'autre coté, un énoncé parallèle dans le monde de la géométrie symplectique dérivée est un des théorèmes principaux de [PTVV], où bien sûr les structures coisotropes sont remplacées par des structures lagrangiennes. Le Théorème 0.0.27 est donc une généralisation à la fois des résultats de [BG] et du théorème sur les intersections de lagrangiennes de [PTVV].

La fin du chapitre 4 est dédiée à l'étude de la relation entre les structures coisotropes et les structures lagrangiennes. Dans [CPTVV], les auteurs montrent qu'un structure n-symplectique sur un champ d'Artin dérivé induit de manière naturelle une structure de n-Poisson, qui de plus est non-dégénérée dans un sens approprié. Il est important de souligner que ce résultat est loin d'être trivial dans le monde dérivé. D'abord, il faut utiliser l'approche de la localisation formelle pour réduire la question à l'algèbre $\mathcal{P}_X$. Même ici, la nature homotopique des structures considérée pose des problèmes: la définition de structure symplectique donne juste un quasi-isomorphisme entre le complexe tangent et le complexe cotangent. Si d'un coté il s'agit d'un choix naturel en géométrie dérivée, le fait ne pas disposer d'un vrai isomorphisme empêche toute méthode classique de marcher.

Dans le cas rélatif, on introduit une notion de structure coisotropes non-dégénérée, et on montre le théorème suivant, qui avait été conjecturé dans [JS] et [CPTVV].

Theorem 0.0.28 (see Theorem 4.3.7)**.** *Soit $f : L \to X$ un morphisme de champs d'Artin dérivés, localement de présentation finie. Soit π_X une structure de Poisson n-décalée sur X, dans le sens de [CPTVV], qui induit sur X une structure symplectique n-décalée ω_X. On a une équivalence naturelle d'espaces*

$$\mathrm{Cois}^{nd}(f, \pi_X) \xrightarrow{\sim} \mathrm{Lagr}(f, \omega_X),$$

où $\mathrm{Cois}^{nd}(f, \pi_X)$ est l'espace des structures coisotropes non-dégénérées sur f, et $\mathrm{Lagr}(f, \omega_X)$ est l'espace des structures Lagrangiennes sur f (dans le sens de [PTVV]).

Ce théorème est bien sûr important du point de vue fondationnel, mais il a aussi des conséquences concrètes intéressantes. En effet, il y a beaucoup d'espaces de modules qui apparaissent comme des lagrangiennes dérivées dans des champs dérivés symplectiques (l'article [ST] en donne une liste assez détaillée). Grâce au théorème ci-dessus, ces espaces de modules ont aussi une structure coisotrope, et on peut maintenant espérer d'étendre au contexte dérivé des résultats de quantification des sous-variétés coisotropes (par exemples ceux de [CF] et [OP]). Ce procédé devrait alors produire des quantifications intéressantes pour des espaces de modules liés à la topologie et à la théorie des représentations.

On remarque aussi que le Théorème 0.0.28 permet de retrouver des résultats sur les structures lagrangiennes en utilisant la théorie des structures coisotropes développée dans cette thèse. Par exemple, le Théorème 2.9 dans [PTVV] est maintenant une conséquence du Théorème 0.0.27 et du Théorème 0.0.28.

Le chapitre final est consacré au problème de la comparaison de notre définition de structure coisotrope avec celle proposée dans la section 3.4 de [CPTVV]. On propose deux approches différentes, mais on renvoie le lecteur à l'article [MS] pour plus de détails sur ce sujet.

Chapter 0

Preliminaries and formal localization

In this chapter we present the general framework in which we will work for most of the time. Almost all the material is taken from [CPTVV].

We start by a preliminary section on our categorical conventions. In many situation throughout this this thesis, we will need to be able to do differential calculus in a quite general ∞-categorical setting. This first section is devoted to fix notations and properties of the model and ∞-categories we will work in, much in the spirit of the first chapter of [CPTVV].

The second section deals with the basic definitions of the objects involved in differential calculus. More specifically, we explain how to construct cotangent complexes and de Rham algebras internal to a nice enough symmetric monoidal ∞-category, and we also treat the slightly more delicate case of tangent complexes and algebras of polyvector fields.

Section 3 is devoted to specialize the formalism developed in the first two sections in the case where the base ∞-category is $\epsilon - \mathrm{dgMod}^{gr}$, the category of graded mixed dg modules over the base field k. This is one of the most important situations in which we will use the general theory of differential calculus. We will introduce the so-called *Tate realization*, which is a refined version of the concept of realization defined in section 1.

In section 4 we introduce the main concepts of formal derived geometry. The subject is of course a vast one, and we will just summarize the definitions and the results that will be needed in the following chapters.

We also recall the notion of formal derived stacks, and more importantly we give an outline of the method of formal localization, which is a very strong technical tool developed in [CPTVV]. This is done in section 5. For our purposes, formal localization will play a fundamental role in extending the definitions and the results obtained on affines (chapters 1 and 3) to general derived stacks (chapters 2 and 4). It should be stressed that formal derived geometry and derived formal localization are interesting in their own right, and they will prove useful in many future development of derived algebraic geometry.

We will nevertheless omit any complete proof and avoid technicalities, as they have already been spelled out in the original paper [CPTVV]. The interested reader can also look at the recent survey [PV].

0.1 Categorical setting

Let k be a field of characteristic zero, and let $C(k)$ be the category of unbounded cochain complexes of k-modules. The category $C(k)$ has a standard projective model structure, whose weak equivalences are the quasi-isomorphisms and whose fibrations are degree-wise surjections. Moreover, $C(k)$ is naturally a closed symmetric monoidal category, where the monoidal structure is given by the standard tensor product $\otimes_k$ of cochain complexes, and the unit is k sitting in degree 0. These two structures are compatible, in the sense that the following compatibility condition is satisfied: given two cofibrations $f : A \to B$ and $g : C \to D$, the induced map

$$(A \otimes D) \coprod_{A \otimes C} (B \otimes C) \longrightarrow B \otimes D$$

is again a cofibration. Moreover the above map is a trivial cofibration if f or g is.

Let now M be a symmetric monoidal combinatorial model category. In addiction to this, suppose M is $C(k)$-enriched, or equivalently M is a symmetric monoidal $C(k)$-model algebra in the sense of Hovey (see definition 4.2.20 in [Ho]). As it is proven in Appendix 1 of [CPTVV], such an M becomes a stable model category. Furthermore, we will make the following assumptions on M.

- The unit of M is cofibrant.
- Let $f : A \to B$ be a cofibration, and take C to be an object of M. Then for any morphism $A \otimes C \to D$ the strict pushout of the diagram

$$\begin{tikzcd} A \otimes C \arrow[r] \arrow[d] & D \\ B \otimes C & \end{tikzcd}$$

  is also a model for the homotopy pushout.
- If A is a cofibrant object, then the functor

$$\begin{array}{ccc} M & \longrightarrow & M \\ X & \longmapsto & A \otimes X \end{array}$$

  preserves weak equivalences.
- Finite products and filtered colimits preserve weak equivalences.

In particular, by the results of [SS], if A is a commutative monoid in M then the category $A - \mathrm{mod}_M$ of A-modules in M inherits a structure of symmetric monoidal model category, where weak equivalences and fibrations are detected in M. Moreover, if A and B are weakly equivalent commutative monoids in M, then the two categories $A - \mathrm{mod}_M$ and $B - \mathrm{mod}_M$ are Quillen equivalent.

Given such a model category M, we will often be interested in working in the category of graded mixed objects in M. In order to give an abstract enough definition of such construction, let us start by considering any commutative and cocommutative Hopf dg algebra A. Since M is enriched over $C(k)$, we can construct the category $A - \mathrm{comod}_M$ of A-comodules in M. This category comes equipped with a natural forgetful functor to M; its right adjoint is the functor $M \to A - \mathrm{comod}_M$

sending an object X to the cofree comodule $X \otimes A$. The fact that A was not only a coalgebra, but also an algebra allows us to consider $A - \mathrm{comod}_M$ as a symmetric monoidal category, such that the forgetful functor $A - \mathrm{comod}_M \to M$ becomes naturally a symmetric monoidal functor.

Let us now specialize the above construction for $A = k[t, t^{-1}] \otimes \mathrm{Sym}_k(k[1])$, where the variable t sits in degree 0. If we denote by ϵ the generator of degree -1 of $\mathrm{Sym}_k(k[1])$, the comultiplication in A is defined by sending t to $t \otimes t$, and ϵ to $\epsilon \otimes 1 + t \otimes \epsilon$. The counit sends t to 1 and ϵ to 0. It is immediate to see that with such comultiplication and counit, A becomes a commutative and cocommutative Hopf dg algebra.

Our interest in such A lies in the fact that the category $A - \mathrm{comod}_M$ of A-comodules in M is naturally identified with the category of graded mixed objects in M, and will be denoted $\epsilon - M^{gr}$. More explicitly, an object X in $\epsilon - M^{gr}$ is the collection of the following data:

- a sequence $\{X(p)\}_{p \in \mathbb{Z}}$ of objects of M indexed by $\mathbb{Z}$;
- a sequence of morphisms in M

  $$\{\epsilon_p : X(p) \longrightarrow X(p+1)[1]\}_{p \in \mathbb{Z}}$$

  such that the composition

  $$X(p) \longrightarrow X(p+1)[1] \longrightarrow X(p+2)[2]$$

  is zero for every p.

Alternatively, one can of course think of X as being the direct sum of the various $X(p)$. In the special case in which we take M to be $C(k)$, we will write $\epsilon - M^{gr} = \epsilon - \mathrm{dgMod}^{\mathrm{gr}}$ and call its objects graded mixed complexes.

Notice that in our context, $\epsilon - M^{gr}$ is again a symmetric monoidal model category satisfying all of out starting assumptions on M: in fact, the category M^{gr} of graded objects in M is naturally a symmetric monoidal model category, and we can use it to define a model structure on $\epsilon - M^{gr}$ via the forgetful functor $\epsilon - M^{gr} \to M^{gr}$. In particular, both weak equivalences and cofibrations are detected in M^{gr}.

We will sometimes adopt the more modern point of view of ∞-categories, which are more general objects than model categories. All the text could have possibly have been written keeping the use of model categories to a minimum, but we felt that the use of the more explicit approach of model categories simplifies the understanding of the contents.

For a general model category M, we will denote by $\mathcal{M}$ the associated ∞-category $L(M)$, obtained by formally inverting weak equivalences. If M satisfies the assumptions of the previous section, $\mathcal{M}$ becomes a stable ∞-category in the sense of [Lur3], and it also has an induced symmetric monoidal structure. As usual, a concrete model for $\mathcal{M}$ is the category of fibrant and cofibrant objects in M, together with its standard simplicial enrichment.

Given such an $\mathcal{M}$, we will denote by $\mathrm{CAlg}_{\mathcal{M}}$ the ∞-category of commutative algebras internal to $\mathcal{M}$, in the sense of [Lur3]. One can of course also think of $\mathrm{CAlg}_{\mathcal{M}}$ as the ∞-category associated to the model category of commutative monoids in M.

In analogy with the case of model categories, we define the symmetric monoidal ∞-category of graded mixed objects in $\mathcal{M}$ to be $L(\epsilon - M^{gr})$, and we denote it by $\epsilon - \mathcal{M}^{gr}$. The category of commutative monoids inside $\epsilon - \mathcal{M}^{gr}$ will be denoted $\epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}}$, and its objects will be called graded mixed commutative dg algebras in $\mathcal{M}$. There is a natural forgetful ∞-functor

$$\epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}} \longrightarrow \epsilon - \mathcal{M}^{gr}$$

whose left adjoint is given by the free algebra construction.

Notice that if M is a model category satisfying our assumptions, then in particular M is enriched over $C(k)$ and the monoidal unit 1_M is cofibrant. This means that there is a natural Quillen functor

$$\begin{array}{ccc} M & \longrightarrow & C(k) \\ X & \longmapsto & \underline{Hom}(1_M, X). \end{array}$$

This functor has a left adjoint, which uses the tensor enrichment of M: namely, it is given by

$$\begin{array}{ccc} C(k) & \longrightarrow & M \\ C & \longmapsto & C \otimes 1_M \end{array}$$

At the general ∞-category level, the above right adjoint will be called *realization functor*, and the image of an object X in $\mathcal{M}$ will be denoted $|X|$.

We end this section by noticing that since the unit 1_M is a comonoid object, the realization functor is lax symmetric monoidal, and thus it induces similar right adjoint functors

$$\begin{array}{ccc} \mathrm{cdga}_{\mathcal{M}} & \longrightarrow & \mathrm{cdga}_k \\ \epsilon - \mathrm{cdga}_{\mathcal{M}}^{\mathrm{gr}} & \longrightarrow & \epsilon - \mathrm{cdga}_k^{\mathrm{gr}} \end{array}$$

The image of an object X through any of the realization functors above will always be denoted $|X|$.

0.2 Differential calculus

We now concentrate on algebra objects inside an ∞-category $\mathcal{M}$ as in the previous section. Given such an $A \in \mathcal{M}$, we can define an ∞-category of A-modules in $\mathcal{M}$, which can be realized as the localization along weak equivalences of the model category of A-modules in M, where we have considered A as an algebra in M. Notice that since A is commutative, the category $A - \mathrm{Mod}_{\mathcal{M}}$ of A-modules in $\mathcal{M}$ is itself a symmetric monoidal stable ∞-category.

Our next goal is to define the cotangent complex and the de Rham algebra of an algebra $A \in \mathrm{CAlg}_{\mathcal{M}}$. As usual, given an A-module N, we can consider the trivial square-zero extension $A \oplus N$; this is again an object in $\mathrm{CAlg}_{\mathcal{M}}$, and comes moreover with a canonical projection $A \oplus N \to A$, which is a morphism of commutative algebras in $\mathcal{M}$.

We can the define the space $\mathrm{Der}(A, N)$ of derivations from A to N as in [HAG-II]: it is the space of dotted maps making the following diagram

$$\begin{tikzcd} & A \oplus N \arrow[d] \\ A \arrow[ur, dotted] \arrow[r, "\mathrm{id}"'] & A \end{tikzcd}$$

commute in $\mathrm{cdga}_{\mathcal{M}}$. This defines an ∞-functor

$$\begin{array}{cccc} \mathrm{Der}(A, -) : & A - \mathrm{Mod}_{\mathcal{M}} & \longrightarrow & \mathrm{sSet} \\ & N & \longmapsto & \mathrm{Der}(A, N) \end{array}$$

This functor is corepresentable by an object, denoted $\mathbb{L}_A^{int}$.

Definition 0.2.1. *Given an algebra* $A \in \mathrm{CAlg}_{\mathcal{M}}$*, the object* $\mathbb{L}_A^{int} \in \mathcal{M}$ *is the* internal cotangent complex of A*. Its realization* $\mathbb{L}_A := |\mathbb{L}_A^{int}|$ *is simply called the* cotangent complex of A.

Remark that by general properties of the realization functor, $\mathbb{L}_A$ is naturally an $|A|$-module in $C(k)$. Moreover, in the simple case where $M = C(k)$, then the above definition coincides with the usual one.

We now pass to de Rham algebras. Given any graded mixed algebra $B \in \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}}$, it is immediate to check that its weight zero part $B(0)$ naturally inherits the structure of a commutative algebra in $\mathcal{M}$. This defines an ∞-functor

$$\begin{array}{rccc} (-)(0): & \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}} & \longrightarrow & \mathrm{cdga}_{\mathcal{M}} \\ & B & \longmapsto & B(0) \end{array}$$

It turns out this functor has a left adjoint, which will be denoted

$$\mathrm{DR}^{int} : \mathrm{cdga}_{\mathcal{M}} \longrightarrow \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}}.$$

We refer to [CPTVV], Section 1.3.2 for more details.

Definition 0.2.2. *Given a commutative algebra $A \in \mathrm{cdga}_{\mathcal{M}}$, the object $\mathrm{DR}^{int}(A) \in \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}}$ is the* internal de Rham algebra of A. *Its realization $\mathrm{DR}(A) := |\mathrm{DR}^{int}(A)| \in \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{k}$ will be simply called the* de Rham algebra of A.

The relation between the (internal) cotangent complex and the de Rham algebra is given by the following result, which is Proposition 1.3.12 in [CPTVV].

Proposition 0.2.3. *For every $A \in \mathrm{cdga}_{\mathcal{M}}$, there is a natural equivalence*

$$\mathrm{Sym}_A(\mathbb{L}^{int}_A[-1]) \longrightarrow \mathrm{DR}^{int}(A)$$

in the ∞-category $\mathrm{cdga}^{gr}_{\mathcal{M}}$, which is moreover natural in A.

Notice that this is not an equivalence of graded *mixed* algebras, simply because a priori there is no mixed structure on $\mathrm{Sym}_A(\mathbb{L}^{int}_A[-1])$. One can actually look at this proposition as a way to induce a (weak) mixed structure on the left hand side, giving an abstract construction of the de Rham differential.

A similar construction of the internal cotangent complex and of the internal de Rham algebra applies to the relative context: starting with a morphism $A \to B$ in $\mathrm{cdga}_{\mathcal{M}}$, one can construct an object $\mathbb{L}^{int}_{B/A}$ which lives in $B - \mathrm{Mod}_{\mathcal{M}}$, together with a graded mixed algebra $\mathrm{DR}^{int}(B/A) \in \epsilon - \mathrm{cdga}^{\mathrm{gr}}_{\mathcal{M}}$. Both graded algebras $\mathrm{Sym}_B(\mathbb{L}^{int}_{B/A}[-1])$ and $\mathrm{DR}^{int}(B/A)$ comes equipped with a morphism from A, considered as concentrated in weight 0. Then we have an equivalence

$$\mathrm{Sym}_B(\mathbb{L}^{int}_{B/A}[-1]) \longrightarrow \mathrm{DR}^{int}(B/A)$$

in the comma ∞-category $A/\mathrm{cdga}^{gr}_{\mathcal{M}}$. Again, we refer to Section 1.3.2 in [CPTVV] for more details on the relative version of DR^{int}, which is however completely analogous to the absolute case.

0.2.1 Differential forms and polyvectors

Using the above formalism, we can define differential forms and polyvector fields for general commutative algebras in $\mathcal{M}$.

Definition 0.2.4. *Let $A \in \mathrm{CAlg}_{\mathcal{M}}$ be a commutative algebra in $\mathcal{M}$. Then the* space of p-forms of degree n on A *is the mapping space*

$$\mathcal{A}^p(A,n) := \mathrm{Map}_{\mathcal{M}}(1_{\mathcal{M}}[-n], \wedge^p_A \mathbb{L}^{int}_A).$$

We define moreover the space of closed p-forms of degree n *to be the mapping space*

$$\mathcal{A}^{p,cl}(A,n) := \mathrm{Map}_{\epsilon-\mathcal{M}^{gr}}(1_{\mathcal{M}}(p)[-n-p], \mathrm{DR}^{int}(A)),$$

where $1_{\mathcal{M}}(p)[-n-p]$ is the monoidal unit of $\mathcal{M}$ viewed as a graded mixed object concentrated in weight p, with trivial mixed structure.

Notice that there is an induced map $\mathcal{A}^{p,cl}(A,n) \to \mathcal{A}^p(A,n)$, which sends a closed p-form to the underlying p-form. By definition of realization functors, one has natural equivalences

$$\mathcal{A}^p(A,n) \simeq \mathrm{Map}_{C(k)}(k[-n], \wedge^p_{|A|}\mathbb{L}_A)$$

$$\mathcal{A}^{p,cl}(A,n) \simeq \mathrm{Map}_{\epsilon-\mathrm{dgMod}^{gr}}(k(p)[-n-p], \mathrm{DR}(A))$$

We now pass to symplectic structures. Let $A \in \mathrm{CAlg}_{\mathcal{M}}$, and consider the ∞-category $A-\mathrm{Mod}_{\mathcal{M}}$. This is a closed symmetric monoidal ∞-category, so that in particular we can take dual of A-modules. The dual of an A-module N will be denoted by $N^\vee$.

Definition 0.2.5. *For $A \in \mathrm{CAlg}_{\mathcal{M}}$, the* internal tangent complex of A *is the A-module $\mathbb{T}^{int}_A := (\mathbb{L}^{int}_A)^\vee$. Its realization $\mathbb{T}_A = |\mathbb{T}^{int}_A| \in |A| - \mathrm{Mod}$ will be simply called the* tangent complex of A.

In particular, if $\mathbb{L}^{int}_A$ is dualizable, any 2-form ω of degree n on A induces by adjunction a morphism

$$\omega^\sharp : \mathbb{T}^{int}_A \longrightarrow \mathbb{L}^{int}_A[n].$$

Definition 0.2.6. *Let $A \in \mathrm{CAlg}_{\mathcal{M}}$, and suppose that $\mathbb{L}^{int}_A$ is dualizable. Then we say that a closed 2-form $\omega \in \mathcal{A}^{2,cl}(A,n)$ is* non-degenerate *if the morphism induced by the underlying 2-form gives an equivalences*

$$\mathbb{T}^{int}_A \simeq \mathbb{L}^{int}_A[n]$$

of A-modules. We define the space of n-shifted symplectic structures on A $\mathrm{Symp}(A,n)$ *to be the subspace of $\mathcal{A}^{2,cl}(A,n)$ given by the union of connected components of non-degenerate closed forms.*

The definition of polyvector fields, which is in some sense the dual notion of differential forms, is a bit more delicate in this context. In particular, the ordinary lack of functoriality of the algebra of polyvector fields makes the definition in the ∞-categorical world not entirely trivial.

Let us us thus start with an algebra A in the model category M. We can define $\mathrm{T}^{(p)}(A,n)$, the n-shifted p-multiderivations on A as in section 1.4.2 of [CPTVV]. Note that this is in particular an object of M. Moreover, the symmetric group S_p acts naturally on $\mathrm{T}^{(p)} := (A,n)$, and let us denote by $\mathrm{T}^{(p)}(A,n)^{S_p}$ the object of M of S_p-invariants multiderivations.

Definition 0.2.7. *Let $A \in \mathrm{CAlg}_M$. The* object of symmetric internal n-shifted multiderivations *on A is*

$$Pol^{int}(A,n) := \bigoplus T^{(p)}(A,n)^{S_p}$$

Note that $\mathrm{Pol}^{int}(A,n)$ is in a natural way an object of M^{gr}. In addiction to this, there is a canonical multiplication of weight 0 in $\mathrm{Pol}^{int}(A,n)$, and the composition by insertion gives rise to a shifted Lie structure, which has to be interpreted as an analogue of the classical Schouten-Nijenhuis bracket. These algebraic structures are compatible, so that $Pol^{int}(A,n)$ is a graded $\mathbb{P}_{n+1}$-algebra in M: by this we mean that the commutative product has weight 0, while the Lie bracket has weight -1.

Even if the association $A \mapsto Pol^{int}(A,n)$ is not completely functorial in A, it is still possible to define a restricted functoriality, as in section 1.4.2 of [CPTVV].

Recall that a morphism of $f : A \to B$ inside $\mathrm{CAlg}_{\mathcal{M}}$ is said to be *formally étale* if the induced map

$$\mathbb{L}_A^{int} \otimes_A B \longrightarrow \mathbb{L}_B^{int}$$

is an equivalence of B-modules.

Proposition 0.2.8. *Let* $\mathrm{CAlg}_{\mathcal{M}}^{fet}$ *be the sub-∞-category of* $\mathrm{CAlg}_{\mathcal{M}}$ *consisting of formally étale morphisms. There is a well defined ∞-functor*

$$\mathrm{Pol}^{int}(-,n) : \mathrm{CAlg}_{\mathcal{M}}^{fet} \longrightarrow \mathbb{P}_{n+1} - \mathrm{alg}_{\mathcal{M}}$$

such that if $A \in \mathrm{CAlg}_M$ and B is a fibrant-cofibrant replacement of A, then $\mathrm{Pol}^{int}(A,n) \simeq Pol^{int}(B,n)$.

Definition 0.2.9. *The object* $\mathrm{Pol}^{int}(A,n)$ *of the previous proposition is called the* internal algebra of polyvectors fields on A. *Its realization* $\mathrm{Pol}(A,n) := |\mathrm{Pol}^{int}(A,n)|$ *is simply called* algebra of polyvector fields on A.

The relation between polyvectors and the tangent complex is as expected: if $A \in \mathrm{CAlg}_{\mathcal{M}}$ is such that $\mathbb{L}_A^{int}$ is dualizable, then we have a natural equivalence

$$\mathrm{Pol}^{int}(A,n) \simeq \mathrm{Sym}_A(\mathbb{T}_A^{int}[-n])$$

in the ∞-category $\mathrm{CAlg}_{\mathcal{M}}^{gr}$ of graded algebras in $\mathcal{M}$.

0.3 Tate realizations and twistings

We already introduced early in this chapter the realization functors $\mathcal{M} \to C(k)$, which were defined for any $\mathcal{M}$ satisfying our starting hypothesis. This functor is a right adjoint, and in some sense it forgets part of the information contained in an object $X \in \mathcal{M}$. For example, if $\mathcal{M}$ is the the ∞-category of dg A-modules for some commutative dg algebra A, then in this case the realization is simply the usual functor $\mathcal{M} \to C(k)$ which forgets the A-action. Another very important example is the case in which $\mathcal{M}$ is the ∞-category of sheaves of complexes on an ∞-site: here the realization functor is typically given by global sections.

In this section we study the above formalism of differential calculus in the case where $\mathcal{M}$ is itself the ∞-category of graded mixed k-dg modules. For such $\mathcal{M}$, the realization functor has an explicit description.

Proposition 0.3.1. *Let X be an object of* $\epsilon - \mathrm{dgMod}^{gr}$. *There is a natural equivalence*

$$|X| \simeq \prod_{p \geq 0} X(p)$$

inside the category $C(k)$, where the right hand side is endowed with the total differential obtained as sum of the internal differential and the mixed structure of X.

As usual, we refer to the paper [CPTVV] (and in particular to Proposition 1.5.1) for more details and the proof.

In particular, the above proposition tells us that the standard realization functors does not see the negative weights of a graded mixed dg module X. In our geometric application of interest, this will prove to be too much: forgetting the negative weights loses too much information. In order to overcome such problems, we defined a refined realization functor.

Definition 0.3.2. *Let again* $\mathcal{M} = \epsilon - \mathrm{dgMod}^{gr}$*. The* Tate realization ∞-functor *is defined to be*

$$\begin{array}{rccc} |-|^t : & \mathcal{M} & \longrightarrow & \mathrm{dgMod} \\ & X & \longmapsto & \mathrm{colim}_{i\geq 0} \prod_{p\geq -i} X(p). \end{array}$$

Note that there is a canonical natural transformation of functors $|-| \to |-|^t$. Also, this defines an equivalence $|X| \simeq |X|^t$ as soon as $X(p) \simeq 0$ for every negative p.

Using this alternative realization functor, we can define Tate version of the objects involved in differential calculus.

Definition 0.3.3. *Let* $\mathcal{M}$ *be the* ∞*-category* $\epsilon - \mathrm{dgMod}^{gr}$*, and consider an algebra* $A \in \mathrm{CAlg}_{\mathcal{M}}$*.*

- *The* Tate de Rham algebra of A *is defined by*

$$\mathrm{DR}^t(A) := |\mathrm{DR}^{int}(A)|^t$$

as an object in $\epsilon - \mathrm{cdga}^{\mathrm{gr}}$*.*

- *The* Tate algebra of n-shifted polyvectors of A *is defined by*

$$\mathrm{Pol}^t(A,n) := |\mathrm{Pol}^{int}(A,n)|^t$$

as a graded $\mathbb{P}_{n+1}$*-algebra in complexes.*

By the above observation, if we start with A such that $A(p) = 0$ for every negative p, then there is a natural equivalence $\mathrm{DR}(A) \simeq \mathrm{DR}^t(A)$. The situation for polyvectors is different: the weights of $\mathbb{T}_A^{int}$ are dual to those of A, so that $\mathrm{Pol}^{int}(A,n)$ has in general non-trivial negative weight components. This implies that the map $\mathrm{Pol}(A,n) \to \mathrm{Pol}^t(A,n)$ will not be an equivalence in general.

We end this section by remarking that the Tate realization functors can actually be interpreted as a standard realization functor, provided that we slightly modify the starting category $\mathcal{M} = \epsilon - \mathrm{dgMod}^{gr}$. Namely, consider the following ind-object in $\mathcal{M}$

$$k(\infty) := \{k(0) \longrightarrow k(1) \longrightarrow \dots k(i) \longrightarrow \dots\},$$

where $k(i)$ is the object of $\mathcal{M}$ which has only k sitting in degree 0 and weight i, with trivial differential and mixed structure. Since the category $\mathrm{Ind}(\mathcal{M})$ of ind-objects in $\mathcal{M}$ is still symmetric monoidal, we get a well defined ∞-functor

$$\begin{array}{rccc} (-)(\infty) : & \mathcal{M} & \longrightarrow & \mathrm{Ind}(\mathcal{M}) \\ & A & \longmapsto & A \otimes k(\infty). \end{array}$$

This functor will be interpreted as a twist by $k(\infty)$, and we will use the simpler notation $A(\infty) := A \otimes k(\infty)$. Notice also that $k(\infty)$ is an algebra in $\mathrm{Ind}(\mathcal{M})$, so that if A is an algebra in $\mathcal{M}$, then $A(\infty)$ is an algebra in $\mathrm{Ind}(\mathcal{M})$.

The ∞-category $\mathrm{Ind}(\mathcal{M})$ fits into our categorical framework, so that it has a natural standard realization functor to dgMod. One can check that for every object $X \in \mathcal{M}$, there is an equivalence

$$|X|^t \simeq |X \otimes k(\infty)|.$$

In particular, this implies that the Tate de Rham algebra and the Tate polyvectors can also be interpreted in an alternative way.

Proposition 0.3.4. *Let $A \in \mathrm{CAlg}_{\mathcal{M}}$, and suppose $\mathbb{L}_A^{int}$ is dualizable. Then there is a natural equivalence*

$$\mathrm{DR}^t(A) \simeq \mathrm{DR}(A(\infty)/k(\infty))$$

in the ∞-category of graded mixed k-dg modules. Similarly, there is an equivalence

$$\mathrm{Pol}^{int}(A, n) \simeq \mathrm{Pol}(A(\infty)/k(\infty), n)$$

in the ∞-category of graded $\mathbb{P}_{n+1}$-algebras.

0.4 Formal derived stacks

Recall that $A \in \mathrm{cdga}_k^{\leq 0}$ is *almost finitely presented* if $H^0(A)$ is a k-algebra of finite type, and all the $H^i(A)$ are finitely presented $H^0(A)$-modules. Throughout this chapter, dAff is the opposite ∞-category of almost finitely presented commutative dg algebras (in non-positive degrees). We will just call its element *derived affine stacks*, without recalling the finite presentation condition. This ∞-category has a standard étale topology (as explained in [HAG-II], definition 2.2.2.3), and thus we can construct the associated ∞-topos of derived stacks, which will be denoted simply dSt, again omitting the finitude condition.

The starting point of formal derived geometry is of course the definition of *formal derived stack*.

Definition 0.4.1. *Let $X \in \mathrm{dSt}$. We say that X is a* formal derived stack *if it satisfies the following conditions:*

1. *The stack X is* nilcomplete*, that is to say that for every $SpecA \in \mathrm{dAff}$, the canonical map*

$$X(A) \longrightarrow \lim_n X(A_{\leq n})$$

is an equivalence of space, where $A_{\leq n}$ is the n-th Postnikov tower decomposition of A.

2. *The stack X is* infinitesimally cohesive*, that is to say that for all cartesian squares in* dAff^{op}

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_0 \end{tikzcd}$$

such that both induced maps $H^0(A_1) \to H^0(A_0)$ and $H^0(A_2) \to H^0(A_0)$ are surjective with nilpotent kernels, the induced square of spaces

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & X(A_1) \arrow[d] \\ X(A_2) \arrow[r] & X(A_0) \end{tikzcd}$$

is again cartesian.

Notice that all derived Artin stacks (in the sense of [HAG-II]) are formal: nilcompleteness is a consequence of the representability criterion of Toën and Vezzosi in [HAG-II, Appendix C], while the infinitesimally cohesiveness follows from [Lur1, DAG XIV, Lemma 2.1.7]. Moreover, any limit of formal derived stacks is again a formal derived stack. The main non-trivial examples of formal derived stacks are provided by the following very general categorical construction on the ∞-category of derived affines schemes.

We say that $A \in \mathrm{dAff}^{op}$ is *reduced* if it is discrete and $H^0(A)$ is a reduced (non dg) k-algebra. Denote the ∞-category of reduced cdga by alg^{red}, and consider the natural inclusion functor $i : \mathrm{alg}^{red} \to \mathrm{cdga}^{\leq 0}$. This functor has a left adjoint, explicitly given by

$$\begin{array}{ccc} \mathrm{cdga}^{\leq 0} & \longrightarrow & \mathrm{alg}^{red} \\ A & \longmapsto & A^{red} \end{array}$$

where A^{red} is the reduced k-algebra $H^0(A)/\mathrm{Nilp}(H^0(A))$. One can moreover check that i induces an ∞-functor $i^* : \mathrm{dSt} \to \mathrm{St}_{red}$, where St_{red} is the category of stacks on $(\mathrm{alg}^{red})^{op}$, endowed with the étale topology. The functor i^* has both a right adjoint (denoted i_*) and a left adjoint (denoted $i_!$).

Definition 0.4.2. 1. *The composite $i_*i^* : \mathrm{dSt} \to \mathrm{dSt}$ is called the* de Rham stack functor*. The image of $X \in \mathrm{dSt}$ under the de Rham stack functor is denoted X_{DR}.*

2. *The composite $i_!i^* : \mathrm{dSt} \to \mathrm{dSt}$ is called the* reduced stack functor*. The image of $X \in \mathrm{dSt}$ under the reduced stack functor is denoted X_{red}.*

For more details on the de Rham stack, we refer to the work of Simpson (see [Si1] and [Si2]). Notice that by adjunction, we get natural morphisms of derived stacks $X \to X_{DR}$ and $X_{red} \to X$.

Definition 0.4.3. *Let $f : X \to Y$ be a morphism of derived stacks. The* formal completion of Y along f *is denoted $\widehat{Y}_f$, and it is defined as the pullback*

$$\begin{tikzcd} \widehat{Y}_f \arrow[r] \arrow[d] & X_{DR} \arrow[d] \\ Y \arrow[r] & Y_{DR} \end{tikzcd}$$

in the ∞-category dSt.

We now state the main properties of these constructions. Most of them are easy verification, and the details can be found in [CPTVV].

- The functors i_* and $i_!$ are fully faithful.

- The composite $i_!i^*$ is left adjoint to i_*i^*, or equivalently the reduced stack functor is left adjoint to the de Rham functor.
- For every $A \in \mathrm{cdga}^{\leq 0}$ and $X \in \mathrm{dSt}$, we have $X_{DR}(A) \simeq X(A^{red})$ and $(\mathrm{Spec}A)_{red} \simeq \mathrm{Spec}(A^{red})$.
- X_{DR} is a formal derived stack for every $X \in \mathrm{dSt}$.
- If Y is a formal derived stack and $f : X \to Y$ is any map in dSt, then the formal completion $\widehat{Y}_f$ is again a formal derived stack.

Let X be a derived stack, and let us now consider the canonical map $X \to X_{DR}$. This map will be the main object of study by formal localization. The first observation is that $X \to X_{DR}$ can be viewed as the family of formal completions of X at its points: this is essentially the content of the following proposition.

Proposition 0.4.4. *Let X be a derived stack, and let $SpecA \in \mathrm{dAff}$. Suppose we have an A-point of X_{DR}, given by a map $SpecA \to X_{DR}$. By definition of the de Rham stack, this corresponds to an A^{red}-point of X. Consider the induced morphism*

$$f : SpecA^{red} \longrightarrow SpecA \times X.$$

Then the fiber product $X \times_{X_{DR}} SpecA$ is equivalent to the formal completion $(\widehat{SpecA \times X})_f$.

In particular, taking $A = k$ in the above proposition tells us that the fiber of $X \to X_{DR}$ over a k-point x is exactly the formal completion $\widehat{X}_x$.

Notice that the formal derived stack X_{DR} has always a cotangent complex, for any derived stack X. In fact, if $A \in \mathrm{cdga}^{\leq 0}$ and M is an A-module, then

$$X_{DR}(A \oplus M) \simeq X((A \oplus M)^{red}) \simeq (A^{red}) \simeq X_{DR}(A).$$

This means that $\mathbb{L}_{X_{DR}} \simeq 0$, so that using the transitivity sequence associated to $X \to X_{DR}$ we get

$$\mathbb{L}_X \simeq \mathbb{L}_{X/X_{DR}}.$$

0.5 Formal localization

As already mentioned, formal localization is a very general tool in derived algebraic geometry. In this text we will limit ourselves to the specific application to derived Poisson geometry. Again, the general theory can be found in section 2 of [CPTVV]. In this section, X will be a derived Artin stack, locally of finite presentation.

Consider the map $q : X \to X_{DR}$. Let $\mathrm{Spec}A \to X_{DR}$ be an A-point of X_{DR}, and let X_A be the fiber product

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}A \arrow[r] & X_{DR} \end{tikzcd}$$

We already saw that X_A is equivalent to the formal completion of the map $\mathrm{Spec}A^{red} \to \mathrm{Spec}A \times X$. This is easily seen to imply that $(X_A)_{red} \simeq \mathrm{Spec}A^{red}$. In particular, the map $\mathrm{Spec}A \to X_{DR}$

corresponds to a map $\mathrm{Spec}A^{red} \to X$, which is induced by the canonical map $\mathrm{Spec}A^{red} \simeq (X_A)_{red} \to X_A$, so that we get a diagram

$$\begin{tikzcd} & X_A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}A^{red} \arrow[ur] \arrow[r] & \mathrm{Spec}A \arrow[r] & X_{DR} \end{tikzcd}$$

Notice also that by hypothesis, X has a perfect cotangent complex, and moreover we have $\mathbb{L}_X \simeq \mathbb{L}_{X/X_{DR}}$. By base change, we also get that $\mathbb{L}_{X_A/A}$ is perfect.

We now pass to the main construction of the section. Consider the ∞-functor

$$\begin{array}{rccc} \mathbb{D}: & \mathrm{cdga}^{\leq 0} & \longrightarrow & \epsilon-\mathrm{cdga}^{\mathrm{gr}} \\ & A & \longmapsto & \mathrm{DR}(A^{red}/A) \end{array}$$

where $\mathrm{DR}(A^{red}/A)$ is the relative de Rham algebra of A^{red} over A. As a graded cdga, $\mathbb{D}(A) = \mathrm{DR}(A^{red}/A)$ is equivalent to $\mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/A}[-1])$. This functor satisfies descent for the étale topology, and thus we get an induced ∞-functor

$$\begin{array}{rccc} \mathbb{D}: & \mathrm{dSt} & \longrightarrow & \epsilon-\mathrm{cdga}^{\mathrm{gr}} \\ & X & \longmapsto & \lim_{\mathrm{Spec}A\to X}\mathbb{D}(A). \end{array}$$

The functor $\mathbb{D}$ can be used to define two important prestacks of graded mixed cdga on dAff/X_{DR}.

Definition 0.5.1. 1. *With notations as above, the* crystalline structure sheaf *of X is defined to be the prestack*

$$\begin{array}{rccc} \mathbb{D}_{X_{DR}}: & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \epsilon-\mathrm{cdga}^{\mathrm{gr}} \\ & (Spec A \to X_{DR}) & \longmapsto & \mathbb{D}(A). \end{array}$$

2. *The* prestack of principal parts *of X is defined as*

$$\begin{array}{rccc} \mathcal{P}_X: & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \epsilon-\mathrm{cdga}^{\mathrm{gr}} \\ & (Spec A \to X_{DR}) & \longmapsto & \mathbb{D}(X_A). \end{array}$$

where as before $X_A = Spec A \times_{X_{DR}} X$.

Note that as a (non-mixed) graded cdga, $\mathbb{D}(X_A)$ is equivalent to $\mathrm{Sym}_{A^{red}}(\mathbb{L}_{\mathrm{Spec}A^{red}/X_A}[-1])$. This means that $\mathbb{D}(A)$ and $\mathbb{D}(X_A)$ can be interpreted as Chevalley-Eilenberg complexes of $\mathbb{T}_{A/A^{red}}$ and $\mathbb{T}_{\mathrm{Spec}A^{red}/X_A}$, seen as Lie algebroids over $\mathrm{Spec}A^{red}$.

Consider the ∞-category $\mathcal{M}$ of prestacks of graded mixed modules on dAff/X_{DR}. The objects $\mathbb{D}_{X_{DR}}$ and $\mathcal{P}_X$ are then algebras in $\mathcal{M}$. Remark that the morphism $X_A \to \mathrm{Spec}A$ induces a canonical map of prestacks $\mathbb{D}_{X_{DR}} \to \mathcal{P}_X$, which can be interpreted as a $\mathbb{D}_{X_{DR}}$-linear structure on the algebra $\mathcal{P}_X$. It follows that $\mathcal{P}_X$ is an algebra in the category of $\mathbb{D}_{X_{DR}}$-modules. As such we can consider its polyvectors and de Rham algebras. In particular, we can define

$$\begin{array}{rccc} \mathrm{Pol}^t(\mathcal{P}_X, n): & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \mathbb{P}_{n+1}-\mathrm{alg}^{gr} \\ & (\mathrm{Spec}A \to X_{DR}) & \longmapsto & |\mathrm{Pol}(\mathcal{P}_X(A)/\mathbb{D}(A), n)|^t \end{array}$$

and

$$\begin{array}{rccc} \mathrm{DR}^t(\mathcal{P}_X, n): & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \epsilon-\mathrm{cdga}^{\mathrm{gr}} \\ & (\mathrm{Spec}A \to X_{DR}) & \longmapsto & |\mathrm{DR}(\mathcal{P}_X(A)/\mathbb{D}(A))|^t \end{array}$$

which are now stacks on X_{DR}.

It turns out that $\mathcal{P}_X$, considered as a $\mathbb{D}_{X_{DR}}$-algebra, knows a lot about the geometry of X. Consider a morphism of derived Artin stack $X \to Y$ locally of finite presentation. We can define

$$\mathrm{DR}(X/Y) := \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_{X/Y}[-1]),$$

as a graded commutative dg algebra over k. Similarly, we define

$$\mathrm{Pol}(X/Y, n) := \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{T}_{X/Y}[-n])$$

which is again a graded commutative dg algebra over k.

Theorem 0.5.2. *Let again X be a derived Artin stack.*

1. *There is a natural equivalence of graded mixed cdgas*

$$\mathrm{DR}(X/X_{DR}) \simeq \mathrm{DR}(X) \simeq \Gamma(X_{DR}, \mathrm{DR}^t(\mathcal{P}_X)).$$

2. *For every integer n, there is a natural equivalence of graded complexes*

$$\mathrm{Pol}(X/X_{DR}, n) \simeq \mathrm{Pol}(X, n) \simeq \Gamma(X_{DR}, \mathrm{Pol}^t(\mathcal{P}_X, n)).$$

3. *There is an equivalence of ∞-categories*

$$\mathrm{Perf}(X) \simeq \mathcal{P}_X - \mathrm{mod}^{perf}$$

where $\mathcal{P}_X - \mathrm{mod}^{perf}$ is the full sub-∞-category of $\mathcal{P}_X$-modules inside $\mathbb{D}_{X_{DR}}$-modules, formed by prestacks F of $\mathcal{P}_X$-modules on dAff/X_{DR} such that

- *For every $SpecA \to X_{DR}$, the $\mathcal{P}_X(A)$-module $F(A)$ is of the form $\mathcal{P}_X(A) \otimes_{A^{red}} F_0$, for some $F_0 \in \mathrm{Perf}(A^{red})$.*
- *For every map $SpecA \to SpecB$ inside dAff/X_{DR}, we have $F(A) \simeq F(B) \otimes_{\mathcal{P}_X(B)} \mathcal{P}_X(A)$.*

This theorem allows us to translate most of the questions about the geometry of X into algebraic questions about the prestack $\mathcal{P}_X$, together with its $\mathbb{D}_{X_{DR}}$-linear structure. For our purpose, notice in particular that it suggests that there could be a definition of Poisson structure given in purely algebraic terms, since the algebra $\mathcal{P}_X$ recovers much of the geometry of the derived Artin stack X. Of course, one needs to understand which is the correct definition of Poisson structure in the algebraic setting: the next chapter addresses precisely this issue.

Chapter 1

Shifted Poisson structures on derived affine stacks

As we have seen in the previous chapter, formal localization allows to reduce general question about the global geometry of X to algebraic structures on its prestack of principal parts $\mathcal{P}_X$. In order to define shifted Poisson structures in general, one thus needs to understand the affine case. The goal of this chapter is to study in detail what happens on affines. There are at least two sensible definitions of Poisson structures on an algebra, and the main result of this chapter proves that they actually coincide.

We work in commutative algebras in cochain complexes, but the arguments extend immediately to a general model category M, satisfying our starting assumptions. We will come back to this point in the next chapter.

Let A be a commutative dg algebra concentrated in degrees $(-\infty, m]$, and let $\operatorname{Spec} A$ be the associated derived stack. We give two proofs of the existence of a canonical map from the moduli space of shifted Poisson structures (as they were initially introduced in the paper [PTVV]) on $\operatorname{Spec} A$ to the moduli space of homotopy (shifted) Poisson algebra structures on A. The first makes use of a more general description of the Poisson operad and of its cofibrant models, while the second is more computational and involves an explicit resolution of the Poisson operad.

Let us first of all recall the setting we are working in. In classical Poisson geometry, one defines a Poisson structure on a smooth manifold to be a Poisson bracket on the algebra of global functions, which is just a Lie bracket compatible with the product of functions. This notion (which is of algebraic nature) has a more geometric version. The geometric analog of skew-symmetric biderivations are bivector fields, and quite expectedly one can define a Poisson structure to be a bivector field satisfying some additional property. The equivalence of the two definitions of Poisson structure is a well-known fact in classical algebraic or differential geometry.

Recently, in their paper [PTVV] Pantev, Toën, Vaquié and Vezzosi introduced the notion of symplectic and Poisson structures in the context of derived algebraic geometry. Informally speaking, derived algebraic geometry is the study of spaces whose local models are *derived commutative algebras*, that is to say simplicial commutative algebras. If we suppose to be working over a base field k of characteristic zero, the local models can also be taken to be non-positively graded commutative dg-algebras. See [To1] for a recent survey, or [HAG-I], [HAG-II], [Lur2] for a complete treatment of the subject.

In attempting to extend Poisson structures to derived algebraic geometry, there are thus two natural approaches: either via bivector fields or via Poisson brackets on the algebra of functions. We will show that these two approaches indeed agree for a huge and geometrically meaningful class of derived stacks. To do so, we will need to use techniques very different from the arguments in the non-derived setting. To be a bit more specific, [PTVV] proposed to use the bivector approach to define a n-Poisson structure on a (nice enough) derived algebraic stack. We refer to Section 1 for the precise definitions of the objects appearing below.

Definition ([PTVV], [To1])**.** *Let X be a derived Artin stack locally of finite presentation over k, and let $n \in \mathbb{Z}$. The space of* n-shifted Poisson structures on X *is the simplicial set*

$$\mathrm{Pois}(X, n) := \mathrm{Map}_{\mathsf{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(X, n)[n+1])$$

where $k[-1](2)$ is concentrated in degree 1, pure of weight 2, and has the trivial bracket. The graded complex $\mathrm{Pol}(X, n)$ *is the complex of* n-shifted polyvector fields.

The purpose of this chapter is to show that, at least for a nice enough affine derived stack $\mathrm{Spec}\, A$ (where A is a derived commutative algebra), the equivalence between Poisson bivectors and Poisson brackets remains true. Our result is further evidence that for nice derived stacks the definition in [PTVV] is the correct derived generalization of Poisson geometry. As we are working in an inherently homotopical context, Poisson brackets have to be given up to homotopy: these are basically $P_{n,\infty}$-structures on A whose (weakly) commutative product is (equivalent to) the one given on A.

With this goal in mind, after having fixed our notational conventions in Section 1, we study in Section 2 the relation between the categories of dg-operads and of graded dg-Lie algebras. In particular, we would like to be able to describe the moduli space of Poisson brackets on a given commutative algebra via a mapping space in the category of graded dg-Lie algebras. This is accomplished in greater generality in Theorem 1.2.11.

In Section 3, we apply the results of the previous section to derived algebraic geometry, and we eventually obtain the following result

Theorem. *Let A be a commutative dg algebra concentrated in degrees $(-\infty, m]$, with $m \geq 0$, and let* End_A *be the (linear) endomorphism operad of the dg-module A. Let $X =$* $\mathrm{Spec}\, A$ *be the associated derived stack, and let $P^h_{n+1}(A)$ be the homotopy fiber of the morphism of simplicial sets*

$$\mathrm{Map}_{\mathrm{dgOp}}(P_{n+1}, \mathrm{End}_A) \longrightarrow \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A)$$

taken at the point μ_A corresponding to the given (strict) multiplication in A.

Then there is a natural map in the homotopy category of simplicial sets

$$\mathrm{Pois}(X, n) \longrightarrow P^h_{n+1}(A)\ .$$

Moreover, this is an isomorphism if $\mathbb{L}_X$ is a perfect complex.

This is exactly the result we were looking for, since the simplicial set $P^h_{n+1}(A)$ is the natural moduli space of weak Poisson brackets on A.

Finally in Section 4 we give an alternative proof of this theorem, which is more computational and uses both an explicit resolution of the strict Poisson operad and the classical concrete definition of L_∞-algebra.

The results in this chapter are started as a part of a bigger project aimed at defining and studying higher quantizations of moduli spaces equipped with shifted Poisson structures. This has been recently achieved in the paper [CPTVV], where the authors give a more general definition of shifted Poisson structure using the results of this chapter. For further details on the general project and its goals, we refer to the introductions of [PTVV] and of [CPTVV], or to the surveys [To1], [To2] and [PV].

The problem studied in this chapter was suggested by my advisors, and also raised independently by N. Rozenblyum and J. Lurie.

1.1 Notations

Let us fix the notations used in this chapter.

- k is the base field, which is of characteristic 0.
- $\mathsf{cdga}^{\leq 0}$ denotes the category of (strictly) commutative differential graded algebras, concentrated in non-positive degrees. We adopt the cohomological point of view, and the differential increases the degree by 1. The category $\mathsf{cdga}^{\leq 0}$ has the usual model structure for which weak equivalences are quasi-isomorphisms, and fibrations are surjections in negative degrees.
- $\mathsf{C(k)}$ denotes the category of unbounded cochain complexes over k. Its objects will be called also dg-modules. It has the usual model structure for which weak equivalences are the quasi-isomorphisms and fibrations are surjections. It is also a symmetric monoidal model category for the standard tensor product $\otimes_k$.
- We will use the term *symmetric sequence* to indicate a collection of dg-modules $\{V(m)\}_{m\in\mathbb{N}}$ such that every $V(m)$ has an action of the symmetric group S_m on it. Explicitly, $V(m)$ is a differential graded S_m-module, meaning that for every $p \in Z$ the degree p component $V(m)^p$ is an S_m-module, and that the differential is a map of S_m-modules. Equivalently, one can say that $V(m)$ is a differential graded $k[S_m]$-module, where $k[S_m]$ is the group algebra of S_m. In the literature objects of this kind are sometimes called $\mathbb{S}$-modules, Σ_*-objects or also just collections in $C(k)$ (see for example [BM] or Chapter 5 in [LV]). If V is a symmetric sequence and $f \in V(m)$, we will denote by $f^{\mathbf{s}}$ the image of f under the action of a permutation $\mathbf{s} \in S_m$. We will say that f is *symmetric* if $f^{\mathbf{s}} = f$ for every $\mathbf{s} \in S_m$. Similarly, we will say that f is *anti-symmetric* if $f^{\mathbf{s}} = (-1)^{\mathbf{s}} f$ for every $\mathbf{s}$, where $(-1)^{\mathbf{s}}$ denotes the sign of $\mathbf{s}$. We will use the notation $V^{\mathbb{S}}$ for the symmetric sequence of invariants (i.e. of symmetric elements): explicitly, $V^{\mathbb{S}}(m) = V(m)^{S_m}$. We will allow ourselves to switch quite freely from the point of view of symmetric sequences to the one of graded dg-modules with an action of S_m on the weight m component.

  Any symmetric sequence V can be naturally seen as a functor from $C(k)$ to itself, sending a dg module M to $\bigoplus(V(n) \otimes_{S_n} M^{\otimes n})$, where the S_n-action on $M^{\otimes n}$ is the natural one. Given two symmetric sequences V and W, one can thus consider them as functors and take their composition; it can be shown that this composition comes from a symmetric sequence, denoted $V \circ W$.

- dgOp is the category of monochromatic (i.e. uncolored) operads in the symmetric monoidal category $C(k)$ (dg-operads for short). It carries a model structure with componentwise quasi-isomorphisms as weak equivalences and componentwise surjections as fibrations (see [Hi]). In particular, every dg-operad is fibrant. If $\mathcal{P}$ is a dg-operad, we denote by $\mathcal{P}_\infty$ a cofibrant replacement; then $\mathcal{P}_\infty$-algebras are up-to-homotopy $\mathcal{P}$-algebras. The operads of commutative algebras, of Lie algebras and of Poisson n-algebras will be denoted with Comm, Lie and P_n respectively. Our convention is that a Poisson n-algebra has a Lie bracket of degree $1-n$; with this definition, the cohomology of a E_n-algebra is a P_n-algebra. Notice however that there are other conventions in the literature: for example in [CFL] the authors define a Poisson n-algebra to have Lie bracket of degree $-n$.

- $\mathsf{dgLie}^{\mathsf{gr}}$ is the category whose objects are graded dg-Lie algebras, that is to say graded dg-modules L together with an antisymmetric binary operation $[\cdot,\cdot] : L \otimes L \to L$ satisfying the (graded) Jacobi identity. The additional (i.e. the non-cohomological one) grading will be called *weight*. The bracket must be of cohomological degree 0 and of weight -1. Notice thus that these are not algebras for the trivial graded version of the Lie operad, since we are asking for the bracket to have weight -1.

  The fact that the bracket has weight -1 is purely conventional: one can of course obtain the same results using brackets of weight 0. The seemingly strange choice is motivated by the observation that for an affine derived stack $\operatorname{Spec} A$, the natural bracket on the (shifted) polyvectors fields $\mathrm{Sym}_A(\mathrm{T}_A[-n])$ has weight -1. This is the same convention used for example in [PTVV].

- Given a dg-module V, one defines its suspension $V[1]$ to be the cochain complex $V \otimes k[1]$, where $k[1]$ is the complex who is k in degree -1 and 0 elsewhere. If we do the same on operads, we should be a bit more careful. In fact, given an operad $\mathcal{O}$, the symmetric sequence $\mathcal{O}'(m) = \mathcal{O}(m)[1]$ does not inherit an operad structure. Instead, one defines the suspension of $\mathcal{O}$ to be the symmetric sequence whose terms are $s\mathcal{O}(m) = \mathcal{O}(m)[1-m]$, together with the natural operadic structure on it. A little more abstractly, $s\mathcal{O}$ is just $\mathcal{O} \otimes_H \mathrm{End}_{k[1]}$, where $\otimes_H$ denotes the Hadamard tensor product of operads (see [LV], Section 5.3.3). Note that the arity p component of $\mathrm{End}_{k[1]}$ is $k[1-p]$; as a S_p-module, it is just the sign representation. This operadic suspension is an auto equivalence of the category dgOp, its inverse being a desuspension functor denoted $\mathcal{O} \mapsto s^{-1}\mathcal{O}$, and which sends $\mathcal{O}$ to $\mathcal{O} \otimes_H \mathrm{End}_{k[-1]}$.

1.2 Operads and graded Lie algebras

1.2.1 The operad Lie and some generalizations

In this section we study the dg-operad Lie and its cofibrant resolutions. Namely, we describe what it means to have a map from any of these dg-operads to another dg-operad $\mathcal{O}$.

We start by recalling how we can obtain a graded dg-Lie algebra $\mathcal{L}(\mathcal{O})$ in a natural way starting with a dg-operad $\mathcal{O}$ (see [KM], section 1.7). These are classical results in operad theory, and they play a very important role in the remainder of the paper.

Proposition 1.2.1. *Let $\mathcal{O}$ be a dg-operad. Then the graded dg-module $\mathcal{L}(\mathcal{O}) = \bigoplus_n \mathcal{O}(n)$ has a natural structure of a graded dg-Lie algebra, where the Lie bracket is induced by the following* *pre-Lie product*

$$f \star g = \sum_{i=1}^{p} \sum_{\mathbf{s} \in S^i_{p,q}} (f \circ_i g)^{\mathbf{s}}$$

where f and g are of weight (i.e. arity) p and q respectively, and where $S^i_{p,q}$ is the set of permutations of $p+q-1$ elements such that

$$\mathbf{s}^{-1}(1) < \mathbf{s}^{-1}(2) < \cdots < \mathbf{s}^{-1}(i) < \mathbf{s}^{-1}(i+q) < \cdots < \mathbf{s}^{-1}(p+q-1)$$

and

$$\mathbf{s}^{-1}(i) < \mathbf{s}^{-1}(i+1) < \cdots < \mathbf{s}^{-1}(i+q-1)\ .$$

Recall that one obtains a Lie bracket starting from a pre-Lie structure in a natural way: in our case, $[f,g] = f \star g - (-1)^{|f||g|} g \star f$. One has of course to check that the $\star$ operation defines a pre-Lie product (and therefore a Lie bracket): this is done by direct computation, showing that the so called *associator* $f \star (g \star h) - (f \star g) \star h$ is (graded) symmetric on g and h (see [LV], Section 5.4.6).

The Lie bracket defined above has a first nice property: the following lemma is a straightforward consequence of the definition of the pre-Lie product.

Lemma 1.2.2. *Let $\mathcal{O}$ be a dg-operad, and let $f, g \in \mathcal{L}(\mathcal{O})$ be two symmetric elements. Then their bracket in $\mathcal{L}(\mathcal{O})$ remains symmetric.*

In particular, $\mathcal{L}(\mathcal{O})$ has a sub-Lie algebra of symmetric elements $\mathcal{L}(\mathcal{O})^{\mathbb{S}}$.

Our first goal is to use the construction of $\mathcal{L}(\mathcal{O})$ to find an alternative description to the set $\mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Lie}, \mathcal{O})$.

As an operad, Lie admits a very nice presentation : it is generated by a binary operation of degree 0 which is antisymmetric and satisfies the Jacobi identity. More specifically, if $l \in \mathrm{Lie}(2)_0$ is the generator, it has to satisfy $l \circ_1 l + (l \circ_1 l)^{(123)} + (l \circ_1 l)^{(132)} = 0$.

Thus we can safely say that

$$\mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Lie}, \mathcal{O}) = \{x \in \mathcal{O}(2)_0 | x^{(12)} = -x \text{ and } x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_1 x)^{(132)} = 0\}.$$

In Section 1, we defined the operadic suspension, which is an auto-equivalence of the category of dg-operads. The operad $s\mathrm{Lie}$ has one generator in arity 2 of degree 1, which is now symmetric; the Jacobi relation still holds in the same form, since it only involves even permutations. Note that algebras for this operad are just dg-Lie algebras whose bracket is of degree 1, or equivalently dg-modules V with a dg-Lie algebra structure on $V[-1]$. The operadic suspension being an equivalence, we have in particular $\mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Lie}, \mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgOp}}(s\mathrm{Lie}, s\mathcal{O})$. Maps from the operad $s\mathrm{Lie}$ have a nice description in terms of maps of graded dg-Lie algebras.

Proposition 1.2.3. *Let $\mathcal{O}$ be a dg-operad. Then we have*

$$\mathrm{Hom}_{\mathsf{dgOp}}(s\mathrm{Lie}, \mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathcal{L}(\mathcal{O})^{\mathbb{S}})$$

where $k[-1](2)$ is the graded dg-Lie algebra which has just k in degree 1 and weight 2, with zero bracket, while $\mathcal{L}$ is the functor $\mathsf{dgOp} \to \mathsf{dgLie}^{\mathsf{gr}}$ defined at the beginning of this section.

Proof. It follows from the explicit presentation of $s\mathrm{Lie}$ given before that

$$\mathrm{Hom}_{\mathsf{dgOp}}(s\mathrm{Lie}, \mathcal{O}) = \{x \in \mathcal{O}(2)_1 | x^{(12)} = x \text{ and } x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_1 x)^{(132)} = 0\}$$

so that in order to prove the lemma we are led to show that the Jacobi relation is equivalent to the condition $[x, x] = 0$ in $\mathcal{L}(\mathcal{O})$. This is done by direct calculation, since for any symmetric $x \in \mathcal{O}(2)$ we have

$$\begin{aligned} x \star x &= x \circ_1 x + (x \circ_1 x)^{(23)} + (x \circ_2 x) \\ &= x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_1 x)^{(132)} \end{aligned}$$

where we just use the general identities that describe the relationship between partial composition and the action of the symmetric groups. More specifically, take $f \in \mathcal{O}(p)$ and $g \in \mathcal{O}(q)$. Then for every $\mathbf{s} \in S_q$ one has

$$f \circ_i g^{\mathbf{s}} = (f \circ_i g)^{\mathbf{s}'}$$

where $\mathbf{s}' \in S_{p+q-1}$ acts as $\mathbf{s}$ on the block $\{i, i+1, \dots, i+q-1\}$ and as the identity elsewhere. Moreover, for every $\tau \in S_p$, one has

$$f^{\mathbf{t}} \circ_i g = (f \circ_{\mathbf{t}(i)} g)^{\mathbf{t}'}$$

where $\mathbf{t}' \in S_{p+q-1}$ acts as the identity on the block $\{i, i+1, \dots, i+q-1\}$ with values in $\{\mathbf{t}(i), \mathbf{t}(i)+1, \dots, \mathbf{t}(i)+q-1\}$ and as $\mathbf{t}$ elsewhere (sending $\{1, \dots, p+q-1\} \setminus \{i, \dots, i+q-1\}$ to $\{1, \dots, p+q-1\} \setminus \{\mathbf{t}(i), \dots, \mathbf{t}(i)+q-1\}$).

The lemma now follows from the observation that for an element $x \in \mathcal{L}(\mathcal{P})$ of degree 1, one has $[x, x] = 2(x \star x)$. ☐

One immediately has the following consequence.

Corollary 1.2.4. *For any dg-operad $\mathcal{O}$, we have*

$$\mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Lie}, \mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgOp}}(s\mathrm{Lie}, s\mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgLie^{gr}}}(k[-1](2), \mathcal{L}(s\mathcal{O})^{\mathbb{S}}) \ .$$

Next we try to find a result analogous to the last proposition for a cofibrant resolution $\widetilde{s\mathrm{Lie}}_Q$ of the dg-operad $s\mathrm{Lie}$. Our strategy is as follows: given a nice replacement $Q(k[-1](2))$ of $k[-1](2)$ as a graded dg Lie algebra, we find a "lift" of this replacement to a cofibrant approximation $\widetilde{s\mathrm{Lie}}_Q$ of the dg operad $s\mathrm{Lie}$. This construction will actually produce a functor from semi-free graded Lie algebras to semi-free operads, but we will not need this functoriality.

Suppose we have a semi-free resolution $Q(k[-1](2))$ of $k[-1](2)$ as a graded dg-Lie algebra. This means that if we forget the differential $Q(k[-1](2))$ is a free graded Lie algebra, say with generators $\{p_i\}_{i \in I}$, homogeneous of degree d_i and of weight w_i. Then there are of course relations $\{r_j\}_{j \in J}$ that can specify the value of $d(p_i)$, where d is the differential. We can now use this resolution to build a dg-operad $\widetilde{s\mathrm{Lie}}_Q$.

Concretely, for every $i \in I$, take a symmetric generator $\widetilde{p_i}$ of arity w_i and of degree d_i. As for relations, we take the same relations r_j defining $Q(k[-1](2))$; this means that whenever such a relation r_j contains a bracket $[p_{i_1}, p_{i_2}]$, we reinterpret it as the bracket (introduced at the beginning of this section) of elements of an operad $[\widetilde{p}_{i_1}, \widetilde{p}_{i_2}]$, thus getting a relation $\widetilde{r_j}$ for the generators $\widetilde{p_i}$. Let us denote $\widetilde{s\mathrm{Lie}}_Q$ the semi-free operad having all the $\widetilde{p_i}$ as generators and all the $\widetilde{r_j}$ as relations. The definition of the operad $\widetilde{s\mathrm{Lie}}_Q$ allows us to describe quite naturally the set of morphism $\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{s\mathrm{Lie}}_Q, \mathcal{O})$ for an arbitrary operad $\mathcal{O}$. One can in fact prove the following result.

Proposition 1.2.5. *Let $Q(k[-1](2))$ be a semi-free resolution of the graded dg-Lie algebra $k[-1](2)$, and let $\widetilde{s\mathrm{Lie}}_Q$ be the operad defined above, which has the same generators and relations of $Q(k[-1](2))$, and such that all generators are symmetric. Then for every dg-operad $\mathcal{O}$ we have*

$$\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{s\mathrm{Lie}}_Q, \mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(\mathcal{O})^{\mathbb{S}}) \ .$$

Proof. This follows from the definition of $\widetilde{s\mathrm{Lie}}_Q$ in terms of generators and relations. Just like what we said before for sLie, morphisms form $\widetilde{s\mathrm{Lie}}_Q$ are completely determined by the images of the generators $\widetilde{p_i}$, provided that they satisfy the relations defining $\widetilde{s\mathrm{Lie}}_Q$. Every relation can be expressed inside $\mathcal{L}(\widetilde{s\mathrm{Lie}}_Q)$, since they only specify the differentials of the $\widetilde{p_i}$ in terms of their brackets. And by definition these relations of course coincide with those of $Q(k[-1](2))$, giving the desired result. □

In particular, we observe that $\widetilde{s\mathrm{Lie}}_Q$ is a cofibrant approximation of sLie: the weak equivalence $\widetilde{s\mathrm{Lie}}_Q \to s\mathrm{Lie}$ is induced by the weak equivalence $Q(k[-1](2)) \to k[-1](2)$. The fact that it is cofibrant follows from the definition of cofibrations in the model category of dg-operads, given in [Hi].

The operad $\widetilde{s\mathrm{Lie}}_Q$ is therefore weakly equivalent to any cofibrant replacement of sLie. This is just a consequence of the existence of the dotted arrow in the following commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & (s\mathrm{Lie})_\infty \arrow[d] \\ \widetilde{s\mathrm{Lie}}_Q \arrow[r] \arrow[ur, dotted] & s\mathrm{Lie} \end{tikzcd}$$

Since the operadic suspension preserves weak equivalences and fibrations, the map $s(\mathrm{Lie}_\infty) \to s\mathrm{Lie}$ is a trivial fibration, where Lie_∞ is the standard minimal model of the operad Lie, studied for example by Markl in [Mar]. In particular, it follows that $\widetilde{s\mathrm{Lie}}_Q$ is weakly equivalent to $s(\mathrm{Lie}_\infty)$. Once again this is just a consequence of the existence of a model category structure on dgOp, which assures that the dotted arrow in the following diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & s(\mathrm{Lie}_\infty) \arrow[d] \\ \widetilde{s\mathrm{Lie}}_Q \arrow[r] \arrow[ur, dotted] & s\mathrm{Lie} \end{tikzcd}$$

exists, and that it is a weak equivalence.

Note that this does not imply that

$$\mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Lie}_\infty, \mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgOp}}(s\mathrm{Lie}_\infty, s\mathcal{O}) \cong \mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s\mathcal{O})^{\mathbb{S}})$$

since $\widetilde{s\mathrm{Lie}}_Q$ and $s\mathrm{Lie}_\infty$ are not isomorphic in general.

1.2.2 Derivations and multi-derivations

Our goal now is to define shifted Poisson brackets on a commutative algebra, and hence we need to understand derivations of a commutative algebra in an operadic way.

Recall that for a commutative dg-algebra A we have a standard notion of *multi-derivation*. Namely one says that a linear map $\phi : A^{\otimes p} \to A$ is a multi-derivation if for every $i = 1, \dots p$ and for every choice of $a_1, \dots, \hat{a}_i, \dots a_p \in A$ the induced linear map

$$\begin{array}{ccc} A & \longrightarrow & A \\ x & \longmapsto & \phi(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots a_p) \end{array}$$

is a (graded) derivation of A. More generally, for every operadic morphism $\mu : \mathrm{Comm} \to \mathcal{O}$, we can say what it means for any element of $\mathcal{O}$ to be a *derivation* with respect to μ. Notice that the map μ is completely determined by the image in $\mathcal{O}(2)$ of the generator of the operad Comm; in order to simplify the notation, we will also use the letter μ to denote the image of the generator.

Definition 1.2.6. *Let $\mathcal{O}$ be a dg-operad, and let $\mu : \mathrm{Comm} \to \mathcal{O}$ be a morphism of dg-operads. Suppose $f \in \mathcal{O}(p)$ is an element of $\mathcal{O}$ of arity $p \in \mathbb{N}$. We say that f is a p-*derivation with respect to μ *if we have*

$$f \circ_i \mu = (\mu \circ_1 f)^{(p+1\ p\ \dots\ i+2\ i+1)} + (\mu \circ_2 f)^{(1\ 2\ \dots\ i-1\ i)}$$

for every $i = 1, \dots, p$. The symmetric sub-sequence of $\mathcal{O}$ formed by p-derivations will be denoted by $\mathcal{MD}(\mathcal{O}, \mu)$, and its elements will just be called multi-derivations *with respect to μ. If the morphism μ is clear from the context, we will just write $\mathcal{MD}(\mathcal{O})$.*

The definition is coherent with the classical case of derivations of an algebra: if $\mathcal{O}$ is the endomorphism operad of a dg-module V and μ is an actual commutative product on V (so that (V, μ) is just a commutative dg-algebra), then multi-derivations in our sense are exactly multi-derivations in the standard sense.

Let us remark that one could give a definition analogous to Definition 1.2.6 that works for every element $\mu \in \mathcal{O}(2)$, of any degree, and without making any assumption on the symmetry of μ. For example, a derivation with respect to such a μ is just an element $f \in \mathcal{O}(1)$ such that

$$f \circ \mu = (-1)^{|\mu||f|} \mu \circ_1 f + (-1)^{|\mu||f|} \mu \circ_2 f\ .$$

In order to generalize this to multi-derivations, one should keep track of the signs.

Definition 1.2.7. *Let $\mathcal{O}$ be a dg-operad, and let $\mu \in \mathcal{O}(2)$. An element $f \in \mathcal{O}(p)$ is called a* p-derivation with respect to μ *if for every $i = 1, 2, \dots, p$ we have*

$$f \circ_i \mu = (-1)^{|\mu||f|} (\mu \circ_1 f)^{(i+1\ p+1\ p\ \dots\ i+2)} + (-1)^{|\mu||f|} (\mu \circ_2 f)^{(1\ 2\ \dots\ i-1\ i)}$$

The dg-module of derivations of an algebra A is known to be a dg-Lie algebra in a natural way: the (graded) commutator of two derivations is in fact still a derivation. Derivations thus form a sub-Lie algebra of $\mathrm{Hom}_{\mathsf{dgMod}}(A, A)$. More can be said, since actually the graded module of multi-derivations of A is a graded sub-Lie algebra of $\mathcal{L}(\mathrm{End}_A)$. The following lemma tells us that the same remains true in the world of operads.

Proposition 1.2.8. *Let $\mathcal{O}$ be a dg-operad, and let $\mu \in \mathcal{O}(2)$ be a binary operation. The (graded module associated to the) symmetric sequence of multi-derivations with respect to μ of Definition 1.2.7 is closed under the Lie bracket of $\mathcal{L}(\mathcal{O})$.*

Proof. This follows from a straightforward computation: let us give the main ideas without going into all the details. We will suppose that μ is of even degree in order to avoid keeping track of too many signs. The proof for μ of odd degree is exactly the same, with additional signs of course.

Let $f \in \mathcal{O}(p)$ and $g \in \mathcal{O}(q)$ be two multi-derivations with respect to μ. We have

$$[f, g] \circ_i \mu = (f \star g) \circ_i \mu + (-1)^{|f||g|}(g \star f) \circ_i \mu \ ,$$

and we would like to show that this is equal to

$$\begin{aligned}(\mu \circ_1 [f, g])^{(p+q\ \dots\ i+1)} + (\mu \circ_2 [f, g])^{(1\ \dots\ i)} = {} & (\mu \circ_1 (f \star g))^{(p+q\ \dots\ i+1)} + \\ & + (-1)^{|f||g|}(\mu \circ_1 (g \star f))^{(p+q\ \dots\ i+1)} + \\ & + (\mu \circ_2 (f \star g))^{(1\ \dots\ i)} \\ & + (-1)^{|f||g|}(\mu \circ_2 (g \star f))^{(1\ \dots\ i)}\end{aligned}$$

Notice that just as with composition of vector fields, $f \star g$ and $g \star f$ have no hope of being multi-derivations themselves, and one really has to develop the sums in order to prove the result. Using the relations between partial compositions and the action of the symmetric groups, we may write

$$(f \star g) \circ_i \mu = \sum_{j=1}^{p} \sum_{\mathbf{s} \in S^j_{p,q}} (f \circ_j g)^{\mathbf{s}} \circ_i \mu = \sum_{j=1}^{p} \sum_{\mathbf{s} \in S^j_{p,q}} ((f \circ_j g) \circ_{\mathbf{s}(i)} \mu)^{\mathbf{s}'} \ .$$

We now observe that if $\mathbf{s}(i) \notin \{j, j+1, \dots, j+q-1\}$, then we can just use the fact that f is a derivation, and we are done. A similar reasoning applies to $(g \star f) \circ_i \mu$. When $\mathbf{s}(i) \in \{j, j+1, \dots, j+q-1\}$, it gets a bit more complicated. With some care, we can write down what it is left to prove, that is

$$\begin{aligned}&\sum_{j=1}^{p} \sum_{\substack{\mathbf{s} \in S^j_{p,q} \\ j \le \mathbf{s}(i) < j+q}} ((\mu \circ_2 f) \circ_1 g)^{\varphi \cdot (j+q\ j+q-1\ \dots\ \mathbf{s}(i)+1) \cdot \mathbf{s}'} = \\ &\qquad = (-1)^{|f||g|} \sum_{k=1}^{q} \sum_{\substack{\mathbf{t} \in S^k_{q,p} \\ k \le \mathbf{t}(i) < k+p}} ((\mu \circ_1 g) \circ_{q+1} f)^{\psi \cdot (k\ k+1\ \dots\ \mathbf{t}(i)) \cdot \mathbf{t}'}\end{aligned}$$

where $\varphi \in S_{p+q}$ is the permutation that exchanges the blocks $\{1, \dots, j-1\}$ and $\{j, \dots, j+q-1\}$, and $\psi \in S_{p+q}$ is the permutation that exchanges the blocks $\{k+1, \dots, k+p\}$ and $\{k+p+1, \dots, p+q\}$. This last equation is true by direct verification: both sides are equal to the sum of all possible "products" of the form $\mu \circ (f, g)$.

□

1.2.3 The operad $\widetilde{P}_{n,Q}$

Recall (see Section 8.6 of [LV] and references therein) that given two operads $\mathcal{P}$ and $\mathcal{Q}$, if we choose a morphism of symmetric sequences $\Lambda : \mathcal{Q} \circ \mathcal{P} \to \mathcal{P} \circ \mathcal{Q}$ (satisfying a series of axioms), then we can put an operad structure on the composite of the underlying symmetric sequences $\mathcal{P} \circ \mathcal{Q}$. The idea is that in order to define a composition $(\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{P} \circ \mathcal{Q}) \to \mathcal{P} \circ \mathcal{Q}$, we can use the morphism Λ followed by the given compositions $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and $\mathcal{Q} \circ \mathcal{Q} \to \mathcal{Q}$, coming from the operad structures on $\mathcal{P}$ and $\mathcal{Q}$. Informally speaking, Λ specifies how the operations encoded by the operad $\mathcal{P}$ interact with those encoded by $\mathcal{Q}$. Such a Λ is called a *distributive law*, because of the motivating example of the relation between the sum and the multiplication in a ring. When $\mathcal{P}$ and $\mathcal{Q}$ have a nice presentation in terms of generators and relations, we only need a *rewriting rule* for the generators (we refer again to Section 8.6 of [LV] for more details).

We now let $Q(k[-1](2))$ be again a semi-free resolution of the graded dg-Lie algebra $k[-1](2)$: as before, we can associate to it an operad $\widetilde{s\mathrm{Lie}_Q}$, which is quasi-isomorphic to $s\mathrm{Lie}_\infty$. The operad introduced in the following definition will play a central role in the remainder of the paper.

Definition 1.2.9. *Let $Q(k[-1](2))$ be again a semi-free resolution of the graded dg-Lie algebra $k[-1](2)$, and let as before $\widetilde{s\mathrm{Lie}_Q}$ be the operad of Proposition 1.2.5. We define the operad $\widetilde{P}_{n,Q}$ to be the operad obtained by means of a rewriting rule out of $s^{-n}\widetilde{s\mathrm{Lie}_Q}$ and Comm, imposing the condition that every generator of $s^{-n}\widetilde{s\mathrm{Lie}_Q}$ is a multi-derivation with respect to the generator of Comm. Explicitly, if we denote by $s^{-n}\widetilde{p}_i$ the generators of $s^{-n}\widetilde{s\mathrm{Lie}_Q}$ and by μ the generator of Comm, the rewriting rule sends $s^{-n}\widetilde{p}_i \circ_k \mu$ to $(\mu \circ_1 f)^{(k+1\ p+1\ p\ \dots\ i+2)} + (\mu \circ_2 f)^{(1\ 2\ \dots\ k-1\ k)}$.*

It is clear from the definition that $\widetilde{P}_{n,Q}$-algebras are commutative dg-algebras A with a compatible $\widetilde{s\mathrm{Lie}_Q}$-structure on $A[n]$, where the compatibility is given by the condition that the operations defining the $\widetilde{s\mathrm{Lie}_Q}$-structure must be multi-derivations of the commutative dg algebra A. This operad is obviously weakly equivalent to the dg-operad obtained in a similar way out of Comm and $s\mathrm{Lie}_\infty$ (recall that with Lie_∞ we mean the minimal model of the operad Lie). Let us call $\widehat{P}_n$ this latter operad. More specifically, $\widehat{P}_n$-algebras are commutative dg-algebras A together with a Lie_∞-structure on $A[n-1]$. The two structures are compatible, meaning that the multi-brackets defining the (shifted) Lie_∞-structure are multi-derivations on the algebra A.

Remark. The operad $\widehat{P}_n$ (actually a non-shifted version of it) has appeared for instance in [CF], where the authors called its algebras *flat P_∞-algebras*. However, as Cattaneo and Felder correctly remarked in their paper, their notation is a bit misleading, because $\widehat{P}_n$ is not a cofibrant replacement of the operad P_n: in particular the product encoded in $\widehat{P}_n$ is strictly commutative. One could see $\widehat{P}_n$ as an operad standing between the original P_n and its minimal model $P_{n,\infty}$. For this reason, algebras for $\widehat{P}_n$ will be called *semi-strict P_n-algebras*. For an explicit definition of $P_{n,\infty}$-algebras in term of generators and relations in the case $n = 2$ (corresponding to homotopy Gerstenhaber algebras), one can look at [Gi].

By construction, the operad $\widetilde{P}_{n,Q}$ has a natural map from the commutative dg-operad Comm. If $\mathcal{O}$ is any dg-operad, we now describe the fiber of the induced morphism $\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O}) \to \mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$ at a point μ. In particular, if we take $\mathcal{O}$ to be the endomorphism operad of a dg-module V, we are studying the possible ways in which a given commutative structure on V can be extended to a $\widetilde{P}_{n,Q}$-structure. From the very definition of $\widetilde{P}_{n,Q}$, it is clear that what we are missing is a shifted $\widetilde{s\mathrm{Lie}_Q}$-structure made out of multi-derivations. Luckily the preceding results give us exactly a way to compute those structures.

Proposition 1.2.10. *Let $\mathcal{O}$ be a dg-operad, and let $\mu : \mathrm{Comm} \to \mathcal{O}$ be a map of operads. The fiber at μ of the map*

$$\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O}) \to \mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$$

is the set $\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s^n \mathcal{MD}(\mathcal{O}))^{\mathbb{S}})$.

Proof. By definition of the operad $\widetilde{P}_{n,Q}$, the strict fiber we are trying to compute is a subset of $\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{s\mathrm{Lie}_Q}, s^n\mathcal{O})$: in fact, it is composed of morphisms $s^{-n}\widetilde{s\mathrm{Lie}_Q} \to \mathcal{O}$. The condition they must satisfy is that the image of the generators must be multi-derivations with respect to μ. It follows that our fiber is the subset of maps $\widetilde{s\mathrm{Lie}_Q} \to s^n\mathcal{O}$ which send generators to suspensions of multi-derivations. Using Proposition 1.2.5, we thus get that the fiber is exactly $\mathrm{Hom}_{\mathsf{dgLie}^{\mathrm{gr}}}(Q(k[-1](2)), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^{\mathbb{S}})$. Notice that it may seem that we are being a bit inaccurate here, as it is not entirely obvious that the (operadic) suspensions of elements of the sub-Lie algebra $\mathcal{MD}(\mathcal{O})$ are still a sub-Lie algebra of $\mathcal{L}(s^n\mathcal{O})$. This is nonetheless true, and it follows from the observation that elements in $s^n\mathcal{MD}(\mathcal{O})$ are exactly multi-derivations with respect to the n-suspension of the commutative product μ. To see this, take f a multi-derivation of $\mathcal{O}$ of arity p. We want to show that image under the operadic suspension of f is a multi-derivation of $s\mathcal{O}$ with respect to the suspension of μ. This would easily imply our claim, and therefore the theorem.

Recall that the component of arity p of $s\mathcal{O}$ is $\mathcal{O}(p) \otimes k[1-p]$, where $k[1-p]$ is the signature representation of S_p put in degree $p-1$. We denote the generator of $k[1-p]$ by x_{p-1}, so that $|x_{p-1}| = p-1$. By definition of the compositions in $s\mathcal{O}$, we have

$$\begin{aligned}(f \otimes x_{p-1}) \circ_i (\mu \otimes x_1) &= (f \circ_i \mu) \otimes (x_{p-1} \circ_i x_1) \\ &= (\mu \circ_1 f)^{(i+1\ p+1 \ldots i+2)} \otimes (x_{p-1} \circ_i x_1) + \\ &\quad + (\mu \circ_1 f)^{(i\ p+1 \ldots i+1)} \otimes (x_{p-1} \circ_i x_1) \\ &= (-1)^{p-i}((\mu \circ_1 f) \otimes (x_{p-1} \circ_i x_1))^{(i+1\ p+1 \ldots i+2)} + \\ &\quad + (-1)^{p+1-i}((\mu \circ_1 f) \otimes (x_{p-1} \circ_i x_1))^{(i\ p+1 \ldots i+1)}\end{aligned}$$

Now observe that

$$\begin{aligned}(\mu \circ_1 f) \otimes (x_{p-1} \circ_i x_1) &= (-1)^{i-1}(\mu \circ_1 f) \otimes (x_1 \circ_1 x_{p-1}) \\ &= (-1)^{i-1}(-1)^{|f|}(\mu \otimes x_1) \circ_1 (f \otimes x_{p-1})\end{aligned}$$

so that we have

$$\begin{aligned}(f \otimes x_{p-1}) \circ_i (\mu \otimes x_1) = &(-1)^{p-1}(-1)^{|f|}((\mu \otimes x_1) \circ_1 (f \otimes x_{p-1}))^{(i+1\ p+1 \ldots i+2)} - \\ &- (-1)^{p-1}(-1)^{|f|}((\mu \otimes x_1) \circ_1 (f \otimes x_{p-1}))^{(i\ p+1 \ldots i+1)}\end{aligned}$$

which tells us exactly that $f \otimes x_{p-1}$ is a multi-derivation with respect to the binary operation $\mu \otimes x_1$. □

1.2.4 The moduli space of $\widetilde{P}_{n,Q}$-structures

We are now ready to prove our first main result. Given two dg-operads $\mathcal{P}$ and $\mathcal{Q}$, one can form a simplicial space of morphisms from $\mathcal{P}$ to $\mathcal{Q}$, which we will denote by $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathcal{P}, \mathcal{Q})$. Namely, we can construct a simplicial resolution $\mathcal{Q}_\bullet$ of $\mathcal{Q}$ and consider the simplicial set whose n-simplices are $\mathrm{Hom}(\mathcal{P}, \mathcal{Q}_n)$ and whose face and degeneracy maps are the ones induced by the simplicial structure of $\mathcal{Q}_\bullet$. Notice that this is not the derived mapping space between $\mathcal{P}$ and $\mathcal{Q}$ in the model category of dg-operads, since we are not replacing $\mathcal{P}$ with a cofibrant model. If the operad $\mathcal{P}$ is cofibrant, then $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathcal{P}, \mathcal{Q})$ is isomorphic to the mapping space $\mathrm{Map}_{\mathsf{dgOp}}(\mathcal{P}, \mathcal{Q})$ in the homotopy category of simplicial sets.

The simplicial set $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathcal{P}, \mathcal{Q})$ has a nice interpretation if we put $\mathcal{Q} = \mathrm{End}_V$, where V is a dg-module. In this case, $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathcal{P}, \mathrm{End}_V)$ can be thought of as a sort of moduli space of $\mathcal{P}$-algebra structures on V.

We can ask whether Proposition 1.2.10 remains true at the level of simplicial sets. First of all, we remark that the question makes sense: for every operad $\mathcal{O}$, we have a naturally induced map $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O}) \to \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$ (induced by the natural morphism of operads $\mathrm{Comm} \to \widetilde{P}_{n,Q}$) that forgets the additional structure, and we could wonder if we can describe the fiber of a 0-simplex $\mu \in \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$ in terms of some simplicial set of morphisms in the category $\mathsf{dgLie}^{\mathsf{gr}}$. The following theorem answers this question affirmatively.

Theorem 1.2.11. *Let $\mathcal{O}$ be a dg-operad, and let $\mu : \mathrm{Comm} \to \mathcal{O}$ be a map of operads. The (strict) fiber at μ of the morphism of simplicial sets*

$$\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O}) \to \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$$

is the simplicial set $\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s^n \mathcal{MD}(\mathcal{O}))^{\mathbb{S}} \otimes \Omega_*)$, which is a right homotopy function complex from $k[-1](2)$ to $\mathcal{L}(s^n \mathcal{MD}(\mathcal{O}))^{\mathbb{S}}$ in the model category of graded dg-Lie algebras.

Before proving the theorem, we give explicit ways to compute simplicial resolutions and mapping spaces in both model categories dgOp and $\mathsf{dgLie}^{\mathsf{gr}}$.

Let $L \in \mathsf{dgLie}^{\mathsf{gr}}$. We can construct new graded dg-Lie algebras from L by extension of scalars from k to any k-dg-algebra. Let us define Ω_n to be the dg-algebra of algebraic differential forms on $\mathrm{Spec}\left(k[t_0, \dots, t_n] / t_0 + \dots + t_n = 1 \right)$. As an algebra, we have

$$\Omega_n = k[t_0, \dots, t_n, dt_0, \dots, dt_n]/(1 - \sum t_i, \sum dt_i)$$

where the generators t_i have degree 0 and the dt_i have degree 1. The algebras Ω_n define a simplicial object in the category of commutative dg-algebras in a natural way. Then the simplicial graded dg-Lie algebra $L \otimes \Omega_*$ is a simplicial resolution of L. Hence in $\mathsf{dgLie}^{\mathsf{gr}}$, the mapping space between two objects L and M has an explicit representative. Its n-simplices are

$$\mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(L, M)_n = \mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(L), M \otimes_k \Omega_n) ,$$

where Q is a cofibrant replacement of L.

Just as for graded dg-Lie algebras, given an operad $\mathcal{O}$ we can construct new operads by extension of scalars.

Proposition 1.2.12. *For a dg-operad $\mathcal{O}$, the simplicial object $\mathcal{O} \otimes_k \Omega_*$ (defined as above) gives a fibrant simplicial framing of the operad $\mathcal{O}$ (i.e. a fibrant replacement of $\mathcal{O}$ in the Reedy model category of simplicial objects in dgOp).*

This follows directly from [Fr], Part II, Chapter 7 (in particular Theorem 7.3.5).

Proof of Theorem 1.2.11. By construction, the m-simplices of the fiber are the m-simplices of the simplicial set $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O})$ that are sent to μ, viewed as a degenerate m-simplex of $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O})$. Therefore we can use Proposition 1.2.10 in order to compute them: they are the fiber of the function

$$\mathrm{Hom}_{\mathsf{dgOp}}(\widetilde{P}_{n,Q}, \mathcal{O} \otimes \Omega_m) \to \mathrm{Hom}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathcal{O} \otimes \Omega_m)$$

taken at the point μ. Notice that we are being a bit sloppy in order to keep notation as simple as possible, as we are identifying $\mu : \mathrm{Comm} \to \mathcal{O}$ with the composition $\mathrm{Comm} \to \mathcal{O} \to \mathcal{O} \otimes \Omega_n$. So Proposition 1.2.10 tells us that the m-simplices of the fiber are $\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s^n \mathcal{MD}(\mathcal{O} \otimes \Omega_m))^{\mathbb{S}})$.

Observe now that multi-derivations of $\mathcal{O} \otimes \Omega_n$ are just multi-derivations of $\mathcal{O}$ with respect to μ, considered over the dg-algebra Ω_n. Concretely, this means $\mathcal{MD}(\mathcal{O} \otimes \Omega_n) = \mathcal{MD}(\mathcal{O}) \otimes \Omega_n$ as graded dg-Lie algebras. Moreover, the operadic suspension commutes with extension of scalars, as does taking invariants. It follows that the graded dg-Lie of n-simplices is

$$\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^{\mathbb{S}} \otimes \Omega_m) = \mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^{\mathbb{S}})_m$$

where the Map on the right is computed by means of the right homotopy function complex described before. These isomorphisms organize in a natural way to give an isomorphism of simplicial set between the fiber at μ and a right homotopy function complex $\mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^{\mathbb{S}})$, and this proves the theorem. □

1.3 Applications to derived algebraic geometry

Let again $Q(k[-1](2))$ be a semi-free resolution of the dg-Lie algebra $k[-1](2)$. In this section we apply Theorem 1.2.11 to the context of derived Poisson geometry. In particular, we will show in Theorem 1.3.1 that a n-Poisson structure in the sense of [PTVV] on a derived stack of the form $\mathrm{Spec}\, A$ (with A concentrated in degree $(-\infty, m]$, with $m \geq 0$) gives rise to a $\widetilde{P}_{n+1,Q}$-structure on A.

Recall from [PTVV] that for a derived scheme X which is locally of finite presentation, the space $\mathrm{Pois}(X, n)$ of n-Poisson structures on X is the mapping space $\mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathrm{Pol}(X, n)[n+1])$, where $\mathrm{Pol}(X, n)$ is the graded Poisson dg-algebra of n-shifted polyvectors, that is to say

$$\mathrm{Pol}(X, n) = \mathbb{R}\Gamma(X, \mathrm{Sym}_{\mathcal{O}_X} \mathbb{T}_X[-n-1]) \ .$$

If $X = \mathrm{Spec}\, A$ is affine, $\mathrm{Pol}(X, n)$ becomes just $\mathrm{Sym}_A(\mathrm{T}_A[-n-1])$ with the usual Schouten-Nijenhuis bracket.

Recall also from [Qu] that the category of bounded above cochain complexes have a natural model structure, taking as weak equivalences the quasi-isomorphisms and as fibrations the degree-wise surjections. This structure induces in the standard way (via the free-forgetful adjunction) a model structure on bounded above commutative dg algebras.

Theorem 1.3.1. *Let A be a cofibrant object in the model category of commutative dg algebras that are bounded above. Suppose that A, viewed as a derived stack, admits a n-shifted Poisson structure in the sense of [PTVV]. Then A has a structure of an $\widetilde{P}_{n+1,Q}$-algebra, whose commutative product coincide with the given multiplication in A. More precisely, let μ_A be the multiplication in A, and let $\widetilde{P}_{n+1,Q}(A)$ be the fiber of the map of simplicial sets*

$$\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widetilde{P}_{n+1,Q}, \mathrm{End}_A) \longrightarrow \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathrm{End}_A)$$

at the point μ_A.

We have a natural map of simplicial set

$$\mathrm{Pois}(\mathrm{Spec}\, A, n) \longrightarrow \widetilde{P}_{n+1,Q}(A)$$

Moreover, this map is a weak equivalence if the cotangent complex $\mathbb{L}_A$ is perfect.

Proof. The simplicial set $\widetilde{P}_{n+1,Q}(A)$ has an equivalent description given by Theorem 1.2.11, namely we can rewrite it as $\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(Q(k[-1](2)), \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}} \otimes \Omega_*)$, where $\mathcal{MD}(A)$ is the Lie algebra of multi-derivations of the operad End_A, with respect to the natural multiplication $\mu_A : \mathrm{Comm} \to \mathrm{End}_A$ (see Definition 1.2.6); that is to say, the classically defined multi-derivations of the algebra A. By functoriality, in order to prove the theorem it will suffice to build up a map of graded dg-Lie algebras

$$\mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1] \longrightarrow \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}} \ .$$

To construct this morphism, notice that since A is cofibrant, $\mathbb{L}_A$ is just the standard module of Kähler differentials, and multi-derivations of A of arity p are by definition the A-module $\mathrm{Hom}_A(\mathbb{L}_A^{\otimes p}, A)$. Hence the weight p component of the graded dg-Lie algebra $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$ is precisely given by the symmetric elements inside $\mathrm{Hom}_A(\mathbb{L}_A^{\otimes p}, A) \otimes k[1-p]^{\otimes(n+1)}$, where $k[1-p]$ is the signature representation of S_p concentrated in degree $p-1$. As an S_p-module, $k[1-p]^{\otimes n}$ can be either a trivial or a signature representation, depending on the parity of n. Concretely, we have

$$k[1-p]^{\otimes(n+1)} = \begin{cases} \text{the trivial representation of } S_p & \text{if } n \text{ is odd} \\ \text{the signature representation of } S_p & \text{if } n \text{ is even} \end{cases}$$

where the S_p-modules are always concentrated in degree $(n+1)(p-1)$. It follows that as a dg-module, the weight p part of $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$ is isomorphic to $\mathrm{Hom}_A(\mathrm{Sym}^p_A\mathbb{L}_A, A)[(n+1)(1-p)]$ if n is odd, and to $\mathrm{Hom}_A(\Lambda^p_A\mathbb{L}_A, A)[(n+1)(1-p)]$ if n is even.

On the other hand, the weight p component of $\mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1]$ is just $\mathrm{Sym}^p_A(\mathrm{T}_A[-n-1])[n+1]$, and we have a natural map of k-dg-modules (actually of A-dg-modules)

$$\mathrm{Sym}^p_A(\mathrm{T}_A[-n-1])[n+1] \longrightarrow \mathrm{Hom}_A(\mathrm{Sym}^p_A(\mathbb{L}_A[n+1]), A)[n+1]$$

induced by the fact that T_A is by definition the dual of $\mathbb{L}_A$. Notice that this map is not an equivalence in general: it becomes an equivalence however if we suppose that $\mathbb{L}_A$ is perfect. Observe next that we have

$$\mathrm{Sym}^p_A(\mathbb{L}_A[n+1]) = \begin{cases} \mathrm{Sym}^p_A(\mathbb{L}_A)[n(p-1)] & \text{if } n \text{ is odd} \\ \Lambda^p_A(\mathbb{L}_A)[n(p-1)] & \text{if } n \text{ is even} \end{cases}$$

so that for every n, $\mathrm{Hom}_A(\mathrm{Sym}^p_A(\mathbb{L}_A[n+1]), A)[n+1]$ is isomorphic as a dg-module to the weight p component of $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$.

Putting all this together, we do get a map of graded dg-modules

$$\mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1] \longrightarrow \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}} \ .$$

The point is to check that this map is compatible with the two Lie brackets: on the left hand side, we have the Schouten bracket, induced by the natural Lie structure on T_A, while on the right hand side we have the bracket of the Lie algebra associated to the operad $s^{n+1}\mathrm{End}_A = \mathrm{End}_{A[n+1]}$.

This can be done by direct calculation, since both brackets have a known explicit expression. One has just to check that the signs coincide.

More abstractly, we can also observe that there is an adjunction

$$\left\{ \begin{array}{c} A\text{-dg-modules } B \text{ with a compatible} \\ k\text{-linear dg-Lie structure on } B[m] \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{commutative } A\text{-dg-algebras } B \text{ with a} \\ \text{compatible } k\text{-linear dg-Lie structure on } B[m] \end{array} \right\}$$

where on the right hand side, compatible means that if we forget the A-action we are left with a P_{m+1}-algebra. Alternatively, these are just P_{m+1}-algebras in $C(k)$ whose underlying commutative algebra is actually an A-algebra, with no relation between the Poisson bracket and the A-action. Notice that the left hand side is essentially the category of Lie algebroids over A.

The adjunction is thus a "lift" of the usual free-forget adjunction between A-modules and A-algebras to the situation where the underlying k-modules have Lie structures. The right adjoint is the forgetful functor, while the left adjoint sends X to $\mathrm{Sym}_A(X)$. In particular this implies that if we were able to show that $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}[-n-1]$ has a compatible A-algebra structure, then the existence of a Lie algebra map

$$\mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1] \longrightarrow \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}} \ .$$

would follow from the existence of a morphism of Lie algebras (and of A-modules)

$$\mathrm{T}_A \longrightarrow \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}} \ .$$

But it follows from the definitions that the weight one component of $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$ is precisely T_A, and that the restriction of the bracket of $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$ to T_A is the natural one (that is to say the graded commutator).

We are thus left to define an appropriate degree zero product on $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}[-n-1]$. It turns out that it is induced by the natural shuffle product on the multilinear morphisms from $A[n+1]$ to itself, which has the following explicit description. Denote by μ the multiplication of A; for $f \in \mathrm{End}_{A[n+1]}(p)$ and $g \in \mathrm{End}_{A[n+1]}(q)$, we pose

$$f \cdot g = \sum_{\mathbf{s} \in \mathrm{Sh}_{p,q}} (s^{n+1}\mu(f,g))^{\mathbf{s}}$$

where the sum is taken over all permutations $\mathbf{s} \in S_{p+q}$ such that $\mathbf{s}^{-1}(1) < \cdots < \mathbf{s}^{-1}(p)$ and $\mathbf{s}^{-1}(p+1) < \cdots < \mathbf{s}^{-1}(p+q)$. It easy to check that this defines a degree m product, which becomes commutative if regarded on $\mathcal{L}(\mathrm{End}_{A[n+1]})[-n-1]$. Moreover, if f and g are symmetric multi-derivations, then $f \cdot g$ is again a symmetric multi-derivation. Finally, the graded Leibniz identity

$$[f, g \cdot h] = [f,g] \cdot h + (-1)^{|g|(|f|+n+1)} g \cdot [f,h]$$

for $f,g,h \in \mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}[-n-1]$ should be checked to be true. Notice that here the product $g \cdot h$ denotes the operation induced by the shuffle product defined above: this means that there are other signs involved, due to the so-called décalage isomorphism. The verification of the identity is a long but straightforward computation, and we omit the details. To summarize, $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}[-n-1]$ is an A-algebra with a k-linear compatible Lie bracket of degree $-n-1$, and by the discussion above this proves the theorem. □

We can rephrase the results of Theorem 1.3.1 in a different way: we constructed a map of simplicial sets

$$\mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1]) \xrightarrow{\phi} \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widetilde{P}_{n+1,Q}, \mathrm{End}_A)$$

that fits in the following diagram

$$\begin{tikzcd}
\mathrm{Pois}(\mathrm{Spec}\,A,n) \arrow[r] \arrow[rr, bend left, "\phi"] & \widetilde{P}_{n+1,Q}(A) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathrm{dgOp}}(\widetilde{P}_{n+1,Q},\mathrm{End}_A) \arrow[d] \\
& pt \arrow[r, "\mu_A"] & \underline{\mathrm{Hom}}_{\mathrm{dgOp}}(\mathrm{Comm},\mathrm{End}_A)
\end{tikzcd}$$

where the square on the right is a pullback of simplicial sets.

Let us weaken a bit our results in order to express them in a more homotopical language. The following theorem is the main result of this text.

Theorem 1.3.2. *Let A be a commutative dg algebra concentrated in degree $(-\infty,m]$, and let $X=\mathrm{Spec}\,A$ be the derived stack associated to A. Let $P^h_{n+1}(A)$ be the homotopy fiber of the morphism of simplicial sets*

$$\mathrm{Map}_{\mathrm{dgOp}}(P_{n+1},\mathrm{End}_A)\longrightarrow\mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm},\mathrm{End}_A)$$

taken at the point μ_A corresponding to the given (strict) multiplication in A.

Then there is a natural map in the homotopy category of simplicial sets

$$\mathrm{Pois}(X,n)\longrightarrow P^h_{n+1}(A)\ .$$

Moreover, this is an isomorphism if $\mathbb{L}_X$ is a perfect complex.

Proof. Notice that since we are only looking for a morphism in the homotopy category of simplicial sets, we can safely suppose that A is cofibrant: in fact, the homotopy type of both $\mathrm{Pois}(X,n)$ and $P^h_{n+1}(A)$ does not change if we replace A with another algebra quasi-isomorphic to it.

As already mentioned towards the end of Section 2, the mapping space between two operads $\mathcal{P}$ and $\mathcal{Q}$ can be computed by taking a cofibrant replacement of the first one and a simplicial resolution of the second one. Let us denote by C the cofibrant replacement functor in the model category of dg-operads. In particular, one has

$$\mathrm{Map}_{\mathsf{dgOp}}(\mathcal{P},\mathcal{Q})\cong\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(C(\mathcal{P}),\mathcal{Q})\ .$$

Notice that we don't need to replace $\mathcal{Q}$ with a fibrant model, since all operads are fibrant.

In order to compute the homotopy fiber $P^h_{n+1}(A)$, one has thus to take cofibrant models for the operad Comm and P_{n+1}. For example, let us take the minimal model Comm_∞ of Comm, and take $P_{n+1,\infty}$ to be the operad whose algebras are Comm_∞-algebras together with a $\widetilde{s\mathrm{Lie}}_Q$-structure on $A[n]$ made of homotopy derivations, in the sense of [DL], [Do]. This just means that the generators of $\widetilde{s\mathrm{Lie}}_Q$ satisfy the Leibniz identity only up to homotopy.

These are clearly cofibrant models for Comm and P_{n+1}, and there is an obvious forgetful functor $\mathrm{Comm}_\infty\to P_{n+1,\infty}$, that is actually easily seen to be a cofibration in dgOp using the characterization of cofibrations given in [Hi]. This means that the induced morphism

$$\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(P_{n+1,\infty},\mathrm{End}_A)\longrightarrow\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}_\infty,\mathrm{End}_A)$$

is a fibration between fibrant simplicial sets, and therefore its *strict* fiber is weakly equivalent to its homotopy fiber, which in turn is a model for $P^h_{n+1}(A)$, the homotopy fiber of

$$\mathrm{Map}_{\mathsf{dgOp}}(P_{n+1},\mathrm{End}_A)\longrightarrow\mathrm{Map}_{\mathsf{dgOp}}(\mathrm{Comm},\mathrm{End}_A)\ \ .$$

Let us now consider the following diagram of simplicial sets:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\overline{P}_{n+1,Q},\mathrm{End}_A) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(P_{n+1,\infty},\mathrm{End}_A) \arrow[d] \\
\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm},\mathrm{End}_A) \arrow[r] & \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}_\infty,\mathrm{End}_A)
\end{tikzcd}$$

where $\overline{P}_{n+1,Q}$ is the operad whose algebras are strictly commutative algebras together with a $\widetilde{s\mathrm{Lie}_Q}$-structure on $A[n]$ made of homotopy derivations. By definition, this is a pullback diagram of simplicial sets, so that the strict fiber of the map on the right (taken at the point μ) is equivalent to the strict fiber of the map on the left (still taken at μ; this makes sense since μ factors through $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm},\mathrm{End}_A)$).

But now the strict fiber of the map

$$\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\overline{P}_{n+1,Q},\mathrm{End}_A) \longrightarrow \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm},\mathrm{End}_A)$$

is the space of $\widetilde{s\mathrm{Lie}_Q}$-structures on $A[n]$ made of homotopy derivations. Our next goal is now to describe this space, that can actually be quite complicated for a general A.

There is a naturally defined dg module $\mathrm{Der}^h(A)$ of homotopy derivation of A, which can be used to compute the Hochshild cohomology of the algebra A. Namely, instead of resolving A and then computing strict derivations, one can leave A unresolved and compute homotopy derivations (see [Do], section 3). In particular, this shows that for a cofibrant algebra A one has a quasi-isomorphism $\mathrm{Der}(A) \cong \mathrm{Der}^h(A)$, where $\mathrm{Der}(A)$ is the standard complex of strict derivations of A. Let us remark that this result should not come as a surprise, since both $\mathrm{Der}(A)$ and $\mathrm{Der}^h(A)$ are in this case sensible candidates for the tangent complex of the algebra A, and one should expect no ambiguity in the definition of such a geometrically meaningful object.

In particular this tells us that for A cofibrant, the space of $\widetilde{s\mathrm{Lie}_Q}$-structures on $A[n]$ made of homotopy derivations is weakly equivalent to the space of $\widetilde{s\mathrm{Lie}_Q}$-structures on $A[n]$ made of strict derivations; but this last space is by definition $\widetilde{P}_{n+1,Q}(A)$. Now Theorem 1.3.1 gives us a map of simplicial sets from $\mathrm{Pois}(A;n)$ to $\widetilde{P}_{n+1,Q}(A)$, which corresponds to a map in the homotopy category of simplicial sets from $\mathrm{Pois}(A;n)$ to $P^h_{n+1}(A)$.

We conclude by observing that the last statement of the theorem is a direct consequence of the analogous statement in Theorem 1.3.1.

□

1.4 Another proof of the main result

In this last section we will give a more explicit description of our results: we take a particular resolution of the graded dg-Lie algebra $k[-1](2)$ and study the induced resolution of the Lie operad. We check that its algebras are just Lie_∞-algebras in the standard sense, see for example [HS]. These concrete computations also give an alternative proof of Theorem 1.3.2.

The graded dgLie algebra $k[-1](2)$ has a cofibrant resolution L_0 given by the free Lie algebra generated by elements p_i for $i = 2, 3, \ldots$, such that p_i sits in weight i and in cohomological degree 1 ; the differential in L_0 is defined as to satisfy

$$dp_n = -\frac{1}{2}\sum_{i+j=n+1}[p_i,p_j]$$

Notice that in particular that we have $dp_2 = 0$. The map $L_0 \to k[-1](2)$ sends p_2 to the generator of $k[-1](2)$ and the other p_i to zero.

By definition, the space of n-shifted Poisson structures on A is

$$\mathrm{Map}_{\mathsf{dgLie}^{\mathsf{gr}}}(k[-1](2), \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1])$$

and we can use the explicit resolution L_0 to compute its n-simplices: these are just elements in

$$\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1] \otimes \Omega_n) \ .$$

In particular, the points of the space of n-shifted structures on A can be identified with

$$\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1]) \ .$$

If A is cofibrant, then the dg module of derivations of A is a model for T_A. In this case the same argument used in the proof of Theorem 1.3.1 proves that $\mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1]$ maps into the dg Lie algebra of symmetric (shifted) multi-derivations $\mathcal{L}(s^{n+1}\mathcal{MD}(A))^{\mathbb{S}}$, which in turn sits inside $\mathcal{L}(s^{n+1}\mathrm{End}_A)^{\mathbb{S}} \cong \mathcal{L}(\mathrm{End}_{A[n+1]})^{\mathbb{S}}$, the dg Lie of all symmetric multilinear maps of $A[n+1]$.

Putting all together, we get a map from the points of $\mathrm{Pois}(A,n)$ to

$$\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \bigoplus_{i\in\mathbb{N}} \mathrm{Hom}_k(\mathrm{Sym}^i_k(A[n+1]), A[n+1])) \ .$$

So at the level of the vertices, a n-Poisson structure on A gives a sequence of symmetric multilinear maps q_i (the images of the p_i) on $A[n+1]$, such that every q_i is an i-linear map of degree 1.

One of the possible definitions (see for example [Man]) of a L_∞-structures is the following.

Definition 1.4.1. *If V is a graded vector space, an L_∞-structure on V is a sequence of symmetric maps of degree* 1

$$l_n : \mathrm{Sym}^n V[1] \to V[1] \ , \quad n > 0$$

such that for every $n > 0$ we have

$$\sum_{i+j=n+1} [l_i, l_j] = 0 \ ,$$

where the bracket is the Lie bracket we defined before on $\bigoplus_{i\in\mathbb{N}} \mathrm{Hom}_k(\mathrm{Sym}^i_k(V[1]), V[1])$.

So if we want to prove that (still at the level of the vertices) an n-Poisson structure gives us an L_∞-structure on $A[n]$, we could try to find such l_n on $A[n+1]$. Natural candidates are the q_i that come directly from the shifted Poisson structure; these are given for $i > 1$. Notice that our brackets satisfy the graded antisymmetry relation $[x,y] = -(-1)^{|x||y|}[y,x]$; in particular, this relation does not involve the weights of x and y. In our case $|p_i| = |q_i| = 1$, and so it follows $[p_i,p_j] = [q_i,q_j] = [p_j,p_i] = [q_j,q_i]$. Let us take $q_1 = d$, the differential of $A[n+1]$. We should now verify that the symmetric maps q_i satisfy

$$\sum_{i+j=n+1} [q_i, q_j] = 0 \ .$$

The other observation we need to make is that for every multilinear map $f \in \mathrm{Hom}_k(\mathrm{Sym}^i_k(A[n+1]), A[n+1]))$, we have $[q_1, f] = [f, q_1] = d(f)$, where d here is the differential of multilinear maps on $A[n+1]$.

So using these facts we have

$$\sum_{i+j=n+1} [q_i, q_j] = 2d(q_n) + \sum_{\substack{i+j=n+1 \\ i,j>1}} [q_i, q_j] = 0 \ ,$$

which is what we wanted. To summarize, an n-Poisson structure induces an L_∞-structure on $A[n]$.

Now we need to show that the induced L_∞-structure on $A[n]$ is compatible with the algebra structure on A, that is to say that A becomes a semi-strict P_{n+1}-algebra. But the q_i we constructed in the previous step are (by definition) derivations of the given commutative product on A ; this gives A precisely the structure of a semi-strict P_{n+1}-algebra.

The upshot of this discussion is the fact that we got a map

$$\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1]) \longrightarrow \mathrm{Hom}_{\mathsf{dgOp}}(\widehat{P}_{n+1}, \mathrm{End}_A)$$

for which the image is contained in the $\widehat{P}_{n+1}$-structures whose commutative product is the one given on A. Equivalently, we get a function from $\mathrm{Hom}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1])$ to the (non-homotopical) fiber product of the following diagram of sets

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{dgOp}}(\widehat{P}_{n+1}, \mathrm{End}_A) \arrow[d] \\ pt \arrow[r, "\mu_A"] & \mathrm{Hom}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A) \end{tikzcd}$$

where μ_A denotes the given commutative product of A. From here one can proceed in the exact same way as done towards the end of Section 2: namely, we can use Theorem 1.2.11 (and the explicit descriptions of the simplicial framings in dgOp and $\mathsf{dgLie}^{\mathsf{gr}}$) in order to prove that we have a map of simplicial sets from $\underline{\mathrm{Hom}}_{\mathsf{dgLie}^{\mathsf{gr}}}(L_0, \mathrm{Sym}_A(\mathrm{T}_A[-n-1])[n+1])$ to the (strict) fiber of the natural map $\underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\widehat{P}_{n+1}, \mathrm{End}_A) \to \underline{\mathrm{Hom}}_{\mathsf{dgOp}}(\mathrm{Comm}, \mathrm{End}_A)$, taken at μ_A.

Now the same arguments used at the end of Section 3 allow to obtain a map in the homotopy category of simplicial sets

$$\mathrm{Pois}(X, n) \longrightarrow P^h_{n+1}(A)$$

giving a more concrete proof of Theorem 1.3.2.

Chapter 2

Shifted Poisson structures on general derived stacks

In this chapter we put together the results of the two previous sections, giving a definition of shifted Poisson structure on a general derived Artin stack. The material is mostly taken from [Me] and [CPTVV], and we will need it in chapter 4 to extend the results of chapter 3 to the general case.

In the first section, we remark that theorem 1.3.2, the main result of chapter 1, is not perfectly suited to be used in the context arising from formal localization. More specifically, in the proof of Theorem 1.3.2 we needed a finiteness assumption on the algebra, assuring that the cotangent complex will be perfect. This is not a big issue though, since the slightly peculiar definition of polyvectors in chapter 0 does not require one to deal with the tangent complex, and thus there is no need to assume any dualizability condition on the cotangent complex. We end up by stating Theorem 2.1.2, which is the modified version of Thereom 1.3.2 that we will use in the following sections.

In section 2, we attack the problem of defining shifted Poisson structures for general derived stacks. As for the affine case, we give two possible definitions, and we show that Theorem 2.1.2 implies that they are in fact equivalent as expected. Thanks to formal localization, this looks a lot like the affine case, where we already showed this kind of result in chapter 1.

2.1 A slight modification of Theorem 1.3.2

Later in this chapter, we will try to extend the results of the previous chapter to general derived Artin stacks, not necessarily affine. In particular, we will need to prove a statement similar to Theorem 1.3.2, which shows that the two possible definitions of shifted Poisson structures (the one in term of bivectors and the one in term of Poisson brackets) are in fact equivalent.

Recall that in Chapter 1 we worked in the simpler category $C(k)$ of cochain complexes over k. But it is immediate to check that all arguments translates smoothly in the context of any symmetric monoidal ∞-category $\mathcal{M}$ having a well behaved $C(k)$-enrichment. In particular, our starting hypothesis on the model category M in the preliminary chapter are enough to assure that Theorem 1.3.2 stays true in the associated ∞-category.

Using the general formalism of differential calculus in $\mathcal{M}$ presented in chapter 0, we can thus state a general categorical version of Theorem 1.3.2.

Let M be a symmetric monoidal model category, satisfying our starting assumption of chapter

0. Recall that in particular M is $C(k)$-enriched. Let $\mathcal{M}$ be the associated symmetric monoidal ∞-category.

Theorem 2.1.1. *With notations as above, take $A \in \mathrm{CAlg}_{\mathcal{M}}$ to be a commutative algebra in $\mathcal{M}$, and suppose moreover that $\mathbb{L}_A^{int}$ is dualizable as an object of $A - \mathrm{Mod}_{\mathcal{M}}$. Then the homotopy fiber of the map*

$$\mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \longrightarrow \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A),$$

taken at the point given by the commutative structure on A, is naturally equivalent to

$$\mathrm{Map}_{\mathrm{LieAlg}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), \mathrm{Sym}_A(\mathbb{T}_A^{int}[-n-1])[n+1])$$

where $1_{\mathcal{M}}$ is the monoidal unit and $\mathrm{LieAlg}^{gr}_{\mathcal{M}}$ is the ∞-category of graded Lie algebras in $\mathcal{M}$.

Note that this results reduces exactly to theorem 1.3.2 if we start with $M = C(k)$, and as already mentioned the hypothesis on M assures that we do get an interpretation of theorem 1.3.2 also in the general case.

There is still a small issue with Theorem 2.1.1, which prevents us to use it freely for Poisson structures on general derived stacks. Namely, we put the hypothesis that as an A-module, the cotangent complex $\mathbb{L}_A^{int}$ was dualizable. In the affine case treated in chapter 1, this was a natural assumption, since we want to work to geometric objects locally of finite presentation, and in particular this assures that the cotangent complex will be perfect (hence dualizable). Notice that as explained in the proof of theorem 1.3.2, if $\mathbb{L}_A^{int}$ is not dualizable there is no hope to get the desired equivalence, and Theorem 2.1.1 is, as stated, simply false.

The situation for a general derived stack X is different though: the algebra A to which we would like to apply Theorem 2.1.1 is $\mathcal{P}_X(\infty)$, as we saw that formal localization techniques allow to establish a deep link between $\mathcal{P}_X(\infty)$ and the geometry of X itself. Here it is not entirely clear whether $\mathcal{P}_X(\infty)$ has a dualizable cotangent complex, internal to the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules.

It turns out that the conclusions of Theorem 2.1.1 can be slightly modified if one drops the dualizability hypothesis on the cotangent complex.

Theorem 2.1.2. *Let $\mathcal{M}$ be an ∞-category satisfying the hypothesis of chapter 0, and let $A \in \mathrm{CAlg}_{\mathcal{M}}$ be a commutative algebra in $\mathcal{M}$. Then the homotopy fiber of the map*

$$\mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \longrightarrow \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A),$$

taken at the point given by the commutative structure on A, is naturally equivalent to

$$\mathrm{Map}_{\mathrm{LieAlg}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), \mathrm{Pol}^{int}(A, n+1)[n+1]),$$

where $\mathrm{Pol}^{int}(A, n+1)$ is the algebra of internal $(n+1)$-shifted polyvectors on A, as defined in chapter 0.

Proof. The proof is basically the same exact proof of Theorem 1.3.2 (and thus of Theorem 2.1.1). One just needs to notice that during the proof of Theorem 1.3.2 we passed through a canonical identification between the fiber of the morphism

$$\mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \longrightarrow \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A)$$

and the space of Maurer-Cartan elements of weight ≥ 2 in the completed Lie algebra of multiderivations, using the dual of the symmetric algebra on the cotangent complex.

Recall from chapter 0 that $\mathrm{Pol}^{int}(A,n)$ was defined precisely using multiderivations, without ever mentioning the internal tangent complex of A, so that we immediately get the desired result. $\square$

Of course, if the internal cotangent complex $\mathbb{L}_A^{int}$ happens to be dualizable, we already observed that there is a natural equivalence

$$\mathrm{Pol}^{int}(A,n) \simeq \mathrm{Sym}_A(\mathbb{T}_A^{int}[-n]),$$

so that we find back theorem 2.1.1.

2.2 Poisson structures

Our goal in this section is to extend the results of chapter 1 to general stacks. We start by recalling the objects involved in what will follow.

Let X be a derived Artin stack locally of finite presentation. Consider the associated de Rham stack X_{DR}: recall that as a functor on the category $\mathrm{cdga}^{\leq 0}$, X_{DR} sends an algebra A to $X(A_{red})$, where A_{red} is the reduced algebra $H^0(A)_{red}$. The de Rham stack comes equipped with a natural projection $q : X \to X_{DR}$, whose fibers are the formal completions of X at its points. Moreover, there are two naturally defined prestacks of graded mixed algebras on X_{DR}, denoted $\mathbb{D}_{X_{DR}}$ and $\mathcal{P}_X$; they are to be thought as derived version of the crystalline structure sheaf and of the sheaf of principal parts respectively. Just as in the classical case, we have a morphism $\mathbb{D}_{X_{DR}} \to \mathcal{P}_X$, that we think of as a $\mathbb{D}_{X_{DR}}$-linear structure on $\mathcal{P}_X$.

The main result of formal localization, as introduced in [CPTVV] and briefly recalled at the end of our preliminary chapter, tells us that we can recover the geometrically defined polyvectors on a derived stack X in terms of the Tate polyvectors on the $\mathcal{P}_X$. More specifically, we have an equivalence

$$\mathrm{Pol}(X,n) \simeq \Gamma(X_{DR}, \mathrm{Pol}^t(\mathcal{P}_X/\mathbb{D}_{X_{DR}}, n)).$$

Here global sections are defined in terms of limits: given a derived stack Z and a functor

$$\mathcal{F} : (\mathrm{dAff}/Z)^{op} \longrightarrow \mathcal{C}$$

to a nice enough ∞-category $\mathcal{C}$, we set

$$\Gamma(Z, \mathcal{F}) := \lim_{\mathrm{Spec}A \to Z} \mathcal{F}(A).$$

Note that in particular this implies that $\mathrm{Pol}(X,n)$ is naturally endowed with a structure of a graded $\mathbb{P}_{n+1}$-algebra, coming from the natural $\mathbb{P}_{n+1}$-structure on (Tate) polyvectors on an algebra in a general ∞-category $\mathcal{M}$ as in chapter 0. This allows us to give a precise definition of Poisson structures on a derived Artin stack X, in the spirit of what was suggested in [PTVV].

Definition 2.2.1. *Let X be a derived Artin stack, locally of finite presentation. The* space of n-shifted Poisson structures on X *is*

$$\mathrm{Pois}(X,n) := \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(X,n+1)[n+1])$$

where as in chapter 1 we denoted with dgLie^{gr} *the ∞-category of graded dg Lie algebras over k.*

In the simpler case where $X = \mathrm{Spec}\ A$ is affine and $\mathbb{L}_X$ is perfect, it is easy to see that $\mathrm{Pol}(X,n) \simeq \mathrm{Sym}_A(\mathbb{T}_A[-n])$. In particular, the definition above generalized the one used in the previous chapter.

Notice that this is again a definition of geometric nature: we think of a Poisson structure as a shifted bivector π on X, which moreover satisfies $[\pi,\pi]=0$, at least up to homotopy. As already explained in the introduction, one would like to have a more algebraic definition, since one of the main goals of derived Poisson geometry is to be able to use Kontsevich's formality and quantize symplectic/Poisson derived moduli stacks. Therefore, we need precisely the results of the previous chapter.

As explained in chapter 0, Tate polyvectors can be interpreted as standard polyvectors on a twisted version of the algebra. This process was defined in the ∞-category $\epsilon - \mathrm{dgMod}^{gr}$ of graded mixed complexes, but it remains of course valid in any category of diagrams of graded mixed complexes. This means that as algebras in functors to the category of graded mixed modules, both $\mathbb{D}_{X_{DR}}$ and $\mathcal{P}_X$ admit natural twistings $\mathbb{D}_{X_{DR}}(\infty)$ and $\mathcal{P}_X(\infty)$, that are now prestacks of algebras in Ind-objects in the category of graded mixed modules. Explicitly, we have

$$\begin{array}{cccc} \mathbb{D}_{X_{DR}}(\infty): & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \mathrm{Ind}(\epsilon - \mathrm{dgMod}^{gr}) \\ & (\mathrm{Spec}A \to X_{DR}) & \longmapsto & \mathbb{D}(A)(\infty) \end{array}$$

and in a similar way

$$\begin{array}{cccc} \mathcal{P}_X(\infty): & (\mathrm{dAff}/X_{DR})^{op} & \longrightarrow & \mathrm{Ind}(\epsilon - \mathrm{dgMod}^{gr}) \\ & (\mathrm{Spec}A \to X_{DR}) & \longmapsto & \mathbb{D}(X_A)(\infty) \end{array}$$

where X_A is the defined as the homotopy fiber product

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}A \arrow[r] & X_{DR}. \end{tikzcd}$$

Notice that in particular $\mathcal{P}_X(\infty)$ is a commutative algebra in the category of $\mathbb{D}_{X_{DR}}(\infty)$-modules.

With these notations, we can now give an alternative definition of Poisson structures.

Definition 2.2.2. *Let X be a derived Artin stack, locally of finite presentation. The* space of n-shifted Poisson structures $\mathrm{Pois}'(X,n)$ on X *is the space of lifts of the given commutative algebra structure on $\mathcal{P}_X(\infty)$ to a compatible $\mathbb{D}_{X_{DR}}(\infty)$-linear $\mathbb{P}_{n+1}$-structure. Explicitly,* $\mathrm{Pois}'(X,n)$ *is the fiber product*

$$\begin{tikzcd} \mathrm{Pois}'(X,n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_{\mathcal{P}_X(\infty)}) \end{tikzcd}$$

where $\mathcal{P}_X(\infty)$ is viewed as an object inside the $C(k)$-enriched symmetric monoidal ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-modules.

If $X = \mathrm{Spec}A$ is an affine derived stack, this is exactly the homotopy fiber of

$$\mathrm{Map}(\mathbb{P}_{n+1}, \mathrm{End}_A) \to \mathrm{Map}(\mathrm{Comm}, \mathrm{End}_A)$$

taken at the point corresponding to the given commutative structure on A.

Using theorem 2.1.2, we can now prove that the two definitions do coincide.

Theorem 2.2.3. *Let again X be a derived Artin stack, locally of finite presentation. With notations as above, there is a canonical equivalence of spaces*

$$\mathrm{Pois}(X,n) \cong \mathrm{Pois}'(X,n).$$

Proof. Let us denote by $\mathcal{M}$ the ∞-category of functors

$$(\mathrm{dAff}/X_{DR})^{op} \longrightarrow \mathrm{Ind}(\epsilon-\mathrm{dgMod}^{gr}).$$

Then $\mathbb{D}_{X_{DR}}(\infty)$ is an object in $\mathrm{CAlg}_{\mathcal{M}}$, and thus we can consider the associated category of modules $\mathbb{D}_{X_{DR}}(\infty)-\mathrm{Mod}_{\mathcal{M}}$. Let us denote by $\mathcal{C}$ the ∞-category of commutative algebras in $\mathbb{D}_{X_{DR}}$-modules in $\mathcal{M}$, that is to say $\mathcal{C} := \mathrm{CAlg}_{\mathbb{D}_{X_{DR}}(\infty)-\mathrm{Mod}_{\mathcal{M}}}$. By definition, the twisted stack of principal parts $\mathcal{P}_X(\infty)$ lives in the symmetric monoidal ∞-category $\mathcal{C}$. Applying Theorem 2.1.2, we get an equivalence

$$\mathrm{Pois}'(X,n) \simeq \mathrm{Map}_{\mathrm{LieAlg}^{gr}_{\mathcal{C}}}(\mathbb{D}_{X_{DR}}(\infty)[-1](2), \mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n+1)[n+1]),$$

where we used the fact that $\mathbb{D}_{X_{DR}}(\infty)$ is the monoidal unit in $\mathcal{C}$. On the other hand, recall that the realization functor yields by definition a right adjoint

$$|-|: \mathrm{LieAlg}^{\mathrm{gr}}_{\mathcal{C}} \longrightarrow \mathrm{dgLie}^{gr},$$

so that we also have

$$\mathrm{Pois}'(X,n) \simeq \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), |\mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n+1)[n+1]|).$$

Since by definition we set

$$\mathrm{Pois}(X,n) \simeq \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(X, n+1)[n+1]),$$

we are left with proving an equivalence

$$|\mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n)| \simeq \mathrm{Pol}(X,n).$$

But by the results of formal localization exposed in chapter 0, the right hand side can be described in terms of $\mathcal{P}_X$, namely

$$\begin{aligned} \mathrm{Pol}(X,n) &\simeq \lim_{\mathrm{Spec}A \to X_{DR}} \mathrm{Pol}^t(\mathcal{P}_X(A)/\mathbb{D}_{X_{DR}}(A), n) \\ &\simeq \lim_{\mathrm{Spec}A \to X_{DR}} |\mathrm{Pol}^{int}(\mathcal{P}_X(\infty)(A), n)| \end{aligned}$$

where we used the usual relation between Tate realization and twisting by $k(\infty)$. Let us now look more in detail at the explicit form of the realization functor for $\mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n)$. In the ∞-category of $(\mathrm{dAff}/X_{DR})^{op}$-shaped diagrams in $\mathrm{Ind}(\epsilon-\mathrm{dgMod}^{gr})$, the monoidal unit is the constant diagram $\underline{k}$. In general, in any category of I-shaped diagrams in a symmetric monoidal category C, the monoidal unit is the constant diagram $\underline{1}_C$, where 1_C is the unit of C. It is a general result that if C is also $C(k)$-enriched, then the $C(k)$-enrichment satisfies

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(I,C)}(\underline{1}_C, X) \simeq \lim_{i \in I} \underline{\mathrm{Hom}}(1_C, X(i))$$

for every diagram X. Applied in our situation of interest, this implies

$$|F| \simeq \lim_{\mathrm{Spec} A \to X_{DR}} |F(A)|$$

for every F in $\mathcal{C}$. In particular, putting $F = \mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n)$, we get

$$|\mathrm{Pol}^{int}(\mathcal{P}_X(\infty), n)| \simeq \lim_{\mathrm{Spec} A \to X_{DR}} |\mathrm{Pol}^{int}(\mathcal{P}_X(\infty)(A), n)|$$

which is precisely what we needed. $\square$

Chapter 3

Coisotropic structures on affine derived stacks

In this chapter we define and study coisotropic structures on morphisms of derived affine stacks. The theory is parallel to the one developed for Lagrangian structures in [PTVV], though, as already explained, derived Poisson geometry is much less functorial that derived symplectic geometry, so that also coisotropic structures become more complicated than Lagrangian ones.

The chapter is organized as follows. In the first section we fix some notations about the objects involved, and we recall some well-known construction in the operadic world. In particular, we recall some generalities on convolution algebras and the Harrison complex, which will be used in the rest of the chapter. Since we will be interested in algebraic structures on morphisms $A \to B$ of commutative dg algebras, we are forced to use the formalism of colored operads. Most of the monochromatic constructions carry over to the colored case, also due to the standard model structure of Caviglia (see theorem 3.1.5).

In section 2, we start with a dg operad $\mathcal{O}$, and we give a detailed construction of the operad governing pair of homotopy $\mathcal{O}$-algebras A and B, together with an ∞-morphism $A \to B$. Thanks to results in [DW], the moduli space of such algebraic structures can be described in terms of Maurer-Cartan elements in a properly defined L_∞-algebra.

During Section 3, we establish a very general operadic formalism in order to define what we call *Swiss cheese operad* $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are reduced Hopf cooperads, together with a compatibility condition. This is a two-colors operad which is a slight generalization of the well known operad of Voronov (see [Vo]). We explicitly describe algebras over $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$, and we get an expression of $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$-structures on a couple (A, B) in term of Maurer-Cartan element in an appropriately defined L_∞-algebra $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2, A, B)$, again borrowing ideas from the cylinder L_∞-algebra of Dolgushev and Willwacher (see [DW]).

In section 4, we define coisotropic structures on morphisms of cdgas as $\mathbb{P}_{[n+1,n]}$-structure, where $\mathbb{P}_{[n+1,n]}$ is a two-colors operad which is an incarnation of our general swiss-cheese construction in section 1. Moreover, we show that the Maurer-Cartan elements of the L_∞-algebra $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2, A, B)$ are equivalent to Maurer-Cartan elements in a geometrically relevant Lie algebra of what we call relative polyvectors. In particular this gives an alternative (and more intuitive) definition of coisotropic structure.

3.1 Operadic notations and preliminaries

3.1.1 Lie algebras and Maurer-Cartan elements

Recall that given a dg Lie algebra $\mathfrak{g}$ the set of Maurer–Cartan elements is defined to be the set of elements $x \in \mathfrak{g}$ such that

$$\mathrm{d}x + \frac{1}{2}[x, x] = 0.$$

Similarly, we can make sense of Maurer-Cartan elements in more general L_∞-algebras. Recall that an L_∞-algebra $\mathfrak{g}$ is said to be *nilpotent* if its lower central series $F^i\mathfrak{g}$ terminates, where $F^1\mathfrak{g} = \mathfrak{g}$ and we define inductively

$$F^i\mathfrak{g} = \sum_{i_1+\cdots+i_k} [F^{i_1}\mathfrak{g}, \dots, F^{i_k}\mathfrak{g}].$$

We say that a L_∞-algebra $\mathfrak{g}$ is *pro-nilpotent* if it is an inverse limit of nilpotent algebras.

If $\mathfrak{g}$ is a nilpotent or a pro-nilpotent L_∞-algebra we can define the set of Maurer–Cartan elements in $\mathfrak{g}$ to be the set of elements $x \in \mathfrak{g}$ satisfying

$$\mathrm{d}x + \sum_{n\geq 2} \frac{1}{n!}[x, ..., x]_n = 0.$$

Let $\Omega_\bullet$ be the cosimplicial commutative algebra of polynomial differential forms on simplices, which already appeared in chapter 1. For instance, $\Omega_0 = k$ and $\Omega_1 = k[x, y]$ with $\deg(x) = 0$, $\deg(y) = 1$ and $\mathrm{d}x = y$. We define the space of Maurer–Cartan elements $\underline{\mathrm{MC}}(\mathfrak{g})$ to be the simplicial set of Maurer–Cartan elements in $\mathfrak{g} \otimes \Omega_\bullet$, see [Ge] for more details.

Given an L_∞-algebra $\mathfrak{g}$ and a Maurer-Cartan element $x \in \mathfrak{g}$, we can define the L_∞-algebra obtained by *twisting* $\mathfrak{g}$ *by* x: as a vector space, it is still $\mathfrak{g}$, and the brackets are defined to be

$$[x_1, \dots, x_n]_n := \sum_{k\geq 0} \frac{1}{k!}[x, \dots, x, x_1, \dots, x_n]_{n+k}$$

for $x_1, \dots, x_n \in \mathfrak{g}$.

The following lemma is an easy verification.

Lemma 3.1.1. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be pro-nilpotent L_∞ algebras with a pair of morphisms $p\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $i\colon \mathfrak{g}_2 \to \mathfrak{g}_1$ such that $p \circ i = \mathrm{id}_{\mathfrak{g}_2}$. Then the homotopy fiber of*

$$\underline{\mathrm{MC}}(\mathfrak{g}_1) \to \underline{\mathrm{MC}}(\mathfrak{g}_2)$$

at a Maurer–Cartan element $x \in \mathfrak{g}_2$ is equivalent to the space of Maurer–Cartan elements in the L_∞ algebra $\ker p$ twisted by the element $i(x)$.

The space of Maurer–Cartan elements in a dg Lie algebra is homotopy invariant only for a pro-nilpotent dg Lie algebra. One way to ensure pro-nilpotence is to keep track of the grading. Recall that in the previous chapters we already encountered a notion of *graded dg Lie algebra* (see in particular the first section of chapter 1). We recall here the definition for the sake of completeness.

Definition 3.1.2. *A* graded dg Lie algebra *is a graded complex*

$$\mathfrak{g} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{g}^m$$

together with a Lie bracket of cohomological degree 0 *and weight* -1.

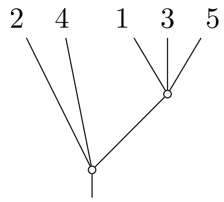

Figure 3.1: The tree $\mathbf{t}_\sigma$ corresponding to a $(2,3)$-shuffle σ.

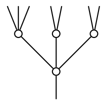

Figure 3.2: A pitchfork in $\mathrm{Isom}_{\pitchfork}(7,3)$.

Similarly, one can define graded L_∞-algebras to be graded complexes with L_∞ operations l_n of weight $1-n$. These form categories dgLie^{gr} and $\mathrm{L}_\infty^{\mathrm{gr}}$.

Given a graded dg Lie algebra or a graded L_∞-algebra $\mathfrak{g}$ we introduce the completions

$$\mathfrak{g}^{\geq m} = \prod_{n\geq m} \mathfrak{g}^n.$$

Note that in particular by our conventions $\mathfrak{g}^{\geq m}$ for any $m \geq 2$ is pro-nilpotent.

As usual, we denote by $k[-1](2)$ the trivial one-dimensional dg Lie algebra concentrated in cohomological degree 1 and weight 2. We have the following statement, which is proved for instance in the last section of chapter 1.

Lemma 3.1.3. *Let $\mathfrak{g}$ be a graded L_∞ algebra. The space $\mathrm{Map}_{\mathrm{L}_\infty^{\mathrm{gr}}}(k[-1](2), \mathfrak{g})$ is equivalent to the space of Maurer–Cartan elements in the pro-nilpotent L_∞ algebra $\mathfrak{g}^{\geq 2}$.*

3.1.2 Operadic notations

Throughout this chapter, we will need at various points to be a tad more explicit with respect to chapter 1. We will therefore need some additional (quite standard) notations. Our conventions about operads mainly follow those of [DR] and [LV]. Unless precisely stated, all operads we consider are operads in cochain complexes.

Recall that a *symmetric sequence* V is a sequence of cochain complexes $V(n)$ together with an action of S_n on $V(n)$. The category of symmetric sequences is monoidal with respect to the composition product and an operad is an algebra in the category of symmetric sequences. Similarly, a cooperad is a coalgebra in the category of symmetric sequences.

We denote by $\mathrm{Tree}_m(n)$ the groupoid of planar trees with labeled n incoming edges and m vertices. The morphisms are not necessarily planar isomorphisms between trees. For instance, the groupoid $\mathrm{Tree}_2(n)$ has components parametrized by $(p, n-p)$-shuffles σ for any p, where a shuffle σ corresponds to the tree $\mathbf{t}_\sigma$ as shown in Figure 3.1.

We will also be interested in the set $\mathrm{Isom}_{\pitchfork}(n,r)$ of *pitchforks* with n incoming edges and $r+1$ vertices, see Figure 3.2 for an example and [DW, Section 2] for more details. The groupoid $\mathrm{Tree}_3(n)$ has trees of two kinds: pitchforks in $\mathrm{Isom}_{\pitchfork}(n,2)$ and the complement $\mathrm{Tree}_3^0(n)$.

Given a tree $\mathbf{t} \in \mathrm{Tree}_m(n)$ and a symmetric sequence $\mathcal{O}$ we define $\mathcal{O}(\mathbf{t})$ to be the tensor product

$$\mathcal{O}(\mathbf{t}) = \bigotimes_i \mathcal{O}(n_i)$$

where the tensor product is taken over the vertices of $\mathbf{t}$ and n_i is the number of incoming edges at vertex i.

Given an operad $\mathcal{O}$, a tree $\mathbf{t} \in \mathrm{Tree}_m(n)$ naturally defines a multiplication map

$$m_{\mathbf{t}} : \mathcal{O}(\mathbf{t}) \to \mathcal{O}(n),$$

corresponding to the formal composition of operations in $\mathcal{O}$. Similarly, for a cooperad $\mathcal{C}$ we have a comultiplication map

$$\Delta_{\mathbf{t}} : \mathcal{C}(n) \to \mathcal{C}(\mathbf{t}).$$

Recall that trees play an important role in the description of the free operad: for a symmetric sequence $\mathcal{P}$, the free operad $\mathrm{Free}(\mathcal{P})$ has operations parametrized by trees $\mathbf{t}$ whose vertices are labeled by operations in $\mathcal{P}$.

All our operads will have either $\mathcal{O}(0) = k$ or $\mathcal{O}(0) = 0$. In the latter case we say that the operad is reduced. Given a non-reduced operad $\mathcal{O}$ we denote by $\mathcal{O}^{nu}$ the same operad with $\mathcal{O}^{nu}(0) = 0$.

All cooperads we consider will have $\mathcal{C}(1) = k$. In particular, they are canonically coaugmented. Let us denote the cokernel of the coaugmentation by $\mathcal{C}_\circ$. Given a coaugmented cooperad $\mathcal{C}$ we define its cobar complex $\Omega\mathcal{C}$ to be the free operad on the symmetric sequence $\mathcal{C}_\circ[-1]$. The differential on the generators $X \in \mathcal{C}(n)[-1]$ for $n > 1$ is given by

$$\mathrm{d}X = -\mathbf{s}\mathrm{d}_1(\mathbf{s}^{-1}X) - \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (\mathbf{s}\otimes\mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(\mathbf{s}^{-1}X)) \tag{3.1}$$

where d_1 is the differential on the symmetric sequence $\mathcal{C}$, and $\mathbf{s}$ stands for suspension.

The following lemma is standard, and its proof can be found for instance in [LV, Proposition 6.5.6].

Lemma 3.1.4. *The cobar differential* d *on* $\Omega\mathcal{C}$ *squares to zero.*

Given an operad $\mathcal{O}$ and a complex A, we define the free $\mathcal{O}$-algebra on A to be

$$\mathcal{O}(A) = \bigoplus_n (\mathcal{O}(n) \otimes A^{\otimes n})^{S_n}.$$

Similarly, for a cooperad $\mathcal{C}$ and a complex A, we define the cofree conilpotent $\mathcal{C}$-coalgebra on A to be

$$\mathcal{C}(A) = \bigoplus_n (\mathcal{C}(n) \otimes A^{\otimes n})_{S_n}.$$

We will also be interested in colored symmetric sequences and colored operads. Let $\mathcal{V}$ be a set. A $\mathcal{V}$-colored symmetric sequence is a collection of complexes $\mathcal{V}(v_1^{\otimes n_1} \otimes ... \otimes v_m^{\otimes n_m}, v_0)$ for every collection of elements $v_0, v_1, ..., v_m \in \mathcal{V}$ together with an action of $S_{n_1} \times ... \times S_{n_m}$. As before, the category of $\mathcal{V}$-colored symmetric sequences has a composition product and a $\mathcal{V}$-colored operad is defined to be an algebra object in the category of $\mathcal{V}$-colored symmetric sequences.

We say that a (colored) dg operad $\mathcal{O}$ is *semi-free* if it is free as a graded (colored) operad. The following results allows us to look for semi-free resolutions of colored dg operads, much like in the case of monochromatic operads.

Theorem 3.1.5 (Caviglia, [Ca])**.** *The category of (colored) dg operads has a model structure with the following properties:*

- *A weak equivalence is the weak equivalence of the underlying (colored) symmetric sequences.*
- *A semi-free (colored) operad is cofibrant.*
- *Every (colored) operad is fibrant.*

3.1.3 Convolution algebras and resolutions

Given a (monochromatic) operad $\mathcal{O}$, we will be mostly interested in working with some cofibrant resolution $\mathcal{O}' \to \mathcal{O}$. One natural object of interest is then the mapping space $\mathrm{Map}_{\mathrm{dgOp}}(\mathcal{O}', \mathrm{End}_X)$, which parametrizes $\mathcal{O}'$-structures on the object X. More generally, one would like to better understand the mapping space $\mathrm{Map}_{dgOp}(\mathcal{O}', \mathcal{P})$, where $\mathcal{P}$ is now any dg operad. We saw in chapter 1 that in the case $\mathcal{O} \simeq \mathrm{Lie}$ this space can be usefully described as the space of Maurer-Cartan elements inside a properly defined dg Lie algebra associated to $\mathcal{P}$.

In this section, we study this problem for a general $\mathcal{O}$, but we suppose moreover that the resolution $\mathcal{O}'$ is of the form $\Omega\mathcal{C}$, where $\mathcal{C}$ is a cooperad. We start by recalling the definition of the convolution Lie algebra, and we describe its Maurer-Cartan elements.

Let $\mathcal{C}$ be a cooperad, and let $\mathcal{P}$ a dg operad. We introduce the convolution algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ as follows. As a complex it is defined to be

$$\mathrm{Conv}(\mathcal{C}, \mathcal{P}) = \prod_{n \geq 2} \underline{\mathrm{Hom}}_{S_n}(\mathcal{C}(n), \mathcal{P}(n)),$$

where the product is taken in the category of dg modules, and $\underline{\mathrm{Hom}}_{S_n}$ is the cochain complex of maps of S_n-modules. In the special case in which $\mathcal{P}$ is an endormorphisms operad, we denote

$$\mathcal{L}(\mathcal{C}; A) := \mathrm{Conv}(\mathcal{C}, \mathrm{End}_A).$$

It is well known that the convolution algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ admits a pre-Lie product. More specifically, let us now introduce a binary operation on $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ by

$$(f \bullet g)(X) = \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} m_{\mathbf{t}}((f \otimes g)\Delta_{\mathbf{t}}(X))$$

for any $f, g \in \mathrm{Conv}(\mathcal{C}, \mathcal{P})$ and $X \in \mathcal{C}(n)$. The following bracket

$$[f, g] = f \bullet g - (-1)^{|f||g|} g \bullet f.$$

satisfies the Jacobi identity, that is to say $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$ becomes a pre-Lie algebra. This is essentially the content of [DR, Proposition 4.1].

Consider the graded piece

$$\mathrm{Conv}(\mathcal{C}, \mathcal{P})^n = \underline{\mathrm{Hom}}_{S_n}(\mathcal{C}(n), \mathcal{P}(n)).$$

It is easy to see that the pre-Lie structure gives a map

$$\mathrm{Conv}(\mathcal{C}, \mathcal{P})^n \otimes \mathrm{Conv}(\mathcal{C}, \mathcal{P})^m \to \mathrm{Conv}(\mathcal{C}, \mathcal{P})^{n+m-1}.$$

Therefore, the completion

$$\mathrm{Conv}(\mathcal{C}, \mathcal{P}) = \prod_{n \geq 2} \mathrm{Conv}(\mathcal{C}, \mathcal{P})^n$$

is a pro-nilpotent pre-Lie algebra.

The Lie structure on the convolution algebra is very useful, as its Maurer-Cartan elements have a nice description in terms of mapping spaces of operads.

Proposition 3.1.6. *Assume $\mathcal{O}$ is a reduced augmented operad with a weak equivalence $\Omega\mathcal{C} \xrightarrow{\sim} \mathcal{O}$. Then the mapping space $\mathrm{Map}_{\mathrm{dgOp}}(\mathcal{O}, \mathcal{P})$ is equivalent to the space of Maurer–Cartan elements in the convolution algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$.*

Proof. We have a sequence of equivalences of spaces

$$\mathrm{Map}_{\mathrm{dgOp}}(\mathcal{O}, \mathcal{P}) \cong \mathrm{Map}_{\mathrm{dgOp}}(\Omega\mathcal{C}, \mathcal{P}) \cong \underline{\mathrm{Hom}}_{\bullet}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}(\Omega\mathcal{C}, \mathcal{P} \otimes \Omega_{\bullet}).$$

An operad morphism $f\colon \Omega\mathcal{C} \to \mathcal{P}$ is uniquely specified by a degree 0 map of symmetric sequences $f_0\colon \mathcal{C}_{\circ}[-1] \to \mathcal{P}$ satisfying the equation

$$\begin{aligned}
\mathrm{d}(f_0(X)) &= f(\mathrm{d}X) \\
&= f(-\mathbf{s}\mathrm{d}(\mathbf{s}^{-1}X) - \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(\mathbf{s}^{-1}X))) \\
&= -f_0(\mathbf{s}\mathrm{d}(\mathbf{s}^{-1}X)) - f(\sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(\mathbf{s}^{-1}X)))
\end{aligned}$$

for any $X \in \mathcal{C}_{\circ}(n)[-1]$. Since f is a morphism of operads, the last term can also be written in terms of f_0, so we obtain

$$\mathrm{d}(f_0(X)) + f_0(\mathbf{s}\mathrm{d}(\mathbf{s}^{-1}X)) + \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(n))} \mu_{\mathbf{t}}((f_0\mathbf{s} \otimes f_0\mathbf{s})\Delta_{\mathbf{t}}(\mathbf{s}^{-1}X)) = 0.$$

Identifying degree 0 maps $f_0\colon \mathcal{C}_{\circ}[-1] \to \mathcal{P}$ with degree 1 maps $f_0\mathbf{s}\colon \mathcal{C}_{\circ} \to \mathcal{P}$ we get exactly the Maurer–Cartan equation in $\mathrm{Conv}(\mathcal{C}, \mathcal{P})$. Since the simplicial set of Maurer–Cartan elements in a dg Lie algebra $\mathfrak{g}$ is defined to be the set of Maurer–Cartan elements in $\mathfrak{g} \otimes \Omega_{\bullet}$ and $\mathrm{Conv}(\mathcal{C}, \mathcal{P} \otimes \Omega_{\bullet}) \cong \mathrm{Conv}(\mathcal{C}, \mathcal{P}) \otimes \Omega_{\bullet}$, we are done. ☐

3.1.4 The Harrison complex

We will now give some explicit examples of our previous constructions. Let us begin with the case $\mathcal{O} = \mathrm{Comm}^{nu}$, the non-unital commutative operad. The operad is Koszul and we have a canonical resolution $\Omega(\mathrm{coLie}\{1\}) \to \mathrm{Comm}^{nu}$, where $\mathrm{coLie}\{1\}$ is the cooperad of shifted Lie coalgebras with the cobracket of degree 1. Define the filtered dg Lie algebra

$$\mathrm{Harr}^{\bullet}(A, A) := \mathrm{Hom}(\mathrm{coLie}(A[1]), A[1]) \subset \prod_{n \geq 1} \mathrm{Hom}(A^{\otimes n}, A)[1-n],$$

of maps $A^{\otimes n} \to A$ vanishing on elements of the form

$$\sum_{\sigma \in S_{r,n-r}} (-1)^{\mathrm{sgn}(\sigma)} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}.$$

The pre-Lie structure is given by

$$(f \circ g)(a_1, \ldots, a_n) = \sum (-1)^{|g| \sum_{k=1}^{i}(|a_k|+1)} f(a_1, \ldots, a_i, g(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_n).$$

We have a natural decreasing filtration on $\mathrm{Harr}^{\bullet}(A, A)$ using the arity filtration on the cofree Lie coalgebra where

$$\mathrm{Harr}^{\geq m}(A, A) \subset \prod_{n \geq m} \mathrm{Hom}(A^{\otimes n}, A)[1-n].$$

Proposition 3.1.6 implies that the space $\mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}^{nu}, \mathrm{End}_A)$ is equivalent to the space of Maurer–Cartan elements in the dg Lie algebra $\mathrm{Harr}^{\geq 2}(A, A)$. Note that $\mathrm{Harr}^{\geq 2}(A, A)$ is naturally pro-nilpotent.

If A is a commutative dg algebra, the multiplication defines a Maurer–Cartan element in $\mathrm{Harr}^\bullet(A, A)$. The induced differential on $\mathrm{Harr}^\bullet(A, A)$ is given by

$$\begin{aligned}(\mathrm{d}f)(a_1, ..., a_n) = \mathrm{d}f(a_1, ..., a_n)\\ &+ \sum_{i=1}^{n}(-1)^{|f|+\sum_{q=1}^{i-1}|a_q|+i} f(a_1, ..., \mathrm{d}a_i, ..., a_n)\\ &+ \sum_{i=1}^{n-1}(-1)^{|f|+\sum_{q=1}^{i}|a_q|+i+1} f(a_1, ..., a_i a_{i+1}, ..., a_n)\\ &+ (-1)^{(|f|+1)(|a_1|+1)} a_1 f(a_2, ..., a_n) + (-1)^{\sum_{q=1}^{n-1}|a_q|+|f|+n+1} f(a_1, ..., a_{n-1})a_n.\end{aligned}$$

We call $\mathrm{Harr}^\bullet(A, A)$ with this differential the *Harrison cochain complex*. Let us also define the Harrison chain complex. As a graded vector space it is defined to be

$$\mathrm{Harr}_\bullet(A, A) = A \otimes \mathrm{coLie}(A[1])[-1].$$

The differential on $\mathrm{Harr}_\bullet(A, A)$ is characterized uniquely by the property that the quotient map $A \otimes \mathrm{T}(A[1])[-1] \to \mathrm{Harr}_\bullet(A, A)$ from the bar complex is compatible with the differential.

Consider the morphism $\mathrm{Harr}_\bullet(A, A) \to \Omega^1_A$ given by sending $f \otimes g \mapsto f\mathrm{d}_{\mathrm{dR}}g$ where $g \in A \subset \mathrm{coLie}(A[1])[-1]$. The following lemma is standard, and is contained for example in [CK].

Lemma 3.1.7. *Suppose A is a cofibrant cdga. Then the morphism $\mathrm{Harr}_\bullet(A, A) \to \Omega^1_A$ is a quasi-isomorphism.*

We have a morphism $\mathrm{Der}(A, A) \to \mathrm{Harr}^\bullet(A, A)$ dual to the previous one given by the inclusion $\mathrm{Der}(A, A) \subset \mathrm{End}(A) \subset \mathrm{Harr}^\bullet(A, A)$.

Proposition 3.1.8. *Let A be a cdga. The Harrison complex $\mathrm{Harr}^\bullet(A, A)$ is a model for the tangent complex $\mathbb{T}_A$. If A is cofibrant, the morphism $\mathrm{Der}(A, A) \to \mathrm{Harr}^\bullet(A, A)$ is a quasi-isomorphism of dg Lie algebras.*

Proof. By Lemma 3.1.7, we have a quasi-isomorphism of A-modules $A\otimes\mathrm{coLie}(A[1])[-1] \cong \mathbb{L}_A$. The first claim follows by taking the A-linear dual.

The fact that $\mathrm{Der}(A, A) \to \mathrm{Harr}^\bullet(A, A)$ is a quasi-isomorphism follows from the same Lemma. Compatibility with the Lie bracket is immediate as the pre-Lie structure on

$$\mathrm{End}(A) \subset \mathrm{Harr}^\bullet(A, A)$$

is given by composition. □

3.2 The operad of ∞-morphisms

Let $\mathcal{O}$ be any operad. We are now going to introduce a colored operad $\mathcal{O} \otimes \Delta^1$ whose set of colors is $\{\mathcal{A}, \mathcal{P}\}$. Algebras over $\mathcal{O} \otimes \Delta^1$ consist of a pair of $\mathcal{O}$-algebras A, B together with a morphism of

$\mathcal{O}$-algebras $A \to B$. Explicitly,

$$(\mathcal{O}\otimes\Delta^1)(\mathcal{A}^{\otimes n},\mathcal{A}) = \mathcal{O}(n), \quad (\mathcal{O}\otimes\Delta^1)(\mathcal{P}^{\otimes n},\mathcal{P}) = \mathcal{O}(n)$$
$$(\mathcal{O}\otimes\Delta^1)(\mathcal{A}^{\otimes n},\mathcal{P}) = \mathcal{O}(n), \quad (\mathcal{O}\otimes\Delta^1)(\mathcal{P}^{\otimes n},\mathcal{A}) = 0.$$

The operad structure on $(\mathcal{O}\otimes\Delta^1)$ comes from the operad structure on $\mathcal{O}$ itself.

Assume we have a resolution $\Omega\mathcal{C} \xrightarrow{\sim} \mathcal{O}$. As usual, $\Omega\mathcal{C}$ can be interpreted as an up-to-homotopy version of $\mathcal{O}$. Our next goal is to use $\mathcal{C}$ to construct an up-to-homotopy version of the operad $\mathcal{O}\otimes\Delta^1$. Let us start by defining a colored symmetric sequence $\mathcal{C}\otimes\Delta^1$ as follows:

$$(\mathcal{C}\otimes\Delta^1)(\mathcal{A}^{\otimes n},\mathcal{A}) = \mathcal{C}_\circ(n), \quad (\mathcal{C}\otimes\Delta^1)(\mathcal{P}^{\otimes n},\mathcal{P}) = \mathcal{C}_\circ(n)$$
$$(\mathcal{C}\otimes\Delta^1)(\mathcal{A}^{\otimes n},\mathcal{P}) = \mathcal{C}(n)[1], \quad (\mathcal{C}\otimes\Delta^1)(\mathcal{P}^{\otimes n},\mathcal{A}) = 0.$$

We then define the colored operad $F = \mathrm{Free}(\mathcal{C}\otimes\Delta^1[-1])$. It has the following explicit description:

$$F(\mathcal{A}^{\otimes n},\mathcal{A}) = (\Omega\mathcal{C})(n), \quad F(\mathcal{P}^{\otimes n},\mathcal{P}) = (\Omega\mathcal{C})(n)$$
$$F(\mathcal{A}^{\otimes n},\mathcal{P}) = (\Omega\mathcal{C}\circ\mathcal{C}\circ\Omega\mathcal{C})(n), \quad F(\mathcal{P}^{\otimes n},\mathcal{A}) = 0.$$

The cooperad structure on $\mathcal{C}$ can be used to define a differential on F. More precisely, the differential on the terms $F(\mathcal{A}^{\otimes n},\mathcal{A})$ and $F(\mathcal{P}^{\otimes n},\mathcal{P})$ is the usual cobar differential. The differential on the generators $X \in \mathcal{C}(n) \subset F(\mathcal{A}^{\otimes n},\mathcal{P})$ has three components:

- The first component $\mathrm{d}_1 X$ comes from the differential on $\mathcal{C}(n)$ itself.
- The second component is

$$\mathrm{d}_2 X = -\sum_{\mathbf{t}\in\mathrm{Tree}_2(n)} (\mathbf{s}\otimes 1)(\mathbf{t},\Delta_{\mathbf{t}}(X))$$

  for any $X \in \mathcal{C}(n) \subset F(\mathcal{A}^{\otimes n},\mathcal{P})$, where we treat the first tensor factor as an element of $F(\mathcal{A}^{\otimes -},\mathcal{A})$, the second tensor factor as an element of $F(\mathcal{A}^{\otimes -},\mathcal{P})$ and the height 1 node is on the right.

- The third component is

$$\mathrm{d}_3 X = \sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{\pitchfork}(n,r)} (1\otimes\ldots\otimes 1\otimes\mathbf{s})(\mathbf{t},\Delta_{\mathbf{t}}(X)),$$

  where the first r factors are treated as elements of $F(\mathcal{A}^{\otimes -},\mathcal{P})$, the last factor is treated as an element of $F(\mathcal{P}^{\otimes -},\mathcal{P})$ and the rightmost tensor factor again comes from the height 1 node.

The operations in the free operad F are parametrized by trees with solid edges corresponding to $\mathcal{A}$ and dashed edges corresponding to $\mathcal{P}$; the vertices of the trees are labeled by the operations in $\mathcal{C}\otimes\Delta^1$. See Figure 3.3 for a pictorial presentation of the differentials.

Lemma 3.2.1. *The total differential on* $\mathrm{Free}(\mathcal{C}\otimes\Delta^1[-1])$ *squares to zero.*

Proof. Let us denote by d_1 the internal differential on $\mathcal{C}$ which gives the first term in the cobar differential (3.1); we denote by $\mathrm{d}_{\mathcal{A}}$ the second term of the cobar differential on $F(\mathcal{A}^{\otimes -},\mathcal{A})$ and by $\mathrm{d}_{\mathcal{P}}$ the second term of the cobar differential on $F(\mathcal{A}^{\otimes -},\mathcal{P})$. The total differentials on $F(\mathcal{A}^{\otimes -},\mathcal{A})$ and $F(\mathcal{P}^{\otimes -},\mathcal{P})$ square to zero by Lemma 3.1.4.

Now consider a generator $X \in \mathcal{C}(n) \subset F(\mathcal{A}^{\otimes n},\mathcal{P})$. Let us split the terms appearing in $\mathrm{d}^2 X$ into the following combinations:

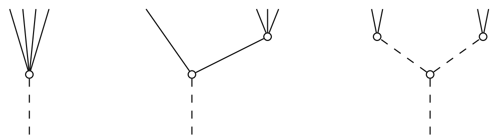

Figure 3.3: Operation X and summands in d_2X and d_3X.

1. d_1^2X,
2. $\mathrm{d}_1\mathrm{d}_2X + \mathrm{d}_2\mathrm{d}_1X$,
3. $\mathrm{d}_1\mathrm{d}_3X + \mathrm{d}_3\mathrm{d}_1X$,
4. $\mathrm{d}_{\mathcal{A}}\mathrm{d}_2X + \mathrm{d}_2^2X$,
5. $\mathrm{d}_3\mathrm{d}_2X + \mathrm{d}_2\mathrm{d}_3X$,
6. $\mathrm{d}_{\mathcal{P}}\mathrm{d}_3X + \mathrm{d}_3^2X$.

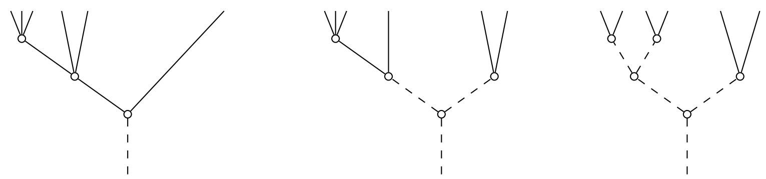

Figure 3.4: Some trees appearing in terms of type (4), (5) and (6) respectively.

We claim that each of these is zero. Indeed, $\mathrm{d}_1^2X = 0$ follows from the fact that the internal differential on $\mathcal{C}$ squares to zero. The terms of type (2) and (3) separately vanish since the cooperad maps are compatible with the internal differential.

Recall the subset $\pi_0(\mathrm{Tree}_3^0(n)) \subset \pi_0(\mathrm{Tree}_3(n))$ from the proof of Lemma 3.1.4 and denote

$$\Delta_{\mathbf{t}}(X) = X^{\mathbf{t}}_{(1)} \otimes X^{\mathbf{t}}_{(2)}.$$

Then

$$\begin{aligned}\mathrm{d}_{\mathcal{A}}\mathrm{d}_2X &= -\sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (\mathrm{d}_{\mathcal{A}}\mathbf{s}X^{\mathbf{t}}_{(1)} \otimes X^{\mathbf{t}}_{(2)}) \\ &= \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_3^0(n))} (\mathbf{s}\otimes\mathbf{s}\otimes 1)(X^{\mathbf{t}}_{(1)} \otimes X^{\mathbf{t}}_{(2)} \otimes X^{\mathbf{t}}_{(3)})\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{d}_2^2 X &= \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (-1)^{|X^{\mathbf{t}}_{(1)}|}(\mathbf{s}X^{\mathbf{t}}_{(1)})\otimes(\mathrm{d}_2 X^{\mathbf{t}}_{(2)}) \\
&= -\sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}^0_3(n))} (-1)^{|X^{\mathbf{t}}_{(1)}|}(\mathbf{s}X^{\mathbf{t}}_{(1)})\otimes(\mathbf{s}X^{\mathbf{t}}_{(2)})\otimes X^{\mathbf{t}}_{(3)} \\
&\quad -\sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,2)} (-1)^{|X^{\mathbf{t}_l}_{(1)}|}(\mathbf{s}X^{\mathbf{t}_l}_{(1)})\otimes(\mathbf{s}X^{\mathbf{t}_l}_{(2)})\otimes X^{\mathbf{t}_l}_{(3)} \\
&\quad -\sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,2)} (-1)^{|X^{\mathbf{t}_r}_{(1)}|}(\mathbf{s}X^{\mathbf{t}_r}_{(1)})\otimes(\mathbf{s}X^{\mathbf{t}_r}_{(2)})\otimes X^{\mathbf{t}_r}_{(3)}.
\end{aligned}$$

As in the proof of Lemma 3.1.4, the terms involving sums over pitchforks cancel. But the sums over $\pi_0(\mathrm{Tree}_3(n))$ have opposite signs, so terms of type (4) vanish.

We have

$$\begin{aligned}
\mathrm{d}_2\mathrm{d}_3 X &= \sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} (\mathrm{d}_2\otimes 1)(1^{\otimes r}\otimes\mathbf{s})(X^{\mathbf{t}}_{(1)}\otimes\ldots\otimes X^{\mathbf{t}}_{(r)}\otimes X^{\mathbf{t}}_{(r+1)}) \\
&= -\sum_{r,i} \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} (1^{\otimes r}\otimes\mathbf{s})(1^{\otimes(i-1)}\otimes\mathrm{d}_2\otimes 1^{\otimes(r+1)})(X^{\mathbf{t}}_{(1)}\otimes\ldots\otimes X^{\mathbf{t}}_{(r)}\otimes X^{\mathbf{t}}_{(r+1)}) \\
&= -\sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} \sum_{i=1}^r (-1)^{\sum_{j=1}^{i-1}|X^{\mathbf{t}}_{(j)}|}(1^{\otimes r}\otimes\mathbf{s})(X^{\mathbf{t}}_{(1)}\otimes\ldots\otimes\mathrm{d}_2X^{\mathbf{t}}_{(i)}\otimes\ldots\otimes X^{\mathbf{t}}_{(r+1)}) \\
&= -\sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} \sum_{i=1}^r \sum_{k=0}^{n^{\mathbf{t}}_i} \sum_{\sigma\in S_{k,n^{\mathbf{t}}_i-k}} (1^{\otimes(i-1)}\otimes\mathbf{s}\otimes 1^{\otimes(r-i+1)}\otimes\mathbf{s})\Delta_{\mathbf{t}_\sigma\bullet_i\mathbf{t}}(X),
\end{aligned}$$

where $\mathbf{t}_\sigma\bullet_i\mathbf{t}$ is the tree obtained by inserting $\mathbf{t}_\sigma$ into the i-th nodal vertex of $\mathbf{t}$.

Similarly,

$$\begin{aligned}
\mathrm{d}_3\mathrm{d}_2 X &= -\sum_{k=0}^n \sum_{\sigma\in S_{k,n-k}} (1\otimes\mathrm{d}_3)(\mathbf{s}\otimes 1)\Delta_{\mathbf{t}_\sigma}(X) \\
&= \sum_{k=0}^n \sum_{\sigma\in S_{k,n-k}} (\mathbf{s}\otimes 1)(1\otimes\mathrm{d}_3)\Delta_{\mathbf{t}_\sigma}(X) \\
&= \sum_{k=0}^n \sum_{\sigma\in S_{k,n-k}} \sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} (\mathbf{s}\otimes 1^{\otimes r}\otimes\mathbf{s})\Delta_{\mathbf{t}_\sigma\bullet_1\mathbf{t}}(X),
\end{aligned}$$

where the sum goes over pitchforks $\mathbf{t}$ with $n^{\mathbf{t}}_1\geq k$. We can identify terms in $\mathrm{d}_2\mathrm{d}_3X$ with those in $\mathrm{d}_3\mathrm{d}_2X$ if we permute the nodes of $\mathbf{t}$ so that the tree $\mathbf{t}_\sigma$ is always attached to the vertex $i=1$. Then we get terms with opposite signs, so terms of type (5) vanish.

For terms of type (6) we have

$$\begin{aligned}
\mathrm{d}_{\mathcal{P}}\mathrm{d}_3 X &= \sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} (1^{\otimes r}\otimes\mathrm{d}_{\mathcal{P}}\mathbf{s})\Delta_{\mathbf{t}}(X) \\
&= -\sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{⋔}(n,r)} \sum_k \sum_{\sigma\in S_{k,r-k}} (1^{\otimes r}\otimes\mathbf{s}\otimes\mathbf{s})\Delta_{\mathbf{t}\bullet_0\mathbf{t}_\sigma}(X),
\end{aligned}$$

where $\mathbf{t} \bullet_0 \mathbf{t}_\sigma$ is the tree obtained by inserting $\mathbf{t}_\sigma$ into the unique height 1 node of $\mathbf{t}$.

We also have

$$\begin{aligned}
\mathrm{d}_3^2 X &= \sum_q \sum_{\mathbf{t}\in \mathrm{Isom}_{\pitchfork}(n,q)} (\mathrm{d}_3 \otimes 1)(1^{\otimes q} \otimes \mathbf{s})\Delta_{\mathbf{t}}(X) \\
&= -\sum_q \sum_{\mathbf{t}\in \mathrm{Isom}_{\pitchfork}(n,q)} \sum_{i=1}^{q} (1^{\otimes q} \otimes \mathbf{s})(1^{i-1} \otimes \mathrm{d}_3 \otimes 1^{q-i})\Delta_{\mathbf{t}}(X) \\
&= -\sum_q \sum_{\mathbf{t}\in \mathrm{Isom}_{\pitchfork}(n,q)} \sum_{i=1}^{q} \sum_k \sum_{\mathbf{t}'\in \mathrm{Isom}_{\pitchfork}(n_i^{\mathbf{t}},k)} (1^{\otimes (k+q-1)} \otimes \mathbf{s})(1^{i-1} \otimes 1^{\otimes k} \otimes \mathbf{s} \otimes 1^{q-i} \otimes 1)\Delta_{\mathbf{t}\bullet_i \mathbf{t}'}(X) \\
&= \sum_q \sum_{\mathbf{t}\in \mathrm{Isom}_{\pitchfork}(n,q)} \sum_{i=1}^{q} \sum_k \sum_{\mathbf{t}'\in \mathrm{Isom}_{\pitchfork}(n_i^{\mathbf{t}},k)} (1^{i-1} \otimes 1^{\otimes k} \otimes \mathbf{s} \otimes 1^{q-i} \otimes \mathbf{s})\Delta_{\mathbf{t}\bullet_i \mathbf{t}'}(X)
\end{aligned}$$

The trees of the type $\mathbf{t} \bullet_i \mathbf{t}'$ can be identified with the trees of the type $\mathbf{t} \bullet_0 \mathbf{t}_\sigma$ once we set $k+q-1=r$. The corresponding terms in $\mathrm{d}_{\mathcal{P}}\mathrm{d}_3$ and d_3^2 have opposite signs, so terms of type (6) vanish as well. □

We denote by $\Omega(\mathcal{C}\otimes\Delta^1)$ the colored operad $\mathrm{Free}(\mathcal{C}\otimes\Delta^1[-1])$ equipped with the above differential. There is a morphism of colored operads $\Omega(\mathcal{C}\otimes\Delta^1) \to \mathcal{O}\otimes\Delta^1$ coming from the morphism $\Omega\mathcal{C} \to \mathcal{O}$ in arities $(\mathcal{A}^{\otimes n}, \mathcal{A})$ and $(\mathcal{P}^{\otimes n}, \mathcal{P})$ and the morphism $\mathcal{C} \to \mathbf{1}$ in arity $(\mathcal{A}^{\otimes n}, \mathcal{P})$.

Lemma 3.2.2. *The morphism $\Omega(\mathcal{C}\otimes\Delta^1) \to \mathcal{O}\otimes\Delta^1$ is a weak equivalence.*

Proof. The claim in arities $(\mathcal{A}^{\otimes n}, \mathcal{A})$ and $(\mathcal{P}^{\otimes n}, \mathcal{P})$ follows from the assumption that $\Omega\mathcal{C} \to \mathcal{O}$ is a weak equivalence.

The claim in arity $(\mathcal{A}^{\otimes n}, \mathcal{P})$ follows from the fact that the symmetric sequence $\Omega\mathcal{C}\circ\mathcal{C}$ with the differential d_2 is quasi-isomorphic to $\mathbf{1}$ [LV, Lemma 6.5.14]. Therefore, the symmetric sequence $\Omega\mathcal{C}\circ\mathcal{C}\circ\Omega\mathcal{C}$ is weakly equivalent to $\mathbf{1}\circ\Omega\mathcal{C}$ which in turn is weakly equivalent to $\mathcal{O}$. □

We thus have a resolution of the colored operad $\mathcal{O}\otimes\Delta^1$; moreover, recall from the Caviglia model structure of theorem 3.1.5 that this resolution is cofibrant, since it is manifestly semi-free. We now go on to describe its algebras.

To this purpose, let us introduce the L_∞-algebra $\mathcal{L}(\mathcal{C}; A, B)$ which controls deformations of a pair of $\Omega\mathcal{C}$-algebras A and B together with an ∞-morphism $A \to B$. This construction appeared in work of Dolgushev and Willwacher under the name of *cylinder L_∞-algebra* (see [DW, Section 3]). As a complex it is

$$\mathcal{L}(\mathcal{C}; A, B) = \mathcal{L}(\mathcal{C}; A) \oplus \mathcal{L}(\mathcal{C}; B) \oplus \mathrm{Hom}(\mathcal{C}(A), B)[-1]$$

with the differentials coming from $\mathcal{C}$, A and B.

The L_∞ brackets are given as follows:

- The Lie structure on the first and second terms are those on the convolution algebras $\mathcal{L}(\mathcal{C}; A)$ and $\mathcal{L}(\mathcal{C}; B)$.
- The Lie bracket between an element $P \in \mathcal{L}(\mathcal{C}; A)$ and $\mathbf{s}T \in \mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(A^{\otimes -}, B))[-1]$ lands in $\mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(A^{\otimes -}, B))[-1]$ and is given by

$$[\mathbf{s}T, P](X; a_1, ..., a_n) = \sum_{p=0}^{n} \sum_{\sigma\in S_{p,n-p}} \pm \mathbf{s}T(X_{(1)}; P(X_{(2)}; a_{\sigma(1)}, ..., a_{\sigma(p)}), a_{\sigma(p+1)}, ..., a_{\sigma(n)})$$

for $X \in \mathcal{C}(n)$ and where $\Delta_{\mathbf{t}_\sigma}(X) = X_{(1)} \otimes X_{(2)}$. Here the sign is the Koszul sign associated to the permutation of $\{T, PX_{(1)}, X_{(2)}, a_1, \dots, a_n\}$.

- The L_∞ brackets between an element $R \in \mathcal{L}(\mathcal{C}; B)$ and $\mathbf{s}T_i \in \mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(A^{\otimes -}, B))[-1]$ land in $\mathrm{Hom}(\mathcal{C}, \mathrm{Hom}(A^{\otimes -}, B))[-1]$ and are given by

$$[R, \mathbf{s}T_1, ..., \mathbf{s}T_r](X; a_1, ..., a_n) = -\sum_{\sigma \in S_r} \sum_{\mathbf{t} \in \mathrm{Isom}_{\pitchfork}(n,r)} \pm\, \mathbf{s}R(X_{(0)}; T_{\sigma(1)}(X_{(1)}; a_{\lambda_{\mathbf{t}}(1)}, ..., a_{\lambda_{\mathbf{t}}(n_1^{\mathbf{t}})}), ..., T_{\sigma(r)}(X_{(r)}; a_{\lambda_{\mathbf{t}}(n-n_r^{\mathbf{t}}+1)}, ..., a_{\lambda_{\mathbf{t}}(n)})).$$

  Here again the sign is the Koszul sign associated to the permutation of the set of graded elements $\{T_1, \dots T_r, X_{(1)}, X_{(2)}, a_1, \dots, a_n\}$.

The remaining brackets are either extended in the obvious way by symmetry or declared to be zero.

The following statement is proved in [DW, Claim 3.1] and essentially follows from Lemma 3.2.1:

Lemma 3.2.3. *These brackets define an L_∞ structure on $\mathcal{L}(\mathcal{C}; A, B)$.*

Moreover, Dolgushev and Willwacher give the following description of the Maurer-Cartan elements in the cylinder algebra [DW, Claim 3.2].

Lemma 3.2.4. *The Maurer-Cartan equation for the L_∞-algebra $\mathcal{L}(\mathcal{C}, A, B)$ is well defined, and Maurer-Cartan elements in $\mathcal{L}(\mathbb{C}, A, B)$ correspond to $\Omega\mathcal{C}$-structures on A and B together with an ∞-morphism of $\Omega\mathcal{C}$-algebras from A to B.*

We know show that Maurer-Cartan elements in $\mathcal{L}(\mathcal{C}, A, B)$ also correspond to algebras for the operad $\Omega(\mathcal{C} \otimes \Delta^1)$, giving our desired description of its algebras.

Proposition 3.2.5. *The mapping space* $\mathrm{Map}(\mathcal{O} \otimes \Delta^1, \mathrm{End}_{A,B})$ *is equivalent to the space of Maurer–Cartan elements in the L_∞ algebra $\mathcal{L}(\mathcal{C}; A, B)$.*

Proof. As in Proposition 3.1.6, we have a sequence of weak equivalences

$$\mathrm{Map}(\mathcal{O} \otimes \Delta^1, \mathrm{End}_{A,B}) \cong \underline{\mathrm{Hom}}(\Omega(\mathcal{C} \otimes \Delta^1), \mathrm{End}_{A,B}) = \mathrm{Hom}(\Omega(\mathcal{C} \otimes \Delta^1), \mathrm{End}_{A,B} \otimes \Omega_\bullet).$$

A morphism of colored operads $f \colon \Omega(\mathcal{C} \otimes \Delta^1) \to \mathcal{P}$ is uniquely determined by degree 0 maps

$$f^{\mathcal{A}\mathcal{A}} \colon \mathcal{C}_\circ[-1] \to \mathcal{P}(\mathcal{A}^{\otimes -}, \mathcal{A}), \quad f^{\mathcal{P}\mathcal{P}} \colon \mathcal{C}_\circ[-1] \to \mathcal{P}(\mathcal{P}^{\otimes -}, \mathcal{P}), \quad f^{\mathcal{A}\mathcal{P}} \colon \mathcal{C} \to \mathcal{P}(\mathcal{A}^{\otimes -}, \mathcal{P}).$$

These have to satisfy the equations

$$\begin{aligned} \mathrm{d}f^{\mathcal{A}\mathcal{A}}(X) &= f(\mathrm{d}X), \quad X \in \mathcal{C}_\circ(n)[-1] \\ \mathrm{d}f^{\mathcal{P}\mathcal{P}}(X) &= f(\mathrm{d}X), \quad X \in \mathcal{C}_\circ(n)[-1] \\ \mathrm{d}f^{\mathcal{A}\mathcal{P}}(X) &= f(\mathrm{d}X), \quad X \in \mathcal{C}(n), \end{aligned}$$

where in the first line we consider X as a generator of $\Omega(\mathcal{C} \otimes \Delta^1)$ in arity $(\mathcal{A}^{\otimes n}, \mathcal{A})$, in the second line as a generator in arity $(\mathcal{P}^{\otimes n}, \mathcal{P})$ and in the last line as a generator in arity $(\mathcal{A}^{\otimes n}, \mathcal{P})$.

As in Proposition 3.1.6, the first two equations imply that the elements $f_0^{\mathcal{A}\mathcal{A}}\mathbf{s}$ and $f_0^{\mathcal{A}\mathcal{P}}\mathbf{s}$ satisfy the Maurer–Cartan equations in the convolution algebras $\mathrm{Conv}(\mathcal{C}, \mathcal{P}(\mathcal{A}^{\otimes -}, \mathcal{A}))$ and $\mathrm{Conv}(\mathcal{C}, \mathcal{P}(\mathcal{P}^{\otimes -}, \mathcal{P}))$. The last equation becomes

$$\begin{aligned} \mathrm{d}f^{\mathcal{A}\mathcal{P}}(X) &= f_0^{\mathcal{A}\mathcal{P}}(\mathrm{d}_1 X) - f\left(\sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} (\mathbf{s}\otimes 1)(\mathbf{t}, \Delta_{\mathbf{t}}(X))\right) \\ &\quad + f\left(\sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{\pitchfork}(n,r)} (1^{\otimes r}\otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(X))\right) \\ &= f^{\mathcal{A}\mathcal{P}}(\mathrm{d}_1 X) - \sum_{\mathbf{t}\in\pi_0(\mathrm{Tree}_2(n))} \mu_{\mathbf{t}}((f^{\mathcal{A}\mathcal{A}}\mathbf{s}\otimes f^{\mathcal{A}\mathcal{P}})\Delta_{\mathbf{t}}(X)) \\ &\quad + \sum_r \sum_{\mathbf{t}\in\mathrm{Isom}_{\pitchfork}(n,r)} \mu_{\mathbf{t}}(((f^{\mathcal{A}\mathcal{P}})^{\otimes r}\otimes f^{\mathcal{P}\mathcal{P}}\mathbf{s})\Delta_{\mathbf{t}}(X)) \end{aligned}$$

This implies that $(f^{\mathcal{A}\mathcal{A}}\mathbf{s}, f^{\mathcal{A}\mathcal{P}}, f^{\mathcal{P}\mathcal{P}}\mathbf{s})$ defines a Maurer–Cartan element. □

3.3 Swiss-cheese operads

The goal of this section is to define a generalized version of the swiss-cheese operad of Voronov (see [Vo]). The construction uses ideas taken from [CW].

Suppose $\mathcal{C}$ is a Hopf cooperad, i.e. a cooperad in dg algebras. Calaque and Willwacher proved that for any cochain complex A, the natural pre-Lie structure on $\mathcal{L}(\mathcal{C}, A)$ can be lifted to a so-called *$\mathcal{C}$-pre-Lie structure*. More specifically, the operad $\mathrm{preLie}_{\mathcal{C}}$ has operation parametrized by rooted trees where each vertex is labeled by an operation of $\mathcal{C}$ whose arity is equal to the number of incoming edges at the given vertex. Explicitly, let $\mathcal{T}(n)$ be the set of rooted trees with n vertices; then we have

$$\mathrm{preLie}_{\mathcal{C}}(n) = \bigoplus_{\mathbf{t}\in\mathcal{T}(n)} \left(\bigotimes_{i=1}^{n} \mathcal{C}(\mathbf{t}_i)\right),$$

where $\mathbf{t}_i$ is the number of incoming edges at the vertex i (see [CW, Section 3] for more details).

As mentioned, the for any complex A the convolution algebra $\mathcal{L}(\mathcal{C}, A)$ is in a natural way a $\mathrm{preLie}_{\mathcal{C}}$-algebra. Suppose now we are given an $\Omega\mathcal{C}$-structure on A. We have already seen that this corresponds to a Maurer-Cartan element in $\mathcal{L}(\mathcal{C}, A)$, which can in turn be used to twist its differential in order to obtain a twisted convolution algebra $\mathcal{L}(\mathcal{C}, A)'$.

Definition 3.3.1. *The twisted convolution algebra $\mathcal{L}(\mathcal{C}, A)'$ is called the* center *of the $\Omega\mathcal{C}$-algebra A, and it will be denoted $Z_{\mathcal{C}}(A)$ (or simply $Z(A)$, if there is no ambiguity).*

The operad $\mathrm{preLie}_{\mathcal{C}}$ does not act on $Z_{\mathcal{C}}(A)$, but an appropriately twisted version of it indeed does. By the general formalism of operadic twisting (see [DW], [CW]), one can in fact construct an operad $\mathrm{TwpreLie}_{\mathcal{C}}$. The operad $\mathrm{TwpreLie}_{\mathcal{C}}$ has operations parametrized by rooted trees where some "external" vertices are labeled by elements of $\mathcal{C}$ and the rest of the vertices, "internal" vertices, are unlabeled. In the pictures external vertices are colored white and internal vertices are colored black. The operad $\mathrm{TwpreLie}_{\mathcal{C}}$ then acts naturally on the twisted convolution algebra $\mathcal{L}(\mathcal{C}; A)'$, where the internal vertices are assigned the Maurer-Cartan element itself. We refer again to [CW, Section 3] for more details on this constructions.

As an example, consider the case $\mathcal{O} = \mathbb{P}_n^{nu}$, the operad of non-unital $\mathbb{P}_n$-algebras. Then we have a weak equivalence $\Omega(\mathrm{co}\mathbb{P}_n^{nu}\{n\}) \to \mathbb{P}_n^{nu}$, where $\mathrm{co}\mathbb{P}_n^{nu}\{n\}$ is the cooperad of shifted non-counital $\mathbb{P}_n$-coalgebras with comultiplication of degree n and cobracket of degree 1.

Given a homotopy $\mathbb{P}_n$-algebra B we define the *Poisson center* $Z(B)$ to be

$$Z(B) = \mathrm{Hom}(\mathrm{co}\mathbb{P}_n^{un}\{n\}, \mathrm{End}_B)[-n]$$

with the differential twisted by the Maurer–Cartan element π_B defining the homotopy $\mathbb{P}_n$-algebra structure.

Tamarkin [Ta] defined a homotopy $\mathbb{P}_{n+1}$-structure on the Poisson center $Z(B)$ which we now sketch following Calaque–Willwacher [CW]. Recall that $\mathrm{co}\mathbb{P}_n$ is a Hopf cooperad. Therefore, we have an action of the operad $\mathrm{TwpreLie}_{\mathrm{co}\mathbb{P}_n}$ on $Z(B)$. Explicitly, the homotopy $\mathbb{P}_{n+1}$-action on $\mathrm{Conv}(\mathrm{co}\mathbb{P}_n\{n\}, \mathrm{End}_B)[-n]$ is given by a morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}^{nu}\{n+1\}) \to \mathrm{TwpreLie}_{\mathrm{co}\mathbb{P}_n}\{n\}$$

defined on generators by the following rule:

- The generators
$$\underline{x_1 ... x_k} \in \mathrm{coLie}\{1\}(k) \subset \mathrm{co}\mathbb{P}_{n+1}^{nu}\{n+1\}(k)$$
are sent to the tree drawn in Figure 3.5 with the root labeled by the element
$$\underline{x_1 ... x_k} \in \mathrm{coLie}\{1\}(k) \subset \mathrm{co}\mathbb{P}_n^{nu}\{n\}(k).$$
Here $\underline{x_1 ... x_k}$ is the image of the k-ary comultiplication under the projection
$$\mathrm{coAss}\{1\} \to \mathrm{coLie}\{1\}.$$

- The generator
$$x_1 \wedge x_2 \in \mathrm{coComm}\{n+1\}(2) \subset \mathrm{co}\mathbb{P}_{n+1}^{nu}\{n+1\}(2)$$
is sent to the linear combination of trees shown in Figure 3.6.

- The generators
$$x_1 \wedge \underline{x_2 ... x_k} \in \mathrm{co}\mathbb{P}_{n+1}^{nu}\{n+1\}(k)$$
for $k > 2$ are sent to the tree shown in Figure 3.7 with the root labeled by the element
$$\underline{x_2 ... x_k} \in \mathrm{coLie}\{1\}(k-1) \subset \mathrm{co}\mathbb{P}_n^{nu}\{n\}(k-1).$$

- The rest of the generators are sent to zero.

Note that under the natural inclusion $\Omega(\mathrm{coComm}^{nu}\{n+1\}) \subset \Omega(\mathrm{co}\mathbb{P}_{n+1}^{nu}\{n+1\})$ the homotopy Lie structure on $Z(B)$ is strict and coincides with the Lie bracket on the convolution algebra as easily seen from Figure 3.6.

We are now going to define a colored operad which can be considered as a version of the Swiss–cheese operad. Let $\mathcal{C}_1$ be a reduced cooperad and $\mathcal{C}_2$ a reduced Hopf cooperad together with an operad morphism $F\colon \Omega\mathcal{C}_1 \to \mathrm{TwpreLie}_{\mathcal{C}_2}$. Notice that this map automatically gives an $\Omega\mathcal{C}_1$-structure on $Z_{\mathcal{C}_2}(B)$ for any $\Omega\mathcal{C}_2$-algebra B. From this data we will define a cofibrant colored operad $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$, whose algebras will be couples (A, B) such that

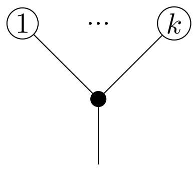

Figure 3.5: Image of $x_1 ... x_k$.

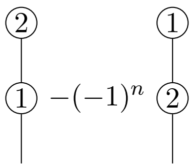

Figure 3.6: Image of $x_1 \wedge x_2$.

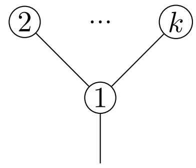

Figure 3.7: Image of $x_1 \wedge \underline{x_2 ... x_k}$

- A is a $\Omega\mathcal{C}_1$-algebra;
- B is a $\Omega\mathcal{C}_2$-algebra;
- there is an ∞-morphism of $\Omega\mathcal{C}_1$-algebras from A to $Z_{\mathcal{C}_2}(B)$.

As before, we will explicitly present $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$, starting from a colored symmetric sequence. The set of colors of $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ is $\{\mathcal{A}, \mathcal{P}\}$. The operad is semi-free on the colored symmetric sequence $P(\mathcal{C}_1, \mathcal{C}_2)$ whose nonzero elements are

$$\begin{aligned} &P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{A}^{\otimes m}, \mathcal{A}) = \mathcal{C}_1(m), \quad m > 1 \\ &P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{P}^{\otimes l}, \mathcal{P}) = \mathcal{C}_2(l), \quad l > 1 \\ &P(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{A}^{\otimes m} \otimes \mathcal{P}^{\otimes l}, \mathcal{P}) = \mathcal{C}_1(m) \otimes \mathcal{C}_2(l)[1], \quad m > 1, l \geq 0. \end{aligned}$$

The colored operad $\mathrm{Free}(P(\mathcal{C}_1, \mathcal{C}_2)[-1])$ has operations parametrized by trees with edges of two types: those of color $\mathcal{A}$ that we denote by solid lines and those of color $\mathcal{P}$ that we denote by dashed lines. The vertices of the trees are labeled by generating operations in $P(\mathcal{C}_1, \mathcal{C}_2)$. We define a differential on $\mathrm{Free}(P(\mathcal{C}_1, \mathcal{C}_2)[-1])$ in the following way. The differentials in arities $(\mathcal{A}^{\otimes -}, \mathcal{A})$ and $(\mathcal{P}^{\otimes -}, \mathcal{P})$ are the usual cobar differentials (3.1). The differential on an element $\mathbf{s}^{-n} X \otimes Y$ for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2(l)$ has four components:

- $$\mathrm{d}_1(X \otimes Y) = \mathrm{d}_1 X \otimes Y + (-1)^{|X|} X \otimes \mathrm{d}_1 Y$$
  where d_1 are the internal differentials on the complexes $\mathcal{C}_1(m)$ and $\mathcal{C}_2(l)$.
- $$\mathrm{d}_2(X \otimes Y) = (1 \otimes \mathbf{s}^{-1}) \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(m))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(X)) \circ Y.$$
  Here we denote by $(\mathbf{t}, \Delta_{\mathbf{t}}(X)) \circ Y$ the tree $\mathbf{t}$ with additional l dashed incoming edges at the root which is labeled by $X_{(0)} \otimes Y$.
- $$\mathrm{d}_3(X \otimes Y) = (1 \otimes \mathbf{s}^{-1}) X \circ \sum_{\mathbf{t} \in \pi_0(\mathrm{Tree}_2(l))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}, \Delta_{\mathbf{t}}(Y)).$$
- $$\mathrm{d}_4(X \otimes Y) = \sum_r \sum_{\mathbf{t} \in \mathrm{Isom}_{\pitchfork}(m, r)} F(\mathbf{t}, \Delta_t(X), Y).$$

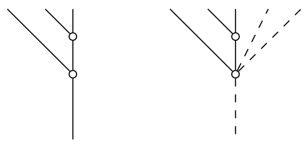

Figure 3.8: A tree $\mathbf{t}$ and $\mathbf{t} \circ Y$ with $l = 2$.

Here $F(\mathbf{t}, \Delta_t(X), Y)$ is defined in the following way. Let $X_{(0)}$ be the label of the root in $\Delta_{\mathbf{t}}(X)$. The image of $\mathbf{s}X_{(0)}$ under $F\colon \Omega\mathcal{C}_1 \to \mathrm{TwpreLie}_{\mathcal{C}_2}$ is a tree $F(\mathbf{s}X_{(0)})$ labeled by r elements of $\mathcal{C}_2$. Consider the composition $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$. We consider the following set of trees $\tilde{\mathbf{t}}$: a tree $\tilde{\mathbf{t}}$ is obtained from $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$ by adding an arbitrary number of incoming dashed edges to vertices so that the total number of incoming dashed edges is l. Note that the vertices of the tree $\mathbf{t}$ are labeled by elements of $\mathcal{C}_1$ while the vertices of $F(\mathbf{s}X_{(0)})$ are labeled by elements Z_i of $\mathcal{C}_2$. The labelings of vertices of $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$ are of two kinds: external vertices are labeled by the tensor product $X_{(i)} \otimes Y_{(i)} Z_{(i)}$ (where $Y_{(i)} Z_{(i)}$ is the product in the Hopf cooperad $\mathcal{C}_2$) and they belong to the operations in $P(\mathcal{A}^{\otimes -} \otimes \mathcal{P}^{\otimes -}, \mathcal{P})$; the internal vertices are simply labeled by elements of $\mathcal{C}_2$ and they belong to the operations in $P(\mathcal{P}^{\otimes -}, \mathcal{P})$. We refer to figure 3.9 for an example. We define $F(\mathbf{t}, \Delta_t(X), Y)$ to be the sum over all such trees $\tilde{\mathbf{t}}$.

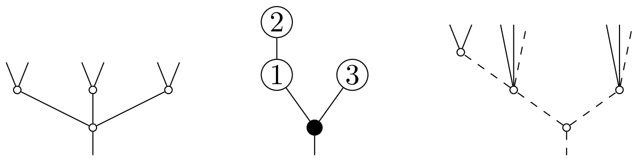

Figure 3.9: A pitchfork $\mathbf{t}$, a rooted tree $F(\mathbf{s}X_{(0)})$ and an example of $\tilde{\mathbf{t}}$.

Lemma 3.3.2. *The total differential* d *on* $\mathrm{Free}(\mathcal{P}(\mathcal{C}_1, \mathcal{C}_2)[-1])$ *squares to zero.*

Proof. The claim in arities $(\mathcal{A}^{\otimes m},\ c A)$ and $(\mathcal{P}^{\otimes l}, \mathcal{P})$ follows from Lemma 3.1.4.

The proof of the claim in arities $(\mathcal{A}^{\otimes m} \otimes \mathcal{P}^{\otimes l}, \mathcal{P})$ is similar to the proof of Lemma 3.2.1, so we only give a sketch of the proof. Let us split the differentials on the generators in arities $(\mathcal{A}^{\otimes -}, \mathcal{A})$ and $(\mathcal{P}^{\otimes -}, \mathcal{P})$ as $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_{\mathcal{A}}$ and $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_{\mathcal{P}}$ respectively.

Given an element $X \otimes Y$ for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2(l)$ the expression $\mathrm{d}^2(X \otimes Y)$ splits into the following combinations:

1. $\mathrm{d}_1^2(X \otimes Y)$,
2. $(\mathrm{d}_1\mathrm{d}_2 + \mathrm{d}_2\mathrm{d}_1)(X \otimes Y)$,
3. $(\mathrm{d}_1\mathrm{d}_3 + \mathrm{d}_3\mathrm{d}_1)(X \otimes Y)$,
4. $(\mathrm{d}_1\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_1)(X \otimes Y)$,
5. $(\mathrm{d}_2^2 + \mathrm{d}_{\mathcal{A}}\mathrm{d}_2)(X \otimes Y)$,
6. $(\mathrm{d}_3^2 + \mathrm{d}_{\mathcal{P}}\mathrm{d}_3)(X \otimes Y)$,
7. $(\mathrm{d}_2\mathrm{d}_3 + \mathrm{d}_3\mathrm{d}_2)(X \otimes Y)$,

8. $(\mathrm{d}_2\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_2)(X \otimes Y)$,
9. $(\mathrm{d}_3\mathrm{d}_4 + \mathrm{d}_4\mathrm{d}_3)(X \otimes Y)$,
10. $(\mathrm{d}_4^2 + \mathrm{d}_{\mathcal{P}}\mathrm{d}_4)(X \otimes Y)$.

We claim that each of these is zero. It is obvious for terms of type (1). Terms of type (2) and (3) vanish due to compatibility of the cooperad structure on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively with the differentials. The vanishing of terms of type (5) and (6) is proved similarly to the vanishing of the terms of type (4) in Lemma 3.2.1. The vanishing of the terms of type (7), (8), (9) is obvious as the corresponding modifications of the trees are independent.

Differentials on both $\Omega\mathcal{C}_1$ and $\mathrm{TwpreLie}_{\mathcal{C}_2}$ have a linear and a quadratic component. Therefore, the compatibility of the morphism $F\colon \Omega\mathcal{C}_1 \to \mathrm{TwpreLie}_{\mathcal{C}_2}$ with differentials has two implications. First, the compatibility of the linear parts of the differentials implies the vanishing of terms of type (4). Second, the compatibility of the quadratic parts of the differentials implies the vanishing of terms of type (10). ☐

We denote by $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ the colored operad $\mathrm{Free}(\mathcal{P}(\mathcal{C}_1, \mathcal{C}_2)[-1])$ equipped with the above differential.

We define the L_∞ algebra $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B)$ as follows. As a complex,

$$\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B) = \mathcal{L}(\mathcal{C}_1; A) \oplus \mathcal{L}(\mathcal{C}_2; B) \oplus \mathrm{Hom}(\mathcal{C}_1(A) \otimes \mathcal{C}_2^{un}(B), B)[-1].$$

The L_∞ operations are given by the following rule:

- The first two terms have the standard convolution algebra brackets.
- The first two terms act on the third term by precomposition.
- Given $R_1, ..., R_m \in \mathcal{L}(\mathcal{C}_2; B)$ and $T_1, ..., T_r \in \mathrm{Hom}(\mathcal{C}_1(A) \otimes \mathcal{C}_2^{un}(B), B)$, their bracket is

  $$[R_1, ..., R_q.T_1, ..., T_r](X \otimes Y; a_1, ..., a_m, b_1, ..., b_l)$$

  for $X \in \mathcal{C}_1(m)$ and $Y \in \mathcal{C}_2(l)$ is given by the sum over pitchforks $\mathbf{t} \in \mathrm{Isom}_{\pitchfork}(m, r)$ where each term is given as follows. Let $\Delta_{\mathbf{t}}(X) = X_{(0)} \otimes ...$ where $X_{(0)}$ is assigned to the root and recall the tree $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$. The value of the bracket is given by the sum over all ways of assigning $T_1, ..., T_r$ to the white external vertices and $R_1, ..., R_q$ to the black internal vertices of $\mathbf{t} \circ_0 F(\mathbf{s}X_{(0)})$.

The proof of the following proposition it entirely analogous to the one given for Proposition 3.2.5.

Proposition 3.3.3. *The space of morphisms* $\mathrm{Map}_{\mathrm{dgOp}}(\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2), \mathrm{End}_{A,B})$ *is equivalent to the space of Maurer–Cartan elements in the* L_∞ *algebra* $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B)$*.*

Notice that a Maurer-Cartan element in $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2; A, B)$ gives a Maurer–Cartan element in $\mathcal{L}(\mathcal{C}_1; A, Z(B))$ where we use the morphism $F\colon \Omega\mathcal{C}_1 \to \mathrm{TwpreLie}_{\mathcal{C}_2}$ to give an action of $\Omega\mathcal{C}_1$ on $Z(B)$. In particular, we get the following consequence.

Corollary 3.3.4. *An algebra over the colored operad* $\mathrm{SC}(\mathcal{C}_1, \mathcal{C}_2)$ *is an* $\Omega\mathcal{C}_1$*-algebra* A*, an* $\Omega(\mathcal{C}_2)$*-algebra* B *and a homotopy morphism of* $\Omega\mathcal{C}_1$*-algebras* $A \to Z(B)$*.*

3.4 Coisotropic structures on affines

We are now ready to apply the constructions and the results of the previous sections in order to give the definition of coisotropic structures.

Recall from chapter 1 that for A a commutative dg algebra, we defined the space $\mathrm{Pois}(A, n)$ of n-shifted Poisson structures on A to be the homotopy fiber of

$$\mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1}, \mathrm{End}_A) \to \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm}, \mathrm{End}_A)$$

taken at the given commutative multiplication. Moreover, Theorem 1.3.2 gave us an alternative way of describing the space $\mathrm{Pois}(A, n)$: we have in fact an equivalence

$$\mathrm{Pois}(A, n) \cong \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k(2)[-1], \mathrm{Pol}(A, n)[n+1]),$$

where $\mathrm{Pol}(A, n)$ is the algebra of shifted polyvectors on A.

Recall that given a homotopy $\mathbb{P}_n$-algebra B we have the Poisson center

$$Z(B) = \mathrm{Hom}(\mathrm{co}\mathbb{P}_n^{un}\{n\}(B), B)[-n]$$

which is a homotopy $\mathbb{P}_{n+1}$-algebra. This structure was obtained from an explicit morphism of operads

$$\Omega(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}) \to \mathrm{TwpreLie}_{\mathrm{co}\mathbb{P}_n}\{n\}$$

defined by Tamarkin and Calaque–Willwacher (see [Ta], [CW]).

Given such morphism, we can now define the operad which will encode coisotropic structures.

Definition 3.4.1. *With notations as in Section 3.3, we define the* $(n+1)$-shifted Poisson swiss-cheese operad *to be*

$$\mathbb{P}_{[n+1,n]} = \mathrm{SC}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_n).$$

We have the following characterization of $\mathbb{P}_{[n+1,n]}$-algebras, which is a special case of Corollary 3.3.4.

Proposition 3.4.2. *A $\mathbb{P}_{[n+1,n]}$-algebra structure on (A, B) consists of the following data:*

- *a homotopy $\mathbb{P}_{n+1}$-structure on A;*
- *a homotopy $\mathbb{P}_n$-structure on B;*
- *an ∞-morphism of $\mathbb{P}_{n+1}$-algebras $A \to Z(B)$, where $Z(B)$ is the Poisson center on B.*

Note that, by definition, we have a natural inclusion $\Omega(\mathrm{coLie}\{1\} \otimes \Delta^1) \to \mathbb{P}_{[n+1,n]}$ which gives a morphism $\mathrm{Comm} \otimes \Delta^1 \to \mathbb{P}_{[n+1,n]}$ in the homotopy category of colored operads. In terms of algebras, this implies that given a $\mathbb{P}_{[n+1,n]}$-algebra (A, B), we get an underlying map of commutative algebras $A \to B$.

Definition 3.4.3. *Let $f\colon A \to B$ be a morphism of commutative dg algebras. The space* $\mathrm{Cois}(f, n)$ *of n*-shifted coisotropic structures on *f is defined to be the homotopy fiber product of the diagram of spaces*

$$\begin{tikzcd}
\mathrm{Cois}(f,n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[d] \\
* \arrow[r] & \mathrm{Map}_{\mathrm{dgOp}}(\mathrm{Comm} \otimes \Delta^1, \mathrm{End}_{A,B})
\end{tikzcd}$$

where the bottom map is induced by the given commutative structure on the morphism f.

We remark that this definition is a homotopy version of the definition of coisotropic structures given in [Sa, Definition 1.4].

We have a canonical pair of forgetful maps

$$\begin{tikzcd} & \mathrm{Cois}(f,n) \arrow[dl] \arrow[dr] & \\ \mathrm{Pois}(B,n-1) & & \mathrm{Pois}(A,n) \end{tikzcd}$$

induced by the morphisms

$$\begin{tikzcd} & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{[n+1,n]},\mathrm{End}_{A,B}) \arrow[dl] \arrow[dr] & \\ \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_n,\mathrm{End}_B) & & \mathrm{Map}_{\mathrm{dgOp}}(\mathbb{P}_{n+1},\mathrm{End}_A). \end{tikzcd}$$

Notice that this definition can be somewhat mysterious. If $f : A \to B$ has a n-shifted coisotropic structure, then in particular the algebras A and B are endowed Poisson structures of different shifts. We stress that this is a purely derived phenomenon, as in the case where both A and B are classical underived k-algebras, they can only admit unshifted (that is to say, 0-shifted) Poisson structures.

As a first evidence to the fact that Definiton 3.4.3 is a sensible generalization of the concept of being coisotropic in ordinary smooth geometry, notice that classically if A is a Poisson algebra and $B \simeq A/I$, then B is coisotropic if the multiplicative ideal I is closed under the Poisson bracket.

In this case, the inclusion $I \to A$ becomes in fact a morphism of non-unital Poisson algebras, which is precisely what will happen for general $\mathbb{P}_{[n+1,n]}$-algebras in Section 3.4.1.

We will discuss the relation between definition 3.4.3 and the classical notion of coisotropic more in detail in the following chapter.

3.4.1 From relative Poisson algebras to Poisson algebras

Recall that for a $\mathbb{P}_n$-algebra B in M, we defined its Poisson center to be the convolution algebra

$$\mathrm{Conv}(\mathrm{co}\mathbb{P}_n^{cu}\{n\}; B)[-n],$$

with differential twisted by the Maurer Cartan element π_B defining the $\mathbb{P}_n$-structure on B. Similarly, let us denote with $\mathrm{Def}(B)$ its *deformation complex*, defined as

$$\mathrm{Conv}(\mathrm{co}\mathbb{P}_n\{n\}; B),$$

again with differential twisted by π_B. Notice that the main difference between $Z(B)$ and $\mathrm{Def}(B)[-n]$ is the absence of the weight 0 part in the convolution algebra. Also, the different shift convention are taken for historical reasons, so that $\mathrm{Def}(B)$ is a Lie algebra. In particular, the natural map

$$\mathrm{Def}(B)[-n] \to Z(B)$$

is a morphism of non-unital $\mathbb{P}_{n+1}$-algebras. Rotating the fiber sequence

$$\mathrm{Def}(B)[-n] \longrightarrow Z(B) \longrightarrow B$$

we find

$$B[-1] \longrightarrow \mathrm{Def}(B)[-n] \longrightarrow Z(B).$$

Since as mentioned the map on the right is a (non-unital) $\mathbb{P}_{n+1}$-map, it follows that the whole fiber sequence can be promoted to a fiber sequence of non-unital $\mathbb{P}_{n+1}$-algebras.

Now suppose that we have a $\mathbb{P}_{[n+1,n]}$-algebra (A, B) in M. In particular, we have a $\mathbb{P}_{n+1}$-morphism $A \to Z(B)$. Therefore, we can construct a commutative diagran of non-unital $\mathbb{P}_{n+1}$-algebras

$$\begin{tikzcd} B[-1] \arrow[r] \arrow[d, equal] & U(A,B) \arrow[r] \arrow[d] & A \arrow[d] \\ B[-1] \arrow[r] & \mathrm{Def}(B)[-n] \arrow[r] & Z(B) \end{tikzcd}$$

where the square on the right is Cartesian, and both rows are fiber sequences of non-unital $\mathbb{P}_{n+1}$-algebras in M. Notice in particular that the connecting morphism of the first row is given by the composite $A \to Z(B) \to B$, which is exactly the underlying morphism in CAlg_M of the $\mathbb{P}_{[n+1,n]}$-algebra (A, B).

We can summarize the above discussion in the following proposition.

Proposition 3.4.4. *Let (A, B) be a $\mathbb{P}_{[n+1,n]}$-algebra in $\mathcal{M}$. Then the fiber $U(A, B)$ of the underlying morphism $A \to B$ in $\mathrm{CAlg}_{\mathcal{M}}$ has a natural structure of a non-unital $\mathbb{P}_{n+1}$-algebra, such that the map $U(A, B) \to A$ preserves such structure.*

One can get a similar result for graded Poisson algebras. More specifically, given a graded $\mathbb{P}_n$-algebra B, one can define a graded $\mathbb{P}_{n+1}$-algebra $Z^{gr}(B)$ (its graded center) and a graded non-unital $\mathbb{P}_{n+1}$-algebra $\mathrm{Def}^{gr}(B)$ (its graded deformation complex). Using the exact same arguments as before, we get the following proposition.

Proposition 3.4.5. *Let (A, B) be a graded $\mathbb{P}_{[n+1,n]}$-algebra in $\mathcal{M}$. Then the fiber $U^{gr}(A, B)$ of the underlying map $A \to B$ has a natural structure of a graded non-unital $\mathbb{P}_{n+1}$-algebra, such that the map $U^{gr}(A, B) \to A$ preserves such structure.*

3.4.2 Graded mixed Poisson algebras

Consider a graded $\mathbb{P}_n$-algebra A in $\mathcal{M}$. The purpose of this section is to introduce the notion of mixed structure on a graded Poisson algebra.

Definition 3.4.6. *With notations as above, the* space $\mathrm{Mix}_{\mathbb{P}_n}(A)$ of mixed structures on A *is the mapping space*

$$\mathrm{Mix}_{\mathbb{P}_n}(A) := \mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), \mathrm{Def}^{gr}(A)),$$

where $1_{\mathcal{M}}$ is the monoidal unit of $\mathcal{M}$, and Def^{gr} *is the graded deformation complex of Section 3.4.1.*

Remark. The space $\mathrm{Mix}_{\mathbb{P}_n}(A)$ can be thought as the space of all possible enhancements of A to a graded mixed $\mathbb{P}_n$-algebra. More specifically, there is a natural forgetful monoidal ∞-functor $\mathcal{M}^{gr,\epsilon} \to \mathcal{M}^{gr}$, which simply forgets the mixed structure. The space $\mathrm{Mix}_{\mathbb{P}_n}(A)$ is then equivalent to the underlying space of the fiber of the ∞-functor

$$\mathbb{P}_n - \mathrm{alg}_{\mathcal{M}^{gr,\epsilon}} \longrightarrow \mathbb{P}_n - \mathrm{alg}_{\mathcal{M}^{gr}}$$

taken at A. As always for our conventions, here the bracket in the graded operad $\mathbb{P}_n$ has weight -1. However, we will not need this alternative characterization of mixed structures in the remainder of the thesis.

Notice that given a graded $\mathbb{P}_n$-algebra A, the results of Section 3.4.1 give us a simple recipe to produce mixed structures on A.

Proposition 3.4.7. *Let $A \in \mathbb{P}_n - \mathrm{alg}^{gr}_{\mathcal{M}}$ be a graded $\mathbb{P}_n$-algebra. There is a natural morphism of spaces*

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), A[-n+1]) \longrightarrow \mathrm{Mix}_{\mathbb{P}_n}(A).$$

Proof. This follows directly from the definition of the space $\mathrm{Mix}_{\mathbb{P}_n}(A)$ and from the existence of a natural morphism $A[-n+1] \to \mathrm{Def}^{gr}(A)$ inside the ∞-category $\mathrm{Lie}^{gr}_{\mathcal{M}}$. $\square$

In particular, given a mixed structure on a graded $\mathbb{P}_n$-algebra A, its graded center $Z^{gr}(A)$ also inherits a mixed structure. In fact, there is a natural morphism $\mathrm{Def}^{gr}(A) \to Z^{gr}(A)$ inside $\mathbb{P}_{n+1} - \mathrm{alg}^{gr}_{\mathcal{M}}$, which by the above proposition gives rise to a natural map

$$\mathrm{Mix}_{\mathbb{P}_n}(A) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{n+1}}(Z^{gr}(A))$$

in the ∞-category of spaces.

Remark. If one interprets mixed structures as in the previous remark, then the existence of the induced mixed structure on $Z(A)^{gr}$ is just a formal consequence of the fact that A is an algebra inside $\mathcal{M}^{gr,\epsilon}$, and therefore its internal center still lives in $\mathcal{M}^{gr,\epsilon}$.

For the coloured case of a graded $\mathbb{P}_{[n+1,n]}$-algebra, we can now give a similar definition of mixed structures, which will be used later on.

Definition 3.4.8. *Let (A, B) be a graded $\mathbb{P}_{[n+1,n]}$-algebra in $\mathcal{M}$. The* space $\mathrm{Mix}_{\mathbb{P}_{[n+1,n]}}(A, B)$ *of* mixed structures on (A, B) *is defined to be the pullback of the following diagram in the ∞-category of spaces*

$$\begin{tikzcd}
\mathrm{Mix}_{\mathbb{P}_{[n+1,n]}}(A,B) \arrow[r] \arrow[d] & \mathrm{Mix}_{\mathbb{P}_n}(B) \arrow[d] \\
\mathrm{Mix}_{\mathbb{P}_{n+1}}(A \to Z^{gr}(B)) \arrow[r] & \mathrm{Mix}_{\mathbb{P}_{n+1}}(Z^{gr}(B))
\end{tikzcd}$$

where $A \to Z^{gr}(B)$ is treated as a graded $\mathbb{P}_{n+1}$-algebra in the category $\mathrm{Mor}(\mathcal{M})$ *of morphisms of* $\mathcal{M}$.

3.4.3 Relative polyvectors

In this subsection we define the complex of relative polyvector fields for derived affine schemes. This will give an alternative way to define coisotropic structures. Given a morphism of commutative dg algebras $f: A \to B$, we have a natural fiber sequence

$$\mathbb{T}_{B/A} \longrightarrow \mathbb{T}_B \longrightarrow \mathbb{T}_A \otimes_A B.$$

Interpreted in ordinary geometry, this suggests that $\mathbb{T}_{B/A}[1]$ can be thought as a derived generalization of the normal bundle of the map f. In particular, for any n we get a map of B-modules

$$\mathbb{T}_A \otimes_A B[-n-1] \longrightarrow \mathbb{T}_{B/A}[n],$$

which induces a map of B-algebras

$$\mathrm{Sym}_B(\mathbb{T}_A \otimes_A B[-n-1]) \longrightarrow \mathrm{Sym}_B(\mathbb{T}_{B/A}[-n]).$$

By precomposing with the natural map $\mathrm{Sym}_A(\mathbb{T}_A[-n-1]) \to \mathrm{Sym}_A(\mathbb{T}[-n-1]) \otimes_A B$, we end up with a canonical morphism of A-algebras

$$\mathrm{Sym}_A(\mathbb{T}_A[-n-1]) \longrightarrow \mathrm{Sym}_B(\mathbb{T}_{B/A}[-n]).$$

Definition 3.4.9. *Let $f : A \to B$ be a morphism of commutative dg algebras. The* complex of relative n-shifted polyvectors $\mathrm{Pol}(f, n)$ *is the homotopy fiber of the map*

$$\mathrm{Pol}(A, n) \longrightarrow \mathrm{Pol}(B/A, n-1)$$

constructed above.

Remark that in particular a model for $\mathrm{Pol}(f, n+1)$ is

$$\mathrm{Pol}(A, n+1) \oplus \mathrm{Pol}(B/A, n)[-1],$$

where the differential is given by the two internal differentials and by the morphism $\mathrm{Pol}(A, n+1) \to \mathrm{Pol}(B/A, n)$.

Our next goal is to enhance the diagram of graded modules

$$\begin{tikzcd}
 & \mathrm{Pol}(f, n+1)[n+1] \arrow[dl] \arrow[dr] & \\
\mathrm{Pol}(A, n+1)[n+1] & & \mathrm{Pol}(B, n)[n].
\end{tikzcd}$$

to a diagram of graded L_∞-algebras. The first problem is of course to endow $\mathrm{Pol}(f, n+1)[n+1]$ with a L_∞-structure. Notice that this is not trivial, since there is for example no easy way to express $\mathrm{Pol}(f, n+1)[n+1]$ as a limit of a diagram of L_∞-algebras.

The L_∞-structure appear however quite naturally if we adopt a slightly different approach to relative polyvectors.

Consider again a morphism of dg modules $f\colon A \to B$. Consider the graded dg module

$$\begin{aligned}
\mathcal{L} \ := \ & \mathrm{Hom}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}(A), A) \\
& \bigoplus \mathrm{Hom}(\mathrm{co}\mathbb{P}_n\{n\}(B), B) \\
& \bigoplus \mathrm{Hom}(\mathrm{co}\mathbb{P}^{nu}_{n+1}\{n+1\}(A) \otimes \mathrm{co}\mathbb{P}_n\{n\}(B), B)[-n-1],
\end{aligned}$$

where the differential is simply given by the internal differentials.

Note that both $\mathrm{co}\mathbb{P}_{n+1}\{n+1\}(A)$ and $\mathrm{co}\mathbb{P}_n\{n\}(B)$ are given by a symmetric coalgebra and we use the natural grading on the symmetric coalgebra to induce a grading on $\mathcal{L}$. Moreover, we already know from the previous sections that there is a natural L_∞-structure on $\mathcal{L}$: explicitly, the L_∞ brackets on $\mathcal{L}$ are defined in the same way as the brackets on $\mathcal{L}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_n; A, B)$.

If A and B are given commutative structures and f becomes a morphism of commutative dg algebras, then this specifies a Maurer-Cartan element in $\mathcal{L}$, which in turn defines a twisted differential on $\mathcal{L}$. We define $\mathrm{Pol}'(f, n+1)[n+1]$ the graded L_∞ algebra obtained by twisting $\mathcal{L}$ by this Maurer-Cartan element.

Let us explicitly describe the twisted differential on $\mathrm{Pol}'(f,n+1)[n+1]$. The Maurer–Cartan element defined by $f\colon A\to B$ has three components: the multiplications m_A, m_B on A and B respectively and the map $A\to B$. The differentials on the first two terms are simply given by twisting the convolution algebras by m_A and m_B respectively. The differential on the last term has the following components:

- Pre-composition with m_A.
- Pre-composition and post-composition with m_B.
- The morphism f gives a morphism $\mathrm{coLie}(A[1])\to\mathrm{coLie}(B[1])$. This gives a differential on $\mathrm{Sym}(\mathrm{coLie}(A[1])[n]\oplus\mathrm{coLie}(B[1])[n-1])$ and thus a differential on the last term of $\mathrm{Pol}(f,n+1)[n+1]$.

The element $m_A\in\mathcal{L}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\};A)$ has Lie brackets with the first term and the last term and has no higher operations. This induces the usual twisted differential

Proposition 3.4.10. *Let $f\colon A\to B$ be a morphism of commutative dg algebras. One has an equivalence*

$$\mathrm{Pol}'(f,n+1)[n+1]\simeq\mathrm{Pol}(f,n+1)[n+1]$$

of graded dg modules.

Proof. Recall that we have a morphism

$$A\otimes\mathrm{coLie}(A[1])[-1]\to\Omega^1_A$$

given in weight 1 by $f\otimes g\mapsto f\mathrm{d}_{\mathrm{dR}}g$.

This induces a morphism

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]),A)[n+1]&\to\mathrm{Hom}_A(A\otimes\mathrm{Sym}(\mathrm{coLie}(A[1])[n]),A)[n+1]\\&\cong\mathrm{Hom}(\mathrm{Sym}(\mathrm{coLie}(A[1])[n]),A)[n+1].\end{aligned}$$

Combining three such morphisms, we get a morphism from

$$\mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]),A)[n+1]\oplus\mathrm{Hom}_B(\mathrm{Sym}_B(\Omega^1_B[n]\oplus f^*\Omega^1_A[n+1]),B)[n]$$

to $\mathrm{Pol}(f,n)[n+1]$.

Introduce the differentials $f^*\Omega^1_A[n+1]\to\Omega^1_B[n]$ given by the pullback of differential forms and

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]),A)\to&\mathrm{Hom}_B(\mathrm{Sym}_B(f^*\Omega^1_A[n+1]),B)[-1])\\\cong&\mathrm{Hom}_A(\mathrm{Sym}_A(\Omega^1_A[n+1]),B)[-1]\end{aligned}$$

given by the post-composition $A\to B$.

Using the explicit description of the differential on $\mathrm{Pol}(f,n+1)[n+1]$ a straightforward check shows that the morphism is compatible with the differentials.

Now suppose A is cofibrant and $A\to B$ is a cofibration. In particular, B is cofibrant. Then each of the morphisms into the Harrison complex are quasi-isomorphisms by Lemma 3.1.7. Since $A\to B$ is a cofibration, the composite

$$\Omega^1_{B/A}\to f^*\Omega^1_A[-1]\to\Omega^1_B\oplus f^*\Omega^1_A[-1]$$

is a quasi-isomorphism (where $\Omega^1_B\oplus f^*\Omega^1_A[-1]$ is considered with the additional differential discussed above), and the claim follows. ∎

The above proposition assures that $\mathrm{Pol}(f,n+1)[n+1]$ does in fact have a L_∞-structure. This is essential to our purposes, since this Lie structure allows us to make sense of the following statement, which is the fundamental result of this chapter.

Theorem 3.4.11. *Given a morphism of commutative dg algebras $f\colon A \to B$ we have an equivalence of spaces*

$$\mathrm{Cois}(f,n) \cong \mathrm{Map}_{\mathrm{L}_\infty^{\mathrm{gr}}}(k(2)[-1], \mathrm{Pol}(f,n+1)[n+1]).$$

Proof. We begin with the following basic observation. We have a homotopy Cartesian diagram

$$\begin{tikzcd}
\mathrm{Map}(\mathbb{P}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathbb{P}^{nu}_{[n+1,n]}, \mathrm{End}_{A,B}) \arrow[d] \\
\mathrm{Map}(\mathrm{Comm} \otimes \Delta^1, \mathrm{End}_{A,B}) \arrow[r] & \mathrm{Map}(\mathrm{Comm}^{nu} \otimes \Delta^1, \mathrm{End}_{A,B}).
\end{tikzcd}$$

Therefore, the homotopy fibers of the two vertical maps are equivalent.

By Proposition 3.2.5 the space $\mathrm{Map}(\mathrm{Comm}^{nu} \otimes \Delta^1, \mathrm{End}_{A,B})$ is equivalent to the space of Maurer–Cartan elements in the L_∞ algebra

$$\begin{aligned}\mathcal{L}(\mathrm{coLie}\{1\}; A, B) \cong &\mathrm{Hom}(\mathrm{coLie}_\circ(A[1]), A)[1] \\ &\oplus \mathrm{Hom}(\mathrm{coLie}_\circ(B[1]), B)[1] \\ &\oplus \mathrm{Hom}(\mathrm{coLie}(A[1]), B).\end{aligned}$$

By Proposition 3.3.3 the space $\mathrm{Map}(\mathbb{P}^{nu}_{[n+1,n]}, \mathrm{End}_{A,B})$ is equivalent to the space of Maurer–Cartan elements in the L_∞ algebra

$$\begin{aligned}\mathcal{L}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_n; A, B) \cong &\mathrm{Hom}(\mathrm{co}\mathbb{P}_{n+1,\circ}\{n+1\}(A), A) \\ &\oplus \mathrm{Hom}(\mathrm{co}\mathbb{P}_{n,\circ}\{n\}(B), B) \\ &\oplus \mathrm{Hom}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}(A) \otimes \mathrm{co}\mathbb{P}^{un}_n\{n\}(B), B)[-n-1].\end{aligned}$$

The morphism $\mathrm{Map}(\mathbb{P}^{nu}_{[n+1,n]}, \mathrm{End}_{A,B}) \to \mathrm{Map}(\mathrm{Comm}^{nu} \otimes \Delta^1, \mathrm{End}_{A,B})$ is induced by the morphism of L_∞ algebras

$$p\colon \mathcal{L}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_n; A, B) \to \mathcal{L}(\mathrm{coLie}\{1\}; A, B)$$

which is given by the inclusion of the space of cogenerators in the reduced symmetric coalgebra. We also have an inclusion

$$i\colon \mathcal{L}(\mathrm{coLie}\{1\}; A, B) \to \mathcal{L}(\mathrm{co}\mathbb{P}_{n+1}\{n+1\}, \mathrm{co}\mathbb{P}_n; A, B)$$

given by projecting the symmetric coalgebras to the space of cogenerators. Therefore, by Lemma 3.1.1 the homotopy fiber of

$$\mathrm{Map}(\mathbb{P}^{nu}_{[n+1,n]}, \mathrm{End}_{A,B}) \to \mathrm{Map}(\mathrm{Comm}^{nu} \otimes \Delta^1, \mathrm{End}_{A,B})$$

is equivalent to the space of Maurer–Cartan elements of

$$\begin{aligned}\mathrm{Pol}(f,n+1)[n+1]^{\geq 2} = &\mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}(A[1])[n]), A)[n+1] \\ &\oplus \mathrm{Hom}(\mathrm{Sym}^{\geq 2}(\mathrm{coLie}(A[1])[n] \oplus \mathrm{coLie}(B[1])[n-1]), B)[n].\end{aligned}$$

The claim then follows from Lemma 3.1.3. $\square$

It is obvious from construction that the natural projections induce a correspondence of graded L_∞ algebras

$$\begin{tikzcd} & \mathrm{Pol}(f, n+1)[n+1] \arrow[dl] \arrow[dr] & \\ \mathrm{Pol}(A, n+1)[n+1] & & \mathrm{Pol}(B, n)[n] \end{tikzcd}$$

Moreover, the relative polyvectors algebra fits in a fiber sequence of L_∞-algebra

$$\mathrm{Pol}(B/A, n)[n] \longrightarrow \mathrm{Pol}(f, n+1)[n+1] \longrightarrow \mathrm{Pol}(A, n+1)[n+1].$$

Notice that an n-shifted Poisson structure on A is given by a Maurer-Cartan element π_A in $\mathrm{Pol}(A, n+1)[n+1]$; by the above discussion, we can interpret a coisotropic structure on a map $A \to B$ as a lift of π_A to a Maurer-Cartan element in $\mathrm{Pol}(f, n+1)[n+1]$.

Chapter 4

Coisotropic structures on derived stacks

The purpose of this chapter is to explain how to generalize Definition 3.4.3 to the case of morphism between derived stacks.

In the first section we give, as in the affine case, two possible definitions of coisotropic structures on a morphism of derived Artin stacks, proving that they are equivalent. We list some examples of derived coisotropic structures, both generalizing classical constructions and exposing intrinsically derived phenomena.

In the second section we prove a general existence theorem for Poisson structures, which is parallel to the Lagrangian intersection theorem of [PTVV]. Given an n-shifted Poisson stack X and two coisotropic structures on a couple of morphisms $L_1 \to X$ and $L_2 \to X$, we show that the derived intersection $L_{12} = L_1 \times_X L_2$ naturally carries a $(n-1)$-shifted Poisson structure, which is in general non-trivial. This generalizes previous (homological, 0-shifted) results of Baranovsky and Ginzburg in [BG], as well as the affine case worked out of [Sa].

The third and final section deals with the comparison with shifted Lagrangian structures, as defined in [PTVV]. We give a definition of what it means for a coisotropic structures to be *non-degenerate*, and show that the space of non-degenerate coisotropic structures on a map f is equivalent to the space of Lagrangian structures on f. This generalize Theorem 3.2.5 of [CPTVV], solving Conjecture 3.4.5 in [CPTVV] and Conjecture 1.1 in [JS].

4.1 Coisotropic structures

Notice that Definition 2.2.2 allows us to work in a almost purely algebraic context, since a Poisson structure is some additional algebraic structure on a prestack. This gives in particular the possibility to deal with coisotropic structures on general derived stacks using ideas from [Sa].

More specifically, let $\mathcal{M}$ be an ∞-category satisfying our starting assumption of chapter 0. Let $\mathbb{P}_{[n+1,n]}$ be the coloured operad whose algebras are pairs of objects (A, B) in $\mathcal{M}$ together with the following additional structure:

- a $\mathbb{P}_{n+1}$-structure on A;
- a $\mathbb{P}_n$-structure on B;
- a morphism of $\mathbb{P}_{n+1}$-algebras $A \to Z(B)$, where $Z(B)$ is again the Poisson center of B, considered with its natural structure of $\mathbb{P}_{n+1}$-algebra.

We refer to the previous chapter for more details on this operad. Since there is a canonical morphism of commutative algebras $Z(B) \to B$, we get a natural forgetful functor

$$\phi : \mathbb{P}_{[n+1,n]} - \mathrm{alg}_{\mathcal{M}} \to \mathrm{Mor}(\mathrm{CAlg}_{\mathcal{M}})$$

to the category of morphisms of commutative algebras, sending a $\mathbb{P}_{[n+1,n]}$-algebra (A, B) to the underlying map $A \to B$. We can use this forgetful functor in order to study $\mathbb{P}_{[n+1,n]}$-structures over a fixed map $A \to B$.

Definition 4.1.1. *Let $f : A \to B$ be a map of commutative algebra objects in the ∞-category $\mathcal{M}$. The* space of $\mathbb{P}_{[n+1,n]}$-structures on f *is the fiber of the forgetful functor ϕ, taken at f. It will be denoted $\mathbb{P}_{[n+1,n]}(f)$.*

We are now ready to give the general definition of coisotropic structure on a map of derived stack. Let $f : L \to X$ be a map of derived Artin stack locally of finite presentation. The map f descends to a map between the de Rham stacks $f_{DR} : L_{DR} \to X_{DR}$, which in turn induces a pullback functor (simply denoted f^*, with a slight abuse of notation) from prestacks on X_{DR} to prestacks on L_{DR}. By definition of $\mathbb{D}_{X_{DR}}$, one gets immediately an equivalence $\mathbb{D}_{L_{DR}} \cong f^*\mathbb{D}_{X_{DR}}$. As for the sheaves of principal parts, f induces a natural map

$$f^*(\mathcal{P}_X) \to \mathcal{P}_L$$

preserving the $\mathbb{D}_{L_{DR}}$-linear structures. It follows that it exists an induced morphism

$$f^*_{\mathcal{P}} : f^*(\mathcal{P}_X(\infty)) \to \mathcal{P}_L(\infty)$$

of $\mathbb{D}_{L_{DR}}(\infty)$-algebras. Now suppose that X is endowed with a n-shifted Poisson structure. This corresponds to a $\mathbb{P}_{n+1}$-structure on $\mathcal{P}_X(\infty)$, so that $f^*(\mathcal{P}_X(\infty))$ becomes a $\mathbb{D}_{L_{DR}}(\infty)$-linear $\mathbb{P}_{n+1}$-algebra.

Definition 4.1.2. *Let $L \to X$ be a map of derived Artin stacks locally of finite presentation, and suppose X is endowed with a n-shifted Poisson structure. Let $\mathbb{P}_{n+1}(f^*_{\mathcal{P}}(\mathcal{P}_X(\infty)))$ be the space of $\mathbb{P}_{n+1}$-structures on $f^*_{\mathcal{P}}(\mathcal{P}_X(\infty))$ that are compatible with the natural commutative algebra structure. Then as above the n-shifted Poisson structure on X gives a point p in $\mathbb{P}_{n+1}(f^*_{\mathcal{P}}(\mathcal{P}_X(\infty)))$. We define the* space of n-shifted coisotropic structures on f *(denoted* $\mathrm{Cois}(f, n)$*) to be the fiber of the forgetful functor*

$$\mathbb{P}_{[n+1,n]}(f^*_{\mathcal{P}}) \longrightarrow \mathbb{P}_{n+1}(f^*_{\mathcal{P}}(\mathcal{P}_X(\infty)))$$

taken at the point p.

Notice that this is just a reformulation of Definition 3.4.3, where we have replaced the category of cochain complexes with the more complicated category of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

Remark. Our definition is in the same spirit as that given in [CPTVV], Section 3.4. The difference is that the authors used a different operad, that they denoted $\mathbb{P}_{(n+1,n)}$, instead of our $\mathbb{P}_{[n+1,n]}$. We strongly believe that the two definitions coincide, since for every $\mathbb{P}_n$-algebra B there should be an equivalence of $\mathbb{P}_{n+1}$-algebras between $Z(B)$ and $\mathrm{End}_{\mathbb{P}_n}(B)$. Notice however that even just proving that $\mathrm{End}_{\mathbb{P}_n}(B)$ is a $\mathbb{P}_{n+1}$-algebra is strictly related to the additivity conjecture for the Poisson operad, as stated for example in Section 3.4 in [CPTVV]. We refer to chapter 5 for more details on the comparison of the two definitions.

As an immediate consequence of the very definition of coisotropic structure, we get the following easy proposition, which links shifted Poisson structures on a stack with coisotropic structures on its natural projection to Spec$k$.

Proposition 4.1.3. *Let X be a derived Artin stack. Let $f : X \to (\mathrm{Spec}\ k, \pi_{n+1})$ be the natural map to the point, taken with its trivial $(n+1)$-shifted Poisson structure π_{n+1}. Then there is an equivalence*

$$\mathrm{Cois}(f, n+1) \cong \mathrm{Pois}(X, n)\ .$$

Notice that the same proposition stays true for symplectic and Lagrangian structures (see for instance [Cal]).

4.1.1 Relative polyvectors for derived stacks

In this subsection we give an alternative definition of coisotropic structures on a morphism of derived Artin stacks, using the notion of relative polyvectors. The goal is to prove an analogue of Theorem 3.4.11 in the case of derived stacks.

For every map of derived stack $f : L \to X$, the map we introduced in the previous subsection

$$f^*\mathcal{P}_X \to \mathcal{P}_L$$

induces a morphism between the Tate polyvectors

$$\mathrm{Pol}^t(f^*\mathcal{P}_X/\mathbb{D}_{L_{DR}}, n+1) \longrightarrow \mathrm{Pol}^t(\mathcal{P}_L/f^*\mathcal{P}_X, n)$$

whose fiber at zero will be denoted by $\mathrm{Pol}^t(f, n+1)$, with a slight abuse of notation. Finally, let

$$\mathrm{Pol}(f, n+1) \cong \Gamma(L_{DR}, \mathrm{Pol}^t(f, n+1))\ .$$

We can use $\mathrm{Pol}(f, n)$ to give an alternative definition of coisotropic structure, which is maybe closer to the classical intuition about coisotropic submanifolds.

Definition 4.1.4. *Let $f : L \to X$ be a map of derived Artin stack, locally of finite presentation. Suppose that X is endowed with a n-shifted Poisson structure π. Then the space $\mathrm{Cois}'(f, n)$ of n-shifted coisotropic structures on f is the fiber of*

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \Gamma(L_{DR}, \mathrm{Pol}^t(f^*\mathcal{P}_X, n+1)[n+1]))$$

taken at the point corresponding to π.

Let us immediately address the question of the equivalence of the two definitions of coisotropic structures.

Theorem 4.1.5. *Let again $f : L \to X$ be a map of derived Artin stacks, locally of finite presentation. Suppose moreover that X has a given n-shifted Poisson structure $\pi \in \mathrm{Pois}(X, n)$. Then the space of coisotropic structures on f in the sense of Definition 4.1.2 is equivalent to the space of coisotropic structures in the sense of Definition 4.1.4. In other words, there is a natural equivalnce*

$$\mathrm{Cois}(f, n) \simeq \mathrm{Cois}'(f, n)$$

in the ∞-category of spaces.

Proof. Let M be the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules. Recall from the previous subsection that there is a morphism of algebras in $\mathcal{M}$

$$f^*_{\mathcal{P}}: \ f^*\mathcal{P}_X(\infty) \to \mathcal{P}_L(\infty) \ \ .$$

It follows from Theorem 3.4.11 applied in M that

$$\mathbb{P}_{[n+1,n]}(f^*_{\mathcal{P}}) \cong \mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), \mathrm{Pol}^{int}(f^*_{\mathcal{P}}, n+1)[n+1]) \ \ ,$$

where $1_{\mathcal{M}}$ is the monoidal unit of $\mathcal{M}$, and $\mathrm{Pol}^{int}(f^*_{\mathcal{P}}, n)$ is the fiber at zero of the map in $\mathcal{M}$

$$\mathrm{Pol}^{int}(f^*\mathcal{P}_X(\infty), n+1) \longrightarrow \mathrm{Pol}^{int}(\mathcal{P}_L(\infty)/f^*\mathcal{P}_X(\infty), n) \ .$$

But just as in the proof of Theorem 3.1.2 in [CPTVV], we also have

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2), \mathrm{Pol}^{int}(f^*_{\mathcal{P}}, n+1)[n+1]) \cong \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), |\mathrm{Pol}^{int}(f^*_{\mathcal{P}}, n+1)[n+1]|) \ \ ,$$

where the functor $|\cdot| : \mathrm{Lie}^{gr}_{\mathcal{M}} \to \mathrm{dgla}^{gr}$ is the realization functor of [CPTVV]. Since this functor is by definition a right adjoint, it commutes with limits, so that $|\mathrm{Pol}^{int}(f^*_{\mathcal{P}}, n)|$ is still the homotopy fiber of the induced map

$$|\mathrm{Pol}^{int}(f^*\mathcal{P}_X(\infty), n+1)| \longrightarrow |\mathrm{Pol}^{int}(\mathcal{P}_L(\infty)/f^*\mathcal{P}_X(\infty), n)| \ .$$

From the general properties between Tate realizations and twists by $k(\infty)$, this is exactly the map

$$\Gamma(L_{DR}, \mathrm{Pol}^t(f^*\mathcal{P}_X, n+1)) \longrightarrow \Gamma(L_{DR}, \mathrm{Pol}^t(\mathcal{P}_L/f^*\mathcal{P}_X, n)) \ ,$$

so that by Theorem 3.4.11 its fiber $\Gamma(L_{DR}, \mathrm{Pol}^t(f, n+1)) \cong \mathrm{Pol}(f, n+1)$ inherits a natural structure of graded $(n+1)$-shifted dg Lie algebra. Combing Definition 4.1.2 and Theorem 2.2.3 we get precisely that $\mathrm{Cois}(f, n)$ is equivalent to the fiber of

$$\mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \mathrm{Pol}(f, n+1)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \Gamma(L_{DR}, \mathrm{Pol}^t(f^*\mathcal{P}_X, n+1))[n+1])$$

taken at the point induced by the structure

$$\pi \in \mathrm{Map}_{\mathrm{dgLie}^{gr}}(k[-1](2), \Gamma(X_{DR}, \mathrm{Pol}^t(\mathcal{P}_X, n+1))[n+1])$$

which is what we wanted. □

The alternative characterization of coisotropic structures given by Theorem 4.1.5 is of more geometric nature than Definition 4.1.2. This perhaps helps understanding why this definition is a sensible generalization of the classical notion, as explained in the following examples.

4.1.2 Examples

1. **Smooth schemes.** Let L be a smooth subscheme of a smooth scheme X, and let $f : L \to X$ the corresponding immersion. Suppose X is endowed with a classical Poisson structure π. In our language, this is the same as saying that $\pi \in \mathrm{Pois}(X, 0)$. The shifted polyvectors $\mathrm{Pol}(X, n)$ are just $\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{T}_X[-n])$, and the relative polyvectors $\mathrm{Pol}(f, n)$ coincide with the kernel of the morphism of graded dg modules

$$\mathrm{Pol}(X, n) \longrightarrow \mathrm{Sym}_{\mathcal{O}_L}(\mathbb{T}_{L/X}[-n+1])$$

so that a lift of the 0-shifted Poisson on X to a coisotropic structure on f gives no additional structure, as is the case for general derived objects, but rather just a condition on π: in particular, $\mathrm{Cois}(f)$ is either empty or contractible. Concretely, the $\mathcal{O}_L$-module $\mathbb{T}_{L/X}[1]$ is the algebraic incarnation of the normal bundle $N_{L/X}$; it follows that f has a coisotropic structure if and only if the projection of π in $\Lambda^2_{\mathcal{O}_L}(N_{L/X})$ is zero, that is to say if and only if L is a coisotropic subscheme of X in the classical sense.

2. **Identity.** Let X be a derived Artin stack locally of finite presentation and consider the identity morphism $\mathrm{id}\colon X \to X$. The projection $\mathrm{Pol}(\mathrm{id}, n) \to \mathrm{Pol}(X, n)$ is a quasi-isomorphism for any n since $\mathbb{T}_{X/X} = 0$. Therefore, the natural projection

$$\mathrm{Cois}(\mathrm{id}, n) \to \mathrm{Pois}(X, n)$$

is a weak equivalence, i.e. the identity morphism has a unique coisotropic structure for any n-shifted Poisson structure on X.

An interesting consequence of this statement is that we obtain a forgetful map $\mathrm{Pois}(X, n) \to \mathrm{Pois}(X, n-1)$ given as the composite

$$\mathrm{Pois}(X, n) \cong \mathrm{Cois}(\mathrm{id}, n) \to \mathrm{Pois}(X, n-1).$$

Notice that an algebraic level, this is probably closely related to the decomposition functor appearing in the additivity conjecture. We refer to chapter 5 for a more detailed discussion.

3. **Point.** Let X be a derived Artin stack locally of finite presentation and consider the projection $p\colon X \to \mathrm{pt}$. The projection $\mathrm{Pol}(p, n+1)[n+1] \to \mathrm{Pol}(X, n)[n]$ is a quasi-isomorphism since $\mathbb{T}_{\mathrm{pt}} = 0$ and $\mathbb{T}_{X/\mathrm{pt}} \cong \mathbb{T}_X$. Therefore, the natural morphism

$$\mathrm{Cois}(p, n) \to \mathrm{Pois}(X, n-1)$$

is a weak equivalence. This gives another proof of Proposition 4.1.3.

4. **Poisson maps.** Let X, Y be derived Artin stacks locally of finite presentation, and suppose they are given n-shifted Poisson structure π_X and π_Y. Let also $f : X \to Y$ be a morphism of stacks. The *space of n-shifted Poisson map structures on f* is the space $\mathrm{Pois}(f, n)$ of lifts of the induced morphism of $\mathbb{D}_{X_{DR}}(\infty)$-algebras

$$f^*_{\mathcal{P}} : f^*\mathcal{P}_Y(\infty) \to \mathcal{P}_X(\infty)$$

to a morphism of $\mathbb{D}_{X_{DR}}(\infty)$-linear $\mathbb{P}_{n+1}$-algebras.

Proposition 4.1.6. *With notations as above, let $g : X \to X \times Y$ the graph of f. There is a canonical equivalence of spaces*

$$\mathrm{Pois}(f, n) \cong \mathrm{Cois}(g, n),$$

where $X \times Y$ is considered as an n-shifted Poisson stacks with the structure $(\pi_X - \pi_Y)$.

This result will be a direct consequence of the following analogous statement for affine derived stacks.

Proposition 4.1.7. *Let $f : A \to B$ be a map of dg algebras, and suppose A and B are given $\mathbb{P}_{n+1}$-structures π_A and π_B. Let $\mathrm{Pois}(f, n)$ be the space of lifts of f to a morphism of $\mathbb{P}_{n+1}$-algebras, and denote with $g : A \otimes B \to B$ be the map induced by f and from the identity on B. Then there is a canonical equivalence of spaces*

$$\mathrm{Pois}(f, n) \cong \mathrm{Cois}(g, n),$$

where $A \otimes B$ is considered with the $\mathbb{P}_{n+1}$-structure $\pi_A - \pi_B$.

Proof. There is a natural fiber sequence of dg modules

$$\mathcal{L}(A, B) \to \mathrm{Pol}(A, n+1)[n+1] \oplus \mathrm{Pol}(B, n+1)[n+1] \to \mathrm{Sym}_B(\mathbb{T}_A \otimes_A B[-n-1])[n+1]$$

where $\mathcal{L}(A, B)$ is the cylinder Lie algebra of Proposition 3.2.5, which controls lifts of f to a $\mathbb{P}_{n+1}$-map. This means that $\mathrm{Pois}(f, n)$ is equivalent to the space of dotted arrows making the following diagram of Lie algebras

$$\begin{tikzcd} & k[-1](2) \arrow[dl, dotted] \arrow[d, "(\pi_A, \pi_B)"] \\ \mathcal{L}(A, B) \arrow[r] & \mathrm{Pol}(A, n+1)[n+1] \oplus \mathrm{Pol}(B, n+1)[n+1] \end{tikzcd}$$

commute.

On the other hand, coisotropic structures on g are controlled by the Lie algebra $\mathrm{Pol}(g, n+1)[n+1]$, which fits it the fiber sequence

$$\mathrm{Pol}(g+1, n+1)[n+1] \to \mathrm{Pol}(A \otimes B, n+1)[n+1] \to \mathrm{Sym}_B(\mathbb{T}_{B/A\otimes B}[-n])[n+1] .$$

Similarly, this means that $\mathrm{Cois}(g, n)$ is the space of dotted arrows making the following diagram of Lie algebras

$$\begin{tikzcd} & k[-1](2) \arrow[dl, dotted] \arrow[d, "\pi_A - \pi_B"] \\ \mathrm{Pol}(g, n)[n+1] \arrow[r] & \mathrm{Pol}(A \otimes B, n+1)[n+1] \end{tikzcd}$$

commute. Notice that the map $\pi_A - \pi_B$ above factors through the Lie algebra $\mathrm{Pol}(A, n+1)[n+1] \oplus \mathrm{Pol}(B, n+1)[n+1]$, and that the two cofibers $\mathrm{Sym}_B(\mathbb{T}_{B/A\otimes B}[-n])[n+1]$ and $\mathrm{Sym}_B(\mathbb{T}_A \otimes_A B[-n-1])[n+1]$ are naturally equivalent. In fact, consider the two algebra maps

$$B \longrightarrow A \otimes B \xrightarrow{g} B$$

whose composite is the identity by definition; then we get a natural equivalence $\mathbb{T}_{B/A\otimes B} \cong \mathbb{T}_{A\otimes B/B} \otimes_{A\otimes B} B[-1]$. But $\mathbb{T}_{A\otimes B/B} \cong \mathbb{T}_A \otimes_A (A \otimes B)$, so that we have $\mathbb{T}_{B/A\otimes B} \cong \mathbb{T}_A \otimes_A B[-1]$ as B-modules.

The equivalence of the cofibers implies in particular that the induced diagram of Lie algebras

$$\begin{tikzcd} \mathcal{L}(A, B) \arrow[r] \arrow[d] & \mathrm{Pol}(A, n+1)[n+1] \oplus \mathrm{Pol}(B, n+1)[n+1] \arrow[d] \\ \mathrm{Pol}(g, n+1)[n+1] \arrow[r] & \mathrm{Pol}(A \otimes B, n+1)[n+1] \end{tikzcd}$$

is cartesian, where the map on the right is the difference of the two natural maps from $\mathrm{Pol}(A, n+1)[n+1]$ and from $\mathrm{Pol}(B, n+1)[n+1]$. This means that lifting problems with respect to the map on the top are equivalent to lifting problems for the bottom map, which is exactly what we wanted. $\square$

We now use this result to prove the same statement for general Artin stacks.

Proof of Proposition 4.1.6. Consider the map of $\mathbb{D}_{X_{DR}}(\infty)$-modules

$$f^*_{\mathcal{P}} : f^*\mathcal{P}_Y(\infty) \to \mathcal{P}_X(\infty)$$

By Proposition 4.1.7, we know that $\mathrm{Pois}(f^*_{\mathcal{P}}, n) \cong \mathrm{Cois}(g^*_{\mathcal{P}}, n)$, where $g^*_{\mathcal{P}}$ is the induced map

$$g^*_{\mathcal{P}} : f^*\mathcal{P}_Y(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathcal{P}_X(\infty) \to \mathcal{P}_X(\infty)$$

so that it will suffice to prove that

$$f^*\mathcal{P}_Y(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathcal{P}_X(\infty) \cong g^*\mathcal{P}_{X\times Y}(\infty)$$

as $\mathbb{D}_{X_{DR}}(\infty)$-modules. This can be easily checked directly: for every affine A, given an A-point of X_{DR}, the value of $g^*\mathcal{P}_{X\times Y}(\infty)$ on A is by definition $\mathbb{D}((X \times Y)_A)(\infty)$, where $(X \times Y)_A$ is the fiber product

$$\begin{tikzcd}
(X \times Y)_A \arrow[r] \arrow[d] & X \arrow[r] & X \times Y \arrow[d] \\
\mathrm{Spec} A \arrow[r] & X_{DR} \arrow[r] & X_{DR} \times Y_{DR}
\end{tikzcd}$$

But $(X \times Y)_A$ is naturally equivalent to $X_A \times Y_A$, so that

$$\mathbb{D}((X \times Y)_A)(\infty) \cong \mathbb{D}(X_A)(\infty) \otimes_{\mathbb{D}_{X_{DR}}(\infty)} \mathbb{D}(Y_A)(\infty)$$

which concludes the proof.

$\square$

Notice that Proposition 4.1.6 immediately produces examples of coisotropic structures: for every n-Poisson derived Artin stack X, the map to $\mathrm{Spec} k$ is naturally a Poisson map, where $\mathrm{Spec} k$ is considered with its trivial n-Poisson structure. The graph of this map is the identity map on X, which therefore admits a canonical coisotropic structure, already constructed in example 2 above.

Alternatively, the identity map itself is of course a Poisson map. Its graph is the diagonal $X \to X \times X$, which then admits a canonical coisotropic structure.

4.2 Coisotropic intersections

In this section we state and prove our main theorem, which extends the Lagrangian intersection theorem (see [PTVV], Theorem 2.9) in the context of shifted Poisson structures. We start by recalling the following result, which is Theorem 1.9 in [Sa].

Theorem 4.2.1 (Safronov)**.** *Let A, B_1, B_2 be three commutative algebras in $\mathcal{M}$, and let $f : A \to B_1$ and $g : A \to B_2$ two algebra morphisms. Let $\pi \in \mathrm{Pois}(A, n)$ be a $\mathbb{P}_{n+1}$-structure on A, and let $\gamma_1 \in \mathrm{Cois}(f, n)$ and $\gamma_2 \in \mathrm{Cois}(g, n)$ be coisotropic structures on f and g respectively, in the sense of Definition 4.1.2. Then the (derived) tensor product $B_1 \otimes_A B_2$ carries a natural $\mathbb{P}_n$-structure such that the projection $B_1^{op} \otimes B_2 \to B_1 \otimes_A B_2$ is a Poisson morphism, where B_1^{op} is the algebra B_1 taken with opposite Poisson structure.*

Notice that this theorem recovers in particular the constructions in [BG] for affine schemes. More generally, derived algebraic geometry provides a suited general context to interpret the results of Baranovsky and Ginzburg: we will extend Theorem 4.2.1 for general derived stacks, giving a general conceptual explanation for the Gerstenhaber algebra structure constructed in [BG]. Concretely, we will prove the following result.

Theorem 4.2.2. *Let X, L_1 and L_2 be derived Artin stacks, locally of finite presentation, and let $\pi \in \mathrm{Pois}(X, n)$ be a n-shifted Poisson structure on X. Let $f : L_1 \to X$ and $g : L_2 \to X$ be morphisms of derived stacks, and let γ_1 and γ_2 be coisotropic structures on f and g respectively. The derived intersection $Y = L_1 \times_X L_2$ naturally carries a $(n-1)$-shifted Poisson structure, such that the map $Y \to L_1 \times L_2$ is a morphism of $(n-1)$-Poisson stacks, where L_1 is taken with the opposite Poisson structure.*

Proof. The cartesian diagram of stacks

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "i"] & L_1 \arrow[d, "f"] \\ L_2 \arrow[r, "g"] & X \end{tikzcd}$$

induces a commutative square of $\mathbb{D}_{Y_{DR}}(\infty)$-algebras

$$\begin{tikzcd} j^*f^*\mathcal{P}_X(\infty) \cong i^*g^*\mathcal{P}_X(\infty) \arrow[r] \arrow[d] & j^*\mathcal{P}_{L_1}(\infty) \arrow[d] \\ i^*\mathcal{P}_{L_2}(\infty) \arrow[r] & \mathcal{P}_Y(\infty) \end{tikzcd}$$

By definition, the two coisotropic structures γ_1 and γ_2 produce two $\mathbb{P}^n_{n+1}$-structures on the maps

$$j^*f^*\mathcal{P}_X(\infty) \to j^*\mathcal{P}_{L_1}(\infty) \quad \text{and} \quad i^*g^*\mathcal{P}_X(\infty) \to i^*\mathcal{P}_{L_2}(\infty)$$

so that by Theorem 4.2.1 we obtain a natural $\mathbb{P}_n$-structure on the coproduct

$$j^*\mathcal{P}_{L_1}(\infty) \otimes_{i^*g^*\mathcal{P}_X(\infty)} i^*\mathcal{P}_{L_2}(\infty) \ .$$

Our goal is now to show that this coproduct is actually equivalent to $\mathcal{P}_Y(\infty)$, which would immediately conclude the proof. Notice that the twist by $k(\infty)$ commutes with colimits, so that is enough to show that

$$j^*\mathcal{P}_{L_1} \otimes_{i^*g^*\mathcal{P}_X} i^*\mathcal{P}_{L_2} \cong \mathcal{P}_Y$$

as $\mathbb{D}_{Y_{DR}}$-algebras.

Let Spec$A \to Y_{DR}$ an A-point of Y_{DR}. We want to prove that $j^*\mathcal{P}_{L_1} \otimes_{i^*g^*\mathcal{P}_X} i^*\mathcal{P}_{L_2}$ and $\mathcal{P}_Y$ coincide on the point Spec$A \to Y_{DR}$. By definition, the value of $\mathcal{P}_Y$ on this point is $\mathbb{D}(Y_A)$, where Y_A is the perfect formal derived stack over Spec$A$ constructed as the fiber product

$$\begin{tikzcd} Y_A \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}A \arrow[r] & Y_{DR} \end{tikzcd}$$

Since the $(-)_{DR}$ construction is defined as a right adjoint, it automatically commutes with limits, so that $Y_{DR} \cong {L_1}_{DR} \times_{X_{DR}} {L_2}_{DR}$. In particular any A-point of Y_{DR} has corresponding A-points of ${L_1}_{DR}, {L_2}_{DR}$ and X_{DR}, for which one can define fibers ${L_1}_A, {L_2}_A$ and X_A. Therefore, we need to show that

$$\mathbb{D}(Y_A) \cong \mathbb{D}({L_1}_A) \otimes_{\mathbb{D}(X_A)} \mathbb{D}({L_2}_A)$$

as graded mixed dg algebras.

We start by remarking that the fiber square

$$\begin{tikzcd} Y_A \arrow[r] \arrow[d] & {L_1}_A \arrow[d] \\ {L_2}_A \arrow[r] & X_A \end{tikzcd}$$

induces a map of graded mixed dg algebras

$$\mathbb{D}({L_1}_A) \otimes_{\mathbb{D}(X_A)} \mathbb{D}({L_2}_A) \to \mathbb{D}(Y_A)$$

by universal property of the coproduct. In order to prove that this map is an equivalence, it is enough to check it at the level of algebras, forgetting the graded mixed structures. But the forgetful functor

$$\mathrm{CAlg}(\epsilon - \mathrm{dgMod}^{gr}) \to \mathrm{CAlg}(\mathrm{dgMod})$$

comes by definition from the forgetful functor

$$B - \mathrm{codg}_{C(k)} \to C(k)$$

where B is the Hopf algebra $B = k[t, t^{-1}] \otimes_k k[\epsilon]$ and $B - \mathrm{codg}_{C(k)}$ is the category of B-comodules in $C(k)$. This means in particular that forgetting the graded mixed structure preserves colimits, so that the underlying algebra of the pushout of

$$\begin{tikzcd} \mathbb{D}(X_A) \arrow[r] \arrow[d] & \mathbb{D}({L_1}_A) \\ \mathbb{D}({L_2}_A) & \end{tikzcd}$$

is exactly the tensor product of algebras $\mathbb{D}(L_{1_A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2_A})$.

Recall that following [CPTVV], a formal derived stack F (in the sense of Definition 0.4.1) is called *affine* if it satisfies the following two conditions:

- its reduced stack F_{red} is an affine derived scheme;
- F has a cotangent complex $\mathbb{L}_F$ (in the sense of [HAG-II, Section 1.4]), such that for every $B \in$ dAff and every map $u : \mathrm{Spec}B \to F$, the dg B-module $u^*\mathbb{L}_F$ is coherent and cohomologically bounded above.

Moreover, an affine formal derived stack F is called *algebraisable* if it is equivalent to the formal completion along a map $F_{red} \to G$, where G is an algebraic n-stack for some n.

Since X_A, L_{1_A}, L_{2_A} are all algebraisable, by applying Theorem 2.2.2 of [CPTVV] we have equivalences of algebras

$$\mathbb{D}(L_{1_A}) \otimes_{\mathbb{D}(X_A)} \mathbb{D}(L_{2_A}) \cong \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{1_A}}[-1]) \otimes_{\mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/X_A}[-1])} \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/L_{2_A}}[-1])$$

$$\mathbb{D}(Y_A) \cong \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/Y_A}[-1])$$

The result now follows directly from the following lemma.

Lemma 4.2.1. Consider the following diagram of derived stacks

$$\begin{tikzcd} K \arrow[r, "\phi"] & X \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[d, "g"] \\ & Z \arrow[r, "f"] & W \end{tikzcd}$$

where the right square is cartesian. Then the following diagram of $\mathcal{O}_K$-modules

$$\begin{tikzcd} \mathbb{T}_{K/X} \arrow[r] \arrow[d] & \mathbb{T}_{K/Y} \arrow[d] \\ \mathbb{T}_{K/Z} \arrow[r] & \mathbb{T}_{K/W} \end{tikzcd}$$

is cartesian.

Proof. From the diagram of stacks, one immediately gets two fiber sequences of $\mathcal{O}_K$-modules

$$\mathbb{T}_{K/Y} \longrightarrow \mathbb{T}_{K/W} \longrightarrow \phi^* i^* \mathbb{T}_{Y/W}$$

$$\mathbb{T}_{K/Z} \longrightarrow \mathbb{T}_{K/W} \longrightarrow \phi^* j^* \mathbb{T}_{Z/W}$$

and therefore the limit of

$$\begin{tikzcd} & \mathbb{T}_{K/Y} \arrow[d] \\ \mathbb{T}_{K/Z} \arrow[r] & \mathbb{T}_{K/W} \end{tikzcd}$$

is precisely the fiber of the map $\mathbb{T}_{K/W} \to \phi^* i^* \mathbb{T}_{Y/W} \oplus \phi^* j^* \mathbb{T}_{Z/W}$. But by general properties of cartesian squares, $\mathbb{T}_{X/W} \cong i^* \mathbb{T}_{Y/W} \oplus j^* \mathbb{T}_{Z/W}$, and hence $\phi^* \mathbb{T}_{X/W} \cong \phi^* i^* \mathbb{T}_{Y/W} \oplus \phi^* j^* \mathbb{T}_{Z/W}$. We now conclude by observing that the fiber of the map

$$\mathbb{T}_{K/W} \to \phi^* \mathbb{T}_{X/W}$$

is naturally identified with $\mathbb{T}_{K/X}$. □

We can now just apply the lemma to the diagram of algebraisable stacks

$$\begin{tikzcd} \mathrm{Spec}(A^{red}) \arrow[r] & Y_A \arrow[r] \arrow[d] & {L_1}_A \arrow[d] \\ & {L_2}_A \arrow[r] & X_A \end{tikzcd}$$

and get a cartesian square of A^{red}-modules

$$\begin{tikzcd} \mathbb{T}_{A^{red}/Y_A} \arrow[r] \arrow[d] & \mathbb{T}_{A^{red}/{L_1}_A} \arrow[d] \\ \mathbb{T}_{A^{red}/{L_2}_A} \arrow[r] & \mathbb{T}_{A^{red}/X_A} \end{tikzcd}$$

From this we deduce a pushout diagram of A^{red}-algebras

$$\begin{tikzcd} \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/X_A}[-1]) \arrow[r] \arrow[d] & \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/{L_1}_A}[-1]) \arrow[d] \\ \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/{L_2}_A}[-1]) \arrow[r] & \mathrm{Sym}_{A^{red}}(\mathbb{L}_{A^{red}/Y_A}[-1]) \end{tikzcd}$$

which is exactly what we wanted.

□

Remark. The argument in this section works in the same way if one wants to use the definition of coisotropic structures given in [CPTVV], provided one has a result similar to Theorem 4.2.1. Namely, one needs the following statement.

Proposition 4.2.3. *Let $f_1 : A \to B_1$ and $f_2 : A \to B_2$ be morphisms of cdgas, with A equipped with a $\mathbb{P}_{n+1}$-structure π. Suppose both f_1 and f_2 are endowed with a coisotropic structure relative to π, in the sense of [CPTVV], section 3.4. Then the intersection $B_1 \otimes_A B_2$ has a natural $\mathbb{P}_n$-structure, such that the map $B_1 \otimes_k B_2^{op} \to B_1 \otimes_A B_2$ is a map of $\mathbb{P}_n$-algebras.*

This will be an immediate corollary of the following slightly more general fact.

Proposition 4.2.4. *Let A, B and C three cdgas with $\mathbb{P}_{n+1}$-structures, and let $f_1 : A \otimes B^{op} \to L_1$ and $f_2 : B \otimes C^{op} \to L_2$ be cdgas map equipped with coisotropic structures (in the sense of [CPTVV], section 3.4). Let $L_{12} = L_1 \otimes_B L_2$. Then the map $A \otimes C^{op} \to L_{12}$ has a natural coisotropic structure.*

Proof. By definition of coisotropic structures as given in [CPTVV], L_1 is a left A-module and a right B-module in the monoidal category of $\mathbb{P}_n$-algebras, and similarly L_2 is a left B-module and a right C-module. From this we immediately see that $L_{12} \simeq L_1 \otimes_B L_2$ is a left A-module and a right C-module, which is exactly what we wanted. We conlude by noticing that the canonical morphism $L_1 \otimes_k L_2 \to L_{12}$ is a map of left A-modules and of right C-modules. $\square$

In particular, Theorem 4.2.2 is true for all definitions of derived coisotropic structures, that is to say Definition 4.1.2, Definition 4.1.4 and [CPTVV, Definition 3.4.4].

4.3 Non degenerate coisotropic structures

The purpose of this section is to introduce the notion of non degeneracy of a coisotropic structure. This is a relative version of non-degenerate Poisson structures, as treated in [CPTVV]. Our main result is a proof of conjecture 3.4.5 in [CPTVV], stating that the space of non-degenerate coisotropic structures is equivalent to the space of Lagrangian structures, in the sense of [PTVV]. We will follow the same strategy used in [CPTVV] to prove that non-degenerate Poisson structures are equivalent to symplectic structures.

4.3.1 Definition of non-degeneracy

We start by first looking at the affine case. Recall the following notion, which is taken from [CPTVV, Definition 1.4.18].

Definition 4.3.1. Given an algebra $A \in \mathbb{P}_{n+1} - \mathrm{alg}_{\mathcal{M}}$, we say that A is *non-degenerate* if the morphism

$$\mathrm{DR}^{int}(A) \longrightarrow \mathrm{Pol}^{int}(A, n)$$

induced by the Poisson bracket is an equivalence in $\mathcal{M}^{gr}$.

Let $f\colon A \to B$ a map inside $\mathrm{CAlg}_{\mathcal{M}}$. Using the results of [MS, Section 2.7] we know that there is a natural graded $\mathbb{P}_{[n+2,n+1]}$-structure on the couple

$$(\mathrm{Pol}(A, n+1), \mathrm{Pol}(B/A, n)),$$

such that the underlying morphism of graded commutative algebras is the map

$$\mathrm{Pol}(A, n+1) \longrightarrow \mathrm{Pol}(B/A, n)$$

induced by $\mathbb{L}_A \to \mathbb{L}_A \otimes_A B \to \mathbb{L}_B$.

Moreover, the fiber of the above morphism is $\mathrm{Pol}(f, n+1)$. Then we know from Section 3.4.1 that $\mathrm{Pol}(f, n+1)$ is in fact a graded $\mathbb{P}_{n+2}$-algebra. By definition, an n-shifted coisotropic structure on f is a morphism in the ∞-category of graded dg Lie algebras

$$k(2)[-1] \longrightarrow \mathrm{Pol}(f, n+1)[n+1].$$

Our next goal is to show that one can use such a dg Lie morphism to endow the algebra

$$(\mathrm{Pol}(A, n+1), \mathrm{Pol}(B/A, n))$$

with a mixed structure, in the sense of Section 3.4.2. In other words, we will show that there is a morphism of spaces

$$\mathrm{Cois}(f,n) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(\mathrm{Pol}(A,n+1),\mathrm{Pol}(B/A,n)).$$

This will immediately follow from the following slightly more general statement.

Proposition 4.3.1. *Let (R,S) be a graded $\mathbb{P}_{[n+2,n+1]}$-algebra in $\mathcal{M}$. As in Section 3.4.1, let $U^{gr}(R,S)$ be the fiber of the underlying morphism of graded commutative algebras $R \to S$. Then there is a canonical morphism of spaces*

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2),U^{gr}(R,S)[n+1]) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(R,S).$$

Proof. By Definition 3.4.8, the space of mixed structures on (R,S) fits in a Cartesian square of spaces

$$\begin{tikzcd}
\mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(R,S) \arrow[r] \arrow[d] & \mathrm{Mix}_{\mathbb{P}_{n+2}}(R \to Z^{gr}(S)) \arrow[d] \\
\mathrm{Mix}_{\mathbb{P}_{n+1}}(S) \arrow[r] & \mathrm{Mix}_{\mathbb{P}_{n+2}}(Z^{gr}(S))
\end{tikzcd}$$

where we considered $R \to Z^{gr}(S)$ as a graded $\mathbb{P}_{n+2}$-algebra inside the category $\mathrm{Mor}(\mathcal{M})$ of morphisms of $\mathcal{M}$.

We start by noticing that there is a natural map

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2),U^{gr}(R,S)[n+1]) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{n+2}}(R \to Z^{gr}(S)).$$

In fact, since $U^{gr}(R,S) \to R \to Z^{gr}(S)$ are maps of graded $\mathbb{P}_{n+2}$-algebras, one has an induced map

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2),U^{gr}(R,S)[n+1]) \longrightarrow \mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathrm{Mor}(\mathcal{M})}}(1_{\mathrm{Mor}(\mathcal{M})}[-1](2),(R \to Z^{gr}(S))[n+1]),$$

where $1_{\mathrm{Mor}(\mathcal{M})}$ is the monoidal unit of $\mathrm{Mor}(\mathcal{M})$, that is to say the identity map of $1_{\mathcal{M}}$. Composing this with the map of Proposition 3.4.7, we immediately get our desired morphism to $\mathrm{Mix}_{\mathbb{P}_{n+2}}(R \to Z^{gr}(S))$.

On the other hand, since $U^{gr}(R,S) \to \mathrm{Def}^{gr}(S)[-n-1]$ is a $\mathbb{P}_{n+2}$-map, by definition of mixed structures we get that there is a morphism of spaces

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2),U^{gr}(R,S)[n+1]) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{n+1}}(S).$$

The results of Section 3.4.1 assure that the square

$$\begin{tikzcd}
U^{gr}(R,S) \arrow[r] \arrow[d] & R \arrow[d] \\
\mathrm{Def}^{gr}(S)[-n-1] \arrow[r] & Z^{gr}(S)
\end{tikzcd}$$

is a pullback of graded $\mathbb{P}_{n+2}$-algebras, and thus the mixed structures on $R \to Z^{gr}(S)$ and on S induce the same mixed structure on $Z^{gr}(S)$. But now using the above presentation of $\mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(R,S)$ as a limit, we do get our desired morphism

$$\mathrm{Map}_{\mathrm{Lie}^{gr}_{\mathcal{M}}}(1_{\mathcal{M}}[-1](2),U(R,S)[n+1]) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(R,S),$$

which concludes the proof. □

Corollary 4.3.2. *Let $A \to B$ be a morphism of commutative algebras in $\mathcal{M}$. There is a canonical map of spaces*

$$\mathrm{Cois}(f,n) \longrightarrow \mathrm{Mix}_{\mathbb{P}_{[n+2,n+1]}}(\mathrm{Pol}(A,n+1),\mathrm{Pol}(B/A,n)),$$

where $(\mathrm{Pol}(A,n+1),\mathrm{Pol}(B/A,n))$ *is considered with its canonical graded* $\mathbb{P}_{[n+2,n+1]}$*-algebra structure.*

Proof. This is simply follows from the definition of coisotropic structures and from the previous proposition. □

Recall that there is a natural forgetful ∞-functor

$$\mathbb{P}_{[n+2,n+1]} - \mathrm{alg}^{gr}_{\mathcal{M}} \longrightarrow \mathrm{Mor}(\mathrm{CAlg}^{gr}_{\mathcal{M}}),$$

which induces a map

$$\mathrm{Cois}(f,n) \longrightarrow \mathrm{Mix}_{\mathrm{Comm}}(\mathrm{Pol}(A,n+1) \to \mathrm{Pol}(B/A,n)).$$

In particular, given a $\mathbb{P}_{[n+1,n]}$-algebra (A,B), the map

$$\mathrm{Pol}(A,n+1) \longrightarrow \mathrm{Pol}(B/A,n)$$

becomes a morphism of graded mixed commutative algebras. By definition, the weight zero component of this map is the underlying map $f : A \to B$ in $\mathrm{CAlg}_{\mathcal{M}}$, so that by the universal property of the de Rham algebra (see [CPTVV, Section 1.4]) one gets a commutative square of graded mixed commutative algebras

$$\begin{tikzcd} \mathrm{Pol}(A,n+1) \arrow[r] & \mathrm{Pol}(B/A,n) \\ \mathrm{DR}(A) \arrow[r] \arrow[u] & \mathrm{DR}(B) \arrow[u] \end{tikzcd}$$

where we have $\mathrm{DR}(A) \simeq \mathrm{Sym}_A(\mathbb{L}_A[-1])$ as graded commutative algebras.

Definition 4.3.2. We say that a $\mathbb{P}_{[n+1,n]}$-algebra (A,B) is *non-degenerate* if the two vertical arrows in the diagram above are equivalences.

Let $\mathrm{DR}(f)$ denote the fiber of the map $\mathrm{DR}(A) \to \mathrm{DR}(B)$. From the above diagram, we see that there is a natural arrow of graded mixed algebras $\mathrm{DR}(f) \to \mathrm{Pol}(f,n+1)$. Notice that the space of morphisms in the ∞-category of graded modules

$$k(2) \longrightarrow \mathrm{DR}(f)[n+1]$$

is by definition the space of closed 2-forms of degree $n-1$ on A, having restriction to B homotopic to 0. Equivalently, this can be described as the space of isotropic structures on the map $A \to B$ (where A is only a pre-symplectic algebra). The module $\mathrm{DR}(f)$ fits into a diagram of graded mixed modules

$$\mathrm{DR}(f)[n+1] \longrightarrow \mathrm{Pol}(f,n+1)[n+1] \longleftarrow k(2)$$

Definition 4.3.3. *Let again (A, B) be a $\mathbb{P}_{[n+1,n]}$-algebra. The space of* isotropic structures compatible with the given $\mathbb{P}_{[n+1,n]}$-structure *is the space of dotted arrows making the following diagram commute*

$$\begin{tikzcd} & \mathrm{DR}(f)[n+1] \arrow[d] \\ k(2) \arrow[r] \arrow[ur, dotted] & \mathrm{Pol}(f, n+1)[n+1] \end{tikzcd}$$

in the ∞-category of graded mixed modules.

Now suppose that (A, B) is a non-degenerate $\mathbb{P}_{[n+1,n]}$-algebra. Then the map $\mathrm{DR}(f) \to \mathrm{Pol}(f, n+1)$ is an equivalence, and therefore the space of compatible isotropic structures on $f : A \to B$ is contractible. Notice that in this case the 2-form is automatically non-degenerate: in fact, we have an induced equivalence

$$\mathrm{DR}(A) \simeq \mathrm{Pol}(A, n+1)$$

which by Theorem 3.2.5 of [CPTVV] can be used to turn the given non-degenerate shifted Poisson structure on A to a non-degenerate closed 2-form, so that A is actually a shifted symplectic algebra. Moreover, the fact that also $\mathrm{DR}(B) \to \mathrm{Pol}(B/A, n)$ is an equivalence implies that the isotropic structure is actually Lagrangian.

As a consequence, for every morphism of cdga $f : A \to B$, there is a well-defined map of spaces

$$\mathbb{P}^{nd}_{[n+1,n]}(f) \longrightarrow \mathrm{Lagr}(f)$$

where $\mathbb{P}^{nd}_{[n+1,n]}(f)$ is the space of non-degenerate $\mathbb{P}_{[n+1,n]}$-structure on (A, B) such that the underlying cdga map is f.

Now let us deal with the general case. Let X be a n-shifted Poisson stack, and let $f : L \to X$ be a morphisms of derived Artin stacks, locally of finite presentation. Suppose we are given a point $\gamma \in \mathrm{Cois}(f, n)$. This means in particular that we have a map of graded dg Lie algebras

$$k[-1](2) \longrightarrow \mathrm{Pol}(X, n+1)[n+1]$$

such that the induced map

$$k[-1](2) \longrightarrow \Gamma(L_{DR}, \mathrm{Pol}^t(\mathcal{P}_L/f^*\mathcal{P}_X, n)[n+1]$$

is homotopic to zero. Looking at weight 2 components, the shifted Poisson structure on X induces by adjunction a morphism of perfect complexes on X

$$\pi^{\#} : \mathbb{L}_X \to \mathbb{T}_X[-n] \ ,$$

and the coisotropic condition implies that the induced map $\mathbb{L}_f \to \mathbb{T}_f[-n+1]$ is homotopic to zero. This in turn gives the existence of dotted arrows in the following diagram

$$\begin{tikzcd} \mathbb{L}_f[-1] \arrow[r] \arrow[d, dotted] & f^*\mathbb{L}_X \arrow[r] \arrow[d, "f^*\pi^{\#}"] & \mathbb{L}_L \arrow[d, dotted] \\ \mathbb{T}_L[-n] \arrow[r] & f^*\mathbb{T}_X[-n] \arrow[r] & \mathbb{T}_f[-n+1] \end{tikzcd}$$

where both horizontal rows are fiber sequences of perfect complexes on L.

Definition 4.3.4. *With notations as above, the coisotropic structure γ on f is called* non-degenerate *if the n-Poisson structure on X is non-degenerate, and the previous diagram is an equivalence of fiber sequences. Equivalently, γ is non-degenerate if $\pi^{\#}$ and the dotted arrows are equivalences of perfect complexes.*

The space of non-degenerate coisotropic structures on f will be denoted $\mathrm{Cois}^{nd}(f)$.

By Theorem 4.1.5, the datum of a coisotropic structure on $f : L \to X$ is equivalent to the datum of a compatible $\mathbb{P}_{[n+1,n]}$-structure on the couple $(f^*\mathcal{P}_X(\infty), \mathcal{P}_L(\infty))$ in the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

Corollary 4.3.5. *Let $f : L \to X$ be a map between derived Artin stacks, locally of finite presentation, and suppose X is equipped with a n-shifted Poisson structure. A coisotropic structure $\gamma \in \mathrm{Cois}(f, n)$ is non-degenerate in the sense of Definition 4.3.4 if and only if the corresponding $\mathbb{P}_{[n+1,n]}$-algebra*

$$(f^*\mathcal{P}_X(\infty), \mathcal{P}_L(\infty))$$

in the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules is non-degenerate in the sense on Definition 4.3.2.

This is an immediate consequence of the general correspondence between geometric differential calculus on derived stacks and algebraic differential calculus on the associated prestacks of Tate principal parts, as exposed in [CPTVV].

We also have a similar result for the symplectic case.

Corollary 4.3.6. *Let $f : L \to X$ again be a map between derived Artin stacks, locally of finite presentation, and suppose ω_X is a n-shifted closed 2-form on X, such that $f^*\omega \sim 0$ inside the space of closed 2-forms on L. The form ω canonically induces a n-shifted closed 2-form ω' on the algebra $f^*\mathcal{P}_X(\infty)$, relative to $\mathbb{D}_{L_{DR}}(\infty)$, such that its restriction to $\mathcal{P}_L(\infty)$ is homotopic to zero.*

Then ω is a Lagrangian structure on f if and only if ω' is a Lagrangian on the couple

$$(f^*\mathcal{P}_X(\infty), \mathcal{P}_L(\infty))$$

in the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

Notice that in the classical case (i.e. if X is a smooth Poisson scheme and $f : L \to X$ is a coisotropic sub-scheme), then f is non-degenerate if and only if the Poisson structure on X comes from a symplectic structure, and L is a Lagrangian sub-scheme. Our next goal is to prove a derived extension of this result.

Theorem 4.3.7. *Let X be a non-degenerate n-shifted Poisson stack, and let $f : L \to X$ be a morphism of derived Artin stacks. Let $\mathrm{Cois}^{nd}(f, n)$ be the subspace of $\mathrm{Cois}(f, n)$ of connected components of non-degenerate coisotropic structures on f. Then X is canonically n-shifted symplectic, and there exists an equivalence of spaces*

$$\mathrm{Cois}^{nd}(f, n) \to \mathrm{Lagr}(f, n) \ .$$

The proof of this theorem will closely follow the proof of Theorem 3.2.5 in [CPTVV]. The result will follow from a slightly more general statement. We will need some general constructions on stacks of Lie algebras and of mixed modules; this is already contained in [CPTVV], and we refer to that paper for more details.

4.3.2 Stacks associated with Lie algebras and mixed complexes

Recall from [CPTVV], section 3.3.1, that given some base derived stack Y, and some graded Lie algebra $\mathcal{L}$ inside the category of $\mathcal{O}_Y$-modules, then we can construct the *stack associated with* $\mathcal{L}$ as the ∞-functor

$$\mathbb{V}(\mathcal{L}) : (\mathrm{dAff}/Y)^{op} \longrightarrow \mathrm{sSet}$$

which sends $(\mathrm{Spec}A \to Y)$ to the space

$$\mathbb{V}(\mathcal{L})(A) = \mathrm{Map}_{\mathrm{dgla}^{gr}}(k[-1](2), \mathcal{L}(A)).$$

Let p be an A-point of $\mathbb{V}(\mathcal{L})$.

By definition, the (higher) tangent spaces $T^i_p(\mathbb{V}(\mathcal{L}))$ of $\mathbb{V}(\mathcal{L})$ at the point p are the homotopy fiber of the map

$$\mathbb{V}(\mathcal{L})(A \oplus A[i]) \longrightarrow \mathbb{V}(\mathcal{L})(A)$$

taken at p.

Using the map

$$k[-1](2) \to \mathcal{L}(A)$$

corresponding to the point p, $\mathcal{L}(A)$ becomes a weak mixed graded complex. Let us denote $(\mathcal{L}(A), p)$ the complex $\mathcal{L}(A)$ together with its mixed structure induced by p.

One has then the following lemma, which is proven in [CPTVV].

Lemma 4.3.8. *Suppose that for all i, the morphism*

$$\mathcal{L}(A) \otimes_A (A \oplus A[i]) \longrightarrow \mathcal{L}(A \oplus A[i])$$

is an equivalence of graded dg Lie algebras. Then the tangent spaces of the derived stack $\mathbb{V}(\mathcal{L})$ at a point $p \in \mathbb{V}(\mathcal{L})(A)$ have the following expression

$$T^i_p(\mathbb{V}(\mathcal{L})) \simeq \mathrm{Map}_{\epsilon-\mathrm{dgmod}^{gr}}(k[-1](2), (\mathcal{L}(A), p)).$$

Remark. In [CPTVV], the authors proved the lemma only for A-points on $\mathbb{V}(\mathcal{L})$ that are given by *strict* morphisms of graded dg Lie algebras, that is to say morphisms $k[-1](2) \to \mathcal{L}(A)$ in the usual strict 1-category of Lie algebras. That was enough for their purpose, since Poisson structures can be nicely strictified. In our case, the strictification of coisotropic structures is a bit more complicated, and we cannot necessarily work with strict Maurer-Cartan elements.

In any case the lemma immediately extends to non-strict points. One just has to check that a map $k[-1](2) \to \mathcal{L}(A)$ in the ∞-category produces a weak mixed structure on $\mathcal{L}(A)$, and use the fact that weak mixed structures can be strictified. For this, one can use for example the explicit resolution of $k[-1](2)$ described in [Me].

In a similar spirit, let us now consider a stack $\mathcal{E}$, which is a graded mixed complex in the category of $\mathcal{O}_Y$-modules. We define the derived stack associated to $\mathcal{E}$ as

$$\mathbb{V}(\mathcal{E}) : (\mathrm{dAff}/Y)^{op} \longrightarrow \mathrm{sSet}$$

which sends $(\mathrm{Spec}A \to Y)$ to the space

$$\mathbb{V}(\mathcal{E})(A) = \mathrm{Map}_{\epsilon-\mathrm{dgmod}^{gr}}(k[-1](2), \mathcal{E}(A)).$$

As before, take an A-point p of $\mathbb{V}(\mathcal{E})$, and define the tangent spaces of $\mathbb{V}(\mathcal{E})$ at p as the homotopy fibers of the maps

$$\mathbb{V}(\mathcal{E})(A \oplus A[i]) \longrightarrow \mathbb{V}(\mathcal{E})(A)$$

taken at the point p. The previous lemma has a similar version that applies to this case.

Lemma 4.3.3. Suppose that for all i, the morphism

$$\mathcal{E}(A) \otimes_A (A \oplus A[i]) \longrightarrow \mathcal{E}(A \oplus A[i])$$

is an equivalence of graded mixed complexes. Then the tangent spaces of the derived stack $\mathbb{V}(\mathcal{E})$ at a point $p \in \mathbb{V}(\mathcal{E})(A)$ have the following expression

$$T^i_p(\mathbb{V}(\mathcal{E})) \simeq \mathrm{Map}_{\epsilon-\mathrm{dgmod}^{gr}}(k[-1](2), \mathcal{E}(A)[i]).$$

4.3.3 Sheafified coisotropic and Lagrangian structures

Let us now consider a sheafified version of the space of coisotropic structures. Recall that the map $f : L \to X$ induces a natural map $f^*_{\mathcal{P}} : f^*\mathcal{P}_X(\infty) \to \mathcal{P}_L(\infty)$ of $\mathbb{D}_{L_{DR}}(\infty)$-modules.

Let $\underline{\mathrm{Cois}}(f, n)$ be the stack on L_{DR} defined as $\mathbb{V}(\mathcal{L}_f)$, where $\mathcal{L}_f$ is the dg Lie defined as

$$\mathcal{L}_f : (\mathrm{Spec} A \to L_{DR}) \longmapsto \mathrm{Pol}(f^*_{\mathcal{P}}(A), n+1)[n+1]$$

and $f^*_{\mathcal{P}}(A)$ is the map

$$f^*_{\mathcal{P}}(A) : f^*\mathcal{P}_X(\infty)(A) \longrightarrow \mathcal{P}_L(\infty)(A).$$

By definition, global sections of $\underline{\mathrm{Cois}}(f, n)$ are in correspondence with $\mathbb{P}_{[n+1,n]}$-structures on the couple $(f^*\mathcal{P}_X(\infty), \mathcal{P}_L(\infty))$ in the category of $\mathbb{D}_{L_{DR}}(\infty)$-modules, such that the underlying commutative monoid morphism is the given $f^*_{\mathcal{P}}$. We can also consider the sub-object $\underline{\mathrm{Cois}}^{nd}(f, n)$, consisting only of non-degenerate $\mathbb{P}_{[n+1,n]}$-structures.

In a totally similar way, we can define $\underline{\mathrm{Lagr}}(f, n)$: start with the stack $\mathbb{V}(\mathcal{E}_f)$ on L_{DR}, where $\mathcal{E}_f$ is the graded mixed $\mathbb{D}_{L_{DR}}(\infty)$-module

$$\mathcal{E}_f : (\mathrm{Spec} A \to L_{DR}) \longmapsto \mathrm{DR}(f^*_{\mathcal{P}}(A)).$$

Consider the substack $\underline{\mathrm{Lagr}}(f, n)$ of $\mathbb{V}(\mathcal{E}_f)$ consisting of non-degenerate isotropic structures.

The following is a slightly finer statement than Theorem 4.3.7.

Theorem 4.3.9. *Let $f : L \to X$ be a map of derived stacks. Then there is an equivalence of stacks over L_{DR}*

$$\phi : \underline{\mathrm{Cois}}^{nd}(f, n) \to \underline{\mathrm{Lagr}}(f, n).$$

We start by proving that the map ϕ is an isomorphism on all higher homotopy sheaves.

We now take a derived affine Spec$A$, and consider an A-point x of $\underline{\mathrm{Cois}}(f)$. As before, the point x induces a mixed structure on the relative polyvectors $\mathrm{Pol}(f^*_{\mathcal{P}}(A), n+1)$. Let us denote this graded mixed complex as $\mathrm{Pol}^{\epsilon}(f^*_{\mathcal{P}}(A), n+1)$.

Using lemma 3.3.4 of [CPTVV], we get an explicit description of the based loop stack of $\underline{\mathrm{Cois}}(f)$ at its point x.

Corollary 4.3.10. *With notations as above, we have an equivalence of derived stack over SpecA*

$$\Omega_x \underline{\mathrm{Cois}}(f) \simeq \mathbb{V}(\mathrm{Pol}^\epsilon(f^*_{\mathcal{P}}(A), n+1))$$

We can use this result to obtain a first step in the proof of Theorem 4.3.9.

Proposition 4.3.11. *The morphism ϕ of Theorem 4.3.9 induces equivalences on based loop stacks. In other words, for every point*

$$x : SpecA \longrightarrow \underline{\mathrm{Cois}}^{nd}(f, n)$$

the induced morphism

$$\Omega_x \underline{\mathrm{Cois}}^{nd}(f, n) \longrightarrow \Omega_{\phi(x)} \underline{\mathrm{Lagr}}(f, n)$$

is an equivalence of derived stacks over SpecA.

Proof. The previous corollary describes the loop stack $\Omega_x \underline{\mathrm{Cois}}^{nd}(f, n)$ as $\mathbb{V}(\mathrm{Pol}^\epsilon(f^*_{\mathcal{P}}(A), n+1))$. But by universal property of the de Rham algebra, there is an induced morphism

$$\psi_x : \mathrm{DR}(f^*_{\mathcal{P}}(A)) \to \mathrm{Pol}^\epsilon(f^*_{\mathcal{P}}(A), n+1)$$

of graded mixed complexes. Since the coisotropic structure is taken to be non-degenerate, ψ_x is actually an equivalence, so that we get

$$\Omega_x \underline{\mathrm{Cois}}^{nd}(f) \simeq \mathbb{V}(\mathrm{DR}(f^*_{\mathcal{P}}(A)).$$

Using again Lemma 3.3.4 in [CPTVV], the stack $\mathbb{V}(\mathrm{DR}(f^*_{\mathcal{P}}(A))$ is identified with the loop stack $\Omega_{\phi(x)} \underline{\mathrm{Lagr}}(f, n)$. Therefore we end up with an equivalence of based loop stacks, which can be easily checked to be the morphism induced by ϕ. □

We now have to show that ϕ of Theorem 4.3.7 induces an isomorphism also on the π_0-sheaves.

4.3.4 Infinitesimal theory

This is parallel to section 3.3.3 in [CPTVV]. Our goal is to reduce the proof of the equivalence of the π_0-sheaves of $\underline{\mathrm{Cois}}^{nd}(f, n)$ and $\underline{\mathrm{Lagr}}(f, n)$ to a question over reduced base rings, in order to be able to use a form of Darboux lemma and to explicitly prove the result.

Let again $f : L \to X$ be a morphism of derived stacks. Consider the inclusion of ∞-categories

$$j : \mathrm{dAff}^{red}/L_{DR} \longrightarrow \mathrm{dAff}/L_{DR}.$$

Proposition 4.3.12. *The morphism ϕ of Theorem 4.3.9 is an equivalence of stacks if and only if the induced morphism*

$$j^*\phi : j^* \underline{\mathrm{Cois}}^{nd}(f, n) \longrightarrow j^* \underline{\mathrm{Lagr}}(f, n)$$

is an equivalence of stacks over $\mathrm{dAff}^{red}/L_{DR}$.

Proof. Let $\mathrm{Spec}A \to L_{DR}$ be an object in dAff/L_{DR}. We will show that

$$\phi_A : \underline{\mathrm{Cois}}^{nd}(f, n)(A) \longrightarrow \underline{\mathrm{Lagr}}(f, n)(A)$$

is an equivalence as soon as

$$\phi_{A_{red}} : \underline{\mathrm{Cois}}^{nd}(f, n)(A_{red}) \longrightarrow \underline{\mathrm{Lagr}}(f, n)(A_{red})$$

is an equivalence.

This will follow from exactly the same argument of Proposition 3.3.7 in [CPTVV]. Thus we are left with showing that the stacks of coisotropic and lagrangian structures satisfy some nice infinitesimal properties.

1. We say that a derived stack F over L_{DR} is *nilcomplete* if for every Spec$A \to L_{DR}$, the canonical map
$$F(B) \longrightarrow \lim_k (F(A_{\leq k}))$$
is an equivalence, where $A_{\leq k}$ is the k-th Postnikov truncation of A.

2. A derived stack F over L_{DR} is *infinitesimally cohesive* if for every cartesian square of almost finite presented objects in cdga$^{\leq 0}$
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_0 \end{tikzcd}$$
such that both $H^0(A_1) \to H^0(A_0)$ and $H^0(A_2) \to H^0(A_0)$ are surjective with nilpotent kernels, the induced diagram of spaces
$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(A_1) \arrow[d] \\ F(A_2) \arrow[r] & F(A_0) \end{tikzcd}$$
is again cartesian.

Lemma 4.3.13. *The stacks $\underline{\mathrm{Cois}}(f,n)$ and $\underline{\mathrm{Lagr}}(f,n)$ are nilcomplete and infinitesimally cohesive.*

Proof of the lemma. Recall that by definition, we have
$$\underline{\mathrm{Cois}}(f,n) \simeq \mathbb{V}(\mathcal{L}_f) \quad \text{and} \quad \underline{\mathrm{Lagr}}(f,n) \simeq \mathbb{V}(\mathcal{E}_f)$$
with $\mathcal{L}_f$ and $\mathcal{E}_f$ two stacks of complexes on L_{DR}. The lemma will be proven if we are able to show that both $\mathcal{L}_f$ and $\mathcal{E}_f$ are nilcomplete and infinitesimally cohesive, as stacks of complexes. But both stacks are defined as homotopy fibers of stacks that were showed to be nilcomplete and infinitesimally cohesive in [CPTVV], Lemma 3.3.8. Since both properties are clearly stable under limits, we are done. ☐

☐

4.3.5 Conclusion of the proof

Thanks to the previous sections, we can reduce the question of the equivalence of the stacks $\underline{\mathrm{Cois}}^{nd}(f,n)$ and $\underline{\mathrm{Lagr}}(f,n)$ to a question on the equivalence of the π_0-sheaves which can be actually checked on *reduced* algebras.

Recall that we have started with a map of derived Artin stacks $f\colon L \to X$. Let A be a reduced discrete algebra, together with a morphism $\mathrm{Spec}A \to L_{DR}$. Since A is reduced, this corresponds to an A-point of L. As before, consider the stacks

$$L_A := L \times_{L_{DR}} \mathrm{Spec}A \qquad X_A := X \times_{X_{DR}} \mathrm{Spec}A,$$

where the map $\mathrm{Spec}A \to X_{DR}$ is obtained by composing the given $\mathrm{Spec}A \to L_{DR}$ with the induced $f_{DR}\colon L_{DR} \to X_{DR}$. By functoriality of $\mathbb{D}$, we get a natural map $\varphi\colon \mathbb{D}(X_A) \to \mathbb{D}(L_A)$ of graded mixed A-cdgas. By the previous discussion, it will be enough to show that the morphism of spaces

$$\mathrm{Map}^{nd}_{\mathrm{Lie}^{gr}_k}(k(2)[-1], \mathrm{Pol}^t(\varphi, n+1)[n+1]) \longrightarrow \mathrm{Map}^{nd}_{C(k)^{\epsilon,gr}}(k(2)[-n-2], \mathrm{DR}(\varphi))$$

induces an isomorphism on the π_0 sets, where Map^{nd} denotes the subspace of the mapping space corresponding to non-degenerate coisotropic or isotropic structures on ϕ. The above map can be easily sheafified over $\mathrm{Spec}A$: namely, we can construct two sheaves $\mathcal{C}$ and $\mathcal{L}$ sending an open $\mathrm{Spec}A' \subset \mathrm{Spec}A$ to

$$\mathrm{Map}^{nd}_{\mathrm{Lie}^{gr}_k}(k(2)[-1], \mathrm{Pol}^t(\varphi, n+1)[n+1] \otimes_A A') \ \text{ and } \ \mathrm{Map}^{nd}_{C(k)^{\epsilon,gr}}(k(2)[-n-2], \mathrm{DR}(\varphi) \otimes_A A')$$

respectively. It will thus suffice to show that the induced map $\pi_0\mathcal{C} \to \pi_0\mathcal{L}$ is an isomorphism an sheaves of sets on the small Zariski site of A. In order to prove this, we can show that for every point $p \in \mathrm{Spec}A$, the induced map between the stalks $\pi_0\mathcal{C}_p \to \pi_0\mathcal{L}_p$ is in fact an isomorphism of set.

Let us thus fix such a point p. Notice that since A is reduced, $\mathbb{D}(A) \simeq A$, and thus both $\mathbb{D}(X_A)$ and $\mathbb{D}(L_A)$ are simply graded mixed A-cdgas. Forgetting the mixed structure, we have isomorphisms

$$\mathbb{D}(X_A) \simeq \mathrm{Sym}_A(u^*f^*\mathbb{L}_X), \qquad \mathbb{D}(L_A) \simeq \mathrm{Sym}_A(u^*\mathbb{L}_L),$$

where $u\colon \mathrm{Spec}A \to L$ is the given A-point of L. The map $\varphi : \mathbb{D}(X_A) \to \mathbb{D}(L_A)$ is then the one induced by the natural map of $\mathcal{O}_L$-modules $f^*\mathbb{L}_X \to \mathbb{L}_L$.

We can now choose a model $M_X \to M_L$ for the map $u^*f^*\mathbb{L}_X \to u^*\mathbb{L}$ which is a surjective morphism of bounded complexes of projective A-modules of finite rank. Let us introduce the graded A-cdgas $B_X = \mathrm{Sym}_A(M_X)$ and $B_L = \mathrm{Sym}_A(M_L)$. Exactly as in [CPTVV, Section 3.3.4], the structure of a morphism of graded mixed algebras on $\mathbb{D}(X_A) \to \mathbb{D}(L_A)$ induces a structure of an explicit map of weak graded mixed algebras on our explicit models $\phi\colon B_X \to B_L$. Notice that since $M_X \to M_L$ is a fibration (hence a surjection), the induced map ϕ is again a surjection of A-algebras. Let us denote by $M_f[-1]$ the strict kernel of $M_X \to M_L$. As $M_X \to M_L$ is a fibration, $M_f[-1]$ is also a model for the homotopy fiber of $M_X \to M_L$. We can now use the fact that B_X and B_L are explicit: they are free as graded commutative A-algebras, and thus the tangent and the cotangent complex are relatively easy to compute. The following lemma is a direct consequence of [CPTVV, Lemma 3.3.10], and it gives a concrete expression for the de Rham and polyvectors algebras of the map ϕ.

Lemma 4.3.14. *Let B_X and B_L defined as above, and let M_X^* and M_f^* be the A-linear duals of the complexes M_X and M_f respectively. We have the following equivalences:*

$$\mathrm{DR}^t(\mathbb{D}(X_A)) \simeq \mathrm{DR}^t(B_X) \simeq |B_X| \otimes_A \mathrm{Sym}_A(M_X[-1])$$

$$\mathrm{DR}^t(\mathbb{D}(L_A)) \simeq \mathrm{DR}^t(B_L) \simeq |B_L| \otimes_A \mathrm{Sym}_A(M_L[-1])$$

$$\mathrm{Pol}^t(\mathbb{D}(X_A), n+1) \simeq \mathrm{Pol}^t(B_X, n+1) \simeq |B_X| \otimes_A \mathrm{Sym}_A(M_X^*[-n-1])$$
$$\mathrm{Pol}^t(\mathbb{D}(L_A)/\mathbb{D}(X_A), n) \simeq |B_L| \otimes_A \mathrm{Sym}_A(M_f^*[-n])$$

Proof. The only difference with [CPTVV, Lemma 3.3.10] is the last identification. But notice that the map $B_X \to B_L$ induces a fiber sequence of B_L-modules

$$\mathbb{T}_{B_L} \longrightarrow \mathbb{T}_{B_X} \otimes_{B_X} B_L \longrightarrow \mathbb{T}_\phi[1],$$

which in turn gives rise to a cofiber sequence of graded commutative algebras

$$\mathrm{Sym}_{B_L}(\mathbb{T}_{B_L}[-n-1]) \to \mathrm{Sym}_{B_X}(\mathbb{T}_{B_X}[-n-1]) \otimes B_L \to \mathrm{Sym}_{B_L}(\mathbb{T}_\phi[-n]).$$

It follows that $\mathrm{Pol}(B_L/B_X, n-1)$ is equivalent to the cofiber of the morphism

$$|B_L| \otimes_A \mathrm{Sym}_A(M_L^*[-n-1]) \longrightarrow |B_L| \otimes_A \mathrm{Sym}_A(M_X^*[-n-1]),$$

which is easily seen to be $|B_L| \otimes_A \mathrm{Sym}_A(M_f^*[-n])$. □

The above lemma tells us in particular that the space of isotropic structures on ϕ is

$$\mathrm{Map}_{C(k)^{gr,\epsilon}}(k(2)[-n-2], \mathrm{DR}(\phi)),$$

where $\mathrm{DR}(\phi)$ fits in the homotopy fiber sequence of graded mixed dg algebras

$$\mathrm{DR}(\phi) \longrightarrow |B_X| \otimes_A \mathrm{Sym}_A(M_X[-1]) \longrightarrow |B_L| \otimes_A \mathrm{Sym}_A(M_L[-1]).$$

By assumption, $M_X \to M_L$ is surjective, so that also the map

$$\mathrm{Sym}_A(M_X[-1]) \longrightarrow |B_L| \otimes_A \mathrm{Sym}_A(M_L[-1])$$

is a surjection of graded mixed dg algebras. In particular, $\mathrm{DR}(\phi)$ can be taken to be the strict kernel of the above morphism.

Similarly, the space of coisotropic structures on ϕ is equivalent to

$$\mathrm{Map}_{\mathrm{Lie}_k^{gr}}(k(2)[-1], \mathrm{Pol}(\phi, n)[n+1])$$

where $\mathrm{Pol}(\phi, n)$ fits in the homotopy fiber sequence of graded dg modules

$$\mathrm{Pol}(\phi, n) \longrightarrow |B_X| \otimes_A \mathrm{Sym}_A(M_X^*[-n-1]) \longrightarrow |B_L| \otimes_A \mathrm{Sym}_A(M_f^*[-n]).$$

By definition the map $M_f[-1] \to M_X$ is injective, so that its dual $M_X^* \to M_f^*[1]$ is surjective. This means that the strict kernel of

$$\mathrm{Sym}_A(M_X^*[-n-1]) \longrightarrow |B_L| \otimes_A \mathrm{Sym}_A(M_f^*[-n])$$

is a model for $\mathrm{Pol}(\phi, n)$.

Since we are now working locally (around the point p), we can suppose without loss of generality that the complexes M_X and M_L are minimal at p, in the sense that their differentials vanish on $M_X \otimes_A k(p)$ and $M_L \otimes_A k(p)$. The Darboux lemma in [CPTVV] immediately allows us to restrict our attention to very simple Lagrangian and coisotropic structures.

Lemma 4.3.15. *Suppose both M_X and M_L are minimal at p.*

(1) Every morphism in the ∞-category of graded mixed complexes

$$k(2)[-n-2] \longrightarrow \mathrm{DR}(\phi) \otimes_A k(p)$$

is equivalent to a strict morphism of graded mixed complexes, that is to say to a strict morphism

$$k(2)[-n-2] \longrightarrow \mathrm{DR}(B_X)$$

whose composite with $\mathrm{DR}(B_X) \to \mathrm{DR}(B_L)$ is strictly equal to zero.

(2) Every non-degenerate morphism in the ∞-category of graded Lie algebras

$$k(2)[-1] \longrightarrow \mathrm{Pol}(\phi, n+1)[n+1] \otimes_A k(p)$$

is equivalent to a strict morphism of graded Lie algebras, that is to say to a strict morphism

$$k(2)[-1] \longrightarrow \mathrm{Pol}(B_X, n+1) \otimes_A k(p)$$

whose composition with $\mathrm{Pol}(B_X, n+1) \to \mathrm{Pol}(B_L/B_X, n)$ is strictly equal to zero.

Proof. Using the Darboux lemma [CPTVV, Lemma 3.3.11], we can show that the symplectic and the non-degenerate Poisson (local) structures on B_X can be taken to be strict. We conclude by noticing that by our assumptions on $M_X \to M_L$ we can take both $\mathrm{DR}(\phi)$ and $\mathrm{Pol}(\phi, n+1)$ to be strict kernels. $\square$

The above Lemma tells us that locally at a point p, we can suppose both Lagrangians and coisotropic structures to be given by strict symplectic/Poisson structures on B_X whose restriction to $\mathrm{DR}(B_L)$ and $\mathrm{Pol}(B_L/B_X, n)$ is strictly zero. We will now separately prove that $\pi_0(\mathcal{C}) \to \pi_0(\mathcal{L})$ is surjective and injective.

For surjectivity, consider a strict Lagrangian structure given by a strictly closed two-form ω_X of degree n on B_X which restricts to zero on B_L. Since $\omega_X\colon \mathrm{T}_{B_X} \to \Omega^1_{B_X}[n]$ is a quasi-isomorphism and M_X is minimal at p, after passing to a Zariski cover we can assume ω_X in fact induces an isomorphism. Let $N^*_{B_L/B_X}$ be the strict fiber of $\phi^*\Omega^1_{B_X} \to \Omega^1_{B_L}$. Since ω_X restricts to zero on B_L, it induces a morphism $\mathrm{T}_{B_L} \to N^*_{B_L/B_X}[n]$ which is assumed to be a quasi-isomorphism by the Lagrangian condition. But again by minimality after passing to a Zariski cover we can assume $\mathrm{T}_{B_L} \to N^*_{B_L/B_X}[n]$ is a strict isomorphism. Let π_X be the bivector on B_X which is obtained by inverting ω_X. It is a classical computation that the condition $\mathrm{d}_{\mathrm{dR}}\omega_X = 0$ is equivalent to $[\pi_X, \pi_X] = 0$. Now consider a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{T}_{B_L} \arrow[r] \arrow[d, "\sim"] & \phi^*\mathrm{T}_{B_X} \arrow[r] \arrow[d, "\sim"] & N_{B_L/B_X} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & N^*_{B_L/B_X}[n] \arrow[r] & \phi^*\Omega^1_{B_X}[n] \arrow[r] & \Omega^1_{B_L}[n] \arrow[r] & 0
\end{tikzcd}$$

where the two rows are exact sequences. Therefore, the composite

$$N^*_{B_L/B_X}[n] \longrightarrow \phi^*\Omega^1_{B_X}[n] \xrightarrow{\pi_X} \phi^*\mathrm{T}_{B_X} \longrightarrow N_{B_L/B_X}$$

is zero and hence π_X defines a coisotropic structure compatible with the Lagrangian structure given by ω_X which is strictly non-degenerate.

Running the same argument in reverse we deduce injectivity. We therefore conclude that the map of sheaves $\mathcal{C} \to \mathcal{L}$ is an equivalence, and hence Theorem 4.3.7 is finally proved.

Chapter 5

Comparison with the CPTVV definition

This chapter deals with the natural question on comparing various possible definitions of derived coisotropic structures in the literature. In this text we proposed two different definitions: one with a more "algebraic" flavor (Definition 4.1.2) and another one using polyvectors, which is probably more intuitive and geometric (Definition 4.1.4). We already proved that these two are equivalent: this is the content of Theorem 4.1.5. In the paper [CPTVV], the authors proposed a different definition of coisotropic structure on a morphism; the goal of this chapter is precisely to discuss the comparison of their definition with the one treated in this thesis.

In the first section, we start by recalling the [CPTVV] definition. It is based on the existence of a rather non-explicit additivity functor for the Poisson operad, whose existence is at the moment somewhat conjectural. One of the main advantages of the definition of coisotropic structures discussed in the previous chapters is precisely the fact that it avoids the use of the additivity property of the Poisson operad.

In the second section we give two proposals for comparing the various definitions of coisotropic structures. The first one is based on showing that the Poisson center $Z(B)$ of a $\mathbb{P}_n$-algebra B (see chapter 3) is in fact a model for the internal endomorphisms object of B inside $\mathbb{P}_n - \text{alg}$. This could be possibly proven combining results of Lurie [Lur3] and Tamarkin [Ta], but we refer to [MS] for the details. Notice that this approach somehow avoids the need of understanding the additivity functor. The second idea concentrates on the construction of a such additivity functor: using results in [Sa] and a conjecture contained in [FG], we produce a functor with which we can make sense of the CPTVV definition. Moreover, we show that in this case any coisotropic in the sense of this thesis is also a coisotropic in the sense of CPTVV, providing further evidence of the equivalence of the two definitions.

5.1 The CPTVV definition

The content of this section appeared in [CPTVV], Section 3.4.

We start by observing that the operad $\mathbb{P}_n$ is a Hopf operad. As such, its category of algebras has a naturally induced symmetric monoidal structure. More specifically, given two $\mathbb{P}_n$-algebras A and B in a category $\mathcal{M}$, we can define their tensor product as $\mathbb{P}_n$-algebras: the underlying object in $\mathcal{M}$ is the tensor product $A \otimes B$ taken in $\mathcal{M}$, and the $\mathbb{P}_n$-structure is defined by the composition

$$\mathbb{P}_n(p) \otimes (A \otimes B)^{\otimes p} \longrightarrow \mathbb{P}_n(p) \otimes \mathbb{P}_n(p) \otimes A^{\otimes p} \otimes B^{\otimes p} \longrightarrow A \otimes B,$$

where we used the Hopf structure on $\mathbb{P}_n$ and the $\mathbb{P}_n$-structures on A and B. In particular, it makes sense to consider the category $\mathrm{Alg}(\mathbb{P}_n - \mathrm{alg}_{\mathcal{M}})$ of associative algebra objects inside $\mathbb{P}_n - \mathrm{alg}_{\mathcal{M}}$.

The additivity theorem for the Poisson operad gives a nice interpretation of these algebras, and it can be stated in the following form.

Theorem 5.1.1. *Let $n \geq 1$, and let $\mathcal{M}$ be a symmetric monoidal ∞ category. There exists an equivalence of ∞-categories*

$$\mathrm{Dec}_{n+1} : \ \mathbb{P}_{n+1} - \mathrm{alg}_{\mathcal{M}} \longrightarrow \mathrm{Alg}(\mathbb{P}_n - \mathrm{alg}_{\mathcal{M}})$$

satisfying the following two properties:

1. *The ∞-functor Dec_{n+1} is functorial in the variable $\mathcal{M}$ with respect to symmetric monoidal ∞-functors.*

2. *The ∞-functor Dec_{n+1} commutes with the two forgetful ∞-functors to $\mathcal{M}$.*

Notice that by formality, one has $\mathbb{E}_n \simeq \mathbb{P}_n$, and thus the additivity property for the Poisson operad follows from the proof of the Deligne's conjecture given by Lurie in [Lur3]. However this proof is not explicit and depends on the choices of formality equivalences. A more direct argument has being announced by Rozenblyum.

In [CPTVV] the authors use Theorem 5.1.1 to give a definition of coisotropic structure on a morphism. Namely, consider the two-colored operad $\mathbb{P}_{(n+1,n)}$, whose algebras in a category $\mathcal{M}$ are pairs (A, B) of objects of $\mathcal{M}$, together with

- a $\mathbb{P}_{n+1}$-structure on A;
- a $\mathbb{P}_n$-structure on B;
- an A-module structure on B in the category $\mathbb{P}_n$-alg, where A is now seen as an algebra in $\mathbb{P}_n$-alg through Theorem 5.1.1.

By construction, there is a natural forgetful functor

$$\mathbb{P}_{(n+1,n)} - \mathrm{alg}_{\mathcal{M}} \to \mathrm{Mor}(\mathrm{CAlg}_{\mathcal{M}})$$

to the category of morphisms of commutative algebras in $\mathcal{M}$.

Now let $f : X \to Y$ be a morphism of derived Artin stacks, locally of finite presentation, and consider the induced morphism

$$f^*_{\mathcal{P}} : \mathcal{P}_Y(\infty) \longrightarrow f^*\mathcal{P}_X(\infty)$$

in the ∞-category of $\mathbb{D}_{X_{DR}}(\infty)$-algebras. Let us denote by $\mathcal{C}$ the category of $\mathbb{D}_{X_{DR}}(\infty)$-modules. Using the operad $\mathbb{P}_{(n+1,n)}$, they give the following definition.

Definition 5.1.2 ([CPTVV], Section 3.4)**.** *With notations as above, the space $\widetilde{\mathrm{Cois}}(f, n)$ of n-*shifted coisotropic structures *on f is the fiber product*

$$\begin{tikzcd}
\widetilde{\mathrm{Cois}}(f,n) \arrow[r] \arrow[d] & \mathbb{P}_{(n+1,n)} - \mathrm{alg}_{\mathcal{C}} \arrow[d] \\
* \arrow[r] & \mathrm{Mor}(\mathrm{CAlg}_{\mathcal{C}})
\end{tikzcd}$$

where the bottom map corresponds to the morphisms $f^*_{\mathcal{P}}$.

This definition is far from being explicit: it uses in particular a forgetful functor $\mathbb{P}_{n+1}-\text{alg} \longrightarrow \mathbb{P}_n-\text{alg}$, whose existence is not obvious. Notice that the easiest such functor is of the one factorizing by the ∞-category of commutative algebras: more specifically, one has of course a natural forgetful functor $\mathbb{P}_{n+1}-\text{alg} \to \text{CAlg}$. On the other hand, there is an inclusion $\text{CAlg} \to \mathbb{P}_n$, which adds the zero bracket to a commutative algebra. This is naturally too simplistic, as we completely forget about the $\mathbb{P}_{n+1}$-bracket.

Philosophically speaking, the additivity forgetful functor $\mathbb{P}_{n+1}-\text{alg} \longrightarrow \mathbb{P}_n-\text{alg}$ should send a $\mathbb{P}_{n+1}$-algebra A to a $\mathbb{P}_n$-algebra A' whose bracket is homotopic to zero, and the datum of the starting $\mathbb{P}_{n+1}$-bracket on A should be used to construct the homotopy witnessing the triviality of the bracket on A'.

As a consequence, it is not easy to produce concrete examples of coisotropic structures in the sense of the CPTVV definition above.

5.2 Two proposals for proving the equivalence

As explained in the previous section, Definition 5.1.2 express a coisotropic structure on a map $A \to B$ as an action of A on B in the category of $\mathbb{P}_n$-algebras. Notice that the ∞-category $\mathbb{P}_n-\text{alg}$ is a closed symmetric monoidal ∞-category, so that it makes sense to talk about internal hom-objects. In particular, given any $\mathbb{P}_n$-algebra X, there is a well defined endomorphism object $\text{End}_{\mathbb{P}_n}(X)$; apart from being by definition a $\mathbb{P}_n$-algebra, $\text{End}_{\mathbb{P}_n}(X)$ is also in a natural way an $\mathbb{E}_1$-algebra inside the category $\mathbb{P}_n-\text{alg}$. By the additivity theorem 5.1.1, $\text{End}_{\mathbb{P}_n}(X)$ is thus a $\mathbb{P}_{n+1}$-algebra.

Said in another way, suppose we are given a $\mathbb{P}_{n+1}$-algebra A and a $\mathbb{P}_n$-algebra B. To complete this data to a $\mathbb{P}_{(n+1,n)}$-structure we need a morphism

$$A \longrightarrow \text{End}_{\mathbb{P}_n(B)}.$$

of $\mathbb{P}_{n+1}$-algebras. Recall that on the other hand the missing piece in order to obtain a $\mathbb{P}_{[n+1,n]}$-structure is a morphism

$$A \longrightarrow Z(B),$$

again in the category of $\mathbb{P}_{n+1}$-algebras, where $Z(B)$ is the Poisson center of chapter 3.

It is therefore natural to propose the following.

Conjecture 5.2.1. *Let $\mathcal{M}$ be an ∞-category satisfying our starting assumptions of chapter 0, and let $B \in \mathbb{P}_n-\text{alg}_{\mathcal{M}}$ be a $\mathbb{P}_n$-algebra in $\mathcal{M}$. Then there is an equivalence*

$$\text{End}_{\mathbb{P}_n}(B) \simeq Z(B)$$

of $\mathbb{P}_{n+1}$-algebras.

We remark that this conjecture is quite natural: the Poisson center is supposed to encode $\mathbb{P}_n$-deformations of B, and in the case of an $\mathbb{E}_n$-algebra it is indeed true that $\mathbb{E}_n$-deformations are controlled by the endomorphisms object.

Rephrased in another way, the conjecture states that the Poisson center $Z(B)$ is a concrete model for the internal object of endomorphism of the $\mathbb{P}_n$-algebra B.

Notice that as a consequence of the above conjecture, one would immediately get that, given a map $f : X \to Y$ of derived Artin stack locally of finite presentation, there is an equivalence

$$\text{Cois}(f,n) \simeq \widetilde{\text{Cois}}(f,n)$$

of spaces.

Alternatively, we end this chapter by sketching a proposal for the additivity functor of Theorem 5.1.1.

Let A be $\mathbb{P}_{n+1}$-algebra: in particular, it is of course a simple $\mathbb{E}_1$-algebra, and as such we can consider its bar complex $B(A)$. As a graded k-module, it is defined to be

$$B(A) \simeq \bigoplus_{k \in \mathbb{N}} A^{\otimes k}[k]$$

and it has the standard bar differential. With the usual deconcatenation coproduct, $B(A)$ has the structure of a coassociative coalgebra. Since the multiplication on A is also commutative, we can endow $B(A)$ with an additional commutative product, which is compatible with the coproduct in the sense that

$$B(A) \in \mathbb{E}_1 - \mathrm{CoAlg}(\mathrm{CAlg}),$$

where $\mathbb{E}_1 - \mathrm{CoAlg}(\mathrm{CAlg})$ is the category of coassociative coalgebras inside the monoidal category of commutative algebras. See Section 1 in [GJ] for more details on these classical constructions.

The additional data of the shifted Lie bracket on A can be used to give $B(A)$ a compatible structure of a $\mathbb{P}_n$-algebra. More specifically, we have the following result.

Proposition 5.2.2. *Let A be a $\mathbb{P}_{n+1}$-algebra. Then its bar complex $B(A)$ is in a natural way a coassociative coalgebra inside the category of $\mathbb{P}_n$-algebras, that is to say*

$$B(A) \in \mathbb{E}_1 - \mathrm{CoAlg}(\mathbb{P}_n - \mathrm{alg}).$$

The proof of this proposition can be found in [Sa], section 1.4. Assuming Conjecture 3.4.5 in [FG], the above proposition yields the existence of an ∞-functor

$$\mathbb{P}_{n+1} - \mathrm{alg} \longrightarrow \mathbb{E}_1 - \mathrm{alg}(\mathbb{P}_n - \mathrm{alg})$$

which is a candidate for the additivity functor of Theorem 5.1.1.

If we interpret Definition 5.1.2 using the above additivity functor, we can concretely compare the various definition of coisotropic structures. In fact, the following result is an immediate consequence of Proposition 1.8 in [Sa].

Proposition 5.2.3. *Let (A, B) be a $\mathbb{P}_{[n+1,n]}$-algebra. Then using the functor above A can be regarded as an associative algebra A' inside the category of $\mathbb{P}_n$-algebras, and B becomes naturally an A'-module. Said in another way, there is a functor*

$$\mathbb{P}_{[n+1,n]} - \mathrm{alg} \longrightarrow \mathbb{P}_{(n+1,n)} - \mathrm{alg}$$

where we make sense of the left hand side using the functor constructed thanks to proposition 5.2.2.

In particular, this shows that any coisotropic structure in the sense of chapter 4 is also a coisotropic structure in the sense of definition 5.1.2. We plan to complete the proof of the expected equivalence between the two definitions in the near future.

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