UNIVERSITE PARIS DIDEROT (Paris 7)

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Victoria LEBED

Objets tressés : une étude unificatrice de structures algébriques et une catégorification des tresses virtuelles

---

dirigée par Marc ROSSO

Soutenue le 13 décembre 2012 devant le jury composé de :

M. Frédéric Chapoton	Université Lyon 1	examinateur
M. Claude Cibils	Université Montpellier 2	examinateur
M$^{\text{me}}$ Muriel Livernet	Université Paris 13	examinateur
M. Paul-André Melliès	Université Paris 7	examinateur
M. Frédéric Patras	Université Nice	rapporteur
M. Marc Rosso	Université Paris 7	directeur

Rapporteur absent lors de la soutenance :
M. Józef Przytycki George Washington University

Institut de Mathématiques de Jussieu
175, rue du Chevaleret
75013 Paris

École Doctorale Paris Centre
Case 188 4 place Jussieu
75252 Paris cedex 05

Aux sourires des gens qui m'entourent

The mathematical world is "connected".
A.Connes. Advice to the beginner

Remerciements

En respectant les traditions des remerciements, devenus déjà un véritable genre littéraire, je commence par exprimer ma profonde gratitude à mon directeur de thèse, Marc Rosso, qui a accepté de me guider dans ce voyage initiatique à la recherche, sans que la destination finale soit très claire pour aucun d'entre nous. Merci de m'avoir donné des points de repère dans mon mouvement brownien de début de thèse, d'avoir su placer mes avancements dans le système de coordonnées (à beaucoup de dimensions) des mathématiques actuelles, et, ce qui est moins banal que cela puisse paraître, de m'avoir soutenue dans toutes les démarches administratives parfois bien enchevêtrées.

Je remercie d'autres mathématiciens qui m'ont inspirée, écoutée ou encouragée pendant ce long trajet. Merci à Frédéric Patras et à Józef Przytycki pour le travail titanesque de rapporter une thèse de 180 pages. Merci encore à Józef Przytycki pour l'occasion unique de parler à Knots in Washington, et surtout pour la découverte, très rassurante, d'une personne qui se pose des questions auxquelles je sais répondre, au moins partiellement. Merci à Muriel Livernet d'avoir accepté de faire partie du jury, et d'organiser le séminaire de topologie algébrique de Paris 13 avec tant de souplesse. Merci à Frédéric Chapoton pour le double crédit de confiance témoigné en venant à ma soutenance à Paris et en m'invitant à parler à Lyon. Merci à Paul-André Melliès pour cette ouverture vers le monde des logiciens, qui a renforcé en moi la conviction que le monde mathématique est simplement connexe. Merci à Claude Cibils pour la participation dans le jury et pour l'algèbre X. Merci à J. Scott Carter pour son intérêt dans ma recherche et pour ses travaux qui m'ont souvent inspirée. Merci à Bernhard Keller qui a encadré mon mémoire de maîtrise, et qui m'a montré que les mathématiques peuvent être rigoureuses et accessibles même aux niveaux avancés. Merci à David Hernandez de m'avoir montré que les anciens étudiants de Marc Rosso réussissent parfaitement dans la vie mathématique et sociale, de m'avoir invitée à parler au Séminaire d'algèbre et d'y inviter régulièrement des gens fort intéressants. Merci à Christian Kassel et à ses livres attisant l'intérêt des jeunes mathématicien-ne-s vers des sujets quantiques et tressés. Merci à Christian Blanchet pour son rôle dans la vie topologique de Paris 7, et dans l'intégration des jeunes dans cette vie bien fascinante.

Mes remerciements chaleureux vont également à tous les mathématiciens qui organisent les olympiades de mathématiques en Biélorussie et qui préparent notre équipe pour les compétitions internationales. Ce sont eux qui m'ont appris le principe que je suis toujours, élégamment formulé par Michael Atiyah : "search for beauty and find truth along the way". Merci ensuite aux nombreux professeurs de l'ENS et de Paris 7 qui par leur façon d'enseigner ont confirmé ce principe, entre autres à Marc Rosso, Patrick Dehornoy, Michel Broué, Ivan Marin, Bernhard Keller, Christian Blanchet, Julien Marché.

J'ai une pensée particulière pour Stéphane Vassout et Olivier Bokanowski qui ont accompagné mes premiers pas en enseignement et ont partagé avec moi leur riche expérience pédagogique. Merci également à mes élèves qui m'ont certainement appris bien plus sur l'enseignement que j'ai pu leur apprendre sur les mathématiques.

Mes remerciements vont aux thésards que j'ai côtoyés à l'IMJ pour la bonne ambiance mathématique, pour tous les groupes de travail plus ou moins formels qu'on a organisés, et pour le soutien permanent dans les moments difficiles. Un merci particulier va à Xin Fang, mon "frère de thèse", pour toutes les questions posées pendant mes exposés.

Je voudrais terminer la partie "mathématique" de ces remerciements en évoquant le nom de Jean-Louis Loday qui nous a quittés récemment. Sans avoir eu le bonheur de le connaître personnellement, j'ai toujours eu l'impression d'avoir un contact avec lui à travers ses travaux, où je retrouvais souvent mes idées récentes, mais en plus beau, plus développé et plus éclairci.

Pour éviter une liste interminable des gens que je voudrais remercier dans la partie "non-mathématique" de ce texte, je vais simplement évoquer quelques endroits où cette thèse a été ruminée et rédigée, et où j'ai pu puiser constamment des forces pour accomplir ce travail. Les gens qui y sont associés se reconnaîtront.

L'IMJ est sans doute le premier endroit à mentionner, avec une vie mathématique bouillonnante, avec des secrétaires capables de résoudre des problèmes a priori insolubles (là je ne peux pas m'empêcher d'évoquer un nom – celui de Pascal Chiettini, notre "secrétaire universel"), avec des pauses thé ou café à volonté, avec la cantine et son personnel accueillant et toujours prêt pour une blague. Mes co-bureaux de 8C24 et 7C8, mes co-couloirs, et les gens du 3ème qu'on appelle quand on a faim, sont bien sûr à évoquer ici. Je vous remercie pour les croissants secrets du mercredi / jeudi matin (chut), pour les leçons de français et de jeux de mots à la française, pour mon anniversaire le jour où je le veux ;) , pour les délires du vendredi après-midi, pour toutes les créations collectives plus ou moins artistiques et plus ou moins podes ;) , pour mon plaid bien chaud, pour mes plantes qui pouvaient toujours squatter à Chevaleret en vacances sans mourir de soif, pour les matches de foot du dimanche, et par-dessus tout pour les sourires, car sans cela la vie n'est pas marrante. Bon courage et beaucoup de patience à ceux d'entre vous qui ne se sont pas encore posé les questions incontournables de la composition du jury, du pot de thèse, de la couleur de la couverture etc. Et bonne chance à ceux qui sont, comme moi, en quête de poste pour l'année prochaine.

Je tiens à mentionner ici l'Institut de Mathématiques de Toulouse où j'ai pu travailler occasionnellement dans de très bonnes conditions ; Gwatt-Zentrum en Suisse ; notre petit appartement familial à Minsk presque sans Internet – ce qui invite bien à la méditation scientifique ; le chalet de ma belle-famille aux Rousses, avec les maths entre le ski et un bon repas façon "mamie" ; la maison de mes beaux-parents à Thionville, avec un jardin propice au travail ; et l'aéroport de Moscou, où, grâce aux prises électriques, on peut bien travailler même si on y reste coincé toute une nuit :) . Des parties de ma thèse viennent de tous ces endroits éclectiques, et je suis reconnaissante à tous les gens qui m'y ont entourée.

Je pense aussi à la salle de sport de l'ENS qui a vu des matchs de basket Biélorussie-Chine (passons sous silence qui gagnait tout le temps :)), à toutes les écoles de danse que j'ai découvertes à Paris, à tous les voyages chez des/entre amis, aux sorties escalade, aux sorties culturelles, à toutes sortes de découvertes culinaires. Merci à tous ceux qui ont partagé ces moments de joie et de détente avec moi. Mes remerciements particuliers vont à mes chères colocataires pour avoir une oreille et une tasse de thé toujours prêtes ;) .

Je termine, toujours en respectant les traditions du genre, par un grand merci à mon chéri qui a su supporter une femme qui trouve toujours des contre-exemples à ses idées mathématiques :p, qui lui donne à corriger son anglais et français (y compris dans ces remerciements :)) et en plus corrige sans cesse son russe, qui dérive souvent vers la question mathématique la plus interdisciplinaire : "À quoi ça sert ?..", et qui n'arrive pas à admettre que dans la vie on ne peut pas atteindre la perfection mathématique.

Résumé

Résumé

Dans cette thèse on développe une théorie générale des objets tressés et on l'applique à une étude de structures algébriques et topologiques.

La partie I contient une *théorie homologique des espaces vectoriels tressés et modules tressés*, basée sur le coproduit de battage quantique. La construction d'un *tressage structurel* qui caractérise diverses structures – auto-distributives (AD), associatives, de Leibniz – permet de généraliser et unifier des homologies familières. Les hyper-bords de Loday, ainsi que certaines opérations homologiques, apparaissent naturellement dans cette interprétation.

On présente ensuite des concepts de *système tressé* et *module multi-tressé*. Appliquée aux bigèbres, bimodules, produits croisés et (bi)modules de Hopf et de Yetter-Drinfel′d, cette théorie donne leurs interprétations tressées, homologies et actions *adjointes*. La notion de *produits tensoriels multi-tressés* d'algèbres donne un cadre unificateur pour les doubles de Heisenberg et Drinfel′d, ainsi que les algèbres X de Cibils-Rosso et Y et Z de Panaite.

La partie III est orientée vers la topologie. On propose une *catégorification des groupes de tresses virtuelles* en termes d'objets tressés dans une catégorie symétrique (CS). Cette approche de *double tressage* donne une source de représentations de VB_n et un traitement catégorique des racks virtuels de Manturov et de la représentation de Burau tordue. On définit ensuite des structures AD dans une CS arbitraire et on les munit d'un tressage. Les techniques tressées de la partie I amènent alors à une théorie homologique des *structures AD catégoriques*. Les algèbres associatives, de Leibniz et de Hopf rentrent dans ce cadre catégorique.

Mots-clefs

objet tressé ; homologie algébrique ; caractère ; module tressé ; algèbre de battage quantique ; complexe de Koszul ; homologie de quandle/rack ; homologie de Hochschild ; algèbre de Leibniz ; hyper-bord de Loday ; système tressé ; produit tensoriel multi-tressé ; produit croisé ; module de Yetter-Drinfel′d ; (bi)module de Hopf ; double de Heisenberg ; double de Drinfel′d ; algèbre X ; R-matrice ; groupes de tresses virtuelles ; rack virtuel ; auto-distributivité catégorique ; représentation de Burau (tordue) ; structures auto-distributives libres.

Braided Objects: Unifying Algebraic Structures and Categorifying Virtual Braids

Abstract

This thesis is devoted to an abstract theory of braided objects and its applications to a study of algebraic and topological structures.

Part I presents our general *homology theory for braided vector spaces and braided modules*, based on the quantum co-shuffle coproduct. The construction of *structural braidings* characterizing different algebraic structures – self-distributive (SD) structures, associative / Leibniz algebras, their representations – allows then to generalize and unify familiar homologies. Loday's hyper-boundaries and certain homology operations are efficiently treated via our braided tools.

We further introduce a concept of *braided system* and *multi-braided module* over it. This enables a thorough study of bialgebras, crossed products, bimodules, Yetter-Drinfel′d and Hopf (bi)modules: their braided interpretation, homologies and *adjoint* actions. A theory of *multi-braided tensor products* of algebras gives a unifying context for Heisenberg and Drinfel′d doubles, the algebras X of Cibils-Rosso and Y and Z of Panaite.

Part III is topology-oriented. We start with a *hom-set type categorification of virtual braid groups* in terms of braided objects in a symmetric category (SC). This *double braiding* approach provides a source of representations of VB_n and a new categorical treatment for Manturov's virtual racks and the twisted Burau representation. We then define SD structures in an arbitrary SC and endow them with a braiding. The associativity and Jacobi identities in an SC are interpreted as SD conditions. Hopf algebras enter in the SD framework as well. Braided techniques from part I give a homology theory of *categorical SD structures.*

Keywords

braided object; algebraic homology; character; braided module; quantum shuffle algebra; Koszul complex; rack/quandle homology; Hochschild homology; Leibniz algebra; Loday's hyper-boundaries; braided system; multi-braided tensor product; crossed product; Yetter-Drinfel′d module; Hopf (bi)module; Heisenberg double; Drinfel′d double; algebra X; R-matrix; virtual braid groups; virtual rack; categorical self-distributivity; (twisted) Burau representation; free (virtual) shelf; free (virtual) quandle.

Contents

Chapter 1

Introduction

This thesis is devoted to several rather unexpected (at least to the author) interactions between the vast concepts of algebraic structure and braiding, their virtual and categorical versions, and homological applications.

In the first part, our starting point is the following procedure, which is at the heart of the homological algebra and which has become omnipresent in modern mathematics:

$$\boxed{\text{algebraic structure} \rightsquigarrow \text{chain complex.}}$$

Figure 1.1: Homology of algebraic structures

The step $\rightsquigarrow$ is far from being canonical, and can be dictated by motivations of very different nature: one can think in terms of

- structure deformations and obstructions (in the sense of M.Gerstenhaber, cf. [28]),
- or classification questions (of the derivations of an algebra for instance),
- or derived functors (the famous Ext and Tor functors for example),
- or generalizations of the "hole-counting" homologies of topological objects (for instance, regarding the notion of algebra as a generalization of the algebra of functions on a space),
- or topological applications (trying to devise a state-sum knot invariant using the fundamental quandle of a knot, cf. [11]).

Here we propose to forget all these motivations and to regard the step $\rightsquigarrow$ from a purely combinatorial viewpoint, in the spirit of operad theory. The complexes one associates in practice to basic algebraic structures on a vector space V usually have the same flavor: they are all **signed** sums $d_n = \sum_{i=1}^{n}(-1)^{i-1}d_{n;i} : V^{\otimes n} \to V^{\otimes(n-1)}$ of **terms of the same nature** $d_{n;i}$, one for each component $1, 2, \ldots, n$ of $V^{\otimes n}$.

The examples we have in mind are the following:

$$\begin{array}{rcl} \text{vector space} & \rightsquigarrow & \text{Koszul complex,} \\ \text{associative algebra} & \rightsquigarrow & \text{bar and Hochschild complexes,} \\ \text{Lie algebra} & \rightsquigarrow & \text{Chevalley-Eilenberg complex,} \\ \text{self-distributive structure} & \rightsquigarrow & \text{rack complex.} \end{array}$$

Verifying that one has indeed a differential, i.e. $d_{n-1} \circ d_n = 0$, can be reduced to checking some **local** algebraic identities (which mysteriously coincide with the defining properties for our algebraic structure!) coupled with a sign manipulation, no less mysterious.

For many algebraic structures, their chain complexes can be refined by introducing a (weakly) (pre)(bi)**simplicial structure** on $T(V)$ (see section 3.2 or J.-L.Loday's book [46] for the simplicial vocabulary). Moreover, the degree -1 differentials can be generalized to **Loday's hyperboundaries** of arbitrary degree (see the definitions from section 3.3, or exercise E.2.2.7 in [46], from which this notion takes inspiration). Certain **homology operations**, similar for different algebraic structures, are also to be mentioned here. Some of such common features are presented, for the example of associative and self-distributive structures, in J.Przytycki's paper [67].

In this work, we propose to interpret and partially explain these parallels (typed in bold letters above) and mysteries by adding a new step to the scheme in figure 1.1:

algebraic structure $\overset{\text{case by case}}{\rightsquigarrow}$ **pre-braiding** $\overset{\text{theorem 2}}{\rightsquigarrow}$ chain complex.

Figure 1.2: Homology of algebraic structures via pre-braidings

After a short reminder on braided structures in chapter 2, we proceed to describing in detail the right part of this new scheme. More precisely, given a vector space endowed with a ***pre-braiding*** $\sigma : V \otimes V \to V \otimes V$ satisfying the ***Yang-Baxter equation*** (=YBE)

$$(\sigma \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma) \circ (\sigma \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes \sigma) \circ (\sigma \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma) \in \mathrm{End}(V^{\otimes 3}),$$

we associate in theorem 2 a bidifferential on $T(V)$ to any couple of ***braided characters*** (= elements of V^* "respecting" the pre-braiding σ) ϵ and ζ, using ***quantum co-shuffle comultiplication*** techniques (cf. M.Rosso's pioneer papers [71],[72]). We call such (bi)differentials *braided*.

In theorem 3 we refine these braided bidifferential structures: we show that they come from a pre-bisimplicial structure on $T(V)$, completed to a weakly bisimplicial one if V is moreover endowed with a "nice" ***comultiplication*** Δ (= coassociative, σ-cocommutative, and compatible with σ). This is done using the graphical calculus (in the spirit of J.C.Baez [2], S.Majid [53] and other authors), appearing naturally due to our use of "braided" techniques. For us, the ***graphical calculus*** is an illustrating tool, a convenient method of presenting some proofs and also an important source of inspiration. Here are for example the components of the weakly bisimplicial structure from the theorem (all diagrams are to be read from bottom to top here):

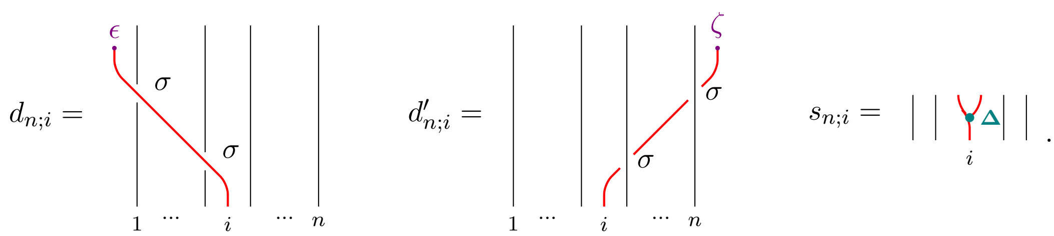


Figure 1.3: Weakly bi-simplicial structure for braided homology

See table 3.1 for a comparison of the quantum co-shuffle and the graphical approaches to braided differentials.

Note that we never demand σ to be **invertible**, which is emphasized by the prefix *pre*-braiding. Rare in literature, this elementary generalization of the notion of braiding allows interesting examples.

Armed with this general homology theory for pre-braided vector spaces, we are now interested in the left part of figure 1.2. Unfortunately we do not know any systematic way of associating a pre-braiding to an algebraic structure. So we do it by hand in chapter 4 for each of the four structures in the above list. In each case, the "*structural*" pre-braiding σ we propose **encodes** surprisingly well the structure in question, in the sense that

- ✓ YBE for σ is equivalent to the defining relation for the structure (e.g. the associativity for an algebra), under some mild assumptions concerning units;
- ✓ the invertibility condition for σ, when this makes sense, translates important algebraic properties (e.g. the rack condition);
- ✓ usual characters for these structures become braided characters for σ;
- ✓ the comultiplication necessary for constructing the degeneracies turns out to be quite characteristic of the structures;
- ✓ morphisms respecting the structural braidings are essentially the same as morphisms preserving the original structure.

Thus the left part of the scheme in figure 1.2 can be informally stated in a stronger way:

$$\boxed{\text{"algebraic structure = pre-braiding".}}$$

Note that the pre-braidings we propose vary from well-known ones (that for self-distributive structures) to original ones (that for associative algebras).

Our braided complexes form a **unifying framework for studying homologies** of different algebraic structures. Besides its generality, this approach has other advantages:

- ⊞ Our construction produces two compatible differentials – a left and a right one – for each braided character, these differentials often being compatible even for different characters. One can combine these differentials, obtaining a **family of homology theories** for the same algebraic structure. A nice illustration is given by self-distributive structures, with
  - (a) usual shelf (or one-term distributive; cf. [68], [67]), rack ([27]) and quandle ([11]) homology theories;
  - (b) the partial derivatives of M.Niebrzydowski and J.Przytycki ([63]);
  - (c) and the twisted rack homology of J.S.Carter, M.Elhamdadi and M.Saito ([9]).
- ⊞ The technical sign manipulation (especially heavy for the Chevalley-Eilenberg complex) is controlled either by using the negative pre-braiding $-\sigma$ in the quantum co-shuffle comultiplication, or by counting the number of intersections in the graphical interpretation.
- ⊞ The identities $d_{n-1} \circ d_n = 0$, which are of "global" nature, are replaced with the YBE for the corresponding pre-braiding, which is "local" and thus easier to verify.
- ⊞ The decomposition $d_n = \sum_{i=1}^{n} (-1)^{i-1} d_{n;i}$ becomes natural when one reasons in terms of braids and strands.
- ⊞ So do some homology operations – for instance, the generalizations of the homology operations for shelves, defined by M.Niebrzydowski and J.Przytycki in [63].
- ⊞ Subscript chasing (in the relations defining simplicial structures for example) is substituted with the more transparent "strand chasing" (cf. remark 3.2.4).
- ⊞ J.-L.Loday's hyper-boundaries of degree $-i$ arise naturally in the co-shuffle interpretation: one simply replaces the $V^{\otimes n} \to V \otimes V^{\otimes (n-1)}$ component of the quantum co-shuffle comultiplication with the $V^{\otimes n} \to V^{\otimes i} \otimes V^{\otimes (n-i)}$ component.

We thus recover all the common features of different algebraic homology theories observed above. Moreover, we obtain a simplified and conceptual way of proving $d^2 = 0$, as well as of "guessing" the right boundary map.

As an illustration to the last remark, note that the "braided" considerations have naturally lead us to lifting the Chevalley-Eilenberg complex from the external to the tensor algebra of a Lie algebra, and to observing that this construction works even for ***Leibniz algebras*** (= "non-anticommutative Lie algebras"). We thus reinterpret the results of C.Cuvier and J.-L.Loday (cf. [16], [17], [46],[47],[48]) and recover for Leibniz algebras the braiding studied in the Lie algebra case by A.Crans (cf. [15], [8]). In particular, one automatically obtains the signs and the element positions in the lift of Chevalley-Eilenberg differential, which are otherwise difficult to guess.

The only approaches to homologies of braided spaces we have found in literature are:

1. the *homology theory for solutions to the set-theoretic Yang-Baxter equation*, developed by J.S.Carter, M.Elhamdadi and M.Saito in [10];
2. the *braided-differential calculus* of S.Majid ([50]);
3. M.Eisermann's *Yang-Baxter cochain complex* ([22]).

We recall them briefly in this work, explaining how our constructions generalize the first two approaches. As for the last one, in spite of being of different nature, it seems (in a sense still obscure for us) connected to our braided homology theory.

The structural pre-braidings allow to define a unifying notion of modules over pre-braided spaces. Concretely, a ***braided module*** over a pre-braided space (V, σ) is a space M equipped with a linear map $\rho : M \otimes V \to M$, satisfying

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes \sigma) : M \otimes V \otimes V \to M.$$

One recovers the usual notions of modules over associative/Leibniz algebras and other familiar structures in the examples above. These braided modules are natural candidates for coefficients in the braided complexes, leading to ***braided homologies with coefficients***, studied in chapter 6. As usual, this braided construction allows to recover familiar algebraic homologies with coefficients.

In an attempt to interpret homology theories for *bialgebras, Hopf (bi)modules and Yetter-Drinfel′d modules* in terms of braided complexes (the well-known pre-braiding σ_{YD} on the category of Yetter-Drinfel′d modules being very suggestive of such an interpretation), one feels that the formalism of pre-braided vector spaces does not have enough flexibility for encoding all the complexity of these structures. The tool we propose in part II is the notion of ***pre-braided system*** of vector spaces. It consists of a finite collection of spaces $V_1, V_2, \ldots, V_r$ endowed with morphisms

$$\sigma_{i,j} : V_i \otimes V_j \longrightarrow V_j \otimes V_i \quad \forall\, 1 \leqslant \underline{i \leqslant j} \leqslant r,$$

satisfying the YBE on all the tensor products $V_i \otimes V_j \otimes V_k$ with $1 \leqslant \underline{i \leqslant j \leqslant k} \leqslant r$. The notion of braided module generalizes to that of ***multi-braided module*** over a pre-braided system in a natural way. Braided complexes (including complexes with coefficients) also generalize to the setting of pre-braided systems.

We present pre-braided systems encoding the structures of bialgebra, module-algebra and Yetter-Drinfel′d (=YD) module, automatically recovering their homology theories. Only <u>finite-dimensional</u> bialgebras (or graded and finite-dimensional in every degree) are considered here, since we need their dual bialgebras as well. Note that the existence of the

antipode is equivalent to the **invertibility** of one of the components of the pre-braiding for bialgebras.

As a by-product of the braided interpretation of YD modules, we naturally recover two definitions of **tensor products of YD modules**, proposed by L.A.Lambe and D.E.Radford in [44].

As for Hopf (bi)modules, they are treated as multi-braided modules over appropriate pre-braided systems. The same is done for YD modules, giving an alternative "braided" viewpoint on this structure (the first one being that of a part of a pre-braided system).

The last interpretations can make one think of the common treatment of "complicated" structures over (bi)algebras as "simple" algebra module structures over more "complicated" associative algebras:

"complicated" structure	**corresponding "complicated" algebra**
bimodule over an algebra A	enveloping algebra $A \otimes A^{op}$
YD module over a bialgebra H	Drinfel′d double $\mathscr{D}(H) := H^* \otimes H^{op}$
Hopf module over a bialgebra H	Heisenberg double $\mathscr{H}(H) := H^* \otimes H$
Hopf bimodule over a Hopf algebra H	algebras $\mathscr{X}(H)$, $\mathscr{Y}(H)$ and $\mathscr{Z}(H)$

Table 1.1: Algebras encoding Hopf and Yetter-Drinfel′d (bi)module structures

The algebra

$$\mathscr{X}(H) := (H^*)^{op} \otimes H^* \otimes H^{op} \otimes H$$

was introduced by C.Cibils and M.Rosso in [14], and its isomorphic versions $\mathscr{Y}(H)$ and $\mathscr{Z}(H)$ were suggested by F.Panaite in [65]. These interpretations were efficiently used by R.Taillefer ([77] and [78]) in a comparison of different homology theories for bialgebras and Hopf bimodules.

Note that the algebra structures on all the tensor products above are not the usual ones, but the braided ones, with carefully chosen pre-braidings. Namely, the multiplication on the tensor product of algebras (A, μ_A) and (B, μ_B) endowed with a linear map $\sigma_{B,A} : B \otimes A \to A \otimes B$ is defined by

$$\mu_{A \otimes B} := (\mu_A \otimes \mu_B) \circ (\mathrm{Id}_A \otimes \sigma_{B,A} \otimes \mathrm{Id}_B) : (A \otimes B) \otimes (A \otimes B) \to A \otimes B,$$

with an obvious generalization for a tensor product of r algebras $A_1, \ldots, A_r$ endowed with $\frac{r(r-1)}{2}$ maps. We call the resulting algebra the ***multi-braided tensor product*** of the algebras $A_1, \ldots, A_r$.

This braided tensor product construction is at the heart of the *braided geometry*, introduced by S.Majid in a long series of papers in the 1990's (cf. for example [50], [51], [52]). S.Majid's motivation was to develop an algebra analogue of the product of spaces in noncommutative geometry. A pleasant consequence of his work was the construction of new examples of non-commutative non-cocommutative Hopf algebras via the *bicrossproduct* construction (which is a particular case of braided tensor product).

Returning to the <u>multi</u>-braided tensor products, we interpret in theorem 7 the compatibilities of the maps σ_{A_i,A_j}, necessary for the unambiguity of the definition of the multi-braided tensor product on $A_1 \otimes \cdots \otimes A_r$ and for its associativity, in terms of YBEs making out of the σ_{A_i,A_j}'s a pre-braiding on the system $A_r, \ldots, A_1$. We recover in particular the results of P.Jara Martínez, J.López Peña, F.Panaite and F. van Oystaeyen, cf. [32] (their notion of *iterated twisted tensor product* coincides with our notion of multi-braided tensor product). The category of modules over a multi-braided tensor product of algebras is then shown to be equivalent to the category of multi-braided modules over the corresponding pre-braided system. Schematically, these results can be summarized as

"multi-braided $\otimes$ of algebras	$=$	pre-braided system of algebras",
$\mathbf{Mod}_{A_1\otimes\cdots\otimes A_r}$	$\simeq$	$\mathbf{Mod}_{(A_r,\ldots,A_1)}$.

We also prove that in general one can permute the factors V_i and V_{i+1} of a pre-braided system if $\sigma_{i,i+1}$ is invertible, with a suitable change in the pre-braiding of the system. In the case of a pre-braided system of algebras, this results in an explicit algebra isomorphism (note the inverse order of subscripts)

$$\mathrm{Id}_{A_1\otimes\ldots\otimes A_{i-1}}\otimes\sigma_{i+1,i}^{-1}\otimes\mathrm{Id}_{A_{i+2}\otimes\ldots\otimes A_r},$$

inducing an equivalence of their representation categories. In particular, one automatically gets explicit isomorphisms between the algebra $\mathscr{X}(H)$, $\mathscr{Y}(H)$ and $\mathscr{Z}(H)$, including them into a family of $4! = 24$ pairwise isomorphic braided tensor products of algebras, since all the $\sigma_{j,i}$'s from the corresponding pre-braiding with $j > i$ are invertible.

Another application of the pre-braided system theory proposed here is a study of the ***generalized two-sided crossed products*** $A\blacktriangleright\!\!<C>\!\!\blacktriangleleft B$, defined by D.Bulacu, F.Panaite and F.Van Oystaeyen in [5].

Combining the "braided" vision of Hopf bimodules with the theory of ***adjoint modules*** which we develop in the multi-braided settings, one gets pleasant **homological consequences**. In particular, we recover, without tedious verifications, the Hopf bimodule structure on the bar complex of a bialgebra with coefficients in a Hopf bimodule. This structure was used by R.Taillefer for defining a cohomology theory of a pair of Hopf bimodules ([77], [78]).

Table 1.2 presents braided structures encoding the algebraic structures mentioned above, and the familiar complexes recovered as particular cases of our braided complexes. Let us note the importance of these concrete examples: for us they were a guideline for building the braided homology theory. On the other hand, the developed theory allowed the author to recover several homology structures she was not aware of, for example that for Leibniz algebras or that for YD modules.

In the table, the component σ_{H,H^*} of the pre-braided system encoding the bialgebra structure is inspired by the pre-braiding for YD modules. Explicitly,

$$\sigma_{H,H^*} = \tau\circ(\mathrm{Id}_H\otimes ev\otimes Id_{H^*})\circ(\Delta\otimes\mu^*) : H\otimes H^*\to H^*\otimes H,$$

where the *flip* τ simply transposes the components H and H^*, and $ev : H\otimes H^*\to R$ is the usual evaluation map.

structure	**pre-braiding**	**invertibility**	**braided characters**
vector space V	flip τ: $v\otimes w\mapsto w\otimes v$	$\tau^{-1}=\tau$	any $\epsilon\in V^*$
unital associative algebra $(V,\mu,\mathbf{1})$	σ_μ: $v\otimes w\mapsto$ $\mathbf{1}\otimes\mu(v\otimes w)$	no inverse in general	algebra character: $\epsilon(\mu(v\otimes w))=\epsilon(v)\epsilon(w)$, $\epsilon(\mathbf{1})=1$
unital Leibniz algebra $(V,[,],\mathbf{1})$	$\sigma_{[,]}$: $v\otimes w\mapsto w\otimes v$ $+\mathbf{1}\otimes[v,w]$	$\exists\ \sigma_{[,]}^{-1}$	Lie character: $\epsilon([v,w])=0$, $\epsilon(\mathbf{1})=1$
shelf $(S,\lhd)$, $V:=\Bbbk S$	$\sigma_\lhd$: $(a,b)\mapsto(b,a\lhd b)$	$\exists\ \sigma_\lhd^{-1}$ iff S is a rack	shelf character: $\epsilon(a\lhd b)=\epsilon(a)$
bialgebra H	(H,H^*) σ_μ, σ_{Δ^*}, σ_{H,H^*}	iff H is a Hopf algebra	ε_H & ε_{H^*}
YD module M over H	(H,M,H^*) σ_μ, σ_{Δ^*}, σ_{H,H^*}, σ_{YD}	iff H is a Hopf algebra	ε_H & ε_{H^*}

structure	braided module	Δ	complexes
vector space	space endowed with commutative operators	–	Koszul
unital associative algebra	algebra module: $m \cdot \mu(v \otimes w) = (m \cdot v) \cdot w$	$\Delta(v) = \mathbf{1} \otimes v$	bar, Hochschild
unital Leibniz algebra	Leibniz module ([46]): $m \cdot [v, w] = (m \cdot v) \cdot w - (m \cdot w) \cdot v$	$\Delta(v) = v \otimes \mathbf{1} + \mathbf{1} \otimes v$, $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$	Leibniz, Chevalley-Eilenberg
shelf	shelf module ([13]): $(m \cdot a) \cdot b = (m \cdot b) \cdot (a \lhd b)$	$\Delta(a) = (a, a)$	shelf ([67],[68]), rack ([27]), quandle ([11])
bialgebra	right-right Hopf module	–	Gerstenhaber-Schack ([29]), Panaite-Ştefan ([66])
YD module	–	–	Panaite-Ştefan ([66])

Table 1.2: Main braided homology ingredients in concrete algebraic settings

Besides the pre-braiding on the category of YD modules, another popular source of concrete pre-braidings (and thus, potentially, of braided homologies) is the one on the representation category of a quasi-triangular Hopf algebra H. We show in section 7.7 that the second pre-braiding is a particular case of the first one. This fact is probably well-known, but the author has not found it in literature. Concretely, one has a pre-braided category inclusion

$$i_R : {}_H\mathbf{Mod} \hookrightarrow {}_H\mathbf{YD}^H,$$
$$(M, \lambda) \mapsto (M, \lambda, \delta_R),$$

where the pre-braiding on ${}_H\mathbf{Mod}$ is given by the R-matrix R, and the comodule structure δ_R is defined using R and the module structure λ. Moreover, a weak version of the notion of R-matrix on H is shown to suffice for i_R to be a well-defined functor (not monoidal in general) and to respect the pre-braidings.

Our braided homology theory, as well as the pre-braidings for associative and Leibniz algebras, are raised to the **categorical level** in chapter 5; from that chapter on, we mostly work in the categorical setting. Several typical applications of the categorical approach are presented, obtained by changing the underlying category or using different types of categorical dualities:

✓ Leibniz <u>super</u>algebra homology;

✓ <u>co</u>homology theories for pre-braided objects;

✓ (co)chain complexes for dual structures (e.g. cobar and Cartier complexes for <u>co</u>algebras);

✓ right-left duality for braidings;

✓ right-left duality for braided differentials.

An important feature of our categorification of the notion of braiding, besides relaxing the **invertibility** condition, is its **"local"** character: instead of demanding the whole category to be pre-braided, one imposes a pre-braiding for a single object or a family of objects only, omitting in particular the naturality condition. In the case of categorified pre-braided systems, a third non-conventional point appears: the notion of braiding generalized this way becomes **"partial"**, i.e. it can be defined on $V \otimes W$ without being defined on $W \otimes V$, making the whole construction highly non-commutative.

A categorification of self-distributive (=SD) structures and of the corresponding pre-braiding is less straightforward. This is done in chapter 11, and constitutes one of the main results of part III of this thesis. Our approach is different from the one proposed by J.S.Carter, A.S.Crans, M. Elhamdadi, and M.Saito in [8] in that we make the **diagonal map** (necessary for writing down the SD condition) **a part of the categorical SD structure**, instead of requiring it on the level of the underlying category. The term ***categorical self-distributive structure*** is abbreviated as CSD here. Schematically,

CSD =	comultiplication Δ	+	binary operation $\lhd$
	✓ coassociative,		✓ self-distributive, with Δ as diagonal,
	✓ central cocommutative;		✓ respects Δ, in the braided bialgebra sense.

Table 1.3: Categorical self-distributivity

This choice of categorification is explained in chapter 11. One of the motivations comes from the connection with virtual braid group representations.

We further recover categorical **associative, Leibniz and Hopf algebras as CSD** for particular choices of the comultiplication, and the pre-braidings defined on them in part I as particular cases of the pre-braiding for CSD. A braided homology theory for CSD is also developed, with a particular role of ***categorical spindles***.

The remaining chapters of part III are devoted to a **categorification of the notion of virtual braids** and some other aspects of virtual braid group theory. We present here some motivations for such a categorification.

The almost century-old *braid theory* is nowadays quite vast and entangled, with unexpected connections with different areas of mathematics still emerging. Patrick Dehornoy's spectacular results intertwining self-distributivity, set theory and braid group ordering ([18]) provide a good example. *Virtual braid theory* (cf. section 8.2 for definitions and illustrations) dates from the pioneer work of L.H.Kauffman ([41]) and V.V.Vershinin ([80]) in the late 1990s, and it still reserves a lot of unexplored questions, in spite of numerous results already obtained. This shows that this theory is far from being an elementary variation of that of usual braids.

Virtual braids are most often considered in the context of virtual knots and links. The topological aspects of these objects are thus naturally in the spotlight. Our aim is on the contrary to clarify some *categorical and representational aspects.*

The flow of ideas related to the objects and concepts we are interested in here can be represented – very schematically – by the following chart:

Categories

$$\text{Topology} \underset{\text{intuition}}{\overset{\text{tools}}{\leftrightarrows}} \text{Algebra} \underset{\text{intuition}}{\overset{\text{tools}}{\leftrightarrows}} \text{Representation theory}$$

One has thus a sort of a triptych of picturesque mathematical areas, with category theory as a unifying background. Such unifications are precisely the raison d'etre of categories. This chart, certainly subjective and simplified, is quite adequate for the content of part III of this thesis.

Part III starts with an extensive reminder on braid groups B_n in section 8.1, where we extract from the large scope of existing results the ones to be extended to virtual braid groups VB_n in the rest of the part. Section 8.2 is a survey of the steps of this vast *virtualization program* which have already been effectuated. Particular attention is given to usual and *virtual SD structures*. Virtuality means here the additional datum of a shelf automorphism f, following V.O.Manturov ([54]). Subsequent chapters contain original patches to the still very fragmentary virtual braid theory.

Chapter 9 is devoted to ***free virtual SD structures***. The faithfulness of the VB_n^+ (or VB_n) action on these structures is discussed, including a (reformulation of a) conjecture of V.O.Manturov ([56]). Some arguments in favor of the faithfulness in the free monogenerated virtual shelf case are presented, however without a definite answer.

Theorem 13 is the heart of the topological part of this thesis. It suggests categorifying the VB_n's by **"locally" braided objects** in a **"globally" braided symmetric category $\mathcal{C}$**. Here is the correspondence between the algebraic notion and its categorification:

category level	"global" symmetric braiding on $\mathcal{C}$	"local" braiding for V
VB_n **level**	S_n part	B_n part

Table 1.4: A categorification of VB_n

One thus recovers the recurrent situation encountered in parts I and II: some pre-braidings were associated there to algebraic objects, often living themselves in a symmetric category. Thus our "structural" pre-braidings provide an unexpected source of representations of virtual braid groups.

One more feature inherited from parts I and II is the attention to non-invertible situations. We thus study ***positive virtual braid monoids*** VB_n^+, their categorification in terms of pre-braided objects in a symmetric category, and representations given by shelves – in particular by free shelves.

Our categorification of VB_n is quite different from that proposed by L.H.Kauffman and S.Lambropoulou in [42]. Their inspiration comes from representation theory (they discover strong connections between virtual braid groups and the algebraic Yang-Baxter equation), while our starting point is an attemps to "virtualize" the interpretation of usual braid groups as hom-sets of a free monogenerated braided category (theorem 12).

Among the advantages of our "double braiding approach" is its high **flexibility**, having two consequences:

1. Manturov's virtual racks are interpreted via a deformation of the underlying symmetric category structure;
2. the twisted Burau representation of D.S.Silver and S.G.Williams ([75]) is recovered by *twisting* both the "local" and the "global" braidings with the help of another symmetric braiding.

We finish the introduction by mentioning some of numerous new ***research directions*** continuing the ideas of this thesis.

The first direction concerns the behavior of our braided bidifferentials with respect to different operations on the complexes, for example the *cyclic* and the *shuffle* ones. We have in mind the cyclic homologies for associative and Hopf algebras, as well as the Harrison homology for associative algebras. In spite of some evidences in favor of a braided interpretation of these homologies, the author can not present a satisfactory braided treatment.

Another structure which is likely to have a braided or quantum shuffle interpretation is the *Gerstenhaber structure* on the Hochschild homology.

Further, a homology theory for *Zinbiel algebras* and, more generally, a braided version of the *operad duality* do not seem impossible. A braided treatment of the homology of *Poisson algebras* would also be of interest. The author's dream is to get new homologies for certain algebraic structures with the help of the braided tools presented here.

The last questions concern the naturality aspects of our structural pre-braidings. Namely, according to the *Schur-Weyl duality*, the symmetric group S_n is precisely the centralizer of the group GL_r acting diagonally on $V^{\otimes n}$, where V is an r-dimensional vector space, and vice versa. In our settings, the action of the positive braid monoid B_n^+ on the tensor powers of an algebra V commutes with all the algebra endomorphisms of V, acting diagonally. We would like to understand how far these two monoids are from being full mutual centralizers.

Notations and conventions

Linear algebra

We systematically use notation R for a commutative unital ring, and $\Bbbk$ for a field. The word "linear" means R- (or $\Bbbk$-) linear, and all tensor products are over R (or $\Bbbk$), unless we work in the settings of a general monoidal category.

Notation

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$$

is used for the tensor algebra of an R-module V, with $V^{\otimes 0} := R$. A simplified notation is used for its elements:

$$\overline{v} = v_1 v_2 \ldots v_n := v_1 \otimes v_2 \otimes \ldots \otimes v_n \in V^{\otimes n},$$

leaving the tensor product sign for

$$v_1 v_2 \ldots v_n \otimes w_1 w_2 \ldots w_m \in V^{\otimes n} \otimes W^{\otimes m}.$$

We often call the R-module $T(V)$ the ***tensor module*** of V, emphasizing that it can be endowed with a multiplication different from the usual concatenation. We talk about the ***tensor vector space*** of V in the $\Bbbk$-linear setting. The tensor module/space $T(V)$ is endowed with a *grading* by putting

$$\deg(v_1 v_2 \ldots v_n) = n. \tag{1.1}$$

The *dual* of an R-module V is denoted by

$$V^* := \operatorname{Hom}_R(V, R).$$

Sweedler's notation

Sweedler's notation, often with the summation sign omitted, is systematically used. For example, a comultiplication, an iterated comultiplication, a left and a right coaction are denoted, respectively, by

$$\begin{aligned}\Delta(v) &= \sum_{(v)} v_{(1)} \otimes v_{(2)} = v_{(1)} \otimes v_{(2)},\\ \Delta^n(v) &= v_{(1)} \otimes \cdots \otimes v_{(n+1)},\\ \delta_L(v) &= v_{(-1)} \otimes v_{(0)},\\ \delta_R(v) &= v_{(0)} \otimes v_{(1)}.\end{aligned}$$

Dualities

Given R-modules V, W and a pairing $B : V \otimes W \to R$ (for example the *evaluation map*

$$\begin{aligned} ev : H^* \otimes H &\longrightarrow R\\ f \otimes a &\longmapsto f(a) \end{aligned} \tag{1.2}$$

for a module and its dual), there are two common ways of extending it to

$$B : V^{\otimes n} \otimes W^{\otimes n} \to R :$$

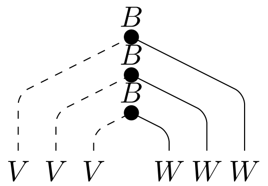
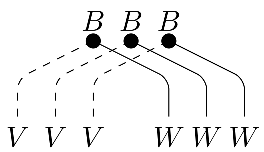
$B(v_1v_2\dots v_n \otimes w_1w_2\dots w_n) :=$	
$B(v_1 \otimes w_n)\cdots B(v_n \otimes w_1)$	$B(v_1 \otimes w_1)\cdots B(v_n \otimes w_n)$
	
"rainbow"	***"arched"***

Table 1.5: Rainbow and arched dualities

The "arched" version is more common in literature, but it is the "rainbow" version we mostly use in this work, avoiding unnecessary flips (in the diagram it is reflected by the absence of crossings). Similar conventions are used in the dual situation, i.e. for Casimir elements, and in the general monoidal settings. In particular, the ***induced bialgebra structure*** on the dual of a finite-dimensional $\Bbbk$-linear bialgebra H is defined in this thesis via the evaluation map ev, extended to $H \otimes H$ and $H^* \otimes H^*$ using the "rainbow" pattern. The multiplication on H^* is given for instance by

$$(l_1 l_2)(h) = l_1(h_{(2)}) l_2(h_{(1)}) \qquad \forall h \in H,\ l_1, l_2 \in H^*,$$

or, graphically,

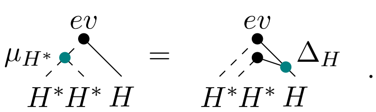


Figure 1.4: Dual structures via the "rainbow" duality

Analyzing the graphical interpretation, one sees that, on the level of structures, the "rainbow" duality corresponds to the ***central symmetry***, while the "arched" duality – to the ***horizontal mirror symmetry***.

Note that the same structure on H^* can be obtained via the dual *coevaluation map* *coev* or via "twisted versions" $ev \circ \tau : H \otimes H^* \to R$ and $\tau \circ coev : R \to H \otimes H^*$, still with the "rainbow" extension on tensor products. Here τ is simply the transposition of factors H and H^*. It is common to simplify notations, writing just *ev* and *coev* for the latter maps, which we do systematically when it does not lead to confusion.

Notations in a strict monoidal category

For an object V in a strict monoidal category (e.g. for an R-module), the notation $V^{\otimes n}$ is often reduced to V^n, and $\mathrm{Id}_{V^{\otimes n}}$ to Id_n . Further, given a morphism $\varphi : V^l \to V^r$, the following notations are repeatedly used:

$$\varphi_i := \mathrm{Id}_V^{\otimes(i-1)} \otimes \varphi \otimes \mathrm{Id}_V^{\otimes(k-i+1)} : V^{k+l} \to V^{k+r}, \tag{1.3}$$

$$\varphi^n := (\varphi_1)^{\circ n} = \varphi_1 \circ \cdots \circ \varphi_1 : V^k \to V^{k+n(r-l)}, \tag{1.4}$$

where φ_1 is composed with itself n times. Similar notations are used for morphisms on tensor products of different objects.

"Differential" terminology

By a *differential* on a graded R-module (for example $T(V)$) we mean a square zero endomorphism of degree $+1$ or -1, while a *bidifferential* is a pair of anticommuting differentials. The word *complex* always means a differential (co)chain complex here, i.e. a graded R-module endowed with a differential. Similarly, a *bicomplex* is a graded R-module endowed with a bidifferential.

Symmetric and braid groups

The symmetric and braid groups on n elements are denoted by S_n and B_n respectively. Inclusions $S_n \subset S_m$ and $B_n \subset B_m$ for $n < m$, implicit in what follows, are obtained by letting an $s \in S_n$ act on the first n elements of an m-tuple, and, respectively, by adding $m-n$ untangled strands on the right of an n-braid. We use the usual action of S_n on $V^{\otimes n}$ for an R-module V:

$$\sigma(v_1 v_2 \ldots v_n) := v_{\sigma^{-1}(1)} v_{\sigma^{-1}(2)} \ldots v_{\sigma^{-1}(n)}.$$

The cyclic group $\mathbb{Z}_n$ is often identified with the subgroup of S_n generated by the cycle

$$t_n = \left(\begin{smallmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & 1 \end{smallmatrix}\right). \tag{1.5}$$

We have $(t_n)^n = 1$.

The non-trivial element of S_2 is denoted by τ and is called *a flip*.

Part I

Homologies of Basic Algebraic Structures via Braidings and Quantum Shuffles

Chapter 2

Braided world: a short reminder

We recall here various facts about *braided vector spaces* necessary for subsequent chapters. Different aspects of the notions of braid groups and positive braid monoids are recalled in more detail in part III. For a more systematic treatment of braid groups, [40] is an excellent reference. A particular focus is made here on *quantum (co-)shuffles*, introduced and studied by M.Rosso in [72] and [73]. These structures will provide an important tool for constructing braided space (co)homologies in the next chapter. The *graphical calculus* is also presented and justified in this chapter.

All the notions defined here for vector spaces are directly generalized for R-modules. We prefer the language of vector spaces for its familiarity.

Pre-braided vector spaces

Definition 2.0.1.

- A *pre-braiding* on a $\Bbbk$-vector space V is a linear map $\sigma : V\otimes V \to V \otimes V$ satisfying the *Yang-Baxter equation* (abbreviated as YBE)

$$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2 : V \otimes V \otimes V \longrightarrow V \otimes V \otimes V, \tag{YB}$$

  where σ_i is the braiding σ applied to components i and $i+1$ of $V^{\otimes 3}$ (cf. notation (1.3)).
- A *braiding* is an invertible pre-braiding.
- A braiding is called *symmetric* if $\sigma^2 = \mathrm{Id}_{V\otimes V}$.
- A vector space endowed with a (pre-)braiding is called *(pre-)braided.*
- A *braided morphism* between pre-braided spaces (V, σ_V) and (W, σ_W) is a $\Bbbk$-linear map $f : V \to W$ respecting the pre-braidings:

$$(f \otimes f) \circ \sigma_V = \sigma_W \circ (f \otimes f) : V \otimes V \to W \otimes W.$$

Unlike most authors ***we mostly work with pre-braidings***, giving interesting highly non-invertible examples. One of the rare papers admitting non-invertible σ's is [74].

Remark 2.0.2. A (pre-)braiding on a set is defined similarly: tensor products $\otimes$ are simply replaced by Cartesian products $\times$. These two settings are particular cases of a more abstract one – that of ***(pre-)braided categories***, studied in detail in chapter 5.

Example 2.0.3. The most familiar braidings are the *flip*, the *signed flip* and their generalization for graded vector spaces, the *Koszul flip*:

$$\begin{aligned} \tau : v \otimes w &\longmapsto w \otimes v, \\ -\tau : v \otimes w &\longmapsto -w \otimes v, \\ \tau_{Koszul} : v \otimes w &\longmapsto (-1)^{\deg v \deg w} w \otimes v \end{aligned} \tag{2.1}$$

for homogeneous v and w. The last braiding explains the ***Koszul sign convention*** in many settings.

Remark 2.0.4. In general for a (pre-)braiding σ, its opposite $-\sigma : v \otimes w \mapsto -\sigma(v \otimes w)$ is also a (pre-)braiding.

Braid monoid action and graphical calculus

A pre-braiding gives an action of the positive braid monoid B_n^+ on $V^{\otimes n}$, i.e. a monoid morphism

$$\begin{aligned} \rho : B_n^+ &\longrightarrow \mathrm{End}_{\Bbbk}(V^{\otimes n}), \\ b &\longmapsto b^{\sigma} \end{aligned} \tag{2.2}$$

defined on the generators σ_i of B_n^+ by

$$\sigma_i \mapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes(n-i-1)}. \tag{2.3}$$

This action is best depicted in the graphical form

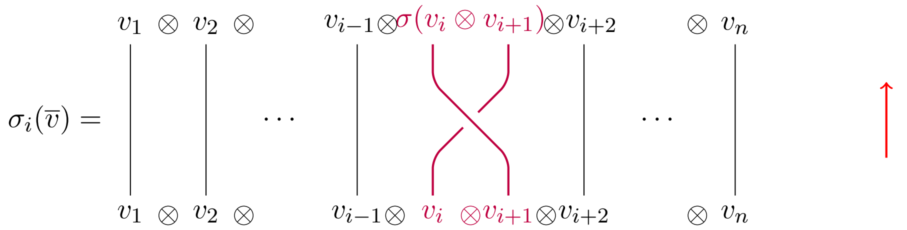


Figure 2.1: B_n^+ acts via pre-braidings

All diagrams in this work are to be read <u>from bottom to top</u>, as indicated by the arrow on the diagram above. One could have presented the crossing as , which is often done in literature. It is just a matter of convention, and the one used here comes from rack theory (section 4.2).

For braidings, the action above is in fact an action of the braid group B_n, and for symmetric braidings it is an action of the symmetric group S_n.

The graphical translation of the Yang-Baxter equation (YB) for pre-braidings is the third Reidemeister move, which is at the heart of knot theory:

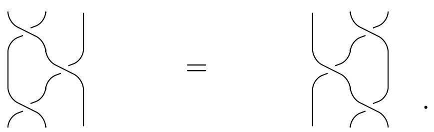


Figure 2.2: Yang-Baxter equation = Reidemeister move III

Pre-braiding extended to tensor powers

Numerous constructions become natural in the graphical settings. For instance,

Remark 2.0.5. A (pre-)braiding σ on V naturally extends to a (pre-)braiding $\boldsymbol{\sigma}$ on its tensor space $T(V)$ by

$$\boldsymbol{\sigma}(\overline{v} \otimes \overline{w}) = (\sigma_k \cdots \sigma_1) \cdots (\sigma_{n+k-2} \cdots \sigma_{n-1})(\sigma_{n+k-1} \cdots \sigma_n)(\overline{vw}) \in V^{\otimes k} \otimes V^{\otimes n}$$

for $\overline{v} \in V^{\otimes n}, \overline{w} \in V^{\otimes k}$, or graphically:

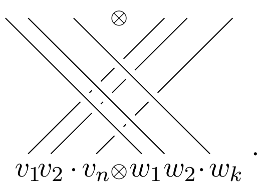


Figure 2.3: Pre-braiding extended to $T(V)$

Here $\overline{vw}$ is simply the concatenation of pure tensors $\overline{v}$ and $\overline{w}$.

Lifting permutations to positive braids

Recall the famous inclusion

$$\begin{aligned} S_n &\hookrightarrow B_n \\ s = \tau_{i_1}\tau_{i_2} \cdots \tau_{i_k} &\mapsto T_s := \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k} \end{aligned} \tag{2.4}$$

where

- ✓ $\tau_i \in S_n$ are transpositions of neighboring elements i and $i+1$, called *simple transpositions*,
- ✓ σ_i are the corresponding generators of B_n,
- ✓ $\tau_{i_1}\tau_{i_2} \cdots \tau_{i_k}$ is one of the shortest words representing s.

It is well defined, since any shortest word representing s can be obtained from any other one by applying YBE a finite number of times, and it is indeed an inclusion, since, followed by the projection

$$\begin{aligned} B_n &\twoheadrightarrow S_n, \\ \sigma_i^{\pm 1} &\mapsto \tau_i, \end{aligned}$$

it gives identity. This inclusion factorizes through

$$S_n \hookrightarrow B_n^+ \hookrightarrow B_n.$$

This is a **set inclusion** not preserving the monoid structure. More precisely,

Lemma 2.0.6. One has $T_{s_1 s_2} = T_{s_1} T_{s_2}$ if and only if, for each pair of elements (i, j) reversed by s_2, their images $(s_2(i), s_2(j))$ are not reversed by s_1.

Shuffles

The following subsets of symmetric groups deserve particular attention:

Definition 2.0.7. The permutation sets

$$Sh_{p,q} := \left\{ s \in S_{p+q} \text{ s.t.} \middle| \begin{array}{l} s(1) < s(2) < \ldots < s(p), \\ s(p+1) < s(p+2) < \ldots < s(p+q) \end{array} \right\}$$

or, more generally,

$$Sh_{p_1,p_2,\ldots,p_k} := \left\{ s \in S_{p_1+p_2+\cdots+p_k} \text{ s.t.} \middle| \begin{array}{l} s(1) < s(2) < \ldots < s(p_1), \\ s(p_1+1) < \ldots < s(p_1+p_2), \\ \ldots, \\ s(p+1) < s(p+2) < \ldots < s(p+p_k) \end{array} \right\}$$

where $p = p_1 + p_2 + \cdots + p_{k-1}$, are called *shuffle sets*.

The conditions from this definition mean that one permutes $p_1 + p_2 + \cdots + p_k$ elements preserving the order within k consecutive blocks of size $p_1, p_2, \ldots, p_k$, just like when shuffling cards, which explains the name. The set $Sh_{p_1,p_2,\ldots,p_k}$ consists of $\binom{p_1+p_2+\cdots+p_k}{p_1,p_2,\ldots,p_k}$ elements. Shuffles and their diverse modifications appear, sometimes quite unexpectedly, in various areas of mathematics.

The first basic result about shuffles is

Lemma 2.0.8. Take $p, q, r \in \mathbb{N}$ and put $n = p + q + r$. Viewing $Sh_{p,q} \subseteq S_{p+q}$ and $Sh_{q,r} \subseteq S_{q+r}$ as subsets of S_n by letting $Sh_{p,q}$ permute the <u>first</u> $p + q$ elements of an n-tuple, and, similarly, by letting $Sh_{q,r}$ permute the <u>last</u> $q + r$ elements of an n-tuple, one has the following decomposition:

$$Sh_{p,q,r} = Sh_{p+q,r} Sh_{p,q} = Sh_{p,q+r} Sh_{q,r}.$$

That is, an element of $Sh_{p,q,r}$ can be seen, in a unique way, as an element of $Sh_{p,q}$ followed by one from $Sh_{p+q,r}$, and similarly for the second decomposition.

Quantum shuffle Hopf algebra

Everything is now ready for defining quantum shuffle algebras. This structure originated in the work of M.Rosso ([71],[72]) and was rediscovered several times since then, with different motivations.

Definition 2.0.9. The *quantum shuffle multiplication* on the tensor space $T(V)$ of a pre-braided vector space (V, σ) is the $\Bbbk$-linear extension of the map

$$\begin{aligned} \underset{\sigma}{\sqcup\!\sqcup} = \underset{\sigma}{\sqcup\!\sqcup}^{p,q} : V^{\otimes p} \otimes V^{\otimes q} &\longrightarrow V^{\otimes (p+q)} \\ \overline{v} \otimes \overline{w} &\longmapsto \overline{v} \underset{\sigma}{\sqcup\!\sqcup} \overline{w} := \sum_{s \in Sh_{p,q}} T_s^{\sigma}(\overline{v}\overline{w}). \end{aligned} \tag{2.5}$$

The expression $\overline{v}\overline{w}$ means simply the concatenation of pure tensors $\overline{v}$ and $\overline{w}$. Notation T_s^{σ} stands for the lift $T_s \in B_n^+$ (cf. (2.4)) acting on $V^{\otimes n}$ via the pre-braiding σ (cf. (2.2)).

The algebra $\mathrm{Sh}_\sigma(V) := (T(V), \underset{\sigma}{\sqcup\!\sqcup})$ is called the *quantum shuffle algebra* of (V, σ).

The symbol $\sqcup\!\sqcup$ comes from a Cyrillic letter pronounced as "sh" in English.

In the case of the trivial braiding ($\sigma =$ flip), one speaks simply about the *shuffle algebra* of V, and a simplified notation $\sqcup\!\sqcup$ is used. This structure has a much longer history. For instance, it was used by S.Eilenberg and S.MacLane in order to give an explicit formula for the equivalence of complexes of the Eilenberg-Zilber theorem.

By a ***(pre-)braided Hopf algebra*** (in the sense of S.Majid, cf. Definition 2.2 in [52] for example) we mean an additional structure on a (pre-)braided vector space satisfying all the axioms of a Hopf algebra except for the compatibility between the multiplication and the comultiplication, which is replaced by the braided compatibility (this last notion is recalled in the following theorem). More generally, it is a Hopf algebra in a (pre-)braided category; see chapter 5 for the categorical notions, and the categorical definition 7.4.1.

The quantum shuffle multiplication can be upgraded to an interesting pre-braided Hopf algebra structure (braided commutative if the initial pre-braiding is symmetric):

Theorem 1. *Let (V, σ) be a pre-braided vector space.*

1. *The multiplication $\underset{\sigma}{\sqcup\!\sqcup}$ of $\mathrm{Sh}_\sigma(V)$ is associative.*
2. *If $\sigma^2 = \mathrm{Id}$, then the multiplication $\underset{\sigma}{\sqcup\!\sqcup}$ is **$\boldsymbol{\sigma}$-commutative**, i.e.*

$$\underset{\sigma}{\sqcup\!\sqcup}(\overline{v} \otimes \overline{w}) = \underset{\sigma}{\sqcup\!\sqcup}(\boldsymbol{\sigma}(\overline{v} \otimes \overline{w}))$$

   (with the extension $\boldsymbol{\sigma}$ of σ to $T(V)$ from remark 2.0.5).
3. *The element $1 \in R$ is a unit for $\mathrm{Sh}_\sigma(V)$.*
4. *The **deconcatenation** map*

$$\begin{aligned} \Delta : v_1 v_2 \dots v_n &\longmapsto \sum_{p=0}^{n} v_1 v_2 \dots v_p \otimes v_{p+1} \dots v_n, \\ 1 &\longmapsto 1 \otimes 1, \end{aligned}$$

   *(where an empty product means 1), and the **augmentation** map*

$$\begin{aligned} \varepsilon : v_1 v_2 \dots v_n &\longmapsto 0, \\ 1 &\longmapsto 1, \end{aligned}$$

   define, after a linearization, a counital coalgebra structure on $T(V)$.
5. *These algebra and coalgebra structures are **$\boldsymbol{\sigma}$-compatible**, in the sense that*

$$\Delta \circ \underset{\sigma}{\sqcup\!\sqcup} = (\underset{\sigma}{\sqcup\!\sqcup} \otimes \underset{\sigma}{\sqcup\!\sqcup}) \circ \boldsymbol{\sigma}_2 \circ (\Delta \otimes \Delta).$$

6. *An antipode can be given on $\mathrm{Sh}_\sigma(V)$ by linearizing the map*

$$\begin{aligned} s : \overline{v} &\longmapsto (-1)^n T^{\sigma}_{\Delta_n}(\overline{v}), \qquad \overline{v} \in V^{\otimes n}, \\ 1 &\longmapsto 1, \\ \text{where} \qquad \Delta_n &:= \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right) \in S_n. \end{aligned} \tag{2.6}$$

The pre-braided vector space $(\mathrm{Sh}_\sigma(V), \boldsymbol{\sigma})$ becomes thus a pre-braided Hopf algebra.

The operator $T^{\sigma}_{\Delta_n}$ defining the antipode is graphically depicted (for $n = 4$) as follows:

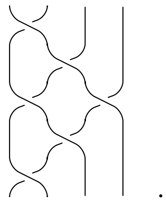


Figure 2.4: Antipode for $\mathrm{Sh}_\sigma(V)$

Note that the lift T_{Δ_n} of Δ_n in B_n^+ is the *Garside element*.

Proof. We only give the proof of the most difficult statements.

1. Let $\overline{v} \in V^p, \overline{w} \in V^q, \overline{u} \in V^r$. We have

$$\begin{aligned}(\overline{v} \underset{\sigma}{\sqcup\!\sqcup} \overline{w}) \underset{\sigma}{\sqcup\!\sqcup} \overline{u} &= \sum_{s \in Sh_{p+q,r}, t \in Sh_{p,q}} T^{\sigma}_s \circ (T^{\sigma}_t \otimes \mathrm{Id}_r)(\overline{vwu}) \\ &\overset{\text{Lemma 2.0.6}}{=} \sum_{\substack{s \in Sh_{p+q,r},\\ t \in Sh_{p,q} \subseteq S_{p+q} \subseteq S_{p+q+r}}} T^{\sigma}_{sot}(\overline{vwu}) \\ &\overset{\text{Lemma 2.0.8}}{=} \sum_{s \in Sh_{p,q,r}} T^{\sigma}_s(\overline{vwu}).\end{aligned}$$

The same reasoning gives

$$\overline{v} \underset{\sigma}{\sqcup\!\sqcup} (\overline{w} \underset{\sigma}{\sqcup\!\sqcup} \overline{u}) = \sum_{s \in Sh_{p,q,r}} T^{\sigma}_s(\overline{vwu}),$$

so these two expressions are equal.

6. Take a $\overline{v} \in V^{\otimes n}, n \geqslant 1$. There are two types of signed summands in the expression of $\underset{\sigma}{\sqcup\!\sqcup} \circ (Id \otimes s) \circ \Delta(\overline{v})$: those where the last element in the quantum shuffle multiplication comes from the first component of $\Delta(\overline{v}) \in T(V) \otimes T(V)$, and those where it comes from the second one. Each summand appears exactly once in each type, and with different signs due the sign $(-1)^{\cdots}$ in the formula for the antipode. The overall sum is therefore zero. □

The above theorem is well-known for invertible braidings ([73]); we point out that it still holds when the pre-braiding admits no inverse.

Quantum co-shuffle Hopf algebra

Dually (in the sense to be specified in section 5.4), the tensor space of a pre-braided vector space (V, σ) can be endowed with the ***quantum co-shuffle comultiplication***:

$$\begin{aligned}\underset{\sigma}{\overline{\sqcup\!\sqcup}}|_{V^{\otimes n}} &:= \sum_{p+q=n;\, p,q \geqslant 0} \underset{\sigma}{\overline{\sqcup\!\sqcup}}^{p,q}, \\ \underset{\sigma}{\overline{\sqcup\!\sqcup}}^{p,q} &:= \sum_{s \in Sh_{p,q}} T^{\sigma}_{s^{-1}} : V^{\otimes n} \longrightarrow V^{\otimes p} \otimes V^{\otimes q},\end{aligned} \tag{2.7}$$

which can be upgraded to a pre-braided Hopf algebra structure "dual" to that described in theorem 1, and denoted by $\overline{\mathrm{Sh}}_\sigma(V)$.

Chapter 3

(Co)homologies of braided vector spaces

We introduce here a homology and, dually, a cohomology theory of braided vector spaces, which are at the heart of this thesis. Different aspects of these theories are studied in detail. A comparison with existing "braided" (co)homology constructions is made.

We propose two different viewpoints on our "braided" differentials (all the notions and properties are explained in this chapter):

approach	**section**	**advantages**
quantum (co-)shuffle (co)multiplication and *square zero (co)elements*	3.1	⊞ the sign manipulation is hidden in the choice of the negative braiding $-\sigma$, ⊞ a subscript-free approach, ⊞ compact formulas;
graphical calculus: diagrams, braids	3.2	⊞ a tool easy to manipulate, ⊞ a finer structure of a pre-bisimplicial complex, ⊞ an intuitive definition of degenerate and normalized complexes via braided coalgebras.

Table 3.1: Two approaches to "braided" differentials

Besides the above-mentioned advantages of our construction, there are some more useful features, common for the two approaches:

- ⊞ Given a pre-braided vector space (V, σ), our construction associates to every *braided (co)character* (which are quite numerous in practice) two compatible differentials on $T(V)$, which are often also compatible with differentials coming from other (co)characters. One thus gets a rich family of differentials for every (V, σ).
- ⊞ The only properties needed to make everything work is the YBE for the pre-braiding and the defining equation (3.1) (or (3.2)) for the braided (co)character. This simplifies the verification of the equation $d^2 = 0$ in concrete examples.
- ⊞ One obtains (section 3.3) a new interpretation and a generalization of J.-L.Loday's hyper-boundaries, automatically calculating all their compositions, some of which are given in [46], exercise E.2.2.7.
- ⊞ One gets almost for free a generalization of some of the homology operations studied by M.Niebrzydowski and J.Przytycki ([63], [67]) for quandle homology.

All these properties are illustrated with concrete examples in chapter 4.

Everything described here can be translated verbatim to the setting of R-modules, and even raised to the categorical level (chapter 5). We prefer working with vector spaces for the sake of clarity.

The tools created here will be applied in various familiar algebraic situations in subsequent chapters.

Fix a pre-braided $\Bbbk$-vector space (V, σ).

3.1 Pre-braiding + character $\longmapsto$ homology

We present here a systematic way of constructing differentials on the tensor vector space $T(V)$ of a pre-braided vector space. The compatibility of such differentials is also studied. A survey of "braided" homology theories existing in literature concludes this section.

Braided (co)characters

We start with distinguishing elements of V and V^* which behave with respect to the pre-braiding σ as if it were just a flip τ.

Definition 3.1.1. ➻ Two elements $v, w \in V$ are called *σ-compatible* if

$$\sigma(v \otimes w) = w \otimes v, \qquad \text{and} \qquad \sigma(w \otimes v) = v \otimes w.$$

➻ A *braided co-character* is an element $e \in V$ which is σ-compatible with itself, i.e.

$$\sigma(e \otimes e) = e \otimes e, \tag{3.1}$$

or, in the shuffle form,

$$e \underset{-\sigma}{\sqcup\!\sqcup} e = 0.$$

➻ Two co-elements $f, g \in V^*$ are called *σ-compatible* if

$$(f \otimes g) \circ \sigma = g \otimes f, \qquad \text{and} \qquad (g \otimes f) \circ \sigma = f \otimes g.$$

➻ A *braided character* is an element $\epsilon \in V^*$ σ-compatible with itself, i.e.

$$(\epsilon \otimes \epsilon) \circ \sigma = \epsilon \otimes \epsilon, \tag{3.2}$$

or, in the co-shuffle form,

$$(\epsilon \otimes \epsilon) \circ \overline{\underset{-\sigma}{\sqcup\!\sqcup}}^{1,1} = 0.$$

The definition of braided (co-)characters takes a simple graphical form:

Figure 3.1: Braided (co-)characters

The labels e, ϵ are often omitted when clear from the context.

In chapter 4, we recover familiar notions of (co-)characters for algebraic structures (such as associative algebras) as examples of braided (co-)characters for corresponding "algebraic" braidings, which justifies our choice of the term. In what follows the part "braided" of the term is omitted when it does not lead to confusion. Units and counits often turn out to be braided (co-)characters, hence the notations e, ϵ.

Remark 3.1.2. Braided characters can also be regarded as ***braided morphisms*** $\epsilon : V \to \Bbbk$, where $\Bbbk$ is endowed with the ***trivial braiding***

$$\Bbbk \otimes \Bbbk \simeq \Bbbk \xrightarrow{\mathrm{Id}_{\Bbbk}} \Bbbk \simeq \Bbbk \otimes \Bbbk.$$

This is consistent with the interpretation of usual characters for algebraic structures as homomorphisms to trivial structures.

Braided (co)homologies: the quantum shuffle approach

A pre-braiding and a braided (co-)character are sufficient for constructing (co)homologies:

Theorem 2. *Let (V,σ) be a pre-braided vector space.*

1. *For a braided co-character e, the maps*
$$\begin{array}{rccc} {}_ed: & V^{\otimes n} & \longrightarrow & V^{\otimes(n+1)} \\ & \overline{v} & \longmapsto & e \underset{-\sigma}{\sqcup\!\sqcup} \overline{v} \end{array} \quad \text{and} \quad \begin{array}{rccc} d_e: & V^{\otimes n} & \longrightarrow & V^{\otimes(n+1)} \\ & \overline{v} & \longmapsto & (-1)^n \overline{v} \underset{-\sigma}{\sqcup\!\sqcup} e \end{array}$$
define differentials on $T(V)$.
2. *For two braided co-characters e and f, one gets a differential bicomplex $(T(V), {}_ed, d_f)$. If the co-characters are moreover σ-compatible, then one also gets differential bicomplexes $(T(V), {}_ed, {}_fd)$ and $(T(V), d_e, d_f)$.*
3. *Similarly, for a braided character ϵ, the maps*
$$\begin{aligned} & V^{\otimes n} \longrightarrow V^{\otimes(n-1)} \\ {}^{\epsilon}d : \ & \overline{v} \longmapsto (\epsilon \otimes \mathrm{Id}_{n-1}) \underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{1,n-1} (\overline{v}) \\ d^{\epsilon} : \ & \overline{v} \longmapsto (-1)^{n-1} (\mathrm{Id}_{n-1} \otimes \epsilon) \underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{n-1,1} (\overline{v}) \end{aligned}$$
define differentials on $T(V)$.
4. *For two braided characters ϵ and ζ, one gets a differential bicomplex $(T(V), {}^{\epsilon}d, d^{\zeta})$. If the braided characters are moreover σ-compatible, then one also gets differential bicomplexes $(T(V), {}^{\epsilon}d, {}^{\zeta}d)$ and $(T(V), d^{\epsilon}, d^{\zeta})$.*

Proof. Easy verifications using the associativity of $\underset{-\sigma}{\sqcup\!\sqcup}$, the coassociativity of $\underset{-\sigma}{\overline{\sqcup\!\sqcup}}$ and the defining property of (co-)characters. For example,

$${}_ed^2(\overline{v}) = e \underset{-\sigma}{\sqcup\!\sqcup} (e \underset{-\sigma}{\sqcup\!\sqcup} \overline{v}) = (e \underset{-\sigma}{\sqcup\!\sqcup} e) \underset{-\sigma}{\sqcup\!\sqcup} \overline{v} = 0 \underset{-\sigma}{\sqcup\!\sqcup} \overline{v} = 0,$$

since a co-character e is defined by $e \underset{-\sigma}{\sqcup\!\sqcup} e = 0$. Similarly,

$$({}_ed \circ {}_fd + {}_fd \circ {}_ed)(\overline{v}) = e \underset{-\sigma}{\sqcup\!\sqcup} (f \underset{-\sigma}{\sqcup\!\sqcup} \overline{v}) + f \underset{-\sigma}{\sqcup\!\sqcup} (e \underset{-\sigma}{\sqcup\!\sqcup} \overline{v}) = (e \underset{-\sigma}{\sqcup\!\sqcup} f + f \underset{-\sigma}{\sqcup\!\sqcup} e) \underset{-\sigma}{\sqcup\!\sqcup} \overline{v},$$

which, since the pre-braiding σ coincides with the flip τ on $e \otimes f$ and $f \otimes e$ for compatible e and f, equals

$$(e \underset{-\tau}{\sqcup\!\sqcup} f + f \underset{-\tau}{\sqcup\!\sqcup} e) \underset{-\sigma}{\sqcup\!\sqcup} \overline{v} = (ef - fe + fe - ef) \underset{-\sigma}{\sqcup\!\sqcup} \overline{v} = 0. \quad \square$$

This proof can be understood as follows: the multiplication by a square zero element in $\mathrm{Sh}_{-\sigma}$ is a square zero operator. An interpretation in terms of simplicial modules, as well as a graphical translation, are postponed until the next section.

The differentials ${}_{\epsilon}d$ and d_{ϵ} increase the degree (1.1) by 1, thus defining cohomologies, while ${}^{\epsilon}d$ and d^{ϵ} decrease the degree and therefore define homologies. We mostly work with homologies in what follows.

The theorem gives for two braidings two compatible differentials on $T(V)$. Their linear combinations are then also differentials; ${}^{\epsilon}d - d^{\epsilon}$ is a recurrent example in practice. All such (bi)differentials, corresponding (bi)complexes and (co)homologies are called ***braided*** in what follows.

A survey of existing "braided" homologies

In [10], J.S.Carter, M.Elhamdadi and M.Saito develop a homology theory for solutions (S, σ) of the set-theoretic Yang-Baxter equation using combinatorial and geometric methods completely different from ours. They also provide applications to virtual knot invariants. It can be checked that their differential on $(\mathbb{Z}S)^{\otimes n}$ coincides with our ${}^{\varepsilon}d - d^{\varepsilon}$, where ε is the linearization of the map

$$\begin{aligned} \varepsilon : S &\longrightarrow \mathbb{Z}, \\ a &\longmapsto 1 \qquad \forall a \in S. \end{aligned}$$

Our applications ${}^{\varepsilon}d$, where $\varepsilon \in V^*$ are not necessarily braided characters, also recover the *braided-differential calculus* of S.Majid ([50]). He introduces an addition law (related to the quantum co-shuffle comultiplication) on the quantum plane associated to a braiding, and defines a differentiation as an infinitesimal translation. In particular, taking

- ✓ one-variable polynomials $T(V) = \Bbbk[x]$ (i.e. $V = \Bbbk x$),
- ✓ the opposite of the q-flip $x \otimes x \mapsto qx \otimes x$ (with $q \in \Bbbk^*$) as a braiding,
- ✓ and the linearization of the map $\varepsilon(x) = 1$,

one gets the famous q-differentials

$${}^{\varepsilon}d(x^{\otimes n}) = (n)_q x^{\otimes (n-1)}, \qquad (n)_q := \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1.$$

The last approach to "braided" cohomologies to be mentioned here is M.Eisermann's *Yang-Baxter cochain complex*, cf. [22]. Motivated by the study of deformations of Yang-Baxter operators, he defines a degree 1 differential on $\mathrm{Hom}_{\Bbbk}(V^{\otimes n}, V^{\otimes n})$. His second cohomology groups classify infinitesimal Yang-Baxter deformations. We do not know precisely how his construction is related to ours, but the parallels between the graphical versions of the two are very suggestive.

3.2 Comultiplication $\longmapsto$ degeneracies

The aim of this section is to better understand the structure of braided (bi)complexes from theorem 2. The simplicial approach proves to be particularly helpful for such a study. The contents of this section is categorified in section 5.1 and dualized in section 5.4.

Simplicial vocabulary

First, recall the notion of simplicial vector spaces (cf. [46] for details and [67] for weak simplicial notions; note that our definition is a shifted version of theirs, and that our definition of bisimplicial vector spaces is different from the usual one):

Definition 3.2.1. Consider a collection of $\Bbbk$-vector spaces V_n, $n \geqslant 0$, equipped with linear maps $d_{n;i} : V_n \to V_{n-1}$ (and $d'_{n;i} : V_n \to V_{n-1}$ and/or $s_{n;i} : V_n \to V_{n+1}$ when necessary) with $1 \leqslant i \leqslant n$, denoted simply by d_i, d'_i, s_i when the subscript n is clear from the context. This datum is (slightly abusively) called

- *a presimplicial vector space* if

$$d_i d_j = d_{j-1} d_i \qquad \forall 1 \leqslant i < j \leqslant n; \tag{3.3}$$

- *a very weakly simplicial vector space* if moreover

$$s_i s_j = s_{j+1} s_i \qquad \forall 1 \leqslant i \leqslant j \leqslant n, \tag{3.4}$$

$$d_i s_j = s_{j-1} d_i \qquad \forall 1 \leqslant i < j \leqslant n, \tag{3.5}$$

$$d_i s_j = s_j d_{i-1} \qquad \forall 1 \leqslant j+1 < i \leqslant n; \tag{3.6}$$

- *a weakly simplicial vector space* if moreover

$$d_i s_i = d_{i+1} s_i \qquad \forall 1 \leqslant i \leqslant n; \tag{3.7}$$

- *a simplicial vector space* if moreover

$$d_i s_i = \mathrm{Id}_{V_n} \qquad \forall 1 \leqslant i \leqslant n; \tag{3.8}$$

- *a pre-bisimplicial vector space* if (3.3) holds for the d_i's, the d'_i's and their mixture:

$$d_i d'_j = d'_{j-1} d_i \qquad \forall 1 \leqslant i < j \leqslant n, \tag{3.9}$$

$$d'_i d_j = d_{j-1} d'_i \qquad \forall 1 \leqslant i < j \leqslant n; \tag{3.10}$$

- *a (weakly / very weakly) bisimplicial vector space* if it is pre-bisimplicial, with both $(V_n, d_{n;i}, s_{n;i})$ and $(V_n, d'_{n;i}, s_{n;i})$ giving (weakly / very weakly) simplicial structures.

The omitted subscripts $n, n \pm 1$ are those which guarantee that the source of all the above mentioned morphisms is V_n. The d_i's and the s_i's are called *face* (resp. *degeneracy*) maps.

Simplicial vector spaces are interesting because of the following properties (see [46] for most proofs):

Proposition 3.2.2. *1. For any presimplicial vector space $(V_n, d_{n;i})$, the map*

$$\partial_n := \sum_{i=1}^{n} (-1)^{i-1} d_{n;i}$$

is a differential (called the ***total differential****) for the graded vector space*

$$\widetilde{V} := \bigoplus_{n \geqslant 0} V_n.$$

2. For any pre-bisimplicial vector space $(V_n, d_{n;i}, d'_{n;i})$, the differentials ∂_n and

$$\partial'_n := \sum_{i=1}^{n} (-1)^{i-1} d'_{n;i}$$

give a bidifferential structure on $\widetilde{V}$.

3. *For any weakly simplicial vector space $(V_n, d_{n;i}, s_{n;i})$, the complex (V_n, ∂_n) contains a subcomplex (called* ***the degenerate subcomplex****)*

$$D_n := \sum_{i=1}^{n-1} s_{n-1;i}(V_{n-1}).$$

4. *If our vector space turns out to be simplicial, then the degenerate subcomplex is acyclic, hence V_* is quasi-isomorphic to the* ***normalized complex***

$$N_* := V_*/D_*.$$

5. *In the weakly <u>bi</u>simplicial case, D_* is a sub-<u>bi</u>complex of V_*, acyclic in the bisimplicial setting.*

In practice, one usually works with $V_n = V^{\otimes n}$ for a chosen space V, i.e. $\widetilde{V} = T(V)$.

Pre-braided coalgebras

We will soon show that the (bi)complexes from theorem 2 come from pre-(bi)simplicial structures. As for degeneracies, they arise from the following structure:

Definition 3.2.3. ➺ A pre-braided vector space (V, σ) endowed with a comultiplication $\Delta : V \to V \otimes V$ is called *a pre-braided coalgebra* if

✓ Δ is *co-associative*:

$$(\Delta \otimes \mathrm{Id}_V) \circ \Delta = (\mathrm{Id}_V \otimes \Delta) \circ \Delta : V \to V \otimes V \otimes V, \tag{3.11}$$

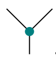
✓ and Δ is *compatible* with the pre-braiding – i.e., using notation φ_i from (1.3),

$$\Delta_2 \circ \sigma = \sigma_1 \circ \sigma_2 \circ \Delta_1 : V^{\otimes 2} \to V^{\otimes 3}, \tag{3.12}$$

$$\Delta_1 \circ \sigma = \sigma_2 \circ \sigma_1 \circ \Delta_2 : V^{\otimes 2} \to V^{\otimes 3}. \tag{3.13}$$

➺ One talks about *semi-pre-braided coalgebras* if only (3.12) holds.
➺ A (semi-)pre-braided coalgebra is called *σ-cocommutative* if

$$\sigma \circ \Delta = \Delta : V \to V \otimes V. \tag{3.14}$$

Representing the comultiplication Δ as , the properties from the definition become

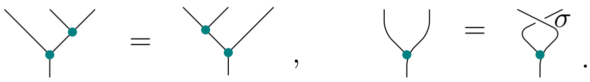


Figure 3.2: Coassociativity and σ-cocommutativity

Figure 3.3: Braided coalgebras

Braided (co)homologies: a simplicial interpretation via the graphical approach

Theorem 3. *Let (V, σ) be a pre-braided vector space.*

1. *For braided characters ϵ and ζ, the maps*
$$d_{n;i}(\overline{v}) := \epsilon_1 \circ T^{\sigma}_{p_{i,n}}(\overline{v}) : V^{\otimes n} \to V^{\otimes(n-1)}, \tag{3.15}$$
$$d'_{n;i}(\overline{v}) := \zeta_n \circ T^{\sigma}_{p'_{i,n}}(\overline{v}) : V^{\otimes n} \to V^{\otimes(n-1)}, \tag{3.16}$$
define a pre-bisimplicial structure on $T(V)$. Here $p_{i,n} \in S_n$ (resp. $p'_{i,n} \in S_n$) is the permutation moving the ith element to the leftmost (resp. rightmost) position, and the notation T^{σ}_s comes from (2.4) and (2.2).
The total differentials ∂ and ∂' coincide with the "shuffle" differentials ${}^{\epsilon}d$ and, respectively, d^{ζ} from theorem 2.
2. *If the braided characters are moreover σ-compatible, then the d_i's for ϵ and the d_i's for ζ define a pre-bisimplicial structure on $T(V)$.*
3. *If a comultiplication Δ endows (V, σ) with a pre-braided coalgebra structure, then the preceding structures are completed into very weakly bisimplicial ones by*
$$s_{n;i} := \Delta_i : V^{\otimes n} \to V^{\otimes(n+1)}. \tag{3.17}$$
4. *If Δ endows (V, σ) with a <u>semi</u>-pre-braided coalgebra structure only, then the data $(V^{\otimes n}, d_{n;i}, s_{n;i})$ described above give a very weakly simplicial vector space.*
5. *If Δ is moreover σ-cocommutative, then the above structures on $T(V)$ are weakly (bi)simplicial.*

See figure 1.3 for a graphical version of the face and degeneracy maps from the theorem.

Proof. One has to deduce the "simplicial" relations of definition 3.2.1 from the properties of the structures on V, which were conceived precisely for these relations to hold. This can be done graphically, using the pictorial interpretation of face and degeneracy maps, and the graphical definitions of a (σ-cocommutaive) pre-braided coalgebra presented above, as well as the pictorial versions of the Yang-Baxter equation (figure 2.2) and of the definition of braided (co)characters (figure 3.1).

For instance,

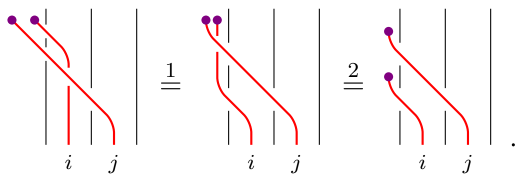


Figure 3.4: Graphical proof of $d_i d_j = d_{j-1} d_i \; \forall 1 \leqslant i < j \leqslant n$.

Here

1. is a repeated application of YBE;
2. follows from the definition 3.2 of a braided character (cf. figure 3.1). □

Remark 3.2.4. When checking the axioms of different types of simplicial structures in the theorem, one can get rid of the tiresome index chasing by reasoning in terms of strands. For example, pulling a strand to the left commutes with applying the branching Δ to any other strand if a strand can pass over a branching.

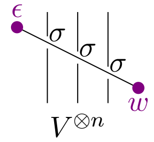
Concatenation and arrow operations on braided complexes

The last face map $d_{n+1;n+1}$ on $V^{\otimes(n+1)}$ is of particular interest. It defines a useful operation on $T(V)$:

Definition 3.2.5. Take a pre-braided vector space (V, σ) endowed with a braided character ϵ. For an element w of V, we call an *arrow operation* on $T(V)$ the map

$$\overline{v} \overset{\epsilon}{\nwarrow} w := d_{n+1;n+1}(\overline{v}w) = \epsilon_1 \circ T^{\sigma}_{p_{n+1,n+1}}(\overline{v}w), \qquad \forall \overline{v} \in V^{\otimes n}.$$

The notation and the name come from the graphical presentation:



.

Figure 3.5: Arrow operation

This map will be interpreted in terms of modules over pre-braided vector spaces and adjoint maps in proposition 6.1.4. Here we study arrow and concatenation operations, and get a generalization of the homology operations of M.Niebrzydowski and J.Przytycki ([63], [67]). Our constructions are deeply inspired by their work.

Start with some technical definitions:

Definition 3.2.6. Take a pre-braided vector space (V, σ).

- A *normalized pair* is an element $w \in V$ and a co-element $\psi \in V^*$ satisfying

$$\psi(w) = 1.$$

- A $w \in V$ and a $\psi \in V^*$ are called *right σ-compatible* if

$$(\mathrm{Id}_V \otimes \psi) \circ \sigma \circ (v \otimes w) = \psi(v) w \qquad \forall v \in V. \tag{3.18}$$

  Left σ-compatible pairs are defined similarly.

- The pre-braiding σ is called *natural* with respect to a $w \in V$ if

$$\sigma \circ (w \otimes v) = v \otimes w \qquad \forall v \in V, \tag{3.19}$$
$$\sigma \circ (v \otimes w) = w \otimes v \qquad \forall v \in V, \tag{3.20}$$

  and *semi-natural* (or *demi-natural*) if only (3.19) (resp. (3.20)) holds.

These three notions – ***normalization***, ***compatibility*** and ***naturality*** – are recurrent in this work. Graphically the last two definitions mean

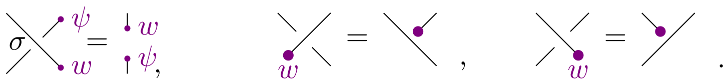


Figure 3.6: Right σ-compatibility and naturality with respect to an element

The naturality can be interpreted as follows: the element w can "pass through a crossing" to the left / to the right. In general, all the σ-compatibilities and the naturality of σ with respect to different structures, often encountered in this thesis, should be thought of as something that "does not distinguish the pre-braiding σ from the flip τ".

Note that condition (3.20) implies (3.18) for any ψ.

The compatibility of arrow operations $\overset{\epsilon}{\nwarrow} w$ with the braided differential ${}^{\epsilon}d$ from theorem 2 follows from theorem 3, where we interpret ${}^{\epsilon}d$ as a total differential for which $\overset{\epsilon}{\nwarrow}$ is the last face map. This remark inspires the following analysis of the behavior of our braided differentials with respect to arrow operations and ***concatenation operations***

$$\overline{v} \mapsto \overline{v}w$$

on $T(V)$, for a fixed $w \in V$.

Proposition 3.2.7. *Let (V, σ) be a pre-braided vector space with braided characters ϵ, ξ, ζ, the first two being σ-compatible, and the last one being right σ-compatible with a $w \in V$.*

1. *The map $\overset{\epsilon}{\nwarrow} w$ is a bicomplex map for $(T(V), {}^{\xi}d, d^{\zeta})$, i.e.*

$$\begin{aligned} {}^{\xi}d(\overline{v} \overset{\epsilon}{\nwarrow} w) =& {}^{\xi}d(\overline{v}) \overset{\epsilon}{\nwarrow} w, \\ d^{\zeta}(\overline{v} \overset{\epsilon}{\nwarrow} w) =& d^{\zeta}(\overline{v}) \overset{\epsilon}{\nwarrow} w. \end{aligned}$$

2. *The following relations hold between the concatenation operations and the braided differentials:*

$$\begin{aligned} {}^{\epsilon}d(\overline{v}w) &= {}^{\epsilon}d(\overline{v})w + (-1)^n \overline{v} \overset{\epsilon}{\nwarrow} w, \\ d^{\zeta}(\overline{v}w) &= d^{\zeta}(\overline{v})w + (-1)^n \zeta(w)\overline{v}. \end{aligned}$$

3. *If σ is demi-natural with respect to w, then the map $\overset{\epsilon}{\nwarrow} w$ is a multiplication by a scalar on $T(V)$:*

$$* \overset{\epsilon}{\nwarrow} w = \epsilon(w) \operatorname{Id}_{T(V)}.$$

Here the notation $\overline{v}$ stays for any pure tensor in $V^{\otimes n}$.

This result admits an evident "left" version (with respect to w).

Proof. Point 1 follows, in the same way as the proof of theorem 3, from the YBE for σ and from the σ-compatibilities (use for instance the graphical calculus).

Point 2 can be checked using the pre-braided Hopf algebra structure on $\overline{\mathrm{Sh}}_{-\sigma}(V)$. For instance, for the left differentials one has

$$\begin{aligned} {}^{\epsilon}d(\overline{v}w) &= (\epsilon \otimes \operatorname{Id}_n) \circ \underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{1,n}(\overline{v}w) \\ &\overset{(*)}{=} (\epsilon \otimes \operatorname{Id}_n)(\underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{1,n-1}(\overline{v})w + T^{-\sigma}_{p_{n+1,n+1}}(\overline{v}w)) \\ &= (\epsilon \otimes \operatorname{Id}_n)(\underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{1,n-1}(\overline{v})w) + (-1)^n(\epsilon \otimes \operatorname{Id}_n) \circ T^{\sigma}_{p_{n+1,n+1}}(\overline{v}w) \\ &= {}^{\epsilon}d(\overline{v})w + (-1)^n \overline{v} \overset{\epsilon}{\nwarrow} w. \end{aligned}$$

Equality $(*)$ is the compatibility between the multiplication and the comultiplication in the quantum co-shuffle Hopf algebra $\overline{\mathrm{Sh}}_{-\sigma}(V)$.

Point 3 is straightforward. □

Corollary 3.2.8. *1. In the settings of proposition 3.2.7, the arrow operation $\overset{\epsilon}{\nwarrow} w$ is homotopic to zero on the complex $(T(V), {}^{\epsilon}d)$, and to $\zeta(w)\operatorname{Id}_{T(V)}$ on $(T(V), {}^{\epsilon}d - d^{\zeta})$.*

2. The complex $(T(V), {}^{\epsilon}d)$ is acyclic if σ is demi-natural with respect to w and the pair (w, ϵ) is normalized.

3. The complex $(T(V), d^{\zeta})$ is acyclic if the pair (w, ζ) is normalized.

Proof. 1. The contracting homotopies are given by the concatenation map $\overline{v} \mapsto \overline{v}w$; use relations from point 2 of the previous proposition.

2. The conditions on w imply, according to point 3 of the proposition, that $\overset{\epsilon}{\nwarrow} w$ is an identity map. Then use the previous point.

3. Use the last relation from point 2 of the proposition. □

3.3 Loday's hyper-boundaries

Our quantum shuffle setting provides a natural interpretation for J.-L.Loday's hyper-boundaries (see [46], exercise E.2.2.7), which we redefine as generalizations of the "shuffle" differentials from theorem 2.

Definition 3.3.1. Let (V, σ) be a pre-braided vector space with a braided character ϵ. The maps

$$\begin{aligned}
V^{\otimes n} &\longrightarrow V^{\otimes(n-k)},\\
{}^{\epsilon,(k)}d : \overline{v} &\longmapsto (\epsilon^{\otimes k} \otimes \operatorname{Id}_{n-k}) \circ \underset{-\sigma}{\overline{\sqcup\!\sqcup}}{}^{k,n-k}(\overline{v}),\\
d^{\epsilon,(k)} : \overline{v} &\longmapsto (-1)^{kn-\frac{k(k+1)}{2}}(\operatorname{Id}_{n-k} \otimes \epsilon^{\otimes k}) \circ \underset{-\sigma}{\overline{\sqcup\!\sqcup}}{}^{n-k,k}(\overline{v})
\end{aligned}$$

are called *hyper-boundaries* on $T(V)$.

The last sign should be understood as $(-1)^{n-1}(-1)^{n-2}\cdots(-1)^{n-k}$.
For $k = 1$ one recovers the braided differentials ${}^{\epsilon}d$ and d^{ϵ}.
The next step is to understand compositions of hyper-boundaries, generalizing

$$d^{(1)} \circ d^{(1)} = 0 = {}^{(1)}d \circ {}^{(1)}d.$$

We start with a kind of a special case. This result seems to be well-known, but we prove it here since the proof is difficult to find in literature.

Lemma 3.3.2. Consider a vector space W and an element $w \in W$. One has

$$w^{\otimes m} \underset{-\tau}{\sqcup\!\sqcup} w^{\otimes k} = \tbinom{m+k}{k}_{-1} w^{\otimes(m+k)},$$

where

$$\binom{m+k}{k}_{-1} = \begin{cases} 0 & \text{if } mk \text{ is odd},\\ \binom{[(m+k)/2]}{[k/2]} & \text{otherwise,} \end{cases}$$

and the brackets $[\,\cdot\,]$ stand for the lower integral part of a number.

Proof. By definition,

$$w^{\otimes m} \underset{-\tau}{\sqcup\!\sqcup} w^{\otimes k} = \sum_{s \in Sh_{m,k}} T_s^{-\tau} w^{\otimes(m+k)} = \sum_{s \in Sh_{m,k}} \operatorname{sign}(s) w^{\otimes(m+k)},$$

where $\operatorname{sign}(s)$ is the sign of a permutation s. Now for each negative permutation in $Sh_{m,k}$ we will associate a positive one in an injective way, counting the remaining positive permutations in $Sh_{m,k}$.

Given a negative permutation $s \in Sh_{m,k}$, choose, if it exists, the least i such that one of the preimages $s^{-1}(2i-1), s^{-1}(2i)$ lies in the set $\{1, \ldots, m\}$, while the other one lies in $\{m+1, \ldots, m+k\}$. Such i's will be called *split*. To such an s one associates $\overline{s}$ with

$$\overline{s}^{-1} := (s(1)^{-1}, \ldots, s(2i-2)^{-1}, s(2i)^{-1}, s(2i-1)^{-1}, s(2i+1)^{-1}, \ldots, s(m+k)^{-1}),$$

i.e. it is our s with the preimages of $2i-1$ and $2i$ interchanged. This constructs a bijection between negative and positive permutations for which a split i exists. It remains to count permutations without split i's (we call such permutations *coupled*) and to check that they are all positive.

- ✓ If $m+k$ is even, a coupled permutation divides the elements $1, \ldots, m+k$ into consecutive pairs with preimages by s lying in the same set $\{1, \ldots, m\}$ or $\{m+1, \ldots, m+k\}$. It is possible only when both m and k are even, giving $\binom{(m+k)/2}{k/2}$ possibilities for the values of s^{-1} on $(m+k)/2$ pairs.
- ✓ If $m+k$ is odd – say, m is even and k is odd – then, similarly, a coupled permutation divides the elements $1, \ldots, m+k-1$ into consecutive pairs with preimages by s lying in the same set, and $s^{-1}(m+k)$ lies automatically in $\{m+1, \ldots, m+k\}$, since only k is odd. This gives $\binom{(m+k-1)/2}{(k-1)/2} = \binom{(m+k-1)/2}{m/2}$ possibilities.

To conclude, notice that all the coupled permutations obtained are positive, since, for any i, the sign coming from the element $s^{-1}(2i-1)$ is "killed" by the sign coming from the element $s^{-1}(2i)$. □

This lemma is crucial in the calculations giving

Theorem 4. *Let (V, σ) be a pre-braided vector space with a braided character ϵ. One has*

$${}^{\epsilon,(m)}d \circ {}^{\epsilon,(k)}d = \binom{m+k}{k}_{-1} {}^{\epsilon,(m+k)}d,$$

$$d^{\epsilon,(m)} \circ d^{\epsilon,(k)} = \binom{m+k}{k}_{-1} d^{\epsilon,(m+k)}.$$

Proof. We prove the first formula only. By definition,

$${}^{\epsilon,(m)}d \circ {}^{\epsilon,(k)}d(\overline{v}) = (\epsilon \otimes \cdots \otimes \epsilon \otimes \mathrm{Id}_{n-k-m}) \circ (\epsilon \otimes \cdots \otimes \epsilon \otimes \underset{-\sigma}{\sqcup\!\sqcup}^{m,n-k-m}) \circ \underset{-\sigma}{\sqcup\!\sqcup}^{k,n-k}(\overline{v}).$$

By the coassociativity of the co-shuffle comultiplication, it equals

$$(\epsilon \otimes \cdots \otimes \epsilon \otimes \mathrm{Id}_{n-k-m}) \circ (\underset{-\sigma}{\sqcup\!\sqcup}^{k,m} \otimes \mathrm{Id}_{n-k-m}) \circ \underset{-\sigma}{\sqcup\!\sqcup}^{m+k,n-m-k}(\overline{v}).$$

Now ϵ is a braided character, so

$$(\epsilon \otimes \epsilon) \circ \sigma = \epsilon \otimes \epsilon = (\epsilon \otimes \epsilon) \circ \tau,$$

thus

$${}^{\epsilon,(m)}d \circ {}^{\epsilon,(k)}d(\overline{v}) = (\epsilon \otimes \cdots \otimes \epsilon \otimes \mathrm{Id}_{n-k-m}) \circ (\underset{-\tau}{\sqcup\!\sqcup}^{k,m} \otimes \mathrm{Id}_{n-k-m}) \circ \underset{-\sigma}{\sqcup\!\sqcup}^{m+k,n-m-k}(\overline{v}).$$

The dual version of the previous lemma calculates

$$(\epsilon \otimes \cdots \otimes \epsilon) \circ \underset{-\tau}{\overline{\sqcup\!\sqcup}}^{k,m} = \binom{m+k}{k}_{-1} \epsilon \otimes \cdots \otimes \epsilon,$$

and the previous expression becomes $\binom{m+k}{k}_{-1}{}^{\epsilon,(m+k)}d$. $\square$

The relations from J.-L.Loday's exercise, which are particular cases of the above theorem for several values of m and k, are thus easily proved and generalized thanks to our quantum co-shuffle interpretation.

Chapter 4

Basic examples: familiar (co)homologies recovered

Now we consider a $\Bbbk$-vector space (or an R-module) V with some algebraic structure, and we look for a ***pre-braiding σ encoding the properties of this structure***. Such pre-braidings are informally called *structural.* Certain algebraic properties of the initial structure are coded by the ***invertibility*** condition for the corresponding pre-braiding. In each case, ***braided (co)characters*** are determined, always up to scalar multiples, recovering the usual algebraic notions of (co)characters. Theorem 2 then gives numerous bicomplex structures on $T(V)$. We calculate explicitly some of the differentials obtained this way, recognizing many familiar (co-)homologies. ***Arrow operations*** are also considered, showing the triviality of some of the appearing (co)homologies (cf. corollary 3.2.8). In some cases, V is endowed with a (semi-)pre-braided coalgebra structure, giving, according to theorem 3, a (very) weakly bisimplicial structure on $T(V)$. The ***comultiplications*** Δ we use always arise naturally from the original algebraic structure.

A typical section of this chapter contains five main lemmas, one for each question emphasized above, followed by propositions explicitly describing the bidifferential or simplicial structures obtained. Graphical calculus is extensively used.

We give here a table summarizing the algebraic counterparts of our braided notions in the concrete examples from this chapter. Everything is explained in detail in what follows. Several recovered familiar complexes are also mentioned. The fact that we get many known constructions and results is not very surprizing: we were inspired by these concrete examples of homologies of algebraic structures when developing our general braided theory.

structure	**pre-braiding**	**inverse**	**characters**
vector space V	flip τ: $v \otimes w \mapsto w \otimes v$	$\tau^{-1} = \tau$	any $\epsilon \in V^*$
unital associative algebra $(V, \cdot, \mathbf{1})$	σ_μ: $v \otimes w \mapsto$ $\mathbf{1} \otimes v \cdot w$	no inverse in general	algebra character: $\epsilon(v \cdot w) = \epsilon(v)\epsilon(w)$, $\epsilon(\mathbf{1}) = 1$
unital Leibniz algebra $(V, [,], \mathbf{1})$	$\sigma_{[,]}$: $v \otimes w \mapsto w \otimes v$ $+\mathbf{1} \otimes [v, w]$	$\exists\ \sigma_{[,]}^{-1}$	Lie character: $\epsilon([v, w]) = 0$, $\epsilon(\mathbf{1}) = 1$
shelf $(S, \lhd)$, $V := \Bbbk S$	$\sigma_\lhd$: $(a, b) \mapsto (b, a \lhd b)$	$\exists\ \sigma_\lhd^{-1}$ iff S is a rack	shelf character: $\epsilon(a \lhd b) = \epsilon(a)$

structure	**arrow operations**	Δ	**complexes**
vector space	multiplication by scalars	–	Koszul
unital associative algebra	peripheral: $v_1 \dots v_{n-1}v_n \overset{\epsilon}{\nwarrow} w = v_1 \dots v_{n-1}\mu(v_n \otimes w)$	$\Delta(v) = \mathbf{1} \otimes v$	bar, Hochschild
unital Leibniz algebra	adjoint: $v_1 \dots v_n \overset{\epsilon}{\nwarrow} w = \sum_{i=1}^{n} v_1 \dots [v_i, w] \dots v_n$	$\Delta(v) = v \otimes \mathbf{1} + \mathbf{1} \otimes v$, $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$	Leibniz, Chevalley-Eilenberg
shelf	diagonal: $(a_1, \dots, a_n) \overset{\epsilon}{\nwarrow} b = (a_1 \lhd b, \dots, a_n \lhd b)$	$\Delta(a) = (a, a)$	shelf ([67],[68]), rack ([27]), quandle ([11])

Table 4.1: Main ingredients of braided homology theories in basic algebraic settings

Pre-braidings for vector spaces and self-distributive structures are classical; that for Leibniz algebras was used in the Lie case by A.Crans in [15] (cf. also [8]), but does not seem to be widely known; the author has never met the pre-braiding for associative algebras elsewhere.

All the constructions for associative and Leibniz algebras can easily be effectuated in any preadditive monoidal (symmetric in the Leibniz case) category, cf. chapter 5. A categorification of self-distributive structures is more subtle; it is presented in chapter 11.

4.1 Koszul complex

Following a nice mathematical tradition, the first example we consider is the trivial one: that of an "empty" structure. Take any vector space V and the flip

$$\tau : v \otimes w \longmapsto w \otimes v$$

as its braiding. Each $e \in V$ is automatically a braided co-character, and each $\epsilon \in V^*$ is a character. In particular,

$${}^{\epsilon}d = d^{\epsilon} : v_1 \dots v_n \longmapsto \sum_{i=1}^{n} (-1)^{i-1} \epsilon(v_i) v_1 \dots \widehat{v_i} \dots v_n$$

gives the well-known ***Koszul differential***, in its simplest form.

Further, a (τ-cocommutative) braided coalgebra structure on (V, τ) is precisely a (resp. cocommutative) comultiplication in the usual sense. The corresponding very weakly simplicial structure on $T(V)$ is simplicial if and only if ϵ is the *counit* for the comultiplication Δ, i.e.

$$(\epsilon \otimes \mathrm{Id}_V) \circ \Delta = (\mathrm{Id}_V \otimes \epsilon) \circ \Delta = \mathrm{Id}_V .$$

In the last case the cocommutativity is not necessary for the structure to be simplicial. Thus, according to theorem 3, one can quotient the Koszul complex by the images of $s_{n;i} := \Delta_i$ without changing the homology.

4.2 Rack complex

A pre-braiding encoding self-distributivity

The simplest non-trivial example of a braiding naturally coming from an algebraic structure is the following. Take a set S with a binary operation $\lhd : S \times S \to S$. Define an application

$$\begin{aligned} \sigma = \sigma_{\lhd} : S \times S &\longrightarrow S \times S, \\ (a, b) &\longmapsto (b, a \lhd b). \end{aligned} \tag{4.1}$$

It is very familiar to topologists, since it can be interpreted in terms of the fundamental group of the complement of a knot. See for instance the seminal paper [34], or [35] for a very readable introduction. Graphically $\sigma_{\lhd}$ looks as follows:

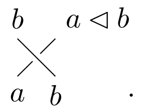


Figure 4.1: Pre-braiding for shelves

All the "braided" notions are to be understood in the ***set-theoretic sense*** in this section (cf. remark 2.0.2).

The structure for which $\sigma_{\lhd}$ is a pre-braiding is well-known:

Lemma 4.2.1. The map $\sigma_{\lhd}$ is a pre-braiding if and only if $\lhd$ is ***self-distributive***:

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c) \qquad \forall a, b, c \in S. \tag{SD}$$

Proof. Here and in subsequent lemmas we content ourselves with graphical proofs.

Let us see what the Yang-Baxter equation (YB) means for $\sigma = \sigma_{\lhd}$:

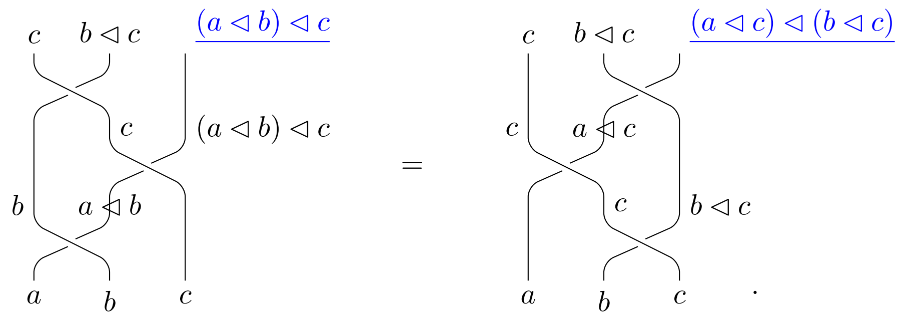


Figure 4.2: Pictorial proof of lemma 4.2.1

In other words,

$$\begin{aligned} \sigma_1 \circ \sigma_2 \circ \sigma_1(a, b, c) &= (c, b \lhd c, (a \lhd b) \lhd c), \\ \sigma_2 \circ \sigma_1 \circ \sigma_2(a, b, c) &= (c, b \lhd c, (a \lhd c) \lhd (b \lhd c)). \end{aligned}$$

The equality of these two triples is equivalent to (SD). $\square$

Definition 4.2.2. A pair $(S, \lhd)$ satisfying (SD) is called a *shelf* (the term is coined by Alissa Crans, see [15]), or a *self-distributive system.*

The "if and only if" formulation of the lemma shows that the pre-braiding $\sigma_{\lhd}$ ***encodes*** the defining property of a shelf, just as we wanted.

Fix a <u>shelf</u> $(S, \lhd)$ until the end of this section.

Invertibility and the rack condition

Let us now study the invertibility conditions for $\sigma_\lhd$. They are also well-known:

Lemma 4.2.3. The pre-braiding $\sigma_\lhd$ is a braiding if and only if the application $a \mapsto a \lhd b$ is a bijection on S for every $b \in S$, that is if there exists an application $\widetilde{\lhd} : S \times S \longrightarrow S$ such that

$$(a \lhd b)\widetilde{\lhd} b = (a \widetilde{\lhd} b) \lhd b = a \qquad \forall a, b \in S. \tag{R}$$

Definition 4.2.4. A triple $(S, \lhd, \widetilde{\lhd})$ satisfying (SD) and (R) is called *a rack* (the term originates from J.H.Conway and G.Wraith's correspondence).

Shelf characters, spindles and quandles

Linearize a shelf $(S, \lhd)$: put

$$V := \Bbbk S,$$

where $\Bbbk$ is a field, and extend the braiding $\sigma_\lhd$ to V linearly. One gets a pre-braided vector space $(V, \sigma_\lhd)$. As usual, most statements remain true over a commutative unital ring R.

Let us describe all the braided (co)characters in this linearized setting.

Lemma 4.2.5. 1. Co-characters $e = \sum_{i \in I} \alpha_i a_i \in V$, where $\{a_i\}_{i \in I}$ is a finite set of pairwise distinct elements of S, and $\alpha_i \in \Bbbk^*$, are characterized by

$$e \lhd a_i = e \qquad \forall i \in I.$$

2. Characters $\epsilon \in V^*$ are characterized by

$$\epsilon(a \lhd b) = \epsilon(a) \tag{4.2}$$

for all $a, b \in S$ such that $\epsilon(b) \neq 0$.

In the R-linear setting, these conditions are sufficient but not necessary in general.

Here are some examples of braided (co)characters:

Example 4.2.6. 1. All $a \in S$ are co-characters if and only if S is a ***spindle*** (one more term coined by A.Crans), i.e. a shelf with idempotent elements:

$$a \lhd a = a \qquad \forall a \in S.$$

2. The linearization of

$$\varepsilon : a \mapsto 1 \ \forall a \in S \tag{4.3}$$

is always a character.

3. The linearization of ***"Dirac maps"***

$$\varphi_a(b) := \delta_{a,b} = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{for other } b \in S \end{cases} \tag{4.4}$$

(here $\delta_{a,b}$ is the Kronecker delta) are characters precisely for idempotent a's such that $b \lhd a \neq a$ for $b \neq a$. In particular, if S is a ***quandle***, i.e. a rack which is also a spindle, then all the φ_a's are characters. (The term "quandle" was introduced by D.Joyce in [34]; he deliberately chose a word not existing in English.)

We finish with a more conceptual construction of a class of braided characters. Recall that a character for an algebraic structure is usually defined as a morphism to the trivial structure. Here it is natural (having in mind the conjugation quandle) to define the ***trivial shelf structure*** on a set X by

$$x \lhd y = x \qquad \forall x, y \in X.$$

Definition 4.2.7. A *shelf character* for a shelf $(S, \lhd)$ is a shelf morphisms $\epsilon : S \to \Bbbk$, where $\Bbbk$ is endowed with the trivial shelf structure. In other words, it is a map $\epsilon : S \to \Bbbk$ satisfying (4.2) for all $a, b \in S$.

Lemma 4.2.5 then implies

Lemma 4.2.8.
1. The linearization of a shelf character is always a braided character for the pre-braiding $\sigma_\lhd$.
2. Moreover, two braided characters coming from shelf characters are automatically $\sigma_\lhd$-compatible.

Diagonal comultiplication

The last ingredient we need is a comultiplication. The one proposed here is quite classical in the self-distributive world:

Lemma 4.2.9. Let $\Delta_D : V \to V \otimes V$ be the linearization of the ***diagonal map***

$$\begin{aligned} D : S &\longrightarrow S \times S, \\ a &\longmapsto (a, a) \qquad \forall a \in S. \end{aligned}$$

Then

1. $(V, \sigma_\lhd, \Delta_D)$ is a semi-pre-braided coalgebra;
2. this coalgebra is pre-braided if and only if

$$a \lhd b = (a \lhd b) \lhd b \qquad \forall a, b \in S;$$

3. Δ_D is $\sigma_\lhd$-cocommutative if and only if $(S, \lhd)$ is a spindle.

Remark 4.2.10. The image of the map

$$s_{n;i} := \Delta_i = \mathrm{Id}_V^{\otimes(i-1)} \otimes \Delta_D \otimes \mathrm{Id}_V^{\otimes(n-i)}$$

is the linear span of the elements $(a_1, \ldots, a_{n+1}) \in S^{\times(n+1)}$ with $a_i = a_{i+1}$.

Shelf, rack and quandle homologies

It is now time to put together all the ingredients and to make some concrete calculations of bidifferentials. Only the case of braided characters and chain complexes is considered here, the co-case being similar.

Start with the character ε defined by (4.3).

Proposition 4.2.11. *Take a shelf* $(S, \lhd)$.

1. *The pre-braiding* $\sigma_\lhd$ *from* (4.1) *and the character* $\varepsilon : a \mapsto 1 \ \forall a \in S$ *define the following bicomplex structure on* $T(\Bbbk S)$ *(or* $T(RS)$*):*

$$
\begin{aligned}
{}^{\varepsilon}d(a_1, \ldots, a_n) &= \sum_{i=1}^{n} (-1)^{i-1}((a_1 \lhd a_i), \ldots, (a_{i-1} \lhd a_i), a_{i+1}, \ldots, a_n),\\
d^{\varepsilon}(a_1, \ldots, a_n) &= \sum_{i=1}^{n} (-1)^{i-1}(a_1, \ldots \widehat{a_i} \ldots, a_n).
\end{aligned}
$$

2. *This bidifferential comes from a pre-bisimplicial structure given by*

$$
\begin{aligned}
d_{n;i}(a_1, \ldots, a_n) &= ((a_1 \lhd a_i), \ldots, (a_{i-1} \lhd a_i), a_{i+1}, \ldots, a_n),\\
d'_{n;i}(a_1, \ldots, a_n) &= (a_1, \ldots \widehat{a_i} \ldots, a_n).
\end{aligned}
$$

3. *If our shelf is moreover a spindle, then* $(T(\Bbbk S), d_{n;i}, d'_{n;i}, s_{n;i} := \Delta_i)$ *is a weakly bisimplicial vector space. As a consequence, the linear span* $C^D_*(S)$ *of the elements* $(a_1, \ldots, a_n) \in S^{\times n}$ *with* $a_i = a_{i+1}$ *for an* $1 \leqslant i \leqslant n-1$ *forms a sub-bicomplex of* $(T(\Bbbk S), {}^{\varepsilon}d, d^{\varepsilon})$*. The same holds in* $T(RS)$.

Proof. Points 1 and 2 are direct applications of theorems 2 and 3 respectively, combined with the lemmas from this section.

As for point 3, theorem 3 gives only a half of this assertion: $(T(\Bbbk S), d_{n;i}, s_{n;i})$ is a weakly simplicial vector space, hence $C^D_*(S)$ is a subcomplex of $(T(\Bbbk S), \partial = {}^{\varepsilon}d)$. Since $(\Bbbk S, \sigma_\lhd, \Delta_D)$ is only a **semi**-braided coalgebra in general, the compatibilities (3.5) between the $d'_{n;i}$'s and the $s_{n;i}$'s should be verified by hand, which is an easy exercise. Finally, the explicit description of the degenerate sub-bicomplex follows from remark 4.2.10. □

Let us point out familiar complexes recovered in this proposition:

Example 4.2.12.

1. The complex
$$C^R_*(S) := (T(\mathbb{Z}S), {}^{\varepsilon}d - d^{\varepsilon})$$
gives what is known as the ***rack homology***.
2. The complex
$$C^{\lhd}_*(S) := (T(\mathbb{Z}S), {}^{\varepsilon}d)$$
gives the ***shelf, or one-term distributive, homology***.
3. The quotient $C^Q_*(S)$ of $C^R_*(S)$ by the subcomplex $C^D_*(S)$ gives what is known as the ***quandle homology***.

The rack homology was first defined by R.Fenn, C.Rourke and B.Sanderson ([27], 1995; according to Roger Fenn, "Unusually in the history of mathematics, the discovery of the homology and classifying space of a rack can be precisely dated to 2 April 1990"), and the quandle homology was later suggested by J.S.Carter, D.Jelsovsky, S.Kamada, L.Langford and M.Saito ([11], 2003). The one-term distributive homology was recently introduced by J.H.Przytycki and A.S.Sikora ([67], [68], 2011), with a multi-term generalization. Cycles from the complexes defining these homologies provide an efficient tool for producing knot invariants. Numerous computations can be found in literature.

Arrow operations are diagonal

The map $\overset{\varepsilon}{\nwarrow}$ takes the familiar ***diagonal*** form here:

$$(a_1, \ldots, a_n) \overset{\varepsilon}{\nwarrow} b = (a_1 \lhd b, \ldots, a_n \lhd b).$$

Moreover, ε is right $\sigma_\lhd$-compatible with any $b \in S$. Proposition 3.2.7 and corollary 3.2.8 are then applicable, recovering some results on homology operations from [63], [68] and [67] and their consequences:

Proposition 4.2.13. *1. The complex $(T(\Bbbk S), d^\varepsilon)$ is acyclic.*

2. If there exists an element $b \in S$ such that the application $a \mapsto a \lhd b$ is a bijection on S, then the complex $(T(\Bbbk S), {}^\varepsilon d)$ is acyclic.

3. If there exists an $a \in S$ stable by all the inner shelf morphisms, i.e.

$$a \lhd b = a \qquad \forall b \in S, \tag{4.5}$$

then the complex $(T(\Bbbk S), {}^\varepsilon d)$ is acyclic.

All the assertions are still valid for RS.

Proof. Point 1 follows from corollary 3.2.8 (point 3) and the observation that every $b \in S$ forms a normalized pair with ε.

Point 2 follows from corollary 3.2.8 (point 1), since the arrow operation $\overset{\varepsilon}{\nwarrow} b$, shown there to be homotopic to zero, is now invertible: the inverse is given by

$$(a_1, \ldots, a_n) \longmapsto (a_1 \widetilde{\lhd} b, \ldots, a_n \widetilde{\lhd} b),$$

where $a \mapsto a \widetilde{\lhd} b$ denotes the map inverse to $a \mapsto a \lhd b$.

Point 3 follows from the "right" version of corollary 3.2.8 (point 3): condition (4.5) means precisely that a is left $\sigma_\lhd$-compatible with ε, and the normalization condition is automatic. □

Thus, the complex $(T(\Bbbk S), {}^\varepsilon d)$ is acyclic for a rack. However, it can be highly non-trivial for shelves (cf. ([67], [68]).

Dirac maps and partial derivatives

Further, let us turn to the characters given by Dirac maps defined by (4.4).

Proposition 4.2.14. *1. Take a quandle $(S, \lhd, \widetilde{\lhd})$ with a fixed element a. Theorem 2 applied to the character φ_a gives the following bicomplex structure on $T(RS)$:*

$$ {}^{\varphi_a}d(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i-1} \delta_{a,a_i}((a_1 \lhd a_i), \ldots, (a_{i-1} \lhd a_i), a_{i+1}, \ldots, a_n),$$

$$d^{\varphi_a}(a_1 \ldots a_n) = \sum_{i=1}^{n} (-1)^{i-1} \delta_{a,(a_i \lhd a_{i+1}) \lhd \cdots) \lhd a_n}(a_1, \ldots \widehat{a_i} \ldots, a_n).$$

2. According to theorem 3, this bidifferential comes from the pre-bisimplicial structure

$$d_{n;i}(a_1, \ldots, a_n) = \delta_{a,a_i}((a_1 \lhd a_i), \ldots, (a_{i-1} \lhd a_i), a_{i+1}, \ldots, a_n),$$

$$d'_{n;i}(a_1, \ldots, a_n) = \delta_{a,(a_i \lhd a_{i+1}) \lhd \cdots) \lhd a_n}(a_1, \ldots \widehat{a_i} \ldots, a_n).$$

3. *The face maps $d_{n;i}$ combined with degeneracies $s_{n;i} := \Delta_i$ give a weakly simplicial structure. Differential ${}^{\varphi_a}d$ thus descends to the normalized complex $T(RS)/C_*^D(S)$.*

The differentials ${}^{\varphi_a}d$ are called ***partial derivatives*** and are denoted by $\frac{\partial^1}{\partial a}$ in [63]. Our general setting thus contains some results of [63].

Remark 4.2.15. One can not talk about weakly bisimplicial structure here, since the coalgebra $(RS, \sigma_\lhd, \Delta_D)$ is only semi-pre-braided, and the compatibilities (3.5) between the $d'_{n;i}$'s and the $s_{n;i}$'s, which are automatical for pre-braided coalgebras and happen to hold for the character ε, are no longer true for the φ_a's. However, one checks that $C_*^D(S)$ is still a sub-bicomplex of $(T(RS), {}^{\varphi_a}d, d^{\varphi_a})$: indeed, $d'_{n+1;i} \circ s_{n;j}(a_1, \ldots, a_n)$ is proportional to $s_{n-1;j-1}(a_1, \ldots, \widehat{a_i}, \ldots, a_n)$ and is thus still in the image of $s_{n-1;j-1}$ for all $1 \leqslant i < j \leqslant n$.

Twisted rack homology

We finish with an example where different characters are used on the right and on the left. It is inspired by the work of J.S.Carter, M. Elhamdadi, and M.Saito, cf. [9].

Proposition 4.2.16. *Take a shelf $(S, \lhd)$ and work with its linearization ΛS, $\Lambda = \mathbb{Z}[T^{\pm 1}]$. The pre-braiding $\sigma_\lhd$, combined with characters ε and*

$$\begin{aligned} \epsilon_T : \Lambda S &\longrightarrow \Lambda, \\ a &\longmapsto T \quad \forall a \in S \end{aligned}$$

define, via theorem 2, a bicomplex structure on $T(\Lambda S)$ by

$$\begin{aligned} {}^{\varepsilon}d(a_1, \ldots, a_n) &= \sum_{i=1}^{n} (-1)^{i-1} ((a_1 \lhd a_i), \ldots, (a_{i-1} \lhd a_i), a_{i+1}, \ldots, a_n), \\ d^{\epsilon_T}(a_1, \ldots, a_n) &= \sum_{i=1}^{n} (-1)^{i-1} T(a_1, \ldots \widehat{a_i} \ldots, a_n). \end{aligned}$$

These differentials come from a pre-bisimplicial or weakly bisimplicial structure analogous to those from proposition 4.2.11.

The differential ${}^{\varepsilon}d - d^{\epsilon_T}$ defines the ***twisted rack homology*** from [9].

4.3 Bar complex

A pre-braiding encoding associativity

Take a $\Bbbk$-vector space (or an R-module) V endowed with a bilinear operation $\mu : V \otimes V \longrightarrow V$ and a distinguished element $\mathbf{1} \in V$, sometimes regarded as a linear map

$$\begin{aligned} \nu : \Bbbk &\longrightarrow V, \\ \alpha &\longmapsto \alpha \mathbf{1}. \end{aligned}$$

Morally, one should think about modeling ***unital associative algebras***. In this section we construct quite an exotic non-invertible pre-braiding on V which encodes the associativity of μ.

Consider the bilinear application

$$\begin{aligned} \sigma = \sigma_\mu : V \otimes V &\longrightarrow V \otimes V, \\ v \otimes w &\longmapsto \mathbf{1} \otimes \mu(v \otimes w) \end{aligned} \tag{4.6}$$

or, graphically,

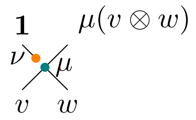


Figure 4.3: Pre-braiding for associative algebras

Lemma 4.3.1. Suppose that $\mathbf{1}$ is a right unit for μ, i.e.

$$\mu(v \otimes \mathbf{1}) = v \quad \forall v \in V.$$

Then the map σ_μ is a pre-braiding if and only if μ is associative on V.

Proof. Graphically, YBE for σ_μ means

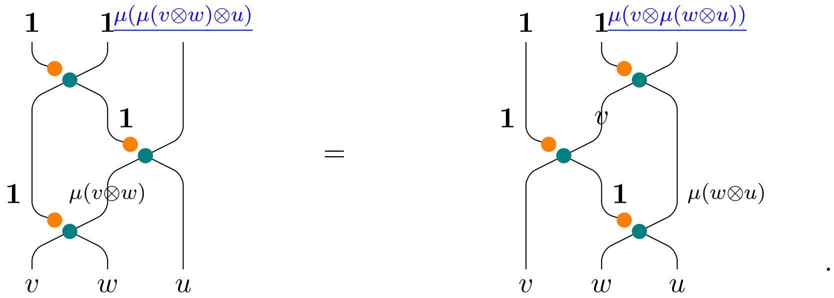


Figure 4.4: Pictorial proof of lemma 4.3.1

This is equivalent to the associativity condition

$$\mu(\mu(v \otimes w) \otimes u) = \mu(v \otimes \mu(w \otimes u)) \qquad \forall v, w, u \in V. \tag{Ass}$$

$\square$

The associativity condition is graphically depicted as follows:

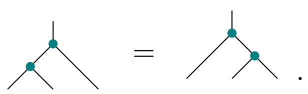


Figure 4.5: Associativity

One thus gets, like in the case of shelves, a pre-braiding subtly encoding the algebraic structure “associative algebra”.

Remark 4.3.2. The braiding σ_μ is **highly non-invertible**. More precisely,

$$\sigma_\mu^2 = \sigma_\mu$$

if $\mathbf{1}$ is moreover a left unit.

Fix an associative $\Bbbk$-algebra (V, μ) with a right unit $\mathbf{1}$ until the end of this section. Such algebras are called ***right-unital*** here.

Algebra characters are braided characters

Now let us look for braided (co)characters. We work, as usual, up to scalar multiples. Natural candidates are certainly the "structural" characters:

Definition 4.3.3. An *algebra character* is a unital algebra morphism $\epsilon : V \to \Bbbk$, where $\Bbbk$ is endowed with the trivial algebra structure. In other words, it is an $\epsilon \in V^*$ satisfying

$$\begin{aligned} \epsilon(\mu(v \otimes w)) &= \epsilon(v)\epsilon(w) \qquad \forall v, w \in V, \\ \epsilon(\mathbf{1}) &= 1. \end{aligned} \tag{4.7}$$

A *non-unital algebra character* satisfies the first condition only.

Graphically, it means

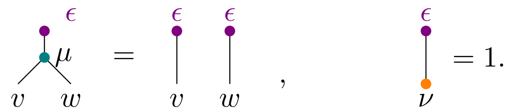


Figure 4.6: Algebra character

Lemma 4.3.4. Take a right-unital associative $\Bbbk$-algebra $(V, \mu, \mathbf{1})$.

1. The only braided co-character for (V, σ_μ) is the right unit $\mathbf{1}$.
2. Braided characters are precisely maps $\epsilon \in V^*$ satisfying

$$\epsilon(\mathbf{1})\epsilon(\mu(v \otimes w)) = \epsilon(v)\epsilon(w) \quad \forall v, w \in V. \tag{4.8}$$

3. In particular, every algebra character is a braided character.
4. Any non-zero solution of (4.8) is a scalar multiple of an algebra character.

Working over a commutative unital ring R instead of a field $\Bbbk$, one has to drop the last statement and the uniqueness assertion from the first one.

An exotic comultiplication

We present now a comultiplication for our pre-braided vector space.

Lemma 4.3.5.

1. The linear map

$$\begin{aligned} \Delta_{\mathbf{1}} : V &\longrightarrow V \otimes V, \\ v &\longmapsto \mathbf{1} \otimes v \end{aligned}$$

endows (V, σ_μ) with a pre-braided coalgebra structure.

2. The comultiplication $\Delta_{\mathbf{1}}$ is σ_μ-cocommutative if and only if $\mathbf{1}$ is a left unit.

Arrow operations are peripheral

The last remarks concern arrow operations and the special role of the right unit $\mathbf{1}$ in our braided story. Recall definition 3.2.6 and corollary 3.2.8.

Lemma 4.3.6.

1. The arrow operations give peripheral actions:

$$v_1 \ldots v_{n-1} v_n \overset{\epsilon}{\nwarrow} w = \epsilon(\mathbf{1}) v_1 \ldots v_{n-1} \mu(v_n \otimes w) \qquad \forall v_i, w \in V.$$

2. In particular, the right unit $\mathbf{1}$ acts by identity if ε is an algebra character:

$$* \overset{\varepsilon}{\nwarrow} \mathbf{1} = \mathrm{Id}_{T(V)}\,.$$

3. The pre-braiding σ_μ is demi-natural with respect to $\mathbf{1}$. Consequently, $\mathbf{1}$ is right σ_μ-compatible with any $f \in V^*$.

Hochschild homology with trivial coefficients

We now turn to concrete computations. Cochain complexes obtained in our situation are not very interesting, while chain complexes are:

Proposition 4.3.7. *Take a right-unital associative algebra $(V, \mu, \mathbf{1})$ over a commutative unital ring R and two algebra characters ϵ and ζ.*

1. *The pre-braiding σ_μ from (4.6) and the braided characters ϵ and ζ define, via theorem 2, the following bicomplex structure on $T(V)$:*

$$\begin{aligned} {}^{\epsilon}d(v_1 \ldots v_n) &= \epsilon(v_1)v_2 \ldots v_n + \sum_{i=1}^{n-1} (-1)^i v_1 \ldots v_{i-1}\mu(v_i \otimes v_{i+1})v_{i+2} \ldots v_n, \\ d^{\zeta}(v_1 \ldots v_n) &= (-1)^{n-1}\zeta(v_n)v_1 \ldots v_{n-1} \\ &\qquad + \sum_{i=0}^{n-2} (-1)^i \zeta(v_{i+1}) \cdots \zeta(v_n) v_1 \ldots v_i \mathbf{1}\mathbf{1} \ldots \mathbf{1}. \end{aligned}$$

2. *According to theorem 3, this bidifferential comes from the pre-bisimplicial structure*

$$\begin{aligned} d_{n;1}(v_1 \ldots v_n) &= \epsilon(v_1)v_2 \ldots v_n, \\ d_{n;i+1}(v_1 \ldots v_n) &= v_1 \ldots v_{i-1}\mu(v_i \otimes v_{i+1})v_{i+2} \ldots v_n, \qquad 1 \leqslant i \leqslant n-1, \\ d'_{n;i}(v_1 \ldots v_n) &= \zeta(v_i) \cdots \zeta(v_n) v_1 \ldots v_{i-1}\mathbf{1}\mathbf{1} \ldots \mathbf{1}, \qquad 1 \leqslant i \leqslant n-1, \\ d'_{n;n}(v_1 \ldots v_n) &= \zeta(v_n)v_1 \ldots v_{n-1}. \end{aligned}$$

3. *The complex $(T(V), {}^{\epsilon}d)$ is acyclic.*
4. *If $\mathbf{1}$ is a two-sided unit, then the above structure can be completed into a weakly bisimplicial one by putting*

$$s_{n;i}(v_1 \ldots v_n) = v_1 \ldots v_{i-1}\mathbf{1}v_i \ldots v_n, \qquad 1 \leqslant i \leqslant n.$$

5. *In this case the structure $(T(V), d_{n;i}, s_{n;i})$ is even simplicial.*
6. *In the normalized bicomplex, $d'_{n;i} = 0$ for $i < n-1$.*
7. *Still supposing the unit $\mathbf{1}$ two-sided, the differential ${}^{\epsilon}d - d^{\zeta}$ descends to $T(V')$, where*

$$V' := V/R\mathbf{1},$$

giving the differential

$$\begin{aligned} {}^{\epsilon}d^{\zeta}(v_1 \ldots v_n) := \epsilon(v_1)v_2 \ldots v_n& \\ + \sum_{i=1}^{n-1} (-1)^i v_1 \ldots v_{i-1}\mu(v_i \otimes v_{i+1})v_{i+2} \ldots v_n&, \\ + (-1)^n \zeta(v_n)v_1 \ldots v_{n-1}&. \end{aligned}$$

Proof. Most of the assertions follow from theorems 2 and 3, combined with preceding lemmas.

Point 3 is the corollary 3.2.8 applied to the element $\mathbf{1}$, posseding the "nice" properties described in lemma 4.3.6.

Point 5 also follows from the properties of $\mathbf{1}$.

More work is needed for proving point 7. Point 4 ensures that tensors $v_1 \dots v_{i-1}\mathbf{1}v_i \dots v_n$ with $1 \leqslant i \leqslant n$ generate a sub-bicomplex of $T(V)$, hence a subcomplex of $(T(V), {}^{\epsilon}d - d^{\zeta})$. Further, proposition 3.2.7 implies that the concatenation map $\overline{v} \mapsto \overline{v}\mathbf{1}$ is a differential complex endomorphism of $(T(V), {}^{\epsilon}d - d^{\zeta})$, thus its image $T(V) \otimes \mathbf{1}$ is a subcomplex. Forming the quotient by these two subcomplexes, one gets the desired differential on $T(V')$. □

Differential ${}^{\epsilon}d^{\zeta}$ defines a (generalization of a) homology sometimes called the ***group homology***, which can also be regarded as the Hochschild homology with trivial coefficients.

A "non-unital" remark

Remark 4.3.8. In the non-unital case, i.e. when V is endowed with a bilinear operation μ only, one enriches V with a formal two-sided unit: $\widetilde{V} := V \oplus R\mathbf{1}$, extending μ by

$$\mu(\mathbf{1} \otimes v) = \mu(v \otimes \mathbf{1}) = v \qquad \forall v \in \widetilde{V}.$$

Due to the equivalence of the associativity of μ on V and on $\widetilde{V}$, lemma 4.3.1 asserts that σ_μ is a pre-braiding $\underline{\text{on } \widetilde{V}}$ if and only if μ is associative $\underline{\text{on } V}$. Take the character $\varepsilon(V) \equiv 0, \varepsilon(\mathbf{1}) = 1$ on $\widetilde{V}$. The differential ${}^{\varepsilon}d^{\varepsilon}$ descends to $T((\widetilde{V})') \simeq T(V)$, as explained in the previous proposition. One recovers the well-known ***bar (or standard) differential***:

$$d_{bar}(v_1 \dots v_n) = \sum_{i=1}^{n-1} (-1)^i v_1 \dots v_{i-1} \mu(v_i \otimes v_{i+1}) v_{i+2} \dots v_n.$$

Moreover, a non-unital algebra character $\epsilon \in V^*$ extends to an algebra character on $\widetilde{V}$ by imposing $\epsilon(\mathbf{1}) = 1$. Two such non-unital algebra characters then define a differential ${}^{\epsilon}d^{\zeta}$ on $T((\widetilde{V})') \simeq T(V)$.

This trick of adding formal elements will often be handy in what follows.

Remark 4.3.9. One can also obtain the bar differential without doing this formal unit gymnastics. It suffices to replace the total differential with a "cut version"

$$\partial := \sum_{i=2}^{n} d_{n;i}$$

for the pre-simplicial structure from point 2 of proposition 4.3.7.

4.4 Leibniz complex

Leibniz algebras are "non-commutative" versions of Lie algebras. They were discovered by A.Bloh in 1965, but it were J.-L.Loday and his student C.Cuvier who woke the general interest in this structure around 1989 by, firstly, lifting the classical Chevalley-Eilenberg boundary map from the exterior to the tensor algebra, which yields a new interesting chain complex, and, secondly, by observing that the antisymmetry condition could be omitted (cf. [46],[16],[47],[48],[17]). Here we recover their complex guided by our "braided" considerations. Our interpretation explains the somewhat mysterious element ordering and signs in their formula.

A braiding encoding the Leibniz condition

The braiding we construct for Leibniz algebras is inspired by the pre-braiding for associative algebras and is quite exotic as well. It also appears, in the Lie algebra context, in A.Crans's work ([15]). She attributes this construction to James Dolan. See also [8].

Like in the previous section, let V be a $\Bbbk$-vector space (or an R-module) with a bilinear operation, denoted by $[,] : V \otimes V \longrightarrow V$ this time, and a distinguished element $\mathbf{1} \in V$. Morally, think about modeling ***Lie or Leibniz algebras***.

Consider the bilinear application

$$\begin{aligned} \sigma = \sigma_{[,]} : V \otimes V &\longrightarrow V \otimes V, \\ v \otimes w &\longmapsto w \otimes v + \mathbf{1} \otimes [v, w]. \end{aligned} \tag{4.9}$$

It is a kind of a mixture of the flip τ and the pre-braiding σ_μ for associative algebras.

Lemma 4.4.1. Suppose that $\mathbf{1}$ is a ***Lie unit***, i.e. a central element, in V:

$$[\mathbf{1}, v] = [v, \mathbf{1}] = 0 \qquad \forall v \in V. \tag{4.10}$$

Then the map $\sigma_{[,]}$ is a pre-braiding if and only if

$$[v, [w, u]] = [[v, w], u] - [[v, u], w] \qquad \forall v, w, u \in V. \tag{Lei}$$

Proof. We omit the details of the calculations here; they are quite easy with our graphical calculus. One gets:

$$\begin{aligned} \sigma_1\sigma_2\sigma_1(v \otimes w \otimes u) = u \otimes w \otimes v + \mathbf{1} \otimes w \otimes [v, u] + \mathbf{1} \otimes [w, u] \otimes v + \\ + u \otimes \mathbf{1} \otimes [v, w] + \mathbf{1} \otimes \mathbf{1} \otimes \underline{[[v, w], u]}, \end{aligned}$$

$$\begin{aligned} \sigma_2\sigma_1\sigma_2(v \otimes w \otimes u) = u \otimes w \otimes v + \mathbf{1} \otimes w \otimes [v, u] + \mathbf{1} \otimes [w, u] \otimes v + \\ + u \otimes \mathbf{1} \otimes [v, w] + \mathbf{1} \otimes \mathbf{1} \otimes \underline{([v, [w, u]] + [[v, u], w])}. \end{aligned}$$

So YBE for $\sigma_{[,]}$ is equivalent to (Lei) for $[,]$. □

Note that for the "only if" part of the statement, it is essential to work over a field $\Bbbk$, or to demand another technical condition (cf. lemma 5.2.3).

The condition (Lei) is graphically depicted as follows:

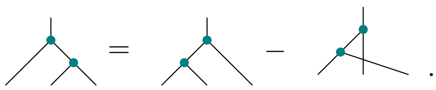


Figure 4.7: Leibniz condition

Definition 4.4.2. A pair $(V, [,])$ satisfying (Lei) is a *Leibniz algebra*, called *unital* if endowed with a Lie unit $\mathbf{1}$.

One gets the notion of *Lie algebra* when adding the antisymmetry condition.

Lemma 4.4.1 means that once again one gets a pre-braiding encoding an algebraic structure – it is the Leibniz algebra structure this time.

Fix a unital Leibniz algebra $(V, [,], \mathbf{1})$ until the end of this section.

Lemma 4.4.3. The pre-braiding $\sigma_{[,]}$ is invertible, the inverse given by

$$\sigma_{[,]}^{-1} : v \otimes w \longmapsto w \otimes v - [w, v] \otimes \mathbf{1}.$$

Remark 4.4.4. The invertibility of $\sigma_{[,]}$ means that this braiding allows to construct braid invariants out of any unital Leibniz algebra. It would be interesting to explore the nature of these invariants.

Lie characters

Now let us look for braided (co)characters. As usual, we start with the "structural" characters.

Definition 4.4.5. A *Lie (or Leibniz) character* is a unital Leibniz algebra morphism $\epsilon : V \to \Bbbk$, where $\Bbbk$ is endowed with the trivial (i.e. identically zero) unital Lie algebra structure, with $1 \in \Bbbk$ as the Lie unit. In other words, it is an $\epsilon \in V^*$ satisfying

$$\begin{aligned} \epsilon([v,w]) &= 0 \qquad \forall v, w \in V, \\ \epsilon(\mathbf{1}) &= 1. \end{aligned} \tag{4.11}$$

A *non-unital Lie character* satisfies the first condition only.

Lemma 4.4.6. Take a unital Leibniz $\Bbbk$-algebra $(V, [,], \mathbf{1})$.

1. An $e \in V$ is a braided co-character if and only if

$$[e, e] = 0.$$

2. Braided characters have to satisfy one of the following conditions:
   - ✓ either $\epsilon(\mathbf{1}) = 0$,
   - ✓ or ϵ is a Lie character.

In the R-linear setting, only the "if" parts of the statements hold.

"Primitive" comultiplication

The comultiplication we choose for Leibniz algebras is what one expects:

Lemma 4.4.7. Suppose that $\Bbbk\mathbf{1}$ is a direct factor of V, i.e. one has a Leibniz sub-algebra V' of V and a Leibniz algebra decomposition

$$V \simeq V' \oplus \Bbbk\mathbf{1}. \tag{4.12}$$

Then the linear map

$$\begin{aligned} \Delta_{pr} : V &\longrightarrow V \otimes V, \\ v &\longmapsto \mathbf{1} \otimes v + v \otimes \mathbf{1} \qquad \forall v \in V', \\ \mathbf{1} &\longmapsto \mathbf{1} \otimes \mathbf{1} \end{aligned}$$

endows $(V, \sigma_{[,]})$ with a semi-braided $\sigma_{[,]}$-cocommutative coalgebra structure.

This comultiplication turns all the elements of V' into primitive ones.

Definition 4.4.8. We call a unital Leibniz algebra which admits a decomposition (4.12) *split.*

Arrow operations are adjoint

Like in the associative algebra case, the Lie unit **1** enjoys important properties with respect to arrow operations:

Lemma 4.4.9. 1. The arrow operations give adjoint actions:

$$v_1 \dots v_n \overset{\epsilon}{\nwarrow} w = \epsilon(\mathbf{1}) \sum_{i=1}^{n} v_1 \dots [v_i, w] \dots v_n + \epsilon(w) v_1 \dots v_n \qquad \forall v_i, w \in V.$$

2. In particular, the Lie unit **1** acts by scalars:

$$* \overset{\varepsilon}{\nwarrow} \mathbf{1} = \varepsilon(\mathbf{1}) \, \mathrm{Id}_{T(V)},$$

which is simply $\mathrm{Id}_{T(V)}$ if ε is a Lie character.

3. The pre-braiding $\sigma_{[,]}$ is natural with respect to the Lie unit **1**. Thus **1** is right $\sigma_{[,]}$-compatible with any $f \in V^*$.

Leibniz complex

Everything is now ready for explicit calculations of differentials. Only the left ones give something interesting:

Proposition 4.4.10. *Take a unital Leibniz algebra $(V, [,], \mathbf{1})$ over a commutative unital ring R.*

1. *The braiding $\sigma_{[,]}$ from* (4.9) *and a braided character ϵ (for instance, a Lie character) define, via theorem 2, the following differential on $T(V)$:*

$$\begin{aligned} {}^{\epsilon}d(v_1 \dots v_n) = \epsilon(\mathbf{1}) &\sum_{1 \leqslant i < j \leqslant n} (-1)^{j-1} v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v_j} \dots v_n + \\ &+ \sum_{1 \leqslant j \leqslant n} (-1)^{j-1} \epsilon(v_j) v_1 \dots \widehat{v_j} \dots v_n. \end{aligned}$$

2. *According to theorem 3, it comes from a pre-simplicial structure given by*

$$\begin{aligned} d_{n;j}(v_1 \dots v_n) = \epsilon(\mathbf{1}) &\sum_{1 \leqslant i < j} v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v_j} \dots v_n \\ &+ \epsilon(v_j) v_1 \dots \widehat{v_j} \dots v_n. \end{aligned}$$

3. *The complex $(T(V), {}^{\epsilon}d)$ is acyclic if $\epsilon(\mathbf{1}) = 1$.*

4. *If V is split, then the above structure can be completed into a weakly simplicial one by putting*

$$s_{n;i}(v_1 \dots v_n) = \begin{cases} v_1 \dots v_{i-1} \mathbf{1} v_i \dots v_n + v_1 \dots v_i \mathbf{1} v_{i+1} \dots v_n & \text{if } v_i \in V', \\ v_1 \dots v_{i-1} \mathbf{1}\mathbf{1} v_{i+1} \dots v_n & \text{if } v_i = \mathbf{1}. \end{cases}$$

Proof. Most of the assertions follow from theorems 2 and 3, combined with preceding lemmas. Point 3 is the corollary 3.2.8 applied to **1**. □

A "non-unital" remark

Remark 4.4.11. Like for associative algebras, in the non-unital case one enriches V with a formal unit: $\widetilde{V} := V \oplus R\mathbf{1}$, extending the bracket by imposing $\mathbf{1}$ to be a Lie unit. Due to the equivalence of the Leibniz condition for $[,]$ on V and on $\widetilde{V}$, lemma 4.4.1 asserts that $\sigma_{[,]}$ is a braiding $\underline{\text{on } \widetilde{V}}$ if and only if $[,]$ is Leibniz $\underline{\text{on } V}$. Taking the Lie character $\varepsilon(V) \equiv 0, \varepsilon(\mathbf{1}) = 1$ on $\widetilde{V}$ and restricting ${}^{\varepsilon}d$ to the subcomplex $T(V) \subset T(\widetilde{V})$, one recovers the familiar ***Leibniz differential***:

$${}^{\varepsilon}d(v_1 \dots v_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^{j-1} v_1 \dots v_{i-1}[v_i, v_j]v_{i+1} \dots \widehat{v_j} \dots v_n.$$

A summary

Let us summarize the last two sections before proceeding to their categorical and then dual versions. First, we recall all the ingredients for braided homology theories identified in the associative and Leibniz settings:

Theorem 5. *1. A right-unital associative (or unital Leibniz) algebra V can be endowed with a pre-braiding σ_μ (resp. a braiding $\sigma_{[,]}$) defined by the formula* (4.6) *(resp.* (4.9)*).*

2. *Comultiplication $\Delta_{\mathbf{1}}$ (resp. Δ_{pr}) completes this pre-braiding into a σ-cocommutative pre- (resp. semi-)braided coalgebra structure if V is moreover unital (resp. split).*
3. *Any (Lie) character φ is a braided character for this pre-braiding.*
4. *Our pre-braiding is demi-natural (even natural in the Leibniz case) with respect to the unit $\mathbf{1}$.*

We then apply the general constructions of theorem 3 and corollary 3.2.8 to these ingredients:

Corollary 4.4.12. *1. A pair of algebra (resp. Lie) characters ϵ, ζ on a right-unital associative (resp. unital Leibniz) algebra V allows to construct a pre-bisimplicial structure, hence a bidifferential, on $T(V)$.*

2. *If V is moreover unital (resp. split), then $T(V)$ can be endowed with a weakly simplicial structure, giving normalized quotient complexes.*
3. *Concatenation $\overline{v} \mapsto \overline{v}\mathbf{1}$ provides a contracting homotopy for both the left and the right differentials ${}^{\epsilon}d$ and d^{ζ}.*

Chapter 5

An upper world: categories

This chapter is devoted to a categorification of our braided (co)homology theory and of the pre-braidings and other "braided" ingredients for associative and Leibniz algebras. We work in the settings of a preadditive monoidal category, symmetric for Leibniz algebras. This categorification is rather straightforward. A (more subtle and technical) categorical version of shelves and racks and of the corresponding braided differentials is postponed until chapter 11.

The advantages of the categorical approach are illustrated with concrete examples, where we automatically get pre-braidings and homology theories for the corresponding algebraic structures:

- ✓ Leibniz superalgebra homology;
- ✓ homologies of dual structures: coassociative/co-Leibniz coalgebras etc.;
- ✓ right Leibniz algebra homology.

One of the essential features of our categorical constructions is the "local" nature of the pre-braidings we use, in contrast with the classical "global" notion of braided category. Concretely, our pre-braiding is defined for one object and not for all the objects of a category, and no naturality is imposed.

This local/global distinction becomes especially important in chapter 10, where the categorification presented here re-emerges in the context of virtual braid theory. In a few words, the construction of a pre-braiding on an object already living in a symmetric category (and thus automatically braided) leads to a "double-braiding" situation, which we will show to be precisely the categorical counterpart of virtual braids, with their two types of crossings (usual and virtual).

Only basic tools of category theory are used here; S.MacLane's and V.G.Turaev's famous books [49] and [79] are excellent references for the general and, respectively, "braided" aspects of category theory. We also recommend the preprint [81] by Q.Westrich, where most of the categorical notions used here are nicely presented and illustrated. In order to make this work as self-contained as possible, we recall here all the necessary definitions.

5.1 Categorifying braided differentials

Basic categorical notions

We start with recalling some classical definitions from category theory.

Definition 5.1.1. ➜ A strict *monoidal (or tensor) category* is a category $\mathcal{C}$ endowed with

✓ a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the associativity condition;

✓ an object $\mathbf{I}$ which is a left and right identity for $\otimes$.

➜ A strict monoidal category $\mathcal{C}$ is called *pre-braided* if it is endowed with a *pre-braiding (or a commutativity constraint)*, i.e. a natural family of morphisms

$$c = \{c_{V,W} : V \otimes W \to W \otimes V\} \quad \forall V, W \in \mathrm{Ob}(\mathcal{C}),$$

satisfying

$$c_{V,W\otimes U} = (\mathrm{Id}_W \otimes c_{V,U}) \circ (c_{V,W} \otimes \mathrm{Id}_U), \tag{5.1}$$

$$c_{V\otimes W,U} = (c_{V,U} \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_V \otimes c_{W,U}) \tag{5.2}$$

for any triple of objects V, W, U. "Natural" means here

$$c_{V',W'} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W} \tag{5.3}$$

for all $V, W, V', W' \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Hom}_{\mathcal{C}}(V, V'), g \in \mathrm{Hom}_{\mathcal{C}}(W, W')$. One talks about *braidings* and *braided categories* if the $c_{V,W}$'s are moreover isomorphisms.

➜ A braided category $\mathcal{C}$ is called *symmetric* if its braiding is symmetric:

$$c_{V,W} \circ c_{W,V} = \mathrm{Id}_{W\otimes V}, \quad \forall V, W \in \mathrm{Ob}(\mathcal{C}). \tag{5.4}$$

We omit the part "monoidal" of the usual terms "braided monoidal" and "symmetric monoidal" in what follows.

➜ A category $\mathcal{C}$ is called *preadditive* if all its morphism sets $\mathrm{Hom}_{\mathcal{C}}(V, W)$ are abelian groups, the composition of morphisms being $\mathbb{Z}$-bilinear. For a preadditive and monoidal category to be called *preadditive monoidal*, its tensor product should be bilinear on morphisms. The same condition is imposed on pre-braided (and in particular symmetric) preadditive categories.

➜ A preadditive category $\mathcal{C}$ is called *additive* if all finite collections of objects $V_1, \ldots, V_n$ of $\mathcal{C}$ have a biproduct $V_1 \oplus \cdots \oplus V_n$ in $\mathcal{C}$.

➜ A *(unital) associative algebra* in a strict monoidal category $\mathcal{C}$, abbreviated as (U)AA, is an object V together with morphisms $\mu : V \otimes V \to V$ (and $\nu : \mathbf{I} \to V$), satisfying the associativity (and the unit) conditions:

$$\mu \circ (\mathrm{Id}_V \otimes \mu) = \mu \circ (\mu \otimes \mathrm{Id}_V) : V^{\otimes 3} \to V,$$
$$\mu \circ (\mathrm{Id}_V \otimes \nu) = \mu \circ (\nu \otimes \mathrm{Id}_V) = \mathrm{Id}_V .$$

For a *right-unital associative algebra* we demand only the $\mu \circ (\mathrm{Id}_V \otimes \nu) = \mathrm{Id}_V$ part of the last condition.

The dual notion (in the sense of section 5.4) is that of a *(counital) coassociative coalgebra*.

- A *(unital) Leibniz algebra* in a symmetric preadditive category $\mathcal{C}$, abbreviated as (U)LA, is an object V together with morphisms $[,] : V \otimes V \to V$ (and $\nu : \mathbf{I} \to V$) satisfying the generalized Leibniz (and the Lie unit) conditions:

$$[,] \circ (\mathrm{Id}_V \otimes [,]) = [,] \circ ([,] \otimes \mathrm{Id}_V) - [,] \circ ([,] \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes c_{V,V}) : V^{\otimes 3} \to V,$$

$$[,] \circ (\mathrm{Id}_V \otimes \nu) = [,] \circ (\nu \otimes \mathrm{Id}_V) = 0 : V \to V.$$

See for instance [2] and [52] for the definition of algebras in a monoidal category, and [30] for a survey on Lie algebras in a symmetric preadditive category.

We work only with ***strict*** monoidal categories here for the sake of simplicity; according to a theorem of S.MacLane ([49]), any monoidal category is monoidally equivalent to a strict one. This justifies in particular notations $V \otimes W \otimes U$ and $V^{\otimes n}$. The word "strict" is omitted but always implied in what follows.

Note that to define a unital Leibniz algebra, one needs more structure on the underlying category than for associative algebras.

A list of categories

Here are some basic examples sufficient for what follows. See chapter 2 for the braidings used here.

Example 5.1.2. 1. The category of sets **Set** is monoidal, with the Cartesian product $\times$ as its tensor product, and a one-element set $\mathbf{I}$ as its identity object. An identification of $(A \times B) \times C$ with $A \times (B \times C)$ and of $\mathbf{I} \times A$ with $A \times \mathbf{I}$ and with A for any sets A, B, C, which is implicitly done in what follows, gives a strict monoidal category. This category is symmetric, with the braiding provided by the usual flip isomorphism.

2. The category of R-modules and R-linear maps $\mathbf{Mod}_R$ (and, in particular, that of $\Bbbk$-vector spaces $\mathbf{Vect}_\Bbbk$) can be regarded as symmetric in two ways. Firstly, it can be endowed with the usual tensor product over R, the free one-dimensional module R as an identity object, and the flip

$$\tau : v \otimes w \longmapsto w \otimes v$$

as a braiding. This structure is symmetric additive. Secondly, one can take the direct sum $\oplus$ as a tensor product, the zero module as an identity object, and the flip

$$\tau : v \oplus w \longmapsto w \oplus v$$

as a symmetric braiding. Notation $\mathbf{Mod}_R^\oplus$ will be used for the second structure. Identifications similar to those for sets are implicit in both cases to assure the strictness.

The linearization map gives a functor of symmetric categories

$$\begin{aligned} Lin : \mathbf{Set} &\to \mathbf{Mod}_R, \\ S &\mapsto RS. \end{aligned} \tag{5.5}$$

One more functor of symmetric categories, the forgetful one, is of interest:

$$\begin{aligned} For : \mathbf{Mod}_R^\oplus &\to \mathbf{Set}, \\ V &\mapsto V. \end{aligned} \tag{5.6}$$

Observe that both functors are faithful but not full in general.

Note also the full subcategory $\mathbf{vect}_\Bbbk$ of $\mathbf{Vect}_\Bbbk$ consisting of finite-dimensional vector spaces.

3. The category of graded R-modules $\mathbf{ModGrad}_R$ is symmetric additive, with the usual graded tensor product and direct sum, the one-dimensional zero-graded space R as its identity object and the Koszul flip (2.1) as its braiding. Necessary identifications are effectuated to assure the strictness.
4. One can replace the sign $(-1)^{\deg v \deg w}$ in the definition of the Koszul flip with any other antisymmetric bicharacter. Concretely, take a finite abelian group Γ endowed with an antisymmetric bicharacter χ. The category ${}_\Gamma\mathbf{Mod}_R$ of R-modules graded over Γ is symmetric, with the usual Γ-graded tensor product, the zero-graded R as its identity object and, as a braiding, the so-called *color flip*

$$\tau_{color} : v \otimes w \longmapsto \chi(f, g) w \otimes v$$

   for homogeneous v and w graded over f and $g \in \Gamma$ respectively.
5. Any monoidal category $\mathcal{C}$ has two interesting subcategories:
   (a) The subcategory $\mathbf{UAlg}(\mathcal{C})$ of UAAs and unital algebra morphisms (i.e. morphisms respecting μ and ν) in $\mathcal{C}$. It is a monoidal subcategory if $\mathcal{C}$ is pre-braided: it is stable by tensor products since

$$\mu_{V\otimes W} := (\mu_V \otimes \mu_W) \circ (\mathrm{Id}_V \otimes c_{W,V} \otimes \mathrm{Id}_W), \tag{5.7}$$
$$\nu_{V\otimes W} := \nu_V \otimes \nu_W \tag{5.8}$$

   give a UAA structure on $V \otimes W$ for UAAs V and W, and it includes $\mathbf{I}$ with identities as algebra structures, which will be the default UAA structure on $\mathbf{I}$ in what follows. $\mathbf{UAlg}(\mathcal{C})$ is moreover a symmetric subcategory if $\mathcal{C}$ is symmetric, since the symmetry of the braiding guarantees that it respects algebra structures. We write $\mathbf{Alg}$ when dealing with non-unital algebras.

   For a preadditive $\mathcal{C}$, $\mathbf{UAlg}(\mathcal{C})$ is not a preadditive subcategory in general, since $f + g$ is not necessarily an algebra morphism even if f and g are.
   (b) If $\mathcal{C}$ is moreover symmetric preadditive, one also has the subcategory $\mathbf{ULei}(\mathcal{C})$ of ULAs and unital Leibniz algebra morphisms (i.e. morphisms respecting $[,]$ and ν) in $\mathcal{C}$. It is neither preadditive nor even monoidal in general. It includes $\mathbf{I}$ with the zero bracket and $\nu = \mathrm{Id}_{\mathbf{I}}$, which will be the default ULA structure on $\mathbf{I}$ in what follows. We write $\mathbf{Lei}$ when dealing with non-unital Leibniz algebras.

   In particular, $\mathbf{UAlg}(\mathbf{Mod}_R)$ and $\mathbf{ULei}(\mathbf{Mod}_R)$ are the familiar categories of R-linear unital associative and Leibniz algebras respectively.

Braided objects, families and algebras

Now we introduce several new categorical notions, necessary for categorifying our constructions. Notations φ_i from (1.3) are widely used here.

Start with a "local" notion of braiding:

Definition 5.1.3. ⤳ An *object* V in a monoidal category $\mathcal{C}$ is called *pre-braided* if it is endowed with a "local" *pre-braiding*, i.e. a morphism

$$\sigma = \sigma_V : V \otimes V \to V \otimes V,$$

satisfying a categorical version of (YB):

$$(\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) = (Id_V \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \sigma_V).$$

- A *family* $\mathcal{F}$ of objects in a monoidal category $\mathcal{C}$ is called *pre-braided* if it is endowed with a "local" *pre-braiding*, i.e. a morphism

$$\sigma_{V,W} : V \otimes W \to W \otimes V$$

for each $V, W \in \mathcal{F}$ satisfying a categorical version of (YB) on $V \otimes W \otimes U$ for each triple $V, W, U \in \mathcal{F}$.
- A pre-braided family $\mathcal{F}$ is said to be *natural with respect to a morphism* $\varphi : V \to W$, with $V, W \in \mathcal{F}$, if, for any $U \in \mathcal{F}$, one has

$$\sigma_{W,U} \circ (\varphi \otimes \mathrm{Id}_U) = (\mathrm{Id}_U \otimes \varphi) \circ \sigma_{V,U}, \tag{5.9}$$

$$\sigma_{U,W} \circ (\mathrm{Id}_U \otimes \varphi) = (\varphi \otimes \mathrm{Id}_U) \circ \sigma_{U,V}. \tag{5.10}$$

One talks about *semi-naturality* if only (5.9) holds, and *demi-naturality* if only (5.10) holds.
- A *pre-braided (unital) algebra* V in a monoidal category $\mathcal{C}$ is a pre-braided object (V, σ) endowed with a (U)AA structure μ (resp. (μ, ν)) compatible with the pre-braiding:

$$\sigma \circ \mu_1 = \mu_2 \circ (\sigma_1 \circ \sigma_2) : V^{\otimes 3} \to V^{\otimes 2}, \tag{5.11}$$

$$\sigma \circ \mu_2 = \mu_1 \circ (\sigma_2 \circ \sigma_1) : V^{\otimes 3} \to V^{\otimes 2}; \tag{5.12}$$

$$\sigma \circ \nu_1 = \nu_2 : V = \mathbf{I} \otimes V = V \otimes \mathbf{I} \to V^{\otimes 2}, \tag{5.13}$$

$$\sigma \circ \nu_2 = \nu_1 : V = \mathbf{I} \otimes V = V \otimes \mathbf{I} \to V^{\otimes 2}. \tag{5.14}$$

One talks about *semi-pre-braided (unital) algebras* if only (5.11) (and (5.13)) hold. The dual notions are those of a *(semi-)(pre-)braided (counital) coalgebra* (cf. (3.12), (3.13)).
- One talks about *braided* objects/families/algebras etc. if all the pre-braidings involved are invertible.
- In a monoidal category, a morphism $\epsilon : V \to \mathbf{I}$ is called a *braided character* for a pre-braided object V if

$$\epsilon \otimes \epsilon = (\epsilon \otimes \epsilon) \circ \sigma_V : V \otimes V \to \mathbf{I} \otimes \mathbf{I} = \mathbf{I}.$$

In other words, it is a homomorphism between pre-braided objects (V, σ_V) and $(\mathbf{I}, \mathrm{Id}_{\mathbf{I}})$.

For a graphical interpretation of pre-braided unital algebras, see (the horizontally symmetric version of) figure 3.3, and figure 3.6.

Example 5.1.4. Every object in a pre-braided category $\mathcal{C}$ is pre-braided, with $\sigma_V = c_{V,V}$, since the YBE is automatic in $\mathcal{C}$ (take $V' = V$, $W' = W = V \otimes V$, $f = \mathrm{Id}_V$ and $g = c_{V,V}$ in the condition (5.3) expressing naturality).

However, the most interesting situation is that of a pre-braiding proper to an object.

The idea of working with ***"local"*** pre-braidings on V instead of demanding the whole category $\mathcal{C}$ to be ***"globally"*** braided is similar to what is done in [30], where self-invertible YB operators are considered in order to define YB-Lie algebras in an additive monoidal category $\mathcal{C}$. Note that, contrary to their operator, our pre-braiding is **not necessarily invertible**.

Continuing the local/global considerations, we remark that a pre-braiding σ on $V \in \mathrm{Ob}(\mathcal{C})$ "globalizes" to a pre-braiding on a certain subcategory of $\mathcal{C}$:

Lemma 5.1.5. 1. Take a (pre-)braided object (V, σ) in a monoidal category $\mathcal{C}$. The family of its tensor powers $\mathcal{F} = \{V^{\otimes n}\}_{n \geqslant 0}$, where $V^{\otimes 0} := \mathbf{I}$, can be endowed with the following (pre-)braiding:

$$\boldsymbol{\sigma}_{V^{\otimes n}, V^{\otimes k}} := (\sigma_k \cdots \sigma_1) \cdots (\sigma_{n+k-2} \cdots \sigma_{n-1})(\sigma_{n+k-1} \cdots \sigma_n). \tag{5.15}$$

2. This (pre-)braiding on $\mathcal{F}$ endows the monoidal subcategory of $\mathcal{C}$ generated by the object V and the morphism σ with a (pre-)braided structure.
3. Further, given a (pre-)braided family $\mathcal{F}$ in $\mathcal{C}$, one can add to $\mathcal{F}$ the tensor products of all the n-tuples of its elements, for all the $n \in \mathbb{N}$, extending the (pre-)braiding thanks to formulas analogous to (5.15). This extended family is denoted by $\mathcal{F}^{\otimes}$.

See remark 2.0.5 for a graphical version of the extended pre-braiding $\boldsymbol{\sigma}$.

Definition 5.1.6. A pre-braiding for an object V (or a family $\mathcal{F}$) in a monoidal category $\mathcal{C}$ is said to be *natural with respect to a morphism* $\varphi : V^{\otimes l} \to V^{\otimes r}$ (resp. $\varphi : V_1 \otimes \ldots \otimes V_l \to W_1 \otimes \ldots \otimes W_r$, with $V_i, W_j \in \mathcal{F}$) if the pre-braided family $\{V\}^{\otimes} = \{V^{\otimes n}\}_{n \geqslant 0}$ (resp. $\mathcal{F}^{\otimes}$) from the preceding lemma is natural with respect to φ, and similarly with semi- and demi-naturality.

A pre-braided (unital) algebra can now be seen as a (unital) algebra structure and a pre-braiding natural with respect to it, and similarly for coalgebras and for semi- or demi-pre-braided (co)algebras.

Normalizations

Definition 5.1.7. ⇝ In a monoidal category, a pair $(\eta : \mathbf{I} \to V, \epsilon : V \to \mathbf{I})$ is said to be *normalized* if

$$\epsilon \circ \eta = \mathrm{Id}_{\mathbf{I}} .$$

⇝ An *algebra character* for an object V of $\mathbf{UAlg}(\mathcal{C})$ is a unital algebra morphism $\epsilon \in \mathrm{Hom}_{\mathbf{UAlg}(\mathcal{C})}(V, \mathbf{I})$. *A Lie character* for an object V of $\mathbf{ULei}(\mathcal{C})$ is a unital Leibniz algebra morphism $\epsilon \in \mathrm{Hom}_{\mathbf{ULei}(\mathcal{C})}(V, \mathbf{I})$, i.e. it satisfies $\epsilon \circ [,] = 0$ and $\epsilon \circ \nu = \mathrm{Id}_{\mathbf{I}}$. The characters are called *non-unital* if they are maps in **Alg** or **Lei** only.

⇝ *A normalized morphism* $\varphi : V \to W$ for $V, W \in \mathbf{UAlg}(\mathcal{C})$ or $\in \mathbf{ULei}(\mathcal{C})$ is a morphism in $\mathcal{C}$ respecting the units, i.e.

$$\varphi \circ \nu_V = \nu_W . \tag{5.16}$$

For $W = \mathbf{I}$ this means that (ν_V, φ) is a normalized pair:

$$\varphi \circ \nu_V = \mathrm{Id}_{\mathbf{I}} . \tag{5.17}$$

Tensor (bi)differentials

Definition 5.1.8. ⇝ *A degree* -1 *differential* for a family of objects $\{V_n\}_{n \geqslant 0}$ of a preadditive category $\mathcal{C}$ is a family of morphisms $\{d_n : V_n \to V_{n-1}\}_{n > 0}$, satisfying

$$d_{n-1} \circ d_n = 0 \qquad \forall n > 1.$$

- *A bidegree -1 bidifferential* is a pair of families of morphisms $\{d_n, d'_n : V_n \to V_{n-1}\}_{n>0}$, satisfying

$$d_{n-1} \circ d_n = d'_{n-1} \circ d'_n = d'_{n-1} \circ d_n + d_{n-1} \circ d'_n = 0 \qquad \forall n > 1. \tag{5.18}$$

- *A degree -1 tensor (bi)differential* for an object V of a preadditive monoidal category $\mathcal{C}$ is defined as a degree -1 (bi)differential for the family of objects $\{V^{\otimes n}\}_{n \geqslant 0}$.
- Given a degree -1 differential $\{d_n\}_{n>0}$ for a family of objects $\{V_n\}_{n \geqslant 0}$ of a preadditive category $\mathcal{C}$, one defines a *contracting homotopy* as a family of morphisms $\{h_n : V_n \to V_{n+1}\}_{n \geqslant 0}$, satisfying

$$h_{n-1} \circ d_n + d_{n+1} \circ h_n = \mathrm{Id}_{V_n} \qquad \forall n > 0.$$

- Different types of *simplicial objects* in a category $\mathcal{C}$ are defined by replacing the words "vector space" by "object in $\mathcal{C}$" in the definition 3.2.1.

Note that points 1 - 3 of proposition 3.2.2 remain valid for simplicial objects in a preadditive category $\mathcal{C}$.

The presence of a contracting homotopy means, in the category $\mathcal{C} = \mathbf{Mod}_R$, that the complex (V_n, d_n) is acyclic.

Observe that any monoidal (and braided and/or preadditive when necessary) functor preserves all the structures from the previous four definitions.

Braided bidifferentials: a categorified version

One more tool is missing for a categorification of theorems 2 and 3. It is a ***categorical quantum (co)shuffle (co)multiplication***, which we define here.

Any pre-braided object (V, σ) in a monoidal category comes with an action of the monoid B_n^+ on $V^{\otimes n}$ for each $n \geqslant 1$, defined by formula (2.3). If the category is moreover preadditive, one can mimic the construction of the quantum (co)shuffle (co)multiplication to get morphisms

$$\underset{\sigma}{\sqcup\!\sqcup}^{p,q} : V^{\otimes n} = V^{\otimes p} \otimes V^{\otimes q} \to V^{\otimes n}$$

and

$$\underset{\sigma}{\overline{\sqcup\!\sqcup}}^{p,q} : V^{\otimes n} \to V^{\otimes p} \otimes V^{\otimes q} = V^{\otimes n}.$$

Here $n = p + q$. Still in the preadditive context, $-\sigma$ is well defined and gives a new pre-braiding for V.

Theorems 2 and 3 and proposition 3.2.7 (with their proofs!) are now generalized as follows (we freely use the notations from those theorems here):

Theorem 6. *In a preadditive monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$, take a pre-braided object (V, σ) endowed with braided characters ϵ and ζ.*

1. *The families of morphisms*

$$({}^{\epsilon}d)_n := \epsilon_1 \circ \underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{1,n-1},$$
$$(d^{\zeta})_n := (-1)^{n-1} \zeta_n \circ \underset{-\sigma}{\overline{\sqcup\!\sqcup}}^{n-1,1}$$

define a bidegree -1 tensor bidifferential for V.

2. *This bidifferential can be derived from a pre-bisimplicial structure on* $(V^{\otimes n})_{n \geqslant 0}$ *given by*

$$d_{n;i} := \epsilon_1 \circ T^{\sigma}_{p_{i,n}},$$
$$d'_{n;i} := \zeta_n \circ T^{\sigma}_{p'_{i,n}}.$$

3. *If a comultiplication* Δ *endows* (V, σ) *with a pre-braided coalgebra structure, then the maps*

$$s_{n;i} := \Delta_i$$

complete the preceding structure into a very weakly bisimplicial one.

4. *If a comultiplication* Δ *endows* (V, σ) *with a semi-pre-braided coalgebra structure, then the data* $(V^{\otimes n}, d_{n;i}, s_{n;i})$ *described above give a very weakly simplicial object only.*

5. *If* Δ *is moreover* σ***-cocommutative****, i.e.*

$$\sigma \circ \Delta = \Delta : V \to V \otimes V,$$

then the above structures on $(V^{\otimes n})_{n \geqslant 0}$ *are weakly (bi)simplicial.*

6. *Take a morphism* $\eta : \mathbf{I} \to V$*. The family*

$$h_n = (-1)^n \operatorname{Id}_{V^{\otimes n}} \otimes \eta : V^{\otimes n} \longrightarrow V^{\otimes (n+1)}$$

- *is a contracting homotopy for* $(V^{\otimes n}, ({}^{\epsilon}d)_n)$ *if the pair* (η, ϵ) *is normalized, and* σ *is demi-natural with respect to* η*.*
- *is a contracting homotopy for* $(V^{\otimes n}, (d^{\zeta})_n)$ *if the pair* (η, ζ) *is normalized and* ***right*** σ***-compatible****, i.e.*

$$(\operatorname{Id}_V \otimes \zeta) \circ \sigma \circ (\operatorname{Id}_V \otimes \eta) = \eta \circ \zeta : V \to V. \tag{5.19}$$

The versions of the theorem for braided co-characters and for "right" differentials are obtained in subsequent sections via different types of categorical dualities.

Note that the demi-naturality of σ with respect to η implies (5.19) for any ζ.

5.2 Basic examples revisited

Shelves

According to lemma 4.2.1, every shelf $S \in \mathrm{Ob}(\mathbf{Set})$ is endowed with a pre-braiding $\sigma_{\lhd} : (a, b) \mapsto (b, a \lhd b)$. Since the one-element set $\mathbf{I}$ is a final object in $\mathbf{Set}$, the unique morphism $S \to \mathbf{I}$ is necessarily a braided character. The diagonal map $D : a \mapsto (a, a)$ gives a semi-pre-braided coalgebra structure on S, $\sigma_{\lhd}$-cocommutative if S is a spindle. Given a commutative unital ring R, the monoidal functor Lin_R provides then the linearization RS of our shelf in the additive category $\mathbf{Mod}_R$. The induced semi-pre-braided coalgebra structure and braided character on RS give, according to theorem 6, a pre-bisimplicial and a weakly simplicial (in the spindle case) structures, and thus a bidegree -1 bidifferential.

Associative and Leibniz algebras

Next we consider more complicated "structural" braidings. We categorify theorem 5 and study several related questions.

Theorem $5^{\mathbf{cat}}$.

1. *Take a right-unital associative algebra (V,μ,ν) in a monoidal category $(\mathcal{C},\otimes,\mathbf{I})$.*
   (a) *V can be endowed with a pre-braiding*
   $$\sigma_{Ass} := \nu\otimes\mu : V\otimes V = \mathbf{I}\otimes V\otimes V \to V\otimes V. \tag{5.20}$$
   (b) *Comultiplication*
   $$\Delta_{Ass} := \nu\otimes \mathrm{Id}_V : V = \mathbf{I}\otimes V \to V\otimes V$$
   completes this pre-braiding into a σ_{Ass}-cocommutative pre-braided coalgebra structure if ν is moreover a two-sided unit.
   (c) *Any algebra character $\epsilon \in \mathrm{Hom}_{\mathbf{UAlg}(\mathcal{C})}(V,\mathbf{I})$ is a braided character for (V,σ_{Ass}).*
   (d) *The pre-braiding σ_{Ass} is demi-natural with respect to the unit ν. Moreover, for any algebra character ϵ, the pair (ν,ϵ) is normalized.*
2. *Take a unital Leibniz algebra $(V,[,],\nu)$ in a symmetric preadditive category $(\mathcal{C},\otimes,\mathbf{I},c)$.*
   (a) *V can be endowed with an invertible braiding*
   $$\sigma_{Lei} := c_{V,V} + \nu\otimes[,]. \tag{5.21}$$
   (b) *Comultiplication*
   $$\Delta_{Lei}|_{V'} := \nu\otimes \mathrm{Id}_{V'} + \mathrm{Id}_{V'}\otimes\nu : V' \to V\otimes V,$$
   $$\Delta_{Lei}|_{\mathbf{I}} := \nu\otimes\nu$$
   completes this braiding into a σ_{Lei}-cocommutative semi-braided coalgebra structure if $\mathcal{C}$ is additive and one has a Leibniz algebra decomposition $V \simeq V'\oplus\mathbf{I}$.
   (c) *Any Lie character $\epsilon \in \mathrm{Hom}_{\mathbf{ULei}(\mathcal{C})}(V,\mathbf{I})$ is a braided character for (V,σ_{Lei}).*
   (d) *The braiding σ_{Lei} is natural with respect to the unit ν. Moreover, for any Lie character ϵ, the pair (ν,ϵ) is normalized.*

Observe that in the Leibniz algebra setting, the naturality (with respect to morphisms ν and $[,]$ in particular) and the symmetry of the braiding c are essential in proving that σ_{Lei} is indeed a braiding, while the naturality of c with respect to ϵ shows that ϵ is a braided character for $(V,c_{V,V})$ (which implies that it is a braided character for (V,σ_V) if it preserves the Leibniz structure).

Remark 5.2.1. According to the theorem, a ULA V provides an example of a ***"doubly braided"*** object: σ_V and $c_{V,V}$ are indeed two distinct braidings for V. One can say more: the two braidings endow tensor powers of V with an action of the ***virtual braid group*** (cf. the foundational paper of virtual knot theory [41]; see also [80], where the virtual braid group was introduced and studied). The close connections between (pre-)braided objects and virtual braid groups are studied in detail in part III.

Theorems 6 and $5^{\mathbf{cat}}$ put together give categorical versions of propositions 4.3.7 and 4.4.10, as well as of the "non-unital" remarks 4.3.8 and 4.4.11:

Corollary 5.2.2. *1. Any algebra character $\epsilon : V \to \mathbf{I}$ for a UAA (V, μ, ν) in a preadditive monoidal category $\mathcal{C}$ produces a degree -1 tensor differential for V, given by*

$$({}^{\epsilon}d)_n := \epsilon_1 + \sum_{i=1}^{n-1} (-1)^i \mu_i,$$

with a contracting homotopy

$$h_n = (-1)^n \nu_{n+1}.$$

2. Any non-unital algebra characters $\epsilon, \zeta : V \to \mathbf{I}$ for an associative algebra (V, μ) in an additive monoidal $\mathcal{C}$ produce a degree -1 tensor differential for V, given by

$$({}^{\epsilon}d^{\zeta})_n := \epsilon_1 + \sum_{i=1}^{n-1} (-1)^i \mu_i + (-1)^n \zeta_n.$$

3. Any Lie character $\epsilon : V \to \mathbf{I}$ for a ULA $(V, [,], \nu)$ in a symmetric preadditive category $\mathcal{C}$ produces a degree -1 tensor differential for V, given by

$$({}^{\epsilon}d)_n := \epsilon_1 \circ \underset{-c}{\sqcup\!\sqcup}^{1,n-1} + \sum_{1 \leqslant i < j \leqslant n} (-1)^{j-1} [,]_i \circ (\mathrm{Id}_V^{\otimes i} \otimes c_{V^{\otimes (j-i-1)}, V} \otimes \mathrm{Id}_V^{\otimes (n-j)}),$$

with a contracting homotopy

$$h_n = (-1)^n \nu_{n+1}.$$

4. Any non-unital Lie character $\epsilon : V \to \mathbf{I}$ for a Leibniz algebra $(V, [,])$ in a symmetric additive category $\mathcal{C}$ produces a degree -1 tensor differential for V, given by

$$({}^{\epsilon}d)_n := \sum_{1 \leqslant i < j \leqslant n} (-1)^{j-1} [,]_i \circ (\mathrm{Id}_V^{\otimes i} \otimes c_{V^{\otimes (j-i-1)}, V} \otimes \mathrm{Id}_V^{\otimes (n-j)}).$$

An "if and only if" result

Working in $\mathbf{Vect}_{\Bbbk}$ in chapter 4, we noticed that the pre-braidings obtained for associative and Leibniz algebras encode the underlying algebraic structures (lemmas 4.3.1 and 4.4.1). It still holds, with some additional technical assumptions, in the categorical setting:

Lemma 5.2.3. 1. Take an object V in a monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$ endowed with two morphisms $\mu : V \otimes V \to V$ and $\nu : \mathbf{I} \to V$, with ν being a two-sided unit for μ. The morphism σ_{Ass} defined by (5.20) is a pre-braiding **if and only if** μ is associative.

2. Take an object V in a symmetric preadditive category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ endowed with two morphisms $[,] : V \otimes V \to V$ and $\nu : \mathbf{I} \to V$, with ν being a Lie unit for $[,]$. Additionally suppose the existence of a normalized morphism $\gamma : V \to \mathbf{I}$ (in the sense of (5.17)). The morphism σ_{Lei} defined by (5.21) is a braiding **if and only if** $[,]$ satisfies the Leibniz condition.

Proof. One repeats the proofs of lemmas 4.3.1 and 4.4.1. The only non-trivial step is to show that

$$\nu \otimes \nu \otimes f = \nu \otimes \nu \otimes g : V^{\otimes 3} \to V^{\otimes 3}$$

implies

$$f = g : V^{\otimes 3} \to V.$$

When ν is a left unit for μ, this is done by applying $\mu \circ (\mathrm{Id}_V \otimes \mu)$ to both sides of the first identity. In the Leibniz case, apply $\gamma \otimes \gamma \otimes \mathrm{Id}_V$. □

Naturality

The pre-braidings constructed above enjoy a naturality property, providing moreover a characterization of associative/Leibniz algebra morphisms:

Proposition 5.2.4. *1. In the settings of theorem $5^{\mathbf{cat}}$, one has*

$$(f \otimes f) \circ \sigma_V = \sigma_W \circ (f \otimes f) : V \otimes V \to W \otimes W \tag{5.22}$$

for any morphism $f : V \to W$ in $\mathbf{UAlg}(\mathcal{C})$ (resp. $\mathbf{ULei}(\mathcal{C})$), where σ_V and σ_W are the pre-braidings σ_{Ass} (resp. σ_{Lei}) for V and W.

2. Suppose additionally, for the algebra case, that ν is a two-sided unit, and, for the Leibniz case, the existence of a normalized morphism $\gamma : V \to \mathbf{I}$.
Then any normalized morphism $f : V \to W$ (cf. (5.16)) in $\mathcal{C}$, compatible with the σ's in the sense of (5.22), necessarily respects the multiplications. In other words, such an f is a morphism in $\mathbf{UAlg}(\mathcal{C})$ (resp. $\mathbf{ULei}(\mathcal{C})$).

Proof. The first point is easy. For the second one, since f is normalized, (5.22) means

$$\nu_W \otimes (f \circ \mu_V) = \nu_W \otimes (\mu_W \circ (f \otimes f)),$$

and similarly – with μ replaced by $[,]$ – in the Leibniz case, since the braiding c is natural. Now, like in the proof of lemma 5.2.3, apply μ_W (resp. $\gamma \otimes \mathrm{Id}_W$) to both sides. □

Note that, contrary to the naturality of the pre-braiding in a pre-braided category, one can not take two distinct morphisms $f, g : V \to W$ here.

5.3 The super trick

The first bonus one generally gains when passing to abstract symmetric categories is the possibility to derive graded and super versions of algebraic results for free, thanks to the Koszul flip τ_{Koszul} from (2.1). One clearly sees where to put signs, which is otherwise quite difficult to guess. Here is a typical example.

Take a ***graded unital Leibniz algebra*** $(V, [,], \nu)$, i.e. an object of $\mathbf{ULei}(\mathbf{ModGrad}_R)$. Recall that the category $\mathbf{ModGrad}_R$ comes with the symmetric braiding τ_{Koszul}. Leibniz condition in this setting is

$$[v, [w, u]] = [[v, w], u] - (-1)^{\deg u \deg w}[[v, u], w]$$

for any homogeneous elements $v, w, u \in V$. On the figure 4.7 illustrating (Lei), the crossing on the right corresponds to the "internal" braiding $c_{V,V} = \tau_{Koszul}$.

Theorem $5^{\mathbf{cat}}$ gives a braiding for V:

$$\sigma_V : v \otimes w \longmapsto (-1)^{\deg v \deg w} w \otimes v + \mathbf{1} \otimes [v, w],$$

which, together with a Lie character $\epsilon : V_0 \to R$ (it has to respect degrees, and thus to be zero on other components of V), can be fed into the machinery from theorem 6 to give

Proposition 5.3.1. *1. An R-linear graded unital Leibniz algebra $(V, [,], \nu)$ with a Lie character ϵ can be endowed with the degree -1 tensor differential*

$$\begin{aligned}{}^{\epsilon}d(v_1 \dots v_n) = &\sum_{1 \leqslant i < j \leqslant n} (-1)^{j-1+\alpha_{i,j}} v_1 \dots v_{i-1}[v_i, v_j]v_{i+1} \dots \widehat{v_j} \dots v_n + \\ &+ \sum_{1 \leqslant j \leqslant n} (-1)^{j-1+\alpha_{0,j}} \epsilon(v_j) v_1 \dots \widehat{v_j} \dots v_n,\end{aligned}$$

where

$$\alpha_{i,j} := \deg(v_j) \sum_{i<k<j} \deg(v_k).$$

2. *An R-linear graded Leibniz algebra* $(V, [,])$ *with a non-unital Lie character* ϵ *can be endowed with the degree* -1 *tensor differential*

$$^{\epsilon}d(v_1 \dots v_n) = \sum_{1 \leqslant i<j \leqslant n} (-1)^{j-1+\alpha_{i,j}} v_1 \dots v_{i-1}[v_i, v_j]v_{i+1} \dots \widehat{v_j} \dots v_n.$$

All the v_i*'s are taken homogeneous here.*

Observe that the $(-1)^{\alpha_{i,j}}$ part of the sign comes from the Koszul braiding, while $(-1)^{j-1}$ appears because we take the opposite braiding when defining $(^{\epsilon}d)_n := \epsilon_1 \circ \underset{-\sigma}{\sqcup\!\sqcup}^{1,n-1}$.

Leibniz superalgebras are treated similarly: one has just to work in the category of super modules over R. The reader is sent to [45] and other papers on the subject for details. One thus recovers the ***Leibniz superalgebra homology***, which is a lift of the Lie superalgebra homology.

Similarly, one gets for free the ***color Leibniz algebra homology*** (cf. [20], or [70] for a Lie version), since color Leibniz algebras are particular cases of Leibniz algebras in the symmetric additive category ${}_{\Gamma}\mathbf{Mod}_R$ (cf. example 5.1.2).

See also [81] for an excellent survey of different types of braided Lie algebras.

5.4 Co-world, or the world upside down

One more nice feature of the categorical approach is an automatic treatment of ***dualities***. The most common notion of duality, the "upside-down" one, is described here, with the cobar complex for coalgebras (first defined by Cartier in [12]; cf. also [19] and [77]) providing an example. In the monoidal context, one has two more dualities, the "right-left" and the combined ones, treated in the next section.

Generalities on co-categories

Definition 5.4.1. Given a category $\mathcal{C}$, its *dual (or opposite) category* $\mathcal{C}^{\mathrm{op}}$ is constructed by keeping the objects of $\mathcal{C}$ and reversing all the arrows. In other words, the domain and codomain of any morphism change places. One writes $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W, V)$ for the morphism in $\mathcal{C}^{\mathrm{op}}$ corresponding to an $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$.

We sometimes call $\mathcal{C}^{\mathrm{op}}$ a *co-category* in order to avoid confusion with other notions of duality. Observe that this construction is involutive: $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Example 5.4.2. A well-known example comes from the full subcategory $\mathbf{vect}_{\Bbbk}$ of $\mathbf{Vect}_{\Bbbk}$ consisting of finite dimensional vector spaces. The usual duality functor sending V to $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and f to f^* gives an equivalence of symmetric preadditive categories $\mathbf{vect}_{\Bbbk}$ and $(\mathbf{vect}_{\Bbbk})^{\mathrm{op}}$.

The ***duality principle*** (cf. [49], section II.2) tells that a "categorical" <u>theorem</u> for $\mathcal{C}$ implies a dual theorem for $\mathcal{C}^{\mathrm{op}}$ by reversing all arrows and the order of arrows in every composition. Our aim here is to apply this principle to theorems 6 and $5^{\mathbf{cat}}$.

Dualities for structures

To get a notion of duality for categorical structures, it suffices to place them to the co-category. For example,

Definition 5.4.3. A *counital coassociative coalgebra* (abbreviated as co-UAA) in a strict monoidal category $\mathcal{C}$ is an object V together with morphisms $\Delta : V \to V \otimes V$ and $\varepsilon : V \to \mathbf{I}$, such that $(V, \Delta^{\mathrm{op}}, \varepsilon^{\mathrm{op}})$ is a UAA in $\mathcal{C}^{\mathrm{op}}$.

The associativity condition is then "reversed" to the *coassociativity* condition (3.11) (cf. figure 3.2), and the unit condition to the counit condition.

Counital co-Leibniz coalgebras (abbreviated as co-ULA) are defined similarly; cf. [59] where Lie coalgebras are introduced. The subcategory of co-UAAs and co-ULAs in $\mathcal{C}$ are denoted by $\mathbf{coUAlg}(\mathcal{C})$ and $\mathbf{coULei}(\mathcal{C})$ respectively. *Coalgebra and co-Lie co-characters, braided co-characters, degree* 1 *tensor differentials* d^n and *bidegree* 1 *tensor bidifferentials* (d^n, d'^n) are also defined via dualities. A braided co-character $e : \mathbf{I} \to V$ is described for example by the familiar condition

$$e \otimes e = \sigma_V \circ (e \otimes e) : \mathbf{I} = \mathbf{I} \otimes \mathbf{I} \to V \otimes V.$$

A convenient way to handle the "upside-down" duality is the graphical one: changing from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$ consists simply in turning all the diagrams upside down, i.e. taking a ***horizontal mirror image***. By "diagrams" we mean those scattered throughout this work. Here is the example for the *co-Leibniz* condition

$$(\mathrm{Id}_V \otimes \partial) \circ \partial = (\partial \otimes \mathrm{Id}_V) \circ \partial - (\mathrm{Id} \otimes c_{V,V}) \circ (\partial \otimes \mathrm{Id}) \circ \partial :$$

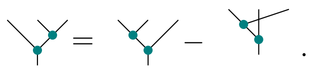


Figure 5.1: Co-Leibniz condition

Now let us make a list of dualities for categorical structures.

monoidal structure on $\mathcal{C}$	monoidal structure on $\mathcal{C}$
(pre-)braiding on $\mathcal{C}$	(pre-)braiding on $\mathcal{C}$
symmetric braiding on $\mathcal{C}$	symmetric braiding on $\mathcal{C}$
(pre)additive structure on $\mathcal{C}$	(pre)additive structure on $\mathcal{C}$
unital associative algebra (V, μ, ν)	co-UAA $(V, \mu^{\mathrm{op}}, \nu^{\mathrm{op}})$
unital Leibniz algebra $(V, [,], \nu)$	co-ULA $(V, [,]^{\mathrm{op}}, \nu^{\mathrm{op}})$
algebra character φ for (V, μ, ν)	coalgebra co-char. φ^{op} for $(V, \mu^{\mathrm{op}}, \nu^{\mathrm{op}})$
Lie character φ for $(V, [,], \nu)$	co-Lie co-char. φ^{op} for $(V, [,]^{\mathrm{op}}, \nu^{\mathrm{op}})$
pre-braiding σ for V	pre-braiding σ^{op} for V
braided character ϵ for (V, σ)	braided co-character ϵ^{op} for $(V, \sigma^{\mathrm{op}})$
(bi)degree -1 tensor (bi)bidifferential for V	(bi)degree 1 tensor (bi)differential for V

Table 5.1: Categorical duality

Note also that for a pre-braided object (V, σ) and the action of B_n^+ on $V^{\otimes n}$ coming from σ, one has

$$(T_s^{\sigma})^{\mathrm{op}} = T_{s^{-1}}^{(\sigma^{\mathrm{op}})} \in End_{\mathcal{C}^{\mathrm{op}}}(V^{\otimes n}) \;\; \forall s \in S_n,$$

since a decomposition of s^{-1} into simple transpositions can be obtained from one for s by reversing the order in the decomposition. Thus, assuming the category preadditive, the definition (2.7) of ***quantum co-shuffle comultiplication*** is translated as

$$(\underset{\sigma}{\sqcup\!\sqcup}^{p,q})^{\mathrm{op}} = \underset{\sigma^{\mathrm{op}}}{\overline{\sqcup\!\sqcup}}^{p,q}.$$

In particular, all the properties of the quantum co-shuffle comultiplication follow from this duality.

Categorical braided degree 1 differentials

Everything is now ready for dualizing theorems 6 and $5^{\mathbf{cat}}$. We present only short versions of these results here, leaving the dualization of the points concerning simplicial and pre-braided coalgebra structures to the reader.

Theorem $6^{\mathbf{co}}$. *Let $(\mathcal{C}, \otimes, \mathbf{I})$ be a preadditive monoidal category. For any pre-braided object (V, σ) with braided co-characters e and c, the morphisms*

$$({}_e d)^n := \underset{-\sigma}{\sqcup\!\sqcup}^{1,n} \circ (e \otimes \mathrm{Id}_{V^{\otimes n}}),$$
$$(d_c)^n := (-1)^n \underset{-\sigma}{\sqcup\!\sqcup}^{n,1} \circ (\mathrm{Id}_{V^{\otimes n}} \otimes c)$$

define a bidegree 1 tensor bidifferential for V.

Pre-braidings for coassociative and co-Leibniz algebras

Theorem $5^{\mathbf{co}}$.

1. *Take a counital coassociative coalgebra (V, Δ, ε) in a monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$.*
   (a) *V can be endowed with a pre-braiding*
   $$\sigma_{coAss} := \varepsilon \otimes \Delta : V \otimes V \to \mathbf{I} \otimes V \otimes V = V \otimes V.$$
   (b) *Any coalgebra co-character $e \in \mathrm{Hom}_{\mathbf{coUAlg}(\mathcal{C})}(\mathbf{I}, V)$ is a braided co-character for (V, σ_{coAss}).*
2. *Take a counital co-Leibniz coalgebra $(V, \partial, \varepsilon)$ in a symmetric preadditive category $(\mathcal{C}, \otimes, \mathbf{I}, c)$.*
   (a) *V can be endowed with a braiding*
   $$\sigma_{coLei} := c_{V,V} + \varepsilon \otimes \partial.$$
   (b) *Any Lie co-character $e \in \mathrm{Hom}_{\mathbf{coULei}(\mathcal{C})}(\mathbf{I}, V)$ is a braided co-character for (V, σ_{coLei}).*

A graphical depiction of, for instance, σ_{coAss} is by construction the horizontal mirror image of the diagram one had for UAAs:

Figure 5.2: $\sigma_{coAss} = HorMirror(\sigma_{Ass})$

A co-version of corollary 5.2.2 is then formulated in the evident way, with dual explicit formulas. Lemma 5.2.3 and proposition 5.2.4 are also dualized directly. In particular, the pre-braidings from the previous theorem encode the co-associativity (resp. co-Leibniz) condition.

Cobar differential

We finish this section with some remarks proper to our favorite category $\mathbf{Mod}_R$.

Lemma 5.4.4. In $\mathbf{Mod}_R$, a map $e : R \to V, \alpha \mapsto \alpha\mathbf{e}$ for a co-UAA (V, Δ, ϵ) is a non-unital coalgebra co-character if and only if $\mathbf{e} \in V$ is ***group-like***, i.e. $\Delta(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e}$, while a non-unital Lie co-character for a co-ULA (V, ∂, ϵ) corresponds to an $\mathbf{e} \in \mathrm{Ker}(\partial)$.

Further, "**non-unital**" remarks 4.3.8 and 4.4.11 admit co-versions. To create a counit for a coassociative or co-Leibniz coalgebra (V, δ) (resp. (V, ∂)), one extends it by adding a formal element: $\widetilde{V} := V \oplus R\mathbf{1}$, modifying the comultiplication:

$$\Delta(v) = \delta(v) + \mathbf{1} \otimes v + v \otimes \mathbf{1} \qquad \forall v \in V,$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$$

in the coassociative coalgebra case, and

$$\partial(\mathbf{1}) = 0,$$

keeping the original ∂ on V, in the co-Leibniz case. Thus the application $\varepsilon \in \widetilde{V}^*$ given by $\varepsilon(V) \equiv 0, \varepsilon(\mathbf{1}) = 1$ is a (Lie) counit for Δ (resp. ∂), and $\mathbf{1}$ is a group-like element (resp. $\mathbf{1} \in \mathrm{Ker}(\partial)$). One easily checks the following

Lemma 5.4.5. The new comultiplication Δ (resp. ∂) is coassociative (resp. co-Leibniz) if and only if the original δ (resp. ∂) is.

To conclude, we write down the left braided differentials obtained in this particular setting:

Proposition 5.4.6. *Given an R-linear coalgebra (V, δ), extend it to a counital one $(\widetilde{V}, \Delta, \varepsilon)$ as described above. Then the group-like $\mathbf{1}$ gives, via theorem $6^{\mathbf{co}}$, the following differential on $T(\widetilde{V})$:*

$${}_{\mathbf{1}}d(v_1 \ldots v_n) = \mathbf{1}v_1 \ldots v_n + \sum_{i=1}^{n} (-1)^i v_1 \ldots v_{i-1} \Delta(v_i) v_{i+1} \ldots v_n.$$

The ideal $I_{\mathbf{1}}$ of the tensor algebra $T(\widetilde{V})$ generated by the element $\mathbf{1}$ is ${}_{\mathbf{1}}d$-stable. The differential induced on $T(V) \simeq T(\widetilde{V}/R\mathbf{1}) \simeq T(\widetilde{V})/I_{\mathbf{1}}$ is

$$\widetilde{{}_{\mathbf{1}}d}(v_1 \ldots v_n) = \sum_{i=1}^{n} (-1)^i v_1 \ldots v_{i-1} \delta(v_i) v_{i+1} \ldots v_n.$$

One eagerly recognizes the ***cobar differential*** for coalgebras.

5.5 Right-left duality

One more notion of duality is available for a monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$. One can simply change its tensor product to the opposite one:

$$V \otimes^{\mathrm{op}} W := W \otimes V$$

for objects, and similarly for morphisms. We call this new monoidal category ***monoidally dual*** to $\mathcal{C}$, denoting it by $\mathcal{C}^{\otimes^{\mathrm{op}}}$ (there seem to be no universally accepted notation, some authors even using $\mathcal{C}^{\mathrm{op}}$ here and another notation for co-categories). Graphically, the categories $\mathcal{C}$ and $\mathcal{C}^{\otimes^{\mathrm{op}}}$ differ by the ***vertical mirror symmetry*** for all diagrams.

Applying monoidal duality to a co-category $\mathcal{C}^{\mathrm{op}}$, one gets

$$\mathcal{C}^{\mathrm{op},\otimes^{\mathrm{op}}} := (\mathcal{C}^{\mathrm{op}})^{\otimes^{\mathrm{op}}} \simeq (\mathcal{C}^{\otimes^{\mathrm{op}}})^{\mathrm{op}}.$$

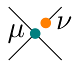
Graphically, it corresponds to the ***central symmetry***.

Similarly to what we have seen for $\mathcal{C}^{\mathrm{op}}$, all "categorical" notions and theorems have monoidally dual versions in $\mathcal{C}^{\otimes^{\mathrm{op}}}$. This gives in particular ***right differentials*** $(d^{\epsilon})_n, (d_{\epsilon})^n$, monoidally dual to the left ones $({}^{\epsilon}d)_n, ({}_{\epsilon}d)^n$. Note that these differentials should be endowed with a sign (cf. theorem 2) if one wants a bidifferential structure.

One also has ***right braidings***, monoidally dual to those from theorems $5^{\mathbf{cat}}$ and $5^{\mathbf{co}}$. In particular, a new braiding emerges for UAAs:

$$\sigma^r_{Ass} := \mu \otimes \nu : V \otimes V = V \otimes V \otimes \mathbf{I} \to V \otimes V.$$

Its diagram is a vertical mirror symmetry of what one had in the "left" case:



.

Figure 5.3: $\sigma^r_{Ass} = VertMirror(\sigma_{Ass})$

Remark that the Leibniz algebra structure is not right-left symmetric: a Leibniz algebra in $\mathcal{C}^{\otimes^{\mathrm{op}}}$ is in fact a ***left Leibniz algebra*** in $\mathcal{C}$ (cf. [48]). Thus one automatically obtains braided homology theories for left Leibniz algebras.

Chapter 6

Braided modules and homologies with coefficients

In this chapter we present two approaches to braided (co)homologies with coefficients. The first one is quite conceptual. It consists in defining a suitable notion of modules and bimodules over braided objects, specializing to the usual notions of (bi)modules for concrete examples of pre-braidings encoding algebraic structures. The second one is rather "handiwork": we simply remark that, for instance, a module M over an algebra V is the same thing as a special algebra structure on $V \oplus M$, and thus one can simply study the braided complexes for this latter algebra, and its reasonable subcomplexes. See the introduction to each section for more details on each method.

The two approaches will be extensively used in part II in the context of Hopf and Yetter-Drinfel′d structures.

6.1 Modules and bimodules over braided objects

We introduce here the notions of modules and bimodules over a pre-braided object V in a monoidal category (in particular over a pre-braided vector space). These "braided" modules generalize, in quite an unexpected manner, the following structures:

- ✓ modules and bimodules over associative algebras;
- ✓ modules over Leibniz algebras (cf. [46]);
- ✓ rack modules (= the rack-sets of S.Kamada [37], or the shadows of W.Chang and S.Nelson [13]), having knot-theoretical motivation.

Since at the same time a braided module generalizes a braided character, one naturally arrives to homologies of pre-braided objects with coefficients. As particular cases, we point out Hochschild and Chevalley-Eilenberg complexes.

We also endow each tensor power $V^{\otimes n}$ with an "adjoint" braided V-module structure, generalizing the tensor powers of the adjoint representation of Leibniz algebras. Our braided differentials then turn out to be braided V-module morphisms for the adjoint modules $V^{\otimes n}$, recovering in particular some properties of the bar complex.

All these facts speak in favor of our notion of braided modules.

Fix a <u>monoidal category</u> $(\mathcal{C}, \otimes, \mathbf{I})$.

Braided modules: definition and examples

Definition 6.1.1. ➜ A *right module* over a pre-braided object (V, σ) is an object $M \in \mathrm{Ob}(\mathcal{C})$ equipped with a morphism $\rho : M \otimes V \to M$ satisfying

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes \sigma) : M \otimes V \otimes V \to M. \tag{6.1}$$

We talk about *braided* V*-modules* when the pre-braiding σ is clear from the context.

➜ A *left module* is a right one in $\mathcal{C}^{\otimes \mathrm{op}}$.

➜ A right (or left) *comodule* is a right (resp. left) module in $\mathcal{C}^{\mathrm{op}}$.

➜ A *braided* V*-module morphism* is a morphism φ between braided V-modules (M, ρ) and (N, π) such that

$$\varphi \circ \rho = \pi \circ (\varphi \otimes \mathrm{Id}_V) : M \otimes V \to N. \tag{6.2}$$

Condition (6.1) is graphically depicted as follows:

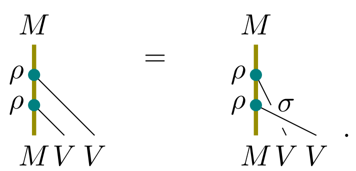


Figure 6.1: Braided module

Start as usual with a trivial example: in a preadditive category, any object M equipped with the zero map $M \otimes V \to M$ is a module over any pre-braided object (V, σ). We further interpret our new notion in more complicated settings from chapter 4.

Example 6.1.2. 1. When the braiding is simply a (signed) flip, one recovers the notion of (anti)commuting operators on M.

2. Take $\mathcal{C} = \mathbf{Set}$, and as a pre-braiding on a set S take $\sigma_{\lhd}$ from (4.1), coming from a self-distributive operation $\lhd$. Condition (6.1) becomes

$$(m \lhd a) \lhd b = (m \lhd b) \lhd (a \lhd b) \qquad \forall m \in M, a, b \in S,$$

which defines precisely a ***rack module*** (cf. [37] and [13]).

3. Any UAA (V, μ, ν) in $\mathcal{C}$ comes with the pre-braiding σ_{Ass} from (5.20). Take a right module (M, ρ) which we suppose ***normalized*** here, i.e.

$$\rho \circ (\mathrm{Id}_M \otimes \nu) = \mathrm{Id}_M \tag{6.3}$$

(morally, "the unit acts by identity"). Condition (6.1) becomes

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\mathrm{Id}_M \otimes \mu).$$

One recognizes the familiar notion of ***right modules over associative algebras***.

4. Take a ULA $(V, [,], \nu)$ in a symmetric preadditive category $\mathcal{C}$. Endow V with the braiding σ_{Lei} from (5.21). Take a normalized right module (M, ρ). Condition (6.1) becomes

$$\rho \circ (\rho \otimes \mathrm{Id}_V) = \rho \circ (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes c_{V,V}) + \rho \circ (\mathrm{Id}_M \otimes [,]).$$

One recognizes the familiar notion of ***right modules over Leibniz algebras*** (cf. [46]), raised to the categorical level.

Note that, dually, left modules over associative or left Leibniz algebras are particular cases of left modules over pre-braided objects.

Trivial and adjoint braided modules

Now, returning to the general monoidal category setting, try a special choice of M, putting $M = \mathbf{I}$.

Lemma 6.1.3. Take a pre-braided object (V, σ) in $\mathcal{C}$. For a morphism $\epsilon : V = \mathbf{I} \otimes V = V \otimes \mathbf{I} \to \mathbf{I}$, the following conditions are equivalent:

1. ϵ defines a right braided V-module;
2. ϵ defines a left braided V-module;
3. ϵ is a braided character.

Thus a braided character for V defines a right and left braided V-module structure on $\mathbf{I}$. This observation can be generalized to endow each tensor power of V with a braided V-module structure. Recall notations φ_i from (1.3).

Proposition 6.1.4. *Given a pre-braided object (V, σ) with a braided character ϵ, the map*

$${}^{\epsilon}\pi := \epsilon_1 \circ \boldsymbol{\sigma}_{V^{\otimes n}, V} : V^{\otimes n} \otimes V \to V^{\otimes n}$$

defines a right braided V-module structure on $V^{\otimes n}$. The braiding σ is extended here to arbitrary powers of V as in lemma 5.1.5.

Proof. The definition of ${}^{\epsilon}\pi$ and repeated application of the YBE give

$$\begin{aligned}
{}^{\epsilon}\pi \circ ({}^{\epsilon}\pi \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V^{\otimes n} \otimes \sigma) = \\
(\epsilon \otimes \epsilon \otimes \mathrm{Id}_V^{\otimes n}) \circ \boldsymbol{\sigma}_{V^{\otimes n}, V^{\otimes 2}} \circ (\mathrm{Id}_V^{\otimes n} \otimes \sigma) = \\
((\epsilon \otimes \epsilon) \circ \sigma) \otimes \mathrm{Id}_V^{\otimes n}) \circ \boldsymbol{\sigma}_{V^{\otimes n}, V^{\otimes 2}}
\end{aligned}$$

which, by the definition of braided character, is the same as

$$(\epsilon \otimes \epsilon \otimes \mathrm{Id}_V^{\otimes n}) \circ \boldsymbol{\sigma}_{V^{\otimes n}, V^{\otimes 2}} = {}^{\epsilon}\pi \circ ({}^{\epsilon}\pi \otimes \mathrm{Id}_V).$$

The reader is advised to draw some diagrams to better follow the proof. □

Definition 6.1.5. We call the modules $(V^{\otimes n}, {}^{\epsilon}\pi)$ *adjoint*.

One recognizes the map $\overset{\epsilon}{\nwarrow}$ from section 3.2. See that section for a diagrammatic depiction and some properties. In particular, proposition 3.2.7 gives the following

Proposition 6.1.6. *The action ${}^{\epsilon}\pi$ on $T(V)$ intertwines the left braided differential ${}^{\xi}d$ for σ-compatible braided characters ϵ and ξ.*

In other words, ${}^{\xi}d$ is a braided V-module morphism, for the adjoint braided V-module structure on $V^{\otimes n}$.

The motivation for our term comes from examples, where one recognizes familiar actions on $T(V)$:

Example 6.1.7. 1. Take a shelf S in $\mathcal{C} = \mathbf{Set}$ and, as a braided character, the only map from S to the one-element set $\mathbf{I}$. Then $S^{\times n}$ becomes a braided S-module, hence a rack module, via the ***diagonal*** action

$$(a_1, \ldots, a_n) \lhd b = (a_1 \lhd b, \ldots, a_n \lhd b).$$

2. For a UAA V in $\mathcal{C}$ with an algebra character ϵ, only the rightmost component of $V_{\otimes n}$ is affected by the adjoint action:
$$^{\epsilon}\pi = \mathrm{Id}_V^{\otimes(n-1)} \otimes \mu, \qquad n > 0.$$
One gets the ***peripheral*** action.
3. For a ULA V in $\mathcal{C}$ with a Lie character ϵ, one gets
$$^{\epsilon}\pi = \sum_{i=1}^{n} [,]_i \circ (\mathrm{Id}_V^{\otimes i} \otimes c_{V^{\otimes(n-i)},V}) + \epsilon_1 \circ c_{V^{\otimes n},V}.$$
Starting with a non necessarily unital Leibniz algebra in an additive category, adding a formal unit, taking the character ε and then restricting everything to $T(V)$ (cf. remark 4.4.11), one gets rid of the last term and arrives to the usual ***adjoint*** action of a Lie algebra V on $T(V)$.

Braided differentials and adjoint modules with coefficients

We have seen that a module over a pre-braided object is a generalization of a braided character. Observe that this generalization picks the right property for a generalized version of theorem 6 (where we replace the braided V-module $\mathbf{I}$ by arbitrary braided modules) to hold:

Theorem 6$^{\text{coeffs}}$. *Let $(\mathcal{C}, \otimes, \mathbf{I})$ be a preadditive monoidal category, (V, σ) a pre-braided object in $\mathcal{C}$, and (M, ρ) and (N, λ) a right and a left braided V-modules respectively. Then two families of morphisms*
$$({}^{\rho}d)_n := (\rho \otimes \mathrm{Id}_V^{\otimes(n-1)} \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_M \otimes \underset{-\sigma}{\sqcup\!\sqcup}^{1,n-1} \otimes \mathrm{Id}_N),$$
$$(d^{\lambda})_n := (-1)^{n-1}(\mathrm{Id}_M \otimes \mathrm{Id}_V^{\otimes(n-1)} \otimes \lambda) \circ (\mathrm{Id}_M \otimes \underset{-\sigma}{\sqcup\!\sqcup}^{n-1,1} \otimes \mathrm{Id}_N),$$
define a bidegree -1 tensor bidifferential for V with coefficient in M and N.

The complicated expression *a bidegree -1 tensor bidifferential for V with coefficient in M and N* hides what one naturally expects: it means two families of morphisms d_n, d'_n : $M \otimes V^n \otimes N \to M \otimes V^{n-1} \otimes N$, satisfying (5.18).

Pictorially, $({}^{\rho}d)_n$ for example is a signed sum of terms of the form

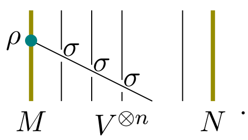


Figure 6.2: Braided differentials with coefficients

The proof of this result is a direct generalization of that of theorem 6. Moreover, all the remaining points of that theorem can be generalized to "coefficient" versions.

Remark 6.1.8. Taking as M or N the unit object $\mathbf{I}$ with a zero module structure, one obtains a degree -1 tensor differential for V with coefficient in the left braided V-module N (resp. right braided V-module M) only.

As usual, everything described here can be dualized, in any of the three senses described in sections 5.4 and 5.5.

Adjoint modules also admit a version with coefficients. Only left coefficients are considered here.

Proposition 6.1.9. *Given a pre-braided object (V,σ) and a right braided V-module (M,ρ), the morphisms*

$$^{\rho}\pi := \rho_1 \circ (\mathrm{Id}_M \otimes \boldsymbol{\sigma}_{V^{\otimes n},V}) : M \otimes V^{\otimes n} \otimes V \to M \otimes V^{\otimes n}$$

define a right braided V-module structure on $M \otimes V^{\otimes n}$, intertwining the left differential $^{\rho}d$.

In other words, $^{\rho}d$ is a braided V-module morphism.

Definition 6.1.10. We call $(M \otimes V^{\otimes n}, {}^{\rho}d)$ *adjoint modules with coefficient.*

Braided bimodules

Having the Hochschild homology in mind, one should also categorify the notion of bimodules.

Definition 6.1.11. A *bimodule* over a pre-braided object (V,σ) is an object $M \in \mathrm{Ob}(\mathcal{C})$ equipped with two morphisms $\rho : M \otimes V \to M$ and $\lambda : V \otimes M \to M$, turning M into a right and left modules respectively and satisfying the following compatibility condition:

$$\rho \circ (\lambda \otimes \mathrm{Id}_V) = \lambda \circ (\mathrm{Id}_V \otimes \rho) : V \otimes M \otimes V \to M.$$

Another interpretation of bimodules – in terms of modules over appropriate pre-braided systems – will be given in chapter 7.

The bidifferential structure from theorem $6^{\mathbf{coeffs}}$ can be nicely adapted to bimodules:

Proposition 6.1.12. *Let $(\mathcal{C},\otimes,\mathbf{I},c)$ be a symmetric preadditive category, (V,σ) a pre-braided object in $\mathcal{C}$, and (M,ρ,λ) a bimodule over V. Then the families of morphisms*

$$({}^{\rho}d)_n := (\rho \otimes \mathrm{Id}_V^{n-1}) \circ (Id_M \otimes \underset{-\sigma}{\sqcup\!\sqcup}^{1,n-1}),$$
$$(d^{\lambda})_n := (-1)^{n-1} c^{-1}_{M,V^{n-1}} \circ (\mathrm{Id}_V^{n-1} \otimes \lambda) \circ (\underset{-\sigma}{\sqcup\!\sqcup}^{n-1,1} \otimes \mathrm{Id}_M) \circ c_{M,V^n},$$

define a bidegree -1 tensor bidifferential for V with coefficients in M on the left.

By definition, $(d^{\lambda})_n$ is a signed sum of terms of the form

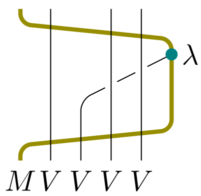


Figure 6.3: Braided differentials with bimodule coefficients

Proof. Relations $({}^{\rho}d)_{n-1} \circ ({}^{\rho}d)_n = 0$ and

$$(d^{\lambda})_{n-1} \circ (d^{\lambda})_n = c^{-1}_{M,V^{n-2}} \circ (d'^{\lambda})_{n-1} \circ c_{M,V^{n-1}} \circ c^{-1}_{M,V^{n-1}} \circ (d'^{\lambda})_n \circ c_{M,V^n}$$
$$= c^{-1}_{M,V^{n-2}} \circ (d'^{\lambda})_{n-1} \circ (d'^{\lambda})_n \circ c_{M,V^n} = 0,$$

with $(d'^{\lambda})_n := (-1)^{n-1}(\mathrm{Id}_V^{n-1} \otimes \lambda) \circ (\underset{-\sigma}{\sqcup\!\sqcup}^{n-1,1} \otimes \mathrm{Id}_M)$, follow directly from the corresponding identities in theorem $6^{\mathbf{coeffs}}$.

To prove the compatibility between $({}^{\rho}d)_n$ and $(d^{\lambda})_n$, observe that

$$(d^{\lambda})_n = (-1)^{n-1}((\lambda \circ c_{M,V}) \otimes \mathrm{Id}_V^{n-1}) \circ (\mathrm{Id}_M \otimes c^{-1}_{V,V^{n-1}}) \circ (\mathrm{Id}_M \otimes \underset{-\sigma}{\sqcup\!\sqcup}^{n-1,1}),$$

then use the defining property of a bimodule, the naturality of the braiding c and the YBE for σ. □

Remark 6.1.13. We have kept the notation c^{-1}, redundant for symmetric c, to be able to treat the non symmetric situation. In this case, on the picture showing $(d^{\lambda})_n$ the thick line (corresponding to M) should go behind all normal lines, in order to distinguish c from c^{-1}. One should be careful to differentiate two braidings, c and σ, which is difficult to do pictorially. For the above theorem to be still valid, one should change the compatibility condition defining a bimodule to the following one, different from the old one in general:

$$\lambda \circ (\mathrm{Id}_V \otimes \rho) \circ c_{M\otimes V,V} = \rho \circ (\lambda \otimes \mathrm{Id}_V) \circ c_{M,V} \circ c^{-1}_{V,V} : M \otimes V \otimes V \to M.$$

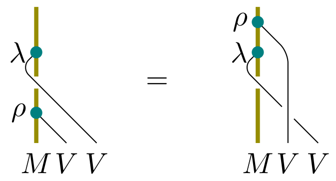


Figure 6.4: Bimodules in the non symmetric case

All the crossings correspond to the braiding c here.

A more elegant solution for the non symmetric case would be welcome.

Recovering classical homologies with coefficients

Example 6.1.14.

1. Taking a vector space V with a simple flip as a braiding and, for instance, its symmetric algebra $S(V)$ as a module over V (with the action coming from concatenation, as usual), one obtains more complicated versions of the Koszul complex.
2. In the case of shelves, one recovers the ***shelf and rack homologies with coefficients***, hinted at in [13].
3. For Leibniz algebras, our machinery gives the ***Leibniz homology with coefficients***, generalizing the Chevalley-Eilenberg homology (cf. [46]).

   In these three cases one generally puts the coefficients only on the left (cf. remark 6.1.8).
4. Coefficients on both sides turn out to be particularly useful for associative algebras in a symmetric preadditive category. In this setting, proposition 6.1.12 gives the following differential for an algebra bimodule (M, ρ, λ):

$$({}^{\rho}d - d^{\lambda})_n := \rho \otimes \mathrm{Id}_V^{n-1} + \sum_{i=1}^{n-1} (-1)^i \mu_i + (-1)^n (\lambda \otimes \mathrm{Id}_V^{n-1}) \circ c_{M\otimes V^{n-1},V}$$
$$+ \text{ some terms involving } \nu.$$

   For $\mathcal{C} = \mathbf{Mod}_R$, one can get rid of the terms with ν as it was done in the proof of point 7 of proposition 4.3.7, getting the ***Hochschild differential***.
5. The co-version of the previous differential is the ***Cartier differential*** for coalgebras (cf. [12], where it was first introduced). It is easily obtained by duality.

6.2 Structure mixing techniques

Another approach to studying (bi)modules over an associative / Leibniz algebra consists in interpreting these structures as an associative / Leibniz multiplication on a larger object, mixing the module structure and the multiplication on the acting algebra. It resembles what is often done when studying Hochschild or Leibniz extensions (see [1] and [48] for example).

We work in $\mathbf{Mod}_R$ here for simplicity, but everything presented in this section remains valid in an additive monoidal category.

Only the example of a ***bimodule*** $M \in {}_V\mathbf{Mod}_W$ over associative algebras is studied in detail here.

Take three R-modules V, W, M with four bilinear operations

$$\begin{array}{llll} \mu_V : V \otimes V & \longrightarrow V & \quad \mu_W : W \otimes W & \longrightarrow W \\ \lambda : V \otimes M & \longrightarrow M & \quad \rho : M \otimes W & \longrightarrow M. \end{array}$$

These operations are denoted by a dot, e.g. $v \cdot a = \lambda(v \otimes a)$, when it does not lead to confusion.

Now mix these structures:

$$M' := V \oplus W \oplus M$$

and defining a bilinear operation μ on M' by

$$\mu|_{V\otimes V} = \mu_V, \qquad \mu|_{W\otimes W} = \mu_W,$$

$$\mu|_{V\otimes M} = \lambda, \qquad \mu|_{M\otimes W} = \rho,$$

extended by zero for other couples of modules. One easily checks the following

Lemma 6.2.1. The associativity of μ is equivalent to a set of conditions:

- V and W are associative algebras;
- λ is a left action of V on M;
- ρ is a right action of W on M;
- these actions are compatible, in the sense that

$$(v \cdot a) \cdot w = v \cdot (a \cdot w), \quad \forall v \in V, w \in W, a \in M.$$

Add a formal unit

$$\widetilde{M} := M' \oplus R\mathbf{1}$$

and consider the application σ_μ from section 4.3. Combining the preceding lemma with lemma 4.3.1, one gets:

Corollary 6.2.2. *The application σ_μ is a braiding on $\widetilde{M}$ if and only if the maps $\mu_V, \mu_W, \lambda, \rho$ define a structure of two associative algebras V and W and a bimodule $M \in {}_V\mathbf{Mod}_W$.*

Thus our pre-braiding encodes the structure of a bimodule.

Proceeding as in the "non-unital" remark 4.3.8, consider the braided character

$$\varepsilon(M') \equiv 0, \varepsilon(\mathbf{1}) = 1.$$

Trying to reasonably restrict the differential ${}^{\varepsilon}d$, one notices that the submodule

$$T(V; M; W) := T(V) \otimes M \otimes T(W) \subseteq T(\widetilde{M})$$

is ${}^{\varepsilon}d$-stable. Explicit calculations give

Proposition 6.2.3. *Take a bimodule $M \in {}_V\mathbf{Mod}_W$. The restriction of the differential ${}^{\varepsilon}d$ described above to $T(V;M;W)$ gives a differential*

$$\begin{aligned}
&{}^{\varepsilon}d(v_1 \dots v_n a w_1 \dots w_m) = \\
&= \sum_{i=1}^{n-1} (-1)^i v_1 \dots v_{i-1}(v_i \cdot v_{i+1}) v_{i+2} \dots v_n a w_1 \dots w_m \\
&+ (-1)^n v_1 \dots v_{n-1}(v_n \cdot a) w_1 \dots w_m + (-1)^{n+1} v_1 \dots v_n (a \cdot w_1) w_2 \dots w_m \\
&+ \sum_{i=1}^{m-1} (-1)^{n+1+i} v_1 \dots v_n a w_1 \dots w_{i-1}(w_i \cdot w_{i+1}) w_{i+2} \dots w_m.
\end{aligned}$$

The differential from the proposition can be used to construct the Hochschild differential and the cyclic homology, using some “cyclic” considerations. This gives an aproach alternative to the one presented in the previous section. This will be done in a subsequent publication.

Part II

Hopf and Yetter-Drinfel$'$d Structures via Braided Systems

Chapter 7

Braided systems: general theory and examples

The aim of this chapter is twofold.

On the one hand, we construct braided (co)homology theories for bialgebras, Hopf algebras, Hopf modules, Hopf bimodules and Yetter-Drinfel′d modules, recovering familiar (co)homologies (for instance, the bialgebra cohomology of M.Gerstenhaber and S.D.Schack, cf. [29]). For this we use "braided" techniques generalizing those from part I. The particularity of the structures in question is that one has to deal with components of different nature (for example, an algebra, its dual and several Hopf modules over it) at the same time. One can try to amalgamate all the structures into one, as it was done in section 6.2 for bimodules over associative algebras, but in this chapter we present a more elegant and flexible tool, which we call *a (pre-)braided system.* This generalization turns out to be sufficient for encoding the algebraic structures listed above, just like the notion of pre-braided object which we used for encoding simpler structures in preceding chapters.

On the other hand, we focus on the presentations of Hopf bimodules and other "complicated" structures as "simpler" structures – algebra modules – over certain "complicated" algebras (for example, the algebra $\mathscr{X}$ of C.Cibils and M.Rosso, cf. [14]), which are *braided* (or *twisted* in some sources) *tensor products* of some "simpler" algebras. See table 1.1 for the concrete examples we are interested in. In this chapter, we introduce an intermediate interpretation of such structures as *multi-braided modules* over appropriate pre-braided systems. This gives a convenient tool for a systematic study of braided tensor products of algebras, describing in particular

- ✓ their associativity conditions;
- ✓ interchanging rules for their components;
- ✓ modules over such braided tensor product algebras.

Our theory recovers the iterated twisted tensor products of algebras, studied by P.Jara Martínez, J.López Peña, F.Panaite and F. van Oystaeyen (cf. [32]).

Continuing the example of the algebra $\mathscr{X}$, we automatically recover its $\mathscr{Y}$ and $\mathscr{Z}$ versions, introduced by F.Panaite in [65], as well as the explicit isomorphisms between the three. Moreover, we include these three algebras into a family of 24 pairwise isomorphic braided tensor product algebras. Our systematic method allows to minimize the technical verifications necessary to establish this kind of results.

Feeding the multi-braided-module interpretation of the "complicated" structures above into the general multi-braided homology theory with coefficients we develop here (generalizing the contents of section 6.1), we reinterpret different "complicated" structures on concrete braided differential complexes (for instance, the Hopf bimodule structure on the bar complex of a bialgebra with coefficients in a Hopf bimodule).

In the concrete settings of Hopf and Yetter-Drinfel′d structures, the building blocks for our pre-braided systems are the pre-braiding σ_{Ass} encoding the associativity, and the pre-braiding σ_{YD} for two Yetter-Drinfel′d modules, both with several "twisted" modifications. See formulas (5.20), (7.12) and figure 7.12.

In the first two sections we create a general abstract framework for dealing with the two types of questions we are interested in here, while in the remaining sections we consider more or less general examples.

7.1 General recipe

Fix a monoidal category $\mathcal{C}$.

Pre-braided systems and braided characters

Start with making the concept of a pre-braided family of objects (definition 5.1.3) more precise:

Definition 7.1.1. A *pre-braided system* in $\mathcal{C}$ is an ordered finite family $V_1, V_2, \ldots, V_r \in \mathrm{Ob}(\mathcal{C})$ endowed with morphisms

$$\sigma_{i,j} : V_i \otimes V_j \longrightarrow V_j \otimes V_i \quad \forall 1 \leqslant i \leqslant j \leqslant r,$$

satisfying the Yang-Baxter equation (YB) on all the tensor products $V_i \otimes V_j \otimes V_k$ with $1 \leqslant i \leqslant j \leqslant k \leqslant r$.

Such a system is denoted by $((V_i)_{1\leqslant i\leqslant r}, (\sigma_{i,j})_{1\leqslant i\leqslant j\leqslant r})$ or briefly $(r, \overline{V}, \overline{\sigma})$.

We call the family *braided* if all the $\sigma_{i,j}$'s are invertible.

For given $1 \leqslant s \leqslant t \leqslant r$, the *pre-braided* (s,t)*-subsystem* of $(r, \overline{V}, \overline{\sigma})$, denoted by $(r, \overline{V}, \overline{\sigma})[s,t]$, is the subfamily $V_s, \ldots, V_t$ with the $\sigma_{i,j}$'s from $\overline{\sigma}$.

Thinking pictorially, one allows a strand to overcross only the strands colored with a smaller or equal index $i \in \{1, 2, \ldots, r\}$.

The difference from the notion of a pre-braided family consists in two points:

1. the finiteness condition;
2. the definition of braiding for the ordered couples of objects only.

Note that one has $\binom{r+2}{3}$ YBEs to verify.

The pre-braiding constructed earlier for a bimodule $M \in {}_V\mathbf{Mod}_W$ does not fit directly to these settings, since $\sigma_\mu(v \otimes a) = \mathbf{1} \otimes v \cdot a \in \mathbf{1} \otimes M$ for $a \in M, v \in V$, i.e. the element of M stays on the right instead of passing to the left as it happens in pre-braided systems.

As for positive examples, the simplest one is the following:

Lemma 7.1.2. Pre-braided objects (V_i, σ_i), $1 \leqslant i \leqslant r$ in a pre-braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ form a pre-braided system when endowed with the pre-braiding

$$\begin{aligned} \sigma_{i,i} &:= \sigma_i, \\ \sigma_{i,j} &:= c_{V_i,V_j}, \qquad i < j. \end{aligned}$$

Proof. There are three kinds of tensor products on which one should check (YB):

1. $V_i \otimes V_i \otimes V_i$. Use YBE for σ_i here.

2. $V_i \otimes V_i \otimes V_j$ or $V_i \otimes V_j \otimes V_j$ for $i < j$. Use the naturality of the pre-braiding c with respect to σ_i or σ_j.
3. $V_i \otimes V_j \otimes V_k$ for $i < j < k$. Use YBE for the pre-braiding c. □

Here are two useful elementary properties of pre-braided systems:

Lemma 7.1.3. Take a pre-braided system $(r, \overline{V}, \overline{\sigma})$ in $\mathcal{C}$.

1. For any subset $I \subseteq \{1, \dots, r\}$, one has a pre-braided system
$$((V_i)_{i \in I}, (\sigma_{i,j})_{i \leqslant j \in I \times I}),$$
called *a pre-braided subsystem of* $(r, \overline{V}, \overline{\sigma})$.
2. For any $1 \leqslant i < r$, the pre-braiding $\overline{\sigma}$ gives a pre-braiding on
$$(V_1, \dots, V_{i-1}, V_i \otimes V_{i+1}, V_{i+2}, \dots, V_r)$$
by choosing the identity or the zero (if $\mathcal{C}$ is preadditive) pre-braiding on $(V_i \otimes V_{i+1})^{\otimes 2}$ and by using the formulas for extending a pre-braiding to tensor products (cf. lemma 5.1.5) on $(V_i \otimes V_{i+1}) \otimes V_j$ and $V_k \otimes (V_i \otimes V_{i+1})$.

The notions of compatible co-elements and braided characters are inherited from the ones we had in the context of pre-braided vector spaces:

Definition 7.1.4. ➜ Families of morphisms $f_i, g_i : V_i \to \mathbf{I}$, $1 \leqslant i \leqslant r$ are called $\overline{\sigma}$*-compatible* if
$$(f_j \otimes g_i) \circ \sigma_{i,j} = g_i \otimes f_j,$$
$$(g_j \otimes f_i) \circ \sigma_{i,j} = f_i \otimes g_j$$
on $V_i \otimes V_j$ for all $i \leqslant j$.

➜ A *braided character* for $(r, \overline{V}, \overline{\sigma})$ is a family $\overline{\epsilon}$ of morphisms $\epsilon_i : V_i \to \mathbf{I}$ which is $\overline{\sigma}$-compatible with itself.

Braided differentials: a "multi-version"

From now on, suppose $\mathcal{C}$ additive monoidal. In particular, one can interpret the collection $\overline{\sigma}$ as a ***partial braiding*** σ_{part} on
$$V := V_1 \oplus V_2 \oplus \cdots \oplus V_r.$$

We then show that the collection $\overline{\sigma}$ suffices for defining a partial version of ***quantum shuffle structures***.

Definition 7.1.5. ➜ An *ordered tensor product* for a pre-braided system $(r, \overline{V}, \overline{\sigma})$ in $\mathcal{C}$ is a tensor product of the form
$$V_1^{\otimes m_1} \otimes V_2^{\otimes m_2} \otimes \cdots \otimes V_r^{\otimes m_r}, \qquad m_i \geqslant 0.$$

➜ A *reversely ordered tensor product* is one of the form
$$V_r^{\otimes m_r} \otimes V_{r-1}^{\otimes m_{r-1}} \otimes \cdots \otimes V_1^{\otimes m_1}, \qquad m_i \geqslant 0.$$

- The *degree* of such a tensor product is simply the sum $\sum_{i=1}^{r} m_i$.
- The direct sum of (reversely) ordered tensor products of degree n is denoted by $T(\overline{V})^{\rightarrow}_n$ (resp. $T(\overline{V})^{\leftarrow}_n$).

In $\mathbf{Mod}_R$, the $T(\overline{V})^{\rightarrow}_n$'s sum up to

$$T(\overline{V})^{\rightarrow} := T(V_1) \otimes T(V_2) \otimes \cdots \otimes T(V_r).$$

Lemma 7.1.6. Let $(r, \overline{V}, \overline{\sigma})$ be a pre-braided system in $\mathcal{C}$. The (categorical version of the) quantum co-shuffle comultiplication (2.7) is well defined on ordered tensor products. It gives a coassociative comultiplication denoted by

$$\underset{\overline{\sigma}}{\overline{\sqcup\!\sqcup}}^{p,q} : T(\overline{V})^{\rightarrow}_{p+q} \longrightarrow T(\overline{V})^{\rightarrow}_p \otimes T(\overline{V})^{\rightarrow}_q.$$

Proof. It is sufficient to observe that if an ordered tensor product is fed into the formula (2.7) defining $\underset{\overline{\sigma}}{\overline{\sqcup\!\sqcup}}$, then the braiding $\overline{\sigma}$ is applied only to components $V_i \otimes V_j$ with $i \leqslant j$. $\square$

Dualizing, one gets an associative multiplication

$$\underset{\overline{\sigma}}{\sqcup\!\sqcup}^{p,q} : T(\overline{V})^{\leftarrow}_p \otimes T(\overline{V})^{\leftarrow}_q \longrightarrow T(\overline{V})^{\leftarrow}_{p+q}.$$

Note that even when its source is an ordered tensor product, the target of $\underset{\overline{\sigma}}{\overline{\sqcup\!\sqcup}}^{p,q}$ is not a single tensor product of ordered tensor products, but their direct sum in general. This explains why we need additive categories here.

In the additive setting, a braided character for $(r, \overline{V}, \overline{\sigma})$ can be seen as a morphism $\epsilon : V \to \mathbf{I}$ satisfying

$$(\epsilon \otimes \epsilon) \circ \underset{-\sigma_{i,j}}{\sqcup\!\sqcup} = 0 : V_i \otimes V_j \longrightarrow \mathbf{I}, \qquad \forall i \leqslant j.$$

An example of such braided characters is given by ***"partial characters"***:

Lemma 7.1.7. A braided character ϵ_i for the pre-braided object $(V_i, \sigma_{i,i})$, extended to the other V_j's by zero, is a braided character for $(r, \overline{V}, \overline{\sigma})$.

Further, the notion of ***(bi)degree*** -1 ***tensor (bi)differentials*** for $(r, \overline{V}, \overline{\sigma})$ is obtained from that for an object V (cf. definition 5.1.8) by replacing all the occurencies of $V^{\otimes n}$ with its ordered substitute $T(\overline{V})^{\rightarrow}_n$ (or, in the dual situation, $T(\overline{V})^{\leftarrow}_n$).

Everything is now ready for a multi-version of theorem 6:

Theorem 6 $^{\mathbf{multi}}$. *Let $(\mathcal{C}, \otimes, \mathbf{I})$ be an additive monoidal category. For a pre-braided system $(r, \overline{V}, \overline{\sigma})$ with two braided characters $\overline{\epsilon}$ and $\overline{\zeta}$, the morphisms*

$$({}^{\epsilon}d)_n := (\overline{\epsilon} \otimes \mathrm{Id}_{T(\overline{V})^{\rightarrow}_{n-1}}) \circ \underset{-\overline{\sigma}}{\overline{\sqcup\!\sqcup}}^{1,n-1} : \ T(\overline{V})^{\rightarrow}_n \longrightarrow T(\overline{V})^{\rightarrow}_{n-1},$$
$$(d^{\zeta})_n := (-1)^{n-1} (\mathrm{Id}_{T(\overline{V})^{\rightarrow}_{n-1}} \otimes \overline{\zeta}) \circ \underset{-\overline{\sigma}}{\overline{\sqcup\!\sqcup}}^{n-1,1} : \ T(\overline{V})^{\rightarrow}_n \longrightarrow T(\overline{V})^{\rightarrow}_{n-1}$$

define a bidegree -1 tensor bidifferential. So do the families $({}^{\epsilon}d)_n$ and $({}^{\zeta}d)_n$ if $\overline{\epsilon}$ and $\overline{\zeta}$ are $\overline{\sigma}$-compatible.

This theorem comes with a co-version, generalizing theorem $6^{\mathbf{co}}$. Recall that one should work with $T(\overline{V})^{\leftarrow}_n$ in the dual settings, since a pre-braiding on the system $(V_1, \ldots, V_r)$ in $\mathcal{C}^{\mathrm{op}}$ is the same thing as a pre-braiding on the reversed sytem $(V_r, \ldots, V_1)$ in $\mathcal{C}$.

As usual, other points of theorem 6 are easily generalized to the "multi"-setting.

Remark 7.1.8. In practice, some sub-bicomplexes of the above-mentioned bicomplexes are often useful. A typical example (cf. section 6.2) is

$$T(V_1; V_2; V_3) := T(V_1) \otimes V_2 \otimes T(V_3),$$

which is ${}^{\epsilon}d$- and d^{ϵ}-stable if $\overline{\epsilon}$ is the zero map on the component V_2 of $\overline{V}$. Note that in this situation one never has two neighboring V_2 components, hence the pre-braiding $\sigma_{2,2}$ for V_2 does not matter and can be chosen identity for simplicity, automatically giving YBE on all the triple tensor products $V_i \otimes V_j \otimes V_k$ with at least two consecutive V_2's.

Adding coefficients: multi-braided modules

The notion of braided module has a particularly fruitful generalization for pre-braided systems:

Definition 7.1.9. ➜ A *right multi-module* over a pre-braided system $(r, \overline{V}, \overline{\sigma})$ in $\mathcal{C}$ is an object $M \in \mathrm{Ob}(\mathcal{C})$ equipped with morphisms

$$\rho_i : M \otimes V_i \to M \qquad \forall\, 1 \leqslant i \leqslant r$$

satisfying, for all $1 \leqslant i \leqslant j \leqslant r$,

$$\rho_j \circ (\rho_i \otimes \mathrm{Id}_{V_j}) = \rho_i \circ (\rho_j \otimes \mathrm{Id}_{V_i}) \circ (\mathrm{Id}_M \otimes \sigma_{i,j}) : M \otimes V_i \otimes V_j \to M. \tag{7.1}$$

➜ Left modules and comodules, as well as multi-module morphisms, are defined in the usual way.

➜ Denote by $\mathbf{Mod}_{(r,\overline{V},\overline{\sigma})}$ the category of such right multi-modules and multi-module morphisms.

➜ We talk about *multi-braided $\overline{V}$-modules* and use the notation $\mathbf{Mod}_{(V_1,\ldots,V_r)}$ when the pre-braided system structure is clear from the context.

Remark 7.1.10. A multi-braided $\overline{V}$-module can be seen as a braided $(V_i, \sigma_{i,i})$-module $\forall\, 1 \leqslant i \leqslant r$, these structures being compatible in the sense of (7.1).

As usual, a left or right $\overline{V}$-module structure on the unit object $\mathbf{I}$ of $\mathcal{C}$ is the same thing as a braided character for $\overline{V}$.

In the following sections, we interpret algebra bimodules, Hopf (bi)modules and Yetter-Drinfel′d modules as multi-braided modules over appropriate pre-braided systems.

Theorem 6 $^{\mathbf{multi}}$ clearly admits a version with coefficients:

Theorem 6 $^{\mathbf{multi,coeffs}}$. *Let $(\mathcal{C}, \otimes, \mathbf{I})$ be an additive monoidal category. For a pre-braided system $(r, \overline{V}, \overline{\sigma})$ and a multi-braided $\overline{V}$-module $(M, \overline{\rho} := (\rho_i)_{1\leqslant i\leqslant r})$, the family of morphisms*

$$({}^{\rho}d)_n := (\overline{\rho} \otimes \mathrm{Id}_{T(\overline{V})^{\rightarrow}_{n-1}}) \circ (\mathrm{Id}_M \otimes \underset{-\overline{\sigma}}{\sqcup\!\sqcup}^{1,n-1}) : \; M \otimes T(\overline{V})^{\rightarrow}_n \longrightarrow M \otimes T(\overline{V})^{\rightarrow}_{n-1}$$

defines a degree -1 tensor differential.

The theory of adjoint modules, including its version with coefficients and the homological consequences (propositions 6.1.6 and 6.1.9), have a natural multi-version. We treat directly the version with coefficients here.

Proposition 7.1.11. *Take numbers $1 \leqslant s \leqslant t \leqslant r$, a pre-braided system $(r, \overline{V}, \overline{\sigma})$ and a multi-braided $\overline{V}$-module $(M, \overline{\rho})$. Denote by $(t-s+1, \overline{V}', \overline{\sigma})$ the pre-braided (s,t)-subsystem of $(r, \overline{V}, \overline{\sigma})$.*

1. *For any $n \in \mathbb{N}$, $M \otimes T(\overline{V}')^{\rightarrow}_n$ becomes a multi-braided $(r, \overline{V}, \overline{\sigma})[t, r]$-module via the morphisms*

$$^{\rho}\pi_i := (\rho_i \otimes \mathrm{Id}_{T(\overline{V}')^{\rightarrow}_n}) \circ (\mathrm{Id}_M \otimes \overline{\boldsymbol{\sigma}}_{T(\overline{V}')^{\rightarrow}_n, V_i}) : M \otimes T(\overline{V}')^{\rightarrow}_n \otimes V_i \to M \otimes T(\overline{V}')^{\rightarrow}_n$$

for all $t \leqslant i \leqslant r$.

2. *Moreover, the left differentials $(^{\rho}d)_n$ are multi-braided module morphisms for the multi-braided module structure $^{\rho}\pi$ over $(r, \overline{V}, \overline{\sigma})[t, r]$.*

Thus, for instance, for components V_i and V_{i+1} of a pre-braided system, the differential $^{\epsilon_i}d$ on $T(V_i)$ is a morphism of braided $(V_{i+1}, \sigma_{i+1,i+1})$-modules, with the module structure $^{\epsilon_{i+1}}\pi$, if ϵ_i and ϵ_{i+1} are braided characters on $(V_i, \sigma_{i,i})$ and $(V_{i+1}, \sigma_{i+1,i+1})$ respectively, compatible in the following sense:

$$(\epsilon_{i+1} \otimes \epsilon_i) \circ \sigma_{i,i+1} = \epsilon_i \otimes \epsilon_{i+1} : V_i \otimes V_{i+1} \to \mathbf{I}.$$

Invertibility questions

The invertibility of some of the $\sigma_{i,j}$'s, often encountered in practice, can be helpful in extending pre-braided – and thus differential – structures:

Proposition 7.1.12. *Let $(r, \overline{V}, \overline{\sigma})$ be a pre-braided system in $\mathcal{C}$, with $\sigma_{i,j}$ invertible for $s \leqslant i < j \leqslant t$. Then one can glue the objects $V_s, \ldots, V_t$ together into $V_{s:t} := \bigoplus_{i=s}^{t} V_i$ and extend the pre-braiding onto $(V_1, \ldots, V_{s-1}, V_{s:t}, V_{t+1}, \ldots, V_r)$, putting*

$$\sigma|_{V_j \otimes V_i} := \sigma_{i,j}^{-1} \quad \forall s \leqslant i < j \leqslant t.$$

Note that the invertibility of the $\sigma_{i,i}$'s is not required here even for $s \leqslant i \leqslant t$.

Proof. One has to check additional YBEs appearing when passing to $V_{s:t}$, i.e. (YB) on all the $V_i \otimes V_j \otimes V_k$ with

- ✓ $s \leqslant i, j, k \leqslant t$ and any order on $\{i, j, k\}$;
- ✓ $s \leqslant i, j \leqslant t < k$ and any order on $\{i, j\}$;
- ✓ $i < s \leqslant j, k \leqslant t$ and any order on $\{j, k\}$.

In other words, one wants

$$\sigma_{|i,j|}^{\varepsilon_{i,j}} \sigma_{|i,k|}^{\varepsilon_{i,k}} \sigma_{|j,k|}^{\varepsilon_{j,k}} = \sigma_{|j,k|}^{\varepsilon_{j,k}} \sigma_{|i,k|}^{\varepsilon_{i,k}} \sigma_{|i,j|}^{\varepsilon_{i,j}} : V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i \qquad \text{(YB}\pm\text{)}$$

where $\varepsilon_{\alpha,\beta} := -1$ if $\alpha > \beta$ (allowed only when $s \leqslant \alpha, \beta \leqslant t$) and 1 otherwise, $|\alpha, \beta| = (\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$, and the braidings σ are tensored with the identity on the left or on the right in the evident manner.

The condition (YB±) for the signs $\overline{\varepsilon} := (\varepsilon_{i,j}, \varepsilon_{i,k}, \varepsilon_{j,k}) = (-1, \alpha, \beta)$ results from (YB±) for $\overline{\varepsilon} = (1, \beta, \alpha)$: multiply the latter by $\sigma_{i,j}^{-1}$ on the left and on the right, and permute the subscripts i, j, k. (We do not speak about equivalence here since the σ's are not necessarily invertible.) The same works for $(\alpha, \beta, -1) \Leftarrow (\beta, \alpha, 1)$. This allows to forget the instances of (YB±) with the minus signs on the left or on the right, leaving just two cases:

1. $\overline{\varepsilon} = (1, 1, 1)$, where (YB±) holds by the definition of pre-braided system;
2. $\overline{\varepsilon} = (1, -1, 1)$, meaning $k < i \leqslant j \leqslant k$, which is impossible. □

On the level of multi-braided modules, the invertibility of $\sigma_{i,i+1}$ allows to interchange the components V_i and V_{i+1} of a pre-braided system without changing the module category:

Proposition 7.1.13. *Let $(r, \overline{V}, \overline{\sigma})$ be a pre-braided system in $\mathcal{C}$, with $\sigma_{i,i+1}$ invertible for a given i between 1 and $r-1$. Then the following categories of multi-braided modules are equivalent:*

$$\mathbf{Mod}_{(V_1,\ldots,V_i,V_{i+1},\ldots,V_r)} \simeq \mathbf{Mod}_{(V_1,\ldots,V_{i+1},V_i,\ldots,V_r)},$$

where the second pre-braided system inherits the pre-braidings $\overline{\sigma}$ except on $V_{i+1} \otimes V_i$, where we choose $\sigma_{i,i+1}^{-1}$.

Proof. The existence of the second pre-braided system from the proposition, which we denote by $\overline{V}'$, is guaranteed by proposition 7.1.12, since the set of occurrencies of YBE one has to check is a subset of those one has for the system $(V_1, \ldots, V_i \oplus V_{i+1}, \ldots, V_r)$. Further, given an object $M \in \mathrm{Ob}(\mathcal{C})$ equipped with morphisms $\rho_j : M \otimes V_j \to M$, the list of compatibility conditions (7.1) one has to check for $\overline{V}$ differs from the list for $\overline{V}'$ only in the conditions for components $i, i+1$:

$$\rho_{i+1} \circ (\rho_i \otimes \mathrm{Id}_{V_{i+1}}) = \rho_i \circ (\rho_{i+1} \otimes \mathrm{Id}_{V_i}) \circ (\mathrm{Id}_M \otimes \sigma_{i,i+1})$$

versus

$$\rho_i \circ (\rho_{i+1} \otimes \mathrm{Id}_{V_i}) = \rho_{i+1} \circ (\rho_i \otimes \mathrm{Id}_{V_{i+1}}) \circ (\mathrm{Id}_M \otimes \sigma_{i,i+1}^{-1}).$$

These two conditions are clearly equivalent. So the identity functor of $\mathcal{C}$ gives the demanded category equivalence. □

7.2 A protoexample: pre-braided systems of algebras

This section is devoted to a study of pre-braided systems whose components V_i have a structure of unital associative algebras, the pre-braidings $\sigma_{i,i}$ being our algebra pre-braiding σ_{Ass} or its right version σ^r_{Ass}. Such systems are proved to be in one-to-one correspondence with multi-braided tensor products of algebras, and multi-braided modules over such systems are shown to coincide with modules over the corresponding tensor product algebras. As a consequence, "invertibility" propositions 7.1.12 and 7.1.13 can be applied. Concrete examples illustrating the advantages of our braided system approach follow in subsequent sections.

Multi-braided tensor products of algebras

We start with showing that the tensor product of UAAs in a pre-braided category (cf. (5.7) - (5.8)) can be generalized to the setting of a pre-braided system. Recall the notion of naturality from definition 5.1.6, which we extend to families of morphisms which are not necessarily pre-braidings in the evident manner.

Theorem 7. *Take r UAAs $(V_i, \mu_i, \nu_i), 1 \leqslant i \leqslant r$, in a monoidal category $\mathcal{C}$, each unit ν_i being a part of a normalized pair (ν_i, ϵ_i), and, for each couple of subscripts $1 \leqslant i < j \leqslant r$, a morphism $\xi_{i,j}$ natural with respect to ν_i and ν_j. The following statements are then equivalent:*

1. *The morphisms*

$$\sigma_{i,i} := \sigma_{Ass} \qquad \forall i,$$
$$\sigma_{i,j} := \xi_{i,j} \qquad \forall i < j$$

define a pre-braided system structure on $\overline{V}$.

2. *Each $\xi_{i,j}$ is natural with respect to μ_i and μ_j, and, for each triple $i < j < k$, the ξ's satisfy the categorical Yang-Baxter equation on $V_i \otimes V_j \otimes V_k$.*
3. *A UAA structure on*
$$\overleftarrow{V} := V_r \otimes V_{r-1} \otimes \cdots \otimes V_1$$
can be defined by
$$\mu_{\overleftarrow{V}} := (\mu_r \otimes \mu_{r-1} \otimes \cdots \otimes \mu_1) \circ T^{\xi}_{\omega_{2r}}, \tag{7.2}$$
$$\nu_{\overleftarrow{V}} := \nu_r \otimes \nu_{r-1} \otimes \cdots \otimes \nu_1 \tag{7.3}$$
(cf. notations (2.2) *and* (2.4)*), with*
$$\omega_{2r} := \left(\begin{smallmatrix} 1 & 2 & \ldots & r & r+1 & r+2 & \ldots & 2r \\ 1 & 3 & \ldots & 2r-1 & 2 & 4 & \ldots & 2r \end{smallmatrix}\right) \in S_{2r}. \tag{7.4}$$

Proof. We show that points 1 and 3 are both equivalent to the (intermediate) point 2.

Start with 1. YBE on the $V_i \otimes V_i \otimes V_i$'s is automatic via theorem $5^{\mathbf{cat}}$. On $V_i \otimes V_i \otimes V_j, i < j$, YBE becomes
$$(\xi_{i,j} \otimes \mathrm{Id}_{V_i}) \circ (\mathrm{Id}_{V_i} \otimes \xi_{i,j}) \circ (\nu_i \otimes \mu_i \otimes \mathrm{Id}_{V_j}) =$$
$$(\mathrm{Id}_{V_j} \otimes \nu_i \otimes \mu_i) \circ (\xi_{i,j} \otimes \mathrm{Id}_{V_i}) \circ (\mathrm{Id}_{V_i} \otimes \xi_{i,j}),$$
or, graphically,

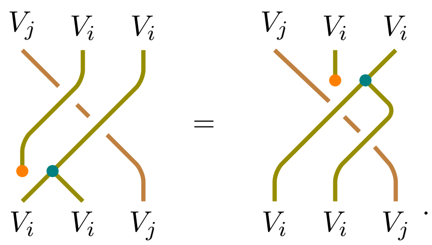


Figure 7.1: YBE for $V_i \otimes V_i \otimes V_j$

The naturality of $\xi_{i,j}$ with respect to the units permits to "pull" the short strand out of the crossing on the left diagram. The equation obtained is equivalent to $\xi_{i,j}$ being natural with respect to μ_i:

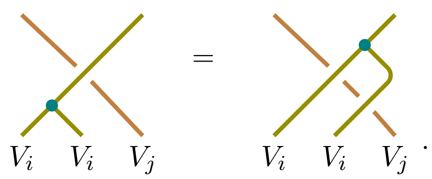


Figure 7.2: Naturality with respect to μ_i

(compose the equation above with $\mathrm{Id}_{V_j} \otimes \mu_i$, like in the proof of lemma 5.2.3, to get one of the implications).

Similarly, YBE on $V_i \otimes V_j \otimes V_j, i < j$, is equivalent to $\xi_{i,j}$ being natural with respect to μ_j. This terminates the proof of the equivalence $1 \Leftrightarrow 2$.

Let us now prove $3 \Leftrightarrow 2$. We use shortcut notations
$$\iota_j := \nu_r \otimes \cdots \otimes \nu_{j+1} \otimes \mathrm{Id}_{V_j} \otimes \nu_{j-1} \otimes \cdots \nu_1 : V_j \to \overleftarrow{V} \qquad \forall 1 \leqslant j \leqslant r. \tag{7.5}$$

Given a collection of $\xi_{i,j}$'s satisfying the consitions of point 2, one verifies (for instance graphically) that the morphisms from 3 define a UAA structure on $\overleftarrow{V}$. This is a generalization of the verifications usually made while defining the tensor product of algebras in a symmetric category. To show that all the conditions from 2 are indeed necessary, consider the associativity condition for $\mu_{\overleftarrow{V}}$ composed with

- either $\iota_i \otimes \iota_j \otimes \iota_k : V_i \otimes V_j \otimes V_k \to \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_t's on all the positions except for i, j, k on the left;
- or $\iota_i \otimes \iota_i \otimes \iota_j : V_i \otimes V_i \otimes V_j \to \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_t's on all the positions except for i, j on the left;
- or $\iota_i \otimes \iota_j \otimes \iota_j : V_i \otimes V_j \otimes V_j \to \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_t's on all the positions except for i, j on the left.

Using the naturality of the ξ's with respect to the units and the defining property of a normalized pair, in the first case one gets YBE for the ξ's on $V_i \otimes V_j \otimes V_k$ with $i < j < k$, in the second and third cases – the naturality of $\xi_{i,j}$ with respect to μ_i and μ_j respectively, with $i < j$. □

This proposition gives a "braided" (point 1), an "algebraic" (point 3) and a "mixed "(point 2) interpretations of the same phenomenon. In practice, it is often convenient to use points 1 or 2 in order to check the associativity of the multiplication $\mu_{\overleftarrow{V}}$.

Definition 7.2.1. A pre-braided system of the type described in the above theorem is called a *pre-braided system of UAAs*, and the UAA from the theorem is called the *multi-braided tensor product* of the UAAs $V_1, \ldots, V_r$, denoted by

$$\overleftarrow{V} = V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1.$$

Our notion of multi-braided tensor products recovers the ***iterated twisted tensor products*** of P.Jara Martínez, J.López Peña, F.Panaite and F. van Oystaeyen (cf. [32]). They show in particular that, using the language of our theorem, $2 \Rightarrow 3$. The role of the naturality of the ξ's with respect to the μ's is underlined in [7]. We make their results more precise, raise all the structures to an arbitrary monoidal category (they work in $\mathbf{Vect}_{\Bbbk}$), and, the most importantly, add a "fully braided" interpretation (point 1), necessary later on for studying homologies.

Remark 7.2.2. In the above proposition, one can replace the existence of the ϵ_i's, used only to prove $3 \Rightarrow 2$, by demanding the point 3 to hold for all subsystems of $\overline{V}$. In this case, while proving $3 \Rightarrow 2$, one can work with the appropriate subsystem instead of composing with the ϵ_i's in order to get to the desired tensor product. In particular, the existence of the ϵ_i's is not necessary for $r = 2$.

Remark 7.2.3. Some or all of the maps $\sigma_{i,i}$ can be replaced with a *right version* (in the sense of section 5.5)

$$\sigma_{i,i} := \sigma^r_{Ass} = \mu_i \otimes \nu_i,$$

or, in the graphical form, . The previous theorem still holds, with analogous proof.

Example 7.2.4. According to lemma 7.1.2, for a pre-braided category $\mathcal{C}$, the choice $\xi_{i,j} := c_{V_i,V_j}$ in the theorem above gives a pre-braided system. In addition, the c_{V_i,V_j}'s are natural with respect to everything hence in particular the units. The UAA structure on $\overleftarrow{V}$ is the usual tensor product of algebras in a pre-braided category in this case. We use the undecorated notation $\otimes$ in this setting.

Multi-braided modules as modules over algebras

The structure equivalence from theorem 7 has an important counterpart on the level of modules. A normalized version of multi-braided modules is necessary to formulate the result:

Definition 7.2.5. A multi-module M over a pre-braided system $(r, \overline{V}, \overline{\sigma})$ of UAAs is called *normalized* if the corresponding braided module structures (M, ρ_i) over each UAA V_i are normalized in the sense of (6.3). We use notation $\overline{\mathbf{Mod}}$ for the category of normalized multi-modules.

Proposition 7.2.6. *In the settings of theorem 7, the following categories are equivalent:*

$$\overline{\mathbf{Mod}}_{(V_1,\ldots,V_r)} \simeq \mathbf{Mod}_{V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \ldots \underset{\xi}{\otimes} V_1}.$$

The second category is the usual category of modules over a UAA.

Proof. According to remark 7.1.10 combined with example 6.1.2, a normalized multi-module M over the pre-braided system described in theorem 7 is a module (M, ρ_i) over each UAA V_i, these structures being compatible in the sense of (7.1). But this is the same thing as a module (M, ρ) over the UAA $V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1$: the correspondence is given by (using notation (7.5))

$$\rho_j := \rho \circ (\mathrm{Id}_M \otimes \iota_j),$$

$$\rho := \rho_r \circ (\rho_{r-1} \otimes \mathrm{Id}_{V_r}) \circ \cdots \circ (\rho_1 \otimes \mathrm{Id}_{V_2} \otimes \cdots \otimes \mathrm{Id}_{V_r}).$$

The identity functor of $\mathcal{C}$ and this structure correspondence give thus the desired category equivalence. □

This proposition recovers a result from [32]. As usual, our main contribution, besides generalizing the context to that of a monoidal category, consists in the interpretation of module structures over the algebras V_i in terms of braided modules, while in the cited paper the braidings appear only in the study of the interactions of the V_i- and the V_j-module structures for different i and j. The advantages of our approach will be visible on the homology level.

Consider now the situation when one of the $\xi_{i,i+1}$'s is invertible. In particular, propositions 7.1.12 and 7.1.13 are applicable.

Proposition 7.2.7. *In the settings of theorem 7, suppose one of the $\xi_{i,i+1}$'s invertible. Then*

1. *UAAs $V_1, \ldots, V_{i-1}, V_{i+1}, V_i, V_{i+2} \ldots, V_r$ endowed with the ξ's one had for the system $\overline{V}$, completed by $\xi_{i,i+1}^{-1}$ on $V_{i+1} \otimes V_i$, still form a pre-braided system of UAAs.*
2. *Further, the map*

$$\mathrm{Id}_{V_r} \otimes \ldots \otimes \mathrm{Id}_{V_{i+2}} \otimes \xi_{i,i+1}^{-1} \otimes \mathrm{Id}_{V_{i-1}} \otimes \ldots \otimes \mathrm{Id}_{V_1},$$

abusively denoted by $\xi_{i,i+1}^{-1}$, gives an algebra isomorphism between the multi-braided UAA tensor products $\overleftarrow{V}$ and

$$\tau_i(\overleftarrow{V}) := V_r \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_{i+2} \underset{\xi}{\otimes} V_i \underset{\xi^{-1}}{\otimes} V_{i+1} \underset{\xi}{\otimes} V_{i-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1$$

(the notation is abusive as well).

3. The last isomorphism is compatible with the category equivalence

$$\mathbf{Mod}_{\overleftarrow{V}} \simeq \overline{\mathbf{Mod}}_{(V_1,\ldots,V_i,V_{i+1},\ldots,V_r)} \simeq \overline{\mathbf{Mod}}_{(V_1,\ldots,V_{i+1},V_i,\ldots,V_r)} \simeq \mathbf{Mod}_{\tau_i(\overleftarrow{V})},$$
$$(M, \rho_{\overleftarrow{V}}) \leftrightarrow (M, \rho_{\tau_i(\overleftarrow{V})}),$$

in the sense that

$$\rho_{\overleftarrow{V}} = \rho_{\tau_i(\overleftarrow{V})} \circ (\mathrm{Id}_M \otimes \xi_{i,i+1}^{-1}).$$

Proof. 1. Proposition 7.1.12 allows to interchange the components V_i and V_{i+1} of the pre-braided system $(V_1, \ldots, V_r)$ from point 1 of theorem 7. The new pre-braided system $(V_1, \ldots, V_{i+1}, V_i, \ldots, V_r)$ then satisfies again the conditions of point 1 from theorem 7. Moreover, $\xi_{i,i+1}^{-1}$ is natural with respect to the units since so is $\xi_{i,i+1}$. One thus gets the desired pre-braided system of UAAs.

2. Theorem 7 (point 3) then gives the multi-braided UAA tensor product $\tau_i(\overleftarrow{V})$. Applying YBE several times, one shows that, in order to see that $\xi_{i,i+1}^{-1}$ is an algebra morphism, it is sufficient to work with V_i and V_{i+1} only. Namely, one has to prove

$$\xi_{i,i+1}^{-1} \circ (\nu_{i+1} \otimes \nu_i) = \nu_i \otimes \nu_{i+1},$$

which follows from the naturality with respect to the units, and

$$(\mu_i \otimes \mu_{i+1}) \circ (\mathrm{Id}_i \otimes \xi_{i,i+1}^{-1} \otimes \mathrm{Id}_{i+1}) \circ (\xi_{i,i+1}^{-1} \otimes \xi_{i,i+1}^{-1}) =$$
$$\xi_{i,i+1}^{-1} \circ (\mu_{i+1} \otimes \mu_i) \circ (\mathrm{Id}_{i+1} \otimes \xi_{i,i+1} \otimes \mathrm{Id}_i) :$$
$$(V_{i+1} \otimes V_i)^{\otimes 2} \to V_i \otimes V_{i+1},$$

or, graphically,

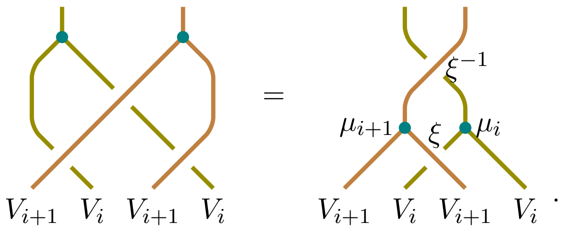


Figure 7.3: $\xi_{i,i+1}^{-1}$ is an algebra morphism

This relation follows from the naturality of $\xi_{i,i+1}$ (and hence $\xi_{i,i+1}^{-1}$) with respect to μ_i and μ_{i+1} (point 2 of theorem 7).

3. The equivalence of module categories is a consequence of propositions 7.2.6 and 7.1.13, and their proofs. □

7.3 A toy example: algebra bimodules

We illustrate the general theory from previous sections by the quite elementary example of algebra bimodules and enveloping algebras. It is certainly much faster to verify the results obtained here directly, but in more complicated settings (that of Hopf bimodules for instance) similar ideas allow to avoid technical calculations and give useful intuitions.

Algebra bimodules as modules over the enveloping algebra

Take two UAAs (V, μ, ν) and (V', μ', ν') in a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. The associated pre-braided object for V is (V, σ_{Ass}), and similarly for V'. Remark further that $(\mu \circ c, \nu)$ is another UAA structure on V, giving a pre-braiding

$$\sigma_{Ass}^{op} := \nu \otimes (\mu \circ c).$$

Denote by V^{op} the object V endowed with this modified UAA structure. This "twisted" multiplication provides a useful transition between left and right module structures:

Lemma 7.3.1. Given a UAA (V, μ, ν) in a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, the following functors give an equivalence of module categories:

$$\mathbf{Mod}_{V^{op}} \simeq {}_V\mathbf{Mod},$$
$$(M, \rho) \mapsto (M, \lambda(\rho) := \rho \circ c^{-1}_{M,V}), \tag{7.6}$$
$$(M, \rho(\lambda) := \lambda \circ c_{M,V}) \leftarrow\!\shortmid (M, \lambda). \tag{7.7}$$

Next, according to lemma 7.1.2, the data

$$V_1 := V, \qquad V_2 := V'^{op},$$
$$\sigma_{1,1} := \sigma_{Ass},$$
$$\sigma_{2,2} := \sigma_{Ass}^{op},$$
$$\sigma_{1,2} := c_{V,V'}$$

define a pre-braided system structure.

Remark 7.1.10 combined with example 6.1.2 show that two morphisms

$$\rho : M \otimes V_1 = M \otimes V \to M,$$
$$\rho' : M \otimes V_2 = M \otimes V' \to M,$$

both normalized in the sense of (6.3), define a right (V, V'^{op})-module M if and only if they are both algebra actions, compatible in the sense of (7.1):

$$\rho' \circ (\rho \otimes \mathrm{Id}_{V'^{op}}) = \rho \circ (\rho' \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes c_{V,V'}).$$

In terms of the correspondence from lemma 7.3.1, it means precisely that ρ and $\lambda(\rho')$ define an *algebra bimodule* in the usual categorical sense, $M \in {}_{V'}\mathbf{Mod}_V$.

One thus gets an interpretation of algebra bimodules in terms of multi-modules over a pre-braided system of UAAs. Recall proposition 7.2.6, which suggests another interpretation via modules over a multi-braided tensor product of UAAs, and proposition 7.2.7 allowing to interchange the components V_1 and V_2. Put together, these results give

Proposition 7.3.2. *Take two UAAs (V, μ, ν) and (V', μ', ν') in a braided category $\mathcal{C}$. Recall the pre-braided system (V, V'^{op}) defined above. The following categories are equivalent:*

$$\mathbf{Mod}_{V'^{op} \underset{c}{\otimes} V} \simeq \overline{\mathbf{Mod}}_{(V, V'^{op})} \simeq {}_{V'}\mathbf{Mod}_V \simeq \overline{\mathbf{Mod}}_{(V'^{op}, V)} \simeq \mathbf{Mod}_{V \underset{c^{-1}}{\otimes} V'^{op}}.$$

The case $V' = V$ gives the familiar ***enveloping algebra*** of a UAA V:

$$V^e := V \otimes V^{op}.$$

Bar complex with coefficients in a bimodule

We finish by applying proposition 7.1.11 to our bimodule context, choosing $s = t = 1$. Recall notations φ_i from (1.3).

Proposition 7.3.3. *Take a bimodule $(M, \rho : M \otimes V \to M, \lambda : V' \otimes M \to M)$ over two UAAs V and V' in a braided category $\mathcal{C}$. The bar complex for V with coefficients in (M, ρ) on the left, i.e. $(M \otimes T(V), {}^{\rho}d)$, is a complex in ${}_{V'}\mathbf{Mod}_V$, i.e. the differentials $({}^{\rho}d)_n$ are bimodule morphisms, the bimodule structure on $M \otimes V^{\otimes n}$ being given by*

$$\rho_{bar} := \mu_{n+1} : M \otimes V^{\otimes n} \otimes V \to M \otimes V^{\otimes n},$$
$$\lambda_{bar} := \lambda_1 : V' \otimes M \otimes V^{\otimes n} \to M \otimes V^{\otimes n}.$$

Proof. Plug the pre-braiding for the system (V, V'^{op}) decribed in the beginning of this section (with, in particular, $\sigma_{V,V} = \sigma_{Ass}$ and $\sigma_{V,V'} = c_{V,V'}$) into the formulas from proposition 7.1.11. Further, recall the correspondence between bimodules and normalized right multi-modules (lemma 7.3.1). This gives

$$\begin{aligned}
{}^{\rho}\pi_1 &= \rho_1 \circ (\mathrm{Id}_M \otimes \overline{\boldsymbol{\sigma}}_{V^{\otimes n},V}) \\
&= \mathrm{Id}_M \otimes \mathrm{Id}_V^{\otimes(n-1)} \otimes \mu : \; M \otimes V^{\otimes n} \otimes V \to M \otimes V^{\otimes n}, \\
\lambda &= {}^{\rho'}\pi_2 \circ c^{-1}_{M \otimes V^{\otimes n}, V'} \\
&= (\lambda \circ c_{M,V'})_1 \circ (\mathrm{Id}_M \otimes \overline{\boldsymbol{\sigma}}_{V^{\otimes n},V'}) \circ c^{-1}_{M \otimes V^{\otimes n}, V'} \\
&= (\lambda \circ c_{M,V'})_1 \circ (\mathrm{Id}_M \otimes c_{V^{\otimes n},V'}) \circ c^{-1}_{M \otimes V^{\otimes n}, V'} \\
&= \lambda_1 : \; V' \otimes M \otimes V^{\otimes n} \to M \otimes V^{\otimes n}.
\end{aligned}$$
□

This bimodule structure on the bar complex is important for one of the methods of obtaining the ***Hochschild cohomology***.

7.4 The first real example: two-sided crossed products

We reinterpret here F.Panaite's example (cf. [65]), consisting in applying some kind of "braided" techniques to a study of two-sided crossed products $A\#H\#B$ (or the generalized two-sided crossed products $A \blacktriangleright\!\!< C >\!\!\blacktriangleleft B$, defined by D.Bulacu, F.Panaite and F.Van Oystaeyen in. [5]). We recover his techniques as a particular case of our general tools, and we automatically obtain (via proposition 7.2.7) six equivalent versions of the algebra $A\#H\#B$. Moreover, we raise all the constructions to an arbitrary symmetric category. Pursuing further the "braided" ideas and using our results on adjoint multi-braided modules, we get, in the same settings, a bimodule structure

$$C^{\otimes n} \in {}_B\mathbf{Mod}_A,$$

applied in section 7.6 to the study of bialgebras.

Categorical bialgebras and module algebras

We need the categorical versions of some familiar algebraic notions:

Definition 7.4.1. ⇝ A *bialgebra* structure in a pre-braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, sometimes called a *(pre-)braided bialgebra*, is a unital associative algebra structure (μ, ν)

and a counital coassociative coalgebra structure (Δ, ε) for an object H, compatible in the following sense:

$$\begin{aligned} \Delta \circ \mu &= (\mu \otimes \mu) \circ c_2 \circ (\Delta \otimes \Delta) : H \otimes H \to H \otimes H, \qquad (7.8)\\ \Delta \circ \nu &= \nu \otimes \nu : \mathbf{I} \to H \otimes H,\\ \varepsilon \circ \mu &= \varepsilon \otimes \varepsilon : H \otimes H \to \mathbf{I},\\ \varepsilon \circ \nu &= \mathrm{Id}_{\mathbf{I}} : \mathbf{I} \to \mathbf{I}. \end{aligned}$$

- If moreover H has an *antipode*, i.e. a morphism $s : H \to H$ satisfying

$$\mu \circ (s \otimes \mathrm{Id}_H) \circ \Delta = \mu \circ (\mathrm{Id}_H \otimes s) \circ \Delta = \nu \circ \varepsilon, \qquad (\mathrm{s})$$

  then it is called a *Hopf algebra* in $\mathcal{C}$.
- For a bialgebra H in $\mathcal{C}$, a left H*-module algebra* is a UAA structure (M, μ_M, ν_M) and a left H-module structure $(M, \lambda : H \otimes M \to M)$ on an $M \in \mathrm{Ob}(\mathcal{C})$, such that μ_M and ν_M are morphisms of left H-modules:

$$\begin{aligned} \lambda \circ (\mathrm{Id}_H \otimes \mu_M) &= \mu_M \circ (\lambda \otimes \lambda) \circ c_2 \circ (\Delta \otimes \mathrm{Id}_M^{\otimes 2}), \qquad (7.9)\\ \lambda \circ (\mathrm{Id}_H \otimes \nu_M) &= \nu_M \circ \varepsilon. \qquad (7.10) \end{aligned}$$

  Right H-module algebras, left and right H-comodule algebras, and H-bi(co)module algebras are defined similarly.

Graphically, the bialgebra compatibility condition (7.8) means

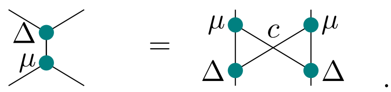


Figure 7.4: Bialgebra relation

See figure 7.11 for a graphical depiction of (7.9) and (7.10).

In $\mathbf{Mod}_R$, (7.8) takes the familiar form

$$(hg)_{(1)} \otimes (hg)_{(2)} = h_{(1)}g_{(1)} \otimes h_{(2)}g_{(2)} \qquad \forall h, g \in H,$$

and (7.9) becomes

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b) \qquad \forall h \in H,\ a, b \in M,$$

where the notations of the multiplication μ_M and of the action λ are omitted.

Two-sided crossed products as multi-braided tensor products

Everything is now ready for handling generalized two-sided crossed product.

Proposition 7.4.2. *Take a bialgebra H, a left H-module algebra (A, λ), a right H-module algebra (B, ρ) and an H-bicomodule algebra $(C, \delta_l : C \to H \otimes C, \delta_r : C \to C \otimes H)$ in a* <u>*symmetric*</u> *category $\mathcal{C}$. Then*

1. *The UAAs (B, C, A) together with morphisms*

$$\begin{aligned} \xi_{1,2} &= (\mathrm{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_B \otimes \delta_r),\\ \xi_{2,3} &= (\lambda \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_H \otimes c_{C,A}) \circ (\delta_l \otimes \mathrm{Id}_A),\\ \xi_{1,3} &= c_{B,A} \end{aligned}$$

   form a pre-braided system of UAAs, and formulas (7.2)-(7.3) define a UAA structure on $A \otimes C \otimes B$.

2. *The category of modules over the algebra defined this way is equivalent to the category of multi-braided normalized modules* $\overline{\mathbf{Mod}}_{(B,C,A)}$*:*

$$\overline{\mathbf{Mod}}_{(B,C,A)} \simeq \mathbf{Mod}_{A \underset{\xi}{\otimes} C \underset{\xi}{\otimes} B}. \tag{7.11}$$

Proof. The key point is to notice that the ξ's satisfy the conditions of the point 2 of theorem 7. Indeed,

- ✓ YBE on $B \otimes C \otimes A$ follows from the compatibility between the left and the right H-coactions on C;
- ✓ the naturality of the ξ's with respect to μ_C is a consequence of the defining properties of H-bicomodule algebras for C;
- ✓ the naturality of the ξ's with respect to μ_A and μ_B can be deduced from the defining properties of H-module algebras for A and B.

As an example, we show in detail that $\xi_{1,2}$ is natural with respect to μ_B:

$$\begin{aligned}
&\xi_{1,2} \circ (\mu_B \otimes \mathrm{Id}_C)\\
\overset{1}{=}\ &(\mathrm{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \mathrm{Id}_H) \circ (\mu_B \otimes \delta_r)\\
\overset{2}{=}\ &(\mathrm{Id}_C \otimes \rho) \circ (\mathrm{Id}_C \otimes \mu_B \otimes \mathrm{Id}_H) \circ (c_{B\otimes B,C} \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_B^{\otimes 2} \otimes \delta_r)\\
\overset{3}{=}\ &(\mathrm{Id}_C \otimes \mu_B) \circ (\mathrm{Id}_C \otimes \rho \otimes \rho) \circ (\mathrm{Id}_{C\otimes B} \otimes c_{B,H} \otimes \mathrm{Id}_H) \circ\\
&(c_{B\otimes B,C} \otimes \Delta_H) \circ (\mathrm{Id}_B^{\otimes 2} \otimes \delta_r)\\
\overset{4}{=}\ &(\mathrm{Id}_C \otimes \mu_B) \circ (\mathrm{Id}_C \otimes \rho \otimes \mathrm{Id}_B) \circ (c_{B,C} \otimes \mathrm{Id}_{H\otimes B}) \circ (\mathrm{Id}_B \otimes \delta_r \otimes \rho) \circ\\
&(\mathrm{Id}_B \otimes c_{B,C} \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_B^{\otimes 2} \otimes \delta_r)\\
\overset{5}{=}\ &(\mathrm{Id}_C \otimes \mu_B) \circ (\xi_{1,2} \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_B \otimes \xi_{1,2}),
\end{aligned}$$

where we use

1. the definition of $\xi_{1,2}$,
2. the naturality of c,
3. the defining property of right H-module algebra for B,
4. the defining property of right H-comodule for C and the naturality of c,
5. the definition of $\xi_{1,2}$.

The reader is advised to draw diagrams in order to better follow these verifications.

Further, the naturality with respect to units follows from the defining properties of H-(co)module algebras as well. Point 1 from theorem 7 then confirms that the ξ's together with the σ_{Ass}'s form a pre-braiding, while point 3 proves the associativity of the multiplication (7.2).

Finally, proposition 7.2.6 gives the required category equivalence □

The tensor product algebra from the proposition is known as the ***generalized two-sided crossed product*** (cf. [5])

$$A \blacktriangleright\!\!< C >\!\!\blacktriangleleft B := A \underset{\xi}{\otimes} C \underset{\xi}{\otimes} B.$$

The choice $C = H$ (with both comodule structures given by Δ_H) gives the ***two-sided crossed product*** of F.Hausser and F.Nill (cf. [31]), usually denoted by

$$A \# H \# B := A \underset{\xi}{\otimes} H \underset{\xi}{\otimes} B.$$

Two-component multi-braided tensor products of UAAs

$$A\#H := A \underset{\xi}{\otimes} H, \qquad H\#B := H \underset{\xi}{\otimes} B$$

are called left and right ***crossed (or smash) products*** respectively, with a generalized version for an arbitrary left and, respectively, right H-comodule C.

We have thus obtained an alternative conceptual proof of the associativity of $A{\blacktriangleright\!\!<}C{>\!\!\blacktriangleleft}B$ and of the category equivalence (7.11), otherwise very technical.

Remark 7.4.3. If H is a Hopf algebra with an invertible antipode s, then all the ξ's are invertible:

$$\begin{aligned}
\xi_{1,2}^{-1} &= ((\rho \circ c_{H,B}) \otimes \mathrm{Id}_C) \circ (s^{-1} \otimes c_{C,B}) \circ ((c_{C,H} \circ \delta_r) \otimes \mathrm{Id}_B), \\
\xi_{2,3}^{-1} &= (\mathrm{Id}_C \otimes (\lambda \circ c_{A,H})) \circ (c_{A,C} \otimes s^{-1}) \circ (\mathrm{Id}_A \otimes (c_{H,C} \circ \delta_l)), \\
\xi_{1,3}^{-1} &= c_{A,B}.
\end{aligned}$$

Proposition 7.2.7 then allows to permute components of $A \underset{\xi}{\otimes} C \underset{\xi}{\otimes} B$, giving six pairwise isomorphic UAAs, these isomorphisms being compatible with the equivalences of their module categories. In particular, one recovers the algebra isomorphisms from [31]:

$$A\#H\#B \simeq (A \otimes B) \bowtie H.$$

Remark 7.4.4. Supposing the category $\mathcal{C}$ moreover additive, one can start with an H-bimodule (C', δ_l, δ_r) and introduce an artificial trivial UAA structure by adding the formal unit

$$C := C' \oplus \mathbf{I},$$

taking the zero multiplication on C' and making $\nu_C := \mathrm{Id}_{\mathbf{I}} : \mathbf{I} \to C$ a unit. The bicomodule structure on C' extended to C by putting

$$\delta_l|_{\mathbf{I}} := \nu_H \otimes \nu_C, \qquad \delta_r|_{\mathbf{I}} := \nu_C \otimes \nu_H$$

endows C with an H-bicomodule algebra structure.

This formal construction will be useful in what follows.

Adjoint actions

We finish this example by applying the theory of adjoint multi-modules (cf. proposition 7.1.11) to the pre-braided system of UAAs from proposition 7.4.2, choosing trivial coefficients ($M = \mathbf{I}$).

Start with a preliminary general observation:

Lemma 7.4.5. Take a pre-braided system $((V_1, \ldots, V_r), \overline{\sigma})$ in a symmetric additive category $\mathcal{C}$, with the component $\sigma_{1,r}$ being simply the underlying symmetric braiding c_{V_1,V_r} of $\mathcal{C}$. Take further two braided characters $\overline{\epsilon}$ and $\overline{\zeta}$ for this pre-braided system. Then the right braided V_r-module structure ${}^{\epsilon}\pi_r$ and the left braided V_1-module structure π_1^{ζ} on $T(\overline{V})_n^{\rightarrow}$, $n \in \mathbb{N}$, commute:

$${}^{\epsilon}\pi_r \circ (\pi_1^{\zeta} \otimes \mathrm{Id}_{V_r}) = \pi_1^{\zeta} \circ (\mathrm{Id}_{V_1} \otimes {}^{\epsilon}\pi_r) : V_1 \otimes T(\overline{V})_n^{\rightarrow} \otimes V_r \to T(\overline{V})_n^{\rightarrow}.$$

Proof. The symmetric braiding c is natural with respect to everything, in particular to the components of $\overline{\epsilon}$ and $\overline{\zeta}$. This is sufficient to show that the two morphisms from the desired equality coincide with

$$(\epsilon_r \otimes \mathrm{Id}_{T(\overline{V})_{\overrightarrow{n}}} \otimes \zeta_1) \circ T^{\overline{\sigma}}_{p_n},$$

where $p_n := \left(\begin{smallmatrix} 1 & 2 & \dots & n & n+1 & n+2 \\ n+2 & 2 & \dots & n & n+1 & 1 \end{smallmatrix}\right) \in S_{n+2}$ (recall notations (2.4) and (2.2)). □

Now return to the two-sided crossed products. Recall notations (1.3) and (1.4).

Proposition 7.4.6. *In a symmetric additive category $\mathcal{C}$, take a bialgebra H, a left H-module algebra (A, λ) and a right H-module algebra (B, ρ), endowed with algebra characters ϵ_A and ϵ_B respectively. Take moreover an H-bicomodule algebra (C, δ_l, δ_r). The tensor powers of C become bimodules, $C^{\otimes n} \in {}_B\mathbf{Mod}_A$ $\forall n \in \mathbb{N}$, via the formulas*

$$\begin{aligned} {}^{\epsilon_A}\pi &= (\epsilon_A)_1 \circ \lambda_1 \circ (\mathrm{Id}_H \otimes c_{C^{\otimes n},A}) \circ \mu^{n-1} \circ ((\omega_{2n}^{-1} \circ \delta_l^{\otimes n}) \otimes \mathrm{Id}_A) : \\ &\quad C^{\otimes n} \otimes A \to C^{\otimes n}, \\ \pi^{\epsilon_B} &= (\epsilon_B)_{n+1} \circ \rho_{n+1} \circ (c_{B,C^{\otimes n}} \otimes \mathrm{Id}_H) \circ (\mu^{n-1})_{n+2} \circ (\mathrm{Id}_B \otimes (\omega_{2n}^{-1} \circ \delta_r^{\otimes n})) : \\ &\quad B \otimes C^{\otimes n} \to C^{\otimes n}, \end{aligned}$$

where $\omega_{2n}^{-1} \in S_{2n}$ from (7.4) acts on tensor products of copies of C and H via the symmetric braiding c.

These actions are graphically depicted as

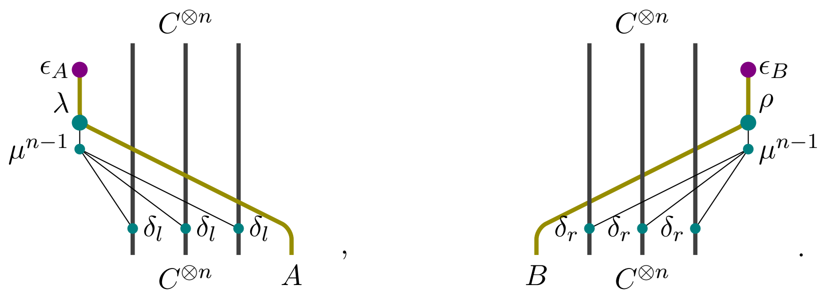


Figure 7.5: ${}_B\mathbf{Mod}_A$ structure on $C^{\otimes n}$

Proof. Proposition 7.1.11 and its right version applied to the pre-braided system (B, C, A) from proposition 7.4.2 and to the algebra characters (hence braided characters) ϵ_A and ϵ_B give a right braided A-module structure and a left braided B-module structure on $C^{\otimes n}$. One verifies that they coincide with the structures given here. Further, since the $\xi_{1,2}$ and $\xi_{2,3}$ components of the pre-braiding on (B, C, A) are natural with respect to the units, these braided modules are normalized, and thus, according to example 6.1.2, they are algebra modules over the corresponding UAAs. It remains to show that the actions of A and B commute. But this is precisely the assertion of lemma 7.4.5 in our setting. □

7.5 Yetter-Drinfel′d systems

Here we describe quite a general pre-braided system including as particular cases pre-braided systems for the following structures: bialgebras, Hopf algebras, Hopf and Yetter-Drinfel′d modules. The key idea is to take up theorem 7 and to choose the well-known braiding for Yetter-Drinfel′d modules as the $\xi_{i,j}$ components of a pre-braiding for a system of UAAs.

Fix a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$.

A reminder on Yetter-Drinfel′d modules

Yetter-Drinfel′d modules were introduced by D.Yetter in [85] under the name of "crossed bimodules" and rediscovered later by different authors under different names. The pre-braiding on the category of Yetter-Drinfel′d modules over a fixed bialgebra has also seen multiple inventors, and no fixed name in literature. We propose the name ***Woronowicz pre-braiding***, since S.L.Woronowicz was (probably) the first to discover this structure (cf. [83]).

Concretely,

Definition 7.5.1. A *left-right Yetter-Drinfel′d (or YD) module* structure over a bialgebra H in $\mathcal{C}$ consists of a left H-module structure λ and a right H-comodule structure δ on an object V, satisfying the *Yetter-Drinfel′d (or YD) compatibility condition*

$$(\mathrm{Id}_V \otimes \mu_H) \circ (\delta \otimes \mathrm{Id}_H) \circ c_{H,V} \circ (\mathrm{Id}_H \otimes \lambda) \circ (\Delta_H \otimes \mathrm{Id}_V) = \tag{YD}$$

$$(\lambda \otimes \mu_H) \circ (\mathrm{Id}_H \otimes c_{H,V} \otimes \mathrm{Id}_H) \circ (\Delta_H \otimes \delta).$$

The category of left-right YD modules over a bialgebra H (with, as morphisms, those preserving the H-module and H-comodule structures) is denoted by ${}_H\mathbf{YD}^H$.

In the graphical form, (YD) becomes

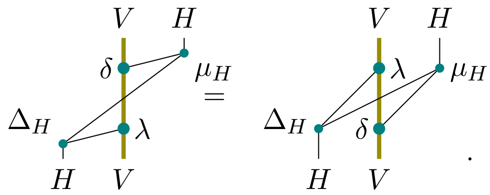


Figure 7.6: Left-right Yetter-Drinfel′d compatibility condition

It takes in $\mathbf{Mod}_R$ the familiar form

$$(h_{(2)}v)_{(0)} \otimes (h_{(2)}v)_{(1)}h_{(1)} = h_{(1)}v_{(0)} \otimes h_{(2)}v_{(1)} \quad \forall h \in H, v \in V.$$

The importance of YD modules in the theory of YBE solutions comes from the following

Lemma 7.5.2. The category ${}_H\mathbf{YD}^H$ can be endowed with the pre-braiding

$$\sigma_{YD} := c_{V,W} \circ (\mathrm{Id}_V \otimes \lambda_W) \circ (\delta_V \otimes \mathrm{Id}_W) : V \otimes W \to W \otimes V, \tag{7.12}$$

where $V, W \in \mathrm{Ob}({}_H\mathbf{YD}^H)$.

Definition 7.5.3. We call the pre-braiding σ_{YD} *Woronowicz pre-braiding*.

In the graphical form, σ_{YD} is depicted as

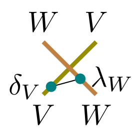
$\sigma_{YD} =$  .

Figure 7.7: Woronowicz pre-braiding for left-right YD modules

In $\mathbf{Mod}_R$, this pre-braiding becomes

$$\sigma_{YD} : v \otimes w \longmapsto v_{(1)} w \otimes v_{(0)}, \qquad \forall v \in V, w \in W.$$

The notion of Yetter-Drinfel′d module is often considered in the ***left-left version***: it is a left H-module and a left H-comodule structures (V, λ, δ) over a Hopf algebra H in a symmetric category $\mathcal{C}$, with the following compatibility condition:

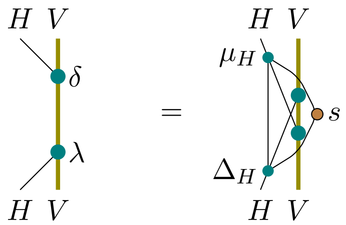


Figure 7.8: Left-left YD module

or, in an equivalent but less familiar antipode-free form,

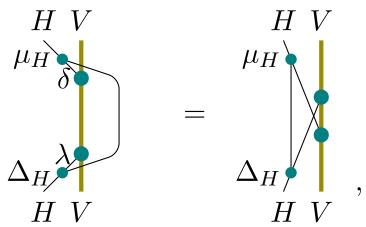


Figure 7.9: Left-left YD module II

which in $\mathbf{Mod}_R$ become

$$(hv)_{(-1)} \otimes (hv)_{(0)} = h_{(1)} v_{(-1)} s(h_{(3)}) \otimes h_{(2)} v_{(0)} \quad \forall h \in H, v \in V,$$

and, respectively,

$$(h_{(1)} v)_{(-1)} h_{(2)} \otimes (h_{(1)} v)_{(0)} = h_{(1)} v_{(-1)} \otimes h_{(2)} v_{(0)} \quad \forall h \in H, v \in V.$$

The category of left-left YD modules ${}^H_H\mathbf{YD}$ is also pre-braided, with the pre-braiding given by

$$(\lambda_W \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_H \otimes c_{V,W}) \circ (\delta_V \otimes \mathrm{Id}_W), \qquad V, W \in \mathrm{Ob}({}^H_H\mathbf{YD}), \tag{7.13}$$

or, in the graphical form,

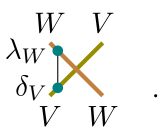
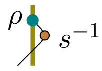


Figure 7.10: Woronowicz pre-braiding for left-left YD modules

Right-right and right-left Yetter-Drinfel′d modules, with the corresponding pre-braidings, can be defined by applying the right-left duality from section 5.5 to the preceding structures.

Remark 7.5.4. If H is a Hopf algebra with an invertible antipode s, then, say, the right-right and left-right YD module structures are equivalent, with the following relation between left and right actions:

$$\lambda = \rho \circ (\mathrm{Id}_V \otimes s^{-1}) \circ c_{H,V} : H \otimes V \to V \;=$$



One checks that the pre-braiding for right-right H-YD modules gives under this equivalence a pre-braiding for left-right H-YD modules which is the inverse of the σ_{YD} from lemma 7.5.2.

Note that in the category $\mathbf{vect}_{\Bbbk}$, the antipode of a Hopf algebra is always invertible (cf. [76]). However, even in the finite-dimensional $\Bbbk$-linear setting, we will often explicitly demand the invertibility of s when we use it, having in mind possible generalizations.

Yetter-Drinfel′d system as an example of a pre-braided system of algebras

Let $H \in \mathrm{Ob}(\mathcal{C})$ be endowed with a unital associative algebra structure (μ, ν) and a counital coassociative coalgebra structure (Δ, ε). No compatibility conditions are required.

In the pictures that follow, thin lines stand for H, thick lines for the V_i's, and dashed lines for the dual H^*. All crossings depict the symmetric braiding c. Notation $\mathrm{Id}_i := \mathrm{Id}_{V_i}$ is often used for brevity.

Definition 7.5.5. A *(left-right) Yetter-Drinfel′d system* over H (or an H-YD system) in $\mathcal{C}$ is an ordered finite family $V_1, V_2, \ldots, V_r$ of objects endowed with the following structure:

- unital associative algebra structures (μ_i, ν_i) on all the V_i's;
- left H-module structures $\lambda_i : H \otimes V_i \to V_i$ on $V_2, \ldots, V_r$;
- right H-comodule structures $\delta_i : V_i \to V_i \otimes H$ on $V_1, \ldots, V_{r-1}$;

satisfying the following conditions:

- ✓ algebra structures on the V_i's are compatible with H-module and H-comodule structures (when defined), in the sense of left H^{cop}*-module algebras* and right H*-comodule algebras* (cf. definition 7.4.1);
- ✓ H-module and H-comodule structures on each V_i, $2 \leqslant i \leqslant r-1$ satisfy the left-right Yetter-Drinfel′d compatibility condition.

For the reader's convenience, we give here a graphical form of the compatibilites of the algebra structures and the H-(co)module structures demanded in the definition:

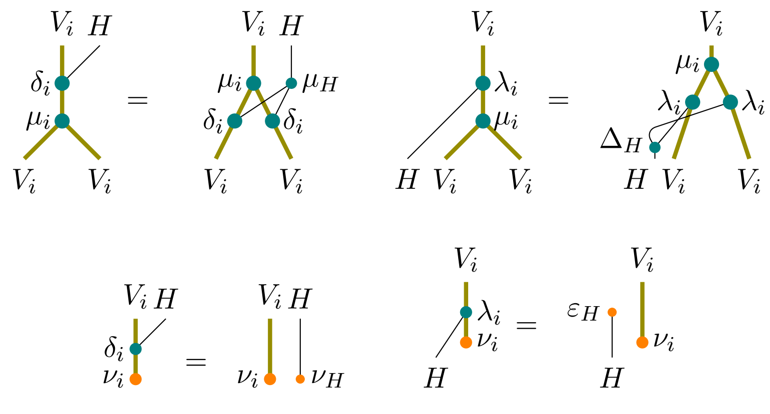


Figure 7.11: Compatibilities between UAA and YD structures

or, in the language of formulas,

$$\delta_i \circ \mu_i = (\mu_i \otimes \mu_H) \circ (\mathrm{Id}_i \otimes c_{H,V_i} \otimes \mathrm{Id}_H) \circ (\delta_i \otimes \delta_i),$$
$$\lambda_i \circ (\mathrm{Id}_H \otimes \mu_i) = \mu_i \circ (\lambda_i \otimes \lambda_i) \circ (c_{H,H \otimes V_i} \otimes \mathrm{Id}_i) \circ (\Delta_H \otimes \mathrm{Id}_i \otimes \mathrm{Id}_i),$$
$$\delta_i \circ \nu_i = \nu_i \otimes \nu_H,$$
$$\lambda_i \circ (\mathrm{Id}_H \otimes \nu_i) = \nu_i \circ \varepsilon_H.$$

Now we show how to endow a YD system with a pre-braiding.

Theorem 8. *A pre-braiding can de defined on a Yetter-Drinfel′d system* $(V_1, \dots V_r)$ *over* H *by*

$$\sigma_{i,i} := \sigma_{Ass} = \nu_i \otimes \mu_i : V_i \otimes V_i \longrightarrow V_i \otimes V_i$$
$$\sigma_{i,j} := \sigma_{YD} := c_{V_i,V_j} \circ (\mathrm{Id}_i \otimes \lambda_j) \circ (\delta_i \otimes \mathrm{Id}_j) : V_i \otimes V_j \longrightarrow V_j \otimes V_i, \ i < j.$$

The graphical form of this pre-braiding is

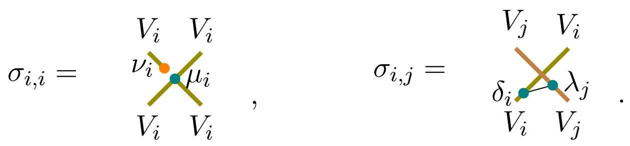


Figure 7.12: Pre-braiding for a Yetter-Drinfel′d system

The following lemma is used in the proof and afterwards:

Lemma 7.5.6. The Woronowicz pre-braiding σ_{YD} is natural with respect to the units.

Proof. Use the compatibility of the H-(co)module structures with the units ν_i, the triviality of the action by ν and of the coaction composed with ε, and the naturality of c. □

Proof of the theorem. According to theorem 7 and taking lemma 7.5.6 into consideration, only three properties remain to be verified.

➻ $\sigma_{i,j}$ is natural with respect to μ_i, $\quad i < j$.
Graphically it means

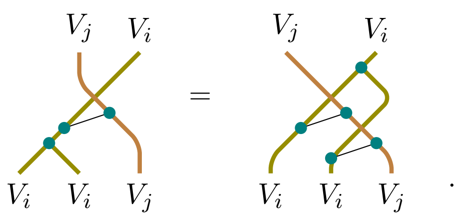


Figure 7.13: Naturality with respect to μ_i.

We give a detailed graphical proof of this relation, leaving the details of the proofs of the remaining two properties to the reader. The labels V_i, V_j are omitted here for compactness.

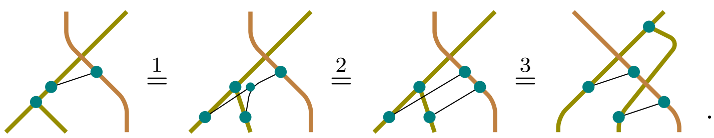


Figure 7.14: Naturality with respect to μ_i: details

We apply here

1. the defining property of right H-comodule algebras for V_i,
2. the defining property of left H-modules for V_j,
3. the naturality of the symmetric braiding c.

➻ $\sigma_{i,j}$ is natural with respect to μ_j, $\quad i < j$.
Graphically it means

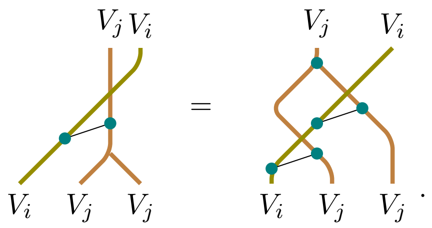


Figure 7.15: Naturality with respect to μ_j.

To prove this, one needs

1. the defining property of left H^{cop}-module algebras for V_j,
2. the defining property of right H-comodules for V_i,
3. the naturality of c.

➻ For each triple $i < j < k$, the σ_{YD}'s satisfy YBE on $V_i \otimes V_j \otimes V_k$.
Present (YB) graphically:

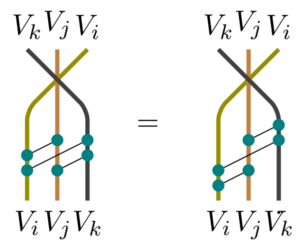


Figure 7.16: Yang-Baxter equation for Yetter-Drinfel′d modules

To prove this, one needs

1. the defining property of right H-comodules for V_i,
2. the defining property of left H^{cop}-modules for V_k,
3. the Yetter-Drinfel′d property for V_j. $\square$

Note that, although one can not say that the maps defined in the theorem give a pre-braiding **if and only if** all the conditions in the definition of YD system are satisfied (the kind of equivalences encountered in previous chapters), each of these conditions is **essential** in the proof. The necessity questions are discussed in more detail for the concrete example of a two component YD system in proposition 7.6.4.

Remark 7.5.7. According to remark 7.2.3, some or all of the maps $\sigma_{i,i}$ can be replaced by a right version $\sigma^r_{Ass} = \mu_i \otimes \nu_i$.

Remarks on the definition: precisions and alternative versions

We now comment on three non-conventional points in the definition of YD system, explaining their effect on the theorem.

Remark 7.5.8. One would expect a slightly different form of compatibility between the multiplication μ_i and the H-action λ_i on V_i, in the spirit of H-module algebras:

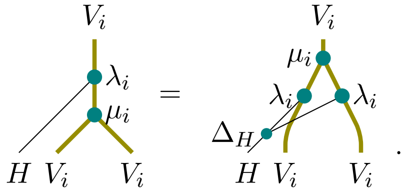


Figure 7.17: Alternative compatibility condition

However, our "twisted" form is needed for theorem 8 to hold. A nice way to interpret our choice is to take the symmetric category $\mathcal{C} = \mathbf{vect}_{\Bbbk}$ and to consider the induced left H^*-coaction on V_i (cf. table 1.5 and subsequent remarks concerning the (co)evaluation maps and the dual (co)algebra structure on H^*):

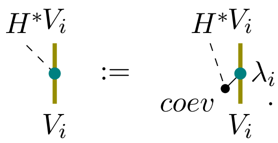


Figure 7.18: The duals come into play

The induced compatibility condition is the familiar

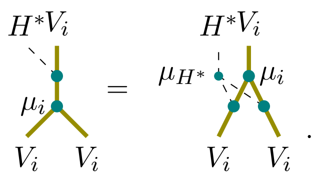


Figure 7.19: Compatibility for the left H^*-coaction

Recall the categorical definition of a bialgebra (definition 7.4.1). We note that it appears naturally in the YD system context:

Remark 7.5.9. No compatibility between the algebra and coalgebra structures on H are demanded explicitly. However, other properties of a YD system dictate that H should be not too far from a bialgebra, at least as far as (co)actions are concerned. For example,

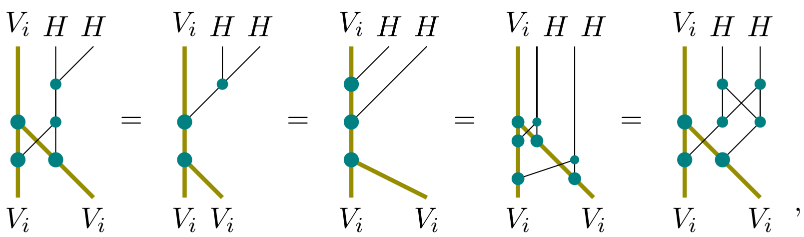


Figure 7.20: Almost a bialgebra

and similarly for module structures.

The last observations concern alternative notions of YD modules.

Remark 7.5.10. A ***left-left Yetter-Drinfel′d system*** over H can be defined similarly to how it was done for a left-right H-YD system, with the following differences:

✓ right H-comodule structures on $V_1, \ldots, V_{r-1}$ are replaced by the left ones;

✓ the compatibilities of algebra structures on the V_i's with H-module and H-comodule structures are no longer "twisted" (cf. remark 7.5.8);

✓ the module-comodule compatibility is the one from figure 7.9.

An analogue of theorem 8 holds for left-left YD systems with only one change: components $\sigma_{i,j} = \sigma_{YD}$ of the pre-braiding for $i < j$ should be replaced by that from equation 7.13. Using the pre-braiding σ^r_{Ass} on some of the V_i's (cf. remark 7.5.7) is still possible in this setting.

One can also define ***right-left and right-right Yetter-Drinfel′d system***, applying the right-left duality from section 5.5. If H is a Hopf algebra with an invertible antipode s, then remark 7.5.4 explains how to pass from right-right to left-right YD systems.

Characters

We now turn to the last ingredient missing for producing braided differentials: characters.

Definition 7.5.11. Take an H-YD system $(V_1, \ldots, V_r)$. A collection of algebra characters $\epsilon_i : V_i \to \mathbf{I}$ for $i \in I \subseteq \{1, 2, \ldots, r\}$ is called *a YD system character* if

$$(\epsilon_i \otimes \epsilon_j) \circ (\mathrm{Id}_i \otimes \lambda_j) \circ (\delta_i \otimes \mathrm{Id}_j) = \epsilon_i \otimes \epsilon_j : V_i \otimes V_j \longrightarrow \mathbf{I} \qquad \forall i < j \in I. \tag{7.14}$$

Graphically, the compatibility of the ϵ_i's looks as follows:

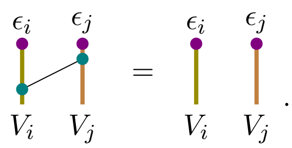


Figure 7.21: YD system character

The simplest example is given by a single algebra character (i.e. $|I| = 1$), which will often be our choice. Another example is given by the following

Lemma 7.5.12. Take an H-YD system $(V_1, \ldots, V_r)$. A collection of algebra characters $\epsilon_i : V_i \to \mathbf{I}$ for $i \in I \subseteq \{1, 2, \ldots, r\}$ forms a YD system character if

1. either all the ϵ_i's with $i > 1$ respect the H-module structures:
$$\epsilon_i \circ \lambda_i = \varepsilon \otimes \epsilon_i : H \otimes V_i \to \mathbf{I},$$
2. or all the ϵ_i's with $i < r$ respect the H-comodule structures:
$$(\epsilon_i \otimes \mathrm{Id}_H) \circ \delta_i = \nu \circ \epsilon_i : V_i \to H = \mathbf{I} \otimes H.$$

The notion of YD system character is designed for producing braided characters:

Lemma 7.5.13. Take an H-YD system $(V_1, \ldots, V_n)$ in a symmetric preadditive category $\mathcal{C}$. A YD system character $\overline{\epsilon}_I := (\epsilon_i)_{i \in I}$ completed by zeroes on the V_j's with $j \notin I$ is a braided character for the pre-braiding from theorem 8.

Notation 7.5.14. We denote the completed braided character from the lemma by $\overline{\epsilon}$.

Summary

Applying the general theory of section 7.1 to a pre-braided system coming from an H-YD system, one concludes:

Corollary 7.5.15. *In a symmetric additive category* $(\mathcal{C}, \otimes, \mathbf{I}, c)$, *choose*

1. *a UAA and coUAA H;*
2. *an H-YD system $(V_1, \ldots, V_r)$;*
3. *for each i, a left or right version of the braiding σ_{Ass};*
4. *two YD system characters $(\epsilon_i)_{i \in I \subseteq \{1,2,\ldots,r\}}$ and $(\zeta_j)_{j \in J \subseteq \{1,2,\ldots,r\}}$.*

The formulas from theorem $6^{\mathbf{multi}}$, applied to the pre-braided structure from theorem 8 and to the braided characters $\overline{\epsilon}$ and $\overline{\zeta}$ from lemma 7.5.13, give a bidegree -1 tensor bidifferential for $T(V)^{\rightarrow}_{*}$.

Now we move on to concrete – and familiar – examples. Explicit calculations of braided differentials in subsequent sections are representative enough to give an idea of what they look like in the general setting of the above corollary.

7.6 Bialgebras

The first examples of H-Yetter-Drinfel′d systems we consider are two-component systems. In particular, we do not work with the YD compatibility condition in this section. As components, we choose our bialgebra H and its dual and/or opposite versions, postponing the work with "external" modules and/or comodules over H until further sections.

The main results of this section are summarized in the table which continues table 4.1:

structure	system	$\overline{\sigma}$	invertibility	characters	complexes
bialgebra H	(H, H^*)	σ_{Ass} & σ_{YD}	iff H is a Hopf algebra	ε_H & ε_{H^*}	Gerstenhaber-Schack, [29]

Table 7.1: Main ingredients of braided homology theory for a bialgebra

Like for basic algebraic structures in part I, the results in this section are of the "if and only if" type – i.e. we obtain "braided" **characterizations** of the bialgebra compatibility condition and of the existence of the antipode in terms of the YBE and, respectively, in terms of the braiding invertibility.

Here we work in $\underline{\mathcal{C} = \mathbf{vect}_{\Bbbk}}$. Note however that one could stay in the general setting of a symmetric additive category and choose a braided bialgebra in $\mathcal{C}$ admitting a dual.

A pre-braiding encoding the bialgebra structure

Let H be a *finite-dimensional* $\Bbbk$-*bialgebra* (we consider only unital, counital, associative and coassociative bialgebras here). Recall the evaluation map $ev : H^* \otimes H \to \Bbbk$ from (1.2) and its dual coevaluation map $coev : \Bbbk \to H^* \otimes H$, as well as their "twisted versions" $ev \circ \tau : H \otimes H^* \to \Bbbk$ and $\tau \circ coev : \Bbbk \to H \otimes H^*$, still denoted by ev and $coev$ for simplicity. Note that in the general settings of a symmetric additive category with dualities, the flip τ should be replaced with the braiding c.

The dual H^* of H has an induced bialgebra structure via the evaluation map ev extended to $H \otimes H$ and $H^* \otimes H^*$ using the **"rainbow" pattern**, cf. table 1.5. Note that because of this non-conventional choice, we sometimes get formulas slightly different from the ones found in literature.

Proposition 7.6.1. *For a finite-dimensional* $\Bbbk$*-bialgebra* $(H, \mu, \nu, \Delta, \varepsilon)$*, there is an* H*-YD system structure on* (H, H^*) *given by:*

- *the usual unital associative algebra structure on* H *and the induced structure on* H^**;*
- *right* H*-comodule structure on* H *given by the comultiplication* Δ*;*
- *left* H*-module structure on* H^* *given by*

$$h \cdot l := h(l_{(1)})l_{(2)} \qquad \forall h \in H, l \in H^*, \tag{7.15}$$

for the induced comultiplication on H^**.*

See remark 7.5.8 for the last structure. It is graphically depicted as

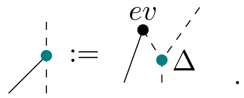


Figure 7.22: The action of H on H^*

Proof. Verifications are easy. The least trivial condition – the compatibility of the algebra structure on H^* (H) with the H-(co)module structure – is precisely the compatibility condition (7.8) defining a bialgebra. □

Observe that the "rainbow" pairing between $H \otimes H$ and $H^* \otimes H^*$ is necessary for (7.15) to define an action.

Choose the **right braiding for** H and the **left one for** H^* (cf. section 5.5 and remark 7.5.7). With this choice, theorem 8 applied to the previous proposition gives

Proposition 7.6.2. *A pre-braiding can be given on the* H*-YD system* (H, H^*) *by*

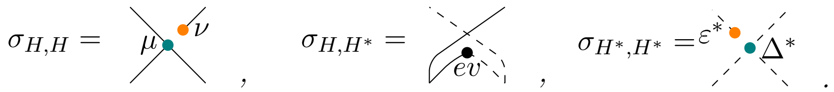


Figure 7.23: Pre-braiding for the system (H, H^*)

Notation 7.6.3. We denote this pre-braided system (H, H^*) by $\overline{H}$.

Like for most algebraic structures considered in previous chapters, we get a pre-braiding **characterizing** the structure of a bialgebra:

Proposition 7.6.4. *Take an* $H \in \mathbf{vect}_\Bbbk$ *endowed with a (not necessarily associative) multiplication* $\mu : H \otimes H \to H$ *with a unit* $\nu : H \to \mathbf{I}$ *and a comultiplication* $\Delta : H \to H \otimes H$ *with a counit* $\varepsilon : H \to \mathbf{I}$*. Consider the "rainbow"-dual structures on* H^**. The structure* $(H, \mu, \nu, \Delta, \varepsilon)$ *describes a bialgebra if and only if the three maps from proposition 7.6.2 define a pre-braiding on* (H, H^*)*.*

Proof. According to lemma 4.3.1, YBE on $H^{\otimes 3}$ is equivalent to μ being associative. By duality, YBE on $(H^*)^{\otimes 3}$ is equivalent to Δ being coassociative. Further, due to the preceding proposition 7.6.2, one has YBE on $H \otimes H \otimes H^*$ and $H \otimes H^* \otimes H^*$ if H is a bialgebra. On the contrary, applying $\nu^* \otimes \mathrm{Id}_H \otimes \varepsilon$ (or $\nu^* \otimes \mathrm{Id}_{H^*} \otimes \varepsilon$) and the evaluation-coevaluation duality to the YBE on $H \otimes H \otimes H^*$ (resp. $H \otimes H^* \otimes H^*$), one recovers the bialgebra compatibility relation. □

Remark 7.6.5. We could have started just with a multiplication and a comultiplication on H, adding a **formal unit** and upgrading the structures on H as usual. In this case one gets a partial braiding on $T(\widetilde{H}) \otimes T((\widetilde{H})^*)$ if and only if these new structures define a bialgebra, which is not the same as imposing the bialgebra compatibility relation on the original multiplication and comultiplication!

Twisted and dual variations

Interesting YD systems can be obtained by applying the above construction to the dual and/or the "twisted" version of our bialgebra H. Since these constructions will be used below in the study of bialgebra homologies, we present them in detail.

First, via the ***evaluation-coevaluation duality***, proposition 7.6.1 endows (H^*, H) with an H^*-YD system structure, hence with a pre-braiding analogous to that from proposition 7.6.2.

Let us further consider ***"twisted" versions*** of H. A classical result (which we state here in full generality, istead of restricting ourselves to $\mathcal{C} = \mathbf{vect}_{\Bbbk}$) says

Lemma 7.6.6. Take a bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ in a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. Then

1. $H^{op} := (H, \mu \circ c^{-1}, \nu, \Delta, \varepsilon)$ is a bialgebra in $(\mathcal{C}, \otimes, \mathbf{I}, c^{-1})$.
2. $H^{cop} := (H, \mu, \nu, c^{-1} \circ \Delta, \varepsilon)$ is a bialgebra in $(\mathcal{C}, \otimes, \mathbf{I}, c^{-1})$.
3. $H^{op,cop} := (H, \mu \circ c^{-1}, \nu, c \circ \Delta, \varepsilon)$ and $H^{cop,op} := (H, \mu \circ c, \nu, c^{-1} \circ \Delta, \varepsilon)$ are bialgebras in $(\mathcal{C}, \otimes, \mathbf{I}, c)$.
4. If the bialgebra H turns out to be a Hopf algebra with an antipode s, then so are $H^{op,cop}$ and $H^{cop,op}$, with the same antipode s. If s is invertible, then s^{-1} becomes the antipode for H^{op} and H^{cop}.
5. Moreover, one has the following bialgebra or Hopf algebra isomorphisms:

$$(H^{op})^* \simeq (H^*)^{cop}, \qquad (H^{cop})^* \simeq (H^*)^{op}, \qquad (H^{op,cop})^* \simeq H^{cop,op}.$$

Return now to our bialgebra H in $\mathcal{C} = \mathbf{vect}_{\Bbbk}$. In particular, the bialgebras $H^{op,cop}$ and $H^{cop,op}$ coincide. Proposition 7.6.1 can be applied to each of the bialgebras H^{op}, H^{cop} and $H^{op,cop}$. Theorem 8 then gives three new pre-braided system structures on (H, H^*).

Summarizing, one gets

Proposition 7.6.7. *For a finite-dimensional* $\Bbbk$*-bialgebra* $(H, \mu, \nu, \Delta, \varepsilon)$*, one can construct the following pre-braided systems:*

1. $\overline{H}^{op} := (H, H^*)$*, with*

$$\begin{aligned}
\sigma_{1,1} &= (\mu \circ \tau) \otimes \nu, \\
\sigma_{2,2} &= \varepsilon^* \otimes \Delta^*, \\
\sigma_{1,2} &= \sigma_{YD}^{op} := \tau \circ (\mathrm{Id}_H \otimes ev \otimes Id_{H^*}) \circ (\Delta \otimes (\tau \circ \mu^*));
\end{aligned}$$

2. $\overline{H}^{cop} := (H, H^*)$*, with*

$$\begin{aligned}
\sigma_{1,1} &= \mu \otimes \nu, \\
\sigma_{2,2} &= \varepsilon^* \otimes (\Delta^* \circ \tau), \\
\sigma_{1,2} &= \sigma_{YD}^{cop} := \tau \circ (\mathrm{Id}_H \otimes ev \otimes Id_{H^*}) \circ ((\tau \circ \Delta) \otimes \mu^*);
\end{aligned}$$

3. $\overline{H}^{op,cop} := (H, H^*)$*, with*

$$\begin{aligned}
\sigma_{1,1} &= (\mu \circ \tau) \otimes \nu, \\
\sigma_{2,2} &= \varepsilon^* \otimes (\Delta^* \circ \tau), \\
\sigma_{1,2} &= \sigma_{YD}^{op,cop} := \tau \circ (\mathrm{Id}_H \otimes ev \otimes Id_{H^*}) \circ ((\tau \circ \Delta) \otimes (\tau \circ \mu^*));
\end{aligned}$$

4. $\overline{H}^r := (H^*, H)$, *with*

$$\begin{aligned}
\sigma_{1,1} &= \Delta^* \otimes \varepsilon^*,\\
\sigma_{2,2} &= \nu \otimes \mu,\\
\sigma_{1,2} &= \sigma^r_{YD} := \tau \circ (\mathrm{Id}_{H^*} \otimes ev \otimes Id_H) \circ (\mu^* \otimes \Delta),
\end{aligned}$$

and the three "twisted" versions of the last structure.

Notations $\overline{H}^r$ and σ^r_{YD} come from the interpretation of the system $\overline{H}^r$ as a "right" version of $\overline{H}$, in the sense of category $\mathbf{vect}_{\Bbbk}^{\otimes \mathrm{op}}$ (cf. section 5.5). Remark that the notion of bialgebra is stable with respect to this duality.

Here are graphical versions of the "twisted" Woronowicz braidings:

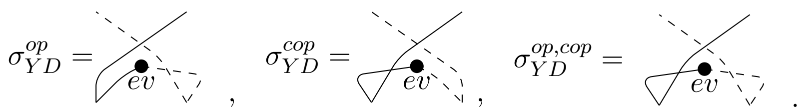


Figure 7.24: "Twisted" Woronowicz braidings

Bialgebra homology of Gerstenhaber and Schack

Our next goal is to write down explicit braided differentials for pre-braided systems from propositions 7.6.2 and 7.6.7. We do it for ***partial characters*** ε_H (the counit of H extended to H^* by zero) and ε_{H^*} (the counit of H^*, i.e. $(1_H)^*$, extended to H by zero); cf. lemma 7.1.7.

In this section the letters h_i always stay for elements of H, l_j – for elements of H^*, the pairing $\langle , \rangle$ is the evaluation, and the multiplications μ and Δ^* on H and H^* respectively are denoted by $\cdot$ for simplicity.

Start with some preliminary observations. The first ones concern ***adjoint actions*** of H on the tensor powers of H^*, and vice versa.

Recall the left H-module structure on H^* given by (7.15). Together with the usual multiplication and unit on H^*, they form a left H^{cop}-module algebra structure (cf. remark 7.5.8 concerning the necessity of twisting the comultiplication of H). By the right-left symmetry (cf. section 5.5), H^* is also a right H^{cop}-module algebra. The triple ($A = H^*, C = H^{cop}, B = H^*$) and the character $\varepsilon_{H^*} = (1_H)^*$ can thus be fed into proposition 7.4.6 (note that all the (bi)(co)module structures are now defined over H^{cop} and not over H). One gets

Lemma 7.6.8. The tensor powers of a finite-dimensional $\Bbbk$-bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ can be endowed with an H^*-bimodule structure via formulas

$$\begin{aligned}
\pi^{H^*} := \pi^{\varepsilon_{H^*}} = ev_1 \circ ev_2 \cdots ev_n \circ ((\mu^*)^{n-1} \otimes (\omega_{2n}^{-1} \circ \Delta^{\otimes n})) :\\
H^* \otimes H^{\otimes n} \to \otimes H^{\otimes n},\\
{}^{H^*}\pi := {}^{\varepsilon_{H^*}}\pi = ev_{n+1} \circ ev_{n+2} \cdots ev_{2n} \circ ((\omega_{2n}^{-1} \circ \Delta^{\otimes n}) \otimes (\mu^*)^{n-1}) :\\
H^{\otimes n} \otimes H^* \to \otimes H^{\otimes n}.
\end{aligned}$$

The H^*-actions are graphically depicted as

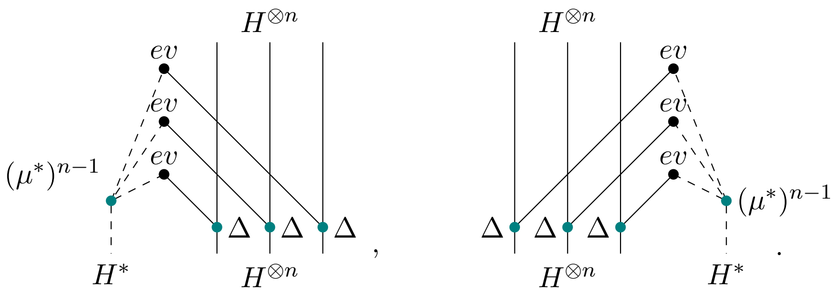


Figure 7.25: $H^{\otimes n}$ as an H^*-bimodule

On the level of elements, the formulas can be written as

$$\pi^{H^*}(l \otimes h_1 \dots h_n) = \left\langle l_{(1)}, h_{n(1)} \right\rangle \left\langle l_{(2)}, h_{n-1(1)} \right\rangle \cdots \left\langle l_{(n)}, h_{1(1)} \right\rangle h_{1(2)} \dots h_{n(2)},$$
$$^{H^*}\pi(h_1 \dots h_n \otimes l) = \left\langle l_{(1)}, h_{n(2)} \right\rangle \left\langle l_{(2)}, h_{n-1(2)} \right\rangle \cdots \left\langle l_{(n)}, h_{1(2)} \right\rangle h_{1(1)} \dots h_{n(1)}.$$

Interchanging the roles of H and H^*, one gets an H-bimodule $((H^*)^{\otimes m}, \pi^H, {}^H\pi)$. By abuse of notation, we define, for all $m, n \in \mathbb{N}$ for which this makes sense, the following morphisms from $H^{\otimes n} \otimes (H^*)^{\otimes m}$ to $H^{\otimes(n-1)} \otimes (H^*)^{\otimes m}$ or $H^{\otimes n} \otimes (H^*)^{\otimes(m-1)}$:

$$\begin{aligned}
{}^{H^*}\pi &:= {}^{H^*}\pi \otimes \mathrm{Id}_{H^*}^{\otimes(m-1)}, \\
\pi^{H^*} &:= (\pi^{H^*} \otimes \mathrm{Id}_{H^*}^{\otimes(m-1)}) \circ \tau_{H^{\otimes n} \otimes (H^*)^{\otimes(m-1)}, H^*}, \\
\pi^H &:= \mathrm{Id}_H^{\otimes(n-1)} \otimes \pi^H, \\
{}^H\pi &:= (\mathrm{Id}_H^{\otimes(n-1)} \otimes {}^H\pi) \circ \tau_{H, H^{\otimes(n-1)} \otimes (H^*)^{\otimes m}}.
\end{aligned}$$

Lemma 7.6.9. The endomorphisms ${}^{H^*}\pi, \pi^{H^*}, \pi^H$ and ${}^H\pi$ of $T(H) \otimes T(H^*)$ pairwise commute.

Proof. Lemma 7.6.8 implies the commutativity of ${}^{H^*}\pi$ and π^{H^*}. The commutativity of ${}^H\pi$ and π^H follows by duality. Next, returning to the braided interpretation of the adjoint actions, π^H corresponds to pulling the rigtmost H strand to the right of all the H^* strands (using the pre-braiding on $\overline{H}$) and applying ε_H, while ${}^{H^*}\pi$ corresponds to pulling the leftmost H^* strand to the left of all the H strands and applying ε_{H^*}. Thus π^H and ${}^{H^*}\pi$ clearly commute, and so do π^{H^*} and ${}^H\pi$ by duality.

In order to prove the commutativity of the two remaining pairs, consider the linear isomorphism

$$\Delta_n \otimes \mathrm{Id}_{H^*}^{\otimes m} : H^{\otimes n} \otimes (H^*)^{\otimes m} \xrightarrow{\sim} (H^{op})^{\otimes n} \otimes ((H^{op})^*)^{\otimes m}, \tag{7.16}$$

where $\Delta_n \in S_n$, defined by (2.6), acts on $H^{\otimes n}$ via the flip τ. This isomorphism, extended to $T(H) \otimes T(H^*)$ by linearity, is denoted by Δ_* by abuse of notation (unfortunately, the common notation for Garside elements coincides with that for the comultiplication on H). One checks that Δ_* transports the endomorphisms ${}^{H^*}\pi, \pi^{H^*}, \pi^H$ and ${}^H\pi$ of $H^{\otimes n} \otimes (H^*)^{\otimes m}$ to, respectively, ${}^{(H^{op})^*}\pi, \pi^{(H^{op})^*}, {}^{H^{op}}\pi$ and $\pi^{H^{op}}$. Thus the commutativity of ${}^{(H^{op})^*}\pi$ and $\pi^{H^{op}}$ induces that of ${}^{H^*}\pi$ and ${}^H\pi$, and similarly for π^{H^*} and π^H. $\square$

Further, recall the ***bar differential***

$$d_{bar}(h_1 \dots h_n l_1 \dots l_m) = \sum_{i=1}^{n-1} (-1)^i h_1 \dots h_{i-1}(h_i \cdot h_{i+1}) h_{i+2} \dots h_n l_1 \dots l_m,$$

and the ***cobar differential***

$$d_{cob}(h_1 \dots h_n l_1 \dots l_m) = \sum_{i=1}^{m-1} (-1)^i h_1 \dots h_n l_1 \dots l_{i-1}(l_i \cdot l_{i+1}) l_{i+2} \dots l_m$$

on $T(H) \otimes T(H^*)$; cf. remark 4.3.8 and proposition 5.4.6. Note that we use the evaluation-coevaluation duality in order to transform the degree 1 cobar differential on $\mathrm{End}_{\Bbbk}(T(H))$ into a degree -1 differential on $T(H) \otimes T(H^*)$.

Putting everything together, one gets

Proposition 7.6.10. *For a finite-dimensional $\Bbbk$-bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$, one has the following bidifferential structures on $T(H) \otimes T(H^*)$:*

1.	d_{bar}	$(-1)^n d_{cob}$
2.	$d_{bar} + (-1)^n \pi^H$	$(-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi)$
3.	$d_{bar} + {}^H\pi$	$(-1)^n d_{cob} + (-1)^{n+m} \pi^{H^*}$
4.	$d_{bar} + (-1)^n \pi^H + {}^H\pi$	$(-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi) + (-1)^{n+m} \pi^{H^*}$

Table 7.2: Bidifferential structures on $T(H) \otimes T(H^*)$

The signs $(-1)^n$ etc. here are those one chooses on the component $H^{\otimes n} \otimes (H^*)^{\otimes m}$ of $T(H) \otimes T(H^*)$.

Proof. We prove the assertion for each pair of morphisms separately, keeping the order from the statement.

1. We have seen that d_{bar} and d_{cob} are differentials. They affect different components of $T(H) \otimes T(H^*)$ ($T(H)$ and, respectively, $T(H^*)$), and thus commute. The sign $(-1)^n$ then assures the anticommutativity.
2. Return to the pre-braided system $\overline{H}$. One calculate the braided differentials:
$$\begin{aligned} {}^{\varepsilon_{H^*}}d &= (-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi), \\ d^{\varepsilon_H} &= -(d_{bar} + (-1)^n \pi^H), \end{aligned}$$
obtaining the desired bidifferential.
3. Dually, one gets a bidifferential $((-1)^m d_{bar} + (-1)^m ({}^H\pi), d_{cob} + (-1)^m \pi^{H^*})$. Observe that multiplying the first differential by $(-1)^m$ and the second one by $(-1)^n$, one still gets a bidifferential, coinciding with the desired one.
4. The last point follows from the three preceding ones thanks to the following elementary observation:

   Lemma 7.6.11. Take an abelian group $(S, +, 0, a \mapsto -a)$ endowed with an operation $\cdot$ distributive with respect to $+$. Then, for any $a, b, c, d, e, f \in S$ such that
$$(a+b) \cdot (d+e) = (a+c) \cdot (d+f) = a \cdot d = b \cdot f + c \cdot e = 0,$$
one has
$$(a+b+c) \cdot (d+e+f) = 0.$$

   Proof.
$$(a+b+c) \cdot (d+e+f) = (a+b) \cdot (d+e) + (a+c) \cdot (d+f) - a \cdot d + (b \cdot f + c \cdot e).$$
□

Now take $S = \mathrm{End}_R(T(H) \otimes T(H^*))$ with the usual addition and the operation $a \cdot b := a \circ b$ (for proving that the two maps from the fourth line of our table are differentials), or the operation $a \cdot b := a \circ b + b \circ a$ (for proving that the two maps anti-commute). The equalities of the type $b \cdot f + c \cdot e = 0$ follow from the pairwise anti-commutativity of $(-1)^n({}^{H^*}\pi), (-1)^{n+m}\pi^{H^*}, (-1)^n\pi^H$ and ${}^H\pi$ (which is a consequence of lemma 7.6.9), and the remaining ones from the preceding points of the proposition. □

One recognizes in $d_{bar} + (-1)^n\pi^H + {}^H\pi$ the Hochschild differential of H with the (right) coefficients in the H-bimodule $T(H^*)$ (cf. the dual version of lemma 7.6.8), and similarly for $(-1)^n d_{cob} + (-1)^n({}^{H^*}\pi) + (-1)^{n+m}\pi^{H^*}$. Thus the last bidifferential from the proposition defines the ***bialgebra homology*** of M.Gerstenhaber and S.D.Schack; cf. [29] where it was first introduced, R.Taillefer's thesis [77] for detailed calculations and a comparison with other bialgebra homologies, and M.Mastnak and S.Witherspoon's paper [57] for explicit formulas and the passage from $\mathrm{Hom}_{\Bbbk}(H^{\otimes m}, H^{\otimes n})$ to $H^{\otimes n} \otimes (H^*)^{\otimes m}$.

The existence of an antipode as an invertibility condition

Here we return to the general setting of an H-YD system $(V_1, V_2, \ldots, V_r)$ in a symmetric preadditive category $\mathcal{C}$.

The pre-braidings $\sigma_{i,i}$ are highly non-invertible, as was pointed out in section 4.3. It is however interesting to explore when the pre-braidings $\sigma_{i,j}$, $i < j$, have inverses, keeping in mind proposition 7.1.12. A well-known sufficient condition is the following:

Lemma 7.6.12. If the bialgebra H is moreover a Hopf algebra, then all the $\sigma_{i,j}$'s with $i < j$ are invertible, the inverse given by

$$\sigma_{i,j}^{-1} = \sigma_{YD}^{-1} = (\mathrm{Id}_i \otimes \lambda_j) \circ (\mathrm{Id}_i \otimes s \otimes \mathrm{Id}_j) \circ (\delta_i \otimes \mathrm{Id}_j) \circ c_{V_j,V_i}. \tag{7.17}$$

The inverse is graphically depicted as

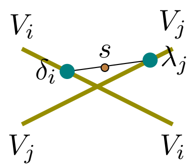
$\sigma_{i,j}^{-1} =$ 

Figure 7.26: The inverse for the Woronowicz braiding

and, in $\mathbf{Mod}_R$, takes the form

$$\sigma_{i,j}^{-1}(v \otimes u) = u_{(0)} \otimes s(u_{(1)})v.$$

Returning to proposition 7.1.12, one gets the following

Corollary 7.6.13. *Any YD system $(V_1, V_2, \ldots, V_r)$ over a Hopf algebra H in a symmetric additive category $\mathcal{C}$ comes with a **total pre-braiding** on $V := \bigoplus V_i$.*

In particular, given a finite-dimensional Hopf $\Bbbk$-algebra H, this method provides the vector space $H \oplus H^*$ with a pre-braiding, extending for instance the bidifferential $({}^{\varepsilon_{H^*}}d, d^{\varepsilon_H})$ from proposition 7.6.10 from $T(H, H^*)$ to $T(H \oplus H^*)$.

Now let us see to what extent the invertibility of $\sigma_{i,j}$ **distinguishes** Hopf algebras among other bialgebras. We return here to $\mathcal{C} = \mathbf{vect}_{\Bbbk}$ (or to a more general setting described at the beginning of this section).

Proposition 7.6.14. *Let H be a finite-dimensional $\Bbbk$-bialgebra. Consider the pre-braided system $\overline{H}$ from proposition 7.6.2. The component $\sigma_{1,2}$ of the partial pre-braiding on $\overline{H}$ is invertible **if and only if** H is a Hopf algebra.*

Proof. The "if" part follows from the previous lemma. For the "only if" part, suppose the existence of $\sigma_{1,2}^{-1}$ and put

$$\widetilde{s} := (((\varepsilon_H \otimes \epsilon_{H^*}) \circ \sigma_{1,2}^{-1}) \otimes \mathrm{Id}_H) \circ \tau_2 \circ (coev \otimes \mathrm{Id}_H) : H \to H$$

or, graphically,

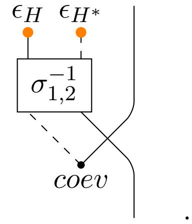
$\widetilde{s} :=$ 

Figure 7.27: A candidate for the antipode

Let us prove that $\widetilde{s}$ is the antipode. The part

$$\widetilde{s}(h_{(1)})h_{(2)} = \varepsilon(h)1_H \tag{7.18}$$

of the defining relation (s) is a direct consequence of $\sigma_{1,2}^{-1} \circ \sigma_{1,2} = \mathrm{Id}$ and the evaluation-coevaluation duality (graphical calculus is the easiest way to check this). One would expect to deduce the second part of (s) from $\sigma_{1,2} \circ \sigma_{1,2}^{-1} = \mathrm{Id}$, but surprisingly it does not seem to work. Some algebraic tricks come into play instead. Mimicking the formula (7.17) for the inverse Woronowicz braiding, set

$$\widetilde{\sigma}(l \otimes h) := h_{(1)} \otimes \widetilde{s}(h_{(2)}) \cdot l,$$

where $\cdot$ is the action of H on H^* defined by (7.15). Relation (7.18) implies

$$\widetilde{\sigma} \circ \sigma_{1,2} = \mathrm{Id}_{H \otimes H^*}.$$

But $\sigma_{1,2}^{-1}$ is the inverse of $\sigma_{1,2}$, so

$$\widetilde{\sigma} = \widetilde{\sigma} \circ (\sigma_{1,2} \circ \sigma_{1,2}^{-1}) = (\widetilde{\sigma} \circ \sigma_{1,2}) \circ \sigma_{1,2}^{-1} = \sigma_{1,2}^{-1}.$$

This gives $\sigma_{1,2} \circ \widetilde{\sigma} = \mathrm{Id}_{H^* \otimes H}$, i.e.

$$(h_{(2)}\widetilde{s}(h_{(3)})) \cdot l \otimes h_{(1)} = l \otimes h \qquad \forall h \in H, l \in H^*,$$

or, writing explicitly the H-action $\cdot$ on H^*,

$$\left\langle l_{(1)}, h_{(2)}\widetilde{s}(h_{(3)}) \right\rangle l_{(2)} \otimes h_{(1)} = l \otimes h \qquad \forall h \in H, l \in H^*.$$

Applying $\varepsilon_{H^*} \otimes \epsilon_H$ to both sides and using $\varepsilon_{H^*}(l) = \langle l, 1_H \rangle$, one gets

$$\left\langle l, h_{(1)}\widetilde{s}(h_{(2)}) \right\rangle = \langle l, \varepsilon(h)1_H \rangle \qquad \forall h \in H, l \in H^*,$$

and thus the second part of (s) for $\widetilde{s}$. □

In part I, we have recovered the defining properties of different algebraic structure as instances of YBE. The equation (s) defining the antipode seems quite difficult to encode by a YBE, but the previous proposition shows how to do it in terms of the invertibility of a braiding.

7.7 Yetter-Drinfel′d modules

This section is devoted to two "braided" approaches to Yetter-Drinfel′d (=YD) modules (see the reminder at the beginning of section 7.5).

The first one consists in viewing a YD module as a part of a Yetter-Drinfel′d system. This is very natural, since in a YD system all the central components V_i, i.e. those with $1 < i < r$, are by definition YD modules. This viewpoint is particularly appropriate for dealing with two kinds of questions:

1. defining tensor products of YD modules (cf. the constructions of L.A.Lambe and D.E.Radford in [44]);
2. constructing homology theories for a pair of YD modules, in the spirit of proposition 7.6.10; one recovers in particular the deformation cohomology for YD modules, defined by F.Panaite and D.Ştefan in [66].

In the second approach, we see a YD module as a multi-braided module over an appropriate pre-braided system of algebras. Thus, it becomes a module of coefficients rather than a component of a pre-braided system. We interpret the YD compatibility between an action and a coaction in terms of a braiding, allowing one to use the language of multi-braided modules and, consequently, the language of multi-braided tensor products of UAAs (cf. section 7.2), recovering the notion of the Drinfel′d double of a bialgebra. The same method is applied to Hopf (bi)modules in section 7.8.

We finish this section with a digression: we recover braidings coming from the R-matrix of a quasi-triangular Hopf algebra as a particular case of the Woronowicz pre-braiding, confirming the central place of YD modules in the study of solutions to the YBE.

A pre-braided system encoding the Yetter-Drinfel′d module structure

Let (M, λ, δ) be a left-right Yetter-Drinfel′d module over a finite-dimensional bialgebra H in $\mathbf{vect}_{\Bbbk}$. Add a formal unit

$$\widetilde{M} := M \oplus \Bbbk\mathbf{1}$$

and define a trivial unital associative multiplication m on $\widetilde{M}$ by

$$m|_{M\otimes M} = 0, \quad m(\mathbf{1} \otimes a) = m(a \otimes \mathbf{1}) = a \qquad \forall a \in \widetilde{M} \tag{7.19}$$

(cf. remark 7.4.4). Let H act on $\mathbf{1}$ by the counit:

$$\lambda(h)(\mathbf{1}) = \varepsilon_H(h)\mathbf{1} \qquad \forall h \in H,$$

and extend the coaction by

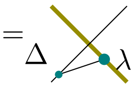
$$\delta(\mathbf{1}) = \mathbf{1} \otimes 1_H.$$

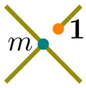
This extends the YD module structure to $\widetilde{M}$. Moreover, this extended YD module structure trivially respects the multiplication and the unit of $\widetilde{M}$ in the sense of definition 7.5.5. Thus one can "insert" $\widetilde{M}$ to the H-YD system $\overline{H}$ from the proposition 7.6.1:

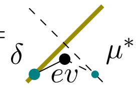
Proposition 7.7.1. *Given a left-right Yetter-Drinfel′d module M over a finite-dimensional bialgebra H in $\mathbf{vect}_{\Bbbk}$, there is an H-YD system structure on $(H, \widetilde{M}, H^*)$ given by the structures of proposition 7.6.1, the trivial multiplication m from (7.19) on $\widetilde{M}$ and the extended left-right YD module structure on $\widetilde{M}$.*

Like in the bialgebra case, we choose the right braiding for H and the left one for H^*. The choice for $\widetilde{M}$ is arbitrary and does not matter for what follows.

Corollary 7.7.2. *Theorem 8 gives a pre-braiding on the H-YD system $(H, \widetilde{M}, H^*)$, with the graphical presentation from figure 7.23 completed by*

$\sigma_{H,\widetilde{M}} =$ 

, $\sigma_{\widetilde{M},\widetilde{M}} =$ 

, $\sigma_{\widetilde{M},H^*} =$ 

.

Figure 7.28: Pre-braiding for the system $(H, \widetilde{M}, H^*)$

Remark 7.7.3. More generally, for a family of left-right Yetter-Drinfel′d modules $M_1, \dots, M_r$ over a finite-dimensional bialgebra H in $\mathbf{vect}_{\Bbbk}$, the structures above extend to an H-YD system – and thus to a pre-braided system – structure on $(H, \widetilde{M_1}, \dots, \widetilde{M_r}, H^*)$.

Using the same type of arguments as for bialgebras (proposition 7.6.4), one shows that the pre-braiding above **characterizes** the structure of a YD module over a bialgebra:

Proposition 7.7.4. *Take an $H \in \mathbf{vect}_{\Bbbk}$ endowed with a (not necessarily associative) multiplication $\mu : H \otimes H \to H$ with a unit $\nu : H \to \mathbf{I}$ and a comultiplication $\Delta : H \to H \otimes H$ with a counit $\varepsilon : H \to \mathbf{I}$. Consider the "rainbow"-dual structures on H^*. Further, take an $M \in \mathbf{vect}_{\Bbbk}$ and two linear morphisms $\lambda : H \otimes M \to M$ and $\delta : M \to M \otimes H$, normalized in the sense of* (6.3). *Then this structure describes a bialgebra H and a left-right YD module over H if and only if the six maps from the above corollary define a pre-braiding on $(H, \widetilde{M}, H^*)$.*

Homologies

In what follows, the letters h_i always stay for elements of H, l_j – for elements of H^*, $a \in M$, $b \in N^*$, the pairing $\langle , \rangle$ is the evaluation, and the multiplications μ and Δ^* on H and H^* respectively are denoted by $\cdot$ for simplicity.

Like for bialgebras, interesting differentials appear for the characters ε_H and ε_{H^*} extended by zero elsewhere. These differentials are easily seen to preserve the subspace

$$T(H; M; H^*) := T(H) \otimes M \otimes T(H^*) \subset T(H) \otimes T(\widetilde{M}) \otimes T(H^*)$$

(cf. remark 7.1.8), giving

Proposition 7.7.5. *For a left-right YD module (M, λ, δ) over a finite-dimensional $\Bbbk$-bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$, there is a bidifferential on $T(H) \otimes M \otimes T(H^*)$ given by*

$$\begin{aligned}
{}^{\varepsilon_{H^*}}d(h_1 \dots h_n \otimes a \otimes l_1 \dots l_m) = {} & \\
& (-1)^{n+1} \left\langle l_{1(1)}, a_{(1)} \right\rangle \left\langle l_{1(2)}, h_{n(2)} \right\rangle \left\langle l_{1(3)}, h_{n-1(2)} \right\rangle \dots \left\langle l_{1(n+1)}, h_{1(2)} \right\rangle \times \\
& \times h_{1(1)} \dots h_{n(1)} \otimes a_{(0)} \otimes l_2 \dots l_m \\
& + \sum_{i=1}^{m-1} (-1)^{n+i+1} h_1 \dots h_n \otimes a \otimes l_1 \dots l_{i-1}(l_i \cdot l_{i+1}) l_{i+2} \dots l_m,
\end{aligned}$$

$$\begin{aligned}
d^{\varepsilon_H}(h_1 \dots h_n \otimes a \otimes l_1 \dots l_m) = {} & \\
& (-1)^{n-1} \left\langle l_{1(1)}, h_{n(m)} \right\rangle \left\langle l_{2(1)}, h_{n(m-1)} \right\rangle \dots \left\langle l_{m(1)}, h_{n(1)} \right\rangle \times \\
& \times h_1 \dots h_{n-1} \otimes (h_{n(m+1)} \cdot a) \otimes l_{1(2)} \dots l_{m(2)} \\
& + \sum_{i=1}^{n-1} (-1)^{i-1} h_1 \dots h_{i-1}(h_i \cdot h_{i+1}) h_{i+2} \dots h_n \otimes a \otimes l_1 \dots l_m.
\end{aligned}$$

These differentials admit evident contracting homotopies

$$\overline{h} \otimes a \otimes \overline{l} \longmapsto 1_H \overline{h} \otimes a \otimes \overline{l},$$
$$\overline{h} \otimes a \otimes \overline{l} \longmapsto \overline{h} \otimes a \otimes \overline{l} 1_{H^*}.$$

So, to get non-trivial homologies, we now try to "cycle" this bidifferential, in the spirit of proposition 7.6.10 for bialgebras.

Start with an observation concerning dualities:

Remark 7.7.6. The notion of left-right YD module is self-dual, with the duality in the sense of the category $\mathcal{C}^{\mathrm{op}, \otimes^{\mathrm{op}}}$ (cf. section 5.5). It means that the structures of YD module in $\mathcal{C}$ and in $\mathcal{C}^{\mathrm{op}, \otimes^{\mathrm{op}}}$ coincide. Recall that graphically this duality corresponds to a central symmetry.

As a consequence, one gets (using, as usual, the "rainbow" duality on tensor products):

Lemma 7.7.7. The data (M, λ, δ) give a left-right H-YD module structure in $\mathbf{vect}_{\Bbbk}$ if and only if $(M^*, \delta^*, \lambda^*)$ give a left-right H^*-YD module structure.

With this in mind, we propose the following setting for adding coefficients to proposition 7.6.10. Take a left-right YD module (M, λ_M, δ_M) and a finite-dimensional left-right YD module (N, λ_N, δ_N) over a finite-dimensional bialgebra H in $\mathbf{vect}_{\Bbbk}$. Our aim is to endow the graded vector space

$$T(H) \otimes M \otimes T(H^*) \otimes N^*$$

with four bidifferentials analogous to those from proposition 7.6.10.

Recall the bar and cobar differentials

$$d_{bar}(h_1 \dots h_n a l_1 \dots l_m b) = \sum_{i=1}^{n-1} (-1)^i h_1 \dots h_{i-1} (h_i \cdot h_{i+1}) h_{i+2} \dots h_n a l_1 \dots l_m b,$$
$$d_{cob}(h_1 \dots h_n a l_1 \dots l_m b) = \sum_{i=1}^{m-1} (-1)^i h_1 \dots h_n a l_1 \dots l_{i-1} (l_i \cdot l_{i+1}) l_{i+2} \dots l_m b.$$

To introduce a generalization of maps $\pi^{H^*}, \pi^H, \dots$ which were defined in the bialgebra setting on $T(H) \otimes T(H^*)$, the following observation is useful:

Lemma 7.7.8. A pre-braiding can be given on the system

$$\overline{V}_{H,M,N} := (H^*, \widetilde{N^*}, H, \widetilde{M}, H^*)$$

by combining the pre-braiding on $(H, \widetilde{M}, H^*)$ from corollary 7.7.2, its dual version (in the sense of lemma 7.7.7) on $(H^*, \widetilde{N^*}, H)$, and flips τ as components $\sigma_{1,4}, \sigma_{1,5}, \sigma_{2,4}$ and $\sigma_{2,5}$.

Proof. We have seen that formula (7.15) and its dual version endow H^* with a left and right H^{cop}-module algebra structures. Further, $\widetilde{M}$ is a left H^{cop}-module algebra by construction, and, dually, $\widetilde{N^*}$ is a right H^{cop}-module algebra. Together with the H^{cop}-comodule algebra H^{cop}, this gives four triples to be fed into proposition 7.4.2. One checks that the pre-braidings obtained this way are precisely those imposed on the corresponding components of $\overline{V}_{H,M,N}$ in the statement of the lemma. Thus one gets the instances of the YBE where all the components belong to one of these triples. On the other hand, corollary 7.7.2 guarantees the YBEs where all the components are from the triples $(H, \widetilde{M}, H^*)$ or $(H^*, \widetilde{N^*}, H)$. The remaining triples combine the first two and the last two components of $\overline{V}_{H,M,N}$. For such triples, the YBE trivially follows from the naturality of the flip. □

Lemma 7.4.5 now gives an H^*-bimodule structure on $T(\overline{V}_{H,M,N})_n^{\rightarrow}$ and, by restriction, on $N^* \otimes H^n \otimes M$ via formulas

$$\begin{aligned}
\pi^{H^*} := \pi^{\varepsilon_{H^*}} &= ev_2 \circ ev_3 \cdots ev_{n+1} \circ \tau_{(H^*)^n,N^*} \circ (\lambda_{N^*})_{n+1} \circ \\
&\quad ((\mu^*)^n \otimes \mathrm{Id}_{N^*} \otimes (\omega_{2n}^{-1} \circ \Delta^{\otimes n}) \otimes \mathrm{Id}_M) : \\
&\quad H^* \otimes N^* \otimes H^n \otimes M \to N^* \otimes H^n \otimes M, \\
{}^{H^*}\pi := {}^{\varepsilon_{H^*}}\pi &= ev_{n+2} \circ ev_{n+3} \cdots ev_{2n+1} \circ \\
&\quad (\mathrm{Id}_{N^*} \otimes (\omega_{2(n+1)}^{-1} \circ (\Delta^{\otimes n} \otimes \delta_M)) \otimes (\mu^*)^n) : \\
&\quad N^* \otimes H^n \otimes M \otimes H^* \to N^* \otimes H^n \otimes M.
\end{aligned}$$

We use the same notation π^{H^*} for the action on $N^* \otimes H^n$ obtained by restricting oneself to the corresponding pre-braided subsystem, and similarly for other actions.

Dually, one gets an H-bimodule structure $(M \otimes (H^*)^m \otimes N^*, \pi^H, {}^H\pi)$. By abuse of notation, we define, for all $m, n \in \mathbb{N}$ for which this makes sense, the following morphisms from $H^n \otimes M \otimes (H^*)^m \otimes N^*$ to $H^{n-1} \otimes M \otimes (H^*)^m \otimes N^*$ or $H^n \otimes M \otimes (H^*)^{m-1} \otimes N^*$:

$$\begin{aligned}
{}^{H^*}\pi &:= {}^{H^*}\pi \otimes \mathrm{Id}_{H^*}^{m-1} \otimes \mathrm{Id}_{N^*}, \\
\pi^{H^*} &:= \tau_{N^*,H^n \otimes M \otimes (H^*)^{m-1}} \circ (\pi^{H^*} \otimes \mathrm{Id}_{H^*}^{\otimes(m-1)}) \circ \tau_{H^n \otimes M \otimes (H^*)^{(m-1)},H^* \otimes N^*}, \\
\pi^H &:= \mathrm{Id}_H^{\otimes(n-1)} \otimes \pi^H, \\
{}^H\pi &:= (\mathrm{Id}_H^{\otimes(n-1)} \otimes {}^H\pi) \circ \tau_{H,H^{n-1} \otimes M \otimes (H^*)^m \otimes N^*}.
\end{aligned}$$

The bidifferential from proposition 7.7.5 can be written with our new notations as

$$\begin{aligned}
{}^{\varepsilon_{H^*}}d &= (-1)^{n+1}({}^{H^*}\pi) + (-1)^{n+1} d_{cob}, \\
d^{\varepsilon_H} &= (-1)^{n-1}\pi^H - d_{bar}.
\end{aligned}$$

Generalizing lemma 7.6.9, one gets

Lemma 7.7.9. The endomorphisms ${}^{H^*}\pi, \pi^{H^*}, \pi^H$ and ${}^H\pi$ of $T(H) \otimes M \otimes T(H^*) \otimes N^*$ pairwise commute.

We have thus generalized all the ingredients of the proof of proposition 7.6.10 to the settings with components, obtaining

Proposition 7.7.10. *For a left-right YD module M and a finite-dimensional left-right YD module N over a finite-dimensional bialgebra H in* $\mathbf{vect}_{\Bbbk}$, *one has the following bidifferential structures on $T(H) \otimes M \otimes T(H^*) \otimes N^*$:*

1.	d_{bar}	$(-1)^n d_{cob}$
2.	$d_{bar} + (-1)^n \pi^H$	$(-1)^n d_{cob} + (-1)^n({}^{H^*}\pi)$
3.	$d_{bar} + {}^H\pi$	$(-1)^n d_{cob} + (-1)^{n+m}\pi^{H^*}$
4.	$d_{bar} + (-1)^n \pi^H + {}^H\pi$	$(-1)^n d_{cob} + (-1)^n({}^{H^*}\pi) + (-1)^{n+m}\pi^{H^*}$

Table 7.3: Bidifferential structures on $T(H) \otimes M \otimes T(H^*) \otimes N^*$

The signs $(-1)^n$ etc. here are those one chooses on the component $H^{\otimes n} \otimes M \otimes (H^*)^{\otimes m} \otimes N^*$ of $T(H) \otimes M \otimes T(H^*) \otimes N^*$.

Substituting the graded vector space $T(H) \otimes M \otimes T(H^*) \otimes N^*$ we work in with its alternative version $\mathrm{Hom}_{\Bbbk}(N \otimes T(H), T(H) \otimes M)$, we obtain (the dual of a mirror version of) the deformation cohomology for YD modules, defined by F.Panaite and D.Ştefan in [66]. We have thus developed a conceptual framework for this cohomology theory, replacing case by case verifications (for instance, when proving that one has indeed a bidifferential) with a structure study, facilitated by graphical tools.

Remark 7.7.11. Working with a YD module M, one does not exploit the power of YD systems to the full, since one uses the trivial algebra structure on M. A situation where this part of the YD system structure becomes important is that of an H-module algebra M. In this case, one has an H^{cop}-YD system (H^{cop}, M). Studying the associated braided differentials, one recovers the deformation bicomplex of module algebras, introduced by D.Yau in [84]. The work on this question is in progress.

Tensor products of YD modules

The interpretation of a YD module as a part of a YD system is particularly useful while working with tensor products of YD modules. For example, it suggests and explains the YD structure on such a tensor product, which is not so intuitive: for instance, the usual diagonal action and coaction fail to satisfy (YD) in general.

Proposition 7.7.12. *Take two left-right YD modules (M, λ_M, δ_M) and (N, λ_N, δ_N) over a finite-dimensional $\Bbbk$-bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$. Then the maps*

$$\lambda_{M\otimes N} = (\lambda_M \otimes \lambda_N) \circ (c_{H,H\otimes M} \otimes \mathrm{Id}_N) \circ (\Delta \otimes \mathrm{Id}_{M\otimes N}),$$
$$\delta_{M\otimes N} = (\mathrm{Id}_{M\otimes N} \otimes \mu) \circ (\mathrm{Id}_M \otimes c_{H,N} \otimes \mathrm{Id}_H) \circ (\delta_M \otimes \delta_N)$$

give a YD module structure on $M \otimes N$. On the element level, taking $h \in H, m \in M, n \in N$, and denoting the H-actions and the multiplication μ by $\cdot$ for simplicity, it means

$$h \cdot (m \otimes n) = h_{(2)} \cdot m \otimes h_{(1)} \cdot n,$$
$$\delta(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)} \cdot n_{(1)}.$$

In other words, one uses the diagonal coaction and the <u>twisted</u> diagonal action.

Proof. We use the "braided" characterization of YD modules from proposition 7.7.4 and some technical formal unit gymnastics. Remark 7.7.3 describes a YD system, and thus a pre-braided system, $(H, \widetilde{M}, \widetilde{N}, H^*)$. Lemma 7.1.3 extracts a pre-braided system $(H, \widetilde{M} \otimes \widetilde{N}, H^*)$ from it, with $\sigma_{2,2} = 0$. The pre-braiding further restricts to $(H, M \otimes N, H^*)$. Extending the zero pre-braiding on $M \otimes N$ to σ_{Ass} on $\widetilde{M \otimes N} := (M \otimes N) \oplus \Bbbk\mathbf{1}$, with the trivial multiplications on the latter space, one gets precisely the type of pre-braiding demanded in proposition 7.7.4, with $\lambda = \lambda_{M\otimes N}$ and $\delta = \delta_{M\otimes N}$. The latter maps then define a left-right YD module structure on $M \otimes N$. □

Note that the formulas from the proposition can be easily generalized to the case of n YD modules.

Remark 7.7.13. As was noticed in remark 7.7.6, the notion of YD module is self-dual. However, the structures from proposition 7.7.12 are not self-dual. One thus obtains an ***alternative notion of tensor product for YD modules***. Namely, one should take the twisted diagonal coaction and the diagonal action:

$$h \cdot (m \otimes n) = h_{(1)} \cdot m \otimes h_{(2)} \cdot n,$$
$$\delta(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes n_{(1)} \cdot m_{(1)}.$$

Both tensor products of YD modules presented here were introduced by L.A.Lambe and D.E.Radford in [44].

Yetter-Drinfel′d modules and the Drinfel′d double

Now we turn to a multi-braided module interpretation of a left-right Yetter-Drinfel′d module (M,λ,δ) over a bialgebra H in $\mathbf{vect}_{\Bbbk}$. An easy preliminary lemma is first necessary:

Lemma 7.7.14. The following functors give an equivalence of categories:

$$\mathbf{Mod}^H \simeq \mathbf{Mod}_{H^*},$$

$$(M,\delta) \mapsto (M,(\mathrm{Id}_M \otimes ev)\circ(\delta\otimes \mathrm{Id}_{H^*})), \tag{7.20}$$

$$(M,(\rho\otimes \mathrm{Id}_H)\circ(\mathrm{Id}_M \otimes coev)) \leftarrow\!\shortmid (M,\rho). \tag{7.21}$$

Cf. remark 7.5.8 and the graphical interpretation therein.

Note that for this lemma to hold, it is essential to use the “rainbow” duality.

Lemmas 7.3.1 and 7.7.14 allow to see our left-right YD module M as a right module (M,ρ_H) over the opposite algebra H^{op} and a right module (M,ρ_{H^*}) over the dual algebra H^*, with the compatibility condition obtained from (YD) (cf. also figure 7.6):

$$\rho_{H^*}\circ(\rho_H\otimes \mathrm{Id}_{H^*})\circ(\mathrm{Id}_M\otimes((\mathrm{Id}_H\otimes ev\otimes \mathrm{Id}_{H^*})\circ((\tau\circ\Delta)\otimes\mu^*)) =$$

$$\rho_H\circ(\rho_{H^*}\otimes \mathrm{Id}_H)\circ(\mathrm{Id}_M\otimes(\tau\circ(\mathrm{Id}_H\otimes ev\otimes \mathrm{Id}_{H^*})\circ(\Delta\otimes(\tau\circ\mu^*))).$$

If H is a Hopf algebra with an invertible antipode s, this condition reads

$$\rho_{H^*}\circ(\rho_H\otimes \mathrm{Id}_{H^*}) =$$

$$\rho_H\circ(\rho_{H^*}\otimes \mathrm{Id}_H)\circ(\mathrm{Id}_M\otimes(((ev\circ(s^{-1}\otimes \mathrm{Id}_{H^*}))\otimes\tau\otimes ev)\circ\omega_6\circ(\Delta^2\otimes(\mu^*)^2)))$$

(recall notations (1.4) and (7.4)), or, graphically,

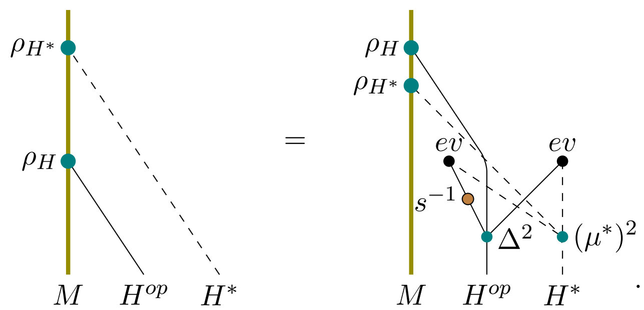


Figure 7.29: YD modules: compatibility between the H^{op} and the H^* actions

On the element level, the compatibility condition becomes

$$(a\cdot h)\cdot l = \left\langle s^{-1}(l_{(1)}),h_{(1)}\right\rangle\left\langle l_{(3)},h_{(3)}\right\rangle(a\cdot l_{(2)})\cdot h_{(2)},$$

where $a\in M$, $h\in H$, $l\in H^*$, the pairing $\langle,\rangle$ is the evaluation, and all the actions are denoted by $\cdot$ for simplicity.

Arguments similar to those leading to proposition 7.3.2 give

Proposition 7.7.15. *Let $(H, \mu, \nu, \Delta, \varepsilon, s)$, be a finite-dimensional $\Bbbk$-linear Hopf algebra (consequently, with an invertible antipode). The following categories are equivalent:*

$$\mathbf{Mod}_{H^{op} \underset{\sigma^{-1}}{\otimes} H^*} \simeq \overline{\mathbf{Mod}}_{(H^*,H^{op})} \simeq {}_H\mathbf{YD}^H \simeq \overline{\mathbf{Mod}}_{(H^{op},H^*)} \simeq \mathbf{Mod}_{H^* \underset{\sigma}{\otimes} H^{op}},$$

where

$$\sigma = ((ev \circ (s^{-1} \otimes \mathrm{Id}_{H^*})) \otimes \tau \otimes ev) \circ \omega_6 \circ (\Delta^2 \otimes (\mu^*)^2).$$

Here the $\sigma_{1,2}$ component of the pre-braiding for the pre-braided systems of UAAs is the one used for the corresponding multi-braided tensor product of UAAs (i.e. σ or σ^{-1}).

Proof. We give a complete proof of a similar statement for Hopf modules (proposition 7.8.2) and thus omit the details here. The only difference with the Hopf module case consists in the necessity of checking the naturality of σ with respect to multiplications, which can be done by easy calculations. □

The multi-braided tensor product of UAAs

$$\mathscr{D}(H) := H^* \underset{\sigma}{\otimes} H^{op}$$

from the proposition coincides, up to some ^{op} signs (due, as usual, to our choice of the "rainbow" pairing), with the familiar ***Drinfel′d double*** of H. We thus add a new viewpoint – that of multi-braided modules – to the well-known interpretation of Yetter-Drinfel′d modules as modules over the Drinfel′d double (cf. for example [51] or [69]).

Note that, if one completes the UAA structure on $\mathscr{D}(H)$ into a bialgebra structure, the category equivalence above gives another method of defining a tensor product of YD modules. Namely, one can transfer the structure from the category $\mathbf{Mod}_{\mathscr{D}(H)}$ of modules over a bialgebra (cf. proposition 7.7.12 and [44]).

Digression: braidings coming from an R-matrix as a particular case of the Woronowicz pre-braiding

The Woronowicz pre-braidings for Yetter-Drinfel′d modules are known to form a very vast family of solutions to the Yang-Baxter equation (YB). According to [23], [24] and [69], this family is complete in the category $\mathbf{vect}_\Bbbk$. This has led L.A.Lambe and D.E.Radford to use the eloquent term *quantum Yang-Baxter module* instead of the more historical term *Yetter-Drinfel′d module*, cf. [44]. Here we recover another famous family of YBE solutions, namely those coming from (a generalization of) the R-matrix of a quasi-triangular Hopf algebra (see for example [38] for an introduction to this theory), as a subfamily of Woronowicz pre-braidings, confirming the central place of Yetter-Drinfel′d modules in the study of YBE solutions. This fact is probably well-known, but the author has not found it in literature. We point out two non-conventional points in our treatment of R-matrices:

1. we do not demand their invertibility, staying in our <u>pre</u>-braided settings;
2. only the "right half" of the usual compatibility relations with the bialgebra structure is required.

We work in an arbitrary <u>symmetric</u> category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ here.

Let (μ, ν) and (Δ, ε) be a UAA and, respectively, a coUAA structures on an object H of $\mathcal{C}$. Take a left module (M, λ) over the algebra H and a morphism $R : \mathbf{I} \longrightarrow H \otimes H$. Put

$$\delta_R := (\lambda \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_H \otimes c_{H,M}) \circ (R \otimes \mathrm{Id}_M) : M \to M \otimes H$$

or, graphically,

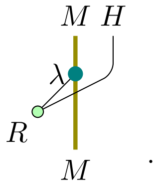
$\delta_R :=$ 

Figure 7.30: Action + R-matrix $\longmapsto$ coaction

Now try to determine conditions on R which make (M, λ, δ_R) a left-right Yetter-Drinfel′d module for any M. One arrives to the following set of conditions:

Definition 7.7.16. A morphism $R : \mathbf{I} \to H \otimes H$ is called a *weak R-matrix* for a UAA and a coUAA object $(H, \mu, \nu, \Delta, \varepsilon)$ in $\mathcal{C}$ if

1. $(\mathrm{Id}_H \otimes \Delta) \circ R = (\mu \otimes \mathrm{Id}_{H \otimes H}) \circ c_2 \circ (R \otimes R)$
2. $(\mathrm{Id}_H \otimes \varepsilon) \circ R = \nu$,
3. $\mu_{H \otimes H} \circ (R \otimes \Delta^{op}) = \mu_{H \otimes H} \circ (\Delta \otimes R)$,

or, graphically,

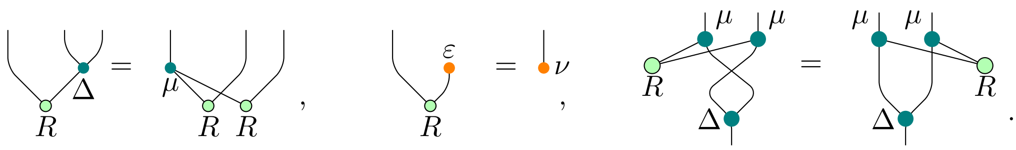


Figure 7.31: A weak R-matrix

Here we use the notation $\mu_{H \otimes H}$ from (5.7), and

$$\Delta^{op} := c \circ \Delta.$$

Note that H is not necessarily a bialgebra in general.

One can informally interpret the first two conditions by saying that R, extended to tensor powers of H in the “arched” way (cf. table 1.5), provides a duality between the UAA (H, μ, ν) on the left and the coUAA (H, Δ, ε) on the right.

As was hinted at above, a weak R-matrix for H allows to **upgrade a module structure over the algebra H into a Yetter-Drinfel′d module structure**:

Proposition 7.7.17. *Take a UAA and a coUAA object $(H, \mu, \nu, \Delta, \varepsilon)$ in $\mathcal{C}$ equipped with a weak R-matrix R.*

1. *For any left H-module (M, λ), the data (M, λ, δ_R) form a left-right YD module.*
2. *Moreover, the Woronowicz pre-braiding*

$$\sigma_{YD} = c_{M,N} \circ (\mathrm{Id}_M \otimes \lambda_N) \circ ((\delta_R)_M \otimes \mathrm{Id}_N) : M \otimes N \to N \otimes M$$

for two such modules (and hence YD modules) (M, λ_M) and (N, λ_N) coincides with the customary pre-braiding given by the R-matrix:

$$\sigma_R := c_{M,N} \circ (\lambda_M \otimes \lambda_N) \circ (\mathrm{Id}_H \otimes c_{H,M} \otimes \mathrm{Id}_N) \circ (R \otimes \mathrm{Id}_{M \otimes N}).$$

Proof. The first two conditions from the definition of weak R-matrix guarantee that δ_R is a counital coalgebra coaction, while the last one implies the YD compatibility (YD). The equality of the two pre-braidings follows from the definition of δ_R. □

The weak notion of R-matrix can be easily seen to **generalize** the well-known notion of **R-matrix in a quasi-triangular Hopf algebra**. The following two lemmas explain where the missing conditions come from.

Lemma 7.7.18. If H is moreover a Hopf algebra with an antipode s, then a weak R-matrix R for H has an inverse,

$$R^{-1} := (\mathrm{Id}_H \otimes s) \circ R,$$

in the sense that

$$\mu_{H\otimes H} \circ (R \otimes R^{-1}) = \mu_{H\otimes H} \circ (R^{-1} \otimes R) = \nu \otimes \nu.$$

Note also that if R is invertible, then the condition 2 from the definition of weak R-matrix follows from 1.

Let now H be a bialgebra with a weak R-matrix R. Take two modules (and hence YD modules) (M, λ_M) and (N, λ_N). According to proposition 7.7.12, $M \otimes N$ has a tensor product YD module structure given by

$$\begin{aligned}\lambda_{M\otimes N} &= (\lambda_M \otimes \lambda_N) \circ (\mathrm{Id}_H \otimes c_{H,M} \otimes \mathrm{Id}_N) \circ (\Delta^{op} \otimes \mathrm{Id}_{M\otimes N}),\\ \delta_{M\otimes N} &= (\mathrm{Id}_{M\otimes N} \otimes \mu) \circ (\mathrm{Id}_M \otimes c_{H,N} \otimes \mathrm{Id}_H) \circ ((\delta_R)_M \otimes (\delta_R)_N).\end{aligned}$$

On the other hand, another coaction is given via the weak R-matrix:

$$(\delta_R)_{M\otimes N} = (\lambda_{M\otimes N} \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_H \otimes c_{H,M\otimes N}) \circ (R \otimes \mathrm{Id}_{M\otimes N}).$$

Similarly, the unit object $\mathbf{I}$ of $\mathcal{C}$ can be endowed with two different coactions ν and $(\delta_R)_{\mathbf{I}}$.

Lemma 7.7.19. If a weak R-matrix R for a bialgebra H in $\mathcal{C}$ satisfies additional conditions

1. $(\Delta \otimes \mathrm{Id}_H) \circ R = (\mathrm{Id}_{H\otimes H} \otimes \mu^{op}) \circ c_2 \circ (R \otimes R)$
2. $(\varepsilon \otimes \mathrm{Id}_H) \circ R = \nu$,

then the two Yetter-Drinfel′d structures on $M \otimes N$ (resp. $\mathbf{I}$) described above coincide.

Definition 7.7.20. A weak R-matrix satisfying the conditions from the previous lemma is called *an R-matrix*.

Note that our conditions on R correspond to the conditions usually imposed on R^{-1}. Remark also that the invertibility of R is not required in our definition.

The preceding lemma leads to a stronger version of proposition 7.7.17 for R-matrices:

Proposition 7.7.21. *Take a bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ in $\mathcal{C}$ equipped with an R-matrix R. Then ${}_H\mathbf{Mod}$ can be seen as a full pre-braided subcategory of ${}_H\mathbf{YD}^H$ via the inclusion*

$$\begin{aligned}i_R : {}_H\mathbf{Mod} &\hookrightarrow {}_H\mathbf{YD}^H,\\ (M, \lambda) &\mapsto (M, \lambda, \delta_R).\end{aligned}$$

Proof. Point 1 of proposition 7.7.17 shows that the map is well defined. Further, a morphism in ${}_H\mathbf{Mod}$ automatically preserves the co-actions δ_R (see the definition of the latter and use the naturality of c), so it is the same thing as a morphism in ${}_H\mathbf{YD}^H$ for the structures from the statement. Lemma 7.7.19 proves that the functor i_R preserves the monoidal structures, and point 2 of proposition 7.7.17 asserts that i_R respects the pre-braidings. □

7.8 Hopf (bi)modules

In this section, we apply our "braided" tools to Hopf (bi)modules.

Similarly to what was done for Yetter-Drinfel′d modules in section 7.7, we interpret the Hopf compatibility condition in terms of pre-braidings, which allows us to use the language of multi-braided modules and, consequently, the language of multi-braided tensor products of UAAs (cf. section 7.2). We thus recover

1. the Heisenberg doubles as a multi-braided tensor product corresponding to the structure of Hopf module,
2. and the algebra X of C.Cibils and M.Rosso (cf. [14]), as well as F.Panaite's algebras Y and Z (cf. [65]), as multi-braided tensor products corresponding to the structure of Hopf bimodule.

Moreover, we include the algebras X, Y and Z into a family of $4! = 24$ algebras, giving explicit isomorhisms between them. One thus avoids tedious verifications and case-by-case studies, made for instance in [65].

As for homologies, we present two theories here:

1. the cohomology of Hopf bimodules, introduced by C.Ospel in the one-module case (cf. [64]) and R.Taillefer (cf. [77] and [78]) in the two-module case;
2. the cohomology of Hopf modules, defined by F.Panaite and D.Ştefan in [66].

Our theory of multi-braided adjoint modules with coefficients turns out to be useful in the first case, giving thus an application of our "braided" interpretation of Hopf bimodules on the homology level. As for the second cohomology theory, we show the relevance of the structure mixing techniques from section 6.2.

Definitions

Recall the categorical definitions of Hopf (bi)modules:

Definition 7.8.1. Take a pre-braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$.

- A right module structure $\rho : M \otimes H \to M$ and a right comodule structure $\delta : M \to M \otimes H$ on M are said to form a *right-right Hopf module* structure over a bialgebra H in $\mathcal{C}$ if they satisfy the *right-right Hopf compatibility condition*

$$\delta \circ \rho = (\rho \otimes \mu) \circ (\mathrm{Id}_M \otimes c_{H,H} \otimes \mathrm{Id}_H) \circ (\delta \otimes \Delta) : M \otimes H \to M \otimes H. \tag{7.22}$$

- A left module structure $\lambda : H \otimes M \to M$ and a right comodule structure $\delta : M \to M \otimes H$ on M are said to form a *left-right Hopf module* structure over a bialgebra H in $\mathcal{C}$ if they satisfy the *left-right Hopf compatibility condition*

$$\delta \circ \lambda = (\lambda \otimes \mu) \circ (\mathrm{Id}_H \otimes c_{H,M} \otimes \mathrm{Id}_H) \circ (\Delta \otimes \delta) : H \otimes M \to M \otimes H. \tag{7.23}$$

- *Right-left* and *left-left* Hopf modules are defined similarly.
- A *Hopf bimodule* structure is a left and right module and a left and right comodule structures satisfying the bimodule, the bicomodule and all the four possible Hopf compatibility conditions.
- The categories of right-right, left-right Hopf modules and Hopf bimodules over H and their morphisms are denoted by, respectively, $\mathbf{Mod}_H^H$, ${}_H\mathbf{Mod}^H$ and ${}_H^H\mathbf{Mod}_H^H$, and similarly for the two remaining "double" structures and for the "single" structures of left or right (co)modules.

Condition (7.22) is graphically depicted as

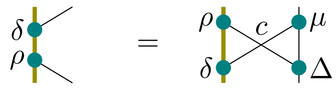


Figure 7.32: Right-right Hopf compatibility condition

In $\mathbf{Mod}_R$, it takes the familiar form

$$(a \cdot h)_{(0)} \otimes (a \cdot h)_{(1)} = a_{(0)} \cdot h_{(1)} \otimes a_{(1)} \cdot h_{(2)} \qquad \forall a \in M, h \in H.$$

Hopf bimodules were introduced by W.D.Nichols in [62] and rediscovered further by S.L.Woronowicz in [82].

We place everything into the category $\mathcal{C} = \mathbf{vect}_{\Bbbk}$; see the remarks in the beginning of section 7.6 concerning a possible higher level of generality.

Fix a bialgebra H in $\mathbf{vect}_{\Bbbk}$.

Hopf modules and the Heisenberg double

We work at first with ***right-right Hopf modules*** and multi-modules over the pre-braided system $\overline{H} = (H, H^*)$ from proposition 7.6.2. This study essentially follows the lines of section 7.3.

Lemma 7.7.14 allows to construct a chain of category equivalences in the spirit of proposition 7.3.2:

Proposition 7.8.2. *1. The following categories are equivalent:*

$$\mathbf{Mod}_H^H \simeq \overline{\mathbf{Mod}}_{(H,H^*)} \simeq \mathbf{Mod}_{H^* \underset{\sigma}{\otimes} H},$$

where

$$\sigma = \sigma_{YD} = \tau \circ (\mathrm{Id}_H \otimes ev \otimes Id_{H^*}) \circ (\Delta \otimes \mu^*) : H \otimes H^* \to H^* \otimes H$$

(cf. figure 7.23).

2. If the bialgebra H turns out to be a Hopf algebra with an antipode s, then this chain of category equivalences can be continued on the left:

$$\mathbf{Mod}_{H \underset{\sigma^{-1}}{\otimes} H^*} \simeq \overline{\mathbf{Mod}}_{(H^*,H)} \simeq \mathbf{Mod}_H^H,$$

where

$$\sigma^{-1} = (\sigma_{YD})^{-1} = (\mathrm{Id}_H \otimes (ev \circ (s \otimes \mathrm{Id}_{H^*}) \otimes Id_{H^*}) \circ (\Delta \otimes \mu^*) \circ \tau.$$

Proof. According to remark 7.1.10 combined with example 6.1.2, a right normalized multi-braided (H, H^*)-module is a right algebra H-module and a right algebra H^*-module structures ρ_H and ρ_{H^*} on M, compatible in the sense of (7.1):

$$\rho_{H^*} \circ (\rho_H \otimes \mathrm{Id}_{H^*}) = \rho_H \circ (\rho_{H^*} \otimes \mathrm{Id}_H) \circ (\mathrm{Id}_M \otimes (\tau \circ (\mathrm{Id}_H \otimes ev \otimes Id_{H^*}) \circ (\Delta \otimes \mu^*))).$$

Further, according to lemma 7.7.14, a right-right Hopf module structure over H is the same thing as a right algebra H-module and a right algebra H^*-module structures, with the compatibilty condition obtained by

- applying $\mathrm{Id}_M \otimes ev$ to relation (7.22), tensored with Id_{H^*} on the right,
- and then using equation (7.20) in order to transform H-comodule structures into H^*-module structures.

The two compatibilty conditions coincide, implying $\mathbf{Mod}_H^H \simeq \overline{\mathbf{Mod}}_{(H,H^*)}$.

Next, since the Woronowicz pre-braiding σ_{YD} is natural with respect to the units (lemma 7.5.6), proposition 7.2.6 gives $\overline{\mathbf{Mod}}_{(H,H^*)} \simeq \mathbf{Mod}_{H^* \underset{\sigma}{\otimes} H}$ (remark 7.2.2 relieves us from exhibiting normalized pairs for units, since $r = 2$).

In the Hopf algebra case, proposition 7.1.13 combined with the invertibility lemma 7.6.12 give $\overline{\mathbf{Mod}}_{(H,H^*)} \simeq \overline{\mathbf{Mod}}_{(H^*,H)}$, the last category equivalent to $\mathbf{Mod}_{H \underset{\sigma^{-1}}{\otimes} H^*}$ again via proposition 7.2.6. $\square$

Denote by

$$\mathscr{H}'(H) := H \underset{\sigma^{-1}}{\otimes} H^*$$

one of the multi-braided tensor products of UAAs from the proposition. Then

$$\mathscr{H}(H) := \mathscr{H}'(H^*) = H^* \underset{\sigma^{-1}}{\otimes} H$$

is the well-known ***Heisenberg double*** of the Hopf algebra H (cf. for example [60] or [14]). Note that some authors use this name for one of the other multi-braided tensor products of UAAs described in the preceding and the following propositions. Moreover, because of our use of the "rainbow" pairing between $H \otimes H$ and $H^* \otimes H^*$, our definitions may differ from the conventional ones by some ^{op} signs.

Hopf bimodules and the algebras X, Y and Z

Continuing in the same vein, we are now heading towards an interpretation of ***Hopf bimodules*** in terms of multi-modules over a pre-braided system of UAAs, and, consequently (via proposition 7.2.6), in terms of modules over a multi-braided tensor product of UAAs. Since a Hopf bimodule is simultaneously an algebra bimodule, a coalgebra bicomodule, and a Hopf module for the four possible left/right choices, the key ideas are

1. to mix constructions from propositions 7.3.2 and 7.8.2,
2. and to study the behavior of category equivalences from proposition 7.8.2 with respect to twisting the multiplication and/or the comultiplication of our bialgebra in the sense of lemma 7.6.6 (recall that such twists allow the left-right passage, cf. lemma 7.3.1).

We start with point 2. Recall the notations of type σ_{YD}^{op} from proposition 7.6.7.

Proposition 7.8.3. *The following categories are equivalent:*

1. ${}_H\mathbf{Mod}^H \simeq \overline{\mathbf{Mod}}_{(H^{op},H^*)} \simeq \mathbf{Mod}_{H^* \underset{\sigma}{\otimes} H^{op}}$, *where* $\sigma = \sigma_{YD}^{op}$*;*
2. ${}^H\mathbf{Mod}_H \simeq \overline{\mathbf{Mod}}_{(H,(H^*)^{op})} \simeq \mathbf{Mod}_{(H^*)^{op} \underset{\sigma}{\otimes} H}$, *where* $\sigma = \sigma_{YD}^{cop}$*;*
3. ${}_H^H\mathbf{Mod} \simeq \overline{\mathbf{Mod}}_{(H^{op},(H^*)^{op})} \simeq \mathbf{Mod}_{(H^*)^{op} \underset{\sigma}{\otimes} H^{op}}$, *where* $\sigma = \sigma_{YD}^{op,cop}$.

If the bialgebra H turns out to be a Hopf algebra with an invertible antipode s, then the H and H^* components of each pre-braided system and of each multi-braided tensor product of UAAs can be interchanged, with σ replaced by σ^{-1}, given explicitly by, respectively,

1. $(\sigma_{YD}^{op})^{-1} = (\mathrm{Id}_H \otimes (ev \circ (s^{-1} \otimes \mathrm{Id}_{H^*}) \otimes Id_{H^*}) \circ (\Delta \otimes (\tau \circ \mu^*)) \circ \tau$;

2. $(\sigma_{YD}^{cop})^{-1} = (\mathrm{Id}_H \otimes (ev \circ (s^{-1} \otimes \mathrm{Id}_{H^*}) \otimes Id_{H^*}) \circ ((\tau \circ \Delta) \otimes \mu^*) \circ \tau;$
3. $(\sigma_{YD}^{op,cop})^{-1} = (\mathrm{Id}_H \otimes (ev \circ (s \otimes \mathrm{Id}_{H^*}) \otimes Id_{H^*}) \circ ((\tau \circ \Delta) \otimes (\tau \circ \mu^*)) \circ \tau.$

Proof. Apply proposition 7.8.2 to

1. ${}_H\mathbf{Mod}^H \simeq \mathbf{Mod}_{H^{op}}^H \simeq \mathbf{Mod}_{H^{op}}^{H^{op}}$;
2. ${}^H\mathbf{Mod}_H \simeq \mathbf{Mod}_H^{H^{cop}} \simeq \mathbf{Mod}_{H^{cop}}^{H^{cop}}$;
3. ${}_H^H\mathbf{Mod} \simeq \mathbf{Mod}_{H^{op}}^{H^{cop}} \simeq \mathbf{Mod}_{H^{op,cop}}^{H^{op,cop}}$.

In the Hopf algebra case, use the antipodes for the "twisted" structures given in lemma 7.6.6, and apply proposition 7.1.13 combined with the invertibility lemma 7.6.12. □

Now, according to point 1 of our program, we mix all the "bi"-structures into a "quadri"-structure of a Hopf bimodule:

Theorem 9. *1. The following categories are equivalent:*

$$ {}_H^H\mathbf{Mod}_H^H \simeq \overline{\mathbf{Mod}}_{(H,H^{op},H^*,(H^*)^{op})} \simeq \mathbf{Mod}_{(H^*)^{op} \underset{\sigma}{\otimes} H^* \underset{\sigma}{\otimes} H^{op} \underset{\sigma}{\otimes} H}, $$

where the pre-braiding $\overline{\sigma}$ on the UAA system

$$ \overline{H}_4 := (H, H^{op}, H^*, (H^*)^{op}) $$

is given by σ_{Ass} on each component and, on the pairs of distinct components, by

$$ \begin{array}{lll} \sigma_{1,2} = \tau, & \sigma_{1,3} = \sigma_{YD}, & \sigma_{1.4} = \sigma_{YD}^{cop}, \\ \sigma_{3,4} = \tau, & \sigma_{2,3} = \sigma_{YD}^{op}, & \sigma_{2,4} = \sigma_{YD}^{op,cop}. \end{array} $$

2. *If the bialgebra H turns out to be a Hopf algebra with an invertible antipode s, then the components of $\overline{H}_4$, and thus of*

$$ \mathscr{X}(H) := (H^*)^{op} \underset{\sigma}{\otimes} H^* \underset{\sigma}{\otimes} H^{op} \underset{\sigma}{\otimes} H, $$

can be arranged in an arbitrary order, with the components of $\overline{\sigma}$ replaced by their inverses when necessary. This gives $4! = 24$ isomorphic multi-braided tensor products of UAAs, these isomorphisms being compatible with the equivalences of corresponding module categories. Explicitly, given an $s \in S_4$, the algebra morphism $T_s^{\overline{\sigma^{-1}}}$ effectuates the permutation s of the 4 components of the $\mathscr{X}(H)$.

Proof. 1. Lemma 7.3.1 and its dual version, together with lemma 7.7.14, show that a Hopf bimodule is the same thing as a module over four UAAs H, H^{op}, H^* and $(H^*)^{op}$, with a compatibility condition for each of the six pairs of algebras. Propositions 7.3.2 and its dual version, and proposition 7.8.2 and its twisted version 7.8.3, translate each compatibility condition into the "braided" language, giving the family $\overline{\sigma}$ from the statement of the theorem. Thus, to get the first category equivalence, it remains to check that $\overline{\sigma}$ is indeed a pre-braiding. Since each two-component subsystem of $(4, \overline{H}_4, \overline{\sigma})$ is a pre-braided system according to the above cited propositions, one has to check the YBE on tensor products of three distinct components only. We study in detail the case $H \otimes H^{op} \otimes H^*$ here, the three other triples being similar. The left actions of H and H^{op} on H^* are given by formula (7.15) and, respectively, the left versions of formulas (7.15) and (7.7). They commute because of the coassociativity of H^*. The form of the Woronowicz pre-braiding σ_{YD} allows to conclude.

The second equivalence follows from proposition 7.2.6, remark 7.2.2 and the naturality of all the pre-braidings in the story with respect to the units (which is a consequence of lemma 7.5.6 and of the naturality of τ).

2. In the Hopf algebra case, use, as usual, propositions 7.1.13 and 7.2.7. $\square$

Note that in the Hopf algebra case, the multi-braided tensor product algebra $\mathscr{X}(H)$ coincides, up to a permutation (of the kind described in the theorem) and some ^{op} signs (due, as usual, to our choice of the "rainbow" pairing), with the algebra X of C.Cibils and M.Rosso (cf. [14]). We recover thus their interpretation of Hopf bimodules as modules over an algebra, adding to it one more viewpoint – that of multi-braided modules. This gives in particular an alternative proof of the associativity of the algebra X. Further, among the 24 algebras isomorphic to $\mathscr{X}(H)$, one recovers F.Panaite's algebras $\mathscr{Y}(H)$ and $\mathscr{Z}(H)$ (cf. [65]) and explicit isomorhisms between them. One thus avoids tedious verifications and case-by-case study.

Homological consequences

Having interpreted Hopf bimodules as multi-braided $\overline{H}_4$-modules, we can use them as coefficients for braided differentials. As an example, we apply proposition 7.1.11 to our Hopf algebra context and the pre-braided system $\overline{H}_4$, choosing $s = t = 1$. Recall notations (1.3) and (1.4).

Proposition 7.8.4. *Take a Hopf bimodule $(M, \rho : M \otimes H \to M, \lambda : H \otimes M \to M, \delta : M \to M \otimes H, \gamma : M \to H \otimes M)$ over a bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ in $\mathbf{vect}_{\Bbbk}$. The bar complex for H with coefficients in (M, ρ), i.e. $(M \otimes T(H), {}^{\rho}d)$, is a complex in ${}^{H}_{H}\mathbf{Mod}^{H}_{H}$, i.e. the differentials $({}^{\rho}d)_n$ are Hopf bimodule morphisms, the Hopf bimodule structure on $M \otimes H^{\otimes n}$ being given by*

$$\begin{aligned}
\rho_{bar} &:= \mu_{n+1} : M \otimes H^{\otimes n} \otimes H \to M \otimes H^{\otimes n},\\
\lambda_{bar} &:= \lambda_1 : H \otimes M \otimes H^{\otimes n} \to M \otimes H^{\otimes n},\\
\delta_{bar} &:= (\mu^n)_{n+2} \circ \omega_{2(n+1)}^{-1} \circ (\delta \otimes \Delta^{\otimes n}) : M \otimes H^{\otimes n} \to M \otimes H^{\otimes n} \otimes H,\\
\gamma_{bar} &:= \mu^n \circ \omega_{2(n+1)}^{-1} \circ (\gamma \otimes \Delta^{\otimes n}) : M \otimes H^{\otimes n} \to H \otimes M \otimes H^{\otimes n},
\end{aligned}$$

where $\omega_{2(n+1)} \in S_{2(n+1)}$ is defined by (7.4), and $S_{2(n+1)}$ acts on $M \otimes H^{\otimes(2n+1)}$ by the flip τ.

The proof essentially repeats that of proposition 7.3.3.

The Hopf bimodule structure on $M \otimes H^{\otimes n}$ thus combines the "peripheral" bimodule and the codiagonal bicomodule structures, the latter graphically depicted as

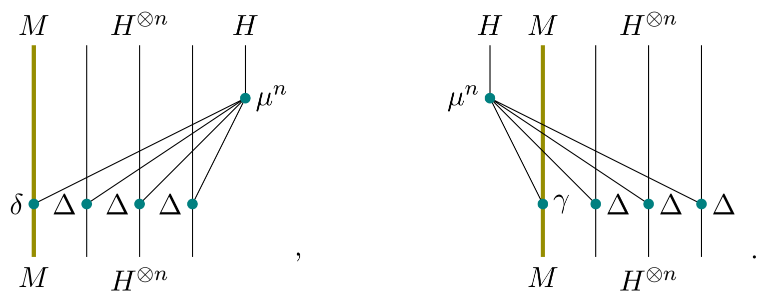


Figure 7.33: H-bicomodule structure on the bar complex with coefficients

This Hopf bimodule structure on the bar complex, as well as its dual one on the cobar complex, are essential in defining the cohomology of Hopf bimodules, introduced by C.Ospel in the one-module case (cf. [64]) and R.Taillefer (cf. [77] and [78]) in the two-module case.

Structure mixing techniques for Hopf modules

We finish this section with an interpretation of F.Panaite and D.Ştefan's cohomology of Hopf modules (cf. [66]) using the tools from section 6.2.

In $\mathbf{vect}_{\Bbbk}$, take two left-right Hopf modules (M, λ_M, δ_M) and (N, λ_N, δ_N) over a bialgebra (H, μ_H, Δ_H). Amalgamate all these structures together:

$$V := H \oplus M \oplus N$$

and define a multiplication μ and a comultiplication Δ on V by

$$\begin{aligned} \mu|_{H\otimes H} &= \mu_H, & \Delta|_H &= \Delta_H,\\ \mu|_{H\otimes M} &= \lambda_M, & \Delta|_M &= \delta_M,\\ \mu|_{H\otimes N} &= \lambda_N, & \Delta|_N &= \delta_N, \end{aligned}$$

extending μ by zero for other couples of spaces. One easily checks the following

Lemma 7.8.5. The applications μ and Δ above define a (non-unital non-counital) bialgebra structure on V if and only if the following conditions hold:

- H is a bialgebra;
- M and N are left-right Hopf modules over H.

If H is moreover unital and counital, then 1_H and ε_H become, respectively, the left unit and the right counit of V.

Proposition 7.6.10 now gives bidifferentials on $T(V) \otimes T(V^*)$ and, in particular, on its sub-bicomplex (for any of the four structures from the proposition)

$$T(H) \otimes M \otimes T(H^*) \otimes N^*.$$

Note that we cheat a little here, since the bialgebra V has only one-sided unit and counit. Some technical work is necessary in order to see that on the above sub-bicomplex, this partial structure suffices.

Writing explicitly the last bidifferential from proposition 7.6.10 in our setting, one gets

Proposition 7.8.6. *For two finite-dimensional left-right Hopf modules* (M, λ_M, δ_M) *and* (N, λ_N, δ_N) *over a finite-dimensional* $\Bbbk$*-bialgebra* $(H, \mu, \nu, \Delta, \varepsilon)$*, there is a bidifferential* *structure on $T(H)\otimes M\otimes T(H^*)\otimes N^*$ given, using our usual notations, by*

$$\begin{aligned}
& d(h_1\dots h_n\otimes a\otimes l_1\dots l_m\otimes b) = \\
& \quad (-1)^{n+1}\left\langle l_{1(1)},a_{(1)}\right\rangle\left\langle l_{1(2)},h_{n(2)}\right\rangle\left\langle l_{1(3)},h_{n-1(2)}\right\rangle\dots\left\langle l_{1(n+1)},h_{1(2)}\right\rangle\times \\
& \quad \times h_{1(1)}\dots h_{n(1)}\otimes a_{(0)}\otimes l_2\dots l_m\otimes b \\
& \quad +\sum_{i=1}^{m-1}(-1)^{n+i+1}h_1\dots h_n\otimes a\otimes l_1\dots l_{i-1}(l_i\cdot l_{i+1})l_{i+2}\dots l_m\otimes b \\
& \quad +(-1)^{n+m+1}h_1\dots h_n\otimes a\otimes l_1\dots l_{m-1}\otimes(l_m\cdot b), \\
& d'(h_1\dots h_n\otimes a\otimes l_1\dots l_m\otimes b) = \\
& \quad -\left\langle l_{1(2)},h_{1(m+1)}\right\rangle\dots\left\langle l_{m(2)},h_{1(2)}\right\rangle\left\langle b_{(1)},h_{1(1)}\right\rangle\times \\
& \quad \times h_2\dots h_n\otimes a\otimes l_{1(1)}\dots l_{m(1)}\otimes b_{(0)} \\
& \quad +\sum_{i=1}^{n-1}(-1)^{i-1}h_1\dots h_{i-1}(h_i\cdot h_{i+1})h_{i+2}\dots h_n\otimes a\otimes l_1\dots l_m\otimes b \\
& \quad +(-1)^{n-1}h_1\dots h_{n-1}\otimes(h_n\cdot a)\otimes l_1\dots l_m\otimes b.
\end{aligned}$$

Substituting the graded vector space $T(H)\otimes M\otimes T(H^*)\otimes N^*$ we work in with its alternative version $\mathrm{Hom}_{\Bbbk}(N\otimes T(H),T(H)\otimes M)$, we obtain (the dual of a mirror version of) the cohomology of Hopf modules from [66]. Note that the theory of multi-braided adjoint modules with coefficients and other "braided" tools can now be applied to this cohomology theory.

Part III

A Categorification of Virtuality and Self-distributivity

Chapter 8

A survey of braid and virtual braid theories

This chapter is a short and very selective introduction to the theory of braids and virtual braids. We present only the concepts and results which will be virtualized and/or categorified in subsequent chapters. Different aspects of braid theory are involved in our story. Our vision of braids can thus be described, somewhat poetically, as that of ***crossings at the intersection of algebra, topology, representation theory and category theory***.

8.1 Different avatars of braids

The notion of braids, completely intuitive from the topological viewpoint, was first introduced by Emil Artin in 1925, although it was implicitly used by many XIXth century mathematicians. Its algebraic counterpart, the notion of braid groups, accompanies its "twin brother" from the birth. Representation theory methods have been extensively applied to braids since then, with, as two major examples, the Burau representation (and, later, quantum invariants) and Artin action on free groups, both recalled in this section (see for instance [4] for more details). Braided categories, a natural categorification of the braid group, appeared in 1993 (A.Joyal and R.H.Street, [33]), long after symmetric categories, corresponding to symmetric groups (1965, S.Eilenberg and G.M.Kelly, [21]). The aim of this section is to recall all those different viewpoints on braids, before proceeding to their virtualization in the rest of this part.

Almost no proofs are given here. For a more detailed exposition, the reader is sent to the wonderfully written books [4] and [40] for the general aspects of braid theory, and [79] for the categorical aspects.

Topology: the birth

Topologically, a braid can be thought of as a C^1 embedding of n copies $I_1, \ldots, I_n$ of $I = [0,1]$ into $\mathbb{R}^2 \times I$, with the left ends of the I_j's being sent bijectively to points $(l, 0, 0)$, $1 \leqslant l \leqslant n$; the right ends being sent bijectively to $(r, 0, 1)$, $1 \leqslant r \leqslant n$; the tangents being vertical at the endpoints of the I_j's; and the images of the I_j's always looking "up" (i.e. the embedding $I_j \hookrightarrow \mathbb{R}^2 \times I$ composed with the projection $\mathbb{R}^2 \times I \twoheadrightarrow I$ is a homeomorphism). One could also consider smooth or piece-wise linear embeddings, with the same resulting theory. Such embeddings, considered up to isotopy, are called

braids on n strands. They are represented by ***diagrams*** corresponding to the projection "forgetting" the second coordinate of $\mathbb{R}^2 \times I$, with the under/over information for each crossing:

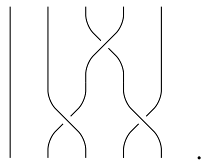


Figure 8.1: A braid on 5 strands

The vertical "stacking" of braids on n strands, followed by an obvious contraction, defines a group structure on them:

$$\boxed{\xi_2} \cdot \boxed{\xi_1} = \boxed{\begin{array}{c}\xi_2\\ \hline \xi_1\end{array}}.$$

Figure 8.2: Composition of braids

This group is denoted by $\mathcal{B}_n$.

The topological interpretation of braids is particularly useful in knot theory due to the closure operation, effectuated by passing to $\mathbb{R}^3 \supset \mathbb{R}^2 \times I$ and connecting all the pairs of points $((j,0,0),(j,0,1))$, with $1 \leqslant j \leqslant n$, by untangled arcs living "outside" the braid.

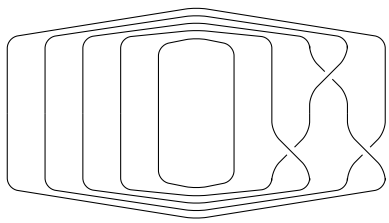


Figure 8.3: The closure of a braid

Alexander's theorem (1923) assures that all links and knots are obtained this way, and Markov's theorem (1935) explains which braids give the same link.

Algebra: a twin brother

Algebraically, ***the (Artin) braid group*** B_n is a generalization of the symmetric group S_n. It is defined by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, 1 \leqslant i,j \leqslant n-1, \qquad (Br_C)$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \forall\, 1 \leqslant i \leqslant n-2. \qquad (Br_{YB})$$

The first equation means partial commutativity. The second one is a form of the ***Yang-Baxter equation***, or briefly ***YBE***.

The symmetric group is then the quotient of B_n by

$$\sigma_i^2 = 1 \qquad \forall\, 1 \leqslant i \leqslant n-1. \qquad (Symm)$$

Other quotients of (the group rings of) Artin braid groups are extensively studied by representation theorists. Hecke algebras give a rich example.

The link between algebraic and topological viewpoints was suggested by Emil Artin already in 1925:

Theorem 10. *There exists an isomorphism φ between the groups B_n and $\mathcal{B}_n$, given by*

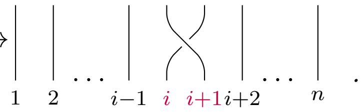
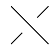
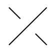
$\sigma_i \mapsto$ 

Figure 8.4: $B_n \simeq \mathcal{B}_n$

Thus the Yang-Baxter equation (Br_{YB}) is simply the algebraic translation of the third Reidemeister move for braid diagrams (cf. figure 2.2).

Positive braids: little brothers

In some contexts it is interesting to regard braid diagrams having crossings  only. The braids represented by such diagrams are called ***positive***. Their interest resides, among other properties, in the fact that every braid is a (non-commutative) quotient of two positive braids. Admitting no inverses, they form a monoid only. The algebraic counterpart is ***the positive braid monoid*** B_n^+. It is generated – as a monoid this time – by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, with the same relations as those defining B_n. One can show that this is a submonoid of B_n. A "positive" analogue of theorem 10 is obvious.

Remark 8.1.1. Considering crossings  instead, one gets the notions of negative braids and negative braid monoid B_n^-, isomorphic to B_n^+ via the obvious monoid map $\sigma_i^{-1} \mapsto \sigma_i$. Note that for most authors our positive braids are negative, and vice versa. We prefer our terminology for the sake of compatibility with the previous parts.

Representations: a full wardrobe

Algebraic structures, even those admitting easy descriptions, are often difficult to study using algebraic tools only. Even comparing two elements of a group defined by generators and relations can be a hard task. A recurrent solution consists in exploring representations (linear or more general) of algebraic objects instead, i.e., in a metaphorical language, in looking for fitting clothes. Free actions are of particular interest, since they allow one to easily distinguish different elements. The corresponding concept on the topological level is that of invariants.

Note that symmetric groups are even <u>defined</u> via their action on a set. This suggests the importance of braid group representations, two of which are recalled here.

The first one was discovered by W.Burau as early as in 1936 ([6]).

Proposition 8.1.2. *An action of the braid group B_n on $\mathbb{Z}[t^{\pm 1}]^{\oplus n}$ can be given, in the matrix form, by*

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 1-t & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}, \tag{8.1}$$

$$\rho(\sigma_i^{-1}) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t^{-1} & t^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

This representation is interesting from the topological viewpoint: the Alexander polynomial, a famous knot invariant, can be interpreted as $\det(\mathbf{I} - \rho_*(b))$, where b is any braid whose closure gives the knot in question, and ρ_* is (a reduced version of) the Burau representation.

The Burau representation, conjectured to be faithful for a long time, turns out not to be so for $n \geqslant 5$. A faithful linear representation was found later by R.Lawrence, D.Krammer and S.Bigelow.

Another representation – a faithful one this time – was already known by E.Artin. See [4] or [26] for a topological proof, or [18] for a short algebraic one.

Theorem 11. *Denote by F_n the free group with n generators $x_1, \ldots, x_n$. The braid group B_n faithfully acts on F_n according to the formulas*

$$\sigma_i(x_j) = \begin{cases} x_i & \text{if } j = i+1, \\ x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_j & \text{otherwise;} \end{cases} \qquad \sigma_i^{-1}(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

These two seemingly different representations can be interpreted as particular cases of a much more general one, which we describe next.

Shelves and racks: arranging the wardrobe

Here we give an example of a topological idea inspiring important algebraic structures – that of shelves, racks and quandles – with an extremely rich representation theory.

For a detailed introduction to the theory of self-distributive structures, as well as for numerous examples, we send the reader to the seminal papers of D.Joyce [34] and S.Matveev [58], or to [35] and [15] for very readable surveys. The connections between racks and braids are explored in detail in [26]. For free self-distributive structures applied to braids, [18] and [39] are nice sources. Some related notions, with historical and bibliographical remarks, were already introduced in section 4.2, but we recall them here for the reader's convenience.

Definition 8.1.3.
- A *shelf* is a set S with a binary operation $\lhd : S \times S \longrightarrow S$ satisfying the *self-distributivity condition*

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c) \qquad \forall a, b, c \in S. \tag{SD}$$

- If moreover the application $a \mapsto a \lhd b$ is a bijection on S for every $b \in S$, that is if there exists an "inverse" application $\widetilde{\lhd} : S \times S \longrightarrow S$ such that

$$(a \lhd b) \widetilde{\lhd} b = (a \widetilde{\lhd} b) \lhd b = a \qquad \forall a, b \in S, \tag{R}$$

  then the couple $(S, \lhd)$ is called *a rack*.
- A *quandle* is a rack satisfying moreover

$$a \lhd a = a \qquad \forall a \in S. \tag{Q}$$

We use the term *SD structures* to refer to any of these three structures, emphasizing the importance of relation (SD).

There are numerous examples of SD structures coming from various areas of mathematics. Only several of them are relevant here, allowing us to recover the two braid group actions described above.

Example 8.1.4. 1. The one-element shelf is necessarily a quandle, called the *trivial quandle*.

2. A group G can be endowed with, among others, a *conjugation quandle* structure:

$$\begin{aligned} a \lhd b &:= b^{-1}ab, \\ a \widetilde{\lhd} b &:= bab^{-1}. \end{aligned}$$

   This quandle is denoted by $\mathrm{Conj}(G)$. Morally, the quandle structure captures the properties of conjugation in a group, forgetting the multiplication structure it comes from. This is the algebraic motivation for studying self-distributivity.

3. The *Alexander quandle* is the set $\mathbb{Z}[t^{\pm 1}]$ with the operations

$$\begin{aligned} a \lhd b &:= ta + (1-t)b, \\ a \widetilde{\lhd} b &:= t^{-1}a + (1-t^{-1})b. \end{aligned} \tag{8.2}$$

4. The *cyclic rack* CR is the set of integers $\mathbb{Z}$ with the operations

$$\begin{aligned} n \lhd m &:= n+1 \qquad \forall n, m \in \mathbb{Z}, \\ n \widetilde{\lhd} m &:= n-1 \qquad \forall n, m \in \mathbb{Z}. \end{aligned}$$

   Note that it is very far from being a quandle: the property (Q) is false for all the elements. Moreover, the quotient of CR by (Q) is the trivial quandle. This somewhat strange structure will be interpreted in the context of free racks.

The theory of racks and quandles owes its rising popularity to topological applications: it allows to upgrade the fundamental group of the complement of a knot to a *complete knot invariant* (up to a symmetry; cf. [34]). The connection to groups is clear from the example of the conjugation quandle. The connection to knots and braids is illustrated by the following well-known result, which will be better explained later.

Proposition 8.1.5. *Take a shelf $(S, \lhd)$. An action of the positive braid monoid B_n^+ on $S^{\times n}$ can be given as follows:*

$$\sigma_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i \lhd a_{i+1}, a_{i+2}, \ldots, a_n).$$

If S is a rack, then this action becomes a braid group action:

$$\sigma_i^{-1}(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1} \widetilde{\lhd} a_i, a_i, a_{i+2}, \ldots, a_n).$$

More precisely, these formulas define an action of B_n^+ (resp. B_n) if and only if the couple $(S, \lhd)$ satisfies the shelf (resp. rack) axioms.

SD structures are thus the "right" structures for carrying a braid group/monoid action. They give a rich source of representations of B_n^+ and B_n.

The action from the proposition is diagrammatically depicted via braid coloring:

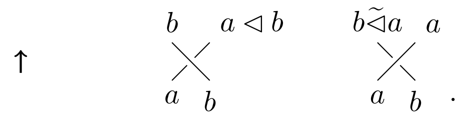


Figure 8.5: Braid group action for a rack

Thus when a strand passes over another one, the corresponding element of S acts by $\lhd$ (when going to the left) or $\widetilde{\lhd}$ (when going to the right) on the element on the lower strand. In some references it happens the other way round (i.e. it is the element on the lower strand that acts), which is simply a matter of choice; cf. also remark 8.1.1.

Note that, with our choice of composing braids from bottom to top, the actions are always depicted from bottom to top here, which is indicated above by the arrow.

Free shelves and racks: basic wardrobe

We now make a short survey of free SD structures.

Notation 8.1.6. Given a set X, the *free shelf, rack* and *quandle* on X are denoted by $FS(X), FR(X), FQ(X)$ respectively. This is abbreviated to

$$\begin{aligned} FS_n &:= FS(\{x_1, \ldots, x_n\}), \\ FS_{\mathbb{Z}} &:= FS(\{x_i, i \in \mathbb{Z}\}), \end{aligned}$$

and similar for racks and quandles.

Start with ***monogenerated*** free SD structures:

1. Free shelves, even generated by one element only, are extremely complicated structures. They have in particular allowed Patrick Dehornoy to construct, in the early 90's, a total left-invariant group order on B_n (see [39] or [18]).
2. Monogenerated quandles are trivial, since $x \lhd x = x$ for the generator x.
3. As for racks, which are intermediate objects between shelves and quandles, the monogenerated free structure is quite simple but not trivial: one has a rack isomorphism

$$\begin{aligned} FR_1 &\xrightarrow{\sim} CR, \\ ((x \lhd\ x) \lhd \cdots) \lhd x &\longmapsto n, \\ ((x \widetilde{\lhd}\ x) \widetilde{\lhd} \cdots) \widetilde{\lhd} x &\longmapsto -n, \end{aligned} \tag{8.3}$$

   where n is the number of operations $\lhd$ (resp. $\widetilde{\lhd}$) in the expression.

For larger sets X, the free quandle $FQ(X)$ becomes interesting. In particular,

Lemma 8.1.7. The free quandle FQ_n can be described via the quandle injection

$$\begin{aligned} FQ_n &\hookrightarrow \mathrm{Conj}(F_n), \\ x_i &\mapsto x_i. \end{aligned} \tag{8.4}$$

The image of this injection is the sub-quandle of $\mathrm{Conj}(F_n)$ generated by the x_i's.

Proof. Remark that

- any element of FQ_n can be written in the form

  $$((x_{i_0} \lhd^{\varepsilon_1} x_{i_1}) \lhd^{\varepsilon_2} \cdots) \lhd^{\varepsilon_k} x_{i_k},$$

  where the values of the ε_j's are ± 1's, with notations

  $$\lhd^1 = \lhd, \qquad \lhd^{-1} = \widetilde{\lhd};$$

- in $\mathrm{Conj}(G)$, one has $a \widetilde{\lhd} b = a \lhd b^{-1}$. □

We finish with some faithfulness remarks for free SD structures.

Proposition 8.1.8. *The action of the positive braid monoid B_n^+ on $FS_1^{\times n}$ is free.*

Proof. This is an easy consequence of P.Dehornoy's results on the ordering of free shelves and braid groups (cf. [39] and [18]). He shows that FS_1 has a total order $<$ generated by the partial order $\prec$:

$$a = c \lhd b \qquad \Longrightarrow \qquad b \prec a.$$

This induces an anti-lexicographic order $<$ on $FS_1^{\times n}$. (The prefix "anti" is needed here since we consider the right version of self-distributivity, while P.Dehornoy works with the left one.) Then he proves that this order is sufficiently nice to induce a total order on the group B_n by declaring, for $\alpha, \beta \in B_n$,

$$\alpha < \beta \qquad \Longleftrightarrow \qquad (\alpha(\overline{a}) < \beta(\overline{a}) \quad \forall \overline{a} \in FS_1^{\times n}).$$

Here $\alpha(\overline{a})$ and $\beta(\overline{a})$ denote a partial extension to B_n of the action of B_n^+, and one takes only those $\overline{a}$ for which $\alpha(\overline{a})$ and $\beta(\overline{a})$ are both defined (one shows that such $\overline{a}$'s exist). Thus, if α and β act on an element of $FS_1^{\times n}$ in the same way, this means precisely $\alpha = \beta$. $\square$

Observe that a monogenerated free rack is not sufficient to produce a faithful action: passing to CR, via the isomorphism (8.3), one sees that the action of B_n on $CR^{\times n}$ simply counts the algebraic number of times a strand passes under other strands (with the sign "+" when moving to the right and "−" when moving to the left); thus the following two braids are indistinguishable by their action:

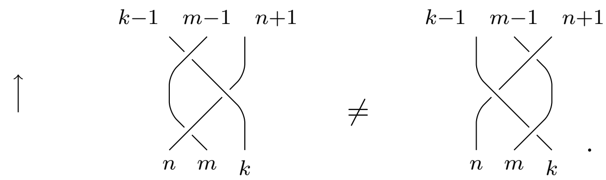


Figure 8.6: Distinct braids acting on $CR^{\times 3}$ in the same way

An example of indistinguishable positive braids can be given by $\sigma_2\sigma_2\sigma_1\sigma_1$ and $\sigma_1\sigma_1\sigma_2\sigma_2$.

However, more complicated racks and quandles can give faithful actions:

Proposition 8.1.9. *The action of the braid group B_n on $FQ_n^{\times n}$ is faithful.*

Proof. One remarks, for a braid $\alpha \in B_n$, the identity

$$\alpha(x_1, \ldots, x_n) = (\alpha^{-1}(x_1), \ldots, \alpha^{-1}(x_n)),$$

where the action on the right is that of theorem 11, which is faithful, thus permitting to conclude. $\square$

Note that this action is not free, since, for instance, the elements 1 and σ_1 of B_n act in the same way on diagonal elements $(a, a, \ldots, a) \in FQ_n^{\times n},\ a \in FQ_n$.

Since $FQ(X)$ is the quotient of $FR(X)$ by (Q), one has

Corollary 8.1.10. *The action of B_n on $FR_n^{\times n}$ is faithful.*

Categories: maturity

The notion of braids is very "categorical" – more than that of knots for example. Braids naturally "correspond" (in the sense to be specified here) to the notion of ***braided monoidal category***. See section 5.1 for this and other categorical notions used here.

Two important classical results express a deep connection between the notions of braids and braided categories.

Theorem 12. *Denote by $\mathcal{C}_{br}$ the free braided category generated by a single object V. Then for each n one has a group isomorphism*

$$\begin{aligned} \psi : B_n &\xrightarrow{\sim} \mathrm{End}_{\mathcal{C}_{br}}(V^{\otimes n}), \\ \sigma_i^{\pm 1} &\mapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes c_{V,V}^{\pm 1} \otimes \mathrm{Id}_V^{\otimes(n-i-1)}. \end{aligned} \tag{8.5}$$

Thus braid groups describe hom-sets of a free monogenerated braided category.

The second result is the following:

Corollary 8.1.11. *For any object V in a braided category $\mathcal{C}$, the map defined by formula* (8.5) *endows $V^{\otimes n}$ with an action of the group B_n.*

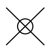
This corollary is a major source of representations of the braid group.

Theorems 10 and 12 put together give

Corollary 8.1.12. *The category $\mathcal{C}_{br}$ is equivalent, as a braided category, to the category $\mathcal{B}r$ of braids (objects $= \mathbb{N}$, $\mathrm{End}_{\mathcal{B}r}(n)$ = braids on n strands, $c_{1,1} =$ ).*

8.2 Virtual braids and virtual racks

Some history

The concept of virtuality was born in the topological framework in L.Kauffman's pioneer 1999 paper [41] (announced in 1996). See also [61] for an express introduction. The original idea is very natural. One tries to encode a knot by writing down the sequence of its crossings encountered when moving along a diagram of the knot, with additional under/over and orientation information for each crossing. This code, called ***Gauss code***, is unambiguous but not surjective: some sequences do not correspond to any knot, since while decoding them one may be forced to intersect the part of the diagram drawn before. L.Kauffman's idea was to introduce in this situation a new, ***virtual*** type of crossings in a diagram. They are depicted like this:  . Such crossings "are not here", they come from the necessity to draw in the plane a diagram given abstractly by its Gauss code. The same happens when one has to draw an abstract non-planar graph in $\mathbb{R}^2$. Note that the "under/over" distinction is no longer relevant for virtual crossings.

We call the non-virtual crossings ***usual*** here, while in literature one encounters the terms *real* and *classical* for the same notion.

Another situation where ***virtual knots***, i.e. knots with both usual and virtual crossings, naturally emerge is when one wants to depict in $\mathbb{R}^2$ knot diagrams living on surfaces other than the plane (for instance, on a torus).

A virtual theory parallel to that of classical knots has been developed in numerous papers. We extract from it only the part concerning ***virtual braids***, essentially due to V.V.Vershinin (see his 1998 paper [80]). To emphasize the connection to virtual knots, one notes the Alexander-Markov type result of S.Kamada ([36]) describing the closure operation for virtual braids.

Definitions

Unfortunately there seem to be no purely topological elementary definition of virtual braids. The common definition is combinatorial: one considers braid diagrams with usual and virtual crossings up to certain relations, which are versions of Reidemeister moves and which are dictated by the Gauss coding. Here is an example of a relation involving both usual and virtual crossings – the mixed Yang-Baxter relation:

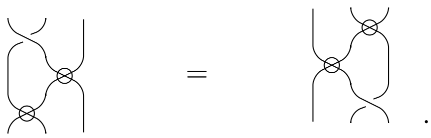


Figure 8.7: Mixed Yang-Baxter relation

It is thus natural to start from the algebraic viewpoint.

Definition 8.2.1. The *virtual braid group* VB_n is defined by a set of generators $\{\sigma_i, \zeta_i,\ 1 \leqslant i \leqslant n-1\}$, and the following relations:

1. (Br_C) and (Br_{YB}) for the σ_i's;
2. (Br_C), (Br_{YB}) and $(Symm)$ for the ζ_i's;
3. *mixed relations*

$$\sigma_i\zeta_j = \zeta_j\sigma_i \qquad \text{if } |i-j|>1, 1 \leqslant i,j \leqslant n-1, \qquad (Br^m_C)$$
$$\sigma_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\sigma_{i+1} \qquad \forall\, 1 \leqslant i \leqslant n-2. \qquad (Br^m_{YB})$$

In other words, the group VB_n is the direct product $B_n * S_n$ factorized by the relations (Br^m_C) and (Br^m_{YB}). This explains the name *braid-permutation group* used in [25] for a slightly different, but closely related structure.

Now *virtual braids on n strands* can be (rather informally) defined as the monoid of braid diagrams with usual and virtual crossings up to ambient isotopy, factorized by the kernel of the monoid surjection

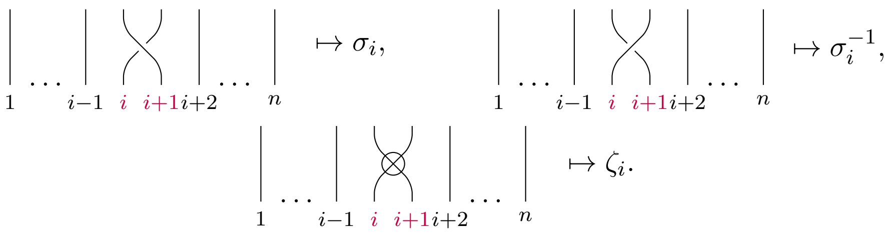


Figure 8.8: A topological version of VB_n

The (evident) definition of the monoid of braid diagrams with usual and virtual crossings is omitted here for the sake of concision.

In particular, figure 8.7 becomes a graphical translation of the equation (Br^m_{YB}).

Observe that virtual braids inherit a group structure from VB_n. Note also that theorem 10 becomes a definition in the virtual world. In what follows, virtual braids are identified with corresponding elements of VB_n.

Remark 8.2.2. In VB_n one automatically has two other versions of Yang-Baxter relation with one σ and two ζ's. On the contrary, YB relations with one ζ and two σ's do not hold. It comes from the fact Gauss decoding process unambiguously prescribes the pattern of usual crossings and leaves a certain liberty only in placing virtual crossings (recall that the definition of virtual knots was motivated by Gauss coding). Such YB relations are called *forbidden*. Here is an example:

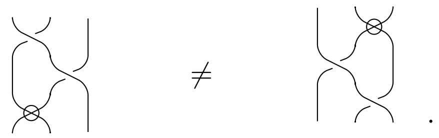


Figure 8.9: A forbidden mixed Yang-Baxter relation

Virtual SD structures

As for representations, shelves and racks remain relevant in the virtual world:

Proposition 8.2.3. *Given a rack $(S, \lhd)$, the virtual braid group VB_n acts on $S^{\times n}$ by*

$$\sigma_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i \lhd a_{i+1}, a_{i+2}, \ldots, a_n), \tag{8.6}$$
$$\zeta_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n). \tag{8.7}$$

Note that one can not hope such actions to be faithful, since virtual braids $\sigma_i\sigma_{i+1}\zeta_i$ and $\zeta_{i+1}\sigma_i\sigma_{i+1}$ act on $S^{\times n}$ in the same way, implying the forbidden YB relation depicted above.

V.O.Manturov proposed in 2002 (cf. [54]) a structure more adequate for the virtual world, namely a virtual quandle. We recall it here, as well as its non-idempotent and non-invertible versions.

Definition 8.2.4. A *virtual shelf* is a shelf $(S, \lhd)$ endowed with a shelf automorphism $f : S \longrightarrow S$, i.e.

1. f admits an inverse f^{-1},
2. $f(a \lhd b) = f(a) \lhd f(b) \quad \forall a, b \in S$.

If moreover $(S, \lhd)$ is a rack or a quandle, then the triple $(S, \lhd, f)$ is called a *virtual rack/quandle*.

Note that for a virtual rack, one automatically has

$$f(a \widetilde{\lhd} b) = f(a) \widetilde{\lhd} f(b).$$

The actions from proposition 8.2.3 can now be upgraded as follows:

Proposition 8.2.5. *Given a virtual rack $(S, \lhd, f)$, the virtual braid group VB_n acts on $S^{\times n}$ by* (8.6) *and*

$$\zeta_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, f^{-1}(a_{i+1}), f(a_i), a_{i+2}, \ldots, a_n). \tag{8.8}$$

We finish this section with two examples proposed by V.O.Manturov (cf. [54]).

Example 8.2.6. In an arbitrary rack R, right adjoint action by an element a, given by $b \mapsto b \lhd a$ for all $b \in R$, is a rack automorphism. Its inverse is $b \mapsto b \widetilde{\lhd} a$. Thus $\mathrm{Conj}(F_n)$ can be given a virtual quandle structure by

$$f(a) := a \lhd x_n \qquad \forall a \in F_n.$$

This virtual quandle appears in the context of free virtual SD structures.

Example 8.2.7. The Alexander quandle (8.2) can be endowed with two virtual quandle structures.

1. The first one is obtained by fixing an element $\varepsilon \in \mathbb{Z}[t^{\pm 1}]$ and putting

$$f(a) := a + \varepsilon \qquad \forall a \in \mathbb{Z}[t^{\pm 1}]. \tag{8.9}$$

   This virtual quandle is used by V.O.Manturov to define the virtual Alexander polynomial, carrying extremely rich topological information about a link. Recall that in the classical setting the Alexander quandle leads to the usual Alexander polynomial, for instance through the Burau representation (cf. example 10.1.5).

   Note that morphism (8.9) is not linear.

2. The second virtual structure is more interesting in a slightly generalized context: one replaces $\mathbb{Z}[t^{\pm 1}]$ with an arbitrary $\mathbb{Z}[t^{\pm 1}]$-module A, keeping the quandle structure from (8.2). If A is moreover a $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$-module, then

$$f(a) := sa \qquad \forall a \in A \tag{8.10}$$

   defines a virtual quandle structure, linear this time.

The aim of the rest of this part is to add some patches to the patchwork of concepts and results around virtual braids presented here. We do it by virtualizing a part of the content of section 8.1.

Positive virtual braid monoids

The first patch, quite a small one, is the notion of positive virtual braid monoid. This concept seems absent in the literature, but we need it here for a study of virtual shelves.

Definition 8.2.8. The *positive virtual braid monoid* VB_n^+ is defined by the set of monoid generators $\{\sigma_i, \zeta_i,\ 1 \leqslant i \leqslant n-1\}$, and relations identical to those from definition 8.2.1.

One gets a submonoid of VB_n.

Like in the real world, the structure of shelf bears an action of this monoid:

Proposition 8.2.9.
- ➜ *Given a shelf* $(S, \lhd)$, *formulas* (8.6) *and* (8.7) *define an action of the positive virtual braid monoid* VB_n^+ *on* $S^{\times n}$.
- ➜ *Given a virtual shelf* $(S, \lhd, f)$, *formulas* (8.6) *and* (8.8) *define an action of the positive virtual braid monoid* VB_n^+ *on* $S^{\times n}$.

Chapter 9

Free virtual self-distributive structures

The next patch we add to virtual braid theory consists in some steps towards understanding the structure of free virtual SD structures and the action of the VB_n's or the VB_n^+'s on them. Some methods from the theory of usual SD structures (section 8.1) are adapted to the virtual context. However, many open question are left.

We start with some observations allowing one to see ***free virtual SD structures as free SD structures on a larger set of generators***. Thus, the results on free SD structures can be trasnsported to the virtual world. However, the monogenerated free virtual shelves and quandles, which are the most interesting for us, correspond to infinitely generated free shelves and quandles, and very little is known about the latter.

Section 9.2 is a study of *free virtual shelves.* Some of P.Dehornoy's "ordering" methods are adapted to the virtual situation. In particular, this allows to recover some information about positive virtual braids (such as their ***linking numbers*** and their ***projection on*** S_n) from their action on products of the monogenerated free virtual shelf, and to show the ***faithfulness*** of this last action for VB_2^+. We also prove that the S_n and the B_n^+ parts of the positive virtual monoid VB_n^+ are indeed its submonoids, extending the braid-permutation interpretation of virtual braid groups to the positive setting.

Section 9.3 contains a reformulation of V.O.Manturov's conjecture on the structure of *free virtual quandles* in terms of the conjugation virtual quandle of a free group.

Notation FVS_n stands here for a free virtual shelf on n generators, and similarly for racks and quandles.

9.1 Adding virtual copies of elements

First, one easily verifies

Proposition 9.1.1. *The morphism of shelves defined by*

$$\begin{aligned} FS_{\mathbb{Z}} &\longrightarrow FVS_1, \\ x_k &\longmapsto f^k(x), \end{aligned}$$

where $x := x_1$ is the generator of FVS_1, is an isomorphism. Analogous isomorphisms take *place for racks and quandles:*

$$FR_{\mathbb{Z}} \xrightarrow{\sim} FVR_1,$$
$$FQ_{\mathbb{Z}} \xrightarrow{\sim} FVQ_1.$$

Notation 9.1.2. In what follows, we implicitly use this isomorphism, writing x_k instead of $f^k(x)$ when working in FVS_1.

Similarly, FVS_n can be seen as a free shelf with separate "virtual" copies $x_{i,k}$ of x_i for all $k \in \mathbb{Z}$.

Summarizing, a free virtual SD structure can be seen as a free SD structure (on a larger set of generators).

9.2 Free virtual shelves and P.Dehornoy's methods

Let us now work with shelves, trying to understand how nice the VB_n^+-actions on $FS_1^{\times n}$ and $FVS_1^{\times n}$ are (cf. proposition 8.2.9). These actions are called ***real*** and ***virtual*** respectively for brevity. The generator x_1 of FS_1 is denoted by x.

The following ***devirtualization*** shelf morphism is systematically used here to extend known results for FS_1 to the virtual world (cf. notation 9.1.2):

$$FVS_1 \simeq FS_{\mathbb{Z}} \overset{devirt}{\twoheadrightarrow} FS_1,$$
$$x_k \longmapsto x.$$

Preliminary remarks

First, the **freeness** result from proposition 8.1.8 does not hold in the virtual context:

Lemma 9.2.1. The virtual action of VB_n^+ on $FVS_1^{\times n}$ is not free.

Proof. It is sufficient to notice that all the ζ_i's act as identities on n-tuples of the form $(x_k, x_{k+1}, \dots, x_{k+n-1}) \in FVS_1^{\times n}$. □

The author does not know if the virtual action is **faithful**. Here are some arguments giving hope for it.

Recall that choosing flips as the actions corresponding to the ζ_i's can lead to a forbidden YB relation, thus implying non-faithfulness. For the virtual action there is no such danger, as one can easily check

Lemma 9.2.2. The virtual action distinguishes the two sides of each forbidden YB relation from remark 8.2.2.

Characteristics of elements of FVS_1

A more refined study of the structure of FVS_1 is needed to prove further results.

The following definition is inspired by [18].

Definition 9.2.3. Fix an alphabet X. The *free magma* T_X on X is the closure of the set X under the formal (non-associative!) operation $(t_1, t_2) \mapsto t_1 * t_2$. The elements of T_X are called *terms*. Notations $T_{\{x_1,\dots,x_n\}}$ and $T_{\{x_i, i\in\mathbb{Z}\}}$ are abbreviated as T_n and $T_{\mathbb{Z}}$ respectively.

Consider the maps

$$\begin{aligned} \mathcal{D} : T_{\mathbb{Z}} &\twoheadrightarrow FVS_1, \\ x_i &\mapsto x_i, \\ * &\mapsto \lhd; \end{aligned} \qquad\qquad \begin{aligned} \mathcal{D} : T_1 &\twoheadrightarrow FS_1, \\ x_1 &\mapsto x, \\ * &\mapsto \lhd. \end{aligned}$$

Concretely, one simply factorizes by the relation (SD). The notation $\mathcal{D}$ comes from the word "distributivity".

Definition 9.2.4. Take a term $t = ((x_f * t_1) * \cdots) * t_k$ in $T_{\mathbb{Z}}$ or T_n.

- Its *first subscript*, denoted by $\mathcal{F}(t)$, is defined to be f.
- Its *sequence/multiset of first subscripts* is the sequence/multiset formed by $\mathcal{F}(t_1), \dots, \mathcal{F}(t_k)$. The multiset of first subscripts is denoted by $\overline{\mathcal{F}}(t)$.
- Finally, $l(t) := k$ is called the *length* of t.

Note that for $n = 1$ only the length function l is relevant.

Playing with the relation (SD), one gets

Lemma 9.2.5. 1. Take two terms giving the same shelf element, i.e. $t, t' \in T_{\mathbb{Z}}$ (or T_n) such that $\mathcal{D}(t) = \mathcal{D}(t')$. One then has $l(t) = l(t')$, $\mathcal{F}(t) = \mathcal{F}(t')$ and $\overline{\mathcal{F}}(t) = \overline{\mathcal{F}}(t')$.

2. Moreover, given a $t \in T_{\mathbb{Z}}$ with $l(t) = k$ and a permutation $\theta \in S_k$, there exists a $t' \in T_{\mathbb{Z}}$ such that $\mathcal{D}(t) = \mathcal{D}(t')$ and their sequences of first subscripts differ precisely by the permutation θ.

Thus one gets several simple characteristics of elements of FVS_1:

Definition 9.2.6. One defines the functions $l(a)$, $\mathcal{F}(a)$ and $\overline{\mathcal{F}}(a)$ for any $a \in FVS_1$ as $l(t)$, $\mathcal{F}(t)$ and $\overline{\mathcal{F}}(t)$ for any term t representing a. The length $l(a)$ of an $a \in FS_1$ is defined similarly.

Recovering the linking numbers of positive virtual braids and their projection on the plane

The combinatorics of first subscripts permits to extract useful information from the virtual action. For this, consider the forgetful monoid morphism

$$\begin{aligned} \mathrm{For} : VB_n^+ &\twoheadrightarrow S_n, \\ \zeta_i, \sigma_i &\mapsto \zeta_i. \end{aligned}$$

The ζ_i's on the right denote the standard generators of S_n. This morphism can be seen as a projection of a virtual braid on the plane, with a loss of the usual/virtual crossing distinction and the under/over information.

Proposition 9.2.7. *1. The real action of a $\theta \in VB_n^+$ on $FS_1^{\times n}$ permits to recover* $\mathrm{For}(\theta)$ *and the number of the σ's in θ.*

2. *The virtual action of a $\theta \in VB_n^+$ on $FVS_1^{\times n}$ permits to recover* $\mathrm{For}(\theta)$ *and the number of the σ's in θ.*

3. *The virtual action of $\theta \in VB_n^+$ on $FVS_1^{\times n}$ permits to recover, for each strand of the virtual braid θ, the multiset of strands passing (non-virtually) under it.*

Note that the number of the σ's in θ and the multisets from the last point are stable under all authorized virtual braid relations and are thus well-defined. The multisets give in particular the ***linking number*** of any strands i and j, i.e. the number of times the strand i passes under j.

Proof. 1. Put $(a_1, a_2 \ldots, a_n) := \theta(x, x, \ldots, x)$ and $l_j := l(a_j)$. Changing the ith element x in the n-tuple $(x, x, \ldots, x)$ to $x \lhd x$ increases exactly one of the l_j's by one. This j is precisely the value of $\mathrm{For}(\theta)(i)$.
Further, the real action of a ζ_i on $FS_1^{\times n}$ does not change the total length of the elements of an n-tuple, whereas the real action of a σ_i increases it by 1. Thus one recovers the number M of the σ's in θ.

2. Follows from the previous point by devirtualizing.
3. In the virtual context, the $\mathcal{F}$'s and $\overline{\mathcal{F}}$'s refine the information given by the length function.
The virtual action of a $\zeta \in S_n$ seen as an element of VB_n^+ (via the intuitive injection $\zeta_i \mapsto \zeta_i$, rigorously studied later) can be written for "simple" n-tuples like this:

$$\zeta(x_{i_1}, x_{i_2}, \ldots) = (x_{i_{\zeta^{-1}(1)}+1-\zeta^{-1}(1)}, x_{i_{\zeta^{-1}(2)}+2-\zeta^{-1}(2)}, \ldots).$$

Note that $|k - \zeta^{-1}(k)| \leqslant n-1$ for all the k's. In general, working with first subscripts, one sees that ζ applied to a general n-tuple changes the first subscript $\mathcal{F}$ of the element on each strand at most by $n-1$. Recall the number M of the σ's in θ determined in the previous point. Remark also that each σ_i simply switches the first subscripts of two of the elements in an n-tuple.
Summarizing, the action of our θ, as well as of its subterms, changes the $\mathcal{F}$ of the element on any strand at most by $(n-1)(M+1)$. Put

$$N := (n-1)(M+1) + 1$$

and

$$(y_1, y_2, \ldots, y_n) := \theta(x_{2N}, x_{4N}, \ldots, x_{2nN}).$$

The ith strand of θ will be called $2iN$ for simplicity.
For any i, replacing each number in $\overline{\mathcal{F}}(y_i)$ by the closest multiple of $2N$, one recovers the multiset of strands passing over the strand corresponding to the closest to $\mathcal{F}(y_i)$ multiple of $2N$. This follows from the observations $\overline{\mathcal{F}}(a \lhd b) = \overline{\mathcal{F}}(a) \cup \mathcal{F}(b)$ and $\mathcal{F}(a \lhd b) = \mathcal{F}(a)$, from the explicit formulas defining the virtual action, from the estimations for subscript modifications above, and from the independence of $\mathcal{F}$ and $\overline{\mathcal{F}}$ from the choice of term representing the braid. □

Precisions on the braid-permutation interpretation of VB_n^+

We now return to the interpretation of VB_n^+ as the direct product $B_n^+ * S_n$ factorized by relations (Br_C^m) and (Br_{YB}^m).

Consider two monoid morphisms

$$\begin{aligned} S_n &\xrightarrow{i_S} VB_n^+, \qquad & B_n^+ &\xrightarrow{i_B} VB_n^+, \\ \zeta_i &\longmapsto \zeta_i; & \sigma_i &\longmapsto \sigma_i. \end{aligned} \tag{9.1}$$

Proposition 9.2.8. *The action of S_n (resp. B_n^+) on $FS_1^{\times n}$ induced by the real action of VB_n^+ via morphism i_S (resp. i_B) is faithful (resp. free).*

Proof. The statement about B_n^+ follows from proposition 8.1.8.

As for S_n, its induced action is the usual action by permutations. Elements

$$a_k := ((x \lhd x) \lhd \cdots) \lhd x,$$

with k occurrences of x, are of different lengths ($l(a_k) = k - 1$) and are hence pairwise distinct. Therefore a permutation $\zeta \in S_n$ is completely defined by $\zeta(a_1, a_2, \ldots, a_n)$. □

Devirtualizing, as usual, one gets the same statement for the virtual action.

Results of this kind allow one to easily get a useful

Corollary 9.2.9. *The submonoid of VB_n^+ generated by the ζ_i's (resp. σ_i's) is isomorphic to S_n (resp. B_n^+).*

Proof. The submonoids in question are images of i_S and i_B, which are monoid injections according to the preceding proposition. □

On a virtual Dehornoy order

It is now time for some remarks on a generalization of the Dehornoy order to FVS_1. Define a partial order on FVS_1, as usual, by

$$a = c \lhd b \qquad \Longrightarrow \qquad b \prec a.$$

Devirtualizing and using the acyclicity of the Dehornoy order on FS_1 ([18]), one sees that only one of the relations $a = b, b \prec a, a \prec b$ can hold for given $a, b \in FVS_1$. Thus,

Lemma 9.2.10. The partial order $<$ on FVS_1 generated by $\prec$ is **acyclic**.

The order $<$ is unfortunately far from being **total**: the x_i's are all minimal elements (since $b < a$ entails $l(a) > 0$, whereas $l(x_i) = 0$) hence mutually incomparable.

The author knows no reasonable total order either on FVS_1 or on VB_n^+. Note that one can not hope for a left- or right-invariant order on VB_n^+ since it has torsion ($\zeta_i^2 = 1$).

Case $\mathbf{n = 2}$

The last result concerns the case $n = 2$.

Proposition 9.2.11. *The real action of VB_2^+ on $FS_1^{\times 2}$ is faithful.*

Proof. Put $\zeta := \zeta_1, \sigma := \sigma_1$. An element θ of VB_2^+ can be uniquely written, after applying $\zeta^2 = 1$ several times, in its shortest form $\theta = \zeta^{\varepsilon_k}\sigma \cdots \sigma\zeta^{\varepsilon_1}\sigma\zeta^{\varepsilon_0}$, where $\varepsilon_i \in \{0, 1\}$.

We first prove that the value of

$$(a, b) := \theta(x, x) \in FS_1^{\times 2}$$

allows one to determine whether $k > 0$ and, if so, to calculate ε_k and

$$(a', b') := \theta'(x, x) \in FS_1^{\times 2},$$

where $\theta = \zeta^{\varepsilon_k}\sigma\theta'$. Indeed, consider three possibilities and their consequences:

1. $k = 0 \implies (a, b) = (x, x)$;
2. $k > 0, \varepsilon_k = 0 \implies a \prec b$;
3. $k > 0, \varepsilon_k = 1 \implies b \prec a$.

Here $\prec$ is the partial Dehornoy order. Its acyclicity proves that only one of the relations $a = b, b \prec a, a \prec b$ can hold. Thus the pair (a, b) tells whether $k > 0$ and, if so, calculates ε_k. To determine (a', b'), recall the *right cancellativity* of FS_1:

$$a \lhd b = a' \lhd b \qquad \Rightarrow \qquad a = a'$$

(cf. [18]). Thus, for instance, relation $a \prec b$ implies that there exists a unique $c \in FS_1$ with $b = c \lhd a$, so $(a', b') = (c, a)$. Case $b \prec a$ is similar.

Proceeding by induction, one gets the value of k and all the ε_i's for $i > 0$. This determines θ up to the rightmost ζ. To conclude, observe that the presence or absence of this rightmost ζ determines $\mathrm{For}(\theta) \in S_2 = \{\mathrm{Id}, \zeta\}$, which, according to proposition 9.2.7, can be read from the real action. $\square$

Remark 9.2.12. In fact, we have proved a more precise result: VB_2^+ acts freely on couples of the form $(a, b) \in FS_1^{\times 2}$ with distinct a and b which are not directly comparable (i.e. one has neither $b \prec a$ nor $a \prec b$). An example of such a and b is given by x and $x \lhd (x \lhd x)$: relation (SD) can not be applied to any of these two terms, hence they both have a unique presentation, in which the desired properties are easily verified.

Devirtualizing, one gets

Corollary 9.2.13. *The virtual action of VB_2^+ on $FVS_1^{\times 2}$ is faithful.*

Remark 9.2.14. The author does not know whether FVS_1 is right cancellative. If it were true, the preceding proof could be easily adapted to show that VB_2^+ acts freely on couples $(a, b) \in FVS_1^{\times 2}$ which are not directly comparable (i.e. $b \neq f(a)$ and, for all $k \in \mathbb{Z}$, one has neither $b \prec f^k(a)$ nor $a \prec f^k(b)$). An example of such a and b is given by x_i and x_j with $j - i \neq 1$.

9.3 Free virtual quandles and a conjecture of V.O.Manturov

Let us now turn to free virtual quandles. Developing example 8.2.6, where $\mathrm{Conj}(F_{n+1})$ was endowed with the virtual quandle structure

$$f(a) := a \lhd x_{n+1} \qquad \forall a \in F_{n+1},$$

one gets a virtual analogue of the quandle injection (8.4):

Proposition 9.3.1. *The virtual quandle morphism defined on the generators by*

$$\begin{aligned} FVQ_n &\longrightarrow \mathrm{Conj}(F_{n+1}), \\ x_i &\longmapsto x_i \qquad \forall 1 \leqslant i \leqslant n, \end{aligned}$$

is injective.

Notation 9.3.2. The image of this injection is denoted by $VConj_n$.

The virtual quandle $VConj_n$ consists of all the conjugates in $\mathrm{Conj}(F_{n+1})$ of the x_i's with $1 \leqslant i \leqslant n$. In particular, $x_{n+1} \in Conj(F_{n+1})$ plays a role different from that of the other x_i's: it is not a generator of the virtual quandle $VConj_n$, but is it here to give the "virtualizing" morphism f.

A conjecture raised by V.O.Manturov in [55] (cf. [56] for an English version) is equivalent to the following:

Conjecture 1. *The virtual braid group VB_n acts freely on $(x_1, x_2, \ldots, x_n) \in FVQ_n^{\times n}$, thus generalizing theorem 11 and proposition 8.1.9.*

Manturov formulated his conjecture in terms of cosets

$$E_i := \{x_i\} \backslash F_{n+1} = F_{n+1}/(a = x_i a \,\forall a).$$

He endowed $E := \sqcup_{i=1}^n E_i$ with the operations

$$\begin{aligned} a * b &:= ab^{-1} x_j b \in E_i \qquad \forall a \in E_i, b \in E_j, \\ f(a) &:= a x_{n+1} \in E_i \qquad \forall a \in E_i. \end{aligned}$$

To see that Manturov's conjecture is equivalent to the one given above, note that $(E, *, f)$ is a virtual quandle, and that one has a virtual quandle isomorphism

$$\begin{aligned} E &\xrightarrow{\sim} VConj_n, \\ a \in E_i &\longmapsto a^{-1} x_i a. \end{aligned}$$

Chapter 10

Categorical aspects of virtuality

Now let us look for a categorical counterpart – in the sense of theorem 12 – of the notion of virtual braids. Interesting results in this direction were obtained by L.H.Kauffman and S.Lambropoulou in [42]. They introduced the ***String Category*** and explored its tight relationship with, on the one hand, the algebraic Yang-Baxter equation, and, on the other hand, virtual braid groups. Morally, passing from the usual to the algebraic YBE requires, besides a braiding, a (substitute for the) flip, thus suggesting connections with virtual braid groups. This point of view turns out to be very fruitful, in particular when working with pure braid groups. In this chapter, we present our categorification of VB_n which is quite different from the one from [42]. It is closer in spirit to the categorification of B_n, and it produces a convenient machine for constructing representations of VB_n.

Concretely, the categorical counterpart we propose for virtual braids is the notion of ***a (pre-)braided object in a symmetric category***. In particular, we show that virtual braid groups are isomorhic to the hom-sets of a free symmetric category generated by a single braided object, and similarly for positive virtual braid monoids and free symmetric category generated by a single pre-braided object. As a consequence, our constructions from parts I and II, where we endowed an object of a category, often symmetric from the very beginning, with another, more compicated "structural" (pre-)braiding, provide interesting examples of representations of VB_n or VB_n^+.

This categorification of VB_n is thus another patch we add to the theory of virtual braids.

One of the advantages of the categorical vision of virtual braid group actions is an enhanced ***flexibility***. In particular, if one has a braided object (V, σ_V) in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, then one can change the associated action of VB_n on $V^{\otimes n}$ by changing either the braiding σ_V of V, or the symmetric braiding c on $\mathcal{C}$. This gives new actions for free. The same is true for pre-braided objects and the actions of VB_n^+. Section 10.2 is devoted to two applications of this flexibility:

1. we interpret V.O.Manturov's virtual racks via a deformation of the underlying symmetric structure;

2. we recover the twisted Burau representation of D.S.Silver and S.G.Williams ([75]) by ***twisting*** both braidings with the help of another symmetric braiding.

This gives two more patches to the theory of virtual braids, this time without analogues for usual braids.

10.1 A categorical counterpart of virtual braids

Virtual braid groups as hom-sets

Two types of braiding present in the definition of VB_n suggest looking at categories with two distinct braided structures. But this approach is too naive to work: the naturality of the braidings would imply that one can pass any braiding "through" the other one, meaning one of the forbidden YB relations (cf. remark 8.2.2). Thus one needs an adequate **non-functorial** notion of braiding. The right notion turns out to be precisely that of a *pre-braiding* from definition 5.1.3.

Theorem 13. *Denote by $\mathcal{C}_{2br}$ the free symmetric category generated by a single braided object (V, σ_V). Then for each n one has a group isomorphism*

$$\psi_{virt} : VB_n \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}_{2br}}(V^{\otimes n})$$

$$\zeta_i \longmapsto c_i := \mathrm{Id}_V^{\otimes(i-1)} \otimes c_{V,V} \otimes \mathrm{Id}_V^{\otimes(n-i-1)}, \tag{10.1}$$

$$\sigma_i^{\pm 1} \longmapsto \sigma_i^{\pm 1} := \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma_V^{\pm 1} \otimes \mathrm{Id}_V^{\otimes(n-i-1)}. \tag{10.2}$$

Notation $\mathcal{C}_{2br}$ emphasizes that two different braidings are present in the story.

Proof.

1. To check that ψ_{virt} is well defined, one should check three instances of the YB relation in $\mathcal{C}_{2br}$, the other verifications being trivial. YB relation for σ_V is a part of the definition of braiding. That for $c_{V,V}$ was proved in example 5.1.4. The mixed one, with one occurrence of σ_V and two of $c_{V,V}$, is a consequence of the naturality of c: take $W = V \otimes V$ and $g = \sigma_V$ in (5.3).
2. To see that the c_i's and the σ_i's generate the whole $\mathrm{End}_{\mathcal{C}_{2br}}(V^{\otimes n})$, remark that the braiding c on tensor powers of V, which is the only part of the structure not described yet, is automatically expressed, due to (5.1) and (5.2), via the c_i's:
$$c_{V^{\otimes n}, V^{\otimes k}} = (c_k \cdots c_1) \cdots (c_{n+k-2} \cdots c_{n-1})(c_{n+k-1} \cdots c_n). \tag{10.3}$$
3. It remains to show that all the relations in $\mathrm{End}_{\mathcal{C}_{2br}}(V^{\otimes n})$ follow from those which are images by ψ_{virt} of some relations from VB_n.
Equations (5.1) and (5.2) for c on tensor powers of V are guaranteed by (10.3). So is the symmetry of c. Naturality of $c_{V^{\otimes n}, V^{\otimes k}}$ is the only condition left. According to point 2, it suffices to check it for the generating morphisms c_i's and σ_i's only. But then everything follows from the appropriate versions of YB relation, discussed in point 1. $\square$

Thus virtual braid groups describe hom-sets of a free symmetric category generated by a single braided object.

Note that a free monogenerated braided category is the same thing as a free monoidal category generated by a single braided object, so the local/global distinction is not relevant for the categorification of the B_n's.

Remark 10.1.1. Observe that checking that one has a symmetric category with a braided object is somewhat easier than verifying directly that one actually has a VB_n-action. It comes from the fact that some of the relations in VB_n are already "built-in" on the categorical level:

- commutation relations (Br_C) and (Br_C^m) are "hidden" in the definition of the action ψ_{virt};
- YB relations (Br_{YB}^m) and (Br_{YB}) for the ζ_i's are consequences of the naturality of the braiding c.

Consequences on the representation level

The categorical vision of VB_n offers, as usual, a machine for constructing its representations:

Corollary 10.1.2. *For any braided object V in a symmetric category $\mathcal{C}$, the morphisms defined by formulas* (10.1) *and* (10.2) *endow $V^{\otimes n}$ with an action of the group VB_n.*

Concrete examples will be given later. The following notation will shorten their presentation:

Definition 10.1.3. Given an object V in a monoidal category $\mathcal{C}$, we briefly say that a pair (ξ, ϑ) of endomorphisms of $V \otimes V$ *defines a VB_n (or VB_n^+) action* if the assignments $\zeta_i \mapsto \xi_i$ and $\sigma_j \mapsto \vartheta_j$ (recall notation (1.3)) define an action of VB_n (or VB_n^+) on $V^{\otimes n}$.

The above corollary says for instance that $(c_{V,V}, \sigma_V)$ defines a VB_n action.

A positive version

Positive virtual braid monoids can be categorified similarly:

Theorem 13$^+$. *Denote by $\mathcal{C}_{2br}^+$ the free symmetric category generated by a single pre-braided object (V, σ_V). Then for each n one has a monoid isomorphism*

$$\begin{aligned} \psi_{virt,pos} : VB_n^+ &\xrightarrow{\sim} \mathrm{End}_{\mathcal{C}_{2br}^+}(V^{\otimes n}) \\ \zeta_i &\longmapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes c_{V,V} \otimes \mathrm{Id}_V^{\otimes(n-i-1)}, \\ \sigma_i &\longmapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma_V \otimes \mathrm{Id}_V^{\otimes(n-i-1)}. \end{aligned}$$

As usual, on the level of representations one deduces that, for any pre-braided object (V, σ_V) in a symmetric category $(\mathcal{C}, c)$, the pair $(c_{V,V}, \sigma_V)$ defines a VB_n^+ action.

Examples: "structural" braidings

Recall the (pre-)braidings constructed for basic algebraic structures in chapter 4, with a categorification in chapter 5. In those chapters, we associated pre-braidings to simple algebraic structures (e.g. an algebra). Such braidings involved the defining morphisms of the structures (e.g. the multiplication in the case of an algebra) and encoded the defining properties of these morphisms (e.g. associativity) as YB relations. While the purpose of part I was to recover basic homologies of algebraic structures as the braided homologies for the corresponding pre-braidings, in this part we discover an **independent interest of the "structural" braidings in the virtual world**.

We list here the "structural" (pre-)braidings from part I for the reader's convenience, specifying the corresponding action of VB_n (resp. VB_n^+) given by corollary 10.1.2 (resp. its non-invertible version).

Proposition 10.1.4. *1. A rack (or a shelf) S endowed with the map*

$$\begin{aligned} \sigma = \sigma_{\lhd} : S \times S &\longrightarrow S \times S \\ (a, b) &\longmapsto (b, a \lhd b). \end{aligned}$$

is a (pre-)braided object in the symmetric category **Set**. *The pair $(\tau, \sigma_{\lhd})$ defines a VB_n (resp. VB_n^+) action on $S^{\times n}$, recovering that from proposition 8.2.3 (resp. 8.2.9).*

2. *A unital associative algebra* (V, μ, ν) *in a monoidal category* $(\mathcal{C}, \otimes, \mathbf{I})$ *is a pre-braided object in* $\mathcal{C}$, *with the pre-braiding given by*

$$\sigma_{Ass} := \nu \otimes \mu : V \otimes V = \mathbf{I} \otimes V \otimes V \to V \otimes V.$$

If the category $\mathcal{C}$ *is moreover symmetric, with the symmetric braiding* c, *then the pair* $(c_{V,V}, \sigma_\mu)$ *defines a* VB_n^+ *action.*

3. *A unital Leibniz algebra* $(V, [,], \nu)$ *in a symmetric preadditive category* $(\mathcal{C}, \otimes, \mathbf{I}, c)$ *is a braided object in* $\mathcal{C}$, *with the braiding given by*

$$\sigma_{Lei} := c_{V,V} + \nu \otimes [,].$$

The pair $(c_{V,V}, \sigma_{[,]})$ *defines a* VB_n *action.*

Here is a concrete example.

Example 10.1.5. Take the Alexander quandle (8.2). Recall the faithful forgetful functor For from (5.6). Observe that the braiding $\sigma_{S,\lhd}$ can be pulled back to $\mathbf{Mod}_R^\oplus$, with $R = \mathbb{Z}[t^{\pm 1}]$, since this pull-back turns out to be an R-linear automorphism of $R \oplus R$. One recovers ***the virtual Burau representation*** (cf. proposition 8.1.2), studied in detail by V.V.Vershinin in [80]. The σ_i's act by (8.1), like in the case of usual braid groups, and the action of the "virtual" ζ_i's can be written in the matrix form as

$$\rho(\zeta_i) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

10.2 Flexibility of the categorical construction

Virtuality as a choice of the "right" world

A nice illustration of the flexibility of our categorification of VB_n is given by the "real" and "virtual" actions of VB_n on a virtual rack, cf. propositions 8.2.3 and 8.2.5. More precisely, we interpret here the "virtualization" of the action as moving to a new symmetric category rather that adding extra structure (the "virtualization morphism" f) to a rack.

The symmetric category we suggest is a particular case of the following general construction.

Theorem 14. *Take a symmetric category* $(\mathcal{C}, \otimes, \mathbf{I}, c)$, *and fix an object* V *with an automorphism* $f \in \mathrm{Aut}_{\mathcal{C}}(V)$ *in it.*

1. *A monoidal subcategory of* $\mathcal{C}$ *can be defined by taking as objects tensor powers* $V^{\otimes n}, n \geqslant 1$ *and* $V^{\otimes 0} := \mathbf{I}$, *and as morphisms all the morphisms in* $\mathcal{C}$ *compatible with* f*:*

$$\mathrm{Hom}^f(V^{\otimes n}, V^{\otimes m}) := \{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes m}) | f^{\otimes m} \circ \varphi = \varphi \circ f^{\otimes n}\}.$$

2. *This subcategory admits, besides* c, *a new symmetric braiding given on* $V \otimes V$ *by*

$$c^f_{V,V} := (f^{-1} \otimes f) \circ c_{V,V}.$$

Before proving the result, note that, thanks to the naturality of c, one has an alternative expression

$$c^f_{V,V} = c_{V,V} \circ (f \otimes f^{-1})$$

and, more generally, one can push any occurrence of $f^{\pm 1}$ from one side of $c_{V,V}$ to the other. Remark also that some basic morphisms are automatically in Hom^f, such as

- identities $\mathrm{Id}_{V^{\otimes n}}$;
- morphisms $f^{\pm 1}$;
- the original braiding c (thanks, as usual, to its naturality).

Proof. 1. One easily checks that all the $\mathrm{Id}_{V^{\otimes n}}$'s are in Hom^f, and that the latter is stable by composition and tensor product.

2. First, the previous point and remarks preceding the proof guarantee that $c^f_{V,V} \in \mathrm{Hom}^f$. Next, extend c^f to other powers $V^{\otimes n}$ by formula (10.3). This extension remains in Hom^f. Such an extension ensures relations (5.1) and (5.2) for c^f. Remaining properties (5.3) and (5.4) for c^f follow from the corresponding properties for c by pushing all the instances of $f^{\pm 1}$ on the left of each expression, using the naturality of c and the compatibility of the morphisms in Hom^f with f. □

Notation 10.2.1. The monoidal category constructed in theorem 14 is denoted by $\mathcal{C}_{V,f}$.

Now take $\mathcal{C} = \mathbf{Set}$ and let $(S, \lhd)$ be a rack. We have seen that $\sigma_\lhd$ is a braiding for S. One checks whether it remains so in new categories of type $\mathbf{Set}_{S,f}$:

Lemma 10.2.2. Given an $f \in \mathrm{Aut}_{\mathbf{Set}}(S)$, the map $\sigma_\lhd$ is a morphism in the subcategory $\mathbf{Set}_{S,f}$ of $\mathbf{Set}$ if and only if f is a rack morphism.

This observation leads to the following

Proposition 10.2.3. *Take a virtual rack (S, f). The action of VB_n on the braided object $(S, \sigma_\lhd)$ of the symmetric category $(\mathbf{Set}_{S,f}, \tau^f)$ is precisely the "virtual" action from proposition 8.2.5.*

Proof. According to theorem 14, one can change the symmetric braiding τ of $\mathbf{Set}_{S,f}$ to τ^f. Further, the previous lemma shows that $\sigma_\lhd$ is a morphism in $\mathbf{Set}_{S,f}$, since f is a rack morphism. Thus $(S, \sigma_\lhd)$ remains a braided object in the symmetric category $(\mathbf{Set}_{S,f}, \tau^f)$. One concludes by writing down explicit formulas for τ^f and comparing them with (8.8). □

Virtually twisted braidings

Changing the (pre-)braiding σ_V for an object V while keeping the underlying symmetric braiding c can also be interesting. In particular, one can twist σ_V using $c_{V,V}$:

Theorem 15. *Take a (pre-)braided object (V, σ_V) in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. Then V can be endowed with another (pre-)braiding*

$$\overline{\sigma}_V := c_{V,V} \circ \sigma_V \circ c_{V,V}.$$

Proof. One should check equation (YB) for $\overline{\sigma}_V$, and show that it is invertible if σ_V is. We treat only invertible braidings here, the pre-braided case being similar.

Consider the *"twisting map"*

$$\begin{aligned} t : VB_n &\longrightarrow VB_n, \\ \theta &\longmapsto \Delta_n \theta \Delta_n, \end{aligned}$$

where Δ_n is the Garside element, i.e. the total twist $\Delta_n := \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right) \in S_n$, seen as an element of VB_n via inclusion (9.1). Since $\Delta_n\Delta_n = 1$, the map t is extremely nice:

1. t is a group map;
2. t is involutive, hence an isomorphism.

On the generators, t gives

$$\begin{aligned} t(\sigma_i) &= \overline{\sigma}_{n-i}, \\ t(\sigma_i^{-1}) &= \overline{\sigma_{n-i}^{-1}}, \\ t(\zeta_i) &= \zeta_{n-i}, \end{aligned}$$

where $\overline{\sigma}_j := \zeta_j\sigma_j\zeta_j$, $\overline{\sigma_j^{-1}} := \zeta_j\sigma_j^{-1}\zeta_j$ $\forall j$. Relation (Br_{YB}) for the $\overline{\sigma}$'s is now a consequence of (Br_{YB}) for the σ's. Observing that $(\overline{\sigma}_V)_j$ is precisely the action, according to corollary 10.1.2, of the element $\overline{\sigma}_j$ of VB_n, one sees that (Br_{YB}) for the $\overline{\sigma}$'s implies (YB) for $\overline{\sigma}_V$.

Further, relation $\overline{\sigma_1^{-1}}\overline{\sigma}_1 = \overline{\sigma}_1\overline{\sigma_1^{-1}} = 1$ in VB_2 implies that the action $\overline{\sigma_V^{-1}} := c_{V,V} \circ \sigma_V^{-1} \circ c_{V,V}$ of $\overline{\sigma_1^{-1}}$ on $V \otimes V$ is the inverse of $\overline{\sigma}_V$, the latter being the action of $\overline{\sigma}_1$. $\square$

Definition 10.2.4. We call the (pre-)braiding from the previous theorem *virtually twisted.*

The element $\overline{\sigma}_i$ of VB_n (or VB_n^+) is graphically depicted as

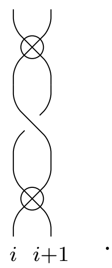


Figure 10.1: Virtually twisted braiding

This element is quite famous in virtual knot theory, since many invariants do not distinguish it from the original σ_i.

In theorem 14, we encountered a category with two distinct symmetric braidings. With this in mind, one can state a stronger version of the previous theorem, based on similar observations:

Proposition 10.2.5. *Take a braided object (V, σ_V) in a category $(\mathcal{C}, \otimes, \mathbf{I})$ admitting two symmetric braidings b and c. Put*

$$\begin{aligned} \sigma''_V &:= c_{V,V} \circ b_{V,V} \circ \sigma_V \circ b_{V,V} \circ c_{V,V}, \\ b'_{V,V} &:= c_{V,V} \circ b_{V,V} \circ c_{V,V}. \end{aligned}$$

Then the pair $(\sigma''_V, b'_{V,V})$ defines a VB_n action, isomorphic to the action given by $(\sigma_V, b_{V,V})$.

Proof. Put $\sigma'_V := b_{V,V} \circ \sigma_V \circ b_{V,V}$.

The involutive action of $\Delta_n := \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{smallmatrix}\right) \in S_n$ on $V^{\otimes n}$ via the symmetric braiding b intertwines $(\sigma_V)_i$ and $(\sigma'_V)_{n-i}$, as well as $(b_{V,V})_i$ and $(b_{V,V})_{n-i}$. Further, the involutive action of Δ_n on $V^{\otimes n}$ via the second symmetric braiding c intertwines $(\sigma'_V)_i$ and $(\sigma''_V)_{n-i}$, as well as $(b_{V,V})_i$ and $(b'_{V,V})_{n-i}$. Their composition yields the announced isomorphism. $\square$

An analogue of this result for positive virtual braid actions can easily be formulated.

Let us now see what this proposition gives in the setting of theorem 14, which is a source of two symmetric braidings coexisting in a category. Taking $b = c^f$, one gets

Proposition 10.2.6. *In a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, take a braided object (V, σ_V) endowed with an automorphism f compatible with the braiding, i.e.*

$$\sigma_V \circ (f \otimes f) = (f \otimes f) \circ \sigma_V.$$

Then the pairs $(\sigma_V, c^f_{V,V})$ and $((f \otimes f^{-1}) \circ \sigma_V \circ (f^{-1} \otimes f), c^{f^{-1}}_{V,V})$ give VB_n actions, which are isomorphic.

Applying this proposition to categories $\mathcal{C}_{V,f^k}$, one obtains

Corollary 10.2.7. *In the settings of the preceding proposition, the pairs $(\sigma_V, c^{f^{k+1}}_{V,V})$ and $((f \otimes f^{-1}) \circ \sigma_V \circ (f^{-1} \otimes f), c^{f^{k-1}}_{V,V})$ give isomorphic VB_n actions for any $k \in \mathbb{Z}$.*

Example 10.2.8. Consider the second virtual quandle structure from example 8.2.7. The automorphism $f(a) = sa$ is compatible with the braiding $\sigma_\lhd$ since f is a quandle morphism (cf. lemma 10.2.2). Then the preceding corollary establishes an isomorphism between VB_n actions given by the pairs $(\sigma_\lhd, \tau^{f^{k+1}})$ and $(\sigma''_\lhd, \tau^{f^{k-1}})$, where

$$\sigma''_\lhd(a, b) = (s^2 b, ts^{-2}a + (1-t)b).$$

Note that A is also a $\mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$-module, with u acting by s^2 and v by ts^{-2}. The matrix form of $\sigma''_\lhd$ is then

$$\begin{pmatrix} 0 & u \\ v & 1 - uv \end{pmatrix},$$

which is precisely the ***twisted Burau matrix*** (cf. [75], or [43], where it is recovered via Alexander biquandles).

Further, the isomorphism of actions for $k = 1$ can be interpreted, in this example, as follows: **virtualizing the VB_n action on a rack**, in the sense of proposition 8.2.5 (for a new quandle morphism $\tilde{f} : a \mapsto f^2(a) = s^2 a = ua$) **is equivalent to "double-twisting" the braiding** $\sigma_\lhd$, in the sense of proposition 10.2.5 (i.e., concretely, passing to $\sigma''_\lhd$). This was noticed in [3].

The equivalence of actions observed in the previous example holds, more generally, in the settings of proposition 10.2.6, whenever the automorphism f of V is a square of another automorphism, which is still compatible with σ_V. In particular, one obtains

Example 10.2.9. One more result from [3] admits a natural interpretation using the tools developed here. It is the possibility to transform a matrix solution $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the YBE over a commutative unital ring R into a solution $\left(\begin{smallmatrix} A & tB \\ t^{-1}C & D \end{smallmatrix}\right)$, with $t \in R^*$, and the equivalence of the two induced representations of the braid group B_n. This, as well as their theorem 7.1, follows from corollary 10.2.7 by taking $\mathcal{C} = \mathbf{Mod}^{\oplus}_R$, $k = 1$ and $f(v) = sv$, with $s^2 = t$ (one formally adds a square root of t to R if necessary).

Chapter 11

Categorical aspects of self-distributivity

As we have seen in chapter 5, associative and Leibniz algebras in monoidal categories are defined very naturally and provide a rich source of (pre-)braided objects in symmetric categories – and thus of representations of virtual braid groups and positive virtual braid monoids. Since shelves and quandles are braided objects par excellence, it would be interesting to categorify them and to look for new examples of braidings emerging in this generalized setting. Such a categorification is given in section 11.1, with several examples in section 11.2, including – quite unexpectedly – associative, Leibniz and Hopf algebras.

Our notion of *categorical, or generalized, self-distributivity* (briefly, GSD) can be presented as follows:

GSD =	comultiplication Δ	+	binary operation $\lhd$	+	compatibility
	coassociative, weakly cocommutative		self-distributive, with Δ as diagonal		in the braided bialgebra sense

Note in particular that the **comultiplication Δ becomes a part of the GSD structure**, which is the main difference between our approach and that of J.S.Carter, A.S.Crans, M. Elhamdadi, and M.Saito (cf. [8]). We discuss the role of this distinction in detail in section 11.1.

The braided homology theories for GSD structures are studied in section 11.3. In particular, this study naturally leads to a notion of ***categorical spindle***.

Note that, having the non-cocommutative comultiplication $\Delta_{Ass} := \nu \otimes \mathrm{Id}_V$ for associative algebras in mind, we choose not to impose the cocommutativity in the definition of GSD. This entails some technical weaker notions of *central* and *left cocommutativity.*

11.1 A categorified version of self-distributivity

Motivations

The main difficulty in defining the self-distributivity in a monoidal category resides in interpreting the ***diagonal map***

$$\Delta_D : a \mapsto (a, a), \tag{11.1}$$

implicit on the right side of (SD). The ***flip***, equally implicit in (SD) (moving one of the c's before b), is also to be appropriately interpreted. Two approaches are proposed by J.S.Carter, A.S.Crans, M. Elhamdadi, and M.Saito in [8] (see also [15]):

1. generally, one can work in an additive category, which admits binary products and hence diagonal and transposition morphisms;
2. a concrete example of an additive category is $\mathbf{Coalg}_{\Bbbk}$, the category of counital coassociative cocommutative coalgebras over a field $\Bbbk$, with the comultiplication as a diagonal map and the flip as a transposition map.

The approach presented here is different. We make the comultiplication Δ, which generalizes the diagonal map, a part of the GSD structure, instead of requiring it on the categorical level. A bialgebra-like compatibility with the "multiplication" $\lhd$ is imposed on Δ. As for the flip, we leave it on the categorical level, working in a symmetric category. Our choice is motivated by the virtual braid group action we want to obtain, via theorem 13, from the (pre-)braiding we hope to extract from the GSD structure via a generalization of lemma 4.2.1. We reserve the underlying symmetric category structure only for the "virtual part" of VB_n (the ζ_i's) and the (pre-)braided object structure (which, according to lemma 4.2.1, uses $\lhd$ and Δ) for the "real part" (σ_i's). This seems to us more consistent with the topological interpretation, where virtual crossings are just artefacts of depicting a diagram in the plane, while usual crossings come from the intrinsic knot structure.

One more argument in favor of a "local" rather than "global" comultiplication is its special role in the homology theory of (pre-)braided objects (cf. section 3.2 for details, or theorem 6 for a categorical version): together with a (pre-)braiding (which we hope to obtain from a categorical SD structure) and a braided character, they are used in the construction of a weakly simplicial structure.

Here is a list of other advantages of our approach:

- we work in a general monoidal rather than $\Bbbk$-linear setting;
- no counit is demanded (note that counits cause some problems in [8], and they do not exist in one of the examples given below);
- cocommutativity, often demanded in [8], is replaced by a weaker notion – again, with an example when it is necessary;
- the flexibility in the choice of the underlying symmetric category allows to treat, among other structures, Leibniz superalgebras.

On the negative side, our definition is quite heavy, since, for example, one has to replace conditions like "a morphism in **Coalg**" by their concrete meaning. The reader is advised to draw pictures, in the spirit of parts I and II or [8], to better manipulate all the notions.

Shelves and racks in other worlds

Recall notations (1.3) and (1.4).

Definition 11.1.1. Take a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$.

- An object V of $\mathcal{C}$ is called *a shelf* in $\mathcal{C}$ if it is endowed with two morphisms $\Delta : V \longrightarrow V \otimes V$ and $\lhd : V \otimes V \longrightarrow V$ satisfying the following conditions (where the braiding $c_{V,V}$ is denoted simply by c for succinctness):
  1. Δ is a coassociative *central-cocommutative* comultiplication, i.e.
  $$\Delta_1 \circ \Delta = \Delta_2 \circ \Delta : V \longrightarrow V^{\otimes 3},$$
  $$c_2 \circ \Delta^3 = \Delta^3 : V \longrightarrow V^{\otimes 4};$$
  2. $\lhd$ is *self-distributive in the generalized sense* (abbreviated as GSD):
  $$\lhd^2 = \lhd \circ (\lhd \otimes \lhd) \circ c_2 \circ \Delta_3 : V^{\otimes 3} \longrightarrow V; \qquad \text{(GSD)}$$

3. the two morphisms are *compatible in the braided bialgebra sense*:

$$\Delta \circ \lhd = (\lhd \otimes \lhd) \circ c_2 \circ (\Delta \otimes \Delta) : V^{\otimes 2} \longrightarrow V^{\otimes 2}. \tag{11.2}$$

➻ A shelf V is called *a rack* in $\mathcal{C}$ if moreover it is endowed with

1. a right counit $\varepsilon : V \longrightarrow \mathbf{I}$, i.e.

$$\varepsilon_2 \circ \Delta = \mathrm{Id}_V : V \longrightarrow V,$$

2. a morphism $\widetilde{\lhd} : V \otimes V \longrightarrow V$ which is the "twisted inverse" of $\lhd$:

$$\widetilde{\lhd} \circ \lhd_1 \circ c_2 \circ \Delta_2 = \lhd \circ \widetilde{\lhd}_1 \circ c_2 \circ \Delta_2 = \mathrm{Id}_V \otimes \varepsilon : V^{\otimes 2} \longrightarrow V.$$

Note that usual cocommutativity implies the central one, and the converse holds if, for example, there exists a counit for Δ. We prefer keeping our weaker condition in order to allow non-cocommutative examples in the next section.

Our definition is designed for a generalized version of lemma 4.2.1 to hold:

Proposition 11.1.2. *1. A shelf $(V, \Delta, \lhd)$ in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ admits a pre-braiding*

$$\sigma = \sigma_{SD} := \lhd_2 \circ c_1 \circ \Delta_2.$$

2. This pre-braiding is invertible if V is moreover a rack, the inverse given by

$$\sigma^{-1} = \widetilde{\lhd}_1 \circ c_2 \circ c_1 \circ c_2 \circ \Delta_1.$$

The verifications are easy but lengthy, so they are not given here. Diagrammatic proof is probably the least tiresome. Here is for instance the graphical form of the (pre-)braiding:

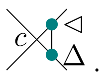


Figure 11.1: Pre-braiding for a categorical shelf

Corollary 11.1.3. *In the settings of the previous proposition, the pair $(\sigma_{SD}, c_{V,V})$ gives a VB_n^+ or VB_n action.*

Alternative definitions

One could have started with proposition 11.1.2 and asked oneself what conditions on morphisms Δ and $\lhd$ make σ_{SD} a (pre-)braiding. In fact, the conditions from definition 11.1.1 are very far from being unique, unlike the "if and only if" results from parts I and II. We cite just two more of multiple alternative sets of conditions here.

1. The central cocommutativity can be transformed to

$$c_2 \circ c_3 \circ \Delta^3 = \Delta^3 : V \longrightarrow V^{\otimes 4},$$

and condition (GSD) to

$$\lhd^2 = \lhd^2 \circ c_2 \circ \lhd_2 \circ \Delta_3 : V^{\otimes 3} \longrightarrow V.$$

Note that in the cocommutative case this coincides with the original definition.

2. The condition (GSD) can be substituted with the usual associativity, and the compatibility condition can be made *Yetter-Drinfel′d-like*:

$$\lhd_2 \circ c_1 \circ \Delta_2 \circ \lhd_2 \circ c_1 \circ \Delta_2 = (\lhd \otimes \lhd) \circ c_2 \circ (\Delta \otimes \Delta) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$$

(cf. the right version of figure 7.9).

Morally, starting with a bialgebra structure, one should substitude either the compatibility condition with a Yetter-Drinfel′d-like, or the associativity condition with the GSD.

Observe that one could also work with a slightly different morphism in proposition 11.1.2:

$$\sigma := \sigma'_{SD} := c \circ \lhd_1 \circ \Delta_2,$$

which coincides with σ_{SD} in the cocommutative case. Conditions similar to those for σ_{SD} guarantee that it is a (pre-)braiding. This choice makes the rack case less "twisted": we demand $\widetilde{\lhd}$ to be simply the inverse of $\lhd$ and omit the occurrences of c_2 in the defining property for $\widetilde{\lhd}$.

Our choice in definition 11.1.1 is motivated by concrete examples which follow.

Remark 11.1.4. The GSD can be efficiently expressed with the help of $\sigma = \sigma_{SD}$:

$$\lhd^2 = \lhd^2 \circ \sigma_2.$$

In other words, $(V, \lhd)$ is right module over the pre-braided object (V, σ_{SD}), in the sense of section 6.1.

Recovering usual shelves and racks

Now let us move to examples. The first one is naturally that of usual SD structures. Choose **Set** as the underlying symmetric category. Recall the diagonalization map (11.1). Further, for a set S, denote by ε the map from S to $\mathbf{I}$, unique since the one-element set $\mathbf{I}$ is a final object. One easily sees that $(S, \Delta_D, \varepsilon)$ is a counital cocommutative coalgebra in **Set**. This ensures some of the properties of definition 11.1.1. Analyzing the remaining ones, one gets

Proposition 11.1.5. *Take a set S endowed with a map $\lhd : S \to S \times S$.*

1. *The triple $(S, \Delta_D, \lhd)$ is a shelf in the symmetric category* **Set** *if and only if $(S, \lhd)$ is a usual shelf.*
2. *The datum $(S, \Delta_D, \varepsilon, \lhd, \widetilde{\lhd})$ is a rack in the symmetric category* **Set** *if and only if $(S, \lhd, \widetilde{\lhd})$ is a usual rack.*
3. *Moreover, for a shelf $(S, \lhd)$, the pre-braiding σ_{SD} from proposition 11.1.2 coincides with $\sigma_{\lhd}$ from lemma 4.2.1.*

Thus generalized self-distributivity includes the usual one. Examples from the next section show that the generalized notion is in fact much wider.

11.2 Associative, Leibniz and Hopf algebras are shelves

The aim of this section is to recover associative, Leibniz and Hopf algebras under the guise of categorical shelves, choosing suitable comultiplications.

Associative algebras

Start with UAAs. The following result allows to see **associativity as a particular case of generalized self-distributivity**:

Proposition 11.2.1. *Take an object V in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, equipped with two morphisms $\mu : V \otimes V \longrightarrow V$ and a right unit $\nu : \mathbf{I} \longrightarrow V$ for μ, i.e. $\mu \circ \nu_2 = \mathrm{Id}_V$. Put*

$$\Delta_{Ass} := \nu \otimes \mathrm{Id}_V .$$

1. *The triple (V, Δ_{Ass}, μ) satisfies all the conditions from definition 11.1.1 but* (GSD)*, which is equivalent to the associativity of μ.*
2. *Moreover, for a UAA (V, μ, ν) and the GSD structure above, the pre-braiding σ_{SD} from proposition 11.1.2 coincides with σ_{Ass} from theorem $5^{\mathbf{cat}}$.*

This example is somewhat exotic. It explains why we were quite demanding in choosing the conditions in definition 11.1.1. In particular,

- Δ_{Ass} is cocommutative in the central but not in the usual sense;
- Δ_{Ass} admits only a left counit in general;
- (V, Δ_{Ass}, μ) is not a rack in general;
- the pre-braiding for (V, Δ_{Ass}, μ) is not invertible in general.

Leibniz algebras

The case of ULAs is somewhat trickier. A natural candidate for comultiplication is

$$\Delta = \nu \otimes \mathrm{Id}_V + \mathrm{Id}_V \otimes \nu,$$

but to recover the Leibniz condition (Lei) as a GSD one, one wants the "right multiplication by one" (i.e. $[,] \circ \nu_2 : V \longrightarrow V$) to be identity and not zero, as the definition of ULA imposes. A standard solution is to start with a (not necessarily unital) Leibniz algebra V' and to introduce a "formal unit", i.e. to work in $V := V' \oplus \mathbf{I}$. This ***"unit problem"*** turns out to be the only one in interpreting Leibniz algebras via GSD, as witnesses the next result.

Take an object V' in a symmetric additive category $\mathcal{C}$, and a morphism $[,] : V' \otimes V' \to V'$. Put

$$V := V' \oplus \mathbf{I}$$

and denote by ν the identity $\mathrm{Id}_\mathbf{I}$ seen as a morphism from $\mathbf{I}$ to V. Define a comultiplication Δ_{Lei} and a counit ε_{Lei} on V by

$$\begin{aligned} \Delta_{Lei}|_{V'} &:= \nu \otimes \mathrm{Id}_{V'} + \mathrm{Id}_{V'} \otimes \nu : V' \to V \otimes V, & \varepsilon_{Lei}|_{V'} &:= 0, \\ \Delta_{Lei}|_{\mathbf{I}} &:= \nu \otimes \nu : \mathbf{I} \to V \otimes V, & \varepsilon_{Lei}|_{\mathbf{I}} &:= \mathrm{Id}_{\mathbf{I}}, \end{aligned}$$

and binary operations $\lhd_{Lei}, \widetilde{\lhd}_{Lei}$ on V by

$$\begin{aligned} \lhd_{Lei}|_{V' \otimes V'} &= -\widetilde{\lhd}_{Lei}|_{V' \otimes V'} := [,], \\ \lhd_{Lei}|_{V \otimes \mathbf{I}} &= \widetilde{\lhd}_{Lei}|_{V \otimes \mathbf{I}} := \mathrm{Id}_V, \\ \lhd_{Lei}|_{\mathbf{I} \otimes V'} &= \widetilde{\lhd}_{Lei}|_{\mathbf{I} \otimes V'} := 0. \end{aligned}$$

Proposition 11.2.2. *1. The datum $(V, \Delta_{Lei}, \varepsilon_{Lei}, \lhd_{Lei}, \widetilde{\lhd}_{Lei})$ satisfies all the conditions from definition 11.1.1 but the GSD, which is equivalent to the Leibniz condition for $[,]$.*

2. *Moreover, for a Leibniz algebra* $(V', [,])$ *and the GSD structure above, the braiding* σ_{SD} *on* V *from proposition 11.1.2 coincides with* σ_{Lei} *from theorem* $5^{\mathbf{cat}}$*, where* $[,]$ *is extended to* V *by declaring* ν *a Lie unit.*

The GSD structure found here turns out to be the same as in [8].

Note that the map $\lhd_{Lei}$ is neither Leibniz nor anti-symmetric when one of the components is $\mathbf{I}$. The advantage of the treatment of ULAs proposed in section 4.4 was that one always stayed within the Leibniz world. Another nice feature was a simple compact formula $\sigma_{Lei} = c + \nu \otimes [,]$ for the braiding, whereas its analogue here $\sigma = c + \nu \otimes \lhd_{Lei}$ is false on $V \otimes \mathbf{I}$.

Working in different symmetric additive categories ($\mathbf{Mod}_R$, $\mathbf{ModGrad}_R$ etc.), one treats the case of Leibniz (super-/color) algebras and other types of structures.

The role of the comultiplication

In the two preceding examples, it is the particular choice of the comultiplication that dictated the nature of the multiplicative structure. More precisely, the comultiplication Δ_{Ass} or Δ_{Lei} imposed the equivalence between the GSD and the associativity or, respectively, the Leibniz condition. For usual shelves this “control” is even stronger:

Lemma 11.2.3. Take the linearization RS of a set S, where R is a commutative unital ring without zero divisors. Consider the comultiplication on RS which is the linearization of the diagonal map Δ_D on S. Suppose that, together with a multiplication $\lhd$, it endows RS with a GSD structure. Then, for any $a, b \in S$, the product $a \lhd b$ is either zero or an element of S.

Proof. Put $a \lhd b = \sum_i \gamma_i c_i$, with $\gamma_i \in R$, and $c_i \in S$ pairwise distinct, in the compatibility condition (11.2). One gets

$$\gamma_i \gamma_j = \begin{cases} 0 & \text{if } i \neq j, \\ \gamma_i & \text{if } i = j. \end{cases}$$

Since R has no zero devisors, the coefficients γ_i are either all zero, or they are zero except one which equals 1. $\square$

Thus $\lhd$ “almost comes from a shelf structure on S”. This is a generalization of lemma 3.8 from [8].

Another example is that of the ***trigonometric coalgebra*** $T = \mathbb{C}a \oplus \mathbb{C}b$ with

$$\Delta_{tr}(a) = a \otimes a - b \otimes b,$$
$$\Delta_{tr}(b) = a \otimes b + b \otimes a.$$

It was also considered in [8]. The elements $x = a + \imath b, y = a - \imath b$ being group-like (i.e. $\Delta_{tr}(x) = x \otimes x$, $\Delta_{tr}(y) = y \otimes y$), all the GSD structures with trigonometric Δ_{tr} are isomorphic to GSD structures with linearized diagonal Δ_D. In particular, lemma 3.9 from [8] is just a reformulation of their lemma 3.8.

Hopf algebras

The last example of “hidden” self-distributivity, studied in [8] as well, is that of a Hopf algebra.

Proposition 11.2.4. *Let $(H, \mu, \Delta, \nu, \varepsilon, S)$ be a cocommutative Hopf algebra in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. Define*

$$\lhd_H = \mu^2 \circ S_1 \circ c_1 \circ \Delta_2 : H \otimes H \longrightarrow H,$$

$$\widetilde{\lhd}_H = \lhd_H \circ S_2^{-1} : H \otimes H \longrightarrow H.$$

The datum $(H, \Delta, \varepsilon, \lhd_H, \widetilde{\lhd}_H)$ satisfies all the conditions from definition 11.1.1. Proposition 11.1.2 then endows H with a braiding.

In $\mathbf{Mod}_R$, the definition of $\lhd_H$ is written, using Sweedler's notation, as

$$x \lhd_H y = S(y_{(1)}) x y_{(2)},$$

$$x \widetilde{\lhd}_H y = y_{(2)} x S^{-1}(y_{(1)}),$$

which are the well-known adjoint actions. The braiding becomes

$$\sigma_{SD}(x \otimes y) = y_{(1)} \otimes S(y_{(2)}) x y_{(3)}.$$

This is precisely the braiding obtained viewing H as a Yetter-Drinfel′d module over itself, cf. [83]. Note in particular that the cocommutativity condition is redundant, since it is not used in the Yetter-Drinfel′d approach.

11.3 Homologies of categorical shelves and spindles

We finish the study of GSD structures by generalizing the cohomology constructions from section 4.2. The main ingredient – a pre-braiding – was already obtained in proposition 11.1.2. Here we consider the remaining ingredients: braided characters and a compatible comultiplication.

Note that another cohomology theory of categorical self-distributivity was proposed in [8]. Their definition was inspired by the bialgebra cohomology and extension-deformation-obstruction ideas. The approach developed here is different. Our motivation is a direct generalization of rack and Chevalley-Eilenberg homologies, with potential applications to topology.

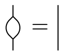
Let us now fix a shelf $(V, \Delta, \lhd)$ in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. Endow it with the pre-braiding σ_{SD} from proposition 11.1.2.

Categorical spindle as a cocommutative braided coalgebra

The intrinsic comultiplication Δ of our shelf is a natural candidate for a comultiplication giving degeneracy maps in theorem 6. Analyzing its cocommutativity and compatibility with the pre-braiding σ_{SD} (in the braided coalgebra sense), one arrives to a categorical version of the notion of spindle. Recall notations (1.3) and (1.4).

Definition 11.3.1. A shelf $(V, \Delta, \lhd)$ in a symmetric category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ is a *spindle* if

1. Δ is *left-cocommutative*:
$$c_1 \circ \Delta^2 = \Delta^2$$
2. and $\lhd$ is Δ-*idempotent*:
$$\lhd \circ \Delta = \mathrm{Id}_V .$$

The first condition is rather technical, while the second one is really essential. It is its graphical form  that explains the term. It was coined, together with the term "shelf", by Alissa Crans in her thesis [15].

Note that the left cocommutativity is stronger than the central one and weaker than the usual one.

Proposition 11.3.2. *Take a shelf* $(V, \Delta, \lhd)$ *in a symmetric category* $(\mathcal{C}, \otimes, \mathbf{I}, c)$. *The data* $(V, \sigma = \sigma_{SD}, \Delta)$ *define a semi-braided coalgebra,* σ*-cocommutative if* V *is a spindle.*

Proof. Compatibility relation (3.12) follows from the coassociativity of Δ and the bialgebra-type compatibility between Δ and $\lhd$. As for σ-cocommutativity, it is a consequence of the two properties defining spindles. $\square$

The additional conditions in the definition of a spindle turn out not to be too restrictive:

Lemma 11.3.3. The following GSD structures are spindles:

1. usual spindles in **Set**;
2. UAAs (for which Δ-idempotence is equivalent to ν being a left unit);
3. ULAs;
4. cocommutative Hopf algebras.

Characters

Another ingredient missing for applying theorem 6 is a source of characters. Here are nice candidates:

Definition 11.3.4. A *GSD character* for a shelf $(V, \Delta, \lhd)$ in $\mathcal{C}$ is a morphism $\epsilon : V \to \mathbf{I}$ compatible with Δ and $\lhd$:

$$\begin{aligned}(\epsilon \otimes \epsilon) \circ \Delta = \epsilon : \qquad & V \longrightarrow \mathbf{I},\\ \epsilon \circ \lhd = \epsilon \otimes \epsilon : \qquad & V \otimes V \longrightarrow \mathbf{I}.\end{aligned}$$

One easily checks

Lemma 11.3.5. A GSD character for a shelf $(V, \Delta, \lhd)$ in $\mathcal{C}$ is a braided character for the pre-braided object (V, σ_{SD}).

Simplicial structures

Everything is now ready for applying theorem 6:

Theorem 16. *Let* ϵ *and* ζ *be two GSD characters for a shelf* $(V, \Delta, \lhd)$ *in a symmetric preadditive category* $(\mathcal{C}, \otimes, \mathbf{I}, c)$. *Morphisms*

$$\begin{aligned}({}^{\epsilon}d)_{n;i} &:= ((\epsilon \otimes \lhd^{\otimes(i-1)}) \circ \omega_{(2i-1)} \circ (\Delta^{i-1})_i) \otimes \mathrm{Id}_V^{\otimes(n-i)} : V^n \to V^{n-1},\\ (d^{\zeta})_{n;i} &:= \mathrm{Id}_V^{i-1} \otimes \zeta \otimes \chi^{\otimes(n-i)} : V^n \to V^{n-1},\end{aligned}$$

define then a pre-bisimplicial structure on $C_n := V^n$, *where*

$$\chi := (\mathrm{Id}_V \otimes \zeta) \circ \Delta : V \longrightarrow V,$$

and $\omega_{(2i-1)} = \left(\begin{smallmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & 2i-1 \\ 2 & 4 & \cdots & 2(i-1) & 1 & 3 & \cdots & 2i-1 \end{smallmatrix}\right) \in S_{2i-1}$ *acts on the tensor powers of* V *via the symmetric braiding* c.

Further, $(C_n, ({}^{\epsilon}d)_{n,i}, s_{n,i} := \Delta_i)$ *is a very weakly simplicial object, becoming weakly simplicial if* V *is a spindle in* $\mathcal{C}$.

As a consequence, any linear combination of total differentials ${}^{\epsilon}d$ and d^{ζ} defines a differential for V, and thus a homology theory if the category $\mathcal{C}$ is sufficiently nice.

For usual shelves and categorical UAAs and ULAs, the homology theories above coincide, for a suitable choice of characters, with the braided homology theories from chapters 4 and 5, and thus recover many known homologies. Moreover, for usual spindles and unital associative algebras, the weakly simplicial structures $(V^n, ({}^{\epsilon}d)_{n;i}, \Delta_i)$ are precisely the familiar ones.

Remark 11.3.6. In fact in the settings of the theorem one has a (very) weakly <u>bi</u>simplicial structure if ζ is a GSD character: although the second braided coalgebra condition (3.13) does not hold in general, one checks directly the relations between the $(d^{\zeta})_{n;i}$'s and the $s_{n,i}$'s, using the central or left cocommutativity of Δ and its compatibility with the GSD character ζ.

Note also that if ζ is a counit for Δ, then the structure $(V^n, (d^{\zeta})_{n;i}, \Delta_i)$ is (trivially) simplicial.

Bibliography

[1] A. Ardizzoni, C. Menini, and D. Ştefan. Hochschild cohomology and "smoothness" in monoidal categories. *J. Pure Appl. Algebra*, 208(1):297–330, 2007.

[2] John C. Baez. Hochschild homology in a braided tensor category. *Trans. Amer. Math. Soc.*, 344(2):885–906, 1994.

[3] Andrew Bartholomew and Roger Fenn. Quaternionic invariants of virtual knots and links. *J. Knot Theory Ramifications*, 17(2):231–251, 2008.

[4] Joan S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82.

[5] Daniel Bulacu, Florin Panaite, and Freddy Van Oystaeyen. Generalized diagonal crossed products and smash products for quasi-Hopf algebras. Applications. *Comm. Math. Phys.*, 266(2):355–399, 2006.

[6] Werner Burau. Uber Zopfgruppen und gleichsinnig verdrillte Verkettungen. *Abh. Math. Sem. Hamburg*, 11:171–178, 1936.

[7] Andreas Cap, Hermann Schichl, and Jiří Vanžura. On twisted tensor products of algebras. *Comm. Algebra*, 23(12):4701–4735, 1995.

[8] J. Scott Carter, Alissa S. Crans, Mohamed Elhamdadi, and Masahico Saito. Cohomology of categorical self-distributivity. *J. Homotopy Relat. Struct.*, 3(1):13–63, 2008.

[9] J. Scott Carter, Mohamed Elhamdadi, and Masahico Saito. Twisted quandle homology theory and cocycle knot invariants. *Algebr. Geom. Topol.*, 2:95–135 (electronic), 2002.

[10] J. Scott Carter, Mohamed Elhamdadi, and Masahico Saito. Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles. *Fund. Math.*, 184:31–54, 2004.

[11] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. *Trans. Amer. Math. Soc.*, 355(10):3947–3989, 2003.

[12] Pierre Cartier. Cohomologie des coalgèbres. *Sém. Sophus Lie*, 5, 1955-1956.

[13] Wesley Chang and Sam Nelson. Rack shadows and their invariants. *J. Knot Theory Ramifications*, 20(9):1259–1269, 2011.

[14] Claude Cibils and Marc Rosso. Hopf bimodules are modules. *J. Pure Appl. Algebra*, 128(3):225–231, 1998.

[15] Alissa Susan Crans. *Lie 2-algebras.* ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–University of California, Riverside.

[16] Christian Cuvier. Homologie de Leibniz et homologie de Hochschild. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(9):569–572, 1991.

[17] Christian Cuvier. Algèbres de Leibnitz: définitions, propriétés. *Ann. Sci. École Norm. Sup. (4)*, 27(1):1–45, 1994.

[18] Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen, and Bert Wiest. *Ordering braids*, volume 148 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[19] Yukio Doi. Homological coalgebra. *J. Math. Soc. Japan*, 33(1):31–50, 1981.

[20] A. S. Dzhumadil′daev. Cohomologies of colour Leibniz algebras: pre-simplicial approach. In *Lie theory and its applications in physics, III (Clausthal, 1999)*, pages 124–136. World Sci. Publ., River Edge, NJ, 2000.

[21] Samuel Eilenberg and G. Max Kelly. Closed categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562. Springer, New York, 1966.

[22] Michael Eisermann. Yang-Baxter deformations of quandles and racks. *Algebr. Geom. Topol.*, 5:537–562 (electronic), 2005.

[23] L. Faddeev, N. Reshetikhin, and L. Takhtajan. Quantum groups. In *Braid group, knot theory and statistical mechanics*, volume 9 of *Adv. Ser. Math. Phys.*, pages 97–110. World Sci. Publ., Teaneck, NJ, 1989.

[24] L. D. Faddeev, N. Yu. Reshetikhin, and L. A. Takhtajan. Quantization of Lie groups and Lie algebras. In *Algebraic analysis, Vol. I*, pages 129–139. Academic Press, Boston, MA, 1988.

[25] Roger Fenn, Richárd Rimányi, and Colin Rourke. The braid-permutation group. *Topology*, 36(1):123–135, 1997.

[26] Roger Fenn and Colin Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1(4):343–406, 1992.

[27] Roger Fenn, Colin Rourke, and Brian Sanderson. Trunks and classifying spaces. *Appl. Categ. Structures*, 3(4):321–356, 1995.

[28] Murray Gerstenhaber. On the deformation of rings and algebras. *Ann. of Math. (2)*, 79:59–103, 1964.

[29] Murray Gerstenhaber and Samuel D. Schack. Bialgebra cohomology, deformations, and quantum groups. *Proc. Nat. Acad. Sci. U.S.A.*, 87(1):478–481, 1990.

[30] I. Goyvaerts and J. Vercruysse. A Note on the categorification of Lie algebras. *ArXiv e-prints*, February 2012.

[31] Frank Hausser and Florian Nill. Diagonal crossed products by duals of quasi-quantum groups. *Rev. Math. Phys.*, 11(5):553–629, 1999.

[32] Pascual Jara Martínez, Javier López Peña, Florin Panaite, and Freddy van Oystaeyen. On iterated twisted tensor products of algebras. *Internat. J. Math.*, 19(9):1053–1101, 2008.

[33] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[34] David Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23(1):37–65, 1982.

[35] Seiichi Kamada. Knot invariants derived from quandles and racks. In *Invariants of knots and 3-manifolds (Kyoto, 2001)*, volume 4 of *Geom. Topol. Monogr.*, pages 103–117 (electronic). Geom. Topol. Publ., Coventry, 2002.

[36] Seiichi Kamada. Braid presentation of virtual knots and welded knots. *Osaka J. Math.*, 44(2):441–458, 2007.

[37] Seiichi Kamada. Quandles with good involutions, their homologies and knot invariants. In *Intelligence of low dimensional topology 2006*, volume 40 of *Ser. Knots Everything*, pages 101–108. World Sci. Publ., Hackensack, NJ, 2007.

[38] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[39] Christian Kassel. L'ordre de Dehornoy sur les tresses. *Astérisque*, (276):7–28, 2002. Séminaire Bourbaki, Vol. 1999/2000.

[40] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[41] Louis H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.

[42] Louis H. Kauffman and Sofia Lambropoulou. A categorical structure for the virtual braid group. *Comm. Algebra*, 39(12):4679–4704, 2011.

[43] Louis H. Kauffman and David Radford. Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links. In *Diagrammatic morphisms and applications (San Francisco, CA, 2000)*, volume 318 of *Contemp. Math.*, pages 113–140. Amer. Math. Soc., Providence, RI, 2003.

[44] Larry A. Lambe and David E. Radford. Algebraic aspects of the quantum Yang-Baxter equation. *J. Algebra*, 154(1):228–288, 1993.

[45] Dong Liu and Naihong Hu. Leibniz superalgebras and central extensions. *J. Algebra Appl.*, 5(6):765–780, 2006.

[46] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[47] Jean-Louis Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math. (2)*, 39(3-4):269–293, 1993.

[48] Jean-Louis Loday and Teimuraz Pirashvili. Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.*, 296(1):139–158, 1993.

[49] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[50] S. Majid. Free braided differential calculus, braided binomial theorem, and the braided exponential map. *J. Math. Phys.*, 34(10):4843–4856, 1993.

[51] Shahn Majid. Quasitriangular Hopf algebras and Yang-Baxter equations. *Internat. J. Modern Phys. A*, 5(1):1–91, 1990.

[52] Shahn Majid. Algebras and Hopf algebras in braided categories. In *Advances in Hopf algebras (Chicago, IL, 1992)*, volume 158 of *Lecture Notes in Pure and Appl. Math.*, pages 55–105. Dekker, New York, 1994.

[53] Shahn Majid. *Foundations of quantum group theory*. Cambridge University Press, Cambridge, 1995.

[54] V. O. Manturov. On invariants of virtual links. *Acta Appl. Math.*, 72(3):295–309, 2002.

[55] V. O. Manturov. On the recognition of virtual braids. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 299(Geom. i Topol. 8):267–286, 331–332, 2003.

[56] V. O. Manturov. Recognition of virtual braids. *Journal of Mathematical Sciences*, 131(1):5409–54192, 2005.

[57] Mitja Mastnak and Sarah Witherspoon. Bialgebra cohomology, pointed Hopf algebras, and deformations. *J. Pure Appl. Algebra*, 213(7):1399–1417, 2009.

[58] S. V. Matveev. Distributive groupoids in knot theory. *Mat. Sb. (N.S.)*, 119(161)(1):78–88, 160, 1982.

[59] Walter Michaelis. Lie coalgebras. *Adv. in Math.*, 38(1):1–54, 1980.

[60] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[61] Sam Nelson. Virtual knots. http://www.esotericka.org/cmc/vknots.html.

[62] Warren D. Nichols. Bialgebras of type one. *Comm. Algebra*, 6(15):1521–1552, 1978.

[63] M. Niebrzydowski and J. H. Przytycki. Homology operations on homology of quandles. *J. Algebra*, 324(7):1529–1548, 2010.

[64] Cyrille Ospel. *Tressages et théories cohomologiques pour les algèbres de Hopf. Application aux invariants des 3-variétés.* Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 1999/2. Université Louis Pasteur Département de Mathématique Institut de Recherche Mathématique Avancée, Strasbourg, 1999. Thèse, Université Louis Pasteur (Strasbourg I), Strasbourg, 1999.

[65] Florin Panaite. Hopf bimodules are modules over a diagonal crossed product algebra. *Comm. Algebra*, 30(8):4049–4058, 2002.

[66] Florin Panaite and Dragoş Ştefan. Deformation cohomology for Yetter-Drinfel′d modules and Hopf (bi)modules. *Comm. Algebra*, 30(1):331–345, 2002.

[67] Józef H. Przytycki. Distributivity versus associativity in the homology theory of algebraic structures. *Demonstratio Math.*, 44(4):823–869, 2011.

[68] Józef H. Przytycki and Adam S. Sikora. Distributive products and their homology. *ArXiv e-prints*, May 2011.

[69] David E. Radford. Solutions to the quantum Yang-Baxter equation and the Drinfel′d double. *J. Algebra*, 161(1):20–32, 1993.

[70] V. Rittenberg and D. Wyler. Generalized superalgebras. *Nuclear Phys. B*, 139(3):189–202, 1978.

[71] Marc Rosso. Groupes quantiques et algèbres de battage quantiques. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(2):145–148, 1995.

[72] Marc Rosso. Integrals of vertex operators and quantum shuffles. *Lett. Math. Phys.*, 41(2):161–168, 1997.

[73] Marc Rosso. Quantum groups and quantum shuffles. *Invent. Math.*, 133(2):399–416, 1998.

[74] Peter Schauenburg. On the braiding on a Hopf algebra in a braided category. *New York J. Math.*, 4:259–263 (electronic), 1998.

[75] Daniel S. Silver and Susan G. Williams. A generalized Burau representation for string links. *Pacific J. Math.*, 197(1):241–255, 2001.

[76] Moss E. Sweedler. *Hopf algebras.* Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[77] Rachel Taillefer. Théories homologiques des algèbres de hopf. http://math.univ-bpclermont.fr/ taillefer/papers/thesedf.pdf, 2001. Thesis (Ph.D.)– Univ. Montpellier II, Montpellier.

[78] Rachel Taillefer. Cohomology theories of Hopf bimodules and cup-product. *Algebr. Represent. Theory*, 7(5):471–490, 2004.

[79] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[80] Vladimir V. Vershinin. On homology of virtual braids and Burau representation. *J. Knot Theory Ramifications*, 10(5):795–812, 2001. Knots in Hellas '98, Vol. 3 (Delphi).

[81] Quinton Westrich. Lie algebras in braided monoidal categories. *Preprint at http://www.scribd.com/QuintonWestrich*, 2006.

[82] S. L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Comm. Math. Phys.*, 122(1):125–170, 1989.

[83] S. L. Woronowicz. Solutions of the braid equation related to a Hopf algebra. *Lett. Math. Phys.*, 23(2):143–145, 1991.

[84] Donald Yau. Deformation bicomplex of module algebras. *Homology, Homotopy Appl.*, 10(1):97–128, 2008.

[85] David N. Yetter. Quantum groups and representations of monoidal categories. *Math. Proc. Cambridge Philos. Soc.*, 108(2):261–290, 1990.