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Etude du pseudo-spectre d'opérateurs non auto-adjoints liés à la mécanique des fluides

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献给我的父母

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Résumé

Le tourbillon d’Oseen est une solution auto-similaire de l’équation d’évolution du tourbillon dans \mathbb{R}^2 et il est stable pour toutes les valeurs de nombre de Reynolds de circulation. La linéarisation du système autour d’un tourbillon d’Oseen intervient naturellement un opérateur non auto-adjoint, pour lequel nous étudions dans cette thèse les propriétés pseudo-spectrales dans la limite de rotation rapide.

On étudie dans le chapitre 2 un modèle bidimensionnel en négligeant un terme non local dans l’opérateur original et on démontre des estimations résolvantes optimales sur l’axe imaginaire. Le chapitre 3 est consacré à l’étude de l’opérateur linéarisé complet autour du tourbillon d’Oseen et on établit des estimations résolvantes qui nous permettent de caractériser le pseudo-spectre de l’opérateur, en utilisant la méthode des multiplicateurs. Enfin dans le chapitre 4, on démontre un résultat sur la dépendance des constantes provenant du calcul pseudo-différentiel dans le cadre de Weyl.

Mots-clefs

pseudo-spectre, opérateur non auto-adjoint, tourbillon d’Oseen, estimation résolvante, méthode des multiplicateurs, calcul de Weyl, métrique sur l’espace des phases, constantes de structure

Study of pseudospectrum for some non-self-adjoint operators linked to fluid mechanics

Abstract

Oseen vortices are self-similar solutions to the vorticity equation in \mathbb{R}^2 and they are stable for any value of the circulation Reynolds number. The linearization of the system around an Oseen vortex gives rise to a non-self-adjoint operator, whose pseudospectral properties in the fast rotation limit are the object of study in this thesis.

We study in chapter 2 a two-dimensional model by neglecting a non-local term in the original operator and we prove some optimal resolvent estimates along the imaginary axis. Chapter 3 is devoted to the study of the complete linearized operator around the Oseen vortex and we establish resolvent estimates which allow us to characterize the pseudospectrum of the operator, by using the multiplier method. Finally in chapter 4, we prove some results about the dependence of constants coming from pseudodifferential calculus in the framework of Weyl calculus.

Keywords

pseudospectrum, non-self-adjoint operator, Oseen vortex, resolvent estimate, multiplier method, Weyl calculus, metric on the phase space, structure constants

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Chapitre 1

Introduction

1.1 Présentation des questions

Dans cette thèse, on étudie des opérateurs non auto-adjoints provenant d'une question de stabilité dans la mécanique des fluides.

1.1.1 Equations du tourbillon

Considérons le mouvement d'un fluide incompressible visqueux dans le plan \mathbb{R}^2 , décrit par les équations de Navier-Stokes. En notant $x \in \mathbb{R}^2$ la variable d'espace et $t \in \mathbb{R}_+$ la variable de temps, le système s'écrit de la manière suivante

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \nu \Delta v - \nabla p, \\ \operatorname{div} v = 0, \end{cases} \quad (1.1.1)$$

où $v(x, t) \in \mathbb{R}^2$ désigne le champ de vitesse du fluide au point $x \in \mathbb{R}^2$ à l'instant t , et $p(x, t) \in \mathbb{R}$ son champ de pression. Le paramètre $\nu > 0$ désigne la viscosité cinématique.

Pour les fluides incompressibles, il est plus commode d'étudier le rotationnel du champ de vitesse. Dans plusieurs situations, nous nous intéressons aux champs de vitesse v nuls à l'infini. Avec la condition d'incompressibilité de divergence nulle, on peut les reconstituer à partir de leurs rotationnels. En particulier, lorsque la dimension est égale à 2, on peut identifier les matrices anti-symétriques avec les réels, donc le rotationnel est en fait scalaire. C'est la différence capitale entre le cas où la dimension vaut 2 et celui où elle est supérieure à 3. Plus précisément, on appelle *tourbillon* d'un champ de vecteurs v dans \mathbb{R}^2 son rotationnel

$$\omega = \partial_1 v_2 - \partial_2 v_1, \quad (1.1.2)$$

et l'on notera $\omega = \operatorname{curl} v$. En supposant que le tourbillon ω décroît suffisamment vite à l'infini, on peut reconstruire le champ à partir du tourbillon via la *loi de Biot-Savart*

$$v(x) = (K_{BS} * \omega)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{(x-y)^2} \omega(y) dy, \quad (1.1.3)$$

où $x^\perp = (-x_2, x_1)$ si $x = (x_1, x_2) \in \mathbb{R}^2$.

On s'intéresse maintenant à l'évolution du tourbillon du fluide. En appliquant l'opérateur rotationnel aux équations (1.1.1) et en utilisant $\operatorname{curl}(\nabla p) = 0$, on arrive à une équation plus simple

$$\partial_t \omega + v \cdot \nabla \omega = \nu \Delta \omega, \quad (1.1.4)$$

où le champ de vitesse v est reconstruit à partir du tourbillon ω par la loi de Biot-Savart (1.1.3). L'équation (1.1.4) est appelée formulation tourbillon de l'équation de Navier-Stokes, qui est une équation non locale.

Il est bien connu que le problème de Cauchy pour l'équation (1.1.4) est globalement bien posé dans $L^1(\mathbb{R}^2)$ (voir [BA94], [Kat94]) : pour toute donnée initiale ω_0 dans $L^1(\mathbb{R}^2)$, (1.1.4) possède une unique solution globale $\omega \in C^0([0, \infty), L^1(\mathbb{R}^2)) \cap C^0((0, \infty), L^\infty(\mathbb{R}^2))$. La masse totale du tourbillon est préservée par l'évolution

$$\int_{\mathbb{R}^2} \omega(x, t) dx = \int_{\mathbb{R}^2} \omega_0(x) dx, \quad t \geq 0. \quad (1.1.5)$$

La quantité ci-dessus est aussi appelée *circulation totale* de la vitesse v , puisque

$$\int_{\mathbb{R}^2} \omega(x, t) dx = \lim_{R \rightarrow \infty} \oint_{|x|=R} v(x, t) \cdot dl.$$

Le premier moment de la solution est aussi préservé si la solution est supposée suffisamment régulière

$$\int_{\mathbb{R}^2} x_j \omega(x, t) dx = \int_{\mathbb{R}^2} x_j \omega_0(x) dx, \quad j = 1, 2.$$

Le problème de Cauchy de (1.1.4) est aussi globalement bien posé dans un espace plus gros $\mathcal{M}(\mathbb{R}^2)$, l'espace des mesures réelles finies. Remarquons que les solutions de (1.1.4) correspondent aux solutions d'énergie infinie des équations de Navier-Stokes (1.1.1). Plus précisément, si $\omega(x, t)$ est une solution de (1.1.4) telle que $\int_{\mathbb{R}^2} \omega(x, t) dx \neq 0$, alors le champ de vitesse $v(x, t)$ donné par la loi de Biot-Savart (1.1.3) vérifie $\|v(\cdot, t)\|_{L^2} = +\infty$ pour tout temps $t \geq 0$.

Le but est d'étudier le comportement asymptotique en temps long des solutions de l'équation (1.1.4).

1.1.2 Tourbillon d'Oseen

Il y a une famille de solutions explicites de (1.1.4), appelées *tourbillon d'Oseen* ou *de Lamb-Oseen*, données par

$$\omega(x, t) = \frac{\alpha}{\nu t} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad v(x, t) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right), \quad (1.1.6)$$

avec les profils

$$G(x) = \frac{1}{4\pi} e^{-|x|^2/4}, \quad v^G(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-|x|^2/4}), \quad x \in \mathbb{R}^2, \quad (1.1.7)$$

et le paramètre $\alpha \in \mathbb{R}$ désignant le *nombre de Reynolds de circulation*. Notons que $\omega(x, t)$ est une fonction radiale en $x \in \mathbb{R}^2$ et ω est strictement positive partout pour tout temps $t > 0$. La loi de Biot-Savart implique que les trajectoires du champ de vitesse $v(x, t)$ sont des cercles pour tout temps $t > 0$. En particulier on voit $v(x, t) \cdot \nabla \omega(x, t) \equiv 0$, donc $\omega(x, t)$ données par (1.1.6) sont en fait solutions de l'équation de la chaleur linéaire. En outre, le paramètre α est égal à la circulation totale du tourbillon, car G est normalisée telle que $\int_{\mathbb{R}^2} G(x) dx = 1$.

Le tourbillon d'Oseen joue un rôle important en mécanique des fluides bidimensionnelle. Un résultat assez surprenant, démontré par T. Gallay et C.E. Wayne [GW05], dit que dès que le tourbillon initial est intégrable sur le plan, la solution de l'équation (1.1.4) converge vers un tourbillon d'Oseen lorsque le temps t tend vers l'infini.

Théorème 1.1.1 (Gallay-Wayne [GW05]). Soit $\omega_0 \in L^1(\mathbb{R}^2)$. Alors la solution de l'équation (1.1.4) avec donnée initiale ω_0 vérifie

$$\lim_{t \rightarrow +\infty} t^{1-\frac{1}{p}} \|\omega(\cdot, t) - \frac{\alpha}{\nu t} G\left(\frac{\cdot}{\sqrt{\nu t}}\right)\|_{L^p(\mathbb{R}^2)} = 0, \quad \text{pour } 1 \leq p \leq \infty,$$

où $\alpha = \int_{\mathbb{R}^2} \omega_0(x) dx$ est la circulation initiale. Si $v(x, t)$ est la solution de (1.1.1) obtenue à partir de $\omega(x, t)$ via la loi de Biot-Savart, alors

$$\lim_{t \rightarrow +\infty} t^{\frac{1}{2}-\frac{1}{q}} \|v(\cdot, t) - \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{\cdot}{\sqrt{\nu t}}\right)\|_{L^q(\mathbb{R}^2)} = 0, \quad \text{pour } 2 < q \leq \infty.$$

En d'autres termes, les solutions de (1.1.4) dans $L^1(\mathbb{R}^2)$ se comportent asymptotiquement comme les solutions de l'équation de la chaleur linéaire $\partial_t \omega = \nu \Delta \omega$ avec la même donnée initiale. Ce résultat a des conséquences importantes. D'une part, il implique que les tourbillons d'Oseen sont les seules solutions auto-similaires des équations de Navier-Stokes dans \mathbb{R}^2 dont le tourbillon est intégrable. D'autre part, il en résulte que les tourbillons d'Oseen sont globalement stables pour toutes les valeurs du nombre de Reynolds de circulation α . Contrairement aux plusieurs situations dans l'hydrodynamique, par exemple le flot de Poiseuille et le flot de Taylor-Couette, aucune instabilité ne se produira en augmentant le nombre de Reynolds de circulation.

Le Théorème 1.1.1 est établi par une étude de l'équation du tourbillon dans des variables auto-similaires. En s'inspirant de la forme de (1.1.6), on introduit les variables auto-similaires, soit $T > 0$,

$$\tilde{x} = x/\sqrt{\nu t}, \quad \tilde{t} = \log(t/T). \quad (1.1.8)$$

Si $\omega(x, t)$ est une solution de (1.1.4) et si $v(x, t)$ est le champ de vitesse correspondant, on définit des nouvelles fonctions $\tilde{\omega}(\tilde{x}, \tilde{t})$, $\tilde{v}(\tilde{x}, \tilde{t})$ de la façon suivante

$$\omega(x, t) = \frac{1}{t} \tilde{\omega}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right), \quad v(x, t) = \sqrt{\frac{\nu}{t}} \tilde{v}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right).$$

Puisqu'on va travailler dans toute la suite dans les variables auto-similaires, on remplace \tilde{x} par x , $\tilde{\omega}$ par ω etc, le système s'écrit de la manière suivante

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \Delta \omega + \frac{1}{2} x \cdot \nabla \omega + \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (1.1.9)$$

où $\omega(x, t) \in \mathbb{R}$ est le tourbillon, la vitesse $v(x, t) \in \mathbb{R}^2$ est reliée à ω par la loi de Biot-Savart (1.1.3). Par construction, pour tout $\alpha \in \mathbb{R}$, le tourbillon d'Oseen $\omega = \alpha G$ est une solution stationnaire de (1.1.9), qui est juste une distribution gaussienne.

1.1.3 Opérateur linéarisé

Afin d'étudier la stabilité du tourbillon d'Oseen, on commence par linéariser l'équation (1.1.9) autour de αG . En remplaçant ω par $\alpha G + \omega^0$, v par $\alpha v^G + v^0$ dans (1.1.9), on obtient l'équation perturbée

$$\frac{\partial \omega^0}{\partial t} + v^0 \cdot \nabla \omega^0 = -(\mathcal{L} + \alpha \Lambda) \omega^0, \quad (1.1.10)$$

où v^0 est le champ de vitesse associé au tourbillon perturbé ω^0 via la loi de Biot-Savart,

$$\mathcal{L} \omega^0 = -\Delta \omega^0 - \frac{1}{2} x \cdot \nabla \omega^0 - \omega^0, \quad (1.1.11)$$

$$\Lambda \omega^0 = v^G \cdot \omega^0 + (K_{BS} * \omega^0) \cdot \nabla G. \quad (1.1.12)$$

L'opérateur linéarisé $\mathcal{L} + \alpha\Lambda$ sera notre objet principal des études de cette thèse.

Il y a déjà beaucoup d'études pour cet opérateur linéarisé dans la littérature. On présente maintenant quelques propriétés spectrales élémentaires.

Propriétés spectrales de \mathcal{L}

En comparant (1.1.10) avec (1.1.9), on voit que l'opérateur \mathcal{L} est le laplacien dans \mathbb{R}^2 dans les variables auto-similaires (1.1.8). Ses propriétés spectrales sont bien connues, voir [GW05], [GW02]. T. Gallay et C.E. Wayne y ont travaillé dans des espaces L^2 avec poids polynomiaux et ils ont obtenu des estimations spectrales de \mathcal{L} qui leur permettent d'obtenir une convergence exponentielle de solution de (1.1.9) vers αG lorsque le temps rescaled \tilde{t} tend vers l'infini. En revenant aux variables originales, une convergence polynomiale de solution de (1.1.4) vers le tourbillon d'Oseen (1.1.6) a été établie.

On s'intéresse à un espace de Hilbert particulier $Y = L^2(\mathbb{R}^2; G^{-1}dx)$ muni du produit scalaire

$$\langle \omega_1, \omega_2 \rangle_Y = \int_{\mathbb{R}^2} \omega_1(x) \overline{\omega_2(x)} G(x)^{-1} dx, \quad \omega_1, \omega_2 \in Y.$$

Une propriété remarquable c'est que l'opérateur \mathcal{L} avec domaine maximal

$$D(\mathcal{L}) = \{\omega \in Y; \mathcal{L}\omega \in Y\}$$

est auto-adjoint et positif sur Y . En effet, en conjuguant l'opérateur \mathcal{L} avec le poids $G^{1/2}$, on trouve l'oscillateur harmonique bidimensionnel

$$L = G^{-1/2} \mathcal{L} G^{1/2} = -\Delta + \frac{|x|^2}{16} - \frac{1}{2}. \quad (1.1.13)$$

Grâce à cette conjugaison, on sait que l'opérateur \mathcal{L} a résolvant compact dans Y et son spectre est composé d'une suite de valeurs propres

$$\sigma(\mathcal{L}) = \left\{ \frac{k}{2}, \quad k = 0, 1, 2, \dots \right\}.$$

La valeur propre $k/2$ est de multiplicité $k+1$ et l'espace propre associé est engendré par les fonctions d'Hermite de degré k . En particulier, $\mathcal{L}G = 0$ et $\mathcal{L}\partial_j G = \frac{1}{2}\partial_j G$ pour $j = 1, 2$. Lorsque l'on travaille dans l'espace Y , on s'intéresse à la stabilité du tourbillon d'Oseen par rapport aux perturbations de décroissance gaussienne à l'infini.

Propriétés spectrales de Λ

L'opérateur Λ (1.1.12) est la somme de deux termes Λ_1, Λ_2 , qui ont des interprétations physiques différentes [Gal11]. L'opérateur d'advection $\Lambda_1\omega^0 = v^G \cdot \nabla \omega^0$ est un opérateur différentiel local d'ordre 1, qui décrit comment le tourbillon perturbé ω^0 est transporté par le champ de vitesse v^G du tourbillon non perturbé. Par contre, le terme $\Lambda_2\omega^0 = (K_{BS} * \omega^0) \cdot \nabla G$ est un opérateur intégral-différentiel qui est non local, et il décrit l'advection du tourbillon non perturbé G par le champ de vitesse perturbé $v^0 = K_{BS} * \omega^0$.

L'avantage de travailler dans l'espace Y est que les deux termes Λ_1, Λ_2 sont séparément anti-symétriques sur Y

$$\langle \Lambda_j \omega_1, \omega_2 \rangle_Y = -\langle \omega_1, \Lambda_j \omega_2 \rangle_Y, \quad \omega_1, \omega_2 \in D(\Lambda_j) \subset Y, \quad j = 1, 2.$$

En plus, si Λ est défini sur son domaine maximal, on peut démontrer que Λ est anti-adjoint sur Y . Le noyau de l'opérateur Λ est caractérisé par Y. Maekawa [Mae11]

$$\text{Ker}(\Lambda) = \{\text{fonctions radiales dans } \mathbb{R}^2\} \oplus \{\beta_1 \partial_1 G + \beta_2 \partial_2 G, \quad \beta_1, \beta_2 \in \mathbb{C}\}.$$

Propriétés spectrales de $\mathcal{L} + \alpha\Lambda$

Il est démontré dans [GW05] que l'opérateur Λ est une perturbation relativement compacte de \mathcal{L} . Par la théorie de perturbation classique (voir Chapter V dans [Kat95]), le spectre de l'opérateur non auto-adjoint $\mathcal{L} + \alpha\Lambda$ sur Y est composé d'une suite de valeurs propres. Cet opérateur a une structure particulier :

$$A + iB, \quad A \text{ auto-adjoint et positif}, \quad iB \text{ anti-adjoint}, \quad [A, iB] \neq 0,$$

qu'on discutera dans la Section 1.2.2.

Introduisons quelques sous-espaces de Y

$$\begin{aligned} Y_0 &= \{\omega \in Y; \int_{\mathbb{R}^2} \omega(x)dx = 0\} = \{G\}^\perp, \\ Y_1 &= \{\omega \in Y_0; \int_{\mathbb{R}^2} x_j \omega(x)dx = 0 \text{ for } j = 1, 2\} = \{G; \partial_1 G; \partial_2 G\}^\perp, \\ Y_2 &= \{\omega \in Y_1; \int_{\mathbb{R}^2} |x|^2 \omega(x)dx = 0\} = \{G; \partial_1 G; \partial_2 G; \Delta G\}^\perp. \end{aligned}$$

Ces sous-espaces sont tous invariants pour \mathcal{L} et Λ , et ils sont même laissés invariants par l'évolution non linéaire (1.1.10). Il n'est pas difficile de démontrer les bornes spectrales de $\mathcal{L} + \alpha\Lambda$.

Proposition 1.1.2 (Gallay-Wayne [GW05]). *Soit $\alpha \in \mathbb{R}$. Alors*

$$\begin{aligned} \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \text{Re}(z) \geq 0\} \quad \text{sur } Y, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \text{Re}(z) \geq \frac{1}{2}\} \quad \text{sur } Y_0, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \text{Re}(z) \geq 1\} \quad \text{sur } Y_1, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \text{Re}(z) > 1\} \quad \text{sur } Y_2, \text{ si } \alpha \neq 0. \end{aligned}$$

Ces bornes spectrales nous permettent de déduire des estimations sur le semi-groupe associé à $\mathcal{L} + \alpha\Lambda$. Celles-ci peuvent être utilisées à démontrer que le tourbillon d'Oseen αG est une solution stable de l'équation du tourbillon dans les variables auto-similaires (1.1.9) pour n'importe quelle valeur du nombre de Reynolds de circulation $\alpha \in \mathbb{R}$.

Pourtant, les bornes inférieures spectrales dans la Proposition 1.1.2 ne sont pas précises et surtout la dernière est même très loin d'être optimale. Il existe évidemment des valeurs propres de \mathcal{L} qui ne bougent pas sous l'action de la perturbation anti-adjointe $\alpha\Lambda$, et les fonctions propres associées sont donc dans le noyau de Λ . À part d'eux, toutes les valeurs propres de $\mathcal{L} + \alpha\Lambda$ correspondant aux fonctions propres dans le complément orthogonal du noyau de Λ , ont une partie réelle qui tend vers $+\infty$ lorsque le paramètre $|\alpha| \rightarrow \infty$. Ce fait a été numériquement observé par A. Prochazka et D. Pullin [PP95] et a récemment été affirmé par Y. Maekawa [Mae11]. Autrement dit, la présence de la perturbation anti-adjointe de taille très grande augmente la borne spectrale de l'opérateur, donc le semi-groupe associé décroît vite lorsque $|\alpha|$ devient grand. En outre, des simulations numériques dans [PP95] ont indiqué que le taux de divergence est égal à $\mathcal{O}(|\alpha|^{1/2})$, mais aucun résultat quantitatif n'est valable jusqu'à présent.

On introduit quelques notations pour présenter deux conjectures sur le spectre et le pseudo-spectre de l'opérateur non auto-adjoint $\mathcal{L} + \alpha\Lambda$, formulées par T. Gallay dans [Gal11].

Définition 1.1.3. Soit Y_\perp le complément orthogonal de $\text{Ker}(\Lambda)$ dans Y . Soit \mathcal{H}_α la restriction de l'opérateur $\mathcal{L} + \alpha\Lambda$ au sous-espace invariant $Y_\perp \subset Y$. Soit $\sigma(\mathcal{H}_\alpha)$ son spectre. On définit deux quantités

$$\text{borne inférieure spectrale} \quad \Sigma_\alpha = \inf \{\text{Re}(z); z \in \sigma(\mathcal{H}_\alpha)\}, \quad (1.1.14)$$

$$\text{borne inférieure pseudo-spectrale} \quad \Psi_\alpha = \left(\sup_{\lambda \in \mathbb{R}} \|(\mathcal{H}_\alpha - i\lambda)^{-1}\| \right)^{-1}. \quad (1.1.15)$$

Une relation élémentaire

D'abord, on a une relation élémentaire entre les deux quantités introduites dans la Définition 1.1.3

$$\frac{1}{2} \leq \Psi_\alpha \leq \Sigma_\alpha, \quad \text{pour tout } \alpha \in \mathbb{R}. \quad (1.1.16)$$

Démonstration. En effet, puisque \mathcal{L} est supérieur à $1/2$ sur Y_\perp , on a pour tout $\lambda \in \mathbb{R}$

$$\text{Re}\langle (\mathcal{H}_\alpha - i\lambda)u, u \rangle_Y \geq \frac{1}{2}\|u\|_Y^2, \quad \forall u \in Y_\perp.$$

Il en résulte que $\|(\mathcal{H}_\alpha - i\lambda)u\|_Y \geq \frac{1}{2}\|u\|_Y$ pour tout $u \in Y_\perp$. Notons que $i\lambda$ n'est pas dans le spectre de \mathcal{H}_α , on a alors $\|(\mathcal{H}_\alpha - i\lambda)^{-1}\| \leq 2$. Soient z_α la valeur propre de \mathcal{H}_α qui a la plus petite partie réelle et u_α la fonction propre associée. Alors

$$(\mathcal{H}_\alpha - i\lambda)u_\alpha = (z_\alpha - i\lambda)u_\alpha, \quad \|(\mathcal{H}_\alpha - i\lambda)u_\alpha\|_Y \geq \text{Re}(z_\alpha)\|u_\alpha\|_Y,$$

ce qui implique

$$\|(\mathcal{H}_\alpha - i\lambda)^{-1}\| \geq \frac{1}{\text{Re}(z_\alpha)} = \frac{1}{\Sigma_\alpha}.$$

En prenant le supremum en $\lambda \in \mathbb{R}$, on obtient $2 \geq \Psi_\alpha^{-1} \geq \Sigma_\alpha^{-1}$, qui est bien (1.1.16). \square

Dans le cas auto-adjoint, c'est-à-dire que la perturbation anti-adjointe $\alpha\Lambda$ n'est pas présente, les deux quantités dans la Définition 1.1.3 coïncident, grâce au théorème spectral. En revanche, dans le cas non auto-adjoint, l'inégalité à droite dans (1.1.16) pourrait devenir stricte. Les quantités Σ_α et Ψ_α ont un lien avec la norme du semi-groupe associé, voir Lemme 1.1 dans [GGN09].

Sur la quantité pseudo-spectrale

Il faudrait expliquer pourquoi la quantité Ψ_α est reliée au pseudo-spectre de \mathcal{H}_α . Nous donnons deux points de vue. Rappelons la définition classique de pseudo-spectre pour les opérateurs semiclassiques, voir par exemple [PS04], [DSZ04]. Pour une famille d'opérateurs $\{P_\alpha\}_{|\alpha|>1}$ sur un espace de Hilbert X , son pseudo-spectre est défini comme le complément de l'ensemble complexe de points $z \in \mathbb{C}$ tels que

$$\exists N \in \mathbb{N}, \quad \limsup_{|\alpha| \rightarrow +\infty} |\alpha|^{-N} \|(P_\alpha - z)^{-1}\|_{\mathcal{L}(X)} < +\infty. \quad (1.1.17)$$

Supposons que l'on a déjà démontré que pour certaines constantes $C, \gamma > 0$,

$$\Psi_\alpha \geq C|\alpha|^\gamma \quad \text{pour tout } |\alpha| \geq 1. \quad (1.1.18)$$

Alors on voit facilement que le pseudo-spectre de la famille d'opérateurs $\{|\alpha|^{-\gamma}\mathcal{H}_\alpha\}_{|\alpha| \geq 1}$ est contenu dans le demi-plan $\{z \in \mathbb{C}; \text{Re}(z) \geq C^{-1}\}$, en utilisant la formule de résolvant.

Démonstration. Si $\operatorname{Re}(z) \leq 0$, on a pour tout $u \in D(\mathcal{H}_\alpha)$ avec $\|u\|_Y = 1$,

$$|\langle (\mathcal{H}_\alpha - z)u, u \rangle_Y| = |\langle \mathcal{H}_\alpha u, u \rangle_Y - z| \geq \frac{1}{2} - \operatorname{Re}(z) \geq \frac{1}{2},$$

donc $\|(\mathcal{H}_\alpha - z)^{-1}\| \leq 2$ pour $\operatorname{Re}(z) \leq 0$. D'autre part, par la formule de résolvant, pour $\mu, \lambda \in \mathbb{R}$, on a

$$(\mathcal{H}_\alpha - \mu - i\lambda)^{-1} - (\mathcal{H}_\alpha - i\lambda)^{-1} = \mu(\mathcal{H}_\alpha - i\lambda)^{-1}(\mathcal{H}_\alpha - \mu - i\lambda)^{-1}.$$

Si $\mu > 0$, $\mu\|(\mathcal{H}_\alpha - i\lambda)^{-1}\| < 1$ et $\mu < C^{-1}|\alpha|^\gamma$, alors on déduit de l'hypothèse (1.1.18)

$$\|(\mathcal{H}_\alpha - \mu - i\lambda)^{-1}\| \leq \frac{\|(\mathcal{H}_\alpha - i\lambda)^{-1}\|}{1 - \mu\|(\mathcal{H}_\alpha - i\lambda)^{-1}\|} \leq \frac{C|\alpha|^{-\gamma}}{1 - C\mu|\alpha|^{-\gamma}}.$$

En résumé, on a démontré que pour tout $\kappa \in (0, 1)$, $z \in \mathbb{C}$ tel que $\operatorname{Re}(z) < \kappa C^{-1}|\alpha|^\gamma$, $\|(\mathcal{H}_\alpha - z)^{-1}\| \leq C_\kappa$, qui est équivalent à la suivante

$$\forall \zeta \in \mathbb{C}, \operatorname{Re}(\zeta) < \kappa C^{-1}, \quad \|(|\alpha|^{-\gamma} \mathcal{H}_\alpha - \zeta)^{-1}\| \leq C_\kappa |\alpha|^\gamma.$$

Par conséquent, si $\operatorname{Re}(\zeta) < C^{-1}$, alors ζ n'est pas dans le pseudo-spectre de $\{|\alpha|^{-\gamma} \mathcal{H}_\alpha\}_{\alpha \geq 1}$, ce qui termine la démonstration. \square

Dans l'article [GGN09], au lieu de définir le pseudo-spectre d'une famille d'opérateurs comme un sous-ensemble de \mathbb{C} , une notion plus flexible est introduite.

Définition 1.1.4. Soit $(\omega_\alpha)_{|\alpha| \geq 1}$ une famille de domaines complexes, i.e. $\omega_\alpha \subset \mathbb{C}$ pour tout $|\alpha| \geq 1$. On dit que ω_α évite le pseudo-spectre de \mathcal{H}_α lorsque $|\alpha| \rightarrow +\infty$ s'il existe $C > 0$ et $N \in \mathbb{N}$ tels que

$$\limsup_{|\alpha| \rightarrow +\infty} |\alpha|^{-N} \sup_{z \in \omega_\alpha} \|(\mathcal{H}_\alpha - z)^{-1}\| \leq C.$$

Au contraire, on dit que ω_α rencontre le pseudo-spectre de \mathcal{H}_α lorsque $|\alpha| \rightarrow +\infty$, si pour tout $N \in \mathbb{N}$,

$$\limsup_{|\alpha| \rightarrow +\infty} |\alpha|^{-N} \sup_{z \in \omega_\alpha} \|(\mathcal{H}_\alpha - z)^{-1}\| = +\infty.$$

On peut démontrer le lemme suivant, voir Lemma 1.3 dans [GGN09].

Lemme 1.1.5. i) Pour tout $\kappa \in (0, 1)$, le domaine $\{z \in \mathbb{C}; \operatorname{Re}(z) \leq \kappa \Psi_\alpha\}$ évite le pseudo-spectre de \mathcal{H}_α lorsque $|\alpha| \rightarrow +\infty$.

ii) Si $\mu_\alpha \gg \Psi_\alpha(1 + \log \Psi_\alpha + \log |\alpha|)$ dans le sens que le rapport tend vers $+\infty$ lorsque $|\alpha| \rightarrow +\infty$, alors le domaine $\{z \in \mathbb{C}; \operatorname{Re}(z) \leq \mu_\alpha\}$ rencontre le pseudo-spectre de \mathcal{H}_α lorsque $|\alpha| \rightarrow +\infty$.

Comme nous avons expliqué, la présence de la perturbation anti-adjointe $\alpha \Lambda$ augmente la borne inférieure spectrale de \mathcal{H}_α . Lorsque le nombre de Reynolds de circulation $|\alpha|$ devient très grand, le spectre est emporté beaucoup plus loin de l'axe imaginaire. On veut visualiser cet effet sur les deux quantités introduites dans la Définition 1.1.3. Le travail de Y. Maekawa [Mae11] nous dit que la quantité $\Sigma_\alpha \rightarrow +\infty$ lorsque $|\alpha| \rightarrow +\infty$, et son résultat peut en fait être modifié à arriver à la conclusion plus forte : $\Psi_\alpha \rightarrow +\infty$ lorsque $|\alpha| \rightarrow +\infty$. Néanmoins, sa preuve est basée sur un argument de contradiction donc il n'a fourni aucun résultat quantitatif. Une question naturelle est de trouver le taux de divergence en fonction de α .

Conjecture 1.1.6 ([Gal11], sur le pseudo-spectre).

$$\Psi_\alpha \approx |\alpha|^{1/3}, \quad \text{lorsque } |\alpha| \rightarrow +\infty.$$

Basé sur un modèle unidimensionnel introduit par I. Gallagher, T. Gallay et F. Nier dans [GGN09] qu'on va présenter dans la Section 1.1.4, et un modèle en dimension 2 étudié dans [Den10a], on croit que la puissance 1/3 soit optimale si la Conjecture 1.1.6 est vraie.

Conjecture 1.1.7 ([Gal11], sur le spectre). *La valeur propre de \mathcal{H}_α qui a la plus petite partie réelle vérifie*

$$\lambda_\alpha \approx \left(\frac{|\alpha|}{16\pi}\right)^{1/2}(1+i), \quad \text{lorsque } |\alpha| \rightarrow +\infty. \quad (1.1.19)$$

La fonction propre associée a une expression suivante

$$\omega_\alpha(r \sin \theta, r \cos \theta) \approx e^{-\frac{1}{4}(r-z_\alpha)^2} e^{i\theta}, \quad \text{où } z_\alpha \approx \left(\frac{8i|\alpha|}{\pi}\right)^{1/4}.$$

En particulier, (1.1.19) implique

$$\Sigma_\alpha \approx |\alpha|^{1/2}, \quad \text{lorsque } |\alpha| \rightarrow +\infty,$$

ce qui coïncide avec les observations numériques dans [PP95]. Notons que la fonction propre ω_α est asymptotiquement concentrée dans une couronne située d'une distance $\mathcal{O}(|\alpha|^{1/4})$ de l'origine, donc très loin du centre du tourbillon.

Remarque 1.1.8. Rappelons que dans le cas auto-adjoint, Ψ et Σ sont égales. Pour le problème original non auto-adjoint, on estime que les taux de divergence de Ψ_α et de Σ_α soient différents, c'est un effet non auto-adjoint.

Conjugaison

Au lieu d'étudier l'opérateur \mathcal{H}_α dans l'espace à poids $Y = L^2(\mathbb{R}^2; G^{-1}dx)$, on préfère travailler dans l'espace usuel sans poids $L^2(\mathbb{R}^2; dx)$. On définit

$$\mathcal{H}_\alpha = G^{-1/2} \mathcal{H}_\alpha G^{1/2} = L + \alpha M, \quad (1.1.20)$$

où

$$L\omega = G^{-1/2} \mathcal{L} G^{1/2} \omega = -\Delta \omega + \frac{|x|^2}{16} \omega - \frac{1}{2} \omega, \quad (1.1.21)$$

$$M\omega = G^{-1/2} \Lambda G^{1/2} \omega = v^G \cdot \nabla \omega - \frac{1}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega)), \quad (1.1.22)$$

pour $\omega \in L^2(\mathbb{R}^2; dx)$. Rappelons que l'on utilise les notations (1.1.7), (1.1.3). Le terme non local donné par la loi de Biot-Savart devient le deuxième membre dans M . Remarquons le premier terme dans M est

$$v^G \cdot \nabla \omega = \frac{1}{8\pi} \sigma(x) \partial_\theta, \quad \text{où } \sigma(x) = \frac{1 - e^{-|x|^2/4}}{|x|^2/4}, \quad \partial_\theta = x_1 \partial_2 - x_2 \partial_1, \quad (1.1.23)$$

un champ de vecteurs de divergence nulle. Le noyau de M est

$$\begin{aligned} \text{Ker}(M) = & \{ \text{fonctions radiales dans } \mathbb{R}^2 \} \oplus \\ & \{ \beta_1 x_1 G(x)^{1/2} + \beta_2 x_2 G(x)^{1/2}; \beta_1, \beta_2 \in \mathbb{C} \}. \end{aligned} \quad (1.1.24)$$

On se restreint comme précédent au complément orthogonal de $\text{Ker}(M)$ dans $L^2(\mathbb{R}^2; dx)$. Le point clé est que les opérateurs L et M sont invariants par rotations autour de l'origine, donc on peut travailler dans les coordonnées polaires et développer la variable angulaire en série de Fourier. La question bidimensionnelle se réduit donc à une famille d'opérateurs unidimensionnels dans la variable radiale.

L'étude des propriétés spectrales et pseudo-spectrales de \mathcal{H}_α est difficile, à cause du terme non local dans la partie anti-adjointe. Puisque ce terme est compact, on pourrait espérer qu'il ne joue pas de rôle important.

1.1.4 Un modèle opérateur unidimensionnel

Le premier essai pour étudier ce problème non auto-adjoint est dû à I. Gallagher, T. Gallay et F. Nier [GGN09]. Motivé par l'opérateur bidimensionnel (1.1.20) qui nous intéresse, ils ont proposé un modèle unidimensionnel général, mais sans terme non local

$$H_\alpha = -\partial_x^2 + x^2 + i\alpha f(x), \quad x \in \mathbb{R}, \quad (1.1.25)$$

où $\alpha \geq 1$ est un grand paramètre, $f: \mathbb{R} \rightarrow \mathbb{R}$ est une fonction lisse, bornée et à valeurs réelles. Le domaine de H_α est $\{u \in H^2(\mathbb{R}); |x|^2 u \in L^2(\mathbb{R})\}$. Ce modèle a une partie imaginaire extrêmement simple, juste une multiplication par une fonction purement imaginaire.

Les auteurs de [GGN09] ont étudié des propriétés spectrales et pseudo-spectrales de ce modèle dans la limite $\alpha \rightarrow +\infty$. Sous des conditions appropriées imposées sur la fonction f , ils veulent savoir comment la présence de la partie imaginaire affecte-t-elle les propriétés spectrales et pseudo-spectrales de l'opérateur H_α . Définissons deux quantités comme celles introduites dans la Définition 1.1.3

$$\tilde{\Sigma}_\alpha = \inf \{\text{Re}(z), z \in \sigma(H_\alpha)\}, \quad (1.1.26)$$

$$\tilde{\Psi}_\alpha = \left(\sup_{\lambda \in \mathbb{R}} \|(H_\alpha - i\lambda)^{-1}\| \right)^{-1}. \quad (1.1.27)$$

On a $\tilde{\Sigma}_\alpha \geq \tilde{\Psi}_\alpha \geq 1$ pour tout $\alpha > 0$.

Les premiers résultats sont qualitatifs : dès que la fonction f est non constante, alors $\tilde{\Psi}_\alpha > 1$, ce qui implique en particulier que l'état fondamental de l'oscillateur harmonique, ainsi que le spectre, bouge vers la droite sous l'action de la perturbation anti-adjointe ; en outre, si toutes les lignes de niveau de la fonction f sont d'intérieur vide, alors $\tilde{\Psi}_\alpha$ tend vers l'infini lorsque α tend vers $+\infty$. La question qui suit naturellement est de chercher des résultats quantitatifs. A ce moment-là, il faudrait imposer des conditions plus précises sur la fonction f . Les auteurs ont proposé deux approches pour étudier ce problème.

La première approche repose sur la méthode d'hypocoercivité, développée par C. Villani [Vil09], [Vil06]. Cette méthode s'applique en particulier aux opérateurs de la forme $L = A^*A + B$ sur un espace de Hilbert X , où B est anti-symétrique. Sous certaines conditions, ceci permet de comparer les propriétés spectrales de L avec celles de l'opérateur auto-adjoint $\hat{L} = A^*A + C^*C$ avec $C = [A, B]$. Le problème (1.1.25) rentre dans ce cadre : si l'on prend $X = L^2(\mathbb{R})$, $A = \partial_x + x$ et $B = i\alpha f(x)$, on a $H_\alpha - 1 = A^*A + B$. Puisque $C = [A, B] = i\alpha f'(x)$, l'opérateur auto-adjoint associé \hat{H}_α défini par $\hat{H}_\alpha - 1 = A^*A + C^*C$ a la forme suivante

$$\hat{H}_\alpha = -\partial_x^2 + x^2 + \alpha^2 f'(x)^2, \quad x \in \mathbb{R}.$$

Il est assez standard de déduire la borne inférieure spectrale de l'opérateur \hat{H}_α , en utilisant des méthodes classiques. La méthode hypocoercive leur permet d'estimer le semi-groupe

associé à l'opérateur non auto-adjoint (1.1.25). Voir le Théorème 1.5 dans [GGN09] pour l'énoncé détaillé.

La deuxième approche est plus classique. On s'intéressent à une classe de fonctions spécifiques :

Hypothèse 1.1.9 ([GGN09, Hypothesis 1.6]). *Supposons que f est une fonction de classe C^3 vérifiant les conditions suivantes.*

- (i) *f est une fonction de Morse, c'est-à-dire $f'(x) = 0$ implique $f''(x) \neq 0$.*
- (ii) *Il existe des constantes $k > 0$, $C > 0$ telles que pour tout $|x| \geq 1$, on a*

$$\left| \partial_x^l \left(f(x) - \frac{1}{|x|^k} \right) \right| \leq \frac{C}{|x|^{k+l+1}}, \quad \text{pour } l = 0, 1, 2, 3.$$

Grosso modo, on considère des fonctions de Morse qui se comportent comme $|x|^{-k}$ à l'infini, ainsi que leurs dérivées jusqu'à l'ordre 3. Cette classe de fonctions est bien motivée par la fonction radiale $\sigma(x)$ dans (1.1.23), qui a un seul point critique non dégénéré localisé à l'origine et décroît comme $|x|^{-2}$ à l'infini. Pour une fonction vérifiant l'Hypothèse 1.1.9, l'estimation optimale pour la quantité pseudo-spectrale a été démontrée.

Théorème 1.1.10 ([GGN09, Theorem 1.8]). *Supposons la fonction f vérifiant l'Hypothèse 1.1.9 pour un certain k . Alors il existe une constante $M \geq 1$ telle que pour tout $\alpha \geq 1$*

$$\frac{1}{M} \alpha^\nu \leq \tilde{\Psi}_\alpha \leq M \alpha^\nu, \quad \text{avec } \nu = \frac{2}{k+4}.$$

Remarque 1.1.11. Ce modèle unidimensionnel suggère que la bonne puissance de croissance pour la quantité pseudo-spectrale soit égale à $1/3$ pour le problème initial (1.1.20), puisque la fonction $\sigma(x)$ dans (1.1.23) vérifie l'Hypothèse 1.1.9 avec $k = 2$. Ceci coïncide avec la Conjecture 1.1.6.

La partie difficile de ce théorème est de trouver une borne inférieure de $\tilde{\Psi}_\alpha$. Le résultat est établi en utilisant une méthode classique, et elle consiste à ramener le problème aux modèles microlocaux, pour lesquels on sait des estimations sous-elliptiques semiclassiques, en localisant la variable x près de points critiques de f et près de l'infini.

L'analyse pour la quantité spectrale $\tilde{\Sigma}_\alpha$ est plus difficile. Ils l'ont examinée sur un exemple concret

$$f(x) = (1 + |x|^2)^{-k/2}, \quad x \in \mathbb{R}.$$

Alors il existe $C > 0$ tel que pour tout $\alpha \geq 1$, (voir Proposition 1.9 dans [GGN09])

$$\tilde{\Sigma}_\alpha \geq C \alpha^{\nu'}, \quad \text{avec } \nu' = \min \left\{ \frac{1}{2}, \frac{2}{k+2} \right\}. \quad (1.1.28)$$

Ce résultat est basé sur une méthode de déformation complexe, et en plus ils ont donné des simulations numériques pour assurer que l'estimation (1.1.28) est optimale au sens où la puissance ν' ne pourrait pas être améliorée. En particulier, lorsque $k = 2$, $\tilde{\Sigma}_\alpha \approx \alpha^{1/2}$, qui coïncide avec la puissance dans la Conjecture 1.1.7.

1.2 Opérateur non auto-adjoint et Pseudo-spectre

Les opérateurs non auto-adjoints apparaissent naturellement dans des problèmes de la physique mathématique. Il est bien connu que pour les opérateurs non auto-adjoints, il

peut y avoir des phénomènes d'instabilité, observés numériquement et étudiés théoriquement. L'opérateur de Orr-Sommerfeld, provenant de la mécanique des fluides, est l'un des premiers exemples traités numériquement, pour lequel l'étude spectrale donne une explication de l'instabilité du flot de Poiseuille dans le plan. Pourtant, dans certains cas, les prédictions suggérées par la théorie spectrale ne correspondent pas aux simulations numériques. La difficulté dans l'étude des opérateurs non auto-adjoints consiste au fait que la norme de la résolvante peut être grande même très loin du spectre.

1.2.1 Pseudo-spectre

La notion de pseudo-spectre est introduite par L.N. Trefethen [Tre97] [TE05], et elle a été étudiée numériquement depuis une dizaine d'années. L'idée est de considérer non seulement le spectre où la résolvante d'opérateur n'est pas définie, mais aussi l'ensemble des points où la résolvante est très grande en norme. Soit A un opérateur linéaire (fermé) sur un espace d'Hilbert X . Pour $\epsilon > 0$, le ϵ -pseudo-spectre de A est défini par

$$\sigma_\epsilon(A) = \{z \in \mathbb{C}; \| (z - A)^{-1} \| > \frac{1}{\epsilon}\}, \quad (1.2.1)$$

avec la convention $\| (z - A)^{-1} \| = +\infty$ pour z appartenant au spectre de A . En d'autres termes, l'étude du pseudo-spectre d'un opérateur est exactement l'étude des courbes de niveau de la norme de la résolvante.

Une façon équivalente de caractériser le ϵ -pseudo-spectre est en termes du spectre des opérateurs perturbés (théorème de S.Roch et B.Silbermann, [RS96]). On a

$$\sigma_\epsilon(A) = \{z \in \mathbb{C}; z \in \sigma(A + \Delta A) \text{ pour } \Delta A \in \mathcal{L}(X) \text{ avec } \|\Delta A\|_{\mathcal{L}(X)} < \epsilon\}. \quad (1.2.2)$$

où $\mathcal{L}(X)$ est l'espace des opérateurs linéaires bornés sur X . Autrement dit, un point z appartient au ϵ -pseudo-spectre de A si et seulement si z appartient au spectre d'un certain opérateur perturbé $A + \Delta A$ avec perturbation de taille plus petite que ϵ . En réalité, lorsqu'on veut calculer numériquement le spectre d'un opérateur, on commence toujours par le discréteriser, et ensuite on calcule les valeurs propres d'une certaine matrice à l'aide des logiciels numériques. Par conséquent, les valeurs propres qu'on obtient sont exactement dans un certain ϵ -pseudo-spectre, mais probablement en dehors du spectre de l'opérateur original. Ceci explique l'importance de comprendre la notion de pseudo-spectre, puisque elle est liée à la stabilité ou l'instabilité du spectre.

Le concept de pseudo-spectre n'est intéressant que pour les opérateurs non auto-adjoints, ou plus précisément pour les opérateurs non normaux. En effet, si A est un opérateur normal, le théorème spectral (voir (V.3.31) dans [Kat95]) nous dit

$$\forall z \notin \sigma(A), \quad \| (z - A)^{-1} \| \leq \frac{1}{d(z, \sigma(A))}, \quad (1.2.3)$$

où $d(z, \sigma(A))$ désigne la distance entre le point z et le spectre $\sigma(A)$ de l'opérateur A . Ceci implique immédiatement que le ϵ -pseudo-spectre d'un opérateur normal est exactement égal au ϵ -voisinage du spectre

$$\sigma_\epsilon(A) = \{z \in \mathbb{C}; d(z, \sigma(A)) < \epsilon\}.$$

A un point loin du spectre d'un opérateur normal, la résolvante ne peut pas être grande en norme.

La difficulté principale dans l'étude des opérateurs non normaux réside au fait que, l'inégalité (1.2.3) n'est plus vraie en général. Pour un opérateur non normal, la norme de la résolvante peut être assez grande même très loin du spectre. Ce fait est connu depuis longtemps. Le ϵ -pseudo-spectre peut être beaucoup plus gros que le ϵ -voisinage du spectre, ce qui induit une instabilité spectrale très forte sous des petites perturbations. Ce type d'instabilité a été beaucoup étudié, voir par exemple [PS06b].

Le problème d'évolution pour les opérateurs non auto-adjoints peuvent être instable. Considérons une équation d'évolution linéaire

$$\begin{cases} \partial_t u(x, t) = Au(x, t), \\ u(x, 0) = u_0(x). \end{cases} \quad (1.2.4)$$

Supposons que A est le générateur d'un semi-groupe C^0 , notons par e^{tA} et la solution de (1.2.4) est $u(t, x) = e^{tA}u_0(x)$. L'abscisse spectrale de A est définie comme

$$\alpha(A) = \sup_{z \in \sigma(A)} \operatorname{Re}(z).$$

D'abord on a toujours

$$\|e^{tA}\| \geq e^{t\alpha(A)}, \quad \forall t \geq 0.$$

Donc si le spectre de A contient des points dont la partie réelle est positive, le semi-groupe augmente exponentiellement. Cependant, dans le cas où A est non normal, même si l'abscisse spectrale est strictement négatif, il y aurait des phénomènes d'instabilité. Ceci est relié aux pseudo-spectres. Pour $\epsilon > 0$, définissons la ϵ -abscisse pseudo-spectrale

$$\hat{\alpha}_\epsilon(A) = \sup_{z \in \sigma_\epsilon(A)} \operatorname{Re}(z),$$

et le taux de croissance du semi-groupe

$$\hat{\omega}_0(A) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\exp(tA)\|.$$

En considérant les fonctions propres, on a

$$\hat{\omega}_0(A) \geq \sup_{z \in \sigma(A)} \operatorname{Re}(z) = \alpha(A),$$

mais l'égalité est fausse en général. Le théorème de Gearhart-Prüss nous dit la relation inverse.

Théorème 1.2.1 (Gearhart-Prüss [EN00], [HS10]). *Soit A un opérateur fermé avec domaine dense sur un espace de Hilbert X tel que A engendre un semi-groupe de contraction. Alors on a*

$$\lim_{\epsilon \rightarrow 0} \hat{\alpha}_\epsilon(A) = \hat{\omega}_0(A).$$

Dans le cas non normal, bien que le spectre soit contenu dans le demi plan complexe où la partie réelle est strictement négative, des abscisses ϵ -pseudo-spectrales peuvent être strictement positives pour ϵ petit strictement positif. La référence [Tre97] nous donne des illustrations intéressantes de ces quantités.

Les exemples d'opérateurs non auto-adjoints sont très variés. L'exemple le plus simple est le bloc de Jordan, étudié par J. Sjöstrand et M. Zworski [SZ07], dans leur article tous les calculs concernant le spectre, pseudo-spectre et semi-groupe sont explicites et élémentaires.

D'autres exemples, comme l'oscillateur harmonique non auto-adjoint, étudié par L. Boulton [Bou02] et ensuite K. Pravda-Starov [PS06a], l'opérateur de Krammers-Fokker-Planck étudié par F. Hérau, J. Sjöstrand et C. Stolk [HSS05], les spectres et pseudo-spectres sont caractérisés. Le livre [TE05] donne un aperçu historique très complet et beaucoup d'illustrations sur le sujet de l'opérateur non normal et le pseudo-spectre.

1.2.2 Une classe d'opérateurs non auto-adjoints

Les opérateurs non auto-adjoints qu'on étudie dans cette thèse ont une structure particulière

$$H = A + iB, \quad (1.2.5)$$

agissant sur un espace de Hilbert X , où A est auto-adjoint et non négatif, iB est anti-adjoint tels que A et iB ne commutent pas, d'où provient la non-normalité de H . Dans beaucoup de problèmes venant de la physique mathématique, on peut rencontrer des équations d'évolution linéaire avec un générateur non auto-adjoint de ce type. La partie A décrit la dissipation du système et la partie iB décrit la conservation.

En particulier, dans la théorie des équations cinétiques, il y a beaucoup de modèles ayant la structure (1.2.5). Il s'agit de décrire l'évolution de la densité de probabilité de présence des particules dans l'espace, qui est une fonction dépendant du temps t , de la position des particules $x \in \mathbb{R}^n$ et de leur impulsion $v \in \mathbb{R}^n$. Le système a une structure suivante :

diffusion dans la variable v + transport dans les variables (t, x) .

Parmi eux, sont l'équation de Kolmogorov, l'opérateur de Fokker-Planck [HSS05], l'opérateur de Boltzmann sans troncature angulaire [LMPS12], l'opérateur de type Landau linéaire [HPS11], etc. La partie dissipative de ces équations est dégénérée et elle ne concerne que la direction v . Néanmoins, la présence de la partie transport et surtout l'interaction entre les deux parties produisent des effets dissipatifs ou effets régularisants dans toutes les variables du système. Un problème important intervenu dans l'étude d'un tel système est la dérivation des estimations de régularités optimales de leurs solutions.

L'opérateur de Kolmogorov, hypoellipticité

Passons aux quelques exemples. L'exemple le plus simple ayant la forme (1.2.5) est l'opérateur de Kolmogorov,

$$K = \partial_t + x\partial_y - \partial_x^2, \quad (t, x, y) \in \mathbb{R}^3. \quad (1.2.6)$$

En introduisant les champs de vecteurs $X_0 = \partial_t + x\partial_y$ et $X_1 = \partial_x$, on peut écrire

$$K = X_0 + X_1^* X_1.$$

La relation de commutation

$$[X_1, X_0] = [\partial_x, \partial_t + x\partial_y] = \partial_y$$

nous dit que l'algèbre de Lie engendrée par les champs de vecteurs X_0, X_1 est égale à l'espace tangent entier \mathbb{R}^3 . L'opérateur de Kolmogorov K n'est pas elliptique, et a priori il n'est elliptique que dans la variable x . Pourtant, il est hypoelliptique, c'est-à-dire que, si Ku est dans C^∞ , alors u est dans C^∞ . Ce résultat est obtenu par une construction

directe de la solution fondamentale, qui est une fonction C^∞ en dehors de la diagonale, voir [Hör83, page 210]. La non-commutation de la partie réelle et la partie imaginaire produit un effet régularisant dans la variable y .

Il est intéressant de se demander combien de dérivées qu'on gagne dans la variable y dans cette procédure. En utilisant un changement de variables et en résolvant une équation différentielle ordinaire, on peut arriver à une estimation hypoelliptique : il existe une constante $C \geq 1$ telle que pour tout $u \in C_0^\infty(\mathbb{R}^3)$,

$$C\|Ku\|_{L^2(\mathbb{R}^3)} + C\|u\|_{L^2(\mathbb{R}^3)} \geq \||D_y|^{2/3}u\|_{L^2(\mathbb{R}^3)} + \|D_x^2u\|_{L^2(\mathbb{R}^3)}. \quad (1.2.7)$$

Cette estimation est aussi optimale.

L'opérateur de Kolmogorov était la motivation de l'analyse de Hörmander sur l'hypoellipticité pour les opérateurs différentiels linéaires d'ordre 2 [Hör67].

Définition 1.2.2. Soient X_0, X_1, \dots, X_p , p champs de vecteurs C^∞ dans un ouvert Ω de \mathbb{R}^n . (X_0, X_1, \dots, X_p) sont dites vérifiant la **condition de Hörmander** dans Ω , s'il existe $r \geq 0$ tel que à tout point dans Ω , l'espace vectoriel engendré par les crochets de Lie itérés d'ordre inférieur que $r - 1$ de X_0, \dots, X_p est égal à \mathbb{R}^n .

On peut trouver le résultat d'hypoellipticité dans [Hör67] ainsi que dans le chapitre 2 de [HN05].

Théorème 1.2.3. *Supposons les champs de vecteurs (X_0, X_1, \dots, X_p) vérifiant la condition de Hörmander pour un certain r dans Ω . Alors l'opérateur (de Hörmander du type 2)*

$$L = X_0 + \sum_{j=1}^p X_j^* X_j$$

est hypoelliptique dans Ω .

L'opérateur de Fokker-Planck

Le deuxième exemple typique est l'opérateur de (Krammers-)Fokker-Planck, provenant de la physique statistique

$$P_{\text{FP}} = -\partial_v^2 + v^2 + v \cdot \partial_x - \nabla V(x) \cdot \partial_v, \quad (x, v) \in \mathbb{R}^{2n}, \quad (1.2.8)$$

où le potentiel $V(x)$ est une fonction à valeurs réelles définie sur \mathbb{R}^n . La partie réelle de P_{FP} est l'oscillateur harmonique dans la variable v , et la partie imaginaire est un champ de vecteurs de divergence nulle. Il y a beaucoup de travaux concernant l'opérateur de Fokker-Planck, voir par exemple [HN05], [HSS05]. Nous nous intéressons ici notamment aux estimations hypoelliptiques, qui ont un lien avec le retour vers l'équilibre du système.

Théorème 1.2.4 ([HN05]). *Supposons pour tout $|\alpha| = 2$,*

$$\forall x \in \mathbb{R}^n, \quad |\partial_x^\alpha V(x)| \leq C_\alpha (1 + |\nabla V(x)|)^{1-\rho_0},$$

avec $\rho_0 > \frac{1}{3}$, et

$$|\nabla V(x)| \rightarrow +\infty, \quad |x| \rightarrow \infty.$$

Alors l'opérateur de Fokker-Planck P_{FP} (1.2.8) vérifie l'estimation suivante : il existe une constante $C \geq 1$ telle que pour tout $u \in C_0^\infty(\mathbb{R}^{2n})$,

$$\begin{aligned} C\|P_{\text{FP}}u\|_{L^2(\mathbb{R}^{2n})} + C\|u\|_{L^2(\mathbb{R}^{2n})} \\ \geq \|(D_x^2 + |\nabla V(x)|^2)^{1/3}u\|_{L^2(\mathbb{R}^{2n})} + \|(D_v^2 + v^2)u\|_{L^2(\mathbb{R}^{2n})}. \end{aligned} \quad (1.2.9)$$

De plus, l'opérateur P_{FP} a résolvant compact.

L'estimation (1.2.9) est optimale.

Estimations résolvantes améliorées

D'un autre côté, la présence d'une grande partie imaginaire peut améliorer les estimations résolvantes. On voit ce fait facilement sur le modèle suivant

$$L_\gamma = -\partial_x^2 + x^2 + i\gamma x, \quad x \in \mathbb{R}, \quad (1.2.10)$$

où $\gamma \geq 1$ est un grand paramètre. La partie réelle de L_γ est l'oscillateur harmonique, qui est supérieur à 1. Donc l'axe imaginaire et le spectre de L_γ sont disjoints, puisque son image numérique est inclus dans le demi plan complexe où la partie réelle est supérieure à 1. En effet, on a des estimations résolvantes meilleures sur l'axe imaginaire quand γ est très grand, voir par exemple la Section 3.2.2 dans [GGN09],

$$\forall \gamma \geq 1, \quad \sup_{\lambda \in \mathbb{R}} \|(L_\gamma - i\lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq C\gamma^{-2/3}. \quad (1.2.11)$$

Même si la partie imaginaire de L_γ n'a pas de signe, la non-commutation améliore les estimations résolvantes sur l'axe imaginaire. De plus, les points sur l'axe imaginaire sont emportés loin du pseudo-spectre de l'opérateur L_γ lorsque le paramètre γ devient très grand. Notons que dans (1.2.11) la puissance 2/3 est optimale.

1.2.3 Méthode des multiplicateurs

Nous utilisons dans cette thèse la méthode des multiplicateurs (ou la méthode des commutateurs), qui est une méthode classique et déjà utilisée par plusieurs auteurs. En travaillant dans l'espace des phases, cette méthode s'applique particulièrement aux opérateurs ayant la forme (1.2.5),

$$H = A + iB, \quad A^* = A, \quad (iB)^* = -iB, \quad A \geq 0, \quad [A, iB] \neq 0.$$

Pour un tel opérateur, en multipliant par l'identité, on obtient une égalité

$$\operatorname{Re}\langle Hu, u \rangle = \langle Au, u \rangle. \quad (1.2.12)$$

La positivité de A donne évidemment une partie de régularité, au moins dans une certaine région de l'espace des phases. Cependant, dans la plupart de cas, l'égalité (1.2.12) n'est pas suffisante, du fait que A peut être dégénéré ou que A peut s'annuler dans une certaine région, comme les modèles que nous avons vus dans la section précédente. Pour retirer la partie de régularité manquante, que A ne peut pas produire, l'idée est d'exploiter la non-commutation entre la partie réelle et la partie imaginaire. Remarquons que pour deux opérateurs $J^* = J$, $K^* = -K$, on a l'égalité suivante

$$2\operatorname{Re}\langle Ju, Ku \rangle = \langle [J, K]u, u \rangle. \quad (1.2.13)$$

En multipliant par un opérateur auto-adjoint M , on trouve

$$2\operatorname{Re}\langle Hu, Mu \rangle = 2\langle Au, Mu \rangle + \langle [M, iB]u, u \rangle,$$

donc on obtient un commutateur $[M, iB]$ qui est auto-adjoint, d'où on espère gagner la partie de régularité qu'on recherche. L'objet est de construire le multiplicateur M , qui est un opérateur (pseudo-différentiel) auto-adjoint, pour lequel on peut profiter de la non-commutation en appliquant les calculs pseudo-différentiels, et bien évidemment il faut savoir contrôler les autres termes.

Un modèle opérateur

Soyons plus précis sur un exemple concret. Soit $\gamma \geq 1$. On considère

$$P_\gamma = D_t^2 - i\gamma t, \quad t \in \mathbb{R}. \quad (1.2.14)$$

L'opérateur P_γ rentre dans ce cadre qu'on vient de discuter. On veut examiner l'estimation suivante, en utilisant la méthode des multiplicateurs : il existe une constante $C \geq 1$ telle que pour tout $u \in C_0^\infty(\mathbb{R})$ et pour tout $\gamma \geq 1$,

$$\|P_\gamma u\|_{L^2(\mathbb{R})} \geq C\gamma^{2/3}\|u\|_{L^2(\mathbb{R})}. \quad (1.2.15)$$

Remarquons d'abord que (1.2.15) est une estimation très classique, on peut l'obtenir par un changement de variables et du fait que l'opérateur P_1 a résolvant compact.

On travaille sur l'espace des phases $\mathbb{R}_t \times \mathbb{R}_\tau$, où τ désigne la variable duale de t . Le premier étape est de multiplier par l'identité

$$\operatorname{Re}\langle P_\gamma u, u \rangle_{L^2(\mathbb{R})} = \langle D_t^2 u, u \rangle_{L^2(\mathbb{R})} \geq \gamma^{2/3}\|u\|_{L^2(\mathbb{R})}^2.$$

où la dernière inégalité est "vraie" si u est supportée (microlocalement) dans la zone $\{|\tau| \geq \gamma^{1/3}\}$ de l'espace des phases. La zone problématique est donc $\{|\tau| \leq \gamma^{1/3}\}$.

Le but est de construire un multiplicateur pour cette zone difficile. Pour cela, choisissons une fonction décroissante ψ telle que

$$\psi \in C^\infty(\mathbb{R}; [0, 1]), \quad \psi|_{(-\infty, 2]} = 1, \quad \psi|_{[2, +\infty)} = 0, \quad \psi'|_{[-1, 1]} \leq -\frac{1}{10}. \quad (1.2.16)$$

Le multiplicateur de Fourier $M = \psi(\gamma^{-1/3}D_t)$ défini par

$$\mathcal{F}(\psi(\gamma^{-1/3}D_t)u)(\tau) = \psi(\gamma^{-1/3}\tau)\hat{u}(\tau),$$

est un opérateur auto-adjoint et borné sur $L^2(\mathbb{R})$, où l'on note par $\mathcal{F}u = \hat{u}$ la transformée de Fourier de la fonction u . On calcule $2\operatorname{Re}\langle P_\gamma u, Mu \rangle_{L^2(\mathbb{R})}$,

$$2\operatorname{Re}\langle P_\gamma u, Mu \rangle_{L^2(\mathbb{R})} = \underbrace{2\operatorname{Re}\langle D_t^2 u, Mu \rangle_{L^2(\mathbb{R})}}_{\geq 0} + \langle [M, -i\gamma t]u, u \rangle_{L^2(\mathbb{R})}.$$

Puisque $i\gamma t$ est affine, on trouve facilement

$$[M, -i\gamma t] = -\gamma^{2/3}\psi'(\gamma^{-1/3}D_t),$$

Donc on a

$$\operatorname{Re}\langle P_\gamma u, (1 + 2M)u \rangle_{L^2(\mathbb{R})} \geq \langle (D_t^2 - \gamma^{2/3}\psi'(\gamma^{-1/3}D_t))u, u \rangle_{L^2(\mathbb{R})} \geq c_0\gamma^{2/3}\|u\|_{L^2(\mathbb{R})}^2,$$

où la dernière inégalité résulte de

$$\forall \tau \in \mathbb{R}, \quad \tau^2 - \gamma^{2/3}\psi'(\gamma^{-1/3}\tau) \geq c_0\gamma^{2/3}.$$

Puisque M est borné sur $L^2(\mathbb{R})$, on conclut par une inégalité de Cauchy-Schwarz

$$\|P_\gamma u\|_{L^2(\mathbb{R})} \geq C\gamma^{2/3}\|u\|_{L^2(\mathbb{R})}.$$

Un modèle général

La méthode des multiplicateurs est adaptée à la plupart des modèles (1.2.5), quand on démontre des estimations hypoelliptiques. Dans l'article [HSS05], un modèle assez général, l'opérateur de Fokker-Planck y compris, est traité en utilisant la méthode des multiplicateurs dans le cadre semiclassique. On présente ici une version simplifiée. Il s'agit de considérer les opérateurs pseudo-différentiels avec symbole de Weyl

$$p = p_1 + ip_2, \quad p_1 \geq 0, \quad (1.2.17)$$

où p_1, p_2 sont des fonctions C^∞ à valeurs réelles définies sur l'espace des phases $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$. Supposons que les symboles sont dans un calcul de Weyl associé à une métrique admissible sur l'espace des phases qui nous permet de faire des calculs symboliques, c'est-à-dire qu'il existe un poids admissible $\Lambda \geq 1$ tel que $\Gamma_0 = |dx|^2 + \Lambda^{-2} |d\xi|^2$ est une métrique admissible, et p est dans $S(\Lambda^2, \Gamma_0)$, un symbole d'ordre 2 vérifiant une condition de crochet de Poisson (voir (H1-H4) dans [HSS05] pour les hypothèses complètes sur les symboles et le poids).

Hypothèse 1.2.5. *Il existe $\epsilon_0 > 0$ tel que*

$$p_1 + \epsilon_0 H_{p_2}^2(p_1) \approx \Lambda^2,$$

où H_{p_2} est le champ hamiltonien associé à p_2 , $H_{p_2}(p_1) = \{p_2, p_1\}$ et $H_{p_2}^2(p_1) = \{p_2, \{p_2, p_1\}\}$.

On a une estimation suivante.

Théorème 1.2.6 ([HSS05]). *Supposons le symbole p vérifiant (1.2.17) et l'Hypothèse 1.2.5. Il existe une constante $C \geq 1$ telle que pour tout $u \in C_0^\infty(\mathbb{R}^n)$,*

$$C \|p^w u\|_{L^2(\mathbb{R}^n)} + C \|u\|_{L^2(\mathbb{R}^n)} \geq \|(\Lambda^{2/3})^w u\|_{L^2(\mathbb{R}^n)}, \quad (1.2.18)$$

où p^w désigne la quantification de Weyl du symbole p .

La zone difficile dans l'espace des phases est

$$\{(x, \xi) \in \mathbb{R}^{2n}, \quad p_1 \lesssim \Lambda^{2/3}\},$$

pour laquelle on cherche un multiplicateur. Le multiplicateur utilisé pour déduire l'estimation est de symbole

$$b = \frac{H_{p_2} p_1}{\Lambda^{4/3}} \phi\left(\frac{p_1}{\Lambda^{2/3}}\right),$$

où $\phi \in C_0^\infty(\mathbb{R}; [0, 1])$ est telle que $\phi = 1$ sur $[-1, 1]$, ϕ est à support dans $[-2, 2]$. Alors b est à support dans $\{p_1 \leq 2\Lambda^{2/3}\}$ et b^w est auto-adjoint. En multipliant par $(1 - \epsilon b)^w$ avec $\epsilon > 0$ à choisir et en faisant (formellement) des calculs symboliques, on trouve

$$\begin{aligned} \operatorname{Re} \langle p^w u, (1 - b)^w u \rangle &= \operatorname{Re} \langle (1 - \epsilon b)^w p^w u, u \rangle = \langle (\operatorname{Re} (1 - \epsilon b) \# p)^w u, u \rangle \\ &= \langle \left((1 - \epsilon b) p_1 + \frac{\epsilon}{2} \{p_2, b\} \right)^w u, u \rangle + \text{terme de reste}. \end{aligned}$$

On retrouve le crochet de Poisson $\{p_2, b\}$ et en plus, en utilisant l'Hypothèse 1.2.5 et en choisissant $\epsilon > 0$ assez petit, on a

$$(1 - \epsilon b) p_1 + \frac{\epsilon}{2} \{p_2, b\} \gtrsim \Lambda^{2/3} - \text{terme de reste},$$

ce qui donne une astuce de la preuve du Théorème 1.2.6.

L'opérateur de Fokker-Planck (1.2.8) rentre dans ce cadre. En effet, on a $P_{\text{FP}} = p_1^w + ip_2^w$ avec

$$p_1 = v^2 + \eta^2, \quad p_2 = v \cdot \xi - \nabla V(x) \cdot \eta,$$

où (ξ, η) désigne les variables duales de (x, v) . On vérifie facilement

$$\{p_2, p_1\} = -2(\eta \cdot \xi + v \cdot \nabla V(x)),$$

$$\{p_2, \{p_2, p_1\}\} = 2(|\nabla V(x)|^2 + |\xi|^2) - 2(V''(x)v^2 + V''(x)\eta^2).$$

Introduisant le poids

$$\Lambda^2 = \Lambda(x, v, \xi, \eta)^2 = 1 + |v|^2 + |\eta|^2 + |\nabla V(x)|^2 + |\xi|^2.$$

En supposant V'' borné, on voit l'Hypothèse 1.2.5 vérifiée par le symbole de l'opérateur de Fokker-Planck. Bien entendu il faut imposer des conditions supplémentaires sur le potentiel V pour garantir un calcul symbolique, et sous ces conditions on retrouve l'estimation hypoelliptique (1.2.9), qui est optimale. On renvoie à l'article [HSS05] pour plus de détails.

Remarque 1.2.7. La méthode d'hypocoercivité [Vil09], [Vil06] permet aussi d'étudier les modèles de la forme (1.2.5), sur le problème de convergence de solution vers l'équilibre. On l'a présentée sur un modèle simple dans la Section 1.1.4.

1.3 Enoncé des résultats

Cette thèse est consacrée à l'étude du pseudo-spectre des opérateurs non auto-adjoints liés à la question de stabilité du tourbillon d'Oseen présentée dans la Section 1.1.

1.3.1 Enoncé des résultats du chapitre 2

Dans le chapitre 2, on reprend le contenu de l'article [Den10a], *Resolvent estimates for a two-dimensional non-self-adjoint operator*, qui a été accepté pour publication dans le journal COMMUNICATIONS ON PURE AND APPLIED ANALYSIS.

On étudie un opérateur bidimensionnel qui est la motivation de l'article [GGN09], en négligeant le terme non local dans l'opérateur original

$$\tilde{H}_\alpha = -\Delta + \frac{|x|^2}{16} - \frac{1}{2} + \frac{\alpha}{8\pi} \sigma(x) \partial_\theta, \quad x \in \mathbb{R}^2, \quad (1.3.1)$$

agissant sur $L^2(\mathbb{R}^2; dx)$, où l'on note

$$\sigma(x) = \frac{1 - e^{-|x|^2/4}}{|x|^2/4}, \quad \partial_\theta = x_1 \partial_2 - x_2 \partial_1. \quad (1.3.2)$$

Avec domaine $D(\tilde{H}_\alpha) = \{\omega \in H^2(\mathbb{R}^2); |x|^2 \omega \in L^2(\mathbb{R}^2)\}$, \tilde{H}_α est un opérateur fermé.

La partie imaginaire $\alpha \sigma(x) \partial_\theta$ s'annule sur des fonctions radiales. En particulier, l'état fondamental de l'oscillateur harmonique $e^{-|x|^2/8}$ restent toujours une fonction propre de \tilde{H}_α correspondant à la valeur propre 0, pour tout $\alpha \in \mathbb{R}$. Donc on se restreint sur le complément orthogonal de la partie imaginaire de \tilde{H}_α pour obtenir des estimations intéressantes. En utilisant les coordonnées polaires dans \mathbb{R}^2 , on introduit

$$\tilde{L}^2(\mathbb{R}^2) = \left\{ \omega \in L^2(\mathbb{R}^2; dx); \omega(r \cos \theta, r \sin \theta) = \sum_{k \neq 0} \omega_k(r) e^{ik\theta} \right\}.$$

Alors en restreignant l'opérateur \tilde{H}_α à $\tilde{L}^2(\mathbb{R}^2)$, son spectre est inclus dans le demi-plan où la partie réelle est supérieure à $1/2$. Nous démontrons des estimations résolvantes uniformes sur l'axe imaginaire.

Théorème 1.3.1 (Theorem 2.2.2). *Il existe des constantes $C_0 > 0$, $\alpha_0 > 0$ telles que pour tout $|\alpha| \geq \alpha_0$, on a*

$$\sup_{\lambda \in \mathbb{R}} \|(\tilde{H}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(\tilde{L}(\mathbb{R}^2))} \leq C_0^{-1} |\alpha|^{-1/3}, \quad (1.3.3)$$

où $\mathcal{L}(\tilde{L}^2(\mathbb{R}^2))$ désigne l'espace des applications linéaires bornées sur $\tilde{L}^2(\mathbb{R}^2)$.

L'estimation (1.3.3) donne une borne inférieure de la quantité pseudo-spectrale Ψ_α (1.1.15) lorsque le terme non local est négligé, qui est la partie difficile de cette analyse. En fait, ce qu'on démontre dans ce chapitre est une estimation plus fine, qui implique le théorème précédent et l'optimalité de la puissance $1/3$ pour la quantité pseudo-spectrale.

Théorème 1.3.2 (Theorem 2.2.3). *Il existe $C_0 > 0$ et $\alpha_0 \geq 1$ tels que pour tout $\lambda \in \mathbb{R}$, $|\alpha| \geq \alpha_0$ et $\omega \in D(\tilde{H}_\alpha) \cap \tilde{L}^2(\mathbb{R}^2)$, on a*

$$\|(\tilde{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)} \geq C_0 |\alpha|^{1/3} \|D_\theta|^{1/3}\omega\|_{L^2(\mathbb{R}^2)}, \quad (1.3.4)$$

où \tilde{H}_α est donné par (1.3.1) et

$$(|D_\theta|^{1/3}\omega)(r \cos \theta, r \sin \theta) = \sum_{k \neq 0} |k|^{1/3} \omega_k(r) e^{ik\theta}, \quad \text{pour } \omega = \sum_{k \neq 0} \omega_k(r) e^{ik\theta}.$$

De plus, l'estimation (1.3.4) est optimale au sens où, pour certain $\lambda \in \mathbb{R}$, on peut trouver une fonction $\omega \in D(\tilde{H}_\alpha)$ telle que $\|(\tilde{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(|\alpha|^{1/3})\|D_\theta|^{1/3}\omega\|_{L^2(\mathbb{R}^2)}$ lorsque $|\alpha| \rightarrow +\infty$.

La preuve de l'estimation (1.3.4) consiste à utiliser les coordonnées polaires et la transformation de Fourier dans la variable angulaire pour ramener l'opérateur bidimensionnel $\tilde{H}_\alpha - i\lambda$ en une famille d'opérateurs unidimensionnels indexés par le paramètre du mode de Fourier $k \in \mathbb{Z}$

$$\tilde{H}_{\alpha,k,\lambda} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} + i\frac{\alpha k}{8\pi}\sigma(r) - i\lambda,$$

agissant sur $L^2(\mathbb{R}_+; rdr)$, où l'on note $\sigma(r) = \sigma(x)$ pour $r = |x|$ puisque σ est une fonction radiale (1.3.2). On écrit la partie imaginaire de la manière suivante

$$i\frac{\alpha k}{8\pi}\sigma(r) - i\lambda = i\frac{\alpha k}{8\pi}(\sigma(r) - \nu_k), \quad \text{avec } \nu_k = \frac{8\pi\lambda}{\alpha k}.$$

Puisque l'image de la fonction σ est $[0, 1]$, on discute deux cas selon le changement de signe de la fonction $\sigma(r) - \nu_k$.

- *Cas facile : la partie imaginaire ne change pas de signe.* Ce cas est facile à traiter, en utilisant les multiplicateurs triviaux Id , $\pm i\text{Id}$ et le fait que la fonction σ se comporte comme $4r^{-2}$ à l'infini et $1 - r^2/8$ près de l'origine.
- *Cas non trivial : la partie imaginaire change de signe.* On a $\nu_k = \sigma(r_k) \in (0, 1)$. Dans ce cas-là, la difficulté est de déduire une estimation pour des fonctions supportées près du point r_k , c'est-à-dire dans la zone où le changement de signe a lieu. Nous appliquons la méthode des multiplicateurs, présentée dans la Section 1.2.3. Le multiplicateur qu'on construit est un opérateur pseudo-différentiel, qui est une régularisation d'une fonction de Heaviside bien choisie et dépend d'une métrique de Hörmander. Ensuite, en utilisant des techniques de localisation via une partition de l'unité, on recolle les estimations dans des différentes zones dans l'espace des phases, et on arrive à une estimation globale.

1.3.2 Enoncé des résultats du chapitre 3

Le chapitre 3 reprend le contenu de l'article [Den11b], *Pseudospectrum for Oseen vortices operators*, qui a été accepté pour publication dans le journal INTERNATIONAL MATHEMATICS RESEARCH NOTICES.

Ce travail est consacré à l'étude du pseudo-spectre de l'opérateur linéarisé complet (1.1.20), en tenant compte du terme non local

$$\begin{aligned}\mathcal{H}_\alpha \omega &= L\omega + \alpha M\omega, \quad \omega \in L^2(\mathbb{R}^2; dx) \\ &= (-\Delta\omega + \frac{|x|^2}{16}\omega - \frac{1}{2}\omega) + \alpha \left[v^G \cdot \nabla\omega - \frac{1}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)) \right],\end{aligned}\quad (1.3.5)$$

où v^G, G, K_{BS} sont donnés par (1.1.7), (1.1.3). On espère démontrer une estimation ressemblant à (1.3.3) pour \mathcal{H}_α sur le complément orthogonal du noyau de la partie imaginaire M (1.1.24). En utilisant les coordonnées polaires dans \mathbb{R}^2 , pour $k_0 \geq 1$, on introduit un sous-espace de $L^2(\mathbb{R}^2; dx)$

$$X_{k_0} = \left\{ \omega \in L^2(\mathbb{R}^2; dx); \omega(r \cos \theta, r \sin \theta) = \sum_{|k| \geq k_0} \omega_k(r) e^{ik\theta} \right\}. \quad (1.3.6)$$

Muni de la norme $\|\cdot\|_{L^2(\mathbb{R}^2)}$, X_{k_0} est un espace de Hilbert et invariant de l'opérateur \mathcal{H}_α . En prenant $D = \{\omega \in L^2(\mathbb{R}^2); \omega \in H^2(\mathbb{R}^2), |x|^2\omega \in L^2(\mathbb{R}^2)\}$ comme domaine, \mathcal{H}_α est un opérateur fermé. Pour tout $k_0 \geq 1$, \mathcal{H}_α est un opérateur fermé sur X_{k_0} avec domaine $D \cap X_{k_0}$ et son image numérique définie par

$$\Theta(\mathcal{H}_\alpha; X_{k_0}) = \{\langle \mathcal{H}_\alpha \omega, \omega \rangle_{L^2(\mathbb{R}^2)} \in \mathbb{C}; \omega \in D \cap X_{k_0}, \|\omega\|_{L^2(\mathbb{R}^2)} = 1\},$$

ainsi que son spectre, est inclus dans le demi plan

$$\{z \in \mathbb{C}; \operatorname{Re}(z) \geq \frac{k_0}{2}\}.$$

Donc l'axe imaginaire est contenu dans l'ensemble résolvant de \mathcal{H}_α . Notre résultat est le suivant.

Théorème 1.3.3 (Theorem 3.2.2). *Il existe des constantes $C_0 > 0$, $k_0 \geq 3$, $\alpha_0 \geq 8\pi$ telles que pour tout $|\alpha| \geq \alpha_0$, $\lambda \in \mathbb{R}$, $\omega \in C_0^\infty(\mathbb{R}^2) \cap X_{k_0}$, on a*

$$\|(\mathcal{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)} \geq C_0 |\alpha|^{1/3} \|D_\theta|^{1/3}\omega\|_{L^2(\mathbb{R}^2)}, \quad (1.3.7)$$

où l'opérateur $|D_\theta|^{1/3}$ est défini de la même manière que celle dans Théorème 1.3.2. En particulier, on a

$$\sup_{\lambda \in \mathbb{R}} \|(\mathcal{H}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq C_0^{-1} \alpha^{-1/3} k_0^{-1/3}. \quad (1.3.8)$$

On doit traiter soigneusement le terme non local, puisqu'il possède un grand coefficient de taille $|\alpha|$. Comme \mathcal{H}_α est invariant par rotation, nous utilisons les coordonnées polaires et la transformée de Fourier dans la variable angulaire, et on réduit le problème en une famille d'opérateurs unidimensionnel agissant sur la demie droite \mathbb{R}_+ , indexé par le paramètre du mode de Fourier k . Ensuite, on va utiliser un changement de variables $r = e^t$ et on va multiplier un poids e^{2t} pour transformer le problème sur la droite entière. On se ramène à étudier l'opérateur suivant, sur $L^2(\mathbb{R}; dt)$,

$$\begin{aligned}\mathcal{L}_k &= -\partial_t^2 + k^2 + \frac{1}{16}e^{4t} - \frac{1}{2}e^{2t} \\ &\quad + i\frac{\alpha k}{8\pi}e^{2t}(\sigma(e^t) - \nu_k) - \underbrace{i\frac{\alpha k}{8\pi}e^{2t}g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)}_{\text{le terme non local}},\end{aligned}\quad (1.3.9)$$

où σ est donné par (1.3.2) et $g(r) = e^{-r^2/8}$. Après ces transformations, les propriétés auto-adjointe et anti-adjointe de chaque partie de \mathcal{H}_α sont préservées. Le terme non local devient un opérateur pseudo-différentiel, avec $\mathcal{L}(L^2(\mathbb{R}; dt))$ -norme majorée par $|\alpha||k|^{-1}$. Donc si le paramètre k est très grand, le terme non local est petit.

On va utiliser une méthode perturbative, i.e. traiter le terme non local comme une perturbation par rapport au terme qui peut avoir un changement de signe. Comme dans le chapitre 2, on discutera différents cas selon le changement de signe de la fonction $\sigma(e^t) - \nu_k$. Le cas difficile est toujours lorsqu'il y a un changement de signe, pour lequel on applique la méthode des multiplicateurs. Le multiplicateur qu'on utilise est essentiellement le même que celui pour le modèle sans terme non local. Mais cette fois-ci, on ne peut pas appliquer directement une partition de l'unité pour localiser dans des zones différentes de l'espace des phases comme dans le chapitre 2, à cause de la présence du terme non local. C'est un opérateur pseudo-différentiel dépendant de la variable t et la variable duale τ en même temps, donc il ne commute pas avec la partition de l'unité, et ceci produise un commutateur de taille $|\alpha|$ pour lequel on ne sait pas contrôler. La stratégie est de construire un multiplicateur global. Le terme non local est traité comme une perturbation, et il est absorbé par le terme principal en prenant $|k| \geq k_0$, avec $k_0 \geq 3$ une constante indépendante du paramètre α .

La valeur de k_0 est calculable. En effet, il suffit de calculer avec soin tous les coefficients venant du terme principal et du terme non local. On donne aussi un exemple de calculs numériques [Den11a] à l'Appendice C, indiquant que le Théorème 1.3.3 est vrai pour $k_0 \simeq 84$.

1.3.3 Enoncé des résultats du chapitre 4

Le chapitre 4 de cette thèse reprend l'article [Den10b], *Structure constants of the Weyl calculus*, qui a été accepté pour publication dans le journal MATHEMATICA SCANDINAVICA.

Dans ce chapitre, on se tourne vers des questions provenant du calcul pseudo-différentiel. Soit $m \in \mathbb{R}$. La classe des symboles classique $S_{1,0}^m$ est constituée des fonctions C^∞ a définies sur l'espace des phases \mathbb{R}^{2n} telles que pour tout multi-indice $\alpha, \beta \in \mathbb{N}^n$,

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m - |\alpha|}. \quad (1.3.10)$$

Les meilleures constantes $C_{\alpha\beta}$ dans (1.3.10) sont appelées semi-normes du symbole a dans l'espace de Fréchet $S_{1,0}^m$. Les deux propriétés sont classiques :

L^2 -continuité

Si le symbole a appartient à $S_{1,0}^0$, alors $a(x, D)$ est un opérateur borné sur $L^2(\mathbb{R}^n)$, c'est-à-dire qu'il existe une constante $C > 0$ telle que

$$\|a(x, D)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C. \quad (1.3.11)$$

Inégalité de Fefferman-Phong

Si a est un symbole non négatif appartenant à $S_{1,0}^2$, alors il existe une constante $C > 0$ telle que pour tout $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\operatorname{Re} \langle a(x, D)u, u \rangle_{L^2(\mathbb{R}^n)} + C\|u\|_{L^2(\mathbb{R}^n)}^2 \geq 0. \quad (1.3.12)$$

On pourrait adresser quelques questions très naturelles.

Question 1.3.4. Quelle est la dépendance des constantes C dans les inégalités (1.3.11) et (1.3.12) ? Les constantes C dépendent-elles une semi-norme du symbole a ? Si oui, laquelle semi-norme ?

Des questions similaires peuvent être posées dans d'autres classes des symboles, par exemples les symboles semi-classiques, la classe de Shubin, etc.

Dans le chapitre 4, on donne une réponse à ces questions dans un cadre plus large. On va considérer la quantification de Weyl pour les opérateurs pseudo-différentiels et on va travailler avec une métrique de Hörmander g sur l'espace des phases, c'est-à-dire que g est à variation lente, vérifie le principe d'incertitude et est tempérée (voir la Définition 4.2.6). Les *constantes de structures* de la métrique g sont étroitement liées à ces propriétés. Pour une telle métrique g et un g -poids admissible m (voir la Définition 4.2.7), on peut définir des classes de symboles très générales $S(m, g)$, les opérateurs associés ont de bonnes propriétés et on dispose d'un calcul symbolique très efficace, décrit dans le chapitre 18 de [Hör85] et le chapitre 2 de [Ler10]. En particulier, les propriétés suivantes sont bien connues :

$$L^2\text{-continuité : } a \in S(1, g) \implies \|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C, \quad (1.3.13)$$

$$\text{Inégalité de Fefferman-Phong : } a \in S(\lambda_g^2, g), a \geq 0 \implies a^w + C \geq 0. \quad (1.3.14)$$

Le résultat du chapitre 4 est le “théorème” suivant (voir les Théorèmes 4.3.2, 4.4.1 pour les énoncés précis), qui précise la dépendance des constantes dans les inégalités (1.3.13) et (1.3.14).

Théorème 1.3.5. *Les constantes C dans (1.3.13) et (1.3.14) ne dépendent que de la dimension n , des constantes de structure de la métrique g et d'une semi-norme du symbole a , dont l'ordre ne dépend que de la dimension n et des constantes de structure de g .*

Le Théorème 1.3.5 est particulièrement utile lorsque la métrique de Hörmander g dépend d'un paramètre non compact, par exemple dans la classe de symboles utilisée pour les estimations de Carleman. Dans notre situation, nous travaillons dans les chapitres 2 et 3 avec des métriques dépendant de plusieurs paramètres, $\alpha \geq 1$ et $k \in \mathbb{Z}$, où k est le paramètre du mode de Fourier dans la variable angulaire. Pour pouvoir revenir à la dimension 2, il faudrait démontrer des estimations indépendantes de k dans la variable radiale. Afin d'obtenir une estimation uniforme sur tous les modes de Fourier, il est très important d'assurer que toutes les constantes provenant du calcul symbolique ne dépendent pas de k . D'après le Théorème 1.3.5, il suffit de vérifier que les constantes de structure de la métrique utilisée et les semi-normes de symboles intervenus dans le calcul symbolique soient toutes bornées supérieurement indépendamment du paramètre k .

Chapter 2

Resolvent estimates for a two-dimensional non-self-adjoint operator

The result of this chapter is taken from the article [Den10a], *Resolvent estimates for a two-dimensional non-self-adjoint operator*, which has been accepted for publication in the journal COMMUNICATIONS ON PURE AND APPLIED ANALYSIS, see Section 2.7 for the letter of acceptance.

We consider a two-dimensional non-self-adjoint differential operator, originated from a stability problem in the two-dimensional Navier-Stokes equation, given by $\mathcal{L}_\alpha = -\Delta + |x|^2 + \alpha\sigma(|x|)\partial_\theta$, where $\sigma(r) = r^{-2}(1-e^{-r^2})$, $\partial_\theta = x_1\partial_2 - x_2\partial_1$ and α is a positive parameter tending to $+\infty$. We give a complete study of the resolvent of \mathcal{L}_α along the imaginary axis in the fast rotation limit $\alpha \rightarrow +\infty$ and we prove $\sup_{\lambda \in \mathbb{R}} \|(\mathcal{L}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(\tilde{L}^2(\mathbb{R}^2))} \leq C\alpha^{-1/3}$, which is an optimal estimate. Our proof is based on a multiplier method, metrics on the phase space and localization techniques.

2.1 Introduction

2.1.1 Motivation

In this paper, we consider a problem coming from the study of the long-time behavior of solutions to the two-dimensional Navier-Stokes equation, see [GW02, GW05]. In 2 dimensions where the vorticity is scalar, it is more convenient to study the evolution of the vorticity which is given by

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \nu \Delta \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (2.1.1)$$

where $\omega(x, t) \in \mathbb{R}$ is the vorticity distribution and $v(x, t) \in \mathbb{R}^2$ is the divergence-free velocity field reconstructed from ω via the Biot-Savart law. It is well-known that the equation (2.1.1) is globally well-posed in $L^1(\mathbb{R}^2)$, i.e. for any initial data $\omega_0 \in L^1(\mathbb{R}^2)$, the equation (2.1.1) has a unique global solution $\omega \in C^0([0, +\infty); L^1(\mathbb{R}^2))$ such that $\omega(0) = \omega_0$. The *total circulation* of the velocity field

$$\int_{\mathbb{R}^2} \omega(x, t) dx = \lim_{R \rightarrow +\infty} \oint_{|x|=R} v(x, t) \cdot dl$$

is a quantity conserved by the semi-flow defined by (2.1.1) in $L^1(\mathbb{R}^2)$. The equation (2.1.1) has a family of explicit self-similar solutions, called *Oseen vortices*, which is given by

$$\omega(x, t) = \frac{\alpha}{\nu t} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad v(x, t) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right), \quad (2.1.2)$$

where

$$G(x) = \frac{1}{4\pi} e^{-|x|^2/4}, \quad v^G(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-|x|^2/4}), \quad x \in \mathbb{R}^2, \quad (2.1.3)$$

and the parameter $\alpha \in \mathbb{R}$ is referred to as the *circulation Reynolds number*. These solutions are the only self-similar solutions to the Navier-Stokes equations in \mathbb{R}^2 whose vorticity is integrable. Moreover, it is proved by T. Gallay and C.E. Wayne in [GW05] that if the initial vorticity ω_0 is in $L^1(\mathbb{R}^2)$, then the solution $\omega(x, t)$ of (2.1.1) satisfies

$$\lim_{t \rightarrow +\infty} \|\omega(\cdot, t) - \frac{\alpha}{\nu t} G\left(\frac{\cdot}{\sqrt{\nu t}}\right)\|_{L^1(\mathbb{R}^2)} = 0, \quad (2.1.4)$$

where $\alpha = \int_{\mathbb{R}^2} \omega_0(x) dx$ is the initial circulation. In physical terms, this means that the Oseen vortices are globally stable for any value of the circulation Reynolds number α . In contrast to many situations in hydrodynamics, such as the Poiseuille or the Taylor-Couette flows, increasing the Reynolds number does not produce any instability.

In order to study the stability of Oseen vortices, we introduce some self-similar variables $\tilde{x} = x/\sqrt{\nu t}$, $\tilde{t} = \log(t/T)$ and we set

$$\omega(x, t) = \frac{1}{t} \tilde{\omega}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right), \quad v(x, t) = \sqrt{\frac{\nu}{t}} \tilde{v}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right).$$

The rescaled system reads (replacing $\tilde{\omega}$ by ω , \tilde{x} by x and so on)

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \Delta \omega + \frac{1}{2} x \cdot \nabla \omega + \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (2.1.5)$$

where ω is the rescaled vorticity and v is the rescaled velocity field again given by the Biot-Savart law. Then by construction, for all $\alpha \in \mathbb{R}$, the Oseen vortex αG is a stationary solution of (2.1.5). Linearizing the equation (2.1.5) at αG , one finds a linear evolution equation

$$\frac{\partial \omega}{\partial t} = -(A + \alpha iB)\omega,$$

where A is a self-adjoint, non-negative operator in the weighted space $L^2(\mathbb{R}^2; G^{-1}dx)$ and iB is a skew-adjoint perturbation, so that the linearized operator $A + \alpha iB$ is non-self-adjoint. By conjugating $A + \alpha iB$ with the Gaussian weight $G^{1/2}$ and by neglecting a nonlocal, lower-order term in the perturbation iB , the linearized operator becomes

$$\tilde{H}_\alpha = -\Delta + \frac{|x|^2}{16} - \frac{1}{2} + \alpha \tilde{f}(x) \partial_\theta, \quad x \in \mathbb{R}^2, \quad (2.1.6)$$

where $\partial_\theta = x_1 \partial_2 - x_2 \partial_1$ and $\tilde{f}(x) = (2\pi|x|^2)^{-1}(1 - e^{-|x|^2/4})$. In this paper, we aim to study the resolvent of the operator \tilde{H}_α along the imaginary axis, in the fast rotating limit $\alpha \rightarrow +\infty$.

2.1.2 General framework

In many problems arising in Mathematical Physics, one encounters the following type of linear evolution equation

$$\frac{du}{dt} + Hu = 0, \quad u(0) = u_0, \quad (2.1.7)$$

where H is a linear operator of the form $A + iB$, where A is a self-adjoint, non-negative operator and iB is a skew-adjoint operator that does not commute with A . A is usually called the dissipative term and iB the conservative term. In such a situation, the conservative term can affect and sometimes enhance the dissipative effects or the regularizing properties of the whole system. A typical example is the Kolmogorov equation $\partial_t u - \partial_x^2 u - x\partial_y u = 0$, which is of the form (2.1.7) with $A = -\partial_x^2$, $iB = -x\partial_y$. The non-commutation of A and iB can be expressed as $[\partial_x, x\partial_y] = \partial_y$, so that the Lie algebra generated by ∂_x and $x\partial_y$ spans the whole tangent space \mathbb{R}^2 . The operator $-\partial_x^2 - x\partial_y$ is not elliptic, however, a fundamental solution constructed in [Hör83, P.210] which is smooth outside the diagonal, gives that the Kolmogorov operator is hypoelliptic. Another example is the one-dimensional operator $-\frac{d^2}{dx^2} + x^2 + i\lambda x$, where $\lambda \geq 1$ is a large parameter. The presence of the large skew-adjoint perturbation $i\lambda x$ enhances the resolvent estimate, we have $\|(-\frac{d^2}{dx^2} + x^2 + i\lambda x)^{-1}\| \leq C\lambda^{-2/3}$ for $\lambda \geq 1$, see e.g. [GGN09, Sect.3.2.2]. In the present paper, we want to explore the size of the resolvent in a typical two-dimensional setting $(\tilde{H}_\alpha - i\lambda)^{-1}$, where \tilde{H}_α is the two-dimensional non-self-adjoint operator given in (2.1.6).

2.1.3 Results on a one-dimensional operator

In 2009, I. Gallagher, T. Gallay and F. Nier [GGN09] investigated a one-dimensional operator given by

$$H_\epsilon = -\partial_x^2 + x^2 + \frac{i}{\epsilon}f(x), \quad x \in \mathbb{R}, \quad (2.1.8)$$

where f is a bounded real-valued function and $\epsilon > 0$ is a small parameter. The operator H_ϵ is a one-dimensional analogue of \tilde{H}_α given by (2.1.6), and the limit $\epsilon \rightarrow 0$ corresponds to the fast rotating limit $\alpha \rightarrow +\infty$. They studied the asymptotics of two quantities related to the spectral and pseudospectral properties in the limit $\epsilon \rightarrow 0$. More precisely, they define $\Sigma(\epsilon)$ as the infimum of the real part of the spectrum of H_ϵ and

$$\Psi(\epsilon)^{-1} := \sup_{\lambda \in \mathbb{R}} \|(H_\epsilon - i\lambda)^{-1}\|$$

as the supremum of the norm of the resolvent of H_ϵ along the imaginary axis. Under appropriate conditions on f , both quantities $\Sigma(\epsilon)$, $\Psi(\epsilon)$ go to infinity as $\epsilon \rightarrow 0$ and lower bounds are given by using an ‘‘hypocoercive’’ method [Vil06], [Vil09]. Furthermore, they focused on a specific class of functions.

Hypothesis 2.1.1. *Assume that $f \in C^3(\mathbb{R}; \mathbb{R})$ has the following properties:*

- i) All critical points of f are non-degenerate, i.e., $f'(x) = 0$ implies $f''(x) \neq 0$.
- ii) There exist positive constants C and k such that, for all $x \in \mathbb{R}$ with $|x| \geq 1$,

$$\left| \partial_x^l \left(f(x) - \frac{1}{|x|^k} \right) \right| \leq \frac{C}{|x|^{k+l+1}}, \quad \text{for } l = 0, 1, 2, 3.$$

For f verifying Hypothesis 2.1.1, some precise and optimal estimates on $\Psi(\epsilon)$ are provided (Theorem 1.8 in [GGN09]): there exists $M_4 \geq 1$ such that for $\epsilon \in (0, 1]$,

$$\frac{1}{M_4 \epsilon^{\bar{\nu}}} \leq \Psi(\epsilon) \leq \frac{M_4}{\epsilon^{\bar{\nu}}}, \quad \text{where } \bar{\nu} = \frac{2}{k+4}.$$

Their proof is based on localization techniques and some semiclassical subelliptic estimates.

The authors of [GGN09] left open the original question about the operator (2.1.6), which is precisely the problem we are dealing with in the present paper. In particular, we shall prove an estimate

$$\sup_{\lambda \in \mathbb{R}} \|(\tilde{H}_\alpha - i\lambda)^{-1}\| \leq C\alpha^{-1/3},$$

see Section 2 for a precise statement. This kind of resolvent estimates will allow us to localize the pseudospectrum of the non-self-adjoint operator \tilde{H}_α , and in particular, the presence of the large skew-adjoint perturbation stabilizes the pseudospectrum of the whole operator.

2.2 The results

2.2.1 Preliminaries and statements

We consider the two-dimensional operator defined by

$$\mathcal{L}_\alpha := -\Delta + |x|^2 + \alpha\sigma(|x|)\partial_\theta, \quad x \in \mathbb{R}^2, \quad (2.2.1)$$

where $\alpha \geq 1$ and

$$\sigma(r) := \frac{1}{r^2}(1 - e^{-r^2}), \quad \text{for } r \geq 0. \quad (2.2.2)$$

Let us first make a few preliminary observations about \mathcal{L}_α . The operator \mathcal{L}_α is unitarily equivalent to the original operator \tilde{H}_α given in (2.1.6), up to some constants (see Appendix 2.6.2). The operator \mathcal{L}_α has the form $A + iB$ where $A = -\Delta + |x|^2$ is positive and $X := iB = \alpha\sigma(r)\partial_\theta$ is a real divergence-free vector field. Let us calculate the iterated commutators $[A, X]$ and $[[A, X], X]$: in polar coordinates,

$$\begin{aligned} [A, X] &= -\alpha\left(\sigma''(r) + \frac{1}{r}\sigma'(r) + 2\sigma'(r)\partial_r\right)\partial_\theta, \\ [[A, X], X] &= -2\alpha^2(\sigma'(r))^2\partial_\theta^2 \geq 0. \end{aligned}$$

This kind of “double-bracket” structure of \mathcal{L}_α will allow us to obtain some subelliptic estimates, which we now explain on a simpler example. Let us consider the operator

$$P_\lambda = \lambda\partial_t + t^2, \quad \lambda \geq 0, \text{ on } L^2(\mathbb{R}; dt).$$

Then $\text{Re}P_\lambda = t^2$, $i\text{Im}P_\lambda = \lambda\partial_t$ and $[[\text{Re}P_\lambda, i\text{Im}P_\lambda], i\text{Im}P_\lambda] = 2\lambda^2 > 0$. We can prove an estimate $\|P_\lambda u\|_{L^2} \geq C\lambda^{2/3}\|u\|_{L^2}$ for all $u \in \mathcal{S}(\mathbb{R})$. By making a change of variable $t = \lambda^{1/3}s$, it suffices to prove the inequality for $\lambda = 1$. One can construct a parametrix for the operator P_1 . For $f \in \mathcal{S}(\mathbb{R})$, the ordinary differential equation

$$P_1 u = \partial_t u + t^2 u = f, \quad u(-\infty) = 0$$

can be solved directly

$$u(t) = \int_{-\infty}^t e^{-(t^3-s^3)/3} f(s) ds.$$

It is easy to see that $\|u\|_{L^\infty} \leq \|f\|_{L^1}$. Moreover, one can obtain $\|u\|_{L^2} \leq C\|f\|_{L^2}$ by using the Schur's criterion. Here we want to take advantage of the “double-bracket” structure of the operator \mathcal{L}_α .

Note that the skew-adjoint part $\alpha\sigma(|x|)\partial_\theta$ vanishes on radial functions, i.e. for any radial function $v(x)$, $\mathcal{L}_\alpha v = (-\Delta + |x|^2)v = \mathcal{L}_0 v$. In particular, $v_0(x) = e^{-|x|^2/2}$ is an eigenfunction of \mathcal{L}_α corresponding to the eigenvalue 2, for any $\alpha \in \mathbb{R}$. This implies that the ground state of the two-dimensional harmonic oscillator is also an eigenfunction of the operator \mathcal{L}_α for any $\alpha \in \mathbb{R}$, and moreover, the eigenvalue 2 does not move under the large skew-adjoint perturbation $\alpha f(x)\partial_\theta$. We shall thus restrict the domain of \mathcal{L}_α to a smaller Hilbert space $\tilde{L}^2(\mathbb{R}^2)$, described below.

Using polar coordinates in \mathbb{R}^2 and expanding the angular variable θ in Fourier series, we can write for $v \in L^2(\mathbb{R}^2; dx)$,

$$v(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}} u_k(r) e^{ik\theta},$$

where $u_k(r) = \frac{1}{2\pi} \int_0^{2\pi} v(r \cos \theta, r \sin \theta) e^{-ik\theta} d\theta \in L^2(\mathbb{R}_+; rdr).$

We have the identity $\|v\|_{L^2(\mathbb{R}^2; dx)}^2 = \sum_{k \in \mathbb{Z}} 2\pi \|u_k\|_{L^2(\mathbb{R}_+; rdr)}^2$. Note that $v \in L^2(\mathbb{R}^2)$ is a radial function if and only if $v(r \cos \theta, r \sin \theta) = u_0(r)$. We define the subspace

$$\tilde{L}^2(\mathbb{R}^2) := \{v \in L^2(\mathbb{R}^2); u_0 = 0\}.$$

Then $(\tilde{L}^2(\mathbb{R}^2), \|\cdot\|_{L^2(\mathbb{R}^2; dx)})$ is a Hilbert space and has the following decomposition

$$\tilde{L}^2(\mathbb{R}^2) = \bigoplus_{k \in \mathbb{Z}^*} L^2(\mathbb{R}_+; rdr) e^{ik\theta}, \quad (2.2.3)$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. That is, for $v \in \tilde{L}^2(\mathbb{R}^2)$, $v(r \cos \theta, r \sin \theta) = \sum_{k \neq 0} u_k(r) e^{ik\theta}$.

Definition 2.2.1 (Domain of \mathcal{L}_α). We define the domain of \mathcal{L}_α to be the set

$$D(\mathcal{L}_\alpha) := \{v \in \tilde{L}^2(\mathbb{R}^2); v \in H^2(\mathbb{R}^2), |x|^2 v \in L^2(\mathbb{R}^2)\}.$$

We prove in Lemma 2.6.4 that $(\mathcal{L}_\alpha, D(\mathcal{L}_\alpha))$ is a closed operator and its spectrum is a sequence of eigenvalues contained in the half-plane $\{z \in \mathbb{C}; \operatorname{Re}(z) \geq 2\}$. The goal of the present article is to study the norm of the resolvent of \mathcal{L}_α along the imaginary axis, i.e. the quantities

$$\|(\mathcal{L}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(\tilde{L}^2(\mathbb{R}^2))}, \quad \lambda \in \mathbb{R}.$$

Now let us state our main result, which answers a question in [GGN09].

Theorem 2.2.2. *There exist constants $C > 0$ and $\alpha_0 \geq 1$, such that for all $\lambda \in \mathbb{R}$ and $\alpha \geq \alpha_0$, the following estimate holds*

$$\|(\mathcal{L}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(\tilde{L}^2(\mathbb{R}^2))} \leq C\alpha^{-1/3}. \quad (2.2.4)$$

Observe that the estimate (2.2.4) is equivalent to the following

$$\forall v \in D(\mathcal{L}_\alpha), \quad \|(\mathcal{L}_\alpha - i\lambda)v\|_{L^2(\mathbb{R}^2)} \geq C^{-1}\alpha^{1/3}\|v\|_{L^2(\mathbb{R}^2)}, \quad (2.2.5)$$

since we know that $i\lambda$ is not in the spectrum of \mathcal{L}_α , see Lemma 2.6.5. In fact in this paper, we will prove a theorem which implies Theorem 2.2.2.

Theorem 2.2.3. *There exist $C > 0$ and $\alpha_0 \geq 1$ such that for all $\lambda \in \mathbb{R}$, $\alpha \geq \alpha_0$ and $v \in D(\mathcal{L}_\alpha)$,*

$$\|(\mathcal{L}_\alpha - i\lambda)v\|_{L^2(\mathbb{R}^2)} \geq C\alpha^{1/3}\||D_\theta|^{1/3}v\|_{L^2(\mathbb{R}^2)}. \quad (2.2.6)$$

Moreover, the estimate (2.2.6) is optimal in the sense that for some $\lambda \in \mathbb{R}$ we can find a function $v \in D(\mathcal{L}_\alpha)$ such that $\|(\mathcal{L}_\alpha - i\lambda)v\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\alpha^{1/3})\||D_\theta|^{1/3}v\|_{L^2(\mathbb{R}^2)}$ as $\alpha \rightarrow +\infty$.

In Theorem 2.2.3, we denote $D_\theta = i^{-1}\partial_\theta$ and define the operator $|D_\theta|^{1/3}$ on $L^2(\mathbb{R}^2)$, using polar coordinates, by the formula

$$(|D_\theta|^{1/3}v)(r \cos \theta, r \sin \theta) = \sum_k |k|^{1/3} u_k(r) e^{ik\theta}, \quad \text{for } v = \sum_k u_k(r) e^{ik\theta}.$$

On $\tilde{L}^2(\mathbb{R}^2)$, both operators $|D_\theta|$, $|D_\theta|^{1/3}$ are bounded from below by 1. Using the equivalence of (2.2.4) and (2.2.5), we conclude that Theorem 2.2.3 implies Theorem 2.2.2.

2.2.2 Pseudospectrum

The notion of pseudospectrum was introduced by L.N. Trefthen and has been studied numerically in recent years [Tre97], [TE05]. The study of pseudospectrum is closely related to the stability of the spectrum and is important for non-self-adjoint operators, since some spectral instability might occur for non-self-adjoint operators. For a family of operators, we introduce the following definition of pseudospectrum, see [PS04].

Definition 2.2.4. The pseudospectrum of a family of operators $\{P_\alpha\}_{\alpha \geq 1}$ is defined as the complement of the set of all $z \in \mathbb{C}$ such that

$$\exists N_0 \in \mathbb{N}, \quad \limsup_{\alpha \rightarrow +\infty} \|(P_\alpha - z)^{-1}\| \alpha^{-N_0} < +\infty.$$

The resolvent estimates in Theorem 2.2.2 gives us information about the location of the pseudospectrum of $\{\alpha^{-1/3}\mathcal{L}_\alpha\}_{\alpha \geq 1}$. We have the following corollary, which tells us in particular that the pseudospectrum is strongly stabilized, as $\alpha \rightarrow +\infty$.

Corollary 2.2.5. *The pseudospectrum of the family of operators $\{\alpha^{-1/3}\mathcal{L}_\alpha\}_{\alpha \geq 1}$ is included in the half plane $\{z \in \mathbb{C}; \operatorname{Re}(z) \geq C^{-1}\}$.*

Proof. If $\operatorname{Re}(z) \leq 0$, we have

$$\|(\mathcal{L}_\alpha - z)^{-1}\| \leq \operatorname{dist}(z, \Theta(\mathcal{L}_\alpha))^{-1} \leq 1/2,$$

where $\Theta(\mathcal{L}_\alpha)$ is the numerical range of \mathcal{L}_α defined by (2.6.17), since for any $u \in D(\mathcal{L}_\alpha)$ with $\|u\|_{L^2(\mathbb{R}^2)} = 1$,

$$|\langle (\mathcal{L}_\alpha - z)u, u \rangle_{L^2(\mathbb{R}^2; dx)}| = |\langle \mathcal{L}_\alpha u, u \rangle_{L^2(\mathbb{R}^2; dx)} - z| \geq \operatorname{dist}(z, \Theta(\mathcal{L}_\alpha)) \geq 2.$$

On the other hand, by the resolvent formula, for $\mu, \lambda \in \mathbb{R}$,

$$(\mathcal{L}_\alpha - \mu - i\lambda)^{-1} - (\mathcal{L}_\alpha - i\lambda)^{-1} = \mu(\mathcal{L}_\alpha - i\lambda)^{-1}(\mathcal{L}_\alpha - \mu - i\lambda)^{-1},$$

and we find, using the estimate (2.2.4), if $\mu > 0$, $\mu \|(\mathcal{L}_\alpha - i\lambda)^{-1}\| < 1$ and $\mu < C^{-1}\alpha^{1/3}$,

$$\|(\mathcal{L}_\alpha - \mu - i\lambda)^{-1}\| \leq \frac{\|(\mathcal{L}_\alpha - i\lambda)^{-1}\|}{1 - \mu \|(\mathcal{L}_\alpha - i\lambda)^{-1}\|} \leq \frac{C\alpha^{-1/3}}{1 - C\mu\alpha^{-1/3}}.$$

For $\kappa \in (0, 1)$, $z \in \mathbb{C}$ such that $\operatorname{Re}(z) < \kappa C^{-1} \alpha^{1/3}$, we have $\|(\mathcal{L}_\alpha - z)^{-1}\| \leq C_\kappa$, which is equivalent to the following

$$\forall \zeta \in \mathbb{C}, \operatorname{Re}(\zeta) < \kappa C^{-1}, \quad \|(\alpha^{-1/3} \mathcal{L}_\alpha - \zeta)^{-1}\| \leq C_\kappa \alpha^{1/3}.$$

As a result, if $\operatorname{Re}(\zeta) < C^{-1}$, then ζ is not in the pseudospectrum of $\{\alpha^{-1/3} \mathcal{L}_\alpha\}_{\alpha \geq 1}$. The proof of the corollary is complete. \square

Remark 2.2.6. In [GGN09], another notion for the pseudospectrum is introduced (Definition 1.2), and some complex domains related to the pseudospectrum (Figure 2) are constructed in Corollary 4.2, using the precise estimates of the resolvent. This allows us to localize the pseudospectrum more accurately.

2.2.3 Sketch of the proof

For $\lambda \in \mathbb{R}$, we shall use the Fourier transformation with respect to the angular variable θ via the decomposition (2.2.3) of $\tilde{L}^2(\mathbb{R}^2)$, to reduce the two-dimensional operator $\mathcal{L}_\alpha - i\lambda$ to a family of one-dimensional operators $\mathcal{L}_{\alpha,\lambda,k}$ given by

$$\mathcal{L}_{\alpha,\lambda,k} = \underbrace{-\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + r^2}_{=\operatorname{Re}\mathcal{L}_{\alpha,\lambda,k} \text{ self-adjoint part}} + \underbrace{i(\alpha k \sigma(r) - \lambda)}_{=i\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k} \text{ skew-adjoint part}}$$

acting on the weighted space $L^2(\mathbb{R}_+; rdr)$, indexed by the parameter $k \in \mathbb{Z}^*$. The new operator $\mathcal{L}_{\alpha,k,\lambda}$ has the form $\operatorname{Re}\mathcal{L}_{\alpha,\lambda,k} + i\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k}$, with $\operatorname{Re}\mathcal{L}_{\alpha,\lambda,k}$ self-adjoint, positive and $i\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k}$ skew-adjoint on $L^2(\mathbb{R}_+; rdr)$. Remark that the adjoint of the operator ∂_r in $L^2(\mathbb{R}_+; rdr)$ is given by

$$\partial_r^* = -\partial_r - \frac{1}{r}, \tag{2.2.7}$$

so that $-\partial_r^2 - \frac{1}{r}\partial_r = \partial_r^* \partial_r$ is non-negative. The advantage of using polar coordinates and expanding the angular variable in Fourier series is that the skew-adjoint operator $\alpha \sigma(r) \partial_\theta - i\lambda$ becomes just the operator of multiplication by a purely imaginary function $i(\alpha k \sigma(r) - \lambda)$. The problem is then reduced to a family of one-dimensional problems with several parameters.

Without loss of generality, we may suppose that $\lambda \geq 0$ and

$$\lambda = \alpha |k| \nu_k^2, \tag{2.2.8}$$

with $\nu_k \in \mathbb{R}$. Then the one-dimensional operators can be written as follows

$$\mathcal{L}_{\alpha,\lambda,k} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + r^2 + i\alpha |k| (\operatorname{sign}(k) \sigma(r) - \nu_k^2), \quad k \in \mathbb{Z}^*. \tag{2.2.9}$$

We shall discuss different cases according to the change-of-sign situation of $\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k}$. Notice that the function σ defined by (2.2.2) is decreasing and has range $(0, 1]$. When $i\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k}$ does not change sign, i.e.

$$k < 0 \quad \text{or} \quad k > 0, \quad \nu_k^2 \notin (0, 1), \tag{2.2.10}$$

it is easy to deal with $\mathcal{L}_{\alpha,\lambda,k}$ by using the multipliers Id , $\pm i\operatorname{Id}$ and some asymptotic properties of σ , see Section 2.3.2. When $i\operatorname{Im}\mathcal{L}_{\alpha,\lambda,k}$ changes sign at one point, i.e. the case

$$k > 0, \quad \nu_k^2 \in (0, 1), \tag{2.2.11}$$

the situation is more complicated, which we shall call “non-trivial cases”. The study will be divided into 4 cases, according to the different behaviors of the function σ near the region where the change-of-sign takes place, and the proofs for each case are given in Section 2.4.1, 2.4.2, 2.5.1 and 2.5.2, respectively.

For the non-trivial cases, we shall seek a multiplier method, which allows us to obtain some subelliptic estimates in the zone where the change-of-sign takes places. As a motivation, we first explain the multiplier method on a model operator in Lemma 2.3.5, which is very simple only using some standard Fourier analysis. In our cases, the multiplier that we shall construct for the operator $\mathcal{L}_{\alpha,\lambda,k}$ is a pseudodifferential operator, which is a carefully chosen regularization of some Heaviside-type function and which is uniform with respect to the parameters. The symbols involved in the construction of this multiplier depend on some Hörmander-type metric and these metrics on the phase space will be a key tool for the proof.

The study for the one-dimensional operators $\mathcal{L}_{\alpha,k,\lambda}$ will allow us to tackle the two-dimensional problem (see Section 2.5.3), which was the initial motivation for the article [GGN09]. The optimality of these resolvent estimates will be given in Section 2.5.4. Finally, the definitions and the main properties of Weyl calculus are recalled in Appendix 2.6.1.

2.3 Proof, first part

Throughout Section 3 and 4, we omit the parameters α, λ in the subscript.

2.3.1 First reductions

Recall that we suppose $\lambda = \alpha|k|\nu_k^2 \geq 0$ (2.2.8) and the problem is reduced to the family of operators \mathcal{L}_k given by (2.2.9), which act on the weighted space $L^2(\mathbb{R}_+; rdr)$. We introduce a new notation

$$\beta_k := \alpha|k|. \quad (2.3.1)$$

Then $\beta_k \geq \alpha \geq 1$ and together with (2.2.7) the operator \mathcal{L}_k can be written as (we drop the indices α, λ)

$$\begin{aligned} \mathcal{L}_k &= -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + r^2 + i\beta_k(\text{sign}(k)\sigma(r) - \nu_k^2) \\ &= \underbrace{\partial_r^* \partial_r + \frac{k^2}{r^2} + r^2}_{=\text{Re}\mathcal{L}_k} + \underbrace{i\beta_k(\text{sign}(k)\sigma(r) - \nu_k^2)}_{=i\text{Im}\mathcal{L}_k}. \end{aligned} \quad (2.3.2)$$

2.3.2 Easy cases: the imaginary part does not change sign

This section is devoted to the estimates for the cases (2.2.10) where $i\text{Im}\mathcal{L}_k$ does not change sign. We will use some asymptotic properties of the function σ defined in (2.2.2) near the origin and infinity, see Figure 2.1. We choose a positive constant c_0 such that

$$\left(\frac{8}{9}\right)^2 < c_0 < 1. \quad (2.3.3)$$

From the behavior of σ near infinity, there exist constants $R_1 > 0$, $0 < c_1 < 1$ such that

$$\begin{cases} c_0 r^{-2} \leq \sigma(r) \leq r^{-2}, & \text{for } r > R_1, \\ \sigma(r) \geq c_1, & \text{for } 0 \leq r \leq R_1. \end{cases} \quad (2.3.4)$$

Near 0, we can choose $R_2 > 0$ and $0 < c_2 < 1$ such that

$$\begin{cases} -\frac{1}{2}r^2 \leq \sigma(r) - 1 \leq -\frac{1}{2}c_0r^2, & \text{for } 0 \leq r \leq R_2, \\ \sigma(r) \leq c_2, & \text{for } r > R_2, \end{cases} \quad (2.3.5)$$

since we have the Taylor expansion

$$\sigma(r) = 1 - \frac{r^2}{2!} + \frac{r^4}{3!} - \cdots + \frac{(-1)^l r^{2l}}{(l+1)!} + O(r^{2l+2}), \quad r \rightarrow 0.$$

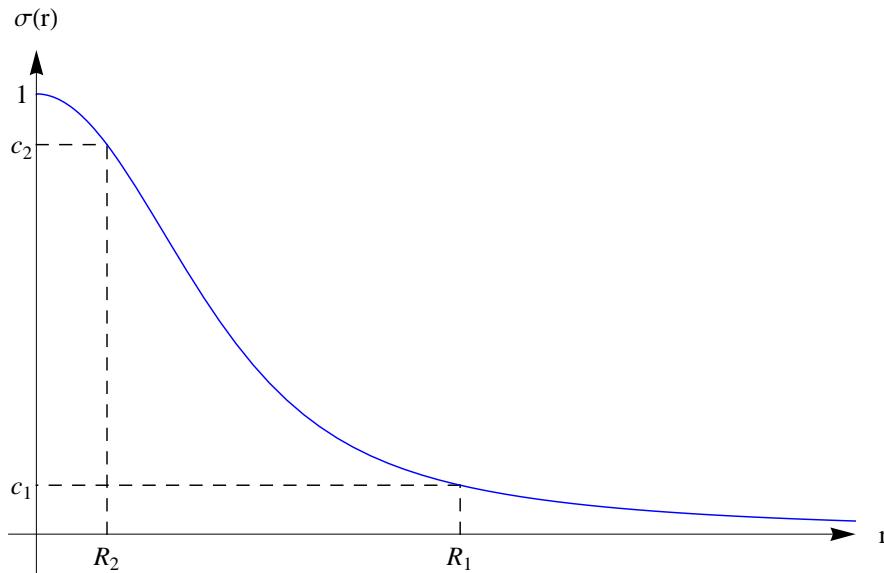


Figure 2.1: THE FUNCTION $\sigma(r) = r^{-2}(1 - e^{-r^2})$.

Now we prove two inequalities which will be used below, using (2.3.4) and (2.3.5).

Lemma 2.3.1. *There exists $C > 0$ such that for any $r > 0$, $k \neq 0$ and $\alpha \geq 1$,*

$$r^2 + \beta_k \sigma(r) \geq C \beta_k^{1/2}, \quad (2.3.6)$$

$$\frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq C \beta_k^{1/2}, \quad (2.3.7)$$

where β_k is given in (2.3.1).

Proof. By (2.3.4) we have,

$$\begin{aligned} \text{if } r \leq R_1, \quad &\text{then } r^2 + \beta_k \sigma(r) \geq c_1 \beta_k, \\ \text{if } r > R_1, \quad &\text{then } r^2 + \beta_k \sigma(r) \geq r^2 + c_0 \beta_k r^{-2} \geq 2c_0^{1/2} \beta_k^{1/2}. \end{aligned}$$

Therefore, for any $r \geq 0$,

$$r^2 + \beta_k \sigma(r) \geq \min(c_1 \beta_k, 2c_0^{1/2} \beta_k^{1/2}) \geq \min(c_1, 2c_0^{1/2}) \beta_k^{1/2}, \quad \text{if } \alpha \geq 1,$$

which proves (2.3.6) with $C = \min(c_1, 2c_0^{1/2})$. By (2.3.5) we have,

$$\begin{aligned} \text{if } r \leq R_2, \quad & \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq \frac{k^2}{r^2} + \beta_k \frac{c_0 r^2}{2} \geq \sqrt{2c_0} |k| \beta_k^{1/2}, \\ \text{if } r > R_2, \quad & \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq (1 - c_2) \beta_k. \quad (0 < c_2 < 1) \end{aligned}$$

Therefore for any $r > 0$, $k \neq 0$,

$$\begin{aligned} \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) & \geq \min(\sqrt{2c_0} |k| \beta_k^{1/2}, (1 - c_2) \beta_k) \\ & \geq \min(\sqrt{2c_0}, 1 - c_2) \beta_k^{1/2}, \quad \text{if } \alpha \geq 1, \end{aligned}$$

which proves (2.3.7) with $C = \min(\sqrt{2c_0}, 1 - c_2)$. \square

Lemma 2.3.2 ($k < 0$). *There exists $C > 0$ such that for all $k < 0$, $\alpha \geq 1$ and $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.3.2) and β_k given in (2.3.1).

Proof. If $k < 0$, $\text{Im}\mathcal{L}_k = -\beta_k(\sigma(r) + \nu_k^2)$ is negative. We use multipliers Id and $-i\text{Id}$, for $u \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k u, u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle (\partial_r^* \partial_r + \frac{k^2}{r^2} + r^2) u, u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \langle r^2 u, u \rangle_{L^2(\mathbb{R}_+; rdr)}, \end{aligned} \tag{2.3.8}$$

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k u, -iu \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k(\sigma(r) + \nu_k^2) u, u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \langle \beta_k \sigma(r) u, u \rangle_{L^2(\mathbb{R}_+; rdr)}. \end{aligned} \tag{2.3.9}$$

Adding (2.3.8) and (2.3.9) together, using (2.3.6), we get

$$\text{Re}\langle \mathcal{L}_k u, (1 - i)u \rangle_{L^2(\mathbb{R}_+; rdr)} \geq \langle (r^2 + \beta_k \sigma(r)) u, u \rangle_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)}^2.$$

By Cauchy-Schwarz inequality, we get the estimate in Lemma 2.3.2. \square

Lemma 2.3.3 ($k > 0$, $\nu_k^2 = 0$). *If $\nu_k^2 = 0$, then there exists $C > 0$ such that for all $k > 0$, $\alpha \geq 1$ and $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.3.2) and β_k given in (2.3.1).

Proof. $\nu_k = 0$ corresponds to $\lambda = 0$, where $\text{Im}\mathcal{L}_k = \beta_k \sigma(r)$ is positive. For $u \in C_0^\infty(\mathbb{R}_+)$, (2.3.8) is unchanged and

$$\text{Re}\langle \mathcal{L}_k u, iu \rangle_{L^2(\mathbb{R}_+; rdr)} = \langle \beta_k \sigma(r) u, u \rangle_{L^2(\mathbb{R}_+; rdr)}. \tag{2.3.10}$$

Adding (2.3.8) and (2.3.10) together, we get

$$\text{Re}\langle \mathcal{L}_k u, (1 + i)u \rangle_{L^2(\mathbb{R}_+; rdr)} \geq \langle (r^2 + \beta_k \sigma(r)) u, u \rangle_{L^2(\mathbb{R}_+; rdr)}.$$

By (2.3.6) and Cauchy-Schwarz inequality, we get the estimate in Lemma 2.3.3. \square

Lemma 2.3.4 ($k > 0, \nu_k^2 \geq 1$). *If $\nu_k^2 \geq 1$, then there exists $C > 0$, such that for all $k > 0$, $\alpha \geq 1$ and $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.3.2) and β_k given in (2.3.1).

Proof. If $\nu_k^2 \geq 1$, $\text{Im}\mathcal{L}_k = \beta_k(\sigma(r) - \nu_k^2)$ is non-positive. For $u \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k u, u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle (\partial_r^* \partial_r + \frac{k^2}{r^2} + r^2)u, u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \langle \frac{k^2}{r^2} u, u \rangle_{L^2(\mathbb{R}_+; rdr)}, \end{aligned} \quad (2.3.11)$$

$$\text{Re}\langle \mathcal{L}_k u, -iu \rangle_{L^2(\mathbb{R}_+; rdr)} = \langle \beta_k(\nu_k^2 - \sigma(r))u, u \rangle_{L^2(\mathbb{R}_+; rdr)}. \quad (2.3.12)$$

Adding (2.3.11) and (2.3.12) together, we get

$$\text{Re}\langle \mathcal{L}_k u, (1 - i)u \rangle_{L^2(\mathbb{R}_+; rdr)} \geq \langle \left(\frac{k^2}{r^2} + \beta_k(\nu_k^2 - \sigma(r)) \right) u, u \rangle_{L^2(\mathbb{R}_+; rdr)}.$$

Since $\nu_k^2 \geq 1$, using (2.3.7) and Cauchy-Schwarz inequality, we get Lemma 2.3.4. \square

2.3.3 Non-trivial cases: the imaginary part does change sign, study of a model operator

We have reviewed the cases $k < 0$; $k > 0, \nu_k^2 = 0$ and $k > 0, \nu_k^2 \geq 1$ in the previous section. It remains to check the case (2.2.11), i.e.

$$k \geq 1, \quad \nu_k^2 \in (0, 1). \quad (2.3.13)$$

In this case, we denote

$$\nu_k^2 = \sigma(r_k) \quad (2.3.14)$$

for some $r_k > 0$. Then $\text{Im}\mathcal{L}_k = \beta_k(\sigma(r) - \sigma(r_k))$ changes sign at $r = r_k$, so that the multipliers Id , $\pm i\text{Id}$ are not appropriate. We will define a multiplier adapted to this change of sign situation and do some symbolic calculus to get a local estimate. Then using a partition of unity we shall obtain a global estimate.

In order to provide some hints about the proof, we first study a simple model to explain the multiplier method which we will use below. Define for $\gamma \in \mathbb{R}$, $t_0 \in \mathbb{R}$,

$$\begin{aligned} P_\gamma &:= D_t^2 - i\gamma(t - t_0) \quad \text{on } L^2(\mathbb{R}; dt), \\ \text{with domain } D(P_\gamma) &= \{u \in L^2(\mathbb{R}; dt); P_\gamma u \in L^2(\mathbb{R}; dt)\}, \end{aligned} \quad (2.3.15)$$

where we denote $D_t = -i\partial_t$. The operator P_γ is closed. Clearly P_γ possesses the same features as \mathcal{L}_k , with real part non-negative and imaginary part changing sign at one point. We prove the following estimate.

Lemma 2.3.5. *There exists $C > 0$ such that for all $\gamma \in \mathbb{R}$ and $u \in C_0^\infty(\mathbb{R})$,*

$$\|P_\gamma u\|_{L^2(\mathbb{R}; dt)} \geq C|\gamma|^{2/3} \|u\|_{L^2(\mathbb{R}; dt)}, \quad (2.3.16)$$

where P_γ is defined in (2.3.15). Moreover, (2.3.16) is optimal.

Remark 2.3.6. The result (2.3.16) is well-known, see e.g. [DSZ04, Thm.1.4]. But we want to expose a robust energy method which can be adapted to our framework.

Proof. For $u \in L^2(\mathbb{R})$, we shall denote by $\hat{u} = \mathcal{F}u$ the Fourier transform of u

$$\hat{u}(\tau) = (\mathcal{F}u)(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-it\tau} dt.$$

Assume $\gamma > 0$. First we have the equality

$$\operatorname{Re}\langle P_\gamma u, u \rangle = \langle D_t^2 u, u \rangle. \quad (2.3.17)$$

Choose a decreasing function $\psi_0 \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\psi_0|_{(-\infty, -2]} = 1, \quad \psi_0|_{[2, \infty)} = 0, \quad \psi_0'|_{[-1, 1]} \leq -\frac{1}{10}. \quad (2.3.18)$$

Then the Fourier multiplier $\psi_0(\gamma^{-1/3} D_t)$ defined by

$$\mathcal{F}(\psi_0(\gamma^{-1/3} D_t)u)(\tau) = \psi(\gamma^{-1/3} \tau) \hat{u}(\tau)$$

is a bounded and self-adjoint operator.

Now let us compute the quantity $2\operatorname{Re}\langle P_\gamma u, \psi_0(\gamma^{-1/3} D_t)u \rangle_{L^2(\mathbb{R}; dt)}$.

$$\begin{aligned} 2\operatorname{Re}\langle D_t^2 u, \psi_0(\gamma^{-1/3} D_t)u \rangle_{L^2(\mathbb{R}; dt)} &= 2\operatorname{Re}\langle \tau^2 \hat{u}, \psi_0(\gamma^{-1/3} \tau) \hat{u} \rangle_{L^2(\mathbb{R}; d\tau)} \\ &= 2 \int_{\mathbb{R}} \tau^2 \psi_0(\gamma^{-1/3} \tau) |\hat{u}(\tau)|^2 d\tau \geq 0. \end{aligned} \quad (2.3.19)$$

Since $-i\gamma(t - t_0)$ is skew-adjoint and $\psi_0(\gamma^{-1/3} D_t)$ is self-adjoint, we have

$$2\operatorname{Re}\langle -i\gamma(t - t_0)u, \psi_0(\gamma^{-1/3} D_t)u \rangle_{L^2(\mathbb{R}; dt)} = \langle [\psi_0(\gamma^{-1/3} D_t), i\gamma(t_0 - t)]u, u \rangle_{L^2(\mathbb{R}; dt)}.$$

We can compute the commutator using Fourier transform and that $i\gamma(t_0 - t)$ is affine,

$$[\psi_0(\gamma^{-1/3} D_t), i\gamma(t_0 - t)] = -\gamma^{2/3} \psi_0'(\gamma^{-1/3} D_t),$$

so that we obtain

$$2\operatorname{Re}\langle -i\gamma(t - t_0)u, \psi_0(\gamma^{-1/3} D_t)u \rangle_{L^2(\mathbb{R}; dt)} = \langle -\gamma^{2/3} \psi_0'(\gamma^{-1/3} D_t)u, u \rangle_{L^2(\mathbb{R}; dt)}. \quad (2.3.20)$$

It follows from (2.3.17), (2.3.19) and (2.3.20) that

$$\begin{aligned} \operatorname{Re}\langle P_\gamma u, (1 + 2\psi_0(\gamma^{-1/3} D_t))u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle (D_t^2 - \gamma^{2/3} \psi_0'(\gamma^{-1/3} D_t))u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &= \langle (\tau^2 - \gamma^{2/3} \psi_0'(\gamma^{-1/3} \tau))\hat{u}, \hat{u} \rangle_{L^2(\mathbb{R}; d\tau)}. \end{aligned}$$

We have for all $\tau \in \mathbb{R}$, $\tau^2 - \gamma^{2/3} \psi_0'(\gamma^{-1/3} \tau) \geq \frac{1}{10} \gamma^{2/3}$. Indeed,

$$\begin{aligned} \text{if } |\tau| > \gamma^{1/3}, \quad \tau^2 - \gamma^{2/3} \psi_0'(\gamma^{-1/3} \tau) &\geq \tau^2 \geq \gamma^{2/3}, \\ \text{if } |\tau| \leq \gamma^{1/3}, \quad \tau^2 - \gamma^{2/3} \psi_0'(\gamma^{-1/3} \tau) &\geq -\gamma^{2/3} \psi_0'(\gamma^{-1/3} \tau) \geq \frac{1}{10} \gamma^{2/3}, \end{aligned}$$

where the last inequality follows from the third condition in (2.3.18). Then

$$\operatorname{Re}\langle P_\gamma u, (1 + 2\psi_0(\gamma^{-1/3} D_t))u \rangle_{L^2(\mathbb{R}; dt)} \geq \langle \frac{1}{10} \gamma^{2/3} \hat{u}, \hat{u} \rangle_{L^2(\mathbb{R}; d\tau)} = \frac{1}{10} \gamma^{2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Using the $L^2(\mathbb{R}; dt)$ -boundedness of the Fourier multiplier $\psi_0(\gamma^{-1/3}D_t)$ and Cauchy-Schwarz inequality, we obtain

$$\|P_\gamma u\|_{L^2(\mathbb{R}; dt)} \geq \frac{1}{30} \gamma^{2/3} \|u\|_{L^2(\mathbb{R}; dt)}.$$

Thus (2.3.16) holds for $\gamma > 0$. The proof for $\gamma < 0$ is essentially the same, we just replace the function ψ_0 by $1 - \psi_0$.

Now let us prove the optimality of (2.3.16). Suppose $\gamma > 0$. For $u \in C_0^\infty(\mathbb{R})$ such that $\|u\|_{L^2(\mathbb{R}; dt)} = 1$, we define

$$w_\gamma(t) = \gamma^{1/6} u(\gamma^{1/3}(t - t_0)), \quad t \in \mathbb{R}.$$

Then $w_\gamma \in C_0^\infty(\mathbb{R})$, $\|w_\gamma\|_{L^2(\mathbb{R}; dt)} = 1$.

$$\begin{aligned} (P_\gamma w_\gamma)(t) &= \gamma^{1/6} \left(-\gamma^{2/3} u''(\gamma^{1/3}(t - t_0)) - i\gamma(t - t_0)u(\gamma^{1/3}(t - t_0)) \right) \\ &= \gamma^{1/6} \gamma^{2/3} \left(-u''(\gamma^{1/3}(t - t_0)) - i\gamma^{1/3}(t - t_0)u(\gamma^{1/3}(t - t_0)) \right) \\ &= \gamma^{2/3} \gamma^{1/6} (P_1 u)(\gamma^{1/3}(t - t_0)), \end{aligned}$$

implying $\|P_\gamma w_\gamma\|_{L^2(\mathbb{R}; dt)} = \gamma^{2/3} \|P_1 u\|_{L^2(\mathbb{R}; dt)}$. Since $\|P_1 u\|_{L^2(\mathbb{R}; dt)} \geq C \|u\|_{L^2(\mathbb{R}; dt)}$, this completes the proof of Lemma 2.3.5. \square

2.4 Proof, second part: change of sign away from 0

Recall (2.3.13), (2.3.14) i.e. we are in the case $k > 0$, $0 < \nu_k^2 < 1$ and

$$\mathcal{L}_k = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + r^2 + i\beta_k(\sigma(r) - \sigma(r_k)). \quad (2.4.1)$$

The operator \mathcal{L}_k is much more complicated than P_γ defined in (2.3.15), and furthermore it acts on a weighted space $L^2(\mathbb{R}_+; rdr)$. We shall also study another model operator L_k on $L^2(\mathbb{R}; dt)$, closely related to \mathcal{L}_k .

Definition 2.4.1. For $k \geq 1$, let L_k be the operator on $L^2(\mathbb{R}; dt)$

$$L_k := D_t^2 + \frac{k^2}{t^2} + t^2 + i\beta_k(\sigma(t) - \sigma(r_k)), \quad (2.4.2)$$

where $D_t = -i\partial_t$, $\sigma(t)$ is given by (2.2.2) and for $t \leq 0$ we define $\sigma(t) = \sigma(-t)$. The operator L_k can be written as $L_k = \text{Re}L_k + i\text{Im}L_k$, where $\text{Im}L_k = \beta_k(\sigma(t) - \sigma(r_k))$ changes sign at $t = r_k$, and $\text{Re}L_k = D_t^2 + \frac{k^2}{t^2} + t^2 \geq 0$. Moreover we have the following equality

$$\text{Re}\langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} = \langle (D_t^2 + \frac{k^2}{t^2} + t^2)u, u \rangle_{L^2(\mathbb{R}; dt)}. \quad (2.4.3)$$

We will use essentially the same multiplier as in Lemma 2.3.5 and do symbolic calculus to give local estimates for L_k . Then we can get local estimates for \mathcal{L}_k by controlling the error terms, and finally obtain a global result by using a partition of unity.

Suppose that $0 < \epsilon_0 < 1$ and $\mu \geq 1$ are constants to be chosen below. We discuss the following four cases according to the different behaviors of σ (see Figure 2.2):

$$r_k > \epsilon_0^{-1}, \quad r_k \in [\epsilon_0, \epsilon_0^{-1}], \quad \mu\beta_k^{-1/4} \leq r_k < \epsilon_0 \quad \text{and} \quad r_k < \mu\beta_k^{-1/4}.$$

For each of the first three cases, we start with studying the operator L_k given in (2.4.2) on $L^2(\mathbb{R}; dt)$ and then we proceed with \mathcal{L}_k given in (2.4.1) on $L^2(\mathbb{R}_+; rdr)$.

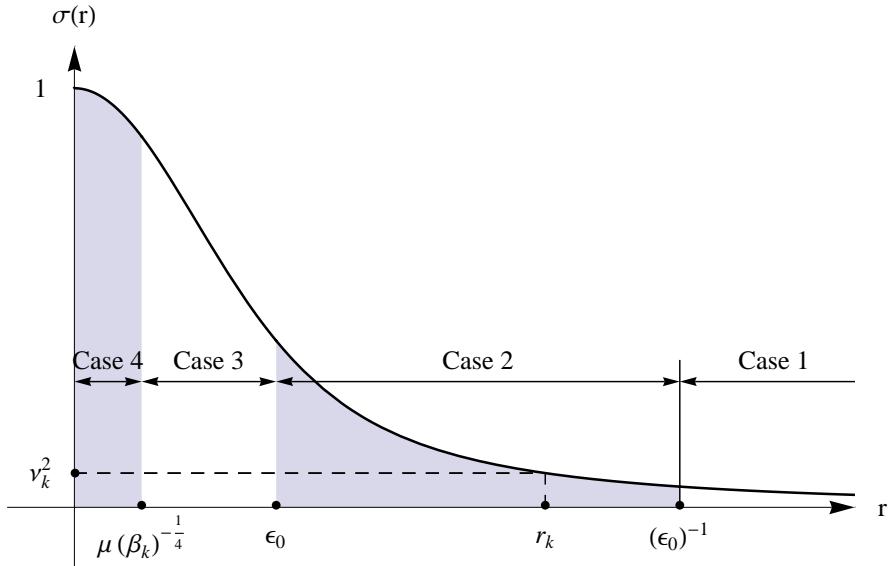


Figure 2.2: THE 4 CASES.

2.4.1 Case 1: crossing near infinity

We first consider the case when r_k is large, where r_k is given in (2.3.14). In order to use the asymptotic properties (2.3.4) of σ at infinity, we make some assumptions about ϵ_0 :

$$\epsilon_0^2 < c_1, \quad \epsilon_0 \leq (4R_1)^{-1}, \quad (2.4.4)$$

where c_1, R_1 are given in (2.3.4). Then according to (2.3.4),

$$\forall r > \frac{1}{4}\epsilon_0^{-1}, \quad c_0 r^{-2} \leq \sigma(r) \leq r^{-2}.$$

We assume also ϵ_0 small enough to ensure

$$\forall r > \frac{1}{4}\epsilon_0^{-1}, \quad -c_4 r^{-3} \leq \sigma'(r) \leq -c_3 r^{-3}, \quad (2.4.5)$$

for some $0 < c_3 < c_4$. We will prove the following estimate.

Theorem 2.4.2. *Suppose $r_k > \epsilon_0^{-1}$, where r_k is given in (2.3.14). Then there exist constants $C > 0$, $\alpha_1 \geq 1$ such that for any $k \geq 1$, $\alpha \geq \alpha_1$, $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{1/3} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Estimates in $L^2(\mathbb{R}; dt)$

Proposition 2.4.3. *Suppose $r_k > \epsilon_0^{-1}$, where r_k is given in (2.3.14). Then there exists $C > 0$ such that for any $k \geq 1$, $\alpha \geq 1$, $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C(\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)},$$

where L_k is given in (2.4.2) and β_k given in (2.3.1).

We prove Proposition 2.4.3 by a series of lemmas. Let us start with a definition.

Definition 2.4.4 (Cutoff functions). We choose two non-negative smooth functions φ_1 and φ_2 , both with values in $[0, 1]$, such that φ_1 is supported in $[\frac{1}{4}, \frac{7}{4}]$ and has value 1 on $[\frac{1}{2}, \frac{3}{2}]$, φ_2 is supported in $[\frac{1}{2}, \frac{3}{2}]$ and has value 1 on $[\frac{3}{4}, \frac{5}{4}]$. See Figure 2.3.

If u has support included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, then $u(t) = \varphi_1(\frac{t}{r_k})u(t) = \varphi_2(\frac{t}{r_k})u(t)$.

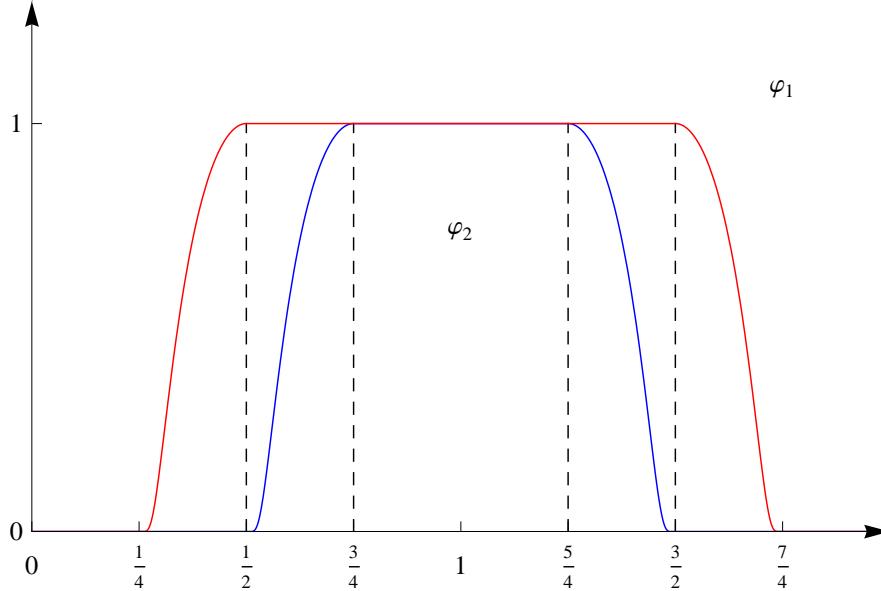


Figure 2.3: CUTOFF FUNCTIONS φ_1 AND φ_2 .

Definition 2.4.5 (Admissible metric). We define a metric on the phase space $\mathbb{R}_t \times \mathbb{R}_\tau$:

$$g := g_{k,(t,\tau)} = \frac{dt^2}{r_k^2} + \frac{d\tau^2}{\beta_k^{2/3} r_k^{-2} + \tau^2}.$$

The quantity λ_g defined by (2.6.4) for the metric g is

$$\lambda_g = r_k(\beta_k^{2/3} r_k^{-2} + \tau^2)^{1/2} \geq \beta_k^{1/3} \geq \alpha^{1/3} \geq 1. \quad (2.4.6)$$

The metric g is of the type studied in Lemma 2.6.1, so that g is admissible and the structure constants C_0 , \tilde{C}_0 , \tilde{N}_0 in (2.6.2) for g are all independent of k , although g depends on k itself.

Definition of the multiplier

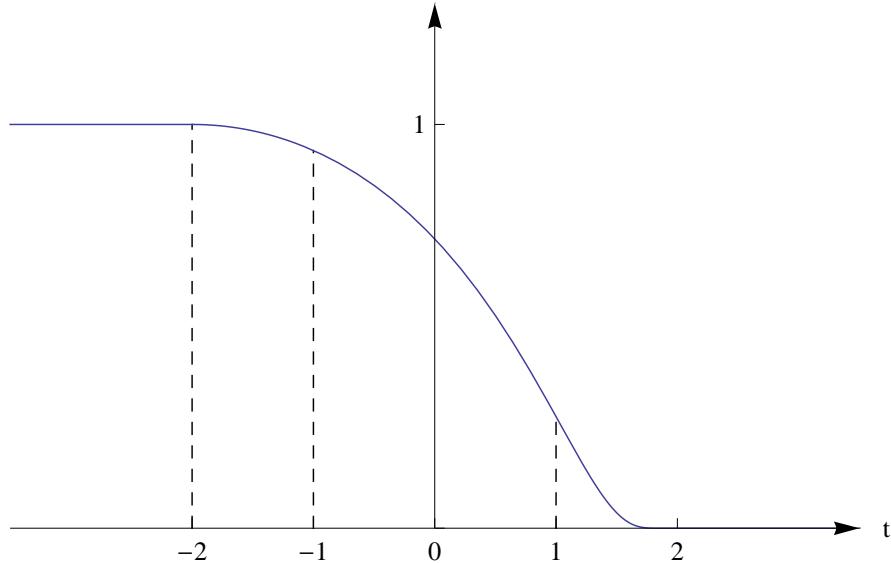
$$b(\tau) := \psi^2(\beta_k^{-1/3} r_k \tau), \quad \tau \in \mathbb{R}, \quad (2.4.7)$$

$$\tilde{b}(\tau) := \psi(\beta_k^{-1/3} r_k \tau), \quad \tau \in \mathbb{R}, \quad (2.4.8)$$

where $\psi \in C^\infty(\mathbb{R}; [0, 1])$ is decreasing and satisfies the following conditions

$$\psi|_{(-\infty, -2]} = 1, \quad \psi|_{[2, \infty)} = 0, \quad (\psi\psi')|_{[-1, 1]} \leq -\frac{1}{20}. \quad (2.4.9)$$

See Figure 2.4. Note that $\text{supp } \psi^{(l)} \subset [-2, 2]$ for any $l \geq 1$.

Figure 2.4: THE FUNCTION ψ .

Lemma 2.4.6. *The symbols b defined in (2.4.7) and \tilde{b} defined in (2.4.8) belong to $S(1, g)$, where g is given in Definition 2.4.5. Moreover, the semi-norms of b, \tilde{b} are independent of k .*

Proof of Lemma 2.4.6. Observe that $|b(\tau)| \leq 1$ and

$$b^{(l)}(\tau) = (\beta_k^{-1/3} r_k)^l (\psi^2)^{(l)}(\beta_k^{-1/3} r_k \tau), \quad \text{for } l \geq 1.$$

If $|\tau| \leq 2\beta_k^{1/3} r_k^{-1}$, then $|b^{(l)}(\tau)| \leq C_l (\beta_k^{-1/3} r_k)^l \leq C_l 5^{l/2} (\beta_k^{2/3} r_k^{-2} + \tau^2)^{-l/2}$, where $C_l = \|(\psi^2)^{(l)}\|_{L^\infty}$. If $|\tau| > 2\beta_k^{1/3} r_k^{-1}$, then $b^{(l)}(\tau) = 0$. This implies that b belongs to $S(1, g)$ and moreover, the semi-norms of b do not depend on k . Using the same computation we get that \tilde{b} is also in $S(1, g)$, with semi-norms independent of k . \square

Remark 2.4.7. As a consequence of Lemma 2.4.6, the Weyl quantizations b^w and \tilde{b}^w are bounded, self-adjoint operators on $L^2(\mathbb{R}; dt)$ with operator norm independent of k (see Appendix 2.6.1). As a matter of fact, $b^w = b(D_t)$ and $\tilde{b}^w = \tilde{b}(D_t)$ are bounded, non-negative Fourier multiplier. Moreover, we have $b^w = (\tilde{b}^w)^2$.

Now let us compute $2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)}$: for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$\begin{aligned} 2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &= 2\operatorname{Re}\underbrace{\langle (D_t^2 + \frac{k^2}{t^2} + t^2)u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_A \\ &\quad + 2\operatorname{Re}\underbrace{\langle i\beta_k(\sigma(t) - \sigma(r_k))u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_B. \end{aligned} \quad (2.4.10)$$

Estimate for the term B

Lemma 2.4.8. *The symbol $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ belongs to $S(\beta_k r_k^{-2}, g)$ and its semi-norms are independent of k , where g is given in Definition 2.4.5.*

Proof of Lemma 2.4.8. It suffices to prove $\forall n \geq 0, \exists c'_n > 0, \forall k \geq 1, \forall t \in \mathbb{R}$,

$$\left| \frac{d^n}{dt^n} \left(i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right) \right| \leq c'_n \beta_k r_k^{-n-2}. \quad (2.4.11)$$

The function is supported in $[\frac{1}{4}r_k, \frac{7}{4}r_k] \subset [\frac{1}{4}\epsilon_0^{-1}, \infty)$, since we are in the case $r_k > \epsilon_0^{-1}$. By the hypothesis $\epsilon_0 < (4R_1)^{-1}$ in (2.4.4), we have

$$|i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right)| \leq \beta_k(t^{-2} + r_k^{-2}) \leq 17\beta_k r_k^{-2}.$$

We get by induction on n that for $n \geq 1$,

$$\sigma^{(n)}(t) = (-1)^n t^{-n-2}((n+1)! - p_n(t)e^{-t^2}),$$

where $p_n(t)$ is a polynomial of degree $2n$. So there exist $c''_n > 0$ such that for all $t > 0$, $|\sigma^{(n)}(t)| \leq c''_n t^{-n-2}$. Then

$$\begin{aligned} \left| \frac{d^n}{dt^n} \left((\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right) \right| &\leq \sum_{1 \leq m \leq n} c_m |\sigma^{(m)}(t)| |\varphi_1^{(n-m)}\left(\frac{t}{r_k}\right)| r_k^{m-n} \\ &\quad + |\sigma(t) - \sigma(r_k)| |\varphi_1^{(n)}\left(\frac{t}{r_k}\right)| r_k^{-n} \\ &\leq C r_k^{-2-n}, \end{aligned}$$

which implies (2.4.11) and completes the proof of Lemma 2.4.8. \square

For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, we write the term B as

$$B = 2\operatorname{Re} \langle i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) u, b^w u \rangle_{L^2(\mathbb{R}; dt)},$$

where φ_1 is defined in Definition 2.4.4. Since $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right)$ is skew-adjoint and b^w is self-adjoint, we have

$$B = \langle \left[b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right] u, u \rangle_{L^2(\mathbb{R}; dt)}.$$

By Weyl's calculus, since $b \in S(1, g)$ and $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \in S(\beta_k r_k^{-2}, g)$, we get that

$$\left[b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right] = b_1^w + d^w, \quad (2.4.12)$$

where b_1 is the Poisson bracket of these two symbols

$$b_1(t, \tau) = \frac{1}{i} \left\{ b(\tau), i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right\} \in S(\beta_k r_k^{-2} \lambda_g^{-1}, g),$$

and d is the remainder belonging to $S(\beta_k r_k^{-2} \lambda_g^{-3}, g)$ with λ_g given in (2.4.6), see Appendix 2.6.1, (2.6.12).

We can compute b_1 directly:

$$\begin{aligned} b_1(t, \tau) &= \beta_k^{2/3} r_k^{-2} (2\psi\psi')(\beta_k^{-1/3} r_k \tau) \\ &\quad \cdot \left(\sigma'(t)\varphi_1\left(\frac{t}{r_k}\right) r_k^3 + (\sigma(t) - \sigma(r_k))\varphi_1'\left(\frac{t}{r_k}\right) r_k^2 \right). \end{aligned} \quad (2.4.13)$$

The symbol b_1 has the following properties.

Lemma 2.4.9 (Properties of b_1). *The symbol b_1 given in (2.4.13) has support included in the set*

$$\{(t, \tau) \in \mathbb{R}^2; |\tau| \leq 2\beta_k^{1/3}r_k^{-1}, t \in [\frac{1}{4}r_k, \frac{7}{4}r_k]\}.$$

There exist $c_5, c_6 > 0$ independent of k , such that

$$\begin{aligned} |\tau| \leq \beta_k^{1/3}r_k^{-1}, t \in [\frac{1}{2}r_k, \frac{3}{2}r_k] \implies b_1(t, \tau) \geq c_5\beta_k^{2/3}r_k^{-2}; \\ \forall \tau \in \mathbb{R}, t \in \mathbb{R}, |b_1(t, \tau)| \leq c_6\beta_k^{2/3}r_k^{-2}. \end{aligned}$$

Proof of Lemma 2.4.9. The first property is a direct consequence of the choices of ψ and φ_1 . If $|\tau| \leq \beta_k^{1/3}r_k^{-1}$ and $t \in [\frac{1}{2}r_k, \frac{3}{2}r_k]$, by the properties of ψ and φ_1 , we get

$$\begin{aligned} (2\psi\psi')(\beta_k^{-1/3}r_k\tau) \leq -\frac{1}{10}, \quad \varphi_1'(\frac{t}{r_k}) = 0, \quad \varphi_1(\frac{t}{r_k}) = 1, \\ \sigma'(t)r_k^3 \leq -c_3t^{-3}r_k^3 \leq -(\frac{2}{3})^3c_3, \end{aligned}$$

where the last inequality follows from (2.4.5) and the fact that $\frac{3}{2}r_k \geq t \geq \frac{1}{2}r_k \geq \frac{1}{4}\epsilon_0^{-1}$. As a result, we get

$$b_1(t, \tau) \geq c_5\beta_k^{2/3}r_k^{-2} \quad \text{for } |\tau| \leq \beta_k^{1/3}r_k^{-1} \text{ and } t \in [\frac{1}{2}r_k, \frac{3}{2}r_k],$$

with $c_5 = \frac{1}{10}(\frac{2}{3})^3c_3$. It follows also from (2.3.4) and (2.4.5) that for all $t \in [\frac{1}{4}r_k, \frac{7}{4}r_k]$,

$$|\sigma'(t)\varphi_1(\frac{t}{r_k})r_k^3 + (\sigma(t) - \sigma(r_k))\varphi_1'(\frac{t}{r_k})r_k^2| \leq 16c_4 + 17\|\varphi_1'\|_{L^\infty},$$

which implies that

$$\forall \tau \in \mathbb{R}, \forall t \in \mathbb{R}, |b_1(t, \tau)| \leq \|2\psi\psi'\|_{L^\infty}(16c_4 + 17\|\varphi_1'\|_{L^\infty})\beta_k^{2/3}r_k^{-2},$$

completing the proof. \square

Estimate for d^w . Recall that d is given in (2.4.12). We deduce from Lemma 2.4.6 and Lemma 2.4.8 that the semi-norms of d in $S(\beta_k r_k^{-2} \lambda_g^{-3}, g)$ are independent of k , with λ_g given in (2.4.6). Since $\lambda_g \geq \beta_k^{1/3}$, the class $S(\beta_k r_k^{-2} \lambda_g^{-3}, g)$ is included in $S(r_k^{-2}, g)$. This implies that $r_k^2 d^w$ is a bounded operator on $L^2(\mathbb{R}; dt)$, and its operator norm is bounded by a semi-norm of d in $S(r_k^{-2}, g)$ (see Appendix 2.6.1), so also bounded by a semi-norm of d in $S(\beta_k r_k^{-2} \lambda_g^{-3}, g)$. We get that

$$|\langle r_k^2 d^w u, u \rangle_{L^2(\mathbb{R}; dt)}| \leq \|r_k^2 d^w u\|_{L^2(\mathbb{R}; dt)} \|u\|_{L^2(\mathbb{R}; dt)} \leq C_1 \|u\|_{L^2(\mathbb{R}; dt)}^2,$$

where $C_1 > 0$ is a constant independent of k ¹. Consequently, we get the estimate for the term B which is defined in (2.4.10): for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$B \geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.14)$$

¹. Here we replace the $S(r_k^{-2}, g)$ semi-norm of d appearing in the operator norm of $r_k^2 d^w$ by a $S(\beta_k r_k^{-2} \lambda_g^{-3}, g)$ semi-norm and the latter is independent of k . We will use this kind of point of view several times.

Proposition 2.4.10 (Estimate for the term A). *There exist positive constants C_2 and C_3 such that for all $k \geq 1$, $\alpha \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$A \geq -C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)}^2 - C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)}^2, \quad (2.4.15)$$

where A is given in (2.4.10) and β_k given in (2.3.1).

Proof of Proposition 2.4.10. We divide the term A into two parts

$$A = \underbrace{2\operatorname{Re}\langle D_t^2 u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_1} + \underbrace{2\operatorname{Re}\langle (\frac{k^2}{t^2} + t^2)u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_2}.$$

A_1 is non-negative, since b^w is a non-negative Fourier multiplier,

$$A_1 = 2\operatorname{Re} \int_{\mathbb{R}} D_t^2 u \cdot \overline{b(D_t)u} dt = 2\operatorname{Re} \int |\tau|^2 b(\tau) \cdot |\hat{u}(\tau)|^2 d\tau \geq 0. \quad (2.4.16)$$

For A_2 , using $u(t) = \varphi_1(\frac{t}{r_k})u(t)$, we have

$$\begin{aligned} A_2 &= 2\operatorname{Re}\langle (\frac{k^2}{t^2} + t^2)\varphi_1(\frac{t}{r_k})u, b^w u \rangle_{L^2(\mathbb{R};dt)} \\ &= \langle \left((\frac{k^2}{t^2} + t^2)\varphi_1(\frac{t}{r_k})b^w + b^w(\frac{k^2}{t^2} + t^2)\varphi_1(\frac{t}{r_k}) \right)u, u \rangle_{L^2(\mathbb{R};dt)}. \end{aligned}$$

Since $b^w = (\tilde{b}^2)^w = (\tilde{b}^w)^2$, where \tilde{b} is given in (2.4.8) (see Remark 2.4.7), we have

$$\begin{aligned} \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) b^w &= \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w \tilde{b}^w \\ &= \left[\left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right] \tilde{b}^w + \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w, \\ b^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) &= \tilde{b}^w \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \\ &= -\tilde{b}^w \left[\left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right] + \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w, \end{aligned}$$

by adding these two terms together, we obtain

$$\left[\left[\left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] + 2\tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w,$$

so that A_2 can be written as

$$\begin{aligned} A_2 &= \underbrace{\langle \left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{21}} + \underbrace{\langle \left[\left[t^2 \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{22}} \\ &\quad + \underbrace{2\langle \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{23}}. \end{aligned}$$

Estimate for A_{23} .

$$A_{23} = 2 \int_{\mathbb{R}} \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) |\tilde{b}^w u|^2 dt \geq 0. \quad (2.4.17)$$

Estimate for A_{21} . By Lemma 2.6.2, the symbol $\frac{k^2}{t^2}\varphi_1(\frac{t}{r_k})$ is in $S(k^2r_k^{-2}, g)$ with semi-norms independent of k . By Lemma 2.4.6, the symbol \tilde{b} is in $S(1, g)$ with semi-norms independent of k , then

$$\begin{aligned} \text{Symbol}\left[\frac{k^2}{t^2}\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right] &\in S(k^2r_k^{-2}\lambda_g^{-1}, g), \\ \text{Symbol}\left[\left[\frac{k^2}{t^2}\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right], \tilde{b}^w\right] &\in S(k^2r_k^{-2}\lambda_g^{-2}, g), \end{aligned}$$

where λ_g is given in (2.4.6), and moreover, the semi-norms of these two symbols are independent of k . Since by (2.4.6) $\lambda_g \geq \beta_k^{1/3}$, we have $S(k^2r_k^{-2}\lambda_g^{-2}, g) \subset S(k^2r_k^{-2}\beta_k^{-2/3}, g)$, implying that

$$\left\| \left[\left[\frac{k^2}{t^2}\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u \right\|_{L^2(\mathbb{R}; dt)} \leq C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)},$$

where C_2 is a constant independent of k . Then

$$|A_{21}| \leq C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.18)$$

Estimate for A_{22} . By Lemma 2.6.2, the symbol $t^2\varphi_1(\frac{t}{r_k})$ is in $S(r_k^2, g)$ with semi-norms independent of k . The symbol \tilde{b} is in $S(1, g)$, then

$$\begin{aligned} \text{Symbol}\left[t^2\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right] &\in S(r_k^2\lambda_g^{-1}, g), \\ \text{Symbol}\left[\left[t^2\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right], \tilde{b}^w\right] &\in S(r_k^2\lambda_g^{-2}, g), \end{aligned}$$

and moreover, the semi-norms of the two symbols above are independent of k . Since $\lambda_g \geq \beta_k^{1/3}$, we have $S(r_k^2\lambda_g^{-2}, g) \subset S(r_k^2\beta_k^{-2/3}, g)$, implying that

$$\left\| \left[\left[t^2\varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u \right\|_{L^2(\mathbb{R}; dt)} \leq C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)},$$

where $C_3 > 0$ is a constant independent of k . Hence

$$|A_{22}| \leq C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.19)$$

The estimate (2.4.15) follows immediately from (2.4.16), (2.4.17), (2.4.18) and (2.4.19). \square

Estimates for $\text{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)}$

From (2.4.10), (2.4.14) and (2.4.15), we obtain

$$\begin{aligned} 2\text{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\quad - C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2 - C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \end{aligned} \quad (2.4.20)$$

On the other hand, (2.4.3) gives the following inequality for u supported in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$

$$\text{Re}\langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} \geq \langle D_t^2 u, u \rangle_{L^2(\mathbb{R}; dt)} + \frac{1}{4}(k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.21)$$

Multiplying (2.4.21) by $4M$, where $M = \max\{C_1 + C_2, C_3\} + 1$, and adding (2.4.20), we get

$$\begin{aligned} \operatorname{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R};dt)} &\geq \underbrace{\langle (4MD_t^2 + b_1^w)u, u \rangle_{L^2(\mathbb{R};dt)}}_I \\ &+ \underbrace{(Mk^2r_k^{-2} + Mr_k^2 - C_1r_k^{-2} - C_2k^2r_k^{-2}\beta_k^{-2/3} - C_3r_k^2\beta_k^{-2/3})}_{\geq k^2r_k^{-2} + r_k^2, \text{ since } \beta_k \geq \alpha \geq 1} \|u\|_{L^2(\mathbb{R};dt)}^2. \end{aligned} \quad (2.4.22)$$

Estimate for I. Since $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, we have

$$I = \langle \left(4MD_t^2 + b_1^w + 2c_6\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k})) \right) u, u \rangle_{L^2(\mathbb{R};dt)},$$

where φ_2 is given in Definition 2.4.4 and c_6 is given in Lemma 2.4.9. Remark that by Lemma 2.4.9, the function $b_1(t, \tau) + 2c_6\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k}))$ is always non-negative and for all $t \in \mathbb{R}$, $|\tau| \leq \beta_k^{1/3}r_k^{-1}$,

$$b_1(t, \tau) + 2c_6\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k})) \geq \min(c_5, c_6)\beta_k^{2/3}r_k^{-2}.$$

Let us denote

$$a(t, \tau) := 4M\tau^2 + b_1(t, \tau) + 2c_6\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k})). \quad (2.4.23)$$

We prove an estimate for $\langle a^w u, u \rangle_{L^2(\mathbb{R};dt)}$, which is equal to I for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$.

Lemma 2.4.11. *There exist constants $C_4 > 0$, $C'_4 > 0$ such that for all $k \geq 1$, $\alpha \geq 1$ and $u \in L^2(\mathbb{R};dt)$,*

$$\langle a^w u, u \rangle_{L^2(\mathbb{R};dt)} \geq (C_4\beta_k^{2/3}r_k^{-2} - C'_4r_k^{-2})\|u\|_{L^2(\mathbb{R};dt)}^2.$$

Proof. First we prove that the symbol a belongs to $S(r_k^{-2}\lambda_g^2, g)$, where g is given in Definition 2.4.5 and λ_g given in (2.4.6). Recall that $\lambda_g^2 = \beta_k^{2/3} + r_k^2\tau^2 \geq \beta_k^{2/3}$. We check the three terms in the right hand side of (2.4.23) separately.

$$b_1 \in S(\beta_k r_k^{-2} \lambda_g^{-1}, g) \subset S(r_k^{-2} \lambda_g^2, g).$$

$$2c_4\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k})) \in S(\beta_k^{2/3}r_k^{-2}, g) \subset S(r_k^{-2} \lambda_g^2, g), \text{ since } \varphi_2(\frac{t}{r_k}) \in S(1, g).$$

$$\tau^2 \in S(r_k^{-2} \lambda_g^2, g): \text{ we have } \tau^2 \leq \beta_k^{2/3}r_k^{-2} + \tau^2 = r_k^{-2}\lambda_g^2 \text{ and we need to check if}$$

$$\begin{aligned} |\tau| &\leq Cr_k^{-2}\lambda_g^2(\beta_k^{2/3}r_k^{-2} + \tau^2)^{-1/2}, \\ 1 &\leq Cr_k^{-2}\lambda_g^2(\beta_k^{2/3}r_k^{-2} + \tau^2)^{-1}. \end{aligned}$$

These inequalities hold with $C = 1$ due to the expression of λ_g . Hence $a \in S(r_k^{-2}\lambda_g^2, g)$. Since the semi-norms of b_1 in $S(\beta_k r_k^{-2} \lambda_g^{-1}, g)$, those of $\varphi_2(\frac{t}{r_k})$ in $S(1, g)$ and those of τ^2 in $S(r_k^{-2} \lambda_g^2, g)$ are all bounded from above independently of k , we deduce that each semi-norm of a in $S(r_k^{-2} \lambda_g^2, g)$ can be bounded above independently of k .

There exists $C > 0$ such that $a(t, \tau) \geq C\beta_k^{2/3}r_k^{-2}$ for all t and τ . Indeed,

$$\begin{aligned} \text{if } |\tau| \leq \beta_k^{1/3}r_k^{-1}, \quad a(t, \tau) &\geq b_1(t, \tau) + 2c_6\beta_k^{2/3}r_k^{-2}(1 - \varphi_2(\frac{t}{r_k})) \\ &\geq \min(c_5, c_6)\beta_k^{2/3}r_k^{-2}, \\ \text{if } |\tau| > \beta_k^{1/3}r_k^{-1}, \quad a(t, \tau) &\geq 4M\tau^2 \geq 4M\beta_k^{2/3}r_k^{-2}. \end{aligned}$$

It follows from Fefferman-Phong inequality that the operator a^w satisfies

$$a^w \geq C\beta_k^{2/3}r_k^{-2} - C'r_k^{-2},$$

where $C' > 0$ depends on a semi-norm of a in $S(r_k^{-2}\lambda_g^2, g)$ (see Proposition 2.6.3) so that we can assume C' independent of k . The proof of Lemma 2.4.11 is completed. \square

Final estimate. By Lemma 2.4.11, now (2.4.22) becomes

$$\begin{aligned} \operatorname{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R};dt)} &\geq \langle a^w u, u \rangle_{L^2(\mathbb{R};dt)} + (k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R};dt)}^2 \\ &\geq (C_4 \beta_k^{2/3} r_k^{-2} - C'_4 r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R};dt)}^2. \end{aligned}$$

Then with (2.4.21), we have

$$\operatorname{Re}\langle L_k u, (4M + 4C'_4 + 2b^w)u \rangle_{L^2(\mathbb{R};dt)} \geq (C_4 \beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R};dt)}^2.$$

Since the operator $4M + 4C'_4 + 2b^w$ is bounded on $L^2(\mathbb{R};dt)$, with operator norm independent of k , we deduce from Cauchy-Schwarz inequality that for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$

$$\|L_k u\|_{L^2(\mathbb{R};dt)} \geq C_5 (\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R};dt)}, \quad (2.4.24)$$

where $C_5 > 0$ is a constant independent of k . In particular, since $\beta_k^{2/3} r_k^{-2} + r_k^2 \geq 2\beta_k^{1/3}$, we have

$$\|L_k u\|_{L^2(\mathbb{R};dt)} \geq 2C_5 \beta_k^{1/3} \|u\|_{L^2(\mathbb{R};dt)}. \quad (2.4.25)$$

This ends the proof of Proposition 2.4.3.

Estimates in $L^2(\mathbb{R}_+; rdr)$

We prove estimates in $L^2(\mathbb{R}_+; rdr)$ for the operator \mathcal{L}_k which is defined in (2.4.1). First we have the local result.

Proposition 2.4.12 (Local estimate for \mathcal{L}_k). *Suppose $r_k > \epsilon_0^{-1}$. There exist $C_6 > 0$, $\tilde{\alpha}_1 \geq 1$ such that for all $k \geq 1$, $\alpha \geq \tilde{\alpha}_1$, $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 (\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}, \quad (2.4.26)$$

where β_k is given in (2.3.1).

Proof. The first order term. For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, the term $\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R};dt)}$ can be controlled by the $L^2(\mathbb{R};dt)$ -norm of $L_k u$. Indeed,

$$\begin{aligned} \|L_k u\|_{L^2(\mathbb{R};dt)}^2 &\geq 2C_5 \beta_k^{1/3} \|L_k u\|_{L^2(\mathbb{R};dt)} \|u\|_{L^2(\mathbb{R};dt)} \quad \text{by Proposition 2.4.3, (2.4.25)} \\ &\geq 2C_5 \beta_k^{1/3} \operatorname{Re}\langle L_k u, u \rangle_{L^2(\mathbb{R};dt)} \\ &\geq 2C_5 \beta_k^{1/3} \|D_t u\|_{L^2(\mathbb{R};dt)}^2 \quad \text{by (2.4.3),} \end{aligned}$$

so that

$$\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R};dt)} \leq \frac{4}{3r_k} \|D_t u\|_{L^2(\mathbb{R};dt)} \leq \frac{4}{3} (2C_5)^{-1/2} \beta_k^{-1/6} r_k^{-1} \|L_k u\|_{L^2(\mathbb{R};dt)}.$$

We choose $\tilde{\alpha}_1 \geq 1$ such that $\frac{4}{3}(2C_5)^{-1/2}\tilde{\alpha}_1^{-1/6} \leq 1/2$, then for all $\alpha \geq \tilde{\alpha}_1$ and $k \geq 1$, we have $\frac{4}{3}(2C_5)^{-1/2}\beta_k^{-1/6} \leq 1/2$; since $r_k^{-1} \leq \epsilon_0 \leq 1$, we get an estimate in $L^2(\mathbb{R}; dt)$:

$$\begin{aligned}\|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} &\geq \|L_k u\|_{L^2(\mathbb{R}; dt)} - \left\| \frac{1}{t} D_t u \right\|_{L^2(\mathbb{R}; dt)} \\ &\geq \frac{1}{2} \|L_k u\|_{L^2(\mathbb{R}; dt)} \geq \frac{C_5}{2} (\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}.\end{aligned}$$

The measure rdr . If u is supported in the interval $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, then

$$\frac{3}{4}r_k \|u\|_{L^2(\mathbb{R}; dt)}^2 \leq \|u\|_{L^2(\mathbb{R}_+; rdr)}^2 \leq \frac{5}{4}r_k \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Since $\mathcal{L}_k u$ also has support in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, we have

$$\begin{aligned}\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} &\geq \left(\frac{3}{4}r_k\right)^{1/2} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{4}r_k\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}.\end{aligned}$$

Hence we obtain the estimate (2.4.26) with $C_6 = \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2}$. In particular, we have

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq 2C_6 \beta_k^{1/3} \|u\|_{L^2(\mathbb{R}_+; rdr)}. \quad \square$$

We now prove Theorem 2.4.2, i.e. the global estimate for \mathcal{L}_k .

Proof of Theorem 2.4.2. We choose a partition of unity on the whole real line:

$$\theta_0(s)^2 + \theta_1(s)^2 + \theta_{-1}(s)^2 \equiv 1, \quad s \in \mathbb{R}, \quad (2.4.27)$$

where θ_j are smooth functions and satisfy the following support conditions

$$\text{supp} \theta_0 \subset [-\frac{1}{4}, \frac{1}{4}], \quad \text{supp} \theta_1 \subset [\frac{1}{8}, \infty), \quad \text{supp} \theta_{-1} \subset (-\infty, -\frac{1}{8}].$$

Define for $j = 0, 1, -1$, $k \geq 1$,

$$\chi_{j,k}(r) := \theta_j\left(\frac{r - r_k}{r_k}\right). \quad (2.4.28)$$

Then $\{\chi_{j,k}, j = -1, 0, 1\}$ is a partition of unity, see Figure 2.5.

Estimates for $\mathcal{L}_k \chi_{j,k} u$.

- For $\mathcal{L}_k \chi_{0,k} u$: the support of $\chi_{0,k}$ is included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$. Assuming $\alpha \geq \tilde{\alpha}_1$, we can apply the result (2.4.26) in Proposition 2.4.12,

$$\|\mathcal{L}_k \chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq 2C_6 \beta_k^{1/3} \|\chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- For $\mathcal{L}_k \chi_{1,k} u$: the support of $\chi_{1,k}$ is included in $\{r \geq \frac{9}{8}r_k\}$. Since we have assumed $\epsilon_0 \leq (4R_1)^{-1}$ in (2.4.4), by (2.3.4) and the monotonicity of σ , we have for $r \geq \frac{9}{8}r_k$,

$$\sigma(r) - \sigma(r_k) \leq \sigma\left(\frac{9}{8}r_k\right) - \sigma(r_k) \leq \left(\frac{9}{8}r_k\right)^{-2} - c_0 r_k^{-2} = \underbrace{\left(c_0 - \left(\frac{8}{9}\right)^2\right) r_k^{-2}}_{>0 \text{ by (2.3.3)}}.$$

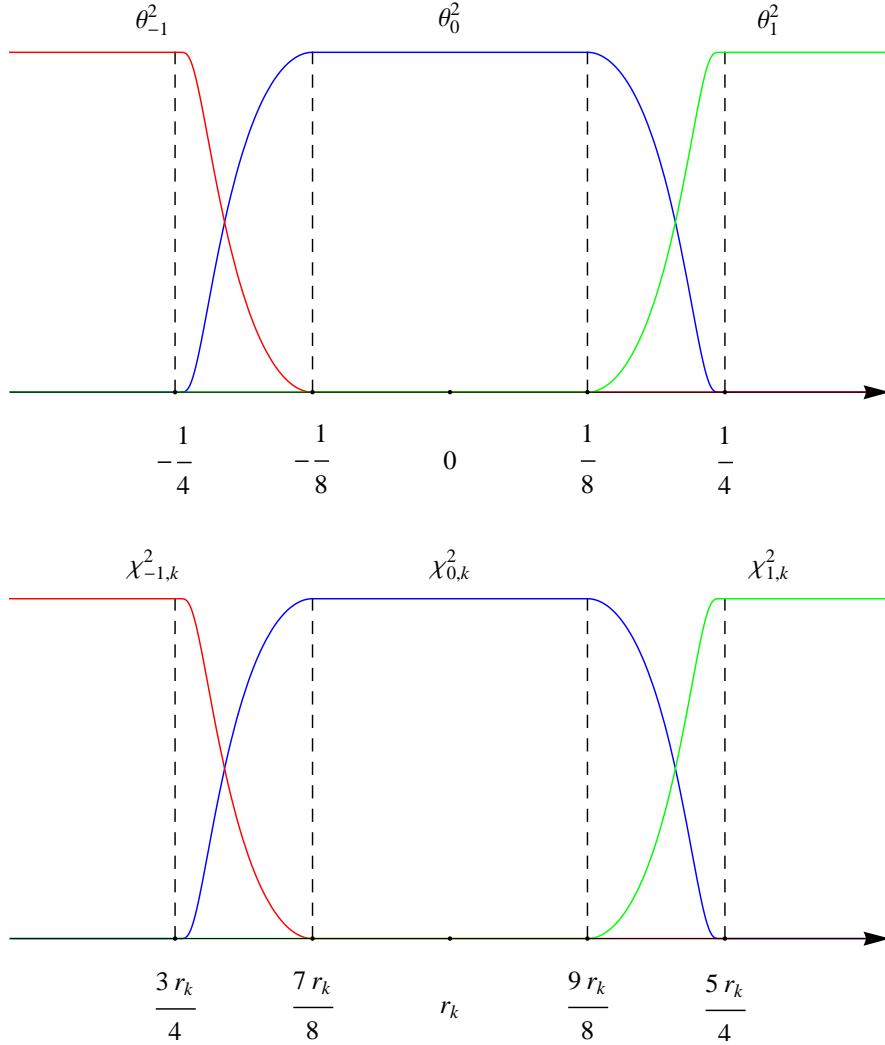


Figure 2.5: PARTITIONS OF UNITY.

We apply the multipliers Id and $-i\text{Id}$,

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k \chi_{1,k} u, -i \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k (\sigma(r_k) - \sigma(r)) \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq (c_0 - (\frac{8}{9})^2) \beta_k r_k^{-2} \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2, \end{aligned}$$

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &\geq \langle r^2 \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq r_k^2 \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2, \end{aligned}$$

this implies that

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k \chi_{1,k} u, (1-i) \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &\geq \left((c_0 - (\frac{8}{9})^2) \beta_k r_k^{-2} + r_k^2 \right) \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ &\geq 2 \sqrt{c_0 - (\frac{8}{9})^2} \beta_k^{1/2} \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

Then by Cauchy-Schwarz inequality, we get

$$\|\mathcal{L}_k \chi_{1,k} u\|_{L^2(\mathbb{R}_+;rdr)} \geq \sqrt{c_0 - (\frac{8}{9})^2} \beta_k^{1/2} \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+;rdr)}.$$

- For $\mathcal{L}_k \chi_{-1,k} u$: the support of $\chi_{-1,k}$ is included in $\{r \leq \frac{7}{8} r_k\}$. For $r \leq \frac{7}{8} r_k$, $\sigma(r_k) \leq \sigma(\frac{8}{7}r)$. Using the multipliers $i\text{Id}$ and Id , we get

$$\operatorname{Re} \langle \mathcal{L}_k \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+;rdr)} \geq \langle r^2 \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+;rdr)},$$

$$\operatorname{Re} \langle \mathcal{L}_k \chi_{-1,k} u, i \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+;rdr)} = \langle \beta_k (\sigma(r) - \sigma(r_k)) \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+;rdr)}.$$

We have $r^2 + \beta_k (\sigma(r) - \sigma(r_k)) \geq c \beta_k^{1/2}$ for all $r \leq \frac{7}{8} r_k$. Indeed, by (2.3.4),

$$\begin{aligned} \text{if } r \leq R_1, \quad r^2 + \beta_k (\sigma(r) - \sigma(r_k)) &\geq \beta_k (c_1 - r_k^{-2}) \\ &\geq (c_1 - \epsilon_0^2) \beta_k, \quad \text{since } r_k \geq \epsilon_0^{-1} \\ \text{if } r > R_1, \quad r^2 + \beta_k (\sigma(r) - \sigma(r_k)) &\geq r^2 + \beta_k (\sigma(r) - \sigma(\frac{8}{7}r)) \\ &\geq r^2 + \beta_k (c_0 r^{-2} - (\frac{8}{7}r)^{-2}) \\ &\geq 2 \sqrt{c_0 - (\frac{7}{8})^2} \beta_k^{1/2}, \quad \text{since } c_0 > (\frac{8}{9})^2 > (\frac{7}{8})^2. \end{aligned}$$

Since we have supposed in (2.4.4) that $\epsilon_0^2 < c_1$, the desired inequality holds with $c = \min(c_1 - \epsilon_0^2, 2\sqrt{c_0 - (7/8)^2})$. Thus by Cauchy-Schwarz inequality we get the estimate

$$\|\mathcal{L}_k \chi_{-1,k} u\|_{L^2(\mathbb{R}_+;rdr)} \geq \frac{c}{2} \beta_k^{1/2} \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+;rdr)}.$$

- We have proved for some $C_7 > 0$, for $j = 0, 1, -1$, $k \geq 1$, $u \in C_0^\infty(\mathbb{R}_+)$,

$$\|\mathcal{L}_k \chi_{j,k} u\|_{L^2(\mathbb{R}_+;rdr)} \geq C_7 \beta_k^{1/3} \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+;rdr)}. \quad (2.4.29)$$

Lemma 2.4.13 (Localization formula). *Take the finite partition of unity $\{\chi_j, j = -1, 0, 1\}$ on \mathbb{R}_+ , where $\chi_j \in C^\infty(\mathbb{R}_+; \mathbb{R})$, such that*

$$\sum_{j=-1}^1 \chi_j(r)^2 = 1, \quad \text{for all } r \in \mathbb{R}_+,$$

and

$$m_1^2 := \sup_r \sum_{j=-1}^1 |\chi_j'(r)|^2 < \infty, \quad m_2^2 := \sup_r \sum_{j=-1}^1 |\chi_j''(r)|^2 + \left| \frac{1}{r} \chi_j'(r) \right|^2 < \infty. \quad (2.4.30)$$

Then the estimate

$$3\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 + (3m_2^2 + 16m_1^4) \|u\|_{L^2(\mathbb{R}_+;rdr)}^2 \geq \sum_{j=-1}^1 \|\mathcal{L}_k \chi_j u\|_{L^2(\mathbb{R}_+;rdr)}^2, \quad (2.4.31)$$

holds for $u \in C_0^\infty(\mathbb{R}_+)$, where \mathcal{L}_k is the operator given in (2.4.1).

Proof. First recall (2.2.7) that the adjoint of ∂_r in $L^2(\mathbb{R}_+; rdr)$ is given by $\partial_r^* = -\partial_r - \frac{1}{r}$ and remark that the adjoint of \mathcal{L}_k in $L^2(\mathbb{R}_+; rdr)$ is given by

$$\mathcal{L}_k^* = \partial_r^* \partial_r + \frac{k^2}{r^2} + r^2 - i(\alpha k \sigma(r) - \lambda).$$

For any real-valued smooth function χ ,

$$\begin{aligned}\mathcal{L}_k^* \chi^2 \mathcal{L}_k &= \mathcal{L}_k^* \chi \mathcal{L}_k \chi + \mathcal{L}_k^* \chi [\chi, \mathcal{L}_k] \\ &= \chi \mathcal{L}_k^* \mathcal{L}_k \chi + [\mathcal{L}_k^*, \chi] \mathcal{L}_k \chi + \mathcal{L}_k^* \chi [\chi, \mathcal{L}_k] \\ &= \chi \mathcal{L}_k^* \mathcal{L}_k \chi + [\mathcal{L}_k^*, \chi] \chi \mathcal{L}_k + \mathcal{L}_k^* \chi [\chi, \mathcal{L}_k] + [\mathcal{L}_k^*, \chi] [\mathcal{L}_k, \chi].\end{aligned}$$

We define the operator

$$R_\chi := [\mathcal{L}_k, \chi] = [\partial_r^* \partial_r, \chi] = [\mathcal{L}_k^*, \chi]. \quad (2.4.32)$$

We see that R_χ is skew-adjoint and can be computed in the following way

$$\begin{aligned}R_\chi &= [\partial_r^* \partial_r, \chi] = -[\partial_r^2, \chi] - [\frac{1}{r} \partial_r, \chi] = -\chi'' - 2\chi' \partial_r - \frac{1}{r} \chi' \\ &\quad = \chi'' - 2\partial_r \chi' - \frac{1}{r} \chi',\end{aligned}$$

which implies

$$\begin{aligned}\mathcal{L}_k^* \chi^2 \mathcal{L}_k &= \chi \mathcal{L}_k^* \mathcal{L}_k \chi + [\mathcal{L}_k^*, \chi] \chi \mathcal{L}_k + \mathcal{L}_k^* \chi [\chi, \mathcal{L}_k] + [\mathcal{L}_k^*, \chi] [\mathcal{L}_k, \chi] \\ &= \chi \mathcal{L}_k^* \mathcal{L}_k \chi + (\chi'' - 2\partial_r \chi' - \frac{1}{r} \chi') \chi \mathcal{L}_k + \mathcal{L}_k^* \chi (\chi'' + 2\chi' \partial_r + \frac{1}{r} \chi') - R_\chi^* R_\chi \\ &= \chi \mathcal{L}_k^* \mathcal{L}_k \chi - \partial_r (\chi^2)' \mathcal{L}_k + \chi'' \chi \mathcal{L}_k - \frac{1}{2r} (\chi^2)' \mathcal{L}_k \\ &\quad + \mathcal{L}_k^* \chi \chi'' + \mathcal{L}_k^* (\chi^2)' \partial_r + \mathcal{L}_k^* \frac{1}{2r} (\chi^2)' - R_\chi^* R_\chi.\end{aligned}$$

We now apply this identity with $\chi = \chi_j$ and sum over j , then in view of the equality of the partition of unity, the left-hand side reduces to $\mathcal{L}_k^* \mathcal{L}_k$, and the second, fourth, sixth and seventh terms in the right-hand side vanish, so that

$$\mathcal{L}_k^* \mathcal{L}_k = \sum_{j=-1}^1 \chi_j \mathcal{L}_k^* \mathcal{L}_k \chi_j + \sum_{j=-1}^1 (\chi_j'' \chi_j \mathcal{L}_k + \mathcal{L}_k^* \chi_j \chi_j'') - \sum_{j=-1}^1 R_{\chi_j}^* R_{\chi_j}.$$

Thus for any $u \in C_0^\infty(\mathbb{R}_+)$, we have

$$\begin{aligned}
\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 &= \sum_{j=-1}^1 \langle \chi_j \mathcal{L}_k^* \mathcal{L}_k \chi_j u, u \rangle_{L^2(\mathbb{R}_+;rdr)} \\
&\quad + \sum_{j=-1}^1 \langle (\chi_j'' \chi_j \mathcal{L}_k + \mathcal{L}_k^* \chi_j \chi_j'') u, u \rangle_{L^2(\mathbb{R}_+;rdr)} - \sum_{j=-1}^1 \langle R_{\chi_j}^* R_{\chi_j} u, u \rangle_{L^2(\mathbb{R}_+;rdr)} \\
&= \sum_{j=-1}^1 \|\mathcal{L}_k \chi_j u\|_{L^2(\mathbb{R}_+;rdr)}^2 + \sum_{j=-1}^1 2\operatorname{Re} \langle \chi_j \mathcal{L}_k u, \chi_j'' u \rangle_{L^2(\mathbb{R}_+;rdr)} \\
&\quad - \sum_{j=-1}^1 \|R_{\chi_j} u\|_{L^2(\mathbb{R}_+;rdr)}^2 \\
&\geq \sum_{j=-1}^1 \|\mathcal{L}_k \chi_j u\|_{L^2(\mathbb{R}_+;rdr)}^2 - \sum_{j=-1}^1 \left(\|\chi_j \mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 + \|\chi_j'' u\|_{L^2(\mathbb{R}_+;rdr)}^2 \right) \\
&\quad - \sum_{j=-1}^1 \left(2\|(\chi_j'' + \frac{1}{r}\chi_j') u\|_{L^2(\mathbb{R}_+;rdr)}^2 + 8\|\chi_j' \partial_r u\|_{L^2(\mathbb{R}_+;rdr)}^2 \right) \\
&\geq \sum_{j=-1}^1 \|\mathcal{L}_k \chi_j u\|_{L^2(\mathbb{R}_+;rdr)}^2 - \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 - 3m_2^2 \|u\|_{L^2(\mathbb{R}_+;rdr)}^2 \\
&\quad - 8m_1^2 \|\partial_r u\|_{L^2(\mathbb{R}_+;rdr)}^2. \tag{2.4.33}
\end{aligned}$$

For the term $8m_1^2 \|\partial_r u\|_{L^2(\mathbb{R}_+;rdr)}^2$ in (2.4.33), we use the following estimate, by recalling that $\mathcal{L}_k = \partial_r^* \partial_r + \frac{k^2}{r^2} + r^2 + i(\alpha k \sigma(r) - \lambda)$,

$$\begin{aligned}
8m_1^2 \|\partial_r u\|_{L^2(\mathbb{R}_+;rdr)}^2 &\leq 8m_1^2 \operatorname{Re} \langle \mathcal{L}_k u, u \rangle_{L^2(\mathbb{R}_+;rdr)} \\
&\leq 8m_1^2 \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)} \|u\|_{L^2(\mathbb{R}_+;rdr)} \\
&\leq \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 + 16m_1^4 \|u\|_{L^2(\mathbb{R}_+;rdr)}^2. \tag{2.4.34}
\end{aligned}$$

From (2.4.33) and (2.4.34), we can deduce the localization formula (2.4.31). \square

End of the proof of Theorem 2.4.2. It remains to estimate the quantities defined in (2.4.30). In our case, $m_1^2 \leq cr_k^{-2}$, $m_2^2 \leq cr_k^{-4}$, with $c > 0$ a constant depending only on the functions θ_j defined in (2.4.27). It follows from (2.4.29) and the localization formula (2.4.31) that for all $u \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned}
3\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)}^2 + (3c + 16c^2)r_k^{-4} \|u\|_{L^2(\mathbb{R}_+;rdr)}^2 &\geq \sum_{j=-1}^1 C_7^2 \beta_k^{2/3} \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+;rdr)}^2 \\
&= C_7^2 \beta_k^{2/3} \|u\|_{L^2(\mathbb{R}_+;rdr)}^2.
\end{aligned}$$

Choosing $\alpha_1 \geq \tilde{\alpha}_1$ such that $(3c + 16c^2)\epsilon_0^4 \leq \frac{1}{2}C_7^2 \alpha_1^{2/3}$, then for all $\alpha \geq \alpha_1$, $k \geq 1$ and $r_k \geq \epsilon_0^{-1}$, we have $(3c + 16c^2)r_k^{-4} \leq \frac{1}{2}C_7^2 \beta_k^{2/3}$. Then we get the estimate

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+;rdr)} \geq \frac{1}{\sqrt{6}} C_7 \beta_k^{1/3} \|u\|_{L^2(\mathbb{R}_+;rdr)},$$

and this completes the proof of Theorem 2.4.2.

2.4.2 Case 2: crossing at a finite distance

We turn to the second case when r_k belongs to a fixed compact set $[\epsilon_0, \epsilon_0^{-1}]$ with r_k given in (2.3.14), see Figure 2.2. Notice that we can find a constant $c_7 > 0$ depending only on ϵ_0 such that

$$\forall r \in [\frac{1}{4}\epsilon_0, \frac{7}{4}\epsilon_0^{-1}], \quad \sigma'(r) \leq -c_7. \quad (2.4.35)$$

We will prove the following estimate.

Theorem 2.4.14. *Suppose $r_k \in [\epsilon_0, \epsilon_0^{-1}]$. There exist $C > 0$, $\alpha_2 \geq 1$ such that for all $k \geq 1$, $\alpha \geq \alpha_2$, $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C \beta_k^{2/3} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Estimates in $L^2(\mathbb{R}; dt)$

Proposition 2.4.15. *Suppose $r_k \in [\epsilon_0, \epsilon_0^{-1}]$. There exists $C > 0$ such that for all $k \geq 1$, $\alpha \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, the following estimate holds*

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C(\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)},$$

where L_k is given in (2.4.2) and β_k given in (2.3.1).

We prove Proposition 2.4.15 by a series of lemmas. Let us start with a definition.

Definition 2.4.16 (Admissible metric). Define a metric on the phase space $\mathbb{R}_t \times \mathbb{R}_\tau$:

$$g := g_{k,(t,\tau)} = dt^2 + \frac{d\tau^2}{\beta_k^{2/3} + \tau^2}.$$

The quantity λ_g defined by (2.6.4) for the metric g is

$$\lambda_g = (\beta_k^{2/3} + \tau^2)^{1/2} \geq \beta_k^{1/3} \geq \alpha^{1/3} \geq 1. \quad (2.4.36)$$

The metric g is of the type studied in Lemma 2.6.1, so that g is admissible and the structure constants $C_0, \tilde{C}_0, \tilde{N}_0$ in (2.6.2) are all independent of k , although the metric g depends on k itself.

We will use again the cutoff functions φ_1, φ_2 given in Definition 2.4.4 (Figure 2.3) and the function ψ given in (2.4.9) (Figure 2.4). If u has support included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, we have always $u(t) = \varphi_1(\frac{t}{r_k})u(t) = \varphi_2(\frac{t}{r_k})u(t)$.

Definition of the multiplier

$$b(\tau) := \psi^2(\beta_k^{-1/3}\tau), \quad \tau \in \mathbb{R}, \quad (2.4.37)$$

$$\tilde{b}(\tau) := \psi(\beta_k^{-1/3}\tau), \quad \tau \in \mathbb{R}, \quad (2.4.38)$$

where ψ is given in (2.4.9), Figure 2.4.

Lemma 2.4.17. *The symbols b defined in (2.4.37) and \tilde{b} defined in (2.4.38) belong to $S(1, g)$, where g is given in Definition 2.4.16. Moreover, their semi-norms are independent of k .*

Proof of Lemma 2.4.17. Observe that $|b(\tau)| \leq 1$ and

$$b^{(l)}(\tau) = \beta_k^{-l/3} (\psi^2)^{(l)}(\beta_k^{-1/3}\tau), \quad \text{for } l \geq 1.$$

If $|\tau| \leq 2\beta_k^{1/3}$, then $|b^{(l)}(\tau)| \leq C_l \beta_k^{-l/3} \leq C_l 5^{l/2} (\beta_k^{2/3} + \tau^2)^{-l/2}$, where $C_l = \|(\psi^2)^{(l)}\|_{L^\infty}$. If $|\tau| > 2\beta_k^{1/3}$, then $b^{(l)}(\tau) = 0$. This implies that b belongs to $S(1, g)$ and moreover, the semi-norms of b do not depend on k . Using the same computation, we get that the symbol $\tilde{b}(\tau)$ is also in $S(1, g)$ with semi-norms independent of k . \square

Remark 2.4.18. As a consequence of Lemma 2.4.17, the Weyl quantizations b^w and \tilde{b}^w are bounded, self-adjoint operators on $L^2(\mathbb{R}; dt)$. As a matter of fact, $b^w = b(D_t)$ and $\tilde{b}^w = \tilde{b}(D_t)$ are both bounded, non-negative Fourier multiplier. Moreover, we have $b^w = (\tilde{b}^w)^2$.

Now let us compute $2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)}$: for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$\begin{aligned} 2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &= \underbrace{2\operatorname{Re}\langle (D_t^2 + \frac{k^2}{t^2} + t^2)u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_A \\ &\quad + \underbrace{2\operatorname{Re}\langle i\beta_k(\sigma(t) - \sigma(r_k))u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_B. \end{aligned} \quad (2.4.39)$$

Estimate for the term B

Lemma 2.4.19. *The symbol $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ belongs to $S(\beta_k, g)$ with semi-norms independent of k , where g is given in Definition 2.4.16.*

Proof of Lemma 2.4.19. The function $(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ is supported in $[\frac{1}{4}r_k, \frac{7}{4}r_k]$ and bounded as well as all its derivatives, since r_k belongs to $[\epsilon_0, \epsilon_0^{-1}]$, a fixed compact set in $(0, \infty)$. This implies that $(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ belongs to $S(1, g)$, with semi-norms depending only on ϵ_0 , σ and φ_1 , so that Lemma 2.4.19 is proved. \square

For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, we write the term B as

$$B = 2\operatorname{Re}\langle i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})u, b^w u \rangle_{L^2(\mathbb{R}; dt)},$$

where φ_1 is defined in Definition 2.4.4. Since $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ is skew-adjoint and b^w is self-adjoint, we have

$$B = \langle [b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})]u, u \rangle_{L^2(\mathbb{R}; dt)}.$$

By Weyl's calculus, since $b \in S(1, g)$ and $i\beta_k(\sigma(t) - \sigma(r_k)) \in S(\beta_k, g)$, we get that

$$[b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})] = b_1^w + d^w, \quad (2.4.40)$$

where b_1 is the Poisson bracket of these two symbols

$$b_1(t, \tau) = \frac{1}{i} \left\{ b(\tau), i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right\} \in S(\beta_k\lambda_g^{-1}, g),$$

and d is the remainder belonging to $S(\beta_k\lambda_g^{-3}, g)$ with λ_g given in (2.4.36), see Appendix 2.6.1, (2.6.12).

We can compute b_1 directly:

$$b_1(t, \tau) = \beta_k^{2/3} (2\psi\psi')(\beta_k^{-1/3}\tau) \left(\sigma'(t)\varphi_1\left(\frac{t}{r_k}\right) + (\sigma(t) - \sigma(r_k))\varphi'_1\left(\frac{t}{r_k}\right)r_k^{-1} \right). \quad (2.4.41)$$

The symbol b_1 has the following properties.

Lemma 2.4.20 (Properties of b_1). *The support of the symbol b_1 given in (2.4.41) is included in the set*

$$\{(t, \tau) \in \mathbb{R}^2; |\tau| \leq 2\beta_k^{1/3}, t \in [\frac{1}{4}r_k, \frac{7}{4}r_k]\}.$$

There exist $c_8, c_9 > 0$ independent of k , such that

$$\begin{aligned} |\tau| \leq \beta_k^{1/3}, t \in [\frac{1}{2}r_k, \frac{3}{2}r_k] &\implies b_1(t, \tau) \geq c_8\beta_k^{2/3}, \\ \forall \tau \in \mathbb{R}, \forall t \in \mathbb{R}, |b_1(t, \tau)| &\leq c_9\beta_k^{2/3}. \end{aligned}$$

Proof of Lemma 2.4.20. The first property is a direct consequence of the choices of ψ and φ_1 . If $|\tau| \leq \beta_k^{1/3}$ and $t \in [\frac{1}{2}r_k, \frac{3}{2}r_k]$, by the properties of ψ and φ_1 , we have

$$(2\psi\psi')(\beta_k^{-1/3}\tau) \leq -\frac{1}{10}, \quad \varphi'_1\left(\frac{t}{r_k}\right) = 0, \quad \varphi_1\left(\frac{t}{r_k}\right) = 1.$$

By (2.4.35) we have $\sigma'(t) \leq -c_7$, since $t \geq \frac{1}{2}r_k \geq \frac{1}{2}\epsilon_0$. As a result, we get

$$b_1(t, \tau) \geq c_8\beta_k^{2/3}, \text{ for } |\tau| \leq \beta_k^{1/3}, t \in [\frac{1}{2}r_k, \frac{3}{2}r_k],$$

with $c_8 = \frac{1}{10}c_7$. For $t \in [\frac{1}{4}r_k, \frac{7}{4}r_k] \subset [\frac{1}{4}\epsilon_0, \frac{7}{4}\epsilon_0^{-1}]$, since we are in the case $r_k \in [\epsilon_0, \epsilon_0^{-1}]$,

$$|\sigma'(t)\varphi_1\left(\frac{t}{r_k}\right) + (\sigma(t) - \sigma(r_k))\varphi'_1\left(\frac{t}{r_k}\right)r_k^{-1}| \leq \|\sigma'\|_{L^\infty} + 2\|\varphi'_1\|_{L^\infty}\epsilon_0^{-1},$$

implying $\forall \tau \in \mathbb{R}, t \in \mathbb{R}, |b_1(t, \tau)| \leq \beta_k^{2/3} \|2\psi\psi'\|_{L^\infty} (\|\sigma'\|_{L^\infty} + 2\|\varphi'_1\|_{L^\infty}\epsilon_0^{-1})$,

which completes the proof of the lemma. \square

Estimate for d^w . Recall that d is given in (2.4.40). We deduce from Lemma 2.4.17 and Lemma 2.4.19 that the semi-norms of d in $S(\beta_k\lambda_g^{-3}, g)$ are independent of k , where λ_g is given in (2.4.36). Since $\lambda_g \geq \beta_k^{1/3}$, the class $S(\beta_k\lambda_g^{-3}, g)$ is included in $S(1, g)$. This implies that d^w is a bounded operator on $L^2(\mathbb{R}; dt)$, and its operator norm is bounded by a semi-norm of d in $S(1, g)$ (see Appendix 2.6.1), so also bounded by a semi-norm of d in $S(\beta_k\lambda_g^{-3}, g)$. We get

$$|\langle d^w u, u \rangle_{L^2(\mathbb{R}; dt)}| \leq \|d^w u\|_{L^2(\mathbb{R}; dt)} \|u\|_{L^2(\mathbb{R}; dt)} \leq C_1 \|u\|_{L^2(\mathbb{R}; dt)}^2,$$

where $C_1 > 0$ is a constant independent of k . Consequently, we get the estimate for the term B which is defined in (2.4.39): for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$B \geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.42)$$

Proposition 2.4.21 (Estimate for the term A). *There exist positive constants C_2, C_3 such that for all $k \geq 1$, $\alpha \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$A \geq -C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)}^2 - C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)}^2, \quad (2.4.43)$$

where A is defined in (2.4.39) and β_k given in (2.3.1).

Proof of Proposition 2.4.21. We divide the term A into two parts

$$A = \underbrace{2\operatorname{Re}\langle D_t^2 u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_1} + \underbrace{2\operatorname{Re}\langle (\frac{k^2}{t^2} + t^2) u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_2}.$$

A_1 is non-negative, since b^w is a non-negative Fourier multiplier. For A_2 , using $u(t) = \varphi_1(\frac{t}{r_k})u(t)$ and doing exactly the same computation as in the previous case in page 51, we get that

$$\begin{aligned} A_2 &= \underbrace{\langle \left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{21}} + \underbrace{\langle \left[\left[t^2 \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{22}} \\ &\quad + \underbrace{2 \langle \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w u, u \rangle_{L^2(\mathbb{R};dt)},}_{A_{23}} \end{aligned}$$

where \tilde{b} is given in (2.4.38).

Estimate for A_{23} .

$$A_{23} = 2 \int_{\mathbb{R}} \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) |\tilde{b}^w u|^2 dt \geq 0. \quad (2.4.44)$$

Estimate for A_{21} . The symbol $\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right)$ is in $S(k^2 r_k^{-2}, g)$ with semi-norms independent of k , by Lemma 2.6.2. The symbol \tilde{b} is in $S(1, g)$ with semi-norms independent of k by Lemma 2.4.17, then

$$\begin{aligned} \operatorname{Symbol} \left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right] &\in S(k^2 r_k^{-2} \lambda_g^{-1}, g), \\ \operatorname{Symbol} \left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] &\in S(k^2 r_k^{-2} \lambda_g^{-2}, g), \end{aligned}$$

where λ_g is given in (2.4.36), and moreover, the semi-norms of these two symbols are independent of k . Since by (2.4.36) $\lambda_g \geq \beta_k^{1/3}$, we have $S(k^2 r_k^{-2} \lambda_g^{-2}, g) \subset S(k^2 r_k^{-2} \beta_k^{-2/3}, g)$, implying that

$$\left\| \left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u \right\|_{L^2(\mathbb{R};dt)} \leq C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)},$$

where C_2 is a constant independent of k . Then

$$|A_{21}| \leq C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R};dt)}^2. \quad (2.4.45)$$

Estimate for A_{22} . By Lemma 2.6.2, the symbol $t^2\varphi_1(\frac{t}{r_k})$ is in $S(r_k^2, g)$ with semi-norms independent of k . Since the symbol \tilde{b} is in $S(1, g)$ with semi-norms independent of k by Lemma 2.4.17, we have

$$\begin{aligned}\text{Symbol}[t^2\varphi_1(\frac{t}{r_k}), \tilde{b}^w] &\in S(r_k^2\lambda_g^{-1}, g), \\ \text{Symbol}\left[t^2\varphi_1(\frac{t}{r_k}), \tilde{b}^w\right] &\in S(r_k^2\lambda_g^{-2}, g),\end{aligned}$$

and moreover, the semi-norms of the two symbols above are independent of k . Since $\lambda_g \geq \beta_k^{1/3}$, we have $S(r_k^2\lambda_g^{-2}, g) \subset S(r_k^2\beta_k^{-2/3}, g)$, implying that

$$\|\left[t^2\varphi_1(\frac{t}{r_k}), \tilde{b}^w\right] u\|_{L^2(\mathbb{R}; dt)} \leq C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)},$$

where $C_3 > 0$ is a constant independent of k . Hence

$$|A_{22}| \leq C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.4.46)$$

The estimate (2.4.43) follows immediately from (2.4.44), (2.4.45), (2.4.46) and the fact that $A_1 \geq 0$. \square

Estimates for $\text{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)}$

From (2.4.39), (2.4.42) and (2.4.43), we obtain

$$\begin{aligned}2\text{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\quad - C_2 k^2 r_k^{-2} \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2 - C_3 r_k^2 \beta_k^{-2/3} \|u\|_{L^2(\mathbb{R}; dt)}^2.\end{aligned} \quad (2.4.47)$$

Recall the inequality (2.4.21) for u supported in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$

$$\text{Re}\langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} \geq \|D_t u\|_{L^2(\mathbb{R}; dt)}^2 + \frac{1}{4}(k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Multiplying (2.4.21) by $4M$, where $M = \max\{C_2, C_3\} + C_1 + 1$, and adding (2.4.47), we get

$$\begin{aligned}\text{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)} &\geq \underbrace{\langle (4MD_t^2 + b_1^w)u, u \rangle_{L^2(\mathbb{R}; dt)}}_I \\ &\quad + \underbrace{(Mk^2 r_k^{-2} + Mr_k^2 - C_1 - C_2 k^2 r_k^{-2} \beta_k^{-2/3} - C_3 r_k^2 \beta_k^{-2/3})}_{\geq k^2 r_k^{-2} + r_k^2, \text{ since } \beta_k \geq \alpha \geq 1} \|u\|_{L^2(\mathbb{R}; dt)}^2.\end{aligned} \quad (2.4.48)$$

Estimate for I . Since $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, we have

$$I = \langle \left(4MD_t^2 + b_1^w + 2c_9 \beta_k^{2/3} (1 - \varphi_2(\frac{t}{r_k}))\right) u, u \rangle_{L^2(\mathbb{R}; dt)},$$

where φ_2 is given in Definition 2.4.4 and c_9 is given in Lemma 2.4.20. Remark that by Lemma 2.4.20, the function $b_1(t, \tau) + 2c_9 \beta_k^{2/3} (1 - \varphi_2(\frac{t}{r_k}))$ is always non-negative and

$$\forall t \in \mathbb{R}, \forall |\tau| \leq \beta_k^{1/3}, \quad b_1(t, \tau) + 2c_9 \beta_k^{2/3} (1 - \varphi_2(\frac{t}{r_k})) \geq \min(c_8, c_9) \beta_k^{2/3}.$$

Let us denote

$$a(t, \tau) := 4M\tau^2 + b_1(t, \tau) + 2c_9\beta_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})). \quad (2.4.49)$$

We now prove an estimate for $\langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)}$, which is equal to I if $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$.

Lemma 2.4.22. *There exist constants $C_4 > 0$, $C'_4 > 0$ such that for all $k \geq 1$, $\alpha \geq 1$ and $u \in L^2(\mathbb{R}; dt)$,*

$$\langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)} \geq (C_4\beta_k^{2/3} - C'_4)\|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Proof. First we prove that the symbol a belongs to $S(\lambda_g^2, g)$, where g is given in Definition 2.4.16 and λ_g given in (2.4.36). Recall that $\lambda_g^2 = \beta_k^{2/3} + \tau^2 \geq \beta_k^{2/3}$. We check the three terms in the right-hand side of (2.4.49) separately.

$$b_1 \in S(\beta_k \lambda_g^{-1}, g) \subset S(\lambda_g^2, g).$$

$$2c_9\beta_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})) \in S(\beta_k^{2/3}, g) \subset S(\lambda_g^2, g), \text{ since } \varphi_2(\frac{t}{r_k}) \in S(1, g).$$

$$\tau^2 \in S(\lambda_g^2, g): \text{ we have } \tau^2 \leq \beta_k^{2/3} + \tau^2 = \lambda_g^2 \text{ and we need only to check if}$$

$$|\tau| \leq C\lambda_g^2(\beta_k^{2/3} + \tau^2)^{-1/2},$$

$$1 \leq C\lambda_g^2(\beta_k^{2/3} + \tau^2)^{-1}.$$

These inequalities hold with $C = 1$ due to the expression of λ_g . Hence $a \in S(\lambda_g^2, g)$. Since the semi-norms of b_1 in $S(\beta_k \lambda_g^{-1}, g)$, those of $\varphi_2(\frac{t}{r_k})$ in $S(1, g)$ and those of τ^2 in $S(\lambda_g^2, g)$ are all bounded from above independently of k , we deduce that each semi-norm of a in $S(\lambda_g^2, g)$ can be bounded by a constant independent of k .

There exists $C > 0$ such that $a(t, \tau) \geq C\beta_k^{2/3}$ for all t and τ . Indeed,

$$\text{if } |\tau| \leq \beta_k^{1/3}, \quad a(t, \tau) \geq b_1(t, \tau) + 2c_9\beta_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})) \geq \min(c_8, c_9)\beta_k^{2/3},$$

$$\text{if } |\tau| > \beta_k^{1/3}, \quad a(t, \tau) \geq 4M\tau^2 \geq 4M\beta_k^{2/3}.$$

It follows from Fefferman-Phong inequality that the operator a^w satisfies

$$a^w \geq C\beta_k^{2/3} - C',$$

where $C' > 0$ depends on a semi-norm of a in $S(\lambda_g^2, g)$ so that we can assume C' independent of k , see Proposition 2.6.3. The proof of Lemma 2.4.22 is complete. \square

Final estimate. By Lemma 2.4.22, now (2.4.48) becomes

$$\begin{aligned} \operatorname{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)} + (k^2 r_k^{-2} + r_k^2)\|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\geq (C_4\beta_k^{2/3} - C'_4 + k^2 r_k^{-2} + r_k^2)\|u\|_{L^2(\mathbb{R}; dt)}^2. \end{aligned}$$

Then with (2.4.21), we have

$$\operatorname{Re}\langle L_k u, (4M + C'_4 + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)} \geq (C_4\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2)\|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Since the operator $4M + C'_4 + 2b^w$ is bounded on $L^2(\mathbb{R}; dt)$ with operator norm independent of k , we deduce that for all $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C_5(\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2)\|u\|_{L^2(\mathbb{R}; dt)}, \quad (2.4.50)$$

where $C_5 > 0$ is a constant independent of k . In particular, we have

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C_5\beta_k^{2/3}\|u\|_{L^2(\mathbb{R}; dt)}. \quad (2.4.51)$$

This ends the proof of Proposition 2.4.15.

Estimates in $L^2(\mathbb{R}_+; rdr)$

We prove estimates in $L^2(\mathbb{R}_+; rdr)$ for the operator \mathcal{L}_k which is defined in (2.4.1). First we have the local result.

Proposition 2.4.23 (Local estimate for \mathcal{L}_k). *There exist $C_6 > 0$, $\tilde{\alpha}_2 \geq 1$ such that for all $k \geq 1$, $\alpha \geq \tilde{\alpha}_2$, $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 (\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}, \quad (2.4.52)$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Proof. The first order term. For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, the term $\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R}; dt)}$ can be controlled by the $L^2(\mathbb{R}; dt)$ -norm of $L_k u$. Indeed,

$$\begin{aligned} \|L_k u\|_{L^2(\mathbb{R}; dt)}^2 &\geq C_5 \beta_k^{2/3} \|L_k u\|_{L^2(\mathbb{R}; dt)} \|u\|_{L^2(\mathbb{R}; dt)} \quad \text{by Proposition 2.4.15, (2.4.51),} \\ &\geq C_5 \beta_k^{2/3} \operatorname{Re} \langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &\geq C_5 \beta_k^{2/3} \|D_t u\|_{L^2(\mathbb{R}; dt)}^2 \quad \text{by (2.4.3),} \end{aligned}$$

so that

$$\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R}; dt)} \leq \frac{4}{3r_k} \|D_t u\|_{L^2(\mathbb{R}; dt)} \leq \frac{4}{3} C_5^{-1/2} \beta_k^{-1/3} r_k^{-1} \|L_k u\|_{L^2(\mathbb{R}; dt)}.$$

We choose $\tilde{\alpha}_2 \geq 1$ such that $\frac{4}{3} C_5^{-1/2} \tilde{\alpha}_2^{-1/3} \epsilon_0^{-1} \leq 1/2$, then for all $\alpha \geq \tilde{\alpha}_2$ and $k \geq 1$, we have $\frac{4}{3} C_5^{-1/2} \beta_k^{-1/3} \epsilon_0^{-1} \leq 1/2$; since $r_k^{-1} \leq \epsilon_0^{-1}$, we get an estimate in $L^2(\mathbb{R}; dt)$:

$$\begin{aligned} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} &\geq \|L_k u\|_{L^2(\mathbb{R}; dt)} - \|\frac{1}{t} D_t u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \frac{1}{2} \|L_k u\|_{L^2(\mathbb{R}; dt)} \geq \frac{C_5}{2} (\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}. \end{aligned}$$

The measure rdr . Since u and $\mathcal{L}_k u$ both have support in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, we have

$$\begin{aligned} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} &\geq \left(\frac{3}{4}r_k\right)^{1/2} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{4}r_k\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}. \end{aligned}$$

Hence we obtain the estimate (2.4.52) with $C_6 = \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2}$. In particular, we have

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 \beta_k^{2/3} \|u\|_{L^2(\mathbb{R}_+; rdr)}. \quad \square$$

Now we prove Theorem 2.4.14, i.e. the global estimate for \mathcal{L}_k .

Proof of Theorem 2.4.14. We use again the partition of unity on the positive-half line $\chi_{j,k}$ defined in (2.4.28) (Figure 2.5). We need only to estimate $\|\mathcal{L}_k \chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}$ for $j = -1, 0, 1$, and the quantities m_1^2 , m_2^2 defined in (2.4.30), then apply the localization formula (2.4.31).

Estimates for $\mathcal{L}_k \chi_{j,k} u$.

- For $\mathcal{L}_k \chi_{0,k} u$: the support of $\chi_{0,k}$ is included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$. Assuming $\alpha \geq \tilde{\alpha}_2$, we can apply the result (2.4.52) in Proposition 2.4.23.

$$\|\mathcal{L}_k \chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 \beta_k^{2/3} \|\chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- For $\mathcal{L}_k \chi_{1,k} u$: the support of $\chi_{1,k}$ is included in $\{r \geq \frac{9}{8}r_k\}$. Let us recall the inequality (2.4.35)

$$\forall r \in [\frac{1}{4}\epsilon_0, \frac{7}{4}\epsilon_0^{-1}], \quad \sigma'(r) \leq -c_7.$$

Using the monotonicity of σ , the mean value theorem and (2.4.35), we have for $r \geq \frac{9}{8}r_k$,

$$\sigma(r) - \sigma(r_k) \leq \sigma(\frac{9}{8}r_k) - \sigma(r_k) = \sigma'(s_k)(\frac{9}{8}r_k - r_k), \leq -\frac{1}{8}c_7r_k \leq -\frac{1}{8}c_7\epsilon_0,$$

for some $s_k \in [r_k, \frac{9}{8}r_k] \subset [\epsilon_0, \frac{9}{8}\epsilon_0^{-1}]$. We apply the multiplier $-i\text{Id}$ and get that

$$\begin{aligned} \operatorname{Re} \langle \mathcal{L}_k \chi_{1,k} u, -i\chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k (\sigma(r_k) - \sigma(r)) \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \frac{1}{8} c_7 \epsilon_0 \beta_k \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2, \end{aligned}$$

which implies by Cauchy-Schwarz inequality that

$$\|\mathcal{L}_k \chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq \frac{1}{8} c_7 \epsilon_0 \beta_k \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- For $\mathcal{L}_k \chi_{-1,k} u$: the support of $\chi_{-1,k}$ is included in $\{r \leq \frac{7}{8}r_k\}$. It follows from (2.4.35) that for $r \leq \frac{7}{8}r_k$,

$$\sigma(r) - \sigma(r_k) \geq \sigma(\frac{7}{8}r_k) - \sigma(r_k) = \sigma'(s'_k)(\frac{7}{8}r_k - r_k) \geq \frac{1}{8}c_7r_k \geq \frac{1}{8}c_7\epsilon_0,$$

for some $s'_k \in [\frac{7}{8}r_k, r_k] \subset [\frac{7}{8}\epsilon_0, \epsilon_0^{-1}]$. Using the multiplier $i\text{Id}$, we get

$$\begin{aligned} \operatorname{Re} \langle \mathcal{L}_k \chi_{-1,k} u, i\chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k (\sigma(r) - \sigma(r_k)) \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \frac{1}{8} c_7 \epsilon_0 \beta_k \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\|\mathcal{L}_k \chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq \frac{1}{8} c_7 \epsilon_0 \beta_k \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- We have proved for some $C_7 > 0$, for all $k \geq 1$, $j = 0, 1, -1$, for all $u \in C_0^\infty(\mathbb{R}_+)$,

$$\|\mathcal{L}_k \chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_7 \beta_k^{2/3} \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}. \quad (2.4.53)$$

Estimates for m_1^2 , m_2^2 . Recall that m_1^2 , m_2^2 are given in (2.4.30). We have $m_1^2 \leq cr_k^{-2}$, $m_2^2 \leq cr_k^{-4}$, with $c > 0$ a constant depending only on the functions θ_j defined in (2.4.27). It follows from (2.4.53) and the localization formula (2.4.31) that for any $u \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned} 3\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)}^2 + (3c + 16c^2)r_k^{-4}\|u\|_{L^2(\mathbb{R}_+; rdr)}^2 &\geq \sum_j C_7^2 \beta_k^{4/3} \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ &= C_7^2 \beta_k^{4/3} \|u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

Choosing $\alpha_2 \geq \tilde{\alpha}_2$ such that $(3c + 16c^2)\epsilon_0^{-4} \leq \frac{1}{2}C_7^2\alpha_2^{4/3}$, then for all $\alpha \geq \alpha_2$, $k \geq 1$ and $r_k \in [\epsilon_0, \epsilon_0^{-1}]$, we have $(3c + 16c^2)r_k^{-4} \leq \frac{1}{2}C_7^2\beta_k^{4/3}$. Then we get the estimate

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq \frac{1}{\sqrt{6}} C_7 \beta_k^{2/3} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

and this completes the proof of Theorem 2.4.14.

2.5 Proof, third part: change of sign close to 0

2.5.1 Case 3: crossing near 0

Now let us consider the third case when $r_k \in [\mu\beta_k^{-1/4}, \epsilon_0]$, where $\mu \geq 1$ is a large constant to be chosen below, see Figure 2.2. We suppose throughout Section 2.5.1

$$\alpha > \mu^4 \epsilon_0^{-4} \quad (2.5.1)$$

such that for any $k \geq 1$, the interval $[\mu\beta_k^{-1/4}, \epsilon_0]$ is not empty. We assume also

$$\frac{7}{4}\epsilon_0 \leq R_2, \quad \epsilon_0^2 < 1 - c_2, \quad (2.5.2)$$

where R_2 and c_2 are given in (2.3.5). Then according to (2.3.5),

$$\forall r \leq \frac{7}{4}\epsilon_0, \quad -\frac{1}{2}r^2 \leq \sigma(r) - 1 \leq -\frac{1}{2}c_0r^2.$$

Moreover, we suppose ϵ_0 small enough to ensure

$$\forall r \leq \frac{7}{4}\epsilon_0, \quad -c_{10}r \leq \sigma'(r) \leq -c_{11}r, \quad (2.5.3)$$

for some $c_{10} > c_{11} > 0$. We will prove the following estimate in this section.

Theorem 2.5.1. *There exist $\mu \geq 1$, $C > 0$ such that if $r_k \in [\mu\beta_k^{-1/4}, \epsilon_0]$, then for all $k \geq 1$ and $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C\beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Estimates in $L^2(\mathbb{R}; dt)$

Proposition 2.5.2. *There exists $C > 0$ such that if $\mu \geq 1$ and $r_k \in [\mu\beta_k^{-1/4}, \epsilon_0]$, for all $k \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, the following estimate holds:*

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C(\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)},$$

where L_k is given in (2.4.2) and β_k given in (2.3.1). In particular, we have

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C\beta_k^{1/2} \|u\|_{L^2(\mathbb{R}; dt)}.$$

Remark that the second estimate in Proposition 2.5.2 is a consequence of the first one, just noticing

$$\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} \geq (\beta_k^{2/3} r_k^{2/3})^{3/4} (k^2 r_k^{-2})^{1/4} = k^{1/2} \beta_k^{1/2} \geq \beta_k^{1/2}.$$

We prove Proposition 2.5.2 by a series of lemmas. Let us start with a definition.

Definition 2.5.3 (Admissible metric). We define a metric on the phase space $\mathbb{R}_t \times \mathbb{R}_\tau$:

$$g := g_{k,(t,\tau)} = \frac{dt^2}{r_k^2} + \frac{d\tau^2}{\beta_k^{2/3} r_k^{2/3} + \tau^2}.$$

The quantity λ_g defined by (2.6.4) for the metric g is

$$\lambda_g = r_k(\beta_k^{2/3}r_k^{2/3} + \tau^2)^{1/2} \geq \beta_k^{1/3}r_k^{4/3} \geq \mu^{4/3} \geq 1. \quad (2.5.4)$$

Since $r_k(\beta_k r_k)^{1/3} \geq \mu^{4/3} \geq 1$, the metric g is of the type studied in Lemma 2.6.1, so that g is admissible and the structure constants $C_0, \tilde{C}_0, \tilde{N}_0$ in (2.6.2) for g are all independent of k , although the metric g depends on k itself.

We will use again the cutoff functions φ_1, φ_2 given in Definition 2.4.4 (Figure 2.3) and the function ψ given in (2.4.9) (Figure 2.4). If u has support included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, we have always $u(t) = \varphi_1(\frac{t}{r_k})u(t) = \varphi_2(\frac{t}{r_k})u(t)$.

Definition of the multiplier

$$b(\tau) := \psi^2(\beta_k^{-1/3}r_k^{-1/3}\tau), \quad \tau \in \mathbb{R}, \quad (2.5.5)$$

$$\tilde{b}(\tau) := \psi(\beta_k^{-1/3}r_k^{-1/3}\tau), \quad \tau \in \mathbb{R}, \quad (2.5.6)$$

where ψ is given in (2.4.9) (Figure 2.4).

Lemma 2.5.4. *The symbols b given in (2.5.5) and \tilde{b} given in (2.5.6) belong to $S(1, g)$, where g is given in Definition 2.5.3. Moreover, their semi-norms are all independent of k .*

Proof of Lemma 2.5.4. Observe that $|b(\tau)| \leq 1$ and

$$b^{(l)}(\tau) = (\beta_k^{-1/3}r_k^{-1/3})^l(\psi^2)^{(l)}(\beta_k^{-1/3}r_k^{-1/3}\tau), \quad \text{for } l \geq 1.$$

If $|\tau| \leq 2\beta_k^{1/3}r_k^{1/3}$, then $|b^{(l)}(\tau)| \leq C_l(\beta_k^{-1/3}r_k^{-1/3})^l \leq C_l 5^{l/2}(\beta_k^{2/3}r_k^{2/3} + \tau^2)^{-l/2}$, with $C_l = \|(\psi^2)^{(l)}\|_{L^\infty}$. If $|\tau| > 2\beta_k^{1/3}r_k^{1/3}$, then $b^{(l)}(\tau) = 0$. This implies that b belongs to $S(1, g)$ and moreover, the semi-norms of b do not depend on k . Using the same computation we get that \tilde{b} is also in $S(1, g)$, with semi-norms independent of k . \square

Remark 2.5.5. As a consequence of Lemma 2.5.4, the Weyl quantizations b^w and \tilde{b}^w are bounded, self-adjoint operators on $L^2(\mathbb{R}; dt)$. As a matter of fact, $b^w = b(D_t)$, $\tilde{b}^w = \tilde{b}(D_t)$ are both bounded, non-negative Fourier multipliers. Furthermore, we have $b^w = (\tilde{b}^w)^2$.

Now let us compute $2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)}$: for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$\begin{aligned} 2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &= \underbrace{2\operatorname{Re}\langle (D_t^2 + \frac{k^2}{t^2} + t^2)u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_A \\ &\quad + \underbrace{2\operatorname{Re}\langle i\beta_k(\sigma(t) - \sigma(r_k))u, b^w u \rangle_{L^2(\mathbb{R}; dt)}}_B. \end{aligned} \quad (2.5.7)$$

Estimate for the term B

Lemma 2.5.6. *The symbol $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ belongs to $S(\beta_k r_k^2, g)$ with semi-norms independent of k , where g is given in Definition 2.5.3.*

Proof of Lemma 2.5.6. The function is supported in $[\frac{1}{4}r_k, \frac{7}{4}r_k] \subset (0, \frac{7}{4}\epsilon_0)$. By our assumption $\frac{7}{4}\epsilon_0 \leq R_2$ in (2.5.2), we have for $t \in [\frac{1}{4}r_k, \frac{7}{4}r_k]$,

$$|\sigma(t) - \sigma(r_k)| \leq |\sigma(t) - 1| + |\sigma(r_k) - 1| \leq \frac{1}{2}(t^2 + r_k^2) \leq \frac{17}{2}r_k^2,$$

$$\text{hence } |i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})| \leq \frac{17}{2}\beta_k r_k^2.$$

To estimate its derivatives, we notice (2.5.3) and that the derivatives of order ≥ 2 of σ are bounded, then

$$\begin{aligned} & \left| \frac{d^n}{dt^n} \left((\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right) \right| \\ & \leq C \sum_{0 \leq m \leq n-2} \left| \sigma^{(n-m)}(t)\varphi_1^{(m)}\left(\frac{t}{r_k}\right)r_k^{-m} \right| + C|\sigma'(t)\varphi_1^{(n-1)}\left(\frac{t}{r_k}\right)r_k^{-n+1}| \\ & \quad + \left| (\sigma(t) - \sigma(r_k))\varphi_1^{(n)}\left(\frac{t}{r_k}\right)r_k^{-n} \right| \\ & \leq C' \sum_{0 \leq m \leq n-2} r_k^{-m} + Cr_k^{-n+2} + C'r_k^{-n+2} \\ & \leq C''r_k^{-n+2}, \quad \text{since } r_k < \epsilon_0 \leq 1, \end{aligned}$$

which proves Lemma 2.5.6. \square

For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, we write the term B as

$$B = 2\operatorname{Re} \langle i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right)u, b^w u \rangle_{L^2(\mathbb{R}; dt)},$$

where φ_1 is defined in Definition 2.4.4. Since $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k})$ is skew-adjoint and b^w is self-adjoint, we have

$$B = \langle \left[b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right] u, u \rangle_{L^2(\mathbb{R}; dt)}.$$

By Weyl's calculus, since $b \in S(1, g)$ and $i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1(\frac{t}{r_k}) \in S(\beta_k r_k^2, g)$, we get that

$$\left[b^w, i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right] = b_1^w + d^w, \tag{2.5.8}$$

where b_1 is the Poisson bracket of these two symbols

$$b_1(t, \tau) = \frac{1}{i} \left\{ b(\tau), i\beta_k(\sigma(t) - \sigma(r_k))\varphi_1\left(\frac{t}{r_k}\right) \right\} \in S(\beta_k r_k^2 \lambda_g^{-1}, g),$$

and d is the remainder belonging to $S(\beta_k r_k^2 \lambda_g^{-3}, g)$ with λ_g given in (2.5.4), see Appendix 2.6.1, (2.6.12).

We can compute b_1 directly:

$$\begin{aligned} b_1(t, \tau) &= \beta_k^{2/3} r_k^{2/3} (2\psi\psi')(\beta_k^{-1/3} r_k^{-1/3} \tau) \\ &\quad \times \left(\sigma'(t)\varphi_1\left(\frac{t}{r_k}\right)r_k^{-1} + (\sigma(t) - \sigma(r_k))\varphi_1'\left(\frac{t}{r_k}\right)r_k^{-2} \right) \end{aligned} \tag{2.5.9}$$

The symbol b_1 has the following properties.

Lemma 2.5.7 (Properties of b_1). *The support of the symbol b_1 given in (2.5.9) is included in the set*

$$\{(t, \tau) \in \mathbb{R}^2; |\tau| \leq 2\beta_k^{1/3}r_k^{1/3}, t \in [\frac{1}{4}r_k, \frac{7}{4}r_k]\}.$$

There exist $c_{12}, c_{13} > 0$ independent of k , such that

$$\begin{aligned} |\tau| \leq \beta_k^{1/3}r_k^{1/3}, t \in [\frac{1}{2}r_k, \frac{3}{2}r_k] \implies b_1(t, \tau) \geq c_{12}\beta_k^{2/3}r_k^{2/3}; \\ \forall \tau \in \mathbb{R}, \forall t \in \mathbb{R}, |b_1(t, \tau)| \leq c_{13}\beta_k^{2/3}r_k^{2/3}. \end{aligned}$$

Proof of Lemma 2.5.7. The first property is a direct consequence of the choices of ψ and φ_1 . When $|\tau| \leq \beta_k^{1/3}r_k^{1/3}$, $t \in [\frac{1}{2}r_k, \frac{3}{2}r_k]$, we have

$$(2\psi\psi')(\beta_k^{-1/3}r_k^{-1/3}\tau) \leq -\frac{1}{10}, \quad \varphi'_1(\frac{t}{r_k}) = 0, \quad \varphi_1(\frac{t}{r_k}) = 1, \quad \sigma'(t) \leq -c_{11}t,$$

where the last inequality follows from (2.5.3) and that $\frac{1}{2}r_k \leq t \leq \frac{3}{2}r_k \leq \frac{3}{2}\epsilon_0$. As a result, we have

$$b_1(t, \tau) \geq c_{12}\beta_k^{2/3}r_k^{2/3}, \text{ for } |\tau| \leq \beta_k^{1/3}r_k^{1/3}, t \in [\frac{1}{2}r_k, \frac{3}{2}r_k].$$

with $c_{12} = \frac{1}{20}c_{11}$. It also follows from (2.3.5) and (2.5.3) that, for $t \in [\frac{1}{4}r_k, \frac{7}{4}r_k] \subset (0, \frac{7}{4}\epsilon_0)$,

$$\begin{aligned} |\sigma'(t)\varphi_1(\frac{t}{r_k}) + (\sigma(t) - \sigma(r_k))\varphi'_1(\frac{t}{r_k})r_k^{-1}| \leq \frac{7}{4}\epsilon_0c_{10} + \frac{17}{2}\|\varphi'_1\|_{L^\infty}\epsilon_0, \\ \text{implying } \forall \tau \in \mathbb{R}, t \in \mathbb{R}, |b_1(t, \tau)| \leq \beta_k^{2/3}r_k^{2/3}\|2\psi\psi'\|_{L^\infty}(\frac{7}{4}\epsilon_0c_{10} + \frac{17}{2}\|\varphi'_1\|_{L^\infty}\epsilon_0), \end{aligned}$$

which completes the proof. \square

Estimate for d^w . Recall that d is given in (2.5.8). We deduce from Lemma 2.5.4 and Lemma 2.5.6 that the semi-norms of d in $S(\beta_k r_k^2 \lambda_g^{-3}, g)$ are independent of k , where λ_g is given in (2.5.4). Since $\lambda_g \geq \beta_k^{1/3}r_k^{4/3}$, the class $S(\beta_k r_k^2 \lambda_g^{-3}, g)$ is included in $S(r_k^{-2}, g)$. This implies that $r_k^2 d^w$ is a bounded operator on $L^2(\mathbb{R}; dt)$, and its operator norm is bounded by a semi-norm of d in $S(r_k^{-2}, g)$ (see Appendix 2.6.1), so also bounded by a semi-norm of d in $S(\beta_k r_k^2 \lambda_g^{-3}, g)$. We get

$$|\langle r_k^2 d^w u, u \rangle_{L^2(\mathbb{R}; dt)}| \leq \|r_k^2 d^w u\|_{L^2(\mathbb{R}; dt)} \|u\|_{L^2(\mathbb{R}; dt)} \leq C_1 \|u\|_{L^2(\mathbb{R}; dt)}^2,$$

where $C_1 > 0$ is a constant independent of k . Consequently, we get the estimate for the term B which is defined in (2.5.7): for $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$B \geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.5.10)$$

Proposition 2.5.8 (Estimate for the term A). *There exist positive constants C_2, C_3 such that for all $k \geq 1$, $\mu \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$A \geq -C_2 k^2 r_k^{-2} \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)}^2 - C_3 r_k^2 \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)}^2, \quad (2.5.11)$$

where A is defined in (2.5.7).

Proof of Proposition 2.5.8. We divide the term A into two parts

$$A = \underbrace{2\operatorname{Re}\langle D_t^2 u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_1} + \underbrace{2\operatorname{Re}\langle (\frac{k^2}{t^2} + t^2)u, b^w u \rangle_{L^2(\mathbb{R};dt)}}_{A_2}.$$

A_1 is non-negative, since b^w is a non-negative Fourier multiplier. For A_2 , using $u(t) = \varphi_1(\frac{t}{r_k})u(t)$ and doing exactly the same computation as in Case 1 in page 51, we get that

$$\begin{aligned} A_2 &= \underbrace{\langle \left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)} + \langle \left[t^2 \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right] u, u \rangle_{L^2(\mathbb{R};dt)}}_{A_{21}} \\ &\quad + \underbrace{2\langle \tilde{b}^w \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) \tilde{b}^w u, u \rangle_{L^2(\mathbb{R};dt)},}_{A_{23}} \end{aligned}$$

where \tilde{b} is given in (2.5.6).

Estimate for A_{23} .

$$A_{23} = 2 \int_{\mathbb{R}} \left(\frac{k^2}{t^2} + t^2 \right) \varphi_1\left(\frac{t}{r_k}\right) |\tilde{b}^w u|^2 dt \geq 0. \quad (2.5.12)$$

Estimate for A_{21} . The symbol $\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right)$ is in $S(k^2 r_k^{-2}, g)$ with semi-norms independent of k by Lemma 2.6.2. The symbol \tilde{b} is in $S(1, g)$ with semi-norms independent of k by Lemma 2.5.4, then

$$\begin{aligned} \operatorname{Symbol}\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right] &\in S(k^2 r_k^{-2} \lambda_g^{-1}, g), \\ \operatorname{Symbol}\left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w\right] &\in S(k^2 r_k^{-2} \lambda_g^{-2}, g), \end{aligned}$$

where λ_g is given in (2.5.4), and moreover, the semi-norms of these two symbols are independent of k . Since by (2.5.4) $\lambda_g \geq \beta_k^{1/3} r_k^{4/3} \geq \mu^{4/3}$, we have $S(k^2 r_k^{-2} \lambda_g^{-2}, g) \subset S(k^2 r_k^{-2} \mu^{-8/3}, g)$, implying that

$$\| \left[\left[\frac{k^2}{t^2} \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w \right] u \|_{L^2(\mathbb{R};dt)} \leq C_2 k^2 r_k^{-2} \mu^{-8/3} \|u\|_{L^2(\mathbb{R};dt)},$$

where C_2 is a constant independent of k . Then

$$|A_{21}| \leq C_2 k^2 r_k^{-2} \mu^{-8/3} \|u\|_{L^2(\mathbb{R};dt)}^2. \quad (2.5.13)$$

Estimate for A_{22} . The symbol $t^2 \varphi_1\left(\frac{t}{r_k}\right)$ is in $S(r_k^2, g)$ with semi-norms independent of k by Lemma 2.6.2. The symbol \tilde{b} is in $S(1, g)$ with semi-norms independent of k by Lemma 2.5.4, then

$$\begin{aligned} \operatorname{Symbol}\left[t^2 \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w\right] &\in S(r_k^2 \lambda_g^{-1}, g), \\ \operatorname{Symbol}\left[\left[t^2 \varphi_1\left(\frac{t}{r_k}\right), \tilde{b}^w \right], \tilde{b}^w\right] &\in S(r_k^2 \lambda_g^{-2}, g), \end{aligned}$$

and moreover, the semi-norms of the two symbols above are independent of k . Since $\lambda_g \geq \beta_k^{1/3} r_k^{4/3} \geq \mu^{4/3}$, we have $S(r_k^2 \lambda_g^{-2}, g) \subset S(r_k^2 \mu^{-8/3}, g)$, implying that

$$\| [t^2 \varphi_1(\frac{t}{r_k}), \tilde{b}^w], \tilde{b}^w \|_{L^2(\mathbb{R}; dt)} \leq C_3 r_k^2 \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)},$$

where $C_3 > 0$ is a constant independent of k . Hence

$$|A_{22}| \leq C_3 r_k^2 \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \quad (2.5.14)$$

The estimate (2.5.11) follows immediately from (2.5.12), (2.5.13), (2.5.14) and the fact that $A_1 \geq 0$. \square

Estimates for $\operatorname{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)}$

From (2.5.7), (2.5.10) and (2.5.11), we obtain

$$\begin{aligned} 2\operatorname{Re}\langle L_k u, b^w u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle b_1^w u, u \rangle_{L^2(\mathbb{R}; dt)} - C_1 r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\quad - C_2 k^2 r_k^{-2} \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)}^2 - C_3 r_k^2 \mu^{-8/3} \|u\|_{L^2(\mathbb{R}; dt)}^2. \end{aligned} \quad (2.5.15)$$

Recall the inequality (2.4.21) for u supported in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$

$$\operatorname{Re}\langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} \geq \|D_t u\|_{L^2(\mathbb{R}; dt)}^2 + \frac{1}{4}(k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Multiplying (2.4.21) by $4M$, where $M = \max\{C_1 + C_2, C_3\} + 1$, and adding (2.5.15), we get

$$\begin{aligned} \operatorname{Re}\langle L_k u, (4M + 2b^w)u \rangle_{L^2(\mathbb{R}; dt)} &\geq \underbrace{\langle (4MD_t^2 + b_1^w)u, u \rangle_{L^2(\mathbb{R}; dt)}}_I \\ &\quad + \underbrace{(Mk^2 r_k^{-2} + Mr_k^2 - C_1 r_k^{-2} - C_2 k^2 r_k^{-2} \mu^{-8/3} - C_3 r_k^2 \mu^{-8/3})}_{\geq k^2 r_k^{-2} + r_k^2, \text{ since } \mu \geq 1} \|u\|_{L^2(\mathbb{R}; dt)}^2. \end{aligned} \quad (2.5.16)$$

Estimate for I. Since $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$I = \langle (4MD_t^2 + b_1^w + 2c_{13}\beta_k^{2/3}r_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})))u, u \rangle_{L^2(\mathbb{R}; dt)},$$

where φ_2 is given in Definition 2.4.4 and c_{13} is given in Lemma 2.5.7. Remark that by Lemma 2.5.7, the function $b_1(t, \tau) + 2c_{13}\beta_k^{2/3}r_k^{2/3}(1 - \varphi_2(\frac{t}{r_k}))$ is always non-negative and for all $t \in \mathbb{R}$, $|\tau| \leq \beta_k^{1/3} r_k^{1/3}$,

$$b_1(t, \tau) + 2c_{13}\beta_k^{2/3}r_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})) \geq \min(c_{12}, c_{13})\beta_k^{2/3}r_k^{2/3}.$$

Let us denote

$$a(t, \tau) := 4M\tau^2 + b_1(t, \tau) + 2c_{13}\beta_k^{2/3}r_k^{2/3}(1 - \varphi_2(\frac{t}{r_k})). \quad (2.5.17)$$

We prove an estimate for $\langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)}$, which is equal to I if $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$.

Lemma 2.5.9. *There exist constants $C_4, C'_4 > 0$, such that for all $k \geq 1$, $\mu \geq 1$, for all $u \in L^2(\mathbb{R}; dt)$,*

$$\langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)} \geq (C_4 \beta_k^{2/3} r_k^{2/3} - C'_4 r_k^{-2}) \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Proof. We first prove that the symbol a belongs to $S(r_k^{-2} \lambda_g^2, g)$, where g is given in Definition 2.5.3 and λ_g given in (2.5.4). Recall that $\lambda_g^2 = r_k^2 (\beta_k^{2/3} r_k^{2/3} + \tau^2) \geq \beta_k^{2/3} r_k^{8/3}$. We check the three terms in the right hand side of (2.5.17) separately.

$$b_1 \in S(\beta_k r_k^2 \lambda_g^{-1}, g) \subset S(r_k^{-2} \lambda_g^2, g).$$

$$2c_{13} \beta_k^{2/3} r_k^{2/3} (1 - \varphi_2(\frac{t}{r_k})) \in S(\beta_k^{2/3} r_k^{2/3}, g) \subset S(r_k^{-2} \lambda_g^2, g), \text{ since } \varphi_2(\frac{t}{r_k}) \in S(1, g).$$

$\tau^2 \in S(r_k^{-2} \lambda_g^2, g)$. We have $\tau^2 \leq \beta_k^{2/3} r_k^{2/3} + \tau^2 = r_k^{-2} \lambda_g^2$ and we need only to check if

$$|\tau| \leq C r_k^{-2} \lambda_g^2 (\beta_k^{2/3} r_k^{2/3} + \tau^2)^{-1/2},$$

$$1 \leq C r_k^{-2} \lambda_g^2 (\beta_k^{2/3} r_k^{2/3} + \tau^2)^{-1}.$$

These inequalities hold with $C = 1$ due to the expression of λ_g . Thus $a \in S(r_k^{-2} \lambda_g^2, g)$. Since the semi-norms of b_1 in $S(\beta_k r_k^2 \lambda_g^{-1}, g)$, those of $\varphi_2(\frac{t}{r_k})$ in $S(1, g)$ and those of τ^2 in $S(\lambda_g^2, g)$ are all bounded from above independently of k , we deduce that each semi-norm of a in $S(r_k^{-2} \lambda_g^2, g)$ can be bounded by a constant independent of k .

There exists $C > 0$ such that $a(t, \tau) \geq C \beta_k^{2/3} r_k^{2/3}$ for all t and τ . Indeed,

$$\begin{aligned} \text{if } |\tau| \leq \beta_k^{1/3} r_k^{1/3}, \quad a(t, \tau) &\geq b_1(t, \tau) + 2c_{13} \beta_k^{2/3} r_k^{2/3} (1 - \varphi_2(\frac{t}{r_k})) \\ &\geq \min(c_{12}, c_{13}) \beta_k^{2/3} r_k^{2/3}, \\ \text{if } |\tau| > \beta_k^{1/3} r_k^{1/3}, \quad a(t, \tau) &\geq 4M\tau^2 \geq 4M\beta_k^{2/3} r_k^{2/3}. \end{aligned}$$

It follows from Fefferman-Phong inequality that the operator a^w satisfies

$$a^w \geq C \beta_k^{2/3} r_k^{2/3} - C' r_k^{-2},$$

where $C' > 0$ depends on a semi-norm of a in $S(r_k^{-2} \lambda_g^2, g)$ (see Proposition 2.6.3) so that we can assume C' independent of k . The proof of Lemma 2.5.9 is complete. \square

Final estimate. By Lemma 2.5.9, now (2.5.16) becomes

$$\begin{aligned} \operatorname{Re} \langle L_k u, (4M + 2b^w) u \rangle_{L^2(\mathbb{R}; dt)} &\geq \langle a^w u, u \rangle_{L^2(\mathbb{R}; dt)} + (k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\geq (C_4 \beta_k^{2/3} r_k^{2/3} - C'_4 r_k^{-2} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2. \end{aligned}$$

Then with (2.4.21), we have

$$\operatorname{Re} \langle L_k u, (4M + 4C'_4 + 2b^w) u \rangle_{L^2(\mathbb{R}; dt)} \geq (C_4 \beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}^2.$$

Since the operator $4M + 4C'_4 + 2b^w$ is bounded on $L^2(\mathbb{R}; dt)$ with operator norm independent of k , we deduce that for all $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C_5 (\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}, \quad (2.5.18)$$

where $C_5 > 0$ is a constant independent of k . Since $\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} \geq \beta_k^{1/2}$, we have in particular

$$\|L_k u\|_{L^2(\mathbb{R}; dt)} \geq C_5 \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}; dt)}. \quad (2.5.19)$$

This ends the proof of Proposition 2.5.2.

Estimates in $L^2(\mathbb{R}_+; rdr)$

We prove estimates in $L^2(\mathbb{R}_+; rdr)$ for the operator \mathcal{L}_k which is defined in (2.4.1). First we have the local result.

Proposition 2.5.10 (Local estimate for \mathcal{L}_k). *There exist $\mu_0 \geq 1$, $C_6 > 0$ such that for any $\mu \geq \mu_0$, $\alpha > \mu^4 \epsilon_0^{-4}$, if $r_k \in [\mu \beta_k^{-1/4}, \epsilon_0]$, then for all $k \geq 1$ and $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 (\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}, \quad (2.5.20)$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Proof. The first order term. For $u \in C_0^\infty([\frac{3}{4}r_k, \frac{5}{4}r_k])$, the term $\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R}; dt)}$ can be controlled by the $L^2(\mathbb{R}; dt)$ -norm of $L_k u$. Indeed,

$$\begin{aligned} \|L_k u\|_{L^2(\mathbb{R}; dt)}^2 &\geq C_5 \beta_k^{1/2} \|L_k u\|_{L^2(\mathbb{R}; dt)} \|u\|_{L^2(\mathbb{R}; dt)} \quad \text{by Proposition 2.5.2, (2.5.19),} \\ &\geq C_5 \beta_k^{1/2} \operatorname{Re} \langle L_k u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &\geq C_5 \beta_k^{1/2} \|D_t u\|_{L^2(\mathbb{R}; dt)}^2 \quad \text{by (2.4.3),} \end{aligned}$$

so that

$$\|\frac{i}{t} D_t u\|_{L^2(\mathbb{R}; dt)} \leq \frac{4}{3r_k} \|D_t u\|_{L^2(\mathbb{R}; dt)} \leq \frac{4}{3} C_5^{-1/2} \beta_k^{-1/4} r_k^{-1} \|L_k u\|_{L^2(\mathbb{R}; dt)}.$$

We choose $\mu_0 \geq 1$ large enough such that $\frac{4}{3} C_5^{-1/2} \mu_0^{-1} \leq 1/2$, then for $\mu \geq \mu_0$, if $r_k \geq \mu \beta_k^{-1/4}$, we have $\frac{4}{3} (2C_5)^{-1/2} \beta_k^{-1/4} r_k^{-1} \leq \frac{4}{3} C_5^{-1/2} \mu^{-1} \leq 1/2$. We get an estimate in $L^2(\mathbb{R}; dt)$:

$$\begin{aligned} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} &\geq \|L_k u\|_{L^2(\mathbb{R}; dt)} - \|\frac{1}{t} D_t u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \frac{1}{2} \|L_k u\|_{L^2(\mathbb{R}; dt)} \geq \frac{C_5}{2} (\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)}. \end{aligned}$$

The measure rdr . Since u and $\mathcal{L}_k u$ both have support in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$, we have

$$\begin{aligned} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} &\geq \left(\frac{3}{4}r_k\right)^{1/2} \|\mathcal{L}_k u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{4}r_k\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}; dt)} \\ &\geq \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2} (\beta_k^{2/3} r_k^{2/3} + k^2 r_k^{-2} + r_k^2) \|u\|_{L^2(\mathbb{R}_+; rdr)}. \end{aligned}$$

Hence we obtain the estimate (2.4.52) with $C_6 = \left(\frac{3}{5}\right)^{1/2} \frac{C_5}{2}$. In particular,

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)}. \quad \square$$

Now we prove Theorem 2.5.1 i.e. the global estimate for \mathcal{L}_k .

Proof of Theorem 2.5.1. We use again the partition of unity on the positive-half line $\chi_{j,k}$ defined in (2.4.28) (Figure 2.5). We need only to estimate $\|\mathcal{L}_k \chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}$ for $j = -1, 0, 1$, and the quantities m_1^2 , m_2^2 defined in (2.4.30), then apply the localization formula (2.4.31).

Estimates for $\mathcal{L}_k \chi_{j,k} u$.

- For $\mathcal{L}_k \chi_{0,k} u$: the support of $\chi_{0,k}$ is included in $[\frac{3}{4}r_k, \frac{5}{4}r_k]$. Assuming $\mu \geq \mu_0$ we can apply the result (2.4.52) in Proposition 2.5.10.

$$\|\mathcal{L}_k \chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_6 \beta_k^{1/2} \|\chi_{0,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- For $\mathcal{L}_k \chi_{1,k} u$: the support of $\chi_{1,k}$ is included in $\{r \geq \frac{9}{8}r_k\}$. Let us recall the inequality (2.3.5):

$$\begin{cases} -\frac{1}{2}r^2 \leq \sigma(r) - 1 \leq -\frac{1}{2}c_0 r^2, & \text{if } r \leq R_2; \\ \sigma(r) \leq c_2, & \text{if } r > R_2. \end{cases}$$

By the monotonicity of σ and (2.3.5), for $r \geq \frac{9}{8}r_k$, since $r_k < \epsilon_0$,

$$\begin{aligned} \text{if } r \leq R_2, \sigma(r) - \sigma(r_k) &\leq \sigma(r) - \sigma\left(\frac{8}{9}r\right) \leq -\frac{1}{2}c_0 r^2 + \underbrace{\frac{1}{2}\left(\frac{8}{9}r\right)^2}_{>0 \text{ by (2.3.3)}} = -\frac{1}{2}\left(c_0 - \left(\frac{8}{9}\right)^2\right)r^2, \\ \text{if } r > R_2, \sigma(r) - \sigma(r_k) &\leq c_2 - \left(1 - \frac{1}{2}r_k^2\right) \leq -(1 - c_2) + \frac{1}{2}\epsilon_0^2 \leq -\frac{1}{2}(1 - c_2), \end{aligned}$$

where the last inequality is due to the hypothesis $\epsilon_0^2 \leq 1 - c_2$ in (2.5.2). So we have proved the following inequality for $r \geq \frac{9}{8}r_k$,

$$\beta_k(\sigma(r_k) - \sigma(r)) + k^2 r^{-2} \geq \min(\sqrt{2(c_0 - (8/9)^2)}k\beta_k^{1/2}, \frac{1}{2}(1 - c_2)\beta_k) \geq c\beta_k^{1/2},$$

with $c = \min(\sqrt{2(c_0 - (8/9)^2)}, (1 - c_2)/2)$. Now apply the multipliers $-i\text{Id}$ and Id :

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k \chi_{1,k} u, -i\chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k(\sigma(r_k) - \sigma(r))\chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ \text{Re}\langle \mathcal{L}_k \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &\geq \langle k^2 r^{-2} \chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \end{aligned}$$

which implies that

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k \chi_{1,k} u, (1 - i)\chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &\geq \langle (\beta_k(\sigma(r_k) - \sigma(r)) + k^2 r^{-2})\chi_{1,k} u, \chi_{1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq c\beta_k^{1/2} \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2 \end{aligned}$$

We get by Cauchy-Schwarz inequality that

$$\|\mathcal{L}_k \chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq \frac{c}{2} \beta_k^{1/2} \|\chi_{1,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- For $\mathcal{L}_k \chi_{-1,k} u$: the support of $\chi_{-1,k}$ is included in $\{r \leq \frac{7}{8}r_k\}$. We have for $r \leq \frac{7}{8}r_k$,

$$\sigma(r) - \sigma(r_k) \geq \sigma\left(\frac{7}{8}r_k\right) - \sigma(r_k) \geq -\frac{1}{2}\left(\frac{7}{8}r_k\right)^2 + \frac{c_0}{2}r_k^2 = \frac{1}{2}(c_0 - (\frac{7}{8})^2)r_k^2,$$

recalling that $c_0 > (\frac{8}{9})^2$ is given in (2.3.3). Using the multipliers $i\text{Id}$ and Id , we get

$$\begin{aligned} \text{Re}\langle \mathcal{L}_k \chi_{-1,k} u, i\chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &= \langle \beta_k(\sigma(r) - \sigma(r_k))\chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq \frac{1}{2}(c_0 - (\frac{7}{8})^2)r_k^2 \beta_k \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2, \\ \text{Re}\langle \mathcal{L}_k \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} &\geq \langle k^2 r^{-2} \chi_{-1,k} u, \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ &\geq k^2 r_k^{-2} \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2, \quad \text{since } r \leq r_k, \end{aligned}$$

which imply that

$$\begin{aligned} & \operatorname{Re} \langle \mathcal{L}_k \chi_{-1,k} u, (1-i) \chi_{-1,k} u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ & \geq \left(\left(\frac{1}{2} (c_0 - (\frac{7}{8})^2) \beta_k r_k^2 + k^2 r_k^{-2} \right) \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2 \right) \\ & \geq \sqrt{2(c_0 - (\frac{7}{8})^2) k \beta_k^{1/2}} \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

We get by Cauchy-Schwarz inequality that

$$\|\mathcal{L}_k \chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq \sqrt{c_0 - (\frac{7}{8})^2} \beta_k^{1/2} \|\chi_{-1,k} u\|_{L^2(\mathbb{R}_+; rdr)}.$$

- As a result, we have proved for some $C_7 > 0$, for all $k \geq 1$, $j = 0, 1, -1$, for all $u \in C_0^\infty(\mathbb{R}_+)$,

$$\|\mathcal{L}_k \chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_7 \beta_k^{1/2} \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}. \quad (2.5.21)$$

Estimates for m_1^2 , m_2^2 . For the quantities m_1^2 , m_2^2 defined in (2.4.30), we have $m_1^2 \leq cr_k^{-2}$, $m_2^2 \leq cr_k^{-4}$, where $c > 0$ is a constant depending only on the functions θ_j defined in (2.4.27). It follows from (2.5.21) and the localization formula (2.4.31) that for any $u \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned} 3\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)}^2 + (3c + 16c^2)r_k^{-4}\|u\|_{L^2(\mathbb{R}_+; rdr)}^2 & \geq \sum_j C_7^2 \beta_k \|\chi_{j,k} u\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ & = C_7^2 \beta_k \|u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

Choose $\mu \geq \mu_0$ large enough such that $(3c + 16c^2)\mu^{-4} \leq \frac{1}{2}C_7^2$, then for $\alpha > \mu^4 \epsilon_0^{-4}$ and if $r_k \in [\mu \beta_k^{-1/4}, \epsilon_0]$,

$$(3c + 16c^2)r_k^{-4} \leq (3c + 16c^2)\mu^{-4}\beta_k \leq \frac{1}{2}C_7^2 \beta_k.$$

We get the estimate

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq \frac{1}{\sqrt{6}} C_7 \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

and this completes the proof of Theorem 2.5.1.

2.5.2 Case 4: crossing “at” 0

Now we deal with the last case when $r_k < \mu \beta_k^{-1/4}$, where μ is already chosen in Theorem 2.5.1. Here we use a direct method without metric. As in Section 2.5.1, suppose $\alpha > \mu^4 \epsilon_0^{-4}$ such that for any $k \geq 1$, $\mu \beta_k^{-1/4} < \epsilon_0$.

Lemma 2.5.11. *If $r_k < \mu \beta_k^{-1/4}$, then there exist $C_\mu > 0$ and $\alpha_\mu \geq 1$ such that for all $k \geq 1$, $\alpha > \alpha_\mu$, $u \in C_0^\infty(\mathbb{R}_+)$,*

$$\|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C_\mu \beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

where \mathcal{L}_k is given in (2.4.1) and β_k given in (2.3.1).

Proof. First recall the inequality (2.3.5) gives

$$\sigma(r_k) \geq 1 - \frac{1}{2}r_k^2 > 1 - \frac{1}{2}\mu^2\beta_k^{-1/2}, \quad (2.5.22)$$

since $r_k < \mu\beta_k^{-1/4} < \epsilon_0 < R_2$. We can prove an inequality similar to (2.3.7) by using the same method, for $\tilde{\mu} > 0$, $r > 0$, $k \geq 1$ and $\alpha \geq \tilde{\mu}^4$,

$$\tilde{\mu}^4 \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq c\tilde{\mu}^2\beta_k^{1/2}. \quad (2.5.23)$$

Indeed, by (2.3.5),

$$\begin{aligned} \text{if } r \leq R_2, \quad & \tilde{\mu}^4 \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq \tilde{\mu}^4 \frac{k^2}{r^2} + \beta_k \frac{c_0}{2}r^2 \geq \sqrt{2c_0}|k|\tilde{\mu}^2\beta_k^{1/2}, \\ \text{if } r > R_2, \quad & \tilde{\mu}^4 \frac{k^2}{r^2} + \beta_k(1 - \sigma(r)) \geq (1 - c_2)\beta_k \geq (1 - c_2)\tilde{\mu}^2\beta_k^{1/2}, \end{aligned}$$

so that we get (2.5.23) with $c = \min(\sqrt{2c_0}, 1 - c_2)$. Now using the multipliers $\tilde{\mu}^4\text{Id}$ and $i\text{Id}$, we get

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k u, \tilde{\mu}^4 u \rangle_{L^2(\mathbb{R}_+; rdr)} & \geq \langle \tilde{\mu}^4 \frac{k^2}{r^2} u, u \rangle_{L^2(\mathbb{R}_+; rdr)}, \\ \operatorname{Re}\langle \mathcal{L}_k u, -iu \rangle_{L^2(\mathbb{R}_+; rdr)} & = \langle \beta_k(\sigma(r_k) - \sigma(r))u, u \rangle_{L^2(\mathbb{R}_+; rdr)}. \end{aligned}$$

Adding together and using (2.5.22), (2.5.23), we obtain, if $\alpha \geq \tilde{\mu}^4$,

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k u, (\tilde{\mu}^4 - i)u \rangle_{L^2(\mathbb{R}_+; rdr)} & \geq \langle \left(\tilde{\mu}^4 \frac{k^2}{r^2} + \beta_k(\sigma(r_k) - \sigma(r)) \right) u, u \rangle_{L^2(\mathbb{R}_+; rdr)} \\ & \geq (c\tilde{\mu}^2 - \frac{1}{2}\mu^2)\beta_k^{1/2} \|u\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

We take $\tilde{\mu}^2 = c^{-1}\mu^2$, then $c\tilde{\mu}^2 - \frac{1}{2}\mu^2 = \frac{1}{2}\mu^2$. By Cauchy-Schwarz inequality we get Lemma 2.5.11 with $C_\mu = \frac{1}{4}c\mu^{-2}$ and $\alpha_\mu = \tilde{\mu}^4$. \square

2.5.3 End of the proof of Theorem 2.2.3

Summarizing the estimates in Lemma 2.3.2, 2.3.3, 2.3.4, Theorem 2.4.2, 2.4.14, 2.5.1 and Lemma 2.5.11, we have proved the following estimate for the operator $\mathcal{L}_k := \mathcal{L}_{\alpha, \lambda, k}$ defined by (2.3.2): there exist constants $C > 0$ and $\alpha_0 \geq 1$ such that for all $k \neq 0$, $\lambda \geq 0$, $\alpha \geq \alpha_0$ and for all $u \in L^2(\mathbb{R}_+; rdr)$,

$$\|\mathcal{L}_{\alpha, \lambda, k} u\|_{L^2(\mathbb{R}_+; rdr)} \geq C\beta_k^{1/3} \|u\|_{L^2(\mathbb{R}_+; rdr)} = C\alpha^{1/3} |k|^{1/3} \|u\|_{L^2(\mathbb{R}_+; rdr)}. \quad (2.5.24)$$

For $v \in \mathcal{D}(\mathcal{L}_\alpha) \subset \tilde{L}^2(\mathbb{R}^2)$, we write $v = \sum_{k \neq 0} u_k e^{ik\theta}$, with $u_k \in L^2(\mathbb{R}_+; rdr)$. For $\lambda \geq 0$, we have

$$(\mathcal{L}_\alpha - i\lambda)v = \sum_{k \neq 0} (\mathcal{L}_{\alpha, \lambda, k} u_k) e^{ik\theta}.$$

Therefore by (2.5.24) we have

$$\begin{aligned} \|(\mathcal{L}_\alpha - i\lambda)v\|_{L^2(\mathbb{R}^2)}^2 & = \sum_{k \neq 0} 2\pi \|\mathcal{L}_{\alpha, \lambda, k} u_k\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ & \geq \sum_{k \neq 0} 2\pi C^2 \alpha^{2/3} |k|^{2/3} \|u_k\|_{L^2(\mathbb{R}_+; rdr)}^2. \end{aligned}$$

On the other hand, we know that $|D_\theta|^{1/3}v = \sum_{k \neq 0} |k|^{1/3} u_k e^{ik\theta}$ for $v \in \tilde{L}^2(\mathbb{R}^2)$ and

$$\| |D_\theta|^{1/3} v \|_{L^2(\mathbb{R}^2)}^2 = \sum_{k \neq 0} 2\pi \| |k|^{1/3} u_k \|_{L^2(\mathbb{R}_+; rdr)}^2.$$

As a result we get for $v \in \mathcal{D}(\mathcal{L}_\alpha)$,

$$\| (\mathcal{L}_\alpha - i\lambda) v \|_{L^2(\mathbb{R}^2)} \geq C\alpha^{1/3} \| |D_\theta|^{1/3} v \|_{L^2(\mathbb{R}^2)}, \quad (2.5.25)$$

and in particular, since $|D_\theta|^{1/3} \geq 1$ on $\tilde{L}^2(\mathbb{R}^2)$, we have

$$\| (\mathcal{L}_\alpha - i\lambda) v \|_{L^2(\mathbb{R}^2)} \geq C\alpha^{1/3} \| v \|_{L^2(\mathbb{R}^2)}. \quad (2.5.26)$$

2.5.4 Optimality

It remains to prove the optimality of (2.2.6), i.e. (2.5.25).

Lemma 2.5.12. *The estimates (2.5.25) and (2.5.26) are optimal in the sense that we can find some function $v \in D(\mathcal{L}_\alpha)$ such that the converse inequalities hold as $\alpha \rightarrow \infty$.*

Proof. It suffices to prove the optimality of the estimate for the operator \mathcal{L}_k defined in (2.4.1) in the case $r_k > \epsilon_0^{-1}$ (Section 2.4.1), i.e.

$$\forall u \in C_0^\infty(\mathbb{R}_+), \quad \|\mathcal{L}_k u\|_{L^2(\mathbb{R}_+; rdr)} \geq C\beta_k^{1/3} \|u\|_{L^2(\mathbb{R}_+; rdr)},$$

more precisely, for any $k \geq 1$, find function $u_k \in L^2(\mathbb{R}_+; rdr)$ such that

$$\|\mathcal{L}_k u_k\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(\beta_k^{1/3}) \|u\|_{L^2(\mathbb{R}_+; rdr)} \text{ as } \alpha \rightarrow +\infty.$$

If this claim is true, then the function $v = u_k(r)e^{ik\theta}$ gives the optimality of (2.5.25) and (2.5.26). For $k \geq 1$ fixed, suppose $r_k^2 = \beta_k^{1/3}$. Take $u_0 \in C^\infty([-\frac{1}{4}, \frac{1}{4}])$ with $\|u_0\|_{L^2(\mathbb{R}; dt)} = 1$ and define

$$u_k(r) := r_k^{-1/2} \beta_k^{1/12} u_0(\beta_k^{1/6}(r - r_k)).$$

Then $u_k(r)$ has support in $\{r; |r - r_k| \leq \frac{1}{4}\beta_k^{-1/6}\} \subset [\frac{3}{4}r_k, \frac{5}{4}r_k]$. We have

$$\|u_k\|_{L^2(\mathbb{R}_+; rdr)}^2 = \int_0^{+\infty} \beta_k^{1/6} |u_0(\beta_k^{1/6}(r - r_k))|^2 \underbrace{r_k^{-1} r}_{\in [\frac{3}{4}, \frac{5}{4}]} dr \in [\frac{3}{4}, \frac{5}{4}].$$

We have also

$$\begin{aligned} -\partial_r^2 u_k &= -\beta_k^{1/3} \cdot r_k^{-1/2} \beta_k^{1/12} u_0''(\beta_k^{1/6}(r - r_k)), \\ -\frac{1}{r} \partial_r u_k &= -r^{-1} \beta_k^{1/6} \cdot r_k^{-1/2} \beta_k^{1/12} u_0'(\beta_k^{1/6}(r - r_k)), \end{aligned}$$

and in the support of u_k , we have, according to (2.4.5),

$$|\sigma(r) - \sigma(r_k)| \leq c_4 \left(\frac{3}{4}r_k\right)^{-3} |r - r_k| \leq \frac{16}{27} c_4 r_k^{-3} \beta_k^{-1/6},$$

so that, with $r_k = \beta_k^{1/6}$,

$$\| -\partial_r^2 u_k \|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(\beta_k^{1/3}), \quad \left\| \frac{1}{r} \partial_r u_k \right\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(r_k^{-1} \beta_k^{1/6}) = \mathcal{O}(1),$$

$$\left\| \frac{k^2}{r^2} u_k \right\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(r_k^{-2}) = \mathcal{O}(\beta_k^{-1/3}), \quad \|r^2 u_k\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(r_k^2) = \mathcal{O}(\beta_k^{1/3}),$$

$$\|i\beta_k(\sigma(r) - \sigma(r_k))u_k\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(\beta_k r_k^{-3} \beta_k^{-1/6}) = \mathcal{O}(\beta_k^{1/3}),$$

which implies $\|\mathcal{L}_k u_k\|_{L^2(\mathbb{R}_+; rdr)} = \mathcal{O}(\beta_k^{1/3})$ and completes the proof. \square

2.6 Appendix

2.6.1 Weyl calculus

We present some facts about the Weyl calculus, which can be found in [Hör85, Chap.18] as well as in [Ler10, Chap.2]. The Weyl quantization associates to a symbol a the operator a^w defined by

$$(a^w u)(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (2.6.1)$$

Consider the symplectic space \mathbb{R}^{2n} equipped with the symplectic form $\sigma = \sum_{i=1}^n d\xi^i \wedge dx^i$. Given a positive definite quadratic form g on \mathbb{R}^{2n} , we define

$$g^\sigma(T) = \sup_{g(Y)=1} \sigma(T, Y)^2,$$

which is also a positive quadratic form. We say that g is an admissible metric if there exist $C_0, \tilde{C}_0, \tilde{N}_0 > 0$ such that for all $X, Y \in \mathbb{R}^{2n}$,

$$\begin{cases} \text{uncertainty principle: } g_X \leq g_X^\sigma, \\ \text{slowness: } g_X(X - Y) \leq C_0^{-1} \implies (g_Y/g_X)^{\pm 1} \leq C_0, \\ \text{temperance: } g_X \leq \tilde{C}_0 g_Y (1 + g_X^\sigma(X - Y))^{\tilde{N}_0}. \end{cases} \quad (2.6.2)$$

An admissible weight is a positive function m on the phase space \mathbb{R}^{2n} , such that there exist $C'_0, \tilde{C}'_0, \tilde{N}'_0 > 0$ so that for all $X, Y \in \mathbb{R}^{2n}$,

$$\begin{cases} \text{slowness: } g_X(X - Y) \leq C'_0^{-1} \implies (m(Y)/m(X))^{\pm 1} \leq C'_0, \\ \text{temperance: } m(X) \leq \tilde{C}'_0 m(Y) (1 + g_X^\sigma(X - Y))^{\tilde{N}'_0}. \end{cases} \quad (2.6.3)$$

In particular, the function defined by

$$\lambda_g(X) := \inf_{T \in \mathbb{R}^{2n}, T \neq 0} (g_X^\sigma(T)/g_X(T))^{1/2} \quad (2.6.4)$$

is an admissible weight for g . The uncertainty principle is equivalent to $\lambda_g \geq 1$.

We prove the admissibility of a special type of metrics, including those we have used in the proof, given in Definition 2.4.5, 2.4.16, 2.5.3.

Lemma 2.6.1. *Suppose that $t_0, t_1 > 0$ are constants satisfying $t_0 t_1 \geq 1$. The following metric on $\mathbb{R}_t \times \mathbb{R}_\tau$ is admissible:*

$$\Gamma = \frac{dt^2}{t_0^2} + \frac{d\tau^2}{t_1^2 + \tau^2}. \quad (2.6.5)$$

Moreover, the structure constants C_0, \tilde{C}_0 and \tilde{N}_0 given in (2.6.2) are independent of t_0 and t_1 .

Proof of Lemma 2.6.1. First we notice that

$$\lambda_\Gamma = \inf_{T \neq 0} (\Gamma_X^\sigma(T)/\Gamma_X(T))^{1/2} = [t_0^2(t_1^2 + \tau^2)]^{1/2} \geq t_0 t_1 \geq 1,$$

so that Γ satisfies the uncertainty principle.

Slowness. it suffices to prove $\Gamma_Y \leq C_0 \Gamma_X$. For $X = (x, \xi)$, $Y = (y, \eta)$, $T = (t, \tau)$, if $\Gamma_X(X - Y) \leq s^2$ then $|\xi - \eta|^2 \leq s^2(t_1^2 + \xi^2)$, and we obtain

$$\xi^2 \leq 2(\xi - \eta)^2 + 2\eta^2 \leq 2s^2(t_1^2 + \xi^2) + 2\eta^2,$$

$$\text{thus } (1 - 2s^2)(t_1^2 + \xi^2) \leq 2(t_1^2 + \eta^2).$$

By choosing $0 < s < 1/\sqrt{2}$ and $C_0 = 2(1 - 2s^2)^{-1} > 1$, we get

$$t_1^2 + \xi^2 \leq C_0(t_1^2 + \eta^2).$$

$$\text{Then } \Gamma_Y(T) = \frac{t^2}{t_0^2} + \frac{\tau^2}{t_1^2 + \eta^2} \leq \frac{t^2}{t_0^2} + \frac{C_0\tau^2}{t_1^2 + \xi^2} \leq C_0\Gamma_X(T).$$

Temperance. We have

$$\Gamma_X^\sigma = (t_1^2 + \eta^2)dr^2 + t_0^2d\eta^2,$$

$$\frac{\Gamma_X(T)}{\Gamma_Y(T)} \leq \max\left\{1, \frac{t_1^2 + \eta^2}{t_1^2 + \xi^2}\right\}.$$

If $|\eta| \leq 2|\xi|$ or $|\eta| \leq t_1$, the right-hand side of the last inequality is bounded from above by 4. If $|\eta| > 2|\xi|$ and $|\eta| \geq t_1$, then $|\xi - \eta| \geq \frac{1}{2}|\eta|$, which implies that $\Gamma_X^\sigma(X - Y) \geq t_0^2(\xi - \eta)^2 \geq \frac{1}{4}t_0^2\eta^2$; on the other hand, we have

$$\frac{t_1^2 + \eta^2}{t_1^2 + \xi^2} \leq \frac{t_1^2 + \eta^2}{t_1^2} = 1 + t_1^{-2}\eta^2,$$

since $t_0t_1 \geq 1$, we have

$$\frac{\Gamma_X(T)}{\Gamma_Y(T)} \leq 1 + 4\Gamma_X^\sigma(X - Y).$$

So the inequality $\Gamma_X(T)/\Gamma_Y(T) \leq 4(1 + \Gamma_X^\sigma(X - Y))$ holds for any X, Y, T . As a result, we have proved that Γ is admissible. From the proof above, we see that the constants C_0 , \tilde{C}_0 , \tilde{N}_0 given in (2.6.2) are independent of t_0 and t_1 , and this ends the proof of Lemma 2.6.1. \square

Now we introduce some spaces of symbols. We say that a is a symbol in $S(m, g)$ if $a \in C^\infty(\mathbb{R}^{2n})$ and the following semi-norms for all $k \in \mathbb{N}$

$$\sup_{g_X(T_j) \leq 1} |a^{(k)}(X)(T_1, \dots, T_k)|m(X)^{-1} < +\infty. \quad (2.6.6)$$

The following lemma is used in the proof of Proposition 2.4.10, 2.4.21 and 2.5.8.

Lemma 2.6.2. *Suppose $\phi \in C_0^\infty(\mathbb{R})$ and $t_0 > 0$. Then the symbol $\phi(\frac{t}{t_0})$ is in $S(1, \Gamma)$ with semi-norms depending only on ϕ , where Γ is defined in (2.6.5).*

Proof. For $n \geq 0$, we have

$$\left| \frac{d^n}{dt^n} \left(\phi\left(\frac{t}{t_0}\right) \right) \right| = \left| \phi^{(n)}\left(\frac{t}{t_0}\right) t_0^{-n} \right| \leq \|\phi^{(n)}\|_{L^\infty} t_0^{-n},$$

which implies the Lemma. \square

We define the composition law \sharp by $a^w b^w = (a \sharp b)^w$, and we have

$$(a \sharp b)(X) = \exp\left(\frac{i}{2}\sigma(D_X, D_Y)\right)a(X)b(Y)|_{Y=X}. \quad (2.6.7)$$

For $a \in S(m_1, g)$, $b \in S(m_2, g)$, we have the asymptotic expansion

$$(a \sharp b)(x, \xi) = \sum_{0 \leq k < N} w_k(a, b) + r_N(a, b), \quad (2.6.8)$$

$$\text{with } w_k(a, b) = 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha! \beta!} D_\xi^\alpha \partial_x^\beta a D_\xi^\beta \partial_x^\alpha b \in S(m_1 m_2 \lambda_g^{-k}, g), \quad (2.6.9)$$

$$r_N(a, b)(X) = R_N(a(X) \otimes b(Y))|_{X=Y} \in S(m_1 m_2 \lambda_g^{-N}, g), \quad (2.6.10)$$

$$R_N = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} \exp \frac{\theta}{2i} [\partial_X, \partial_Y] d\theta \left(\frac{1}{2i} [\partial_X, \partial_Y] \right)^N. \quad (2.6.11)$$

We use here the notation $D = i^{-1}\partial$. The $w_k(a, b)$ with k even are symmetric in a, b and skew-symmetric for k odd. In particular, we have

$$a \sharp b - b \sharp a = \frac{1}{i} \{a, b\} + \tilde{r}, \quad \tilde{r} \in S(m_1 m_2 \lambda_g^{-3}, g), \quad (2.6.12)$$

where $\{ , \}$ is the Poisson bracket, implying that $[a^w, b^w] = \frac{1}{i} \{a, b\}^w + \tilde{r}^w$.

The symbols in $S(1, g)$ are quantified in bounded operators on $L^2(\mathbb{R}^n)$, with operator norm depending on the structure constants $C_0, \tilde{C}_0, \tilde{N}_0$ given in (2.6.2) and a semi-norm (2.6.6) of the symbol in $S(1, g)$, whose order depends only on the dimension and the structure constants $C_0, \tilde{C}_0, \tilde{N}_0$ in (2.6.2).

Next let us recall the Fefferman-Phong inequality which is used in the proof of Lemma 2.4.11, 2.4.22 and 2.5.9.

Proposition 2.6.3 (Fefferman-Phong inequality). *If $a \in S(\lambda_g^2, g)$ and $a \geq 0$, then a^w is bounded from below by a constant depending on $C_0, \tilde{C}_0, \tilde{N}_0$ given in (2.6.2) and a semi-norm (2.6.6) of the symbol a in $S(\lambda_g^2, g)$, whose order depends only on the dimension and the structure constants $C_0, \tilde{C}_0, \tilde{N}_0$.*

We use the ellipticity to give a second proof of Lemma 2.4.11. Similar proofs can be found for Lemma 2.4.22 and 2.5.9.

Another proof of Lemma 2.4.11. Recall that a is defined by (2.4.23). We prove first that a is bounded below by $Cr_k^{-2}\lambda_g^2$.

$$\begin{aligned} \text{For } |\tau| \leq \beta_k^{1/3} r_k^{-1}, \quad a(t, \tau) &\geq \min(c_5, c_6) \beta_k^{2/3} r_k^{-2} \geq \frac{1}{2} \min(c_5, c_6) (\beta_k^{2/3} r_k^{-2} + \tau^2), \\ \text{for } |\tau| > \beta_k^{1/3} r_k^{-1}, \quad a(t, \tau) &\geq 4M\tau^2 \geq 2M(\beta_k^{2/3} r_k^{-2} + \tau^2). \end{aligned}$$

On the other hand $a \in S(r_k^{-2}\lambda_g^2, g)$, we get

$$\forall t \in \mathbb{R}, \forall \tau \in \mathbb{R}, \quad Cr_k^{-2}\lambda_g^2 \leq a(t, \tau) \leq C'r_k^{-2}\lambda_g^2.$$

Let us denote

$$\tilde{a} := a - \frac{1}{2}Cr_k^{-2}\lambda_g^2,$$

then $\frac{1}{2}Cr_k^{-2}\lambda_g^2 \leq \tilde{a} \in S(r_k^{-2}\lambda_g^2, g)$ (with semi-norms bounded by constants independent of k). We apply the Faà di Bruno formula

$$\frac{(g \circ f)^{(n)}}{n!} = \sum_{1 \leq m \leq n} \frac{g^{(m)} \circ f}{m!} \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{f^{(j_l)}}{j_l!}$$

with $f = \tilde{a}$ and $g(x) = x^{1/2}$,

$$\begin{aligned} |(\tilde{a}^{1/2})^{(n)}(X)T^n| &= \left| \sum_{1 \leq m \leq n} c_m \tilde{a}^{\frac{1}{2}-m}(X) \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} \tilde{a}^{(j_l)}(X) T^{j_l} \right| \\ &\leq \sum_{1 \leq m \leq n} c_m |\tilde{a}^{\frac{1}{2}-m}(X)| \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} r_k^{-2} \lambda_g^2(X) g_X(T)^{j_l/2} \\ &\leq \sum_{1 \leq m \leq n} c'_m (r_k^{-2} \lambda_g^2)^{\frac{1}{2}-m}(X) \cdot (r_k^{-2} \lambda_g^2)^m(X) g_X(T)^{n/2} \\ &\leq c''_n r_k^{-1} \lambda_g(X) g_X(T)^{n/2}, \end{aligned}$$

which implies that $\tilde{a}^{1/2} \in S(r_k^{-1} \lambda_g, g)$ (and each semi-norm can be bounded by a constant independent of k). The composition formula tells us that

$$\tilde{a}^{1/2} \# \tilde{a}^{1/2} = \tilde{a} + S(r_k^{-2}, g).$$

Then $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2} \in S(r_k^{-2}, g)$ and

$$\begin{aligned} \langle \tilde{a}^w u, u \rangle_{L^2(\mathbb{R}; dt)} &= \langle (\tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} + \langle (\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &\geq \|(\tilde{a}^{1/2})^w u\|_{L^2(\mathbb{R}; dt)}^2 - C'' r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\geq -C'' r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2, \end{aligned}$$

where $C'' > 0$ depends on a semi-norm of $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2}$ in $S(r_k^{-2}, g)$ and we can suppose C'' independent of k , see the footnote in page 50. We get the estimate for a^w :

$$a^w = \frac{C}{2} (r_k^{-2} \lambda_g^2)^w + \tilde{a}^w \geq \frac{C}{2} (r_k^{-2} \lambda_g^2)^w - C'' r_k^{-2}.$$

Remark that λ_g is a function depending only on the variable τ , so $(\lambda_g^2)^w = \lambda_g^2(D_t)$ is a Fourier multiplier. We deduce from $\lambda_g^2 \geq \beta_k^{2/3}$ that $(\lambda_g^2)^w \geq \beta_k^{2/3}$ and

$$a^w \geq \frac{C}{2} \beta_k^{2/3} r_k^{-2} - C'' r_k^{-2}.$$

The proof of Lemma 2.4.11 is completed. \square

2.6.2 A closed operator

We present in this appendix some properties of the operator \mathcal{L}_α which is defined in (2.2.1). First the operator \mathcal{L}_α is unitarily equivalent to the original operator \tilde{H}_α given in (2.1.6), up to some constants. Indeed, the Weyl symbols corresponding to the operators \mathcal{L}_α , \tilde{H}_α are given by

$$\begin{aligned} p_\alpha(x, \xi) &= |\xi|^2 + |x|^2 + \frac{\alpha}{|x|^2} (1 - e^{-|x|^2}) (x_1 \xi_2 - x_2 \xi_1), \\ \tilde{p}_\alpha(x, \xi) &= |\xi|^2 + \frac{|x|^2}{16} - \frac{1}{2} + \frac{\alpha}{2\pi|x|^2} (1 - e^{-|x|^2/4}) (x_1 \xi_2 - x_2 \xi_1). \end{aligned}$$

Let κ be a symplectic transform given by $\kappa(x, \xi) = (2x, \frac{1}{2}\xi)$, then we have

$$4(\tilde{p}_{2\pi\alpha} \circ \kappa)(x, \xi) + 2 = p_\alpha(x, \xi),$$

which implies the identity $4\tilde{H}_{2\pi\alpha} + 2 = M^* \mathcal{L}_\alpha M$, where M is the unitary operator on $L^2(\mathbb{R}^2)$ defined by $Mu(x) = 2u(2x)$.

Now we prove that the operator \mathcal{L}_α with domain $D(\mathcal{L}_\alpha)$ given in Definition 2.2.1 is closed.

Lemma 2.6.4. *The operator \mathcal{L}_α given in (2.2.1) is closed with domain $D(\mathcal{L}_\alpha)$ which is dense in $\tilde{L}^2(\mathbb{R}^2)$. Its spectrum is contained in the half plane $\{z \in \mathbb{C}; \operatorname{Re}(z) \geq 2\}$.*

Proof. Remark $\mathcal{L}_\alpha v \in \tilde{L}^2(\mathbb{R}^2)$ for $v \in \tilde{L}^2(\mathbb{R}^2)$. We first prove $D(\mathcal{L}_\alpha) = D$ where

$$D := \{v \in \tilde{L}^2(\mathbb{R}^2); \mathcal{L}_\alpha v \in L^2(\mathbb{R}^2)\}.$$

We have clearly the inclusion $D(\mathcal{L}_\alpha) \subset D$. It suffices to prove that $v \in H^2(\mathbb{R}^2)$ and $|x|^2 v \in L^2(\mathbb{R}^2)$ for $v \in D$. Firstly we have $v \in H^1(\mathbb{R}^2)$ and $|x|v \in L^2(\mathbb{R}^2)$. Indeed,

$$\begin{aligned} \|v\|_{L^2(\mathbb{R}^2)}^2 + \||x|v\|_{L^2(\mathbb{R}^2)}^2 &= \langle (-\Delta + |x|^2)v, v \rangle_{L^2(\mathbb{R}^2)} \\ &= \operatorname{Re} \langle \mathcal{L}_\alpha v, v \rangle_{L^2(\mathbb{R}^2)} \leq \|\mathcal{L}_\alpha v\|_{L^2(\mathbb{R}^2)}^2 + \|v\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

Next we compute $2\operatorname{Re} \langle (-\Delta + |x|^2)v, \alpha\sigma(|x|)\partial_\theta v \rangle_{L^2(\mathbb{R}^2)}$ for $v \in C_0^\infty(\mathbb{R}^2)$. Denote $f(x) = \sigma(|x|)$, since $-\Delta + |x|^2$ is self-adjoint and $\alpha f(x)\partial_\theta$ is skew-adjoint, we have

$$\begin{aligned} 2\operatorname{Re} \langle (-\Delta + |x|^2)v, \alpha f(x)\partial_\theta v \rangle_{L^2(\mathbb{R}^2)} &= \langle [-\Delta + |x|^2, \alpha f(x)\partial_\theta]v, v \rangle_{L^2(\mathbb{R}^2)} \\ &= -\alpha \left(\langle (\Delta f(x))\partial_\theta v, v \rangle_{L^2(\mathbb{R}^2)} + \langle 2\nabla f \cdot \nabla(\partial_\theta v), v \rangle_{L^2(\mathbb{R}^2)} \right). \end{aligned}$$

For the first term in the parentheses, we have the following estimate

$$\begin{aligned} |\langle (\Delta f)\partial_\theta v, v \rangle_{L^2(\mathbb{R}^2)}| &= |\langle (\Delta f)(x_1\partial_{x_2} - x_2\partial_{x_1})v, v \rangle_{L^2(\mathbb{R}^2)}| \\ &= |\langle \partial_{x_2}v, x_1(\Delta f)v \rangle_{L^2(\mathbb{R}^2)} - \langle \partial_{x_1}v, x_2(\Delta f)v \rangle_{L^2(\mathbb{R}^2)}| \\ &\leq 2\|\Delta f\|_{L^\infty}\|v\|_{H^1(\mathbb{R}^2)}\||x|v\|_{L^2(\mathbb{R}^2)}. \end{aligned} \quad (2.6.13)$$

For the second, we have

$$\begin{aligned} \langle 2\nabla f \cdot \nabla(\partial_\theta v), v \rangle_{L^2(\mathbb{R}^2)} &= 2\langle \nabla(\partial_\theta v), \nabla f \cdot v \rangle_{L^2(\mathbb{R}^2)} = -2\langle \partial_\theta v, \nabla \cdot (\nabla f v) \rangle_{L^2(\mathbb{R}^2)} \\ &= -2\langle \partial_\theta v, (\Delta f)v \rangle_{L^2(\mathbb{R}^2)} - 2\langle \partial_\theta v, \nabla f \cdot \nabla v \rangle_{L^2(\mathbb{R}^2)}, \end{aligned} \quad (2.6.14)$$

since $\nabla f(x) = \sigma'(x)|x|^{-1}x$, we have $|x|\nabla f \in L^\infty$, then

$$\begin{aligned} |\langle \partial_\theta v, \nabla f \cdot \nabla v \rangle_{L^2(\mathbb{R}^2)}| &= |\langle \partial_{x_2}v, x_1\nabla f \cdot \nabla v \rangle_{L^2(\mathbb{R}^2)} - \langle \partial_{x_1}v, x_2\nabla f \cdot \nabla v \rangle_{L^2(\mathbb{R}^2)}| \\ &\leq 2\||x|\nabla f\|_{L^\infty}\|v\|_{H^1(\mathbb{R}^2)}^2. \end{aligned} \quad (2.6.15)$$

It follows from (2.6.13), (2.6.14) and (2.6.15) that

$$\begin{aligned} |2\operatorname{Re} \langle (-\Delta + |x|^2)v, \alpha\sigma(|x|)\partial_\theta v \rangle_{L^2(\mathbb{R}^2)}| &\leq 4\|\Delta f\|_{L^\infty}\|v\|_{H^1(\mathbb{R}^2)}\||x|v\|_{L^2(\mathbb{R}^2)} + 2\||x|\nabla f\|_{L^\infty}\|v\|_{H^1(\mathbb{R}^2)}^2. \end{aligned} \quad (2.6.16)$$

Since $C_0^\infty(\mathbb{R}^2) \cap \tilde{L}^2(\mathbb{R}^2)$ is dense in $D(\mathcal{L}_\alpha)$, the inequality (2.6.16) holds for all $v \in D(\mathcal{L}_\alpha)$. On the other hand, we have

$$\begin{aligned}\|\mathcal{L}_\alpha v\|_{L^2(\mathbb{R}^2)}^2 &= \|(-\Delta + |x|^2)v\|_{L^2(\mathbb{R}^2)}^2 + \|\alpha\sigma(|x|)\partial_\theta v\|_{L^2(\mathbb{R}^2)}^2 \\ &\quad + 2\operatorname{Re}\langle (-\Delta + |x|^2)v, \alpha\sigma(|x|)\partial_\theta v \rangle_{L^2(\mathbb{R}^2)},\end{aligned}$$

so that $(-\Delta + |x|^2)v \in L^2(\mathbb{R}^2)$ and $\alpha\sigma(|v|)\partial_\theta v \in L^2(\mathbb{R}^2)$. This gives $v \in H^2(\mathbb{R}^2)$ and $|x|^2v \in L^2(\mathbb{R}^2)$ for all $v \in D(\mathcal{L}_\alpha)$.

Now consider a sequence $(v_n)_{n \in \mathbb{N}} \subset D(\mathcal{L}_\alpha)$, $v, w \in \tilde{L}^2(\mathbb{R}^2)$ such that

$$v_n \rightarrow v \text{ in } L^2(\mathbb{R}^2) \quad \text{and} \quad \mathcal{L}_\alpha v_n \rightarrow w \text{ in } L^2(\mathbb{R}^2), \quad \text{as } n \rightarrow \infty.$$

This implies that

$$v_n \rightarrow v \text{ in } \mathcal{D}'(\mathbb{R}^2) \quad \text{and} \quad \mathcal{L}_\alpha v_n \rightarrow w \text{ in } \mathcal{D}'(\mathbb{R}^2), \quad \text{as } n \rightarrow \infty.$$

We have convergence $\mathcal{L}_\alpha v_n \rightarrow \mathcal{L}_\alpha v$ in $\mathcal{D}'(\mathbb{R}^2)$ as $n \rightarrow \infty$ since \mathcal{L} is a differential operator of order 2 with smooth coefficients. Thus $w = \mathcal{L}_\alpha v$, implying $v \in H^2(\mathbb{R}^2)$ and $|x|^2v \in L^2(\mathbb{R}^2)$. This proves that $v \in D(\mathcal{L}_\alpha)$ and that \mathcal{L}_α is closed.

By classical perturbation theory [Kat95, Chap.V], \mathcal{L}_α has compact resolvent for any α , and its spectrum is a sequence of eigenvalues. The numerical range

$$\Theta(\mathcal{L}_\alpha) := \{\langle \mathcal{L}_\alpha v, v \rangle_{L^2(\mathbb{R}^2; dx)} \in \mathbb{C}; v \in D(\mathcal{L}_\alpha), \|v\|_{L^2(\mathbb{R}^2; dx)} = 1\} \quad (2.6.17)$$

is contained in the half plane $\{z \in \mathbb{C}; \operatorname{Re}(z) \geq 2\}$, since

$$\operatorname{Re}\langle \mathcal{L}_\alpha v, v \rangle_{L^2(\mathbb{R}^2; dx)} = \langle (-\Delta + |x|^2)v, v \rangle_{L^2(\mathbb{R}^2; dx)} \geq 2\|v\|_{L^2(\mathbb{R}^2; dx)}^2.$$

Then the spectrum of \mathcal{L}_α is contained in $\Theta(\mathcal{L}_\alpha)$ and this ends the proof of Lemma 2.6.4. \square

Finally we prove an argument that is used to give the equivalence of the estimates (2.2.4) and (2.2.5).

Lemma 2.6.5. *Let P be an (unbounded) operator with domain $D(P)$ on a Hilbert space \mathcal{H} . Suppose that z is not in the spectrum of P and $\delta > 0$. The following are equivalent,*

- i) for all $v \in D(P)$, $\|(P - z)v\| \geq \delta\|v\|$,
- ii) $\|(P - z)^{-1}\| \leq \delta^{-1}$.

Proof. Clearly ii) \implies i). Assume now that i) holds. We know that $(P - z)^{-1} \in \mathcal{L}(\mathcal{H})$ since z is not in the spectrum of P . For $u \in \mathcal{H}$, let $v = (P - z)^{-1}u$, then $v \in D(P)$ and i) gives the inequality $\|u\| \geq \delta\|(P - z)^{-1}u\|$, implying $\|(P - z)^{-1}\| \leq \delta^{-1}$ and completing the proof. \square

Remark 2.6.6. Without the assumption that z is not in the spectrum of P , i) \implies ii) is possibly wrong. For example, let P be the right shift operator on $l^2(\mathbb{N})$, i.e. for $v = (v_1, v_2, \dots)$, $Pv = (0, v_1, v_2, \dots)$. P verifies the condition i) at $z = 0$ since $\|Pv\| = \|v\|$ for any $v \in l^2(\mathbb{N})$, but it is not invertible since it is not onto.

2.7 Letter of acceptance

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Paper Title: Resolvent estimates for a two-dimensional non-self-adjoint operator

Paper Author: Deng, Wen

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Chapter 3

Pseudospectrum for Oseen vortices operators

The result of this chapter is taken from the article [Den11b], *Pseudospectrum for Oseen vortices operators*, which has been accepted for publication in the journal INTERNATIONAL MATHEMATICS RESEARCH NOTICES, see Section 3.5 for the letter of acceptance.

We give resolvent estimates for the complete linearized operator of the Navier-Stokes equation in \mathbb{R}^2 around the Oseen vortices, in the fast rotating limit $\alpha \rightarrow +\infty$.

3.1 Introduction

3.1.1 The origin of the problem

Consider the motion of a viscous incompressible fluid in the whole plane, which is described by the Navier-Stokes equation in \mathbb{R}^2 . In two dimensions where the vorticity is a scalar, it is more convenient to study the evolution of the vorticity which is given by

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \nu \Delta \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (3.1.1)$$

where ν is the kinematic viscosity, $\omega(x, t) \in \mathbb{R}$ is the vorticity of the fluid, $v(x, t) \in \mathbb{R}^2$ is the divergence-free velocity field reconstructed from ω by the Biot-Savart law

$$v(x, t) = (K_{BS} * \omega)(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(y, t) dy, \quad (3.1.2)$$

where we denote $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The equation (3.1.1) is globally well-posed in $L^1(\mathbb{R}^2)$ ([BA94], [Kat94]), i.e. for any initial data $\omega_0 \in L^1(\mathbb{R}^2)$, (3.1.1) has a unique global solution $\omega \in C^0([0, +\infty); L^1(\mathbb{R}^2))$ such that $\omega(0) = \omega_0$. The *total circulation* of the velocity field

$$\int_{\mathbb{R}^2} \omega(x, t) dx = \lim_{R \rightarrow +\infty} \oint_{|x|=R} v(x, t) \cdot dl \quad (3.1.3)$$

is a quantity conserved by the semi-flow defined by (3.1.1) in $L^1(\mathbb{R}^2)$. It is well-known that the equation (3.1.1) has a family of explicit self-similar solutions, called *Oseen vortices*, which is given by

$$\omega(x, t) = \frac{\alpha}{\nu t} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad v(x, t) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right), \quad (3.1.4)$$

where

$$G(x) = \frac{1}{4\pi} e^{-|x|^2/4}, \quad v^G(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-|x|^2/4}), \quad x \in \mathbb{R}^2, \quad (3.1.5)$$

and the parameter $\alpha \in \mathbb{R}$ is referred to as the *circulation Reynolds number*. In fact these solutions are trivial in the sense that $v(x, t) \cdot \nabla \omega(x, t) \equiv 0$ so that (3.1.1) reduces to the linear heat equation, and the Oseen vortices are the only self-similar solutions to the Navier-Stokes equations in \mathbb{R}^2 whose vorticity is integrable. Moreover, it is proved by T. Gallay and C.E. Wayne in [GW05] that if the initial vorticity ω_0 is in $L^1(\mathbb{R}^2)$, then the solution $\omega(x, t)$ of (3.1.1) satisfies

$$\lim_{t \rightarrow +\infty} \|\omega(\cdot, t) - \frac{\alpha}{\nu t} G\left(\frac{\cdot}{\sqrt{\nu t}}\right)\|_{L^1(\mathbb{R}^2)} = 0, \quad (3.1.6)$$

where $\alpha = \int_{\mathbb{R}^2} \omega_0(x) dx$. In physical terms, this means that the Oseen vortices are globally stable for any value of the circulation Reynolds number α . In contrast to many situations in hydrodynamics, such as the Poiseuille or the Taylor-Couette flows, increasing the Reynolds number does not produce any instability.

In order to investigate the stability of the Oseen vortices, we introduce the self-similar variables $\tilde{x} = x/\sqrt{\nu t}$, $\tilde{t} = \log(t/T)$ and we set

$$\omega(x, t) = \frac{1}{t} \tilde{\omega}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right), \quad v(x, t) = \sqrt{\frac{\nu}{t}} \tilde{v}\left(\frac{x}{\sqrt{\nu t}}, \log \frac{t}{T}\right).$$

Then the rescaled system reads (replacing \tilde{x} by x , $\tilde{\omega}$ by ω and so on)

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \Delta \omega + \frac{1}{2} x \cdot \nabla \omega + \omega, \quad x \in \mathbb{R}^2, \quad t \geq 0, \quad (3.1.7)$$

where $\omega(x, t) \in \mathbb{R}$ is the rescaled vorticity, $v(x, t) \in \mathbb{R}^2$ is the rescaled velocity field again given by the Biot-Savart law (3.1.2). Then for any $\alpha \in \mathbb{R}$, the Oseen vortex $\omega = \alpha G$ is a stationary solution of (3.1.7). Linearizing the equation (3.1.7) at αG , we get a linear evolution equation

$$\frac{\partial \omega}{\partial t} = -(\mathcal{L} + \alpha \Lambda) \omega,$$

where

$$\mathcal{L}\omega = -\Delta \omega - \frac{1}{2} x \cdot \nabla \omega - \omega, \quad \Lambda \omega = v^G \cdot \nabla \omega + (K_{BS} * \omega) \cdot \nabla G. \quad (3.1.8)$$

It turns out that the operator \mathcal{L} is self-adjoint, non-negative on the weighted space $L^2(\mathbb{R}^2; G^{-1} dx)$ and Λ is a relatively compact perturbation of \mathcal{L} , which is the sum of two skew-adjoint operators on $L^2(\mathbb{R}^2; G^{-1} dx)$. The spectrum of $\mathcal{L} + \alpha \Lambda$ is a sequence of eigenvalues by classical perturbation theory ([Kat95]). Introducing the following subspaces of $Y = L^2(\mathbb{R}^2; G^{-1} dx)$:

$$\begin{aligned} Y_0 &= \{\omega \in Y; \int_{\mathbb{R}^2} \omega(x) dx = 0\} = \{G\}^\perp, \\ Y_1 &= \{\omega \in Y_0; \int_{\mathbb{R}^2} x_j \omega(x) dx = 0 \text{ for } j = 1, 2\} = \{G; \partial_1 G; \partial_2 G\}^\perp, \\ Y_2 &= \{\omega \in Y_1; \int_{\mathbb{R}^2} |x|^2 \omega(x) dx = 0\} = \{G; \partial_1 G; \partial_2 G; \Delta G\}^\perp, \end{aligned}$$

which are invariant spaces for \mathcal{L} and Λ , the following spectral bounds for $\mathcal{L} + \alpha\Lambda$ are proved in [GW05],

$$\begin{aligned}\text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \operatorname{Re}(z) \geq 0\} \quad \text{in } Y, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \operatorname{Re}(z) \geq \frac{1}{2}\} \quad \text{in } Y_0, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \operatorname{Re}(z) \geq 1\} \quad \text{in } Y_1, \\ \text{Spec}(\mathcal{L} + \alpha\Lambda) &\subset \{z \in \mathbb{C}; \operatorname{Re}(z) > 1\} \quad \text{in } Y_2, \text{ if } \alpha \neq 0.\end{aligned}$$

These spectral bounds allow us to obtain estimates on the semigroup associated to $\mathcal{L} + \alpha\Lambda$, which can be used to show that Oseen vortex αG is a stable stationary solution of (3.1.7) for any $\alpha \in \mathbb{R}$. However, these bounds are not precise. The eigenvalues that do not move are those which correspond to eigenvectors in the kernel of Λ . All eigenvalues of $\mathcal{L} + \alpha\Lambda$ which correspond to eigenvectors in the orthogonal complement of $\ker(\Lambda)$, have a real part that goes to $+\infty$ as $|\alpha| \rightarrow \infty$, observed numerically by A. Prochazka and D. Pullin [PP95] and recently proved by Y. Maekawa [Mae11].

In this paper, we are interested in pseudospectral properties of this linearized operator. We conjugate the linear operators \mathcal{L} and Λ with $G^{1/2}$, then we obtain two operators on $L^2(\mathbb{R}^2; dx)$

$$L\omega = G^{-1/2}\mathcal{L}G^{1/2}\omega = -\Delta\omega + \frac{|x|^2}{16}\omega - \frac{1}{2}\omega, \quad (3.1.9)$$

$$M\omega = G^{-1/2}\Lambda G^{1/2}\omega = v^G \cdot \nabla\omega - \frac{1}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)). \quad (3.1.10)$$

Up to some numerical constants, L is the two-dimensional harmonic oscillator, which is self-adjoint and non-negative on $L^2(\mathbb{R}^2; dx)$. On the other hand, both terms in M are separately skew-adjoint on $L^2(\mathbb{R}^2; dx)$. Letting

$$\begin{aligned}\mathcal{H}_\alpha\omega &= L\omega + \alpha M\omega, \quad \omega \in L^2(\mathbb{R}^2; dx) \\ &= (-\Delta\omega + \frac{|x|^2}{16}\omega - \frac{1}{2}\omega) + \alpha \left[v^G \cdot \nabla\omega - \frac{1}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)) \right],\end{aligned} \quad (3.1.11)$$

our aim is to give estimates for the resolvent of the non-self-adjoint operator \mathcal{H}_α along the imaginary axis, in the fast rotating limit $\alpha \rightarrow +\infty$.

3.1.2 About non-self-adjoint operators

In many problems originated from mathematical physics, one encounters a linear evolution equation with a non-self-adjoint generator, of the form $H = A + iB$, where A is self-adjoint, non-negative and iB is skew-adjoint such that A, B do not commute. A is usually called the dissipative term and iB the conservative term. The conservative term can affect and sometimes enhance the dissipative effects or the regularizing properties of the whole system. When a large skew-adjoint term iB is present, the spectrum and the pseudospectrum of the whole operator H may be strongly stabilized. In particular, the norm of the resolvent $\|(H - z)^{-1}\|$ may tend to 0 quickly.

In the paper [GGN09], a one-dimensional analogue of \mathcal{H}_α is studied by I. Gallagher, T. Gallay and F. Nier

$$H_\epsilon = -\partial_x^2 + x^2 + \frac{i}{\epsilon}f(x), \quad x \in \mathbb{R}, \quad (3.1.12)$$

where $\epsilon > 0$ is a small parameter, $f: \mathbb{R} \rightarrow \mathbb{R}$ is a bounded smooth function. Here the limit $\epsilon \rightarrow 0$ corresponds to the fast rotating limit $\alpha \rightarrow +\infty$. They studied the asymptotics of

two quantities related to the spectral and pseudospectral properties in the limit $\epsilon \rightarrow 0$. More precisely, they define $\Sigma(\epsilon)$ as the infimum of the real part of the spectrum of H_ϵ and

$$\Psi(\epsilon)^{-1} = \sup_{\lambda \in \mathbb{R}} \|(H_\epsilon - i\lambda)^{-1}\|$$

as the supremum of the norm of the resolvent of H_ϵ along the imaginary axis. Under some appropriate conditions on f , both quantities $\Sigma(\epsilon)$, $\Psi(\epsilon)$ tend to infinity as $\epsilon \rightarrow 0$ and lower bounds are given by using the so-called hypocoercive method. Furthermore, they focused on Morse functions of $C^3(\mathbb{R}; \mathbb{R})$ which are bounded together with their derivatives up to the third order, and which behave like $|x|^{-k}$ as $|x| \rightarrow \infty$ (Hypothesis 1.6 in [GGN09]). For functions verifying these hypotheses, some precise and optimal estimates on $\Psi(\epsilon)$ are proved (Theorem 1.8 in [GGN09]): there exists $M \geq 1$ such that for any $\epsilon \in (0, 1]$,

$$\frac{1}{M\epsilon^\nu} \leq \Psi(\epsilon) \leq \frac{M}{\epsilon^\nu}, \quad \text{with } \nu = \frac{2}{k+4}.$$

Their proof is based on the localization techniques and some semiclassical subelliptic estimates.

In our recent work [Den10a], a two-dimensional non-self-adjoint operator is considered

$$\mathcal{L}_\alpha = -\Delta + |x|^2 + \alpha\sigma(|x|)\partial_\theta, \quad x \in \mathbb{R}^2, \quad (3.1.13)$$

where $\sigma(r) = r^{-2}(1 - e^{-r^2})$, $\partial_\theta = x_1\partial_2 - x_2\partial_1$ and α is a positive parameter tending to infinity. Note that up to some numerical constants, the differential operator \mathcal{L}_α is equal to the operator \mathcal{H}_α given in (3.1.11), by neglecting the second member in the skew-adjoint part αM , which is a non-local, lower-order term. In that paper, we gave a complete study of the resolvent of \mathcal{L}_α along the imaginary axis in the limit $\alpha \rightarrow +\infty$ and proved an estimate of type (Theorem 2.2 in [Den10a])

$$\sup_{\lambda \in \mathbb{R}} \|(\mathcal{L}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(\tilde{L}(\mathbb{R}^2))} \leq C\alpha^{-1/3}, \quad (3.1.14)$$

$$\text{where } \tilde{L}^2(\mathbb{R}^2) = \{\omega \in L^2(\mathbb{R}^2; dx); \omega(r \cos \theta, r \sin \theta) = \sum_{k \neq 0} \omega_k(r) e^{ik\theta}\}.$$

(3.1.14) is also optimal. The result is established by using a multiplier method, metrics on the phase space and localization techniques.

The present paper is devoted to proving resolvent estimates similar to (3.1.14) for the whole linearized operator \mathcal{H}_α in (3.1.11).

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3.2 Statement of the result

3.2.1 The theorem

Using the notations in Section 3.1.1, we consider the operator on $L^2(\mathbb{R}^2; dx)$

$$\begin{aligned} \mathcal{H}_\alpha \omega = & \underbrace{-\Delta \omega + \frac{|x|^2}{16}\omega - \frac{1}{2}\omega}_{\text{self-adjoint and non-negative on } L^2(\mathbb{R}^2)} \\ & + \underbrace{\alpha v^G \cdot \nabla \omega - \frac{\alpha}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega))}_{\text{skew-adjoint on } L^2(\mathbb{R}^2)}, \end{aligned} \quad (3.2.1)$$

where G , v^G are given by (3.1.5), K_{BS} is given in (3.1.2) and $\alpha \geq 1$ is a large parameter. The real part of \mathcal{H}_α is the two-dimensional harmonic oscillator and the imaginary part of \mathcal{H}_α is the sum of a divergence-free vector field and a non-local integral operator, multiplied by the circulation Reynolds number α .

The skew-adjoint part of \mathcal{H}_α vanishes on radial functions and in particular the function $e^{-|x|^2/8}$ is an eigenfunction of \mathcal{H}_α corresponding to the eigenvalue 0, for any $\alpha \in \mathbb{R}$, which implies that the ground state of the two-dimensional harmonic-oscillator does not move under the large skew-adjoint perturbation. Moreover, one can also check that the skew-adjoint part of \mathcal{H}_α vanishes on the functions $x_1 e^{-|x|^2/8}$, $x_2 e^{-|x|^2/8}$. Thus we shall work in some subspaces of $L^2(\mathbb{R}^2; dx)$, defined below.

Using polar coordinates in \mathbb{R}^2 , for $k_0 \geq 1$, we define the subspace of $L^2(\mathbb{R}^2; dx)$

$$X_{k_0} = \left\{ \omega \in L^2(\mathbb{R}^2; dx); \omega(r \cos \theta, r \sin \theta) = \sum_{|k| \geq k_0} \omega_k(r) e^{ik\theta} \right\}, \quad (3.2.2)$$

which is a Hilbert space equipped with the norm $\|\cdot\|_{L^2(\mathbb{R}^2)}$ and which is an invariant space for \mathcal{H}_α .

Definition 3.2.1 (Domain of \mathcal{H}_α). Let

$$D = \{ \omega \in L^2(\mathbb{R}^2); \omega \in H^2(\mathbb{R}^2), |x|^2 \omega \in L^2(\mathbb{R}^2) \}.$$

The non-local term in (3.2.1) is in fact a bounded operator on $L^2(\mathbb{R}^2)$ (see Section 3.4.4 for a proof). Then (\mathcal{H}_α, D) is a closed operator on $L^2(\mathbb{R}^2)$. Moreover, for any $k_0 \geq 1$, \mathcal{H}_α is a closed operator on X_{k_0} with dense domain $D \cap X_{k_0}$ and the numerical range defined by

$$\Theta(\mathcal{H}_\alpha; X_{k_0}) = \{ \langle \mathcal{H}_\alpha \omega, \omega \rangle_{L^2(\mathbb{R}^2)} \in \mathbb{C}; \omega \in D \cap X_{k_0}, \|\omega\|_{L^2(\mathbb{R}^2)} = 1 \}$$

is included in the set $\{z \in \mathbb{C}; \operatorname{Re} z \geq k_0/2\}$ (see Section 3.4.4 for a proof), so that its spectrum is also contained in $\{z \in \mathbb{C}; \operatorname{Re} z \geq k_0/2\}$.

Now let us state our main result.

Theorem 3.2.2. *There exist constants $C_0 > 0$, $k_0 \geq 3$, $\alpha_0 \geq 8\pi$ such that for all $\alpha \geq \alpha_0$, $\lambda \in \mathbb{R}$, for all $\omega \in C_0^\infty(\mathbb{R}^2) \cap X_{k_0}$, we have*

$$\|(\mathcal{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)} \geq C_0 \alpha^{1/3} \|D_\theta|^{1/3} \omega\|_{L^2(\mathbb{R}^2)}, \quad (3.2.3)$$

where $|D_\theta|^{1/3} \omega = \sum_k |k|^{1/3} \omega_k(r) e^{ik\theta}$, for $\omega = \sum_k \omega_k(r) e^{ik\theta}$. In particular, we have

$$\sup_{\lambda \in \mathbb{R}} \|(\mathcal{H}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq C_0^{-1} \alpha^{-1/3} k_0^{-1/3}. \quad (3.2.4)$$

The resolvent estimate (3.2.4) gives information about the pseudospectrum of the family of operators $\{\alpha^{-1/3}\mathcal{H}_\alpha\}_{\alpha \geq 1}$.

Definition 3.2.3. For a family of operators $\{P_\alpha\}_{\alpha \geq 1}$ on a Hilbert space X , we define the pseudospectrum of $\{P_\alpha\}_{\alpha \geq 1}$ as the complement of the set of $z \in \mathbb{C}$ such that

$$\exists N_0 \in \mathbb{N}, \quad \limsup_{\alpha \rightarrow +\infty} \alpha^{-N_0} \|(P_\alpha - z)^{-1}\|_{\mathcal{L}(X)} < +\infty.$$

Corollary 3.2.4. When restricted to X_{k_0} , the pseudospectrum of $\{\alpha^{-1/3}\mathcal{H}_\alpha\}_{\alpha \geq 1}$ is included in the set

$$\{z \in \mathbb{C}; \operatorname{Re} z \geq C_0 k_0^{1/3}\},$$

where k_0 is given in Theorem 3.2.2.

Proof. If $\operatorname{Re} \eta \leq 0$, then for $\omega \in D \cap X_{k_0}$ with $\|\omega\|_{L^2(\mathbb{R}^2)} = 1$,

$$|\langle (\mathcal{H}_\alpha - \eta)\omega, \omega \rangle_{L^2(\mathbb{R}^2)}| \geq |\operatorname{Re} \langle \mathcal{H}_\alpha \omega, \omega \rangle_{L^2(\mathbb{R}^2)} - \operatorname{Re} \eta| \geq \frac{k_0}{2},$$

implying $\|(\mathcal{H}_\alpha - \eta)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq 2k_0^{-1}$.

Let $\kappa \in (0, 1)$. For $\eta = \mu + i\lambda$ with $0 < \mu \leq \kappa C_0 \alpha^{1/3} k_0^{1/3}$ and $\lambda \in \mathbb{R}$, we infer from the resolvent formula

$$(\mathcal{H}_\alpha - \mu - i\lambda)^{-1} - (\mathcal{H}_\alpha - i\lambda)^{-1} = \mu(\mathcal{H}_\alpha - i\lambda)^{-1}(\mathcal{H}_\alpha - \mu - i\lambda)^{-1}$$

and the resolvent estimate (3.2.4) that for $\alpha \geq \alpha_0$,

$$\|(\mathcal{H}_\alpha - \mu - i\lambda)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq \frac{\|(\mathcal{H}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(X_{k_0})}}{1 - \mu \|(\mathcal{H}_\alpha - i\lambda)^{-1}\|_{\mathcal{L}(X_{k_0})}} \leq \frac{C^{-1} \alpha^{-1/3} k_0^{-1/3}}{1 - \kappa}.$$

As a result, for $\kappa \in (0, 1)$, for all $\eta \in \mathbb{C}$ such that $\operatorname{Re} \eta \leq \kappa C_0 \alpha^{1/3} k_0^{1/3}$, we have $\|(\mathcal{H}_\alpha - \eta)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq C_\kappa$. This is equivalent to the following

$$\forall z \in \mathbb{C}, \text{s.t. } \operatorname{Re} z \leq \kappa C_0 k_0^{1/3}, \quad \sup_{\alpha \geq \alpha_0} \alpha^{-1/3} \|(\alpha^{-1/3} \mathcal{H}_\alpha - z)^{-1}\|_{\mathcal{L}(X_{k_0})} \leq C_\kappa.$$

According to the definition, we know that the set $\{z \in \mathbb{C}; \operatorname{Re} z \leq \kappa C_0 k_0^{1/3}\}$ is included in the complement of the pseudospectrum of $\{\alpha^{-1/3} \mathcal{H}_\alpha\}_{\alpha \geq 1}$, so that the corollary is proved. \square

3.2.2 Comments

The nonlocal term

The term

$$\frac{\alpha}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega))$$

is an integral operator which is non-local and skew-adjoint. This term should be carefully treated as it has a large coefficient α .

A weight

We shall reduce the two-dimensional operator $\mathcal{H}_\alpha - i\lambda$ to a family of one-dimensional operators acting on the positive-half real line \mathbb{R}_+ by using polar coordinates and expanding the angular variable θ in Fourier series, indexed by the Fourier mode parameter $k \in \mathbb{Z}$. Then we transform the problem onto the whole real line \mathbb{R} by making a change of variable $r = e^t$ and multiplying by a weight e^{2t} . After these transformations, the properties of self-adjointness and skew-adjointness are preserved (see Section 3.3.1), and the non-local term turns out to be a skew-adjoint pseudodifferential operator with $\mathcal{L}(L^2(\mathbb{R}; dt))$ -norm bounded above by $\alpha|k|^{-1}$. The discussion is divided into different cases according to a change-of-sign situation.

Multiplier method

The proof relies on a classical multiplier method. For the non-trivial cases where the change-of-sign takes place (see Section 3.3.3, 3.3.4), we shall construct a multiplier bounded on $L^2(\mathbb{R}; dt)$, which is a pseudodifferential operator associated to a Hörmander-type metric. The non-local term will be treated as a perturbation and will be absorbed by the main term letting $|k| \geq k_0$, with k_0 a constant independent of the circulation parameter α .

The value of k_0

Given $\epsilon_0, \epsilon_1 \in (0, 1)$, we shall discuss 4 different cases given in (3.3.22). Then k_0 can be expressed as a function of (ϵ_0, ϵ_1) . For example, if we take $\epsilon_0 \simeq 0.462$, $\epsilon_1 \simeq 0.426$, then Theorem 3.2.2 holds with $k_0 = 84$, see [Den11a]. (In fact, we can obtain $k_0 = 51$ if we do some improvements.)

3.3 The proof

3.3.1 First reductions

The operator \mathcal{H}_α in (3.2.1) is invariant under rotations with respect to the origin in \mathbb{R}^2 . We can reduce the problem to a family of one-dimensional operators by using polar coordinates and expanding the angular variable θ in Fourier series.

Polar coordinates

We can write for $\omega \in L^2(\mathbb{R}^2)$ and $v = K_{BS} * \omega$ given by (3.1.2) as

$$\begin{aligned}\omega(r \cos \theta, r \sin \theta) &= \sum_{k \in \mathbb{Z}} \omega_k(r) e^{ik\theta}, \\ v(r \cos \theta, r \sin \theta) &= \sum_{k \in \mathbb{Z}} \left(\frac{u_k(r)}{r} \mathbf{e}_r + \frac{w_k(r)}{r} \mathbf{e}_\theta \right) e^{ik\theta},\end{aligned}$$

where $\mathbf{e}_r = (\cos \theta, \sin \theta)$ and $\mathbf{e}_\theta = (-\sin \theta, \cos \theta)$. The relations $\partial_1 v_1 + \partial_2 v_2 = 0$, $\partial_1 v_2 - \partial_2 v_1 = \omega$ become

$$u'_k + \frac{ik}{r} w_k = 0, \quad w'_k - \frac{ik}{r} u_k = r\omega_k, \tag{3.3.1}$$

(see Section 3.4.4) so that $-\Delta_k u_k = ik\omega_k$, where

$$-\Delta_k = -\partial_r^2 - \frac{1}{r} \partial_r + \frac{k^2}{r^2}.$$

If $k \neq 0$, the Poisson equation $-\Delta_k \Omega = f$ has the explicit solution $\Omega = \mathcal{K}_k[f]$, where

$$\mathcal{K}_k[f](r) = \frac{1}{2|k|} \int_0^{+\infty} \left(\left(\frac{r}{s}\right)^{|k|} H(r)H(s-r) + \left(\frac{s}{r}\right)^{|k|} H(s)H(r-s) \right) f(s) s ds, \quad (3.3.2)$$

where $H(r)$ is the Heaviside function. We thus have

$$u_k = ik\mathcal{K}_k[\omega_k] \quad \text{and} \quad w_k = -r\mathcal{K}_k[\omega_k]'$$

if $k \neq 0$. For $k = 0$, we find $u_0 = 0$ and $w'_0 = r\omega_0$, hence $w_0(r) = \int_0^r s\omega_0(s) ds$.

By using the following notations:

$$\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}, \quad \text{for } r > 0, \quad (3.3.3)$$

and observing that $v^G = \frac{1}{8\pi}r\sigma(r)\mathbf{e}_\theta$, we rewrite the skew-adjoint part of \mathcal{H}_α in polar coordinates as

$$\begin{aligned} \alpha v^G \cdot \nabla \omega &= \sum_{k \neq 0} \frac{i\alpha k}{8\pi} \sigma(r) \omega_k(r) e^{ik\theta}, \\ \frac{\alpha}{2} G^{1/2} x \cdot (K_{BS} * (G^{1/2} \omega)) &= \sum_{k \neq 0} \frac{i\alpha k}{8\pi} g(r) \mathcal{K}_k[g\omega_k](r) e^{ik\theta}. \end{aligned}$$

Thus we find that for \mathcal{H}_α given by (3.2.1), $\lambda \in \mathbb{R}$ and for $\omega = \sum_{k \in \mathbb{Z}^*} \omega_k(r) e^{ik\theta}$,

$$((\mathcal{H}_\alpha - i\lambda)\omega)(r \cos \theta, r \sin \theta) = \sum_{k \in \mathbb{Z}^*} (\mathcal{H}_{\alpha,k,\lambda}\omega_k)(r) e^{ik\theta}, \quad (3.3.4)$$

where $\mathcal{H}_{\alpha,k,\lambda}$ acts on $L^2(\mathbb{R}_+; rdr)$ and is given by

$$\mathcal{H}_{\alpha,k,\lambda} v = -\partial_r^2 v - \frac{1}{r} \partial_r v + \frac{k^2}{r^2} v + \frac{r^2}{16} v - \frac{1}{2} v + \frac{ik\alpha}{8\pi} \left(\sigma(r)v - g\mathcal{K}_k[gv] \right) - i\lambda v. \quad (3.3.5)$$

Introducing two new notations

$$\beta_k = \frac{\alpha k}{8\pi}, \quad \lambda = \beta_k \nu_k, \quad \nu_k \in \mathbb{R}, \quad (3.3.6)$$

we are led to study the resolvent of the one-dimensional operator $\mathcal{H}_{\alpha,k,\lambda}$ on $L^2(\mathbb{R}_+; rdr)$ for $|\beta_k| \rightarrow +\infty$, where (we omit the indices α, λ in $\mathcal{H}_{\alpha,k,\lambda}$)

$$\begin{aligned} \mathcal{H}_k v &= \underbrace{-\partial_r^2 v - \frac{1}{r} \partial_r v + \frac{k^2}{r^2} v + \frac{r^2}{16} v - \frac{1}{2} v}_{\text{self-adjoint and non-negative on } L^2(\mathbb{R}_+; rdr)} \\ &\quad + \underbrace{i\beta_k (\sigma(r) - \nu_k) v - i\beta_k g \mathcal{K}_k[gv]}_{\text{skew-adjoint on } L^2(\mathbb{R}_+; rdr)}. \end{aligned} \quad (3.3.7)$$

Note that the non-local term is transformed to $i\beta_k g \mathcal{K}_k g$ with \mathcal{K}_k given by (3.3.2). Moreover, $C_0^\infty((0, +\infty))$ is a core for the closed operator \mathcal{H}_k with domain

$$D(\mathcal{H}_k) = \{v \in L^2(\mathbb{R}_+; rdr); \partial_r^2 v, \frac{1}{r} \partial_r v, \frac{1}{r^2} v, r^2 v \in L^2(\mathbb{R}_+; rdr)\}, \quad \text{if } |k| \geq 2,$$

$$\text{and } D(\mathcal{H}_k) = \{v \in L^2(\mathbb{R}_+; rdr); \partial_r^2 v, \partial_r \left(\frac{v}{r}\right), r^2 v \in L^2(\mathbb{R}_+; rdr)\}, \quad \text{if } |k| = 1.$$

One can check that the function $rg(r) = re^{-r^2/8}$ is in $D(\mathcal{H}_{\pm 1})$ but not in $D(\mathcal{H}_k)$ for any $|k| \geq 2$.

Change of variables

We wish to transform the operator \mathcal{H}_k in (3.3.7) acting on the positive half-line into an operator acting on the whole real line, by making the change of variables $r = e^t$. A simple but key observation is

Lemma 3.3.1. *For $v \in L^2(\mathbb{R}_+; rdr)$, define $u(t) = v(e^t)$. Then*

$$\|e^t u\|_{L^2(\mathbb{R}; dt)}^2 = \int_{\mathbb{R}} |e^t u(t)|^2 dt = \int_0^{+\infty} |v(r)|^2 r dr = \|v\|_{L^2(\mathbb{R}_+; rdr)}^2. \quad (3.3.8)$$

Moreover, for $v \in C_0^\infty((0, +\infty))$, multiplying $(\mathcal{H}_k v)(e^t)$ by the weight e^{2t} , we have

$$e^{2t}(\mathcal{H}_k v)(e^t) = (\widetilde{\mathcal{L}}_k u)(t), \quad \text{for } u(t) = v(e^t), \quad (3.3.9)$$

where

$$\begin{aligned} \widetilde{\mathcal{L}}_k = & -\partial_t^2 + k^2 + \frac{1}{16}e^{4t} - \frac{1}{2}e^{2t} \\ & + i\beta_k e^{2t}(\sigma(e^t) - \nu_k) - i\beta_k e^{2t}g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t). \end{aligned} \quad (3.3.10)$$

Proof. Indeed, we have

$$r^2(\partial_r^2 + r^{-1}\partial_r) = (r\partial_r)^2 = \partial_t^2 \quad \text{for } r = e^t.$$

On the other hand, by the definition (3.3.2) of \mathcal{K}_k , we have for $v \in C_0^\infty((0, +\infty))$,

$$\begin{aligned} e^{2t}(g\mathcal{K}_k[gv])(e^t) &= e^{2t}g(e^t)\mathcal{K}_k[gv](e^t) \\ &= \frac{1}{2|k|} \int_0^{+\infty} e^{2t}g(e^t) \left[\left(\frac{e^t}{s} \right)^{|k|} H(e^t)H(s - e^t) + \left(\frac{s}{e^t} \right)^{|k|} H(s)H(e^t - s) \right] g(s)v(s)sds \\ &= \frac{1}{2|k|} \int_{\mathbb{R}} e^{2t}g(e^t) \left[\left(\frac{e^t}{e^s} \right)^{|k|} H(e^s - e^t) + \left(\frac{e^s}{e^t} \right)^{|k|} H(e^t - e^s) \right] g(e^s)v(e^s)e^{2s}ds \\ &= \frac{1}{2|k|} \int_{\mathbb{R}} e^{2t}g(e^t) \left[e^{-|k|(s-t)} H(s-t) + e^{-|k|(t-s)} H(t-s) \right] e^{2s}g(e^s)v(e^s)ds \\ &= \frac{1}{2|k|} \int_{\mathbb{R}} e^{2t}g(e^t) e^{-|k||t-s|} e^{2s}g(e^s)u(s)ds. \end{aligned}$$

For $k \neq 0$, we have

$$\frac{1}{2|k|} \int_{\mathbb{R}} e^{-|k||t|} e^{it\tau} dt = \frac{1}{k^2 + \tau^2},$$

(see Lemma 3.4.4) so that the non-local term $g\mathcal{K}_k g$ becomes

$$e^{2t}(g\mathcal{K}_k[gv])(e^t) = (e^{2t}g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)u)(t), \quad \text{for } u(t) = v(e^t),$$

which is a self-adjoint, positive (non-local) pseudodifferential operator on $L^2(\mathbb{R}; dt)$. The proof of the lemma is complete. \square

When $\widetilde{\mathcal{L}}_k$ given by (3.3.10) is viewed as an operator on $L^2(\mathbb{R}; dt)$, we see that

$$\begin{aligned} \widetilde{\mathcal{L}}_k = & \underbrace{-\partial_t^2 + k^2 + \frac{1}{16}e^{4t} - \frac{1}{2}e^{2t}}_{\text{self-adjoint and non-negative on } L^2(\mathbb{R}; dt)} \\ & + \underbrace{i\beta_k e^{2t}(\sigma(e^t) - \nu_k)}_{\text{skew-adjoint on } L^2(\mathbb{R}; dt)} - \underbrace{i\beta_k e^{2t}g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)}_{\text{skew-adjoint on } L^2(\mathbb{R}; dt)}. \end{aligned}$$

After the change of variables $r = e^t$ and the multiplication by the weight e^{2t} , the self-adjoint (resp. skew-adjoint) part of \mathcal{H}_k in (3.3.7) does not lose its self-adjointness (resp. skew-adjointness), and in particular, the non-local term $i\beta_k g \mathcal{K}_k g$ stays skew-adjoint. Moreover, the power 2 in the weight is the only power to keep these properties unchanged.

In view of (3.3.8) and (3.3.9) in Lemma 3.3.1, the problem is reduced to prove estimates for the operator $\widetilde{\mathcal{L}}_k$ in (3.3.10) of type

$$\|e^{-t} \widetilde{\mathcal{L}}_k u\|_{L^2(\mathbb{R};dt)} \geq C|\beta_k|^a \|e^t u\|_{L^2(\mathbb{R};dt)} \quad (3.3.11)$$

for some $a > 0$, which correspond to the estimates for the operator \mathcal{H}_k given in (3.3.7)

$$\|\mathcal{H}_k v\|_{L^2(\mathbb{R}_+;rdr)} \geq C|\beta_k|^a \|v\|_{L^2(\mathbb{R}_+;rdr)}, \quad (3.3.12)$$

where $u(t) = v(e^t)$, since we have exactly

$$\|e^{-t} \widetilde{\mathcal{L}}_k u\|_{L^2(\mathbb{R};dt)} = \|\mathcal{H}_k v\|_{L^2(\mathbb{R}_+;rdr)}, \quad \|e^t u\|_{L^2(\mathbb{R};dt)} = \|v\|_{L^2(\mathbb{R}_+;rdr)}.$$

Furthermore, we need only to prove estimates (3.3.11) for $u \in C_0^\infty(\mathbb{R})$, since it is enough to get (3.3.12) for $v \in C_0^\infty((0, +\infty))$.

As in [Den10a], we divide our discussion into different cases, according to the change-of-sign situation of $\sigma(e^t) - \nu_k$, where the function σ is given in (3.3.3). Note that $\sigma(e^t)$ is a decreasing function of the variable t and has range $(0, 1)$. When $\sigma(e^t) - \nu_k$ does not change sign, it is easy to deal with by using the multipliers Id , $\pm i\text{Id}$ (see Section 3.3.2). If $\sigma(e^t) - \nu_k$ changes sign at one point, it is more complicated (see Section 3.3.3, 3.3.4). In this case, we will construct a multiplier well-adapted to this change-of-sign situation, which is a pseudodifferential operator depending on a Hörmander metric on the phase space. Compared with the method in [Den10a], the multiplier that we shall construct is a global one, because of the existence of the non-local term, which possesses a large coefficient and would produce a commutator of size $|\beta_k|$ if we just used a partition of unity on \mathbb{R}_t as done in [Den10a].

Notations

In Section 3.3.2, 3.3.3 and 3.3.4, we shall always assume that $k \geq 1$ hence $\beta_k > 0$, and we denote by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$ the $L^2(\mathbb{R};dt)$ -norm, inner-product respectively. We shall also be able to neglect the term $-\frac{1}{2}e^{2t}$ in the real part of $\widetilde{\mathcal{L}}_k$ and by introducing two notations,

$$\langle D_k \rangle^{-2} = (D_t^2 + k^2)^{-1}, \quad \gamma(t) = e^{2t}g(e^t) = e^{2t}e^{-e^{2t}/8}, \quad (3.3.13)$$

we shall study

$$\mathcal{L}_k = D_t^2 + k^2 + \frac{1}{16}e^{4t} + i\beta_k e^{2t}(\sigma(e^t) - \nu_k) - i\beta_k \gamma(t) \langle D_k \rangle^{-2} \gamma(t). \quad (3.3.14)$$

In fact, as soon as we prove (3.3.11) for \mathcal{L}_k in (3.3.14) with $a > 0$, we have for the operator $\widetilde{\mathcal{L}}_k$ given in (3.3.10)

$$\begin{aligned} \|e^{-t} \widetilde{\mathcal{L}}_k u\|_{L^2(\mathbb{R};dt)} &= \|e^{-t} (\mathcal{L}_k - \frac{1}{2}e^{2t}) u\|_{L^2(\mathbb{R};dt)} \\ &\geq C\beta_k^a \|e^t u\|_{L^2(\mathbb{R};dt)} - \frac{1}{2} \|e^t u\|_{L^2(\mathbb{R};dt)}, \end{aligned}$$

so that it suffices to let α large enough since $k \geq 1$, $\beta_k \geq \alpha/8\pi$.

We present in Appendix 3.4.3 some inequalities concerning the functions σ and g given in (3.3.3) that will be used in the proof. We have for all $u \in C_0^\infty(\mathbb{R})$,

$$\operatorname{Re}\langle \mathcal{L}_k u, u \rangle_{L^2(\mathbb{R}; dt)} = \langle (D_t^2 + k^2 + \frac{1}{16}e^{4t})u, u \rangle_{L^2(\mathbb{R}; dt)}, \quad (3.3.15)$$

$$0 \leq \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle_{L^2(\mathbb{R}; dt)} \leq k^{-2} \|\gamma u\|_{L^2(\mathbb{R}; dt)}^2, \quad \text{by Lemma 3.4.3,} \quad (3.3.16)$$

where \mathcal{L}_k is given in (3.3.14) and $\gamma, \langle D_k \rangle^{-2}$ are given in (3.3.13).

3.3.2 Easy cases

In this section, we study the cases where $\sigma(e^t) - \nu_k$ does not change sign, that is $\nu_k \geq 1$ or $\nu_k \leq 0$.

Lemma 3.3.2. *Suppose $\nu_k \geq 1$. There exists $C > 0$ such that for all $k \geq 1$, $\alpha \geq 8\pi$ and for $u \in C_0^\infty(\mathbb{R})$,*

$$\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad (3.3.17)$$

where \mathcal{L}_k is given in (3.3.14) and β_k is given in (3.3.6).

Proof. If $\nu_k \geq 1$, then $\sigma(e^t) - \nu_k$ is non-positive. Using the multiplier $-i\operatorname{Id}$ and by (3.3.16), we have

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k u, -iu \rangle &= \beta_k \langle e^{2t} (\nu_k - \sigma(e^t)) u, u \rangle + \beta_k \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle \\ &\geq \beta_k \langle e^{2t} (1 - \sigma(e^t)) u, u \rangle. \end{aligned} \quad (3.3.18)$$

Adding (3.3.15), (3.3.18) together, we obtain

$$\operatorname{Re}\langle \mathcal{L}_k u, (1 - i)u \rangle \geq \langle (k^2 e^{-2t} + \beta_k (1 - \sigma(e^t))) e^{2t} u, u \rangle.$$

Then using the second inequality in (3.4.17), we get

$$\operatorname{Re}\langle e^{-t} \mathcal{L}_k u, e^t (1 - i)u \rangle \geq C \beta_k^{1/2} \langle e^{2t} u, u \rangle = C \beta_k^{1/2} \|e^t u\|^2.$$

By Cauchy-Schwarz inequality, the estimate (3.3.17) is proved. \square

Lemma 3.3.3. *Suppose $\nu_k \leq 0$. There exists $C > 0$ such that for all $k \geq 2$, $\alpha \geq 8\pi$ and for $u \in C_0^\infty(\mathbb{R})$,*

$$\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad (3.3.19)$$

where \mathcal{L}_k is given in (3.3.14) and β_k in (3.3.6).

Proof. If $\nu_k \leq 0$, then $\sigma(e^t) - \nu_k$ is non-negative. Using the multiplier $i\operatorname{Id}$ and by (3.3.16), we have

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k u, iu \rangle &= \beta_k \langle e^{2t} (\sigma(e^t) - \nu_k) u, u \rangle - \beta_k \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle \\ &\geq \beta_k \left(\langle e^{2t} \sigma(e^t) u, u \rangle - k^{-2} \|e^{2t} g(e^t) u\|^2 \right). \end{aligned}$$

Using (3.4.12), we get for $k \geq 2$,

$$\operatorname{Re}\langle \mathcal{L}_k u, iu \rangle \geq (1 - (4\delta)^{-1}) \beta_k \langle e^{2t} \sigma(e^t) u, u \rangle, \quad (3.3.20)$$

with $1 - (4\delta)^{-1} > 0$. Adding (3.3.15), (3.3.20) together we obtain

$$\operatorname{Re}\langle \mathcal{L}_k u, (1+i)u \rangle \geq \langle \left(\frac{1}{16} e^{2t} + (1 - (4\delta)^{-1}) \beta_k \sigma(e^t) \right) e^{2t} u, u \rangle, \quad k \geq 2.$$

Using the first inequality in (3.4.17), we get

$$\operatorname{Re}\langle e^{-t} \mathcal{L}_k u, e^t (1+i)u \rangle \geq C \beta_k^{1/2} \langle e^{2t} u, u \rangle = C \beta_k^{1/2} \|e^t u\|^2, \quad k \geq 2.$$

By Cauchy-Schwarz inequality, the estimate (3.3.19) is proved. \square

Remark 3.3.4. When $k = 1$ and $\nu_k = 0$, the imaginary part of \mathcal{H}_1 vanishes on the function $v(r) = rg(r) \in D(\mathcal{H}_1)$, i.e. we have $g\mathcal{K}_1[gv] = \sigma v$. Consequently, when $\nu_k = 0$, the imaginary part of \mathcal{L}_1 vanishes on the function $u(t) = e^t g(e^t)$.

3.3.3 Nontrivial cases

We turn to study the cases where the change-of-sign of $\sigma(e^t) - \nu_k$ takes place, that is $\nu_k \in (0, 1)$. We have thus $\nu_k = \sigma(e^{t_k})$ for some $t_k \in \mathbb{R}$. Then the operator \mathcal{L}_k can be written as

$$\mathcal{L}_k = D_t^2 + k^2 + \frac{1}{16} e^{4t} + i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) - i\beta_k \gamma(t) \langle D_k \rangle^{-2} \gamma(t). \quad (3.3.21)$$

Suppose $\epsilon_0, \epsilon_1 \in (0, 1)$. We discuss four cases according to the behavior of the function σ near the point e^{t_k} :

$$e^{t_k} > \epsilon_0^{-1} \quad \text{or} \quad e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}] \quad \text{or} \quad e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) \quad \text{or} \quad e^{t_k} \leq \beta_k^{-1/4}. \quad (3.3.22)$$

Before going through the proofs for each case, let us first choose some functions that will be used to construct the multipliers. Suppose that $c_0 \in (0, 1)$ is the constant chosen in Proposition 3.4.7. Let $\chi_j \in C^\infty(\mathbb{R}; [0, 1])$, $j = 0, +, -$, satisfying that

$$\begin{cases} \operatorname{supp} \chi_0 \subset [-c_0, c_0], \\ \chi_+ = 1 \text{ on } [c_0, +\infty), \quad \operatorname{supp} \chi_+ \subset [\frac{c_0}{2}, +\infty), \\ \chi_- = 1 \text{ on } (-\infty, -c_0], \quad \operatorname{supp} \chi_- \subset (-\infty, -\frac{c_0}{2}], \\ \chi_0(\theta)^2 + \chi_+(\theta)^2 + \chi_-(\theta)^2 = 1, \quad \forall \theta \in \mathbb{R}. \end{cases} \quad (3.3.23)$$

See Figure 3.1.

Choose a function $\tilde{\chi}_0 \in C_0^\infty(\mathbb{R}; [0, 1])$ such that

$$\tilde{\chi}_0 = 1 \text{ on } [-2c_0, 2c_0], \quad \operatorname{supp} \tilde{\chi}_0 \subset [-3c_0, 3c_0]. \quad (3.3.24)$$

Take a decreasing function $\psi \in C^\infty(\mathbb{R}; [-1, 1])$ such that

$$\psi = 1 \text{ on } (-\infty, -2], \quad \psi = -1 \text{ on } [2, +\infty), \quad \psi' = -\frac{1}{2} \text{ on } [-1, 1]. \quad (3.3.25)$$

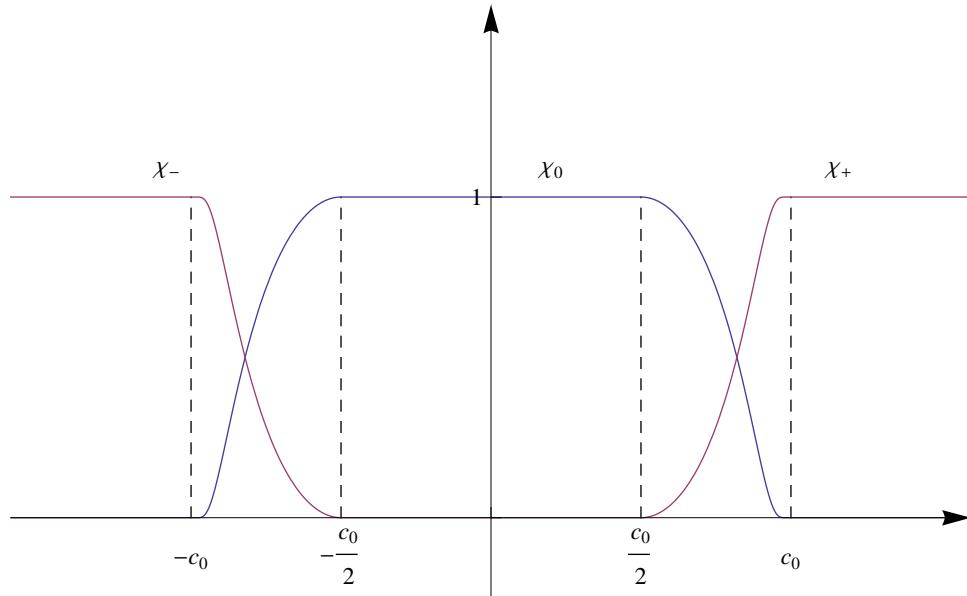
We can assume that ψ has a factorization

$$\psi(\theta) = -e(\theta)\theta, \quad (3.3.26)$$

where $e \in C_b^\infty(\mathbb{R}; [0, 1])$ satisfies¹ that

$$e(\theta) = \frac{1}{2} \text{ for } \theta \in [-1, 1], \quad e(\theta) = |\theta|^{-1} \text{ for } |\theta| \geq 2.$$

1. We denote by $C_b^\infty(\mathbb{R}; [0, 1])$ the set of smooth functions defined on \mathbb{R} with values in $[0, 1]$ such that all their derivatives are bounded.

Figure 3.1: THE FUNCTIONS χ_0, χ_{\pm} .

Plan of the paragraph

The sections 3.3.3, 3.3.4 are organized as follows. Recall the four cases given in (3.3.22) and we give in Proposition 3.4.7 inequalities about the function σ that will be used in the proof for the first three cases.

Section 3.3.3.a) is devoted to the proof for Case 1 where $e^{t_k} > \epsilon_0^{-1}$. We shall construct a multiplier adapted to the change-of-sign situation. Moreover, there is a special localization effect in this case (see Remark 3.3.13).

In Section 3.3.3.b), we prove estimates for Case 2 where $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. The multiplier to be used in this case is the same as that in Case 1.

In Section 3.3.4.c), we prove estimates for Case 3 where $e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1)$. The multiplier will be different from that in the previous cases and the condition $e^{t_k} > \beta_k^{-1/4}$ is required such that the metric verifies the uncertainty principle.

Finally, Section 3.3.4.d) is devoted to proving estimates for the last case where $e^{t_k} \leq \beta_k^{-1/4}$ and estimates are easily obtained by using the multipliers Id , $-i\text{Id}$.

3.3.3.a) Case 1: $e^{t_k} > \epsilon_0^{-1}$

We present in Proposition 3.4.7,(1) some inequalities about the function σ that will be used in this case.

Theorem 3.3.5. Suppose $e^{t_k} > \epsilon_0^{-1}$. There exist $C > 0$, $k_0 \geq 1$ such that for all $k \geq k_0$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

$$\|e^{-t}\mathcal{L}_k u\| \geq C\beta_k^{1/3} \|e^t u\|, \quad (3.3.27)$$

where \mathcal{L}_k is given in (3.3.21) and β_k is given in (3.3.6).

a. **Definition of the multiplier.** We first give the definition of the Hörmander-type metric that we shall work with (see Appendix 3.4.1).

Definition 3.3.6. Define a metric on the phase space $\mathbb{R}_t \times \mathbb{R}_\tau$

$$\Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + \beta_k^{2/3}},$$

which is admissible with

$$\lambda_\Gamma = (\tau^2 + \beta_k^{2/3})^{1/2} \geq \beta_k^{1/3} \geq \left(\frac{\alpha}{8\pi}\right)^{1/3} \geq 1, \quad \text{provided } \alpha \geq 8\pi. \quad (3.3.28)$$

Remark 3.3.7. We give a proof for the uniform admissibility (w.r.t. $k \geq 1, \alpha \geq 8\pi$) of the metric Γ in Lemma 3.4.1. Moreover, the function $f(\beta_k^{-1/3}\tau)$ belongs to $S(1, \Gamma)$ whenever $f \in S(1, \frac{|d\theta|^2}{1+\theta^2})$, since for any $n \in \mathbb{N}$,

$$\begin{aligned} \left| \frac{\partial^n}{\partial \tau^n} (f(\beta_k^{-1/3}\tau)) \right| &= |f^{(n)}(\beta_k^{-1/3}\tau) \beta_k^{-n/3}| \\ &\leq C_n (1 + |\beta_k^{-1/3}\tau|^2)^{-n/2} \beta_k^{-n/3} = C_n (\beta_k^{2/3} + \tau^2)^{-n/2}. \end{aligned}$$

Now we can construct the multiplier, using the functions that we have chosen in (3.3.23), (3.3.25).

Definition 3.3.8.

$$M_k = m_{0,k}^w + m_{+,k}^w + m_{-,k}^w, \quad (3.3.29)$$

where

$$\begin{aligned} m_{0,k}(t, \tau) &= \chi_0(t - t_k) \# \psi(\beta_k^{-1/3}\tau) \# \chi_0(t - t_k), \\ m_{+,k}(t, \tau) &= -i\beta_k^{-1/3} \chi_+(t - t_k)^2, \\ m_{-,k}(t, \tau) &= i\beta_k^{-1/3} \chi_-(t - t_k)^2, \end{aligned}$$

where a^w stands for the Weyl quantization for the symbol a and $\#$ denotes the composition law in Weyl calculus. (See Appendix 3.4.1 for Weyl calculus.)

Remark 3.3.9. The functions $\chi_0(t - t_k)$, $\chi_\pm(t - t_k)$, $\psi(\beta_k^{-1/3}\tau)$ are real-valued symbols in $S(1, \Gamma)$. Then M_k given in Definition 3.3.8 is a bounded operator on $L^2(\mathbb{R}; dt)$. Moreover, we see that

$$m_{0,k}^w = \chi_0(t - t_k) \psi(\beta_k^{-1/3} D_t) \chi_0(t - t_k) \quad (3.3.30)$$

and M_k can be written as

$$M_k = \chi_0(t - t_k) \psi(\beta_k^{-1/3} D_t) \chi_0(t - t_k) - i\beta_k^{-1/3} \chi_+(t - t_k)^2 + i\beta_k^{-1/3} \chi_-(t - t_k)^2.$$

Furthermore, the operator $e^t M_k e^{-t}$ is bounded on $L^2(\mathbb{R}; dt)$, since

$$e^t m_{0,k}^w e^{-t} = [e^{t-t_k} \chi_0(t - t_k)] \psi(\beta_k^{-1/3} D_t) [\chi_0(t - t_k) e^{-(t-t_k)}], \quad (3.3.31)$$

and $|e^{\pm(t-t_k)} \chi_0(t - t_k)| \leq e^{c_0}$.

The three parts in M_k are used to handle different zones in the phase space. We use $m_{0,k}^w$ to localize near the point t_k , where the change-of-sign of $\sigma(e^t) - \sigma(e^{t_k})$ happens. The Fourier multiplier $\psi(\beta_k^{-1/3} D_t)$ allows us to obtain some subelliptic estimate in this zone, acting with the skew-adjoint part of \mathcal{L}_k . As we shall see in the computations, it is important to put the cutoff function $\chi_0(t - t_k)$ on both sides of $\psi(\beta_k^{-1/3} D_t)$, so that we

are able to do symbolic calculus with the exponential functions since they are all localized near t_k .

The other two multipliers $m_{\pm,k}^w$ are used for dealing with the zones where there is no change-of-sign of $\sigma(e^t) - \sigma(e^{t_k})$, that is t away from the point t_k , and the sign of $m_{+,k}, m_{-,k}$ corresponds exactly to the sign of $\sigma(e^t) - \sigma(e^{t_k})$ on their supports. If the non-local term $i\beta_k\gamma\langle D_k \rangle^{-2}\gamma$ were not present, then we could remove the factor $\beta_k^{-1/3}$ to get better estimates in these zones, as we have already done in [Den10a]. However, we see that the non-local term has a large coefficient β_k and it does not commute with $\chi_{\pm}(t-t_k)$, so that we would obtain a commutator of size β_k that we would not know how to control. Our strategy is to weaken the multiplier in these regions by multiplying a factor $\beta_k^{-1/3}$.

The method that we use here is perturbative: the non-local term is treated as a perturbation with respect to the main term $\sigma(e^t) - \sigma(e^{t_k})$. Thanks to the operator $\langle D_k \rangle^{-2}$ and the nice function $\gamma(t)$ (see (3.3.13)), this perturbation is controlled by the main term with an extra factor k^{-2} . Letting $k \geq k_0$, with $k_0 \geq 1$ a constant independent of the parameter α , we can get the desired result.

However, it is of course impossible to consider the non-local term as a “global” perturbation, i.e. to absorb it by a term controlled by $\|e^{-t}\mathcal{L}_{k,0}u\|_{L^2(\mathbb{R};dt)}$, where $\mathcal{L}_{k,0}$ is the unperturbed part of \mathcal{L}_k : in fact the size of that perturbation is β_k and the best estimate we can hope is controlling a factor $\beta_k^{1/3}$. We have instead to follow our multiplier method to check the effect of the perturbation.

b. Computations. Now let us compute $2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle$.

Proposition 3.3.10. *Suppose $e^{t_k} > \epsilon_0^{-1}$. There exist $c, C > 0$ such that for all $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,*

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq c\beta_k^{2/3} \langle \rho(t, t_k)u, u \rangle - C\beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2 \\ &\quad - 2\beta_k^{2/3}k^{-2}\|e^{2t}g(e^t)\chi_-(t-t_k)u\|^2 - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned} \quad (3.3.32)$$

where \mathcal{L}_k is given in (3.3.21), M_k in Definition 3.3.8, χ_0, χ_\pm in (3.3.23), σ, g in (3.3.3), β_k in (3.3.6),

$$\rho(t, t_k) = \chi_0(t-t_k)^2 + e^{2t}\sigma(e^{t_k})\chi_+(t-t_k)^2 + e^{2t}\sigma(e^t)\chi_-(t-t_k)^2 \quad (3.3.33)$$

$$\text{and } \kappa(e^{t_k}) = \sup \left\{ g(s)^{1/2} \max \left(1, \left| 2 - \frac{1}{4}s^2 \right|, \left| 4 - \frac{3}{2}s^2 + \frac{1}{16}s^4 \right| \right); s \in [e^{t_k-c_0}, e^{t_k+c_0}] \right\}. \quad (3.3.34)$$

Proof of Proposition 3.3.10. First recall that for all $u \in C_0^\infty(\mathbb{R})$,

$$\operatorname{Re}\langle \mathcal{L}_k u, u \rangle = \|D_t u\|^2 + k^2\|u\|^2 + \frac{1}{16}\|e^{2t}u\|^2. \quad (3.3.35)$$

In the following computations, we omit the dependence of $\chi_j(t-t_k)$ on $t-t_k$ for the sake of brevity.

Estimates for $2\operatorname{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle$.

$$\begin{aligned} A := 2\operatorname{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, m_{0,k}^w u \rangle \\ &\quad - 2\operatorname{Re}\langle i\beta_k \gamma\langle D_k \rangle^{-2}\gamma u, m_{0,k}^w u \rangle \\ &\quad + 2\operatorname{Re}\langle (D_t^2 + k^2 + \frac{1}{16}e^{4t})u, m_{0,k}^w u \rangle \\ &=: A_1 + A_2 + A_3. \end{aligned} \quad (3.3.36)$$

Noticing $\chi_0 \tilde{\chi}_0 = \chi_0$ and (3.3.30), we have

$$\begin{aligned} A_1 &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, \chi_0 \psi(\beta_k^{-1/3} D_t) \chi_0 u \rangle \\ &= 2\operatorname{Re}\langle i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \chi_0 u, \psi(\beta_k^{-1/3} D_t) \chi_0 u \rangle \\ &= \langle [\psi(\beta_k^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \chi_0 u, \chi_0 u \rangle. \end{aligned}$$

By (3.4.19), we know that the symbol $\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))$ belongs to $S(\beta_k, \Gamma)$ and we get

$$\left[\psi(\beta_k^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right] = b_1^w + r_1^w,$$

where $b_1 \in S(\beta_k \lambda_\Gamma^{-1}, \Gamma)$ is a Poisson bracket and $r_1 \in S(\beta_k \lambda_\Gamma^{-3}, \Gamma) \subset S(1, \Gamma)$, with λ_Γ given in (3.3.28) (see (3.4.11)). More precisely, we have

$$\begin{aligned} b_1(t, \tau) &= \frac{1}{i} \left\{ \psi(\beta_k^{-1/3} \tau), i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right\} \\ &= \beta_k^{2/3} \psi'(\beta_k^{-1/3} \tau) \frac{d}{dt} \left(\tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right) \in S(\beta_k^{2/3}, \Gamma). \end{aligned}$$

By (3.3.24), (3.3.25) and (3.4.18), we have in the zone $\{|t - t_k| \leq 2c_0, |\tau| \leq \beta_k^{1/3}\}$

$$b_1(t, \tau) = \beta_k^{2/3} \psi'(\beta_k^{-1/3} \tau) \frac{d}{dt} \left(e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right) \geq \frac{C_1}{2} \beta_k^{2/3}.$$

This implies for all $t, \tau \in \mathbb{R}$,

$$\frac{C_1}{2} \beta_k^{2/3} \leq b_1(t, \tau) + \frac{C_1}{2} \tau^2 + \tilde{C}_1 \beta_k^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \in S(\lambda_\Gamma^2, \Gamma), \quad (3.3.37)$$

where $\tilde{C}_1 = 2\|b_1\|_{0, S(\beta_k^{2/3}, \Gamma)}$. Indeed, the function

$$b_1(t, \tau) + \tilde{C}_1 \beta_k^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \geq \frac{C_1}{2} \beta_k^{2/3} \text{ for all } t \in \mathbb{R} \text{ and } |\tau| \leq \beta_k^{1/3},$$

and it is non-negative for all $t, \tau \in \mathbb{R}$; if $|\tau| \geq \beta_k^{1/3}$, then $\tau^2 \geq \beta_k^{2/3}$, which proves the inequality in (3.3.37). Moreover, each term in the right hand side of (3.3.37) is in $S(\lambda_\Gamma^2, \Gamma)$. The Fefferman-Phong inequality (Proposition 3.4.2) implies

$$b_1(t, \tau)^w + \frac{C_1}{2} D_t^2 + \tilde{C}_1 \beta_k^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \geq \frac{C_1}{2} \beta_k^{2/3} - C'.$$

Applying to $\chi_0 u$ and noting $\chi_0(\cdot) \tilde{\chi}_0(2\cdot) = \chi_0(\cdot)$, we obtain

$$\begin{aligned} A_1 + \frac{C_1}{2} \langle D_t^2 \chi_0 u, \chi_0 u \rangle &= \langle \left(\frac{C_1}{2} D_t^2 + b_1^w \right) \chi_0 u, \chi_0 u \rangle + \langle r_1^w \chi_0 u, \chi_0 u \rangle \\ &\geq \frac{C_1}{2} \beta_k^{2/3} \|\chi_0 u\|^2 - C'' \|\chi_0 u\|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle D_t^2 \chi_0 u, \chi_0 u \rangle &= \|D_t \chi_0 u\|^2 \leq 2\|\chi_0 D_t u\|^2 + 2\|\chi'_0 u\|^2 \\ &\leq C\|D_t u\|^2 + C\|u\|^2, \end{aligned}$$

which gives

$$A_1 \geq \frac{C_1}{2} \beta_k^{2/3} \|\chi_0 u\|^2 - C \|D_t u\|^2 - C \|u\|^2. \quad (3.3.38)$$

For the term A_2 defined in (3.3.36), we have

$$\begin{aligned} A_2 &= -2\operatorname{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, m_{0,k}^w u \rangle \\ &= -2\operatorname{Re}\langle i\beta_k \langle D_k \rangle^{-2} \gamma u, \gamma m_{0,k}^w u \rangle \\ &= -2\operatorname{Re}\langle i\beta_k \langle D_k \rangle^{-2} \gamma u, m_{0,k}^w \gamma u \rangle - 2\operatorname{Re}\langle i\beta_k \langle D_k \rangle^{-2} \gamma u, [\gamma, m_{0,k}^w] u \rangle \\ &=: A_{21} + A_{22}. \end{aligned} \quad (3.3.39)$$

For A_{21} in (3.3.39), since $i\langle D_k \rangle^{-2}$ is skew-adjoint and $m_{0,k}^w$ is self-adjoint, we have

$$A_{21} = i\beta_k \langle [\langle D_k \rangle^{-2}, m_{0,k}^w] \gamma u, \gamma u \rangle. \quad (3.3.40)$$

Recalling (3.3.30) and noting that $\langle D_k \rangle^{-2}$ commutes with $\psi(\beta_k^{-1/3} D_t)$, we get

$$[\langle D_k \rangle^{-2}, m_{0,k}^w] = [\langle D_k \rangle^{-2}, \chi_0] \psi(\beta_k^{-1/3} D_t) \chi_0 + \chi_0 \psi(\beta_k^{-1/3} D_t) [\langle D_k \rangle^{-2}, \chi_0]. \quad (3.3.41)$$

We compute the commutator as follows

$$\begin{aligned} [\langle D_k \rangle^{-2}, \chi_0] &= \langle D_k \rangle^{-2} \chi_0 - \chi_0 \langle D_k \rangle^{-2} \\ &= \langle D_k \rangle^{-2} \left(\chi_0 (D_t^2 + k^2) - (D_t^2 + k^2) \chi_0 \right) \langle D_k \rangle^{-2} \\ &= \langle D_k \rangle^{-2} \left([\chi_0, D_t] D_t + D_t [\chi_0, D_t] \right) \langle D_k \rangle^{-2} \\ &= i \langle D_k \rangle^{-2} \chi'_0 D_t \langle D_k \rangle^{-2} + i \langle D_k \rangle^{-2} D_t \chi'_0 \langle D_k \rangle^{-2}, \end{aligned}$$

which implies

$$[\langle D_k \rangle^{-2}, \chi_0] D_t = i \underbrace{\langle D_k \rangle^{-2} \chi'_0}_{\leq k^{-2}} \underbrace{D_t \langle D_k \rangle^{-2} D_t}_{\leq 1} + i \underbrace{\langle D_k \rangle^{-2} D_t}_{\leq (2k)^{-1}} \underbrace{\chi'_0 \langle D_k \rangle^{-2} D_t}_{\leq (2k)^{-1}},$$

$$\text{and } \|[\langle D_k \rangle^{-2}, \chi_0] D_t\|_{\mathcal{L}(L^2(\mathbb{R}; dt))} \leq \frac{5}{4} \|\chi'_0\|_{L^\infty} k^{-2}.$$

The factorization (3.3.26) of ψ gives

$$\psi(\beta_k^{-1/3} D_t) = -\beta_k^{-1/3} D_t e(\beta_k^{-1/3} D_t) = -\beta_k^{-1/3} e(\beta_k^{-1/3} D_t) D_t,$$

so that

$$[\langle D_k \rangle^{-2}, \chi_0] \psi(\beta_k^{-1/3} D_t) \chi_0 = -\beta_k^{-1/3} \underbrace{[\langle D_k \rangle^{-2}, \chi_0] D_t}_{\text{has norm } \leq Ck^{-2}} \underbrace{e(\beta_k^{-1/3} D_t)}_{\text{bounded}} \chi_0,$$

$$\text{and } | \langle [\langle D_k \rangle^{-2}, \chi_0] \psi(\beta_k^{-1/3} D_t) \chi_0 \gamma u, \gamma u \rangle | \leq C \beta_k^{-1/3} k^{-2} \|\chi_0 \gamma u\| \|\gamma u\|.$$

Similarly we can get

$$| \langle \chi_0 \psi(\beta_k^{-1/3} D_t) [\langle D_k \rangle^{-2}, \chi_0] \gamma u, \gamma u \rangle | \leq C \beta_k^{-1/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma u\|.$$

By (3.3.40) and (3.3.41), we obtain

$$|A_{21}| \leq C\beta_k^{2/3}k^{-2}\|\chi_0\gamma u\|\|\gamma u\|. \quad (3.3.42)$$

For the term A_{22} defined in (3.3.39), we have, using (3.3.30)

$$\begin{aligned} A_{22} &= -2\operatorname{Re}\langle i\beta_k\langle D_k\rangle^{-2}\gamma u, [\gamma, m_{0,k}^w]u\rangle \\ &= -2\operatorname{Re}\langle i\beta_k\langle D_k\rangle^{-2}\gamma u, \chi_0[\gamma, \psi(\beta_k^{-1/3}D_t)]\chi_0u\rangle, \end{aligned}$$

so that we should compute the commutator $[\gamma, \psi(\beta_k^{-1/3}D_t)]$, for which we will do some symbolic calculus with the metric Γ given in Definition 3.3.6. The symbol $\gamma(t)$ is in $S(1, \Gamma)$ since $\gamma(t) = e^{2t}e^{-e^{2t}/8}$ belongs to $C_b^\infty(\mathbb{R})$, and we get

$$[\gamma, \psi(\beta_k^{-1/3}D_t)] = b_2^w + r_2^w,$$

where $b_2 \in S(\lambda_\Gamma^{-1}, \Gamma)$ is a Poisson bracket and $r_2 \in S(\lambda_\Gamma^{-3}, \Gamma) \subset S(\beta_k^{-1}, \Gamma)$, with λ_Γ given in (3.3.28) (see (3.4.11)). By direct computation, we have

$$\begin{aligned} b_2 &= \frac{1}{i}\{\gamma(t), \psi(\beta_k^{-1/3}\tau)\} \\ &= -\frac{1}{i}\beta_k^{-1/3}\psi'(\beta_k^{-1/3}\tau)\gamma'(t) \in S(\beta_k^{-1/3}, \Gamma) \\ &= -\frac{1}{i}\beta_k^{-1/3}\psi'(\beta_k^{-1/3}\tau)\sharp\gamma'(t) + b_3 + r_3, \end{aligned}$$

where $b_3 \in S(\beta_k^{-1/3}\lambda_\Gamma^{-1}, \Gamma)$ is again a Poisson bracket and r_3 belongs to $S(\beta_k^{-1/3}\lambda_\Gamma^{-2}, \Gamma)$ thus to $S(\beta_k^{-1}, \Gamma)$, since $\lambda_\Gamma \geq \beta_k^{1/3}$. We continue to expand b_3

$$\begin{aligned} b_3 &= -\frac{1}{2i}\left\{-\frac{1}{i}\beta_k^{-1/3}\psi'(\beta_k^{-1/3}\tau), \gamma'(t)\right\} \\ &= -\frac{1}{2}\beta_k^{-2/3}\psi''(\beta_k^{-1/3}\tau)\gamma''(t) \in S(\beta_k^{-2/3}, \Gamma) \\ &= -\frac{1}{2}\beta_k^{-2/3}\psi''(\beta_k^{-1/3}\tau)\sharp\gamma''(t) + r_4, \end{aligned}$$

with $r_4 \in S(\beta_k^{-2/3}\lambda_\Gamma^{-1}, \Gamma) \subset S(\beta_k^{-1}, \Gamma)$. Thus we get for $w \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} [\gamma, \psi(\beta_k^{-1/3}D_t)]w &= -\frac{1}{i}\beta_k^{-1/3}\psi'(\beta_k^{-1/3}D_t)\gamma'(t)w - \frac{1}{2}\beta_k^{-2/3}\psi''(\beta_k^{-1/3}D_t)\gamma''(t)w \\ &\quad + (r_2^w + r_3^w + r_4^w)w, \end{aligned}$$

where $r_2, r_3, r_4 \in S(\beta_k^{-1}, \Gamma)$. Since $\psi'(\beta_k^{-1/3}D_t)$ and $\psi''(\beta_k^{-1/3}D_t)$ are bounded on $L^2(\mathbb{R}; dt)$, we deduce that for $w \in C_0^\infty(\mathbb{R})$,

$$\|[\gamma(t), \psi(\beta_k^{-1/3}D_t)]w\| \leq C\beta_k^{-1/3}\|\gamma'w\| + C\beta_k^{-2/3}\|\gamma''w\| + C\beta_k^{-1}\|w\|.$$

Applying the above inequality to χ_0u , we get the estimate for A_{22} defined in (3.3.39):

$$\begin{aligned} |A_{22}| &\leq 2\beta_k\|\langle D_k\rangle^{-2}\gamma u\|\|\chi_0[\gamma, \psi(\beta_k^{-1/3}D_t)]\chi_0u\| \\ &\leq 2\beta_kk^{-2}\|\gamma u\| \times \left(C\beta_k^{-1/3}\|\gamma'\chi_0u\| + C\beta_k^{-2/3}\|\gamma''\chi_0u\| + C\beta_k^{-1}\|\chi_0u\|\right). \end{aligned} \quad (3.3.43)$$

It follows from (3.3.39), (3.3.42) and (3.3.43) that

$$\begin{aligned} |A_2| &\leq C\beta_k^{2/3}k^{-2}\|\gamma u\| \times (\|\chi_0\gamma u\| + \|\chi_0\gamma' u\|) \\ &\quad + C\beta_k^{1/3}k^{-2}\|\gamma u\|\|\chi_0\gamma'' u\| + Ck^{-2}\|\gamma u\|\|\chi_0 u\|. \end{aligned} \quad (3.3.44)$$

Recall $\gamma(t) = e^{2t}g(e^t) = e^{2t}e^{-e^{2t}/8} \leq 8/e$ and $g(e^t) = e^{-e^{2t}/8}$, then

$$\gamma'(t) = e^{2t}g(e^t)(2 - \frac{1}{4}e^{2t}), \quad \gamma''(t) = e^{2t}g(e^t)(4 - \frac{3}{2}e^{2t} + \frac{1}{16}e^{4t}). \quad (3.3.45)$$

Define for $r \geq 0$,

$$\kappa(r) = \sup \left\{ g(s)^{1/2} \max \left(1, \left| 2 - \frac{1}{4}s^2 \right|, \left| 4 - \frac{3}{2}s^2 + \frac{1}{16}s^4 \right| \right); s \in [re^{-c_0}, re^{c_0}] \right\}. \quad (3.3.46)$$

Using the fact that $\chi_0(\cdot - t_k)$ supported in $|t - t_k| \leq c_0$, we have

$$\forall t \in \mathbb{R}, \quad \max(|\chi_0\gamma|, |\chi_0\gamma'|, |\chi_0\gamma''|) \leq \kappa(e^{t_k})e^{2t}g(e^t)^{1/2},$$

and we deduce from (3.3.44) that

$$|A_2| \leq C\beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2 + Ck^{-2}\|u\|^2. \quad (3.3.47)$$

Remark 3.3.11. The function $\kappa(r)$ is bounded and decreasing to 0 at infinity. If ϵ_0 is taken small, then for all $e^{t_k} > \epsilon_0^{-1}$, $\kappa(e^{t_k})$ can be bounded above by $\kappa(\epsilon_0^{-1})$, which is very small.

For the term A_3 defined in (3.3.36), we have

$$\begin{aligned} A_3 &= 2\operatorname{Re}\langle D_t^2 u, m_{0,k}^w u \rangle + 2\operatorname{Re}\langle k^2 u, m_{0,k}^w u \rangle + \frac{1}{8}\operatorname{Re}\langle e^{4t} u, m_{0,k}^w u \rangle \\ &=: A_{31} + A_{32} + A_{33}. \end{aligned} \quad (3.3.48)$$

For A_{31} in (3.3.48),

$$\begin{aligned} A_{31} &= 2\operatorname{Re}\langle D_t u, D_t m_{0,k}^w u \rangle \\ &= 2\operatorname{Re}\langle D_t u, m_{0,k}^w D_t u \rangle + 2\operatorname{Re}\langle D_t u, [D_t, m_{0,k}^w] u \rangle \\ &= 2\operatorname{Re}\langle D_t u, m_{0,k}^w D_t u \rangle + \langle [D_t, [D_t, m_{0,k}^w]] u, u \rangle. \end{aligned}$$

Since $m_{0,k} \in S(1, \Gamma)$ and $\tau \in S(\lambda_\Gamma, \Gamma)$, the double commutator $[D_t, [D_t, m_{0,k}^w]]$ has a symbol in $S(1, \Gamma)$. We get

$$|A_{31}| \leq C\|D_t u\|^2 + C\|u\|^2. \quad (3.3.49)$$

Using the $L^2(\mathbb{R}; dt)$ -boundedness of $m_{0,k}^w$, we get for A_{32} defined in (3.3.48)

$$|A_{32}| \leq Ck^2\|u\|^2. \quad (3.3.50)$$

For A_{33} in (3.3.48), we have by (3.3.31)

$$\begin{aligned} 8A_{33} &= \operatorname{Re}\langle e^{4t} u, \chi_0 \psi(\beta_k^{-1/3} D_t) \chi_0 u \rangle \\ &= \operatorname{Re}\langle e^{2t} u, \underbrace{\chi_0 \psi(\beta_k^{-1/3} D_t) \chi_0 e^{-2t}}_{\text{bounded on } L^2(\mathbb{R}; dt)} e^{2t} u \rangle. \end{aligned}$$

Hence

$$|A_{33}| \leq C\|e^{2t}u\|^2. \quad (3.3.51)$$

By (3.3.48), (3.3.49), (3.3.50) and (3.3.51) we get

$$|A_3| \leq C\|D_t u\|^2 + Ck^2\|u\|^2 + C\|e^{2t}u\|^2. \quad (3.3.52)$$

We deduce from (3.3.36), (3.3.38), (3.3.47) and (3.3.52) that

$$\begin{aligned} A &\geq \frac{C_1}{2}\beta_k^{2/3}\|\chi_0 u\|^2 - C\beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned} \quad (3.3.53)$$

with $\kappa(e^{t_k})$ given by (3.3.46).

Estimates for $2\operatorname{Re}\langle \mathcal{L}_k u, m_{+,k}^w u \rangle$. Recall that $m_{+,k}^w = -i\beta_k^{-1/3}\chi_+(t - t_k)^2$.

$$\begin{aligned} B^+ &:= 2\operatorname{Re}\langle \mathcal{L}_k u, m_{+,k}^w u \rangle = 2\operatorname{Re}\langle \mathcal{L}_k u, -i\beta_k^{-1/3}\chi_+^2 u \rangle \\ &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, -i\beta_k^{-1/3}\chi_+^2 u \rangle \\ &\quad - 2\operatorname{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, -i\beta_k^{-1/3}\chi_+^2 u \rangle \\ &\quad + 2\operatorname{Re}\langle D_t^2 u, -i\beta_k^{-1/3}\chi_+^2 u \rangle \\ &=: B_1^+ + B_2^+ + B_3^+. \end{aligned} \quad (3.3.54)$$

The support of $\chi_+(t - t_k)$ is included in the set $\{t - t_k \geq c_0/2\}$. By (3.4.20) we have

$$\begin{aligned} B_1^+ &= 2\beta_k^{2/3}\langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, \chi_+^2 u \rangle \\ &\geq 2c_1\beta_k^{2/3}\langle e^{2t}\sigma(e^{t_k})\chi_+^2 u, u \rangle. \end{aligned} \quad (3.3.55)$$

For B_2^+ in (3.3.54) we have

$$\begin{aligned} B_2^+ &= \beta_k^{2/3}2\operatorname{Re}\langle \chi_+ \langle D_k \rangle^{-2} \gamma u, \chi_+ \gamma u \rangle \\ &= \beta_k^{2/3}2\operatorname{Re}\langle [\chi_+, \langle D_k \rangle^{-2}] \gamma u, \chi_+ \gamma u \rangle + \beta_k^{2/3}2\operatorname{Re}\langle \langle D_k \rangle^{-2} \chi_+ \gamma u, \chi_+ \gamma u \rangle \\ &= \beta_k^{2/3}\langle [\chi_+, [\chi_+, \langle D_k \rangle^{-2}]] \gamma u, \gamma u \rangle + \beta_k^{2/3}2\operatorname{Re}\underbrace{\langle \langle D_k \rangle^{-2} \chi_+ \gamma u, \chi_+ \gamma u \rangle}_{\geq 0}. \end{aligned}$$

The kernel of $[\chi_+, [\chi_+, \langle D_k \rangle^{-2}]]$ is

$$\frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2,$$

vanishing if $\max(t, s) \leq t_k + c_0/2$ and also if $\min(t, s) \geq t_k + c_0$. Then we have

$$\begin{aligned} &|\langle [\chi_+, [\chi_+, \langle D_k \rangle^{-2}]] \gamma u, \gamma u \rangle| \\ &= \left| \iint_{t-t_k \geq c_0/2} \frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2 (\gamma u)(s)\overline{(\gamma u)(t)} dt ds \right. \\ &\quad \left. + \iint_{\substack{t-t_k \leq c_0/2 \\ s-t_k \geq c_0/2}} \frac{1}{2k}e^{-k|t-s|}[\chi_+(t - t_k) - \chi_+(s - t_k)]^2 (\gamma u)(s)\overline{(\gamma u)(t)} dt ds \right| \\ &\leq g(e^{t_k})^{1/2} \iint_{t-t_k \geq c_0/2} \frac{1}{2k}e^{-k|t-s|} \left(e^{2s}g(e^s)|u(s)| \right) \left(e^{2t}g(e^t)^{1/2}|u(t)| \right) dt ds \\ &\quad + g(e^{t_k})^{1/2} \iint_{\substack{t-t_k \leq c_0/2 \\ s-t_k \geq c_0/2}} \frac{1}{2k}e^{-k|t-s|} \left(e^{2s}g(e^s)^{1/2}|u(s)| \right) \left(e^{2t}g(e^t)|u(t)| \right) dt ds \\ &\leq g(e^{t_k})^{1/2}k^{-2}\|e^{2t}g(e^t)^{1/2}u\|^2, \end{aligned}$$

so that we obtain

$$B_2^+ \geq -\beta_k^{2/3} k^{-2} \kappa(e^{t_k}) \|e^{2t} g(e^t)^{1/2} u\|^2, \quad (3.3.56)$$

where $\kappa(e^{t_k})$ is given in (3.3.46). For B_3^+ in (3.3.54), we have

$$\begin{aligned} B_3^+ &= \beta_k^{-1/3} \langle [D_t^2, -i\chi_+] u, u \rangle \\ &= i\beta_k^{-1/3} \langle (2\chi_+ \chi''_+ + 2\chi'_+ \chi'_+) u, u \rangle + i\beta_k^{-1/3} \langle 4\chi_+ \chi'_+ \partial_t u, u \rangle, \end{aligned}$$

which implies

$$|B_3^+| \leq C\beta_k^{-1/3} \|u\|^2 + C\beta_k^{-1/3} \|\partial_t u\| \|u\| \leq C\|D_t u\|^2 + C\|u\|^2. \quad (3.3.57)$$

We get from (3.3.54), (3.3.55), (3.3.56) and (3.3.57) that

$$\begin{aligned} B^+ &\geq 2c_1 \beta_k^{2/3} \langle e^{2t} \sigma(e^{t_k}) \chi_+^2 u, u \rangle \\ &\quad - \beta_k^{2/3} k^{-2} \kappa(e^{t_k}) \|e^{2t} g(e^t)^{1/2} u\|^2 - C\|u\|^2 - C\|D_t u\|^2. \end{aligned} \quad (3.3.58)$$

Estimates for $2\text{Re}\langle \mathcal{L}_k u, m_{-,k}^w u \rangle$. Recall that $m_{-,k}^w = i\beta_k^{-1/3} \chi_- (t - t_k)^2$.

$$\begin{aligned} B^- &:= 2\text{Re}\langle \mathcal{L}_k u, m_{-,k}^w u \rangle = 2\text{Re}\langle \mathcal{L}_k u, i\beta_k^{-1/3} \chi_-^2 u \rangle \\ &= 2\text{Re}\langle i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, i\beta_k^{-1/3} \chi_-^2 u \rangle \\ &\quad - 2\text{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, i\beta_k^{-1/3} \chi_-^2 u \rangle \\ &\quad + 2\text{Re}\langle D_t^2 u, i\beta_k^{-1/3} \chi_-^2 u \rangle \\ &=: B_1^- + B_2^- + B_3^-. \end{aligned} \quad (3.3.59)$$

Recall (3.4.20) and note that the support of $\chi_-(t - t_k)$ is included in $\{t - t_k \leq -c_0/2\}$, then we get

$$\begin{aligned} B_1^- &= 2\beta_k^{2/3} \langle e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, \chi_-^2 u \rangle \\ &\geq 2c_1 \langle \beta_k^{2/3} e^{2t} \sigma(e^t) \chi_-^2 u, u \rangle. \end{aligned} \quad (3.3.60)$$

For B_2^- in (3.3.59) we have

$$\begin{aligned} B_2^- &= -2\beta_k^{2/3} \text{Re}\langle \chi_- \langle D_k \rangle^{-2} \gamma u, \chi_- \gamma u \rangle \\ &= -2\beta_k^{2/3} \text{Re}\langle [\chi_-, \langle D_k \rangle^{-2}] \gamma u, \chi_- \gamma u \rangle - 2\beta_k^{2/3} \text{Re}\langle \langle D_k \rangle^{-2} \chi_- \gamma u, \chi_- \gamma u \rangle \\ &= -\beta_k^{2/3} \langle [\chi_-, [\chi_-, \langle D_k \rangle^{-2}]] \gamma u, \gamma u \rangle - 2\beta_k^{2/3} \text{Re}\langle \langle D_k \rangle^{-2} \chi_- \gamma u, \chi_- \gamma u \rangle \\ &=: B_{21}^- + B_{22}^-. \end{aligned}$$

For B_{22}^- , we have

$$0 \geq B_{22}^- = -2\beta_k^{2/3} \text{Re}\langle \langle D_k \rangle^{-2} \chi_- \gamma u, \chi_- \gamma u \rangle \geq -2\beta_k^{2/3} k^{-2} \|\chi_- \gamma u\|^2.$$

By using the method that is used to estimate the double commutator in B_2^+ , we find

$$\begin{aligned} |B_{21}^-| &\leq g(e^{t_k - c_0})^{1/2} \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 \\ &\leq \beta_k^{2/3} k^{-2} \kappa(e^{t_k}) \|e^{2t} g(e^t)^{1/2} u\|^2, \end{aligned}$$

where $\kappa(e^{t_k})$ is given in (3.3.46), so that

$$B_2^- \geq -2\beta_k^{2/3}k^{-2}\|\chi_- \gamma u\|^2 - \beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2. \quad (3.3.61)$$

For B_3^- in (3.3.59), we have

$$\begin{aligned} B_3^- &= \beta_k^{-1/3}\langle [D_t^2, i\chi_-^2]u, u \rangle \\ &= -i\beta_k^{-1/3}\langle (2\chi_- \chi''_- + 2\chi'^2_-)u, u \rangle - i\beta_k^{-1/3}\langle 4\chi_- \chi'_- \partial_t u, u \rangle, \end{aligned}$$

which implies

$$|B_3^-| \leq C\beta_k^{-1/3}\|u\|^2 + C\beta_k^{-1/3}\|\partial_t u\|\|u\| \leq C\|u\|^2 + C\|D_t u\|^2. \quad (3.3.62)$$

We get from (3.3.59), (3.3.60), (3.3.61) and (3.3.62) that

$$\begin{aligned} B^- &\geq 2c_1\beta_k^{2/3}\langle e^{2t}\sigma(e^t)\chi_-^2 u, u \rangle - C\|u\|^2 - C\|D_t u\|^2 \\ &\quad - 2\beta_k^{2/3}k^{-2}\|\chi_- e^{2t}g(e^t)u\|^2 - \beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2. \end{aligned} \quad (3.3.63)$$

End of the proof of Proposition 3.3.10. By (3.3.53), (3.3.58), (3.3.63) and the definition 3.3.6 of M_k , we get

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &= A + B^+ + B^- \\ &\geq \frac{C_1}{2}\beta_k^{2/3}\|\chi_0 u\|^2 + 2c_1\beta_k^{2/3}\langle e^{2t}\sigma(e^{t_k})\chi_+^2 u, u \rangle + 2c_1\beta_k^{2/3}\langle e^{2t}\sigma(e^t)\chi_-^2 u, u \rangle \\ &\quad - C\beta_k^{2/3}k^{-2}\kappa(e^{t_k})\|e^{2t}g(e^t)^{1/2}u\|^2 - 2\beta_k^{2/3}k^{-2}\|\chi_- e^{2t}g(e^t)u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned}$$

where $\kappa(e^{t_k})$ is given in (3.3.46). This completes the proof of (3.3.32) in Proposition 3.3.10. \square

Recall the definition (3.3.33) of $\rho(t, t_k)$, then (3.3.32) implies

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq \beta_k^{2/3}\langle (c\rho(t, t_k) - Ck^{-2}e^{4t}g(e^t))u, u \rangle \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned} \quad (3.3.64)$$

since $\kappa(e^{t_k})$ is bounded above by a constant depending on ϵ_0 (see Remark 3.3.11). We have the following two estimates for $\rho(t, t_k)$.

Lemma 3.3.12. *There exist $C_4, C_5 > 0$ such that for $e^{t_k} > \epsilon_0^{-1}$, $t \in \mathbb{R}$, $\alpha \geq 8\pi$, $k \geq 1$,*

$$\rho(t, t_k) \geq C_4 e^{4t}g(e^t), \quad (3.3.65)$$

$$\beta_k^{2/3}\rho(t, t_k) + e^{4t} \geq C_5\beta_k^{1/3}e^{2t}, \quad (3.3.66)$$

where $\rho(t, t_k)$ is given in (3.3.33), g is given in (3.3.3) and β_k is given in (3.3.6).

Proof of Lemma 3.3.12. Suppose $e^{t_k} > \epsilon_0^{-1}$, then $\sigma(e^{t_k}) \geq \delta e^{-2t_k}$ for some $\delta > 0$. Note also that the function $r^4 g(r)$ is bounded. If t is in the support of $\chi_0(\cdot - t_k)$, i.e. $|t - t_k| \leq c_0$, we have

$$e^{4t} g(e^t) \leq C \text{ (always true)}, \quad \beta_k^{2/3} + e^{4t} = (\beta_k^{2/3} e^{-2t} + e^{2t}) e^{2t} \geq \beta_k^{1/3} e^{2t},$$

$$\text{implying } \forall t \in \mathbb{R}, \quad C\chi_0^2 \geq e^{4t} g(e^t)\chi_0^2 \quad \text{and} \quad \beta_k^{2/3}\chi_0^2 + e^{4t}\chi_0^2 \geq \beta_k^{1/3} e^{2t}\chi_0^2.$$

When t is in the support of $\chi_+(\cdot - t_k)$, i.e. $t \geq t_k + c_0/2$, we have

$$e^{2t}\sigma(e^{t_k}) \geq \delta e^{2t}e^{-2t_k} \geq \delta e^{c_0} \geq C e^{4t}g(e^t),$$

$$\beta_k^{2/3}e^{2t}\sigma(e^{t_k}) + e^{4t} \geq (\delta\beta_k^{2/3}e^{-2t_k} + e^{2t_k})e^{2t} \geq \delta^{1/2}\beta_k^{1/3}e^{2t},$$

$$\text{implying } \forall t \in \mathbb{R}, \quad e^{2t}\sigma(e^{t_k})\chi_+^2 \geq C e^{4t}g(e^t)\chi_+^2$$

$$\text{and } \forall t \in \mathbb{R}, \quad \beta_k^{2/3}e^{2t}\sigma(e^{t_k})\chi_+^2 + e^{4t}\chi_+^2 \geq \delta^{1/2}\beta_k^{1/3}e^{2t}\chi_+^2.$$

When t is in the support of $\chi_-(\cdot - t_k)$, i.e. $t \leq t_k - c_0/2$, we have, using (3.4.13) and the first inequality in (3.4.17)

$$e^{4t}g(e^t) \leq 16e^{2t}\sigma(e^t),$$

$$\beta_k^{2/3}e^{2t}\sigma(e^t) + e^{4t} \geq (\beta_k^{2/3}\sigma(e^t) + e^{2t})e^{2t} \geq (2\log 2)^{-1}\beta_k^{1/3}e^{2t},$$

$$\text{implying } \forall t \in \mathbb{R}, \quad 16e^{2t}\sigma(e^t)\chi_-^2 \geq e^{4t}g(e^t)\chi_-^2$$

$$\text{and } \forall t \in \mathbb{R}, \quad \beta_k^{2/3}e^{2t}\sigma(e^t)\chi_-^2 + e^{4t}\chi_-^2 \geq (2\log 2)^{-1}\beta_k^{1/3}e^{2t}\chi_-^2.$$

Summing up, this completes the proof of (3.3.65) and (3.3.66). \square

Proof of Theorem 3.3.5. The estimates (3.3.64) and (3.3.65) imply that there exists $k_0 \geq 1$, for all $k \geq k_0$,

$$2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle \geq \frac{c}{2}\beta_k^{2/3}\langle \rho(t, t_k)u, u \rangle - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2.$$

Together with (3.3.35), by choosing $C_6 > 0$ large enough, we have for $k \geq k_0$,

$$\operatorname{Re}\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle \geq \frac{c}{2}\langle (\beta_k^{2/3}\rho(t, t_k) + D_t^2 + k^2 + e^{4t})u, u \rangle. \quad (3.3.67)$$

It follows from (3.3.67) and (3.3.66) that for $k \geq k_0$,

$$\operatorname{Re}\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle \geq C\langle \beta_k^{1/3}e^{2t}u, u \rangle. \quad (3.3.68)$$

Noticing that $e^t(C_6 + 2M_k)e^{-t}$ is bounded on $L^2(\mathbb{R}; dt)$ by (3.3.31) and that

$$\langle \mathcal{L}_k u, (C_6 + 2M_k)u \rangle = \langle e^{-t}\mathcal{L}_k u, (e^t(C_6 + 2M_k)e^{-t})(e^t u) \rangle,$$

we deduce from (3.3.68) and Cauchy-Schwarz inequality that

$$\|e^{-t}\mathcal{L}_k u\| \|e^t u\| \geq C\beta_k^{1/3}\|e^t u\|^2, \quad \text{for } k \geq k_0,$$

which is

$$\|e^{-t}\mathcal{L}_k u\| \geq C\beta_k^{1/3}\|e^t u\|, \quad k \geq k_0,$$

completing the proof of Theorem 3.3.5. \square

Remark 3.3.13. There is a localization effect taking place in this case. We see in (3.3.32) of Proposition 3.3.10 that the coefficient of the term $\|e^{2t}g(e^t)^{1/2}u\|^2$ has a factor $\kappa(e^{t_k})$, which is small if e^{t_k} is taken very large (see Remark 3.3.11). As a result, if we suppose e^{t_k} large enough, this term is negligible (even without taking k large), and the only bad term coming from the nonlocal operator that we need to control is

$$2\beta_k^{2/3}k^{-2}\|e^{2t}g(e^t)\chi_-u\|^2.$$

On the other hand, we can prove that there exists $\epsilon_2 > 0$ such that for all $e^{t_k} > \epsilon_2^{-1}$,

$$\forall k \geq 2, \forall t \leq t_k - \frac{c_0}{2}, \quad 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) > 2k^{-2}e^{4t}g(e^t)^2.$$

This implies that it suffices to take $k \geq 2$ to absorb the remainders and thus Theorem 3.3.5 holds for $k_0 = 2$ and $e^{t_k} > \epsilon_2^{-1}$. Furthermore, we shall see that this localization effect does not present in Case 2 and Case 3.

3.3.3.b) Case 2: $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$

Theorem 3.3.14. Suppose $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. Then there exist $C > 0$, $k_0 \geq 1$ such that for all $k \geq k_0$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

$$\|e^{-t}\mathcal{L}_k u\| \geq C\beta_k^{2/3}\|e^t u\|, \quad (3.3.69)$$

where \mathcal{L}_k is given in (3.3.21) and β_k is given in (3.3.6).

We present some inequalities concerning σ that will be used in Case 2 in Proposition 3.4.7,(2). Note that they are similar to those in the Case 1 (given in Proposition 3.4.7,(1)).

We use the metric Γ and the multiplier M_k in Definition 3.3.6, 3.3.8, and we use the notations A_1, A_2, A_3, B^+, B^- in (3.3.36), (3.3.54), (3.3.59). The estimate (3.3.38) for A_1 is valid with constant C_1 replaced by C_2 (which is given in (3.4.21)) and the estimate (3.3.52) for A_3 holds in Case 2. For A_2 , the estimate (3.3.44) remains true:

$$\begin{aligned} |A_2| &= |2\operatorname{Re}\langle i\beta_k\gamma\langle D_k\rangle^{-2}\gamma u, m_{0,k}^w u\rangle| \\ &\leq C\beta_k^{2/3}k^{-2}\|\gamma u\|(\|\chi_0\gamma u\| + \|\chi_0\gamma' u\|) \\ &\quad + C\beta_k^{2/3}k^{-2}\|\gamma u\|\|\chi_0\gamma'' u\| + Ck^{-2}\|\gamma u\|\|\chi_0 u\|. \end{aligned}$$

In the case where $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$, we have by (3.3.45):

$$|\chi_0(t - t_k)\gamma'(t)| \leq C\gamma(t), \quad |\chi_0(t - t_k)\gamma''(t)| \leq C\gamma(t),$$

for some C depending on $\epsilon_1, \epsilon_0^{-1}$, so that

$$|A_2| \leq C\beta_k^{2/3}k^{-2}\|\gamma u\|^2 + Ck^{-2}\|u\|^2. \quad (3.3.70)$$

For the terms B^+, B^- , we have by (3.4.23),

$$\begin{aligned} B^+ &= 2\operatorname{Re}\langle \mathcal{L}_k u, m_{+,k}^w u\rangle \\ &\geq 2c_2\beta_k^{2/3}\langle e^{2t}\sigma(e^{t_k})\chi_+^2 u, u\rangle - C\|u\|^2 - C\|D_t u\|^2 - 2\beta_k^{2/3}k^{-2}\|\gamma u\|^2, \end{aligned}$$

$$\begin{aligned} B^- &= 2\operatorname{Re}\langle \mathcal{L}_k u, m_{-,k}^w u\rangle \\ &\geq 2c_2\beta_k^{2/3}\langle e^{2t}\sigma(e^t)\chi_-^2 u, u\rangle - C\|u\|^2 - C\|D_t u\|^2 - 2\beta_k^{2/3}k^{-2}\|\gamma u\|^2. \end{aligned}$$

Summarizing, we get that for all $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &= A + B^+ + B^- \\ &\geq \frac{C_2}{2} \beta_k^{2/3} \|\chi_0 u\|^2 + 2c_2 \beta_k^{2/3} \langle e^{2t} \sigma(e^{t_k}) \chi_+^2 u, u \rangle + 2c_2 \beta_k^{2/3} \langle e^{2t} \sigma(e^t) \chi_-^2 u, u \rangle \\ &\quad - C \beta_k^{2/3} k^{-2} \|\gamma u\|^2 - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2, \end{aligned}$$

so that the following proposition is proved:

Proposition 3.3.15. *Suppose $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$. There exist $c > 0, C > 0$ such that for $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,*

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq \beta_k^{2/3} \langle (c\rho(t, t_k) - Ck^{-2}e^{4t}g(e^t)^2)u, u \rangle \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2, \end{aligned} \quad (3.3.71)$$

where \mathcal{L}_k is given in (3.3.21), M_k in Definition 3.3.8, g in (3.3.3), β_k in (3.3.6) and

$$\rho(t, t_k) = \chi_0(t - t_k)^2 + e^{2t} \sigma(e^{t_k}) \chi_+(t - t_k)^2 + e^{2t} \sigma(e^t) \chi_-(t - t_k)^2, \quad (3.3.72)$$

with χ_0, χ_\pm defined in (3.3.23) and σ in (3.3.3).

We have the following estimates for $\rho(t, t_k)$.

Lemma 3.3.16. *There exist $C_7, C_8 > 0$ such that for all $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$, $t \in \mathbb{R}$,*

$$\rho(t, t_k) \geq C_7 e^{4t} g(e^t)^2, \quad (3.3.73)$$

$$\rho(t, t_k) \geq C_8 e^{2t}, \quad (3.3.74)$$

where $\rho(t, t_k)$ is given in (3.3.72) and g is given in (3.3.3).

Proof of Lemma 3.3.16. Firstly, we have for $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$, $\sigma(e^{t_k}) \geq \sigma(\epsilon_0^{-1}) = \delta > 0$, since σ is decreasing. Note also that $r^4 g(r)^2$, $r^2 g(r)^2$ are bounded functions. If t is in the support of $\chi_0(\cdot - t_k)$, we have $|t - t_k| \leq c_0$, then

$$e^{4t} g(e^t)^2 \leq C, \quad e^{2t} \leq \epsilon_0^{-2} e^{2c_0},$$

$$\text{implying } \forall t \in \mathbb{R}, \quad e^{4t} g(e^t)^2 \chi_0^2 \leq C \chi_0^2 \quad \text{and} \quad e^{2t} \chi_0^2 \leq \epsilon_0^{-2} e^{2c_0} \chi_0^2.$$

If t is in the support of $\chi_+(\cdot - t_k)$, we have $t \geq t_k + c_0/2$, (recall $\sigma(e^{t_k}) \geq \delta$ and $r^2 g(r)^2$ bounded)

$$e^{4t} g(e^t)^2 \leq C e^{2t} \leq C \delta^{-1} e^{2t} \sigma(e^{t_k}), \quad e^{2t} \leq \delta^{-1} e^{2t} \sigma(e^{t_k}),$$

$$\text{implying } \forall t \in \mathbb{R}, \quad e^{4t} g(e^t)^2 \chi_+^2 \leq C \delta^{-1} e^{2t} \sigma(e^{t_k}) \chi_+^2 \quad \text{and} \quad e^{2t} \chi_+^2 \leq \delta^{-1} e^{2t} \sigma(e^{t_k}) \chi_+^2.$$

If t is in the support of $\chi_-(\cdot - t_k)$, we have $t \leq t_k - c_0/2$, then $\sigma(e^t) \geq \delta$,

$$e^{4t} g(e^t)^2 \leq 3e^{2t} \sigma(e^t) \quad (\text{by (3.4.12)}), \quad e^{2t} \leq \delta^{-1} e^{2t} \sigma(e^t),$$

$$\text{implying } \forall t \in \mathbb{R}, \quad e^{4t} g(e^t)^2 \chi_-^2 \leq 3e^{2t} \sigma(e^t) \chi_-^2 \quad \text{and} \quad e^{2t} \chi_-^2 \leq \delta^{-1} e^{2t} \sigma(e^t) \chi_-^2.$$

Summing up, the inequalities (3.3.73), (3.3.74) are proved. \square

Proof of Theorem 3.3.14. (3.3.71) and (3.3.73) imply that there exists $k_0 \geq 1$, for $k \geq k_0$,

$$2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle \geq \frac{c}{2} \beta_k^{2/3} \langle \rho(t, t_k) u, u \rangle - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2. \quad (3.3.75)$$

Hence by choosing $C_9 > 0$ large enough, we get for $k \geq k_0$

$$\operatorname{Re}\langle \mathcal{L}_k u, (C_9 + 2M_k) u \rangle \geq \frac{c}{2} \langle (\beta_k^{2/3} \rho(t, t_k) + D_t^2 + k^2 + e^{4t}) u, u \rangle,$$

and in particular by (3.3.74)

$$\operatorname{Re}\langle \mathcal{L}_k u, (C_9 + 2M_k) u \rangle \geq C \langle \beta_k^{2/3} e^{2t} u, u \rangle, \quad \forall k \geq k_0.$$

Finally we obtain the inequality (3.3.69) by using the $L^2(\mathbb{R}; dt)$ -boundedness of the operator $e^t(C_9 + M_k)e^{-t}$ and Cauchy-Schwarz inequality. \square

3.3.4 Nontrivial cases, continued

3.3.4.c) Case 3: $\beta_k^{-1/4} < e^{t_k} < \epsilon_1$

We present in Proposition 3.4.7,(3) the inequalities about the function σ to be used in this case. Moreover, we assume $\alpha_0 \geq 8\pi$ such that the interval $(\beta_k^{-1/4}, \epsilon_1)$ is not empty for any $k \geq 1$, $\alpha \geq \alpha_0$.

Theorem 3.3.17. Suppose $e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1)$. Then there exist $C > 0$, $k_0 \geq 3$ such that for all $k \geq k_0$, $\alpha \geq \alpha_0$, $u \in C_0^\infty(\mathbb{R})$,

$$\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad (3.3.76)$$

where \mathcal{L}_k is given in (3.3.21) and β_k is given in (3.3.6).

We shall modify the metric Γ and the multiplier M_k as follows.

Definition 3.3.18.

$$\Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + (\beta_k e^{4t_k})^{2/3}}, \quad (t, \tau) \in \mathbb{R}_t \times \mathbb{R}_\tau, \quad (3.3.77)$$

$$M_k = m_{0,k}^w + m_{+,k}^w + m_{-,k}^w, \quad (3.3.78)$$

where

$$\begin{aligned} m_{0,k}(t, \tau) &= \chi_0(t - t_k) \# \psi((\beta_k e^{4t_k})^{-1/3} \tau) \# \chi_0(t - t_k), \\ m_{+,k}(t, \tau) &= -i(\beta_k e^{4t_k})^{-1/3} \chi_+(t - t_k)^2, \\ m_{-,k}(t, \tau) &= i(\beta_k e^{4t_k})^{-1/3} \chi_-(t - t_k)^2, \end{aligned}$$

with χ_0, χ_\pm, ψ given in (3.3.23), (3.3.25).

Remark 3.3.19. Since we are in the region $e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1)$, we have

$$\lambda_\Gamma = (\tau^2 + (\beta_k e^{4t_k})^{2/3})^{1/2} \geq (\beta_k e^{4t_k})^{1/3} \geq 1, \quad (3.3.79)$$

so that the metric Γ verifies the uncertainty principle and moreover, Γ is uniformly admissible (see Lemma 3.4.1). Furthermore, the operator $e^t M_k e^{-t}$ is bounded on $L^2(\mathbb{R}; dt)$.

Proposition 3.3.20. Suppose $e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1)$. There exist $c, C > 0$ such that for all $k \geq 3$, $\alpha \geq \alpha_0$, $u \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq \beta_k (\beta_k e^{4t_k})^{-1/3} \langle \left(c\tilde{\rho}(t, t_k) - Ck^{-2}e^{4t}g(e^t)^2 \right) u, u \rangle \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned} \quad (3.3.80)$$

where \mathcal{L}_k is given in (3.3.21), M_k in Definition 3.3.18, g in (3.3.3), β_k in (3.3.6) and

$$\tilde{\rho}(t, t_k) = e^{4t_k} \chi_0(t - t_k)^2 + e^{2t} (1 - \sigma(e^t)) \chi_+(t - t_k)^2 + e^{2t} (1 - \sigma(e^{t_k})) \chi_-(t - t_k)^2, \quad (3.3.81)$$

with χ_0, χ_\pm defined in (3.3.23) and σ given in (3.3.3).

Proof of Proposition 3.3.20.

Estimates for $2\operatorname{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle$.

$$\begin{aligned} A := 2\operatorname{Re}\langle \mathcal{L}_k u, m_{0,k}^w u \rangle &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, m_{0,k}^w u \rangle \\ &\quad - 2\operatorname{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, m_{0,k}^w u \rangle \\ &\quad + 2\operatorname{Re}\langle (D_t^2 + k^2 + \frac{1}{16}e^{4t})u, m_{0,k}^w u \rangle \\ &=: A_1 + A_2 + A_3. \end{aligned} \quad (3.3.82)$$

For A_1 in (3.3.82), we get a commutator

$$\begin{aligned} A_1 &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 u \rangle \\ &= \langle [\psi((\beta_k e^{4t_k})^{-1/3} D_t), i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \chi_0 u, \chi_0 u \rangle, \end{aligned}$$

where $\tilde{\chi}_0$ is given in (3.3.24). We know that, with Γ given in Definition 3.3.18

$$\left[\psi((\beta_k e^{4t_k})^{-1/3} D_t), \underbrace{i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))}_{\in S(\beta_k e^{4t_k}, \Gamma) \text{ by (3.4.25)}} \right] = b_1^w + r_1^w,$$

where b_1 is a Poisson bracket and $r_1 \in S(\beta_k e^{4t_k} \lambda_\Gamma^{-3}, \Gamma) \subset S(1, \Gamma)$, with λ_Γ given in (3.3.79) (see (3.4.11)). More precisely,

$$\begin{aligned} b_1(t, \tau) &= \frac{1}{i} \left\{ \psi((\beta_k e^{4t_k})^{-1/3} \tau), i\beta_k \tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right\} \\ &= \beta_k (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} \tau) \frac{d}{dt} \left(\tilde{\chi}_0 e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right) \\ &\in S((\beta_k e^{4t_k})^{2/3}, \Gamma) \subset S(\lambda_\Gamma^2, \Gamma). \end{aligned}$$

By (3.3.24), (3.3.25) and (3.4.24), we have in the zone $\{|t - t_k| \leq 2c_0, |\tau| \leq (\beta_k e^{4t_k})^{1/3}\}$

$$\begin{aligned} b_1(t, \tau) &= \beta_k (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3} \tau) \frac{d}{dt} \left(e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \right) \\ &\geq \beta_k (\beta_k e^{4t_k})^{-1/3} \times \frac{1}{2} C_3 e^{4t_k} \geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3}. \end{aligned} \quad (3.3.83)$$

This implies for all $t, \tau \in \mathbb{R}$,

$$\frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} \leq b_1(t, \tau) + \frac{C_3}{2} \tau^2 + \tilde{C}_3 (\beta_k e^{4t_k})^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \in S(\lambda_\Gamma^2, \Gamma), \quad (3.3.84)$$

where $\tilde{C}_3 = 2\|b_1\|_{0,S((\beta_k e^{4t_k})^{2/3}, \Gamma)}$. Indeed, the function

$$b_1(t, \tau) + \tilde{C}_3(\beta_k e^{4t_k})^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3}$$

for all $t \in \mathbb{R}$ and $|\tau| \leq (\beta_k e^{4t_k})^{1/3}$, and it is non-negative for all $t, \tau \in \mathbb{R}$; if $|\tau| \geq (\beta_k e^{4t_k})^{1/3}$, then $\tau^2 \geq (\beta_k e^{4t_k})^{2/3}$, which proves the inequality in (3.3.84). Moreover, each term in the right hand side of (3.3.84) is in $S(\lambda_\Gamma^2, \Gamma)$. The Fefferman-Phong inequality (Proposition 3.4.2) implies

$$b_1(t, \tau)^w + \frac{C_3}{2} D_t^2 + \tilde{C}_3(\beta_k e^{4t_k})^{2/3} \left(1 - \tilde{\chi}_0(2(t - t_k)) \right) \geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3} - C'.$$

Applying to $\chi_0 u$, we get

$$\begin{aligned} A_1 + \frac{C_3}{2} \langle D_t^2 \chi_0 u, \chi_0 u \rangle &= \langle \left(\frac{C_3}{2} D_t^2 + b_1^w \right) \chi_0 u, \chi_0 u \rangle + \langle r_1^w \chi_0 u, \chi_0 u \rangle \\ &\geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3} \|\chi_0 u\|^2 - C'' \|\chi_0 u\|^2. \end{aligned}$$

Hence we get the estimate for A_1 :

$$A_1 \geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3} \|\chi_0 u\|^2 - C \|D_t u\|^2 - C \|u\|^2. \quad (3.3.85)$$

For A_2 defined in (3.3.82), we have

$$\begin{aligned} A_2 &= -2\operatorname{Re} \langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, m_{0,k}^w u \rangle \\ &= -2\operatorname{Re} \langle i\beta_k \langle D_k \rangle^{-2} \gamma u, m_{0,k}^w \gamma u \rangle - 2\operatorname{Re} \langle i\beta_k \langle D_k \rangle^{-2} \gamma u, [\gamma, m_{0,k}^w] u \rangle \\ &=: A_{21} + A_{22}. \end{aligned} \quad (3.3.86)$$

For A_{21} in (3.3.86), since $i\langle D_k \rangle^{-2}$ is skew-adjoint and $m_{0,k}^w$ is self-adjoint, we get

$$A_{21} = i\beta_k \langle [\langle D_k \rangle^{-2}, m_{0,k}^w] \gamma u, \gamma u \rangle.$$

Noting that $\langle D_k \rangle^{-2}$ commutes with $\psi((\beta_k e^{4t_k})^{-1/3} D_t)$, we have

$$\begin{aligned} [\langle D_k \rangle^{-2}, m_{0,k}^w] &= [\langle D_k \rangle^{-2}, \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0] \\ &= [\langle D_k \rangle^{-2}, \chi_0] \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 + \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) [\langle D_k \rangle^{-2}, \chi_0]. \end{aligned}$$

By using the method that is used in Case 1, we can get

$$\begin{aligned} |\langle [\langle D_k \rangle^{-2}, \chi_0] \psi((\beta_k e^{4t_k})^{-1/3} D_t) \chi_0 \gamma u, \gamma u \rangle| &\leq C(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\chi_0 \gamma u\| \|\gamma u\|, \\ |\langle \chi_0 \psi((\beta_k e^{4t_k})^{-1/3} D_t) [\langle D_k \rangle^{-2}, \chi_0] \gamma u, \gamma u \rangle| &\leq C(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma u\|, \end{aligned}$$

so that

$$|A_{21}| \leq C\beta_k(\beta_k e^{4t_k})^{-1/3} k^{-2} \|\chi_0 \gamma u\| \|\gamma u\|. \quad (3.3.87)$$

For A_{22} in (3.3.86), we have

$$\begin{aligned} A_{22} &= -2\operatorname{Re} \langle i\beta_k \langle D_k \rangle^{-2} \gamma u, [\gamma, m_{0,k}^w] u \rangle \\ &= -2\operatorname{Re} \langle i\beta_k \langle D_k \rangle^{-2} \gamma u, \chi_0 [\tilde{\chi}_0 \gamma, \psi((\beta_k e^{4t_k})^{-1/3} D_t)] \chi_0 u \rangle, \end{aligned}$$

where $\tilde{\chi}_0$ is given in (3.3.24). Since $\tilde{\chi}_0\gamma = \tilde{\chi}_0(t - t_k)e^{2t}g(e^t) \in S(e^{2t_k}, \Gamma)$, we get

$$[\tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)] = b_2^w + r_2^w,$$

where $b_2 \in S(e^{2t_k}\lambda_\Gamma^{-1}, \Gamma)$ is a Poisson bracket and r_2 belongs to $S(e^{2t_k}\lambda_\Gamma^{-3}, \Gamma) \subset S(e^{2t_k}(\beta_k e^{4t_k})^{-1}, \Gamma)$, with λ_Γ given in (3.3.79) (see (3.4.11)). We compute b_2 as follows

$$\begin{aligned} b_2 &= \frac{1}{i} \left\{ \tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}\tau) \right\} \\ &= -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3}\tau)(\tilde{\chi}_0\gamma)'(t) \in S(e^{2t_k}(\beta_k e^{4t_k})^{-1/3}, \Gamma) \\ &= -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3}\tau) \sharp(\tilde{\chi}_0\gamma)'(t) + b_3 + r_3, \end{aligned}$$

where $b_3 \in S(e^{2t_k}(\beta_k e^{4t_k})^{-2/3}, \Gamma)$ is a Poisson bracket and $r_3 \in S(e^{2t_k}(\beta_k e^{4t_k})^{-1}, \Gamma)$. We continue to expand b_3

$$\begin{aligned} b_3 &= -\frac{1}{2i} \left\{ -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3}\tau), (\tilde{\chi}_0\gamma)'(t) \right\} \\ &= -\frac{1}{2} (\beta_k e^{4t_k})^{-2/3} \psi''((\beta_k e^{4t_k})^{-1/3}\tau)(\tilde{\chi}_0\gamma)''(t) \\ &= -\frac{1}{2} (\beta_k e^{4t_k})^{-2/3} \psi''((\beta_k e^{4t_k})^{-1/3}\tau) \sharp(\tilde{\chi}_0\gamma)''(t) + r_4, \end{aligned}$$

where $r_4 \in S(e^{2t_k}(\beta_k e^{4t_k})^{-1}, \Gamma)$. Thus we get for $w \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} [\tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)]w &= -\frac{1}{i} (\beta_k e^{4t_k})^{-1/3} \psi'((\beta_k e^{4t_k})^{-1/3}D_t)(\tilde{\chi}_0\gamma)'(t)w \\ &\quad - \frac{1}{2} (\beta_k e^{4t_k})^{-2/3} \psi''((\beta_k e^{4t_k})^{-1/3}D_t)(\tilde{\chi}_0\gamma)''(t)w + (r_2^w + r_3^w + r_4^w)w, \end{aligned}$$

where $r_2, r_3, r_4 \in S(e^{2t_k}(\beta_k e^{4t_k})^{-1}, \Gamma)$. Using the boundedness of ψ' and ψ'' , we obtain for $w \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} \|[\tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)]w\| &\leq C(\beta_k e^{4t_k})^{-1/3} \|(\tilde{\chi}_0\gamma)'w\| \\ &\quad + C(\beta_k e^{4t_k})^{-2/3} \|(\tilde{\chi}_0\gamma)''w\| + Ce^{2t_k}(\beta_k e^{4t_k})^{-1} \|w\|. \end{aligned}$$

Now the term A_{22} defined in (3.3.86) can be estimated as follows:

$$\begin{aligned} |A_{22}| &= |2\operatorname{Re} \langle i\beta_k \chi_0 \langle D_k \rangle^{-2} \gamma u, [\tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)] \chi_0 u \rangle| \\ &\leq 2\beta_k \|\chi_0 \langle D_k \rangle^{-2} \gamma u\| \|[\tilde{\chi}_0\gamma, \psi((\beta_k e^{4t_k})^{-1/3}D_t)] \chi_0 u\| \\ &\leq 2\beta_k \|\chi_0 \langle D_k \rangle^{-2} \gamma u\| \times \left(C(\beta_k e^{4t_k})^{-1/3} \|(\tilde{\chi}_0\gamma)' \chi_0 u\| \right. \\ &\quad \left. + C(\beta_k e^{4t_k})^{-2/3} \|(\tilde{\chi}_0\gamma)'' \chi_0 u\| + Ce^{2t_k}(\beta_k e^{4t_k})^{-1} \|\chi_0 u\| \right) \\ &\leq C\beta_k (\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\| \|\gamma' \chi_0 u\| + C\beta_k (\beta_k e^{4t_k})^{-2/3} k^{-2} \|\gamma u\| \|\gamma'' \chi_0 u\| \\ &\quad + Ck^{-2} \|g(e^t)u\| \|\chi_0 u\|, \tag{3.3.88} \end{aligned}$$

where in the last inequality we use the following

$$\|\chi_0 \langle D_k \rangle^{-2} \gamma u\| = \|\chi_0 e^{2t} \underbrace{e^{-2t} \langle D_k \rangle^{-2} e^{2t}}_{\substack{\text{has norm } \leq 3k^{-2} \\ \text{since } k \geq 3}} g(e^t)u\| \leq Ce^{2t_k} k^{-2} \|g(e^t)u\|$$

(see Lemma 3.4.5).

It follows from (3.3.86), (3.3.87) and (3.3.88) that

$$\begin{aligned} |A_2| &\leq C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|(\|\chi_0\gamma u\| + \|\chi_0\gamma' u\|) \\ &\quad + C\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|\gamma u\|\|\chi_0\gamma'' u\| + Ck^{-2}\|u\|^2. \end{aligned} \quad (3.3.89)$$

From (3.3.45) we deduce that for $e^{t_k} < \epsilon_1$,

$$|\chi_0(t - t_k)\gamma'(t)| \leq C\gamma(t), \quad |\chi_0(t - t_k)\gamma''(t)| \leq C\gamma(t),$$

with C depending only on ϵ_1 , so that

$$|A_2| \leq C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2 + Ck^{-2}\|u\|^2. \quad (3.3.90)$$

The estimate for A_3 defined in (3.3.82) is the same as that in Case 1

$$|A_3| \leq C\|D_t u\|^2 + Ck^2\|u\|^2 + C\|e^{2t}u\|^2. \quad (3.3.91)$$

We deduce from (3.3.82), (3.3.85), (3.3.90) and (3.3.91) that

$$\begin{aligned} A &\geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3}\|\chi_0 u\|^2 - C\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2 - Ck^{-2}\|u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2. \end{aligned} \quad (3.3.92)$$

Estimates for $2\operatorname{Re}\langle \mathcal{L}_k u, m_{+,k}^w u \rangle$. Recall $m_{+,k}^w = -i(\beta_k e^{4t_k})^{-1/3}\chi_+(t - t_k)^2$.

$$\begin{aligned} B^+ &:= 2\operatorname{Re}\langle \mathcal{L}_k u, m_{+,k}^w u \rangle = 2\operatorname{Re}\langle \mathcal{L}_k u, -i(\beta_k e^{4t_k})^{-1/3}\chi_+^2 u \rangle \\ &= 2\operatorname{Re}\langle i\beta_k e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, -i(\beta_k e^{4t_k})^{-1/3}\chi_+^2 u \rangle \\ &\quad - 2\operatorname{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, -i(\beta_k e^{4t_k})^{-1/3}\chi_+^2 u \rangle \\ &\quad + 2\operatorname{Re}\langle D_t^2 u, -i(\beta_k e^{4t_k})^{-1/3}\chi_+^2 u \rangle \\ &= B_1^+ + B_2^+ + B_3^+. \end{aligned} \quad (3.3.93)$$

Recall that the support of $\chi_+(t - t_k)$ is included in $\{t - t_k \geq c_0/2\}$ and (3.4.26). Thus

$$\begin{aligned} B_1^+ &= 2\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, \chi_+^2 u \rangle \\ &\geq 2c_3\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(1 - \sigma(e^t))\chi_+^2 u, u \rangle. \end{aligned} \quad (3.3.94)$$

For B_2^+ in (3.3.93) we get

$$\begin{aligned} |B_2^+| &= |2\beta_k(\beta_k e^{4t_k})^{-1/3}\operatorname{Re}\langle \gamma \langle D_k \rangle^{-2} \gamma u, \chi_+^2 u \rangle| \\ &\leq 2\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2. \end{aligned} \quad (3.3.95)$$

For B_3^+ in (3.3.93) we have

$$\begin{aligned} |B_3^+| &= |(\beta_k e^{4t_k})^{-1/3}\langle [D_t^2, -i\chi_+^2]u, u \rangle| \\ &\leq C(\beta_k e^{4t_k})^{-1/3}(\|u\|^2 + \|\partial_t u\|\|u\|) \leq C\|D_t u\|^2 + C\|u\|^2. \end{aligned} \quad (3.3.96)$$

We get from (3.3.93), (3.3.94), (3.3.95) and (3.3.96) that

$$\begin{aligned} B^+ &\geq 2c_3\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(1 - \sigma(e^t))\chi_+^2 u, u \rangle \\ &\quad - C\|u\|^2 - C\|D_t u\|^2 - 2\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2. \end{aligned} \quad (3.3.97)$$

Estimates for $2\operatorname{Re}\langle \mathcal{L}_k u, m_{-,k}^w u \rangle$. Recall that $m_{-,k}^w = i(\beta_k e^{4t_k})^{-1/3} \chi_-(t - t_k)^2$.

$$\begin{aligned} B^- &:= 2\operatorname{Re}\langle \mathcal{L}_k u, m_{-,k}^w u \rangle = 2\operatorname{Re}\langle \mathcal{L}_k u, i(\beta_k e^{4t_k})^{-1/3} \chi_-^2 u \rangle \\ &= 2\operatorname{Re}\langle i\beta_k e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, i(\beta_k e^{4t_k})^{-1/3} \chi_-^2 u \rangle \\ &\quad - 2\operatorname{Re}\langle i\beta_k \gamma \langle D_k \rangle^{-2} \gamma u, i(\beta_k e^{4t_k})^{-1/3} \chi_-^2 u \rangle \\ &\quad + 2\operatorname{Re}\langle D_t^2 u, i(\beta_k e^{4t_k})^{-1/3} \chi_-^2 u \rangle \\ &= B_1^- + B_2^- + B_3^-. \end{aligned} \tag{3.3.98}$$

Recall that the support of $\chi_-(t - t_k)$ is included in $\{t - t_k \leq -c_0/2\}$ and (3.4.26). Thus

$$\begin{aligned} B_1^- &= 2\beta_k (\beta_k e^{4t_k})^{-1/3} \langle e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, \chi_-^2 u \rangle \\ &\geq 2c_3 \beta_k (\beta_k e^{4t_k})^{-1/3} \langle e^{2t} (1 - \sigma(e^{t_k})) \chi_-^2 u, u \rangle. \end{aligned} \tag{3.3.99}$$

For B_2^- in (3.3.98) we have

$$\begin{aligned} |B_2^-| &= |2\beta_k (\beta_k e^{4t_k})^{-1/3} \operatorname{Re}\langle \gamma \langle D_k \rangle^{-2} \gamma u, \chi_-^2 u \rangle| \\ &\leq 2\beta_k (\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\|^2. \end{aligned} \tag{3.3.100}$$

For B_3^- in (3.3.98), we have

$$\begin{aligned} |B_3^-| &= |(\beta_k e^{4t_k})^{-1/3} \langle [D_t^2, i\chi_-^2] u, u \rangle| \\ &\leq C (\beta_k e^{4t_k})^{-1/3} (\|u\|^2 + \|\partial_t u\| \|u\|) \leq C \|u\|^2 + C \|D_t u\|^2. \end{aligned} \tag{3.3.101}$$

We get from (3.3.98), (3.3.99), (3.3.100) and (3.3.101) that

$$\begin{aligned} B^- &\geq 2c_3 \beta_k (\beta_k e^{4t_k})^{-1/3} \langle e^{2t} (1 - \sigma(e^{t_k})) \chi_-^2 u, u \rangle \\ &\quad - C \|u\|^2 - C \|D_t u\|^2 - 2\beta_k (\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\|^2. \end{aligned} \tag{3.3.102}$$

End of the proof of Proposition 3.3.20. By (3.3.92), (3.3.97), (3.3.102) and the definition 3.3.18 of M_k , we get

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &= A + B^+ + B^- \\ &\geq \frac{C_3}{2} (\beta_k e^{4t_k})^{2/3} \|\chi_0 u\|^2 \\ &\quad + 2c_3 \beta_k (\beta_k e^{4t_k})^{-1/3} \langle e^{2t} (1 - \sigma(e^t)) \chi_+^2 u, u \rangle \\ &\quad + 2c_3 \beta_k (\beta_k e^{4t_k})^{-1/3} \langle e^{2t} (1 - \sigma(e^{t_k})) \chi_-^2 u, u \rangle \\ &\quad - C \beta_k (\beta_k e^{4t_k})^{-1/3} k^{-2} \|\gamma u\|^2 \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2 \\ &\geq \beta_k (\beta_k e^{4t_k})^{-1/3} \langle (c\tilde{\rho}(t, t_k) - C k^{-2} e^{4t} g(e^t)^2) u, u \rangle \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2, \end{aligned}$$

where $\tilde{\rho}(t, t_k)$ is given in (3.3.81). This completes the proof of (3.3.80) in Proposition 3.3.20. \square

We have the following estimates for $\tilde{\rho}(t, t_k)$.

Lemma 3.3.21. *There exist $C_{10}, C_{11} > 0$ such that for all $e^{t_k} < \epsilon_1$, $t \in \mathbb{R}$, $\alpha \geq \alpha_0$, $k \geq 1$,*

$$\tilde{\rho}(t, t_k) \geq C_{10} e^{4t} g(e^t)^2, \quad (3.3.103)$$

$$\beta_k (\beta_k e^{4t_k})^{-1/3} \tilde{\rho}(t, t_k) + k^2 \geq C_{11} \beta_k^{1/2} e^{2t}, \quad (3.3.104)$$

where $\tilde{\rho}$ is given in (3.3.81), g is given in (3.3.3) and β_k is given in (3.3.6).

Proof of Lemma 3.3.21. Firstly, there exists $\delta > 0$ such that for $e^{t_k} < \epsilon_1$, $1 - \sigma(e^{t_k}) \geq \delta e^{2t_k}$. If t is in the support of $\chi_0(\cdot - t_k)$, i.e. $|t - t_k| \leq c_0$, we have

$$e^{4t} g(e^t)^2 \leq e^{4t} \leq e^{4c_0} e^{4t_k}, \quad \text{since } g \leq 1,$$

$$\beta_k (\beta_k e^{4t_k})^{-1/3} e^{4t_k} + k^2 \geq ((\beta_k e^{4t_k})^{2/3})^{3/4} (k^2)^{1/4} = \beta_k^{1/2} e^{2t_k} k^{1/2} \geq e^{-2c_0} \beta_k^{1/2} e^{2t}.$$

This implies

$$\forall t \in \mathbb{R}, \quad e^{4t} g(e^t)^2 \chi_0^2 \leq e^{4c_0} e^{4t_k} \chi_0^2,$$

$$\text{and } \forall t \in \mathbb{R}, \quad \beta_k (\beta_k e^{4t_k})^{-1/3} e^{4t_k} \chi_0^2 + k^2 \chi_0^2 \geq \beta_k^{1/2} e^{2t_k} k^{1/2} \geq e^{-2c_0} \beta_k^{1/2} e^{2t} \chi_0^2.$$

If t is in the support of $\chi_-(\cdot - t_k)$, i.e. $t \leq t_k - c_0/2$, we have

$$e^{4t} g(e^t)^2 \leq e^{2t} e^{2t_k} \leq \delta^{-1} e^{2t} (1 - \sigma(e^{t_k})),$$

$$\begin{aligned} & \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t} (1 - \sigma(e^{t_k})) + k^2 \\ & \geq (\delta \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t_k} + k^2 e^{-2t_k}) e^{2t} \\ & \geq C (\beta_k^{2/3} e^{2t_k/3})^{3/4} (k^2 e^{-2t_k})^{1/4} e^{2t} = C \beta_k^{1/2} k^{1/2} e^{2t}, \end{aligned}$$

which implies

$$\forall t \in \mathbb{R}, \quad e^{4t} g(e^t)^2 \chi_-^2 \leq \delta^{-1} e^{2t} (1 - \sigma(e^{t_k})) \chi_-^2,$$

$$\text{and } \forall t \in \mathbb{R}, \quad \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t} (1 - \sigma(e^{t_k})) \chi_-^2 + k^2 \chi_-^2 \geq C \beta_k^{1/2} e^{2t} \chi_-^2.$$

If t is in the support of $\chi_+(\cdot - t_k)$, i.e. $t \geq t_k + c_0/2$, we have, by (3.4.13)

$$e^{4t} g(e^t)^2 \leq 8e^{2t} (1 - \sigma(e^t)).$$

Suppose $t \geq t_k + c_0/2$, if $e^t \leq 2$, we have $1 - \sigma(e^t) \geq e^{2t}/16$ and then

$$\begin{aligned} & \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t} (1 - \sigma(e^t)) + k^2 \geq (\beta_k^{2/3} e^{-4t_k/3} \frac{e^{2t}}{16} + k^2 e^{-2t}) e^{2t} \\ & \geq (\frac{1}{16} \beta_k^{2/3} e^{2t/3})^{3/4} (k^2 e^{-2t})^{1/4} e^{2t} = \frac{1}{8} \beta_k^{1/2} k^{1/2} e^{2t}; \end{aligned}$$

if $e^t \geq 2$, we have $1 - \sigma(e^t) \geq 1 - \sigma(2) = e^{-1}$ and then

$$\beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t} (1 - \sigma(e^t)) \geq e^{-1} \beta_k^{2/3} e^{-4t_k/3} e^{2t} \geq e^{-1} \epsilon_1^{-4/3} \beta_k^{2/3} e^{2t} \geq e^{-1} \epsilon_1^{-4/3} \beta_k^{1/2} e^{2t}.$$

Then we get for all $t \in \mathbb{R}$,

$$e^{4t} g(e^t) \chi_+^2 \leq 8e^{2t} (1 - \sigma(e^t)) \chi_+^2,$$

$$\text{and } \beta_k (\beta_k e^{4t_k})^{-1/3} e^{2t} (1 - \sigma(e^t)) \chi_+^2 + k^2 \chi_+^2 \geq C \beta_k^{1/2} e^{2t} \chi_+^2.$$

Summing up, this completes the proof of (3.3.103), (3.3.104). \square

Proof of Theorem 3.3.17. The estimates (3.3.80) and (3.3.103) imply that there exists $k_0 \geq 3$, for all $k \geq k_0$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq \frac{c}{2} \beta_k (\beta_k e^{4t_k})^{-1/3} \langle \tilde{\rho}(t, t_k) u, u \rangle \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2. \end{aligned}$$

Choosing $C_{12} > 0$ large enough, we have, using (3.3.35)

$$2\operatorname{Re}\langle \mathcal{L}_k u, (C_{12} + M_k) u \rangle \geq \frac{c}{2} \langle (\beta_k (\beta_k e^{4t_k})^{-1/3} \tilde{\rho}(t, t_k) + D_t^2 + k^2 + e^{4t}) u, u \rangle. \quad (3.3.105)$$

We deduce from (3.3.104) and (3.3.105) that for $k \geq k_0$,

$$2\operatorname{Re}\langle \mathcal{L}_k u, (C_{12} + M_k) u \rangle \geq C \langle \beta_k^{1/2} e^{2t} u, u \rangle.$$

Using that $e^t (C_{12} + M_k) e^{-t}$ is bounded on $L^2(\mathbb{R}; dt)$ and Cauchy-Schwarz inequality, we get

$$\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad \forall k \geq k_0,$$

completing the proof of Theorem 3.3.17. \square

3.3.4.d) Case 4: $e^{t_k} \leq \beta_k^{-1/4}$

If $e^{t_k} \leq \beta_k^{-1/4}$, we can get estimate by using the multipliers Id and $i\operatorname{Id}$.

Lemma 3.3.22. Suppose $e^{t_k} \leq \beta_k^{-1/4}$. There exists $C > 0$ such that for all $k \geq 1$, $\alpha \geq 8\pi$, $u \in C_0^\infty(\mathbb{R})$,

$$\|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|, \quad (3.3.106)$$

where \mathcal{L}_k is given in (3.3.21) and β_k is given in (3.3.6).

Proof. At first note that

$$\forall r > 0, \quad 1 - \sigma(r) \leq \frac{1}{8} r^2.$$

We have for $e^{t_k} \leq \beta_k^{-1/4}$,

$$\begin{aligned} \operatorname{Re}\langle \mathcal{L}_k u, -iu \rangle &= \beta_k \langle e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) u, u \rangle + \beta_k \langle \gamma \langle D_k \rangle^{-2} \gamma u, u \rangle \\ &\geq \beta_k \langle e^{2t} ((1 - \sigma(e^t)) - (1 - \sigma(e^{t_k}))) u, u \rangle \\ &\geq \beta_k \langle e^{2t} ((1 - \sigma(e^t)) - \frac{1}{8} \beta_k^{-1/2}) u, u \rangle, \end{aligned}$$

so that with (3.3.35) we get,

$$\operatorname{Re}\langle \mathcal{L}_k u, (1 - i)u \rangle \geq \underbrace{\langle (k^2 e^{-2t} + \beta_k (1 - \sigma(e^t)) - \frac{1}{8} \beta_k^{1/2}) e^{2t} u, u \rangle}_{\geq e^{-1} \beta_k^{1/2}, \text{ by (3.4.17)}},$$

thus

$$\operatorname{Re}\langle \mathcal{L}_k u, (1 - i)u \rangle \geq \left(\frac{1}{e} - \frac{1}{8}\right) \beta_k^{1/2} \langle e^{2t} u, u \rangle.$$

By Cauchy-Schwarz inequality, we complete the proof of (3.3.106). \square

3.3.5 End of the proof of Theorem 3.2.2

The case $k < 0$ can be covered by modifying the multiplier in each cases:

Easy case 1: $\nu_k \geq 1$ (Lemma 3.3.2) use multipliers $i\text{Id}, \text{Id}$,

Easy case 2: $\nu_k \leq 0$ (Lemma 3.3.3) use multipliers $-i\text{Id}, \text{Id}$,

Case 1: $e^{t_k} > \epsilon_0^{-1}$ (Theorem 3.3.5), Case 2: $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$ (Theorem 3.3.14)
use multipliers Id and M_k , where

$$M_k = -\chi_0(t - t_k)\psi(|\beta_k|^{-1/3}D_t)\chi_0(t - t_k) + i|\beta_k|^{-1/3}\chi_+(t - t_k)^2 \\ - i|\beta_k|^{-1/3}\chi_-(t - t_k)^2,$$

Case 3: $e^{t_k} \in (|\beta_k|^{-1/4}, \epsilon_1)$ (Theorem 3.3.17) use multipliers Id and M_k , where

$$M_k = -\chi_0(t - t_k)\psi(|\beta_k e^{4t_k}|^{-1/3}D_t)\chi_0(t - t_k) + i|\beta_k e^{4t_k}|^{-1/3}\chi_+(t - t_k)^2 \\ - i|\beta_k e^{4t_k}|^{-1/3}\chi_-(t - t_k)^2,$$

Case 4: $e^{t_k} \leq |\beta_k|^{-1/4}$ (Lemma 3.3.22) use multipliers $i\text{Id}, \text{Id}$.

The estimates in Lemma 3.3.2, 3.3.3, Theorem 3.3.5, 3.3.14, 3.3.17 and Lemma 3.3.22 hold with β_k replaced by $|\beta_k|$ for $k \leq -k_0$. Summarizing, we have proved the estimate for the operator \mathcal{L}_k given in (3.3.14): There exist $C > 0$, $k_0 \geq 3$, $\alpha_0 \geq 1$ such that for all $|k| \geq k_0$, $\alpha \geq \alpha_0$, $u \in C_0^\infty(\mathbb{R})$,

$$\|e^{-t}\mathcal{L}_k u\|_{L^2(\mathbb{R};dt)} \geq C|\beta_k|^{1/3}\|e^t u\|_{L^2(\mathbb{R};dt)}, \quad (3.3.107)$$

and an estimate of the same type for $\widetilde{\mathcal{L}}_k$ given in (3.3.10) (with different constants C, α_0). This corresponds to the following estimate for the operator $\mathcal{H}_{\alpha,k,\lambda} = \mathcal{H}_k$ given in (3.3.5), (3.3.7) for $v \in C_0^\infty((0, +\infty))$, by the equivalence of (3.3.11) and (3.3.12)

$$\|\mathcal{H}_{k,\alpha,\lambda} v\|_{L^2(\mathbb{R}_+;rdr)} \geq C|\beta_k|^{1/3}\|v\|_{L^2(\mathbb{R}_+;rdr)}. \quad (3.3.108)$$

Then noticing (3.3.4), we get for $\omega = \sum_{|k| \geq k_0} \omega_k(r)e^{ik\theta} \in C_0^\infty(\mathbb{R}^2) \cap X_{k_0}$,

$$\begin{aligned} \|(\mathcal{H}_\alpha - i\lambda)\omega\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{|k| \geq k_0} 2\pi\|\mathcal{H}_{k,\alpha,\lambda}\omega_k\|_{L^2(\mathbb{R}_+;rdr)}^2 \\ &\geq \sum_{|k| \geq k_0} 2\pi C^2|\beta_k|^{2/3}\|\omega_k\|_{L^2(\mathbb{R}_+;rdr)}^2 \\ &= C^2\alpha^{2/3}\||D_\theta|^{1/3}\omega\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

$$\text{since } \||D_\theta|^{1/3}\omega\|_{L^2(\mathbb{R}^2)}^2 = \sum_{|k| \geq k_0} 2\pi|k|^{2/3}\|\omega_k\|_{L^2(\mathbb{R}_+;rdr)}^2.$$

Thus (3.2.3) is proved. Since $k_0 \geq 3$, we know that the imaginary axis does not intersect with the spectrum of \mathcal{H}_α viewed as an operator acting on X_{k_0} , which gives (3.2.4). The proof of Theorem 3.2.2 is complete.

3.4 Appendix

3.4.1 Weyl calculus

We present some facts about the Weyl calculus, which can be found in [Hör85, Chapter 18] as well as in [Ler10, Chapter 2]. The Weyl quantization associates to a symbol a the

operator a^w defined by

$$(a^w u)(x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (3.4.1)$$

Consider the symplectic space \mathbb{R}^{2n} equipped with the symplectic form $\sigma = \sum_{i=1}^n d\xi^i \wedge dx^i$. Given a positive definite quadratic form Γ on \mathbb{R}^{2n} , we define

$$\Gamma^\sigma(T) = \sup_{\Gamma(Y)=1} \sigma(T, Y)^2,$$

which is also a positive quadratic form. We say that Γ is an admissible metric if there exist $C_0, \tilde{C}_0, \tilde{N}_0 > 0$ such that for all $X, Y \in \mathbb{R}^{2n}$,

$$\begin{cases} \text{uncertainty principle: } \Gamma_X \leq \Gamma_X^\sigma, \\ \text{slowness: } \Gamma_X(X - Y) \leq C_0^{-1} \implies (\Gamma_Y/\Gamma_X)^{\pm 1} \leq C_0, \\ \text{temperance: } \Gamma_X \leq \tilde{C}_0 \Gamma_Y (1 + \Gamma_X^\sigma(X - Y))^{\tilde{N}_0}. \end{cases} \quad (3.4.2)$$

$C_0, \tilde{C}_0, \tilde{N}_0$ in (3.4.2) are called *structure constants* of the metric Γ . An admissible weight is a positive function m on the phase space \mathbb{R}^{2n} , such that there exist $C'_0, \tilde{C}'_0, \tilde{N}'_0 > 0$ so that for all $X, Y \in \mathbb{R}^{2n}$,

$$\begin{cases} \text{slowness: } \Gamma_X(X - Y) \leq C'_0^{-1} \implies (m(Y)/m(X))^{\pm 1} \leq C'_0, \\ \text{temperance: } m(X) \leq \tilde{C}'_0 m(Y) (1 + \Gamma_X^\sigma(X - Y))^{\tilde{N}'_0}. \end{cases} \quad (3.4.3)$$

$C'_0, \tilde{C}'_0, \tilde{N}'_0$ in (3.4.3) are called *structure constants* of the weight m . In particular, the function defined by

$$\lambda_\Gamma(X) = \inf_{T \in \mathbb{R}^{2n}, T \neq 0} (\Gamma_X^\sigma(T)/\Gamma_X(T))^{1/2} \quad (3.4.4)$$

is an admissible weight for Γ and its structure constants depend only on the structure constants of Γ (see [Den10b]). The uncertainty principle is equivalent to $\lambda_\Gamma \geq 1$.

We prove the uniform admissibility of a special type of metrics, including those we have used in the proof, given in Definition 3.3.6, 3.3.18.

Lemma 3.4.1. *For $\gamma \geq 1$, the metric on the phase space $\mathbb{R}_t \times \mathbb{R}_\tau$ given by*

$$\Gamma = |dt|^2 + \frac{|d\tau|^2}{\tau^2 + \gamma^2},$$

is admissible. Moreover, the structure constants of Γ defined in (3.4.2) are bounded above independently of γ .

Proof. First we notice that

$$\lambda_\Gamma = (\tau^2 + \gamma^2)^{1/2} \geq \gamma \geq 1,$$

so that Γ satisfies the uncertainty principle.

Slowness. It suffices to prove for $X = (x, \xi)$, $Y = (y, \eta)$, $T = (t, \tau)$, $\Gamma_X(X - Y) \leq s^2$ implies $\Gamma_Y \leq C_0 \Gamma_X$. Indeed, if $\Gamma_X(X - Y) \leq s^2$ then $|\xi - \eta|^2 \leq s^2(\xi^2 + \gamma^2)$, and we obtain

$$\xi^2 \leq 2(\xi - \eta)^2 + 2\eta^2 \leq 2s^2(\xi^2 + \gamma^2) + 2\eta^2,$$

$$\text{thus } (1 - 2s^2)(\xi^2 + \gamma^2) \leq 2(\eta^2 + \gamma^2).$$

By choosing $0 < s < 1/\sqrt{2}$ and $C_0 = 2(1 - 2s^2)^{-1} > 1$, we get

$$\xi^2 + \gamma^2 \leq C_0(\eta^2 + \gamma^2).$$

$$\text{Then } \Gamma_Y(T) = t^2 + \frac{\tau^2}{\eta^2 + \gamma^2} \leq t^2 + \frac{C_0\tau^2}{\xi^2 + \gamma^2} \leq C_0\Gamma_X(T).$$

Temperance. We have

$$\begin{aligned} \Gamma_X^\sigma &= (\xi^2 + \gamma^2)|dt|^2 + |d\tau|^2, \\ \frac{\Gamma_X(T)}{\Gamma_Y(T)} &\leq \max(1, \frac{\eta^2 + \gamma^2}{\xi^2 + \gamma^2}). \end{aligned}$$

If $|\eta| \leq 2|\xi|$ or $|\eta| \leq \gamma$, the right-hand side of the last inequality is bounded from above by 4. If $|\eta| > 2|\xi|$ and $|\eta| \geq \gamma$, then $|\xi - \eta| \geq \frac{1}{2}|\eta|$, which implies that $\Gamma_X^\sigma(X - Y) \geq (\xi - \eta)^2 \geq \frac{1}{4}\eta^2$; on the other hand, we have

$$\frac{\eta^2 + \gamma^2}{\xi^2 + \gamma^2} \leq \frac{\eta^2 + \gamma^2}{\gamma^2} = 1 + \gamma^{-2}\eta^2,$$

since $\gamma \geq 1$, we have

$$\frac{\Gamma_X(T)}{\Gamma_Y(T)} \leq 1 + 4\Gamma_X^\sigma(X - Y).$$

So the inequality $\Gamma_X(T)/\Gamma_Y(T) \leq 4(1 + \Gamma_X^\sigma(X - Y))$ holds for any X, Y, T . As a result, we have proved that Γ is admissible. From the proof above, we see that the structure constants are independent of γ , and this ends the proof of lemma. \square

The space of symbols $S(m, \Gamma)$ is defined as the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that the following semi-norms for all $k \in \mathbb{N}$

$$\sup_{\Gamma_X(T_j) \leq 1} |a^{(k)}(X)(T_1, \dots, T_k)|m(X)^{-1} < +\infty. \quad (3.4.5)$$

The composition law \sharp is defined by $a^w b^w = (a \sharp b)^w$ and we have

$$(a \sharp b)(X) = \exp\left(\frac{i}{2}\sigma(D_X, D_Y)\right)a(X)b(Y)|_{Y=X}. \quad (3.4.6)$$

For $a \in S(m_1, \Gamma)$, $b \in S(m_2, \Gamma)$, we have the asymptotic expansion

$$(a \sharp b)(x, \xi) = \sum_{0 \leq k < N} w_k(a, b) + r_N(a, b), \quad (3.4.7)$$

$$\text{with } w_k(a, b) = 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha! \beta!} D_\xi^\alpha \partial_x^\beta a D_\xi^\beta \partial_x^\alpha b \in S(m_1 m_2 \lambda_\Gamma^{-k}, \Gamma), \quad (3.4.8)$$

$$r_N(a, b)(X) = R_N(a(X) \otimes b(Y))|_{X=Y} \in S(m_1 m_2 \lambda_\Gamma^{-N}, \Gamma), \quad (3.4.9)$$

$$R_N = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} \exp \frac{\theta}{2i} [\partial_X, \partial_Y] d\theta \left(\frac{1}{2i} [\partial_X, \partial_Y] \right)^N. \quad (3.4.10)$$

We use here the notation $D = i^{-1}\partial$. The $w_k(a, b)$ with k even are symmetric in a, b and skew-symmetric for k odd. In particular, we have

$$a \sharp b - b \sharp a = \frac{1}{i} \{a, b\} + \tilde{r}, \quad \tilde{r} \in S(m_1 m_2 \lambda_\Gamma^{-3}, \Gamma), \quad (3.4.11)$$

where $\{ , \}$ is the Poisson bracket, implying that $[a^w, b^w] = \frac{1}{i}\{a, b\}^w + \tilde{r}^w$.

The symbols in $S(1, \Gamma)$ are quantified in bounded operators on $L^2(\mathbb{R}^n)$, with operator norm depending on the structure constants of Γ defined in (3.4.2) and a semi-norm (3.4.5) of the symbol in $S(1, \Gamma)$, whose order depends only on the dimension n and the structure constants of Γ . See [Den10b].

Proposition 3.4.2 (Fefferman-Phong inequality). *If $a \in S(\lambda_\Gamma^2, \Gamma)$ and $a \geq 0$, then a^w is bounded from below by a constant depending on the structure constants of Γ given in (3.4.2) and a semi-norm (3.4.5) of the symbol a in $S(\lambda_\Gamma^2, \Gamma)$, whose order depends only on the dimension n and the structure constants of Γ .*

3.4.2 For the operator $(k^2 + D_t^2)^{-1}$

Lemma 3.4.3. *For $k \geq 1$, we have*

$$(\tau^2 + k^2)^{-1} \in S((\tau^2 + k^2)^{-1}, \frac{|d\tau|^2}{\tau^2 + k^2}) \subset S(k^{-2}, \frac{|d\tau|^2}{\tau^2 + k^2}),$$

with semi-norms bounded above independently of k . Moreover, the Fourier multiplier $\langle D_k \rangle^{-2} = (k^2 + D_t^2)^{-1}$ is bounded on $L^2(\mathbb{R}; dt)$ with $\mathcal{L}(L^2(\mathbb{R}; dt))$ -norm bounded by k^{-2} .

Proof. We see that

$$\frac{1}{\tau^2 + k^2} = \frac{k^{-2}}{(k^{-1}\tau)^2 + 1}.$$

Then for any $m \geq 0$,

$$\left| \frac{d^m}{d\tau^m} \left(\frac{1}{\tau^2 + k^2} \right) \right| \leq C_m k^{-2} ((k^{-1}\tau)^2 + 1)^{-1-m/2} k^{-m} = C_m (k^2 + \tau^2)^{-1-m/2},$$

where C_m is a positive constant depending only on m . This completes the proof of the lemma. \square

We can also compute the kernel of the operator $(k^2 + D_t^2)^{-1}$.

Lemma 3.4.4. *For $k \geq 1$, we have*

$$\frac{1}{2k} \int_{\mathbb{R}} e^{-k|t|} e^{it\tau} dt = \frac{1}{k^2 + \tau^2}.$$

As a consequence, $\langle D_k \rangle^{-2}$ is just the convolution operator with the function $(2k)^{-1}e^{-k|\cdot|}$.

Proof. We have

$$\frac{1}{2k} \int_{\mathbb{R}} e^{-k|t|} e^{it\tau} dt = \frac{1}{k} \int_0^{+\infty} e^{-kt} \cos(t\tau) dt = \frac{1}{k} \operatorname{Re} \int_0^{+\infty} e^{-t(k+i\tau)} = \frac{1}{k^2 + \tau^2}. \quad \square$$

As a corollary of the lemma, the following result is used in the proof of Case 3.

Lemma 3.4.5. *For $k \geq 3$, the operator $e^{-2t} \langle D_k \rangle^{-2} e^{2t}$ is bounded on $L^2(\mathbb{R}; dt)$ with $\mathcal{L}(L^2(\mathbb{R}; dt))$ -norm bounded above by $3k^{-2}$.*

Proof. We deduce from the previous lemma that the operator $e^{-2t}\langle D_k \rangle^{-2}e^{2t}$ has kernel

$$T_k(t, s) = \frac{1}{2k} e^{-2t} e^{-k|t-s|} e^{2s} = \frac{1}{2k} e^{-k|t-s|-2(t-s)}.$$

We have

$$|T_k(t, s)| \leq \frac{1}{2k} e^{-(k-2)|t-s|}.$$

If $k \geq 3$, then the convolution with $\frac{1}{2k}e^{-(k-2)|\cdot|}$ is bounded on $L^2(\mathbb{R}; dt)$ with norm

$$\frac{1}{2k} \|e^{-(k-2)|\cdot|}\|_{L^1(\mathbb{R}; dt)} = \frac{1}{k(k-2)},$$

which is smaller than $3k^{-2}$ since $k \geq 3$. This completes the proof of the lemma. \square

3.4.3 Some inequalities

We present some inequalities that we have used in the proof. Recall the functions σ, g given in (3.3.3)

$$\sigma(r) = \frac{1 - e^{-r^2/4}}{r^2/4}, \quad g(r) = e^{-r^2/8}, \quad r > 0.$$

Firstly, a calculation shows that

$$\inf_{\theta > 0} \theta^{-2}(e^\theta - 1) \simeq 1.54414\dots$$

so that

$$\forall r > 0, \quad \delta r^2 g(r)^2 \leq \sigma(r), \quad \text{with } \delta \simeq \frac{1}{4} \times 1.54414. \quad (3.4.12)$$

We have

$$\forall r > 0, \quad r^2 g(r)^2 \leq 16\sigma(r), \quad r^2 g(r)^2 \leq 8(1 - \sigma(r)). \quad (3.4.13)$$

Indeed, let $\theta = r^2/4$, then (3.4.13) are equivalent to the following

$$\forall \theta > 0, \quad 4\theta e^{-\theta/2} \leq 16 \frac{1 - e^{-\theta}}{\theta}, \quad 4\theta e^{-\theta} \leq 8(1 - \frac{1 - e^{-\theta}}{\theta}),$$

$$\text{and also } \forall \theta > 0, \quad \theta^2 e^{\theta/2} \leq 4(e^\theta - 1), \quad \theta^2 \leq 2(\theta e^\theta - e^\theta + 1).$$

It is easy to check that the functions

$$\theta \mapsto 4(e^\theta - 1) - \theta^2 e^{\theta/2}, \quad \theta \mapsto 2(\theta e^\theta - e^\theta + 1) - \theta^2$$

vanish at $\theta = 0$ and have positive derivative for $\theta > 0$. Hence (3.4.13) are proved.

We can get by induction on $n \in \mathbb{N}$ that

$$\sigma^{(n)}(r) = (-1)^n 4r^{-n-2}((n+1)! - p_n(r)e^{-r^2/4}), \quad (3.4.14)$$

where p_n is a polynomial of degree $2n$. In particular, we have

$$\forall r > 0, \quad \sigma'(r) = -\frac{8}{r^3}(1 - e^{-r^2/4} - \frac{r^2}{4}e^{-r^2/4}) < 0, \quad (3.4.15)$$

so that σ is decreasing. We have the Taylor expansion of σ near 0

$$\sigma(r) = 1 - \frac{1}{2} \cdot \frac{r^2}{4} + \frac{1}{3!} \left(\frac{r^2}{4}\right)^2 - \dots + \frac{(-1)^l}{(l+1)!} \left(\frac{r^2}{4}\right)^l + O(r^{2l+2}), \quad \text{as } r \rightarrow 0. \quad (3.4.16)$$

Lemma 3.4.6. *For all $k \geq 1$, $\beta \geq 1$, $r > 0$, we have*

$$\begin{cases} r^2 + \beta\sigma(r) \geq (2\log 2)^{-1}\beta^{1/2}, \\ \frac{k^2}{r^2} + \beta(1 - \sigma(r)) \geq e^{-1}\beta^{1/2}, \end{cases} \quad (3.4.17)$$

where $\sigma(r)$ is given in (3.3.3).

Proof. By the definition of σ , we have

$$\begin{aligned} \text{if } r \leq 2(\log 2)^{1/2}, & \text{ then } \sigma(r) \geq \sigma(2(\log 2)^{1/2}) = (2\log 2)^{-1}, \\ \text{if } r > 2(\log 2)^{1/2}, & \text{ then } e^{-r^2/4} < \frac{1}{2}, \text{ implying } \sigma(r) > 2r^{-2}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \text{if } r \leq 2(\log 2)^{1/2}, & \text{ then } r^2 + \beta\sigma(r) \geq (2\log 2)^{-1}\beta, \\ \text{if } r > 2(\log 2)^{1/2}, & \text{ then } r^2 + \beta\sigma(r) \geq r^2 + 2\beta r^{-2} \geq 2\sqrt{2}\beta^{1/2}, \end{aligned}$$

which implies for any $r \geq 0$,

$$r^2 + \beta\sigma(r) \geq \min((2\log 2)^{-1}\beta, 2\sqrt{2}\beta^{1/2}) \geq (2\log 2)^{-1}\beta^{1/2}, \quad \text{since } \beta \geq 1,$$

proving the first inequality in (3.4.17). Noting that

$$\forall 0 \leq \theta \leq 1, \quad e^{-\theta} - 1 + \theta \geq \frac{\theta^2}{4},$$

we have

$$\forall 0 < r \leq 2, \quad 1 - \sigma(r) = \frac{4}{r^2}(e^{-r^2/4} - 1 + \frac{r^2}{4}) \geq \frac{r^2}{16}.$$

Then we obtain

$$\begin{aligned} \text{if } r \leq 2, & \quad \frac{k^2}{r^2} + \beta(1 - \sigma(r)) \geq \frac{k^2}{r^2} + \beta\frac{r^2}{16} \geq \frac{1}{2}k\beta^{1/2}, \\ \text{if } r > 2, & \quad \frac{k^2}{r^2} + \beta(1 - \sigma(r)) \geq \beta(1 - \sigma(2)) = e^{-1}\beta. \end{aligned}$$

Therefore for any $r > 0$, $k \geq 1$,

$$\frac{k^2}{r^2} + \beta(1 - \sigma(r)) \geq \min(\frac{1}{2}k\beta^{1/2}, e^{-1}\beta) \geq e^{-1}\beta^{1/2}, \quad \text{since } \beta \geq 1,$$

which proves the second inequality in (3.4.17). \square

Now we prove inequalities about the function σ that are used in the proof of Case 1, 2 and 3.

Proposition 3.4.7. *Given $\epsilon_0, \epsilon_1 \in (0, 1)$, there exist $c_0 \in (0, 1)$, $C_1, C_2, C_3, \tilde{C}_n > 0$, $c_1, c_2, c_3 \in (0, 1)$ satisfying the following. Recall that σ is given in (3.3.3). Let $t_k \in \mathbb{R}$.*

1. *For $e^{t_k} > \epsilon_0^{-1}$, we have*

$$\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_1, \quad (3.4.18)$$

$$\forall |t - t_k| \leq 3c_0, \quad \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \right| \leq \tilde{C}_n, \quad (3.4.19)$$

$$\text{and} \quad \begin{cases} \sigma(e^t) - \sigma(e^{t_k}) \leq -c_1\sigma(e^{t_k}), & \text{for } t - t_k \geq c_0/2, \\ \sigma(e^t) - \sigma(e^{t_k}) \geq c_1\sigma(e^{t_k}), & \text{for } t - t_k \leq -c_0/2. \end{cases} \quad (3.4.20)$$

2. For $\epsilon_1 \leq e^{t_k} \leq \epsilon_0^{-1}$, we have

$$\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_2, \quad (3.4.21)$$

$$\forall |t - t_k| \leq 3c_0, \quad \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \right| \leq \tilde{C}_n, \quad (3.4.22)$$

$$\text{and} \quad \begin{cases} \sigma(e^t) - \sigma(e^{t_k}) \leq -c_2 \sigma(e^{t_k}), & \text{for } t - t_k \geq c_0/2, \\ \sigma(e^t) - \sigma(e^{t_k}) \geq c_2 \sigma(e^{t_k}), & \text{for } t - t_k \leq -c_0/2. \end{cases} \quad (3.4.23)$$

3. For $e^{t_k} < \epsilon_1$, we have

$$\forall |t - t_k| \leq 2c_0, \quad \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_3 e^{4t_k}, \quad (3.4.24)$$

$$\forall |t - t_k| \leq 3c_0, \quad \forall n \in \mathbb{N}, \quad \left| \frac{d^n}{dt^n} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \right| \leq \tilde{C}_n e^{4t_k}, \quad (3.4.25)$$

$$\text{and} \quad \begin{cases} \sigma(e^t) - \sigma(e^{t_k}) \leq -c_3(1 - \sigma(e^t)), & \text{for } t - t_k \geq c_0/2, \\ \sigma(e^t) - \sigma(e^{t_k}) \geq c_3(1 - \sigma(e^{t_k})), & \text{for } t - t_k \leq -c_0/2. \end{cases} \quad (3.4.26)$$

Proof. Step 1. The essential step is to choose $c_0 \in (0, 1)$ such that (3.4.18), (3.4.21) and (3.4.24) hold.

By (3.4.15), given $\epsilon_0, \epsilon_1 \in (0, 1)$, there exist $\mu_1 > \mu_2 > 0$ such that

$$\begin{aligned} \forall r > e^{-2}\epsilon_0^{-1}, \quad -\mu_1 r^{-3} &\leq \sigma'(r) \leq -\mu_2 r^{-3}, \\ \forall r \in [e^{-2}\epsilon_1, e^2\epsilon_0^{-1}], \quad -\mu_1 &\leq \sigma'(r) \leq -\mu_2, \\ \forall r < e^2\epsilon_1, \quad -\mu_1 r &\leq \sigma'(r) \leq -\mu_2 r. \end{aligned} \quad (3.4.27)$$

Let us denote

$$f(t, t_k) := \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] = e^{3t} \sigma'(e^t) + 2e^{2t} (\sigma(e^t) - \sigma(e^{t_k})).$$

The Taylor's formula gives

$$\sigma(e^t) - \sigma(e^{t_k}) = \int_0^1 \sigma'(e^{t_k + \theta(t-t_k)}) e^{t_k + \theta(t-t_k)} (t - t_k) d\theta.$$

Suppose $|t - t_k| \leq 2c_0$ with $c_0 < 1$. Using (3.4.27), we get the following

- if $e^{t_k} > \epsilon_0^{-1}$, then $e^t > e^{-2}\epsilon_0^{-1}$ and

$$f(t, t_k) \leq -\mu_2 + 4\mu_1 e^{4c_0} c_0;$$

- if $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$, then $e^t \in [e^{-2}\epsilon_1, e^2\epsilon_0^{-1}]$ and

$$f(t, t_k) \leq -\mu_2 e^{3t} + 4\mu_1 e^{3t} e^{2c_0} c_0 = -(\mu_2 - 4\mu_1 e^{2c_0} c_0) e^{3t};$$

- if $e^{t_k} < \epsilon_1$, then $e^t < e^2\epsilon_1$ and

$$f(t, t_k) \leq -\mu_2 e^{4t} + 4\mu_1 e^{4t} e^{4c_0} c_0 = -(\mu_2 - 4\mu_1 e^{4c_0} c_0) e^{4t}.$$

Let $c_0 \in (0, 1)$ satisfying

$$4c_0 e^{4c_0} \leq \frac{\mu_2}{2\mu_1},$$

then we get (3.4.18), (3.4.21), (3.4.24) with

$$C_1 = \mu_2/2, \quad C_2 = \mu_2 e^{-6} \epsilon_1^3/2 \quad \text{and} \quad C_3 = \mu_2 e^{-8c_0}/2.$$

Step 2. We prove (3.4.19), (3.4.22) and (3.4.25). Notice for $r \in [e^{t_k-3c_0}, e^{t_k+3c_0}]$,

$$|\sigma(r) - \sigma(e^{t_k})| \leq \begin{cases} Ce^{-2t_k}, & \text{if } e^{t_k} > \epsilon_0^{-1}, \\ C, & \text{if } e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}], \\ Ce^{2t_k}, & \text{if } e^{t_k} < \epsilon_1. \end{cases}$$

It follows from (3.4.14), (3.4.16) that $\sigma^{(n)}(r) = \mathcal{O}(r^{-n-2})$ as $r \rightarrow +\infty$ and $\sigma^{(n)}(r) = \mathcal{O}(1)$ as $r \rightarrow 0$. We have also $\sigma'(r) = \mathcal{O}(r)$ as $r \rightarrow 0$, since σ is even. Denote

$$f(r) = r^2(\sigma(r) - \sigma(e^{t_k})), \quad h(t) = e^t.$$

By Leibniz formula, we have

$$\begin{aligned} f'(r) &= r^2 \sigma'(r) + 2r(\sigma(r) - \sigma(e^{t_k})), \\ f''(r) &= r^2 \sigma''(r) + 4r\sigma'(r) + 2(\sigma(r) - \sigma(e^{t_k})), \\ f^{(n)}(r) &= r^2 \sigma^{(n)}(r) + 2nr\sigma^{(n-1)}(r) + n(n-1)\sigma^{(n-2)}(r), \quad n \geq 3, \end{aligned}$$

so that for $r \in [e^{t_k-3c_0}, e^{t_k+3c_0}]$, $n \geq 0$,

$$|f^{(n)}(r)| \leq \begin{cases} C_n(e^{t_k})^{-n}, & \text{if } e^{t_k} > \epsilon_0^{-1}, \\ C_n, & \text{if } e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}], \\ C_n(e^{t_k})^{\max(4-n, 0)}, & \text{if } e^{t_k} < \epsilon_1, \end{cases} \quad (3.4.28)$$

by using the properties of σ stated above. Now applying the Faà di Bruno formula

$$\frac{(f \circ h)^{(l)}}{l!} = \sum_{1 \leq n \leq l} \frac{f^{(n)} \circ h}{n!} \prod_{\substack{l_1 + \dots + l_n = l \\ l_j \geq 1}} \frac{h^{(l_j)}}{l_j!},$$

we get

$$\left| \frac{d^l}{dt^l} \left[e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \right] \right| = |(f \circ h)^{(l)}(t)| \leq \sum_{1 \leq n \leq l} C_{l,n} |f^{(n)}(e^t)| (e^t)^n.$$

Using (3.4.28) for $e^t \in [e^{t_k-3c_0}, e^{t_k+3c_0}]$, we obtain (3.4.19), (3.4.22) and (3.4.25).

Step 3. It remains to prove (3.4.20), (3.4.23) and (3.4.26). Denoting $r_k = e^{t_k}$ and $r = e^t$, then (3.4.20), (3.4.23) are equivalent to the following

$$\forall r_k > \epsilon_0^{-1}, \quad \begin{cases} \sigma(r) \leq (1 - c_1)\sigma(r_k), & \text{for } r \geq r_k e^{c_0/2}, \\ \sigma(r_k) \leq (1 - c_1)\sigma(r), & \text{for } r \leq r_k e^{-c_0/2}. \end{cases} \quad (3.4.29)$$

$$\forall r_k \in [\epsilon_1, \epsilon_0^{-1}], \quad \begin{cases} \sigma(r) \leq (1 - c_2)\sigma(r_k), & \text{for } r \geq r_k e^{c_0/2}, \\ \sigma(r_k) \leq (1 - c_2)\sigma(r), & \text{for } r \leq r_k e^{-c_0/2}. \end{cases} \quad (3.4.30)$$

The function σ is decreasing, so that in order to prove (3.4.29) and (3.4.30), it suffices to prove the following

$$\exists c \in (0, 1), \forall r \geq \epsilon_1 e^{-c_0/2}, \quad \sigma(r e^{c_0/2}) \leq (1 - c)\sigma(r). \quad (3.4.31)$$

We know that for any $\lambda > 1$, the function

$$[0, 1] \ni \theta \mapsto f_1(\theta; \lambda) = \frac{1 - \theta^\lambda}{\lambda(1 - \theta)}$$

is strictly increasing in $(0, 1)$ and $f_1(1; \lambda) = 1$. Hence for all $\lambda > 1$, there exists $0 < \delta_1(\lambda) < 1$ such that

$$\forall 0 < \theta \leq \exp(-\epsilon_1^2 e^{-c_0}/4), \quad f_1(\theta; \lambda) \leq \delta_1(\lambda).$$

Then we have for $r \geq \epsilon_1 e^{-c_0/2}$,

$$\frac{\sigma(r e^{c_0/2})}{\sigma(r)} = \frac{1 - (e^{-r^2/4})^{e^{c_0}}}{e^{c_0}(1 - e^{-r^2/4})} = f_1(e^{-r^2/4}; e^{c_0}) \leq \delta_1(e^{c_0}), \quad \text{since } e^{c_0} > 1,$$

which proves (3.4.31) with $c = 1 - \delta_1(e^{c_0})$. Thus (3.4.20) and (3.4.23) are proved.

Now we turn to prove (3.4.26), which is equivalent to the following

$$\forall r_k < \epsilon_1, \quad \begin{cases} \sigma(r) - \sigma(r_k) \leq -c_3(1 - \sigma(r)), & \text{for } r \geq r_k e^{c_0/2}, \\ \sigma(r) - \sigma(r_k) \geq c_3(1 - \sigma(r_k)), & \text{for } r \leq r_k e^{-c_0/2}. \end{cases} \quad (3.4.32)$$

Since $1 - \sigma(r)$ is increasing, we need only to prove

$$\exists c_3 \in (0, 1), \forall r < \epsilon_1, \quad 1 - \sigma(r) \leq (1 - c_3)(1 - \sigma(r e^{c_0/2})). \quad (3.4.33)$$

By direct computation, we find that for any $\lambda > 1$, the function

$$(0, +\infty) \ni x \mapsto f_2(x; \lambda) = \frac{\lambda(e^{-x} - 1 + x)}{e^{-\lambda x} - 1 + \lambda x}$$

is continuous on $[0, +\infty)$, $f_2(0; \lambda) = \lambda^{-1}$ and $f_2(x; \lambda) < 1$ for all $x > 0$. Hence for any $\lambda > 1$, there exists $0 < \delta_2(\lambda) < 1$ such that

$$f_2(x; \lambda) < \delta_2(\lambda), \quad \forall 0 < x < 1/4.$$

We get for all $r < \epsilon_1 < 1$,

$$\frac{1 - \sigma(r)}{1 - \sigma(r e^{c_0/2})} = e^{c_0} \frac{e^{-r^2/4} - 1 + r^2/4}{e^{-e^{c_0} r^2/4} - 1 + e^{c_0} r^2/4} = f_2(r^2/4; e^{c_0}) < \delta_2(e^{c_0}),$$

which proves (3.4.33) with $c_3 = 1 - \delta_2(e^{c_0})$ thus (3.4.26) is proved. The proof of Proposition 3.4.7 is now complete. \square

3.4.4 Miscellaneous

The non-local term

The non-local term $\frac{\alpha}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega))$ in (3.2.1) is a bounded operator on $L^2(\mathbb{R}^2)$, see (3.1.5), (3.1.2) for notations. In polar coordinates, we have, for $\omega = \sum \omega_k(r)e^{ik\theta}$ (see Section 3.1.1)

$$\frac{\alpha}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)) = \sum_{k \neq 0} \frac{iak}{8\pi} g(r) \mathcal{K}_k[g\omega_k](r) e^{ik\theta},$$

where g is given in (3.3.3) and \mathcal{K}_k in (3.3.2). Note that for any $f \in L^1(\mathbb{R}_+; rdr)$, $k \neq 0$,

$$\|\mathcal{K}_k[f]\|_{L^\infty} \leq \frac{1}{2|k|} \int_0^\infty |f(s)| s ds$$

so that by Cauchy-Schwarz inequality,

$$\begin{aligned} \|g\mathcal{K}_k[g\omega_k]\|_{L^2(\mathbb{R}_+; rdr)} &\leq \|g\|_{L^2(\mathbb{R}_+; rdr)} \|\mathcal{K}_k[g\omega_k]\|_{L^\infty} \\ &\leq \frac{1}{2|k|} \|g\|_{L^2(\mathbb{R}_+; rdr)} \int_0^\infty |g(s)\omega_k(s)| s ds \\ &\leq \frac{1}{2|k|} \|g\|_{L^2(\mathbb{R}_+; rdr)}^2 \|\omega_k\|_{L^2(\mathbb{R}_+; rdr)}. \end{aligned}$$

Noticing $\|g\|_{L^2(\mathbb{R}_+; rdr)} = \sqrt{2}$, we get

$$\begin{aligned} \left\| \frac{\alpha}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)) \right\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{k \neq 0} 2\pi \left| \frac{\alpha k}{8\pi} \right|^2 \|g\mathcal{K}_k[g\omega_k]\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ &\leq \sum_{k \neq 0} 2\pi \left| \frac{\alpha}{8\pi} \right|^2 \|\omega_k\|_{L^2(\mathbb{R}_+; rdr)}^2 \leq \left| \frac{\alpha}{8\pi} \right|^2 \|\omega\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

so that the non-local operator is bounded on $L^2(\mathbb{R}^2)$.

Remark 3.4.8. By using the change of variables $r = e^t$, we can prove a better inequality for the non-local term. By Lemma 3.3.1, we know

$$e^{2t}(g\mathcal{K}_k[g\omega_k])(e^t) = (e^{2t}g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)u_k)(t), \quad \text{for } u_k(t) = \omega_k(e^t),$$

$$\text{and } \|g\mathcal{K}_k[g\omega_k]\|_{L^2(\mathbb{R}_+; rdr)} = \|e^t g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)u_k\|_{L^2(\mathbb{R}^2; dt)}.$$

Using $\|e^t g(e^t)\|_{L^\infty} = 2/\sqrt{e}$ and $\|(k^2 + D_t^2)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}; dt))} \leq k^{-2}$, we obtain

$$\|e^t g(e^t)(k^2 + D_t^2)^{-1}e^{2t}g(e^t)u_k\|_{L^2(\mathbb{R}^2; dt)} \leq \frac{4}{e} k^{-2} \|e^t u_k\|_{L^2(\mathbb{R}; dt)},$$

so that $\|g\mathcal{K}_k[g\omega_k]\|_{L^2(\mathbb{R}_+; rdr)} \leq 4e^{-1}k^{-2}\|\omega_k\|_{L^2(\mathbb{R}_+; rdr)}$ and

$$\begin{aligned} \left\| \frac{\alpha}{2}G^{1/2}x \cdot (K_{BS} * (G^{1/2}\omega)) \right\|_{L^2(\mathbb{R}^2)}^2 &\leq \sum_{k \neq 0} 2\pi \left| \frac{\alpha}{8\pi} \right|^2 \frac{16}{e^2} k^{-2} \|\omega_k\|_{L^2(\mathbb{R}_+; rdr)}^2 \\ &= \frac{16}{e^2} \left| \frac{\alpha}{8\pi} \right|^2 \||D_\theta|^{-1}\omega\|_{L^2(\mathbb{R}^2)}^2, \end{aligned}$$

$$\text{where } |D_\theta|^{-1}\omega = \sum_{k \neq 0} |k|^{-1} \omega_k(r) e^{ik\theta}, \quad \text{for } \omega = \sum \omega_k(r) e^{ik\theta}.$$

Numerical range of \mathcal{H}_α

The numerical range

$$\Theta(\mathcal{H}_\alpha; X_{k_0}) = \{\langle \mathcal{H}_\alpha \omega, \omega \rangle_{L^2(\mathbb{R}^2)} \in \mathbb{C}; \omega \in D \cap X_{k_0}, \|\omega\|_{L^2(\mathbb{R}^2)} = 1\}$$

is included in the set $\{z \in \mathbb{C}; \operatorname{Re}(z) \geq k_0/2\}$, where \mathcal{H}_α is given in (3.2.1). In other words, we need to prove

$$\operatorname{Re}\langle \mathcal{H}_\alpha \omega, \omega \rangle_{L^2(\mathbb{R}^2)} \geq \frac{k_0}{2} \|\omega\|_{L^2(\mathbb{R}^2)}^2, \quad \forall \omega \in D \cap X_{k_0}.$$

Recall that $\operatorname{Re}\mathcal{H}_\alpha = -\Delta + \frac{|x|^2}{16} - \frac{1}{2}$. Using polar coordinates, it suffices to prove that for all $k \geq 0$, $v \in C_0^\infty((0, +\infty))$

$$\langle \left(-\partial_r^2 - \frac{1}{r} \partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right) v, v \rangle_{L^2(\mathbb{R}_+; rdr)} \geq \frac{k}{2} \|v\|_{L^2(\mathbb{R}_+; rdr)}^2. \quad (3.4.34)$$

Note that the adjoint of ∂_r on $L^2(\mathbb{R}_+; rdr)$ is given by $\partial_r^* = -\partial_r - \frac{1}{r}$. We can check the following identities

$$\partial_r^* \frac{k}{r} + \frac{k}{r} \partial_r = -\frac{k}{r^2} + [\frac{k}{r}, \partial_r] = 0, \quad \partial_r^* \frac{r}{4} + \frac{r}{4} \partial_r = -\frac{1}{4} + [\frac{r}{4}, \partial_r] = -\frac{1}{2},$$

$$\text{so that } (\partial_r^* - \frac{k}{r} + \frac{r}{4})(\partial_r - \frac{k}{r} + \frac{r}{4}) = \partial_r^* \partial_r + \frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2} - \frac{k}{2}.$$

Then the left hand side of (3.4.34) is equal to

$$\|(\partial_r - \frac{k}{r} + \frac{r}{4})v\|_{L^2(\mathbb{R}_+; rdr)}^2 + \frac{k}{2} \|v\|_{L^2(\mathbb{R}_+; rdr)}^2,$$

so that (3.4.34) is proved.

Proof of (3.3.1)

In this section, we provide a computation for (3.3.1). Using $(x_1, x_2) = (r \cos \theta, r \sin \theta)$, we have

$$\partial_1 = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta, \quad \partial_2 = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta.$$

Write the vector field $v = (v_1, v_2)$ as $v = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta$, where $\mathbf{e}_r = (\cos \theta, \sin \theta)$, $\mathbf{e}_\theta = (-\sin \theta, \cos \theta)$. Then

$$v_1 = v_r \cos \theta - v_\theta \sin \theta, \quad v_2 = v_r \sin \theta + v_\theta \cos \theta.$$

We have the following

$$\begin{aligned} \partial_1 v_1 + \partial_2 v_2 &= (\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta)(v_r \cos \theta - v_\theta \sin \theta) \\ &\quad + (\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta)(v_r \sin \theta + v_\theta \cos \theta) \\ &= \partial_r v_r + \frac{1}{r} v_r + \frac{1}{r} \partial_\theta v_\theta = \frac{1}{r} \partial_r(r v_r) + \frac{1}{r} \partial_\theta v_\theta, \\ \partial_1 v_2 - \partial_2 v_1 &= (\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta)(v_r \sin \theta + v_\theta \cos \theta) \\ &\quad - (\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta)(v_r \cos \theta - v_\theta \sin \theta) \\ &= \partial_r v_\theta + \frac{1}{r} v_\theta - \frac{1}{r} \partial_\theta v_r = \frac{1}{r} \partial_r(r v_\theta) - \frac{1}{r} \partial_\theta v_r. \end{aligned}$$

The relations $\partial_1 v_1 + \partial_2 v_2 = 0$, $\partial_1 v_2 - \partial_2 v_1 = \omega$ become

$$\frac{1}{r} \partial_r(rv_r) + \frac{1}{r} \partial_\theta v_\theta = 0, \quad \frac{1}{r} \partial_r(rv_\theta) - \frac{1}{r} \partial_\theta v_r = \omega. \quad (3.4.35)$$

We look for functions of the form

$$v_r = \frac{u_k(r)}{r} e^{ik\theta}, \quad v_\theta = \frac{w_k(r)}{r} e^{ik\theta}, \quad \omega = \omega_k(r) e^{ik\theta}.$$

Then (3.4.35) implies

$$u'_k + \frac{ik}{r} w_k = 0, \quad w'_k - \frac{ik}{r} u_k = r\omega_k,$$

and this ends the proof of (3.3.1).

3.5 Letter of acceptance

From: imrn.editorialoffice@oup.com
Subject: Decision on your IMRN submission - Manuscript ID IMRN-21-10-08.R2
Date: Mon, February 27, 2012 5:21 pm
To: wendeng@math.jussieu.fr
Cc: emmanuel.hebey@math.u-cergy.fr

27-Feb-2012

Dear Miss Deng,

We are pleased to accept your paper IMRN-21-10-08.R2 entitled "Pseudospectrum for Oseen vortices operators" for publication in IMRN/IMRP.

Composition will begin on your article very quickly and you will receive page proofs in a matter of weeks.

The IMRN typographical style is distinctive and we assume you are aware of it. Please inform us in advance if there are unusual composition needs.

Recall that it is a condition of submission of an article that the authors permit editing the article for language correctness, readability and IMRN mathematical style compliance.

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With kind regards,

Emmanuel Hebey
Principal Editor
International Mathematics Research Notices

Chapter 4

Structure constants of the Weyl calculus

The result of this chapter is taken from the article [Den10b], *Structure constants in Weyl calculus*, which has been accepted for publication in MATHEMATICA SCANDINAVICA, see Section 4.5 for the letter of acceptance.

We find some explicit bounds on the $\mathcal{L}(L^2)$ -norm of pseudo-differential operators with symbols defined by a metric on the phase space. In particular, we prove that this norm depends only on the “structure constants” of the metric and a fixed semi-norm of the symbol. Analogous statements are made for the Fefferman-Phong inequality.

4.1 Introduction

The class of symbols $S_{1,0}^m$ consists of smooth functions a defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α, β ,

$$|(\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}. \quad (4.1.1)$$

The best constants $C_{\alpha, \beta}$ in (4.1.1) are called the semi-norms of the symbol a in the Fréchet space $S_{1,0}^m$. We have

Property 4.1.1. *If a is in $S_{1,0}^0$, then $a(x, D)$ defines a bounded operator on $L^2(\mathbb{R}^n)$.*

One might ask some very natural questions: the operator norm $\|a(x, D)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ is bounded by which constant? Is it a semi-norm of the symbol a ? If yes, then which semi-norm? Questions of the same type might be asked for the constant C in the following inequality:

Property 4.1.2 (Fefferman-Phong inequality). *If a is a non-negative symbol belonging to $S_{1,0}^2$, then there exists $C > 0$ such that, for all $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$\operatorname{Re} \langle a(x, D)u, u \rangle_{L^2(\mathbb{R}^n)} + C\|u\|_{L^2(\mathbb{R}^n)}^2 \geq 0. \quad (4.1.2)$$

We can pose similar questions in many other examples of classes of symbols, such as the semi-classical symbols, Shubin’s class, etc. As a particular example, the class Σ^m , defined as the set of smooth functions a on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$ such that for all multi-indices α, β ,

$$\forall x, \xi \in \mathbb{R}^n, \tau \in \mathbb{R}^+, \quad |(\partial_\xi^\alpha \partial_x^\beta a)(x, \xi, \tau)| \leq C_{\alpha, \beta} (1 + |\xi| + \tau)^{m - |\alpha|}, \quad (4.1.3)$$

is useful for Carleman estimates. One would like to check the Property 4.1.1 and Property 4.1.2 independent of the parameter τ .

Several authors like Bony [Bon99], Boulkhemair [Bou08], Lerner-Morimoto [LM07], have already considered these questions and they were able to identify the constants. The constants in Properties 4.1.1, 4.1.2 are always a constant C_n times a semi-norm of the symbol, whose order depends only on the dimension n . Although the problem is well-understood for a single class of pseudo-differential calculus, including the class $S(m, g)$ developed by Hörmander, we want to address a more general and useful question, having in mind the class Σ^m depending on the non-compact parameter $\tau \geq 0$ which is defined in (4.1.3) and is useful for Carleman estimates.

In this paper, we consider the Weyl quantization for pseudodifferential operators and we choose the framework with a metric g on the phase space. The metric g is assumed to be admissible, that is slowly varying, satisfying the uncertainty principle and is temperate (see Definition 4.2.1, 4.2.6 below). The so-called structure constants of g are closely related to these properties. We can define very general classes of symbols $S(m, g)$ attached to the metric g and a g -admissible weight m (see Definition 4.2.3) and we have an effective symbolic calculus. The following results are classical: (see [Hör85, chapter 18], [Ler10, chapter 2])

$$L^2\text{-boundedness: } a \in S(1, g) \implies \|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C, \quad (4.1.4)$$

$$\text{Fefferman-Phong: } a \in S(\lambda_g^2, g), a \geq 0 \implies a^w + C \geq 0. \quad (4.1.5)$$

The question that we would like to address is the following: what happens if we change the metric g but keep the same structure constants?

We intend to show that the constants involved in (4.1.4), (4.1.5) depend only on the structure constants of the metric g and a fixed semi-norm of a . Since it may happen that the metric g depends on a non-compact parameter with uniform structure constants (e.g. the class Σ^m), this fact is useful explicitly or implicitly in many examples where these metrics are used and it seems useful to rely on a more stable argument than referring to “inspection of the proofs”.

Remark 4.1.3. An abstract functional analysis argument does not seem to work. Our method is to follow the proofs, by carefully computing all the constants.

4.2 Metric on the phase space

In this section, we introduce the definitions of the admissible metric and exhibit its properties. We use the Weyl quantization which associates to a symbol a the operator a^w defined by

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (4.2.1)$$

Consider the symplectic space \mathbb{R}^{2n} equipped with the symplectic form $\sigma = \sum_{j=1}^n d\xi^j \wedge dx^j$. Given a positive-definite quadratic form Γ on \mathbb{R}^{2n} , we define

$$\Gamma^\sigma(T) = \sup_{\Gamma(Y)=1} \sigma(T, Y)^2, \quad (4.2.2)$$

which is also a positive-definite quadratic form. Let g be a measurable map from \mathbb{R}^{2n} into the cone of positive-definite quadratic forms on \mathbb{R}^{2n} , i.e. for each $X \in \mathbb{R}^{2n}$, g_X is a positive definite quadratic form on \mathbb{R}^{2n} .

Definition 4.2.1 (Slowly varying metric). We say that g is a slowly varying metric on \mathbb{R}^{2n} , if there exists $C_0 \geq 1$ such that for all $X, Y, T \in \mathbb{R}^{2n}$,

$$g_X(X - Y) \leq C_0^{-1} \implies C_0^{-1} \leq \frac{g_X(T)}{g_Y(T)} \leq C_0. \quad (4.2.3)$$

Definition 4.2.2 (Slowly varying weight). Let g be a slowly varying metric on \mathbb{R}^{2n} . A function $m: \mathbb{R}^{2n} \rightarrow (0, +\infty)$ is called a g -slowly varying weight if there exists $\mu_m \geq 1$ such that for all $X, Y \in \mathbb{R}^{2n}$,

$$g_X(Y - X) \leq \mu_m^{-1} \implies \mu_m^{-1} \leq \frac{m(X)}{m(Y)} \leq \mu_m. \quad (4.2.4)$$

Definition 4.2.3 (Class of symbols). Let g be a slowly varying metric on \mathbb{R}^{2n} and m be a g -slowly varying weight. The class of symbols $S(m, g)$ is defined as the subset of functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying that for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $X, T_1, \dots, T_k \in \mathbb{R}^{2n}$,

$$|a^{(k)}(X)(T_1, \dots, T_k)| \leq C_k m(X) \prod_{1 \leq j \leq k} g_X(T_j)^{1/2}.$$

For $a \in S(m, g)$, $l \in \mathbb{N}$, we denote

$$\|a\|_{S(m,g)}^{(l)} = \max_{0 \leq k \leq l} \sup_{\substack{X, T_j \in \mathbb{R}^{2n} \\ g_X(T_j)=1}} |a^{(k)}(X)(T_1, \dots, T_k)| m(X)^{-1}. \quad (4.2.5)$$

The space $S(m, g)$ equipped with the countable family of semi-norms $(\|\cdot\|_{S(m,g)}^{(l)})_{l \in \mathbb{N}}$ is a Fréchet space.

For a slowly varying metric g on the phase space \mathbb{R}^{2n} , we can introduce some partition of unity related to g . Define the g -ball near $X \in \mathbb{R}^{2n}$

$$U_{X,r} = \{Y, g_X(X - Y) \leq r^2\}, \quad (4.2.6)$$

we have the following theorem, which is Theorem 2.2.7 in [Ler10].

Theorem 4.2.4 (Partition of unity). *Let g be a slowly varying metric on \mathbb{R}^{2n} and $C_0 > 0$ given in (4.2.3). Then for all $r \in (0, C_0^{-1/2}]$, there exists a family $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ of smooth functions supported in $U_{Y,r}$ such that*

$$\forall k \in \mathbb{N}, \quad \sup_{Y \in \mathbb{R}^{2n}} \|\varphi_Y\|_{S(1,g)}^{(k)} \leq C(k, r, n, C_0), \quad (4.2.7)$$

$$\forall X \in \mathbb{R}^{2n}, \quad \int_{\mathbb{R}^{2n}} \varphi_Y(X) |g_Y|^{1/2} dY = 1, \quad (4.2.8)$$

where $C(k, r, n, C_0)$ is a positive constant depending only on k, r, n, C_0 and $|g_Y|$ is the determinant of g_Y with respect to the standard Euclidean norm.

Proof. As in the proof of Theorem 2.2.7 in [Ler10], let $\chi_0 \in C_0^\infty(\mathbb{R}_+; [0, 1])$ non-increasing such that $\chi_0(t) = 1$ on $t \leq 1/2$, $\chi_0(t) = 0$ on $t \geq 1$. Define for $r \in (0, C_0^{-1/2}]$,

$$\omega(X, r) = \int_{\mathbb{R}^{2n}} \underbrace{\chi_0(r^{-2} g_Y(X - Y))}_{=\omega_Y(X)} |g_Y|^{1/2} dY.$$

Since $\omega_Y(X)$ is supported in $U_{Y,r}$ and χ_0 is non-increasing, by (4.2.3) we have

$$\omega(X, r) \geq \int_{\mathbb{R}^{2n}} \chi_0(r^{-2} C_0 g_X(X - Y)) C_0^{-n} |g_X|^{1/2} dY = \int_{\mathbb{R}^{2n}} \chi_0(|Z|^2) dZ C_0^{-2n} r^{2n},$$

and an estimate from above of the same type, i.e. there exists a positive constant $C_1 = C_1(r, n, C_0)$ such that

$$C_1^{-1} \leq \omega(X, r) \leq C_1.$$

Now let us check the derivatives of $\omega_Y(X)$. Using the notation $\langle \cdot, \cdot \rangle_Y$ the inner-product associated to g_Y , we have

$$\omega'_Y(X)T = \chi'_0(r^{-2} g_Y(X - Y)) r^{-2} \langle X - Y, T \rangle_Y,$$

and by induction, for $k \geq 1$, $T \in \mathbb{R}^{2n}$, $\omega_Y^{(k)}(X)T^k$ is a finite sum of terms of type

$$c_{p,k} \chi_0^{(p)}(r^{-2} g_Y(X - Y)) r^{-2p} \langle X - Y, T \rangle_Y^{2p-k} g_Y(T)^{k-p}, \quad (4.2.9)$$

where $c_{p,k}$ is a constant depending only on p, k and $p \in [k/2, k] \cap \mathbb{N}$. Since the support of $\chi_0^{(p)}$ is included in $[0, 1]$ and $r^2 \leq C_0^{-1}$, the term (4.2.9) can be bounded from above by

$$c_{p,k} \|\chi_0^{(p)}\|_{L^\infty} r^{-2p} (r^2)^{(2p-k)/2} C_0^{k/2} g_X(T)^{k/2},$$

so that for all $k \geq 1$, $|\omega_Y^{(k)}(X)T^k| \leq C(k, r, C_0) g_X(T)^{k/2}$. This implies that ω_Y is in $S(1, g)$ and moreover,

$$\forall k \in \mathbb{N}, \quad \sup_{Y \in \mathbb{R}^{2n}} \|\omega_Y\|_{S(1,g)}^{(k)} \leq C(k, r, C_0). \quad (4.2.10)$$

Now we choose a non-negative function $\chi_1 \in C_0^\infty(\mathbb{R}_+; [0, 1])$ such that $\chi_1(t) = 1$ on $t \leq 1$, then

$$\begin{aligned} |\omega^{(k)}(X, r)T^k| &= \left| \int_{\mathbb{R}^{2n}} \omega_Y^{(k)}(X)T^k \chi_1(r^{-2} g_Y(X - Y)) |g_Y|^{1/2} dY \right| \\ &\leq \sup_{Y \in \mathbb{R}^{2n}} \|\omega_Y\|_{S(1,g)}^{(k)} g_X(T)^{k/2} \int_{\mathbb{R}^{2n}} \chi_1(r^{-2} g_Y(X - Y)) |g_Y|^{1/2} dY \\ &\leq C(k, r, n, C_0) g_X(T)^{k/2}, \end{aligned}$$

which implies that $\omega(\cdot, r)$ is a symbol in $S(1, g)$ with $\|\omega(\cdot, r)\|_{S(1,g)}^{(k)} \leq C'(k, r, n, C_0)$. Since ω is bounded from below by C_1^{-1} , the function $\omega(\cdot, r)^{-1}$ is also in $S(1, g)$ and

$$\|\omega(\cdot, r)^{-1}\|_{S(1,g)}^{(k)} \leq C''(k, r, n, C_0). \quad (4.2.11)$$

We define

$$\varphi_Y(X) = \omega_Y(X) \omega(X, r)^{-1},$$

then the estimate (4.2.7) follows from (4.2.10), (4.2.11) and moreover, the family $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ satisfies the requirements of Theorem 4.2.4. \square

A direct consequence of Theorem 4.2.4 is the following.

Proposition 4.2.5. Let g be a slowly varying metric on \mathbb{R}^{2n} and m be a g -slowly varying weight. Let C_0, μ_m be given in (4.2.3), (4.2.4) respectively. Let a be a symbol in $S(m, g)$. Then for all $0 < r \leq \min(C_0^{-1/2}, \mu_m^{-1/2})$,

$$a(X) = \int_{\mathbb{R}^{2n}} a_Y(X) |g_Y|^{1/2} dY,$$

where a_Y has support included in $U_{Y,r}$ and

$$\forall k \in \mathbb{N}, \quad \sup_{Y \in \mathbb{R}^{2n}} \|a_Y\|_{S(m(Y), g_Y)}^{(k)} \leq C(k, r, C_0, n, \mu_m) \|a\|_{S(m, g)}^{(k)}. \quad (4.2.12)$$

Proof. Define $a_Y(X) = a(X)\varphi_Y(X)$. Since φ_Y is supported in $U_{Y,r}$, we have, for $k \geq 0$, $X \in U_{Y,r}$, $T \in \mathbb{R}^{2n}$,

$$\begin{aligned} |a_Y^{(k)}(X)T^k| &= \left| \sum_{0 \leq l \leq k} \binom{k}{l} a^{(l)}(X) T^l \cdot \varphi_Y^{(k-l)}(X) T^{k-l} \right| \\ &\leq \sum_{0 \leq l \leq k} c_{k,l} \|a\|_{S(m,g)}^{(l)} m(X) g_X(T)^{l/2} \|\varphi_Y\|_{S(1,g)}^{(k-l)} g_X(T)^{(k-l)/2} \\ &\leq C(k) \|a\|_{S(m,g)}^{(k)} \|\varphi_Y\|_{S(1,g)}^{(k)} m(X) g_X(T)^{k/2} \\ &\leq C(k) \mu_m C_0^{k/2} \|a\|_{S(m,g)}^{(k)} \|\varphi_Y\|_{S(1,g)}^{(k)} m(Y) g_Y(T)^{k/2}, \end{aligned}$$

which completes the proof. \square

For two positive-definite quadratic forms Γ_1, Γ_2 on \mathbb{R}^{2n} , the harmonic mean $\Gamma_1 \wedge \Gamma_2$ is defined by

$$\Gamma_1 \wedge \Gamma_2 = 2(\Gamma_1^{-1} + \Gamma_2^{-1})^{-1}, \quad (4.2.13)$$

which is also a positive-definite quadratic form on \mathbb{R}^{2n} .

Definition 4.2.6 (Admissible metric). We say that g is an admissible metric on \mathbb{R}^{2n} if g is slowly varying (see Definition 4.2.1) and there exist $C'_0 > 0$, $N_0 \in \mathbb{N}$ such that for all $X, Y, T \in \mathbb{R}^{2n}$,

$$\text{uncertainty principle} \quad g_X(T) \leq g_X^\sigma(T), \quad (4.2.14)$$

$$\text{temperance} \quad g_X(T) \leq C'_0 g_Y(T) (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^{N_0}, \quad (4.2.15)$$

where g^σ is given by (4.2.2) and \wedge given by (4.2.13).

We may suppose $C'_0 = C_0$ in the sequel, where C_0 is given in (4.2.3). Then the constants (C_0, N_0) appearing in (4.2.3), (4.2.15) are called the **structure constants** of the metric g .

Definition 4.2.7 (Admissible weight). Suppose that g is an admissible metric on \mathbb{R}^{2n} . A function $m: \mathbb{R}^{2n} \rightarrow (0, +\infty)$ is called a g -admissible weight if m is a g -slowly varying weight (see Definition 4.2.2) and there exist $\mu_m > 0$, $\nu_m \in \mathbb{N}$ such that for all $X, Y \in \mathbb{R}^{2n}$,

$$m(X) \leq \mu_m m(Y) (1 + (g_X^\sigma \wedge g_Y^\sigma)(X - Y))^{\nu_m}. \quad (4.2.16)$$

The constants (μ_m, ν_m) appearing in (4.2.4), (4.2.16) are called the structure constants of the g -admissible weight m .

Let g be an admissible metric on \mathbb{R}^{2n} . We define for $X \in \mathbb{R}^{2n}$,

$$\lambda_g(X) = \inf_{T \neq 0} \left(\frac{g_X^\sigma(T)}{g_X(T)} \right)^{1/2}. \quad (4.2.17)$$

Then the uncertainty principle (4.2.14) can be expressed by

$$g_X \leq \lambda_g(X)^{-2} g_X^\sigma, \quad \lambda_g(X) \geq 1.$$

Lemma 4.2.8 ([Ler10, Remark 2.2.17]). *For any $s \in \mathbb{R}$, λ_g^s is an admissible weight, with structure constants $(\mu_{\lambda_g^s}, \nu_{\lambda_g^s})$ in (4.2.4), (4.2.16) depending only on the structure constants of the metric g (C_0, N_0).*

Proof. We first verify that λ_g^s is a g -slowly varying weight. For $g_X(X-Y) \leq C_0^{-1}, T \in \mathbb{R}^{2n}$, we have

$$C_0^{-1} g_X(T) \leq g_Y(T) \leq C_0 g_X(T), \quad C_0^{-1} g_X^\sigma(T) \leq g_Y^\sigma(T) \leq C_0 g_X^\sigma(T),$$

which implies

$$C_0^{-2} \frac{g_X^\sigma(T)}{g_X(T)} \leq \frac{g_Y^\sigma(T)}{g_Y(T)} \leq C_0^2 \frac{g_X^\sigma(T)}{g_X(T)}.$$

Taking the infimum with respect to T , we get

$$C_0^{-2} \lambda_g(X)^2 \leq \lambda_g(Y)^2 \leq C_0^2 \lambda_g(X)^2,$$

so that λ_g is g -slowly varying with $\mu_{\lambda_g} = C_0$ and so is λ_g^s with $\mu_{\lambda_g^s} = C_0^{|s|}$. Next we check that λ_g^s is temperate. We have for all $X, Y, T \in \mathbb{R}^{2n}$,

$$\begin{aligned} g_X(T) &\geq C_0^{-1} g_Y(T) (1 + (g_X^\sigma \wedge g_Y^\sigma)(X-Y))^{-N_0}, \\ g_X^\sigma(T) &\leq C_0 g_Y^\sigma(T) (1 + (g_X^\sigma \wedge g_Y^\sigma)(X-Y))^{N_0}, \end{aligned}$$

which gives

$$\lambda_g(X)^2 \leq C_0^2 \lambda_g(Y)^2 (1 + (g_X^\sigma \wedge g_Y^\sigma)(X-Y))^{2N_0}.$$

Thus λ_g is temperate with $\nu_{\lambda_g} = N_0$ and so is λ_g^s with $\nu_{\lambda_g^s} = |s|N_0$. This completes the proof of Lemma 4.2.8. \square

The composition $a \sharp b$ of two symbols is defined by $a^w b^w = (a \sharp b)^w$ and we have, with the notations $[X, Y] = \sigma(X, Y)$, $D = (2i\pi)^{-1}\partial$,

$$(a \sharp b)(X) = 2^{2n} \iint_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} a(Y) b(Z) e^{-4i\pi[X-Y, X-Z]} dY dZ, \quad (4.2.18)$$

$$(a \sharp b)(X) = \exp(i\pi[D_Y, D_Z])(a(Y)b(Z))_{|Y=Z=X}. \quad (4.2.19)$$

For $a \in S(m_1, g)$, $b \in S(m_2, g)$, we have the asymptotic expansion

$$a \sharp b(x, \xi) = \sum_{0 \leq k < p} w_k(a, b)(x, \xi) + r_p(a, b)(x, \xi), \quad (4.2.20)$$

$$\text{with } w_k(a, b) = 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha! \beta!} D_\xi^\alpha \partial_x^\beta a D_\xi^\beta \partial_x^\alpha b \in S(m_1 m_2 \lambda_g^{-k}, g), \quad (4.2.21)$$

$$r_p(a, b)(X) = R_p(a(X) \otimes b(Y))_{|Y=X} \in S(m_1 m_2 \lambda_g^{-p}, g), \quad (4.2.22)$$

$$R_p = \int_0^1 \frac{(1-\theta)^{p-1}}{(p-1)!} \exp \frac{\theta}{4i\pi} [\partial_X, \partial_Y] d\theta \left(\frac{1}{4i\pi} [\partial_X, \partial_Y] \right)^p \quad (4.2.23)$$

Notice $w_1(a, b) = \frac{1}{4i\pi} \{a, b\}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket, so that the asymptotic (4.2.20) at $p = 2$ is

$$a \sharp b = ab + \frac{1}{4i\pi} \{a, b\} + r_2(a, b). \quad (4.2.24)$$

Definition 4.2.9 (The main distance function). Let g be an admissible metric on \mathbb{R}^{2n} . Define the main distance function, for $r > 0$, $X, Y \in \mathbb{R}^{2n}$,

$$\delta_r(X, Y) = 1 + (g_X^\sigma \wedge g_Y^\sigma)(U_{X,r} - U_{Y,r}), \quad (4.2.25)$$

where $U_{X,r}$ is given in (4.2.6) and

$$g(U - V) = \inf_{X \in U, Y \in V} g(X - Y).$$

Lemma 4.2.10 ([Ler10, Lemma 2.2.24], Integrability of δ_r). *Let g be an admissible metric with structure constants (C_0, N_0) . Then there exist positive constants $N_1 = N_1(n, C_0, N_0)$, $C = C(n, C_0, N_0)$ such that for all $r \in (0, C_0^{-1/2}]$,*

$$\sup_{X \in \mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \delta_r(X, Y)^{-N_1} |g_Y|^{1/2} dY \leq C < +\infty, \quad (4.2.26)$$

Proof. Suppose $r \leq C_0^{-1/2}$. Using the slowness and temperance of g , for $X' \in U_{X,r}$, $Y' \in U_{Y,r}$, $T \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} (g_X^\sigma \wedge g_Y^\sigma)(T) &\geq C_0^{-1} (g_{X'}^\sigma \wedge g_{Y'}^\sigma)(T) \geq C_0^{-2} g_{X'}^\sigma(T) (1 + (g_{X'}^\sigma \wedge g_{Y'}^\sigma)(X' - Y'))^{-N_0} \\ &\geq C_0^{-3} g_X^\sigma(T) (1 + C_0(g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{-N_0} \\ &\geq C_0^{-3-N_0} g_X^\sigma(T) (1 + (g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{-N_0}. \end{aligned}$$

Taking the infimum in $X' \in U_{X,r}$, $Y' \in U_{Y,r}$, we get

$$g_X^\sigma(T) \leq C_0^{3+N_0} \delta_r(X, Y)^{N_0} (g_X^\sigma \wedge g_Y^\sigma)(T). \quad (4.2.27)$$

We have also

$$\begin{aligned} \frac{g_X(T)}{g_Y(T)} &\leq C_0^2 \frac{g_{X'}(T)}{g_{Y'}(T)} \leq C_0^3 (1 + (g_{X'}^\sigma \wedge g_{Y'}^\sigma)(X' - Y'))^{N_0} \\ &\leq C_0^3 (1 + C_0(g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{N_0} \\ &\leq C_0^{3+N_0} (1 + (g_X^\sigma \wedge g_Y^\sigma)(X' - Y'))^{N_0}. \end{aligned}$$

By taking the infimum in X', Y' , we get the following inequality

$$\frac{g_X(T)}{g_Y(T)} \leq C_0^{3+N_0} \delta_r(X, Y)^{N_0}. \quad (4.2.28)$$

Then

$$\begin{aligned} 1 + g_X(X - Y) &\leq 1 + 3g_X(X - X') + 3g_X(X' - Y') + 3g_X(Y' - Y) \\ &\leq 3C_0^{3+N_0} \delta_r(X, Y)^{N_0} (1 + g_X(X - X') + g_X(X' - Y') + g_Y(Y' - Y)) \quad \text{by (4.2.28)} \\ &\leq 3C_0^{3+N_0} \delta_r(X, Y)^{N_0} (1 + 2r^2 + g_X^\sigma(X' - Y')) \\ &\leq 9C_0^{6+2N_0} \delta_r(X, Y)^{2N_0} (1 + (g_X^\sigma \wedge g_Y^\sigma)(X' - Y')) \quad \text{by (4.2.27),} \end{aligned}$$

so that $1 + g_X(X - Y) \leq 9C_0^{6+2N_0} \delta_r(X, Y)^{2N_0+1}$. In the other hand, we have

$$\frac{|g_Y|^{1/2}}{|g_X|^{1/2}} \leq C_0^{n(3+N_0)} \delta_r(X, Y)^{nN_0},$$

so that for $N_1 = nN_0 + (n+1)(2N_0+1) > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \delta_r(X, Y)^{-N_1} |g_Y|^{1/2} dY &\leq C(n, C_0, N_0) \int_{\mathbb{R}^{2n}} \delta_r(X, Y)^{-N_1+nN_0} |g_X|^{1/2} dY \\ &\leq C'(n, C_0, N_0) \int_{\mathbb{R}^{2n}} (1 + g_X(X - Y))^{-(n+1)} |g_X|^{1/2} dY \\ &= C'(n, C_0, N_0) \int_{\mathbb{R}^{2n}} (1 + |Z|^2)^{-(n+1)} dZ < +\infty. \end{aligned}$$

The proof of the lemma is complete. \square

4.3 L^2 -boundedness

In this section, we prove the L^2 -boundedness of pseudo-differential operators with symbol in $S(1, g)$ and make precise the operator norms.

4.3.1 The constant metric case

Proposition 4.3.1. *Suppose that g is a positive-definite quadratic form (constant metric) on \mathbb{R}^{2n} with $g \leq g^\sigma$. Then there exists a constant $C(n) > 0$ depending only on the dimension n such that for all $a \in S(1, g)$,*

$$\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C(n) \|a\|_{S(1,g)}^{(2n+1)}.$$

Proof. Since g is a constant metric, according to Lemma 4.4.25 in [Ler10], there exist symplectic coordinates (x, ξ) such that

$$g = \sum_{1 \leq j \leq n} \lambda_j^{-1} (|dx_j|^2 + |d\xi_j|^2), \quad g^\sigma = \sum_{1 \leq j \leq n} \lambda_j (|dx_j|^2 + |d\xi_j|^2),$$

with $\lambda_j > 0$. $g \leq g^\sigma$ is expressed as

$$\min_{1 \leq j \leq n} \lambda_j \geq 1.$$

As a result, we have $g \leq |dx|^2 + |d\xi|^2 := \Gamma_0$, which implies $S(1, g) \subset S(1, \Gamma_0)$ and for all $a \in S(1, g)$,

$$\forall l \in \mathbb{N}, \quad \|a\|_{S(1,\Gamma_0)}^{(l)} \leq \|a\|_{S(1,g)}^{(l)}. \quad (4.3.1)$$

By Theorem 1.1.4 in [Ler10] and $a^w = (J^{1/2}a)(x, D)$, where J^t is introduced in Lemma 4.1.2 in [Ler10], we obtain that

$$\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C(n) \|a\|_{S(1,\Gamma_0)}^{(2n+1)},$$

where $C(n)$ depends only on n . Together with (4.3.1), we complete the proof of the proposition. \square

4.3.2 The general case

Theorem 4.3.2. *Let g be an admissible metric on \mathbb{R}^{2n} with structure constants (C_0, N_0) (see Definition 4.2.6). Then there exist $C = C(n, C_0, N_0) > 0$ and $l = l(n, C_0, N_0) \in \mathbb{N}$ such that for all $a \in S(1, g)$ (see Definition 4.2.3),*

$$\|a^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C \|a\|_{S(1,g)}^{(l)}.$$

Proof. Using the partition in Proposition 4.2.5, we write

$$a^w = \int_{\mathbb{R}^{2n}} a_Y^w |g_Y|^{1/2} dY,$$

where a_Y is supported in $U_{Y,r}$ and satisfies (4.2.12). By Proposition 4.3.1, we have $\sup_Y \|a_Y^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C(r, n, C_0, N_0) \|a\|_{S(1,g)}^{(2n+1)} < +\infty$. The following lemma is useful.

Lemma 4.3.3 (Cotlar). *Let H be a Hilbert space and $(\Omega, \mathcal{A}, \nu)$ a measured space such that ν is a σ -finite positive measure. Let $(A_y)_{y \in \Omega}$ be a measurable family of bounded operators on H such that*

$$\sup_{y \in \Omega} \int_{\Omega} \|A_y^* A_z\|_{\mathcal{L}(H)}^{1/2} d\nu(z) \leq M, \quad \sup_{y \in \Omega} \int_{\Omega} \|A_y A_z^*\|_{\mathcal{L}(H)}^{1/2} d\nu(z) \leq M.$$

Then for all $u \in H$, we have

$$\iint_{\Omega \times \Omega} |\langle A_y u, A_z u \rangle_H| d\nu(y) d\nu(z) \leq M^2 \|u\|_H^2,$$

which implies the strong convergence of $A = \int_{\Omega} A_y d\nu(y)$ and $\|A\|_{\mathcal{L}(H)} \leq M$.

In order to apply Cotlar's lemma, we should estimate $\|\bar{a}_Y^w a_Z^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$, i.e. a semi-norm of $\bar{a}_Y \# a_Z$ in $S(1, g_Y + g_Z)$. Indeed, the following estimate holds.

Lemma 4.3.4. *Let g, a_Y be as above. For any $k, N \in \mathbb{N}$, there exist $C = C(k, N, n) > 0$, $l = l(k, N, n) \in \mathbb{N}$ such that*

$$\|\bar{a}_Y \# a_Z\|_{S(1, g_Y + g_Z)}^{(k)} \leq C \|\bar{a}_Y\|_{S(1, g_Y)}^{(l)} \|a_Z\|_{S(1, g_Z)}^{(l)} \delta_r(Y, Z)^{-N}. \quad (4.3.2)$$

We use some biconfinement estimates, which can be found in [Ler10, section 2.3], to prove Lemma 4.3.4.

Definition 4.3.5 (Confined symbols). *Let g be a positive-definite quadratic form on \mathbb{R}^{2n} such that $g \leq g^\sigma$. Let a be a smooth function on \mathbb{R}^{2n} and $U \subset \mathbb{R}^{2n}$. We say that a is g -confined in U , if for all $k, N \in \mathbb{N}$, there exists $C_{k,N} > 0$ such that for all $X, T \in \mathbb{R}^{2n}$,*

$$|a^{(k)}(X)T^k| \leq C_{k,N} g(T)^{k/2} (1 + g^\sigma(X - U))^{-N/2}.$$

We denote

$$\|a\|_{g,U}^{(k,N)} = \sup_{X,T \in \mathbb{R}^{2n}, g(T)=1} |a^{(k)}(X)T^k| (1 + g^\sigma(X - U))^{N/2}, \quad (4.3.3)$$

$$\text{and } \|a\|_{g,U}^{(l)} = \max_{k \leq l} \|a\|_{g,U}^{(k,l)}. \quad (4.3.4)$$

Theorem 4.3.6 ([Ler10, Theorem 2.3.2], biconfinement estimate). Let g_1, g_2 be two positive-definite quadratic forms on \mathbb{R}^{2n} such that $g_j \leq g_j^\sigma$. Let $a_j, j = 1, 2$ be g_j -confined in U_j , a g_j -ball of radius ≤ 1 . Then for all $k, N \in \mathbb{N}$, for all $X, T \in \mathbb{R}^{2n}$,

$$|(a_1 \# a_2)^{(k)}(X)T^k| \leq A_{k,N}(g_1 + g_2)(T)^{k/2} \left(1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1) + (g_1^\sigma \wedge g_2^\sigma)(X - U_2)\right)^{-N/2}, \quad (4.3.5)$$

with $A_{k,N} = \gamma(k, N, n) \|a_1\|_{g_1, U_1}^{(l)} \|a_2\|_{g_2, U_2}^{(l)}$, $l = 2n + 1 + k + N$.

Now we begin the proof of Lemma 4.3.4.

Proof of Lemma 4.3.4. The symbol a_Y is g_Y -confined in $U_{Y,r}$, since a_Y is supported in the g_Y -ball $U_{Y,r}$. Moreover, we have

$$\forall k, N \in \mathbb{N}, \quad \|a_Y\|_{g_Y, U_{Y,r}}^{(k, N)} = \sup_{\substack{X \in U_{Y,r}, T \in \mathbb{R}^{2n} \\ g_Y(T)=1}} |a^{(k)}(X)T^k|,$$

$$\forall l \in \mathbb{N}, \quad \|a_Y\|_{g_Y, U_{Y,r}}^{(l)} = \max_{k \leq l} \|a_Y\|_{g_Y, U_{Y,r}}^{(k, l)} = \|a_Y\|_{S(1, g_Y)}^{(l)}.$$

Applying (4.3.5) to $\bar{a}_Y \# a_Z$ and using the triangular inequality

$$(g_Y^\sigma \wedge g_Z^\sigma)(X - U_{Y,r}) + (g_Y^\sigma \wedge g_Z^\sigma)(X - U_{Z,r}) \geq \frac{1}{2}(g_Y^\sigma \wedge g_Z^\sigma)(U_{Y,r} - U_{Z,r}),$$

we get

$$\begin{aligned} |(\bar{a}_Y \# a_Z)^{(k)}(X)T^k| &\leq \gamma(k, N, n) \|\bar{a}_Y\|_{S(1, g_Y)}^{(l)} \|a_Z\|_{S(1, g_Z)}^{(l)} (g_Y + g_Z)(T)^{k/2} \\ &\quad \times \left(1 + \frac{1}{2}(g_Y^\sigma \wedge g_Z^\sigma)(U_{Y,r} - U_{Z,r})\right)^{-N/2}. \end{aligned}$$

Using the definition of the distance δ_r , we complete the proof of Lemma 4.3.4. \square

End of the proof of Theorem 4.3.2. Now by Proposition 4.3.1, Lemma 4.3.4 and the estimate (4.2.12), we obtain that for any $N > 0$, there exists $l = l(N, n) \in \mathbb{N}$ such that

$$\begin{aligned} \|\bar{a}_Y^w a_Z^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))} &\leq C(n) \|\bar{a}_Y^w a_Z^w\|_{S(1, g_Y + g_Z)}^{(2n+1)} \\ &\leq C(N, n) \|\bar{a}_Y\|_{S(1, g_Y)}^{(l)} \|a_Z\|_{S(1, g_Z)}^{(l)} \delta_r(Y, Z)^{-N} \\ &\leq C(N, n, C_0) (\|a\|_{S(1, g)}^{(l)})^2 \delta_r(Y, Z)^{-N} \end{aligned}$$

The same inequality holds for $a_Y \# \bar{a}_Z$. Choose $N = 2N_1$, where N_1 is given in (4.2.26), so that

$$\max \{\|\bar{a}_Y^w a_Z^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}^{1/2}, \|a_Y^w \bar{a}_Z^w\|_{\mathcal{L}(L^2(\mathbb{R}^n))}^{1/2}\} \leq C \|a\|_{S(1, g)}^{(l)} \delta_r(Y, Z)^{-N_1},$$

where $C = C(n, C_0, N_1) > 0$, $l = l(n, N_1) \in \mathbb{N}$. Then together with Lemma 4.2.10, the assumptions of Cotlar's lemma are fulfilled with $M = C \|a\|_{S(1, g)}^{(l)}$, and this completes the proof of Theorem 4.3.2. \square

4.4 Fefferman-Phong inequality

In this section, we prove that the constant in the Fefferman-Phong inequality depends only on the structure constants of the metric and a fixed semi-norm of the symbol.

Theorem 4.4.1 (Fefferman-Phong inequality). *Let g be an admissible metric on \mathbb{R}^{2n} with structure constants (C_0, N_0) (see Definition 4.2.6). Let a be a non-negative symbol in $S(\lambda_g^2, g)$ (see Definition 4.2.3 and (4.2.17)). Then the operator a^w on $L^2(\mathbb{R}^n)$ is semi-bounded from below. More precisely, there exist $l = l(n, C_0, N_0) \in \mathbb{N}$, $C = C(n, C_0, N_0) > 0$ such that*

$$a^w + C \|a\|_{S(\lambda_g^2, g)}^{(l)} \geq 0. \quad (4.4.1)$$

4.4.1 The constant metric case

For the constant metric case, we use the results of Sjöstrand and refer the readers to [Ler10, page 116] for the detailed proof.

Let $1 = \sum_{j \in \mathbb{Z}^{2n}} \chi_0(X - j)$ be a partition of unity, $\chi_0 \in C_c^\infty(\mathbb{R}^{2n})$. Denote $\chi_j(X) = \chi_0(X - j)$.

Proposition 4.4.2 ([Ler10, Proposition 2.5.6]). *Suppose $a \in \mathcal{S}(\mathbb{R}^{2n})$. We say that a belongs to the class \mathcal{A} if $\omega_a \in L^1(\mathbb{R}^{2n})$, with $\omega_a(\Xi) = \sup_{j \in \mathbb{Z}^{2n}} |\mathcal{F}(\chi_j a)(\Xi)|$, where \mathcal{F} is the Fourier transform. We have*

$$S_{0,0}^0 \subset S_{0,0;2n+1} \subset \mathcal{A} \subset C^0(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}),$$

where $S_{0,0}^0 = C_b^\infty(\mathbb{R}^{2n})$ is the space of C^∞ functions on \mathbb{R}^{2n} which are bounded as well as all their derivatives, $S_{0,0;2n+1}$ is the set of functions defined on \mathbb{R}^{2n} such that $|(\partial_\xi^\alpha \partial_x^\beta a)(x, \xi)| \leq C_{\alpha\beta}$ for $|\alpha| + |\beta| \leq 2n + 1$. \mathcal{A} is a Banach algebra for the multiplication with the norm $\|a\|_{\mathcal{A}} = \|\omega_a\|_{L^1(\mathbb{R}^{2n})}$.

Theorem 4.4.3 ([Ler10, Theorem 2.5.10]). *For all non-negative function a defined on \mathbb{R}^{2n} satisfying $a^{(4)} \in \mathcal{A}$, then the operator a^w is semi-bounded from below. More precisely,*

$$a^w + C_n \|a^{(4)}\|_{\mathcal{A}} \geq 0,$$

where C_n depends only on the dimension n .

4.4.2 Proof of Theorem 4.4.1

We shall use the partition of unity $(\varphi_Y)_{Y \in \mathbb{R}^{2n}}$ given in Theorem 4.2.4. Let $(\psi_Y)_{Y \in \mathbb{R}^{2n}}$ be a family of real-valued functions supported in $U_{Y,2r}$, equal to 1 on $U_{Y,r}$ and

$$\sup_{Y \in \mathbb{R}^{2n}} \|\psi_Y\|_{S(1,g)}^{(k)} \leq C(k, r, C_0). \quad (4.4.2)$$

Indeed, with the same notations as in the proof of Theorem 4.2.4, the function $\psi_Y(X) = \chi_0(\frac{1}{2}r^{-2}g_Y(X - Y))$ satisfies the requirements. Then with $a_Y = \varphi_Y a$, we write

$$\psi_Y \# a_Y \# \psi_Y = a_Y + r_Y. \quad (4.4.3)$$

Lemma 4.4.4 (Estimate for r_Y). *For all $k, N \in \mathbb{N}$, there exist $C = C(k, N, C_0) > 0$, $l = l(k, N, C_0) \in \mathbb{N}$ such that for all $X \in \mathbb{R}^{2n}$, $T \in \mathbb{R}^{2n}$ with $g_Y(T) \leq 1$,*

$$|r_Y^{(k)}(X)T^k| \leq C \|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l)} (1 + g_Y^\sigma(X - U_{Y,2r}))^{-N}. \quad (4.4.4)$$

Moreover, there exist $C_1 = C_1(n, C_0, N_0) > 0$, $l_1 = l_1(n, C_0, N_0) \in \mathbb{N}$ such that

$$\left\| \int_{\mathbb{R}^{2n}} r_Y^w |g_Y|^{1/2} dY \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_1 \|a\|_{S(\lambda_g^2, g)}^{(l_1)}, \quad (4.4.5)$$

To prove Lemma 4.4.4, we use the biconfinement estimate for the remainders, the proof of which can be found in [Ler10, section 2.3].

Theorem 4.4.5 ([Ler10, Theorem 2.3.4], biconfinement estimate). *Let g_1, g_2 be two positive-definite quadratic forms on \mathbb{R}^{2n} with $g_j \leq g_j^\sigma$. Let $a_j, j = 1, 2$ be g_j -confined in U_j , a g_j -ball of radius ≤ 1 . Recall (4.2.20)*

$$r_p(a_1, a_2)(X) = (a_1 \# a_2)(X) - \sum_{0 \leq k < p} \frac{1}{j!} (i\pi[D_{X_1}, D_{X_2}])^j (a_1(X_1) a_2(X_2))|_{X_1=X_2=X}.$$

Then for all $k, l, p \in \mathbb{N}$, for all $X, T \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} |(r_p(a_1, a_2))^{(k)}(X)T^k| &\leq A_{k,N,p}(g_1 + g_2)(T)^{k/2} \Lambda_{1,2}^{-p} \\ &\times \left(1 + (g_1^\sigma \wedge g_2^\sigma)(X - U_1) + (g_1^\sigma \wedge g_2^\sigma)(X - U_2) \right)^{-N/2} \end{aligned} \quad (4.4.6)$$

with $A_{k,N,p} = C(k, N, p, n) \|a_1\|_{g_1, U_1}^{(l)} \|a_2\|_{g_2, U_2}^{(l)}$, $l = 2n + 1 + k + p + N$ and

$$\Lambda_{1,2} = \inf_{T \in \mathbb{R}^{2n}, T \neq 0} \left(\frac{g_1^\sigma(T)}{g_2^\sigma(T)} \right)^{1/2} = \inf_{T \in \mathbb{R}^{2n}, T \neq 0} \left(\frac{g_2^\sigma(T)}{g_1^\sigma(T)} \right)^{1/2}. \quad (4.4.7)$$

Now we use Theorem 4.4.5 to prove Lemma 4.4.4.

Proof of Lemma 4.4.4. By the asymptotic formula (4.2.24), we have

$$\psi_Y \# a_Y = a_Y + \underbrace{\frac{1}{4i\pi} \{\psi_Y, a_Y\}}_{=0} + r_2(\psi_Y, a_Y),$$

since $\psi_Y = 1$ on the support of a_Y . The symbol ψ_Y is g_Y -confined in $U_{Y,2r}$ and a_Y is g_Y -confined in $U_{Y,r}$, and moreover, we have

$$\forall l \in \mathbb{N}, \quad \|\psi_Y\|_{g_Y, U_{Y,2r}}^{(l)} = \|\psi_Y\|_{S(1, g_Y)}^{(l)}, \quad \|a_Y\|_{g_Y, U_{Y,r}}^{(l)} = \lambda_g(Y)^2 \|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l)}.$$

Applying (4.4.6) to $r_2(\psi_Y, a_Y)$, we have for all $k, N \in \mathbb{N}$, there exist $C(k, N, n) > 0$, $l(k, N, n) \in \mathbb{N}$ such that for all $X, T \in \mathbb{R}^{2n}$,

$$\begin{aligned} &|(r_2(\psi_Y, a_Y))^{(k)}(X)T^k| \\ &\leq C(k, N, n) \|\psi_Y\|_{g_Y, U_{Y,2r}}^{(l)} \|a_Y\|_{g_Y, U_{Y,r}}^{(l)} g_Y(T)^{k/2} \Lambda_{1,2}^{-2} (1 + g_Y^\sigma(X - U_{Y,2r}))^{-N} \\ &\leq C(k, N, n) \|\psi_Y\|_{S(1, g_Y)}^{(l)} \|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l)} g_Y(T)^{k/2} (1 + g_Y^\sigma(X - U_{Y,2r}))^{-N}, \end{aligned} \quad (4.4.8)$$

noticing here $\Lambda_{1,2}$ defined in (4.4.7) is equal to $\lambda_g(Y)$. An analogous estimate as (4.4.8) holds for $r_2(a_Y, \psi_Y)$. In our case, we write r_Y , which is defined in (4.4.3),

$$\begin{aligned} r_Y &= (\psi_Y \# a_Y - a_Y) \# \psi_Y + (a_Y \# \psi_Y - a_Y) \\ &= r_2(\psi_Y, a_Y) \# \psi_Y + r_2(a_Y, \psi_Y). \end{aligned}$$

Then the estimate (4.4.4) follows from (4.4.8) and (4.3.5). Furthermore, for any $k, N \in \mathbb{N}$, there exist $C = C(k, N, n, C_0) > 0$, $l = l(k, N, n, C_0) \in \mathbb{N}$ such that

$$\|\bar{r}_Y \# r_Z\|_{S(1, g_Y + g_Z)}^{(k)} \leq C \|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l)} \|a_Z\|_{S(\lambda_g(Z)^2, g_Z)}^{(l)} \delta_{2r}(Y, Z)^{-N}.$$

Thus we can apply Cotlar's lemma and get the estimate (4.4.5). \square

Lemma 4.4.6 (Estimate for ψ_Y). *For all $k, N \in \mathbb{N}$, there exist $C = C(k, N, C_0) > 0$, $l = l(k, N, C_0) \in \mathbb{N}$ such that for all $X \in \mathbb{R}^{2n}$, $T \in \mathbb{R}^{2n}$ with $g_Y(T) \leq 1$,*

$$|(\psi_Y \# \psi_Y)^{(k)}(X)T^k| \leq C(\|\psi_Y\|_{S(1, g_Y)}^{(l)})^2 (1 + g_Y^\sigma(X - U_{Y, 2r}))^{-N}. \quad (4.4.9)$$

Moreover, there exists $C_2 = C_2(n, C_0, N_0) > 0$ such that

$$\left\| \int \psi_Y^w \psi_Y^w |g_Y|^{1/2} dY \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_2. \quad (4.4.10)$$

Proof. The inequality (4.4.9) follows immediately from (4.3.5). And it follows from (4.3.5), (4.4.2) and (4.4.9) that for all $k, N \in \mathbb{N}$,

$$\|(\psi_Y \# \psi_Y) \# (\psi_Z \# \psi_Z)\|_{S(1, g_Y + g_Z)}^{(k)} \leq C \delta_{2r}(Y, Z)^{-N},$$

for some $C = C(k, N, n, C_0) > 0$. Then by choosing $N = 2N_1$ and using Cotlar's lemma, we get the estimate (4.4.10). \square

End of the proof of Theorem 4.4.1. The symbol a_Y is non-negative and uniformly in $S(\lambda_g(Y)^2, g_Y)$, so that we can apply the Fefferman-Phong inequality (Theorem 4.4.3) for the constant metric g_Y to get

$$a_Y^w + C(n) \|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l(n))} \geq 0.$$

By Proposition 4.2.5 and Lemma 4.2.8, we have

$$\|a_Y\|_{S(\lambda_g(Y)^2, g_Y)}^{(l(n))} \leq C(n, C_0, N_0) \|a\|_{S(\lambda_g^2, g)}^{(l(n))},$$

so that

$$a_Y^w + C_3 \|a\|_{S(\lambda_g^2, g)}^{(l(n))} \geq 0. \quad (4.4.11)$$

where $C_3 = C_3(n, C_0, N_0) > 0$, $l(n) \in \mathbb{N}$ are constants. Combining (4.4.3), (4.4.5), (4.4.10) and (4.4.11), we obtain

$$\begin{aligned} a^w &= \int_{\mathbb{R}^{2n}} a_Y^w |g_Y|^{1/2} dY \\ &= \int_{\mathbb{R}^{2n}} \psi_Y^w a_Y^w \psi_Y^w |g_Y|^{1/2} dY - \int_{\mathbb{R}^{2n}} r_Y^w |g_Y|^{1/2} dY \\ &\geq -C_3 \|a\|_{S(\lambda_g^2, g)}^{(l(n))} \int_{\mathbb{R}^{2n}} \psi_Y^w \psi_Y^w |g_Y|^{1/2} dY - C_1 \|a\|_{S(\lambda_g^2, g)}^{(l_1)} \\ &\geq -C \|a\|_{S(\lambda_g^2, g)}^{(l)}, \end{aligned}$$

for some $C = C(n, C_0, N_0) > 0$ and $l = l(n, C_0, N_0) \in \mathbb{N}$. The proof of Theorem 4.4.1 is complete. \square

4.5 Letter of acceptance

From: Mikael Lindström <mikael.lindstrom@oulu.fi>
Subject: Math.Scand.
Date: Thu, September 22, 2011 4:47 pm
To: wendeng@math.jussieu.fr

Dear Dr Wen Deng,

The editors of Mathematica Scandinavica are pleased to inform you that your manuscript

"STRUCTURE CONSTANTS OF THE WEYL CALCULUS "

has been accepted for publication. In case you have not signed a submission letter, the attached file "submit-erkl.pdf" must be signed and returned to Arhus. Moreover, please send a copy of the LaTeX file of your manuscript to mscand@imf.au.dk. The file should contain all your macros and (non-standard) input files. Figures should be submitted separately as encapsulated postscript (.eps) files.

You can help processing your file by adapting them to our style. The instructions needed and the style file mathscan.cls can be downloaded from

<http://home.imf.au.dk/mscand/mathscan>

Your help is much appreciated.

Sincerely yours

Mikael Lindström
Editor

Attachments:

submit-erkl.pdf	
Size:	15 k
Type:	application/pdf

Annexe A

Un opérateur normal

On s'intéresse à un opérateur différentiel bidimensionnel

$$L_\alpha = -\Delta + |x|^2 + \alpha \partial_\theta, \quad x \in \mathbb{R}^2, \quad \alpha > 0, \quad (\text{A.0.1})$$

où $\partial_\theta = x_1 \partial_2 - x_2 \partial_1$. Muni du domaine

$$D(L_\alpha) = \{u \in H^2(\mathbb{R}^2); |x|^2 u \in L^2(\mathbb{R}^2)\},$$

L_α est un opérateur fermé sur $L^2(\mathbb{R}^2)$. Notons que l'opérateur L_α n'est pas auto-adjoint, mais il est normal puisque sa partie réelle commute avec la partie imaginaire.

Nous démontrons la proposition suivante dans cette annexe.

Proposition A.0.1. *Le spectre de L_α est constitué d'une suite des valeurs propres*

$$\{4m + 4 \pm (2l + 1)\alpha i, 4m + 2 \pm 2l\alpha i; \quad m \in \mathbb{N}, 0 \leq l \leq m\}.$$

Notons l'oscillateur harmonique bidimensionnel

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^2. \quad (\text{A.0.2})$$

On va calculer le spectre de L_α en utilisant deux méthodes différentes. La première méthode consiste à transformer L_α en une famille de matrices à l'aide des fonctions d'Hermite. Dans la deuxième partie de cette annexe, on calcule le spectre de l'oscillateur harmonique bidimensionnel H en coordonnées polaires, en résolvant une équation différentielle ordinaire en dimension 1. Ceci nous permet de diagonaliser l'opérateur L_α , et en particulier, on trouve explicitement les fonctions propres.

A.1 Fonctions d'Hermite

A.1.1 Opérateurs de création et d'annihilation

En notant $D_j = i^{-1} \partial_{x_j}$, on définit

$$\begin{aligned} C_1 &= D_1 + ix_1, & C_2 &= D_2 + ix_2, & \text{opérateurs de création,} \\ A_1 &= D_1 - ix_1, & A_2 &= D_2 - ix_2, & \text{opérateurs d'annihilation.} \end{aligned} \quad (\text{A.1.1})$$

Ces opérateurs satisfont les relations de commutation suivantes

$$\begin{aligned} [A_1, C_1] &= [A_2, C_2] = 2, \\ [C_1, C_2] &= [A_1, A_2] = [C_1, A_2] = [C_2, A_1] = 0. \end{aligned}$$

On peut écrire

$$H = |D|^2 + |x|^2 = C_1 A_1 + C_2 A_2 + 2,$$

ce qui implique

$$[H, C_j] = 2C_j, \quad [H, A_j] = -2A_j, \quad j = 1, 2. \quad (\text{A.1.2})$$

Les fonctions d'Hermite sont définies de la manière suivante, pour $x \in \mathbb{R}^2$,

$$\begin{aligned} \psi_{0,0}(x) &= e^{-|x|^2/2} && \text{l'état fondamental,} \\ \psi_{p,q}(x) &= C_1^p C_2^q \psi_{0,0}(x), && \text{pour } p, q \in \mathbb{N}, \end{aligned} \quad (\text{A.1.3})$$

où C_j sont les opérateurs de création (A.1.1). La fonction $\psi_{p,q}$ est une fonction propre de l'oscillateur harmonique H correspondante à la valeur propre $2(p+q+1)$, c'est-à-dire

$$H\psi_{p,q} = 2(p+q+1)\psi_{p,q}, \quad \text{pour } p, q \in \mathbb{N}.$$

Il est bien connu que l'ensemble des fonctions d'Hermite $\{\psi_{p,q}; p, q \in \mathbb{N}\}$ est une base orthogonale (non normalisée) de $L^2(\mathbb{R}^2)$. Enfin, par récurrence on obtient

$$\begin{aligned} C_1 \psi_{p,q} &= \psi_{p+1,q}, & C_2 \psi_{p,q} &= \psi_{p,q+1}, \\ A_1 \psi_{p,q} &= 2p \psi_{p-1,q}, & A_2 \psi_{p,q} &= 2q \psi_{p,q-1}, \end{aligned} \quad (\text{A.1.4})$$

avec convention évidente $\psi_{-1,p} = \psi_{p,-1} = 0$.

A.1.2 Représentation matricielle

L'opérateur ∂_θ peut être exprimé en termes des opérateurs de création et d'annihilation C_j, A_j (A.1.1)

$$\partial_\theta = ix_1 D_2 - ix_2 D_1 = \frac{1}{2}(C_1 A_2 - C_2 A_1).$$

On déduit de (A.1.4) que pour tout $p, q \in \mathbb{N}$,

$$\partial_\theta \psi_{p,q} = \frac{1}{2}(C_1 A_2 - C_2 A_1) \psi_{p,q} = q \psi_{p+1,q-1} - p \psi_{p-1,q+1}. \quad (\text{A.1.5})$$

Pour tout $n \in \mathbb{N}$, notons E_{2n+2} le sous-espace $(n+1)$ -dimensionnel de $L^2(\mathbb{R}^2)$ engendré par les fonctions d'Hermite d'ordre n

$$\{\psi_{p,q}; p, q \in \mathbb{N}, p+q=n\},$$

qui est l'espace propre de H associé à la valeur propre $2n+2$. La relation (A.1.5) implique que ∂_θ est invariant sur E_{2n+2} pour tout $n \in \mathbb{N}$. En plus, on en déduit une représentation matricielle de ∂_θ sous la base $\{\psi_{0,n}, \psi_{1,n-1}, \dots, \psi_{n,0}\}$ de E_{2n+2}

$$A_n = \begin{pmatrix} 0 & n & & & & & \\ -1 & 0 & n-1 & & & & \\ & -2 & 0 & n-2 & & & \\ & & -3 & 0 & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & 2 & \\ & & & & & -(n-1) & 0 & 1 \end{pmatrix} \in M_{n+1}(\mathbb{R}). \quad (\text{A.1.6})$$

Si l'on normalise la base $\{\psi_{0,n}, \psi_{1,n-1}, \dots, \psi_{n,0}\}$ de E_{2n+2} , alors on arrive à une matrice anti-symétrique

$$B_n = \begin{pmatrix} 0 & \sqrt{n} & 0 & 0 & \cdots & \cdots & 0 \\ -\sqrt{n} & 0 & \sqrt{2(n-1)} & 0 & \cdots & \cdots & 0 \\ 0 & -\sqrt{2(n-1)} & 0 & \sqrt{3(n-2)} & \cdots & 0 & \vdots \\ 0 & 0 & -\sqrt{3(n-2)} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & \sqrt{(n-1)2} & 0 \\ 0 & 0 & 0 & \cdots & -\sqrt{(n-1)2} & 0 & \sqrt{n} \\ 0 & 0 & 0 & \cdots & 0 & -\sqrt{n} & 0 \end{pmatrix}, \quad (\text{A.1.7})$$

qui est semblable à la matrice A_n (A.1.6). En faisant un calcul explicite, on trouve les valeurs propres de A_n et B_n

$$\begin{aligned} & \pm (2l+1)i, \quad 0 \leq l \leq m, \quad \text{si } n = 2m+1 \text{ est impaire,} \\ & \pm 2li, \quad 0 \leq l \leq m, \quad \text{si } n = 2m \text{ est paire.} \end{aligned} \quad (\text{A.1.8})$$

Revenons à l'opérateur L_α (A.0.1). Pour tout $n \in \mathbb{N}$, E_{2n+2} est donc un sous-espace invariant pour L_α . Le spectre de la restriction de L_α au sous-espace E_{2n+2} est

$$\begin{aligned} & 4m+4 \pm (2l+1)\alpha i, \quad 0 \leq l \leq m, \quad \text{si } n = 2m+1 \text{ est impaire,} \\ & 4m+2 \pm 2l\alpha i, \quad 0 \leq l \leq m, \quad \text{si } n = 2m \text{ est paire.} \end{aligned} \quad (\text{A.1.9})$$

Ceci termine la démonstration de la Proposition A.0.1.

A.2 Coordonnées polaires

On peut trouver explicitement les fonctions propres de l'opérateur L_α , en utilisant les coordonnées polaires (r, θ) dans le plan. On écrit l'oscillateur harmonique comme

$$H = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2 + r^2, \quad r > 0, \quad \theta \in [0, 2\pi].$$

En développant la variable angulaire θ en série de Fourier, on obtient une famille d'opérateurs indexés par $k \in \mathbb{Z}$

$$H_k = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2} + r^2, \quad k \in \mathbb{Z}, \quad (\text{A.2.1})$$

qui agissent sur l'espace de Hilbert $L^2(\mathbb{R}_+; rdr)$ avec domaine maximal

$$D(H_k) = \{u \in L^2(\mathbb{R}_+; rdr); H_k u \in L^2(\mathbb{R}_+; rdr)\}.$$

A.2.1 Spectre de H_k

Dans cette section, on étudie le problème des valeurs propres de H_k

$$H_k u = -\partial_r^2 u - \frac{1}{r}\partial_r u + \frac{k^2}{r^2} u + r^2 u = \lambda u. \quad (\text{A.2.2})$$

On cherche des solutions de (A.2.2) ayant la forme $u(r) = P(r)e^{-r^2/2}$, où P est un polynôme. Alors $(\lambda, P(r))$ doit vérifier une équation différentielle ordinaire d'ordre 2

$$-P'' + \left(2r - \frac{1}{r}\right)P' + \left(\frac{k^2}{r^2} + 2 - \lambda\right)P = 0. \quad (\text{A.2.3})$$

Supposons que P est un polynôme de degré n de la forme suivante

$$P(r) = \sum_{j=0}^n a_j r^j, \quad a_n = 1.$$

L'équation (A.2.3) devient

$$\begin{aligned} & -P'' + \left(2r - \frac{1}{r}\right)P' + \left(\frac{k^2}{r^2} + 2 - \lambda\right)P \\ &= (2n+2-\lambda)a_n r^n + (2n-\lambda)a_{n-1}r^{n-1} \\ &+ \sum_{j=0}^{n-2} \left((k^2 - (j+2)^2)a_{j+2} + (2j+2-\lambda)a_j\right)r^j - \frac{(1-k^2)a_1}{r} + \frac{k^2 a_0}{r^2} = 0. \end{aligned} \quad (\text{A.2.4})$$

Par conséquent, les coefficients de P vérifient le système suivant

$$\begin{cases} (2n+2-\lambda)a_n = 0, \\ (2n-\lambda)a_{n-1} = 0, \\ (k^2 - (j+2)^2)a_{j+2} + (2j+2-\lambda)a_j = 0, \quad \forall 0 \leq j \leq n-2, \\ (1-k^2)a_1 = 0, \\ k^2 a_0 = 0. \end{cases} \quad (\text{A.2.5})$$

En résolvant (A.2.5), on obtient les solutions de (A.2.3) :

- si $|k| = 2l$, $l \geq 0$, alors pour tout $m \geq l$, la couple $(4m+2, P_{2m}^{2l})$ est une solution de (A.2.3), où

$$P_{2m}^{2l}(r) = \sum_{j=l}^m \frac{(-1)^{m-j}}{(m-j)!} \frac{(m+l)!(m-l)!}{(j+l)!(j-l)!} r^{2j}. \quad (\text{A.2.6})$$

- si $|k| = 2l+1$, $l \geq 0$, alors pour tout $m \geq l$, la couple $(4m+4, P_{2m+1}^{2l+1})$ est une solution de (A.2.3), où

$$P_{2m+1}^{2l+1}(r) = \sum_{j=l}^m \frac{(-1)^{m-j}}{(m-j)!} \frac{(m+l+1)!(m-l)!}{(j+l+1)!(j-1)!} r^{2j+1}. \quad (\text{A.2.7})$$

On obtient donc les valeurs propres de H_k (A.2.1), ainsi que ses fonctions propres :

- si $|k| = 2l$, $l \geq 0$, le spectre de $H_{\pm 2l}$ est l'ensemble $\{4m+2; m \geq l\}$. La fonction propre associée à $4m+2$ est $P_{2m}^{2l}(r)e^{-r^2/2}$, avec P_{2m}^{2l} donné par (A.2.6) ;
- si $|k| = 2l+1$, $l \geq 0$, le spectre de $H_{\pm(2l+1)}$ est l'ensemble $\{4m+4; m \geq l\}$. La fonction propre associée à $4m+4$ est $P_{2m+1}^{2l+1}(r)e^{-r^2/2}$ avec P_{2m+1}^{2l+1} donné par (A.2.7).

A.2.2 Spectre de L_α

Revenons sur le spectre de L_α (A.0.1). Notons comme précédent E_λ l'espace propre associé à la valeur propre λ de l'oscillateur harmonique bidimensionnel H . Rappelons que dans la Section A.1, pour tout $n \in \mathbb{N}$, on a obtenu une base orthogonale (non normalisée) de E_{2n+2}

$$\{\psi_{p,q}, p + q = n\},$$

où les $\psi_{p,q}$ sont des fonctions d'Hermite (A.1.3). Ici, on obtient une nouvelle base de E_λ en coordonnées polaires, sous laquelle les deux opérateurs H et ∂_θ sont diagonalisés simultanément.

Cas 1 : $\lambda = 4m + 2$ avec $m \geq 0$.

L'espace E_{4m+2} de dimension $(2m + 1)$ est engendré par les fonctions suivantes

$$\{P_{2m}^{2l}(r)e^{-r^2/2}e^{i2l\theta}; -m \leq l \leq m\}, \quad (\text{A.2.8})$$

où $P_{2m}^{2l} = P_{2m}^{|2l|}$ sont donnés par (A.2.6). Il est facile de voir que les fonctions dans (A.2.8) sont deux à deux orthogonales dans $L^2(\mathbb{R}^2)$, et elles font une base de E_{4m+2} . Sous cette base, l'oscillateur harmonique H est égal à $(4m+2)\text{Id}$ et ∂_θ est égal à la matrice diagonalisée

$$\text{diag}\left\{-2mi, -2(m-1)i, \dots, 0, \dots, 2(m-1)i, 2mi\right\}.$$

En particulier, on obtient les valeurs propres et les fonctions propres associées de la restriction à E_{4m+2} de l'opérateur L_α (A.0.1)

$$(4m + 2 + 2l\alpha i, P_{2m}^{2l}(r)e^{-r^2/2}e^{i2l\theta}), \quad -m \leq l \leq m. \quad (\text{A.2.9})$$

Cas 2 : $\lambda = 4m + 4$ avec $m \geq 0$

L'espace E_{4m+4} de dimension $(2m + 2)$ est engendré par les fonctions suivantes

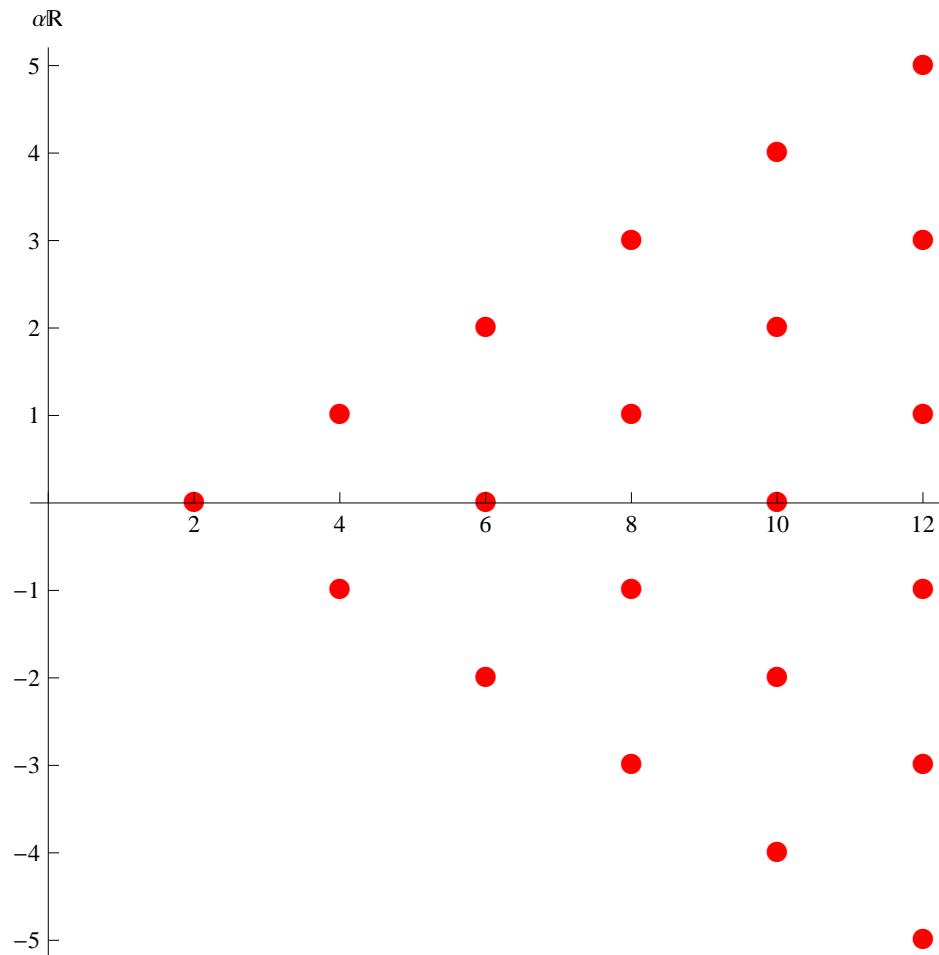
$$\{P_{2m+1}^{2l+1}(r)e^{-r^2/2}e^{i(2l+1)\theta}, -m - 1 \leq l \leq m\}, \quad (\text{A.2.10})$$

où $P_{2m+1}^{2l+1} = P_{2m+1}^{|2l+1|}$ sont donnés par (A.2.7). Comme le cas précédent, les fonctions (A.2.10) forment une base orthogonale de E_{4m+4} . Sous cette base, l'oscillateur harmonique H est égal à $(4m + 4)\text{Id}$ et ∂_θ est égal à la matrice diagonalisée

$$\text{diag}\left\{-(2m+1)i, -(2m-1)i, \dots, -i, i, \dots, (2m+1)i\right\}.$$

En particulier, on obtient les valeurs propres et les fonctions propres associées de la restriction à E_{4m+4} de L_α (A.0.1)

$$(4m + 4 + (2l + 1)\alpha i, P_{2m+1}^{2l+1}(r)e^{-r^2/2}e^{i(2l+1)\theta}), \quad -m - 1 \leq l \leq m. \quad (\text{A.2.11})$$

FIGURE A.1 – LE SPECTRE DE L’OPÉRATEUR L_α .

Appendix B

Miscellaneous

B.1 Alternative proofs

In this section, we provide alternative proofs of Lemma 2.4.22 and Lemma 2.5.9.

Another proof of Lemma 2.4.22. Recall that a is defined by (2.4.49). We first prove that a is bounded from below by $C\lambda_g^2$.

$$\begin{aligned} \text{For } |\tau| \leq \beta_k^{1/3}, \quad a(t, \tau) &\geq \min(c_8, c_9)\beta_k^{2/3} \geq \frac{1}{2}\min(c_8, c_9)(\beta_k^{2/3} + \tau^2), \\ \text{for } |\tau| > \beta_k^{1/3}, \quad a(t, \tau) &\geq 4M\tau^2 \geq 2M(\beta_k^{2/3} + \tau^2). \end{aligned}$$

On the other hand $a \in S(\lambda_g^2, g)$, we get

$$\forall t \in \mathbb{R}, \forall \tau \in \mathbb{R}, \quad C\lambda_g^2 \leq a(t, \tau) \leq C'\lambda_g^2.$$

We denote

$$\tilde{a} := a - \frac{1}{2}C\lambda_g^2,$$

then $\frac{1}{2}C\lambda_g^2 \leq \tilde{a} \in S(\lambda_g^2, g)$ with semi-norms bounded from above by constants independent of k . We apply the Faà di Bruno formula

$$\frac{(g \circ f)^{(n)}}{n!} = \sum_{1 \leq m \leq n} \frac{g^{(m)} \circ f}{m!} \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{f^{(j_l)}}{j_l!}$$

with $f = \tilde{a}$ and $g(x) = x^{1/2}$,

$$\begin{aligned} |(\tilde{a}^{1/2})^{(n)}(X)T^n| &= \left| \sum_{1 \leq m \leq n} c_m \tilde{a}^{\frac{1}{2}-m}(X) \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} \tilde{a}^{(j_l)}(X) T^{j_l} \right| \\ &\leq \sum_{1 \leq m \leq n} c_m |\tilde{a}^{\frac{1}{2}-m}(X)| \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} \lambda_g^2(X) g_X(T)^{j_l/2} \\ &\leq \sum_{1 \leq m \leq n} c'_m (\lambda_g^2)^{\frac{1}{2}-m}(X) \cdot (\lambda_g^2)^m(X) g_X(T)^{n/2} \\ &\leq c''_n \lambda_g(X) g_X(T)^{n/2}, \end{aligned}$$

which implies $\tilde{a}^{1/2} \in S(\lambda_g, g)$ (and each semi-norm can be bounded by a constant independent of k). The composition formula tells us that

$$\tilde{a}^{1/2} \# \tilde{a}^{1/2} = \tilde{a} + S(1, g).$$

Then $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2} \in S(1, g)$ and

$$\begin{aligned} \langle \tilde{a}^w u, u \rangle_{L^2(\mathbb{R}; dt)} &= \langle (\tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} + \langle (\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &\geq \|(\tilde{a}^{1/2})^w u\|_{L^2(\mathbb{R}; dt)}^2 - C'' \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\geq -C'' \|u\|_{L^2(\mathbb{R}; dt)}^2, \end{aligned}$$

where $C'' > 0$ depends on a semi-norm of $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2}$ in $S(1, g)$, so that we can suppose C'' independent of k , see the footnote in page 50. We get the estimate for a^w :

$$a^w \geq \frac{C}{2} (\lambda_g^2)^w - C'', \quad \text{on } L^2(\mathbb{R}; dt).$$

Remark that λ_g is a function depending only on the variable τ , so $(\lambda_g^2)^w = \lambda_g^2(D_t)$ is a Fourier multiplier. We deduce from $\lambda_g^2 \geq \beta_k^{2/3}$ that $(\lambda_g^2)^w \geq \beta_k^{2/3}$ and

$$a^w \geq \frac{C}{2} \beta_k^{2/3} - C'', \quad \text{on } L^2(\mathbb{R}; dt).$$

Lemma 2.4.22 is proved. \square

Another proof of Lemma 2.5.9. Recall that a is defined by (2.5.17). We first prove that a is bounded below by $Cr_k^{-2}\lambda_g^2$.

$$\begin{aligned} \text{For } |\tau| \leq \beta_k^{1/3} r_k^{1/3}, \quad a(t, \tau) &\geq \min(c_{12}, c_{13}) \beta_k^{2/3} r_k^{2/3} \geq \frac{1}{2} \min(c_{12}, c_{13}) (\beta_k^{2/3} r_k^{2/3} + \tau^2), \\ \text{for } |\tau| > \beta_k^{1/3} r_k^{1/3}, \quad a(t, \tau) &\geq 4M\tau^2 \geq 2M(\beta_k^{2/3} r_k^{2/3} + \tau^2). \end{aligned}$$

On the other hand $a \in S(r_k^{-2}\lambda_g^2, g)$, we get

$$\forall t \in \mathbb{R}, \forall \tau \in \mathbb{R}, \quad Cr_k^{-2}\lambda_g^2 \leq a(t, \tau) \leq C'r_k^{-2}\lambda_g^2.$$

We denote

$$\tilde{a} := a - \frac{1}{2} Cr_k^{-2}\lambda_g^2,$$

then $\frac{1}{2} Cr_k^{-2}\lambda_g^2 \leq \tilde{a} \in S(r_k^{-2}\lambda_g^2, g)$ with semi-norms bounded from above by constants independent of k . We apply the Faà di Bruno formula

$$\frac{(g \circ f)^{(n)}}{n!} = \sum_{1 \leq m \leq n} \frac{g^{(m)} \circ f}{m!} \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{f^{(j_l)}}{j_l!}$$

with $f = \tilde{a}$ and $g(x) = x^{1/2}$,

$$\begin{aligned} |(\tilde{a}^{1/2})^{(n)}(X)T^n| &= \left| \sum_{1 \leq m \leq n} c_m \tilde{a}^{\frac{1}{2}-m}(X) \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} \tilde{a}^{(j_l)}(X) T^{j_l} \right| \\ &\leq \sum_{1 \leq m \leq n} c_m |\tilde{a}^{\frac{1}{2}-m}(X)| \prod_{\substack{j_1 + \dots + j_m = n, \\ j_l \geq 1}} \frac{1}{j_l!} r_k^{-2} \lambda_g^2(X) g_X(T)^{j_l/2} \\ &\leq \sum_{1 \leq m \leq n} c'_m (r_k^{-2} \lambda_g^2)^{\frac{1}{2}-m}(X) \cdot (r_k^{-2} \lambda_g^2)^m(X) g_X(T)^{n/2} \\ &\leq c''_n r_k^{-1} \lambda_g(X) g_X(T)^{n/2}, \end{aligned}$$

which implies $\tilde{a}^{1/2} \in S(r_k^{-1}\lambda_g, g)$ (and each semi-norm can be bounded by a constant independent of k). The composition formula tells us that

$$\tilde{a}^{1/2} \# \tilde{a}^{1/2} = \tilde{a} + S(r_k^{-2}, g).$$

Then $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2} \in S(r_k^{-2}, g)$ and

$$\begin{aligned} \langle \tilde{a}^w u, u \rangle_{L^2(\mathbb{R}; dt)} &= \langle (\tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} + \langle (\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2})^w u, u \rangle_{L^2(\mathbb{R}; dt)} \\ &\geq \|(\tilde{a}^{1/2})^w u\|_{L^2(\mathbb{R}; dt)}^2 - C'' r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2 \\ &\geq -C'' r_k^{-2} \|u\|_{L^2(\mathbb{R}; dt)}^2, \end{aligned}$$

where $C'' > 0$ depends on a semi-norm of $\tilde{a} - \tilde{a}^{1/2} \# \tilde{a}^{1/2}$ in $S(r_k^{-2}, g)$, so that we can suppose C'' independent of k , see the footnote in page 50. We get the estimate

$$a^w \geq \frac{C}{2} (r_k^{-2} \lambda_g^2)^w - C'' r_k^{-2}, \quad \text{on } L^2(\mathbb{R}; dt).$$

Remark that λ_g is a function depending only on the variable τ , so $(\lambda_g^2)^w = \lambda_g^2(D_t)$ is a Fourier multiplier. We deduce from $\lambda_g^2 \geq \beta_k^{2/3} r_k^{8/3}$ that $(\lambda_g^2)^w \geq \beta_k^{2/3} r_k^{8/3}$ and

$$a^w \geq \frac{C}{2} \beta_k^{2/3} r_k^{2/3} - C'' r_k^{-2}, \quad \text{on } L^2(\mathbb{R}; dt),$$

which completes the proof. \square

B.2 A Fréchet space

We give a proof for that the symbol class $S(m, g)$ defined in Definition 4.2.3 is a Fréchet space.

Proposition B.2.1. *Let g be a slowly varying metric on \mathbb{R}^{2n} and m be a g -slowly varying weight (see Definition 4.2.1, 4.2.2). The space $S(m, g)$ equipped with the countable family of semi-norms $(\|\cdot\|_{S(m,g)}^{(l)})_{l \in \mathbb{N}}$ (see Definition 4.2.3) is a Fréchet space.*

Proof. It suffices to show that $S(m, g)$ is complete. First remark that the semi-norm defined in (4.2.5) has an equivalent form

$$\|a\|_{S(m,g)}^{(l)} = \max_{0 \leq k \leq l} \sup_{X, T \in \mathbb{R}^{2n}, T \neq 0} |a^{(k)}(X)T^k| g_X(T)^{-k/2} m(X)^{-1}. \quad (\text{B.2.1})$$

Suppose that $\{a_j\}_{j \geq 1}$ is a Cauchy sequence with respect to each semi-norm $\|\cdot\|_{S(m,g)}^{(l)}$, then for any $\epsilon > 0$, for any $k \geq 0$, there exists $N = N(\epsilon, k) \geq 1$ such that for any $j_1, j_2 \geq N$,

$$\forall X, T \in \mathbb{R}^{2n}, T \neq 0, \quad |(a_{j_1}^{(k)} - a_{j_2}^{(k)})(X)T^k| g_X(T)^{-k/2} m(X)^{-1} \leq \epsilon. \quad (\text{B.2.2})$$

For $X_0 \in \mathbb{R}^{2n}$, we define

$$U_{X_0} = \{X \in \mathbb{R}^{2n}; g_{X_0}(X - X_0) \leq \min(C_0^{-1}, \mu_m^{-1})\}, \quad (\text{B.2.3})$$

where C_0 is given in (4.2.3) and μ_m given in (4.2.4). Then by using the slowness of the metric g and the weight m , we have for all $X \in U_{X_0}$,

$$\left(\frac{g_X}{g_{X_0}}\right)^{\pm 1} \leq C_0, \quad \left(\frac{m(X)}{m(X_0)}\right)^{\pm 1} \leq \mu_m. \quad (\text{B.2.4})$$

We deduce from (B.2.2) that, for $\epsilon > 0$, $k \geq 0$, $j_1, j_2 \geq N$,

$$\forall X \in U_{X_0}, \forall T \in \mathbb{R}^{2n}, \quad |(a_{j_1}^{(k)} - a_{j_2}^{(k)})(X)T^k|g_{X_0}(T)^{-k/2} \leq C_0^{k/2} \mu_m m(X_0)\epsilon, \quad (\text{B.2.5})$$

which implies that for any $k \geq 0$, $\{a_j^{(k)}\}_{j \geq 1}$ is a Cauchy sequence in $L^\infty(U_{X_0}; S_k(X_0))$, where $S_k(X_0)$ denotes the set of symmetric k -multilinear forms on the normed space $(\mathbb{R}^{2n}, \|\cdot\|_{X_0})$, with $\|\cdot\|_{X_0}$ denoting the norm associated to the positive-definite quadratic form g_{X_0} . Hence there exists $b_k \in L^\infty(U_{X_0}; S_k(X_0))$ such that

$$a_j^{(k)} \rightarrow b_k \text{ in } L^\infty(U_{X_0}; S_k(X_0)), \quad \text{as } j \rightarrow +\infty.$$

Since the a_j 's are smooth, we have $b_k \in C(U_{X_0}; S_k(X_0))$. Moreover, b_k is well-defined in the whole space \mathbb{R}^{2n} .

Let us now prove that for each X_0 , b_0 is $C^1(U_{X_0})$ and $b'_0 = b_1$ in U_{X_0} . Indeed, we use the Taylor formula for a_j , $X \in U_{X_0}$,

$$a_j(X) = a_j(X_0) + \int_0^1 a'_j(X_0 + \theta(X - X_0))(X - X_0)d\theta.$$

Since $a'_j \rightarrow b_1$ in $S_1(X_0)$ uniformly in U_{X_0} as $j \rightarrow +\infty$ and $X_0 + \theta(X - X_0) \in U_{X_0}$ for $0 \leq \theta \leq 1$, we get

$$b_0(X) = b_0(X_0) + \int_0^1 b_1(X_0 + \theta(X - X_0))(X - X_0)d\theta.$$

Thus b_0 is C^1 and $b'_0 = b_1$. By induction, we can show that b_0 is $C^\infty(\mathbb{R}^{2n})$ and $b_k = b_0^{(k)}$ for all $k \geq 0$.

It remains to show $a_j \rightarrow b_0$ in $S(m, g)$ as $j \rightarrow +\infty$. For $k \geq 0$, taking the limit $j_2 \rightarrow +\infty$ in (B.2.5), we get for $j \geq N(\epsilon, k)$,

$$\forall X \in U_{X_0}, \forall T \in \mathbb{R}^{2n}, \quad |(a_j^{(k)} - b_0^{(k)})(X)T^k|g_{X_0}(T)^{-k/2}m(X_0)^{-1} \leq C_0^{k/2} \mu_m \epsilon.$$

Using (B.2.4), we get

$$\forall X \in U_{X_0}, \forall T \in \mathbb{R}^{2n}, \quad |(a_j^{(k)} - b_0^{(k)})(X)T^k|g_X(T)^{-k/2}m(X)^{-1} \leq C_0^k \mu_m^2 \epsilon,$$

$$\implies \sup_{X, T \in \mathbb{R}^{2n}, T \neq 0} |(a_j^{(k)} - b_0^{(k)})(X)T^k|g_X(T)^{-k/2}m(X)^{-1} \leq C_0^k \mu_m^2 \epsilon,$$

completing the proof of the proposition. \square

Appendix C

Numerical computations for the value of k_0

We provide some numerical computations for Chapter 3 (i.e. article [Den11b]) and the purpose of this appendix is to give a bound for the constant k_0 in Theorem 3.2.2. This appendix are taken from [Den11a], all the computations are done by using MATHEMATICA. We also give some discussions that can be used to improve the value of k_0 at the end of this appendix.

For the non trivial cases where a change-of-sign takes place, we have discussed 4 cases in Section 3.3.3, 3.3.4: given $\epsilon_0, \epsilon_1 \in (0, 1)$,

$$\begin{cases} \text{Case 1: } e^{t_k} > \epsilon_0^{-1} & \text{Theorem 3.3.5,} \\ \text{Case 2: } e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}] & \text{Theorem 3.3.14,} \\ \text{Case 3: } e^{t_k} \in (\beta_k^{-1/4}, \epsilon_1) & \text{Theorem 3.3.17,} \\ \text{Case 4: } e^{t_k} \leq \beta_k^{-1/4} & \text{Lemma 3.3.22.} \end{cases}$$

For each of the first 3 cases, we need a condition $k \geq k_0$ and the value of k_0 could be expressed as a function of (ϵ_0, ϵ_1) . In this note, we shall compute with

$$\epsilon_0 \simeq 0.461558, \quad \epsilon_1 \simeq 0.426072 \tag{C.0.1}$$

and the result that we shall obtain is

$$k_0 = 84. \tag{C.0.2}$$

We shall keep all the notations that have been used in Chapter 3. In order to obtain a better bound on k_0 , we would like to make a few changes, including some improvements, described below.

The 4 cases

The 4 cases are slightly changed and we aim to get the following estimates: Suppose $\mu_0 \geq 1$ a large constant.

- Case 1: $e^{t_k} > \epsilon_0^{-1}$.

$$\begin{aligned} \exists C > 0, \quad k_0 \geq 1, \quad \alpha_0 \geq 8\pi, \quad \text{s.t. } \forall k \geq k_0, \quad \alpha \geq \alpha_0, \quad u \in C_0^\infty(\mathbb{R}), \\ \|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/3} \|e^t u\|. \end{aligned} \tag{C.0.3}$$

– Case 2: $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$.

$$\begin{aligned} \exists C > 0, \quad k_0 \geq 1, \quad \alpha_0 \geq 8\pi, \quad \text{s.t. } \forall k \geq k_0, \quad \alpha \geq \alpha_0, \quad u \in C_0^\infty(\mathbb{R}), \\ \|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{2/3} \|e^t u\|. \end{aligned} \quad (\text{C.0.4})$$

– Case 3: $e^{t_k} \in (\mu_0 \beta_k^{-1/4}, \epsilon_1)$.

$$\begin{aligned} \exists C > 0, \quad k_0 \geq 1, \quad \alpha_0 \geq 8\pi, \quad \text{s.t. } \forall k \geq k_0, \quad \alpha \geq \alpha_0, \quad u \in C_0^\infty(\mathbb{R}), \\ \|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|. \end{aligned} \quad (\text{C.0.5})$$

– Case 4: $e^{t_k} \leq \mu_0 \beta_k^{-1/4}$.

$$\begin{aligned} \exists C > 0, \quad \alpha_0 \geq 8\pi, \quad \text{s.t. } \forall k \geq 1, \quad \alpha \geq 8\pi, \quad u \in C_0^\infty(\mathbb{R}), \\ \|e^{-t} \mathcal{L}_k u\| \geq C \beta_k^{1/2} \|e^t u\|. \end{aligned} \quad (\text{C.0.6})$$

The constant μ_0 will be chosen in Case 3. It is clear that the estimates (C.0.3), (C.0.4), (C.0.5), (C.0.6) together with the estimates for easy cases given in Lemma 3.3.2, 3.3.3 can imply the estimate (3.2.3) thus Theorem 3.2.2.

The constant c_0

Recall that, in each of the first 3 cases, we have constructed a multiplier with the help of a partition of unity on \mathbb{R}_t given in (3.3.23), which allows us to localize near t_k and which depends on a small constant c_0 . The constant c_0 was chosen according to Proposition 3.4.7 and was common for the 3 cases for the purpose of simplification of notations.

Here for each of the first 3 cases, we shall choose a constant c_0 :

$$\text{Case 1: } c_0 = 0.04, \quad \text{Case 2: } c_0 = 0.04, \quad \text{Case 3: } c_0 = 0.08.$$

Some bounds

We can suppose that the functions χ_0, ψ given in (3.3.24), (3.3.25) satisfy the following bounds

$$\|\chi'_0\|_{L^\infty} \leq \frac{2}{c_0} + \varepsilon, \quad \|\psi'\|_{L^\infty} \leq \frac{1}{2} + \varepsilon, \quad (\text{C.0.7})$$

for any $\varepsilon > 0$ (fixed). Then the function e given in (3.3.26) verifies

$$\|e\|_{L^\infty} \leq \frac{1}{2} + \varepsilon, \quad (\text{C.0.8})$$

noting that

$$e(\theta) = -\frac{\psi(\theta) - \psi(0)}{\theta} = -\int_0^1 \psi'(t\theta) dt, \quad \text{since } \psi(0) = 0.$$

Notations

Recall the notations

$$\sigma(r) = \frac{4}{r^2} (1 - e^{-r^2/4}), \quad g(r) = e^{-r^2/8}, \quad r \geq 0.$$

C.1 Case 1

In this section we treat the case where $e^{t_k} > \epsilon_0^{-1} \simeq 2.16657$. We choose

$$c_0 \simeq 0.04. \quad (\text{C.1.1})$$

Estimate for A_1

Using the definition of σ

$$\sigma(r) = \frac{4}{r^2}(1 - e^{-r^2/4}), \quad \sigma'(r) = -\frac{8}{r^3}(1 - e^{-r^2/4} - \frac{r^2}{4}e^{-r^2/4}),$$

we have for all $r \geq 2$,

$$-8r^{-3} \leq \sigma'(r) \leq -8(1 - 2e^{-1})r^{-3}.$$

Using the above inequality and the Taylor's formula, for $e^t, e^{t_k} \geq 2$, we have

$$|\sigma(e^t) - \sigma(e^{t_k})| \leq \int_0^1 |\sigma'(e^{t+\theta(t_k-t)})e^{t+\theta(t_k-t)}| |t - t_k| d\theta \leq 8e^{-2t}e^{2|t-t_k|} |t - t_k|,$$

which implies

$$\begin{aligned} \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] &= e^{3t}\sigma'(e^t) + 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \\ &\leq -8(1 - 2e^{-1}) + 16e^{2|t-t_k|} |t - t_k|. \end{aligned}$$

Then we have,

$$\begin{aligned} \forall e^{t_k} > \epsilon_0^{-1} = 2.16657, \quad \forall |t - t_k| \leq 2c_0, \quad (\text{which implies } e^t \geq 2) \\ \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] &\leq -C_1, \quad \text{with } C_1 \simeq 0.611835. \end{aligned} \quad (\text{C.1.2})$$

Then the estimate (3.3.38) holds for A_1 with C_1 given in (C.1.2).

Estimate for A_2

We have to compute the precise coefficients in the estimate for the term A_2 , especially for terms of size $\beta_k^{2/3}k^{-2}$. With the notations there, the precised inequalities (3.3.42), (3.3.43) are

$$\begin{aligned} |A_{21}| &\leq \left(\frac{5}{2}\|\chi'_0\|_{L^\infty}\|e\|_{L^\infty}\right)\beta_k^{2/3}k^{-2}\|\chi_0\gamma u\|\|\gamma u\|, \\ |A_{22}| &\leq 2\|\psi'\|_{L^\infty}\beta_k^{2/3}k^{-2}\|\gamma u\|\|\gamma'\chi_0 u\| \\ &\quad + \|\psi''\|_{L^\infty}\beta_k^{1/3}k^{-2}\|\gamma u\|\|\gamma''\chi_0 u\| + Ck^{-2}\|\gamma u\|\|\chi_0 u\|, \end{aligned}$$

so that (3.3.44) becomes

$$\begin{aligned} |A_2| &\leq \left(\frac{5}{2}\|\chi'_0\|_{L^\infty}\|e\|_{L^\infty}\right)\beta_k^{2/3}k^{-2}\|\gamma u\|\|\chi_0\gamma u\| + \left(2\|\psi'\|_{L^\infty}\right)\beta_k^{2/3}k^{-2}\|\gamma u\|\|\chi_0\gamma' u\| \\ &\quad + \|\psi''\|_{L^\infty}\beta_k^{1/3}k^{-2}\|\gamma u\|\|\chi_0\gamma'' u\| + Ck^{-2}\|\gamma u\|\|\chi_0 u\|. \end{aligned} \quad (\text{C.1.3})$$

Recall (3.3.45) and define

$$\kappa_1(e^{t_k}) = \frac{5}{2} \|\chi'_0\|_{L^\infty} \|e\|_{L^\infty} g(e^{t_k - c_0})^{1/2} + 2\|\psi'\|_{L^\infty} \max_{|t-t_k| \leq c_0} g(e^t)^{1/2} \left| \frac{1}{4} e^{2t} - 2 \right|, \quad (\text{C.1.4})$$

$$\text{and } \kappa_2(e^{t_k}) = \|\psi''\|_{L^\infty} \max_{|t-t_k| \leq c_0} g(e^t)^{1/2} \left| 4 - \frac{3}{2} e^{2t} + \frac{1}{16} e^{4t} \right|, \quad (\text{C.1.5})$$

(note that κ_1, κ_2 are more precise than the function κ) then we deduce from (C.1.3) that

$$\begin{aligned} |A_2| &\leq \kappa_1(e^{t_k}) \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 \\ &\quad + \kappa_2(e^{t_k}) \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 + 2C_2 k^{-2} \|u\|^2. \end{aligned} \quad (\text{C.1.6})$$

Estimates for the other terms

Instead of applying the inequalities (3.4.20) for the terms B_1^+, B_1^- , we keep the equalities for the moment

$$\begin{aligned} B_1^+ &= 2\beta_k^{2/3} \langle e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) u, \chi_+^2 u \rangle, \\ B_1^- &= 2\beta_k^{2/3} \langle e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) u, \chi_-^2 u \rangle. \end{aligned}$$

For B_2^+ and B_2^- , we use the following

$$\begin{aligned} B_2^+ &\geq -g(e^{t_k+c_0/2})^{1/2} \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2, \\ B_2^- &\geq -2\beta_k^{2/3} k^{-2} \|\chi_- \gamma u\|^2 - g(e^{t_k-c_0/2})^{1/2} \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2. \end{aligned}$$

The estimates (3.3.52), (3.3.57), (3.3.62) for the terms A_3, B_3^+, B_3^- remain unchanged. Adding these estimates together, we get the following

$$\begin{aligned} 2\text{Re} \langle \mathcal{L}_k u, M_k u \rangle &\geq \frac{C_1}{2} \beta_k^{2/3} \|\chi_0 u\|^2 + 2\beta_k^{2/3} \langle e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) \chi_+^2 u, u \rangle \\ &\quad + 2\beta_k^{2/3} \langle e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \chi_-^2 u, u \rangle - \phi(e^{t_k}) \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 \\ &\quad - 2\beta_k^{2/3} k^{-2} \|\chi_- e^{2t} g(e^t) u\|^2 - \kappa_2(e^{t_k}) \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2, \end{aligned} \quad (\text{C.1.7})$$

where C_1 is given in (C.1.2), κ_2 is given in (C.1.5) and

$$\begin{aligned} \phi(e^{t_k}) &= \kappa_1(e^{t_k}) + g(e^{t_k+c_0/2})^{1/2} + g(e^{t_k-c_0/2})^{1/2} \\ &= \frac{5}{2} \|\chi'_0\|_{L^\infty} \|e\|_{L^\infty} g(e^{t_k-c_0})^{1/2} + 2\|\psi'\|_{L^\infty} \max_{|t-t_k| \leq c_0} g(e^t)^{1/2} \left| \frac{1}{4} e^{2t} - 2 \right| \\ &\quad + g(e^{t_k+c_0/2})^{1/2} + g(e^{t_k-c_0/2})^{1/2}. \end{aligned} \quad (\text{C.1.8})$$

Using (C.0.7), (C.0.8) (with $\varepsilon \simeq 0.000001$), $e^{t_k} > 2.16657$ and the following

$$\max_{r \geq 2} g(r)^{1/2} \left| \frac{1}{4} r^2 - 2 \right| \leq 1,$$

we get an upper bound for $\phi(e^{t_k})$ given in (C.1.8):

$$\phi(e^{t_k}) \leq \tilde{C}_1 \simeq 50.1458. \quad (\text{C.1.9})$$

Computations for k_0

The main task is to absorb the two negative terms of size $\beta_k^{2/3}k^{-2}$ in the rhs of (C.1.7), for which we have to take k large. We need the following inequality: for $e^{t_k} > 2.16657$, $t \in \mathbb{R}$,

$$\begin{aligned} \frac{C_1}{2}\chi_0(t-t_k)^2 + 2e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))\chi_+(t-t_k)^2 + 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))\chi_-(t-t_k)^2 \\ > k^{-2}(\phi(e^{t_k})e^{4t}g(e^t) + 2e^{4t}g(e^t)^2)\chi_-(t-t_k)^2, \end{aligned}$$

which can be inferred from the following

$$\begin{cases} \frac{C_1}{2} > k^{-2}\phi(e^{t_k})e^{4t}g(e^t), & |t-t_k| \leq c_0, \\ 2\sigma(e^{t_k}) - 2\sigma(e^t) > k^{-2}\phi(e^{t_k})e^{2t}g(e^t), & t-t_k \geq c_0/2, \\ 2\sigma(e^t) - 2\sigma(e^{t_k}) > k^{-2}(\phi(e^{t_k})e^{2t}g(e^t) + 2e^{2t}g(e^t)^2), & t-t_k \leq -c_0/2, \end{cases} \quad (\text{C.1.10})$$

with C_1 given in (C.1.2) and $c_0 = 0.04$.

By making a change of variables $r_k = e^{t_k}$, $r = e^t$ and by using the bound (C.1.9) for $\phi(e^{t_k})$, we know that (C.1.10) can be inferred from the following: for $r_k > 2.16657$,

$$C_1 > 2k^{-2}\tilde{C}_1r^4g(r), \quad e^{-c_0}r_k \leq r \leq e^{c_0}r_k, \quad (\text{C.1.11})$$

$$2\sigma(r_k) - 2\sigma(r) > k^{-2}\tilde{C}_1r^2g(r), \quad r \geq e^{c_0/2}r_k, \quad (\text{C.1.12})$$

$$2\sigma(r) - 2\sigma(r_k) > k^{-2}(\tilde{C}_1r^2g(r) + 2r^2g(r)^2), \quad 0 < r \leq e^{-c_0/2}r_k. \quad (\text{C.1.13})$$

Now we compute a k_0 such that (C.1.11), (C.1.12), (C.1.13) hold for $k \geq k_0$.

- For (C.1.11). It suffices to take

$$k^2 > 2C_1^{-1}\tilde{C}_1 \max\{r^4g(r); r_k > 2.16657, r_k e^{-c_0} \leq r \leq r_k e^{c_0}\}.$$

Note that the function $r^4g(r)$ has maximum $256e^{-2}$ at $r = 4$, so that it suffices to take

$$k^2 > 2C_1^{-1}\tilde{C}_1 \times 256e^{-2} \simeq 5679.12, \quad \text{i.e. } k > 75.3599.$$

- For (C.1.12). We rewrite (C.1.12) as follows

$$\forall r_k > 2.16657, r \geq r_k e^{c_0/2}, \quad k^2 > \frac{\tilde{C}_1}{2} \frac{r^2g(r)}{\sigma(r_k) - \sigma(r)}.$$

Note that

$$r \geq r_k e^{c_0/2} \implies \frac{r^2g(r)}{\sigma(r_k) - \sigma(r)} \leq \frac{r^2g(r)}{\sigma(re^{-c_0/2}) - \sigma(r)} =: b(r; c_0),$$

so that it suffices to take

$$k^2 \geq \frac{\tilde{C}_1}{2} \max_{r>0} b(r; c_0).$$

By computing the maximum of $b(r; c_0)$, it suffices to take $k \geq 80$ to ensure (C.1.12).

- For (C.1.13). (C.1.13) can be rewritten as follows

$$\forall r_k > 2.16657, r \leq r_k e^{-c_0/2}, \quad k^2 > \frac{\tilde{C}_1r^2g(r) + 2r^2g(r)^2}{2(\sigma(r) - \sigma(r_k))}.$$

Note that

$$r \leq r_k e^{-c_0/2} \implies \begin{cases} \frac{r^2 g(r)}{\sigma(r) - \sigma(r_k)} \leq \frac{r^2 g(r)}{\sigma(r) - \sigma(re^{c_0/2})} =: c(r; c_0), \\ \frac{r^2 g(r)^2}{\sigma(r) - \sigma(r_k)} \leq \frac{r^2 g(r)^2}{\sigma(r) - \sigma(re^{c_0/2})} =: d(r; c_0), \end{cases}$$

then to ensure (C.1.13) it suffices to take

$$k^2 > \max_{r>0} \left(\frac{\tilde{C}_1}{2} c(r; c_0) + d(r; c_0) \right).$$

By computing the maximum, it suffices to take $k \geq 81$ to ensure (C.1.13).

As a result, the equalities in (C.1.10) are true for $k \geq 81$.

Final estimates

Using the estimates (3.4.20) and the notation $\rho(t, t_k)$ given in (3.3.33), we deduce from (C.1.7) that there exists $c > 0$ such that for $k \geq 81$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq c \beta_k^{2/3} \langle \rho(t, t_k) u, u \rangle - \kappa_2(e^{t_k}) \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2 \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2. \end{aligned} \quad (\text{C.1.14})$$

It remains to control the term of size $\beta_k^{1/3} k^{-2}$ in the rhs of (C.1.14). Note that $\kappa_2(e^{t_k})$ is bounded and recall (3.3.65), the term

$$-\kappa_2(e^{t_k}) \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)^{1/2} u\|^2,$$

can be absorbed by

$$\frac{c}{2} \beta_k^{2/3} \langle \rho(t, t_k) u, u \rangle$$

by letting $\alpha \geq \alpha_0$ with α_0 taken large. Continuing the proof in Section 3.3.3.a), we can get the estimate (C.0.3).

C.2 Case 2

We turn to the case where $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$ with

$$\epsilon_1 \simeq 0.426072, \quad \epsilon_0^{-1} \simeq 2.16657.$$

We choose

$$c_0 \simeq 0.04. \quad (\text{C.2.1})$$

Estimate for A_1

We have

$$e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}], \quad |t - t_k| \leq 2c_0 \implies e^t \in [0.393314, 2.34702],$$

$$\text{and } \forall r \in [0.393314, 2.34702], \quad -0.267226 \leq \sigma'(r) \leq -0.0958297.$$

For $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$ such that $|t - t_k| \leq 2c_0$, we get by Taylor's formula

$$\begin{aligned} \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] &= e^{3t}\sigma'(e^t) + 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \\ &\leq -0.0958297e^{3t} + 2e^{2t} \times 0.267226e^{t+2c_0}2c_0, \end{aligned}$$

so that

$$\forall e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}], |t - t_k| \leq 2c_0, \quad \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_2 e^{3t_k} \quad \text{with } C_2 = e^{-6c_0}(0.0958297 - 2 \times 0.267226 \times 2c_0 e^{2c_0}) \simeq 0.038948. \quad (\text{C.2.2})$$

Note $e^{3t_k} \in [\epsilon_1^3, \epsilon_0^{-3}]$. Then slightly changing the proof, one can get the following estimate for A_1 :

$$A_1 \geq \frac{C_2 e^{3t_k}}{2} \|\chi_0 u\|^2 - C \|D_t u\|^2 - C \|u\|^2, \quad (\text{C.2.3})$$

with C_2 given in (C.2.2). Note that it is better to keep the factor e^{3t_k} .

Estimate for A_2

For A_2 , we have the following (the same as (C.1.3))

$$\begin{aligned} |A_2| &\leq \left(\frac{5}{2} \|\chi'_0\|_{L^\infty} \|e\|_{L^\infty} \right) \beta_k^{2/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma u\| + \left(2 \|\psi'\|_{L^\infty} \right) \beta_k^{2/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma' u\| \\ &\quad + \|\psi''\|_{L^\infty} \beta_k^{1/3} k^{-2} \|\gamma u\| \|\chi_0 \gamma'' u\| + C k^{-2} \|\gamma u\| \|\chi_0 u\|. \end{aligned}$$

Recall (3.3.45) and note that the support of $\chi_0(\cdot - t_k)$ is included in the interval $[0.39, 2.4]$, we know that

$$|\chi_0(t - t_k)\gamma'(t)| \leq 2\gamma(t), \quad |\chi_0(t - t_k)\gamma''(t)| \leq 4\gamma(t),$$

so that

$$\begin{aligned} |A_2| &\leq \left(\frac{5}{2} \|\chi'_0\|_{L^\infty} \|e\|_{L^\infty} + 4 \|\psi'\|_{L^\infty} \right) \beta_k^{2/3} k^{-2} \|\gamma u\|^2 \\ &\quad + 4 \|\psi''\|_{L^\infty} \beta_k^{1/3} k^{-2} \|\gamma u\|^2 + C k^{-2} \|\gamma u\| \|\chi_0 u\|. \end{aligned} \quad (\text{C.2.4})$$

Estimate for the other terms

We keep the equalities for the terms B_1^+, B_1^- :

$$\begin{aligned} B_1^+ &= 2\beta_k^{2/3} \langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, \chi_+^2 u \rangle, \\ B_1^- &= 2\beta_k^{2/3} \langle e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, \chi_-^2 u \rangle, \end{aligned}$$

and we use the following bounds for B_2^+, B_2^-

$$B_2^+ \geq -2\beta_k^{2/3} k^{-2} \|\gamma u\|^2, \quad B_2^- \geq -2\beta_k^{2/3} k^{-2} \|\gamma u\|^2.$$

The estimates for A_3, B_3^+, B_3^- remain unchanged. Summarizing, we obtain

$$\begin{aligned} 2\operatorname{Re} \langle \mathcal{L}_k u, M_k u \rangle &\geq \frac{C_2 e^{3t_k}}{2} \beta_k^{2/3} \|\chi_0 u\|^2 + 2\beta_k^{2/3} \langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))\chi_+^2 u, u \rangle \\ &\quad + 2\beta_k^{2/3} \langle e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))\chi_-^2 u, u \rangle \\ &\quad - \tilde{C}_2 \beta_k^{2/3} k^{-2} \|e^{2t} g(e^t)u\|^2 - 4 \|\psi''\|_{L^\infty} \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)u\|^2 \\ &\quad - C \|D_t u\|^2 - C k^2 \|u\|^2 - C \|e^{2t} u\|^2, \end{aligned} \quad (\text{C.2.5})$$

where C_2 is given in (C.2.2) and

$$\tilde{C}_2 = \frac{5}{2} \|\chi'_0\|_{L^\infty} \|e\|_{L^\infty} + 4 \|\psi'\|_{L^\infty} + 4. \quad (\text{C.2.6})$$

We use (C.0.7), (C.0.8) (with $\varepsilon = 0.000001$) to obtain

$$\tilde{C}_2 \leq 6 + \frac{5}{2c_0} \simeq 68.5. \quad (\text{C.2.7})$$

Computations for k_0

The object is to absorb the negative term of size $\beta_k^{2/3} k^{-2}$ in the rhs of (C.2.5), for which we have to take k large enough. We need the following: for $e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}]$, $t \in \mathbb{R}$,

$$\frac{C_2 e^{3t_k}}{2} \chi_0^2 + 2e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) \chi_+^2 + 2e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) \chi_-^2 > \tilde{C}_2 k^{-2} e^{4t} g(e^t)^2,$$

which can be inferred from the following

$$\begin{cases} \frac{C_2}{2} e^{3t_k} > \tilde{C}_2 k^{-2} e^{4t} g(e^t)^2, & |t - t_k| \leq c_0, \\ 2e^{2t} (\sigma(e^{t_k}) - \sigma(e^t)) > \tilde{C}_2 k^{-2} e^{4t} g(e^t)^2, & t - t_k \geq c_0/2, \\ 2e^{2t} (\sigma(e^t) - \sigma(e^{t_k})) > \tilde{C}_2 k^{-2} e^{4t} g(e^t)^2, & t - t_k \leq -c_0/2, \end{cases} \quad (\text{C.2.8})$$

where C_2 is given in (C.2.2), \tilde{C}_2 is given in (C.2.6) and $c_0 = 0.04$.

By making a change of variables $r_k = e^{t_k}$ and $r = e^t$, (C.2.8) are equivalent to the following: for $r_k \in [\epsilon_1, \epsilon_0^{-1}]$,

$$\frac{C_2}{2} r_k^3 > \tilde{C}_2 k^{-2} r^4 g(r)^2, \quad r_k e^{-c_0} \leq r \leq r_k e^{c_0}, \quad (\text{C.2.9})$$

$$2(\sigma(r_k) - \sigma(r)) > \tilde{C}_2 k^{-2} r^2 g(r)^2, \quad r \geq r_k e^{c_0/2}, \quad (\text{C.2.10})$$

$$2(\sigma(r) - \sigma(r_k)) > \tilde{C}_2 k^{-2} r^2 g(r)^2, \quad r \leq r_k e^{-c_0/2}. \quad (\text{C.2.11})$$

Now we compute a k_0 such that (C.2.9), (C.2.10), (C.2.11) hold for $k \geq k_0$.

• For (C.2.9). It suffices to take

$$\begin{aligned} k^2 &> \frac{2\tilde{C}_2}{C_2} \sup \left\{ \frac{r^4 g(r)^2}{r_k^3}; r_k \in [\epsilon_1, \epsilon_0^{-1}], r_k e^{-c_0} \leq r \leq r_k e^{c_0} \right\}, \\ \text{or } k^2 &> \frac{2\tilde{C}_2}{C_2} e^{3c_0} \sup_{r>0} r g(r)^2. \end{aligned}$$

The function $r g(r)^2$ has a maximum $\sqrt{2/e}$ at $r = \sqrt{2}$, then it suffices to take

$$k^2 > \frac{2\tilde{C}_2}{C_2} e^{3c_0} \times \sqrt{2/e} \simeq 3401.88, \quad \text{i.e. } k > 58.3256.$$

• For (C.2.10). It suffices to take

$$k^2 > \frac{\tilde{C}_2}{2} \sup \left\{ \frac{r^2 g(r)^2}{\sigma(r e^{-c_0/2}) - \sigma(r)}; r \geq \epsilon_1 e^{c_0/2} \right\},$$

by computing the supremum, we get $k \geq 84$.

• For (C.2.11). It suffices to take

$$k^2 > \frac{\tilde{C}_2}{2} \sup \left\{ \frac{r^2 g(r)^2}{\sigma(r) - \sigma(r e^{c_0/2})}; r \leq \epsilon_0^{-1} e^{-c_0/2} \right\},$$

by computing the supremum, we get $k \geq 82$.

As a result, the inequalities in (C.2.8) hold for $k \geq 84$.

Final estimates

Using the estimate (3.4.23) and the notation $\rho(t, t_k)$ given in (3.3.72), we get the following: there exists $c > 0$ such that for $k \geq 84$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq c\beta_k^{2/3} \langle \rho(t, t_k)u, u \rangle - 4\|\psi''\|_{L^\infty} \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2 \|u\|^2 - C\|e^{2t} u\|^2. \end{aligned} \quad (\text{C.2.12})$$

It remains to control the negative term of size $\beta_k^{1/3} k^{-2}$ in the rhs of (C.2.12). Noticing (3.3.73), the term

$$-4\|\psi''\|_{L^\infty} \beta_k^{1/3} k^{-2} \|e^{2t} g(e^t)u\|^2$$

can be absorbed by

$$\frac{c}{2} \beta_k^{2/3} \langle \rho(t, t_k)u, u \rangle,$$

letting $\alpha \geq \alpha_0$ with α_0 taken large enough. Continuing the proof in Section 3.3.3.b), we can complete the proof of (C.0.4).

C.3 Case 3

In this section, we deal with the case where $e^{t_k} \in (\mu_0 \beta_k^{-1/4}, \epsilon_1)$ with $\epsilon_1 \simeq 0.426072$. We take

$$c_0 \simeq 0.08. \quad (\text{C.3.1})$$

We shall choose μ_0 in the end of this section and the condition $\beta_k e^{4t_k} > \mu_0$ is used to absorb a lower order term.

Estimate for A_1

We have for $0 < r \leq 1/2$,

$$\frac{0.979r^2}{8} \leq 1 - \sigma(r) \leq \frac{r^2}{8}, \quad -\frac{r}{4} \leq \sigma'(r) \leq -\frac{0.959r}{4}.$$

Using these bounds, for $e^t, e^{t_k} \leq 1/2$ such that $|t - t_k| \leq 2c_0$, we have

$$\begin{aligned} \frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] &= e^{3t}\sigma'(e^t) + 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) \\ &\leq -\frac{0.959}{4}e^{4t} + 2e^{2t} \left(-\frac{0.979}{8}e^{2t} + \frac{1}{8}e^{2t_k} \right) \leq -e^{4t}(0.4845 - \frac{1}{4}e^{4c_0}), \end{aligned}$$

Then $\forall e^{t_k} < 0.426072, \forall |t - t_k| \leq 2c_0$, (which implies $e^t \leq \frac{1}{2}$)

$$\frac{d}{dt} [e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_3 e^{4t_k}, \quad \text{with } C_3 \simeq 0.0739359. \quad (\text{C.3.2})$$

The estimate (3.3.85) for A_1 holds with C_3 given in (C.3.2).

Estimates for A_2

The precise versions of (3.3.87), (3.3.88) are

$$\begin{aligned} |A_{21}| &\leq \left(\frac{5}{2}\|\chi'_0\|_{L^\infty}\|e\|_{L^\infty}\right)\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\chi_0\gamma u\|\|\gamma u\|, \\ |A_{22}| &\leq 2\|\psi'\|_{L^\infty}\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|\|\gamma'\chi_0 u\| \\ &\quad + \|\psi''\|_{L^\infty}\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|\gamma u\|\|\gamma''\chi_0 u\| + Ck^{-2}\|g(e^t)u\|\|\chi_0 u\|. \end{aligned}$$

Recall (3.3.45). In the support of $\chi_0(\cdot - t_k)$, we have $e^t \leq 1/2$, then

$$|\chi_0(t - t_k)\gamma'(t)| \leq 2\gamma(t), \quad |\chi_0(t - t_k)\gamma''(t)| \leq 4\gamma(t),$$

so that

$$\begin{aligned} |A_2| &\leq \left(\frac{5}{2}\|\chi'_0\|_{L^\infty}\|e\|_{L^\infty} + 4\|\psi'\|_{L^\infty}\right)\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2 \\ &\quad + 4\|\psi''\|_{L^\infty}\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|\gamma u\|^2 + Ck^{-2}\|u\|^2. \end{aligned} \quad (\text{C.3.3})$$

Estimates for the other terms

We keep the equalities for B_1^+, B_1^-

$$\begin{aligned} B_1^+ &= 2\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, \chi_+^2 u \rangle, \\ B_1^- &= 2\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))u, \chi_-^2 u \rangle, \end{aligned}$$

and we use the following for B_2^+, B_2^-

$$B_2^+ \geq -2\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2, \quad B_2^- \geq -2\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|\gamma u\|^2.$$

The estimates (3.3.91), (3.3.96), (3.3.101) for A_3, B_3^+, B_3^- remain unchanged. Summarizing, we obtain the following

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq \frac{C_3}{2}(\beta_k e^{4t_k})^{2/3}\|\chi_0 u\|^2 + 2\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))\chi_+^2 u, u \rangle \\ &\quad + 2\beta_k(\beta_k e^{4t_k})^{-1/3}\langle e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))\chi_-^2 u, u \rangle \\ &\quad - \tilde{C}_3\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}\|e^{2t}g(e^t)u\|^2 - 4\|\psi''\|_{L^\infty}\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}\|e^{2t}g(e^t)u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2\|u\|^2 - C\|e^{2t}u\|^2, \end{aligned} \quad (\text{C.3.4})$$

where C_3 is given in (C.3.2) and

$$\tilde{C}_3 = 4 + 4\|\psi'\|_{L^\infty} + \frac{5}{2}\|\chi'_0\|_{L^\infty}\|e\|_{L^\infty}. \quad (\text{C.3.5})$$

We use the bounds in (C.0.7), (C.0.8) (with $\varepsilon = 0.000001$) to obtain

$$\tilde{C}_3 \leq 6 + \frac{5}{2c_0} \simeq 37.25. \quad (\text{C.3.6})$$

Computations for k_0

The main task is to absorb the negative term of size $\beta_k(\beta_k e^{4t_k})^{-1/3}k^{-2}$ in the rhs of (C.3.4), for which we have to take k large. We need the following: for $e^{t_k} < 0.426072$, $t \in \mathbb{R}$,

$$\frac{C_3}{2}e^{4t_k}\chi_0^2 + 2e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))\chi_+^2 + 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k}))\chi_-^2 > \tilde{C}_3 k^{-2} e^{4t} g(e^t)^2,$$

which can be inferred from the following

$$\begin{cases} \frac{C_3}{2}e^{4t_k} > \tilde{C}_3 k^{-2} e^{4t} g(e^t)^2, & |t - t_k| \leq c_0, \\ 2e^{2t}(\sigma(e^{t_k}) - \sigma(e^t)) > \tilde{C}_3 k^{-2} e^{4t} g(e^t)^2, & t - t_k \geq c_0/2, \\ 2e^{2t}(\sigma(e^t) - \sigma(e^{t_k})) > \tilde{C}_3 k^{-2} e^{4t} g(e^t)^2, & t - t_k \leq -c_0/2, \end{cases} \quad (\text{C.3.7})$$

where C_3 is given in (C.3.2), \tilde{C}_3 is given in (C.3.5) and $c_0 = 0.08$.

By making a change of variables $r_k = e^{t_k}$ and $r = e^t$, (C.3.7) are equivalent to the following: for $r_k < 0.426072$,

$$C_3 r_k^4 > 2\tilde{C}_3 k^{-2} r^4 g(r)^2, \quad r_k e^{-c_0} \leq r \leq r_k e^{c_0}, \quad (\text{C.3.8})$$

$$2(\sigma(r_k) - \sigma(r)) > \tilde{C}_3 k^{-2} r^2 g(r)^2, \quad r \geq r_k e^{c_0/2}, \quad (\text{C.3.9})$$

$$2(\sigma(r) - \sigma(r_k)) > \tilde{C}_3 k^{-2} r^2 g(r)^2, \quad r \leq r_k e^{-c_0/2}. \quad (\text{C.3.10})$$

Now let us compute k_0 such that (C.3.8), (C.3.9), (C.3.10) hold for $k \geq k_0$.

• For (C.3.8). It suffices to take

$$k^2 > \frac{2\tilde{C}_3}{C_3} \sup \left\{ \frac{r^4 g(r)^2}{r_k^4} ; r_k < 0.426072, r_k e^{-c_0} \leq r \leq r_k e^{c_0} \right\},$$

$$\text{thus } k^2 > \frac{2\tilde{C}_3 e^{4c_0}}{C_3} \simeq 1387.63, \quad \text{i.e. } k > 37.251.$$

• For (C.3.9). It suffices to take

$$k^2 > \frac{\tilde{C}_3}{2} \sup \left\{ \frac{r^2 g(r)^2}{\sigma(re^{-c_0/2}) - \sigma(r)} ; r > 0 \right\},$$

by computing the supremum, we get $k \geq 45$.

• For (C.3.10). It suffices to take

$$k^2 > \frac{\tilde{C}_3}{2} \sup \left\{ \frac{r^2 g(r)^2}{\sigma(r) - \sigma(re^{c_0/2})} ; r \leq \epsilon_1 e^{c_0/2} \right\},$$

by computing the supremum, we get $k \geq 43$.

As a result, the inequalities in (C.3.7) are true for $k \geq 45$.

Final estimates

Using the estimates (3.4.26) and the notation $\tilde{\rho}(t, t_k)$ given in (3.3.81), we deduce from (C.3.4) that there exists $c > 0$ such that for $k \geq 45$,

$$\begin{aligned} 2\operatorname{Re}\langle \mathcal{L}_k u, M_k u \rangle &\geq c\beta(\beta_k e^{4t_k})^{-1/3} \langle \tilde{\rho}(t, t_k)u, u \rangle - 4\|\psi''\|_{L^\infty} \beta_k(\beta_k e^{4t_k})^{-2/3} k^{-2} \|e^{2t} g(e^t)u\|^2 \\ &\quad - C\|D_t u\|^2 - Ck^2 \|u\|^2 - C\|e^{2t} u\|^2. \end{aligned} \quad (\text{C.3.11})$$

It remains to control the negative term of size $\beta_k(\beta_k e^{4t_k})^{-2/3}k^{-2}$ in the rhs of (C.3.11), for which we take a large constant μ_0 so that the quantity $\beta_k e^{4t_k}$ is very large and we do not need k large. Using (3.3.103) and letting $\mu_0 \geq 1$ large enough, the term

$$-4\|\psi''\|_{L^\infty}\beta_k(\beta_k e^{4t_k})^{-2/3}\|e^{2t}g(e^t)u\|^2$$

can be absorbed by

$$\frac{c}{2}\beta_k(\beta_k e^{4t_k})^{-1/3}\langle\tilde{\rho}(t, t_k)u, u\rangle.$$

Continuing the proof in Section 3.3.4.c), we can complete the proof of (C.0.5).

C.4 Case 4

It remains to give a proof for (C.0.6), the estimate for the case where $e^{t_k} \leq \mu_0\beta_k^{-1/4}$ with μ_0 chosen in the previous section. We use the multiplier $\mu_0^4\text{Id}$ and $i\text{Id}$. Using $1-\sigma(r) \leq r^2/8$, we have for $e^{t_k} \leq \mu_0\beta_k^{-1/4}$,

$$\begin{aligned} \operatorname{Re}\langle\mathcal{L}_k u, -iu\rangle &= \beta_k\langle e^{2t}(\sigma(e^{t_k}) - \sigma(e^t))u, u\rangle + \beta_k\langle\gamma\langle D_k\rangle^{-2}\gamma u, u\rangle \\ &\geq \beta_k\langle e^{2t}\left((1 - \sigma(e^t)) - (1 - \sigma(e^{t_k}))\right)u, u\rangle \\ &\geq \beta_k\langle e^{2t}\left((1 - \sigma(e^t)) - \frac{\mu_0^2}{8}\beta_k^{-1/2}\right)u, u\rangle, \end{aligned}$$

so that

$$\operatorname{Re}\langle\mathcal{L}_k u, (\mu_0^4 - i)u\rangle \geq \langle\left(\mu_0^4 k^2 e^{-2t} + \beta_k(1 - \sigma(e^t)) - \frac{\mu_0^2}{8}\beta_k^{1/2}\right)e^{2t}u, u\rangle.$$

We claim that there exists $\alpha_0 \geq 8\pi$ such that for all $k \geq 1, \alpha \geq \alpha_0, t \in \mathbb{R}$,

$$\mu_0^4 k^2 e^{-2t} + \beta_k(1 - \sigma(e^t)) \geq \frac{\mu_0^2}{4}\beta_k^{1/2}. \quad (\text{C.4.1})$$

Indeed, $1 - \sigma(e^t) \geq e^{2t}/16$ for $e^t < 2$, and $1 - \sigma(e^t) \geq e^{-1}$ for $e^t \geq 2$. Then we have for $k \geq 1, \alpha \geq \alpha_0$,

$$\begin{aligned} \text{if } e^t < 2, \quad &\mu_0^4 k^2 e^{-2t} + \beta_k(1 - \sigma(e^t)) \geq \mu_0^4 k^2 e^{-2t} + \frac{1}{16}\beta_k e^{2t} \geq \frac{\mu_0^2}{2}k\beta_k^{1/2}; \\ \text{if } e^t \geq 2, \quad &\mu_0^4 k^2 e^{-2t} + \beta_k(1 - \sigma(e^t)) \geq e^{-1}\beta_k \geq e^{-1}\sqrt{\frac{\alpha_0 k}{8\pi}}\beta_k^{1/2}, \end{aligned}$$

so that (C.4.1) holds with $\alpha_0 = \pi e^2 \mu_0^4 / 2$. thus

$$\operatorname{Re}\langle\mathcal{L}_k u, (\mu_0^4 - i)u\rangle \geq \frac{\mu_0^2}{8}\beta_k^{1/2}\langle e^{2t}u, u\rangle.$$

By Cauchy-Schwarz inequality, we complete the proof of (C.0.6).

Improvements

The bound $k_0 = 84$ can be improved. For example, when estimating for the term A_1 , instead of requiring a control for the derivative of the function σ of type (see (3.4.18), (3.4.21), (3.4.24))

$$\begin{aligned} \forall e^{t_k} > \epsilon_0^{-1}, \quad |t - t_k| < 2c_0, \quad \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_1; \\ \forall e^{t_k} \in [\epsilon_1, \epsilon_0^{-1}], \quad |t - t_k| < 2c_0, \quad \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_2; \\ \forall e^{t_k} < \epsilon_1, \quad |t - t_k| < 2c_0, \quad \frac{d}{dt} [e^{2t} (\sigma(e^t) - \sigma(e^{t_k}))] \leq -C_3 e^{4t_k}, \end{aligned} \quad (\text{C.4.2})$$

in fact we need only these inequalities above for t such that $|t - t_k| < c_0 + \varepsilon$ for $\varepsilon > 0$ small fixed, since (C.4.2) are applied on the function $\chi_0(\cdot - t_k)u$ which is supported in $[t_k - c_0, t_k + c_0]$. As the small constant c_0 appears in the denominator of the coefficients of estimates for the term A_2 , the requirement $|t - t_k| < c_0 + \varepsilon$ will allow us to get a better control of coefficients for A_2 if c_0 could be taken slightly larger. And also, the lower bounds for the terms B_1^+ and B_1^- can be improved since they are localized in the zone away from t_k of a distance $c_0/2$. Certainly, the requirement $|t - t_k| < c_0 + \varepsilon$ will change the coefficients in the remainders and the lower order terms (which will depend on ε), but we need not to take k large to control them.

On the other hand, note that the constant in each of the inequalities in (C.4.2) is uniform on e^{t_k} , depending only on the choice of (ϵ_0, ϵ_1) . One may divide the discussion on e^{t_k} into small intervals to localize e^{t_k} more precisely, by following the behavior of the derivative of the function σ , instead of just distinguishing 4 cases as that we have done. This will also allow us to deduce accurate estimates and to improve the upper bound on k_0 .

Bibliographie

- [BA94] Matania Ben-Artzi. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal.*, 128(4) :329–358, 1994.
- [BC94] Jean-Michel Bony and Jean-Yves Chemin. Espaces fonctionnels associés au calcul de Weyl-Hörmander. *Bull. Soc. Math. France*, 122(1) :77–118, 1994.
- [BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [Bon99] Jean-Michel Bony. Sur l’inégalité de Fefferman-Phong. In *Séminaire : Équations aux Dérivées Partielles, 1998–1999*, Sémin. Équ. Dériv. Partielles, pages Exp. No. III, 16. École Polytech., Palaiseau, 1999.
- [Bou02] Lyonell S. Boulton. Non-self-adjoint harmonic oscillator, compact semigroups and pseudospectra. *J. Operator Theory*, 47(2) :413–429, 2002.
- [Bou08] Abdesslam Boulkhemair. On the Fefferman-Phong inequality. *Ann. Inst. Fourier (Grenoble)*, 58(4) :1093–1115, 2008.
- [Che98] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
- [Den06] Nils Dencker. The resolution of the Nirenberg-Treves conjecture. *Ann. of Math.* (2), 163(2) :405–444, 2006.
- [Den10a] Wen Deng. Resolvent estimates for a two-dimensional non-self-adjoint operator. preprint, 2010.
- [Den10b] Wen Deng. Structure constants of the Weyl calculus. preprint, 2010.
- [Den11a] Wen Deng. Numerical computations for the value of k_0 . <http://www.math.jussieu.fr/~wendeng/>, 2011.
- [Den11b] Wen Deng. Pseudospectrum for Oseen vortices operators. preprint, 2011.
- [DSZ04] Nils Dencker, Johannes Sjöstrand, and Maciej Zworski. Pseudospectra of semi-classical (pseudo-) differential operators. *Comm. Pure Appl. Math.*, 57(3) :384–415, 2004.
- [EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

- [Gal11] Thierry Gallay. Nonselfadjoint operators in fluid mechanics : a case study. lecture notes for the summer school “Spectral analysis of non-selfadjoint operators and applications”, Rennes, 2011.
- [GGN09] Isabelle Gallagher, Thierry Gallay, and Francis Nier. Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator. *Int. Math. Res. Not. IMRN*, 12(12) :2147–2199, 2009.
- [GW02] Thierry Gallay and C. Eugene Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on \mathbf{R}^2 . *Arch. Ration. Mech. Anal.*, 163(3) :209–258, 2002.
- [GW05] Thierry Gallay and C. Eugene Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, 255(1) :97–129, 2005.
- [HN05] Bernard Helffer and Francis Nier. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, volume 1862 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.
- [Hör67] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119 :147–171, 1967.
- [Hör83] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.
- [Hör85] Lars Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. Pseudo-differential operators.
- [HPS11] F. Hérau and K. Pravda-Starov. Anisotropic hypoelliptic estimates for Landau-type operators. *J. Math. Pures Appl.* (9), 95(5) :513–552, 2011.
- [HS10] Bernard Helffer and Johannes Sjöstrand. From resolvent bounds to semigroup bounds. arXiv :1001.4171, 2010.
- [HSS05] Frédéric Hérau, Johannes Sjöstrand, and Christiaan C. Stolk. Semiclassical analysis for the Kramers-Fokker-Planck equation. *Comm. Partial Differential Equations*, 30(4-6) :689–760, 2005.
- [Kat94] Tosio Kato. The Navier-Stokes equation for an incompressible fluid in \mathbf{R}^2 with a measure as the initial vorticity. *Differential Integral Equations*, 7(3-4) :949–966, 1994.
- [Kat95] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [Ler10] Nicolas Lerner. *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, volume 3 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010.
- [LM07] Nicolas Lerner and Yoshinori Morimoto. On the Fefferman-Phong inequality and a Wiener-type algebra of pseudodifferential operators. *Publ. Res. Inst. Math. Sci.*, 43(2) :329–371, 2007.
- [LMPS12] Nicolas Lerner, Yochinori Morimoto, and Karel Pravda-Starov. Hypoelliptic estimates for a linear model of the boltzmann equation without angular cutoff. *Comm. Partial Differential Equations*, 37(2) :234–284, 2012.

- [Mae11] Yasunori Maekawa. Spectral properties of the linearization at the Burgers vortex in the high rotation limit. *J. Math. Fluid Mech.*, 13(4) :515–532, 2011.
- [PP95] A. Prochazka and D. I. Pullin. On the two-dimensional stability of the axisymmetric Burgers vortex. *Phys. Fluids*, 7(7) :1788–1790, 1995.
- [PS04] Karel Pravda-Starov. A general result about the pseudo-spectrum of Schrödinger operators. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2042) :471–477, 2004.
- [PS06a] Karel Pravda-Starov. A complete study of the pseudo-spectrum for the rotated harmonic oscillator. *J. London Math. Soc.* (2), 73(3) :745–761, 2006.
- [PS06b] Karel Pravda-Starov. *Etude du pseudo-spectre d'opérateurs non auto-adjoints*. PhD thesis, Université de Rennes I, 2006.
- [RS72] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.
- [RS96] Steffen Roch and Bernd Silbermann. C^* -algebra techniques in numerical analysis. *J. Operator Theory*, 35(2) :241–280, 1996.
- [SZ07] Johannes Sjöstrand and Maciej Zworski. Elementary linear algebra for advanced spectral problems. *Ann. Inst. Fourier (Grenoble)*, 57(7) :2095–2141, 2007. Festival Yves Colin de Verdière.
- [TE05] Lloyd N. Trefethen and Mark Embree. *Spectra and pseudospectra*. Princeton University Press, Princeton, NJ, 2005. The behavior of nonnormal matrices and operators.
- [Tre97] Lloyd N. Trefethen. Pseudospectra of linear operators. *SIAM Rev.*, 39(3) :383–406, 1997.
- [Vil06] Cédric Villani. Hypocoercive diffusion operators. In *International Congress of Mathematicians. Vol. III*, pages 473–498. Eur. Math. Soc., Zürich, 2006.
- [Vil09] Cédric Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950) :iv+141, 2009.

