Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Xavier Blot

La série de Witten-Kontsevich quantique

dirigée par Dimitri Zvonkine

Soutenue le 3 juillet 2020 devant le jury composé de :

M. Alexandr Buryak	Higher School of Economics, Moscow	examinateur
M. Guido Carlet	Institut de mathématiques de Bourgogne	rapporteur
M. Alessandro Chiodo	Institut de mathématiques de Jussieu-PRG	examinateur
M. Paolo Rossi	Università degli Studi di Padova	rapporteur
M. Sergey Shadrin	Korteweg-de-Vries Institute, Amsterdam	examinateur
M. Dimitri Zvonkine	Laboratoire de mathématiques de Versailles	directeur

Institut de mathématiques de Jussieu-Paris Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Université Pierre et Marie Curie.
École doctorale de sciences mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

À mon père,

Remerciements

Mes premiers remerciements vont à mon directeur de thèse Dimitri Zvonkine qui m'a guidé tout au long de cette belle aventure. Merci de m'avoir proposé un sujet de recherche aussi riche et passionnant, pour tes explications toujours très éclairantes et pour m'avoir aidé à trouver la voie dans des calculs sans fins. Travailler avec toi m'a fait incroyablement progresser. Merci aussi d'avoir eu la patience de m'enseigner la rigueur de l'écriture d'un article. J'ai découvert pendant ma thèse le plaisir de l'exploration et de la recherche, merci de m'y avoir initié.

Je tiens aussi à remercier Paolo Rossi de m'avoir introduit aux fonctions tau quantiques qui sont, à l'heure actuelle, l'objet principal de mes recherches. Merci aussi de m'avoir proposé, avec Renzo Cavalieri, de présenter mon travail à San Jose. C'était mon premier exposé et malgré le stresse de l'instant, j'en garde en très beau souvenir.

Mes sincères remerciements vont à Guido Carlet et Paolo Rossi pour leur travail de rapporteur ainsi qu'à Alexandr Buryak, Alessandro Chiodo et Sergey Shadrin pour me faire l'honneur d'être membres du jury de ma thèse.

J'aimerais aussi remercier quelques membres de l'IMJ avec qui j'ai pris plaisir à discuter. Alessandro, les quelques discussions informelles que nous avons eu m'ont à chaque fois réjoui et conforté dans mon travail, merci pour ton enthousiasme. Merci aussi à Cyril pour avoir suivi le déroulé de ma thèse. Merci enfin à Alexandru pour l'intérêt que tu as porté à mes travaux.

Je remercie aussi Vincent Bouchard, qui, pendant une nuit à Moscou, m'a donné de précieux conseils qui m'ont suivi jusqu'à aujourd'hui.

Merci aux doctorants de l'IMJ qui rendent le quotidien aussi plaisant, Léo, Louis, Nicolina, Adrien, Hugo, Thomas, Justin, Leon, Sylvain, Sudarshan, Grace, Mahya, Thibault, Mathieu, Alex, Benoit, Ilias, Florian, Christina, Haowen, Vadim, Christophe, Charles. Un remerciement particulier aux occupants de mon bureau, Maelyss, Mathieu pour ces quatre années ensemble et Malo parti bien trop tôt. Merci à Vincent pour ces discussions tard le soir dans les couloirs de l'IMJ, elles m'ont fait le plus grand bien. Merci aussi à Michou pour ton rire communicatif. Merci enfin à Sebastian pour nos pauses toujours très chouettes en ta compagnie.

Merci aussi aux doctorants d'ailleurs qui rendent les conférences si agréables Danilo, Ann, Reinier, Alessandro, Campbell, Elba, Raphaël, Elder et Séverin.

Pour finir, j'aimerais remercier ma famille et mes amis. Vous êtes essentiels à mon travail, mon bonheur et mon épanouissement.

Merci à ma mère et mon frère pour leur soutient inébranlable et pour la solidité si rassurante de notre trio. Merci à Patrick pour tous ces bon moments, ces belles randonnées et ce cochon rôti. À mes grands parents, mes oncles, mes tantes et mes cousines pour tous ces délicieux moments en famille. Un remerciement spécial à Colette et Yves pour m'avoir bichonné et requinqué à plusieurs reprises pendant d'intenses révisions.

Merci aux 6 nains et à blanche neige pour notre belle amitié à laquelle je crois très fort. Vous m'accompagnez depuis tant d'années que je me sens indissociable de vous. J'ai hâte de voir ce que le futur nous réserve. Merci à Clément qui a un place toute particulière dans mon coeur. Alice, merci pour ton enthousiasme, c'est un bonheur de te voir à chaque fois. À mes anciens colocataires, Xavier, Morgane, Fiona et Tony, merci pour ces quelques mois de bonheur avec vous. Enfin, merci à Enora de me faire tant rire.

Abstract

The main theorem of this PhD thesis states the following: the genus 0 coefficients of the quantum Witten-Kontsevich series defined by Buryak, Dubrovin, Guéré, and Rossi are equal to the coefficients of the polynomials defined by Goulden, Jackson, and Vakil in their study of double Hurwitz numbers. We also prove several other results on the quantum Witten-Kontsevich series.

The classical Witten-Kontsevich series is a generating series of intersection numbers on the moduli spaces of stable curves. The Witten conjecture, proved by Kontsevich, asserts that this series is the logarithm of a tau function of the KdV hierarchy. In 2016, Buryak and Rossi introduced a new way to construct quantum integrable hierarchies, including a quantum KdV hierarchy. Buryak, Dubrovin, Guéré and Rossi then defined quantum tau functions, one of which is the quantum Witten-Kontsevich series. This series depends on two parameters: the genus parameter ϵ and the quantization parameter $\hbar$. It reduces to the Witten-Kontsevich series when we plug $\hbar = 0$.

One-part double Hurwitz numbers count non-equivalent holomorphic maps from a Riemann surface of genus g to $\mathbb{P}^1$ with a prescribed ramification profile over 0, a complete ramification over ∞, and a given number of simple ramifications elsewhere. Goulden, Jackson and Vakil proved that these numbers are polynomial in the orders of ramification over 0. We show that the coefficients of these polynomials are equal to the coefficients of the quantum Witten-Kontsevich series with $\epsilon = 0$.

In Chapter 1, we present the setting of the classical and quantum integrable hierarchies that we will use. We also present the construction of classical and quantum tau functions.

In Chapter 2, we present the moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$ and their tautological rings. We briefly review the Witten conjecture. Then we introduce the double ramification cycle and discuss various methods for computing it. This cycle is needed to define the Hamiltonians of quantum integrable hierarchies.

In Chapter 3, we present the quantum KdV hierarchy and some of its properties. We then define the quantum Witten-Kontsevich series as a particular quantum tau function of this hierarchy.

In Chapter 4, we introduce Hurwitz numbers. We first present a remarkable link between the quantum KdV hierarchy and the cut-and-join equation. Then we introduce the so-called one-part double Hurwitz numbers. Their relation with the quantum Witten-Kontsevich series is the main result ot this thesis.

In Chapter 5, we present Eulerian numbers. These numbers appear in the computations of the coefficients of the quantum Witten-Kontsevich series. Their properties are crucial for our proofs.

In Chapter 6, we formulate and prove our main theorem and other results on the quantum Witten-Kontsevich series.

Keywords: moduli space of curves; classical and quantum integrable systems; deformation quantization; Hurwitz numbers; Eulerian numbers.

Résumé

Le théorème principal de cette thèse établit le lien suivant : les coefficients de genre 0 de la série de Witten-Kontsevich quantique définie par Buryak, Dubrovin, Guéré et Rossi sont égaux aux coefficients des polynômes définis par Goulden, Jackson et Vakil dans leur étude des nombres de Hurwitz doubles. Nous prouvons aussi d'autres résultats concernant la série de Witten-Kontsevich quantique.

La série de Witten-Kontsevich classique est une série génératrice de nombres d'intersections sur les espaces de module des courbes. La conjecture de Witten, prouvée par Kontsevich, affirme que cette série est le logarithme d'une fonction tau de la hiérarchie de KdV. En 2016, Buryak et Rossi ont introduit une nouvelle façon de construire des hiérarchies intégrables quantiques, ils ont en particulier construit une hiérarchie de KdV quantique. Buryak, Dubrovin, Guéré et Rossi ont ensuite défini des fonctions tau quantiques, l'une d'entre elles est la série de Witten-Kontsevich quantique. Cette série dépend de deux paramètres : le paramètre de genre ϵ et le paramètre quantique $\hbar$. Elle se restreint à la série de Witten-Kontsevich lorsque l'on substitue $\hbar = 0$.

Les nombres de Hurwitz double polynomiaux comptent le nombre d'applications holomorphes non équivalentes d'une surface de Riemann de genre g à $\mathbb{P}^1$ avec un profil de ramification fixé au dessus de 0, une ramification complète au dessus de ∞, et un nombre donné de ramifications simples au dessus de $\mathbb{P}^1 \setminus \{0, \infty\}$. Goulden, Jackson et Vakil ont prouvé que ces nombres sont polynomiaux en les ordres de ramification au dessus de 0. Nous montrons que les coefficients de ces polynômes sont égaux aux coefficients de la série de Witten-Kontsevich quantique avec $\epsilon = 0$.

Dans le Chapitre 1, nous présentons le cadre des hiérarchies intégrables classiques et quantiques que nous utiliserons. Nous présentons aussi la construction des fonctions tau classiques et quantiques.

Dans le Chapitre 2, nous présentons les espaces de module des courbes $\overline{\mathcal{M}}_{g,n}$ et leurs anneaux tautologiques. Nous présentons brièvement la conjecture de Witten. Ensuite, nous introduisons le cycle de double ramification et discutons de quelques méthodes pour le calculer. Ce cycle est nécessaire pour définir les hamiltoniens des hiérarchies intégrables quantiques.

Dans le Chapitre 3, nous présentons la hiérarchie de KdV quantique et quelques-unes de ses propriétés. Nous définissons ensuite la série de Witten-Kontsevich quantique comme une certaine fonction tau quantique de cette hiérarchie.

Dans le Chapitre 4, nous introduisons les nombres de Hurwitz. Nous présentons d'abord un lien remarquable entre la hiérarchie de KdV quantique et l'équation de cut-and-join. Ensuite, nous introduisons les nombres de Hurwitz doubles polynomiaux. Leur relation avec la série de Witten-Kontsevich quantique est le résultat principal de cette thèse.

Dans le Chapitre 5, nous présentons les nombres Eulériens. Ces nombres apparaissent dans le calcul des coefficients de la série de Witten-Kontsevich quantique. Leurs propriétés sont cruciales pour nos preuves.

Dans le Chapitre 6, nous formulons et prouvons notre théorème principal ainsi que d'autres résultats sur la série de Witten-Kontsevich quantique.

Mots-clefs: espace de module des courbes; hiérarchie intégrable classique et quantique; quantification par déformation; nombres de Hurwitz; nombres Euleriens.

Contents

Chapter 1

Classical and quantum integrable hierarchies with their tau functions

1.1 Introduction

We first review classical integrable systems on a finite dimensional manifold. We then define some integrable systems on an infinite dimensional space called the loop space. The approach is formal, we define formal objects that mimic the finite dimensional description of integrable systems. We also define the tau functions associated to such integrable systems. We then use deformation quantization to define a quantization of integrable systems on the loop space. By analogy with classical integrable systems, we introduce quantum tau functions.

1.2 Finite dimensional integrable system

1.2.1 Poisson structure

Definition 1. Let P be a smooth manifold. A *Poisson structure* on $\mathcal{C}^{\infty}(P)$ is a Lie bracket

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{F}(P) \times \mathcal{F}(P) &\to \mathcal{F}(P) \\ (f,g) &\to \{f,g\} \end{aligned}$$

that satisfies the Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$.

A manifold P endowed with a Poisson bracket is a *Poisson manifold.*

Let $\left(x^1,\dots,x^n\right)$ be a system of local coordinates on P. In these coordinates, the Poisson structure reads

$$\{f,g\} = \sum_{1\leq i,j\leq n} \pi^{ij}(x) \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j},$$

where π^{ij} satisfies $\pi^{ij} = -\pi^{ji}$ (antisymmetry) and $\sum_{s=1}^{n}\left(\frac{\partial \pi^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial \pi^{jk}}{\partial x^s}\pi^{si} + \frac{\partial \pi^{ki}}{\partial x^s}\pi^{sj}\right) = 0$ (Jacobi identity).

Definition 2. A Poisson structure is *non degenerate* if $\pi(x)$ is invertible for every $x \in P$.

Example 3. On $\mathbb{R}^2$, denote by (q,p) the system of canonical coordinates. Let $f,g \in \mathcal{C}^\infty(\mathbb{R}^2)$, a Poisson structure is given by

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

According to Darboux's theorem, any non degenerate Poisson structure on $\mathbb{R}^2$ locally looks like this one.

1.2.2 The Hamilton equation

Definition 4. A *Hamiltonian system* is the data of a Poisson manifold with the choice of a function $h \in \mathcal{C}^\infty(P)$. The function h is called *the Hamiltonian of the system.*

A Poisson bracket defines an morphism

$$\begin{aligned} \Phi : \mathcal{C}^\infty(P) &\to \Gamma(TP) \\ h &\to X_h := \{\cdot, h\} \end{aligned}$$

which satisfies $[X_h, X_g] = -X_{\{h,g\}}$.

Definition 5. Let (P,h) be an Hamiltonian system. The vector field X_h is *the Hamiltonian vector field of the system.* It defines a first order differential equation called *Hamilton's equation:*

$$\frac{dp}{dt} = X_h(p) = \left\{x^i(p), h\right\}\frac{\partial}{\partial x^i},$$

where $\left(x^1, \dots, x^n\right)$ is a system of local coordinates on P and $x^i(p)$ is the ith coordinate of $p \in P$.

A Hamiltonian system is used to describe the evolution of a physical system. The points of P represent the physical states of the system. In the physics terminology, P is called the *phase space*. The solutions of the system are given by the flow of X_h. Its trajectories describe how the physical system evolves along time.

Example 6. Consider one particle evolving in $\mathbb{R}$. The physical state of the particle is described by the its position q and its momentum p. The space of states is then $\mathbb{R}^2$ and we denote by (q,p) the system of canonical coordinates. We endow this space with the Poisson bracket of Example 3. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, then the equation of motions are

$$\dot{q} = \frac{\partial h}{\partial p} \text{ and } \dot{p} = -\frac{\partial h}{\partial q}.$$

A remark towards deformation quantization

Solving Hamilton's equations is equivalent to solving

$$\frac{df}{dt} = X_h(f) = \{f, h\}, \quad \text{for any } f \in \mathcal{C}^\infty(P).$$

Indeed, by applying this equation to $f = x^i$, where x^i is a local coordinate, we recover Hamilton's equations in coordinates. Conversely, let $(t, p) \to p(t)$ be the flow of X_h on P. The unique solution f^t of $\frac{df}{dt} = \{f, h\}$ starting at f is given by $f(p(t))$.

Hence, the solutions of a Hamiltonian system can be view in two equivalent ways : (i) the flow $(t, p) \to p(t)$ of X_h on the manifold P itself ; (ii) the flow $(t, f) \to f^t$ on the space $\mathcal{F}(P)$ of functions on P, satisfying $f^t(p) = f(p(t))$. The first point of view describes the evolution of the states of the system. The second describes the evolution of quantities (e.g. position, components of the angular momentum...). Deformation quantization uses the second point of view.

1.2.3 Arnold-Liouville integrable systems

1.2.3.1 Conserved quantities

In physics, a function $f \in \mathcal{C}^\infty(P)$ corresponds to a measurable physical quantity. For example, the Hamiltonian function h measures the energy of the Hamiltonian system and the points of $h^{-1}(E)$ are the physical states of energy E. The time evolution of a physical quantity f is given by

$$\frac{df}{dt} = \{f, h\}.$$

Definition 7. A *conserved quantity* of the Hamiltonian system (P, h) is a function $f \in \mathcal{F}(P)$ such that

$$\frac{df}{dt} = \{f, h\} = 0.$$

In other words, f is a conserved quantity if it is constant along the trajectories of X_h.

1.2.3.2 Integrable systems

Definition 8. A Hamiltonian system (P, h) on manifold P of dimension $2n$ with a non degenerate Poisson bracket is *integrable* if there exists n conserved quantities $h_1 := h, h_2, \ldots, h_n$ in involution

$$\{h_i, h_j\} = 0$$

such that the their differentials are linearly independent.

In this case, the following theorem shows that the equations of motion are particularly simple.

Proposition 9 (Arnold-Liouville, [Arn13]). *Suppose $(P, h = h_1, \ldots, h_n)$ is an integrable system. Fix $E = (E_1, \ldots, E_n) \in \mathbb{R}^n$. Suppose $M_E = \{x \in P | h_i(x) = E_i\}$ is compact and connected. Then M_E is diffeomorphic to the torus $T^n = \{(\phi_1, \ldots, \phi_n) \bmod 2\pi\}$. Moreover, M_E is invariant under the flow of X_h and the motion on M_E is given by*

$$\frac{d\phi_i}{dt} = \omega_i(E)$$

for some constants $\omega_i(E) \in \mathbb{R}$, with $1 \leq i \leq n$.

1.3 Integrable systems on the loop space and their tau functions

1.3.1 A formal poisson structure on the loop space

We define an algebra of formal functions and a Poisson structure used to give a formal Hamiltonian presentation of partial differential equations such as the Korteweg-de-Vries (KdV) equation

$$\frac{\partial u}{\partial t} = u\partial_x u + \frac{\epsilon^2}{12}\partial_x^3 u.$$

The intuitive idea behind the following construction is to describe such an equation as a vector field on the space P of loops $u : S^1 \to \mathbb{C}$. We suppose that these periodic maps have a Fourier transform $u(x) = \sum_{a\in\mathbb{Z}} p_a e^{iax}$, this gives a system of coordinates $\{p_a, a \in \mathbb{Z}\}$ on P. We define an algebra of power series in the coefficients p_a and interpret it as the algebra of functions on P. The loop space P is not properly defined but we only need an algebra of functions and a Poisson structure in order to decribe partial differential equation such as the KdV equation.

Definition 10. Let $\mathcal{F}(P)$ be the algebra $\mathbb{C}[p_{>0}][[p_{\leq 0}, \epsilon]]$, where the indeterminates $p_{>0}$ (resp. $p_{\leq 0}$) stands for p_a, with $a \in \mathbb{Z}_{>0}$ (resp. $a \in \mathbb{Z}_{\leq 0}$).

The formal parameter ϵ is not necessary to write the equations, however it will be convenient in the study of tau functions.

As in the finite dimensional case, a Poisson structure on $\mathcal{F}(P)$ is a Lie bracket on $\mathcal{F}(P)$ that satisfies the Leibniz rule.

Definition 11. A *Poisson structure on* $\mathcal{F}(P)$ is given by

$$\{p_a, p_b\} = ia\delta_{a+b,0},$$

and we extend it to $\mathcal{F}(P)$ by the Leibniz rule.

Let $f, g \in \mathcal{F}(P)$, we obtain the following expression for the Poisson bracket

$$\{f, g\} = \sum_{a\in\mathbb{Z}} ia \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial p_{-a}}.$$

Let $f \in \mathcal{F}(P)$, the Hamilton equation associated to f reads in coordinates $\frac{\partial p_a}{\partial t} = \{p_a, f\}$,where $a \in \mathbb{Z}$. To obtain the Hamilton equation on $u(x) = \sum_{a\in\mathbb{Z}} p_a e^{iax}$, we combine these equations

$$\frac{\partial u}{\partial t} = \{p_a, f\} e^{iax}.$$

In order to write the Hamilton equation as a polynomial in u and its derivatives, as for the KdV equation, we have to make a particular choice of Hamiltonians. These Hamiltonians are expressed in term of differential polynomial.

1.3.2 Differential polynomials

We give two equivalent definitions of differential polynomials and explain how to identify them.

Notation 12. From now on, we adopt the notation $u_s = \partial_x^s u$ with $s \geq 0$

Definition 13. A *differential polynomial* is an element of $\mathcal{A} := \mathbb{C}[u_0, u_1, \dots][[\epsilon]]$.

Definition 14. Let d be a positive integer. Let $(\phi_0, \dots, \phi_d)$ be a list where $\phi_k(a_1, \dots, a_k) \in \mathbb{C}[a_1, \dots, a_k][[\epsilon]]$ is a symmetric polynomial in its k indeterminates $a_1, \dots, a_k$ for $0 \leq k \leq d$. The *formal Fourier series associated to* $(\phi_0, \dots, \phi_d)$ is

$$\phi(x) = \sum_{A\in\mathbb{Z}} \left(\sum_{k\geq 0}^{d} \sum_{\substack{a_1,\dots,a_k\in\mathbb{Z} \\ \sum a_i = A}} \phi_k(a_1, \dots, a_k) p_{a_1} \dots p_{a_k} \right) e^{ixA} \in \mathcal{F}(P)\left[\left[e^{-ix}, e^{ix}\right]\right].$$

The set of formal Fourier series associated to any $d \in \mathbb{N}$ and any $(\phi_0, \dots, \phi_d)$ is an algebra that we denote by $\tilde{\mathcal{A}}$.

Lemma 15. *The algebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are isomorphic.*

Proof. By substituting the formal Fourier series $u_s(x) = \sum_{a\in\mathbb{Z}} (ia)^s p_a e^{iax}$, with $s \in \mathbb{N}$, of u and its derivatives in a differential polynomial, we obtain an element of $\tilde{\mathcal{A}}$. By this application, the differential monomial $u_{s_1} \dots u_{s_n}$ yields the formal Fourier series associated to $\phi_n(a_1, \dots, a_n) = \frac{1}{n!} \sum_{\sigma\in S_n} a_{\sigma(1)}^{s_1} \dots a_{\sigma(n)}^{s_n}$ and $\phi_i = 0$ if $i \neq n$. □

The element of $\mathcal{A}$ and of $\tilde{\mathcal{A}}$ will be called differential polynomials. When precision will be needed, we will refer to the elements of $\tilde{\mathcal{A}}$ as Fourier series associated to a differential polynomial. In the rest of the text, we will mainly use the notation $\mathcal{A}$ to refer to differential polynomials and Fourier transform of differential polynomials.

Definition 16. The derivative ∂_x of a differential polynomial ϕ is the differential polynomial obtained by multiplying the A-th mode of ϕ by A.

The integration along S^1 of a differential polynomial ϕ is the 0-th mode of ϕ. We denote by $\int_{S^1} \phi(x)\, dx = \overline{\phi}$ this integral.

The *primitive of a differential polynomial* ϕ is a differential polynomial ψ such that $\partial_x \psi = \phi$.

Poisson structure and differential polynomials

We deduce from the definition of the Poisson structure the following lemma.

Lemma 17. *Let $\phi, \psi \in \mathcal{A}$, then*

$$\left\{\phi, \overline{\psi}\right\} \in \mathcal{A}.$$

Moreover, the constant term vanishes

$$\left\{\phi, \overline{\psi}\right\}\Big|_{p_*=0} = 0.$$

Remark 18. The Poisson bracket of differential polynomials has the following expression in the variables u:

$$\left\{\phi, \overline{\psi}\right\} = \sum_{r,s\geq 0} (-1)^s \frac{\partial \phi}{\partial u_r} \partial_x^{r+s+1} \frac{\partial \overline{\psi}}{\partial u_s},$$

where $\partial_x = \sum_{i\geq 0} u_{i+1} \frac{\partial}{\partial u_i}$. The classical integrable hierarchies are mostly presented in the variables u. However, the variables p offer a direct way to quantize such systems.

1.3.3 The Hamilton equation

From now on, we will only use Hamiltonians that are obtained by integration of a differential polynomial.

Definition 19. Let $h \in \mathcal{A}$. The *Hamilton equation* associated to the Hamiltonian $\overline{h} \in \mathcal{F}(P)$ is

$$\frac{\partial u}{\partial t} = \sum_{a\in\mathbb{Z}} \left\{p_a, \overline{h}\right\} e^{iax}.$$

The differential polynomial h is called *Hamiltonian density.*

Lemma 17 ensures that the RHS of Hamilton's equation is a differential polynomial.

Example 20. Consider the Hamiltonian density $h = \frac{u_0^3}{3!} + \epsilon^2 \left(\frac{u_0 u_2}{12} + \frac{u_1^2}{24}\right) + \epsilon^4 \frac{u_4}{240}$. The associated Hamiltonian is $\overline{h} = \frac{1}{3!} \sum_{a+b+c=0} p_a p_b p_c + \frac{\epsilon^2}{24} \sum_{a\in\mathbb{Z}} (ia)^2 p_a p_{-a}$ and the Hamilton equation is the KdV equation

$$\frac{\partial u}{\partial t} = \sum_{a\in\mathbb{Z}} \left\{p_a, \overline{h}\right\} e^{iax} = u u_1 + \frac{\epsilon^2}{12} u_3.$$

1.3.4 Integrable hierarchies and solutions

In infinite dimension, there is no exact analog of Arnold-Liouville's theorem. In particular, there are different notions of integrability.

Definition 21. An *integrable hierarchy in* $(\mathcal{F}(P), \{\cdot,\cdot\})$ is an infinite collection of Hamiltonians densities $\{h_i \in \mathcal{A},\, i \geq 0\}$ satisfying

$$\left\{\overline{h}_i, \overline{h}_j\right\} = 0.$$

The corresponding equations of the hierarchy are

$$\frac{\partial u}{\partial t_i} = \left\{u, \overline{h}_i\right\}, \quad \text{for any i} \geq 0.$$

According to the commutativity of the Hamiltonians, we can solve these equations simultaneously.

Example 22. The KdV hierarchy is an integrable hierarchy containing the KdV equation. The first Hamiltonian densities of the KdV hierarchy are

$$\begin{aligned}
h_0^{KdV} &= \frac{u_0^2}{2!} + \epsilon^2 \frac{u_2}{12}, \\
h_1^{KdV} &= \frac{u_0^3}{3!} + \epsilon^2 \left(\frac{u_0 u_2}{12} + \frac{u_1^2}{24}\right) + \epsilon^4 \frac{u_4}{240}, \\
h_2^{KdV} &= \frac{u_0^4}{4!} + \epsilon^2 \left(\frac{u_0^2 u_2 + u_0 u_1^2}{24}\right) + \epsilon^4 \left(\frac{u_0 u_4}{240} + \frac{u_2^2}{160} + \frac{u_1 u_3}{120}\right) + \epsilon^6 \frac{u_6}{6720}.
\end{aligned}$$

In Chapter 3, we will present a formula for these Hamiltonian densities and a recursive way to construct them from $\overline{h}_1^{KdV}$.

From these Hamiltonian densities, we obtain the first equations of the KdV hierarchy:

$$\begin{aligned}
u_{t_0} &= u_1, \\
u_{t_1} &= u_0 u_1 + \epsilon^2 \frac{u_3}{12}, \\
u_{t_2} &= \frac{u_0^2 u_1}{2} + \epsilon^2 \left(\frac{u_0 u_3}{12} + \frac{u_1 u_2}{6}\right) + \epsilon^4 \frac{u_5}{240}.
\end{aligned}$$

Remark 23. The algebra of functions $\mathcal{F}(P)$ and the Poisson structure we defined are sufficient to give a Hamiltonian presentation of integrable hierarchies such as the KdV hierarchy. However, there exists a more general definition of the space of functions and various Poisson structures on it, see [DZ05]. These structures are used to define two families of integrable hierachies: the Dubrovin-Zhang hierarchies [BPS12a, BPS$^+$12b, DZ05] and the double ramification hierarchies [Bur15]. The KdV hierarchy is a particular element of each of these two families. More generally, the DR/DZ conjecture [BGR19] asserts that any Dubrovin-Zhang hierarchy is a double ramification hierarchy up to a change of coordinates.

Definition 24. A *solution* of the integrable hierarchy $\{h_i \in \mathcal{A},\ i \geq 0\}$ with initial condition $u^{\text{initial}} \in \mathbb{C}[[x]]$ is an element $u(x, \boldsymbol{t}) \in \mathbb{C}[[\epsilon, x, t_0, t_1, \dots]]$ satisfying the equations of the hierarchy.

We think of $u^{\text{initial}} \in \mathbb{C}[[x]]$ as an element of P. We will only consider solutions with initial conditions in $\mathbb{C}[[x]]$.

Lemma 25. *The unique solution of the integrable hierarchy $\{h_i \in \mathcal{A},\ i \geq 0\}$ with initial condition $u^{\text{initial}} \in \mathbb{C}[[x]]$ is given by*

$$u(x, \boldsymbol{t}) = \exp\left(\sum_{i \geq 0} t_i \left\{\cdot, \overline{h}_i\right\}\right) u \Bigg|_{u = u^{\text{initial}}}. \tag{1.3.1}$$

By this notation, we mean that we evaluate $\exp\left(\sum_{i\geq 0} t_i \left\{\cdot, \overline{h}_i\right\}\right) u \in \mathcal{A}[[\epsilon, t_0, t_1, \dots]]$ *at* $u_k(x) = \partial_x^{(k)} u^{\text{initial}}(x)$.

Proof. It is clear that Eq (1.3.1) is a solution of the hierarchy. The ith equation of the hierarchy is a first order derivative in t_i. Since a solution is a power series, we can construct the solutions by solving degree by degree the equations of the hierarchy in a unique way. $\square$

We are interested in formal solutions even if they do not converge. For example, the coefficients of the following formal solution of KdV are related with intersection numbers on the moduli space of curves. This is the content of the famous Witten conjecture (see Section 2.2).

Example 26 (The string solution)**.** The first terms of the solution of the KdV hierarchy starting at $u(x) = x$ are given by

$$\begin{aligned} u(x, \boldsymbol{t}) &= (x + t_0) + \epsilon^2 \frac{t_3}{24} + \dots \\ &\quad + (x + t_0) t_1 + \frac{(x + t_0) t_2}{2} + \epsilon^2 \left(\frac{(x + t_0) t_4}{24} + \frac{t_1 t_3}{8} + \frac{t_2^2}{12} \right) + \dots \\ &\quad + \dots \end{aligned}$$

This particular solution is called the *string solution* of the KdV hierarchy. This terminology will be explained in Section 2.2.

A remark towards deformation quantization.

The equations describing the evolution of elements of $\mathcal{F}(P)$ are given by

$$\frac{df}{dt_i} = \left\{ f, \overline{h}_i \right\}, \text{ with } i \geq 0.$$

Definition 27. A *solution* of these equations is an element of $\mathcal{F}(P)[[t_0, t_1, \dots]]$ satisfying the equations.

The unique solution starting at $f^{\text{initial}} \in \mathcal{F}(P)$ is given by

$$f^{\boldsymbol{t}} = \exp\left(\sum_{i \geq 0} t_i \left\{ \cdot, \overline{h}_i \right\} \right) f^{\text{initial}} \in \mathcal{F}(P)[[t_0, t_1, \dots]].$$

It follow from the Leibniz property of the Poisson bracket that

$$f^{\boldsymbol{t}} \left(u^{\text{initial}}(x) \right) = f(u(x, \boldsymbol{t})), \tag{1.3.2}$$

where $(x, \boldsymbol{t}) \to u(x, \boldsymbol{t})$ is the solution of the hierarchy starting at u^{initial}. In particular, as in the finite dimensional case, the solutions of an integrable system are equivalently given by a flow on P and a flow on $\mathcal{F}(P)$.

1.3.5 Tau functions

Tau functions were first introduced by the Kyoto school in their study of the Kadomtsev-Petviashvili (KP) hierarchy, see [DJKM82, Sat83] and [MJJD00] for a more recent introduction to the subject. There are now various definitions of tau functions depending on the notion of integrability we consider. We present in this section the definition of tau functions given by Dubrovin and Zhang [DZ05] in the context of Hamiltonian integrable systems.

Tau functions turn out to be key objects at the interplay between integrable systems, the cohomology of $\overline{\mathcal{M}}_{g,n}$ and Hurwitz theory. In this thesis, we will be particularly interested in these relations, example will be given in the next sections.

Definition 28. A *tau structure* is a collection of Hamiltonian densities $\{h_i \in \mathcal{A},\, i \geq -1\}$ such that

1. the Hamiltonians commute
$$\left\{\overline{h}_i, \overline{h}_j\right\} = 0, \text{ for any } i, j \geq -1,$$
2. the Hamiltonian densities are *tau symmetric*
$$\left\{h_{i-1}, \overline{h}_j\right\} = \left\{h_{j-1}, \overline{h}_i\right\}, \text{ for any } i, j \geq 0,$$
3. the evolution given by the Hamiltonian $\overline{h}_{-1}$ is trivial
$$\frac{\partial f}{\partial t_{-1}} = \left\{f, \overline{h}_{-1}\right\} = 0, \text{ for any } f \in \mathcal{F}(P),$$
4. the Hamiltonian $\overline{h}_0$ generates the translations in space
$$\frac{\partial \phi}{\partial t_0} = \left\{\phi, \overline{h}_0\right\} = \partial_x \phi \text{ for any } \phi \in \mathcal{A}.$$

Let $\{h_i \in \mathcal{A},\, i \geq -1\}$ be a tau structure. We present the construction of the tau functions associated to the hierarchy $\left\{\overline{h}_i,\, i \geq -1\right\}$. Note that the third condition of the tau structure ensures that the Hamiltonian $\overline{h}_{-1}$ does not play any role in the hierarchy.

First, there exists a unique differential polynomial $\Omega_{i,j}$, for any $i, j \geq 0$, such that

$$\partial_x \Omega_{i,j} = \left\{h_{i-1}, \overline{h}_j\right\}$$

and $\Omega_{i,j}\big|_{p_*=0} = 0$. This follows from the commutations of the Hamiltonians and Lemma 17. We then define the time dependent two-point function

$$\Omega^{\boldsymbol{t}}_{i,j} := \exp\left(\sum_{k\geq 0} t_k \left\{\cdot, \overline{h}_k\right\}\right) \Omega_{i,j} \in \mathcal{A}\left[\left[\epsilon, t_0, t_1, \ldots\right]\right].$$

Lemma 29. *Let $(x,\boldsymbol{t}) \to u(x,\boldsymbol{t})$ be the solution of the integrable hierarchy with initial condition $u^{\mathrm{initial}} \in \mathbb{C}[[x]]$. There exists a series $\mathscr{F} \in \mathbb{C}[[\epsilon, t_0, t_1, \dots]]$ uniquely defined up to the choice of the constant and linear terms in t_* such that*

$$\frac{\partial^2 \mathscr{F}}{\partial t_i \partial t_j} = \Omega^{\boldsymbol{t}}_{i,j}\Big(u^{initial}(x=0)\Big).$$

The notation $\Omega^{\boldsymbol{t}}_{i,j}\Big(u^{initial}(x=0)\Big)$ means that we substitute $u_k := \partial_x^{(k)} u^{\mathrm{initial}}(x=0)$ in the differential polynomial depending of the times $\Omega^{\boldsymbol{t}}_{i,j}$.

Definition 30. The function $\mathscr{F}$ is the *logarithm of the tau function* ($\log\tau$ function) associated to the solution $(x,\boldsymbol{t}) \to u(x,\boldsymbol{t})$.

The $\log\tau$ function associated to a given solution is not completely determined since we have to specify the constant and linear terms in t_* of $\mathscr{F}$. We usually fix these terms by imposing additional equations on $\mathscr{F}$ depending on the context. Once this choice is made, the exponent of the $\log\tau$ function is the tau function. However, we will only be interested in the $\log\tau$ function.

Proof. We use twice the Poincaré lemma.

1. The 1-forms
$$\alpha_i := \sum_{i\geq 0} \Omega^{\boldsymbol{t}}_{i,j} dt^j$$
are closed. Indeed the commutativity of the Hamiltonians implies that $\partial_x \left\{\Omega_{i,j}, \overline{h}_k\right\} = \partial_x \left\{\Omega_{i,k}, \overline{h}_j\right\}$. Moreover the differential polynomials $\left\{\Omega_{i,j}, \overline{h}_k\right\}$ and $\left\{\Omega_{i,k}, \overline{h}_j\right\}$ have no constant terms according to Lemma 17. We deduce that $\left\{\Omega_{i,j}, \overline{h}_k\right\} = \left\{\Omega_{i,k}, \overline{h}_j\right\}$ and then
$$\frac{\partial \Omega^{\boldsymbol{t}}_{i,j}}{\partial t_k} = \left\{\Omega^{\boldsymbol{t}}_{i,j}, \overline{h}_k\right\} = \left\{\Omega^{\boldsymbol{t}}_{i,k}, \overline{h}_j\right\} = \frac{\partial \Omega^{\boldsymbol{t}}_{i,k}}{\partial t_j},$$
that is the 1-forms are closed. Thus, there exist $f_i \in \mathcal{A}[[\epsilon, t_0, t_1, \dots]]$, with $i \geq 0$, such that $\frac{\partial f_j}{\partial t_j} = \Omega^{\boldsymbol{t}}_{i,j}$.

2. The 1-form
$$\sum_{i\geq 0} f_i t^i$$
is closed. Indeed, the tau symmetry ensures that $\Omega_{i,j} = \Omega_{j,i}$ so that $\frac{\partial f_i}{\partial t_j} = \Omega^{\boldsymbol{t}}_{i,j} = \Omega^{\boldsymbol{t}}_{j,i} = \frac{\partial f_j}{\partial t_i}$. Thus, there exists a series $\tilde{\mathscr{F}} \in \mathcal{A}[[\epsilon, t_0, t_1, \dots]]$ such that $\frac{\partial^2 \tilde{\mathscr{F}}}{\partial t_i \partial t_j} = \Omega^{\boldsymbol{t}}_{i,j}$.

We obtain the series $\mathscr{F} \in \mathbb{C}[[\epsilon, t_0, t_1, \dots]]$ by the evaluation of $\tilde{\mathscr{F}}$ at the point $u^{\mathrm{inital}} \in \mathbb{C}[[x]]$ and then substituting $x = 0$,

$$\mathscr{F} := \tilde{\mathscr{F}}\Big(u^{initial}(x=0)\Big).$$

$\square$

Remark 31. We deduce from Eq. (1.3.2) the equality $\Omega_{i,j}^{\boldsymbol{t}}\Big(u^{initial}\left(x\right)\Big) = \Omega_{i,j}\Big(u\left(x,\boldsymbol{t}\right)\Big)$, where $(x,\boldsymbol{t}) \to u\left(x,\boldsymbol{t}\right)$ is the solution starting at u^{initial}. Thus, we could define the $\log\tau$ function by

$$\frac{\partial^2 \mathscr{F}}{\partial t_i \partial t_j} = \Omega_{i,j}\Big(u\left(x=0,\boldsymbol{t}\right)\Big),$$

that is by evaluating $\Omega_{i,j}$ at the solution $(x,\boldsymbol{t}) \to u\left(x,\boldsymbol{t}\right)$ and then substituting $x=0$. This was the original definition of the $\log\tau$ function associated to the solution $(x,\boldsymbol{t}) \to u\left(x,\boldsymbol{t}\right)$ (see [BDGR18, DZ05]). However, the generalization of the definition of $\log\tau$ functions to quantum integrable systems is more straightforward from our presentation.

Remark 32. The two-point functions with a 0-insertion are closely related to the Hamiltonian densities. Indeed, according to the fourth condition of the tau structure, we have $\partial_x\Omega_{i,0} = \left\{h_{i-1}, \overline{h}_0\right\} = \partial_x h_{i-1}$ so that

$$\Omega_{i,0} = h_{i-1} + C, \ \ \text{where}\, C \in \mathbb{C}.$$

For example the Hamiltonian densities of the KdV hierarchy (and, more generally, the Hamiltonian densities of the double ramification hierarchies, see [BDGR18]) have no constant term in u_* so that the constants vanish.

Remark 33. One may wonder why we forget about the x dependency in the definition of the $\log\tau$ function. Let $\phi^{\boldsymbol{t}}$ be the time-dependent differential polynomial with initial condition given by the differential polynomial ϕ. The fourth condition of the tau structure ensures that $\frac{\partial\phi^{\boldsymbol{t}}}{\partial t_0} = \partial_x\phi^{\boldsymbol{t}}$, hence

$$\phi^{\boldsymbol{t}}\left(x\right) = \exp\left(\sum_{k\geq 0} t_k\left\{\cdot,\overline{H}_k\right\} + x\left\{\cdot,\overline{H}_0\right\}\right)\phi\left(0\right).$$

In particular, the evolutions along x and t_0 are the same. We can then recover the x dependency in the $\log\tau$ function by substituting $t_0 := t_0 + x$. It is then confusing since $x\in S^1$. Once again, the space P is not rigorously defined and only serves as a motivation for the definition of $\mathcal{F}\left(P\right)$, we chose to take $x\in S^1$ in order to make the Fourier transform and the x-integration more natural.

Remark 34. The Hamiltonian densities of the KdV hierarchy together with $h_{-1}^{KdV} = u_0$ form a tau structure (see Section 3.1.2). Let $\mathscr{F}$ be the $\log\tau$ function of KdV associated to the solution $(x,\boldsymbol{t}) \to u\left(x,\boldsymbol{t}\right)$. We obtain this solution from $\mathscr{F}$ by

$$\frac{\partial^2\mathscr{F}}{\partial t_0\partial t_0} = \left(h_{-1}^{KdV}\right)^{\boldsymbol{t}}\left(u^{\text{initial}}\left(x=0\right)\right) = u\left(x=0,\boldsymbol{t}\right).$$

The 0th KdV equation reads $\partial_{t_0} = \partial_x$, thus we can replace ∂_x by ∂_{t_0} in every equation of KdV. We deduce that $\boldsymbol{t} \to \frac{\partial^2\mathscr{F}}{\partial t_0\partial t_0}$ satisfies every equation of the KdV hierarchy except the 0th.

This relation between $\log\tau$ functions and solutions of the hierarchy is quite general, for example it is satisfied by any $\log\tau$ function of the double ramification hierarchy (see [BDGR18]).

Example 35. The $\log\tau$ function $\mathscr{F}^{WK}$ of the KdV hierarchy associated to the string solution is the *Witten-Kontsevich series*. Its first terms are

$$\mathscr{F}^{WK} = \frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \epsilon^2 \frac{t_1}{24} + \epsilon^2 \frac{t_0 t_2}{24} + \epsilon^2 \frac{t_1^2}{24} + \epsilon^2 \frac{t_0^2 t_3}{48} + \epsilon^2 \frac{t_0^2 t_1 t_3}{16} + \dots$$

Dubrovin and Zhang gave a construction of the Hamiltonian densities of the KdV hierarchy from the only input of this power series (see [DZ05, BPS12a, BPS$^+$12b]).

1.4 Quantum integrable systems and their tau functions

By a deformation quantization procedure we define quantum integrable hierarchies and quantum tau functions following [BDGR16].

1.4.1 Idea of deformation quantization

Start with a Hamiltonian system: the space of physical states is given by a manifold P, the physical quantities are given by the commutative algebra of functions $\mathcal{F}(P)$ which is endowed with a Poisson structure, and there is a particular function h giving the equations of motion

$$\frac{df}{dt} = \{f, h\}.$$

In quantum mechanics, a physical system is described by a Hilbert space $\mathcal{H}$ and the physical states of the system are the vectors of norm 1 in $\mathcal{H}$. The physical quantities are related to a non commutative algebra of operators from $\mathcal{H}$ to $\mathcal{H}$ called observables. The choice of a particular observable H yields the evolution of the system. There are two "pictures" giving equivalent formulations of the evolution of the system. In the Schrödinger picture, the equations of motion give the evolution of the physical states on the Hilbert space. In the Heisenberg picture, the physical states do not evolve but the physical quantities (i.e. the observables) do. In our context of deformation quantization, the evolution of the system is given in the Heisenberg picture. The evolution of an observable $\mathcal{O}_t$ is given by

$$\frac{d\mathcal{O}_t}{dt} = \frac{1}{\hbar}\left[\mathcal{O}_t, H\right],$$

where $[\cdot,\cdot]$ is the commutator of the operators. The evolution of $\mathcal{O}_t$ is then given by $\mathcal{O}_t = \exp\left(\frac{t}{\hbar}\left[\cdot, H\right]\right)\mathcal{O}_{t=0}$.

A quantum system is a more accurate version of its classical counterpart. Moreover, the physics that these systems describe should be the same at a "classical scale". This scale is parametrized by $\hbar$. At the classical limit $\hbar \to 0$, a quantum system should look like its classical counterpart: we recover a Poisson manifold and the Hamilton equations. Of course there are many quantum systems corresponding to the same classical system.

An approach to quantize is to find a map $\rho : f \to \rho(f)$ which maps an element of $\mathcal{F}(P)$ to an operator acting on some Hilbert space. In order to recover the Poisson structure at the classical limit, we add the constraint

$$\left[\rho(f), \rho(g)\right] = \hbar\rho\left(\{f,g\}\right) + O\left(\hbar^2\right).$$

If such map is invertible, we can pull-back the the non commutative product on $\mathcal{F}(P)$, this defines a star product

$$f \star g = \rho^{-1}\left(\rho(f)\,\rho(g)\right).$$

This star product is then sufficient to define the dynamics of the observables.

Motivated by this approach, the idea of deformation by quantization is to define a star product on $\mathcal{F}(P)[[\hbar]]$ as power series in $\hbar$

$$f \star g = \sum C_n(f,g)\,\hbar^n$$

such that

- the star product is a deformation of the usual product

$$f \star g = fg + O(\hbar),$$

- we recover the Poisson bracket at the classical limit

$$[f, g] = \hbar \{f, g\} + O\left(\hbar^2\right),$$

- the star product is associative.

1.4.2 A star product on $\mathcal{F}(P)$

Definition 36. Let R be a ring and A a commutative associative R-algebra with a unit. A *formal deformation* of A is an associative algebra $(A[[\hbar]], \star)$ over $R[[\hbar]]$ such that

$$f \star g = fg + \sum_{n \geq 1} C_n(f, g)\, \hbar^n,$$

where $f, g \in A[[\hbar]]$ and $C_n(\cdot, \cdot) : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a bilinear map for every $n \geq 1$. Moreover, we require that the unit $1 \in A$ remains the unit element : $1 \star f = f \star 1 = f$.

Definition 37. Let A be a commutative associative algebra with a unit and endowed with a Poisson structure $\{\cdot, \cdot\}$. A *star product* $\star$ on $A[[\hbar]]$ is such that $(A[[\hbar]], \star)$ is a formal deformation of A satisfying

$$f \star g - g \star f = \hbar \{f, g\} + O\left(\hbar^2\right).$$

We say that the formal deformation is in the direction of the poisson bracket.

The following example describes a finite dimensional version of the star product we will define on $\mathcal{F}(P)$.

Example 38. Let the algebra $\mathbb{C}[q, p]$ endowed with the Poisson structure $\{f, g\} = \frac{1}{i}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right)$. A star product on $\mathbb{C}[q, p]$ is given by

$$f \star g = \sum_{k \geq 0} \frac{(i\hbar)^k}{k!} \frac{\partial^k f}{\partial q^k} \frac{\partial^k g}{\partial p^k} = f \exp\left(\frac{\hbar}{i} \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q}\right) g, \tag{1.4.1}$$

where $f, g \in \mathbb{C}[q, p]$.

We can realize this star product in the following way. Let $\mathcal{H} = \{f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})\, |\mathrm{supp}(f) \text{ is compact}\}$ be the Hilbert space endowed with the hermitian product

$$\langle f, g \rangle := \int_{\mathbb{R}} \overline{f(q)} g(q)\, dq.$$

Let $\mathrm{Diffop}_{poly} = \left\{ \sum_{k=0}^{N} f_k(q) \frac{\partial^k}{\partial q^k},\ f_k(q) \in \mathbb{C}[q] \right\}$ be the space of observables. The star product can be realized via

$$\begin{aligned} \rho : \mathbb{C}[q,p] &\to \mathrm{Diffop}_{poly} \\ q^n p^m &\to q^n \left(i\hbar \frac{\partial}{\partial q} \right)^m . \end{aligned}$$

The map ρ sends q on the operator of multiplication by q and p on the operator $i\hbar\frac{\partial}{\partial q}$ so that $[\rho(p), \rho(q)] = \hbar\rho(\{p,q\}) = i\hbar$. We also deduce that $[\rho(f), \rho(g)] = \hbar\rho(\{f,g\}) + O(\hbar^2)$. The choice of defining $\rho(q^n p^m)$ by the product $\rho(q^n)\rho(p^m)$ i.e. with $\rho(q^n)$ on the left and $\rho(p^m)$ on the right is called a normal ordering.

The application ρ is invertible, we then define the star product on $\mathbb{C}[q,p]$ by

$$f \star g := \rho^{-1}(\rho(f)\rho(g)).$$

That is we organize the product $\rho(f)\rho(g)$ with $\rho(q)$ on the left and $\rho(p)$ on the right using the commutation relation $[\rho(p), \rho(q)] = i\hbar$. This organization is a normal ordering of a unique element of $\mathbb{C}[q,p]$. This element is the star product $f \star g$. One can easily check that Eq (1.4.1) is an explicit expression of this star product.

The star product on $\mathcal{F}(P)$ is defined using the same idea. Let $f, g \in \mathcal{F}(P)[[\hbar]]$ organized with the $p_{<0}$ on the left. Start with the concatenation fg, we commute the $p_{<0}$ of g with the $p_{>0}$ of f using the commutation relation

$$[p_a, p_b] = i\hbar a \delta_{a+b,0}$$

and find an element of $\mathcal{F}(P)[[\hbar]]$ with the $p_{<0}$ on the left, this is the star product. Note that this process is well defined thanks to the polynomiality in the $p_{>0}$ of f.

Explicitly, the star product on $\mathcal{F}(P)$ have the following form.

Lemma 39. *Let $f, g \in \mathcal{F}(P)[[\hbar]]$. The star product on $\mathcal{F}(P)$ is*

$$f \star g = f \exp\left(\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}} \right) g, \tag{1.4.2}$$

where $f \exp\left(\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}} \right) g = fg + i\hbar \sum_{k>0} \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial p_{-k}} + (i\hbar)^2 \sum_{k_1,k_2>0} \frac{\partial^2 f}{\partial p_{k_1} \partial p_{k_2}} \frac{\partial^2 g}{\partial p_{-k_1} \partial p_{-k_2}} + \dots$

Definition 40. We denote by $\mathcal{F}^\hbar(P)$ the deformed algebra obtained by endowing $\mathcal{F}(P)[[\hbar]]$ with this star product.

1.4.3 Quantum differential polynomials

We extend the algebra of differential polynomials to the algebra of *quantum differential polynomials* $\mathcal{A}^\hbar := \mathcal{A}[[\hbar]]$. However, we will also call differential polynomial the elements of $\mathcal{A}^\hbar$ to simplify the reading.

Let $\tilde{\mathcal{A}}^{\hbar}$ be the algebra of formal Fourier series

$$\phi(x) = \sum_{A\in\mathbb{Z}} \left(\sum_{k\geq 0}^{d} \sum_{\substack{a_1,\dots,a_k\in\mathbb{Z} \\ \sum a_i = A}} \phi_k(a_1,\dots,a_k)\, p_{a_1}\dots p_{a_k} \right) e^{ixA} \in \mathcal{F}^{\hbar}(P)\left[\left[e^{-ix}, e^{ix}\right]\right],$$

where $\phi_k(a_1,\dots,a_k) \in \mathbb{C}[a_1,\dots,a_k][[\epsilon,\hbar]]$ and is symmetric in the indeterminates $a_1,\dots,a_k$.

The algebra $\mathcal{A}^{\hbar}$ and $\tilde{\mathcal{A}}^{\hbar}$ are isomorphic. The derivative with respect to x, the integration along S^1 and the primitive of a differential polynomial are defined similarly to Definition 16.

Quantum differential polynomials, commutator and Ehrhart polynomials

The commutator of differential polynomials is expressed in term of the following Ehrhart polynomials studied by Buryak and Rossi.

Lemma 41 ([BR16])**.** *Let $r_1,\dots,r_q$ be nonnegative integers. The quantity*

$$C^{r_1,\dots,r_q}(N) = \sum_{k_1+\dots+k_q=N} k_1^{r_1}\dots k_q^{r_q}$$

is a polynomial in the indeterminate N of degree $n-1+\sum_{i=1}^{n} r_i$. Moreover, if $r_1,\dots,r_q \geq 1$ this polynomial has the parity of $n-1+\sum_{i=1}^{n} r_i$.

We explain how these coefficients appear in the commutator of the star product. Consider two differential polynomials

$$\phi(x) = \sum_{m\geq 0} \sum_{a_1,\dots,a_m\in\mathbb{Z}} \phi_m(a_1,\dots,a_m)\, p_{a_1}\dots p_{a_m} e^{i\sum a_j x}$$

and

$$\psi(y) = \sum_{n\geq 0} \sum_{b_1,\dots,b_n\in\mathbb{Z}} \psi_n(b_1,\dots,b_n)\, p_{b_1}\dots p_{b_m} e^{i\sum b_j y},$$

where ϕ_m and ψ_n are symmetric polynomials. We use the expression of the star product given by Eq. (1.4.2) to get

$$\phi(x) \star \psi(y) = \sum_{q\geq 0} \frac{(i\hbar)^q}{q!} \sum_{k_1,\dots,k_q>0} k_1\dots k_q \frac{\partial^q \phi(x)}{\partial p_{k_1}\dots\partial p_{k_q}} \frac{\partial^q \psi(x)}{\partial p_{-k_1}\dots\partial p_{-k_q}}.$$

We then obtain

$$\begin{aligned}
\left[\phi\left(x\right),\overline{\psi}\right] &= \phi\left(x\right)\star\overline{\psi}-\overline{\psi}\star\phi\left(x\right)\\
&= \sum_{q\geq 0}\sum_{\tilde{m}\geq 0}\sum_{\tilde{n}\geq 0}\sum_{a_1,\dots,a_{\tilde{m}}\in\mathbb{Z}}\sum_{b_1,\dots,b_{\tilde{n}}\in\mathbb{Z}}\frac{(\tilde{m}+q)!\,(\tilde{n}+q)!}{\tilde{m}!\tilde{n}!q!}\\
&\times p_{a_1}\dots p_{a_{\tilde{m}}}p_{b_1}\dots p_{b_{\tilde{n}}}e^{i\left(x\sum_{j=1}^{\tilde{m}}a_j+\sum_{l=1}^{\tilde{n}}b_l\right)}\\
&\times\left(\sum_{\substack{k_1,\dots,k_q>0\\ k_1+\dots+k_q=B}}k_1\dots k_q\phi_m\left(a_1,\dots,a_{\tilde{m}},k_1,\dots,k_q\right)\psi_n\left(b_1,\dots,b_{\tilde{n}},-k_1,\dots,-k_q\right)\right.\\
&\left.-\sum_{\substack{k_1,\dots,k_q>0\\ k_1+\dots+k_q=-B}}k_1\dots k_q\phi_m\left(a_1,\dots,a_{\tilde{m}},-k_1,\dots,-k_q\right)\psi_n\left(b_1,\dots,b_{\tilde{n}},k_1,\dots,k_q\right)\right)
\end{aligned}$$

where $\tilde{m}=m-q$, $\tilde{n}=n-q$ and $B=\sum_{i=1}^{\tilde{n}}b_i$. Note that the condition imposed by the integration with respect to y in ψ translates to the conditions $k_1+\dots+k_q=B$ and $k_1+\dots+k_q=-B$. Since ϕ_m and ψ_n are polynomials, the polynomials of Buryak and Rossi appear in the first line and in the second line of the expression in parenthesis. Hence, the first (resp. second) line is a polynomial in the a_i's and the b_j's defined for $B>0$ (resp. $B<0$). One can check using the parity property of Buryak and Rossi polynomials that these two piecewise polynomials combine in one polynomial in the indeterminates a_i's and b_j's. We deduce the following lemma.

Lemma 42. *Let $\phi,\psi\in\mathcal{A}^{\hbar}$. We have*

$$\left[\phi,\overline{\psi}\right]\in\mathcal{A}^{\hbar}.$$

and this differential polynomial has no constant term

$$\left[\phi,\overline{\psi}\right]\Big|_{p_*=0}=0.$$

The second point of the lemma, i.e. $\left[\phi,\overline{\psi}\right]$ has no constant term, follows from the fact that Buryak and Rossi polynomials has no constant term, which is clear from the definition.

Remark 43. In [BR16], Buryak and Rossi gave an explicit expression of this commutator in term of the variables u, its derivatives and the polynomials $C^{r_1,\dots,r_q}\left(N\right)$. In particular, it is clear from their expression that $[f,\overline{g}]$ belongs to $\mathcal{A}^{\hbar}$ and has no constant term.

1.4.4 Quantum integrable hierarchy

Definition 44. A *quantum integrable hierarchy in $\mathcal{F}^{\hbar}\left(P\right)$* is an infinite collection of Hamiltonian densities $\left\{H_i\in\mathcal{A}^{\hbar},\, i\geq 0\right\}$ satisfying

$$\left[\overline{H}_i,\overline{H}_j\right]=0.$$

The *equations* of the quantum integrable hierarchy $\left\{H_i \in \mathcal{A}^{\hbar},\, i \geq 0\right\}$ are given by

$$\frac{\partial f^{\boldsymbol{t}}}{\partial t_i} = \frac{1}{\hbar}\left[f^{\boldsymbol{t}}, \overline{H}_i\right], \quad \text{for any i} \geq 0.$$

According to the integrability condition, we can solve the equations of the hierarchy simultaneously. Such a quantum hierarchy is a quantization of the classical hierarchy $\frac{\partial u}{\partial t_i} = \left\{u, \overline{H}_i\big|_{\hbar=0}\right\}$, with $i \geq 0$, where the Poisson structure is defined by $\{\cdot,\cdot\} = \frac{1}{\hbar}\left[\cdot,\cdot\right]\Big|_{\hbar=0}$.

Example 45. The first Hamiltonians densities of the quantum KdV hierarchy are

$$\begin{aligned}
H_{-1}^{KdV} &= u_0,\\
H_0^{KdV} &= h_0^{KdV} - \frac{i\hbar}{24},\\
H_1^{KdV} &= h_1^{KdV} - i\hbar\left(\frac{u_0}{24} + \frac{u_2}{12} + \epsilon^2\frac{1}{2880}\right),\\
H_2^{KdV} &= h_2^{KdV} - i\hbar\left(\frac{u_0^2}{48} + \frac{u_1^2}{24} + \frac{u_0u_2}{12} + \epsilon^2\left(\frac{u_0}{2880} + \frac{u_2}{288} + \frac{u_4}{144}\right) + \epsilon^4\frac{1}{120960}\right).
\end{aligned}$$

In Chapter 3, we will present a formula for these Hamiltonian densities and a recursive way to construct them from $\overline{H}_1^{KdV}$.

Remark 46. The algebra $\left(\mathcal{F}^{\hbar}(P), \star\right)$ is sufficient to define the quantum KdV hierarchy. However, there exists a bigger algebra (see [BR16]) used to define a family of quantum integrable hierarchies called the quantum double ramification hierarchies. These hierarchies reduce to the double ramification hierarchies of Remark 23 when $\hbar = 0$.

Definition 47. A *solution* of the quantum integrable hierarchy $\left\{H_i \in \mathcal{A}^{\hbar},\, i \geq 0\right\}$ with initial condition $f^{\text{initial}} \in \mathcal{F}^{\hbar}(P)$ is an element of $f^{\boldsymbol{t}} \in \mathcal{F}^{\hbar}(P)[[t_0, t_1, \dots]]$ satisfying the equations and such that $f^{\boldsymbol{t}=0} = f^{\text{initial}}$.

Lemma 48. *The unique solution of the hierarchy with initial condition* $f^{\text{initial}} \in \mathcal{F}^{\hbar}(P)$ *is*

$$f^{\boldsymbol{t}} = \exp\left(\sum_{k\geq 0}\frac{t_k}{\hbar}\left[\cdot, \overline{H}_k\right]\right) f^{\text{initial}} \in \mathcal{F}^{\hbar}(P)[[t_0, t_1, \dots]].$$

This extends to differential polynomials: $\phi^{\boldsymbol{t}} = \exp\left(\sum_{k\geq 0}\frac{t_k}{\hbar}\left[\cdot, \overline{H}_k\right]\right)\phi^{\text{initial}} \in \mathcal{A}^{\hbar}[[t_0, t_1, \dots]]$ is the unique solution of the hierarchy associated to the initial condition $\phi^{\text{initial}} \in \mathcal{A}^{\hbar}$.

Remark 49. Let $\boldsymbol{t} \to u^{\boldsymbol{t}} = \exp\left(\sum_{k\geq 0}\frac{t_k}{\hbar}\left[\cdot, \overline{H}_k\right]\right) u$ be the solution of the quantum hierarchy associated to $u \in \mathcal{A}^{\hbar}$ (here u is not a point of P, it is an element of $\mathcal{A}^{\hbar}$, that is a collection of elements of $\mathcal{F}^{\hbar}(P)$). We recover the classical solution of the associated classical hierarchy given in Eq. (1.3.1) by

$$u(x, \boldsymbol{t}) = u^{\boldsymbol{t}}\Big|_{u=u^{\text{initial}},\, \hbar=0},$$

that is we evaluate de element $u^{\boldsymbol{t}} \in \mathcal{A}^{\hbar}[[t_0, t_1, \dots]]$ at $u_k = \partial^{(k)} u^{\text{initial}}(x)$ and $\hbar = 0$. We also point out that $u^{\boldsymbol{t}}\Big|_{u=u^{\text{initial}}}$ is not a solution of the quantum hierarchy since a quantum solution is a flow on $\mathcal{F}^{\hbar}(P)$ or $\mathcal{A}^{\hbar}$.

Remark 50. In the quantum setting, the solutions are given by the flow $(\boldsymbol{t}, f) \to f^{\boldsymbol{t}}$ on $\mathcal{F}^{\hbar}(P)$. However, this flow is no more equivalent to a flow on P because

$$f^{\boldsymbol{t}} \neq f\left(u^{\boldsymbol{t}}\right),$$

where $u^{\boldsymbol{t}} = \exp\left(\sum_{k\geq 0} \frac{t_k}{\hbar}\left[\cdot, \overline{H}_k\right]\right) u$ and $u \in \mathcal{A}^{\hbar}$. This quantum behavior is due to the fact that the bracket $\left[\cdot, \overline{H}\right]$ is not a derivation on $\mathcal{F}^{\hbar}(P)$ with respect to the usual product. Conceptually, this comes from the fact that the space P is not the space of states in the quantum setting. Thus, it does not make sense to think of the solutions as a flow on P.

1.4.5 Quantum tau functions

In [BDGR16], Buryak, Dubrovin, Guéré and Rossi generalized the definition of tau functions to quantum integrable systems. We present this definition in this section.

Definition 51. A *quantum tau structure* is a collection of quantum Hamiltonian densities $\left\{h_i \in \mathcal{A}^{\hbar},\, i \geq -1\right\}$ such that

1. the quantum Hamiltonians commute

$$\left[\overline{H}_i, \overline{H}_j\right] = 0, \text{ for any } i, j \geq -1,$$

2. the quantum Hamiltonian densities are *tau symmetric*

$$\left[H_{i-1}, \overline{H}_j\right] = \left[H_{j-1}, \overline{H}_i\right], \text{ for any } i, j \geq 0,$$

3. the evolution given by the quantum Hamiltonian $\overline{H}_{-1}$ is trivial

$$\frac{\partial f}{\partial t_{-1}} = \frac{1}{\hbar}\left[f, \overline{H}_{-1}\right] = 0, \text{ for any } f \in \mathcal{F}^{\hbar}(P),$$

4. the quantum Hamiltonian $\overline{H}_0$ generates the translations in space

$$\frac{\partial f}{\partial t_0} = \frac{1}{\hbar}\left[\phi, \overline{H}_0\right] = \partial_x \phi \text{ for any } \phi \in \mathcal{A}^{\hbar}.$$

The construction of the quantum tau functions from the quantum tau structure is similar the construction of classical tau functions. We point out the differences. The *quantum two-point function* $\Omega^{\hbar}_{i,j}$ is the element of $\mathcal{A}^{\hbar}$ defined by

$$\partial_x \Omega^{\hbar}_{i,j} := \frac{1}{\hbar}\left[H_{i-1}, \overline{H}_j\right],$$

with a choice of the constant $\Omega^{\hbar}_{i,j}|_{u_*=0}$. In the classical case, we chose this constant to be 0. As we pointed out in Remark 32, this choice is justified by the fact that the Hamiltonian densities of interest have no constant term, we then had $\Omega_{i,0} = h_{i-1}$. In the quantum setting this is no longer true. For example, the Hamiltonian densities of the quantum KdV hierarchy given in Example 45 have non constant terms. In Section 3.2, we give a coherent choice of this constant in the example of quantum KdV, this leads to the definition of the quantum Witten-Kontsevich series.

The time dependent quantum two-point functions

$$\Omega^{\hbar,\boldsymbol{t}}_{i,j} = \exp\left(\sum_{i\geq 0} \frac{t_i}{\hbar}\left[\cdot, \overline{H}_i\right]\right)\Omega^{\hbar}_{i,j} \in \mathcal{A}^{\hbar}\left[\left[\epsilon, \hbar, t_0, t_1, \dots\right]\right]$$

satisfies the same symmetries than its classical counterpart: $\Omega^{\hbar,\boldsymbol{t}}_{i,j}$ is symmetric with respect to the exchange i, j and $\frac{\partial \Omega^{\hbar,\boldsymbol{t}}_{i,j}}{\partial t_k}$ is symmetric with respect to the exchange of i, j, k. As in the classical setting, the construction of quantum tau functions follows from these symmetries by using twice the Poincaré lemma.

Lemma 52. *Let $u^{\text{initial}} \in \mathbb{C}\left[[x]\right]$. There exists a series $\mathscr{F}^{\hbar} \in \mathbb{C}\left[\left[\epsilon, \hbar, t_0, t_1, \dots\right]\right]$ uniquely defined up to the constant and linear terms in t_* such that*

$$\frac{\partial^2 \mathscr{F}^{\hbar}}{\partial t_i \partial t_j} = \Omega^{\hbar,\boldsymbol{t}}_{i,j}\Big(u^{\text{initial}}\left(x=0\right)\Big).$$

Definition 53. The *quantum* $\log\tau$ *function* associated to u^{initial} is $\mathscr{F}^{\hbar}$.

The construction of quantum $\log\tau$ functions is intended to satisfy the following proposition.

Proposition 54. *The classical limit of the quantum $\log\tau$ function $\mathscr{F}^{\hbar}$ associated to the point $u^{\text{initial}} \in P$ is the classical $\log\tau$ function $\mathscr{F}$ associated to the solution starting at u^{initial}, that is*

$$\mathscr{F}^{\hbar}\Big|_{\hbar=0} = \mathscr{F}.$$

A quantum $\log\tau$ function is associated to a point $u^{\text{initial}} \in \mathbb{C}\left[[x]\right]$ and not a solution of the hierarchy as it is the case classically. This point u^{initial} is not an initial condition of a quantum solution. We only use this notation in order to match with the classical one.

Remark 55. The Hamiltonian densities of quantum KdV together with $H^{KdV}_{-1} = u_0$ form a tau structure (see Section 3.1.2). Let $u^{\text{initial}} \in \mathbb{C}\left[[x]\right]$ and $\mathscr{F}^{\hbar}$ the quantum $\log\tau$ function associated to it. We still have the relation

$$\frac{\partial^2 \mathscr{F}^{\hbar}}{\partial t_0 \partial t_0} = u^{\boldsymbol{t}}\Big|_{u:=u^{\text{initial}}(x=0)},$$

however the RHS is not a solution of the quantum hierarchy since we evaluate the solution $\boldsymbol{t} \to u^{\boldsymbol{t}}$ at $u^{\text{initial}}\left(x=0\right)$. The fact that $\frac{\partial^2 \mathscr{F}^{\hbar}}{\partial t_0 \partial t_0}$ is not a solution of the hierarchy is a major difference with the classical case.

The definition of quantum $\log\tau$ functions mimics the definition of classical $\log\tau$ functions. The goal of this thesis is to study the first example of quantum $\log\tau$ function. Our results suggest that the other examples deserve a further interest.

Chapter 2

The geometry of $\overline{\mathcal{M}}_{g,n}$

2.1 The moduli space of curves

2.1.1 Definition of the moduli space of curves

2.1.1.1 Introduction

A moduli space $\mathcal{M}$ is roughly speaking a geometrical space that classify geometrical objects of the same type. More precisely, we want that one point represents one object (and all the objects equivalent to it), such that two objects that are closely related should be nearby. In the good cases, there exists a so-called universal family $\pi : \mathcal{F} \to \mathcal{M}$ which is such that $\pi^{-1}(p)$ is a realization of the object p.

Example 56. The moduli space classifying isomorphism classes of finite sets is $\mathbb{N}$. Each point $n \in \mathbb{N}$ corresponds to an isomorphism class of sets of cardinality n. Let $U = \left\{(n,m) \in \mathbb{N}^2 | 1 \leq m \leq n\right\}$, the universal family is the projection on the first coordinate $\pi : U \to \mathbb{N}$.

Example 57. The moduli space classifying complex vectorial lines in $\mathbb{C}^2$ is $\mathbb{P}^1$. The universal family over $\mathbb{P}^1$ is $\pi : \mathcal{O}(-1) \to \mathbb{P}^1$ where $\mathcal{O}(-1) = \left\{(l,x) \in \mathbb{P}^1 \times \mathbb{C}^2 | x \in l\right\}$ is the tautological bundle.

2.1.1.2 The moduli space of curves

We will be interested in the moduli spaces classifying the following objects.

Definition 58. A *curve of genus g with n marked points* is a compact Riemann surface with n distinct points. Two curves of genus g with n marked points $(C, x_1, \dots, x_n)$ and $\left(C', x_1', \dots, x_n'\right)$ are isomorphic if there exists a biholomorphism $\phi : C \to C'$ respecting the marked points $\phi(x_i) = x_i'$, for all $1 \leq i \leq n$.

We denote by $\mathcal{M}_{g,n}$ the set of isomorphism classes of curves of genus g with n marked points and endow it with the following geometrical structure.

Proposition 59 ([ACG11])**.** *Suppose $2g-2+n > 0$. There exists a complex orbifold (or Deligne-Mumford stack) structure on $\mathcal{M}_{g,n}$, of dimension $3g-3+n$, such that points are isomorphism classes of genus g*

curves with n marked points and the stabilizer of a point is the automorphism group of the corresponding marked curve.

Remark 60. The condition $2g-2+n>0$, called the stability condition excludes four cases

$$(g,n) = (0,0)\,,\ (0,1)\,,\ (0,2)\ \text{ and }\ (1,1)\,.$$

These four cases correspond to the marked curves with an infinite automorphism group. Thus, their corresponding moduli spaces can not be endow with a orbifold struture.

Proposition 61 ([ACG11])**.** *There exists a orbifold $\mathcal{C}_{g,n}$, of dimension $3g-2+n$, and a morphism of orbifold $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ such that $\pi^{-1}\left([C,x_1,\dots,x_n]\right)$ is a genus g curve with n marked points in the equivalence class $[C,x_1,\dots,x_n]$.*

Definition 62. The map $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ is called the *universal curve*.

The universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ is endowed with n sections $\sigma_1,\dots,\sigma_n$. The section σ_i maps the equivalence class of curves $[C,x_1,\dots,x_n]$ to the marked point x_i in $\pi^{-1}\left([C]\right)$.

Example 63. We describe the moduli spaces $\mathcal{M}_{0,3}$ and $\mathcal{M}_{0,4}$. By the Riemann-Roch theorem, any genus 0 curve is biholomorphic to $\mathbb{P}^1$. Moreover, the biholomorphisms from $\mathbb{P}^1$ to $\mathbb{P}^1$ are well know: these are the Möbius transformations. From their explicit form, we see that there exists a unique Möbius transformation sending any triplet (x_1,x_2,x_3) of points of $\mathbb{P}^1$ to the triplet $(0,1,\infty)$. Thus, every genus 0 curve with three marked points is isomorphic to $\left(\mathbb{P}^1,0,1,\infty\right)$, the moduli space $\mathcal{M}_{0,3}$ is then a point.

Similarly, $\mathbb{P}^1$ with four marked points (x_1,x_2,x_3,x_4) is isomorphic to $\left(\mathbb{P}^1,0,1,\infty,x\right)$, where x is the image of x_4 by the unique Möbius transformation sending $(x_1,x_2,x_3) \to (0,1,\infty)$. We deduce that $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \backslash \{0,1,\infty\}$. Note that this space is not compact. We introduce a compactification in the next section.

2.1.1.3 The Deligne-Mumford compactification

We introduce a compactification of the moduli space of curves. The idea of this compactification is to add points to the moduli space that correspond to singular curves.

Definition 64. A *nodal curve* is a singular algebraic curve, with a finite number of nodal singularities, i.e. a singularity locally given by $\left\{(x,y)\in\mathbb{C}^2 | xy=0\right\}$. The *genus of a nodal* curve is the genus of the curve obtained by smoothing the singularities. A *nodal curve with n marked* points is such that its marked points are pairwise distinct and distinct of the nodes.

Remark 65. Any Riemann surface is isomorphic to a smooth algebraic curve. In particular, a curve of genus g with n marked points is a nodal curve (with no nodes).

Definition 66. A *stable curve* of genus g with n marked points is a nodal curve of genus g with n marked points satisfying the stability condition

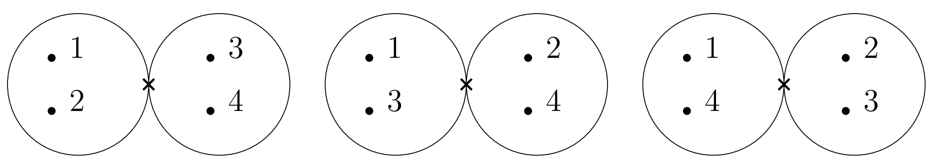
$$2g-2+n<0.$$

Proposition 67 ([ACG11])**.** *There exist two compact smooth complex orbifolds $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{C}}_{g,n}$ and a map $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that*

- *$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ and $\mathcal{C}_{g,n} \subset \overline{\mathcal{C}}_{g,n}$ are open dense subsets such that $\pi|_{\mathcal{C}_{g,n}}$ is the universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$,*
- *if $[C] \in \overline{\mathcal{M}}_{g,n}$, then $\pi^{-1}([C])$ is a stable curve, this curve is nodal if and only if $[C] \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$,*
- *each stable curve is isomorphic to a unique fiber of π,*
- *the stabilizer of $[C] \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of C.*

Definition 68. The orbifold $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne-Mumford compactification* of $\mathcal{M}_{g,n}$ and $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the *universal curve.*

Example 69. We describe the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,4}$. We add three points to $\mathcal{M}_{0,4}$ corresponding to nodal curves defined by two genus 0 smooth components, each of them containing 2 marked points, intersecting in one node:

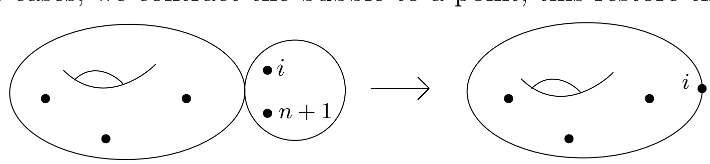


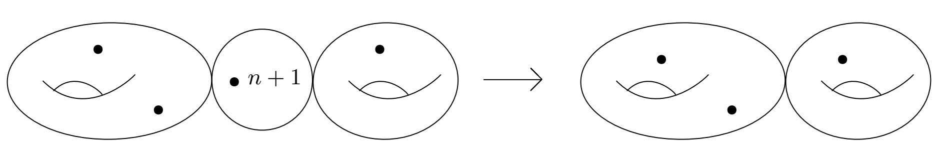
These stable curves describe what happen when marked points collide. For example, the leftmost curve corresponds to the collision of x_3 and x_4. In this case, these two points "jump" on a new genus 0 component.

2.1.2 Natural morphisms between moduli spaces

2.1.2.1 The forgetful map

The forgetful map is the map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ defined in the following way. Start from a curve $([C], x_1, \ldots x_{n+1})$ of $\overline{\mathcal{M}}_{g,n+1}$. We forget the $(n+1)$th marked point and obtain a genus g nodal curve with n marked points. However the stability condition of this curve does not necessary hold. The pathological cases happen when x_{n+1} is in a bubble (i.e. a genus 0 component) with 2 other special points (nodes or marked points). In these cases, we contract the bubble to a point, this restore the stability condition.





We then obtain a stable curve of genus g with n marked points, that is an element of $\overline{\mathcal{M}}_{g,n}$.

2.1.2.2 The glueing maps

We define two glueing maps.

1. The glueing map with one separating node

$$\mathrm{gl}_1 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

attaches a curve of $\overline{\mathcal{M}}_{g_1,n_1+1}$ with a curve of $\overline{\mathcal{M}}_{g_2,n_2+1}$ by identifying their lasts marked points

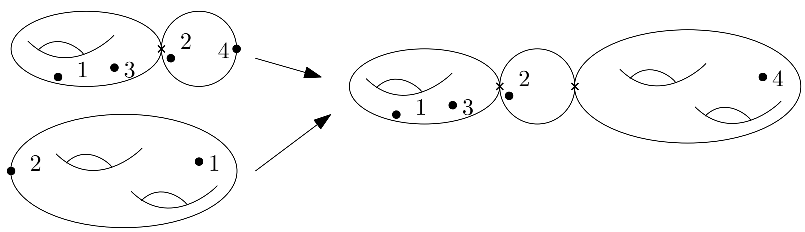


Figure 2.1.1: A glueing map from $\overline{\mathcal{M}}_{1,4} \times \overline{\mathcal{M}}_{2,2}$ to $\overline{\mathcal{M}}_{3,4}$.

2. The glueing map with one non separating node

$$\mathrm{gl}_2 : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

identifies the two last points of a curve of genus g with $n+2$ marked points. It gives a curve of genus $g+1$ with n marked points.

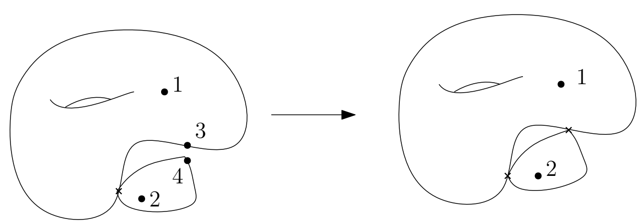


Figure 2.1.2: A glueing map $\overline{\mathcal{M}}_{1,4} \to \overline{\mathcal{M}}_{2,2}$

2.1.3 Stable graphs

We associate to a stable curve the following combinatorial data.

Definition 70. A *stable graph* is the data of

$$\Gamma = (V, H, g : V \to \mathbb{Z}_{\geq 0}, \theta : H \to V, \iota : H \to H)$$

such that

- V is the set of vertices equipped with a genus function $g : V \to \mathbb{Z}_{\geq 0}$,
- H is the set of half-edges equipped with a vertex assignment $\theta : H \to V$ and an involution $\iota : H \to H$,
- the set of edges E is defined by the set of orbits of length 2 of ι,
- the set of legs L is defined by the set of fixed points of ι,
- (V, E) is a connected graph,
- on each vertex, the stability condition holds

$$2g(v) - 2 + n(v) > 0,$$

  where $n(v) = |H(v)|$, where $H(v)$ is the set of half edges assigned to v by the map θ.

The stable graph associated to a stable curve C is the following. Each vertex v corresponds to a connected component of C and $g(v)$ is the genus of the component. Each edge corresponds to a node and each leg corresponds to a marked point.

We now describe the inverse operation which associates stable curves to a stable graph. To each graph Γ we associate the product of moduli spaces

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)}$$

and a map

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g(\Gamma),n(\Gamma)},$$

where $g(\Gamma) = \sum_{v \in V} g(v) + h_1(\Gamma)$ is the genus of the graph and $n(\Gamma) = |L|$. We recall that $h_1(\Gamma) = |E| - |V| + 1$. This map associates to a product of curves of $\overline{\mathcal{M}}_\Gamma$ the stable curve in $\overline{\mathcal{M}}_{g(\Gamma),n(\Gamma)}$ obtained by glueing the curves along the edges of Γ by the two glueing maps.

Remark 71. The image $\xi_\Gamma\left(\overline{\mathcal{M}}_\Gamma\right)$ is a codimension $|E|$ locus called a *boundary stratum.* The set of boundary strata gives a stratification of $\overline{\mathcal{M}}_{g,n}$.

2.1.4 The tautological rings

Definition 72. The minimal family of subrings $R^*\left(\overline{\mathcal{M}}_{g,n}\right) \subset H^*\left(\overline{\mathcal{M}}_{g,n}\right)$ containing $1 \in H^0\left(\overline{\mathcal{M}}_{g,n}\right)$ and stable under push-forwards by the forgetful map and by the two glueing maps is called the family of *tautological ring*s. A class in a tautological ring is called a *tautological class.*

Remark 73. The Poincaré duality works on compact orbifolds, so in particular on $\overline{\mathcal{M}}_{g,n}$. We will sometimes use the same notation for a cohomology class and its Poincaré dual.

Example 74. The class of a boundary stratum associated to a graph Γ is a tautological class. Indeed, it is the image of $\prod_{v \in V}\left[\overline{\mathcal{M}}_{g(v),n(v)}\right]$ by the glueing maps.

The tautological rings contain most of the cohomology classes of $\overline{\mathcal{M}}_{g,n}$ we are interested in. It is a hard task to find a non tautological class, see e.g. [GP01, FP11, PZ18] for the construction of a non tautological classes.

2.1.4.1 The ψ-, κ- and λ-classes

We give three examples of tautological classes: the ψ-classes, the κ-classes and the λ-classes. We first need to explain how to extend the cotangent bundle on stable curves.

Definition 75 ([ACG11])**.** Let $\mathcal{L}$ be the *relative dualizing line bundle* of $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.

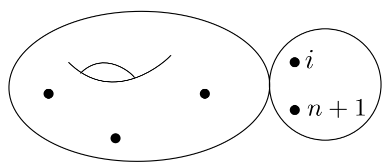
This line bundle $\mathcal{L} \to \overline{\mathcal{C}}_{g,n}$ is identified with the cotangent line bundle over the smooth part of $\overline{\mathcal{C}}_{g,n}$. Moreover, let $[C] \in \overline{\mathcal{M}}_{g,n}$ be a stable curve, a section of $\mathcal{L}$ over $\pi^{-1}([C])$ is a meromorphic differential form on the normalization of C with at most simple poles at the preimages of the nodes with opposite residues.

Let $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ be the section of the universal curve at the ith point. Let $\mathcal{L}_i = \sigma_i^* (\mathcal{L})$ be the pull back on $\overline{\mathcal{M}}_{g,n}$ of the restriction at the ith point of relative dualizing line bundle $\mathcal{L}$.

Definition 76. The class $\psi_i \in H^2 \left(\overline{\mathcal{M}}_{g,n}\right)$ is defined by the first Chern class of the bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$

$$\psi_i := c_1 \left(\sigma_i^* (\mathcal{L})\right).$$

We justify that the ψ-classes are tautological. Let $\delta_{(i,n+1)} := \mathrm{gl}_{1*}\left(\left[\overline{\mathcal{M}}_{g,n}\right] \times \left[\overline{\mathcal{M}}_{0,3}\right]\right) \in H^2\left(\overline{\mathcal{M}}_{g,n+1}\right)$ where $\mathrm{gl}_1 : \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+1}$ and such that the points i and $n+1$ belong to the curve of $\overline{\mathcal{M}}_{0,3}$. The class $\delta_{(i,n+1)}$ is represented by the curves



where the left component contains the genus g and $n-1$ marked points. The self intersection formula gives $\delta^2_{(i,n+1)} = \delta_{(i,n+1)} \cdot c_1\left(\mathcal{N}\left(\delta_{(i,n+1)}\right)\right)$, where $\mathcal{N}\left(\delta_{(i,n+1)}\right)$ is the normal bundle to the divisor $\delta_{(i,n+1)}$. Moreover, we have $\mathcal{N}\left(\delta_{(i,n+1)}\right) = p^* \left(\mathcal{L}_i^*\right)|_{\delta_{(i,n+1)}}$, where $\mathcal{L}_i$ is the cotangent bundle at the ith point on $\overline{\mathcal{M}}_{g,n}$. We deduce that

$$\psi_i = -p_* \left(\delta^2_{(i,n+1)}\right).$$

The ψ-classes are then tautological.

Definition 77. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. The κ-class is defined by $\kappa_j = \pi_*\left(\psi_{n+1}^{j+1}\right) \in H^{2j}\left(\overline{\mathcal{M}}_{g,n}\right)$.

Since the ψ-classes are tautological, the κ-classes are also tautological.

Let $\mathbb{E}_g := \pi_*(\mathcal{L})$ be the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over C is $H^0\left(C, \mathcal{L}|_C\right)$. It is called the *Hodge bundle*.

Definition 78. The λ-*classes* are the $g+1$ Chern classes of the Hodge bundle

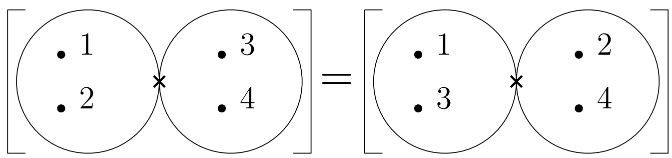
$$\lambda_j = c_j(\mathbb{E}) \in H^{2j}\left(\overline{\mathcal{M}}_{g,n}\right).$$

The Mumford formula [Mum83] expresses the Chern character of the Hodge bundle in term of ψ-classes, κ-classes and push-forwards of stable graphes with one node decorated with ψ-classes. The λ-classes are expressed in term of the Chern character of the Hodge, it follows that the λ-classes are tautological.

2.1.4.2 Tautological relations

The boundary strata decorated with ψ- and λ-classes are known to additively generate the whole tautological ring (see [FP11]). However there are relations between these generators. A relation between tautological classes is called a *tautological relation*.

Example 79. In $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$, the cohomology class of the point does not depend of the point. In particular, we have the following equality in $H^2\left(\overline{\mathcal{M}}_{0,4}\right)$:



This tautological relation is called the WDVV relation. This very simple relation has non trivial consequences, see e.g. [BSSZ15].

Example 80. As a consequence of Mumford formula, one can deduce (see [Mum83]) that

$$\lambda_g^2 = 0 \text{ in } H^*\left(\overline{\mathcal{M}}_{g,n}\right),$$

for all g, n satisfying $2g - 2 + n > 0$.

In [Pix12], Pixton conjectured a collection of tautological relations called the Pixton-Faber-Zagier relations. He also conjectured that these relations contain all the tautological relations. The Pixton-Faber-Zagier relations were proved in [PPZ13].

2.1.5 Intersection numbers

A class $\alpha \in H^{2(3g-3+n)}\left(\overline{\mathcal{M}}_{g,n}\right)$ is given by a number (recall that where $3g-3+n$ is the complex dimension of $\overline{\mathcal{M}}_{g,n}$). By Poincaré duality, this number is given by

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha.$$

We call *intersection numbers* the classes of $H^{2(3g-3+n)}\left(\overline{\mathcal{M}}_{g,n}\right)$. Since we are dealing with numbers, the ring $R^{2(3g-3+n)}\left(\overline{\mathcal{M}}_{g,n}\right)$ is in general the easiest tautological ring to study. Moreover these numbers, whose properties come from the geometry of the moduli space of curves, are related to many other subjects. In this thesis, we will be interested in the relations between intersection numbers, integrable systems and Hurwitz theory. The first and striking example of such relation is given by the Witten conjecture explained in the next section. More recently, Buryak recognized the coefficients of the KdV Hamiltonians as intersection numbers. This is the starting point for the quantization of the KdV hierarchy which leads to the definition of the quantum Witten-Kontsevich series. A direct link between intersection numbers and Hurwitz numbers is given by the ELSV formula, see Section 4.2.

Hodge integrals

As an example of intersection numbers, we will briefly discuss Hodge integrals. The Hodge integrals are the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{k_1} \dots \lambda_{k_m} \psi_1^{d_1} \dots \psi_n^{d_n},$$

where $g, n, m \geq 0$ satisfy $2g - 2 + n > 0$ and $k_1, \dots, k_m, d_1, \dots, d_n \geq 0$. Using the Mumford formula, any Hodge integral can be computed from the knowledge of intersection numbers involving only ψ-classes. These numbers can then be computed using the Witten conjecture. However there are few explicit formulas for Hodge integrals.

In [GP98], Getzler and Pandharipande deduced from the degree 0 of the Virasoro constraints the so-called λ_g-conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi_1^{d_1} \dots \psi_n^{d_n} = \left(\begin{array}{c} 2g-3+n \\ d_1, \dots, d_n \end{array} \right) \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2} \tag{2.1.1}$$

and $1 + \sum_{g \geq 1} z^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2} = \frac{z/2}{\mathrm{sh}(z/2)}$. A direct proof can be found in [FP03a, GJV09, FP98]. The dependency in $d_1, \dots, d_n$ of the intersection number of the LHS is contained in a combinatorial factor and the intersection number $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2}$ only depends of the genus. A similar formula exists for the intersection number

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \lambda_{g-1} \psi_1^{d_1} \dots \psi_n^{d_n} = \frac{(2g-3+n)!\,(2g-1)!!}{(2g-1)!\prod_{i=1}^{n}(2d_i-1)!!} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_{g-1} \psi^{g-1},$$

where $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_{g-1} \psi^{g-1} = \frac{|B_{2g}|(g-1)!}{2^g(2g)!}$, see [Fab99]. We denoted by B_i the Bernoulli numbers.

Remark 81. The Hodge integrals with one and two λ-classes appear as the coefficients of the quantum Witten-Kontsevich series (see Section 6.1.1 for the exact statement). The presence of Hodge integrals with two λ-classes is still unexplained.

2.2 The Witten-Kontsevich series

In [Wit90], by identifying two models of quantum gravity, Witten conjectured that the generating series of intersection numbers of monomials of ψ-classes

$$\mathscr{F}^{WK}(t_0, t_1, \dots) = \sum_{\substack{g,n \geq 0 \\ 2g-2+n>0}} \frac{\epsilon^{2g}}{n!} \sum_{d_1, \dots, d_n \geq 0} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \, t_{d_1} \dots t_{d_n},$$

where $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$, is the $\log \tau$ function of the KdV hierarchy associated to the string solution (recall that the string solution is the solution $u(x, t_0, t_1, \dots)$ of the KdV hierarchy with initial condition $u(x, 0, 0, \dots) = x$). This conjecture was then proved by Kontsevich in [Kon92]. The series

$\mathscr{F}^{WK}$ is called the *Witten-Kontsevich series*. The numbers $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g$ are called the *correlators* of the Witten-Kontsevich series.

This theorem is the first example of a link between the geometry of $\overline{\mathcal{M}}_{g,n}$ and integrable hierarchies. It is the starting point of many beautiful works. For example, Dubrovin and Zhang found a way to build a Hamiltonian presentation of the KdV hierarchy with the only input of $\mathscr{F}^{WK}$. They generalized this procedure by constructing an integrable hierarchy associated to the generating series of intersection numbers of ψ-classes with any Cohomological Field Theory (a CohFT is a family of tautological classes on $\overline{\mathcal{M}}_{g,n}$, for any g and n, compatible with the tautological maps, see [KM94]) that satisfy a certain property called semi-simplicity. These integrable hierarchies associated to any semi-simple CohFT are called the Dubrovin-Zhang hierarchies (see [DZ05, BPS12a, BPS+12b] for their constructions)

Remark 82. Specifying the genus in the notation $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g$ is redundant since this number is non-zero only if $\sum d_i = 3g-3+n$. We use this notation in view of its quantum generalization.

String and dilaton equations

It follows from the pull-back property of ψ-classes

$$\psi_i = p^*\left(\psi_i\right) + \delta_{(i,n+1)}$$

that intersection numbers of ψ-classes satisfy the two following equations.

Proposition 83 (String equation, [Wit90])**.** *The correlators of the Witten-Kontsevich series satisfy the string equation*

$$\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_g = \sum_{i=1}^{n}\langle\tau_{d_1}\dots\tau_{d_i-1}\dots\tau_{d_n}\rangle_{0,g}.$$

This equation can be reformulated as an equation on $\mathscr{F}^{WK}$

$$\frac{\partial}{\partial t_0}\mathscr{F}^{WK} = \sum_{i\geq 0} t_{i+1}\frac{\partial}{\partial t_i}\mathscr{F}^{WK} + \frac{t_0^2}{2}.$$

The Witten-Kontsevich series is related to the string solution by $\frac{\partial^2\mathscr{F}^{WK}}{\partial t_0\partial t_0}\left(x+t_0,t_1,\dots\right) = u^{\text{string}}\left(x,t_0,t_1,\dots\right)$, see Remark 34. As a consequence of the string equation, we get $u^{\text{string}}\left(x,0,0,\dots\right) = x$. Thus, the string solution is the solution of the KdV hierarchy with initial condition given by the string equation. This explains the terminology.

In [Wit90], Witten explained how to use the KdV hierarchy and the string equation in order to compute any intersection intersection number of monomial of ψ-classes.

Example 84. As a consequence of the string equation, we find

$$\int_{\overline{\mathcal{M}}_{0,n}}\psi^{n-2} = 1$$

since $\int_{\overline{\mathcal{M}}_{0,n}}\psi^{n-2} = \int_{\overline{\mathcal{M}}_{0,3}}\psi$ and $\overline{\mathcal{M}}_{0,3}$ is a point.

Proposition 85 (Dilaton equation, [Wit90])**.** *The correlators of the Witten-Kontsevich series satisfy the dilaton equation*

$$\langle \tau_1 \tau_{d_1} \dots \tau_{d_n} \rangle_g = (2g-2+n) \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g .$$

This equation can be reformulated as an equation on $\mathscr{F}^{WK}$

$$\frac{\partial}{\partial t_1} \mathscr{F}^{WK} = \sum_{i \geq 0} t_i \frac{\partial}{\partial t_i} \mathscr{F}^{WK} + \epsilon \frac{\partial}{\partial \epsilon} \mathscr{F}^{WK} - 2 \mathscr{F}^{WK}.$$

Remark 86. Let $\alpha \in H^* \left(\overline{\mathcal{M}}_{g,n+1} \right)$ and $\beta \in H^* \left(\overline{\mathcal{M}}_{g,n} \right)$ such that $\pi^* (\beta) = \alpha$. We can slightly generalize the string and dilaton equation by

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \alpha \psi_1^{d_1} \dots \psi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \beta \psi_1^{d_1} \dots \psi_i^{d_i - 1} \dots \psi_n^{d_n}$$

and

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \alpha \psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \beta \psi_1^{d_1} \dots \psi_n^{d_n}.$$

This is clear from the proof of Witten [Wit90].

Remark 87. These two equations, string and dilaton, are the two first equations of the so-called Virasoro constraints. In the more general context of Gromow-Witten theory, it is conjectured that the Virasoro constraints are satisfied by the generating series of Gromov-Witten numbers, see [EHX97, Get99]. In particular, the Witten-Kontsevich series satisfy the Virasoro constraints. Moreover, it is proved in [DVV93] that being a solution to Virasoro constraints is equivalent to being a solution of the KdV hierarchy and the string equation.

2.3 The double ramification cycle

2.3.1 Definition of the double ramification cycle

Fix two nonnegative integers g, n such that $2g-2+n > 0$ and a list of integer $A = (a_1, \dots, a_n)$ such that $\sum_{i=1}^n a_i = 0$. We define a tautological class depending on g, n and A called the double ramification cycle.

2.3.1.1 The double ramification cycle on $\mathcal{M}_{g,n}$

Consider the locus in $\mathcal{M}_{g,n}$ defined by

$$\mathcal{Z} = \left\{ [C, x_1, \dots, x_n] \, | \mathcal{O}_C \simeq \mathcal{O}_C \left(\sum_{i=1}^{n} a_i x_i \right) \right\}.$$

Definition 88. The Poincaré dual of $\mathcal{Z}$ is an element of $H^* (\mathcal{M}_{g,n})$ called the *double ramification cycle* and denoted by $\mathrm{DR}_g (a_1, \dots, a_n)$.

In order to extend the definition of the double ramification cycle to $\overline{\mathcal{M}}_{g,n}$, we first reformulate this definition. A curve belongs to the locus $\mathcal{Z}$ if there exists a nonzero holomorphic map $f : C \to \mathbb{P}^1$ with $\operatorname{div}(f) = \sum_{i=1}^n a_i x_i$. Moreover two such maps differ by a nonzero constant or equivalently an isomorphism of the target $\mathbb{P}^1$ fixing 0 and ∞. We define the set

$$\left\{ (C, x_1, \ldots, x_n, f) \,|\, f : C \to \mathbb{P}^1 \text{ s.t. } \operatorname{div}(f) = \sum_{i=1}^n a_i x_i \right\} \Big/ \sim,$$

where two maps $f_1 : C_1 \to \mathbb{P}^1$ and $f_2 : C_2 \to \mathbb{P}^1$ are isomorphic if and only if there exist two isomorphisms $\alpha : C_1 \to C_2$ and $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ respecting the markings (the image of a marked point of C by f is a marked point) and such that $\beta \circ f_1 = f_2 \circ \alpha$. We then obtain $\mathcal{Z}$ from this set by forgetting the maps f and keeping the source curves.

2.3.1.2 The moduli space of rubber maps

The definition of the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$ follows the same idea. It is obtained from the moduli space that parametrize the so-called rubber maps by forgetting the maps and keeping the source curve.

We define two unordered lists $\mu = (a \in A,\, a > 0)$ and $\nu = (a \in A,\, a < 0)$. Note that μ and $-\nu$ are two partitions of the same number. We denote by n_0 the cardinal of $A \backslash (\mu \cup \nu)$ which corresponds to the number of marked points with weight 0.

Definition 89. A *prestable rubber map* $(C, x_1, \ldots, x_n) \xrightarrow{f} (L, x_0, x_\infty)$ consists of the following data.

- A connected genus g curve with n marked points $(C, x_1, \ldots, x_n)$ possibly nodal such that the marking are different from the nodes.
- A chain of $\mathbb{P}^1$ denoted by L. We suppose that each sphere is attached by the point ∞ to the point 0 of the next sphere by a nodal singularity. We denote by x_0 the point 0 at one extremity of the chain and x_∞ the point ∞ of the other extremity.
- A degree map of degree $d = \sum_{a \in \mu} a = -\sum_{a \in \nu} a$ such that
  - the ramification profile over x_0 is given by μ,
  - the ramification profile over x_∞ is given by $-\nu$,
  - the pre-images of a node of L are nodes of C. Moreover, the ramification orders of f at these node of C are opposed (kissing condition).

Two maps $f : (C, x_1, \ldots, x_n) \to L$ and $f' : \left(C', x_1', \ldots, x_n'\right) \to L'$ and are isomorphic if there exist two isomorphisms $\alpha : C \to C'$ and $\beta : L \to L'$ respecting the markings such that $\beta \circ f = f' \circ \alpha$.

The automorphism group of $f : (C, x_1, \ldots, x_n) \to L$ is determined by the curves automorphisms $\alpha : C \to C$ and $\beta : L \to L$ commuting with f.

Definition 90. A *rubber map* is prestable rubber map with a finite automorphism group.

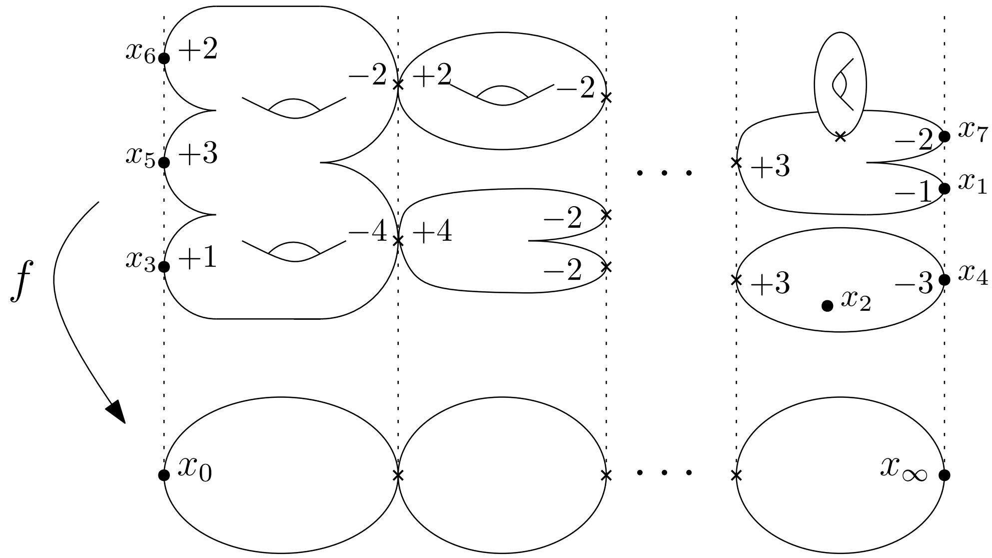


Figure 2.3.1: A rubber map f of degree 6 with $\mu = (2, 3, 1)$, $\nu = (-2, -1, -3)$ and $n = 7$.

We denote by $\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1, \mu, \nu\right)$ the moduli space parametrizing isomorphisms classes of rubber maps. This moduli space, sometimes also called the moduli space of unparameterized relative stable maps to $\mathbb{P}^1$ was studied by Jun Li in [L+01] (see also [Li04]).

Proposition 91 ([L+01])**.** *The moduli space of rubber maps $\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1, \mu, \nu\right)$ has a virtual fundamental class of virtual dimension $2g - 3 + n$.*

The moduli space $\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1, \mu, \nu\right)$ is endowed with two maps.

- The map
$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1, \mu, \nu\right) \to \overline{\mathcal{M}}_{g,n}$$
forgets everything except the marked source curve. If the marked curve have non stable components, i.e. bubbles with less than 3 special points, we contract them.

- The map
$$\rho : \overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1, \mu, \nu\right) \to LM_{r+n_0}/S_r$$
forgets everything except the target curve. The target space LM_{r+n_0} is called the Losev-Manin space. It parametrize stable chains of $\mathbb{P}^1$ with $r + n_0 + 2$ marked points. These $r + n_0 + 2$ marked points are of two types; (i) the two marked points x_0 and x_∞ at the extremities of the chain and (ii) $r + n_0$ other marked points. The marked points of type (ii) are allowed to collide with each others. We refer to [LM00] for the construction of this space.
The marked points of type (ii) of the curves in the image of ρ have two origins: there are n_0 of them which correspond to the image of the marked points of C with weight 0 and r of them which correspond to the branching points of the rubber map.
We finally quotient LM_{r+n_0} by S_r since the branching points are not numbered.

2.3.1.3 The double ramification cycle on $\overline{\mathcal{M}}_{g,n}$

Definition 92. The *double ramification cycle* on $\overline{\mathcal{M}}_{g,n}$ is

$$\mathrm{DR}_g\left(a_1,\ldots,a_n\right)=\epsilon_*\left(\left[\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1,\mu,\nu\right)\right]^{\mathrm{virt}}\right).$$

This cycle has virtual dimension $2g-3+n$. It is proved in [MW13] that the restriction to $\mathcal{M}_{g,n}$ of this cycle is the class $[\mathcal{Z}]$.

Proposition 93 ([FP03b])**.** *The double ramification cycle is a tautological class.*

The double ramification cycle satisfies the following pull-back property.

Proposition 94 ([BHP+20], Invariance II)**.** *We have*

$$\pi^*\left(\mathrm{DR}_g\left(a_1,\ldots,a_n\right)\right)=\mathrm{DR}_g\left(a_1,\ldots,a_n,0\right),$$

where $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ *and* $\sum_{i=1}^n a_i=0$.

2.3.2 Computations of the double ramification cycle

Since the double ramification cycle is tautological, a long standing question was to find an expression of the double ramification cycle in term of the generators of the tautological ring. This question was answered in [JPPZ17]. They found a tautological class denoted by $P_g^{d,r}(A)$, of cohomological degree $2d$, depending on a parameter r and expressed as a sum of stable graphs decorated with ψ-classes. Pixton proved in [Pix] that this class depends polynomially of r, when r is large enough. The double ramification cycle is obtained via the constant term of this polynomial by

$$\mathrm{DR}_g(A)=P_g^{g,0}(A).$$

This formula simplifies when $A=(0,\ldots,0)$.

Proposition 95 ([JPPZ17])**.** *We have*

$$\mathrm{DR}_g(0,\ldots,0)=(-1)^g\lambda_g.$$

In a paper to appear, Pixton and Zagier used the class $P_g^{g,0}(A)$ to prove the polynomiality of the double ramification cycle.

Proposition 96 ([PZ])**.** *There exists a polynomial* $Q_{g,n}(a_1,\ldots,a_n)\in H^{2g}\left(\overline{\mathcal{M}}_{g,n}\right)[a_1,\ldots,a_n]$ *of polynomial degree* $2g$ *such that*

$$\mathrm{DR}_g\left(a_1,\ldots,a_n\right)=Q_{g,n}\left(a_1,\ldots,a_n\right)$$

for all $(a_1,\ldots,a_n)\in\mathbb{Z}^n$ *with* $\sum a_i=0$.

The expression of the double ramification cycle obtained by Janda, Pandharipande, Pixton and Zvonkine is hard to use in general. Indeed, you have to sum over many graphs and $P_g^{d,r}(A)$ is not explicitly polynomilal in r. We give various ways to compute the double ramification cycle.

Remark 97. There exists a computer program made by Schmidt and van Zelm which computes in some cases the double ramification cycle in terms of decorated stable graphs, see [SvZ].

2.3.2.1 The double ramification cycle with ψ-classes

In [BSSZ15], the authors studied intersections of ψ-classes with the double ramification cycle. By pulling back a simple tautological relation of the Losev-Manin space (the ψ-class is equivalent to the divisor with one separating node) to the moduli space of rubber maps, they obtained various expressions of intersections of ψ-classes with the double ramification cycle. In particular, they obtained an expression for the intersection number $\psi_s^{2g-2+n}\mathrm{DR}_g(a_1,\dots,a_n)$, with $0 \le s \le n$.

Notation 98. Denote by $S(z)$ the power series

$$S(z) = \frac{\mathrm{sh}(z/2)}{z/2} = \sum_{l\ge 0} \frac{z^{2l}}{2^{2l}(2l+1)!}.$$

Proposition 99 ([BSSZ15])**.** *We have*

$$\psi_s^{2g-2+n}\mathrm{DR}_g(a_1,\dots,a_n) = \left[z^{2g}\right] \frac{\prod_{i\neq s} S(a_i z)}{S(z)}. \tag{2.3.1}$$

2.3.2.2 The double ramification cycle with the λ_g-class

Denote by $\mathcal{M}_{g,n}^{\mathrm{ct}} \subset \overline{\mathcal{M}}_{g,n}$ the moduli space of stable curves with only separating nodes. The class λ_g vanishes on the complement of $\mathcal{M}_{g,n}^{\mathrm{ct}}$ in $\overline{\mathcal{M}}_{g,n}$. Indeed, let $\mathrm{gl}_2 =: \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the glueing map with one non separating node. We have the following exact sequence of vector bundles over $\overline{\mathcal{M}}_{g-1,n+2}$:

$$0 \to \mathbb{E}_{g-1} \to \mathrm{gl}_2^*(\mathbb{E}_g) \to \mathbb{C} \to 0,$$

where the first map is the identity and the second is the residue at the non separating node. It follows that that $\mathrm{gl}_2^*(\lambda_g) = 0$. We conclude by remarking that the map $\mathrm{gl}_2 : \overline{\mathcal{M}}_{g-1,n-2} \to \left(\mathcal{M}_{g,n}^{\mathrm{ct}}\right)^c \subset \overline{\mathcal{M}}_{g,n}$, where $\left(\mathcal{M}_{g,n}^{\mathrm{ct}}\right)^c$ is the complement of $\mathcal{M}_{g,n}^{\mathrm{ct}}$ in $\overline{\mathcal{M}}_{g,n}$, is an isomorphism.

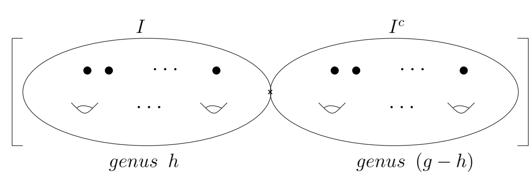
Thus, in order to compute $\lambda_g\mathrm{DR}_g(a_1,\dots,a_n)$, it is enough to use an expression of the DR cycle restricted to $\mathcal{M}_{g,n}^{\mathrm{ct}}$. Hain obtained the following expression.

Proposition 100 ([Hai11])**.** *We have*

$$\mathrm{DR}_g(a_1,\dots,a_n)\Big|_{\mathcal{M}_{g,n}^{\mathrm{ct}}} = \frac{1}{g!}\left(-\frac{1}{4}\sum_{I\subset\{1,\dots,n\}}\sum_{h=0}^{g} a_I^2\delta_h^I\right)^g,$$

where $a_I := \sum_{i\in I} a_i$, $\delta_0^{\{i\}} = -\psi_i$, $\delta_g^{\{1,\dots,\hat{i},\dots n\}} = -\psi_i$ *and*

$\delta_h^I =$



is the class of stable curves with one separating node, the left component is of genus h and contains the marked points of I, the right component is of genus $(g-h)$ and contains the rest of the marked points.

Remark 101. As a consequence of Hain's formula, we get that $\lambda_g \mathrm{DR}_g(a_1,\ldots,a_n)$ is a homogeneous polynomial of degree $2g$ in the indeterminates $a_1,\ldots,a_n$.

Remark 102. Over $\mathcal{M}_{g,n}^{\mathrm{ct}}$, the formula of [JPPZ17] for the double ramification cycle simplifies since $P_g^{d,r}(A)\big|_{\mathcal{M}_{g,n}^{\mathrm{ct}}}$ is a sum over stables trees and a constant polynomial in r. Using the intersection theory for graphs (see [PPZ13]) in Hain's formula, we obtain the same expression for the double ramification cycle restricted on $\mathcal{M}_{g,n}^{\mathrm{ct}}$.

Example 103. We compute $\int_{\mathrm{DR}_1(a,-a)} \lambda_1$. By the Hain formula, we compute $\mathrm{DR}_1(a,-a)\Big|_{\mathcal{M}_{1,2}^{\mathrm{ct}}} = \frac{a^2}{2}(\psi_1+\psi_2)$. Hence

$$\int_{\mathrm{DR}_1(a,-a)} \lambda_1 = a^2 \int_{\overline{\mathcal{M}}_{1,2}} \lambda_1\psi_1 = a^2 \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1,$$

where we used the string equation in the last equality. Finally, using the λ_g-conjecture, we find

$$\int_{\mathrm{DR}_1(a,-a)} \lambda_1 = \frac{a^2}{12}.$$

Chapter 3

The quantum KdV hierarchy and the quantum Witten-Kontsevich series

In [Bur15], Buryak identified the coefficients of the KdV Hamiltonians as intersection numbers on the double ramification cycle. For example, the first Hamiltonian density of the KdV hierarchy is given as a formal Fourier series by

$$h_0 = \frac{1}{2} \sum_{a_1, a_2 \in \mathbb{Z}} \int_{\mathrm{DR}_0(0, a_1, a_2, -a_1-a_2)} \psi_0 \, p_{a_1} p_{a_2} e^{i(a_1+a_2)x} - \epsilon^2 \sum_{a \in \mathbb{Z}} \int_{\mathrm{DR}_1(0,a,-a)} \psi_0 \, p_a e^{iax}.$$

Later, Buryak and Rossi [BR16] used deformation quantization to define a quantum version of the KdV hierarchy. They found a collection of quantum Hamiltonians which commute with respect to the star product of $\mathcal{F}^{\hbar}(P)$ and which restrict to the KdV Hamiltonians when $\hbar = 0$. The coefficients of these quantum Hamiltonians are also given in term of intersection numbers on the double ramification cycle.

The work of Buryak and Rossi is actually more general. They constructed a quantum integrable hierarchy associated to any CohFT. These are the so-called *quantum double ramification hierarchies*. The quantization of the KdV hierarchy we present in this section is the quantum double ramification hierarchy associated to the trivial CohFT.

In [BDGR16], the authors proved that the Hamiltonians densities of the quantum KdV hierarchy form a tau structure. We can then build its quantum tau functions. The quantum Witten-Kontsevich series is a natural choice of such function which restricts to the Witten-Kontsevich series when $\hbar = 0$.

In this section, we present the quantum KdV hierarchy and its properties. These properties are deduced from the geometric properties of the DR cycle. We then present the quantum Witten-Kontsevich series. The study of this series is the main goal of this thesis.

3.1 The quantum KdV hierarchy

3.1.1 Hamiltonians and Hamiltonians densities

We define the Hamiltonian densities as a formal Fourier series of differential polynomials. The symmetric polynomials of these formal Fourier series are defined by an integral of ψ- and λ- classes over the double ramification cycle.

Definition 104. Fix $d \geq -1$. The *quantum Hamiltonian density* H_d of the quantum KdV hierarchy is

$$H_d(x) = \sum_{\substack{g\geq 0, m\geq 0\\ 2g+m>0}} \frac{(i\hbar)^g}{m!} \sum_{a_1,\dots,a_m\in\mathbb{Z}} \left(\int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)} \psi_0^{d+1} \Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)\right) p_{a_1}\dots p_{a_m} e^{ix\sum a_i} \in \mathcal{A}^\hbar, \quad (3.1.1)$$

where $\Lambda\left(\frac{-\epsilon^2}{i\hbar}\right) := 1 + \left(\frac{-\epsilon^2}{i\hbar}\right)\lambda_1 + \dots + \left(\frac{-\epsilon^2}{i\hbar}\right)^g \lambda_g$.

The virtual dimension of $\mathrm{DR}_g(0, a_1, \dots, a_m, -\sum a_i)$ is $2g - 1 + n$, hence the summation over g and n is finite. According to the polynomiality of the double ramification cycle, the Hamiltonian densities are indeed elements of $\mathcal{A}^\hbar$.

Definition 105. The *quantum Hamiltonians* of the quantum KdV hierarchy are obtained by the x-integration of the Hamiltonian densities:

$$\overline{H}_d = \int_{S^1} H_d(x)\, dx \in \mathcal{F}^\hbar(P),$$

for $d \geq -1$.

Notation 106. From now on, we will only use the quantum Hamiltonian densities and quantum Hamiltonians of the quantum KdV hierarchy. We then suppress the upper index KdV we used in Chapter 1.

Remark 107. In [BR16], Buryak and Rossi introduced the Hamiltonian densities of the quantum KdV hierarchy (i.e. the double ramification hierarchy associated to the trivial CohFT) by

$$G_d = \sum_{\substack{g\geq 0, m\geq 0\\ 2g-1+m>0}} \frac{(i\hbar)^g}{m!} \sum_{a_1,\dots,a_m\in\mathbb{Z}} \left(\int_{\mathrm{DR}_g(-\sum a_i,a_1,\dots,a_m)} \psi_0^{d} \Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)\right) p_{a_1}\dots p_{a_m} e^{ix\sum a_i}.$$

A simple use of the string equation shows that $\overline{G}_d = \overline{H}_d$ yielding to the same hierarchy. However only the Hamiltonian densities H_d are tau-symmetric.

In [BDGR16], the authors defined the Hamiltonian density of the quantum KdV hierarchy by $H_d^{BDGR} := \sum_{s\geq 0} (-\partial_x)^s \frac{\partial G_{d+1}}{\partial u_s}$. One can check the equality $\sum_{s\geq 0} (-\partial_x)^s \frac{\partial \phi}{\partial u_s} = \sum_{a\in\mathbb{Z}} e^{-iax} \frac{\partial \overline{\phi}}{\partial p_a}$ for any differential polynomial ϕ. We deduce that the two definitions are equivalent:

$$H_d^{BDGR} = \sum_{b\in\mathbb{Z}} e^{-iax} \frac{\partial \overline{G}_{d+1}}{\partial p_a} = H_d.$$

Remark 108. When we substitute $\hbar = 0$, we obtain the classical Hamiltonians densities $h_d(x) := H_d(x)\big|_{\hbar=0}$ and the classical Hamiltonians $\overline{h}_d := \overline{H}_d\big|_{\hbar=0}$ of the KdV hierarchy. This is proved in the original paper of Buryak [Bur15].

Examples of computations of Hamiltonian densities

We first compute H_{-1} and H_0. We will use these expressions to justify that the Hamiltonian densities of the quantum KdV hierarchy form a tau structure. Then, we compute $\overline{H}_1$. In Section 3.1.4, we will present a recursive formula which allows to compute each Hamiltonian density of the quantum KdV hierarchy from the non constant terms of $\overline{H}_1$ and from H_{-1}.

Example 109. Computation of H_{-1}. We compute

$$\int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)} \lambda_l$$

for every g, m satisfying the stability condition $2g+m>0$ and such that $0 \le l \le g$. The virtual dimension of the double ramification cycle $\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)$ is $2g-1+m$. Thus, the integral vanishes for dimensional reason unless $g+(g-l)+m=1$. The only stable solutions are

$$(g,l,m) = (0,0,0) \text{ and } (1,1,0).$$

- The case $(g,l,m)=(0,0,0)$. In genus 0, the double ramification cycle is identified with the fundamental class of the moduli space of curves. We get $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$.
- The case $(g,l,m)=(1,1,0)$ vanishes since $\mathrm{DR}_1(0,0) = -\lambda_1$ and $\lambda_1^2 = 0$.

We then find

$$H_{-1} = \sum_{a\in\mathbb{Z}} p_a e^{iax} = u_0.$$

We then deduce that $\overline{H}_{-1} = p_0$ commutes with any element of $\mathcal{F}^{\hbar}(P)$.

Example 110. Computation of H_0. We compute

$$\int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)} \psi_0\lambda_l$$

for every g, m satisfying the stability condition $2g+m>0$ and such that $0 \le l \le g$. The integral vanishes for dimensional reason unless $g+(g-l)+m=1$. The only stable solutions are

$$(g,l,m) = (0,0,2)\,,\ (1,0,0)\,,\ (1,1,1) \text{ and } (2,2,0)\,.$$

- The case $(g,l,m)=(0,0,2)$. We get $\int_{\overline{\mathcal{M}}_{0,4}} \psi = 1$.

- The case $(g,l,m)=(1,0,0)$. We use $\mathrm{DR}_1(0,0)=-\lambda_1$ and the string equation to obtain

$$\int_{\mathrm{DR}_1(0,0)}\psi_0=-\int_{\overline{\mathcal{M}}_{1,2}}\lambda_1\psi_0=-\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1.$$

  The λ_g-conjecture (see Section 2.1.5) gives $\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1=\frac{1}{24}$.

- The case $(g,l,m)=(1,1,1)$. We first use the dilaton equation and the pull-back property of the double ramification cycle (see Proposition 94) to obtain

$$\int_{\mathrm{DR}_1(0,a,-a)}\psi_0\lambda_1=2\int_{\mathrm{DR}_1(a,-a)}\lambda_1.$$

  This last integral was computed in Example 103 using Hain's formula, we find $2\int_{\mathrm{DR}_1(a,-a)}\lambda_1=\frac{a^2}{12}$.

- The case $(g,l,m)=(2,2,0)$ vanishes since $\mathrm{DR}_2(0,0)=\lambda_2$ and $(\lambda_2)^2=0$.

We obtain

$$H_0=\frac{u_0^2}{2}+\epsilon^2\frac{u_2}{12}-\frac{i\hbar}{24}.$$

We then deduce that $\overline{H}_0=\sum_{a\in\mathbb{Z}}\frac{p_ap_{-a}}{2}-\frac{i\hbar}{24}$ and it follows that $\frac{1}{\hbar}\left[\phi,\overline{H}_0\right]=\left\{\phi,\overline{H}_0\right\}=\partial_x\phi$, for any $\phi\in\mathcal{A}^\hbar$.

Example 111. Computation of $\overline{H}_1$. We compute

$$\int_{\mathrm{DR}_g(0,a_1,\ldots,a_m,0)}\psi_0^2\lambda_l,\ \text{such that}\sum a_i=0$$

for every g,m satisfying the stability condition $2g+m>0$ and such that $0\leq l\leq g$. The integral vanishes for dimensional reason unless $g+(g-l)+m=3$. There are only a finite number of solutions

$$(g,l,m)=(0,0,3)\,,\ (1,0,1)\,,\ (1,1,2)\,,\ (2,1,0)\,,\ (2,2,1)\ \text{and}\ (3,3,0)\,.$$

- The case $(g,l,m)=(0,0,3)$. We have $\int_{\overline{\mathcal{M}}_{0,5}}\psi^2=1$.

- The case $(g,l,m)=(1,0,1)$. We have

$$\int_{\mathrm{DR}_1(0,0,0)}\psi^2=-\int_{\overline{\mathcal{M}}_{1,3}}\lambda_1\psi^2=-\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1=-\frac{1}{24}.$$

- The case $(g,l,m)=(1,1,2)$. We have

$$\int_{\mathrm{DR}_1(0,a,-a,0)}\lambda_1\psi_0^2=\int_{\mathrm{DR}_1(0,a,-a)}\lambda_1\psi_0=2\int_{\mathrm{DR}_1(a,-a)}\lambda_1=\frac{a^2}{12}.$$

- The case $(g,l,m)=(2,1,0)$. Using the explicit formula for intersection number of ψ-classes with $\lambda_g\lambda_{g-1}$ discussed in the paragraph on Hodge integral, we get

$$\int_{\mathrm{DR}_2(0,0)} \lambda_1\psi^2 = -\frac{1}{2880}.$$

- The case $(g,l,m)=(2,2,1)$ vanishes since $(\lambda_2)^2=0$.
- The case $(g,l,m)=(3,3,0)$ vanishes since $(\lambda_3)^2=0$.

We can then write $\overline{H}_1$ as

$$\overline{H}_1 = \int_{S^1}\left(\frac{u_0^3}{6} - i\hbar\frac{u_0}{24} + \epsilon^2\frac{u_0u_2}{24} - i\hbar\epsilon^2\frac{1}{2880}\right)dx.$$

Of course, this expression is not unique since we can modify the integrand by a ∂_x-term without modifying the expression of $\overline{H}_1$.

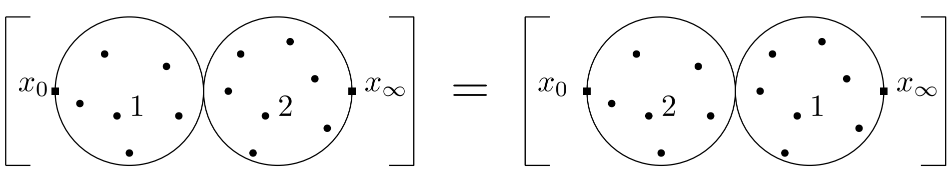
3.1.2 Integrability and tau symmetry

We present the integrability and the tau symmetry of the quantum KdV hierarchy. These properties follow from geometric properties of the double ramification cycle.

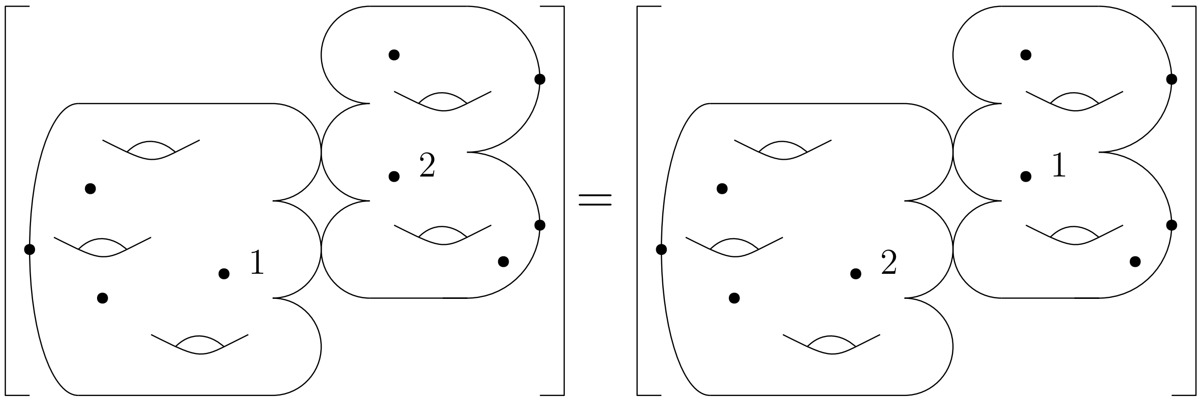
Proposition 112 (Quantum integrability, [BR16]). *We have*

$$\left[\overline{H}_{d_1},\overline{H}_{d_2}\right]=0, \text{ for } d_1,d_2\geq -1.$$

Idea of the proof. We justify a tautological relation involving the double ramification cycle, the quantum integrability follows from this relation. Recall that the double ramification cycle is defined by $\epsilon_*\left(\left[\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1,\mu,\nu\right)\right]^{\mathrm{virt}}\right)$, see Section 2.3.1 for the notations. We also introduced the map $\rho:\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1,\mu,\nu\right)\to LM_{r+n_0}/S_r$. Suppose $n_0\geq 2$. There is a map $p: LM_{r+n_0}\to\overline{\mathcal{M}}_{0,4}$ which forgets every point except x_0,x_∞ and the two first numbered points. We then pull back the WDVV relation on LM_{r+n_0} to obtain the relation



where on each side of the equality, we sum over the ways to distribute the $r+n_0-2$ marked points on the two sides of the separating node. Then we pull-back this relation by $\rho:\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1,\mu,\nu\right)\to LM_{r+n_0}/S_r$ and push it forward by $\epsilon:\overline{\mathcal{M}}^{\sim}_{g,n}\left(\mathbb{P}^1,\mu,\nu\right)\to\overline{\mathcal{M}}_{g,n}$. We obtain a tautological relation of type



where we sum on the two sides over the ways to distribute the marked points different from 1 and 2, the number of bridges between the two components and genera of each component to obtain a curve of genus g. Each side of the equation is a sum of glueings by the glueing maps of products of double ramification cycles.
If we add the class ψ^{d_1} on the marked point 1 and a class ψ^{d_2} on the marked point 2, this formula says that we can exchange the two ψ-classes. This is exactly what is required to prove $\overline{G}_{d_1} \star \overline{G}_{d_2} = \overline{G}_{d_2} \star \overline{G}_{d_1}$ (recall that $\overline{G}_d = \overline{H}_d$). Indeed the qth order expansion of the star product $\sum_{k_1,\dots,k_q>0} \frac{k_1 \dots k_q}{q!} \overleftarrow{\frac{\partial}{\partial p_{k_1}}} \cdots \overleftarrow{\frac{\partial}{\partial p_{k_q}}} \overrightarrow{\frac{\partial}{\partial p_{-k_1}}} \cdots \overrightarrow{\frac{\partial}{\partial p_{-k_q}}}$ glues a double ramification cycle in $\overline{G}_{d_1}$ and in $\overline{G}_{d_2}$ by q bridges. $\square$

Remark 113. The substitution $\hbar = 0$ in $\frac{1}{\hbar}\left[\overline{H}_{d_1}, \overline{H}_{d_2}\right] = 0$ gives the integrability condition of the classical KdV hierarchy.

Proposition 114 (Tau symmetry, [BDGR16])**.** *We have*

$$\left[H_{d_1-1}(x), \overline{H}_{d_2}\right] = \left[H_{d_2-1}(x), \overline{H}_{d_1}\right], \ \textit{for } d_1, d_2 \geq -1.$$

Remark 115. The substitution $\hbar = 0$ in $\frac{1}{\hbar}\left[H_{d_1-1}(x), \overline{H}_{d_2}\right] = \frac{1}{\hbar}\left[H_{d_2-1}(x), \overline{H}_{d_1}\right]$ gives the tau symmetry of the KdV hierarchy.

3.1.3 String equation for the Hamiltonian densities

From the pull-back property of the ψ-classes, we deduce the following property of the Hamiltonian densities.

Proposition 116. *Fix a nonnegative integer d. The Hamiltonian density H_d satisfies the string equation*

$$\frac{\partial}{\partial p_0} H_d = H_{d-1}.$$

The proof is analogous to the one of Lemma 2.7 in [BR16].

Proof. We have

$$\frac{\partial}{\partial p_0} H_d = \sum_{\substack{g \geq 0, m \geq 0 \\ 2g+m+1>0}} \frac{(i\hbar)^g}{m!} \sum_{a_1,\dots,a_m \in \mathbb{Z}} \left(\int_{\mathrm{DR}_g(0,a_1,\dots,a_m,0,-\sum a_i)} \psi_0^{d+1} \Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)\right) p_{a_1} \dots p_{a_m} e^{ix\sum a_i}.$$

Let $\pi : \overline{\mathcal{M}}_{g,m+3} \to \overline{\mathcal{M}}_{g,m+2}$ be the map defined when $(g,m) \neq (0,0)$ that forgets the $(m+1)$-th marked point (we start the numbering at 0). We use that $\pi^* \mathrm{DR}_g\left(0, a_1, \ldots, a_m, -\sum a_i\right) = \mathrm{DR}_g\left(0, a_1, \ldots, a_m, 0, -\sum a_i\right)$, $\pi^*\left(\Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)\right) = \Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)$ and $\psi_0^{d+1} = \pi^*\left(\psi_0^{d+1}\right) + \delta_{(i,n+1)}\pi^*\left(\psi_0^d\right)$, where $\delta_{(i,n+1)}$ is the divisor with a bubble containing the marked points 0 and $(m+1)$, to obtain

$$\int_{\mathrm{DR}_g(0,a_1,\ldots,a_m,0,-\sum a_i)} \psi_0^{d+1}\Lambda\left(\frac{-\epsilon^2}{i\hbar}\right) = \begin{cases} \int_{\mathrm{DR}_g(0,a_1,\ldots,a_m,-\sum a_i)} \psi_0^d \Lambda\left(\frac{-\epsilon^2}{i\hbar}\right) & \text{if } 2g+m>0 \text{ and } d>0 \\ 0 & \text{if } 2g+m>0 \text{ and } d=0 \\ \delta_{d,0} & \text{if } g=0, m=0 \text{ and } d=0. \end{cases}$$

This proves the lemma. □

3.1.4 A recursive construction of the Hamiltonian densities

In [BR16], Buryak and Rossi explained how to recursively construct the differential polynomials G_d introduced in Remark 107 from the non constant terms of $\overline{G}_1 = \overline{H}_1$ and with initial condition $H_{-1} = u_0$. Moreover, the Hamiltonian densities H_d are obtained from G_{d+1} by $H_d = \sum_{a\in\mathbb{Z}} e^{-iax}\frac{\partial \overline{G}_{d+1}}{\partial p_a}$. We construct in this way the Hamiltonian densities H_d. Let us recall their construction.

Proposition 117 ([BR16])**.** *Let $d \geq -1$. We have*

$$\partial_x (D-1) G_{d+1} = \frac{1}{\hbar}\left[G_d, \overline{G}_1\right], \tag{3.1.2}$$

where $D = \sum \epsilon\frac{\partial}{\partial\epsilon} + 2\hbar\frac{\partial}{\partial\hbar} + \sum_{a\in\mathbb{Z}} p_a\frac{\partial}{\partial p_a}$ and $G_{-1} = H_{-1}$.

Recursive construction. Suppose we know G_d. We use Eq. (3.1.2) to construct G_{d+1} up to a constant term. Moreover, the constants commute with any element of $\mathcal{F}^\hbar(P)$. We then use this expression to construct G_{d+2} up to a constant. We can recover the constant term of G_{d+1} using a string equation $\frac{\partial}{\partial p_0}G_{d+2} = G_{d+1}$ (see Lemma 2.7 in [BR16]). However we do not need them since H_d is obtained as a derivative of $\overline{G}_{d+1}$.

3.2 The quantum Witten-Kontsevich series

The Hamiltonian densities of the quantum KdV hierarchy form a tau structure. Indeed, the integrability and tau symmetry are presented in Section 3.1.2. The two last conditions of the tau structure are verified in Examples 109 & 110. We can then build the quantum tau functions and in particular the quantum Witten-Kontsevich. To define this quantum tau function, we have to specify our choice of constant terms for the two-point functions and our choice of constant and linear terms in the $\log\tau$ function.

Let $d_1, d_2 \geq 0$. The two-point function is defined by

$$\partial_x \Omega^\hbar_{d_1,d_2} := \frac{1}{\hbar}\left[H_{d_1-1}, \overline{H}_{d_2}\right],$$

where we fix the constant using the recursive formula $\left.\frac{\partial\Omega^{\hbar}_{d_1,d_2}}{\partial p_0}\right|_{p_*=0} = \left.\Omega^{\hbar}_{d_1-1,d_2}\right|_{p_*=0} + \left.\Omega^{\hbar}_{d_1,d_2-1}\right|_{p_*=0}$ with the initial conditions $\left.\Omega^{\hbar}_{0,d}\right|_{p_*=0} = \left.\Omega^{\hbar}_{d,0}\right|_{p_*=0} = H_{d-1}|_{p_*=0}$, with $d \geq 0$. This convention differs from the one used in [BDGR16]. We made this choice so that the quantum Witten-Kontsevich series satisfies the string equation.

Definition 118. The *quantum Witten-Kontsevich series* $\mathscr{F}^{qWK}$ is the $\log\tau$ function of quantum KdV associated to the point $u(x) = x$ of P. Moreover we impose that the coefficient of $\epsilon^{2l}\hbar^{g-l}t_d$ is the coefficient of $\epsilon^{2l}\hbar^{g-l}t_0t_{d+1}$ for any $0 \leq l \leq g$ and $d \geq 0$. We also impose that the constant coefficient of $\epsilon^{2l}\hbar^{g-l}$ is given by $\frac{1}{2g-2}$ times the coefficient of $\epsilon^{2l}\hbar^{g-l}t_1$.

Let k, g be two non negative integers, and a list $(d_1,\dots,d_n)$ of non negative integers. The *quantum correlators* $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l}$ is the coefficient of the quantum Witten-Kontsevich series written as

$$\mathscr{F}^{qWK} = \sum \frac{\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l}}{n!}\epsilon^{2l}(-i\hbar)^{g-l}t_{d_1}\dots t_{d_n} \in \mathbb{C}[[\epsilon,\hbar,t_0,t_1,\dots]].$$

Remark 119. We made the choice of the linear terms (resp. constant term) of $\mathscr{F}^{qWK}$ so that the series satisfies the string equation (resp. the dilaton equation).

Proposition 120. *The classical limit of the quantum Witten-Kontsevich series is the classical Witten-Kontsevich series*

$$\left.\mathscr{F}^{qWK}\right|_{\hbar=0} = \mathscr{F}^{WK}.$$

Indeed, the series $\left.\mathscr{F}^{qWK}\right|_{\hbar=0}$ is the $\log\tau$ function of the classical KdV hierarchy associated to the string solution (i.e. the solution of the KdV hierarchy starting at $u(x) = x$). It is then Witten-Kontsevich series according to the Witten-Kontsevich theorem (see Section 2.2).

By definition, the correlator $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l}$ is obtained from

$$\left.\left(\frac{\partial^{n-2}\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}}{\partial t_{d_3}\dots\partial t_{d_n}}\right)\right|_{t_*=0} \in \mathcal{A}[[\epsilon,\hbar,t_0,t_1,\dots]]$$

by evaluating at $u(x) = x$, then substituting $x = 0$ and finally extracting the coefficient of $\epsilon^l(-i\hbar)^{g-l}$. This is equivalent to evaluate this expression at $u_0 = 0$, $u_1 = 1$, $u_2 = 0$, $u_3 = 0\dots$ and then extracting the coefficient of $\epsilon^l(-i\hbar)^{g-l}$. By definition of $\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}$, we obtain the following expression for the quantum correlators.

Lemma 121. *The quantum correlators are given by*

$$\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l} = i^{g-l}\left[\epsilon^l\hbar^{g-l+n-2}\right]\left.\left[\dots\left[\Omega^{\hbar}_{d_1,d_2},\overline{H}_{d_3}\right],\dots,H_{d_n}\right]\right|_{u_i=\delta_{1,i}},$$

where the notation $\left[\epsilon^l\hbar^{g-l+n-2}\right]$ *means that we extract the coefficient of* $\epsilon^l\hbar^{g-l+n-2}$.

3.2.1 First terms of $\mathscr{F}^{qWK}$

We give the first terms of the quantum Witten-Kontsevich series. These terms are stored in an array in order to emphasize their structure. This structure will be explained in Section 6.1.1. In the box of line l and column k are stored some coefficients of $\epsilon^{2l}\,(-i\hbar)^k$ of the quantum Witten-Kontsevich series.

In the first column are stored some terms of the classical Witten-Kontsevich series. The box of line l corresponds to the genus l intersection numbers of ψ-classes on the moduli space of curves.

Starting from the second column, that is in the purely quantum part of $\mathscr{F}^{qWK}$, the boxes are divided into levels. This division is realized by dashed lines. This level structure will be explained in the Section 6.1.1.

The reader used to intersection number on the moduli space of curves will recognize typical intersection numbers on the moduli space of genus g curves in the boxes of the diagonal $l + k = g$ of the array.

	$\hbar^0$	$\hbar^1$	$\hbar^2$
ϵ^0	$\frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \frac{t_0^4 t_2}{24} + \dots$	$\frac{t_2}{24} + \frac{t_0 t_3}{24} + \frac{t_1 t_2}{24}$ $+ \frac{t_1^2 t_2}{24} + \frac{t_0 t_2^2}{24} + \dots$ - - - - - - - - - - - - - - - - - - $\frac{t_0}{24} + \frac{t_0 t_1}{24}$ $+ \frac{t_0^2 t_2}{48} + \frac{t_0 t_1^2}{24} + \dots$	$\frac{1}{1920} t_6 + \frac{1}{1920} t_0 t_7 + \frac{1}{480} t_1 t_6 + \dots$ - - - - - - - - - - - - - - - - - - $\frac{1}{576} t_4 + \frac{1}{576} t_0 t_5 + \frac{1}{192} t_1 t_4 + \dots$ - - - - - - - - - - - - - - - - - - $\frac{7}{5760} t_2 + \frac{7}{5760} t_0 t_3 + \frac{7}{1920} t_1 t_2 + \dots$
ϵ^2	$\frac{t_1}{24} + \frac{t_0 t_2}{24} + \frac{t_1^2}{48} + \dots$	$\frac{1}{720} t_5 + \frac{1}{720} t_0 t_6 + \frac{1}{240} t_1 t_5 + \dots$ - - - - - - - - - - - - - - - - - - $\frac{1}{576} t_3 + \frac{1}{576} t_0 t_4 + \frac{t_1 t_3}{192} + \dots$ - - - - - - - - - - - - - - - - - - $\frac{1}{2880} t_1 + \frac{1}{2880} t_0 t_2 + \frac{t_1^2}{1920} + \dots$	
ϵ^4	$\frac{t_4}{1152} + \frac{1}{384} t_1 t_4$ $+ \frac{29}{5760} t_2 t_3 + \dots$		

Chapter 4

Hurwitz numbers

In Section 4.1, we give two equivalent definitions of Hurwitz numbers. Then, we present the so-called cut-and-join equations which allows to recursively compute Hurwitz numbers. We explain how to relate this equation to the restriction at $\epsilon = 0$ of the quantum KdV hierarchy. In the following sections, we present two special cases of Hurwitz numbers. In Section 4.2, we present simple Hurwitz numbers and the ELSV formula. This formula expresses simple Hurwitz numbers in term of intersection numbers on $\overline{\mathcal{M}}_{g,n}$. In Section 4.3, we present one-part double Hurwitz numbers. These numbers are closely related to the quantum Witten-Kontsevich series restricted at $\epsilon = 0$. We introduce the material to precisely state this relation. We also give an explicit formula for one-part double Hurwitz numbers.

4.1 Definitions, the cut-and-join operators and quantum KdV

4.1.1 Definitions of Hurwitz numbers

Definition 122. Two maps of Riemann surfaces $f : X \to Y$ and $\tilde{f} : \tilde{X} \to Y$ are *isomorphic* if there is a biholomorphism between Riemann surfaces $\alpha : X \to \tilde{X}$ such that $f = \tilde{f} \circ \alpha$.

An *automorphism* of f is a biholomorphism $\beta : X \to X$ such that $f = f \circ \beta$. The automorphism group of f will be denoted by $\mathrm{Aut}\,(f)$. Its cardinal will be denoted by $|\mathrm{Aut}\,(f)|$.

Definition 123. Fix two nonnegative integers g, m, a positive integer d and m partitions $\mu^1, \ldots, \mu^m$ of d. We denote by $l\left(\mu^i\right)$ the length of the partition μ^i and we set $r = 2d + 2g - 2 - md + \sum_{i=1}^{m} l\left(\mu^i\right)$. The *Hurwitz number* $H^g_{\mu^1,\ldots,\mu^m}$ is defined by

$$H^g_{\mu^1,\ldots,\mu^m} = \sum_{[f]} \frac{1}{|\mathrm{Aut}\,(f)|},$$

where the summation is over isomorphism classes of degree d holomorphic maps $f : C \to \left(\mathbb{P}^1, b_1, \ldots, b_m, x_1, \ldots, x_r\right)$ such that

- C is a connected Riemann surface of genus g,

- the target curve $\mathbb{P}^1$ has $m+r$ fixed and pairwise distinct marked points $b_1, \ldots, b_m, x_1, \ldots, x_r$,

- f is a ramified covering, its ramification profile over b_i is given by μ^i, it has only one simple ramification over each x_i and no ramification elsewhere..

The map f is called a *Hurwitz cover* of type $\left(g, \mu^1, \ldots, \mu^m\right)$.

Remark 124. From the input of the genus g and the m partitions $\mu^1, \ldots, \mu^m$, we obtain the number of simple ramifications r of the Hurwitz cover of type $\left(g, \mu^1, \ldots, \mu^m\right)$ by the Riemann-Hurwitz formula.

Example 125. We compute the Hurwitz number $H^0_{(d),(d)}$. In the target $\mathbb{P}^1$, we fix $b_1 = 0$ and $b_2 = \infty$. It follows from the Riemann-Roch theorem that any genus 0 curve is isomorphic to $\mathbb{P}^1$. The Riemann-Hurwitz formula gives $r = 0$. Moreover, a holomorphic map from $\mathbb{P}^1$ to $\mathbb{P}^1$ completely ramified over 0 and ∞ and with no ramification elsewhere is isomorphic to $z \to z^d$. The cardinal of the automorphism group of this map is d. Thus, we find $H^0_{(d),(d)} = \frac{1}{d}$.

We now give a combinatorial definition of Hurwitz numbers.

Definition 126. Fix two nonnegative integers g, m, a positive integer d and m partitions $\mu^1, \ldots, \mu^m$ of d. We denote by $l\left(\mu^i\right)$ the length of the partition μ^i and we set $r = 2d + 2g - 2 - md + \sum_{i=1}^m l\left(\mu^i\right)$. Moreover, we denote by (2) the partition of d given by $(2, 1, \ldots, 1)$. The list $(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_r)$ is a *factorization of type* $\left(\mu^1, \ldots, \mu^m, \underbrace{(2), \ldots, (2)}_{r}\right)$ if

- σ_i is permutation of $\mathcal{S}_d$ and its cycle decomposition has $l\left(\mu^i\right)$ cycles of lengths given by $\mu^i_1, \ldots, \mu^i_{l(\mu^i)}$,

- τ_i is a transposition of $\mathcal{S}_d$,

- $\tau_r \ldots \tau_1 \sigma_m \ldots \sigma_1 = Id$,

- the group generated by $(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_r)$ acts transitively on $\{1, \ldots, d\}$.

We denote by $F^g\left(\mu^1, \ldots, \mu^m, (2), \ldots, (2)\right)$ the number of factorization of type $\left(\mu^1, \ldots, \mu^m, (2), \ldots, (2)\right)$. In this notation, the number of partitions of type (2) is not specified since we can obtain it from g and and the partitions $\mu^1, \ldots, \mu^m$.

Proposition 127. *The Hurwitz number* $H^g_{\mu^1,\ldots,\mu^m}$ *is given by*

$$H^g_{\mu^1,\ldots,\mu^m} = \frac{\left|F^g\left(\mu^1, \ldots, \mu^m, (2), \ldots, (2)\right)\right|}{d!}.$$

A proof can be found in [CM16]. It is clear from the combinatorial definition of Hurwitz numbers that these numbers are finite.

4.1.2 The cut-and-join operators and the quantum KdV hierarchy

Fix $m \geq 1$. The generating series of Hurwitz numbers is

$$\mathfrak{H}_m = \sum H^g_{\mu^1,\ldots,\mu^m} q^{(1)}_{\mu^1} \ldots q^{(m)}_{\mu^m} \frac{u^r}{r!} z^{1-g} s^d,$$

where the summation is over the d, g, r and the partitions $\mu^1, \ldots, \mu^m$ of d. If $\mu^i = \left(\mu^i_1, \ldots, \mu^i_{l(\mu^i)}\right)$ is a partition of d, then the notation $q^{(i)}_{\mu^i}$ stands for $q^{(i)}_{\mu^i_1} \ldots q^{(i)}_{\mu^i_{l(\mu^i)}}$. The generating series of disconnected Hurwitz numbers, i.e. Hurwitz numbers where the covering curve is not necessarily connected, is given by

$$\mathfrak{H}^\bullet_m := \exp\left(\mathfrak{H}_m\right).$$

A disconnected Hurwitz number can be computed from disconnected Hurwitz numbers with a smaller number of simple ramifications. This is the content of the cut-and-join equations.

Proposition 128. *Fix an integer i between 1 and m. The ith cut-and-join operator is*

$$M^{(i)} = \frac{1}{2}\left(\sum_{a,b\geq 1} ab q^{(i)}_{a+b} z \frac{\partial^2}{\partial q^{(i)}_a \partial q^{(i)}_b} + (a+b)\, q^{(i)}_a q^{(i)}_b \frac{\partial}{\partial q^{(i)}_{a+b}}\right),$$

The generating series $\mathfrak{H}^\bullet_m$ satisfies the ith cut-and-join equation

$$\frac{\partial}{\partial u}\mathfrak{H}^\bullet_m = M^{(i)} \mathfrak{H}^\bullet_m.$$

A proof can be found in [CM16]. These equations determine any Hurwitz number from Hurwitz numbers with no simple ramifications by

$$\mathfrak{H}^\bullet_m = \exp\left(uM^{(i)}\right) \mathfrak{H}^\bullet_m|_{u=0}.$$

Example 129. Suppose $m = 1$, we then suppress the the upper index of the q variables. We deduce from the Riemann-Hurwitz formula that $\mathfrak{H}_1|_{u=0} = q_1 z s$. Thus $\mathfrak{H}^\bullet_1 = \exp\left(uM\right) e^{q_1 z s}$.

We explain how to relate the Hamiltonian $\left.\overline{H}_1\right|_{\epsilon=0}$ of the quantum KdV hierarchy with a cut-and-join operator. Let the map ρ from $\mathcal{F}\left(P\right)[[\hbar]]$ to the operators acting on $\mathbb{C}\left[[q_a]\right]$ defined by

$$\rho\left(p_0\right) = 0, \quad \rho\left(p_{-a}\right) = q_a, \quad \rho\left(p_a\right) = ia\hbar\frac{\partial}{\partial q_a},$$

for any $a > 0$ and $f\left(q\right) \in \mathbb{C}\left[[q_a]\right]$, moreover we use a normal ordering procedure: we first write a monomial in the p variables with the $p_{<0}$ on the left and then replace the variable p_a, for any $a \in \mathbb{Z}$, by the operator $\rho\left(p_a\right)$.

Proposition 130. *We have*

$$\rho\left(\overline{H}_1\Big|_{\epsilon=0}\right) = (i\hbar)\,\frac{1}{2}\sum_{a,b\geq 1}\left((i\hbar)\,abq_{a+b}\frac{\partial^2}{\partial q_a\partial q_b} + (a+b)\,q_aq_b\frac{\partial}{\partial q_{a+b}}\right).$$

That is the image of $\overline{H}_1\Big|_{\epsilon=0}$ *by* ρ *is a cut-and-join operator multiplied by* $i\hbar$*, with* $z := i\hbar$*.*

Proof. In Example 111, we computed the expression of $\overline{H}_1$. Its restriction at $\epsilon = 0$ is

$$\begin{aligned}\overline{H}_1\Big|_{\epsilon=0} &= \int\left(\frac{u_0^3}{3!} - i\hbar\frac{u_0}{24}\right)dx\\ &= \sum_{a_1+a_2+a_3=0}\frac{p_{a_1}p_{a_2}p_{a_3}}{6} - i\hbar p_0.\end{aligned}$$

□

Remark 131. Let $f, g \in \mathcal{F}(P)[[\hbar]]$, we have

$$\rho(f \star g) = \rho(f)\,\rho(g)\,,$$

hence the commutator $[\rho(f), \rho(g)]$ is given by $\rho(f \star g - g \star f)$. In particular, the image by ρ of the quantum Hamiltonians of the quantum KdV hierarchy commute. We then obtain an alternative quantization of the KdV hierarchy. It is not a deformation quantization procedure, we quantize here in the Schrödinger picture.

Remark 132. This relation between $\overline{H}_1|_{\epsilon=0}$ and a cut-and-join operator could explain the presence of Hurwitz numbers in the quantum Witten-Kontsevich series (see Theorem 1). A further investigation of this remark will be carry out in a future work.

4.2 Simple Hurwitz numbers

Definition 133. Fix two nonnegative integer g, d and a partition μ of d. A *simple Hurwitz number* is a Hurwitz number with only one ramification profile prescribed μ. We denote them by

$$H_\mu^g.$$

In [ELSV00], Ekedahl, Lando, Shapiro and Vainshtein proved a remarkable formula expressing simple Hurwitz numbers in term of Hodge integrals with one λ-class.

Proposition 134 (ELSV formula)**.** *Fix two nonnegative integers* g *and* n *such that* $2g-2+n > 0$*. Fix a positive integer* d *and let* $\mu = (\mu_1, \ldots, \mu_n)$ *be a partition of* d*. We have*

$$H_\mu^g = (2g-2+n+d)!\left(\prod_{i=1}^n\frac{\mu_i^{\mu_i}}{\mu_i!}\right)\int_{\overline{\mathcal{M}}_{g,n}}\frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{\prod_{i=1}^n(1-\mu_i\psi_i)}.$$

Remark 135. This formula also proves the polynomiality of the Hurwitz number H_μ^g in the ramifications $\mu_1, \ldots, \mu_n$.

As a consequence of the ELSV formula, we give two results.

- In [KL07], Kazarian and Lando deduced a proof of Witten's conjecture from the ELSV formula.
- In [Kaz09], Kazarian deduced from the ELSV formula that the generating series of Hodge integrals satisfy the KP hierarchy up to a change of variables. This gives another proof of the Witten conjecture. He also deduced another proof of the Virasoro constraints for the Witten-Kontsevich series.

4.3 One-part double Hurwitz numbers

Definition 136. Fix a nonnegative integer g a positive integer d and a partition $\mu = (\mu_1, \ldots, \mu_n)$ of d. The *one-part double Hurwitz numbers* are Hurwitz numbers with two ramification profiles prescribed. One profile of ramification is given by (d) and the other one by μ, we denote such a number by

$$H^g_{(d),\mu}.$$

These numbers were studied by Goulden, Jackson and Vakil in [GJV05]. They proved the polynomiality of $H^g_{(d),\mu}$ in the ramifications $\mu_1, \ldots, \mu_n$. Moreover, using the expression of Hurwitz numbers in term of characters of the symmetric group, they obtained the following expression for the one-part double Hurwitz numbers.

Proposition 137 (Goulden-Jackson-Vakil [GJV05])**.** *We have*

$$H^g_{(d),\mu} = r! d^{r-1} \left[z^{2g}\right] \frac{\prod_{i=1}^n S\left(\mu_i z\right)}{S\left(z\right)},$$

where $d = \mu_1 + \cdots + \mu_n$ is the degree of the ramified covering, $r = 2g - 1 + n$ is the number of simple ramifications and $S\left(z\right) = \frac{\sinh(z/2)}{z/2}$.

Remark 138. Comparing this formula with Eq (2.3.1), we obtain

$$H^g_{(d),\mu} = r! d^{r-1} \cdot \psi_1^{2g-2+n} \mathrm{DR}_g \left(-\sum \mu_i, \mu_1, \ldots, \mu_n\right).$$

This link between one-part double Hurwitz numbers and intersection of ψ-class on the double ramification cycle is the starting point of the proof of Theorem 1 relating the coefficients of one-part double Hurwitz numbers with the coefficients of the quantum Witten-Kontsevich series.

Goulden, Jackson and Vakil also conjectured an ELSV-type formula for one-part double Hurwitz numbers.

Conjecture 1 ([GJV05]). *Fix two nonnegative integers g and n such that $2g-2+n>0$. Fix a positive integer d and let $\mu=(\mu_1,\ldots,\mu_n)$ be a partition of d. There exists a space $X_{g,n}$ of dimension $4g-3+n$, with n classes $\tilde{\psi}_i \in H^2(X_{g,n})$ and $g+1$ classes $\tilde{\lambda}_i \in H^{4i}(X_{g,n})$ such that*

$$H^g_{(d),\mu} = rd! \int_{X_{g,n}} \frac{\tilde{\lambda}_0 - \tilde{\lambda}_1 + \cdots + (-1)^g \tilde{\lambda}_g}{\prod_{i=1}^n \left(1-\mu_i\tilde{\psi}_i\right)},$$

where $r=2g-1+n$ counts the number of simple ramifications of Hurwitz covers.

This conjecture is still open.

Remark 139. In a paper recently appeared [DL20], Do and Lewanski gave three formulas expressing one-part double Hurwitz numbers in term of intersection numbers on certain moduli spaces. None of these formulas answer the Goulden, Jackson and Vakil conjecture but each of them share some properties with their formula.

Remark 140. A natural question is: can we generalize this conjecture and look for an ELSV type formula for double Hurwitz numbers (i.e. Hurwitz numbers with two ramification profiles prescribed) ? The answer is negative. Indeed, double Hurwitz numbers are not polynomial in their ramifications (see [GJV05]). Thus, we can not hope to get an ELSV type formula for double Hurwitz numbers.

To justify this conjecture, Goulden, Jackson and Vakil studied the coefficients of the polynomial $H^g_{(d),\mu}$ in the indeterminates $\mu_1,\ldots,\mu_n$ and proved that they behave as the simple hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_k \psi_1^{d_1}\ldots\psi_n^{d_n}$ appearing as the coefficients of the ELSV formula. We introduce the following notation for these coefficients.

Definition 141. Fix two nonnegative integers g and n such that $2g-2+n>0$. Let $(d_1,\ldots,d_n)$ be a list of non negative integers. The number

$$\langle\langle \tau_{d_1}\ldots\tau_{d_n}\rangle\rangle_g = (-1)^{\frac{4g-3+n-\sum d_i}{2}} \left[\mu_1^{d_1}\ldots\mu_n^{d_n}\right]\left(\frac{H^g_{(d),\mu}}{r!d}\right) \tag{4.3.1}$$

is called a *Hurwitz correlator.*

Remark 142. The conjectural formula of Goulden, Jackson and Vakil would imply

$$\langle\langle \tau_{d_1}\ldots\tau_{d_n}\rangle\rangle_g = \int_{X_{g,n}} \tilde{\lambda}_k \tilde{\psi}_1^{d_1}\ldots\tilde{\psi}_n^{d_n}, \tag{4.3.2}$$

where $2k=4g-3+n-\sum d_i$.

Remark 143. Fix the nonnegative integers g and n. Form Proposition 137, it is clear that Hurwitz correlators $\langle\langle \tau_{d_1}\ldots\tau_{d_n}\rangle\rangle_g$ vanishes if $\sum d_i$ is outside the interval

$$[2g-3+n, 4g-3+n]$$

or if $\sum d_i$ has the parity of n. Hence the Hurwitz correlators can be non zero only if $\sum d_i$ only takes one of these $g+1$ values, this corresponds to the $g+1$ conjectural classes $\tilde{\lambda}_k$ in Eq. (4.3.2).

Proposition 144 ([GJV05]). *The Hurwitz correlators satisfy the string equation*

$$\langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g = \sum_{i=1}^{n} \langle\langle \tau_{d_1} \dots \tau_{d_i - 1} \dots \tau_{d_n} \rangle\rangle_g .$$

The Hurwitz correlators satisfy the dilaton equation

$$\langle\langle \tau_1 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g = (2g - 2 + n) \langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g .$$

The Hurwitz correlators corresponding to the conjectural intersection with $\tilde{\lambda}_g$ are expressed in term of intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

Proposition 145. *Fix two nonnegative integers g and n such that $2g-2+n > 0$. The Hurwitz correlators satisfying $\sum_{i=1}^{n} d_i = 2g - 3 + n$ are given by*

$$\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi_1^{d_1} \dots \psi_n^{d_n}.$$

In particular, we can compute these numbers using the λ_g-conjecture see Section 2.1.5.

Remark 146. In [Sha08], Shadrin used an approach similar to Kazarian in [Kaz09] to prove that the generating series of Hurwitz correlators properly arranged satisfies the KP hierarchy.

Chapter 5

Eulerian numbers

One of the difficulties to obtain an expression for the quantum Witten-Kontsevich series is to compute the commutator of the star product. In the first chapter, we explained that this amounts to study Buryak and Rossi's polynomials

$$C^{r_1,\dots,r_q}(N) = \sum_{k_1+\cdots+k_q=N} k_1^{r_1}\dots k_q^{r_q} = \left[t^N\right]\prod_{i=1}^{q}\left(\sum_{k\geq 1} k^{r_i}t^k\right),$$

where $r_1,\dots,r_q$ and N are nonnegative integers numbers. In this section, we explain how to obtain an expression for $\sum_{k\geq 1} k^r t^k$ in term of the so-called Eulerian numbers. These numbers enjoy many properties. They will particularly simplify our computation of the commutator of the star product.

In the first section, we present Eulerian numbers and some of their properties. In the second section, we deduce from these properties a combinatorial formula. This formula is a key ingredient of the proof of the main theorem (Theorem 1).

5.1 Generalities on Eulerian numbers

We present Eulerian numbers and prove some of their properties following [Pet15].

Definition 147. Fix two nonnegative integers k,n and a permutation $\sigma\in S_n$. A descent of the permutation σ is a pair $(i,i+1)$ such that $\sigma(i)>\sigma(i+1)$, where $i\in\{1,\dots,n-1\}$. The *Eulerian number* $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ is the number of permutation of S_n with k descents.

Example 148. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in S_4$ has one desent.

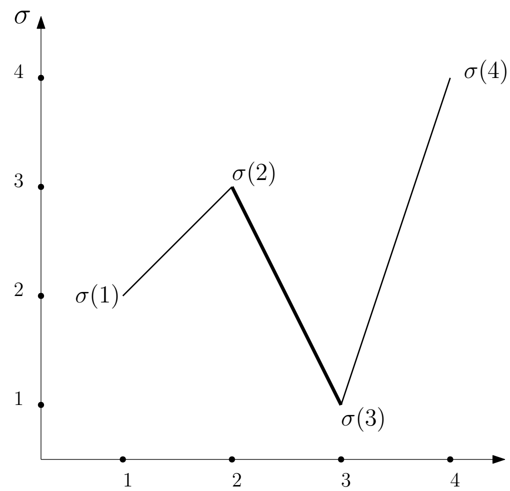


Figure 5.1.1: A descent of σ corresponds to an actual descent in the graph of σ.

Eulerian numbers satisfy the following recursive property.

Proposition 149. *We have*

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle = (n-k) \left\langle \begin{array}{c} n-1 \\ k-1 \end{array} \right\rangle + (k+1) \left\langle \begin{array}{c} n-1 \\ k \end{array} \right\rangle .$$

Proof. Let σ be a permutation of S_n written as a list $(\sigma(1), \dots, \sigma(n))$ and suppose σ has k descents. If we delete n from this list, we obtain a permutation of S_{n-1} with k or $k-1$ descents. We prove the recursive formula in the other way : start from a permutation of S_{n-1} with k or $k-1$ desents, we count the number of possiblilities to add n in order to obtain a permutation of S_n with k descents.

- Let σ be a permutation of S_{n-1} written as a list $(\sigma(1), \dots, \sigma(n-1))$ and suppose σ has k descents. In order to get a permutation of S_n with k descents, you can add n at the end of the list or at the middle of any of the k descents. Thus, there are $k+1$ choices.
- Let σ be a permutation of S_{n-1} written as a list $(\sigma(1), \dots, \sigma(n-1))$ and suppose σ has $k-1$ descents. In order to get a permutation of S_n with k descents, you can add n at the begining of the list or at the middle of any of the $n-k-1$ ascents. Thus there are $n-k$ choices.

□

Definition 150. The *Eulerian polynomial* $E_n(t)$ is the generating polynomial of Eulerian numbers

$$E_n(t) := \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)} = \sum_{k=0}^{n-1} \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle t^k.$$

The recursive property of Eulerian numbers translates to the following identity of the generating polynomial.

Corollary 151. *We have*

$$E_{n+1}(t) = (1+nt)\, E_n(t) + t\,(1-t)\, E_n^{'}(t)\,. \tag{5.1.1}$$

Proposition 152 (Carlitz identity)**.** *Let n be a nonnegative integer. Let t be a formal variable. We have*

$$\sum_{k\geq 1} k^n t^k = \frac{tE_n(t)}{(1-t)^{n+1}}.$$

Proof. Define $e_n(t) = (1-t)^{n+1}\sum_{k\geq 1} k^n t^{k-1}$. This series satisfies Eq 5.1.1 and $e_0(t) = 1 = E_0(t)$, we then deduce $e_n(t) = E_n(t)$ for any $n \geq 0$. $\square$

Remark 153. Eulerian numbers have the nice symmetry $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ n-k-1 \end{matrix} \right\rangle$. Indeed, let $\sigma = (\sigma(1), \ldots, \sigma(n))$ be a permutation of S_n with k descents. We associate to σ the permutation $\tilde{\sigma} = (\sigma(n), \ldots, \sigma(1))$, i.e. we simply read σ in the opposite direction. The descents of $\tilde{\sigma}$ correspond to the ascents of σ : there are $n-k-1$ of them. This map is obviously a bijection.

We deduce from this symmetry of Eulerian numbers that they are palindromic: $t^{n-1}E_n(1/t) = E_n(t)$. We re-prove from this symmetry, the parity property of the polynomials $C^{r_1,\ldots,r_q}(N)$. The proof is moslty inspired by the one of [BR16]. We introduce the coefficients of the polynomial

$$C^{r_1,\ldots,r_q}(N) = \sum_{j\geq 0} C_j^{r_1,\ldots,r_q} N^j.$$

The LHS is given by $\left[t^N\right]\prod_{i=1}^{q}\left(\sum_{k\geq 1} k^{r_i}t^k\right)$ and the RHS by $\left[t^N\right]\sum_{j\geq 0} C_j^{r_1,\ldots,r_q}\left(1+\sum_{k\geq 1} k^j t^k\right)$, so that we have the equality of series

$$\prod_{i=1}^{q}\left(\sum_{k\geq 1} k^{r_i}t^k\right) = \sum_{j\geq 0} C_j^{r_1,\ldots,r_q}\left(1+\sum_{k\geq 1} k^j t^k\right).$$

Using the Carlitz identity, we get

$$\prod_{i=1}^{q}\left(\frac{tE_{r_i}(t)}{(1-t)^{r_i+1}}\right) = \sum_{j\geq 0} C_j^{r_1,\ldots,r_q}\left(1+\frac{tE_j(t)}{(1-t)^{j+1}}\right).$$

The polindromicity of E_r implies that $\frac{tE_r(t)}{(1-t)^{r+1}} = (-1)^{r+1}\frac{tE_r(t)}{(1-t)^{r+1}}$. Using this property in the preceeding equation and extracting the coefficient of t^N, we find

$$(-1)^{\sum r_i+q}\, C^{r_1,\ldots,r_q}(N) = (-1)\, C^{r_1,\ldots,r_n}(N)\,.$$

Thus, $C^{r_1,\ldots,r_q}(N)$ has the parity of $\sum r_i + q - 1$.

The exponential generating series of Eulerian polynomials has the following explicit form.

Proposition 154. *Let t and z be two formal variables. We have*

$$\sum_{n\geq 0}\frac{E_n\left(t\right)}{n!}z^n=\frac{t-1}{t-e^{z(t-1)}}.$$

Proof. We construct a permutation $\sigma=\left(\sigma\left(1\right),\ldots,\sigma\left(n\right)\right)$ with k descent in the following way. First choose the position of n in the list. If n is at the last position, their are $\left\langle \begin{array}{c} n-1 \\ k \end{array} \right\rangle$ ways to construct such a permutation. If n is at the ith position , where $1\leq i\leq n-1$, we choose the $i-1$ integers on the left of n in the list (and then the $n-i$ on the right), there are $\left(\begin{array}{c} n-1 \\ i-1 \end{array} \right)$ possibilities. Since n is at the position i, the pair $(i,i+1)$ is a descent. We then have to distribute the $k-1$ remaining desents : there are $\left[t^{k-1}\right]E_{i-1}\left(t\right)E_{n-i}\left(t\right)$ possibilities. We then find

$$E_n\left(t\right)=E_{n-1}\left(t\right)+t\sum_{i=1}^{n-1}\left(\begin{array}{c} n-1 \\ i-1 \end{array} \right)E_{i-1}\left(t\right)E_{n-1}\left(t\right).$$

This equation translates to the following equation on $E\left(t,z\right)=\sum_{n\geq 0}\frac{E_n(t)}{n!}z^n$,

$$\frac{dE\left(t,z\right)}{dz}=tE^2\left(t,z\right)+\left(1-t\right)E\left(z,t\right).$$

Solving this differential equation with the initial condition $E\left(t,0\right)=1$ gives the result. $\square$

5.2 A combinatorial formula

We deduce from the explicit form of the generating function of Eulerian numbers a combinatorial formula.

Recall Notation 98, we have

$$S\left(z\right)=\frac{\mathrm{sh}\left(z/2\right)}{z/2}=\sum_{l\geq 0}\frac{z^{2l}}{2^{2l}\left(2l+1\right)!}.$$

Proposition 155. *Let A,B,t and z be some formal variables. We have*

$$\begin{aligned}\exp\left(\sum_{k>0}ABz^2kS\left(kAz\right)S\left(kBz\right)t^k\right)&=\frac{\left(1-te^{\frac{A-B}{2}z}\right)\left(1-te^{-\frac{A-B}{2}z}\right)}{\left(1-te^{\frac{A+B}{2}z}\right)\left(1-te^{-\frac{A+B}{2}z}\right)}\\&=1+4\sum_{k>0}\frac{\mathrm{sh}\left(\frac{A}{2}z\right)\mathrm{sh}\left(\frac{B}{2}z\right)}{\mathrm{sh}\left(\frac{A+B}{2}z\right)}\mathrm{sh}\left(k\frac{A+B}{2}z\right)t^k.\end{aligned}$$

The proof of this proposition is divided in two lemmas.

Lemma 156. *Let A, B, t and z be some formal variables. We have*

$$\sum_{k>0} ABz^2 kS(kAz)\, S(kBz)\, t^k = \ln\left(\frac{\left(1-te^{\frac{A-B}{2}z}\right)\left(1-te^{-\frac{A-B}{2}z}\right)}{\left(1-te^{\frac{A+B}{2}z}\right)\left(1-te^{-\frac{A+B}{2}z}\right)}\right).$$

Proof. We start from the LHS. Use the developed expression of S (see Notation 98) to obtain

$$\sum_{k>0} ABz^2 kS(kAz)\, S(kBz)\, t^k = \sum_{k>0}\sum_{l_1,l_2\geq 0} \frac{A^{2l_1+1}B^{2l_2+1}z^{2(l_1+l_2)+2}}{2^{2(l_1+l_2)}(2l_1+1)!\,(2l_2+1)!} k^{2(l_1+l_2)+1}t^k.$$

Then, we use the Carlitz identity (Proposition 152) to compute the sum running over k. We obtain

$$\sum_{l_1,l_2\geq 0} \frac{A^{2l_1+1}B^{2l_2+1}z^{2(l_1+l_2)+2}}{2^{2(l_1+l_2)}(2l_1+1)!\,(2l_2+1)!} \frac{tE_{2(l_1+l_2)+1}(t)}{(1-t)^{2(l_1+l_2)+2}}.$$

Simplifying this expression using

$$\sum_{l_1+l_2=l} \frac{A^{2l_1+1}B^{2l_2+1}}{(2l_1+1)!\,(2l_2+1)!} = \frac{1}{2}\frac{(A+B)^{2l+2}-(A-B)^{2l+2}}{(2l+2)!},$$

we obtain

$$\sum_{l\geq 0} \frac{(A+B)^{2l+2}-(A-B)^{2l+2}}{2^{2l+1}(2l+2)!} z^{2l+2}\frac{tE_{2l+1}(t)}{(1-t)^{2l+2}}.$$

Denote by $F(z,t) := -z - \ln\left(t-e^{-z}\right) + \ln(t-1)$. According to Corollary **??**, we have

$$\sum_{l\geq 0} \frac{(A+B)^{2l+2}}{2^{2l+1}(2l+2)!} z^{2l+2}\frac{tE_{2l+1}(t)}{(1-t)^{2l+2}} = F\left(\frac{A+B}{2}z,t\right) + \left(-\frac{A+B}{2}z,t\right)$$

and

$$\sum_{l\geq 0} \frac{(A-B)^{2l+2}}{2^{2l+1}(2l+2)!} z^{2l+2}\frac{tE_{2l+1}(t)}{(1-t)^{2l+2}} = F\left(\frac{A-B}{2}z,t\right) + \left(-\frac{A-B}{2}z,t\right).$$

Finally, we remark that

$$\begin{aligned} &F\left(\frac{A+B}{2}z,t\right) + \left(-\frac{A+B}{2}z,t\right) - \left(F\left(\frac{A-B}{2}z,t\right) + \left(-\frac{A-B}{2}z,t\right)\right) \\ &= \ln\left(\frac{\left(1-te^{\frac{A-B}{2}z}\right)\left(1-te^{-\frac{A-B}{2}z}\right)}{\left(1-te^{\frac{A+B}{2}z}\right)\left(1-te^{-\frac{A+B}{2}z}\right)}\right)\end{aligned}$$

to obtain the RHS of the Lemma. ∎

Lemma 157. *Let A, B and t be some formal variables. We have*

$$\frac{\left(1-te^{A-B}\right)\left(1-te^{B-A}\right)}{\left(1-te^{A+B}\right)\left(1-te^{-A-B}\right)}=1+4\sum_{k>0}\frac{\mathrm{sh}\left(A\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A+B\right)}\mathrm{sh}\left(k\left(A+B\right)\right)t^{k}.$$

Proof. Start from the LHS of the equality. Develop the two geometric series of the denominators, we obtain

$$\frac{1}{\left(1-te^{A+B}\right)\left(1-te^{-A-B}\right)}=\sum_{n,m\geq 0}e^{(n-m)(A+B)}t^{n+m}=\sum_{k\geq 0}\frac{\mathrm{sh}\left(\left(k+1\right)\left(A+B\right)\right)}{\mathrm{sh}\left(A+B\right)}t^{k}.$$

Express the numerator as $\left(1-te^{A-B}\right)\left(1-te^{B-A}\right)=t^{2}-2t\mathrm{ch}\left(A-B\right)+1$. We then obtain the following expression for $\frac{\left(1-te^{A-B}\right)\left(1-te^{B-A}\right)}{\left(1-te^{A+B}\right)\left(1-te^{-A-B}\right)}$:

$$\frac{1}{\mathrm{sh}\left(A+B\right)}\sum_{k\geq 0}\left(\mathrm{sh}\left(\left(k-1\right)\left(A+B\right)\right)-2\mathrm{ch}\left(A-B\right)\mathrm{sh}\left(k\left(A+B\right)\right)+\mathrm{sh}\left(\left(k+1\right)\left(A+B\right)\right)\right)t^{k}.$$

We use first the hyperbolic identity $\mathrm{sh}\left(\left(k-1\right)\left(A+B\right)\right)+\mathrm{sh}\left(\left(k+1\right)\left(A+B\right)\right)=2\mathrm{ch}\left(A+B\right)\mathrm{sh}\left(k\left(A+B\right)\right)$ and then $\mathrm{ch}\left(A+B\right)\mathrm{ch}\left(A-B\right)=2\mathrm{sh}\left(A\right)\mathrm{sh}\left(B\right)$ to obtain the result. $\square$

The first equality of Proposition 155 is obtained from Lemma 156. The second is given by the formula of Lemma 157 with $A:=\frac{A}{2}z$ and $B:=\frac{B}{2}z$.

Chapter 6

Study of the quantum Witten-Kontsevich series

6.1 Statement of the results

6.1.1 The correlators of the Witten-Kontsevich series

The quantum Witten-Kontsevich series $\mathscr{F}^{qWK}$ is an element of $\mathbb{C}\,[[\epsilon, \hbar, t_0, t_1, \ldots]]$. Its classical part, obtained by plugging $\hbar = 0$, is the Witten-Kontsevich series $\mathscr{F}^{WK}$ of Section 2.2.

The following theorem concerns the restriction $\epsilon = 0$ of the quantum Witten-Kontsevich series. The coefficients of $\mathscr{F}^{qWK}\big|_{\epsilon=0}$ are expressed in terms of one-part double Hurwitz numbers.

Theorem 1. *Fix two nonnegative integers g, n and a list of nonnegative integers $(d_1, \ldots, d_n)$. We have*

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \langle\langle \tau_{d_1} \ldots \tau_{d_n} \rangle\rangle_g \,.$$

Thus we have a geometric interpretation for the coefficients of $\epsilon^0 \hbar^g$ and $\epsilon^{2g} \hbar^0$ of the quantum Witten-Kontsevich series. So far there is no such interpretation for the other coefficients, but we have a conjecture for some of them. Let us first explain some vanishing properties of the correlators.

Level structure

The correlators satisfy some vanishing properties similar to the Hurwitz correlators (see Proposition **??**).

Proposition 158. *Fix three nonnegative integers g, n, l such that $l \leq g$. The correlator $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{l,g-l}$ vanishes if*

$$\sum_{i=1}^{n} d_i > 4g - 3 + n - l \quad \text{or if} \quad \sum d_i \equiv n - l \,(\text{mod } 2)\,,$$

where $(d_1, \ldots, d_n)$ is a list of nonnegative integers.

Conjecture 2. *Fix three nonnegative integers g, n, l such that $l \leq g$. The correlator $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ vanishes if*

$$\sum_{i=1}^{n} d_i < 2g - 3 + n - l,$$

where $(d_1, \dots, d_n)$ is a list of nonnegative integers.

Hence, the correlators are possibly nonzero only when $\sum d_i$ takes the $g+1$ values of the interval $[2g-3+n-l, 4g-3+n-l]$ with the parity of its maximum (or minimum). We say that the correlators $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ are structured in $g+1$ levels.

Minimal level

We have the following geometrical interpretation for the minimal level of the correlators.

Conjecture 3. *Fix three nonnegative integers g, n, l such that $l \leq g$. When $\sum d_i = 2g - 3 + n - l$, the correlators are given by*

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \lambda_l \psi_1^{d_1} \dots \psi_n^{d_n}.$$

Remark 159. Let us check the level structure and the minimal level property when $\epsilon = 0$. In this case, the correlators $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$, with $g \geq 0$ are equal to $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ according to the main theorem. The level structure of the correlators follows from the similar level structure described in Remark 143. The minimal level property in this case reads

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \lambda_0 \psi_1^{d_1} \dots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi_1^{d_1} \dots \psi_n^{d_n},$$

which follows from the analogous equality for Hurwitz correlators [GJV05, Proposition 3.12].

Remark 160. When $l = g$, the correlators $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{g,0}$ are the coefficients of the classical Witten-Kontsevich series. They are given by

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{g,0} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

They correspond to the top level of the level structure : $\sum d_i = 3g - 3 + n$. All the other levels vanish. In particular, the bottom level is given by

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g^2 \psi_1^{d_1} \dots \psi_n^{d_n} = 0.$$

Recap table

The following table presents the structure of the correlators of the quantum Witten-Kontsevich series coming from Theorem 1, Proposition 158, Conjecture 2 and Conjecture 3. Fix three nonnegative integers l, k and n. In the box corresponding to the l-th line and the k-th column we store the correlators

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,k}$$

coming from the coefficients of $\epsilon^{2l} \hbar^k$ of $\mathscr{F}^{qWK}$, where $(d_1, \dots, d_n)$ is a list of nonnegative integers. We set $g := l + k$ in the table.

The first row and column in the table present proved facts, while the cells $k, l \geq 1$ present new results partially proved.

	$\hbar^0$	$\hbar^1$	$\cdots$	$\hbar^k$	
ϵ^0	$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$	$\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_1$ s.t. $\sum d_i = 4-3+n$ $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_1 = \int_{\overline{\mathcal{M}}_{1,n}} \lambda_0\lambda_1\psi_1^{d_1} \dots \psi_n^{d_n}$	$\cdots$	$\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ s.t. $\sum d_i = 4g-3+n$ $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ s.t. $\sum d_i = 4g-5+n$ $\vdots$ $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ s.t. $\sum d_i = 2g-1+n$ $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g\psi_1^{d_1} \dots \psi_n^{d_n}$	$\underbrace{k}_{=g}+1$
ϵ^2	$\int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$ 0	? ? $\int_{\overline{\mathcal{M}}_{2,n}} \lambda_1\lambda_2\psi_1^{d_1} \dots \psi_n^{d_n}$	$\cdots$	? $\vdots$? $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_1\lambda_g\psi_1^{d_1} \dots \psi_n^{d_n}$	$\underbrace{k+1}_{=g}+1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
ϵ^{2l}	$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$ 0 $\vdots$ 0	? $\vdots$? $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_k\lambda_g\psi_1^{d_1} \dots \psi_n^{d_n}$	$\cdots$	? $\vdots$? $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_l\lambda_g\psi_1^{d_1} \dots \psi_n^{d_n}$	$\underbrace{k+l}_{=g}+1$

6.1.2 String and dilaton for the quantum Witten-Kontsevich series

In [GJV05], the authors prove the string and dilaton equations for the Hurwitz correlators. Hence Theorem 1 implies that the $\epsilon = 0$ restriction of the quantum Witten-Kontsevich series satisfies the string and dilaton equations. These equations are actually satisfied by the full quantum Witten-Kontsevich series.

Theorem 2. *The quantum Witten-Kontsevich series satisfies the string equation*

$$\frac{\partial}{\partial t_0}\mathscr{F}^{qWK} = \sum_{i\geq 0} t_{i+1}\frac{\partial}{\partial t_i}\mathscr{F}^{qWK} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.$$

Conjecture 4. *The quantum Witten-Kontsevich series satisfies the dilaton equation*

$$\frac{\partial}{\partial t_1}\mathscr{F}^{qWK} = \sum_{i\geq 0} t_i\frac{\partial}{\partial t_i}\mathscr{F}^{qWK} + \epsilon\frac{\partial}{\partial \epsilon}\mathscr{F}^{qWK} + 2\hbar\frac{\partial}{\partial \hbar}\mathscr{F}^{qWK} - 2\mathscr{F}^{qWK}.$$

6.1.3 A geometric formula for the correlators

We give a formula for the correlator

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l},$$

where $d_1, \dots, d_n, l$ and g are nonnegative integers. This formula expresses $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ as a sum over stable graphs. One can deduce from it an expression for any correlators using the string equation (see Section 6.3.1.1).

Definition 161. Fix the nonnegative integers m, n, g, l and $n+1$ integers $a_0, \dots, a_m$ such that $\sum_{i=0}^m a_i = 0$. Let $G(m, n, g, l, a_0, \dots, a_m)$ be the set of quadruplets

$$\left(\Gamma, \mathfrak{l} : V \to \mathbb{Z}_{\geq 0}, \sigma : V \to \{+,-\}, a : H \to \mathbb{Z}\right)$$

satisfying the following properties.

- $\Gamma = (V, H, g : V \to \mathbb{Z}_{\geq 0}, \theta : H \to V, \iota : H \to H)$ is a stable graph with n vertices (see Section 2.1.3 for the notations concerning stable graphs).
- Γ has $m + n + 1$ legs. We take a partition of the set of legs L in two subsets $L = L_1 \amalg L_2$. There are $m + 1$ legs in L_1 numbered from 0 to m distributed over all the vertices. There are n legs in L_2 numbered from $m + 1$ to $m + n$, one on each vertex. Moreover the leg of L_2 with number $m + 1$ should sit on the same vertex than the leg of L_1 numbered by 0. The vertices are then indexed by the legs of second type. The indexing map is

$$\begin{aligned} p : V &\to \{1, \dots, n\} \\ v &\to i - m, \end{aligned}$$

where i is the index of the unique leg of second type on the vertex v.

- The graph Γ is rooted, its root is the vertex incident to the leg of L_1 numbered by 0 or equivalently the vertex with index 1.
- Each vertex is linked with at least one smaller vertex, i.e. a vertex v with a smaller index $p(v)$.
- There is no loop i.e. edge going from one vertex to itself.
- The genus of Γ is fixed to g:
$$\sum_{v\in V}\mathfrak{g}(v)+h_1(\Gamma)=g.$$
- The function $\mathfrak{l}:V\to\mathbb{Z}_{\geq 0}$ satisfies
$$\mathfrak{l}(v)\leq\mathfrak{g}(v),$$
for any $v\in V$, and
$$\sum_{v\in V}\mathfrak{l}(v)=l.$$
- Each vertex is equipped with a sign $\sigma:V\to\{+,-\}$. The root is, by convention, equipped with a positive sign.
- To each half edge h we associate an integer $a(h)$ such that
  - if $h\in L_1$, we set $a(h)=a_i$, where $i\in\{0,\dots,m\}$ is the index of the half edge h,
  - if $h\in L_2$, we set $a(h)=0$,
  - if $h\in H\backslash L$, we set $a(h)=-a(\iota(h))$, moreover, if h is attatched on a vertex equipped with the sign $+$ (resp. $-$) and connects this vertex to a smaller vertex, we set $a(h)\in\mathbb{Z}_{\leq 0}$ (resp. $a(h)\in\mathbb{Z}_{\geq 0}$),
  - for each vertex v, we impose the condition $\sum_{h\in H(v)}a(h)=0$.

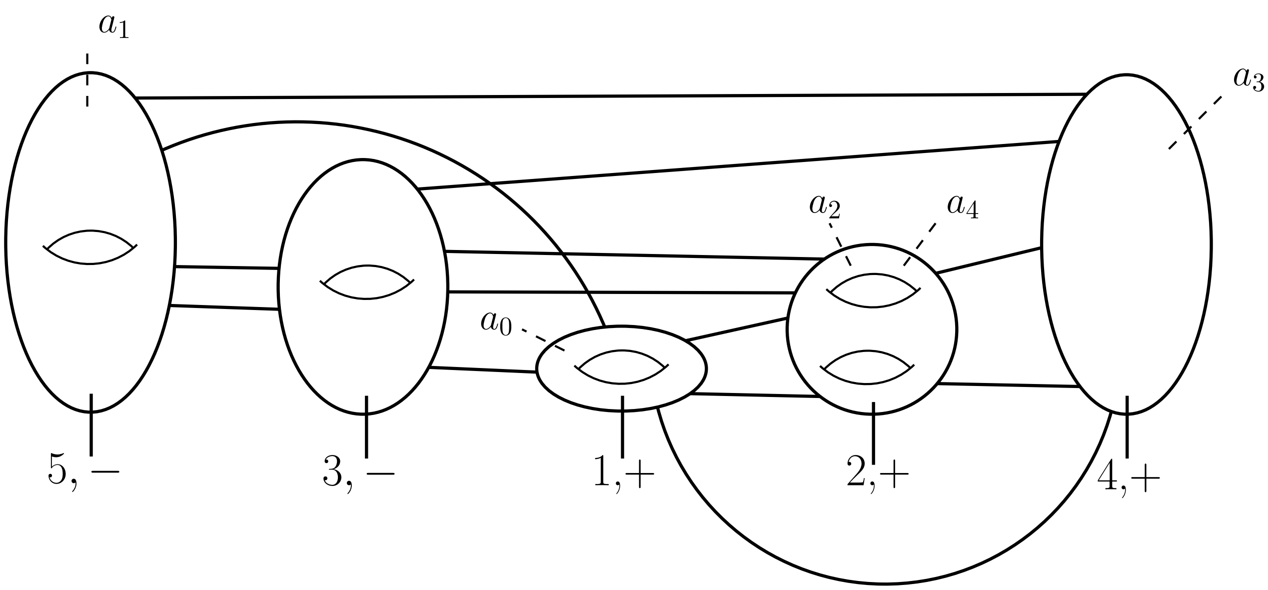


Figure 6.1.1: An example of a graph with $n = 5$ and $m = 4$. The legs of L_1 are represented by dashed lines and the legs of L_2 by full lines. We draw a genus on every vertex, corresponding to the map $\mathfrak{g}$. We did not draw the map $\mathfrak{l}$ or the weightings of the half-edges of $H\backslash L$.

Remark 162. There is a finite choice of weightings for the half edges in $H\backslash L$. Indeed, let v be the last vertex (i.e. v is indexed by $p(v) = n$), if v is equipped with a $+$ sign (resp. $-$ sign), all the half edges in $H\backslash L$ assigned to v are of negative (resp. positive) weight, since the sum of weights on a vertex should be zero, we deduce that there is a finite number of choices of weightings for theses half edges. We repeat this process on each vertex to prove the assertion.

We associate to $(\Gamma, l, \sigma, a) \in G(m, n, g, l, a_0, \ldots, a_n)$ the cohomology class

$$(\lambda \cdot \mathrm{DR})_\Gamma (a_0, a_1, ..., a_n) = \left(\prod_{h\in E} |a(h)|\right) \cdot \xi_{\Gamma *}\left(\prod_{v\in V} \sigma(v)\, \lambda_{l(v)} \mathrm{DR}_{g(v)}\left(a(h)_{h\in H[v]}\right)\right) \in H^{2(l+g+n-1)}\left(\overline{\mathcal{M}}_{g,n+m+1}\right).$$

Proposition 163. *Let g, l and n be nonnegative integers such that $l \leq g$. Let $d_1, \ldots, d_n$ be nonnegative integers. The correlator $\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n}\rangle_{l,g-l}$ is given by*

$$[a_1 \ldots a_m] \sum_{\Gamma \in G(m,n,g,l,a_0=-\sum a_i, a_1,\ldots,a_n)} \frac{i^{2g+n-1-m}}{m!} \int_{\overline{\mathcal{M}}_{g,m+n+1}} (\lambda \cdot \mathrm{DR})_\Gamma \left(-\sum a_i, a_1, \ldots, a_m\right) \psi_{m+1}^{d_1} \ldots \psi_{m+n}^{d_n},$$

where $m = \sum_{i=1}^{n} d_i + l - 2g + 1$.

Remark 164. A reformulation of Proposition 158 and Conjecture 2 about the level structure of the correlators is the correlators $\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n}\rangle_{l,g-l}$ vanish if

$$m > 2g + n - 1, \quad m < n - 1, \quad \text{or} \quad m \equiv n \,(\mathrm{mod}\, 2).$$

Proposition 158 is proved by studying the maximal degree and the parity of the polynomial in $a_1, \ldots, a_m$ in this formula.

Remark 165. At the classical limit, i.e. when $l = g$, this formula reduces to the formula of Lemma 6.9 in [BDGR18]. Indeed, we deduce from the condition $l = g$ that $h_1(\Gamma) = 0$ and $l(v) = g(v)$ for any $v \in V$. Hence, we sum over trees and each vertex of genus $g(v)$ is equipped with a class $\lambda_{g(v)}$. Moreover, since the graph is a tree, the conditions $\sum_{h \in H(v)} a(h) = 0$ on each vertex $v \in V$ uniquely determine all the weights $a(h)$, for any $h \in H$. In particular, the sign of any vertex is determined. Furthermore, the class on any vertex v of genus $g(v)$ is $\lambda_g \mathrm{DR}_g \left(a(h)_{h \in H[v]} \right)$. As a consequence of Hain's formula, this class is a homogenous polynomial of degree $2g$. In addition, each of the $n-1$ edges of the tree carries a weight a. We deduce that

$$(\lambda \cdot \mathrm{DR})_\Gamma \left(-\sum a_i, a_1, ..., a_m \right)$$

is a homogenous polynomial of degree $2\underbrace{(g_1 + \cdots + g_n)}_{g} + n - 1$. Hence m is fixed to its maximum $2g + n - 1$ (or equivalently, $\sum_{i=1}^n d_i$ must be equal to $4g - 3 + n$). We hence recover the summation over the set of admissible modified stable trees.

6.1.4 Plan of the sections

In Section 6.2, we first explain how to substitute $u_i = \delta_{i,1}$ in a differential polynomial written as a Fourier series. We then give an expression for the correlators $\langle \tau_d \rangle_{l,g-l}$, for any nonnegative g, l, d, and check that these correlators satisfy the level structure and the minimal level properties. Finally we give a proof of the string equation that is Theorem 2.

In Section 6.3, we prove the main theorem that is Theorem 1.

in Section 6.4, we prove a combinatorial identity that is used in the last step of the proof of the main theorem.

In Section 6.5, we prove the vanishing properties of the correlators stated in Proposition 158.

6.2 Preliminaries

6.2.1 On the substitution $u_i = \delta_{i,1}$

The construction of the quantum Witten-Kontsevich series uses differential polynomials as elements of $\mathcal{A}^\hbar$ and $\tilde{\mathcal{A}}^\hbar$, indeed you start with some Hamiltonian densities that belong to $\tilde{\mathcal{A}}^\hbar$ and you perform at the end the substitution $u_i = \delta_{i,1}$. We need to explain how to substitute $u_i = \delta_{i,1}$ in an element of $\tilde{\mathcal{A}}^\hbar$. This is the purpose of the following lemma.

Lemma 166. *Let ϕ be a differential polynomial. We write ϕ as a formal Fourier series, that is*

$$\phi(x) = \sum_{k=0}^{d} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} \phi_k(a_1, \ldots, a_k) \, p_{a_1} \ldots p_{a_k} e^{ix(a_1 + \cdots + a_k)}$$

where $\phi_k(a_1,\dots,a_k) \in \mathbb{C}[a_1,\dots,a_k][[\epsilon,\hbar]]$ *is a symmetric polynomial in its* k *indeterminates* $a_1,\dots,a_k$ *for* $0 \le k \le d$*. The substitution* $u_i = \delta_{i,1}$ *in* ϕ *is given by*

$$\phi\Big|_{u_i=\delta_{i,1}} = \sum_{k\geq 0} (-i)^k \left[a_1 \dots a_k\right] \phi_k\left(a_1,\dots,a_k\right).$$

Proof. The Fourier series of u and its s-th derivative is given by $u_s(x) = \sum_{a\in\mathbb{Z}} (ia)^s p_a e^{iax}$. Hence we get

$$\begin{aligned}\phi(x) &= \sum_{k\geq 0} \sum_{a_1,\dots,a_k\in\mathbb{Z}} \left(\sum_{s_1,\dots,s_k\geq 0} a_1^{s_1} \dots a_k^{s_k} \left[a_1^{s_1}\dots a_k^{s_k}\right] \phi_k\left(a_1,\dots,a_k\right)\right) p_{a_1}\dots p_{a_k} e^{ix(a_1+\dots+a_k)} \\ &= \sum_{k\geq 0} \sum_{s_1,\dots,s_k\geq 0} (-i)^{s_1+\dots+s_k} u_{s_1}\dots u_{s_k} \left[a_1^{s_1}\dots a_k^{s_k}\right] \phi_k\left(a_1,\dots,a_k\right).\end{aligned}$$

Hence $\phi\Big|_{u_i=\delta_{i,1}} = \sum_{k\geq 0} (-i)^k \left[a_1\dots a_k\right] \phi_k\left(a_1,\dots,a_k\right)$. □

6.2.2 First example : computation of $\langle\tau_d\rangle_{l,g-l}$

We give an expression for $\langle\tau_d\rangle_{l,g-l}$, for any nonnegative integers g, l and d. From this expression, we check the vanishing properties of the correlators $\langle\tau_d\rangle_{l,g-l}$ leading to the level structure (see Proposition 158 and Conjecture 2), the explicit expression of the minimal level (Conjecture 3) and that the level structure disappears at the classical limit.

By definition of the linear terms in t_* of the quantum Witten-Kontsevich series, we have

$$\langle\tau_d\rangle_{l,g-l} = \langle\tau_0\tau_{d+1}\rangle_{l,g-l}.$$

Moreover, according to Lemma 121, this second correlator is given by

$$\langle\tau_0\tau_{d+1}\rangle_{l,g-l} = i^{g-l}\left[\epsilon^l\hbar^{g-l}\right]\Omega^{\hbar}_{0,d+1}\Big|_{u_i=\delta_{i,1}}.$$

Hence, to obtain an expression for $\langle\tau_d\rangle_{l,g-l}$, we need an expression of $\Omega^{\hbar}_{0,d+1}$. This is the purpose of the following lemma.

Lemma 167. *Fix a nonnegative integer* d*. We have*

$$\Omega^{\hbar}_{0,d} = H_{d-1}.$$

Proof. This is an equality between two differential polynomials written as Fourier series, that is between two lists of polynomials. The definition of the constant term of the two-point function is $\Omega^{\hbar}_{0,d}\Big|_{p_*=0} = H_{d-1}\Big|_{p_*=0}$, this proves the equality between the first polynomial of each list. To obtain the equality of the rest of the polynomials, it is enough to check that $\partial_x\Omega^{\hbar}_{0,d} = \partial_x H_{d-1}$.

Using the definition of the two-point function and the tau symmetry, we find

$$\partial_x \Omega^{\hbar}_{0,d} = \frac{1}{\hbar}\left[H_{-1}, \overline{H}_d\right] = \frac{1}{\hbar}\left[H_{d-1}, \overline{H_0}\right].$$

Since the Hamiltonian density of quantum KdV form a tau structure, the commutator of any differential polynomial with $\overline{H}_0$ corresponds to the derivation with respect to x. Hence we get the equality

$$\partial_x \Omega^{\hbar}_{0,d} = \partial_x H_{d-1}.$$

□

Using the definition of the Hamiltonian density H_d (see Eq (3.1.1)), and the substitution lemma (Lemma 166), we obtain the following expression for $\langle\tau_d\rangle_{l,g-l}$.

Lemma 168. *We have*

$$\langle\tau_d\rangle_{l,g-l} = \frac{(-1)^{\frac{m+2g}{2}}}{m!}\left[a_1 \dots a_m\right] \int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)} \psi_0^{d+1}\lambda_l, \tag{6.2.1}$$

where $m = d + 2 - 2g + l$*.*

Verification of the level structure for the correlators $\langle\tau_d\rangle_{l,g-l}$. The double ramification cycle is an even polynomial $\mathrm{DR}_g(0, a_1, \dots, a_m, -\sum a_i) = \mathrm{DR}_g(0, -a_1, \dots, -a_m, \sum a_i)$ of degree $2g$. In Eq. (6.2.1), we extract the coefficient of $a_1 \dots a_m$ from this polynomial. Thus, this coefficient can be non zero only if m takes the even values between 0 and $2g$. Recalling that $m = d + 2 - 2g + l$, this proves the level structure: $\langle\tau_d\rangle_{l,g-l}$ can be non zero only if d takes the $g+1$ values of the interval

$$[2g - 2 - l, 4g - 2 - l]$$

with the parity of l.

We now verify the explicit expression conjectured for the minimal level of the correlators. The minimal level is given by $m = 0$. Recalling that $\mathrm{DR}_g(0,0) = (-1)^g \lambda_g$, we find using the string equation that

$$\langle\tau_{2g-2-l}\rangle_{l,g-l} = \int_{\overline{\mathcal{M}}_{g,1}} \psi^d \lambda_g \lambda_l.$$

Classical limit : the correlator $\langle\tau_d\rangle_{g,0}$. We verify that the level structure disappears at the classical limit. The classical limit of Eq. (6.2.1) is

$$\langle\tau_d\rangle_{g,0} = (-i)^m \frac{(-1)^g}{m!}\left[a_1 \dots a_m\right] \int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)} \psi_0^{d+1}\lambda_g, \tag{6.2.2}$$

where $m = d + 2 - g$. To compute $\lambda_g \mathrm{DR}_g$, we use the Hain formula. It is clear from this formula that $\lambda_g \mathrm{DR}_g$ is a homogeneous polynomial of degree $2g$. Since we extract the coefficient of $a_1 \dots a_m$, we conclude that the correlator is possibly nonzero only for $m = 2g$. We recover the level sructure in the classical limit.

According to the Witten-Kontsevich theorem, we know that $\langle\tau_d\rangle_{g,0}$ is equal to

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^d.$$

However, is not easy to obtain this expression from Eq. (6.2.2) using Hain's formula.

6.2.3 A proof of the string equation

We prove in this section that the quantum Witten-Kontsevich series satisfies the string equation

$$\frac{\partial}{\partial t_0}\mathscr{F}^{qWK} = \sum_{i\geq 0} t_{i+1}\frac{\partial}{\partial t_i}\mathscr{F}^{qWK} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.$$

The string equation for the quantum Witten-Kontsevich series does not directly come from a comparison between the ψ-classes and their pull-backs as it usually does. Indeed, the quantum Witten-Kontsevich series is defined by various commutations of Hamiltonians, that are themselves defined by integration of ψ-classes over the double ramification cycle, and then the substitution $u_i = \delta_{i,1}$. We have to follow this definition to prove the string equation and any other property of the quantum Witten-Kontsevich series.

Plan of the proof. The string equation is an equality of power series. In order to prove this equation, we verify that the constant and linear terms of the power series on the LHS and RHS correspond, then we show that the second derivative of this equation is true. From the definition of the quantum Witten-Kontsevich series, we find that the derivative with respect to t_{d_1} and t_{d_2} of the string equation yields

$$\frac{\partial}{\partial t_0}\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}\Big|_{u_i=\delta_{i,1}} = \sum_{k\geq 0} t_{k+1}\frac{\partial}{\partial t_k}\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}\Big|_{u_i=\delta_{i,1}} + \Omega^{\hbar,\boldsymbol{t}}_{d_1-1,d_2}\Big|_{u_i=\delta_{i,1}} + \Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2-1}\Big|_{u_i=\delta_{i,1}} + \delta_{0,d_1}\delta_{0,d_2}, \qquad (6.2.3)$$

where d_1 and d_2 are two nonnegative integers.

Before proving Equation (6.2.3), let us focus on the constant and linear terms of the string equation.

Constant term of the string equation. The constant term in the RHS of the string equation is given by $-\frac{i\hbar}{24}$. We then have to show that $-\frac{i\hbar}{24}$ is also the coefficient of t_0 in $\mathscr{F}^{qWK}$. By construction, the coefficient of t_0 is the coefficient of t_0t_1. The coefficient of t_0t_1 in $\mathscr{F}^{qWK}$ is given by $\Omega_{0,1}\Big|_{u_i=\delta_{i,1}} \underset{\text{Lemma 167}}{=} H_0\Big|_{u_i=\delta_{i,1}}$. We use the expression of H_0 computed in Example 110 to conclude.

Linear terms of the string equation. We now focus on the linear terms. The coefficient of t_0 does not appear on the RHS of the string equation. Let us show that it vanishes in the LHS. The coefficient of t_0 in the LHS of string equation is given by the coefficient of t_0t_0 in $\mathscr{F}^{qWK}$. This coefficient is

$$\Omega_{0,0}\Big|_{u_i=\delta_{i,1}} \underset{\text{Lemma 167}}{=} H_{-1}\Big|_{u_i=\delta_{i,1}} = u_0\Big|_{u_i=\delta_{i,1}} = 0.$$

The second equality is just the definition if H_{-1}.

Fix an integer $d \geq 1$. The coefficient of t_d on the LHS of the string equation is given by the coefficient of $t_0 t_d$ in $\mathscr{F}^{qWK}$. The coefficient of t_d on the RHS of the string equation is given by the coefficient of t_{d-1} in $\mathscr{F}^{qWK}$. However, we made the choice that the coefficient of $t_0 t_d$ is the coefficient of t_{d-1} in $\mathscr{F}^{qWK}$.

Some necessary lemmas. We prove two lemmas used for the proof of Equation (6.2.3). The first one is deduced from the string equation for the Hamiltonian densities.

Lemma 169. *Fix d_1, d_2 two positive integers, we have*

$$\frac{\partial \Omega^{\hbar}_{d_1,d_2}}{\partial p_0} = \Omega^{\hbar}_{d_1-1,d_2} + \Omega^{\hbar}_{d_1,d_2-1} + \delta_{0,d_1}\delta_{0,d_2},$$

where we use the convention that $\Omega^{\hbar}$ vanishes if at least one of its indices is negative.

Proof. If $(d_1, d_2) = (0,0)$, we obtain $\Omega_{0,0} = H_{-1} = u_0$ from Lemma 167. Hence the equation is satisfied.

Otherwise, we want to prove an equality of elements of $\tilde{\mathcal{A}}^{\hbar}$, that is an equality of two lists of symmetric polynomials. The equality of the first polynomial of each list follows from the choice of the constant $\Omega^{\hbar}_{d_1,d_2}\Big|_{p_*=0}$ in the definition of $\Omega^{\hbar}_{d_1,d_2}$. To obtain the equality for the rest of the polynomials, it is enough to prove that the x-derivative of the equation is verified. From the definition of $\Omega^{\hbar}_{d_1,d_2}$, we have

$$\begin{aligned}
\partial_x \frac{\partial}{\partial p_0}\Omega^{\hbar}_{d_1,d_2} = \frac{\partial}{\partial p_0}\partial_x \Omega^{\hbar}_{d_1,d_2} &= \frac{\partial}{\partial p_0}\frac{1}{\hbar}\left[H_{d_1-1}, \overline{H}_{d_2}\right] \\
&= \frac{1}{\hbar}\left[\frac{\partial H_{d_1-1}}{\partial p_0}, \overline{H}_{d_2}\right] + \frac{1}{\hbar}\left[H_{d_1-1}, \frac{\partial \overline{H}_{d_2}}{\partial p_0}\right] \\
&= \partial_x \Omega^{\hbar}_{d_1-1,d_2} + \partial_x \Omega^{\hbar}_{d_1,d_2-1}.
\end{aligned}$$

We used the string equation for the Hamiltonian densities (Proposition 116) to obtain the last equality. $\square$

Lemma 170. *Let ϕ be a differential polynomial, we have*

$$\partial_x \phi\Big|_{u_i=\delta_{i,1}} = \frac{\partial \phi}{\partial p_0}\Big|_{u_i=\delta_{i,1}}.$$

Proof. Write ϕ as an element of $\tilde{\mathcal{A}}^{\hbar}$, that is

$$\phi(x) = \sum_{k\geq 0}^{d} \sum_{a_1,\ldots,a_k \in \mathbb{Z}} \phi_k(a_1,\ldots,a_k) p_{a_1}\ldots p_{a_k} e^{ix(a_1+\cdots+a_k)},$$

where $\phi_k(a_1,\dots,a_k) \in \mathbb{C}[a_1,\dots,a_k][[\epsilon,\hbar]]$ is a symmetric polynomial in its k indeterminates $a_1,\dots,a_k$ for $0 \le k \le d$. Then, thanks to Lemma 166, we have

$$\begin{aligned}
\partial_x \phi\Big|_{u_i=\delta_{i,1}} &= \sum_{k\ge 0} (-1)^k\, i^{k+1} \left[a_1 \dots a_k\right] \phi_k(a_1,\dots,a_k)\,(a_1+\dots+a_k) \\
&= \sum_{k\ge 0} (-1)^k\, i^{k+1} \sum_{j=1}^{k} \left[a_1 \dots \hat{a}_j \dots a_k\right] \phi_k\left(a_1,\dots,a_{j-1},0,a_{j+1},\dots,a_k\right) \\
&= \frac{\partial \phi}{\partial p_0}\Big|_{u_i=\delta_{i,1}}.
\end{aligned}$$

□

Proof of Equation (6.2.3). We first recall this equation:

$$\frac{\partial}{\partial t_0}\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}\Big|_{u_i=\delta_{i,1}} = \sum_{k\ge 0} t_{k+1}\frac{\partial}{\partial t_k}\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2}\Big|_{u_i=\delta_{i,1}} + \Omega^{\hbar,\boldsymbol{t}}_{d_1-1,d_2}\Big|_{u_i=\delta_{i,1}} + \Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2-1}\Big|_{u_i=\delta_{i,1}} + \delta_{0,d_1}\delta_{0,d_2}.$$

Proof. We prove this equality at every degree in the indeterminates $(t_0,t_1,\dots)$. Recall that $\Omega^{\hbar,\boldsymbol{t}}_{d_1,d_2} = \exp\left(\sum_{k\ge 0}\frac{t_k}{\hbar}\left[\cdot,\overline{H}_k\right]\right)\Omega^{\hbar}_{d_1,d_2}$. Let $n\ge 0$ and $(d_3,\dots,d_n)$ be a list of nonnegative integers. Then the coefficient of $t_{d_3}\dots t_{d_n}$ of the LHS is given by

$$\begin{aligned}
\frac{1}{\hbar^{n-1}}\left[\left[\dots\left[\Omega^{\hbar}_{d_1,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right],\overline{H}_0\right]\Bigg|_{u_i=\delta_{i,1}} &= \frac{1}{\hbar^{n-2}}\partial_x\left[\dots\left[\Omega^{\hbar}_{d_1,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}} \\
&= \frac{1}{\hbar^{n-2}}\frac{\partial}{\partial p_0}\left[\dots\left[\Omega^{\hbar}_{d_1,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}}.
\end{aligned}$$

We act with $\frac{\partial}{\partial p_0}$ on every elements of the commutators. Then we use Lemmas **??** and 169 to find

$$\begin{aligned}
\frac{1}{\hbar^{n-2}}\Bigg(&\left[\dots\left[\Omega_{d_1-1,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}} + \left[\dots\left[\Omega_{d_1,d_2-1},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}} \\
&+ \left[\dots\left[\delta_{0,d_1}\delta_{0,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}} \\
&+ \left[\dots\left[\Omega_{d_1,d_2},\overline{H}_{d_3-1}\right]\dots,\overline{H}_{d_n}\right]\Bigg|_{u_i=\delta_{i,1}} + \dots + \left[\dots\left[\Omega_{d_1,d_2},\overline{H}_{d_3}\right]\dots,\overline{H}_{d_n-1}\right]\Bigg|_{u_i=\delta_{i,1}}\Bigg).
\end{aligned}$$

We recognize the coefficient of $t_{d_3}\dots t_{d_n}$ of the RHS Equation (6.2.3). □

6.3 Proof of the main theorem

We give in this section the proof of Theorem 1, that is we prove the equality

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g ,$$

where $g, n, d_1, \dots, d_n$ are some nonnegative integers. First, we explain the strategy of the proof.

6.3.1 Computing $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$ and $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$

In the next section, we show that the string equation allows one to express $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$ and $\langle\langle \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ from the the quantum and Hurwitz correlators with a τ_0 insertion. We deduce that it is enough to prove the equality

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g \tag{6.3.1}$$

in order to prove Theorem 1. Then, we explain how to obtain an explicit expression for the RHS, and how to compute the LHS. The two expressions are completely different, but can be used to prove the equality.

6.3.1.1 String Equation

Fix a nonnegative integer g. The correlators of the quantum Witten-Kontsevich and the Hurwitz correlators satisfy the string equation

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \sum_{i=1}^{n} \langle \tau_{d_1} \dots \tau_{d_i - 1} \dots \tau_{d_n} \rangle_{0,g},$$

$$\langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g = \sum_{i=1}^{n} \langle\langle \tau_{d_1} \dots \tau_{d_i - 1} \dots \tau_{d_n} \rangle\rangle_g .$$

The first equation is the statement of Theorem 2 proved in Section 6.2. The second equation is Proposition 3.10 in [GJV05]. Let us define the following generating series

$$\mathring{G}_g^{(q)}(s_1, \dots, s_n) := \sum_{d_1, \dots, d_n \geq 0} \langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} s_1^{d_1} \dots s_n^{d_n}$$

and

$$G_g^{(q)}(s_1, \dots, s_n) := \sum_{d_1, \dots, d_n \geq 0} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} s_1^{d_1} \dots s_n^{d_n}.$$

We also define $\mathring{G}_g^H$ and G_g^H by replacing the quantum correlators by the Hurwitz correlators. According to the string equation, we have

$$\mathring{G}_g^{(q)} = (s_1 + \dots + s_n)\, G_g^{(q)} \text{ and } \mathring{G}_g^H = (s_1 + \dots + s_n)\, G_g^H.$$

We can inverse these two equations in the same way, we then obtain $G_g^{(q)}$ in terms of $\mathring{G}_g^{(q)}$ and G_g^H in terms of $\mathring{G}_g^H$. Hence, proving $\mathring{G}_g^{(q)} = \mathring{G}_g^H$ is equivalent to proving that $G_g^{(q)} = G_g^H$.

6.3.1.2 An explicit expression for $\langle\langle \tau_0\tau_{d_1}\dots\tau_{d_n}\rangle\rangle_g$

In [GJV05], Theorem 3.1 gives the following explicit expression for the one-part double Hurwitz numbers

$$H^g_{(d),\mu} = r! d^{r-1}\left[z^{2g}\right]\frac{\prod_{i=1}^n S\left(\mu_i z\right)}{S\left(z\right)},$$

where $d = \mu_1 + \dots + \mu_n$ is the degree of the ramified cover and $r = 2g-1+n$ is the number of simple ramifications. We also used Notation 98, that is $S\left(z\right) = \frac{\mathrm{sh}(z/2)}{z/2}$. Note that the polynomiality of the one-part double Hurwitz numbers $H^g_{(d),\mu}$ in their ramifications $\mu_1,\dots,\mu_n$ is clear from this expression. From the definition of the Hurwitz correlators given in Eq. (4.3.1), we find

$$\langle\langle \tau_0\tau_{d_1}\dots\tau_{d_n}\rangle\rangle_g = (-1)^{\frac{-2+n-\sum d_i}{2}}\left[\mu_1^{d_1}\dots\mu_n^{d_n}\right]\left[z^{2g}\right]\left(\mu_1+\dots+\mu_n\right)^{2g-2+n}\frac{S\left(\mu_1 z\right)\dots S\left(\mu_n z\right)}{S\left(z\right)}, \quad (6.3.2)$$

where we used $S\left(0\right)=1$.

6.3.1.3 Computing $\langle \tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g}$

From the construction of the quantum Witten-Kontsevich series (see Section **??**), we get the following expression of its correlators

$$\langle \tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g} = i^g\left[\epsilon^0\hbar^g\right]\left(\frac{\partial^{n-1}\Omega^t_{0,d_1}}{\partial t_{d_2}\dots\partial t_{d_n}}\right)\Bigg|_{t_*=0,\,u_i=\delta_{1,i}} = i^g\left[\epsilon^0\hbar^g\right]\frac{1}{\hbar^{n-1}}\left[\dots\left[\Omega_{0,d_1},\overline{H}_{d_2}\right],\dots,H_{d_n}\right]\Big|_{u_i=\delta_{1,i}}.$$

Lemma 167 gives $\Omega_{0,d} = H_{d-1}$. Thus we have to study

$$\langle \tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g} = i^g\left[\hbar^{g+n-1}\right]\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right]\Big|_{u_i=\delta_{1,i},\epsilon=0}. \quad (6.3.3)$$

We will compute this expression in the proof. To do so, we need a computable expression of H_d and a computable expression of the commutator. We give these expressions in the next two paragraphs. The computation will be carried out in $\tilde{\mathcal{A}}^\hbar$. We will then need a way to perform the substitution $u_i = \delta_{i,1}$. To do so, we use Lemma 166.

A computable expression of H_p. In [BSSZ15], Theorem 1 gives an explicit expression for the intersection number of a DR-cycle with the maximal power of a ψ-class. From this theorem we get

$$\int_{DR_g(0,a_1,\dots,a_m,-\sum a_i)}\psi_0^{p+1} = \delta_{p+1,2g-1+m}\left[z^{2g}\right]\frac{S\left(a_1 z\right)\dots S\left(a_m z\right)S\left(\sum_{i=1}^m a_i z\right)}{S\left(z\right)}.$$

We then obtain from the definition of H_p in Eq. (3.1.1),

$$\begin{aligned}H_p\left(x\right)\Big|_{\epsilon=0} &= \sum_{\substack{g\geq 0, m\geq 0\\ 2g+m>0}}\frac{(i\hbar)^g}{m!}\sum_{a_1,\dots,a_m\in\mathbb{Z}}\left(\int_{\mathrm{DR}_g(0,a_1,\dots,a_m,-\sum a_i)}\psi_0^{d+1}\Lambda\left(\frac{-\epsilon^2}{i\hbar}\right)\right)p_{a_1}\dots p_{a_m}e^{ix\sum_{i=1}^m a_i}\\ &= \sum_{g\geq 0}\frac{(i\hbar)^g}{m!}\sum_{a_1,\dots,a_m\in\mathbb{Z}}\left(\left[z^{2g}\right]\frac{S\left(a_1 z\right)\dots S\left(a_m z\right)S\left(\sum_{i=1}^m a_i z\right)}{S\left(z\right)}\right)p_{a_1}\dots p_{a_m}e^{ix\sum_{i=1}^m a_i}, \quad (6.3.4)\end{aligned}$$

with $m = p+2-2g$.

Explicit expression of the star product. Let $f, g \in \mathcal{F}^{\hbar}(P)$. One can check that the star product of these elements is given by

$$f \star g = f \exp\left(\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}}\right) g. \tag{6.3.5}$$

The notations $\overleftarrow{\frac{\partial}{\partial p_k}}$ and $\overrightarrow{\frac{\partial}{\partial p_{-k}}}$ mean that the derivative acts on the left or on the right, that is on f or g.

6.3.2 Proof of the equality $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$

In Section 6.3.2.1, we prove the equality of Equation (6.3.1) for $n = 1$.

In Section 6.3.2.2, we prove the equality of Equation (6.3.1) for $n = 2$. This particular case is included as an example to illuminate the proof of the general case.

In Section 6.3.2.3, we prove the equality of Equation (6.3.1) for $n \geq 2$.

Convention. In the rest of the proof, we focus on the restriction $\epsilon = 0$. Hence, we forget the formal variable ϵ and always suppose it to be zero.

6.3.2.1 Proof for $n = 1$

Fix two nonnegative integers d and g. We prove in this section that

$$\langle \tau_0 \tau_d \rangle_{0,g} = \langle\langle \tau_0 \tau_d \rangle\rangle_g.$$

We start from the LHS, $\langle \tau_0 \tau_d \rangle_{0,g}$. As explained in Eq. (6.3.3), we have $\langle \tau_0 \tau_d \rangle_g = i^g \left[\hbar^g\right] H_{d-1}\Big|_{u_i = \delta_{i,1}}$. Then we use the expression of H_p in Eq. (6.3.4) and perform the evaluation $u_i = \delta_{i,1}$ with Lemma 166. We find

$$\langle \tau_0 \tau_d \rangle_{0,g} = i^g \frac{i^g}{m!} (-i)^m \left[a_1 \dots a_m\right] \left[z^{2g}\right] \frac{S(a_1 z) \dots S(a_m z) S\left(\sum_{i=1}^m a_i z\right)}{S(z)}$$

with $m = d + 1 - 2g$. Note that there is no a-linear term in $S(az)$ because S is an even function, hence the expression of $\langle \tau_0 \tau_d \rangle_{0,g}$ simplifies to

$$\langle \tau_0 \tau_d \rangle_{0,g} = \frac{(-i)^{d+1}}{m!} \left[a_1 \dots a_m\right] \left[z^{2g}\right] \frac{S\left(\sum_{i=1}^m a_i z\right)}{S(z)}.$$

Let $A = \sum_{i=1}^m a_i$. It is easy to check $\left[a_1 \dots a_m\right] S\left(\sum_{i=1}^m a_i z\right) = m! \left[A^m\right] S(Az)$. Hence we get

$$\langle \tau_0 \tau_d \rangle_{0,g} = (-i)^{d+1} \left[z^{2g} A^m\right] \frac{S(Az)}{S(z)} = (-i)^{d+1} \left[z^{2g} A^{d+1-2g}\right] \frac{S(Az)}{S(z)},$$

and we rewrite it as

$$\langle \tau_0 \tau_d \rangle_{0,g} = i^{-1-d} \left[A^d\right] \left[z^{2g}\right] A^{2g-1} \frac{S(Az)}{S(z)}.$$

We recognize the expression of $\langle\langle \tau_0 \tau_d \rangle\rangle_g$ given by Eq. (6.3.2).

6.3.2.2 Proof for $n = 2$

Fix three nonnegative integers d_1, d_2 and g. We prove in this section that

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g .$$

We start from the LHS. As explained in Eq. (6.3.3), we have $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = [\hbar^g] \frac{i^g}{\hbar} \left[H_{d_1}, \overline{H}_{d_2} \right] \big|_{u_i = \delta_{1,i}}$.

In Step 1, we obtain an expression of $\frac{i^g}{\hbar} \left[H_{d_1-1}, \overline{H}_{d_2} \right]$ using the formulas of H_{d_1-1}, $\overline{H}_{d_2}$ and of the star product given in Section 6.3.1.3.

In Step 2, we first extract the coefficient of $\hbar^g$ in $\frac{i^g}{\hbar} \left[H_{d_1-1}, \overline{H}_{d_2} \right]$. Then we perform the substitution $u_i = \delta_{i,1}$ in $[\hbar^g] \frac{i^g}{\hbar} \left[H_{d_1-1}, \overline{H}_{d_2} \right]$ and get a first expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$. However this expression will be totally different from the one of $\langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g$ given by Eq. (6.3.2).

In Steps 3,4 and 5, we will transform this last expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ into the expression of $\langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g$ given by Eq. 6.3.2.

Step 1. We compute $\frac{i^g}{\hbar} \left[H_{d_1-1}, \overline{H}_{d_2} \right] = \frac{i^g}{\hbar} \left(H_{d_1-1} \star \overline{H}_{d_2} - \overline{H}_{d_2} \star H_{d_1-1} \right)$. Recall that

$$H_{d_1-1}(x) = \sum_{\substack{g_1 \geq 0 \\ \text{with } m_1 = d_1+1-2g_1}} \frac{(i\hbar)^{g_1}}{m_1!} \sum_{a_1,\dots,a_{m_1} \in \mathbb{Z}} \left(\left[z^{2g_1} \right] \frac{S(a_1 z) \dots S(a_{m_1} z) S\left(\sum_{i=1}^{m_1} a_i z\right)}{S(z)} \right) p_{a_1} \dots p_{a_{m_1}} e^{ix \sum_{i=1}^{m_1} a_i}$$

and

$$\overline{H}_{d_2} = \sum_{\substack{g_2 \geq 0 \\ \text{with } m_2 = d_2+2-2g_2}} \frac{(i\hbar)^{g_2}}{m_2!} \sum_{\substack{b_1,\dots,b_{m_2} \in \mathbb{Z} \\ \sum b_i = 0}} \left(\left[w^{2g_2} \right] \frac{S(b_1 w) \dots S(b_{m_2} w)}{S(w)} \right) p_{b_1} \dots p_{b_{m_2}} .$$

The expression of $\overline{H}_{d_2}$ is obtained by a formal x-integration along S^1 of H_{d_2}. Hence this imposes the condition $\sum_{i=1}^{m_2} b_i = 0$ and then $S\left(\sum_{i=1}^{m_2} b_i w\right) = 1$.

From the expression of the star product (6.3.5), we get

$$\begin{aligned} H_{d_1-1} \star \overline{H}_{d_2} &= H_{d_1-1} \exp\left(\sum_{k>0} i\hbar k \overleftarrow{\frac{\partial}{\partial p_k}} \overrightarrow{\frac{\partial}{\partial p_{-k}}} \right) \overline{H}_{d_2} \\ &= H_{d_1-1} \sum_{q \geq 0} \frac{(i\hbar)^q}{q!} \left(\sum_{k_1,\dots,k_q > 0} k_1 \dots k_q \overleftarrow{\frac{\partial}{\partial p_{k_1}}} \dots \overleftarrow{\frac{\partial}{\partial p_{k_q}}} \overrightarrow{\frac{\partial}{\partial p_{-k_1}}} \dots \overrightarrow{\frac{\partial}{\partial p_{-k_q}}} \right) \overline{H}_{d_2} . \end{aligned}$$

We first describe the action of the left-derivatives of the star product on H_{d_1-1}. Fix g_1 in H_{d_1-1}. The product of q left-derivatives $\overleftarrow{\frac{\partial}{\partial p_{k_1}}} \dots \overleftarrow{\frac{\partial}{\partial p_{k_q}}}$ acts on the formal power series $\sum_{a_1,\dots,a_{m_1} \in \mathbb{Z}} S(a_1 z) \dots S(a_{m_1} z) S\left(\sum_{i=1}^{m_1} a_i z\right) p_{a_1} \dots p_{a_{m_1}}$ yielding

$$m_1 \dots (\tilde{m}_1 + 1) \sum_{a_1,\dots,a_{\tilde{m}_1} \in \mathbb{Z}} S(a_1 z) \dots S(a_{\tilde{m}_1} z) S(k_1 z) \dots S(k_q z) S\left(\left(\tilde{A} + K\right) z\right) p_{a_1} \dots p_{a_{\tilde{m}_1}}$$

where $\tilde{m}_1 = m_1 - q$, $\tilde{A} = \sum_{i=1}^{\tilde{m}_1} a_i$ and $K = \sum_{i=1}^{q} k_i$. Indeed, each derivative $\frac{\partial}{\partial p_{k_j}}$ may act on each factor p_{a_i}. Without loss of generality we can assume that $i = \tilde{m}_1 + j$, multiplying the result by $m_1 \dots (\tilde{m}_1 + 1)$ to account for the number of equivalent choices. The derivative yields a nonvanishing result if and only if $a_i = k_j$.

Similarly we describe the action of the right-derivatives of the star product on H_{d_2}. Fix g_2 in H_{d_2}. The product of q right-derivatives $\overrightarrow{\frac{\partial}{\partial p_{-k_1}}} \dots \overrightarrow{\frac{\partial}{\partial p_{-k_q}}}$ acts on the formal series $\sum_{\substack{b_1,\dots,b_{m_2} \in \mathbb{Z} \\ \sum b_i = 0}} S(b_1 w) \dots S(b_{m_2} w)\, p_{b_1} \dots p_{b_{m_2}}$ yielding

$$m_2 \dots (\tilde{m}_2 + 1) \sum_{\substack{b_1,\dots,b_{\tilde{m}_2} \in \mathbb{Z} \\ \tilde{B} = K}} S(b_1 w) \dots S(b_{\tilde{m}_2} w)\, S(-k_1 w) \dots S(-k_q w)\, p_{b_1} \dots p_{b_{\tilde{m}_2}}$$

where $\tilde{m}_2 = m_2 - q$ and $\tilde{B} = \sum_{i=1}^{\tilde{m}_2} b_i$.

Recall that S is an even function, hence $S(-k_i w) = S(k_i w)$. Note that the condition $\sum_{i=1}^{m} b_i = 0$ becomes $K = \tilde{B}$.

Finally, the expression of $H_{d_1 - 1} \star \overline{H}_{d_2}$ becomes

$$\begin{aligned} &\sum_{g \geq 0} \sum_{g_1 + g_2 + q = g} \frac{(i\hbar)^g}{\tilde{m}_1! \tilde{m}_2! q!} \left[z^{2g_1} w^{2g_2} \right] \\ &\times \sum_{a_1,\dots,a_{\tilde{m}_1} \in \mathbb{Z}} \sum_{b_1,\dots,b_{\tilde{m}_2} \in \mathbb{Z}} \sum_{\substack{k_1,\dots,k_q > 0 \\ K = \tilde{B}}} k_1 \dots k_q \\ &\times \frac{S(a_1 z) \dots S(a_{\tilde{m}_1} z)\, S(k_1 z) \dots S(k_q z)\, S\left(\left(\tilde{A} + \tilde{B}\right) z\right)}{S(z)} \frac{S(b_1 w) \dots S(b_{\tilde{m}_2} w)\, S(-k_1 w) \dots S(-k_q w)}{S(w)} \\ &\times p_{a_1} \dots p_{a_{\tilde{m}_1}} p_{b_1} \dots p_{b_{\tilde{m}_2}} e^{ix\left(\tilde{A} + \tilde{B}\right)}, \end{aligned} \tag{6.3.6}$$

where $\tilde{m}_1 = d_1 + 1 - 2g_1 - q$ and $\tilde{m}_2 = d_2 + 2 - 2g_2 - q$.

We can re-do this exercise to compute $\overline{H}_{d_2} \star H_{d_1 - 1}$. The main difference is the condition $K = \tilde{B}$ which becomes $K=-\tilde{B}$. Thus, the expression of $\frac{i^g}{\hbar}\left[H_{d_{1-1}},\overline{H}_{d_2}\right]$ is

$$\sum_{g\geq 0}\sum_{\substack{g_1+g_2+q-1=g\\ g_1,g_2\geq 0,\, q\geq 1}}\frac{i^{2g+1}\hbar^g}{\tilde{m}_1!\tilde{m}_2!q!}\left[z^{2g_1}w^{2g_2}\right] \tag{6.3.7}$$
$$\times\sum_{a_1,\dots,a_{\tilde{m}_1}\in\mathbb{Z}}\sum_{b_1,\dots,b_{\tilde{m}_2}\in\mathbb{Z}}\frac{S(a_1z)\dots S(a_{\tilde{m}_1}z)\,S\left(\left(\tilde{A}+\tilde{B}\right)z\right)S(b_1w)\dots S(b_{\tilde{m}_2}w)}{S(z)\,S(w)}$$
$$\times\left(\sum_{k_1+\cdots+k_q=\tilde{B}}k_1\dots k_qS(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw)\right.$$
$$\left.-\sum_{k_1+\cdots+k_q=-\tilde{B}}k_1\dots k_qS(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw)\right),$$
$$\times\, p_{a_1}\dots p_{a_{\tilde{m}_1}}p_{b_1}\dots p_{b_{\tilde{m}_2}}e^{ix\left(\tilde{A}+\tilde{B}\right)}.$$

where $\tilde{m_1}=d_1+1-2g_1-q$ and $\tilde{m}_2=d_2+2-2g_2-q$.

Remark 171. The term $q=0$ of the expression of $H_{d_1-1}\star\overline{H}_{d_2}$ in Eq. (6.3.6) corresponds to the commutative part of the star product. This term disappears in the commutator $\frac{1}{\hbar}\left[H_{d_1-1},\overline{H}_{d_2}\right]$. We can then suppose that $q\geq 1$ in Eq. (6.3.7).

Change of notation. For convenience, we change the notation by removing the tildes, i.e. we set $m_1:=\tilde{m}_1$, $m_2:=\tilde{m}_2$, $A:=\tilde{A}$ and $B:=\tilde{B}$.

Step 2. We first extract the coefficient of $\hbar^g$ from $\frac{i^g}{\hbar}\left[H_{d_{1-1}},\overline{H}_{d_2}\right]$. Then we evaluate this coefficient, which is a differential polynomial, at $u_i=\delta_{i,1}$. We will then get an expression for $\langle\tau_0\tau_{d_1}\tau_{d_2}\rangle_{0,g}=[\hbar^g]\,\frac{i^g}{\hbar}\left[H_{d_1},\overline{H}_{d_2}\right]\big|_{u_i=\delta_{1,i}}$.

We extract the coefficient of $\hbar^g$ in $\frac{i^g}{\hbar}\left[H_{d_{1-1}},\overline{H}_{d_2}\right]$ from its expression obtained in Eq. (6.3.7). This only removes the summation over g. With our new notations, this coefficient is

$$
\begin{aligned}
\left[\hbar^{g}\right] \frac{i^{g}}{\hbar}\left[H_{d_{1-1}}, \overline{H}_{d_2}\right] = & \sum_{\substack{g_1+g_2+q-1=g \\ g_1, g_2 \geq 0,\, q \geq 1}} \frac{i^{2g+1}}{m_1! m_2! q!}\left[z^{2g_1} w^{2g_2}\right] \\
& \times \sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} \sum_{b_1, \ldots, b_{m_2} \in \mathbb{Z}} \frac{S(a_1 z) \ldots S(a_{m_1} z) S((A+B) z) S(b_1 w) \ldots S(b_{m_2} w)}{S(z) S(w)} \\
& \times \left(\sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)\right. \\
& \left.-\sum_{k_1+\cdots+k_q=-B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)\right) \\
& \times p_{a_1} \ldots p_{a_{m_1}} p_{b_1} \ldots p_{b_{m_2}} e^{ix(A+B)},
\end{aligned}
$$

where $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$.

This last expression is a differential polynomial thanks to Proposition **??**. In order to substitute $u_i = \delta_{i,1}$, we use Lemma 166. We get

$$
\begin{aligned}
\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = & \sum_{\substack{g_1+g_2+q-1=g \\ g_1, g_2 \geq 0,\, q \geq 1}} \left[z^{2g_1} w^{2g_2}\right]\left[a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}\right] \\
& \times \frac{i^{-d_1-d_2}}{m_1! m_2! q!} \frac{S(a_1 z) \ldots S(a_{m_1} z) S((A+B) z) S(b_1 w) \ldots S(b_{m_2} w)}{S(z) S(w)} \\
& \times \left(\sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)\right. \\
& \left.-\sum_{k_1+\cdots+k_q=-B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)\right),
\end{aligned}
\tag{6.3.8}
$$

where $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$.

Remark 172. It can look confusing that in this expression, b_i stands for a formal variable and an integer when we write $k_1 + \cdots + k_q = B = \sum_{i=1}^{m_2} b_i$. This is due to the the presence of Ehrhart polynomials. Indeed, the coefficient of any power of z and w in

$$
\sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)
$$

is an Ehrhart polynomial in the indeterminate $B = \sum_{i=1}^{m_2} b_i$, see [BR16, Lemma A.1] for a proof. Hence, when we write B as an integer, we use this lemma to justify that B can also be used as an formal variable. The same phenomenon applies to $\sum_{k_1+\cdots+k_q=-B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z) S(k_1 w) \ldots S(k_q w)$ when $B < 0$.

Plan of Steps $3, 4$ **and** 5. This expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ is completely different from the one of $\langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g$ given by Eq. (6.3.2). Moreover, it is difficult to compute the number $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ from this expression. Let us point out the difficulties. The three last lines of Eq. (6.3.8) form a series depending on the parameters d_1, d_2, g_1, g_2, q in the indeterminates $a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w$. We denote this series by $F_{d_1,d_2,g_1,g_2,q}(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w)$ so that Eq. (6.3.8) becomes

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \sum_{\substack{g_1+g_2+q-1=g \\ g_1, g_2 \geq 0,\, q \geq 1}} \left[z^{2g_1} w^{2g_2}\right] [a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}]\, F_{d_1,d_2,g_1,g_2,q}(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w).$$

Hence, for each choice of the parameters d_1, d_2, g_1, g_2, q, we have to extract the coefficient of $z^{2g_1} w^{2g_2} a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}$ in $F_{d_1,d_2,g_1,g_2,q}$. Then we sum these coefficients over the parameters g_1, g_2, q. The main difficulty is to extract the coefficient of b_i from the expression appearing in parenthesis in $F_{d_1,d_2,g_1,g_2,q}$, that is from

$$\sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z)\, S(k_1 w) \ldots S(k_q w)$$
$$- \sum_{k_1+\cdots+k_q=-B} k_1 \ldots k_q S(k_1 z) \ldots S(k_q z)\, S(k_1 w) \ldots S(k_q w).$$

As we explained in Remark 172, the coefficient of any power of z and w in each of the two sums is an Ehrhart polynomial in the indeterminate $B = \sum_{i=1}^{m_2} b_i$. However we do not have an explicit expression for the coefficients of these Ehrhart polynomials. Luckily, Eulerian numbers appear in the computation of these coefficients, see Remark **??**. The plan is to modify our expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ in order to use known properties of Eulerian numbers.

In Step 3, we will modify our expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ using simplifications arising from extracting the coefficient of $a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}$ in $F_{d_1,d_2,g_1,g_2,q}(a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w)$. These simplifications mainly come from the fact that $z \to S(z)$ is even.

In Step 4, we use some changes of variables in order to get $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ as the coefficient of an exponential generating series. This is the exponential of an expression that can be computed using Eulerian numbers.

In Step 5, we finally we use a property of Eulerian number in order to get a simplified expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$. We then recover from it the expression of $\langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g$ given by Eq. (6.3.2).

Step 3. The evaluation $u_i = \delta_{i,1}$ brings many simplifications that we explain now.

- First recall that S is an even power series so that the coefficient of α in $S(\alpha z) \times F(\alpha)$ where F is a formal power series in α is the coefficient of α in $F(\alpha)$. Hence, Expression (6.3.8) simplifies as

$$\begin{aligned}\langle\tau_0\tau_{d_1}\tau_{d_2}\rangle_{0,g} = &\sum_{\substack{g_1+g_2+q-1=g \\ g_1,g_2\geq 0,\, q\geq 1}} \frac{i^{-d_1-d_2}}{m_1!m_2!q!}\left[z^{2g_1}w^{2g_2}\right] \\ &\times [a_1\dots a_{m_1}b_1\dots b_{m_2}]\,\frac{S\left((A+B)\,z\right)}{S(z)\,S(w)} \\ &\times\left(\sum_{k_1+\dots+k_q=B} k_1\dots k_q S(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw)\right. \\ &\quad\left.-\sum_{k_1+\dots+k_q=-B} k_1\dots k_q S(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw)\right).\end{aligned} \tag{6.3.9}$$

- Fix g_1, g_2 and q in Expression (6.3.9) so that m_1 and m_2 are fixed. We extract the coefficient of $a_1\dots a_{m_1}b_1\dots b_{m_2}$ from a power series that only depends of the sums $A = a_1+\dots+a_{m_1}$ and $B = b_1+\dots+b_{m_2}$. This is equivalent to extracting the coefficient of $\frac{A^{m_1}B^{m_2}}{m_1!m_2!}$ from the same power series.
- Consider the expression in parenthesis in Eq. (6.3.9). This is a power series in the indeterminate B. Moreover, when $B\geq 0$ this power series becomes

$$\sum_{k_1+\dots+k_q=B} k_1\dots k_q S(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw)$$

and when $B<0$ it becomes

$$-\sum_{k_1+\dots+k_q=-B} k_1\dots k_q S(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw).$$

We are interested in the coefficients of this power series, so for simplicity we can suppose $B>0$. Expression (6.3.9) becomes

$$\begin{aligned}\langle\tau_0\tau_{d_1}\tau_{d_2}\rangle_{0,g} = &\sum_{\substack{g_1+g_2+q-1=g \\ g_1,g_2\geq 0,\, q\geq 1}} \frac{i^{-d_1-d_2}}{q!}\left[z^{2g_1}w^{2g_2}\right] \\ &\times [A^{m_1}B^{m_2}]\,\frac{S\left((A+B)\,z\right)}{S(z)\,S(w)} \\ &\times\sum_{k_1+\dots+k_q=B} k_1\dots k_q S(k_1z)\dots S(k_qz)\,S(k_1w)\dots S(k_qw).\end{aligned} \tag{6.3.10}$$

Step 4. We perform the changes of variables $z := Az$ and $w := Bw$ in Expression (6.3.10). Recall that $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$, these changes of variables yield

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = & \sum_{\substack{g_1+g_2+q-1=g \\ g_1,g_2\geq 0,\, q\geq 1}} \frac{i^{-d_1-d_2}}{q!} \left[z^{2g_1} w^{2g_2} A^{d_1+1-q} B^{d_2+2-q} \right] \\ & \times \frac{S\left(\left(A+B\right)Az\right)}{S\left(Az\right)S\left(Bw\right)} \\ & \times \sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S\left(k_1 Az\right) \ldots S\left(k_q Az\right) S\left(k_1 Bw\right) \ldots S\left(k_q Bw\right).\end{aligned}$$

We re-write this as

$$\begin{aligned}&\sum_{q=1}^{g+1} \frac{i^{-d_1-d_2}}{q!} \left[A^{d_1+1-q} B^{d_2+2-q} \right] \\ & \times \sum_{g_1+g_2=g-q+1} \left[z^{2g_1} w^{2g_2} \right] \underbrace{\frac{S\left(\left(A+B\right)Az\right)}{S\left(Az\right)S\left(Bw\right)} \sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S\left(k_1 Az\right) \ldots S\left(k_q Az\right) S\left(k_1 Bw\right) \ldots S\left(k_q Bw\right)}_{G(z,w)}\end{aligned} \tag{6.3.11}$$

Note that for any formal power series $G\left(z,w\right) = \sum_{i,j\geq 0} G_{i,j} z^i w^j$, we have $\sum_{g_1+g_2=h} \left[z^{g_1} w^{g_2}\right] G\left(z,w\right) = \sum_{g_1+g_2=h} G_{g_1,g_2} = \left[z^h\right] G\left(z,z\right)$. Using this remark in Expression (6.3.11) with $h = g + 1 - q$ and using that $G\left(z,w\right)$ is even in z and w , we get the following expression for $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$,

$$\begin{aligned}&\sum_{q=1}^{g+1} \frac{i^{-d_1-d_2}}{q!} \left[A^{d_1+1-q} B^{d_2+2-q} z^{2g-2q+2} \right] \\ & \times \frac{S\left(\left(A+B\right)Az\right)}{S\left(Az\right)S\left(Bz\right)} \sum_{k_1+\cdots+k_q=B} k_1 \ldots k_q S\left(k_1 Az\right) \ldots S\left(k_q Az\right) S\left(k_1 Bz\right) \ldots S\left(k_q Bz\right).\end{aligned}$$

Re-write this expression as

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} =& i^{-d_1-d_2} \left[A^{d_1} B^{d_2} z^{2g} \right] \frac{S\left(\left(A+B\right)Az\right)}{S\left(Az\right)S\left(Bz\right)} \\ & \times \sum_{q=1}^{g+1} \frac{1}{q!} A^{q-1} B^{q-2} z^{2q-2} \sum_{k_1+\cdots+k_q=B} \prod_{i=1}^{q} k_i S\left(k_i Az\right) S\left(k_i Bz\right).\end{aligned} \tag{6.3.12}$$

We can extend the range of summation to q running from 1 to ∞. Indeed, it is clear from the expression that the terms with $q > g+1$ vanishes, since we extract the coefficient of z^{2g} from a power series with a factor z^{2q-2}. Hence, the second line of Expression (6.3.12) can be re-written as the coefficient of an exponential series. Expression (6.3.12) becomes

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} &= i^{-d_1-d_2} \left[A^{d_1} B^{d_2} z^{2g}\right] \frac{S\left(\left(A+B\right) Az\right)}{S\left(Az\right) S\left(Bz\right)} \\ &\times \frac{1}{AB^2 z^2} \left[t^B\right] \left(\exp\left(\sum_{k>0} ABz^2 kS\left(kAz\right) S\left(kBz\right) t^k\right) - 1\right) \end{aligned} \tag{6.3.13}$$

Step 5. We use properties of Eulerian numbers in order to simplify the expression of the exponential power series in Expression (6.3.13). Using first Proposition 155 and then the definition $S\left(z\right) = \frac{\mathrm{sh}(z/2)}{z/2}$, we get

$$\begin{aligned}\exp\left(\sum_{k>0} ABz^2 kS\left(kAz\right) S\left(kBz\right) t^k\right) &= 1 + 4 \sum_{k>0} \frac{\mathrm{sh}\left(\frac{A}{2}z\right) \mathrm{sh}\left(\frac{B}{2}z\right)}{\mathrm{sh}\left(\frac{A+B}{2}z\right)} \mathrm{sh}\left(k \frac{A+B}{2} z\right) t^k \\ &= 1 + \sum_{k>0} ABkz^2 \frac{S\left(Az\right) S\left(Bz\right)}{S\left(\left(A+B\right) z\right)} S\left(k\left(A+B\right) z\right) t^k.\end{aligned}$$

Thus, Expression (6.3.13) becomes after extracting the coefficient of t^B

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} &= i^{-d_1-d_2} \left[A^{d_1} B^{d_2} z^{2g}\right] \frac{S\left(\left(A+B\right) Az\right)}{S\left(Az\right) S\left(Bz\right)} \\ &\times \frac{4}{AB^2 z^2} \frac{\mathrm{sh}\left(\frac{A}{2}z\right) \mathrm{sh}\left(\frac{B}{2}z\right)}{\mathrm{sh}\left(\frac{A+B}{2}z\right)} \mathrm{sh}\left(B \frac{A+B}{2} z\right).\end{aligned}$$

We re-write the second line as $\frac{S(Az)S(Bz)}{S((A+B)z)} S\left(B\left(A+B\right) z\right)$ so that

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = i^{-d_1-d_2} \left[A^{d_1} B^{d_2} z^{2g}\right] \frac{S\left(\left(A+B\right) Az\right) S\left(B\left(A+B\right) z\right)}{S\left(\left(A+B\right) z\right)}.$$

Finally, the change of variable $z := \frac{z}{A+B}$ in this last expression gives $\langle\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle\rangle_g$ as expressed in Eq. (6.3.2), that is

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \left(-1\right)^{\frac{-d_1-d_2}{2}} \left[A^{d_1} B^{d_2} z^{2g}\right] \left(A+B\right)^{2g} \frac{S\left(Az\right) S\left(Bz\right)}{S\left(z\right)}.$$

6.3.2.3 Proof for $n \geq 2$

Convention. Because of the multiple use of the index i, we choose to denote the imaginary unit i as $\sqrt{-1}$ in this section.

Fix $n+1$ nonnegative integers $d_1, \ldots, d_n$ and g, we prove in this section that

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \langle\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle\rangle_g$$

for $n \geq 2$. We start from the LHS. As explained in the strategy of the proof, we have

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = [\hbar^g] \frac{\sqrt{-1}^g}{\hbar^{n-1}} \left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right] \Big|_{u_i = \delta_{1,i}}.$$

We follow the steps of the previous section in this more general setting. The main difference occurs in Step 5, where we need a combinatorial lemma which was obvious when $n = 2$. Let us recall these steps.

In Step 1, we obtain an expression of $\left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right]$ using the formulas of $H_{d_1-1}, \overline{H}_{d_2}, \dots, \overline{H}_{d_n}$ and of the developed expression of the star product.

In Step 2, we first extract the coefficient of $\hbar^g$ in $\frac{\sqrt{-1}^g}{\hbar^{n-1}} \left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right]$. Then we perform the substitution $u_i = \delta_{i,1}$ in it and get a first expression of $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$. However this expression will be totally different from the one of $\langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ given by Eq. (6.3.2).

In Steps 3,4 and 5, we will transform this last expression of $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$ into the expression of $\langle\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle\rangle_g$ given by Eq. (6.3.2).

Step 1. We compute in the first step $\left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right]$. From the expression of the Hamiltonian density Eq. (6.3.4), we set

$$H_{d_1-1}(x) = \sum_{\substack{g_1, m_1 \geq 0 \\ \text{s.t. } 2g_1 + m_1 = d_1 + 1}} \frac{\left(\sqrt{-1}\hbar\right)^{g_1}}{m_1!} \sum_{a_1^1, \dots, a_{m_1}^1 \in \mathbb{Z}} \left(\left[z_1^{2g_1} \right] \frac{S\left(a_1^1 z\right) \dots S\left(a_{m_1}^1 z\right) S\left(A_1 z\right)}{S(z)} \right) p_{a_1^1} \dots p_{a_{m_1}^1} e^{\sqrt{-1} x A_1},$$

and

$$\overline{H}_{d_i} = \sum_{\substack{g_i, m_i \geq 0 \\ \text{s.t. } 2g_i + m_i = d_i + 2}} \frac{\left(\sqrt{-1}\hbar\right)^{g_i}}{m_i!} \sum_{\substack{a_1^i, \dots, a_{m_i}^i \in \mathbb{Z} \\ A_i = 0}} \left(\left[z_i^{2g_i} \right] \frac{S\left(a_1^i z_i\right) \dots S\left(a_{m_i}^i z_j\right)}{S(z_i)} \right) p_{a_1^i} \dots p_{a_{m_i}^i}, \text{ with } 2 \leq i \leq n,$$

where $A_i := \sum_{j=1}^{m_i} a_j^i$ and $1 \leq i \leq n$. In these notations, we use the variables of summations a_j^1 in H_{d_1-1} and the variables of summations a_j^i in $\overline{H}_{d_i}$, with $2 \leq i \leq n$. Note that in the notation a_j^i, i is just an upper index. The expression of $\overline{H}_{d_i}$ is obtained by a formal x-integration along S^1 of H_{d_i}. Hence, the sum over $a_1^i, \dots, a_{m_i}^i$ has the constraint $A_i = 0$.

In Step 1.1, we give an expression for $H_{d_1-1} \star \overline{H}_{d_2} \star \dots \star \overline{H}_{d_n}$. In Step 1.2 we explain why this is the only term of the commutator $\left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right]$ needed to compute $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$.

Step 1.1.

Proposition 173. *We have*

$$H_{d_1-1}\star\overline{H}_{d_2}\star\cdots\star\overline{H}_{d_n}=\sum_{q_I\geq 0,\,I\in\mathcal{C}}\;\sum_{g_1,\dots,g_n\geq 0}\;\sum_{\substack{\tilde{m}_1,\dots,\tilde{m}_n\geq 0\\ \text{with conditions }\alpha}} \tag{6.3.14}$$
$$\prod_{i=1}^{n}\left(\frac{\left(\sqrt{-1}\hbar\right)^{g_i}}{\tilde{m}_i!}\left[z_i^{2g_i}\right]\sum_{a^i_1,\dots,a^i_{\tilde{m}_i}\in\mathbb{Z}}W_i\left(a^i_1,\dots,a^i_{\tilde{m}_1},z_i\right)\,p_{a_1}\dots p_{a_{\tilde{m}_i}}e^{\sqrt{-1}x\tilde{A}_i}\right)$$
$$\times\prod_{I\in\mathcal{C}}\left(\frac{\left(\sqrt{-1}\hbar\right)^{q_I}}{q_I!}\sum_{\substack{k^I_1,\dots,k^I_{q_I}>0\\ \text{with conditions }\beta}}k^I_1\dots k^I_{q_I}W^I\left(k^I_1,\dots,k^I_{q_I},z_I\right)\right),$$

where

- $\mathcal{C}$ *is the set of pairs (2-element subsets) of* $\{1,\dots,n\}$*; we also denote by* $\mathcal{C}_i\subset\mathcal{C}$ *the subset of pairs that contain* i*,*
- *the conditions* α *on the summations running over* $g_1,\dots,g_n,\tilde{m}_1,\dots,\tilde{m}_n$ *and* q_I, $I\in\mathcal{C}$ *are*
$$2g_1+\tilde{m}_1+\sum_{I\in\mathcal{C}_1}q_I=d_1+1$$
and
$$2g_j+\tilde{m}_j+\sum_{I\in\mathcal{C}_j}q_I=d_j+2,\ \text{with}\,2\leq j\leq n,$$
- *the weight* W_i *with* $1\leq i\leq n$ *is defined by*
$$W_1\left(a^1_1,\dots,a^1_{\tilde{m}_1},z_1\right):=\frac{\prod_{j=1}^{\tilde{m}_1}S\left(a^1_jz_1\right)}{S\left(z_1\right)}S\left(\tilde{A}_1z_1+\dots\tilde{A}_nz_1\right)$$
and
$$W_i\left(a^i_1,\dots,a^i_{\tilde{m}_1},z_i\right):=\frac{\prod_{j=1}^{\tilde{m}_i}S\left(a^i_jz_i\right)}{S\left(z_i\right)},\ \textit{for}\ 2\leq i\leq n,$$
- $\tilde{A}_j=\sum_{i=1}^{\tilde{m}_j}a_i$,
- *the conditions* β *are the following* $(n-1)$ *constraints over the summations:*
$$\tilde{A}_i-\sum_{j=1}^{i-1}K^{\{j,i\}}+\sum_{j=i+1}^{n}K^{\{i,j\}}=0,\quad 2\leq i\leq n \tag{6.3.15}$$
where $K^I=k^I_1+\dots+k^I_{q_I}$*, for* $I\in\mathcal{C}$*,*

- *the weight W^I for $I = \{i,j\} \in \mathcal{C}$ is defined by*

$$W^I\left(k_1^I,\dots,k_{q_I}^I,z_I\right) := S\left(k_1^I z_i\right)\dots S\left(k_{q_I}^I z_i\right)\times S\left(k_1^I z_j\right)\dots S\left(k_{q_I}^I z_j\right),$$

note that the notation z_I means z_i, z_j.

Proof. We use the expression of the star product given by

$$f\star g = f\sum_{q\geq 0}\frac{\left(\sqrt{-1}\hbar\right)^q}{q!}\left(\sum_{k_1,\dots,k_q>0} k_1\dots k_q \overleftarrow{\frac{\partial}{\partial p_{k_1}}}\dots\overleftarrow{\frac{\partial}{\partial p_{k_q}}}\overrightarrow{\frac{\partial}{\partial p_{-k_1}}}\dots\overrightarrow{\frac{\partial}{\partial p_{-k_q}}}\right) g. \tag{6.3.16}$$

The star product is associative as one can check from Eq. (6.3.16). We use this associativity in the following way

$$H_{d_1-1}\star\overline{H}_{d_2}\star\dots\star\overline{H}_{d_n} = \left(\cdots\left(H_{d_1-1}\star\overline{H}_{d_2}\right)\star\dots\star\overline{H}_{d_n}\right).$$

Each of the $n-1$ star products has couples of derivatives acting on the left and on the right with opposite indices. Let $2\leq i\leq n$. The $(i-1)$th star product acts on the left on H_{d_1-1}, $\overline{H}_{d_2},\dots,\overline{H}_{d_{i-1}}$ and on the right only on $\overline{H}_{d_i}$. Fix a nonnegative integer q and q positive integers $k_1,\dots,k_q$. Consider the term

$$\frac{\left(\sqrt{-1}\hbar\right)^q}{q!}k_1\dots k_q\overleftarrow{\frac{\partial}{\partial p_{k_1}}}\dots\overleftarrow{\frac{\partial}{\partial p_{k_q}}}\overrightarrow{\frac{\partial}{\partial p_{-k_1}}}\dots\overrightarrow{\frac{\partial}{\partial p_{-k_q}}}$$

in the development in $\hbar$ of the $(i-1)$th star product. Among these q left derivatives, we denote by $q_{\{i,j\}}$, with $j<i$, the number of derivatives acting on $\overline{H}_{d_j}$ (or H_{d_1-1} if $j=1$). We furthermore add an upper index $\{i,j\}$ on the corresponding k variables so that the $q_{\{i,j\}}$ left derivatives coming from the $(i-1)$th star product and acting on the jth Hamiltonian are denoted by

$$\overleftarrow{\frac{\partial}{\partial p_{k_1^{\{i,j\}}}}}\dots\overleftarrow{\frac{\partial}{\partial p_{k^{\{i,j\}}_{q_{\{i,j\}}}}}}.$$

The associate right derivatives $\overrightarrow{\frac{\partial}{\partial p_{-k_1^{\{i,j\}}}}}\dots\overrightarrow{\frac{\partial}{\partial p_{-k^{\{i,j\}}_{q_{\{i,j\}}}}}}$ act on $\overline{H}_{d_i}$ with opposite indices. With this notation, we obtain

$$\begin{aligned}
&H_{d_1-1}\star\overline{H}_{d_2}\star\dots\star\overline{H}_{d_n}\\
&=\sum_{q_I\geq 0,\,I\in\mathcal{C}}\ \sum_{k_1^I,\dots,k^I_{q_I}\geq 0,\,I\in\mathcal{C}}\ \prod_{I\in\mathcal{C}}\frac{\left(\sqrt{-1}\hbar\right)^{q_I}}{q_I!}k_1^I\dots k_{q_I}^I\\
&\times\prod_{J\in\mathcal{C}_1}\frac{\partial^{q_J}}{\partial p_{k_1^J}\dots\partial p_{k^J_{q_J}}}H_{d_1-1}\\
&\times\prod_{i=2}^{n}\prod_{j=1}^{i-1}\left(\frac{\partial^{q_{\{i,j\}}}}{\partial p_{-k_1^{\{i,j\}}}\dots\partial p_{-k^{\{i,j\}}_{q_{\{i,j\}}}}}\right)\prod_{l=i+1}^{n}\left(\frac{\partial^{q_{\{i,l\}}}}{\partial p_{k_1^{\{i,l\}}}\dots\partial p_{k^{\{i,l\}}_{q_{\{i,l\}}}}}\right)\overline{H}_{d_i}.
\end{aligned}\tag{6.3.17}$$

Let us explain this formula. The derivatives acting on $\overline{H}_{d_i}$ have two different origins; the derivatives coming from the $(i-1)$th star product, these are the derivatives with negative indices in the product running over the variable j, and the derivatives coming from the ith to the $(n-1)$th star product, these are the derivatives with positive indices in the product running over the variable l. Similarly, the derivatives acting on H_{d_1-1} come from all the star products and have positive indices. Moreover, when we develop the star products, we have to choose which derivative acts on which Hamiltonian so that multinomial coefficients appear and simplify the factorials.

We now describe the action of the $\sum_{j=1,j\neq i}^{n} q_{\{i,j\}}$ derivatives of the last line of Eq. (6.3.17) on $\overline{H}_{d_i}$. We find

$$\begin{aligned}
&\prod_{j=1}^{i-1}\left(\frac{\partial^{q_{\{i,j\}}}}{\partial p_{-k_1^{\{i,j\}}}\dots\partial p_{-k^{\{i,j\}}_{q_{\{i,j\}}}}}\right)\prod_{l=i+1}^{n}\left(\frac{\partial^{q_{\{i,l\}}}}{\partial p_{k_1^{\{i,l\}}}\dots\partial p_{k^{\{i,l\}}_{q_{\{i,l\}}}}}\right)\overline{H}_{d_i}\\
&=\sum_{\substack{g_i,\tilde{m}_i\geq 0\\ 2g_i+\tilde{m}_i+\sum_{I\in\mathcal{C}_i}q_I=d_i+2}}\frac{\left(\sqrt{-1}\hbar\right)^{g_i}}{\tilde{m}_i!}\left[z^{2g_i}\right]\\
&\times\sum_{\substack{a_1^i,\dots,a^i_{\tilde{m}_i}\in\mathbb{Z}\\ \text{with condition }\beta}} W_i\left(a_1^i,\dots,a^i_{\tilde{m}_1},z_i\right)\,p_{a_1}\dots p_{a_{\tilde{m}_i}}\\
&\times\prod_{j=1}^{i-1}S\left(k_1^{\{i,j\}}z_i\right)\dots S\left(k^{\{i,j\}}_{q_{\{i,j\}}}z_i\right)\prod_{l=i+1}^{n}S\left(k_1^{\{i,l\}}z_i\right)\dots S\left(k^{\{i,l\}}_{q_{\{i,l\}}}z_i\right).
\end{aligned}\tag{6.3.18}$$

Indeed, when the $\sum_{j=1,j\neq i}^{n} q_{\{i,j\}}$ derivatives act on $\frac{S\left(a_1^i z_i\right)\dots S\left(a^i_{m_i}z_j\right)}{S(z_i)}p_{a_1^i}\dots p_{a^i_{m_i}}$ in $\overline{H}_{d_i}$, it remains $\tilde{m}_i = m_i-\sum_{j=1,j\neq i}^{n} q_{\{i,j\}}$ variables p. The part of this expression which is not reached by the derivatives is contained in the third line while the part reached by the derivatives with negative and positive indices is the content of the last line. Finally, the condition $A_i=0$ in $\overline{H}_{d_i}$ becomes $\tilde{A}_i+\sum_{j=i+1}^{n}K^{\{i,j\}}-\sum_{j=1}^{n-1}K^{\{j,i\}}=0$, this is the ith equation of conditions β.

Similarly, there are only derivatives with positive indices acting on H_{d_1-1}. We find

$$\begin{aligned}
\prod_{J\in\mathcal{C}_1}\frac{\partial^{q_J}}{\partial p_{k_1^J}\dots\partial p_{k^J_{q_J}}}H_{d_1-1} = &\sum_{\substack{g_1,m_1\geq 0\\ \text{s.t. }2g_1+\tilde{m_1}+\sum_{I\in\mathcal{C}_1}q_I=d_1+1}}\frac{\left(\sqrt{-1}\hbar\right)^{g_1}}{\tilde{m}_1!}\left[z^{2g_1}\right]\\
&\times\sum_{a_1^1,\dots,a^1_{\tilde{m}_1}\in\mathbb{Z}} W_1\left(a_1^1,\dots,a^1_{\tilde{m}_1},z_1\right)\,p_{a_1}\dots p_{a_{\tilde{m}_1}}e^{\sqrt{-1}x\sum_{i=1}^{n}\tilde{A}_i}\\
&\times\prod_{J\in\mathcal{C}_1}S\left(k_1^J z_1\right)\dots S\left(k^J_{q_J}z_1\right).
\end{aligned}\tag{6.3.19}$$

Note that A_1 becomes after the action of the derivatives

$$\tilde{A}_1-\sum_{j=2}^{n}K^{(1,j)}=\sum_{i=1}^{n}\tilde{A}_i.$$

We obtained this equality by summing the $(n-1)$ equations of conditions β (see Eq. (6.3.15)).

By combining Eq. (6.3.17), (6.3.18) and (6.3.19) we obtain Eq. (6.3.14). This proves the proposition. $\square$

Step 1.2. Similarly to the expression of $H_{d_1-1}\star\overline{H}_{d_2}\star\cdots\star\overline{H}_{d_n}$ obtained in Step 1.1 , we can get the expressions of the $2^{n-1}-1$ others terms appearing in $\left[\ldots\left[H_{d_1-1},\overline{H}_{d_2}\right],\ldots,\overline{H}_{d_n}\right]$. To each term we associate a permutation $\sigma\in\mathcal{S}_n$ such that $\sigma(i)$ is the index of the Hamiltonian appearing at the ith position (the index of H_{d_1-1} is 1 and the index of $\overline{H}_{d_i}$ is i). Then, the expression of the term in $\left[\ldots\left[H_{d_1-1},\overline{H}_{d_2}\right],\ldots,\overline{H}_{d_n}\right]$ corresponding to the permutation σ is given by Expression (6.3.14) with one modifications: conditions β become

$$\tilde{A}_i-\sum_{j=1}^{\sigma^{-1}(i)-1}K^{\{i,\sigma(j)\}}+\sum_{j=\sigma^{-1}(i)+1}^{n}K^{\{i,\sigma(j)\}},\ \ 2\leq i\leq n.$$

However, it is not necessary to compute these $2^{n-1}-1$ other terms in order to obtain $\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{0,g}$. Indeed, $\left[\ldots\left[H_{d_1-1},\overline{H}_{d_2}\right],\ldots,\overline{H}_{d_n}\right]$ is a power series in the indeterminates $a^1_1,\ldots,a^1_{\tilde{m}_1},\ldots,a^n_1\ldots,a^n_{\tilde{m}_n}$. In Step 2 we extract one coefficient of this power series. Then, we can restrict to compute the terms in $\left[\ldots\left[H_{d_1-1},\overline{H}_{d_2}\right],\ldots,\overline{H}_{d_n}\right]$ such that

$$\tilde{A}_i>0,\text{ with } i\geq 2.$$

The only term in $\left[\ldots\left[H_{d_1-1},\overline{H}_{d_2}\right],\ldots,\overline{H}_{d_n}\right]$ satisfying these inequalities is $H_{d_1-1}\star\overline{H}_{d_2}\star\cdots\star\overline{H}_{d_n}$. Indeed, in the other terms, the condition coming from the Hamiltonian on the leftmost in the star product, say $\overline{H}_{d_i}$, is

$$\tilde{A}_i+\sum_{1<j}K^{\{1,j\}}=0,$$

that is $\tilde{A}_j<0$.

Although, we will use one simplification coming from the commutators. A commutator simplifies the constant term in $\hbar$ in the star product, that is the term coming from the commutative product. These terms correspond in the expression of $H_{d_1-1}\star\overline{H}_{d_2}\star\cdots\star\overline{H}_{d_n}$ given by Eq. (6.3.14) to the terms satisfying

$$\sum_{j=1}^{i-1}q_{\{i,j\}}=0,\text{ for } 2\leq i\leq n.$$

Indeed, $\sum_{j=1}^{i-1}q_{\{i,j\}}$ counts the number of left (or right) derivatives coming from the $(i-1)$th star product and the commutative term in the star product is the one without derivatives. We call conditions γ the inequalities

$$\sum_{j=1}^{i-1}q_{\{i,j\}}\geq 1,\text{ for } 2\leq i\leq n.$$

Change of notation. We remove the tildes in our notations, i.e. we set $m_i:=\tilde{m}_i$ and $A_i:=\tilde{A}_i$ for any $1\leq i\leq n$.

Step 2. We first extract the coefficient of $\hbar^g$ from $\frac{\sqrt{-1}^g}{\hbar^{n-1}}\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right]$. Then we evaluate this coefficient, which is a differential polynomial, at $u_i=\delta_{i,1}$. We will then get an expression for $\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g}=[\hbar^g]\,\frac{\sqrt{-1}^g}{\hbar^{n-1}}\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right]\big|_{u_i=\delta_{1,i}}$.

We want to extract the coefficient of $\hbar^g$ in $\frac{\sqrt{-1}^g}{\hbar^{n-1}}\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right]$. As explained in Step 1.2, we only need to study the term $H_{d_1-1}\star\overline{H}_{d_2}\star\dots\star\overline{H}_{d_n}$ in $\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right]$. The coefficient of $\hbar^g$ in $\frac{\sqrt{-1}^g}{\hbar^{n-1}}H_{d_1-1}\star\overline{H}_{d_2}\star\dots\star\overline{H}_{d_n}$ is easily obtained from Expression (6.3.14). We get with our new notations

$$
\begin{aligned}
[\hbar^g]\,\frac{\sqrt{-1}^g}{\hbar^{n-1}}\left[\dots\left[H_{d_1-1},\overline{H}_{d_2}\right],\dots,\overline{H}_{d_n}\right] = &\sum_{\substack{g_1+\dots+g_n+\sum_{I\in\mathcal{C}}q_I=g+n-1\\ \text{with conditions }\gamma}}\sqrt{-1}^{2g+(n-1)}\\
&\times\prod_{i=1}^{n}\left(\frac{1}{m_i!}\left[z_i^{2g_i}\right]\sum_{a_1^i,\dots,a_{m_1}^i\in\mathbb{Z}}W_i\left(a_1^i,\dots,a_{m_1}^i,z_i\right)\,p_{a_1}\dots p_{a_{m_i}}e^{\sqrt{-1}xA_i}\right)\\
&\times\prod_{I\in\mathcal{C}}\left(\frac{1}{q_I!}\sum_{\substack{k_1^I,\dots,k_{q_I}^I>0\\ \text{with conditions }\beta}}k_1^I\dots k_{q_I}^IW^I\left(k_1^I,\dots,k_{q_I}^I,z_I\right)\right)\\
&+2^{n-1}-1\text{ other terms,}
\end{aligned}
$$

where we used conditions α to fix

$$
m_1=d_1+1-2g_1-\sum_{I\in\mathcal{C}_1}q_I,\ \text{and}\ m_i=d_i+2-2g_i-\sum_{I\in\mathcal{C}_i}q_I,\ \text{when}\ 2\le i\le n.
$$

This last expression is a differential polynomial thanks to Proposition **??**. In order to substitute $u_i=\delta_{i,1}$, we use Lemma 166. We get

$$
\begin{aligned}
\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g}= &\sum_{\substack{g_1+\dots+g_n+\sum_{I\in\mathcal{C}}q_I=g+n-1\\ \text{with conditions }\gamma}}\sqrt{-1}^{n-2-|\boldsymbol{d}|}\\
&\times\prod_{i=1}^{n}\left(\frac{1}{m_i!}\left[a_1^i\dots a_{m_1}^i\right]\left[z_i^{2g_i}\right]W_i\left(a_1^i,\dots,a_{m_1}^i,z_i\right)\right)\\
&\times\prod_{I\in\mathcal{C}}\left(\frac{1}{q_I!}\sum_{\substack{k_1^I,\dots,k_{q_I}^I>0\\ \text{with conditions }\beta}}k_1^I\dots k_{q_I}^IW^I\left(k_1^I,\dots,k_{q_I}^I,z_I\right)\right)\\
&+2^{n-1}-1\text{ other terms,}
\end{aligned}
$$

where we simplified the power of $\sqrt{-1}$ using $m_1 := d_1+1-2g_1-\sum_{I\in\mathcal{C}_1} q_I$ and $m_i := d_i+2-2g_i-\sum_{I\in\mathcal{C}_i} q_I$, when $2 \le i \le n$. We used the notation $|\boldsymbol{d}| = \sum_{i=1}^n d_i$.

Remark 174. It can look confusing that in this expression, a^i_j for $2 \le i \le n$ and $1 \le j \le m_i$ stands for a formal variable and an integer when we write the ith constraint

$$A_i - \sum_{l=1}^{i-1} K^{\{l,i\}} + \sum_{l=i+1}^{n} K^{\{i,l\}} = 0$$

of conditions β. This is due to the presence of Ehrhart polynomials. Indeed, according to [BR16, Lemma A.1], for any list of integers $A_2, \dots, A_n$, the coefficient of any power in $z_1, \dots, z_n$ of

$$\prod_{I\in\mathcal{C}} \left(\frac{1}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I>0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I W^I\left(k_1^I,\dots,k_{q_I}^I, z_I\right) \right)$$

is a polynomial in the variables $A_2, \dots, A_n$. We will then use the A_i's and the a^i_j's as integers and formal variables.

Step 3. We use the same simplification than Step 3 in Section 6.3.2.2, that is we consider the simplifications coming from extracting the coefficient of $a^i_1 \dots a^i_{m_1}$ in each factor of the product over i.

- Recall that S an even power series so that the coefficient of α in $\mathrm{S}(\alpha z) \times F(\alpha)$ is the coefficient of α in F. Hence, we can replace, in our expression of $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$, W_1 by $\frac{S\left(\sum_{i=1}^n A_i z_1\right)}{S(z_1)}$ and W_i by $\frac{1}{S(z_i)}$, when $2 \le i \le n$.
- Thanks to these simplifications, we see that we extract the coefficient of $\prod_{i=1}^n a^i_1 \dots a^i_{m_1}$ from a power series which only depends in the a^j_i's through their sums $A_i = \sum_{j=1}^{m_i} a^i_j$, for $1 \le i \le n$. This is equivalent to extracting the coefficient of $\prod_{i=1}^n \frac{A_i^{m_i}}{m_i!}$ from the same power series.
- For simplicity (see Step 1.2), we can suppose that $A_i > 0$, with $i \ge 2$.

Thanks to these three points, we get

$$\begin{aligned} \langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = & \sum_{\substack{g_1+\dots+g_n+\sum_{I\in\mathcal{C}} q_I = g+n-1 \\ \text{with conditions } \gamma}} \sqrt{-1}^{\,n-2-|\boldsymbol{d}|} \\ & \times \prod_{i=1}^n \left(\left[A_i^{m_i}\right] \left[z_i^{2g_i}\right] \frac{1}{S(z_i)} \right) S\left(|\boldsymbol{A}|\, z_1\right) \\ & \times \prod_{I\in\mathcal{C}} \left(\frac{1}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I>0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I W^I\left(k_1^I,\dots,k_{q_I}^I, z_I\right) \right), \end{aligned} \tag{6.3.20}$$

where we used the notations $|\boldsymbol{A}| = \sum_{i=1}^n A_i$.

Step 4. We perform some change of variable in our expression of $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$. We will then organize this expression to see $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g}$ as the coefficient of a product of exponential power series. This will allow us to use known properties of Eulerian numbers.

We perform the change of variables $z_i := A_i z_i$, with $1 \leq i \leq n$ in Expression (6.3.20). We get

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = & \sum_{\substack{g_1+\cdots+g_n+\sum_{I\in\mathcal{C}} q_I = g+n-1 \\ \text{with conditions } \gamma}} \sqrt{-1}^{n-2-|\boldsymbol{d}|} \\ & \times \prod_{i=1}^n \left(\left[A_i^{m_i+2g_i} \right] \left[z_i^{2g_i} \right] \frac{1}{S(A_i z_i)} \right) S(|\boldsymbol{A}| A_1 z_1) \\ & \times \prod_{I\in\mathcal{C}} \left(\frac{1}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I > 0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I \tilde{W}^I \left(k_1^I, \dots, k_{q_I}^I, z_I\right) \right),\end{aligned}$$

where we used the notation

$$\begin{aligned}\tilde{W}^I \left(k_1^I, \dots, k_{q_I}^I, z_I\right) : &= W^I \left(k_1^I, \dots, k_{q_I}^I, A_i z_i, A_j z_j\right) \\ &= S\left(k_1^I A_i z_i\right) \dots S\left(k_{q_I}^I A_i z_i\right) S\left(k_1^I A_j z_j\right) \dots S\left(k_{q_I}^I A_j z_j\right)\end{aligned}$$

for any pair $I = \{i, j\}$ of $\mathcal{C}$.

Using that $m_1 = d_1 + 1 - 2g_1 - \sum_{I\in\mathcal{C}_1} q_I$ and $m_i = d_i + 2 - 2g_i - \sum_{I\in\mathcal{C}_i} q_I$, when $2 \leq i \leq n$, we re-write this expression as

$$\begin{aligned}\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = & \sum_{\substack{\sum_{I\in\mathcal{C}} q_I \leq g+n-1 \\ \text{with conditions } \gamma}} \sqrt{-1}^{n-2-|\boldsymbol{d}|} \left[A_1^{d_1+1-\sum_{I\in\mathcal{C}_1} q_I} A_2^{d_2+2-\sum_{I\in\mathcal{C}_2} q_I} \dots A_n^{d_n+2-\sum_{I\in\mathcal{C}_n} q_I} \right] \\ & \times \sum_{g_1+\cdots+g_n = g+n-1-\sum_{I\in\mathcal{C}} q_I} \left[z_1^{2g_1} \dots z_n^{2g_n} \right] \\ & \times \underbrace{S(|\boldsymbol{A}| A_1 z_1) \prod_{i=1}^n \left(\frac{1}{S(A_i z_i)} \right) \prod_{I\in\mathcal{C}} \left(\frac{1}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I > 0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I \tilde{W}^I \left(k_1^I, \dots, k_{q_I}^I, z_I\right) \right)}_{G(z_1,\dots,z_n)}.\end{aligned}$$

We use in this expression that

$$\sum_{g_1+\cdots+g_n = g+n-1-\sum_{I\in\mathcal{C}} q_I} \left[z_1^{2g_1} \dots z_n^{2g_n} \right] G(z_1, \dots, z_n) = \left[z^{2g+2n-2-\sum_{I\in\mathcal{C}} 2q_I} \right] G(z, \dots, z)$$

to obtain

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \sum_{\substack{\sum_{I\in\mathcal{C}} q_I \le g+n-1 \\ \text{with conditions } \gamma}} \sqrt{-1}^{n-2-|\boldsymbol{d}|} \left[A_1^{d_1+1-\sum_{I\in\mathcal{C}_1} q_I} A_2^{d_2+2-\sum_{I\in\mathcal{C}_2} q_I} \dots A_n^{d_n+2-\sum_{I\in\mathcal{C}_n} q_I} \right]$$
$$\times \left[z^{2g+2n-2-\sum_{I\in\mathcal{C}} 2q_I} \right]$$
$$\times S\left(|\boldsymbol{A}|\, A_1 z\right) \prod_{i=1}^{n} \left(\frac{1}{S\left(A_i z_i\right)} \right) \prod_{I\in\mathcal{C}} \left(\frac{1}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I > 0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I \tilde{W}^I\left(k_1^I,\dots,k_{q_I}^I, z_I\right) \right).$$

Then, we rewrite this expression as

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2-|\boldsymbol{d}|} \left[A_1^{d_1} A_2^{d_2} \dots A_n^{d_n} z^{2g} \right] S\left(|\boldsymbol{A}|\, A_1 z\right) \prod_{i=1}^{n} \left(\frac{1}{S\left(A_i z_i\right)} \right) \frac{1}{A_1 A_2^2 \dots A_n^2 z^{2n-2}}$$
$$\times \sum_{\substack{\sum_{I\in\mathcal{C}} q_I \le g+n-1 \\ \text{with conditions } \gamma}} \prod_{I=\{i,j\}\in\mathcal{C}} \left(\frac{A_i^{q_I} A_j^{q_I} z^{2q_I}}{q_I!} \sum_{\substack{k_1^I,\dots,k_{q_I}^I > 0 \\ \text{with conditions } \beta}} k_1^I \dots k_{q_I}^I \tilde{W}^I\left(k_1^I,\dots,k_{q_I}^I, z_I\right) \right).$$

We can extend the range of summation to $\sum_{I\in\mathcal{C}} q_I$ running from 0 to ∞. Indeed, it is clear from this expression that the terms with $\sum_{I\in\mathcal{C}} q_I > g+n-1$ vanishes, since we extract the coefficient of z^{2g} from a power series with a factor $\frac{z^{2\sum_{I\in\mathcal{C}} q_I}}{z^{2n-2}}$. Hence, we rewrite the second line of the expression in the following way

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2-|\boldsymbol{d}|} \left[A_1^{d_1} A_2^{d_2} \dots A_n^{d_n} z^{2g} \right] S\left(|\boldsymbol{A}|\, A_1 z\right) \prod_{i=1}^{n} \left(\frac{1}{S\left(A_i z\right)} \right) \frac{1}{A_1 A_2^2 \dots A_n^2 z^{2n-2}}$$
$$\times \prod_{i=2}^{n} \left(\prod_{j=1}^{i-1} \left(\sum_{q_{\{i,j\}} \ge 0} \frac{A_i^{q_{\{i,j\}}} A_j^{q_{\{i,j\}}} z^{2q_{\{i,j\}}}}{q_{\{i,j\}}!} \sum_{\substack{k_1^I,\dots,k_{q_{\{i,j\}}}^I > 0 \\ \text{with conditions } \beta}} k_1^{\{i,j\}} \dots k_{q_{\{i,j\}}}^{\{i,j\}} \tilde{W}^{\{i,j\}}\left(k_1^{\{i,j\}},\dots,k_{q_{\{i,j\}}}^{\{i,j\}}, z_i, z_j\right) \right) - 1 \right). \tag{6.3.21}$$

Note that when $\sum_{j=1}^{i-1} q_{\{i,j\}} = 0$, for any $2 \le i \le n$, the product running over j equals 1 so that conditions γ are satisfied. Then we rewrite the last line of this expression as the coefficient of an exponential series.

Using the expression of $\tilde{W}^{\{i,j\}}\left(k_1^{\{i,j\}},\ldots,k_{q_{\{i,j\}}}^{\{i,j\}},z_i,z_j\right)$ and conditions β, we get

$$\prod_{j=2}^{n}\left(\prod_{i=1}^{j-1}\left(\sum_{q_{\{i,j\}}\geq 0}\frac{A_i^{q_{\{i,j\}}}A_j^{q_{\{i,j\}}}z^{2q_{\{i,j\}}}}{q_{\{i,j\}}!}\sum_{\substack{k_1^I,\ldots,k_{q_{\{i,j\}}}^I>0\\ \text{with conditions }\beta}}k_1^{\{i,j\}}\ldots k_{q_{\{i,j\}}}^{\{i,j\}}\tilde{W}^{\{i,j\}}\left(k_1^{\{i,j\}},\ldots,k_{q_{\{i,j\}}}^{\{i,j\}},z_i,z_j\right)\right)-1\right)$$
$$=\left[t_2^{A_2}\ldots t_n^{A_n}\right]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\exp\left(A_iA_jz^2\sum_{k>0}kS\left(kA_iz_i\right)S\left(kA_jz_j\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right)\Bigg|_{t_1=1}.$$

Substituting this expression in Eq. (6.3.21), we get

$$\begin{aligned}\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{0,g}=&\sqrt{-1}^{n-2-|\boldsymbol{d}|}\left[A_1^{d_1}A_2^{d_2}\ldots A_n^{d_n}z^{2g}\right]S\left(|\boldsymbol{A}|A_1z\right)\prod_{i=1}^{n}\left(\frac{1}{S\left(A_iz\right)}\right)\frac{1}{A_1A_2^2\ldots A_n^2z^{2n-2}}\\ &\times\left[t_2^{A_2}\ldots t_n^{A_n}\right]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\exp\left(A_iA_jz^2\sum_{k>0}kS\left(kA_iz_i\right)S\left(kA_jz_j\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right)\Bigg|_{t_1=1}.\end{aligned}\tag{6.3.22}$$

Step 5. The second line of Eq. (6.3.22) is simplified using the following property.

Proposition 175 (The products of exponentials formula)**.** *Fix n positive integers $A_1,...,A_n$. Fix n formal variables $t_1,\ldots,t_n$; by convention, let $t_1=1$. We have*

$$\begin{aligned}&\left[t_2^{A_2}\ldots t_n^{A_n}\right]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\exp\left(A_iA_jz^2\sum_{k>0}kS\left(kA_iz\right)S\left(kA_jz\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right)\\ =&A_1A_2^2\ldots A_n^2z^{2n-2}\left(\sum_{i=1}^{n}A_i\right)^{n-2}\frac{\prod_{i=1}^{n}S\left(A_iz\right)}{S\left(A_1z+\cdots+A_nz\right)}\prod_{r=2}^{n}S\left(A_r\left(A_1z+\cdots+A_nz\right)\right).\end{aligned}$$

This proposition is proved in Section 6.4 using properties of Eulerian numbers.

According to this proposition, we get

$$\begin{aligned}\langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_{0,g}=&\sqrt{-1}^{n-2-|\boldsymbol{d}|}\left[A_1^{d_1}A_2^{d_2}\ldots A_n^{d_n}z^{2g}\right]S\left(|\boldsymbol{A}|A_1z\right)\prod_{i=1}^{n}\left(\frac{1}{S\left(A_iz\right)}\right)\frac{1}{A_1A_2^2\ldots A_n^2z^{2n-2}}\\ &\times A_1A_2^2\ldots A_n^2z^{2n-2}|\boldsymbol{A}|^{n-2}\frac{\prod_{i=1}^{n}S\left(A_iz\right)}{S\left(|\boldsymbol{A}|z\right)}\prod_{r=2}^{n}S\left(A_r|\boldsymbol{A}|z\right)\end{aligned}$$

that we simplify as

$$\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g} = \sqrt{-1}^{n-2-|\boldsymbol{d}|}\left[A_1^{d_1}A_2^{d_2}\dots A_n^{d_n}z^{2g}\right]\frac{|\boldsymbol{A}|^{n-2}}{S\left(|\boldsymbol{A}|\,z\right)}\prod_{r=1}^{n}S\left(A_r\,|\boldsymbol{A}|\,z\right).$$

Finally, with the change of variable $z := \frac{z}{|\boldsymbol{A}|}$, we get

$$\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{0,g} = \sqrt{-1}^{n-2-|\boldsymbol{d}|}\left[A_1^{d_1}A_2^{d_2}\dots A_n^{d_n}z^{2g}\right]|\boldsymbol{A}|^{2g+n-2}\frac{1}{S\left(z\right)}\prod_{r=1}^{n}S\left(A_r z\right)$$

and we recognize the expression of $\langle\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle\rangle_g$ given by Expression (6.3.2).

6.4 Proof of the products of the exponentials formula

The purpose of this section is to prove Proposition 175 which ends the proof of the main theorem. To do so, we first use Corollary 155 which follows from Eulerian numbers properties. We get

$$\begin{aligned}&\left[t_2^{A_2}\dots t_n^{A_n}\right]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\exp\left(A_iA_jz^2\sum_{k>0}kS\left(kA_iz\right)S\left(kA_jz\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right)\\ &=\left[t_2^{A_2}\dots t_n^{A_n}\right]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\left(1+4\sum_{k>0}\frac{\mathrm{sh}\left(\frac{A_i}{2}z\right)\mathrm{sh}\left(\frac{A_j}{2}z\right)}{\mathrm{sh}\left(\frac{A_i+A_j}{2}z\right)}\mathrm{sh}\left(k\frac{A_i+A_j}{2}z\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right),\end{aligned}$$

where we used the convention $t_1 = 1$. Recall that $A_1, \dots, A_n$ are positive integers and $t_2, \dots, t_n, z$ are formal variables.

Proposition 176. *Fix $(n-1)$ positive integers $a_2, ..., a_n$ and n formal variables $\tilde{A}_1, ..., \tilde{A}_n$. Fix $(n-1)$ more formal variables $t_2, \dots, t_n$; by convention, let $t_1 = 1$. We have*

$$\begin{aligned}&[t_2^{a_2}\dots t_n^{a_n}]\prod_{i=2}^{n}\left(\prod_{j=1}^{i-1}\left(1+4\sum_{k>0}\frac{\mathrm{sh}\left(\tilde{A}_i\right)\mathrm{sh}\left(\tilde{A}_j\right)}{\mathrm{sh}\left(\tilde{A}_i+\tilde{A}_j\right)}\mathrm{sh}\left(k\left(\tilde{A}_i+\tilde{A}_j\right)\right)\left(\frac{t_i}{t_j}\right)^k\right)-1\right)\\ &=2^{2(n-1)}\frac{\prod_{i=1}^{n}\mathrm{sh}\left(\tilde{A}_i\right)}{\mathrm{sh}\left(\tilde{A}_1+\dots+\tilde{A}_n\right)}\left(\prod_{r=2}^{n}\left(\mathrm{sh}\left(a_r\left(\tilde{A}_1+...+\tilde{A}_r\right)+\tilde{A}_r\left(a_{r+1}+...+a_n\right)\right)\right)\right).\end{aligned}$$

Before proving this proposition, let us end the proof of the products of exponentials formula. According to this proposition by substituting $\tilde{A}_i := \frac{A_i}{2} z$ and $a_i := A_i$, we find

$$\begin{aligned}
&\left[t_2^{A_2} \dots t_n^{A_n}\right] \prod_{i=2}^{n} \left(\prod_{j=1}^{i-1} \left(1 + 4 \sum_{k>0} \frac{\operatorname{sh}\left(\frac{A_i}{2} z\right) \operatorname{sh}\left(\frac{A_j}{2} z\right)}{\operatorname{sh}\left(\frac{A_i+A_j}{2} z\right)} \operatorname{sh}\left(k \frac{A_i + A_j}{2} z\right) \left(\frac{t_i}{t_j}\right)^k \right) - 1 \right) \\
&= 2^{2(n-1)} \frac{\prod_{i=1}^{n} \operatorname{sh}\left(\frac{A_i}{2} z\right)}{\operatorname{sh}\left(\frac{A_1+\dots+A_n}{2} z\right)} \prod_{r=2}^{n} \operatorname{sh}\left(\frac{A_r \left(A_1 + \dots + A_n\right)}{2} z\right) \\
&= A_1 A_2^2 \dots A_n^2 z^{2n-2} \left(\sum_{i=1}^{n} A_i\right)^{n-2} \frac{\prod_{i=1}^{n} S\left(A_i z\right)}{S\left(A_1 z + \dots + A_n z\right)} \prod_{r=2}^{n} S\left(A_r \left(A_1 z + \dots + A_n z\right)\right),
\end{aligned}$$

where we used the notation $S(z) = \frac{\operatorname{sh}(z/2)}{z/2}$ to obtain this equality. This proves the products of exponentials formula.

Convention. In the rest of this section, we make no use of the positive integers $A_1, \dots, A_n$. However we will intensively use the formal variables $\tilde{A}_1, \dots, \tilde{A}_n$. For convenience, we change the notation by removing the tildes on these formal variables.

Proof of Proposistion 176 . We prove this formula by induction over n. The first step $n = 2$ is obvious. Suppose this induction is proved until step n. We prove the $(n+1)$-th step. Start from the LHS

$$\left[t_2^{a_2} \dots t_{n+1}^{a_{n+1}}\right] \prod_{i=2}^{n+1} \left(\prod_{j=1}^{i-1} \left(1 + 4 \sum_{k>0} \frac{\operatorname{sh}\left(A_i\right) \operatorname{sh}\left(A_j\right)}{\operatorname{sh}\left(A_i + A_j\right)} \operatorname{sh}\left(k \left(A_i + A_j\right)\right) \left(\frac{t_i}{t_j}\right)^k \right) - 1 \right).$$

We decompose the product in order to use the induction hypothesis ; we move the terms corresponding to $i = n+1$ on a second line and get

$$\begin{aligned}
&\left[t_2^{a_2} \dots t_n^{a_n} t_{n+1}^{a_{n+1}}\right] \\
&\times \prod_{i=2}^{n} \left(\prod_{j=1}^{i-1} \left(1 + 4 \sum_{k>0} \frac{\operatorname{sh}\left(A_i\right) \operatorname{sh}\left(A_j\right)}{\operatorname{sh}\left(A_i + A_j\right)} \operatorname{sh}\left(k \left(A_i + A_j\right)\right) \left(\frac{t_i}{t_j}\right)^k \right) - 1 \right) \\
&\times \left(\prod_{s=1}^{n} \left(1 + 4 \sum_{l>0} \frac{\operatorname{sh}\left(A_s\right) \operatorname{sh}\left(A_{n+1}\right)}{\operatorname{sh}\left(A_s + A_{n+1}\right)} \operatorname{sh}\left(k \left(A_s + A_{n+1}\right)\right) \left(\frac{t_{n+1}}{t_s}\right)^l \right) - 1 \right).
\end{aligned}$$

We simplify the series on the second line of the expression using the induction hypothesis. We get

$$\begin{aligned}
&\left[t_2^{a_2} \dots t_n^{a_n} t_{n+1}^{a_{n+1}}\right] \\
&\times 2^{2(n-1)} \frac{\prod_{i=1}^{n} \operatorname{sh}\left(A_i\right)}{\operatorname{sh}\left(A_1 + \dots + A_n\right)} \sum_{i_2, \dots, i_n > 0} \prod_{r=2}^{n} \operatorname{sh}\left(i_r \left(A_1 + \dots + A_r\right) + A_r \left(i_{r+1} + \dots + k_n\right)\right) t_r^{i_r} \\
&\times \left(\prod_{s=1}^{n} \left(1 + 4 \sum_{k>0} \frac{\operatorname{sh}\left(A_s\right) \operatorname{sh}\left(A_{n+1}\right)}{\operatorname{sh}\left(A_s + A_{n+1}\right)} \operatorname{sh}\left(k \left(A_s + A_{n+1}\right)\right) \left(\frac{t_{n+1}}{t_s}\right)^k \right) - 1 \right).
\end{aligned}$$

We obtain the result using the following proposition with

$$u := t_{n+1},\ b := a_{n+1},\ B := A_{n+1},\ \text{and } X_r = 0 \text{ for } 2 \leq r \leq n.$$

Proposition 177 (The sinh formula)**.** *Fix n positive integers $a_2, \ldots, a_n, b$ and $2n$ formal variables $A_1, \ldots, A_n, B, X_2, \ldots$ Fix n more formal variables $t_2, \ldots, t_n, u$; by convention, let $t_1 = 1$. Then the coefficient of $t_2^{a_2} \ldots t_n^{a_n} u^b$ in the formal power series*

$$\sum_{i_2,\ldots,i_n>0} \prod_{r=2}^{n} \operatorname{sh}\left(i_r\left(A_1+\cdots+A_r\right)+A_r\left(i_{r+1}+\cdots+i_n\right)+X_r\right) t_r^{i_r}$$
$$\times \left\{\prod_{s=1}^{n}\left(1+4\sum_{j_s>0}\frac{\operatorname{sh}\left(A_s\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_s+B\right)}\operatorname{sh}\left(j_s\left(A_s+B\right)\right)\left(\frac{u}{t_s}\right)^{j_s}\right)-1\right\}$$

is

$$4\frac{\operatorname{sh}\left(A_1+\cdots+A_n\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_1+\cdots+A_n+B\right)}\prod_{r=2}^{n}\Big(\operatorname{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)\Big)\operatorname{sh}\left(b\left(A_1+\cdots+A_n+B\right)\right).$$

□

The purpose of the rest of this section is to prove the sinh formula. The proof goes by induction. We prove the case $n = 2$ in Section 6.4.1. The heredity is proved in Section 6.4.2.

6.4.1 Proof by induction of the sinh formula: initialization

We prove in this section, the case $n = 2$ of the sinh formula. We begin by a series of lemmas.

Lemma 178. *Let α, β and γ be some formal variables. We have*

$$\operatorname{ch}(\alpha)\operatorname{sh}(\beta+\gamma)-\operatorname{ch}(\beta)\operatorname{sh}(\alpha+\gamma)=\operatorname{sh}(\alpha-\beta)\operatorname{sh}(\gamma).$$

Proof. One can check this formula using the basic hyperbolic identities. □

Lemma 179. *Let μ and ν be some formal variables. We have*

$$\sum_{j=0}^{b}\operatorname{sh}(\mu j+\nu)=\frac{\operatorname{sh}(\mu(b+1)/2)\operatorname{sh}(\mu b/2+\nu)}{\operatorname{sh}(\mu/2)}$$
$$=\frac{\operatorname{ch}(\mu/2)}{\operatorname{sh}(\mu/2)}\operatorname{sh}(b\mu/2)\operatorname{sh}(b\mu/2+\nu)+\operatorname{ch}(b\mu/2)\operatorname{sh}(b\mu/2+\nu).$$

Proof. Using the exponential form of the hyperbolic sine and geometric sums, we obtain the first equality. The second equality is obtained by using $\operatorname{sh}\left(\frac{\mu b}{2}+\frac{\mu}{2}\right)=\operatorname{ch}\left(\frac{\mu}{2}\right)\operatorname{sh}\left(\frac{b\mu}{2}\right)+\operatorname{ch}\left(\frac{b\mu}{2}\right)\operatorname{sh}\left(\frac{\mu}{2}\right)$ and simplifying. □

Lemma 180. *Fix two integers a, b and four formal variables A_1, A_2, B, X. We have*

$$\begin{aligned}
&\sum_{j=0}^{b} \mathrm{sh}\Big((a+j)(A_1+A_2)+X\Big)\mathrm{sh}\Big((b-j)(A_1+B)\Big)\mathrm{sh}\Big(j(A_2+B)\Big)\\
&=\frac{\mathrm{ch}(A_1)}{\mathrm{sh}(A_1)}\mathrm{sh}(bA_1)\,\mathrm{sh}(a(A_1+A_2)-bB+X)\\
&+\frac{\mathrm{ch}(A_2)}{\mathrm{sh}(A_2)}\mathrm{sh}(bA_2)\,\mathrm{sh}(a(A_1+A_2)+b(A_1+A_2+B)+X)\\
&+\frac{\mathrm{ch}(B)}{\mathrm{sh}(B)}\mathrm{sh}(bB)\,\mathrm{sh}(-a(A_1+A_2)-bA_1-X)\\
&+\frac{\mathrm{ch}(A_1+A_2+B)}{\mathrm{sh}(A_1+A_2+B)}\mathrm{sh}(b(A_1+A_2+B))\,\mathrm{sh}(-a(A_1+A_2)-bA_2-X)\,.
\end{aligned}$$

Proof. We start from the LHS of this formula. We first linearize the product of the three hyperbolic sine using

$$\mathrm{sh}(u)\,\mathrm{sh}(v)\,\mathrm{sh}(w)=\mathrm{sh}(u+v+w)+\mathrm{sh}(u-v-w)+\mathrm{sh}(-u+v-w)+\mathrm{sh}(-u-v+w) \qquad (6.4.1)$$

with

$$\begin{aligned}
u&=(a+j)(A_1+A_2)+X\\
v&=(b-j)(A_1+B)\\
w&=j(A_2+B)\,.
\end{aligned}$$

Then, we use the formula of Lemma 179

$$\sum_{j=0}^{b}\mathrm{sh}(\mu j+\nu)=\frac{\mathrm{ch}(\mu/2)}{\mathrm{sh}(\mu/2)}\mathrm{sh}(\mu b/2)\,\mathrm{sh}(\mu b/2+\nu)+\mathrm{ch}(\mu b/2)\,\mathrm{sh}(\mu b/2+\nu)$$

to compute the finite sum of each of the four terms in the RHS of Eq. (6.4.1). We find

$$
\begin{aligned}
\sum_{j=0}^{b} \operatorname{sh}(u-v-w) =& \frac{\operatorname{ch}(A_1)}{\operatorname{sh}(A_1)} \operatorname{sh}(bA_1) \operatorname{sh}(a(A_1+A_2)-bB+X) \\
&+ \operatorname{ch}(bA_1) \operatorname{sh}(a(A_1+A_2)-bB+X)
\end{aligned} \tag{6.4.2}
$$

$$
\begin{aligned}
\sum_{j=0}^{b} \operatorname{sh}(u+v+w) =& \frac{\operatorname{ch}(A_2)}{\operatorname{sh}(A_2)} \operatorname{sh}(bA_2) \operatorname{sh}(a(A_1+A_2)+b(A_1+A_2+B)+X) \\
&+ \operatorname{ch}(bA_2) \operatorname{sh}(a(A_1+A_2)+b(A_1+A_2+B)+X)
\end{aligned} \tag{6.4.3}
$$

$$
\begin{aligned}
\sum_{j=0}^{b} \operatorname{sh}(-u-v+w) =& \frac{\operatorname{ch}(B)}{\operatorname{sh}(B)} \operatorname{sh}(bB) \operatorname{sh}(-a(A_1+A_2)-bA_1-X) \\
&+ \operatorname{ch}(bB) \operatorname{sh}(-a(A_1+A_2)-bA_1-X)
\end{aligned} \tag{6.4.4}
$$

$$
\begin{aligned}
\sum_{j=0}^{b} \operatorname{sh}(-u+v-w) =& \frac{\operatorname{ch}(A_1+A_2+B)}{\operatorname{sh}(A_1+A_2+B)} \operatorname{sh}(b(A_1+A_2+B)) \operatorname{sh}(-a(A_1+A_2)-bA_2-X) \\
&+ \operatorname{ch}(b(A_1+A_2+B)) \operatorname{sh}(-a(A_1+A_2)-bA_2-X).
\end{aligned} \tag{6.4.5}
$$

Finally, we prove that the sum of the seconds terms in the RHS of Equations (6.4.2), (6.4.3), (6.4.4), (6.4.5) vanishes. This will end the proof.

We sum the second term of the RHS of Eq. (6.4.2) and the second term of the RHS of Eq. (6.4.4). Using Lemma 178, that is

$$
\operatorname{ch}(\alpha) \operatorname{sh}(\beta+\gamma) - \operatorname{ch}(\beta) \operatorname{sh}(\alpha+\gamma) = \operatorname{sh}(\alpha-\beta) \operatorname{sh}(\gamma)
$$

with $\alpha = bA_1$, $\beta = -bB$ and $\gamma = a(A_1+A_2)+X$, we find

$$
\begin{aligned}
&\operatorname{ch}(bA_1) \operatorname{sh}(a(A_1+A_2)-bB+X) + \operatorname{ch}(bB) \operatorname{sh}(-a(A_1+A_2)-bA_1-X) \\
&= \operatorname{sh}(b(A_1+B)) \operatorname{sh}(a(A_1+A_2)+X).
\end{aligned}
$$

We sum the second term of the RHS of Eq. (6.4.3) and the second term of the RHS of Eq. (6.4.5). Using Lemma 178 with $\alpha = bA_2$, $\beta = b(A_1+A_2+B)$ and $\gamma = a(A_1+A_2)+X$, we find

$$
\begin{aligned}
&\operatorname{ch}(bA_2) \operatorname{sh}(a(A_1+A_2)+b(A_1+A_2+B)+X) + \operatorname{ch}(b(A_1+A_2+B)) \operatorname{sh}(-a(A_1+A_2)-bA_2-X) \\
&= -\operatorname{sh}(b(A_1+B)) \operatorname{sh}(a(A_1+A_2)+X).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 =& \operatorname{ch}(bA_1) \operatorname{sh}(a(A_1+A_2)-bB+X) + \operatorname{ch}(bB) \operatorname{sh}(-a(A_1+A_2)-bA_1-X) \\
&+ \operatorname{ch}(bA_2) \operatorname{sh}(a(A_1+A_2)+b(A_1+A_2+B)+X) + \operatorname{ch}(b(A_1+A_2+B)) \operatorname{sh}(-a(A_1+A_2)-bA_2-X)
\end{aligned}
$$

$\square$

Lemma 181. *Let α, β and γ be some formal variables. We have*

$$\mathrm{sh}\,(\alpha)\,\mathrm{sh}\,(\beta) + \mathrm{sh}\,(\gamma)\,\mathrm{sh}\,(\alpha+\beta+\gamma) = \mathrm{sh}\,(\alpha+\gamma)\,\mathrm{sh}\,(\beta+\gamma)\,.$$

Proof. One can check this formula using the usual hyperbolic identities. □

Lemma 182. *Let A_1, A_2 and B be some formal variables. We have*

$$\begin{aligned}
&\mathrm{ch}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{sh}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\\
+&\mathrm{sh}\,(A_1)\,\mathrm{ch}\,(A_2)\,\mathrm{sh}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\\
+&\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\\
-&\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{sh}\,(B)\,\mathrm{ch}\,(A_1+A_2+B)\\
=&\mathrm{sh}\,(A_1+B)\,\mathrm{sh}\,(A_2+B)\,\mathrm{sh}\,(A_1+A_2)\,.
\end{aligned}$$

Proof. We start from the LHS. We use the hyperbolic identity $\mathrm{sh}\,(\alpha+\beta) = \mathrm{sh}\,(\alpha)\,\mathrm{ch}\,(\beta) + \mathrm{ch}\,(\alpha)\,\mathrm{sh}\,(\beta)$. We sum the two first lines using $\alpha = A_1$ and $\beta = A_2$, we obtain

$$\mathrm{sh}\,(A_1+A_2)\,\mathrm{sh}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\,.$$

We sum the two last lines using $\alpha = -B$ and $\beta = A_1+A_2+B$, we obtain

$$\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{sh}\,(A_1+A_2)\,.$$

Finally we sum these two terms. We factor $\mathrm{sh}\,(A_1+A_2)$ and use Lemma 181 to obtain the expected result. □

We now prove the case $n=2$ of the sinh formula.

Proposition 183. *Fix two integers a_2, b and four formal variables A_1, A_2, B, X. Fix two more formal variables t_2, u. Then the coefficient of $t_2^{a_2}u^b$ in the formal power series*

$$\begin{aligned}
&\sum_{i>0}\mathrm{sh}\,(i\,(A_1+A_2)+X)\,t_2\\
&\left\{\left(1+4\sum_{j_1>0}\frac{\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(B)}{\mathrm{sh}\,(A_1+B)}\mathrm{sh}\,(j_0\,(A_1+B))\,u^{j_1}\right)\left(1+4\sum_{j_2>0}\frac{\mathrm{sh}\,(A_2)\,\mathrm{sh}\,(B)}{\mathrm{sh}\,(A_2+B)}\mathrm{sh}\,(j_1\,(A_2+B))\left(\frac{u}{t_2}\right)^{j_2}\right)\right.\\
&\quad\left.-1\right\}
\end{aligned}$$

is

$$4\frac{\mathrm{sh}\,(A_1+A_2)\,\mathrm{sh}\,(B)}{\mathrm{sh}\,(A_1+A_2+B)}\left(\mathrm{sh}\,(a_2\,(A_1+A_2)+A_2b+X)\right)\mathrm{sh}\,(b\,(A_1+A_2+B))$$

Proof. We re-write the power series of the LHS of the proposition as

$$\Phi\left\{\left(1+\Delta_1\right)\left(1+\Delta_2\right)-1\right\}$$

with

$$\begin{aligned}
\Phi &= \sum_{i\geq 0} \mathrm{sh}\left(i\left(A_1+A_2\right)+X\right) t_2 \\
\Delta_1 &= 4\sum_{j_1\geq 0} \frac{\mathrm{sh}\left(A_1\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+B\right)} \mathrm{sh}\left(j_0\left(A_1+B\right)\right) u^{j_1} \\
\Delta_2 &= 4\sum_{j_2\geq 0} \frac{\mathrm{sh}\left(A_2\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_2+B\right)} \mathrm{sh}\left(j_1\left(A_2+B\right)\right) \left(\frac{u}{t_2}\right)^{j_2}.
\end{aligned}$$

We begin by expanding the expression

$$\Phi\left\{\left(1+\Delta_1\right)\left(1+\Delta_2\right)-1\right\} = \underbrace{\Delta_1\Phi}_{\text{term }(i)} + \underbrace{\Delta_2\Phi}_{\text{term }(ii)} + \underbrace{\Phi\Delta_1\Delta_2}_{\text{term }(iii)}.$$

Then we extract the coefficient of $t_2^{a_2}u^b$:

$$\begin{aligned}
\text{in term }(i) &= 4\frac{\mathrm{sh}\left(A_1\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+B\right)} \mathrm{sh}\left(b\left(A_1+B\right)\right)\mathrm{sh}\left(a_2\left(A_1+A_2\right)+X\right), \\
\text{in term }(ii) &= 4\frac{\mathrm{sh}\left(A_2\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_2+B\right)} \mathrm{sh}\left(b\left(A_2+B\right)\right)\mathrm{sh}\left(\left(a_2+b\right)\left(A_1+A_2\right)+X\right), \\
\text{in term }(iii) &= 16\frac{\mathrm{sh}\left(A_1\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+B\right)}\frac{\mathrm{sh}\left(A_2\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_2+B\right)} \\
&\quad \times\sum_{j=0}^{b} \mathrm{sh}\left(\left(a_2+j\right)\left(A_1+A_2\right)+X\right)\mathrm{sh}\left(\left(b-j\right)\left(A_1+B\right)\right)\mathrm{sh}\left(j\left(A_2+B\right)\right).
\end{aligned}$$

Observing that $4\mathrm{sh}\left(B\right)$ appears in terms (i), (ii) and (iii) but also in the result, we factor it out. For reasons that will become clear, we also factor out $\frac{1}{\mathrm{sh}(A_1+B)\mathrm{sh}(A_2+B)\mathrm{sh}(A_1+A_2+B)}$. Hence, we re-define our three terms by

$$\begin{aligned}
\text{term }(i) &:= \mathrm{sh}\left(A_1\right)\mathrm{sh}\left(A_2+B\right)\mathrm{sh}\left(A_1+A_2+B\right)\mathrm{sh}\left(b\left(A_1+B\right)\mathrm{sh}\left(a_2\left(A_1+A_2\right)+X\right)\right), \\
\text{term }(ii) &:= \mathrm{sh}\left(A_2\right)\mathrm{sh}\left(A_1+B\right)\mathrm{sh}\left(A_1+A_2+B\right)\mathrm{sh}\left(b\left(A_2+B\right)\mathrm{sh}\left(\left(a_2+b\right)\left(A_1+A_2\right)+X\right)\right), \\
\text{term }(iii) &:= 4\mathrm{sh}\left(A_1\right)\mathrm{sh}\left(B\right)\mathrm{sh}\left(A_2\right)\mathrm{sh}\left(A_1+A_2+B\right) \\
&\quad +\sum_{j=0}^{b} \mathrm{sh}\left(\left(a_2+j\right)\left(A_1+A_2\right)+X\right)\mathrm{sh}\left(\left(b-j\right)\left(A_1+B\right)\right)\mathrm{sh}\left(j\left(A_2+B\right)\right).
\end{aligned}$$

In Step 1, we develop terms (i) and (ii) using a basic hyperbolic identity. We also compute the sum of term (iii) using Lemma 180. We obtain 8 terms from terms (i), (ii) and (iii). Then in Step 2, we combine 7 of these terms using Lemma 181. Finally, in Step 3, we combine all the terms and we use Lemma 182 to obtain the result.

Step 1. **Term** (i) : using the hyperbolic identity $\operatorname{sh}(A_2+B)=\operatorname{sh}(A_2)\operatorname{ch}(B)+\operatorname{sh}(B)\operatorname{ch}(A_2)$, we split term (i) in a sum of two terms. We get

$$\operatorname{sh}(A_1)\operatorname{sh}(A_2)\operatorname{ch}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_1+B))\operatorname{sh}(a_2(A_1+A_2)+X) \tag{ia}$$
$$+\operatorname{sh}(A_1)\operatorname{ch}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_1+B))\operatorname{sh}(a_2(A_1+A_2)+X). \tag{ib}$$

Term (ii) : using the same hyperbolic identity for $\operatorname{sh}(A_1+B)$, we split term (ii) in a sum of two terms. We get

$$\operatorname{sh}(A_1)\operatorname{sh}(A_2)\operatorname{ch}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_2+B))\operatorname{sh}((a_2+b)(A_1+A_2)+X) \tag{iia}$$
$$+\operatorname{ch}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_2+B))\operatorname{sh}((a_2+b)(A_1+A_2)+X). \tag{iib}$$

Term (iii) : we use Lemma 180 to compute the sum. This gives four terms

$$\operatorname{ch}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(bA_1)\operatorname{sh}(a_2(A_1+A_2)-bB+X) \tag{iiia}$$
$$+\operatorname{sh}(A_1)\operatorname{ch}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(bA_2)\operatorname{sh}(a_2(A_1+A_2)+b(A_1+A_2+B)+X) \tag{iiib}$$
$$+\operatorname{sh}(A_1)\operatorname{sh}(A_2)\operatorname{ch}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(bB)\operatorname{sh}(-a_2(A_1+A_2)-bA_1-X) \tag{iiic}$$
$$+\operatorname{sh}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{ch}(A_1+A_2+B)\times\operatorname{sh}(b(A_1+A_2+B))\operatorname{sh}(-a_2(A_1+A_2)-bA_2-X). \tag{iiid}$$

Step 2. We now combine the terms (ia) until $(iiic)$. We will do so using the formula of Lemma 181, that is

$$\operatorname{sh}(\alpha)\operatorname{sh}(\beta)+\operatorname{sh}(\gamma)\operatorname{sh}(\alpha+\beta+\gamma)=\operatorname{sh}(\alpha+\gamma)\operatorname{sh}(\beta+\gamma).$$

- We sum terms $(iiia)$ and (iib). They have the common factor $\operatorname{ch}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)$. We get

$$\begin{aligned}&\operatorname{ch}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\\&\times[\operatorname{sh}(bA_1)\operatorname{sh}(a_2(A_1+A_2)-bB+X)+\operatorname{sh}(b(A_2+B))\operatorname{sh}((a_2+b)(A_1+A_2)+X)].\end{aligned}$$

  Then simplify the expression appearing inside the brackets using Lemma 181 with $\alpha=bA_1$, $\beta=a_2(A_1+A_2)-bB+X$ and $\gamma=b(A_2+B)$. We find

$$\operatorname{ch}(A_1)\operatorname{sh}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_1+A_2+B))\operatorname{sh}(a_2(A_1+A_2)+bA_2+X). \tag{6.4.6}$$

- We sum terms (ib) and $(iiib)$ using the same computation. They have the common factor $\operatorname{sh}(A_1)\operatorname{ch}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)$. We get

$$\begin{aligned}&\operatorname{sh}(A_1)\operatorname{ch}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\\&\times[\operatorname{sh}(b(A_1+B))\operatorname{sh}(a_2(A_1+A_2)+X)+\operatorname{sh}(bA_2)\operatorname{sh}(a_2(A_1+A_2)+b(A_1+A_2+B)+X)].\end{aligned}$$

  Then simplify the expression appearing inside the brackets using Lemma 181 with $\alpha=b(A_1+B)$, $\beta=a_2(A_1+A_2)+X$, $\gamma=bA_2$. We find

$$\operatorname{sh}(A_1)\operatorname{ch}(A_2)\operatorname{sh}(B)\operatorname{sh}(A_1+A_2+B)\times\operatorname{sh}(b(A_1+A_2+B))\operatorname{sh}(a_2(A_1+A_2)+bA_2+X). \tag{6.4.7}$$

- We sum the terms *(ia)* and *(iia)* and *(iiic)*. They have the common factor $\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)$. First re-write $(ia)+(iiic)$ as

$$\begin{aligned}&\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\\ &\times\left[\mathrm{sh}\,(b\,(A_1+B))\,\mathrm{sh}\,(a_2\,(A_1+A_2)+X)+\mathrm{sh}\,(bB)\,\mathrm{sh}\,(-a_2\,(A_1+A_2)-bA_1-X)\right].\end{aligned}$$

We then apply Lemma 181 with $\alpha=b\,(A_1+B)$, $\beta=a_2\,(A_1+A_2)+X$ and $\gamma=-bB$. We get

$$\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\times\mathrm{sh}\,(bA_1)\,\mathrm{sh}\,(a_2\,(A_1+A_2)-bB+X)\,.$$

Then, we add the term *(iia)* to this expression, we get

$$\begin{aligned}&\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\\ &\times\left[\mathrm{sh}\,(bA_1)\,\mathrm{sh}\,(a_2\,(A_1+A_2)-bB+X)+\mathrm{sh}\,(b\,(A_2+B))\,\mathrm{sh}\,((a_2+b)\,(A_1+A_2)+X)\right].\end{aligned}$$

Finally, we use Lemma 181 with $\alpha=bA_1$, $\beta=a_2\,(A_1+A_2)-bB+X$ and $\gamma=b\,(A_2+B)$. We find

$$\mathrm{sh}\,(A_1)\,\mathrm{sh}\,(A_2)\,\mathrm{ch}\,(B)\,\mathrm{sh}\,(A_1+A_2+B)\times\mathrm{sh}\,(b\,(A_1+A_2+B))\,\mathrm{sh}\,(a_2\,(A_1+A_2)+bA_2+X)\,. \tag{6.4.8}$$

Step 3. We sum the three terms (6.4.6), (6.4.7) and (6.4.8) obtained in Step 2 with the remaining term of Step 1, that is term *(iiid)*. These four terms have the common factor $\mathrm{sh}\,(b\,(A_1+A_2+B))\,\mathrm{sh}\,(a_2\,(A_1+A_2)+bA_2+X)$. We factor it. The sum of the four remaining terms is the sum of the LHS of Lemma 182. Using this Lemma, we get

$$\mathrm{sh}\,(A_1+B)\,\mathrm{sh}\,(A_2+B)\,\mathrm{sh}\,(A_1+A_2)\times\mathrm{sh}\,(b\,(A_1+A_2+B))\,\mathrm{sh}\,(a_2\,(A_1+A_2)+bA_2+X)\,.$$

Before re-defining terms $(i)\,,(ii)$ and (iii) we factored out $\frac{4\mathrm{sh}(B)}{\mathrm{sh}(A_1+B)\mathrm{sh}(A_2+B)\mathrm{sh}(A_1+A_2+B)}$. Multiplying this factor with the expression we just obtained, we get the result. $\square$

6.4.2 Proof by induction of the sinh formula: heredity

Let us recall the sinh formula before proving it.

Fix n positive integers $a_2,\dots,a_n,b$ and $2n$ formal variables $A_1,\dots,A_n,B,X_2,\dots,X_n$. Fix n more formal variables $t_2,\dots,t_n,u$; by convention, let $t_1=1$. The coefficient of $t_2^{a_2}\dots t_n^{a_n}u^b$ in the formal power series

$$\begin{aligned}&\sum_{i_2,\dots,i_n>0}\prod_{r=2}^{n}\mathrm{sh}\,(i_r\,(A_1+\cdots+A_r)+A_r\,(i_{r+1}+\cdots+i_n)+X_r)\,t_r^{i_r}\\ &\times\left\{\prod_{s=1}^{n}\left(1+4\sum_{j_s>0}\frac{(A_s)\,\mathrm{sh}\,(B)}{\mathrm{sh}\,(A_s+B)}\mathrm{sh}\,(j_s\,(A_s+B))\left(\frac{u}{t_s}\right)^{j_s}\right)-1\right\}\end{aligned}$$

is

$$4\frac{\mathrm{sh}\,(A_1+\cdots+A_n)\,\mathrm{sh}\,(B)}{\mathrm{sh}\,(A_1+\cdots+A_n+B)}\prod_{r=2}^{n}\Big(\mathrm{sh}\,(a_r\,(A_1+\cdots+A_r)+A_r\,(a_{r+1}+\cdots+a_n+b)+X_r)\Big)\mathrm{sh}\,(b\,(A_1+\cdots+A_n+B))\,.$$

Proof. We prove this formula by induction over n. The first step, for $n = 2$, is proved by Proposition 183 in the preceding section.

We now prove the nth step by induction. We can schematically write the formula of the LHS of the proposition as

$$\Phi\left\{\Psi\left(1+\Sigma\right)-1\right\},$$

with

$$\begin{aligned}
\Phi &= \sum_{i_2,\ldots,i_n>0}\prod_{r=2}^{n}\operatorname{sh}\left(i_r\left(A_1+\cdots+A_r\right)+A_r\left(i_{r+1}+\cdots+i_n\right)+X_r\right)t_r^{i_r}\\
\Psi &= \prod_{s=1}^{n-1}\left(1+4\sum_{j_s>0}\frac{\operatorname{sh}\left(A_s\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_s+B\right)}\operatorname{sh}\left(j_s\left(A_s+B\right)\right)\left(\frac{u}{t_s}\right)^{j_s}\right)\\
\Sigma &= 4\sum_{j_n>0}\frac{\operatorname{sh}\left(A_n\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_n+B\right)}\operatorname{sh}\left(j_n\left(A_n+B\right)\right)\left(\frac{u}{t_n}\right)^{j_n}.
\end{aligned}$$

We split the expression in three terms :

$$\Psi\left(1+\Sigma\right)-1=\underbrace{\Sigma}_{term\,1}+\underbrace{\left(\Psi-1\right)}_{term\,2}+\underbrace{\Sigma\left(\Psi-1\right)}_{term\,3}.$$

We now extract the coefficient $t_2^{a_2}\ldots t_n^{a_n}u^b$ in the three terms coming from this development, that is from $\Phi\Psi$, $\Phi\left(\Psi-1\right)$ and $\Phi\Sigma\left(\Psi-1\right)$. In the second and third term, we will need the $(n-1)$th step of the induction to extract this coefficient. Then, we will sum these three coefficients, see this sum as the coefficient of a series and use the first step of the induction to conclude.

Term 1. We extract the coefficient of $t_2^{a_2}\ldots t_n^{a_n}u^b$ in $\Phi\Sigma$. To do this, we remove the summations and substitute $i_2 := a_2,\ldots,i_{n-1}=a_{n-1}, i_n = a_n+b, j_n = b$, we get

$$\begin{aligned}
&\prod_{r=2}^{n}\operatorname{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)\\
&\times 4\frac{\operatorname{sh}\left(A_n\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_n+B\right)}\operatorname{sh}\left(b\left(A_n+B\right)\right).
\end{aligned}$$

For reasons that will become clear later, we move the factor $r = n$ of the product to the second line :

$$\begin{aligned}
&\prod_{r=2}^{n-1}\operatorname{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)\\
&\times\operatorname{sh}\left(\left(a_n+b\right)\left(A_2+\cdots+A_n\right)+X_n\right)\times 4\frac{\operatorname{sh}\left(A_n\right)\operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_n+B\right)}\operatorname{sh}\left(b\left(A_n+B\right)\right).
\end{aligned}$$

Term 2. We want to extract the coefficient of $t_2^{a_2}\dots t_n^{a_n}u^b$ in $\Phi(\Psi-1)$. First we extract the coefficient of $t_n^{a_n}$. We get

$$\sum_{i_2,\dots,i_{n-1}>0}\prod_{r=2}^{n-1}\mathrm{sh}\left(i_r\left(A_1+\cdots+A_r\right)+A_r\left(i_{r+1}+\cdots+i_{n-1}\right)+A_ra_n+X_r\right)t_r^{i_r}\times\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+X_n\right)$$
$$\times\left\{\prod_{s=1}^{n-1}\left(1+4\sum_{j_s>0}\frac{\mathrm{sh}\left(A_s\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_s+B\right)}\mathrm{sh}\left(k_s\left(A_s+B\right)\right)\left(\frac{u}{t_s}\right)^{k_s}\right)-1\right\}$$

and we re-arrange the product as

$$\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+X_n\right)\times\left[\sum_{i_2,\dots,i_{n-1}>0}\prod_{r=2}^{n-1}\mathrm{sh}\left(i_r\left(A_1+\cdots+A_r\right)+A_r\left(i_{r+1}+\cdots+i_{n-1}\right)+A_ra_n+X_r\right)t_r^{i_r}\right.$$
$$\left.\times\left\{\prod_{s=1}^{n-1}\left(1+4\sum_{j_s>0}\frac{\mathrm{sh}\left(A_s\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_s+B\right)}\mathrm{sh}\left(k_s\left(A_s+B\right)\right)\left(\frac{u}{t_s}\right)^{k_s}\right)-1\right\}\right].$$

We then have to extract the coefficient of $t_2^{a_2}...t_{n-1}^{a_{n-1}}u^b$ of the term in squared the bracket. Using the recursion hypothesis on this term with

$$A_1:=A_1,\dots,\,A_{n-1}:=A_{n-1},\,B:=B,\,X_r:=X_r+A_ra_n$$

we get

$$\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+X_n\right)\times\left[4\frac{\mathrm{sh}\left(A_1+\cdots+A_{n-1}\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_{n-1}+B\right)}\right.$$
$$\left.\prod_{r=2}^{n-1}\Big(\mathrm{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)\Big)\mathrm{sh}\left(b\left(A_1+\cdots+A_{n-1}+B\right)\right)\right].$$

Finally, we re-arrange the product as

$$\prod_{r=2}^{n-1}\mathrm{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)$$
$$\times4\frac{\mathrm{sh}\left(A_1+\cdots+A_{n-1}\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_{n-1}+B\right)}\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+X_n\right)\mathrm{sh}\left(b\left(A_1+\cdots+A_{n-1}+B\right)\right).$$

Term 3. We want to extract the coefficient of $t_2^{a_2}...t_n^{a_n}u^b$ in $\Phi\Sigma(\Psi-1)$. Here we start by re-arranging the product as follows :

$$\left(4\sum_{j_n>0}\frac{\mathrm{sh}\left(A_n\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_n+B\right)}\mathrm{sh}\left(j_n\left(A_n+B\right)\right)\left(\frac{u}{t_n}\right)^{j_n}\right)\left(\sum_{i_n>0}\mathrm{sh}\left(i_n\left(A_1+\cdots+A_n\right)+X_n\right)t_n^{i_n}\right)$$
$$\left[\left(\sum_{i_2,\dots,i_{n-1}>0}\prod_{r=2}^{n-1}\mathrm{sh}\left(i_r\left(A_1+\cdots+A_r\right)+A_r\left(i_{r+1}+\cdots+i_{n-1}\right)+A_ri_n+X_r\right)t_r^{i_r}\right)\right.$$
$$\left.\left\{\prod_{s=1}^{n-1}\left(1+4\sum_{k_s>0}\frac{\mathrm{sh}\left(A_s\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_s+B\right)}\mathrm{sh}\left(k_s\left(A_s+B\right)\right)\left(\frac{u}{t_s}\right)^{k_s}\right)-1\right\}\right].$$

Now we extract the coefficient of $t_n^{a_n} u^b$. Note that t_n is only present in the first line of the previous expression. In Σ, $1/t_n$ appears with the same exponent as u, that is j_n. We then extract the coefficient of $t_2^{a_2} \dots t_{n-1}^{a_{n-1}} u^{b-j_n}$ from the expression in the square brackets. This is done using the recursion hypothesis with

$$A_1 := A_1, \dots, A_{n-1} := A_{n-1},\ B := B,\ X_r := X_r + A_r \left(a_n - j_n\right),$$

we get

$$\begin{aligned}&\sum_{j_n=0}^{b} \left(4 \frac{\operatorname{sh}\left(A_n\right) \operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_n + B\right)} \operatorname{sh}\left(j_n \left(A_n + B\right)\right)\right) \left(\sum_{i_n > 0} \operatorname{sh}\left(\left(a_n + j_n\right)\left(A_1 + \cdots + A_n\right) + X_n\right)\right) \\ &\left[4 \frac{\operatorname{sh}\left(A_1 + \cdots + A_{n-1}\right) \operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_1 + \cdots + A_{n-1} + B\right)}\right. \\ &\left.\prod_{r=2}^{n-1} \left(\operatorname{sh}\left(a_r \left(A_1 + \cdots + A_r\right) + A_r \left(a_{r+1} + \cdots + a_n + b\right) + X_r\right)\right) \operatorname{sh}\left(\left(b - j_n\right)\left(A_1 + \cdots + A_{n-1} + B\right)\right)\right]\end{aligned}$$

Again, we re-arrange the product

$$\begin{aligned}&\prod_{r=2}^{n-1} \operatorname{sh}\left(a_r \left(A_1 + \cdots + A_r\right) + A_r \left(a_{r+1} + \cdots + a_n + b\right) + X_r\right) \\ &\left(16 \frac{\operatorname{sh}\left(A_1 + \cdots + A_{n-1}\right) \operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_1 + \cdots + A_{n-1} + B\right)} \frac{\operatorname{sh}\left(A_n\right) \operatorname{sh}\left(B\right)}{\operatorname{sh}\left(A_n + B\right)}\right. \\ &\left.\sum_{j_n=0}^{b} \operatorname{sh}\left(j_n \left(A_n + B\right)\right) \operatorname{sh}\left(\left(a_n + j_n\right)\left(A_1 + \cdots + A_n\right) + X_n\right) \operatorname{sh}\left(\left(b - j_n\right)\left(A_1 + \cdots + A_{n-1} + B\right)\right)\right).\end{aligned}$$

We now combine terms 1, 2 and 3. We factor out $\prod_{r=2}^{n-1} \mathrm{sh}\left(a_r\left(A_1+\cdots+A_r\right)+A_r\left(a_{r+1}+\cdots+a_n+b\right)+X_r\right)$ in these three terms. We present the rest as the coefficient of a formal series in x and y as follows

$$\begin{aligned}
&\left(\mathrm{sh}\left(\left(a_n+b\right)\left(A_2+\cdots+A_n\right)+X_n\right)\times 4\frac{\mathrm{sh}\left(A_n\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_n+B\right)}\mathrm{sh}\left(b\left(A_n+B\right)\right)\right)\\
+&\left(4\frac{\mathrm{sh}\left(A_1+\cdots+A_{n-1}\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_{n-1}+B\right)}\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+X_n\right)\mathrm{sh}\left(b\left(A_1+\cdots+A_{n-1}+B\right)\right)\right)\\
+&\left(16\frac{\mathrm{sh}\left(A_1+\cdots+A_{n-1}\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_{n-1}+B\right)}\frac{\mathrm{sh}\left(A_n\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_n+B\right)}\right.\\
&\left.\sum_{j_n=0}^{b}\mathrm{sh}\left(j_n\left(A_n+B\right)\right)\mathrm{sh}\left(\left(a_n+j_n\right)\left(A_1+\cdots+A_n\right)+X_n\right)\mathrm{sh}\left(\left(b-j_n\right)\left(A_1+\cdots+A_{n-1}+B\right)\right)\right)\\
=&\\
&\left[x^{a_n}y^b\right]\sum_{i>0}\mathrm{sh}\left(i\left(A_1+\cdots+A_n\right)+X_n\right)x^i\\
&\left\{\left(1+4\sum_{j>0}\frac{\mathrm{sh}\left(A_1+\cdots+A_{n-1}\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_{n-1}+B\right)}\mathrm{sh}\left(j\left(A_1+\cdots+A_{n-1}+B\right)\right)y^j\right)\right.\\
&\left.\left(1+4\sum_{k>0}\frac{\mathrm{sh}\left(A_n\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_n+B\right)}\mathrm{sh}\left(k\left(A_n+B\right)\right)\left(\frac{y}{x}\right)^k\right)-1\right\}.
\end{aligned}$$

We recognize the recursion hypothesis for $n=2$ with $t_1:=x$, $t_2:=y$, $A_1:=a_n$, $A_2:=b$ and

$$A_1:=A_1+\cdots+A_{n-1},\ A_2:=A_n,\ B:=B,\ X_2:=X_n.$$

Thus the expression above simplifies to

$$4\frac{\mathrm{sh}\left(A_1+\cdots+A_n\right)\mathrm{sh}\left(B\right)}{\mathrm{sh}\left(A_1+\cdots+A_n+B\right)}\mathrm{sh}\left(a_n\left(A_1+\cdots+A_n\right)+A_nb+X_n\right)\mathrm{sh}\left(b\left(A_1+\cdots+A_n+B\right)\right).$$

This ends the proof. $\square$

6.5 Proof of the level structure of the correlators

In this section we prove Proposition 158 that is the vanishing of the correlator $\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l}$ if $\sum d_i > 4g-3+n-l$ of if $\sum d_i$ has the parity of $n-l$. In Section 6.5.1, we explain why it is sufficient to prove these vanishings when the correlator has a τ_0 insertion. The correlator $\langle\tau_0\tau_{d_1}\dots\tau_{d_n}\rangle_{l,g-l}$ is expressed in term of Ehrhart polynomials that we study in Section 6.5.2. We then deduce the proof of Proposition 158 in Section 6.5.3.

6.5.1 String equation

We can rewrite the string equation (Theorem 2) as the following infinite system of equations

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} = \sum_{i=1}^{n} \langle \tau_{d_1} \dots \tau_{d_i - 1} \dots \tau_{d_n} \rangle_{l,g-l},$$

where $g, l, d_1, \dots, d_n \geq 0$ and such that a correlator vanishes if τ has a negative index. In these equations, the quantity defined by the sum of the indices of τ minus the number of τ insertions does not depend on the correlator and is equal to $\sum_{i=1}^{n} d_i - n - 1$. Moreover, the correlators of the LHS and RHS depend on the same indices l and $g-l$. We use this system of equations to express any correlator $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ as a sum of correlators with a τ_0 insertion. Our two remarks are still valid: $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ is expressed as a sum of correlators with a τ_0 insertion such that each correlator is indexed by l and $g-l$ and the quantity defined by the sum of the indices of τ minus the number of τ insertions does not depend of the correlator and is equal to $\sum d_i - n$ (see Section 6.3.1.1 to solve explicitly this system using of generating series). It is then sufficient to prove that the correlators $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle$ vanish if $\sum d_i > 4g - 2 + n - l$ or if $\sum d_i$ has the parity of $n - l + 1$ in order to prove Proposition 158.

6.5.2 Properties of the Ehrhart polynomials of Buryak and Rossi

The vanishing of the correlators come from the properties the following Ehrhart polynomials.

Lemma 184 ([BR16])**.** *Fix a list of q positive integers $(r_1, \dots, r_q)$. The function*

$$C^{r_1,\dots,r_q}(N) = \sum_{k_1+\dots+k_q=N} k_1^{r_1} \dots k_q^{r_q}$$

is a polynomial in N of degree $q - 1 + \sum r_i$. Moreover, this polynomial has the parity of $q - 1 + \sum r_i$.

We then deduce the following lemma.

Lemma 185. *Let $P(k_1, \dots, k_q) \in \mathbb{C}[k_1, \dots, k_n]$ be an even (resp. odd) polynomial. Then*

$$\sum_{k_1+\dots+k_q=N} k_1 \dots k_q P(k_1, \dots, k_q)$$

is an odd (resp. even) polynomial in the indeterminate N of degree $2q - 1 + \deg P$.

By induction, we obtain the following lemma.

Lemma 186. *Fix an integer $n \geq 2$ and a list $A_2, \dots, A_n$ of nonnegative integers. Let $\mathcal{C}$ be the set of pairs (2-element subsets) of $\{1, \dots, n\}$. Fix another list of nonnegative integers $(q_I,\, I \in \mathcal{C})$. Let $P\left(k_i^I, I \in \mathcal{C}, 1 \leq i \leq q_I\right)$ be* *an even (resp. odd) polynomial in the inderterminates k_i^I, where $I \in \mathcal{C}$ and $1 \leq i \leq q_I$. Then*

$$\begin{aligned}
&\sum_{\sum_{i=1}^{n-1} K^{\{i,n\}}=A_n} \prod_{I\in\mathcal{C}_n} k_1^I \dots k_{q_I}^I \\
&\times \sum_{\sum_{i=1}^{n-2} K^{\{i,n-1\}}=A_{n-1}+K^{\{n-1,n\}}} \prod_{I\in\mathcal{C}_{n-1}\setminus\mathcal{C}_n} k_1^I \dots k_{q_I}^I \\
&\times \dots \\
&\times \sum_{K^{\{1,2\}}=A_2+\sum_{j=3}^n K^{\{2,j\}}} k_1^{\{1,2\}} \dots k_{q_{\{1,2\}}}^{\{1,2\}} \\
&\times P\left(k_i^I, I \in \mathcal{C}, 1 \leq i \leq q_I\right),
\end{aligned}$$

is a polynomial in the indeterminates $A_2, \dots, A_n$ with the parity of $\deg P - (n-1)$ *(resp.* $\deg P - n$*) of degree* $2\sum_{I\in\mathcal{C}} q_I - (n-1) + \deg P$*. We used the notation $K^I = \sum_{i=1}^{q_I} k_i^I$.*

6.5.3 Proof of the level structure

We now prove Proposition 158. We first obtain an expression of the correlators $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ in term of the polynomials of Lemma 186. The level structure follows from the vanishing properties of these polynomials.

Proposition 187. *Fix three nonnegative integers n, g, l and a list $(d_1, \dots, d_n)$ of nonnegative integers. We have*

$$\begin{aligned}
&\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} \\
&= \sqrt{-1}^{\,n-2-3l-\sum d_i} \sum_{\substack{g_1+\dots+g_n+\sum_{I\in\mathcal{C}} q_I = g+n-1 \\ \text{with conditions } \alpha \text{ and } \gamma}} \sum_{\substack{l_1+\dots+l_n=l \\ l_i \leq g_i}} \left[a_1^1 \dots a_{m_1}^1 \dots a_1^n \dots a_{m_n}^n\right] \\
&\times \sum_{\substack{k_i^I>0,\, I\in\mathcal{C},\, 1\leq i\leq q_I \\ \text{with conditions } \beta}} \prod_{I\in\mathcal{C}} \frac{1}{q_I!} k_1^I \dots k_{q_I}^I \\
&\times \frac{1}{m_1!} P_{d_1-1,g_1,l_1}\left(a_1^1, \dots, a_{m_1}^1, k_1^{\{1,2\}}, \dots, k_{q_{\{1,2\}}}^{\{1,2\}}, \dots, k_1^{\{1,n\}}, \dots, k_{q_{\{1,n\}}}^{\{1,n\}}, -\sum A_i\right) \\
&\times \prod_{i=2}^n \frac{1}{m_i!} P_{d_i,g_i,l_i}\left(a_1^i, \dots, a_{m_1}^i, -k_1^{\{1,i\}}, \dots, -k_{q_{\{1,i\}}}^{\{1,i\}}, \dots, k_1^{\{i,n\}}, \dots, k_{q_{\{i,n\}}}^{\{i,n\}}, 0\right),
\end{aligned} \tag{6.5.1}$$

where

- *$\mathcal{C}$ is the set of pairs (2-element subsets) of $\{1, \dots, n\}$; we also denote by $\mathcal{C}_i \subset \mathcal{C}$ the subset of pairs that contain i,*
- *the conditions α are*

  $$g_1 + (g_1 - l_1) + m_1 + \sum_{I\in\mathcal{C}_1} q_I = d_1 + 1$$

  and

  $$g_i + (g_i - l_i) + m_i + \sum_{I\in\mathcal{C}_i} q_I = d_i + 2, \text{ when } i \geq 2,$$

- *the conditions β are*

$$A_i - \sum_{j=1}^{i-1} K^{\{j,i\}} + \sum_{j=i+1}^{n} K^{\{i,j\}} = 0, \quad 2 \leq i \leq n,$$

where $K^I = \sum_{i=1}^{q_I} k_i^I$*,*

- *the conditions γ are*

$$\sum_{j=1}^{i-1} q_{\{i,j\}} \geq 1, \text{ for } 2 \leq i \leq n,$$

- *the polynomial $P_{d,g,l}(x_1, \ldots, x_{m+1})$, with $d, g, l, m \geq 0$, is of degree $2g$ and defined by*

$$P_{d,g,l}(x_1, \ldots, x_{m+1}) = \int_{\mathrm{DR}_g(0, x_1, \ldots, x_{m+1})} \lambda_l \psi_0^{d+1},$$

where $\sum_{i=1}^{m+1} x_i = 0$ and the ψ-class sits on the marked point with weight 0 of the double ramification cycle.

Proof. To obtain this formula we use the definition of the correlators

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{l,g-l} = \left[\epsilon^{2l} \hbar^{g-l}\right] \frac{\sqrt{-1}^{g-l}}{\hbar^{n-1}} \left[\ldots \left[H_{d_1 - 1}, \overline{H}_{d_2}\right], \ldots, \overline{H}_{d_n}\right] \Big|_{u_i = \delta_{i,1}}$$

and proceed as in Section 6.3.2.2.

We first need an expression of $H_{d_1} \star \overline{H}_{d_2} \star \cdots \star \overline{H}_{d_n}$. Start from the following expression of the Hamiltonians

$$H_{d_i} = \sum_{g_i, m_i, l_i \geq 0} \frac{\epsilon^{2l_i} \left(\sqrt{-1}\hbar\right)^{g_i - l_i}}{m_i!} \sum_{a_1^i, \ldots, a_{m_i}^i \in \mathbb{Z}} P_{d_i, g_i, l_i}\left(a_1^i, \ldots, a_{m_i}^i, -\sum_{j=1}^{m_i} a_j^i\right) p_{a_1^i} \ldots p_{a_{m_i}^i} e^{x\sqrt{-1}\sum_{j=1}^{m_i} a_j^i}, \tag{6.5.2}$$

where the first summation satisfies $g_i + (g_i - l_i) + m_i = d_i + 2$ and $l_i \leq g_i$. We proceed as in the proof of Proposition 173: we use the associativity of the star product and its developed expression to obtain Eq. (6.3.17). We then plug the expression of the Hamiltonian given by Eq (6.5.2) in Eq. (6.3.17). When the $\sum_{j=1, j\neq i}^{n} q_{\{i,j\}}$ derivatives act on $\sum_{a_1^i + \cdots + a_{m_i}^i = 0} P_{d_i, g_i, l_i}\left(a_1^i, \ldots, a_{m_i}^i, 0\right) p_{a_1^i} \ldots p_{a_{m_i}^i}$ in $\overline{H}_{d_i}$, it remains $\tilde{m}_i := m_i - \sum_{j=1, j\neq i}^{n} q_{\{i,j\}}$ variables p. Similarly, it remains $\tilde{m}_1 := m_1 - \sum_{j=2}^{n} q_{\{1,j\}}$ variables p when the derivatives act on $H_{d_1 - 1}$. We then obtain

$$\begin{aligned}
&H_{d_1} \star \overline{H}_{d_2} \star \cdots \star \overline{H}_{d_n} \\
&= \sum_{q_I \geq 0,\, I \in \mathcal{C}} \; \sum_{g_1, \ldots, g_n \geq 0} \; \sum_{\substack{\tilde{m}_1, \ldots, \tilde{m}_n \geq 0 \\ \text{with conditions } \alpha}} \; \sum_{l_1, \ldots, l_n \geq 0} \; \sum_{\substack{k_i^I > 0,\, I \in \mathcal{C},\, 1 \leq i \leq q_I \\ \text{with conditions } \beta}} \\
&\times \prod_{I \in \mathcal{C}} \frac{\left(\sqrt{-1}\hbar\right)^{q_I}}{q_I!} k_1^I \ldots k_{q_I}^I \\
&\times \frac{\epsilon^{2l_1} \left(\sqrt{-1}\hbar\right)^{g_1 - l_1}}{\tilde{m}_1!} P_{d_1 - 1, g_1, l_1}\left(a_1^1, \ldots, a_{\tilde{m}_1}^1, k_1^{\{1,2\}}, \ldots, k_{q_{\{1,2\}}}^{\{1,2\}}, \ldots, k_1^{\{1,n\}}, \ldots, k_{q_{\{1,n\}}}^{\{1,n\}}, -\sum_{i=1}^{n} \tilde{A}_i\right) p_{a_1^1} \ldots p_{a_{\tilde{m}_1}^1} e^{x\sqrt{-1}\sum_{i=1}^{n} \tilde{A}_i} \\
&\times \prod_{i=2}^{n} \frac{\epsilon^{2l_i} \left(\sqrt{-1}\hbar\right)^{g_i - l_i}}{\tilde{m}_i!} P_{d_i, g_i, l_i}\left(a_1^i, \ldots, a_{\tilde{m}_1}^i, -k_1^{\{1,i\}}, \ldots, -k_{q_{\{1,i\}}}^{\{1,i\}}, \ldots, k_1^{\{i,n\}}, \ldots, k_{q_{\{i,n\}}}^{\{i,n\}}, 0\right) p_{a_1^i} \ldots p_{a_{\tilde{m}_i}^i},
\end{aligned} \tag{6.5.3}$$

where

- $\tilde{A}_i = \sum_{j=1}^{\tilde{m}_i} a^i_j$, with $1 \le i \le n$,
- the conditions α are
$$g_1 + (g_1 - l_1) + \tilde{m}_1 + \sum_{I \in \mathcal{C}_1} q_I = d_1 + 1$$
  and
$$g_i + (g_i - l_i) + \tilde{m}_i + \sum_{I \in \mathcal{C}_i} q_I = d_i + 2, \text{ when } i \ge 2,$$
- the conditions β are
$$\tilde{A}_i - \sum_{j=1}^{i-1} K^{\{j,i\}} + \sum_{j=i+1}^{n} K^{\{i,j\}} = 0, \quad 2 \le i \le n.$$

We now modify the notations by removing the tildes, i.e. we set $m_i := \tilde{m}_i$ and $A_i := \tilde{A}_i$ for any $1 \le i \le n$.

In Step 1.2 of Section 6.3.2.2 we explained why such an expression of $H_{d_1} \star \overline{H}_{d_2} \star \cdots \star \overline{H}_{d_n}$ is enough to get an expression for $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$. We also explained that the commutators offer some simplifications given by the conditions γ.

We now proceed as in Step 2 of Section 6.3.2.2: we extract the coefficient of $\epsilon^{2l}\hbar^{g-l}$ in the expression of $\frac{\sqrt{-1}^{g-l}}{\hbar^{n-1}} H_{d_1} \star \overline{H}_{d_2} \star \cdots \star \overline{H}_{d_n}$ given by Eq. (6.5.3), then we use the Lemma 166 to substitute $u_i = \delta_{i,1}$ in this coefficient. We get Eq. (6.5.1). □

We now prove that the correlator $\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ vanishes when $\sum d_i > 4g - 2 + n - l$ or when $\sum d_i$ has the parity of $n + 1 - l$. According to Section 6.5.1, this proves Proposition 158.

Proof of Proposition 158. The three last lines of Eq. (6.5.1) form a polynomial the indeterminates $a^1_1, \dots, a^1_{m_1}, \dots, a^n_1, \dots, a^n_{m_n}$ depending of the set of parameters $\mathcal{S} = \{d_i, g_i, l_i, m_i | 1 \le i \le n\}$, we denote this polynomial by $Q_{\mathcal{S}}$. The parity and the degree of this polynomial is described by Lemma (186). Since the polynomial $P_{d,g,l}$ is even and of degree $2g$, we deduce that $Q_{\mathcal{S}}$ is a polynomial of degree
$$2 \sum_{I \in \mathcal{C}} q_I - (n-1) + \sum_{i=1}^{n} 2g_i = 2g + n - 1$$
and has the parity of $n - 1$. We used the constraint $\sum_{I \in \mathcal{C}} q_I + \sum_{i=1}^{n} g_i = g + n - 1$ of the first summation in formula (6.5.1) to obtain this equality.

We then extract the coefficient of $a^1_1 \dots a^1_{m_1} \dots a^n_1 \dots a^n_{m_n}$ in $\mathcal{Q}_S$. This coefficient vanishes if
$$\sum_{i=1}^{n} m_i > 2g + n - 1 \tag{6.5.4}$$
or if $\sum_{i=1}^{n} m_i$ has the parity of n. However conditions α give $\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} d_i - 2g + l + 1$. Hence, this coefficient vanishes if
$$\sum d_i > 4g - 2 + n - l$$
or if $\sum d_i$ has the parity of $n - l + 1$. This proves the proposition. □

6.6 Proof of the geometric formula

According to Lemma 121, we have

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} = i^{g-l} \left[\epsilon^l \hbar^{g-l+n-1} \right] \left[\dots \left[\Omega^{\hbar}_{0,d_1}, \overline{H}_{d_2} \right], \dots, H_{d_n} \right] \Big|_{u_i = \delta_{1,i}},$$

and we justified that $\Omega^{\hbar}_{0,d} = H_{d-1}$ (see Lemma 167). Moreover, Lemma 166 explains how to extract the coefficient $u_i = \delta_{i,1}$ from a differential polynomial in the p variables. We justify that

$$\frac{1}{\hbar^{n-1}} \left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right] =$$
$$\sum_{m \geq 0} \sum_{g \geq 0} \sum_{k=0}^{g} \sum_{a_1,\dots,a_m} \sum_{\Gamma \in G(k,g,m,n)}$$
$$\left(\frac{-\epsilon^2}{i\hbar} \right)^k \frac{(i\hbar)^g}{m!} i^{n-1} \int_{\overline{\mathcal{M}}_{g,m+n+1}} (\lambda \cdot DR)_\Gamma \left(-\sum a_i, a_1, \dots, a_m \right) \psi^{d_1}_{m+1} \dots \psi^{d_n}_{m+n} \; p_{a_1} \dots p_{a_m} e^{ix \sum a_i}.$$

In view of the previous proofs, this formula should be pretty clear. We briefly justify this formula by induction over n. The expression for $n = 1$ is clear. By induction, suppose we add a $(n+1)$th Hamiltonian. The term

$$\left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right] \star \overline{H}_{d_{n+1}}$$

corresponds to all the graphs such that the vertex v indexed by $p(v) = n+1$ is equipped with a $+$ sign (resp. a $-$ sign for the other term $\overline{H}_{d_{n+1}} \star \left[\dots \left[H_{d_1-1}, \overline{H}_{d_2} \right], \dots, \overline{H}_{d_n} \right]$). Since the commutative term of the commutator vanishes, one can see that this vertex is linked with at least one smaller vertex.

We then use the evaluation lemma (Lemma 166) to obtain

$$\langle \tau_0 \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} =$$
$$\sum_{m \geq 0} [a_1 \dots a_m] \sum_{\Gamma \in G(m,n,g,l,a_0 = -\sum a_i, a_1, \dots, a_n)} \frac{i^{2g+n-1-m}}{m!} \int_{\overline{\mathcal{M}}_{g,m+n+1}} (\lambda \cdot \mathrm{DR})_\Gamma \left(-\sum a_i, a_1, \dots, a_m \right) \psi^{d_1}_{m+1} \dots \psi^{d_n}_{m+n} \,.$$

Then, since $(\lambda \cdot \mathrm{DR})_\Gamma \left(-\sum a_i, a_1, \dots, a_m \right) \in H^{2(l+g+n-1)} \left(\overline{\mathcal{M}}_{g,n+m+1} \right)$, the dimensional condition of the integral (the degree of the integrand is equal to the dimension of the space) yields

$$m = \sum_{i=1}^{n} d_i + l - 2g + 1.$$

Bibliography

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip Griffiths. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*, volume 268. Springer Science & Business Media, 2011.

[Arn13] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.

[BDGR16] Alexandr Buryak, Boris Dubrovin, Jérémy Guéré, and Paolo Rossi. Integrable systems of double ramification type. *arXiv preprint arXiv:1609.04059*, 2016.

[BDGR18] Alexandr Buryak, Boris Dubrovin, Jérémy Guéré, and Paolo Rossi. Tau-structure for the double ramification hierarchies. *Communications in Mathematical Physics*, 363(1):191–260, 2018.

[BGR19] Alexandr Buryak, Jérémy Guéré, and Paolo Rossi. Dr/dz equivalence conjecture and tautological relations. *Geometry & Topology*, 23(7):3537–3600, 2019.

[BHP$^+$20] Bae, Holmes, Pandharipande, Schmitt, and Schwarz. Pixton's formula and abel-jacobi theory on the picard stack. 2020.

[BPS12a] Alexandr Buryak, Hessel Posthuma, and Sergey Shadrin. On deformations of quasi-miura transformations and the dubrovin–zhang bracket. *Journal of Geometry and Physics*, 62(7):1639–1651, 2012.

[BPS$^+$12b] Alexandr Buryak, Hessel Posthuma, Sergey Shadrin, et al. A polynomial bracket for the dubrovin-zhang hierarchies. *Journal of Differential Geometry*, 92(1):153–185, 2012.

[BR16] Alexandr Buryak and Paolo Rossi. Double ramification cycles and quantum integrable systems. *Letters in mathematical physics*, 106(3):289–317, 2016.

[BSSZ15] Alexandr Buryak, Sergey Shadrin, Loek Spitz, and Dimitri Zvonkine. Integrals of ψ-classes over double ramification cycles. *American Journal of Mathematics*, 137(3):699–737, 2015.

[Bur15] Alexandr Buryak. Double ramification cycles and integrable hierarchies. *Communications in Mathematical Physics*, 336(3):1085–1107, 2015.

[CM16] Renzo Cavalieri and Eric Miles. *Riemann surfaces and algebraic curves*, volume 87. Cambridge University Press, 2016.

[DJKM82] Etsuro Date, Michio Jimbo, Masaki Kashiwara, and Tetsuji Miwa. Transformation groups for soliton equations—euclidean lie algebras and reduction of the kp hierarchy—. *Publications of the Research Institute for Mathematical Sciences*, 18(3):1077–1110, 1982.

[DL20] Norman Do and Danilo Lewański. On the goulden-jackson-vakil conjecture for double hurwitz numbers. *arXiv preprint arXiv:2003.08043*, 2020.

[DVV93] Robbert Dijkgraaf, Herman Verlinde, and Erik Verlinde. Loop equations and virasoro constraints in non-perturbative two-dimensional quantum gravity. In *The Large N Expansion In Quantum Field Theory And Statistical Physics: From Spin Systems to 2-Dimensional Gravity*, pages 842–863. World Scientific, 1993.

[DZ05] Boris Dubrovin and Youjin Zhang. Normal forms of hierarchies of integrable pdes. *Frobenius manifolds and Gromov-Witten invariants, a new*, 2005.

[EHX97] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong. Quantum cohomology and virasoro algebra. *Physics Letters B*, 402(1-2):71–80, 1997.

[ELSV00] Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein. Hurwitz numbers and intersections on moduli spaces of curves. *arXiv preprint math/0004096*, 2000.

[Fab99] Carel Faber. A conjectural description of the tautological ring of the moduli space of curves. In *Moduli of curves and abelian varieties*, pages 109–129. Springer, 1999.

[FP98] Carel Faber and Rahul Pandharipande. Hodge integrals and gromov-witten theory. *arXiv preprint math/9810173*, 1998.

[FP03a] Carel Faber and Rahul Pandharipande. Hodge integrals, partition matrices, and the λ g conjecture. *Annals of mathematics*, pages 97–124, 2003.

[FP03b] Carel Faber and Rahul Pandharipande. Relative maps and tautological classes. *arXiv preprint math/0304485*, 2003.

[FP11] Carel Faber and Rahul Pandharipande. Tautological and non-tautological cohomology of the moduli space of curves. *arXiv preprint arXiv:1101.5489*, 2011.

[Get99] Ezra Getzler. The virasoro conjecture for gromov-witten invariants. *Contemporary Mathematics*, 241:147–176, 1999.

[GJV05] IP Goulden, David M Jackson, and Ravi Vakil. Towards the geometry of double hurwitz numbers. *Advances in Mathematics*, 198(1):43–92, 2005.

[GJV09] Ian P Goulden, David Martin Jackson, and Ravi Vakil. A short proof of the λg-conjecture without gromov-witten theory: Hurwitz theory and the moduli of curves. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2009(637):175–191, 2009.

[GP98] Ezra Getzler and Rahul Pandharipande. Virasoro constraints and the chern classes of the hodge bundle. *Nuclear Physics B*, 530(3):701–714, 1998.

[GP01] Tom Graber and Rahul Pandharipande. Constructions of nontautological classes on moduli spaces of curves. *arXiv preprint math/0104057*, 2001.

[Hai11] Richard Hain. Normal functions and the geometry of moduli spaces of curves. *arXiv preprint arXiv:1102.4031*, 2011.

[JPPZ17] Felix Janda, Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine. Double ramification cycles on the moduli spaces of curves. *Publications mathématiques de l'IHÉS*, 125(1):221–266, 2017.

[Kaz09] Maxim Kazarian. Kp hierarchy for hodge integrals. *Advances in Mathematics*, 221(1):1–21, 2009.

[KL07] Maxim Kazarian and Sergei Lando. An algebro-geometric proof of witten's conjecture. *Journal of the American Mathematical Society*, 20(4):1079–1089, 2007.

[KM94] Maxim Kontsevich and Yu Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164(3):525–562, 1994.

[Kon92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. *Communications in Mathematical Physics*, 147(1):1–23, 1992.

[L+01] Jun Li et al. Stable morphisms to singular schemes and relative stable morphisms. *Journal of Differential Geometry*, 57(3):509–578, 2001.

[Li04] Jun Li. Lecture notes on relative gw-invariants. *preprint*, 2004.

[LM00] Andrey Losev and Yu Manin. New moduli spaces of pointed curves and pencils of flat connections. *arXiv preprint math/0001003*, 2000.

[MJJD00] Tetsuji Miwa, Masaki Jinbo, Michio Jimbo, and E Date. *Solitons: Differential equations, symmetries and infinite dimensional algebras*, volume 135. Cambridge University Press, 2000.

[Mum83] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry*, pages 271–328. Springer, 1983.

[MW13] Steffen Marcus and Jonathan Wise. Stable maps to rational curves and the relative jacobian. *arXiv preprint arXiv:1310.5981*, 2013.

[Pet15] T Kyle Petersen. Eulerian numbers. In *Eulerian Numbers*, pages 3–18. Springer, 2015.

[Pix] A Pixton. On combinatorial properties of the explicit expression for double ramification cycles.

[Pix12] Aaron Pixton. Conjectural relations in the tautological ring of mg, n. *arXiv preprint arXiv:1207.1918*, 2, 2012.

[PPZ13] Rahul Pandharipande, Aaron Pixton, and Dimitri Zvonkine. Relations on mbar_ {g, n} via 3-spin structures. *arXiv preprint arXiv:1303.1043*, 2013.

[PZ] Aaron Pixton and Don Zagier. In preparation.

[PZ18] Rahul Pandharipande and Dimitri Zvonkine. Cohomological field theories with non-tautological classes. *arXiv preprint arXiv:1802.08981*, 2018.

[Sat83] Mikio Sato. Soliton equations as dynamical systems on infinite dimensional grassmann manifold. In *North-Holland Mathematics Studies*, volume 81, pages 259–271. Elsevier, 1983.

[Sha08] Sergey Shadrin. On the structure of goulden-jackson-vakil formula. *arXiv preprint arXiv:0810.0729*, 2008.

[SvZ] Schmitt and van Zelm. https://people.math.ethz.ch/ schmittj/admcycles.html.

[Wit90] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys in differential geometry*, 1(1):243–310, 1990.